On the Cohomology of Generalized Kummer Varieties

I n a u g u r a l - D i s s e r t a t i o n

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Michael Britze

aus Saarbrücken

Köln 2004

Berichterstatter: Prof. Dr. Manfred Lehn
Prof. Dr. Daniel Huybrechts

Tag der mündlichen Prüfung: 12. Februar 2002

KURZZUSAMMENFASSUNG. In dieser Arbeit beleuchten wir die Kohomologie verallgemeinerter Kummerscher Varietäten. Diese sind kompakte irreduzible holomorph symplektische Mannigfaltigkeiten. Auf einer solchen Mannigfaltigkeit X der Dimension $2n$ hat nach einem Ergebnis von Huybrechts ([**17**]) die Hirzebruch-Riemann-Roch Formel für ein Geradenbündel L die spezielle Form $\chi(X,L)=\sum_{k=0}^{n} a_{2k} q_X(c_1(L))^k$ mit universellen, d.h. nur von X abhängigen, Konstanten a_{2k} und der Beauville- Bogomolovschen quadratischen Form q_X auf $\mathrm{H}^2(X,\mathbb{Z})$.

Im ersten Teil dieser Arbeit geben wir im Falle der verallgemeinerten Kummerschen Varietäten eine explizite Hirzebruch-Riemann-Roch Formel an und berechnen damit diese Konstanten.

Im folgenden Kapitel beschäftigen wir uns dann mit der singulären Kohomologie. Durch die Betrachtung lokal konstanter Systemen auf dem Hilbertschema $A^{[n]}$ einer abelschen Fläche A, welche durch eine Galois-Überlagerung $A \times K^{(n-1)}A \to A^{[n]}$ induziert werden, und indem wir die Beschreibung der Produktstruktur auf $\mathrm{H}^*(A^{[n]},\mathbb{C})$ von Lehn und Sorger ([**20**]) verwenden, gelangen wir zu einer Beschreibung der Ringstruktur der Kohomologie verallgemeinerter Kummerscher Varietäten.

In einem abschließenden Abschnitt beschäftigen wir uns schließlich mit dem Orbifoldkohomologiering, der von Fantechi und Göttsche in [**7**] berechnet wurde. Wir korrigieren einen kleinen Fehler, der ihnen bei der Berechnung der Produktstruktur im Falle der verallgemeinerten Kummerschen Varietäten unterlaufen ist, und beweisen die Isomorphie des Orbifoldkohomologieringes mit dem singulären Kohomologiering.

ABSTRACT. In this thesis we are dealing with questions about the cohomology of generalized Kummer varieties. These are irreducible holomorphic symplectic manifolds. According to a theorem of Huybrechts ([**17**]), on such a manifold X of dimension $2n$, the Hirzebruch-Riemann-Roch formula for a line bundle L has the special form $\chi(X,L)=\sum_{k=0}^{n} a_{2k} q_X(L)^k$, where the a_{2k} are universal constants, only depending on X, and q_X denotes the Beauville-Bogomolov quadratic form on $\mathrm{H}^2(X,\mathbb{Z})$.

In the first part of this thesis we give an explicit Hirzebruch-Riemann-Roch formula in the case of generalized Kummer varieties and therewith compute these constants.

In the following sections we consider the singular cohomology of generalized Kummer varieties. By using locally constant systems on the Hilbert scheme $A^{[n]}$ of an abelian surface which are induced by a Galois cover $A \times K^{(n-1)}A \to A^{[n]}$ and the description of the ring structure of $\mathrm{H}^*(A^{[n]},\mathbb{C})$ by Lehn and Sorger ([**20**]), we determine the ring structure of the cohomology of the generalized Kummer varieties.

In a last chapter we deal with the orbifold cohomology which was computed by Fantechi and Göttsche in [**7**]. After the correction of a slight error that occurred in their computation of the orbifold cup product in the case of the generalized Kummer varieties, we prove that the orbifold cohomology ring is in fact isomorphic to the ordinary cohomology ring.

Vorwort

Diese Arbeit ist während meiner Zeit an der Universität zu Köln entstanden. Dieses Vorwort möchte ich nutzen, um mich bei all denen zu bedanken, die zu ihrem Entstehen beigetragen haben.

Zuerst möchte ich Manfred Lehn danken, der mich nach meiner Diplomarbeit in Göttingen von dort nach Köln mitgenommen hat. Er hat immer ein offenes Ohr für meine Fragen gehabt. Weiterhin gilt mein Dank Daniel Huybrechts, der mir sozusagen ein zweiter Doktorvater gewesen ist. Auch er hat sich für die Betreuung dieser Arbeit viel Zeit genommen. Eine bessere Betreuung kann man sich nicht wünschen.

Meinem Bürokollegen Marc Nieper-Wißkirchen schulde ich Dank nicht nur für die schöne Zeit, in der wir uns ein gemeinsames Zimmer im Institut teilten, sondern auch für so manche fruchtbare mathematische oder nichtmathematische Diskussion. Aus der Ferne war Jan-Philipp Hoffmann oft bereit, mit mir mathematische Probleme zu wälzen oder bei der einen oder anderen Flasche Wein zu entspannen; auch das war dieser Arbeit sicherlich von Nutzen.

Ferner danke ich meiner Familie, insbesondere meiner Frau Claudia und meinem Sohn Finn Joshua, sowie allen anderen Freunden, die für den nötigen privaten Rückhalt gesorgt haben.

Köln, im Dezember 2002 — Michael Britze

Nachtrag Januar 2004

Ich möchte an dieser Stelle noch einmal meiner Familie, insbesondere meiner Tochter Liv Carlotta, die im Januar 2003 geboren wurde, meinen Freunden, sowie meinen „beiden Doktorvätern“ Prof. Dr. Manfred Lehn und Prof. Dr. Daniel Huybrechts meinen Dank aussprechen.

Köln, im Januar 2004 — Michael Britze

Einleitung

Verallgemeinerte Kummersche Varietäten wurden von Arnaud Beauville eingeführt. Sie bilden in [**1**], neben den Hilbertschemata von K3-Flächen, die zweite Reihe von Beispielen für irreduzible holomorph symplektische Mannigfaltigkeiten. Es handelt sich dabei um einfach zusammenhängende kompakte Kählermannigfaltigkeiten X, deren Raum der globalen holomorphen Zweiformen $\mathrm{H}^0(X, \Omega_X^2)$ von einer überall nichtausgearteten Form σ erzeugt wird. Diese Mannigfaltigkeiten kann man als höherdimensionale Analoga von K3-Flächen ansehen. So wird in oben genannter Arbeit auch eine quadratische Form

$$q_X : \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

heute Beauville-Bogomolovsche quadratische Form genannt, eingeführt, welche die Schnittpaarung auf der zweiten Kohomologie einer K3-Fläche verallgemeinert. Daniel Huybrechts konnte in [**17**] (vgl. auch [**13**]) unter anderem den folgenden Satz über die Eulercharakteristik eines Geradenbündels L auf einer kompakten irreduziblen symplektischen Mannigfaltigkeit beweisen.

THEOREM (Huybrechts, [**17**]). — *Sei X eine kompakte irreduzible holomorph symplektische Mannigfaltigkeit der Dimension $\dim X = 2n$. Dann gibt es Konstanten a_{2k}, $k = 0, \ldots n$, welche nur von X abhängen, derart, dass für jedes Geradenbündel $L \in \mathrm{Pic}(X)$ die Eulercharakteristik durch das Polynom*

$$\chi(X, L) = \sum_{k=0}^{n} a_{2k} q_X(c_1(L))^k$$

gegeben ist, wobei q_X die Beauville-Bogomolovsche quadratische Form bezeichnet.

In [**6**] wird eine explizite Formel für die Eulercharakteristik $\chi(X^{[n]}, L)$ eines Geradenbündels L auf dem Hilbertschema $X^{[n]}$ von n Punkten auf einer K3-Fläche angegeben und somit für die erste Beispielklasse die Koeffizienten in Huybrechts' Theorem bestimmt.

Im ersten Teil der vorliegenden Arbeit, in Kapitel 2, lösen wir dieses Problem im Falle der verallgemeinerten Kummerschen Varietäten. Das Ergebnis ist die folgende Hirzebruch-Riemann-Roch Formel:

THEOREM 6. — *Es sei L ein holomorphes Geradenbündel auf der verallgemeinerten Kummerschen Varietät $K^{(n-1)}A$ der Dimension $2(n-1)$. Weiterhin sei q die Beauville-Bogomolovsche quadratische Form. Dann ist die Eulercharakteristik $\chi(K^{(n-1)}A, L)$ durch das folgende Polynom in $q(c_1(L))$ gegeben:*

$$\chi(K^{(n-1)}A, L) = n\binom{\frac{1}{2}q(c_1(L)) + n - 1}{n-1}.$$

In den darauf folgenden Kapiteln wenden wir uns der gewöhnlichen, d.h. singulären, Kohomologie zu.

In [**12**] benutzen Lothar Göttsche und Wolfgang Soergel schnittkohomologische Methoden, um die Vektorraumstruktur der Kohomologie der Hilbertschemata von Punkten auf Flächen und der verallgemeinerten Kummerschen Varietäten zu bestimmen. (Die Bettizahlen der Hilbertschemata waren schon von Göttsche in [**11**] berechnet worden.)

Aufbauend auf Ergebnisse von Hiraku Nakajima, der in [**23**] auf geometrische Weise eine darstellungstheoretische Deutung der Kohomologie der Hilbertschemata von Flächen gibt, und Manfred Lehn, der auf diesem Raum die zusätzliche Operation der Virasoroalgebra gefunden und geometrisch gedeutet hat ([**19**]), gelang es Manfred Lehn und Christoph Sorger in [**21**] und [**20**] die Ringstruktur der Kohomologie der Hilbertschemata im Falle von $\mathbb{C}^2$, einer K3-Fläche oder eines komplexen Torus' zu bestimmen.

Auf diese Beschreibung aufbauend berechnen wir in Kapitel 3 die Ringstruktur im Falle der verallgemeinerten Kummerschen Varietäten. Wir betrachten dazu lokal konstante Systeme auf dem Hilbertschema $A^{[n]}$, welches durch $A \times K^{(n-1)}A$ überlagert wird. Dies liefert zunächst eine abstrakte Beschreibung der Ringstruktur von $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ in Termen der Kohomologie von $A^{[n]}$ mit Werten in diesen lokal konstanten Systemen. Wir verallgemeinern im Folgenden Nakajimas Operatorbeschreibung auf die Kohomologie der Hilbertschemata $A^{[n]}$ mit Werten in einem lokal konstanten System. Wir geben weiterhin eine konkrete geometrische Beschreibung davon, wie diese Operatoren die Kohomologie des Hilbertschemas erzeugen, so dass wir eine explizite geometrische Beschreibung der Kohomologie des Hilbertschemas $A^{[n]}$ mit Werten in lokal konstanten Systemen erhalten. Durch diese Beschreibung gelingt es uns in Theorem 32 die Struktur des cup-Produktes auf den verallgemeinerten Kummerschen Varietäten auf die bekannte Ringstruktur des Hilbertschemas $A^{[n]}$ zurückzuführen.

Im abschließenden Kapitel 4 vergleichen wir unser Ergebnis mit dem Orbifoldkohomologiering der glatten Quotienten-Orbifold $[A \times (A_0^n)/\mathfrak{S}_n]$, wobei A_0^n die Menge der $(a_i) \in A^n$, deren Summe $\sum a_i = 0$ ist, bezeichnet. Dieser Ring wurde von Barbara Fantechi und Lothar Göttsche in [**7**] berechnet. Sie formulieren dort die Vermutung, dass dieser dadurch beschriebene Orbifold-Kohomologiering von

$[A_0^n/\mathfrak{S}_n]$ mit dem singulären Kohomologiering der verallgemeinerten Kummerschen Varietät $K^{(n-1)}A$, welche die Singularitäten von $A_0^n/\mathfrak{S}_n$ auflöst, übereinstimme.

Zunächst ergänzen wir die Produktformel von Fantechi und Göttsche im Falle der verallgemeinerten Kummerschen Varietäten um einen von ihnen übersehenen Faktor. Nach dieser Berechnung des Orbifold-cup-Produktes beweisen wir, dass der Orbifoldkohomologiering zum gewöhnlichen Kohomologiering isomorph ist.

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CHAPTER 1

Introduction

Generalized Kummer varieties were introduced by Arnaud Beauville. In [**1**], they form — beside the Hilbert schemes of K3 surfaces — the second series of examples of so called compact irreducible holomorphic symplectic manifolds. These are simply connected compact Kähler manifolds X, such that the space of global holomorphic two-forms $\mathrm{H}^0(X, \Omega^2_X)$ is generated by an everywhere non-degenerate form σ. These manifolds should be seen as higher dimensional analogs of K3 surfaces. Accordingly, in the paper cited above, a quadratic form

$$q_X \colon \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

now known as the Beauville-Bogomolov quadratic form, is introduced. It generalizes the intersection pairing on the second cohomology of a K3 surface.
Daniel Huybrechts showed in [**17**] (cf. also [**13**]) the following theorem about the Euler characteristic of a holomorphic line bundle L on a compact irreducible symplectic manifold X:

THEOREM (Huybrechts, [**17**]). — *Let X be a compact irreducible holomorphic symplectic manifold of dimension $\dim X = 2n$. Then, there are constants a_{2k}, $k = 0, \dots, n$, only depending on X, such that for every line bundle $L \in \mathrm{Pic}(X)$ the Euler characteristic is given by the following polynomial:*

$$\chi(X, L) = \sum_{k=0}^{n} a_{2k} q_X(c_1(L))^k,$$

where q_X denotes the Beauville-Bogomolov quadratic form.

In [**6**], an explicit formula for the Euler characteristic $\chi(X^{[n]}, L)$ of a line bundle L on the Hilbert scheme $X^{[n]}$ of n points on a K3 surface is given.
In Chapter 2 of this thesis, we solve the problem of computing these coefficients for the generalized Kummer varieties. The result is the following Hirzebruch-Riemann-Roch formula:

THEOREM 6. — *Let L be a line bundle on the generalized Kummer variety $K^{(n-1)}A$ of dimension $2(n-1)$. Then, the Euler characteristic of L is given by*

$$\chi(K^{(n-1)}A, L) = n \binom{\frac{1}{2} q(c_1(L)) + n - 1}{n-1},$$

where q is the Beauville-Bogomolov quadratic form on $\mathrm{H}^2(K^{(n-1)}A, \mathbb{Z})$.

In the following chapter of this thesis, we consider the ordinary, i.e. singular, cohomology of generalized Kummer varieties.
In [**12**], Lothar Göttsche and Wolfgang Soergel use methods from intersection cohomology to determine the structure of the cohomology of the Hilbert schemes and generalized Kummer varieties as a vector space. (The Betti numbers had been computed by Göttsche in [**11**] already.)
Using the results of Hiraku Nakajima, who gave a representation theoretic interpretation of the cohomology in [**23**], and Manfred Lehn, who found an operation of the Virasoro algebra on this space and gave an geometric interpretation of it in [**19**], Manfred Lehn and Christoph Sorger succeeded in [**21**] and [**20**] in determining the ring structure of the cohomology of the Hilbert scheme in the cases of the affine plane $\mathbb{C}^2$, K3 surfaces and complex tori.
Using this description, in Chapter 3 we compute the ring structure of the cohomology of generalized Kummer varieties. We consider locally constant systems on the Hilbert scheme $A^{[n]}$ which is covered by $A \times K^{(n-1)}A$. We generalize Nakajima's Operators to the cohomology of $A^{[n]}$ with values in locally constant systems and give a concrete geometric interpretation of how the resulting operators generate the cohomology of the Hilbert scheme. We thus get an explicit geometrical description of the cohomology of the Hilbert scheme $A^{[n]}$ with values in locally constant systems. The central result is Theorem 32, in which we express the ring structure of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ in terms of the known cup product on $\mathrm{H}^*(A^{[n]}, \mathbb{C})$.

In the last chapter, we compare our result with the orbifold cohomology ring of the quotient orbifold $[A \times A_0^n/\mathfrak{S}_n]$ which was computed by Barbara Fantechi and Lothar Göttsche in [**7**]. There, they state the conjecture that analogous to the case of the Hilbert scheme, the orbifold cohomology ring in the case of the generalized Kummer varieties should be isomorphic to the actual cohomology. After replenishing their formula for the orbifold cup product with certain factors that were overlooked in [**7**] in the case of the generalized Kummer varieties , we prove that the two rings are in fact isomorphic.

CHAPTER 2

A Hirzebruch-Riemann-Roch formula

1. Basic Facts.

In this section, we will recall the basic definitions and fix the notations used throughout this thesis.

GENERAL NOTATIONS. Let X be a smooth irreducible projective surface over $\mathbb{C}$. The *n-th symmetric product* S^nX is the quotient $X^n/\mathfrak{S}_n$ of the n-fold product by the symmetric group. It is of dimension $\dim S^nX = 2n$ and parameterizes effective zero-cycles of degree n on X.
The *Hilbert scheme* $X^{[n]}$ is the moduli space of zero-dimensional closed subschemes of length n of X. It is a projective scheme by a theorem of Grothendieck ([**15**]).
There is a natural morphism of schemes, the so called *Hilbert-Chow morphism* $\rho\colon X^{[n]} \to S^nX$ which sends a closed subscheme to its (weighted) support:

$$\rho(\xi) = \sum_{x\in X} l(\mathcal{O}_{\xi,x})x,$$

where $l(\mathcal{O}_{\xi,x})$ denotes the length of the structure sheaf $\mathcal{O}_{\xi,x}$ of the subscheme corresponding to the point $\xi \in X^{[n]}$.
Further, we have the following geometric fact due to Fogarty ([**8**]) concerning $X^{[n]}$:

THEOREM (Fogarty, [**8**]). — *The Hilbert scheme $X^{[n]}$ is smooth and projective of dimension $2n$.*

It follows that in this case ρ is a birational morphism and a desingularization of S^nX.

Especially in Chapter 3 we will made use of the following notions for the stratification of the symmetric product S^nX and the Hilbert scheme $X^{[n]}$ of a surface X:
Let $\lambda = (l_1 \geq l_2 \geq \ldots \geq l_s) = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$ be a partition of n. In the second notation of λ, the number α_i just counts the multiplicity with which i occurs in the partition. The length of λ will be denoted by $|\lambda| := s = \sum \alpha_i$.
The zero-cycles of the form $\sum_{1\leq j\leq s} l_jx_j \in S^nX$, with $x_j \in X$ and $x_j \neq x_k$ for $j \neq k$, form a locally closed subset $S^n_\lambda X$ in S^nX: We have $\overline{S^n_\lambda X} = \bigcup_{\mu\leq\lambda} S^n_\mu X$, where the union runs over all partitions $\mu = (m_1 \geq \ldots \geq m_{s'})$ such that there

exists a surjection $\varphi\colon \{1,\dots,s\} \twoheadrightarrow \{1,\dots,s'\}$ with $m_j = \sum_{i\in\varphi^{-1}(j)} l_i$ for all j. That is, $\overline{S^n_\lambda X}$ is the locus consisting of all tuples $\sum_{1\le j\le s} l_j x_j$ without the condition on the x_j's being mutually different.
Denote by $\Delta \subset S^n X$ the big diagonal, i.e. the locus of zero-cycles $\sum x_j$ such that at least two points coincide. We have $\Delta = \overline{S^n_{\lambda_0} X}$, where $\lambda_0 = (2,1,\dots,1)$.
The variety $S^n_\lambda X$ is isomorphic to the open set in $\prod_i (S^{\alpha_i}X - \Delta)$, consisting of tuples $(C_i := \sum_{j=1}^{\alpha_i} x_{ij})_i$ such that C_i and C_j do not meet for $i \ne j$. It follows that $\dim(S^n_\lambda X) = 2|\lambda|$.
Denote further by $S^\lambda X$ the variety $\prod_i S^{\alpha_i} X$. We have a morphism $S^\lambda X \to \overline{S^n_\lambda X}$ given on points by $(\sum_{j=1}^{\alpha_i} x_{ij})_i \mapsto \sum_{i,j} i x_{ij}$. In fact the morphism is a homeomorphism on the level of closed points as it is a bijective continuous map between compact spaces. In particular, on the level of (singular) cohomology, we find $\mathrm{H}^*(S^\lambda X, \mathbb{C}) = \mathrm{H}^*(\overline{S^n_\lambda X}, \mathbb{C})$.
We denote by $X^{[n]}_\lambda := \rho^{-1}(S^n_\lambda X)$ the corresponding locally closed stratum of the Hilbert scheme. The fibre of ρ over a point of $S^n_\lambda X$ is isomorphic to $\rho^{-1}(\sum l_j x_j) = (\rho^{-1}_{l_j}(l_j x_j))_j \subset \prod_j X^{[l_j]}$. By a theorem of Briançon (cf. [**4**]), this fibre is irreducible of dimension $\sum(l_j - 1) = n - |\lambda|$. It follows that $X^{[n]}_\lambda$ is irreducible and $\dim(X^{[n]}_\lambda) = n + |\lambda|$. Furthermore, observe that there is precisely one stratum of codimension one, namely $X^{[n]}_{(2,1,\dots,1)}$. Its closure

$$E := \overline{X^{[n]}_{(2,1,\dots,1)}} = \bigcup_{\lambda \ne (1^n)} X^{[n]}_\lambda$$

is an irreducible divisor on $X^{[n]}$, the exceptional divisor of ρ.

GENERALIZED KUMMER VARIETIES. To define Generalized Kummer varieties, we consider the special case of an abelian surface A. Here, choosing a point $0 \in A$, we have a summation morphism $A^n \longrightarrow A$. Since this is clearly symmetric, it factors through $\Sigma\colon S^n A \longrightarrow A$, and by composing with ρ one gets the 'summation' morphism $s\colon X^{[n]} \longrightarrow A$. The $(n-1)$*-th Generalized Kummer variety* $K^{(n-1)}A$ is defined as the fiber of the summation morphism s over 0.
As was shown by Beauville in [**1**], $K^{(n-1)}A$ is an irreducible holomorphic symplectic manifold of dimension $2(n-1)$. Recall that this means that $K^{(n-1)}A$ is a simply connected compact Kähler manifold such that $\mathrm{H}^0(K^{(n-1)}A, \Omega^2)$ is generated by an everywhere non-degenerate holomorphic two-form σ. In order to introduce some more notations, we briefly show that the Generalized Kummer variety is smooth and independent of the choice of the point 0:
Since the group A acts on itself by translation, there is an induced action on the Hilbert scheme $A^{[n]}$. Let us denote the translation by an element $a \in A$ by t_a in both cases. Let an element $a \in A$ acting on $A^{[n]}$ via t_a while acting on A via t_{na}. With respect to this action s is an equivariant morphism. Since A acts transitively

on itself, all fibers of s are isomorphic. In particular, the definition of $K^{(n-1)}A$ is independent of the choice of $0 \in A$. Since A and $A^{[n]}$ are both smooth, there are smooth fibers, so the Kummer variety is smooth.
Actually, the fibration $A^{[n]} \xrightarrow{s} A$ is isotrivial, i.e. one has the following Cartesian diagram, where n denotes the morphism 'multiplication by n' $A \xrightarrow{\Delta} A^n \xrightarrow{\Sigma} A$:

$$\begin{tikzcd} A \times_{\mathbb{C}} K^{(n-1)}A \arrow[r, "\nu"] \arrow[d, "p_A"'] & A^{[n]} \arrow[d, "s"] \\ A \arrow[r, "n"'] & A. \end{tikzcd} \tag{1}$$

In terms of closed points, the fibre product is given as $A \times_A A^{[n]} = \{(a,\xi) \in A \times A^{[n]} | s(\xi) = na\}$. This is isomorphic to $A \times K^{(n-1)}A$ via

$$A \times_A A^{[n]} \ni (a,\xi) \mapsto (a, t_{-a}(\xi)) \in A \times K^{(n-1)}A.$$

Therefore, on closed points the morphism ν in the above diagram is just the restriction of the translation operation on $A^{[n]}$ to the Generalized Kummer Variety. Observe that since n is a Galois cover with Galois group $A[n]$, the group of n torsion points, so is ν. It therefore realizes the Hilbert scheme $A^{[n]}$ as a quotient $(A \times K^{(n-1)}A)/A[n]$.

EXAMPLE. — *The classical Kummer surface.*
The easiest example — and the reason for the terminology 'Generalized Kummer varieties' — is the Kummer model of a K3 surface. This surface is constructed as follows (for details, cf. e.g. [**2**]):
One starts with the Abelian surface A and considers the singular quotient $A/\sim$ by the involution $(-1)_A$. The singularities are the images of the 16 two-division points. The desingularization, which we again denote by K^1A, is the classical Kummer K3 surface.
Alternatively one can first blow up the 16 points of order 2 of A and let K^1A be the quotient of the induced involution on the blown up surface $\hat{A}$. In other words, one has the following commutative diagram:

$$\begin{tikzcd} K^1A \arrow[d, "\varepsilon"'] & \hat{A} \arrow[l] \arrow[d] \\ A/\sim & A. \arrow[l] \end{tikzcd}$$

The surface K^1A can be identified with the fiber over 0 of the summation morphism $s\colon A^{[2]} \longrightarrow A$ as follows:
In the case $n = 2$ the Hilbert-Chow morphism $\rho\colon A^{[2]} \longrightarrow S^2A$ is simply the blow-up of the diagonal $\Delta \subset S^2A$ (cf. [**8**]). Denote by $\tilde{\Delta}\colon A \longrightarrow S^2A$ the morphism induced by the two isomorphisms id_A and $(-1)_A$. On closed points, $\tilde{\Delta}$ is given

by $a \mapsto (a, -a)$. By definition of S^2A and $A/\!\sim$, it descends to a morphism $A/\!\sim \longrightarrow S^2A$, which we again denote by $\tilde{\Delta}$. From the universal property of the fibre product, we get an isomorphism $A/\!\sim \longrightarrow \Sigma^{-1}(0)$.
Thus, we have the following commutative diagram:

$$\begin{tikzcd} K^1A \arrow[r] \arrow[d, "\varepsilon"'] & A^{[2]} \arrow[d, "\rho"] \\ A/\!\sim \arrow[r, "\tilde{\Delta}"] \arrow[d] & S^2A \arrow[d, "\Sigma"] \\ 0 \arrow[r] & A \end{tikzcd} \tag{2}$$

This shows that $\varepsilon = \rho|_{K^1A}$ and that the two descriptions of K^1A coincide.

In [**1**], the following theorem on the fundamental group of the Hilbert scheme $X^{[n]}$ of a surface X is shown:

THEOREM (Beauville, [**1**]). — *Let X be a compact complex surface and $n \geq 2$. The fundamental group of $X^{[n]}$ is given by $\pi_1(X^{[n]}) = \pi_1(X)^{\mathrm{ab}}$.*

From this fact, it follows that the generalized Kummer varieties are simply connected: It results from Beauville's Theorem that $\pi_1(A^{[n]})$ is isomorphic to $\pi_1(A) \simeq \mathbb{Z}^4$. Considering the long exact sequence

$$\cdots \to \pi_2(A) \longrightarrow \pi_1(K^{(n-1)}A) \longrightarrow \pi_1(A^{[n]}) \longrightarrow \pi_1(A) \longrightarrow 0,$$

and using $\pi_2(A) = 0$, it follows that $K^{(n-1)}A$ is simply connected.

THE BEAUVILLE-BOGOMOLOV QUADRATIC FORM. We will end this section with a few remarks on the Beauville-Bogomolov quadratic form q_X on an irreducible holomorphic symplectic manifold X of dimension $2n$.
As such an X is in particular a compact Kähler manifold, Hodge decomposition holds. Normalize the symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$ by demanding $\int_X \sigma\bar{\sigma} = 1$. Decomposes a class $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ as $\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}$, with $\beta \in \mathrm{H}^{1,1}(X)$. In this notation, the *unnormalized Beauville-Bogomolov quadratic form* f_X is defined by

$$f_X(\alpha) = \lambda\mu + \frac{n}{2}\int_X \beta^2(\sigma\bar{\sigma})^{n-1}.$$

It is shown in [**1**] that f_X is non-degenerate and that it comes — up to a positive scalar factor — from a unique integral form q_X of signature $(3, b_2 - 3)$ on $\mathrm{H}^2(X, \mathbb{Z})$.
In the sequel of this section, we will describe the structure of the second cohomology $\mathrm{H}^2(K^{(n-1)}A, \mathbb{C})$ of the generalized Kummer variety $K^{(n-1)}A$ and the normalized Beauville-Bogomolov quadratic form q on it.

As a first step we recall the following general fact, a proof of which can be found in [**1**]: Let X be a compact surface and let $n \geq 2$. Denote by $\pi\colon X^n \to S^nX$ the canonical projection. Let $E = \overline{X^{[n]}_{(2,1,\dots,1)}}$ be the exceptional divisor in $X^{[n]}$. Then there exists an embedding $j\colon \mathrm{H}^2(X,\mathbb{C}) \hookrightarrow \mathrm{H}^2(X^{[n]},\mathbb{C})$, where for $a \in \mathrm{H}^2(X,\mathbb{C})$ the associated class $\alpha := j(a) \in \mathrm{H}^2(X^{[n]},\mathbb{C})$ is given by

$$\alpha = \rho^*\alpha', \text{ with } \mathrm{H}^2(S^nX,\mathbb{C}) \ni \alpha' \text{ such that } \pi^*\alpha' = \sum_{i=1}^n p_i^* a.$$

The second cohomology of $X^{[n]}$ is given by

$$\mathrm{H}^2(X^{[n]},\mathbb{C}) = j\left(\mathrm{H}^2(X,\mathbb{C})\right) \oplus \left(\bigoplus_{i=1}^n \mathrm{H}^1(X,\mathbb{C})^{\otimes 2}\right)^{\mathfrak{S}_n} \oplus \mathbb{C}[E].$$

Let $\alpha = j(a)$ and $\beta = j(b)$ be two classes in $\mathrm{H}^2(X^{[n]},\mathbb{C})$. One finds for the top intersection product

$$\int_{X^{[n]}} \alpha^{2n} = \int_{S^nX} \alpha'^{2n} = \frac{1}{n!}\int_{X^n} \left(\sum p_i^* a\right)^{2n} = \frac{(2n)!}{2^n n!}\left(a^2\right)^n, \tag{3}$$

where on the surface X, we write a^2 instead of $\int_X a^2$.

By decomposing $\int_{X^{[n]}} (\alpha+\beta)^{2n}$ in the components of the appropriate degree in a and b and using (3) , it follows that

$$\int_{X^{[n]}} \alpha^{2n-2}\beta^2 = \frac{(2n-2)!}{2^{n-1}n!}\left(n(a^2)^{n-1}b^2 + 4\binom{n}{2}(a^2)^{n-2}(a\cdot b)^2\right) \text{ and} \tag{4}$$

$$\int_{X^{[n]}} \alpha^{2n-1}\beta = \frac{(2n)!}{2^n n!}(a^2)^{n-1}(a\cdot b). \tag{5}$$

Now, consider the special case of $X = A$, an abelian surface. Let $n > 2$. Denote by $F = E|_{K^{(n-1)}A}$ the trace of the exceptional divisor. In [**1**], it is shown that

$$\mathrm{H}^2(K^{(n-1)}A,\mathbb{C}) = i\left(\mathrm{H}^2(A,\mathbb{C})\right) \oplus \mathbb{C}[F],$$

where $i\colon \mathrm{H}^2(A,\mathbb{C}) \hookrightarrow \mathrm{H}^2(K^{(n-1)}A,\mathbb{C})$ is given by $i(a) = j(a)|_{K^{(n-1)}A}$.

The following proposition is well-known. We although include it here, since there seems to exist no proof of it in the literature.

PROPOSITION 1. — *The quadratic form $f_{K^{(n-1)}A}$ can be normalized by a positive scalar factor to a quadratic form q, such that one has $q(i(a)) = a^2$. For the class of the exceptional divisor one has $q([F]) = -8n$.*

Further, the decomposition $\mathrm{H}^2(K^{(n-1)}A,\mathbb{C}) = i\left(\mathrm{H}^2(A,\mathbb{C})\right) \oplus \mathbb{C}[F]$ is orthogonal with respect to q.

Proof. The proof will occupy the rest of this section.

1. Denote again by $\nu\colon A \times K^{(n-1)}A \to A^{[n]}$ the $A[n]$-Galois cover of $A^{[n]}$ and by $n\colon A \to A$ the corresponding cover of the torus, given by multiplication with

n. Denote by $p_A\colon A\times K^{(n-1)}A\to A$ and $p_K\colon A\times K^{(n-1)}A\to K^{(n-1)}A$ the projections. Let $\alpha=j(a)\in \mathrm{H}^2(A^{[n]},\mathbb{C})$. Then one has the following

LEMMA 2. — *With the notations introduced above, one has*

$$\nu^*\alpha = n\cdot p_A^* a + p_K^*(i(a)) \in \mathrm{H}^2(A\times K^{(n-1)}A,\mathbb{C}).$$

To see this, recall the following general situation:

LEMMA 3. — *Let X be a simply connected compact manifold and let Y be a connected compact manifold. Let $p_X\colon X\times Y\to X$ and $p_Y\colon X\times Y\to Y$ be the projections. Let x_0 and y_0 be a point in X and Y, respectively. Denote by $i_X\colon X\to X\times\{y_0\}$ and $i_Y\colon Y\to\{x_0\}\times Y$ the corresponding splittings of the projections. Then every class $\alpha\in\mathrm{H}^2(X\times Y,\mathbb{C})$ is given as $\alpha=p_X^*i_X^*\alpha+p_Y^*i_Y^*\alpha$.*

Proof of Lemma 2. Our assertion follows from Lemma 3: Choose an origin $0\in A$. Choose further an element $\xi_0\in K^{(n-1)}A$ that lies over the point $n\cdot 0\in S^nA$. With these points, define the corresponding splittings i_A and i_K of p_A and p_K, respectively. Since $\nu\circ i_K$ is the embedding of $K^{(n-1)}A$ in $A^{[n]}$ as fibre over 0, one gets $\nu^*\alpha=p_A^*\beta+p_K^*(i(a))$. To determine β, recall that α is of the form $\rho^*(\alpha')$ with $\alpha'\in\mathrm{H}^2(S^nA,\mathbb{C})$ by definition. Complete diagram (1) as follows

$$\begin{tikzcd}
A\times K^{(n-1)}A \arrow[r, "\nu"] \arrow[d, "\rho'"'] & A^{[n]} \arrow[d, "\rho"] \\
A\times (S^nA)_0 \arrow[r, "\nu'"] \arrow[d, "p_A"'] & S^nA \arrow[d, "\Sigma"] \\
A \arrow[r, "n"] & A.
\end{tikzcd} \tag{6}$$

Here, $(S^nA)_0$ denotes the fiber over 0 of the addition morphism $\Sigma : S^nA \longrightarrow A$, the morphism $\rho'=\mathrm{id}_A\times\rho|_{K^{(n-1)}A}$ the desingularization of $A\times(S^nA)_0$, and ν' is defined analogously to ν.
Instead of computing $i_A^*\nu^*\alpha$ we compute

$$i_A^*\rho'^*\nu'^*\alpha' = na,$$

since the morphism $\nu'\circ\rho'\circ i_A$ sends a point $x\in A$ to $nx\in S^nA$ and thus is just the diagonal morphism $A\to S^nA$. ☐

REMARK 4. — An analogous statement will be proven in the case of line bundles in Lemma 2 in Section 2.

2. It follows that for $\alpha=j(a)\in\mathrm{H}^2(A^{[n]},\mathbb{C})$, one has the equality

$$n^4\int_{A^{[n]}}\alpha^{2n}=\int_{A\times K^{(n-1)}A}\nu^*\alpha^{2n}=\binom{2n}{2}n^2a^2\int_{K^{(n-1)}A}\left(\alpha|_{K^{(n-1)}A}\right)^{2n-2}$$

Using equation (3), this implies for $\alpha = i(a) \in \mathrm{H}^2(K^{(n-1)}A, \mathbb{C})$:

$$\int_{K^{(n-1)}A} \alpha^{2n-2} = n^2 \frac{(2n-2)!}{2^{n-1}n!} \left(a^2\right)^{n-1}. \tag{7}$$

Further, using the same polarization method as in the derivation of the equations (4) and (5), one finds for two classes $\alpha = i(a)$ and $\beta = i(b)$:

$$\int_{K^{(n-1)}A} \alpha^{2n-4}\beta^2 = n^2 \frac{(2n-4)!}{2^{n-2}n!} \cdot \left((n-1)(a^2)^{n-2}b^2 + 4\binom{n-1}{2}(a^2)^{n-3}(a \cdot b)^2 \right) \tag{8}$$

and

$$\int_{K^{(n-1)}A} \alpha^{2n-3}\beta = n^2 \frac{(2n-2)!}{2^{n-1}n!}(a^2)^{n-2}(a \cdot b). \tag{9}$$

3. For every $\alpha \in \mathrm{H}^2(K^{(n-1)}A, \mathbb{C})$ denote by $v(\alpha) = \int_X \alpha^{2n-2}$ the top intersection product on $X = K^{(n-1)}A$. With this notation one has, as on every irreducible holomorphic symplectic variety, the following formula, relating values of the unnormalized quadratic form $f_X(\alpha)$ and $f_X(\beta)$ of two classes α and $\beta \in \mathrm{H}^2(K^{(n-1)}A, \mathbb{C})$:

$$v(\alpha)^2 f_X(\beta) = f_X(\alpha) \left((2n-3)v(\alpha) \int_X \alpha^{2n-4}\beta^2 - (2n-4) \left(\int_X \alpha^{2n-3}\beta \right)^2 \right) \tag{10}$$

(cf. [**1**]).
Let $\alpha = i(a)$ and $\beta = i(b)$ with $a^2 \neq 0$ (and therefore $v(\alpha) \neq 0$). Using equation (10) above, and the equalities (8) and (9), we find

$$\begin{aligned} f(\beta) &= f(\alpha) \left(\frac{b^2}{a^2} + \frac{2(n-2)(a \cdot b)^2}{(a^2)^2} - (2n-4)\frac{(a \cdot b)^2}{(a^2)^2} \right) \\ &= f(\alpha) \cdot \frac{b^2}{a^2}. \end{aligned}$$

Therefore, it follows directly that by correcting $f_{K^{(n-1)}A}$ by a positive scalar factor, we get a new form q which is normalized such that $q(i(a)) = a^2$. It is this form, we will refer as the *Beauville-Bogomolov quadratic form* to in the sequel.

4. Furthermore, $[F]$ is orthogonal to $i\left(\mathrm{H}^2(A, \mathbb{C})\right)$ with respect to q. This follows directly from

$$\int_{K^{(n-1)}A} (i(a))^{2n-3}[F] = \int_{(S^nA)_0} \left(\alpha'|_{(S^nA)_0} \right)^{2n-3} \rho_*[F] = 0,$$

for degree reasons, where we denoted again by $(S^nA)_0$ the fibre over 0 of the summation morphism $S^nA \to A$, and by α' the class in $\mathrm{H}^2(S^nA, \mathbb{C})$ such that $j(a) = \rho^*\alpha'$.

5. We will now begin to compute $q([F])$. Considering formula (10) and the orthogonality of $[F]$ to $i\left(\mathrm{H}^2(A,\mathbb{C})\right)$, one finds

$$q([F]) = \frac{q(\alpha)}{v(\alpha)}(2n-3)\int_{K^{(n-1)}A}\alpha^{2n-4}[F]^2,$$

for every class $\alpha = i(a)$ such that $a^2 \neq 0$. Thus, to compute $q([F])$, it is enough to compute the integral $\int_{K^{(n-1)}A}\alpha^{2n-4}[F]^2$. Let us first consider the following special situation:
Let X be an arbitrary smooth compact surface again. Then $X^{[2]}$ is the variety obtained by blowing up the diagonal $\Delta \subset S^2X$. Let E be the exceptional divisor. Let $\alpha = j(a)$ be a class in $\mathrm{H}^2(X^{[2]},\mathbb{C})$ associated to $a \in \mathrm{H}^2(X,\mathbb{C})$. In this case $\int_{X^{[2]}}\alpha^2[E]^2$ can be computed as follows:

$$\begin{aligned}\int_{X^{[2]}}\alpha^2[E]^2 &= \int_{X^{[2]}}\alpha^2 c_1(\mathcal{O}(E))^2 = \int_E c_1(\mathcal{O}(E))\big|_E\,\alpha\big|_E^2\\ &= \int_E \underbrace{c_1(\mathcal{O}_E(E))}_{=:\varepsilon}\,\rho^*\alpha'\big|_E^2 = (\rho_*\varepsilon)\int_{\Delta\subset S^2X}\alpha'^2\\ &= -2\int_{D\subset X^2}(p_1^*a+p_2^*a)^2 = -2\cdot\int_X(2a)^2\\ &= -8a^2.\end{aligned}$$

Here, in the third line, the factor (-2) results from integrating ε along the fibres of the blowing up. Furthermore, we have used the fact that in this simple case, one has isomorphisms $\Delta \simeq D \simeq X$, where we denoted by $X^2 \supset D = \{(x,x)\}$ the diagonal of X^2.

6. In the general case of $X^{[n]}$, we have

$$\int_{X^{[n]}}\alpha^{2n-2}[E]^2 = \int_E \underbrace{c_1(\mathcal{O}(E)|_E)}_{=:\varepsilon}\,\alpha\big|_E^{2n-2} = \rho_*\varepsilon\cdot\int_{\Delta\subset S^nX}\alpha'^{2n-2}.$$

Here, we have to integrate the class ε along the fibers of the Hilbert-Chow morphism $\rho|_E\colon E \to \Delta = \overline{S^n_{\lambda_0}A} = \bigcup_{\lambda\neq(1^n)} S^n_\lambda A$, with $\lambda_0 = (2,1,\dots,1)$. As we have seen above, the fibers of ρ restricted to the open stratum $X^{[n]}_\lambda$ have real dimension $2\,(n-|\lambda|)$. The class ε that we want to integrate has cohomological degree 2. Thus, for dimension reasons, only the fibre over $S^n_{\lambda_0}A$ contributes, since λ_0 is the only partition of length $n-1$. But over this open stratum we are in the situation of the blowing up that we have considered in 4. above. Thus, one finds

$$\int_{X^{[n]}}\alpha^{2n-2}[E]^2 = -2\cdot\int_{\Delta\subset S^nX}\alpha'^{2n-2} = -2\cdot\frac{2}{n!}\cdot\int_{D\subset X^n}\left(\sum p_i^*a\right)^{2n-2}.$$

The factor $\frac{2}{n!}$ in the last equation results from the order of the inertia group of the big diagonal $D \subset X^n$ under the action of $\mathfrak{S}_n$: For D, we have the equality $D = \bigcup_{i<j} D_{ij}$, where $D_{ij} = \{(x_k) \in X^n | x_i = x_j\}$. Since D_{ij} has $\langle(ij)\rangle \simeq$

$\mathbb{Z}/2$ as inertia group, we find that the quotient morphism restricted to the diagonal $\pi|_D\colon D \to \Delta$ is of degree $\frac{n!}{2}$, and therefore $[\Delta] = \frac{2}{n!}\pi_*[D]$. Thus, we find by the projection formula

$$\int_{\Delta\subset S^nX} \alpha'^{2n-2} = \frac{2}{n!}\int_{D\subset X^n} \pi^*(\alpha')^{2n-2} = \frac{2}{n!}\int_{D\subset X^n}\left(\sum p_i^* a\right)^{2n-2}.$$

We can now complete the calculation of $\int_{X^{[n]}} \alpha^{2n-2}[E]^2$ by computing

$$\begin{aligned}\int_D \left(\sum\nolimits_{i=1}^n p_i^* a\right)^{2n-2} &= \binom{n}{2}\int_{X^{n-1}}\left(2p_1^*a + \cdots + p_{n-1}^*a\right)^{2n-2}\\ &= \binom{n}{2}\frac{(2n-2)!}{2^{n-1}}4(a^2)^{n-1}.\end{aligned}$$

Putting everything together, we have

$$\int_{X^{[n]}} \alpha^{2n-2}[E]^2 = -16\frac{(2n-2)!}{2^{n-1}n!}\binom{n}{2}(a^2)^{n-1} \tag{11}$$

7. We come back to the case of an abelian surface A and $n > 2$. Observe that for the inverse image of the exceptional divisor E under the Galois cover ν, one has $\nu^{-1}(E) = A \times F$ and thus $\nu^*([E]) = p_K^*[F]$. It follows that

$$\begin{aligned}n^4\int_{A^{[n]}} \alpha^{2n-2}[E]^2 &= \int_{A\times K^{(n-1)}A} \nu^*\left(\alpha^{2n-2}[E]^2\right)\\ &= n^2(a^2)\binom{2n-2}{2}\int_{K^{(n-1)}A} \alpha|_{K^{(n-1)}A}^{2n-4}[F]^2.\end{aligned}$$

Therefore, we have for a class $\alpha = i(a)$

$$\begin{aligned}\int_{K^{(n-1)}A} \alpha^{2n-4}[F]^2 &= \left((a^2)\binom{2n-2}{2}\right)^{-1} n^2\int_{A^{[n]}} (j(a))^{2n-2}[E]^2\\ &= \frac{2n^2}{(2n-2)(2n-3)}\cdot(-16)\frac{(2n-2)!}{2^{n-1}n!}\binom{n}{2}(a^2)^{n-2}\\ &= \frac{-8n^2\cdot(2n-4)!}{2^{n-3}n!}\binom{n}{2}(a^2)^{n-2}\end{aligned}$$

Collecting everything together, we can compute $q([F])$ as

$$\begin{aligned}q([F]) &= \frac{q(\alpha)}{v(\alpha)}(2n-3)\int_{K^{(n-1)}A}\alpha^{2n-4}[F]^2\\ &= \frac{2^{n-1}n!(2n-3)}{(2n-2)!}\cdot\frac{-8\cdot(2n-4)!}{2^{n-3}n!}\binom{n}{2}\\ &= -8n.\end{aligned}$$

This completes the proof of the proposition. □

REMARK 5. —

(1) Equality (11) can be used to show that $q_{X^{[n]}}([E]) = -8(n-1)$, for a K3 surface X. This was stated in [**1**].
(2) In [**1**], Beauville gives a description of the integral second cohomology of the Hilbert scheme of a surface. From this description and Proposition 1, it follows that in the case of the generalized Kummer varieties, one has the following equality of lattices together with quadratic forms

$$\left(\mathrm{H}^2(K^{(n-1)}A,\mathbb{Z}), q_{K^{(n-1)}A}\right) = \left(\mathrm{H}^2(A,\mathbb{Z}),\cup\right) \oplus (-2n)\mathbb{Z}.$$

Here, the two factors on the right hand side are mutually orthogonal and the last factor $(-2n)\mathbb{Z}$ is spanned by a class δ such that $2\delta = [F]$. As we have seen, one has $q(\delta) = -2n$.

2. Explicit Hirzebruch-Riemann-Roch for $K^{(n-1)}A$.

As stated in the last chapter, Generalized Kummer varieties are irreducible holomorphic symplectic manifolds, and we have seen that they carry a natural quadratic form $q\colon \mathrm{H}^2(K^{(n-1)}A,\mathbb{Z}) \longrightarrow \mathbb{Z}$. In [**17**], Huybrechts obtained the following result:

THEOREM (Huybrechts). — *Let X be a compact irreducible holomorphic symplectic manifold of dimension $\dim X = 2n$. Then, there are constants a_{2k}, $k = 0,\dots,n$, only depending on X, such that for every line bundle $L \in \mathrm{Pic}(X)$ the Euler characteristic is given by the following polynomial:*

$$\chi(X,L) = \sum_{k=0}^{n} a_{2k} q_X(c_1(L))^k,$$

where q_X denotes the Beauville-Bogomolov quadratic form.

In this section we will compute this polynomial for the Generalized Kummer varieties by proving the following Hirzebruch-Riemann-Roch formula:

THEOREM 6. — *Let L be a line bundle on $K^{(n-1)}A$. The Euler characteristic of L is given by*

$$\chi(L) = n\binom{\frac{1}{2}q(c_1(L)) + n - 1}{n-1},$$

where q is the Beauville-Bogomolov quadratic form on $\mathrm{H}^2(K^{(n-1)}A,\mathbb{Z})$.

We will prove the theorem as follows: First, observe that since the Hirzebruch-Riemann-Roch formula has the form $\chi(L) = \sum a_{2k} q(c_1(L))^k$ with universal coefficients a_{2k}, it is actually enough to consider a special class of line bundles L with $q(c_1(L)) \neq 0$: Then, Huybrechts' theorem says that each $\chi(L^n)$ is a polynomial in $n^2 q(c_1(L))$ with the same coefficients in each case. Considering these formulas for all natural numbers n determines the polynomial uniquely.

We will prove the theorem using line bundles $K^{(n-1)}L$ on $K^{(n-1)}A$, which are constructed from an invertible sheaf $L \in \mathrm{Pic}(A)$ as follows:
Starting with a line bundle L on the surface A, the sheaf $L^{\boxtimes n} := \bigotimes_{i=1}^n \mathrm{pr}_i^* L$ is an $\mathfrak{S}_n$-invariant line bundle on the n-th product A^n of A. Therefore, we can define the sheaf $S^nL := (\pi_*(L^{\boxtimes n}))^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariant sections of $\pi_*(L^{\boxtimes n})$ on the symmetric product S^nA, where π denotes the quotient morphism $\pi \colon A^n \longrightarrow S^nA$. The pull-back $L_n := \rho^* S^nL$ by the Hilbert-Chow morphism is a line bundle on the Hilbert scheme $A^{[n]}$. Restricting to the generalized Kummer variety $K^{(n-1)}A \subset A^{[n]}$, we get the invertible sheaf $K^{(n-1)}L$ we wanted to construct.
Observe that by definition of $K^{(n-1)}L$, we have for its first Chern class the equality

$$c_1(K^{(n-1)}L) = i(c_1(L)),$$

with the injective homomorphism $i \colon \mathrm{H}^2(A, \mathbb{C}) \to \mathrm{H}^2(K^{(n-1)}A, \mathbb{C})$ described in the last section. It follows that the value of quadratic form $q(c_1(K^nL))$ coincides with the self intersection of $c_1(L) \in \mathrm{H}^2(A, \mathbb{Z})$ on the abelian surface A. Explicitly, we have the equality $q(c_1(K^{(n-1)}L)) = c_1(L)^2$, and thus our theorem is equivalent to the following

PROPOSITION 7. — *The Euler characteristic of $L \in \mathrm{Pic}(A)$ and of $K^{(n-1)}L \in \mathrm{Pic}(K^{(n-1)}A)$ are related by*

$$\chi(K^{(n-1)}L) = n\binom{\frac{c_1(L)^2}{2} + n - 1}{n-1}.$$

The rest of this section is dedicated to the proof of Proposition 7. In the next lemma we will compute the Euler characteristic of the line bundle S^nL.

LEMMA 8. — *Let L be a line bundle on A. Then one has*

$$\chi(S^nL) = \binom{\chi(L) + n - 1}{n}$$

Proof. Let $L, H \in \mathrm{Pic}(A)$. One has

$$S^n(L \otimes H) = \pi_* \left((L \otimes H)^{\boxtimes n}\right)^{\mathfrak{S}_n} = \pi_* \left(L^{\boxtimes n} \otimes H^{\boxtimes n}\right)^{\mathfrak{S}_n},$$

and since the action of the symmetric group does not flip the factors $L^{\boxtimes n}$ and $H^{\boxtimes n}$, and for an arbitrary line bundle M on A we have $\pi^* S^nM = M^{\boxtimes n}$, we get

$$S^n(L \otimes H) = \pi_* \left(L^{\boxtimes n} \otimes \pi^* S^nH\right)^{\mathfrak{S}_n} = \left(\pi_*(L^{\boxtimes n}) \otimes S^nH\right)^{\mathfrak{S}_n} = S^nL \otimes S^nH.$$

Let H be an ample invertible sheaf. It follows that $H^{\boxtimes n}$ is also ample, and so is S^nH, because π is a finite surjective morphism and $\pi^* S^nH = H^{\boxtimes n}$ is ample. Let $N \in \mathbb{N}$ be large enough such that both $L \otimes H^N$ and $S^n(L \otimes H^N) = S^nL \otimes (S^nH)^N$ have no higher cohomology, and thus $\chi(S^n(L \otimes H^N)) = h^0(S^n(L \otimes H^N))$.

For the global sections of the line bundle S^nM built from an invertible sheaf M on A, one has the isomorphism

$$\mathrm{H}^0(S^nA, S^nM) \simeq \mathrm{H}^0(A^n, M^{\boxtimes n})^{\mathfrak{S}_n} = \left(\mathrm{H}^0(A,M)^{\otimes n}\right)^{\mathfrak{S}_n} \simeq S^n\mathrm{H}^0(A,M).$$

Replacing M by $L \otimes H^N$, we find

$$\chi(S^n(L \otimes H^N)) = \binom{\chi(L \otimes H^N) + n - 1}{n} \quad \text{for all } N \gg 0.$$

Since $N \mapsto \chi(S^n(L \otimes H^N)) = \chi(S^nL \otimes (S^nH)^N)$ is a polynomial in N, evaluation in $N = 0$ proves the lemma. □

Consider the Hilbert-Chow morphism $\rho\colon A^{[n]} \longrightarrow S^nA$. Since ρ is a birational proper morphism of normal varieties one has $\rho_*\mathcal{O}_{A^{[n]}} = \mathcal{O}_{S^nA}$. Furthermore, S^nA as a quotient of a smooth variety by a finite group has rational singularities (cf. [**18**]). Therefore its resolution ρ satisfies $R^j\rho_*\mathcal{O}_{A^{[n]}} = 0$ for $j > 0$. Combining the Leray spectral sequence $\mathrm{H}^i(R^j\rho_*(\rho^*S^nL)) \Rightarrow \mathrm{H}^{i+j}(\rho^*S^nL)$ and the projection formula $R^j\rho_*(\rho^*S^nL) = S^nL \otimes R^j\rho_*\mathcal{O}_{A^{[n]}}$, one gets

$$\mathrm{H}^i(S^nL) = \mathrm{H}^i(\rho^*S^nL)$$

So we have proven the following

PROPOSITION 9. — *For a line bundle $L \in \mathrm{Pic}(A)$ one has*

$$\chi(L_n) = \binom{\chi(L) + n - 1}{n}.$$

□

REMARK 10. — This result is proven by a somewhat different method in [**6**].

Next we will attack the cohomology of the restricted bundle $K^{(n-1)}L$. The first step in this direction is the following

LEMMA 11. — *In the notation of diagram* (1)*, one has $\nu^*L_n = L^n \boxtimes K^{(n-1)}L$.*

Proof. The splitting of the sheaf ν^*L_n follows from the seesaw principle (cf. [**22**]): For fixed $a \in A$ we have seen that the restricted morphism $\nu|_{\{a\}\times K^{(n-1)}A}$ is the isomorphism which maps $K^{(n-1)}A$ to the fiber of the summation morphism s over the point na. Since $K^{(n-1)}A$ is simply connected, its Picard group is discrete and it follows that $\nu^*L_n|_{\{a\}\times K^{(n-1)}A} \simeq p^*_{K^{(n-1)}A}K^{(n-1)}L|_{\{a\}\times K^{(n-1)}A}$.
Therefore ν^*L_n is of the form $L_2 \boxtimes K^{(n-1)}L$ with $L_2 \in \mathrm{Pic}(A)$, and we can compute the component L_2 by considering the restrictions of ν to $A \times \{\xi_0\}$.

Recall diagram (6) from the last section

$$\begin{tikzcd} A \times K^{(n-1)}A \arrow[r, "\nu"] \arrow[d, "\rho'"'] & A^{[n]} \arrow[d, "\rho"] \\ A \times (S^nA)_0 \arrow[r, "\nu'"] \arrow[d, "p_A"'] & S^nA \arrow[d, "\Sigma"] \\ A \arrow[r, "n"] & A. \end{tikzcd}$$

Again, $(S^nA)_0$ denotes the fiber over 0 of the addition morphism $\Sigma : S^nA \longrightarrow A$, the morphism $\rho' = \mathrm{id}_A \times \rho|_{K^{(n-1)}A}$ its desingularization, and ν' is defined analogously to ν.
Now consider a point $\xi_0 \in K^{(n-1)}A$ over $n \cdot 0 \in S^nA$. Instead of computing the sheaf $\nu^* L_n|_{A\times\{\xi_0\}}$, we equivalently compute $\rho'^*\nu'^*S^nL|_{A\times\{\xi_0\}}$. But since $\rho'|_{A\times\{\xi_0\}}$ corresponds to the identity on A and $\nu'|_{A\times\{n\cdot 0\}}$ corresponds to the morphism $\Delta\colon A \longrightarrow S^nA$, induced by the diagonal, we find $L_2 = \Delta^*S^nL = L^n$. $\square$

The next lemma describes the structure of the direct image of $\mathcal{O}_A$ under the n^4-fold Galois covering $A \xrightarrow{n} A$.

LEMMA 12. — *The direct image $n_*\mathcal{O}_A$ of the structure sheaf of A splits into a direct sum of line bundles $L_\sigma, \sigma \in A[n]^\vee$, indexed by the characters σ of the n-torsion points of A. Further, for the trivial character $1 \in A[n]^\vee$, we have $L_1 = \mathcal{O}_A$ and $c_1(L_\sigma) = 0 \in \mathrm{H}^2(A, \mathbb{Z})$ for all σ.*

Proof. The splitting of $n_*\mathcal{O}_A$ is a well known fact, cf. [**22**], §7. (See also Chapter 3 of this thesis where we give a proof of the analogous behaviour of the direct image $\nu_*\mathbb{C}$ of the constant sheaf $\mathbb{C}$.) The triviality of the first Chern classes of the line bundles L_σ follows from the relation:

$$L_\sigma^{\otimes n} = L_{\sigma^n} = L_1 = \mathcal{O}.$$

It follows that $nc_1(L_\sigma) = 0$, which proves the lemma, since the cohomology of a torus has no torsion. $\square$

Now we have collected all necessary ingredients for the proof of Proposition 7.

Proof of the proposition. We start again with a line bundle L on A that we twist with a sufficiently ample bundle H^N. By abuse of notation we denote the resulting bundle by L again. By construction, the symmetrized bundle S^nL is still ample and thus the invertible sheaf L_n on $A^{[n]}$ as a pull-back along the birational morphism ρ is nef and big.
The same argument shows that the line bundle $K^{(n-1)}L$ is nef and big: Using the notations of diagram (6) the bundle $(S^nL)_0 := \nu'^*S^nL|_{(S^nA)_0}$ is ample and

$K^{(n-1)}L = \rho'^*(S^nL)_0$ is big and nef. Thus, by the Kawamata-Viehweg vanishing theorem ([**28**]), we have

$$\chi(K^{(n-1)}L) = h^0(K^{(n-1)}A, K^{(n-1)}L)$$

On the one hand, due to the Künneth formula, we have

$$\mathrm{H}^0(\nu^*L_n) = \mathrm{H}^0(K^{(n-1)}L \boxtimes L^n) = \mathrm{H}^0(K^{(n-1)}L) \otimes \mathrm{H}^0(L^n).$$

On the other hand, since ν is finite and n is a flat morphism, we can compute $\mathrm{H}^0(\nu^*L_n)$ alternatively

$$\begin{aligned}\mathrm{H}^0(\nu^*L_n) &= \mathrm{H}^0(L_n \otimes \nu_*\mathcal{O}_{A\times K^{(n-1)}A}) = \mathrm{H}^0(L_n \otimes s^*n_*\mathcal{O}_A)\\ &= \mathrm{H}^0(L_n \otimes \bigoplus_{\sigma\in A[n]^\vee} s^*L_\sigma).\end{aligned}$$

This shows that

$$h^0(A \times K^{(n-1)}A, \nu^*L_n) = \sum_{\sigma\in A[n]^\vee} h^0(A^{[n]}, L_n \otimes L_\sigma)$$

where, by abuse of notation, we denote the line bundles s^*L_σ on $A^{[n]}$ by L_σ again. Since the sheaf $L_n \otimes L_\sigma$ is still nef and big, the vanishing theorem of Kawamata and Viehweg implies that $h^0(A^{[n]}, L_n \otimes L_\sigma)$ equals the Euler characteristic of this line bundle. Therefore, using the classical Hirzebruch-Riemann-Roch theorem on the Hilbert scheme $A^{[n]}$ we have

$$\begin{aligned}h^0(\nu^*L_n) &= \sum_{\sigma\in A[n]^\vee} \chi(A^{[n]}, L_n \otimes L_\sigma)\\ &= \sum_{\sigma\in A[n]^\vee} \int \mathrm{ch}(L_n \otimes L_\sigma)\mathrm{td}(A^{[n]})\\ &= n^4 \int \mathrm{ch}(L_n)\mathrm{td}(A^{[n]}), && \text{since } c_1(L_\sigma) = s^*c_1(L_\sigma) = 0\\ &= n^4\chi(A^{[n]}, L_n)\\ &= n^4 \dim(S^n\mathrm{H}^0(L)) && \text{due to Proposition 9.}\end{aligned}$$

Combining the computations — and noting that $h^0(A, L^n) \neq 0$ — we find

$$\begin{aligned}\chi(K^{(n-1)}A, K^{(n-1)}L) = h^0(K^{(n-1)}L) = \frac{h^0(\nu^*L_n)}{h^0(A, L^n)} &= \frac{n^4\binom{\frac{c_1(L)^2}{2}+n-1}{n}}{n^2\frac{c_1(L)^2}{2}}\\ &= n\binom{\frac{c_1(L)^2}{2}+n-1}{n-1}.\end{aligned}$$

Once again — considering the formula as a polynomial in N and evaluating in $N = 0$ — the formula holds for a general line bundle L. $\square$

EXAMPLE. — In the case of the Kummer surface K^1A, the above formula gives back the classical Riemann-Roch formula for K3 surfaces:
Recall the diagram

$$\begin{tikzcd} K^1A \arrow[d, "\varepsilon"'] & \hat{A} \arrow[l] \arrow[d] \\ A/\!\sim & A. \arrow[l, "p"] \end{tikzcd}$$

Start with a symmetric line bundle L on A, i.e. $L = p^*L'$ with $L' \in \mathrm{Pic}(A/\!\sim)$. Then L induces a line bundle $M = \varepsilon^*L'$ on K^1A. Let $K^1L = \rho^*\pi_*(L \boxtimes L)^{\mathfrak{S}_2}$, as usual. Then one has $K^1L = M^2$:
Considering diagram (2), it suffices to show that $\tilde{\Delta}^*S^2L = L'^2$. But this is clear from the definition of $\tilde{\Delta}$ and $L = p^*L'$.
Our Hirzebruch-Riemann-Roch formula gives

$$\chi(K^1L) = 2\binom{\frac{c_1(L)^2}{2}+1}{1} = c_1(L)^2 + 2$$

Using that ε is birational, p is generically 2:1 and the equality $K^1L = M^2$, one finds

$$\chi(K^1L) = \frac{c_1(K^1L)^2}{2} + 2,$$

which is the classical Riemann-Roch formula for the K3 surface K^1A.

REMARK 13. — In [**25**], Marc Nieper-Wißkirchen gives a formula for the Euler characteristic $\chi(X, L)$ for a line bundle L on an arbitrary irreducible holomorphic symplectic manifold X in terms of the characteristic numbers of X. He uses an other normalization of f_X, which is given for $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ by

$$\lambda(\alpha) := \begin{cases} \frac{24n \int_X \exp(\alpha)}{\int_X c_2(X)\exp(\alpha)} & \text{if defined,} \\ 0 & \text{otherwise.} \end{cases}$$

He defines a twisted Todd genus $\mathrm{td}_\varepsilon(X)$ of X by setting for $\varepsilon \in \mathbb{C}$

$$\mathrm{td}_\varepsilon(X) := \exp\left(-2\sum\nolimits_{k=0}^{\infty} b_{2k}\mathrm{ch}_{2k}(2k)! \cdot T_k(1+\varepsilon)\right),$$

where T_k is the k-th Chebyshev polynomial, defined by $T_k(\cos x) = \cos(kx)$. One has $T_k(1) = 1$, whereas $\mathrm{td}_0(X) = \mathrm{td}(X)$.

With these notations, Nieper-Wißkirchen proves

$$\chi(X,L) = \int_X \mathrm{td}_\varepsilon(X) \text{ with } \varepsilon = \frac{1}{2}\lambda(c_1(L)). \tag{12}$$

In [**5**], Nieper-Wißkirchen and the author show that for $\alpha \in \mathrm{H}^*(K^{(n-1)}A, \mathbb{C})$, one has

$$\lambda(\alpha) = \frac{2}{n} q(\alpha).$$

Combining this with Theorem 6, one gets for a line bundle L on the generalized Kummer variety $K^{(n-1)}A$

$$\chi(K^{(n-1)}A, L) = n\binom{\frac{n}{4}\lambda(c_1(L)) + n - 1}{n-1}.$$

Comparing with formula (12), in [**5**] Nieper-Wißkirchen and the author compute the Chern numbers of generalized Kummer varieties up to dimension ten. Meanwhile Nieper-Wißkirchen succeeded in computing the Chern numbers of generalized Kummer varieties in arbitrary dimension (cf. [**26**]).

CHAPTER 3

The Cohomology Ring of Generalized Kummer Varieties

1. Locally constant systems.

We start this section considering a quite general situation. Let X be a complex variety, G a finite quotient of the fundamental group $\pi_1(X)$ of X and let $\pi\colon Y \longrightarrow X = Y/G$ be the corresponding Galois cover of X, on which we let G act from the right. Let further $\rho\colon G \to \mathrm{GL}(V_\rho)$ be a complex linear representation of G. Then we have a natural G-action on $Y \times V_\rho$, given by $g(y,v) = (yg^{-1}, \rho(g)v)$. The quotient, which is denoted by $Y \times_G V_\rho$, is a fibre bundle E_ρ over X, and is called a *locally constant system*:

$$E_\rho = Y \times_G V_\rho \longrightarrow X, [(y,v)] \mapsto \pi(y).$$

Observe that due to this definition of the G-action, we have $[(yg,v)] = [(y,\rho(g)v)]$. Assume that there is an additional right G-action on V which is compatible with the action defined by ρ. Here, compatible means that for all $g,h \in G$ and a vector $v \in V$ we have

$$(\rho(g)v)h = \rho(g)(vh).$$

Under this assumption, the bundle E_ρ still carries a G-action, defined by $g\,[(y,v)] = [(y,vg)]$.

The following lemma describes the behaviour of the pull-back of a locally constant system.

LEMMA 14. — *Let E_ρ be a locally constant system on $X = Y/G$ as above. Let $\pi'\colon Y' \to X$ be the intermediate Galois cover with Galois group $H := G/\ker(\rho)$ Then, we have a canonical trivialization of its pull-back $\pi'^{*}E_\rho$ to X'.*

Proof. We will prove a more general assertion: Let $\pi'\colon Y' \longrightarrow X$ be an intermediate covering, corresponding to an arbitrary normal subgroup $N \subset G$. Set $H := G/N$. Then $\pi'^{*}E_\rho$ is the locally constant system on Y' corresponding to the representation $\mathrm{res}^G_N(\rho)$ of N. As a diagram, this reads as

$$\begin{tikzcd}
Y \times V_\rho \arrow[r] \arrow[d] & Y \times_N V_{\mathrm{res}^G_N(\rho)} = \pi'^{*}E_\rho \arrow[r] \arrow[d] & E_\rho \arrow[d] \\
Y \arrow[r, "p"] & Y' = Y/N \arrow[r, "\pi'"] & X = Y'/H.
\end{tikzcd}$$

Denote the covering $Y \to Y'$ by p, as in the diagram above. The pull-back of E_ρ is given by

$$\pi'^* E_\rho = \left\{ \left(y', [(x,v)]\right) \middle| \, y' \in Y', [(x,v)] \in E_\rho \text{ and } \pi'(y') = \pi(x) \right\}.$$

With these notations, the isomorphism $Y \times_N V_{\mathrm{res}^G_N(\rho)} \to \pi'^* E_\rho$ is given by

$$[(y,v)] \mapsto (p(y), [(y,v)]) . \tag{13}$$

This is clearly well defined: Indeed, for $n \in N$, we have

$$\left[(yn^{-1}, nv)\right] \mapsto \left(p(yn^{-1}), \left[(yn^{-1}, nv)\right]\right) = (p(y), [(y,v)]) .$$

To construct its inverse, choose a representative $y \in Y$ such that $p(y) = y'$. Since $\pi'(y') = \pi(x)$ there exists a unique $g \in G$ such that $y = xg^{-1}$. With this g, we define the morphism

$$\vartheta \colon \pi'^* E_\rho \to Y \times_N V_{\mathrm{res}^G_N} \text{ by setting } \vartheta((p(y), [(x,v)])) = [(y, gv)] .$$

By definition ϑ is the inverse of (13). It remains to check that this map is independent of the choice of y and of the representative (x,v): Let $n \in N$ and $h \in G$. Then we have

$$(p(y), [(x,v)]) = \left(p(yn^{-1}), \left[(xh^{-1}, hv)\right]\right) .$$

With the element g defined above, we have the equality $yn^{-1} = xh^{-1}(hg^{-1}n^{-1})$. That is, on the new representative our morphism is given by

$$\begin{aligned} \vartheta\left(\left(p(yn^{-1}), \left[(xh^{-1}, hv)\right]\right)\right) &= \left[\left(yn^{-1}, (hg^{-1}n^{-1})^{-1}(hv)\right)\right] \\ &= \left[\left(yn^{-1}, ngv\right)\right] \\ &= [(y, gv)] \\ &= \vartheta((p(y), [(x,v)])) . \end{aligned}$$

This completes the proof of the assertion.

In particular, $\pi'^* E_\rho$ is a trivial fibre bundle if and only if $N \subset \ker(\rho)$: Indeed, in this case, ρ, by definition, factors over H and the restricted representation $\mathrm{res}^G_N(\rho)$ is trivial. Thus, in the diagram above, we have

$$\pi'^* E_\rho = Y \times_N V_{\mathrm{res}^G_N(\rho)} = Y' \times V_\rho .$$

This shows that the locally constant system E_ρ comes with a canonical trivialization of its pull-back to the Galois cover with Galois group $H := G/\ker(\rho)$ and any covering lying over this one. □

Observe that in particular over Y itself, we have the canonical isomorphism $Y \times V_\rho \xrightarrow{\sim} \pi^* E_\rho, (y,v) \mapsto (y, [(y,v)])$.

REMARK 15. — Let us end our considerations of trivializing locally constant systems after pull-back with the following special case of the trivialization described in Lemma 14 above that will be important in the sequel:
Let A be our abelian surface and $A[n]$ be the (abelian) group of n-torsion points on it. Denote by $A[n]^\vee$ the dual group and let $\sigma, \tau \in A[n]^\vee$ be two characters of order s and t, respectively. Denote by $L_{A,\sigma}$, $L_{A,\tau}$ and $L_{A_\sigma\tau}$ the locally constant systems on the abelian surface corresponding to σ, τ and their product $\sigma\tau$.
Let further $l \in \mathbb{N}$ be a natural number that divides n and is a multiple of both s and t. Our considerations made in the proof of Lemma 14 show that under the intermediate Galois cover given by the multiplication-by-l-map

$$A \xrightarrow{\frac{n}{l}} A \xrightarrow{l} A$$

all three $L_{A,\sigma}$, $L_{A,\tau}$ and $L_{A,\sigma\tau}$ become trivial. More explicitly we have a canonical 1-section of $l^*L_{A,\sigma}$ given by

$$u_\sigma \colon A \longrightarrow l^*L_{A,\sigma} = \{(y,[x,z])\,|ly = nx\}\,, y \mapsto \left(y, [\frac{l}{n}y, 1]\right).$$

Here, $\frac{l}{n}y$ denotes a point y' such that $\frac{n}{l}y' = y$. This 1-section is well-defined since another choice y'' for such a point differs from y' just by a translation with a point $a \in A[\frac{n}{l}]$. Thus, one finds

$$[y'', 1] = [y' + a, 1] = [y', \sigma(-a)1] = [y', 1],$$

since the $\frac{n}{l}$-torsion points lie in the kernel of the character σ.
Analogously the 1-sections u_τ and $u_{\sigma\tau}$ of $l^*L_{A,\tau}$ and $l^*L_{A_\sigma\tau}$ are defined.
Interpreting u_σ, u_τ and $u_{\sigma\tau}$ as elements in the spaces of global sections of the corresponding locally constant sheaves of $\mathbb{C}$-modules $L_{A,\sigma}$, $L_{A,\tau}$ and $L_{A,\sigma\tau}$ (again we use the same symbols for both interpretations of locally constant systems) by definition it follows that they are compatible with multiplication:

$$\begin{aligned} \mathrm{H}^0(A, L_{A,\sigma}) \otimes \mathrm{H}^0(A, L_{A,\tau}) &\xrightarrow{\cup} \mathrm{H}^0(A, L_{A,\sigma} \otimes L_{A,\tau}) = \mathrm{H}^0(A, L_{A,\sigma\tau}) \\ u_\sigma \otimes u_\tau &\mapsto u_\sigma \cup u_\tau = u_{\sigma\tau} \end{aligned}$$

Accordingly, the corresponding trivializations of the locally constant $\mathbb{C}$-module sheaves are compatible with the isomorphism $L_{A,\sigma} \otimes L_{A,\tau} = L_{A,\sigma\tau}$ in the sense that the following diagram of sheaves is commutative:

$$\begin{array}{ccc} L_{A,\sigma} \otimes L_{A,\tau} & \xrightarrow{=} & L_{A,\sigma\tau} \\ {\scriptstyle (u_\sigma \otimes u_\tau)}\uparrow & & \uparrow{\scriptstyle u_{\sigma\tau}} \\ \mathbb{C} \otimes \mathbb{C} & \xrightarrow[=]{} & \mathbb{C}. \end{array} \tag{14}$$

Our next aim is to describe the direct image sheaf $\pi_*\mathbb{C}_Y$ as the sheaf of sections of a certain locally constant system.

Denote by $\mathbb{C}^G = \{\varphi\colon G \to \mathbb{C}\}$ the space of complex valued functions on G provided with the ring structure given by pointwise addition and multiplication. The group G operates on $\mathbb{C}^G$ from the left via $(g\varphi)(h) = \varphi(g^{-1}h)$. Analogously, one has a compatible right-action given by $(\varphi g)(h) = \varphi(hg^{-1})$. Obviously, both actions are compatible with the ring structure. The representation $\mathbb{C}^G$ has a basis given by the delta functions ε_g defined by $\varepsilon_g(h) = \delta_{g,h}$. In this basis the multiplication is given by $\varepsilon_g\varepsilon_h = \delta_{g,h}\varepsilon_g$. The left-$G$-action reads $h\varepsilon_g = \varepsilon_{hg}$ and accordingly the right action is given by $\varepsilon_g h = \varepsilon_{gh}$.

Assume now that G is a finite abelian group. In this case, every irreducible representation of G is one dimensional. Denote by $G^\vee$ the dual group of characters of G. This is also a finite abelian group. The characters $\sigma \in G^\vee$ form a basis of $\mathbb{C}^G$. Denote by $\mathbb{C}[G^\vee]$ the group ring of $G^\vee$. Recall that this is the $\mathbb{C}$ algebra with basis e_σ, indexed by the elements of $G^\vee$, and multiplication given by $e_\sigma e_\tau = e_{\sigma\tau}$. In this case, we have a ring isomorphism

$$\mathbb{C}[G^\vee] \xrightarrow{\sim} \mathbb{C}^G, e_\sigma \mapsto \sigma.$$

The induced G-structure on $\mathbb{C}[G^\vee]$ is given by $ge_\sigma = \sigma(g^{-1})e_\sigma$, and thus is immediately seen to be compatible with the ring structure.

Let us come back to the general case. Denote by $\mathcal{R}_X$ the $\mathbb{C}_X$-module sheaf of locally constant sections of the locally constant system $Y \times_G \mathbb{C}^G$. By construction, since the compatible right-G-action and the ring structure survive the transition from $Y \times \mathbb{C}^G$ to the G-quotient, $\mathcal{R}_X$ is a sheaf of $\mathbb{C}$-algebras on which G acts from the right. Observe that it is the same with the sheaf $\pi_*\mathbb{C}_Y$: This is a sheaf of $\mathbb{C}$-algebras by construction and carries the G module structure given for $s \in \Gamma(U, \pi_*\mathbb{C}_Y) = \Gamma(\pi^{-1}(u), \mathbb{C}_Y)$ by $sg = s \circ g$, where the second g denotes the automorphism of $\pi^{-1}(U)$ induced by the right-G-structure of Y. We have the following

LEMMA 16. — *The sheaf $\mathcal{R}_X$ defined above is canonically isomorphic to the direct image sheaf $\pi_*\mathbb{C}_Y$ as a right $\mathbb{C}[G]$-algebra.*

Proof. We shall construct a morphism of sheaves $\pi_*\mathbb{C}_Y \to \mathcal{R}_X$, and check on the level of stalks that it is an isomorphism. Let $U \subset X$ be an open set. Without loss of generality, we assume that U is connected. To a section $s \in \Gamma(U, \pi_*\mathbb{C}_Y) = \Gamma(\pi^{-1}(U), \mathbb{C}_Y)$ we associate a section $S \in \Gamma(U, \mathcal{R}_X)$ as follows:

Choose a point $y \in \pi^{-1}(U)$. The section s defines a complex number s_{yg} for each translated point $yg \in \pi^{-1}(u)$. Set

$$S(u) = \left[\left(y, \textstyle\sum_{g\in G} s_{yg}\varepsilon_g\right)\right].$$

Observe that $S(u)$ is in fact a constant section, since s is constant.
It remains to show that $S(u)$ is well defined: If $y' \in \pi^{-1}(u)$ is another choice, there exists an $h \in G$ such that $y' = yh$. We have

$$\begin{aligned}\left[\left(y', \sum_g s_{y'g}\varepsilon_g\right)\right] &= \left[\left(yh, \sum_g s_{yhg}\varepsilon_g\right)\right] \\ &= \left[\left(y, h\sum_g s_{yhg}\varepsilon_g\right)\right] \\ &= \left[\left(y, \sum_g s_{y(hg)}\varepsilon_{hg}\right)\right] = S(u).\end{aligned}$$

On the level of stalks, this morphism is seen to be an isomorphism: After choosing a point $y \in \pi^{-1}(u)$, the stalk $(\pi_*\mathbb{C})_u$ is isomorphic to $\{(s_{yg})_{g\in G} | s \in \Gamma(V, \pi_*\mathbb{C})\}$, where V is a small simply connected neighborhood of u and thus $\pi^{-1}(V) = V \times G$. The stalk of $\mathcal{R}_X$ at u is given as $\pi^{-1}(u) \times_G \mathbb{C}^G$. The morphism described above reads now

$$(s_{yg})_g \mapsto \left[\left(y, \sum_g s_{yg}e_g\right)\right].$$

This is well defined, as we have seen above. Its inverse morphism is given by $\left[\left(y, \sum_g a_g e_g\right)\right] \mapsto (a_g)_g$.
The fact that this is indeed a morphism of sheaves of rings is also seen immediately on the level of stalks: In both cases the stalk is isomorphic to $\bigoplus_g \mathbb{C}\varepsilon_g$ with the same ring structure given by $\varepsilon_g\varepsilon_h = \delta_{g,h}\varepsilon_g$. Similarly, both stalks carry the same right-G-action given by $\varepsilon_g h = \varepsilon_{gh}$ and thus the lemma follows. □

Due to Lemma 16 above, we will not distinguish between the sheaf $\pi_*\mathbb{C}_Y$ and $\mathcal{R}_X$ in the sequel. More generally we will often identify a locally constant system with the corresponding locally constant $\mathbb{C}$-module sheaf of its sections.
Applying the Leray spectral sequence to π, one finds an isomorphism of rings

$$\mathrm{H}^*(Y, \mathbb{C}) = \mathrm{H}^*(X, \mathcal{R}_X).$$

We shall now investigate the meaning of Lemma 16 in the case of a finite abelian group G. In this case, as we have seen above, we have a ring isomorphism $\mathbb{C}^G = \mathbb{C}[G^\vee]$, and thus $\mathcal{R}_X$ is the sheaf of sections of $Y \times_G \mathbb{C}[G^\vee]$. Since the representation $\mathbb{C}[G^\vee]$ decomposes as $\mathbb{C}[G^\vee] = \bigoplus_{\sigma\in G^\vee} \mathbb{C}e_\sigma$, so does $\mathcal{R}_X$:

$$\mathcal{R}_X = \bigoplus_{\sigma\in G^\vee} L_\sigma,$$

where L_σ is the locally constant $\mathbb{C}$-module of rank 1 corresponding to the locally constant system $Y \times_G \mathbb{C}e_\sigma$. Due to the ring structure of $\mathcal{R}_X$, we have a commutative and associative system of isomorphisms $L_\sigma \otimes L_\tau \xrightarrow{\sim} L_{\sigma\tau}$. Furthermore, we have seen that $\mathcal{R}_X$ still carries a G-action in this case, which is given for $g \in G$ by multiplication with $\sigma(g^{-1})$ on L_σ. Accordingly, the isomorphism of rings given by the Leray spectral sequence reads now

$$\mathrm{H}^*(Y,\mathbb{C}) = \bigoplus_{\sigma\in G^\vee} \mathrm{H}^*(X, L_\sigma).$$

We have seen in Lemma 14 above that L_σ comes together with a canonical trivialization of $\pi^* L_\sigma$ over Y. As L_σ carries the G-action given by σ^{-1} however, its inverse image $\pi^* L_\sigma$ is also a sheaf with a G-action. This action is still given by the character σ^{-1}, thus the sheaf $\pi^* L_\sigma$ is canonically identified with the constant sheaf $\mathbb{C}_\sigma$ on which G acts via σ^{-1}.
On the level of cohomology, we find that the induced action on $\mathrm{H}^*(Y, \pi^* L_\sigma)$, compared to the action on $\mathrm{H}^*(Y,\mathbb{C})$ is also twisted by σ^{-1}. This means that for $g \in G$, $\alpha \in \mathrm{H}^*(Y,\mathbb{C})$ and $\alpha_\sigma \in \mathrm{H}^*(Y, \pi^* L_\sigma)$ the corresponding class in $\mathrm{H}^*(Y, \pi^* L_\sigma)$ given by the trivialization of $\pi^* L_\sigma$, we have the following operation:

$$g\alpha_\sigma = \sigma^{-1}(g) g\alpha.$$

In particular, an invariant class in $\mathrm{H}^*(Y, \mathbb{C}_\sigma)$ corresponds to a class in the σ-weight space of $\mathrm{H}^*(Y,\mathbb{C})$, i.e. the space $\{\alpha \in \mathrm{H}^*(Y,\mathbb{C}) | g\alpha = \sigma(g)\cdot\alpha$ for all $g \in G\}$.
One has $\mathrm{H}^*(X, L_\sigma) = \mathrm{H}^*(Y, \mathbb{C}_\sigma)^G$ by a theorem of Grothendieck (cf. [**14**], §5). It follows that in the isomorphism of cohomology rings induced by the Leray spectral sequence

$$\mathrm{H}^*(Y,\mathbb{C}) = \bigoplus_{\sigma\in G^\vee} \mathrm{H}^*(X, L_\sigma),$$

the component $\mathrm{H}^*(X, L_\sigma)$ is just the σ-weight space with respect to the G-action on the cohomology of Y.
Observe that the cup product on the right hand side of this isomorphism distributes

$$\mathrm{H}^*(X, L_\sigma) \otimes \mathrm{H}^*(X, L_\tau) \to \mathrm{H}^*(X, L_{\sigma\tau}),$$

due to the isomorphism $L_\sigma \otimes L_\tau = L_{\sigma\tau}$.

We will now apply these observations to our situation of the Hilbert scheme. We find the following

LEMMA 17. — *Let X be a compact algebraic surface and let G be an abelian quotient of $\pi_1(X)$. Then we have a G-Galois cover $Y \longrightarrow X^{[n]}$ and an isomorphism*

$$\mathrm{H}^*(Y,\mathbb{C}) = \bigoplus_{\sigma\in G^\vee} \mathrm{H}^*(X^{[n]}, L_\sigma),$$

Proof. As stated in the first chapter, in [**1**], it is shown that $\pi_1(X^{[n]}) = \pi_1(X)^{\mathrm{ab}}$ sucht that G is also a quotient group of $\pi_1(X^{[n]})$, therefore the existence of Y. The rest follows directly from our considerations above. □

Recall diagram (1) from Section 1 in Chapter 2

$$\begin{tikzcd} A \times K^{(n-1)}A \arrow[r, "\nu"] \arrow[d, "p_A"'] & A^{[n]} \arrow[d, "s"] \\ A \arrow[r, "n"'] & A. \end{tikzcd}$$

As we have seen, ν realizes the Hilbert scheme $A^{[n]}$ as a quotient of $A \times K^{(n-1)}A$ by the group $A[n]$ of n-torsion points of the abelian surface, so we can apply the lemma above: Taking the direct image of the constant sheaf $\mathbb{C}_{A\times K^{(n-1)}A}$, we get $\nu_*\mathbb{C} = \bigoplus L_{A^{[n]},\sigma}$, parameterized by the characters $\sigma \in A[n]^\vee$ of $A[n]$. Observe that, since $\nu_*\mathbb{C} = s^*n_*\mathbb{C}$, these local systems are inverse images of corresponding locally constant systems on the surface: $L_{A^{[n]},\sigma} = s^*L_{A,\sigma}$.
Summarizing, we have found a first description of the cohomology ring of generalized Kummer varieties in terms of the cohomology of the Hilbert scheme:

PROPOSITION 18. — *The cohomology ring of $A \times K^{(n-1)}A$ is given by*

$$\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C}) = \bigoplus_{\sigma\in A[n]^\vee} \mathrm{H}^*(A^{[n]}, L_{A^{[n]},\sigma})$$

□

The aim of the next section is to give a decomposition of the cohomology groups $\mathrm{H}^*(A^{[n]}, L_{A^{[n]},\sigma})$ by using intersection cohomology.

2. Intersection cohomology.

Let $\lambda = (l_1 \geq l_2 \geq \ldots \geq l_{|\lambda|}) = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ be a partition of n. Denote by $S^\lambda A$ the space $\prod_i S^{\alpha_i}A$ and set $\gcd(\lambda) := \gcd(l_i)$, the greatest common divisor of the l_i's. In [**12**], Göttsche and Soergel use intersection cohomology to prove the following theorem of the structure of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ as a vector space:

$$\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C}) = \bigoplus_{\lambda} \bigoplus_{x\in A[\gcd(\lambda)]} \mathrm{H}^*(S^\lambda A, \mathbb{C})[2(|\lambda| - n)].$$

In this section, we will give a somehow dual description of the cohomology of $A \times K^{(n-1)}A$ using locally constant systems. The first ingredient for this description is again the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber in a rather weak form (cf. [**12**], Theorem 3 and Proposition 1):
Consider the following diagram:

$$\begin{tikzcd} & A^{[n]} \arrow[d, "\rho"] \\ S^\lambda A \arrow[r, "\kappa_\lambda"] & S^n A \end{tikzcd}$$

Here, ρ denotes — as usual — the Hilbert-Chow-morphism and κ_λ is the composition $S^\lambda A \to \overline{S^n_\lambda A} \hookrightarrow S^n A$. The decomposition theorem gives the following quasi-isomorphism in the derived category of sheaves of $\mathbb{C}$-modules:

$$\rho_*\mathbb{C}_{A^{[n]}} = \bigoplus_{\lambda} \kappa_{\lambda *}\mathbb{C}_{S^\lambda A}[2(|\lambda| - n)]. \tag{15}$$

Recall diagram (6) in Section 1 of Chapter 2:

$$\begin{tikzcd}
A \times K^{(n-1)}A \arrow[r, "\nu"] \arrow[d, "\rho'"'] & A^{[n]} \arrow[d, "\rho"] \\
A \times (S^n A)_0 \arrow[r, "\nu'"] \arrow[d, "p_A"'] & S^n A \arrow[d, "\Sigma"] \\
A \arrow[r, "n"] & A.
\end{tikzcd}$$

With the notations introduced there, we have $\mathcal{R}_{A^{[n]}} = \nu_*\mathbb{C} = \rho^*\nu'_*\mathbb{C} =: \rho^*\mathcal{R}_{S^nA}$. Twisting equation (15) with the sheaf $\mathcal{R}_{A^{[n]}}$ gives the following

LEMMA 19. — *Denote by $\mathcal{R}_{S^\lambda A} := \kappa_\lambda^*\mathcal{R}_{S^nA}$. Then, one has a natural isomorphism of vector spaces*

$$\mathrm{H}^*(A^{[n]}, \mathcal{R}_{A^{[n]}}) = \bigoplus_{\lambda \in P(n)} \mathrm{H}^*(S^\lambda A, \mathcal{R}_{S^\lambda A})[2(|\lambda| - n)].$$

Proof.
With the notations introduced above, we conclude

$$\begin{aligned}
\rho_*\mathcal{R}_{A^{[n]}} &= \rho_*\left(\mathbb{C}_{A^{[n]}} \otimes \rho^*\mathcal{R}_{S^nA}\right) = \rho_*\mathbb{C}_{A^{[n]}} \otimes \mathcal{R}_{S^nA} \\
&= \bigoplus\nolimits_\lambda \kappa_{\lambda *}\mathbb{C}_{S^\lambda A}[2(|\lambda| - n)] \otimes \mathcal{R}_{S^nA} \\
&= \bigoplus\nolimits_\lambda \kappa_{\lambda *}\left(\mathbb{C}_{S^\lambda A} \otimes \kappa_\lambda^*\mathcal{R}_{S^nA}\right)[2(|\lambda| - n)] \\
&= \bigoplus\nolimits_\lambda \kappa_{\lambda *}\mathcal{R}_{S^\lambda A}[2(|\lambda| - n)],
\end{aligned}$$

where have used the projection formula, observing that $\rho_* = \rho_!$ and analogously for κ_λ, since they are morphisms between projective varieties, and the decomposition formula (15).
The lemma now follows by pushing forward to a point. □

The following lemma gives a description of the cohomology of $S^\lambda A$ with values in $\mathcal{R}_{S^\lambda A}$:

LEMMA 20. — *One has:*

$$\mathrm{H}^*(S^\lambda A, \mathcal{R}_{S^\lambda A}) = \mathrm{H}^*(S^\lambda A, \mathbb{C}[A[\gcd(\lambda)]^\vee]),$$

where $\mathbb{C}[A[\gcd(\lambda)]^\vee]$ denotes the group ring of the group of characters on the $\gcd(\lambda)$-torsion points of A, considered as a constant sheaf on $S^\lambda A$.

Proof. We have $\mathcal{R}_{S^\lambda A} = \bigoplus_{\sigma\in A[n]^\vee} L_{S^\lambda A,\sigma}$ with the abbreviation $L_{S^\lambda A,\sigma} := \kappa_\lambda^* L_{S^nA,\sigma}$. As a first step, we prove the following:

$$\mathrm{H}^*(S^\lambda A, L_{S^\lambda A,\sigma}) \simeq \begin{cases} \mathrm{H}^*(S^\lambda A,\mathbb{C}) & \text{if there is an isomorphism } L_{S^\lambda A,\sigma}\simeq\mathbb{C},\\ 0 & \text{otherwise.}\end{cases}$$

By a theorem of Grothendieck (cf. [**14**], §5) , we have

$$\mathrm{H}^*(S^\lambda A, L_\sigma) = \mathrm{H}^*(A^{|\lambda|}, p^*L_\sigma)^{\prod \mathfrak{S}_{\alpha_i}},$$

where $p\colon A^{|\lambda|} \longrightarrow S^\lambda A$ denotes the projection. We will show now that p^*L_σ is trivial as a $\mathbb{C}$-module if and only if L_σ is trivial:
We have to consider the case $p\colon A^n \longrightarrow S^nA$ only. Here, we have $p_* : \pi_1(A^n) = \pi_1(A)^n \twoheadrightarrow \pi_1(S^nA) = \pi_1(A)$ (cf. [**1**]). If we regard σ as a representation of the fundamental group of S^nA, the representation belonging to the local system p^*L_σ comes from the reduction along the surjective group homomorphism p_* (cf. Section 1). This representation is trivial if and only if σ is the trivial character.
Let $T = \mathbb{C}^d/\Gamma$ be a complex torus, and let σ be a non-trivial character $\sigma\colon \Gamma\to\mathbb{C}^\times$ of finite order s. Then due to the étale covering $s\colon T\longrightarrow T$ given by multiplication with s, we find

$$\mathrm{H}^*(T,\mathbb{C}) = \mathrm{H}^*(T, s_*\mathbb{C}) = \mathrm{H}^*(T,\mathbb{C}) \oplus \bigoplus_{1\neq\tau\in(\Gamma/s\Gamma)^\vee} \mathrm{H}^*(T, L_\tau).$$

Since σ was assumed to be non-trivial of order s, it factors over $\Gamma/s\Gamma$. So we can interpret it as $1\neq\sigma\in(\Gamma/s\Gamma)^\vee$. It follows that the cohomology of a torus with coefficients in a (non-trivial) local system is zero.
Thus, we have shown that $\mathrm{H}^*(A^{|\lambda|}, p^*L_\sigma)\neq 0$ if and only if L_σ is trivial.

Since a character $\sigma\in A[n]^\vee$ comes from $A[\gcd(\lambda)]^\vee$ if and only if its order divides $\gcd(\lambda)$, the only thing that remains to show is that the sheaf $L_{S^\lambda A,\sigma}$ on $S^\lambda A$ is trivial if and only if the greatest common divisor of λ is a multiple of the order of the character σ. But this follows directly from our considerations in Section 1: Consider the commutative diagram

$$\begin{tikzcd} S^\lambda A \arrow[r] \arrow[d, "s^\lambda_{\mathrm{red}}"'] & S^n_\lambda A \arrow[r, hook] & S^nA \arrow[d, "\Sigma"] \\ A \arrow[rr, "l"] & & A. \end{tikzcd}$$

Here, l denotes multiplication with the greatest common divisor of the partition λ, $\Sigma\colon S^nA\to A$ is the summation morphism and s^λ_{red} is the 'reduced weighted summation', which is defined by

$$S^\lambda A = \prod S^{\alpha_i}A \ni (\textstyle\sum_{j=1}^{\alpha_i} a_{ij})_i \mapsto \sum_i \frac{i}{l}\Sigma(\sum_{j=1}^{\alpha_i} a_{ij}) \in A.$$

Since the locally constant systems on $S^\lambda A$ are inverse images of the corresponding sheafs on A , we have $L_{S^\lambda A,\sigma} = (s^\lambda_{\mathrm{red}})^* l^* L_{A,\sigma}$. So, $L_{S^\lambda A,\sigma}$ is trivial if and only if $l^*L_{A,\sigma}$ is trivial, which is the case if and only if the covering of A given by multiplication with l factors over $A \xrightarrow{\cdot\mathrm{ord}(\sigma)} A$, i.e. if and only if $\mathrm{ord}(\sigma)$ is a factor of $l = \mathrm{gcd}(\lambda)$. $\square$

We can reformulate the above lemma to get the following

COROLLARY 21. — *We have a natural isomorphism of vector spaces*

$$\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C}) \simeq \bigoplus_{\lambda \in P(n)} \bigoplus_{\sigma \in A[\mathrm{gcd}(\lambda)]^\vee} \mathrm{H}^*(S^\lambda A, \mathbb{C})[2(|\lambda| - n)].$$

$\square$

In the next section we will use a generalization of Nakajima's description of the cohomology of $A^{[n]}$ to give a more geometric version of this decomposition.

3. Nakajima's description

In [**23**] Nakajima constructs in a geometric way a representation of the Heisenberg algebra $\mathfrak{h}$ modelled on the cohomology of a surface X on the space $\mathbb{H} := \bigoplus_n \mathrm{H}^*(X^{[n]})$ and proves that $\mathbb{H}$ is an irreducible $\mathfrak{h}$-module.
In this section, we will recall his construction and generalize it in the case of an abelian surface to cohomology with values in a locally constant system. Further, we give an alternative description of how the cohomology of the Hilbert schemes is generated by Nakajima's operators applied to the vacuum.
Here and in the following we will use the words 'symmetric' and 'commutative' and also all occurring Lie brackets in a $\mathbb{Z}/2$-graded sense. E.g. a bilinear form $\langle -,- \rangle$ on a vector space $\mathfrak{g}$ is symmetric if for all $x, y \in \mathfrak{g}$ we have

$$\langle x, y \rangle = (-1)^{\deg(x)\deg(y)} \langle y, x \rangle.$$

DEFINITION 22. — Let $\mathfrak{g}$ be a complex Lie algebra equipped with an invariant symmetric bilinear form $\langle -,- \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$. Its *Loop algebra* is defined to be $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with the following Lie bracket:

$$[x_n, y_m] = [x, y]_{n+m},$$

where we define $x_n := x \otimes t^n$ for $x \in \mathfrak{g}$.
Its *affinization* $\tilde{\mathfrak{g}}$ is the algebra $\tilde{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ where c is central and the Lie bracket is given by

$$[x_n, y_n] = [x, y]_{n+m} + n\delta_{n,-m} \langle x, y \rangle c.$$

We are interested in the special case $\mathfrak{g} = \mathrm{H}^*(A, \mathbb{C})$, the cohomology of an abelian surface, with the trivial Lie bracket and the bilinear form given by

$$\langle \alpha, \beta \rangle := -\int_A \alpha \cup \beta.$$

The resulting algebra $\mathfrak{h} := \tilde{\mathfrak{g}} = \mathrm{H}^*(A)[t, t^{-1}] \oplus \mathbb{C}c$ is called the *Heisenberg algebra of* $\mathrm{H}^*(A)$ and satisfies the commutator relations

$$[\alpha_n, \beta_m] = n\delta_{n,-m}\langle \alpha, \beta \rangle c.$$

In order to define Nakajima's operators it is convenient to work within the framework of *Borel-Moore homology*. This homology theory can be defined sheaf theoretically, which can be found e.g. in [**3**]. In the case of constant coefficients however, there is an alternative account that can be found in Appendix B of [**9**] and which we want to remark here:
Let X be a topological space that can be embedded as a closed subspace of a Euclidean space $\mathbb{R}^n$. Then its Borel-Moore homology groups can be defined by

$$\mathrm{H}_i(X, \mathbb{C}) := \mathrm{H}^{n-i}(\mathbb{R}^n, \mathbb{R}^n – X; \mathbb{C}).$$

One shows that this definition is independent of the chosen embedding, and that furthermore, for closed subspace of an oriented differentiable manifold M^n one has a canonical isomorphism

$$\mathrm{H}_i(X, \mathbb{C}) = \mathrm{H}^{n-i}(M, M – X; \mathbb{C}).$$

That this is the right way of thinking of Borel-Moore homology in the case of values in a locally constant system follows from the subsequent list of

FACTS 23. — *Let L be a locally constant system on X and denote by $\mathrm{H}_i(X, L)$ the Borel-Moore homology of X with values in L. Then one has*
*1.) H_i is a covariant functor with respect to proper maps, i.e. if $f\colon Y \to X$ is proper, it induces a morphism $f_*\colon \mathrm{H}_i(Y, f^*L) \to \mathrm{H}_i(X, L)$ and for the composition of two proper maps f and g, one has $(f \circ g)_* = f_* \circ g_*$. ([**3**], V.4.5)*
2.) Let $j\colon A \to X$ be a closed embedding of A in a compact complex (weak homology) manifold X of dimension $\dim(X) = n$. Then there is an isomorphism

$$\mathrm{H}_i(A, L) = \mathrm{H}^{2n-i}(X, X – A; L). \tag{16}$$

*In particular for X itself, we have $\mathrm{H}_i(X, L) = \mathrm{H}^{2n-i}(X, L)$. Furthermore, with respect to this isomorphism, the push-forward $j_*\colon \mathrm{H}_i(A, L) \to \mathrm{H}_i(X, L)$ is given by $I^*\colon \mathrm{H}^{2n-i}(X, X – A, L) \to \mathrm{H}^{2n-i}(X, L)$. ([**3**], V.9.3)*
3.) Let X be as in (2.), let L and M locally constant systems on X and let A, B be closed subspaces of X. By the isomorphism (16) and the cup product on relative cohomology

$$\mathrm{H}^{2n-i}(X, X – A; L) \otimes \mathrm{H}^{2n-j}(X, X – B; M) \to \\ \mathrm{H}^{2n-(i+j-2n)}(X, X – (A \cap B); L \otimes M)$$

we get an intersection product

$$\cdot\colon \mathrm{H}_i(A, L) \otimes \mathrm{H}_j(B, M) \to \mathrm{H}_{i+j-2n}(A \cap B, L \otimes M)$$

in homology. Observe that this product depends on X although this is not explicitly denoted in the formula above.
*4.) There is a cap-product between cohomology and homology ([**3**],V.10) that satisfies the usual rules:*

$$\cap\colon \mathrm{H}^j(X, L) \otimes \mathrm{H}_i(X, M) \to \mathrm{H}_{i-j}(X, L \otimes M).$$

*5.) Let $Z \subset X$ be a closed subvariety of the complex manifold X. Assume further that Z has only one irreducible component of top dimension m. Then there exists a fundamental class $[Z] \in \mathrm{H}_{2m}(X, \mathbb{C})$. ([**9**], B.3)*

Observe that the isomorphism (16) shows that the naïve definition of Borel-Moore homology via relative cohomology in the case of constant coefficients is the right way of thinking of this homology theory in our case also.

We are now ready to recall the notion of a correspondence (For a detailed description cf. [**10**].): Let X_1 and X_2 be smooth projective complex varieties. A class u in the Chow group $A_n(X_1 \times X_2)$ of the product of X_1 and X_2 is called a *correspondence* between X_1 and X_2. (We work with rational or even complex coefficients because we are not interested in rationality questions.) The case we are most interested in is that of irreducible correspondences that is a correspondence given by a class $u = [Z]$ induced by an (irreducible) closed subvariety $Z \subset X_1 \times X_2$ of dimension n. We will denote the image of u in $\mathrm{H}_{2n}(X_1 \times X_2)$ by the same symbol. Let p_i, $i = 1, 2$ be the projections from $X_1 \times X_2$ to the factor X_i. Observe that the pull-back $p_1^* y$ of a class $y \in \mathrm{H}^*(X_1, \mathbb{C})$ can be interpreted as a homology class which we denote by the same symbol $p_1^* y \in \mathrm{H}_*(X \times Y, \mathbb{C})$, since $X \times Y$ is a manifold. A correspondence induces a linear map u_* on the level of (co)homology, which is given by

$$u_*\colon \mathrm{H}_i(X_1, \mathbb{C}) \to \mathrm{H}_{i+2n-2\dim(X_1)}(X_2, \mathbb{C}); y \mapsto p_{2*}(p_1^* y \cdot u)$$

or

$$u_*\colon \mathrm{H}^j(X_2, \mathbb{C}) \to \mathrm{H}^{j-2n+2\dim(X_2)}(X_1, \mathbb{C}); y \mapsto \mathrm{PD}^{-1} p_{1*}(u \cap p_2^* y),$$

where $\mathrm{PD}\colon \mathrm{H}^*(X_1) \to \mathrm{H}_*(X_1)$ is the Poincaré duality map. We will call these map also correspondences.
Assume that X_3 is another smooth projective variety and that $v \in A_m(X_2 \times X_3)$ is a correspondence between X_2 and X_3. Denote further by p_{ij} the projection from the triple product $X_1 \times X_2 \times X_3$ to the factors $X_i \times X_j$. The correspondence

$$w := p_{13*}(p_{12}^* u \cdot p_{23}^* v) \in A_{n+m-dim(X_2)}(X_1 \times X_3)$$

is called the *product* of u and v and denoted by $w = u \circ v$. With this definition of a product it is also clear how to define the commutator of two correspondences if it makes sense.

On the level of (co)homology we have

$$(u \circ v)_* = u_* \circ v_*.$$

Suppose now that $Z_{12} \subset X_1 \times X_2$ and $Z_{23} \subset X_2 \times X_3$ are closed subvarieties and that $u = [Z_{12}]$ and $v = [Z_{23}]$. Let further

$$W := p_{13}\left(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})\right). \tag{17}$$

Then the product-correspondence w defined above is already defined in $A_*(W)$. One often uses this fact in order to show the vanishing of a correspondence: If the dimension of the intersection W is smaller then the degree of w the product-correspondence has to be zero.

We will now generalize the map induced by a correspondence on the level of homology under certain assumption to the case of homology with twisted coefficients. Let X and Y be smooth projective complex varieties and let L and M be locally constant systems on X and Y respectively. Denote by p_i, $i = 1, 2$ the projections from $X \times Y$ to X and to Y, respectively. Let $Z \subset X \times Y$ be a closed subvariety and denote by $[Z]$ the correspondence between X and Y induced by Z. We assume further that over Z we have a canonical isomorphism $p_1^* L|_Z = p_2^* M|_Z$ of the restrictions of the locally constant sheaves.

Again, the pull-back $p_1^* y$ of a class $y \in \mathrm{H}^*(X, L)$ can be interpreted as a homology class which we denote by the same symbol $p_1^* y \in \mathrm{H}_*(X \times Y, p_1^* L)$, since $X \times Y$ is a manifold. Via the isomorphism of the restricted locally constant systems over Z the class $(p_1^* y \cdot [Z]) \in \mathrm{H}_*(Z, p_1^* L|_Z)$ corresponds to a class in $\mathrm{H}_*(Z, p_2^* M|_Z)$ wich we denote by the same symbol. Pushing this class forward to $\mathrm{H}_*(Y, M)$ along the second projection, we get a map $[Z]_L^M$ on the level of homology with twisted coefficients that is totally analogous to the map $[Z]_*$ defined above:

$$[Z]_L^M \colon \mathrm{H}_i(X, L) \to \mathrm{H}_{i+2\dim(Z)-2\dim(X)}(Y, M); y \mapsto p_{2*}(p_1^* y \cdot [Z]).$$

More generally, for any class $\alpha \in \mathrm{H}_k(Z, \mathbb{C})$, one has a correspondence

$$\mathrm{H}_i(X, L) \to \mathrm{H}_{i+k-2\dim(X)}(Y, M); y \mapsto p_{2*}(p_1^* y \cdot \alpha). \tag{18}$$

Since we assumed X and Y to be manifolds, these correspondences can be seen as homomorphisms between the cohomology groups of X and Y using the isomorphism given in Facts 23.2.

We will now define Nakajima's operators as certain correspondences. Since we want to generalize them to locally constant coefficients, let $\sigma \colon \pi_1(A) \to \mathbb{C}^\times$ be a character of finite order s of the fundamental group of our abelian surface A. It defines for all $n \in \mathbb{N}_0$ a locally constant system $L_\sigma = L_{A^{[ns]},\sigma}$ on $A^{[ns]}$ given by

$$L_\sigma = \left\{ [a, \eta, z] \in \left(A \times_A A^{[ns]} \right) \times_{A[ns]} \mathbb{C} \right\}.$$

Here, $A \times_A A^{[ns]}$ is the $A[ns]$-Galois cover of $A^{[ns]}$, induced by the morphism 'multiplication with (ns)' $A \to A$. With other words

$$A \times_A A^{[ns]} = \{(a, \eta) | s(\eta) = ns \cdot a\},$$

where $s\colon A^{[ns]} \to A$ is the summation morphism. In this notation, the group of ns-torsion points acts on $A \times_A A^{[ns]}$ via the action on the first factor. Thus two representatives (a, η, z) and (a', η', z') for points in L_σ are equivalent, if $\eta = \eta'$ and if there exists an $y \in A[ns]$ such that $a' = a + y$ and $z' = \sigma^{-1}(y) z$.
Define further $\mathbb{H}^\sigma := \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}^*(A^{[ns]}, L_\sigma))$. Due to the results of the last section, we have

$$\begin{aligned}
\mathrm{H}^i(A^{[ns]}, L_\sigma)[2ns] &= \bigoplus_{\substack{\lambda \in P(ns)\\ s|\gcd(\lambda)}} \mathrm{H}^i(S^\lambda A, \mathbb{C})[2|\lambda|] \\
&= \bigoplus_{\substack{\lambda : \lambda = s\lambda' \\ \text{for } \lambda' \in P(n)}} \mathrm{H}^i(S^\lambda A, \mathbb{C})[2|\lambda|] \\
&= \bigoplus_{\lambda' \in P(n)} \mathrm{H}^i(S^{\lambda'} A, \mathbb{C})[2|\lambda'|] \\
&= \mathrm{H}^i(A^{[n]}, \mathbb{C})[2n],
\end{aligned}$$

where for a partition $\lambda' = (l'_1 \geq \cdots \geq l'_{|\lambda'|})$, we have denoted by $s\lambda'$ the partition $(sl'_1 \geq \cdots \geq sl'_{|\lambda'|})$. Furthermore, we have used that $S^\lambda A = S^{\lambda'} A$ for $\lambda = s\lambda'$, since it depends only on the α's occurring in the partition which are not changed by the transition from λ' to λ. For the same reason, we have $|\lambda| = |\lambda'|$.
Thus, as a vector space, $\mathbb{H}^\sigma$ is just a stretched version of $\mathbb{H}$. We will show that this is true also on the level of representations.
We want now to define for a class $\alpha \in \mathrm{H}^*(A, \mathbb{C})$ and an integer $l \in \mathbb{Z}$ an operator $\alpha^\sigma_{ls}\colon \mathrm{H}^*(A^{[ns]}, L_\sigma) \to \mathrm{H}^*(A^{[(n+l)s]}, L_\sigma)$. In the special case of $\sigma = 1$, this gives us back Nakajima's original construction. In order to do that, we introduce the incidence schemes $A^{[n,n']} \subset A^{[n]} \times A^{[n']}$, where we assume that $n' > n \geq 0$. It is given as

$$A^{[n,n']} = \{(\xi, \xi') | \xi \subset \xi'\},$$

where $\xi \subset \xi'$ means that ξ is a subscheme of ξ'. For such a pair $\xi \subset \xi'$, we have an inclusion $I_{\xi'} \subset I_\xi$ of the ideal sheaves and the quotient $I_\xi / I_{\xi'}$ is an $\mathcal{O}_A$-module of finite length corresponding to the points where ξ' and ξ are different. Accordingly, there is an analog to the Hilbert-Chow morphism $\rho\colon A^{[n,n']} \to S^{n'-n} A$ given on closed points by $\rho(\xi, \xi') = \sum_{x \in A} l((I_\xi / I_{\xi'})_x) x$.

Define $A_0^{[n,n']}$ to be $\rho^{-1}(D)$, where $A \simeq D \subset S^{n'-n}A$ denotes the small diagonal. Finally let $\mathcal{Z} = \mathcal{Z}_{n,n'}$ be the component of $A_0^{[n,n']}$, that contains the locus of points where $\operatorname{supp}(I_\xi/I_{\xi'})$ is disjoint from ξ and give it the reduced scheme structure. Its dimension is $\dim(\mathcal{Z}) = n + n' + 1$: This follows from a theorem of Briançon (cf. [**4**]), where the dimension of $X_x^{[n]}$, the subset of $X_{(n^1)}^{[n]}$ consisting of subschemes centered in a point $x \in X$ is computed to be $\dim\left(X_x^{[n]}\right) = n-1$. We will consider $\mathcal{Z}$ as a subvariety of $A^{[n]} \times A^{[n']}$. By construction, it comes together with a morphism $\rho\colon \mathcal{Z} \to D \simeq A$, mapping a pair (ξ, ξ') to the point, they differ in.

Now, let $\alpha \in \mathrm{H}^*(A, \mathbb{C})$ be a cohomology class on the surface, let $n \in \mathbb{N}_0$ be a natural number, $l \in \mathbb{N}$ be a positive integer, let σ be a character of finite order s and let $\mathcal{Z} \subset A^{[ns]} \times A^{[(n+l)s]}$ be the subvariety defined above. Denote by p_1 and p_2 the projections from $A^{[ns]} \times A^{[(n+l)s]}$ to $A^{[ns]}$ and $A^{[(n+l)s]}$, respectively. Pulling back α to $\mathcal{Z}$ and capping with the fundamental class $[\mathcal{Z}]$ we get a class in the homology of $\mathcal{Z}$. As discussed above, this defines a correspondence

$$\alpha_{-ls}^{\sigma}\colon \mathrm{H}^*(A^{[ns]}, L_\sigma) \to \mathrm{H}^*(A^{[(n+l)s]}, L_\sigma),$$

given by $y \mapsto p_{2*}(p_1^* y \cdot (\rho^*\alpha \cap [\mathcal{Z}]))$, under the assumption that $p_1^* L_\sigma$ and $p_2^* L_\sigma$ are isomorphic over $\mathcal{Z}$. This is the case by the following

LEMMA 24. — *With the notations introduced above, one has*

$$p_1^* L_\sigma|_{\mathcal{Z}} = p_2^* L_\sigma|_{\mathcal{Z}}.$$

Proof. We have

$$p_1^* L_\sigma|_{\mathcal{Z}} = \left\{ (\xi, \xi'; [a, \eta, z]) \,\middle|\, \begin{smallmatrix} \operatorname{supp}(\xi') = \operatorname{supp}(\xi) + ls\cdot x, \\ s(\eta) = ns\cdot a \text{ and } \eta = \xi \end{smallmatrix} \right\}$$

and

$$p_2^* L_\sigma|_{\mathcal{Z}} = \left\{ (\xi, \xi'; [a', \eta', z']) \,\middle|\, \begin{smallmatrix} \operatorname{supp}(\xi') = \operatorname{supp}(\xi) + ls\cdot x, \\ s(\eta') = (n+l)s\cdot a' \text{ and } \eta' = \xi' \end{smallmatrix} \right\}.$$

We define a morphism $\vartheta\colon p_1^* L_\sigma|_{\mathcal{Z}} \to p_2^* L_\sigma|_{\mathcal{Z}}$ by

$$\vartheta((\xi, \xi'; [a, \eta, z])) := (\xi, \xi'; [a', \xi', z']),$$

with $a' := \frac{1}{n+l}(na + lx)$, where $\frac{1}{m}b \in A$ denotes a point b' such that $mb' = b$ and $x \in A$ is the point $\rho(\xi, \xi')$ where ξ is prolonged. With this definition, we have $(n + l)s \cdot a' = nsa + lsx = s(\xi')$, as it should be. We have to check that ϑ is well-defined.

1.) Let $a'' = a' + y$, with $y \in A[n + l]$ be another choice of a'. Then we have $[a'', \xi', z] = [a' + y, \xi', z] = [a', \xi', \sigma(y)z]$. But by assumption, σ is a character of order s, i.e. it factorizes over $A[(n + l)s] \xrightarrow{\cdot(n+l)} A[s]$ and thus $y \in \ker(\sigma)$.

2.) We have to check further that ϑ is independent of the choice of a representative of $[a,\eta,z]$: Let $y\in A[ns]$ be an ns-torsion point. We have

$$\vartheta((\xi,\xi';[a+y,\eta,\sigma^{-1}(y)z])=(\xi,\xi';[a'',\xi',\sigma^{-1}(y)z]),$$

with $a''=a'+y'$, where $y'\in A$ a point with $(n+l)y'=ny$. It follows that $\sigma(y')=\sigma((n+l)y')=\sigma(ny)=\sigma(y)$.
Thus the morphism ϑ is indeed well-defined. Its inverse is given by

$$\vartheta^{-1}((\xi,\xi';[a',\eta',z']))=(\xi,\xi';[a,\xi,z']),$$

with $a:=\frac{1}{n}((n+l)a'-lx)$, where again x is the point where ξ' and ξ differ. That ϑ^{-1} is well-defined is shown analogously. $\square$

We have therewith defined the *creation-operators* α^{σ}_{-ls}. Observe that by definition for a cohomology class $y\in \mathrm{H}^i(A^{[ns]},L_\sigma)$, the cohomological degree of $\alpha^{\sigma}_{-ls}(y)$ is given by

$$\deg(\alpha^{\sigma}_{-ls}(y))=i+2(ls)-2+\deg(\alpha):$$

The class $\rho^*(\alpha)\cap[\mathcal{Z}]$ is of homological degree $2\dim(\mathcal{Z})-\deg(\alpha)=2(2ns+ls+1)-\deg(\alpha)$. The homological degree of the image $\alpha^{\sigma}_{-ls}(y)$ of $y\in\mathrm{H}^i(A^{[ns]},L_\sigma)=\mathrm{H}_{4ns-i}(A^{[ns]},L_\sigma)$ is $4ns-i+2(2ns+ls+1)-\deg(\alpha)-2(2ns)$, according to formula (18). Thus, we have indeed

$$\alpha^{\sigma}_{-ls}(y)\in\mathrm{H}^{i+2(ls)-2+\deg(\alpha)}(A^{[(n+l)s]},L_\sigma).$$

Further, still for positive l, we define the *annihilation-operator* α^{σ}_{ls} by changing the rôle of p_1 and p_2:

$$\alpha^{\sigma}_{ls}\colon \mathrm{H}^*(A^{[(n+l)s]},L_\sigma)\to\mathrm{H}^*(A^{[ns]},L_\sigma);y\mapsto(-1)^{ls}p_{1*}(p_2^*y\cdot(\rho^*\alpha\cap[\mathcal{Z}])).$$

Let σ^{-1} the inverse character to σ. We define the following pairing between $\mathrm{H}^*(A^{[ns]},L_\sigma)$ and $\mathrm{H}^*(A^{[ns]},L_{\sigma^{-1}})$, compatible with the sign convention chosen in the definition of the Heisenberg algebra above: For $y\in\mathrm{H}^*(A^{[ns]},L_\sigma)$ and $z\in\mathrm{H}^*(A^{[ns]},L_{\sigma^{-1}})$ set

$$\langle y,z\rangle:=(-1)^{ns}\int_{A^{[ns]}}yz.$$

With this convention, we have the following easy

LEMMA 25. — *According to the above pairing, the operators α^{σ}_{-ls} and α^{σ}_{ls} are adjoint operators.*

Proof. The assumption follows directly from the projection formula: We have

$$\begin{aligned}
\langle \alpha^\sigma_{-ls}(y), z\rangle &= (-1)^{(n+l)s} \int_{A^{[(n+l)s]}} p_{2*}(p_1^* y \cdot (\rho^*\alpha \cap [\mathcal{Z}]))z \\
&= (-1)^{(n+l)s} \int_{A^{[ns]} \times A^{[(n+l)s]}} p_1^* y \cdot (\rho^*\alpha \cap [\mathcal{Z}]) \cdot p_2^* z \\
&= (-1)^{ns} \int_{A^{[ns]}} y(-1)^{ls} p_{1*}(p_2^* z \cdot (\rho^*\alpha \cap [\mathcal{Z}])) \\
&= \langle y, \alpha^\sigma_{ls}(z)\rangle.
\end{aligned}$$

□

According to our considerations above, we have for every $\alpha \in \mathrm{H}^*(A, \mathbb{C})$ and every $l \in \mathbb{N}$ defined the creation operator α^σ_{-ls} and the annihilation operator $\alpha^\sigma_{ls} \in \mathrm{End}(\mathbb{H}^\sigma)$. Finally, let α^σ_0 be the zero operator in $\mathrm{End}(\mathbb{H}^\sigma)$. The following Theorem is due to Nakajima ([**23**]) in the case of the trivial character:

THEOREM 26. — *Let $\alpha, \beta \in \mathrm{H}^*(A, \mathbb{C})$ and $n, m \in \mathbb{Z}$. The operators α^σ_{ns} and β^σ_{ms} defined above satisfy the following commutator relation:*

$$[\alpha^\sigma_{ns}, \beta^\sigma_{ms}] = ns\delta_{n,-m}\langle \alpha, \beta\rangle \mathrm{id}_{\mathbb{H}^\sigma}.$$

It follows that $\mathbb{H}^\sigma$ is a representation of the Heisenberg algebra $\mathfrak{h}$.

Proof. The theorem follows directly from the proof of Nakajima's original theorem, where it is shown that the section of the considered incidence varieties $\mathcal{Z}$ is of to small dimension to carry a non-zero product-correspondence (cf. equation (17) where the 'support' of a product-correspondence is given) in all cases except when $n + m = 0$ and that in that last case the correspondence is a multiple of the diagonal of multiplicity stated above. (Cf. [**24**] for a very detailed version of the proof.)
Since all these arguments are purely geometric (in the sense that they live on the level of the Chow-groups) and independent of the character σ the only thing we had to check was the existence of an isomorphism of the restricted locally constant sheaves. □

This theorem shows that the $\mathfrak{h}$-representation $\mathbb{H}^\sigma$ is also a stretched version of $\mathbb{H}$. Thus we conclude that, as $\mathbb{H}$, the space $\mathbb{H}^\sigma$ even is an irreducible representation of $\mathfrak{h}$. It is generated from the so called *vacuum* $\mathbb{1} = \mathbb{1}_\sigma = 1 \in \mathbb{C} = \mathrm{H}^*(A^{[0]}, L_\sigma) \subset \mathbb{H}^\sigma$ by the creation operators $\alpha^\sigma_{-ns}, n \geq 0$ (cf. [**23**]).

The following interpretation of Nakajima's approach to the generation of the cohomology of $A^{[ns]}$ from the vacuum $\mathbb{1}$ seems to be well known. But since I could not find a proof of it in the literature, it is included in this thesis:

Let σ be again a character of $\pi_1(A)$ of finite order s. Let further $\lambda = (l_1 \geq l_2 \geq \ldots \geq l_{|\lambda|}) = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, (ns)^{\alpha_{ns}})$ be a partition of ns such that $s|\gcd(\lambda)$. It follows that $\mathrm{H}^*(A^{[ns]}, L_\sigma) \neq 0$.
Recall that $S^\lambda A$ denotes the variety $\prod_{j=1}^{ns} S^{\alpha_j} A$. For $j \in \{i | 1 \leq i \leq ns$ and $\alpha_i \neq 0\}$ and $k \in \{1, \ldots, \alpha_j\}$, let $\beta_{j,k} \in \mathrm{H}^*(A, \mathbb{C})$ be $|\lambda|$ cohomology classes on the surface A. We define a class

$$\beta'_\lambda \in \mathrm{H}^*(S^\lambda A, L_\sigma) = \mathrm{H}^*(S^\lambda A, \mathbb{C}) = \bigotimes_{j=1}^{ns} \left(\bigotimes_{k=1}^{\alpha_j} \mathrm{H}^*(A, \mathbb{C}) \right)^{\mathfrak{S}_{\alpha_j}}$$

by setting

$$\beta'_\lambda := \boxtimes_{j=1}^{ns} \frac{1}{\alpha_j!} \sum_{g \in \mathfrak{S}_{\alpha_j}} \boxtimes_{k=1}^{\alpha_j} \beta_{j,g(k)}. \tag{19}$$

Observe that $\deg(\beta'_\lambda) = \sum_{j,k} \deg(\beta_{j,k})$. Starting from this cohomology class, we produce a new class $\beta_{\lambda,\sigma} \in \mathrm{H}^{2ns-2|\lambda|+\sum \deg(\beta_{j,k})}(A^{[ns]}, L_\sigma)$ by defining

$$\beta_{\lambda,\sigma} = j_{\lambda *} \rho^* \beta'_\lambda. \tag{20}$$

This has to be read as follows: We denote by $\rho^* : \mathrm{H}^*(S^\lambda A, L_\sigma) \longrightarrow \mathrm{H}^*(\overline{A^{[ns]}_\lambda}, L_\sigma)$ the map induced from

$$\begin{array}{ccc} & & \overline{A^{[ns]}_\lambda} \\ & & \big\downarrow \rho \\ S^\lambda A & \longrightarrow & \overline{S^{ns}_\lambda A} : \end{array}$$

As we have seen in section 1 of chapter 2, $S^\lambda A \to \overline{S^{ns}_\lambda A.}$ is a homeomorphism, thus ρ^* is the pull-back along the map

$$\overline{A^{[ns]}_\lambda} \to S^\lambda A; \xi \mapsto (\sum_{j=1}^{\alpha_i} x_{ij})_i,$$

where x_{ij} runs through the points of length i in the support of ξ.
Further, let $j_\lambda \colon \overline{A^{[ns]}_\lambda} \hookrightarrow A^{[ns]}$ be the inclusion of the closed stratum belonging to λ into the Hilbert scheme. This induces a morphism

$$\mathrm{H}^i(\overline{A^{[ns]}_\lambda}, L_\sigma) \xrightarrow{\cap[\overline{A^{[ns]}_\lambda}]} \mathrm{H}_{2(ns+|\lambda|)-i}(\overline{A^{[ns]}_\lambda}, L_\sigma) \xrightarrow{j_{\lambda *}} \mathrm{H}_{2(ns+|\lambda|)-i}(A^{[ns]}, L_\sigma)$$

and this last homology group is isomorphic to $\mathrm{H}^{2(ns-|\lambda|)+i}(A^{[ns]}, L_\sigma)$, since $A^{[ns]}$ is a manifold. The resulting homomorphism in cohomology $\mathrm{H}^i(\overline{A^{[ns]}_\lambda}, L_\sigma) \to \mathrm{H}^{2(ns-|\lambda|)+i}(A^{[ns]}, L_\sigma)$ will be also denoted by $j_{\lambda *}$.

Using the result of Nakajima, we have a second option to define a cohomology class of degree $2ns - 2|\lambda| + \sum \deg(\beta_{j,k})$ on the Hilbert scheme, starting from the classes $\beta_{j,k}$, namely

$$\beta^{\sigma}_{-\lambda}\mathbb{1} := \left(\prod_{j,k}(\beta^{\sigma}_{j,k})_{-j}\right)\mathbb{1} = \underbrace{\left((\beta^{\sigma}_{1,1})_{-1}\circ\cdots\circ(\beta^{\sigma}_{ns,\alpha_{ns}})_{-ns}\right)}_{|\lambda|\text{ factors}}\mathbb{1}. \tag{21}$$

Observe that this operator is indeed defined since all i in the partition with $\alpha_i \neq 0$ are divisible by s by our assumption on $\gcd(\lambda)$. We can now formulate the following

THEOREM 27. — *With the notations introduced above, one has the equality* $\beta_{\lambda,\sigma} = |\prod \mathfrak{S}_{\alpha_i}|^{-1}\beta^{\sigma}_{-\lambda}\mathbb{1}$.

Proof. First, observe that $\beta^{\sigma}_{-\lambda}\mathbb{1}$ has in fact the right degree: Starting from the vacuum, which lives in degree $\deg(\mathbb{1}) = 0$, each operator $(\beta^{\sigma}_{j,k})_{-j}$ increases the degree by $2j-2+\deg(\beta_{j,k})$ and thus altogether, we find

$$\deg(\beta^{\sigma}_{-\lambda}\mathbb{1}) = \sum 2j-2+\deg(\beta_{j,k}) = 2ns-2|\lambda| + \sum\deg(\beta_{j,k}) = \deg(\beta_\lambda).$$

To simplify notations let us enumerate the classes $\beta_{j,k}$ in such a way that β_i belongs to l_i in the partition λ. With this notation, we have

$$\beta^{\sigma}_{-\lambda} = \left((\beta_{|\lambda|})^{\sigma}_{-l_{|\lambda|}}\circ\cdots\circ(\beta_1)^{\sigma}_{-l_1}\right)\mathbb{1}$$

Next, observe that the operator $\beta^{\sigma}_{-\lambda}$ is a composition of correspondences. As such it is itself a correspondence. By definition, the variety $\mathcal{Z}_\lambda \subset A^{[0]}\times A^{[l_1]}\times\cdots\times A^{[ns]}$ defined as

$$\mathcal{Z}_\lambda := p_{12}^{-1}\mathcal{Z}_{0,l_1}\cap\ldots\cap p^{-1}_{(|\lambda|-1)|\lambda|}\mathcal{Z}_{(ns-l_{|\lambda|}),ns}$$

plays an important rôle. Here we have used the symbol p_{ij} to denote the projection from $A^{[0]}\times A^{[l_1]}\times\cdots\times A^{[ns]}$ to the product of the i-th and the j-th factor. $\mathcal{Z}_\lambda$ is given as

$$\mathcal{Z}_\lambda := \left\{(\xi_0,\xi_1,\ldots,\xi_{|\lambda|})\;\middle|\;\begin{array}{c}\text{There is a sequence } \emptyset=\xi_0\subset\xi_1\subset\ldots\subset\xi_{|\lambda|}\\ \text{s.t. for all } 1\leq i\leq|\lambda|\text{: } \operatorname{supp}(\xi_i)=\operatorname{supp}(\xi_{i-1})+l_ix_i \text{ for a } x_i\in A\end{array}\right\}.$$

We thus have again a morphism $\rho\colon \mathcal{Z}_\lambda \to A^{|\lambda|}$ sending $(\xi_0,\ldots,\xi_{|\lambda|})$ to the tuple $(x_0,\ldots,x_{|\lambda|})$ such that $\operatorname{supp}(\xi_i)-\operatorname{supp}(\xi_{i-1}) = l_ix_i$. Denote further by p_1 and p_2 the projections from $A^{[0]}\times A^{[ns]}$ to $A^{[0]}$ and $A^{[ns]}$, respectively.
By the rule of how to compose correspondences, we have

$$\beta^{\sigma}_{-\lambda}\mathbb{1} = p_{2*}(p_1^*\mathbb{1}\cdot p_{1|\lambda|*}(\rho^*(\beta_1\boxtimes\cdots\boxtimes\beta_{|\lambda|})\cap[\mathcal{Z}_\lambda])).$$

Observe that due to the commutator relations every permutation of the classes β_i produces the same class $\beta^{\sigma}_{-\lambda}\mathbb{1}$. Thus, in particular, it follows that

$$\begin{aligned}\beta^{\sigma}_{-\lambda}\mathbb{1} &= p_{2*}\left(\frac{1}{\alpha_1!\cdots\alpha_n!}\sum_{g\in\prod\mathfrak{S}_{\alpha_j}} p_{1|\lambda|*}(\rho^*(\beta_{g(1)}\boxtimes\cdots\boxtimes\beta_{g(s)})\cap[\mathcal{Z}_\lambda])\right)\\ &= p_{2*}(p_{1|\lambda|*}(\rho^*\beta'_\lambda\cap[\mathcal{Z}_\lambda]))\end{aligned}$$

where β'_λ is the cohomology class defined in (19).

But now we are through: The morphism p_2 factors as $p_{1|\lambda|}(\mathcal{Z}_\lambda)\to\overline{A^{[ns]}_\lambda}\xrightarrow{j_\lambda}A^{[ns]}$, and the morphism ρ corresponds to that of the same name $\rho\colon\overline{A^{[ns]}_\lambda}\to S^\lambda A$. Since the degree of the first map, $p_{1|\lambda|}(\mathcal{Z}_\lambda)\to\overline{A^{[ns]}_\lambda}$, is $|\prod\mathfrak{S}_{\alpha_i}|$, we have $p_{2*}p_{1|\lambda|*}(\rho^*\beta'_\lambda\cap[\mathcal{Z}_\lambda]) = |\prod\mathfrak{S}_{\alpha_i}|j_{\lambda*}\rho^*\beta'_\lambda$ which is what we wanted to show. □

COROLLARY 28. — *Let L_σ be a locally constant system on $A^{[n]}$. Then one has*

$$\mathrm{H}^*(A^{[n]},L_\sigma)=\bigoplus_{\lambda\in P(n)} j_{\lambda*}\rho^*\mathrm{H}^*(S^\lambda A,L_\sigma).$$

This decomposition coincides with that induced by Nakajima's operators up to a factor that is independent of the coefficient sheaf.

For $\alpha\in\mathrm{H}^*(S^\lambda A,L_\sigma)$, we will note the element $j_{\lambda*}\rho^*\alpha$ by $\alpha_{\lambda,\sigma}\in\mathrm{H}^*(A^{[ns]},L_\sigma)$ and we will often refer to this decomposition as 'Nakajima's description' in the following.

The next lemma and the following corollary show that Nakajima's description of the cohomology determines $\mathrm{H}^*(A\times K^{(n-1)}A,\mathbb{C})$ as a vector space with the intersection pairing.

LEMMA 29. — *Let $n\in\mathbb{N}$ and let $\lambda=(l_1,\dots,l_t)$ and $\mu=(m_1,\dots,m_r)$ be two partitions of n. Then we have*

$$\langle(\alpha_t)_{-l_t}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle=0 \textit{ if } r\neq t \textit{ and}$$

$$\langle(\alpha_t)_{-l_t}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle\mathrm{id}=\sum_{\pi\in\mathfrak{S}_t}\prod_{j=1}^{t}[(\alpha_j)_{l_j},(\beta_{\pi(j)})_{-m_{\pi(j)}}]$$

otherwise.

Proof. We will first check that the pairing vanishes if the partitions have different length. Without loss of generality we assume $t<r$. In the case $t=1$, we have $\lambda=(n)$ and $m_i\neq n$ for all $i=1,\dots,r$. Thus we have

$$\langle\alpha_{-n}\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle=\langle\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}\underbrace{\alpha_n\mathbb{1}}_{=0}\rangle=0.$$

In the general case, using induction on t, we find

$$\langle(\alpha_t)_{-l_t}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle\mathrm{id}=$$
$$\langle(\alpha_{t-1})_{-l_{t-1}}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots(\beta_1)_{-m_1}(\alpha_t)_{l_t}\mathbb{1}\rangle\mathrm{id}+$$
$$\sum_{i=1}^{t}[(\alpha_t)_{l_t},(\beta_i)_{-m_i}]\langle(\alpha_{t-1})_{-l_{t-1}}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots\widehat{(\beta_i)_{-m_i}}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle$$
$$=0.$$

As usual, we have put a hat over the operator that has to be excepted in the product. The first addend vanishes since an annihilator is applied to the vacuum, the others by the induction hypothesis.

Assume now that μ and λ are of the same length t. In the case $t=1$, the equality follows easily:

$$\langle\alpha_{-n}\mathbb{1},\beta_{-n}\mathbb{1}\rangle\mathrm{id}=\langle\mathbb{1},\beta_{-n}\alpha_n\mathbb{1}\rangle\mathrm{id}+[\alpha_n,\beta_{-n}]\langle\mathbb{1},\mathbb{1}\rangle=[\alpha_n,\beta_{-n}].$$

We conclude again by induction on t. In the general case, we find again

$$\begin{aligned}\langle(\alpha_t)_{-l_t}\cdots(\alpha_1)_{-l_1}\mathbb{1},&(\beta_t)_{-m_t}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle\mathrm{id}\\ &=\sum_{i=1}^{t}[(\alpha_t)_{l_t},(\beta_i)_{-m_i}]\cdot\\ &\qquad\langle(\alpha_{t-1})_{-l_{t-1}}\cdots(\alpha_1)_{-l_1}\mathbb{1},(\beta_r)_{-m_r}\cdots\widehat{(\beta_i)_{-m_i}}\cdots(\beta_1)_{-m_1}\mathbb{1}\rangle\\ &=\sum_{i=1}^{t}[(\alpha_t)_{l_t},(\beta_i)_{-m_i}]\sum_{\pi\in\mathfrak{S}^{(i)}_{t-i}}\prod_{\substack{j=1\\ j\neq i}}^{t}[(\alpha_j)_{l_j},(\beta_{\pi(j)})_{-m_{\pi(j)}}]\\ &=\sum_{\pi\in\mathfrak{S}_t}\prod_{j=1}^{t}[(\alpha_j)_{l_j},(\beta_{\pi(j)})_{-m_{\pi(j)}}],\end{aligned}$$

where we have denoted by $\mathfrak{S}^{(i)}_{t-1}\subset\mathfrak{S}_t$ the subgroup of permutations that fix i and thus altogether we sum over all permutations in $\mathfrak{S}_t$, whereas the last equality. This proves the lemma. □

One can use Nakajima's commutator relations to make the above lemma more explicit. This is done in the following

COROLLARY 30. — *Let* $\lambda=(l_1\geq\cdots\geq l_t)=(1^{a_1},\ldots,n^{a_n})$ *and* $\mu=(m_1\geq\cdots\geq m_r)=(1^{b_1},\ldots,n^{b_n})$ *be two partitions of* n. *Let* σ *be a character of* $A[n]$ *and let*

$$\alpha_{\lambda,\sigma}=|\mathfrak{S}_\lambda|^{-1}\left(\prod_{j,k}(\alpha_{j,k})_{-j}\right)\mathbb{1}_\sigma$$

and

$$\beta_{\mu,\sigma^{-1}} = |\mathfrak{S}_\mu|^{-1}\left(\prod_{j',k'}(\beta_{j',k'})_{-j}\right)\mathbb{1}_{\sigma^{-1}}$$

as in (21) above for classes $\alpha_{j,k} \in \mathrm{H}^*(A,\mathbb{C})$ *for* $j \in \{i | 1 \leq i \leq n$ *and* $a_i \neq 0\}$ *and* $k \in \{1,\dots,a_j\}$ *and analogously for* $\beta_{j',k'}$. *Further, we have denoted by* $\mathbb{1}_\sigma$ *the vacuum* $1 \in \mathrm{H}^0(A^{[0]}, L_\sigma) = \mathbb{C}$ *and analogously for* σ^{-1}.
Then one has

$$\langle \alpha_{\lambda,\sigma}, \beta_{\mu,\sigma^{-1}}\rangle = \delta_{\lambda,\mu}|\mathfrak{S}_\lambda|^{-2}\sum_{\pi\in\mathfrak{S}_\lambda}\prod_{j,k} j\langle\alpha_{j,k},\beta_{j,\pi(k)}\rangle,$$

where we have denoted by $\mathfrak{S}_\lambda = \prod_j \mathfrak{S}_{a_j} \subset \mathfrak{S}_t$ *the group permuting only the indices* k *and analogously for* $\mathfrak{S}_\mu$.

Proof. First observe that we can restrict our attention to classes in the cohomology with coefficients in the constant sheaf $\mathbb{C}$: Indeed if we denote by α_λ the class $|\mathfrak{S}_\lambda|^{-1}\prod_{j,k}(\alpha_{j,k})_{-j}\mathbb{1}$ obtained by the same operators from the vacuum $\mathbb{1} \in \mathrm{H}^0(A^{[0]}, \mathbb{C})$ and analogously for β_μ, we have due to Theorem 27

$$\langle\alpha_\lambda,\beta_\mu\rangle = (-1)^n\int_{A^{[n]}} j_{\lambda*}\rho^*\alpha j_{\mu*}\rho^*\beta,$$

But due to point (2.) in the list of facts given in Fact 23, we have

$$j_{\lambda*}\rho^*\alpha j_{\mu*}\rho^*\beta = I_\lambda^*\rho^*\alpha I_\mu^*\rho^*\beta = I_{\lambda\mu}^*(\rho^*\alpha\rho^*\beta),$$

where we have used that pull-backs are compatible with the product and denoted by I_λ the inclusion of the pair $(A^{[n]},\emptyset) \hookrightarrow (A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda})$, analogously I_μ and $I_{\lambda\mu}\colon (A^{[n]},\emptyset) \hookrightarrow (A^{[n]}, A^{[n]} - \overline{A^{[n]}_{\lambda\mu}})$, with $\overline{A^{[n]}_{\lambda\mu}}$ the closed stratum given by the intersection $\overline{A^{[n]}_\lambda} \cap \overline{A^{[n]}_\mu}$.
Denote by u_σ and $u_{\sigma^{-1}}$ the sections in $\mathrm{H}^0(S^\lambda A, L_\sigma)$ and $\mathrm{H}^0(S^\mu A, L_{\sigma^{-1}})$ corresponding to the canonical trivializations of L_σ and $L_{\sigma^{-1}}$, respectively. We have

$$\begin{aligned}\alpha_\lambda\beta_\mu &= j_{\lambda*}\rho^*\alpha j_{\mu*}\rho^*\beta\\ &= I_{\lambda\mu}^*(\rho^*(\alpha\cup u_\sigma)\rho^*(\beta\cup u_{\sigma^{-1}}))\\ &= j_{\lambda*}\rho^*((\alpha\cup u_\sigma)j_{\mu*}\rho^*(\beta\cup u_{\sigma^{-1}})\\ &= \alpha_{\lambda,\sigma}\beta_{\mu,\sigma^{-1}}.\end{aligned}$$

Thus the equality

$$\langle\alpha_\lambda,\beta_\mu\rangle = \langle\alpha_{\lambda,\sigma},\beta_{\mu,\sigma^{-1}}\rangle$$

follows.

The assertion that two classes belonging to different partitions have trivial intersection follows from Lemma 29: If $\lambda \neq \mu$ we have in every permutation a pair

$l_j \neq m_{\pi(j)}$ and thus the corresponding commutator $[(\alpha_j)_{l_j}, (\beta_{\pi(j)})_{-m_{\pi(j)}}]$ and therefore the whole pairing vanishes.
For $\lambda = \mu$, Lemma 29 and the commutator relations give:

$$\begin{aligned}\langle \alpha_{\lambda,\sigma}, \beta_{\mu,\sigma^{-1}} \rangle &= |\mathfrak{S}_\lambda|^{-2} \sum_{\pi \in \mathfrak{S}_\lambda} \prod_{j,k} [(\alpha_{j,k})_j, (\beta_{j,\pi(k)})_{-j}] \\ &= |\mathfrak{S}_\lambda|^{-2} \sum_{\pi \in \mathfrak{S}_\lambda} \prod_{j,k} j \langle \alpha_{j,k}, \beta_{j,\pi(k)} \rangle.\end{aligned}$$

The permutations lying outside $\mathfrak{S}_\lambda$ do not contribute, since again they produce a vanishing commutator. □

We will use the conclusion of Theorem 27 to reduce the ring structure of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ to that of $\mathrm{H}^*(A^{[n]}, \mathbb{C})$ in Section 5. The cohomology ring of the Hilbert scheme $A^{[n]}$ has been computed by Lehn and Sorger an we will recall their results in the following section.

4. The cohomology ring of $A^{[n]}$

In [**20**], Manfred Lehn and Christoph Sorger determine the ring structure of the cohomology groups $\mathrm{H}^*(X^{[n]}, \mathbb{C})$ for surfaces X with trivial canonical class. Since we will refer to their description in the case of an abelian surface, we will use this section to present the ideas of this paper as far as we need them.
We start with the definition of a graded Frobenius algebra.

DEFINITION 31. — A *graded Frobenius algebra of degree* d is a finite dimensional graded vector space $H = \bigoplus_{i=-d}^{d} H^i$ with a graded commutative and associative multiplication $H \otimes H \to H$ of degree d and unit element 1 (necessarily of degree $-d$) together with a linear form $T\colon H \to \mathbb{C}$ of degree $-d$ such that the induced symmetric bilinear form $\langle a, b \rangle := T(ab)$ is non-degenerate (and of degree 0).
Consider the composite linear map $H \xrightarrow{\Delta_*} H \otimes H \to H$, where the second arrow is multiplication and Δ_* is the adjoint comultiplication. The image of 1 under this map is called the *Euler class* $e = e(H)$ *of* H.

In the applications, H will be the shifted cohomology ring $\mathrm{H}^*(X; \mathbb{Q})[d]$ of a compact complex manifold X of even dimension d. In this case, if X is connected, we have $e(\mathrm{H}^*(X; \mathbb{Q})[d]) = e(X)[\mathrm{pt.}]$, where $[\mathrm{pt.}]$ is the class dual to 1 and $e(X)$ denotes the topological Euler characteristic of X.
Observe that with H the n-fold tensor product $H^{\otimes n}$ is again a Frobenius algebra of degree nd with the product

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_n) = \varepsilon \cdot (a_1 b_1) \otimes \cdots \otimes (a_n b_n),$$

where ε is the sign resulting from the reordering of the a's and b's. The symmetric group $\mathfrak{S}_n$ acts on $H^{\otimes n}$ via

$$g(a_1 \otimes \cdots \otimes a_n) = \varepsilon a_{g^{-1}(1)} \otimes \cdots \otimes a_{g^{-1}(n)}.$$

Here, $\varepsilon = (-1)^{\sum_{i<j, g(i)>g(j)} \deg(a_i)\deg(a_j)}$ is again the sign resulting from interchanging the a's.

The strategy of [**20**] is to define an endofunctor $H \to H^{[n]}$ on the category of graded Frobenius algebras such that applied to $H = \mathrm{H}^*(X, \mathbb{C})[2]$, $H^{[n]}$ is the (shifted) cohomology ring of $X^{[n]}$.

For a set I, Lehn and Sorger recall the natural definition of $H^{\otimes I}$ and construct for a surjective map $\varphi \colon I \twoheadrightarrow J$ of sets the induced ring homomorphism $\varphi^* \colon H^{\otimes I} \to H^{\otimes J}$ and the adjoint module homomorphism $\varphi_* \colon H^{\otimes J} \to H^{\otimes I}$.

For $G \subset \mathfrak{S}_n$ a subgroup and $o \subset \{1, \dots, n\}$ a G-stable subset denote by $G\backslash o$ the set of orbits. For $\{1, \dots, n\}$ we will also write $O(G) := G\backslash\{1, \dots, n\}$ for short. If $G = \langle g \rangle$ is cyclic or $G = \langle g, h \rangle$ is generated by two elements, we omit the brackets in the notation and write $O(g)$ and $O(g, h)$, respectively.

With these conventions, define the *ambient space* $H\{\mathfrak{S}_n\}$ to be

$$H\{\mathfrak{S}_n\} := \bigoplus_{g \in \mathfrak{S}_n} H^{\otimes O(g)}.$$

For $g \in \mathfrak{S}_n$ and an element $a \in H^{\otimes O(g)}$ denote the corresponding element in $H\{\mathfrak{S}_n\}$ by a_g. Define a grading on $H\{\mathfrak{S}_n\}$ by setting $\deg(\alpha_g) := \deg(\alpha)$. There is a natural action of the symmetric group on $H\{\mathfrak{S}_n\}$: For $h \in \mathfrak{S}_n$, the operation is given by

$$h a_g = (h^* a)_{hgh^{-1}}, \tag{22}$$

where h^* is the homomorphism induced by the bijection $O(g) \simeq O(hgh^{-1}), o \mapsto ho$. Define $H^{[n]} := (H\{\mathfrak{S}_n\})^{\mathfrak{S}_n}$ to be the subspace of invariants. By definition, the operation of $\mathfrak{S}_n$ on $H\{\mathfrak{S}_n\}$ preserves the grading. Thus $H^{[n]}$ is still a graded vector space.

The next aim is the definition of a multiplication on the ambient space $H\{\mathfrak{S}_n\}$. For two permutations $g, h \in \mathfrak{S}_n$ Lehn and Sorger define the *graph defect* $\gamma(g, h)$ as the following function on $O(g, h)$:

$$\gamma(g, h) \colon O(g, h) \to \mathbb{Q}, o \mapsto \frac{1}{2}(|o| + 2 - |\langle g \rangle\backslash o| - |\langle h \rangle\backslash o| - |\langle gh \rangle\backslash o|)$$

They prove that $\gamma(g, h)$ is a non-negative integer.

Observe that for two subgroups $G \subset K$ of $\mathfrak{S}_n$, one gets a surjection $f \colon O(G) \twoheadrightarrow O(K)$ which leads to the to morphisms

$$f^{G,K} \colon H^{\otimes O(G)} \to H^{\otimes O(K)} \text{ and } f_{K,G} \colon H^{\otimes O(K)} \to H^{\otimes O(G)},$$

that we have denoted by f^* and f_* above. For $g, h \in \mathfrak{S}_n$ define $m_{g,h}\colon H^{\otimes O(g)} \otimes H^{\otimes O(h)} \to H^{\otimes O(gh)}$ by

$$m_{g,h}(a \otimes b) := f_{\langle g,h\rangle,gh}\left(f^{g,\langle g,h\rangle}(a) f^{h,\langle g,h\rangle}(b) e^{\gamma(g,h)}\right).$$

Here e is the Euler class and one uses the convention

$$e^{\gamma(g,h)} := \bigotimes\nolimits_{o \in O(g,h)} e^{\gamma(g,h)(o)} \in H^{\otimes O(g,h)}.$$

The following two theorems are the main results of [**20**]:

THEOREM ([**20**], Prop. 2.13 and 2.15). —

(1) *The product $H\{\mathfrak{S}_n\} \otimes H\{\mathfrak{S}_n\} \to H\{\mathfrak{S}_n\}$ given by*

$$a_g \cdot b_h := m_{g,h}(a \otimes b)_{gh}$$

is associative $\mathfrak{S}_n$-equivariant and of degree nd.

(2) *$H^{[n]}$ is a subring of the center of $H\{\mathfrak{S}_n\}$.*

As we have seen, on the geometric side, one has a natural isomorphism

$$\mathrm{H}^*(X^{[n]}, \mathbb{C}) = \bigoplus_{\lambda \in P(n)} \mathrm{H}^*(S^\lambda X, \mathbb{C})[2(|\lambda| - n)],$$

where the sum runs over all partitions of n. As in Theorem 27 in the last section, we will interpret this isomorphism in an explicit geometric way: Every cohomology class in $\mathrm{H}^*(X^{[n]}, \mathbb{C})$ has a unique decomposition as $\sum_\lambda \alpha_\lambda$, where α_λ is defined to be

$$\alpha_\lambda := j_{\lambda *}\rho^*\alpha,$$

for a class $\alpha \in \mathrm{H}^*(S^\lambda X, \mathbb{C})$. Observe that the degree shifting $\deg \alpha_\lambda = \deg \alpha + 2(n - |\lambda|)$ vanishes if we center all occurring cohomology groups around the middle degree.

Consider now the graded vector space $(\mathrm{H}^*(X, \mathbb{C})[2])\{\mathfrak{S}_n\}$. By the definition of the ambient space, it is given as

$$(\mathrm{H}^*(X, \mathbb{C})[2])\{\mathfrak{S}_n\} = \bigoplus_{g \in \mathfrak{S}_n} \mathrm{H}^*(X, \mathbb{C})[2]^{\otimes O(g)}.$$

By definition, the symmetric group acts on it as follows: Let α_g be the element in $\bigoplus_{g \in \mathfrak{S}_n} \mathrm{H}^*(X, \mathbb{C})^{\otimes O(g)}$ corresponding to $\alpha \in \mathrm{H}^*(X, \mathbb{C})^{\otimes O(g)}$. Then the action of $h \in \mathfrak{S}_n$ was defined in equation (22) above as

$$ha_g = (h^*a)_{hgh^{-1}}.$$

By the Künneth decomposition theorem, the space $\mathrm{H}^*(X, \mathbb{C})^{\otimes O(g)}$ is isomorphic to $\mathrm{H}^*(X^{O(g)}, \mathbb{C})$. On the other hand, a set of representatives of the conjugacy classes of elements of $\mathfrak{S}_n$ is given by the permutations of n. It follows that the space of invariants is given by

$$\begin{aligned}(\mathrm{H}^*(X;\mathbb{C})[2])^{[n]} &= \left(\bigoplus_{g\in\mathfrak{S}_n} \mathrm{H}^*(X,\mathbb{C})[2]^{\otimes O(g)}\right)^{\mathfrak{S}_n}\\ &\simeq \bigoplus_{\lambda\in P(n)} \mathrm{H}^*(S^\lambda X,\mathbb{C})[2|\lambda|]\\ &= \mathrm{H}^*(X^{[n]},\mathbb{C})[2n].\end{aligned}$$

This shows that one has a natural bijective liner map

$$(\mathrm{H}^*(X;\mathbb{C})[2])^{[n]} \longrightarrow \mathrm{H}^*(X^{[n]};\mathbb{C})[2n].$$

With these definitions, Lehn and Sorger prove the following

THEOREM ([**20**], Theorem 3.2). — *Let X be a smooth projective surface with numerically trivial canonical divisor. Then the bijective linear map described above is a canonical isomorphism of graded rings*

$$(\mathrm{H}^*(X;\mathbb{C})[2])^{[n]} \xrightarrow{\ \cong\ } \mathrm{H}^*(X^{[n]};\mathbb{C})[2n].$$

This gives a rather explicit description of how to multiply two cohomology classes on the Hilbert scheme. Observe that in our case of an abelian surface the Euler class is trivial, so the product of two classes $\alpha_g, \beta_h \in (\mathrm{H}^*(X,\mathbb{C})[2])^{[n]}$ is zero unless the graph defect vanishes for all orbits $o \in O(g,h)$.

5. The ring structure of $\mathrm{H}^*(A\times K^{(n-1)}A)$.

We now have collected all necessary ingredients to compute the ring structure of $\mathrm{H}^*(A\times K^{(n-1)}A)$ and thereby implicitly determine the cohomology ring of the generalized Kummer variety $K^{(n-1)}A$ itself. We start by collecting the facts, we have seen so far:

1.) We have a ring isomorphism

$$\mathrm{H}^*(A\times K^{(n-1)}A,\mathbb{C}) = \mathrm{H}^*(A^{[n]},\mathcal{R}) = \bigoplus_{\sigma\in A[n]^\vee} \mathrm{H}^*(A^{[n]},L_\sigma).$$

2.) For the building blocks $\mathrm{H}^*(A^{[n]},L_\sigma)$ we have a decomposition

$$\mathrm{H}^*(A^{[n]},L_\sigma) = \bigoplus_{\lambda\in P(n)} j_{\lambda*}\rho^*\mathrm{H}^*(S^\lambda A,L_\sigma).$$

For a class $\alpha \in \mathrm{H}^*(S^\lambda A,L_\sigma)$, we write $\alpha_{\lambda,\sigma} = j_{\lambda*}\rho^*(\alpha) \in \mathrm{H}^*(A^{[n]},L_\sigma)$. For $\sigma=1$ and correspondingly $L_\sigma=\mathbb{C}$ we write α_λ for short.

3.) For a locally constant system L_σ over $S^\lambda A$, we have

$$\mathrm{H}^*(S^\lambda A,L_\sigma) = \begin{cases}\mathrm{H}^*(S^\lambda A,\mathbb{C}) & \text{if } L_\sigma \text{ is trivial over } S^\lambda A,\\ 0 & \text{otherwise.}\end{cases}$$

Thereby, the isomorphism is given by a canonical trivialization $\mathbb{C} \xrightarrow{\sim} L_\sigma$, corresponding to a section $u_\sigma \in \mathrm{H}^0(S^\lambda A, L_\sigma)$ (cf. the proof of Lemma 20 in Section 2).
4.) In the case that the locally constant system L_σ over $S^\lambda A$ is non-trivial, we define $u_\sigma := 0 \in \mathrm{H}^0(S^\lambda A, L_\sigma)$. Thus in any case, for $\alpha \in \mathrm{H}^*(S^\lambda A, \mathbb{C})$ and the corresponding class $\alpha_\lambda \in \mathrm{H}^*(A^{[n]}, \mathbb{C})$ we define

$$\alpha_{\lambda,\sigma} = j_{\lambda *}\rho^*(\alpha \cup u_\sigma).$$

By definition $\alpha_{\lambda,\sigma}$ vanishes if L_σ is not constant over $S^\lambda A$.
5.) The other way round, we have for $0 \neq \alpha_{\lambda,\sigma}$ the class $\alpha_\lambda := j_{\lambda *}\rho^*(\alpha \cup u_{\sigma^{-1}}) \in \mathrm{H}^*(A^{[n]}, \mathbb{C})$.

Thus, to describe the ring structure of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ it is enough to describe how to multiply two classes $0 \neq \alpha_{\lambda,\sigma}$ and $0 \neq \beta_{\mu,\tau}$. We will use the analogous considerations as in the proof of Corollary 30 to show the following

THEOREM 32. — *The cup product of two non-trivial classes $\alpha_{\lambda,\sigma} \in \mathrm{H}^*(A^{[n]}, L_\sigma)$ and $\beta_{\mu,\tau} \in \mathrm{H}^*(A^{[n]}, L_\tau)$ as defined above, is given by*

$$\alpha_{\lambda,\sigma} \cup \beta_{\mu,\tau} = \sum \gamma_{\nu,\sigma\tau},$$

where the sum on the right hand side is running over all classes $\gamma_{\nu,\sigma\tau}$ corresponding via (4.) above to the classes γ_ν occurring in the cup product inside $\mathrm{H}^*(A^{[n]}, \mathbb{C})$: $\alpha_\lambda\beta_\mu = \sum \gamma_\nu$.

Proof. We have $\alpha_{\lambda,\sigma} = j_{\lambda *}\rho^*\alpha$ and analogously $\beta_{\mu,\tau} = j_{\mu *}\rho^*\beta$. As we have seen in the list of Facts on Borel-Moore cohomology (Facts 23.2 in Section 3), the push-forward $j_{\lambda *}\colon \mathrm{H}_*(\overline{A^{[n]}_\lambda}, L_\sigma) \to \mathrm{H}_*(A^{[n]}, L_\sigma) = \mathrm{H}^{4n-*}(A^{[n]}, L_\sigma)$ is given by the pull-back I_λ^* along the inclusion of pairs

$$I_\lambda\colon (A^{[n]}, \emptyset) \to (A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda})$$

and analogously for $j_{\mu *}$. Since pull-backs are compatible with the cup product, we find

$$\alpha_{\lambda,\sigma}\beta_{\mu,\tau} = j_{\lambda\mu *}(\rho^*\alpha \cdot \rho^*\beta),$$

where $\cdot$ denotes the intersection product

$$\mathrm{H}_p(\overline{A^{[n]}_\lambda}, L_\sigma) \otimes \mathrm{H}_q(\overline{A^{[n]}_\mu}, L_\tau) \to \mathrm{H}_{4n-p-q}(\overline{A^{[n]}_{\lambda\mu}}, L_{\sigma\tau})$$

induces by the cup-product in relative cohomology

$$\mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda}, L_\sigma) \otimes \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\mu}, L_\tau) \to \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_{\lambda\mu}}, L_{\sigma\tau}).$$

We have again denoted by $\overline{A^{[n]}_{\lambda\mu}}$ the closed stratum given by the intersection $\overline{A^{[n]}_\lambda} \cap \overline{A^{[n]}_\mu}$ and by $j_{\lambda\mu}$ the corresponding inclusion.

By assumption we started with two non-trivial classes. It follows that L_σ and L_τ are trivial over $\overline{A^{[n]}_\lambda}$ and $\overline{A^{[n]}_\mu}$, respectively. Denote the corresponding isomorphisms of sheaves over $\overline{A^{[n]}_\lambda}$ and $\overline{A^{[n]}_\mu}$ by $\varphi_\sigma\colon \mathbb{C} \xrightarrow{\sim} L_\sigma$ and $\varphi_\tau\colon \mathbb{C} \xrightarrow{\sim} L_\tau$. Over the stratum $\overline{A^{[n]}_{\lambda\mu}}$ both sheaves and accordingly $L_\sigma \otimes L_\tau = L_{\sigma\tau}$ are trivial. Furthermore, as we have seen in Remark 15 in Section 1 above the following diagram over $\overline{A^{[n]}_{\lambda\mu}}$(whose analogue over A was numbered as (14) above) is commutative

$$\begin{array}{ccc} L_\sigma \otimes L_\tau & \xrightarrow{=} & L_{\sigma\tau} \\ {\scriptstyle (\varphi_\sigma\otimes\varphi_\tau)|_{\overline{A^{[n]}_{\lambda\mu}}}}\big\uparrow & & \big\uparrow{\scriptstyle \varphi_{\sigma\tau}} \\ \mathbb{C}\otimes\mathbb{C} & \xrightarrow[=]{} & \mathbb{C}. \end{array} \tag{23}$$

The following considerations show that these trivializations are also compatible with the intersection product:

The isomorphism $\mathrm{H}_*(\overline{A^{[n]}_\lambda}, \mathbb{C}) = \mathrm{H}_*(\overline{A^{[n]}_\lambda}, L_\sigma)$ is given by capping with the section $\rho^* u_\sigma \in \mathrm{H}^0(\overline{A^{[n]}_\lambda}, L_\sigma)$. We will now analyze how this isomorphism is transformed by passing to the relative cohomology $\mathrm{H}_p(\overline{A^{[n]}_\lambda}, \mathbb{C}) = \mathrm{H}^{4n-p}(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda}, \mathbb{C})$.

First, observe that $\overline{A^{[n]}_\lambda}$ has an open neighborhood U of which it is an neighborhood retract. This follows from the fact that algebraic spaces can be triangulated (cf. [**16**]). Over U, the sheaf L_σ is still constant by construction. Chose a small closed set V such that $U \supset V \supset \overline{A^{[n]}_\lambda}$. We have again a ring isomorphism $\varphi'_\sigma\colon \mathbb{C}|_V \to L_\sigma|_V$ that expands φ_σ.

Doing the same for L_τ we find a small closed neighborhood V' of $\overline{A^{[n]}_\mu}$ such that φ_τ can be expanded to an isomorphism $\varphi'_\tau\colon \mathbb{C}|_{V'} \to L_\tau|_{V'}$

It follows from the excision theorem for sheaf cohomology (cf. e.g. [**3**], II, 12.9) that we have a natural isomorphism given as the composition

$$\begin{aligned} \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda}; \mathbb{C}) \simeq & \mathrm{H}^*(V, V - \overline{A^{[n]}_\lambda}; \mathbb{C}|_V) \xrightarrow{\Phi'_\sigma} \\ & \mathrm{H}^*(V, V - \overline{A^{[n]}_\lambda}; L_\sigma|_V) \simeq \mathrm{H}^*(A^{[n]}; A^{[n]} - \overline{A^{[n]}_\lambda}; L_\sigma), \end{aligned}$$

where we have denoted by Φ'_σ the isomorphism induced by φ'_σ on the level of cohomology. We have an according isomorphism for L_τ. These isomorphisms are compatible with the cup product, which can be seen as follows:

We can write the cup product in relative cohomology as the composition of the cross product followed by the pull-back along the diagonal:

$$\mathrm{H}^*(X, A) \otimes \mathrm{H}^*(X, B) \xrightarrow{\times} \mathrm{H}^*(X \times X, X \times B \cup A \times X) \xrightarrow{\Delta^*} \mathrm{H}^*(X, A \cup B).$$

In our case the cross product is given by

$$\mathrm{H}^*(V, V - \overline{A^{[n]}_\lambda}) \otimes \mathrm{H}^*(V', V' - \overline{A^{[n]}_\mu}) \to \mathrm{H}^*(V \times V', V \times (V' - \overline{A^{[n]}_\mu}) \cup (V - \overline{A^{[n]}_\lambda}) \times V').$$

Setting $V'' := V\cap V' \supset \overline{A^{[n]}_{\lambda\mu}}$ and using again the excision theorem, we find on the one hand

$$\mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda\mu}}) \simeq \mathrm{H}^*(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}})$$

and on the other hand

$$\begin{aligned}&\mathrm{H}^*(A^{[n]}\times A^{[n]}, A^{[n]}\times(A^{[n]}\text{–}\overline{A^{[n]}_{\mu}})\cup(A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}})\times A^{[n]})\\ &\simeq \mathrm{H}^*(V\times V', V\times(V'\text{–}\overline{A^{[n]}_{\mu}})\cup(V\text{–}\overline{A^{[n]}_{\lambda}})\times V').\end{aligned}$$

These isomorphism are induced by the inclusion of pairs

$$(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}}) \hookrightarrow (A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda\mu}}),$$

and analogously for the second one. Thus, the induced diagonal map

$$\Delta\colon (V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}}) \to (V\times V', V\times(V'\text{–}\overline{A^{[n]}_{\mu}})\cup(V\text{–}\overline{A^{[n]}_{\lambda}})\times V')$$

gives us the following commutative diagram on the level of cohomology:

$$\begin{array}{ccc}\mathrm{H}^*((A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}})\times(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\mu}})) & \xrightarrow{\Delta^*} & \mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda\mu}})\\ \simeq\downarrow & & \downarrow\simeq\\ \mathrm{H}^*(V\times V', V\times(V'\text{–}\overline{A^{[n]}_{\mu}})\cup(V\text{–}\overline{A^{[n]}_{\lambda}})\times V') & \xrightarrow{\Delta^*} & \mathrm{H}^*(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}})\end{array}$$

where we have used the shorter $(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}})\times(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\mu}})$ to denote the pair $(A^{[n]}\times A^{[n]}, A^{[n]}\times(A^{[n]}\text{–}\overline{A^{[n]}_{\mu}})\cup(A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}})\times A^{[n]})$.

Accordingly, we have a cup product

$$\cup := (\Delta^*\circ\times)\colon \mathrm{H}^*(V, V\text{–}\overline{A^{[n]}_{\lambda}})\otimes\mathrm{H}^*(V', V'\text{–}\overline{A^{[n]}_{\mu}}) \to \mathrm{H}^*(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}})$$

that makes the following diagram commutative:

$$\begin{array}{ccc}\mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}};\mathbb{C})\otimes\mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\mu}};\mathbb{C}) & \xrightarrow{\cup} & \mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda\mu}};\mathbb{C})\\ \simeq\downarrow & & \downarrow\simeq\\ \mathrm{H}^*(V, V\text{–}\overline{A^{[n]}_{\lambda}};\mathbb{C})\otimes\mathrm{H}^*(V', V'\text{–}\overline{A^{[n]}_{\mu}};\mathbb{C}) & \xrightarrow{\cup} & \mathrm{H}^*(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}};\mathbb{C})\\ \Phi'_\sigma\otimes\Phi'_\tau\downarrow & & \downarrow\Phi'_{\sigma\tau}\\ \mathrm{H}^*(V, V\text{–}\overline{A^{[n]}_{\lambda}};L_\sigma)\otimes\mathrm{H}^*(V', V'\text{–}\overline{A^{[n]}_{\mu}};L_\tau) & \xrightarrow{\cup} & \mathrm{H}^*(V'', V''\text{–}\overline{A^{[n]}_{\lambda\mu}};L_{\sigma\tau})\\ \simeq\downarrow & & \downarrow\simeq\\ \mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda}};L_\sigma)\otimes\mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\mu}};L_\tau) & \xrightarrow{\cup} & \mathrm{H}^*(A^{[n]}, A^{[n]}\text{–}\overline{A^{[n]}_{\lambda\mu}};L_{\sigma\tau}).\end{array}$$

The upper and the lowest square commute due to our considerations above, which show that the isomorphisms given by the excision theorem are compatible with the cup product. The middle square commutes due to the commutativity of diagram (23).

Summarizing, we find on the level of homology the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_*(\overline{A^{[n]}_\lambda}, L_\sigma) \otimes \mathrm{H}_*(\overline{A^{[n]}_\mu}, L_\tau) & \xrightarrow{\ \cdot\ } & \mathrm{H}_*(\overline{A^{[n]}_{\lambda\mu}}, L_{\sigma\tau}) \\ {\scriptstyle (\cap\rho^*(u_\sigma))\otimes(\cap\rho^*(u_\tau))}\Big\uparrow & & \Big\uparrow{\scriptstyle \cap\rho^*(u_{\sigma\tau})} \\ \mathrm{H}_*(\overline{A^{[n]}_\lambda}, \mathbb{C}) \otimes \mathrm{H}_*(\overline{A^{[n]}_\mu}, \mathbb{C}) & \xrightarrow{\ \cdot\ } & \mathrm{H}_*(\overline{A^{[n]}_{\lambda\mu}}, \mathbb{C}) : \end{array}$$

Since the isomorphism on the level of relative cohomology was that given by the trivialization $\varphi_\sigma \colon \mathbb{C} \to L_\sigma$, the corresponding isomorphism on the level of homology is seen immediately be given by the cap-product with the global section $\rho^*(u_\sigma) \in \mathrm{H}^0(\overline{A^{[n]}_\lambda}, L_\sigma)$:

$$\mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda}, \mathbb{C}) \xrightarrow{\cap\rho^*(u_\sigma)} \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_\lambda}, L_\sigma)$$

and analogously for $\overline{A^{[n]}_\mu}$ and $\overline{A^{[n]}_{\lambda\mu}}$.

According to this, we have

$$\rho^*\alpha\rho^*\beta = \rho^*\alpha'\rho^*\beta' \cap \rho^*(u_{\sigma\tau}),$$

where we have denoted by $\alpha' = \alpha \cup u_{\sigma^{-1}}$ the class in $\mathrm{H}^*(S^\lambda A, \mathbb{C})$ corresponding to α under the isomorphism $\mathrm{H}^*(S^\lambda A, \mathbb{C}) \xrightarrow{\cup u_\sigma} \mathrm{H}^*(S^\lambda A, L_\sigma)$ and analogously for β'.

Everything that remains to show is the following: Given a cohomology class $\delta \in \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_{\lambda\mu}}, L_{\sigma\tau})$ and the corresponding cohomology class with constant coefficients $\delta' = \delta \cap \rho^*(u_{(\sigma\tau)^{-1}}) \in \mathrm{H}^*(A^{[n]}, A^{[n]} - \overline{A^{[n]}_{\lambda\mu}}, \mathbb{C})$, then we have $I^*_{\lambda\mu}(\delta) = (I^*_{\lambda\mu}(\delta'))_{\sigma\tau}$.

But this equality can be checked by means of the intersection pairing: Let $I^*_{\lambda\mu}(\delta') = \sum_\nu \gamma_\nu$ be the decomposition in classes coming from the strata belonging to partitions ν. We have seen in Section 3 that such a γ_ν pairs non-trivially at most with classes coming from the same stratum. But for such a class $\varepsilon_\nu = j_{\nu *}\rho^*(\varepsilon)$, we have on the one hand

$$I^*(\delta')\varepsilon_\nu = I^*(\delta \cap \rho^*(u_{\sigma\tau})\rho^*(\varepsilon) \cap \rho^*(u_{(\sigma\tau)^{-1}}))$$

and thus $\langle I^*(\delta'), \varepsilon_\nu \rangle = \langle I^*(\delta), \varepsilon_{\nu,(\sigma\tau)^{-1}} \rangle$.

On the other hand, we have

$$\langle \gamma_\nu, \varepsilon_\nu \rangle = \langle \gamma_{\nu,\sigma\tau}, \varepsilon_{\nu,(\sigma\tau)^{-1}} \rangle.$$

It follows that indeed $I^*(\delta) = \sum_\nu \gamma_{\nu,\sigma\tau}$. Using this for $\delta = \rho^*(\alpha)\rho^*(\beta)$, our theorem follows. $\square$

We have therewith described the ring structure of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ in terms of the known cup product of $\mathrm{H}^*(A^{[n]}, \mathbb{C})$.

In the next chapter we will show that this ring structure is the same as the one given by the orbifold cup product and therewith prove a conjecture of Fantechi and Göttsche.

CHAPTER 4

Orbifold Cohomology

In their paper [**7**], Barbara Fantechi and Lothar Göttsche introduce a nice description of the orbifold cohomology ring for global quotients. They compute the orbifold cohomology ring structure for $[X^n/\mathfrak{S}_n]$, the orbifold given by the symmetric product of a surface X with trivial canonical bundle. By comparing with the description of the cohomology ring of $X^{[n]}$ computed by Lehn and Sorger, they show that they coincide up to a sign. They state also a conjecture on the ring structure of the cohomology of generalized Kummer varieties.
In this chapter, after briefly recalling the general definitions following [**7**], we will show that in fact the orbifold cohomology is isomorphic to the cohomology of the generalized Kummer varieties. Before doing that, we have to correct a small error that occurred in [**7**] in the computation of the orbifold cohomology ring in this case.

1. The General Concept

Let Y be a complex manifold with an action of a finite group G. Denote by Y^g the fixed locus of an element $g \in G$. Then — analogously to Lehn and Sorger —, Fantechi and Göttsche define an ambient space $\mathrm{H}^*(Y, G) := \bigoplus_{g\in G} \mathrm{H}^*(Y^g, \mathbb{C})$. Denote for $g \in G$ and a class $\alpha \in \mathrm{H}^*(Y^g, \mathbb{C})$ the corresponding element in $\mathrm{H}^*(Y, G)$ by α_g. With this notation, $\mathrm{H}^*(Y, G)$ carries a natural G-action, defined by

$$h\alpha_g := (h_*\alpha)_{hgh^{-1}}.$$

Here, we consider h as an automorphism of Y and denote by h_* the cohomology push-forward.
To introduce a (rational) grading on $\mathrm{H}^*(Y, G)$ one uses the following

DEFINITION 33. — Let Y be a manifold of dimension d with the action of a finite group G. For $g \in G$ and $y \in Y^g$, let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of the action of g on $T_{Y,y}$. Note that they are roots of unity. Write $\lambda_j = e^{2\pi i r_j}$ where r_j is a rational number in the interval $[0, 1)$. The *age of g in y* is the rational number

$$a(g, y) := \sum_{j=1}^{d} r_j.$$

REMARK 34. — The age obviously depends only on the connected component of Z of Y^g which y lies in. Here, one has the equality

$$a(g,Z)+a(g^{-1},Z)=\operatorname{codim}(Z\subset Y).$$

To see this, observe that if the eigenvalues of g are given by $e^{2\pi i r_j}$, the corresponding eigenvalues of g^{-1} are $e^{2\pi i q_j}$, with $q_j=(1-r_j)$, except the cases in which $r_j=0$. In the first case we obtain $r_j+q_j=1$. The second case occurs exactly on the subspace $T_{Z,y}\subset T_{Y,y}$, where g acts as the identity.

According to the above considerations, Fantechi and Göttsche define a (rational) grading on $\mathrm{H}^*(Y,G)$ as follows: Let $g\in G$ and let Z be a connected component of Y^g, and $j\colon Z\hookrightarrow Y^g$ the inclusion. Let $\alpha\in\mathrm{H}^i(Z)$ and assign to $j_*\alpha_g$ the degree

$$\deg(j_*\alpha_g):=i+2a(g,Z). \tag{24}$$

One can define a splitting of $\mathrm{H}^*(Y,G)$ into even and odd part by setting $\mathrm{H}^{\mathrm{ev}}(Y,G)=\bigoplus_{g\in G}\mathrm{H}^{\mathrm{ev}}(Y^g)$ and analogously for $\mathrm{H}^{\mathrm{odd}}$. Observe that $\mathrm{H}^{\mathrm{ev}}(Y,G)$ coincides with the even-graded part of $\mathrm{H}^*(Y,G)$ if and only if for every $g\in G$ and for every $y\in Y^g$ the age of g in y is an integer. Note also that the action of G on $\mathrm{H}^*(Y,G)$ preserves both the splitting into even and odd parts and the grading.
Finally, one defines the orbifold cohomology $\mathrm{H}^*_{\mathrm{orb}}([Y/G]):=\mathrm{H}^*(Y,G)^G$ to be the graded vector space of G-invariant classes of $\mathrm{H}^*(Y,G)$.

To introduce a product structure on $\mathrm{H}^*_{\mathrm{orb}}([Y/G])$, Fantechi and Göttsche define a bilinear map

$$\mu:\mathrm{H}^*(Y,G)\times\mathrm{H}^*(Y,G)\to\mathrm{H}^*(Y,G)$$

by

$$\mu(\alpha_g,\beta_h):=\gamma_{gh},\ \text{where}\ \gamma=i_*\left(\alpha|_{Y^{\langle g,h\rangle}}\cdot\beta|_{Y^{\langle g,h\rangle}}\cdot c(g,h)\right) \tag{25}$$

and $i:Y^{g,h}\to Y^{gh}$ is the natural inclusion.
The class $c(g,h)$ is a correction factor, which makes μ a graded product on the ambient space. It is defined as the top Chern class of the vector bundle $F(g,h):=(T_Y|_{Y^{\langle g,h\rangle}}\otimes_{\mathbb{C}}V)^{\langle g,h\rangle}$ on $Y^{\langle g,h\rangle}$. Here V is a natural representation of the group $\langle g,h\rangle$ which is constructed in a quantum cohomological way.
In the case of a complex torus, where the tangent bundle is trivial, this class will be 0 except the rank of the bundle is 0, and thus we omit its exact definition. We just cite the following lemma stated in **[7]**, which allows one to express the rank of $F(g,h)$ in terms of the ages of the group elements g and h:

LEMMA (**[7]**, Lemma 1.12). — *Let U be a connected component of $Y^{\langle g,h\rangle}$. Then one has*

$$\operatorname{rk}(F(g,h)|_U)=a(g,U)+a(h,U)-a(gh,U)-\operatorname{codim}(U\subset Y^{gh}). \tag{26}$$

From this rank formula, it follows easily that μ preserves the grading: Using the notations from the lemma above, we have

$$\begin{aligned}\deg(\mu(\alpha_g,\beta_h)) &= \deg(\gamma_{gh}) = \deg(\gamma) + 2a(gh,U) + \deg(c(g,h))\\ &= \deg(i_*(\alpha|_{Y^{\langle g,h\rangle}}\cdot\beta|_{Y^{\langle g,h\rangle}}) + 2a(gh,U) + 2\mathrm{rk}(F(g,h)|_U)\\ &= \deg(\alpha) + \deg(\beta) + 2\mathrm{codim}(U\subset Y^{gh}) + 2a(gh,U)\\ &+ 2\left(a(g,U) + a(h,U) - a(gh,U) - \mathrm{codim}(U\subset Y^{gh})\right)\\ &= \deg(\alpha) + 2a(g,U) + \deg(\beta) + 2a(h,U)\\ &= \deg(\alpha_g) + \deg(\beta_h).\end{aligned}$$

The heart of the article of Fantechi and Göttsche is the proof of the following

THEOREM ([**7**], Thm. 1.18 and Thm. 1.29). — *The bilinear map μ is graded, associative, and G-invariant. It induces a graded commutative multiplication, denoted by $\cup_{\mathrm{orb}}$, on the orbifold cohomology $\mathrm{H}^*_{\mathrm{orb}}([Y/G])$.*

2. The case of the Hilbert scheme

In Section 3 of [**7**], Fantechi and Göttsche compute the orbifold cohomology of symmetric products. For brevity, we will restrict our presentation to the case of the symmetric product of an abelian surface.

The length $l(g)$ of a permutation $g\in\mathfrak{S}_n$ is the minimal number of transpositions whose product is g. Denote the set of orbits of g action on the set $\{1,\dots,n\}$ by $O(g)$. Then, one has the equality $|O(g)| = n - l(g)$.

Let us consider the symmetric group acting on the n-fold product A^n of an abelian surface A. The age of an element $g\in\mathfrak{S}_n$ acting on the tangent space $(A^n)^g$ is given by

$$a(g) = l(g) = n - |O(g)|, \tag{27}$$

the length of the permutation g: Indeed, the fixed locus of g is connected and can be identified with $A^{O(g)}$ by sending

$$(A^n)^g \ni (a_1,\dots,a_n)\mapsto (a_o)_{o\in O(g)}\in A^{O(g)},$$

where a_o is defined to be $a_o := a_i$ for an arbitrary $i\in o$. Thus, the codimension of $(A^n)^g$ inside A^n is $2(n-|O(g)|)$. Now, the assertion follows by using the equality proven in Remark 34 and observing that, in the case of the symmetric group, g and g^{-1} have the same age since they are conjugate.

From our considerations above, it follows that in this case the ring $\mathrm{H}^*(A^n,\mathfrak{S}_n)$ is integrally graded and the division in even and odd part is the intuitive one.

Let α_g and β_h be two classes in $\mathrm{H}^*(A^n,\mathfrak{S}_n) = \bigoplus_{g\in\mathfrak{S}_n}\mathrm{H}^*((A^n)^g)$. As said above, the bundle $F(g,h)$ is trivial in our case, so it has trivial top Chern class except when its rank equals 0 and therefore $c(g,h) = 1$ in formula (25). The description of the age as $a(g) = n - |O(g)|$ together with the rank formula (26) for the bundle $F(g,h)$ stated above implies the following

PROPOSITION 35. — *Let A be an abelian surface. Then the ring structure on $\mathrm{H}^*(A^n, \mathfrak{S}_n)$ is given by*

$$\alpha_g \cup_{\mathrm{orb}} \beta_h = \begin{cases} i_* \left(\alpha_g|_{(A^n)^{\langle g,h\rangle}} \beta_h|_{(A^n)^{\langle g,h\rangle}} \right) & \text{if } |O(g)| + |O(h)| + |O(gh)| = \\ & \quad 2|O(g,h)| + n, \\ 0 & \text{otherwise.} \end{cases}$$

□

Observe that in the language of Lehn and Sorger, the condition on the orbit lengths means the vanishing of the graph defects which in their description of the singular cohomology of the Hilbert scheme we have already understood as a necessary condition in Section 4.

The climax of Section 3 of [**7**] is the proof of the fact that the orbifold cohomology ring of $[X/\mathfrak{S}_n]$ and the cohomology ring of the Hilbert scheme $X^{[n]}$ in the case of a surface X with trivial canonical bundle are isomorphic after a slight sign change. We only present the application of this theorem to an abelian surface A.

Let $g, h \in \mathfrak{S}_n$ and set $\varepsilon(g,h) := (l(g) + l(h) - l(gh))$. We change the orbifold product by defining

$$\alpha_g \cup_{\mathrm{orb,dt}} \beta_h := (-1)^{\varepsilon(g,h)} \alpha_g \cup_{\mathrm{orb}} \beta_h$$

and denote the resulting ring structure by $\mathrm{H}^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n])$. (The abbreviation 'dt' stands for 'discrete torsion' which is the physicists term for this sign change.) Then one has the following

THEOREM ([**7**], Thm 3.8). — *The two rings $\mathrm{H}^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n])$ and $\mathrm{H}^*(A^{[n]}, \mathbb{C})$ are naturally isomorphic.* □

In the following section we will prove an according theorem in the case of the generalized Kummer varieties.

3. Generalized Kummer Varieties

In the last section of [**7**], Fantechi and Göttsche compute the orbifold cup product in the case of the generalized Kummer varieties and formulate the conjecture that, like in the case of the Hilbert scheme, their formula will give the right product after an analogous sign change.

In the actual computation, they use isomorphisms which identify for a given subgroup H of $\mathfrak{S}_n$ the cohomology of $A \times (A_0^n)^H$ and $(A^n)^H$. Here and in the following, A_0^n denotes the fibre over 0 of the summation morphism $A^n \longrightarrow A$. That is, we have $A_0^n = \{(a_1, \dots, a_n) | \sum a_j = 0\}$.
These isomorphisms, as we will see, are essentially induced by quotients maps with respect to an operation of the $\frac{n}{\gcd(H)}$-torsion points. As such they are naturally in the context of cohomology but, and this fact is overlooked in [**7**], commute with Poincaré duality only up to a factor, given by the group order. Since there occurs a push-forward in the computation of the orbifold cup product, the formula of Fantechi and Göttsche has to be replenished with this factors. Let us recall the situation:

Our aim is to compute the orbifold cohomology ring of the orbifold $[A \times A_0^n/\mathfrak{S}_n]$, where $\mathfrak{S}_n$ operates on $A \times A_0^n$ by permuting the factors of A_0^n while acting trivially on A. We will first describe the structure of the space of invariants $(A \times A_0^n)^H = A \times (A_0^n)^H$ for a given subgroup $H \subset \mathfrak{S}_n$.
For such an H, denote again by $O(H)$ the set of the orbits of H acting on $\{1, \dots, n\}$. Analogously to the case of partitions, we define the greatest common divisor of the subgroup to be

$$\gcd(H) := \gcd\left(\{\, |o| |\, o \in O(H)\}\right),$$

that is, the greatest common divisor of the orbit lengths of H. In the case of a cyclic subgroup $\langle g \rangle \subset \mathfrak{S}_n$, we will use both notations $\gcd(\langle g \rangle)$ and the shorter $\gcd(g)$. Observe that with this definition, we have for an element $g \in \mathfrak{S}_n$, whose conjugacy type corresponds to a partition $\lambda \in P(n)$, the equality $\gcd(g) = \gcd(\lambda)$.
Furthermore, we will use the abbreviation $A[H]$ for the $\gcd(H)$-torsion points of the surface A.
For $H \subset \mathfrak{S}_n$, let us denote by q the morphism

$$q\colon A \times (A_0^n)^H \longrightarrow (A^n)^H, (a, (b_i)_i) \mapsto (a + b_i)_i.$$

The n-division points $A[n]$ act on $A \times (A_0^n)^H$ by setting for $c \in A[n]$: $c(a, (b_i)_i) = (a - c, (b_i + c)_i)$. The map q is just the quotient map for this action. Let us prove the following lemma (cf. [**12**], p.243).

LEMMA 36. — *In the case that* $\gcd(H) = 1$*, the fixed locus* $A \times (A_0^n)^H$ *is isomorphic to* $(A^n)^H$*. Furthermore, the* n*-division points* $A[n]$ *act trivially on* $\mathrm{H}^*(A \times (A_0^n)^H)$*. It follows that in this case* q^* *is an isomorphism*

$$q^*\colon \mathrm{H}^*((A^n)^H) \xrightarrow{\sim} \mathrm{H}^*(A \times (A_0^n)^H).$$

Proof. Denote by $s^H\colon A^{O(H)} \to A, (b_o)_{o \in O(H)} \mapsto \sum |o| b_o$ the morphism to which the summation $A^n \to A$ corresponds under the identification $(A^n)^H = A^{O(H)}$ described in Section 2. It follows that we can identify A_0^n with $A_0^{O(H)}$, the kernel of s^H.

This yields an exact sequence

$$0 \to A_0^{O(H)} \longrightarrow A^{O(H)} \xrightarrow{s^H} A \to 0.$$

Since we have assumed that $1 = \gcd(H) = \gcd(|o|)$, for $o \in O(H)$, we get a linear combination $1 = \sum j_o|o|$ with $j_o \in \mathbb{Z}$. This defines a splitting of s^H, by sending $A \ni b \mapsto (j_o b)_o \in A^{O(H)}$. It follows that $A^{O(H)} \simeq A \times A_0^{O(H)}$. In particular $A_0^{O(H)}$ is connected in this case.
To see that the n-division points $A[n]$ act trivially on the level of $\mathrm{H}^*(A \times (A_0^n)^H)$, it is enough to embed the action of $A[n]$ in that of a connected group. As we have seen $A \times (A^n)^H$ is a connected group. It acts on itself by

$$(a', (b'_i))(a, (b_i)) = (a - a', (b_i + b'_i)).$$

It follows that the $A[n]$-action can be embedded in that of $A \times (A_0^n)^H$ by sending $A[n] \ni a \mapsto (a, (a)_i)$.
Since q^*, as induced by a quotient map, is an isomorphism of $\mathrm{H}^*((A^n)^H)$ with $\mathrm{H}^*(A \times (A_0^n)^H)^{A[n]}$, the lemma follows. □

It becomes a bit more complicated in the case of a non-trivial greatest common divisor. In this case, we have the following

LEMMA 37. — *Let $H \subset \mathfrak{S}_n$ be a subgroup of $\mathfrak{S}_n$. Then $A \times (A_0^n)^H$ has $|A[H]|$ isomorphic connected components $A \times (A^n)_x^H$, with $x \in A[H]$.*
For each component, we have a natural isomorphism

$$\vartheta^H \colon \mathrm{H}^*((A^n)^H) \xrightarrow{\sim} \mathrm{H}^*(A \times (A^n)_x^H),$$

that is induced by a $A[n/\gcd(H)]$-quotient morphism.

Proof. Let us first determine the connected components of $A \times (A_0^n)^H$: Identify again $(A^n)^H$ with $A^{O(H)}$ and write $s^H \colon A^{O(H)} \to A, (b_o) \mapsto \sum |o| b_o$ for the summation. If $\gcd(H) \neq 1$, this map does not split. But we can decompose s^H via

$$s^H \colon A^{O(H)} \xrightarrow{s^h_{\mathrm{red}}} A \xrightarrow{\gcd(H)} A,$$

where the last arrow is multiplication with $\gcd(H)$ and the first arrow is the reduced summation morphism, defined by

$$s^H_{\mathrm{red}} \colon A^{O(H)} \to A; (b_o) \mapsto \sum \frac{|o|}{\gcd(H)} b_o.$$

Using this decomposition and observing that $\ker(\gcd(H)) = A[H]$, one sees

$$A_0^{O(H)} = \ker(s^H) = (s^H_{\mathrm{red}})^{-1}(A[H]) = \coprod_{x \in A[H]} (A^n)_x^H,$$

where we denoted by $(A^n)_x^H$ the fibre of s^H_{red} over the $\gcd(H)$-division point x.

To see that all components are isomorphic, we group the elements of $\{1,\dots,n\}$ in sets of $\gcd(H)$ elements, each of which is contained in an orbit of H. This defines a surjection $\{1,\dots,n\} \to \{1,\dots,\frac{n}{\gcd(H)}\}$ and gives us a subgroup $\overline{H}$ of $\mathfrak{S}_{(n/\gcd(H))}$ with $\gcd(\overline{H}) = 1$ such that we have a one-to-one correspondence of the orbits of H and $\overline{H}$.
Furthermore any z with $\frac{n}{\gcd(H)} z = x$ induces an isomorphism

$$j\colon A \times (A^n)^H_x \xrightarrow{\sim} A \times (A_0^{n/\gcd(H)})^{\overline{H}}, (a,(b_o)_{o\in O(\overline{H})}) \mapsto (a-z,(b_o - z)_{o\in O(H)}),$$

which shows that all components are isomorphic.
But dealing with $\overline{H} \subset \mathfrak{S}_{(n/\gcd(H))}$ we are in the situation of Lemma 36 again. In particular we have the isomorphism

$$q'^*\colon \mathrm{H}^*((A^{n/\gcd(H)})^{\overline{H}}) \xrightarrow{\sim} \mathrm{H}^*(A \times (A^{n/\gcd(H)})^{\overline{H}}),$$

which is induced by the $A[n/\gcd(H)]$ quotient morphism

$$q'\colon A \times (A^{n/\gcd(H)})^{\overline{H}} \to (A^{n/\gcd(H)})^{\overline{H}}; (a,(b_o)) \mapsto (a+b_o).$$

Moreover, it follows, that the isomorphism j^* on the level of cohomology is independent of the choice of z: Indeed, two choices of a point z over x differ by an element of $A[n/\gcd(H)]$, and we have already seen in Lemma 36 above that this group acts trivially on $\mathrm{H}^*(A \times (A_o^{n/\gcd(H)})^{\overline{H}})$.
Putting everything together, we get an isomorphism ϑ^H, defined as the composition

$$\vartheta^H\colon \mathrm{H}^*((A^n)^H) \xrightarrow[\iota^*]{\sim} \mathrm{H}^*((A^{n/\gcd(H)})^{\overline{H}}) \xrightarrow[q'^*]{\sim} \mathrm{H}^*(A \times (A^{n/\gcd(H)})^{\overline{H}}_0)$$
$$\xrightarrow[j^*]{\sim} \mathrm{H}^*(A \times (A^n)^H_x),$$

where the first arrow is induced by the identification

$$\iota\colon \left(A^{n/\gcd(H)}\right)^{\overline{H}} = A^{O(\overline{H})} = A^{O(H)} = (A^n)^H .$$

This is what we wanted to prove. $\square$

Observe that $\vartheta^H = q^*$ if $\gcd(H) = 1$. Thus Lemma 37 is a generalization of Lemma 36. Observe further that Lemma 37 implies in particular for the cyclic group $H = \langle g \rangle$:

$$\mathrm{H}^*((A \times A_0^n)^g) \simeq \bigoplus_{x\in A[g]} \mathrm{H}^*((A^n)^g). \tag{28}$$

The n-division points $A[n]$ operate on $\mathrm{H}^*((A\times A_0^n)^g)$ via the surjection $p_g\colon A[n] \twoheadrightarrow A[g]$, given by multiplication with $n/\gcd(g)$: As we have seen in the proof, a point $z \in A[n]$ identifies $A \times (A^n)^g_x$ with the fibre $A \times (A^n)^g_{x+p_g(z)}$. On the cohomology the kernel of p_g acts trivially. Thus on the level of cohomology the point $z \in A[n]$ acts by flipping the component $\mathrm{H}^*((A^n)^g) \subset \mathrm{H}^*((A \times A_0^n)^g)$ belonging to x identically to that belonging to $x + p_g(z)$.

The following corollary describes the interaction of ϑ^H with Poincaré duality. Denote by PD: $\mathrm{H}^* \to \mathrm{H}_*$ the Poincaré duality map. The isomorphism $\vartheta^H = (\iota q' j)^*$ induces an isomorphism $\vartheta_H := (\iota q' j)_*$ on the level of homology. We can deduce the following

COROLLARY 38. — *Let $H \subset \mathfrak{S}_n$ be a subgroup of $\mathfrak{S}_n$ and let ϑ^H and ϑ_H be the isomorphisms defined above. Then the following diagram commutes only up to the factor $\frac{|A[n]|}{|A[H]|}$:*

$$\begin{tikzcd} \mathrm{H}^*((A^n)^H) \arrow[r, "\vartheta^H"] \arrow[d, "PD"'] & \mathrm{H}^*(A \times (A^n)^H_z) \arrow[d, "PD"] \\ \mathrm{H}_*((A^n)^H) & \mathrm{H}_*(A \times (A^n)^H_z). \arrow[l, "\vartheta_H"] \end{tikzcd}$$

More specifically, the homomorphism $PD^{-1}\vartheta_H PD \vartheta^H \colon \mathrm{H}^*((A^n)^H) \to \mathrm{H}^*((A^n)^H)$ is multiplication with $\frac{|A[n]|}{|A[H]|}$.

Proof. In general, if $q \colon X \longrightarrow Y := X/G$ is a quotient map between two compact manifolds, one has

$$q_* q^* := \left(\mathrm{PD}^{-1} q_* \mathrm{PD} q^* \colon \mathrm{H}^*(Y) \longrightarrow \mathrm{H}^*(Y)\right) = \cdot|G|$$

which can be seen immediately by pairing a class $q_* q^* \alpha$ with an arbitrary class β:

$$\langle q_* q^* \alpha, \beta \rangle_Y = \langle q^* \alpha, q^* \beta \rangle_X = \int_X q^*(\alpha\beta) = |G| \int_Y \alpha\beta = |G| \langle \alpha, \beta \rangle_Y.$$

In our case, we want to compute $\mathrm{PD}^{-1}(\iota q' j)_* \mathrm{PD}(\iota q' j)^* =: \iota_* q'_* j_* j^* q'^* \iota^*$. But since j and ι are isomorphisms, we have $j_* j^* = \mathrm{id}$ and in the same manner $\iota_* \iota^* = \mathrm{id}$. The remaining morphism q' is an $A[n/\gcd(H)]$-quotient map, whereas

$$q'_* q'^* \colon \mathrm{H}^*(A \times (A^{n/\gcd(H)})^{\overline{H}}_0) \to \mathrm{H}^*(A \times (A^{n/\gcd(H)})^{\overline{H}}_0)$$

is multiplication with $|A[n/\gcd(H)]|$.
Thus, we have $\mathrm{PD}^{-1}\vartheta_H \mathrm{PD}\vartheta^H = \cdot\frac{|A[n]|}{|A[H]|}$, as asserted. □

This corollary becomes essential when we are dealing with the cohomology push-forward used in the definition of the orbifold cup product:
Let $g, h \in \mathfrak{S}_n$ and $H := \langle g, h \rangle$ the subgroup generated by g and h. The following diagram occurs in the last step of the computation of the orbifold cup product, where one has to push-forward a cohomology class $(\alpha_g \cup \beta_h) \in \bigoplus_{z \in A[\langle gh \rangle]} \mathrm{H}^*(A \times (A^n)^H_z)$ along the embedding $i \colon A \times (A^n)^H_z \hookrightarrow A \times (A^n)^{\langle gh \rangle}_z$. Using the isomorphisms ϑ^H and $\vartheta^{\langle gh \rangle}$ one computes the cohomology push-forward along the

embedding $i'\colon (A^n)^H \hookrightarrow (A^n)^{\langle gh\rangle}$ instead. This yields the following diagram:

$$\begin{tikzcd}
\mathrm{H}^*((A^n)^H) \arrow[r, "\vartheta^H"] \arrow[d, "\mathrm{PD}"'] & \mathrm{H}^*(A \times (A^n)_z^H) \arrow[d, "\mathrm{PD}"] \\
\mathrm{H}_*((A^n)^H) \arrow[d, "i'_*"'] & \mathrm{H}_*(A \times (A^n)_z^H) \arrow[l, "\vartheta_H"] \arrow[d, "i_*"] \\
\mathrm{H}_*((A^n)^{\langle gh\rangle}) \arrow[d, "\mathrm{PD}^{-1}"'] & \mathrm{H}_*(A \times (A^n)_z^{\langle gh\rangle}) \arrow[l, "\vartheta_{\langle gh\rangle}"] \arrow[d, "\mathrm{PD}^{-1}"] \\
\mathrm{H}^*((A^n)^{\langle gh\rangle}) \arrow[r, "\vartheta^{\langle gh\rangle}"] & \mathrm{H}^*(A \times (A^n)_z^{\langle gh\rangle})
\end{tikzcd}$$

One actually wants to compute the right vertical arrow of the rectangle and computes the left one instead. But the upper and lowest square commute only up to the factor $|A[n/\gcd(H)]|$ and $|A[n/\gcd(\langle gh\rangle)]|$, respectively, due to Corollary 38. We therewith have corrected the orbifold cup product computed by Fantechi and Göttsche in the case of the generalized Kummer varieties (Proposition 4.1 in [**7**]) by the factor stated the following

PROPOSITION 39. — *The ambient ring* $\mathrm{H}^*(A \times A_0^n, \mathfrak{S}_n)$ *of the orbifold* $[(A \times A_0^n)/\mathfrak{S}_n]$ *is isomorphic to*

$$\bigoplus_{g\in\mathfrak{S}_n} \bigoplus_{x\in A[g]} \mathrm{H}^*((A^n)^g)$$

with the ring structure given by

$$\alpha_{g,x} \cdot \beta_{h,y} = \frac{|A[gh]|}{|A[H]|} \sum_{z\in A[gh]} n_{g,h}(x,y,z)\gamma_{gh,z},$$

where $\gamma \in \mathrm{H}^*((A^n)^{gh})$ *is given by the corresponding product in the Hilbert scheme case (cf. (35)) and*

$$n_{g,h}(x,y,z) = \left|\left\{w \in A[H] \;\middle|\; p_g(w) = x, p_h(w) = y, p_{gh}(w) = z\right\}\right|$$

with the notation $p_g\colon A[H] \twoheadrightarrow A[g]$*, the surjection given by multiplication with* $\frac{\gcd(H)}{\gcd(g)}$*, and analogously for* p_h *and* p_{gh}*.* □

After this correction, it is easy to see that both ring structures coincide, thereby proving the conjecture that the orbifold cohomology ring of $[A \times A_0^n/\mathfrak{S}_n]$ and the singular cohomology ring $\mathrm{H}^*(A \times K^{(n-1)}A)$ coincide after the sign change described above: Denote again by $\varepsilon(g,h) = (l(g) + l(h) - l(gh))$ and change the orbifold cup product by the sign

$$\alpha_{g,x} \cup_{\mathrm{orb,dt}} \alpha_{h,y} := (-1)^{\varepsilon(g,h)} \alpha_{g,x} \cdot \beta_{h,y}.$$

Denote the resulting ring by $\mathrm{H}^*_{\mathrm{orb,dt}}([A \times A_0^n/\mathfrak{S}_n])$. Then, we get the following

THEOREM 40. — *The orbifold cohomology ring* $\mathrm{H}^*_{\mathrm{orb,dt}}([A \times A_0^n/\mathfrak{S}_n])$ *is isomorphic to* $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$.

Proof. We will write down an $A[n]$-equivariant isomorphism of the underlying vector spaces and check its compatibility with the ring structures:
On the one hand, due to Corollary 21 in Section 2 of the last chapter, we have for the cohomology of the product $A \times K^{(n-1)}A$ the equality

$$\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C}) = \bigoplus_{\lambda \in P(n)} \bigoplus_{\sigma \in A[\gcd(\lambda)]^\vee} \mathrm{H}^*(S^\lambda A, \mathbb{C})[2(|\lambda| - n)]$$

with the group $A[n]$ acting on the factor $\mathrm{H}^*(S^\lambda A, \mathbb{C})$ belonging to σ via the character σ.
On the other hand, Proposition 39, the definition (24) of the grading, and equality (27) gives us a degree preserving isomorphism

$$\mathrm{H}^*(A \times A_0^n, \mathfrak{S}_n) = \bigoplus_{g \in \mathfrak{S}_n} \bigoplus_{x \in A[g]} \mathrm{H}^*((A^n)^g, \mathbb{C})[2(|O(g)| - n)].$$

On this ring $y \in A[n]$ operates on the part belonging to a permutation g, by sending a class $\alpha_{g,x}$ to $\alpha_{g,x+p_g(y)}$, where $p_g \colon A[n] \to A[g]$ denotes the surjection given by multiplying with $n/\gcd(g)$.

A choice of a permutation g of conjugacy type $\lambda = (1^{\alpha_1}, \ldots, n^{\alpha_n})$ defines a quotient morphism

$$(A^n)^g \simeq A^{O(g)} \to A^{O(g)}/C(g),$$

where $C(g)$ denotes the centralizer of the element g inside $\mathfrak{S}_n$. Since g is of conjugacy type λ, we can identify $A^{O(g)}$ with $A^{|\lambda|}$ and therewith identify the centralizer of g with $\prod \mathfrak{S}_{\alpha_i}$. Thus, we have for each g of conjugacy type λ a $\prod \mathfrak{S}_{\alpha_i}$-quotient morphism $(A^n)^g \to S^\lambda A$. It induces on the level of cohomology an isomorphism

$$\mathrm{H}^*(S^\lambda A, \mathbb{C}) \simeq \mathrm{H}^*((A^n)^g, \mathbb{C})^{\prod \mathfrak{S}_{\alpha_i}}. \tag{29}$$

Now, let $\alpha_{\lambda,\sigma}$ be a cohomology class lying in the σ-weight space of $\mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C})$ and belonging to the partition λ. Recall that in terms of the cohomology of $A^{[n]}$ with values in locally constant systems, this means that

$$\alpha_{\lambda,\sigma} \in \mathrm{H}^*(S^\lambda A, L_{S^\lambda A,\sigma}) \subset \mathrm{H}^*(A^{[n]}, L_\sigma).$$

For every $g \in \lambda$ and every $x \in A[\lambda]$ the isomorphism (29) applied to $\alpha_{\lambda,\sigma}$ defines a class in $\mathrm{H}^*((A^n)^g, \mathbb{C})^{\prod \mathfrak{S}_{\alpha_i}} \subset \mathrm{H}^*((A^n)^g, \mathbb{C})$, which we denote by $\alpha_{g,x}$.
With these notations, let Θ be the following homomorphism of vector spaces

$$\Theta \colon \mathrm{H}^*(A \times K^{(n-1)}A, \mathbb{C}) \longrightarrow \mathrm{H}^*(A \times A_0^n, \mathfrak{S}_n); \alpha_{\lambda,\sigma} \mapsto \sum_{g \in \lambda} \sum_{x \in A[\lambda]} \sigma^{-1}(x) \alpha_{g,x},$$

where we used the fact that σ comes from the injection $A[\lambda]^\vee \hookrightarrow A[n]^\vee$, because otherwise $\mathrm{H}^*(S^\lambda A, L_{S^\lambda A,\sigma})$ would be trivial according to Lemma 20 of the last chapter, and wrote '$g \in \lambda$' for 'g is of conjugacy type λ'.

Observe that the class $\Theta(\alpha_{\lambda,\sigma})$ lies again in the σ-weight space of $\mathrm{H}^*(A\times A_0^n, \mathfrak{S}_n)$. In fact an element $z \in A[n]$ acts on $\Theta(\alpha_{\lambda,\sigma})$ by

$$\begin{aligned} z\Theta(\alpha_{\lambda,\sigma}) &= z\sum_{g\in\lambda}\sum_{x\in A[\lambda]} \sigma^{-1}(x)\alpha_{g,x} = \sum_{g\in\lambda}\sum_{x\in A[\lambda]} \sigma^{-1}(x)\alpha_{g,x+p_g(z)} \\ &= \sum_{g\in\lambda}\sum_{x\in A[\lambda]} \sigma^{-1}(x-p_g(z))\alpha_{g,x} = \sigma^{-1}(-z)\cdot \sum_{g\in\lambda}\sum_{x\in A[\lambda]} \sigma^{-1}(x)\alpha_{g,x} \\ &= \sigma(z)\cdot\Theta(\alpha_{\lambda,\sigma}). \end{aligned}$$

Thus, Θ is equivariant with respect to the action of the n-division points.

Further, by running through all $g \in \lambda$, we forced that the homomorphism Θ has its image in $\mathrm{H}^*(A\times A_0^n, \mathfrak{S}_n)^{\mathfrak{S}_n} = \mathrm{H}^*_{\mathrm{orb}}([A\times A_0^n/\mathfrak{S}_n])$, the invariant part under the action of the symmetric group. For dimension reasons, it is an isomorphism of the vector spaces $\mathrm{H}^*(A\times K^{(n-1)}A)$ and $\mathrm{H}^*_{\mathrm{orb}}([A\times A_0^n/\mathfrak{S}_n])$.

It remains to check that Θ is compatible with the ring structures. But this can be done by a straightforward calculation. Using the notations of Proposition 39, we have

$$\begin{aligned} \Theta(\alpha_{\lambda,\sigma}) \cup_{\mathrm{orb,dt}} \Theta(\beta_{\mu,\tau}) &= \sum_{\substack{g\in\lambda\\h\in\mu}}\sum_{\substack{x\in A[g]\\y\in A[h]}} \sigma^{-1}(x)\tau^{-1}(y)\alpha_{g,x}\cup_{\mathrm{orb,dt}}\beta_{h,y} \\ &= \sum_{g,h}\sum_{\substack{x\in A[g]\\y\in A[h]}}\sum_{z\in A[gh]} \sigma^{-1}(x)\tau^{-1}(y)\frac{|A[gh]|}{|A[g,h]|}n_{g,h}(x,y,z)\gamma_{gh,z} \\ &\overset{(A)}{=} \sum_{g,h}\sum_{w\in A[g,h]}(\sigma\tau)^{-1}(w)\frac{|A[gh]|}{|A[g,h]|}\gamma_{gh,p_{gh}(w)} \\ &\overset{(B)}{=} \sum_{g,h}\sum_{z\in A[gh]}(\sigma\tau)^{-1}(z)\gamma_{gh,z} \\ &= \sum_{\substack{\nu\in P(n)\\ \exists g,h: gh\in\nu}}\sum_{z\in A[\nu]}(\sigma\tau)^{-1}(z)\gamma_{gh,z} \\ &= \sum_{\substack{\nu\in P(n)\\ \exists g,h: gh\in\nu}}\Theta(\gamma_{\nu,\sigma\tau}) \\ &= \Theta(\alpha_{\lambda,\sigma}\cdot\beta_{\mu,\tau}). \end{aligned}$$

We will explain the calculation step by step: The first two equalities are just the definition of the orbifold cup product.

Equality (A) follows from the definition of the multiplicity $n_{g,h}(x,y,z)$: It counts the number of points $w \in A[g,h]$ lying simultaneously over x, y and z. Instead of summing over all $x \in A[g]$ and $y \in A[h]$ we sum over all $w \in A[g,h]$ each of which gives its contribution for $x = p_g(w)$, $y = p_h(w)$ and $z = p_{gh}(w)$, respectively. Since, via the surjections p_g and p_h, we can treat σ and τ as characters of the bigger group $A[g,h]$ we have replaced $\sigma^{-1}(x)$ by $\sigma^{-1}(w)$ and analogously for τ, which completes the proof of (A).

Equality (B) can be seen as follows: Denote by $\iota\colon K \to A[g,h]$ the kernel of $p_{gh}\colon A[g,h] \twoheadrightarrow A[gh]$. This gives a homomorphism $\iota^\vee\colon A[g,h]^\vee \twoheadrightarrow K^\vee$ of the character groups.

Assume that $(\sigma\tau)^{-1} \in A[g,h]^\vee$ is not the image under $p_{gh}^\vee$ of a character of the smaller group $A[gh]$. In this case, via $\iota^\vee$, we can consider $(\sigma\tau)^{-1}$ as a nontrivial element in $K^\vee$. Running through the set $p_{gh}^{-1}(z)$ for a fixed point $z \in A[gh]$, produces a contribution

$$\sum_{w\in p_{gh}^{-1}(z)} (\sigma\tau)^{-1}(w)\gamma_{gh,p_{gh}(w)} = \gamma_{gh,z} \underbrace{\sum_{k\in K}(\sigma\tau)^{-1}(k)}_{=0},$$

where the last sum vanishes since $(\sigma\tau)^{-1}$ was assumed to be nontrivial. From this it follows that we can change the index set of the inner sum from $A[g,h]$ to $A[gh]$ by just introducing the factor $\frac{|A[g,h]|}{|A[gh]|}$ which cancels out with the factor occurring in the orbifold cup product.

The next two equalities are given just by the separation of the occurring permutations in conjugacy classes and the definition of Θ. For the last equality, one has to observe that due to the description of the orbifold cup product in Proposition 39, the sum runs over all partitions ν that occur in the cup product in $\mathrm{H}^*(A^{[n]},\mathbb{C}) \simeq \mathrm{H}^*_{\mathrm{orb,dt}}([A^n/\mathfrak{S}_n])$. This coincides with our description of the singular cup product given in Theorem 32 of the previous chapter, whereas the last equality follows.

This completes the proof of Theorem 40. $\square$

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Teilpublikationen

(mit Marc A. Nieper-Wißkirchen.) Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. *arXiv:math.AG/0101062.*

Lebenslauf

Name:	Michael Britze, geb. Lang
Adresse:	Kyllburger Straße 9 50937 Köln Deutschland
Geburt:	am 29. Januar 1974 in Saarbrücken
Staatsangehörigkeit:	deutsch
Eltern:	Ursula Lang, geb. Streckel und Reinhold Lang
Familienstand:	seit 14. Juli 2000 verheiratet mit Claudia Britze
Kinder:	am 28. Oktober 2000 Geburt von Finn Joshua Britze am 12. Januar 2003 Geburt von Liv Carlotta Britze
Schulausbildung:	von 1980 bis 1984 an der Grundschule “Im Vogelsang”, Saarlouis von 1984 bis 1993 am Saarlouiser Gymnasium am Stadtgarten im Juli 1993 Abitur
Studium und Beruf:	vom Wintersemester 1993/1994 an Studium der Mathematik (Diplom) mit Nebenfach Philosophie an der Georg-August-Universität zu Göttingen; von Wintersemester 1995/1996 bis Sommersemester 1999 studentische Hilfskraft am Mathematischen Institut und an der Philosophischen Fakultät der Universität zu Göttingen; Diplomabschluss in Mathematik am 28. Oktober 1999; vom Wintersemester 1999/2000 an wissenschaftlicher Mitarbeiter am Mathematischen Institut der Universität zu Köln.