Open books for contact five-manifolds and applications of contact homology

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Abstract

In the first half of this thesis, we use Giroux's construction of contact open books to construct contact structures on simply connected five-manifolds. This allows us to reprove a theorem of Geiges concerning the existence of contact structures in all homotopy classes of almost contact structures on simply-connected five-manifolds.

In the second part of this thesis, we give an algorithm for computing the contact homology of some Brieskorn manifolds. As an application, we construct infinitely many contact structures on the class of simply connected contact manifolds that admit nice contact forms (i.e. no Reeb orbits of degree $-1, 0$ or 1) and have index positivity with trivial first Chern class. In particular we give examples of simply connected five-manifolds with infinitely many contact structures.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Existenz und Eindeutigkeit von Kontaktstrukturen in Dimension 5. Wir untersuchen die Existenz von Kontaktstrukturen mittels der sogenannten offenen Buchzerlegung. Giroux hat gezeigt, daß jede Kontaktmannigfaltigkeit sich als offenes Buch darstellen läßt. Umgekehrt schlägt er eine einfache Konstruktion vor, um gewisse offene Bücher für die Konstruktion von Kontaktmannigfaltigkeiten zu benutzen. Wir verwenden diese Konstruktion, um einen neuen Beweis des folgenden Satzes von Geiges [**18**] zu geben.

Satz 1. *Sei M eine einfach zusammenhängende Fünfmannigfaltigkeit. Dann hat M eine Kontaktstruktur in jeder Homotopieklasse von Fastkontaktstrukturen.*

Unser Beweis gibt eine explizite Konstruktion dieser Kontaktstrukturen. Außerdem hat man genügend Freiheit, um möglicherweise verschiedene Kontaktstrukturen in einer Homotopieklasse zu konstruieren.

Im zweiten Teil dieser Arbeit studieren wir eine Invariante von Kontaktmannigfaltigkeiten, nämlich Kontakthomologie. Die Theorie ist von Eliashberg und Hofer entwickelt worden [**14**], und mit Hilfe dieser Theorie kann man unter anderem manchmal die Existenz von unendlich vielen Kontaktstrukturen in einer Homotopieklasse zeigen. Wir präsentieren einen Algorithmus, um die Kontakthomologie von sogenannten Brieskornmannigfaltigkeiten zu berechnen. Der Algorithmus funktioniert für Brieskornmannigfaltigkeiten der Dimension größer als 3. Die meisten unserer Anwendungen sind allerdings in Dimension 5. Zum Beispiel beweisen wir die folgende Behauptung.

Sei B_{p^k} die einfach zusammenhängende Fünfmannigfaltigkeit mit zweiter Homologiegruppe isomorph zu $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ für $k \in \mathbb{N}$. Diese Mannigfaltigkeit ist nach einem Satz von Barden [**2**] eindeutig bis auf Diffeomorphismus. Wir schreiben $B_\infty = S^2 \times S^3$.

Satz 2. *Für jedes $k \in \mathbb{N}$ und jede Primzahl $p > 3$ besitzt die Mannigfaltigkeit B_{p^k} unendlich viele Kontaktstrukturen mit der gleichen Homotopieklasse von Fastkontaktstrukturen. Ebenfalls besitzen zusammenhängende Summen der Gestalt*

$$S^2 \times S^3 \underset{k\ mal}{\#\ldots\#} S^2 \times S^3 \# B_2 \underset{l\ mal}{\#\ldots\#} B_2 \# B_3 \underset{m\ mal}{\#\ldots\#} B_3 \# B_5 \underset{n\ mal}{\#\ldots\#} B_5$$

unendlich viele Kontaktstrukturen mit der gleichen Homotopieklasse von Fastkontaktstrukturen für $k, l, m, n \geq 0$.

Hiermit verbessern wir ein Ergebnis von Ustilovsky [**50**], der das gleiche für Sphären der Dimension $4k + 1$ gezeigt hat.

Contents

CHAPTER 1

Introduction

This thesis concerns itself with some issues in contact topology. It is divided into two parts. The first part is about open book decompositions, which we use as a way of constructing contact manifolds. Although it has been known for a long time, due to a paper of Thurston and Winkelnkemper [**49**], that there is a relation between open book decompositions and 3-dimensional contact manifolds, only recently has this relation been explored in higher dimensions. Key to this new interest in open books was Giroux's announcement that not only do certain open book decompositions give rise to contact manifolds, but that the converse holds true as well. Every contact structure is carried in some sense by an open book decomposition.

In one direction, this correspondence is easy to understand. Suppose we are given a compact Stein manifold P, i.e. a compact subset of a Stein manifold whose boundary is a level set of a plurisubharmonic function, and a symplectomorphism of P that is the identity near the boundary ∂P. Such a symplectomorphism gives rise to a mapping torus that carries a contact structure. In order to get a compact contact manifold and to complete the open book, we still need to glue in the binding of the open book. Since the boundary of the mapping torus looks like $\partial P \times S^1$ (the symplectomorphism was assumed to be the identity on the boundary), we can glue in the binding $\partial P \times D^2$. The binding carries a contact structure as well and can be made compatible with the contact structure on the pages. Thus we obtain a contact structure on a compact manifold which is induced by an open book.

We give one example to illustrate the procedure. Consider the disk D^2 with its standard exact symplectic structure, which is given by $2r dr \wedge d\vartheta$ in polar coordinates. This is the simplest example of a compact Stein manifold; it is a compact subset of $\mathbb{C}$ satisfying the above condition. We will use the identity as the monodromy for the mapping torus. The mapping torus is the solid torus $D^2 \times S^1$ with contact form $d\varphi + r^2 d\vartheta$. The binding is also a solid torus and the contact form there has a similar shape. Gluing the pages and binding together gives us a compact contact manifold, which in fact is isomorphic to S^3 with its standard structure. We can represent the open book graphically by removing one point from S^3 so that we can draw the situation in $\mathbb{R}^3$. This is done in Figure 1.2 and it explains the name "open book structure".

In view of the fact that A'Campo [**1**] gave a construction for open books on simply-connected 5-manifolds, it is interesting to see how Giroux's construction works in that case. Unfortunately, it turns out that most of A'Campo's open books are not admissible for Giroux's construction, but we can still use Giroux's construction for contact five-manifolds. Indeed, we follow Giroux's most basic observation to construct contact 5-manifolds. More specifically, one of our results can be regarded as an alternative proof of a theorem of Geiges.

THEOREM 1.1 (Geiges). *Let M be a simply-connected five-manifold. Then M admits a contact structure in every homotopy class of almost contact structures.*

Roughly speaking, we reprove this theorem by constructing an explicit contact structure in every single case. This procedure is greatly simplified by Barden's classification of simply-connected 5-manifolds, which allows us to write every simply-connected 5-manifold as a connected sum of model 5-manifolds. Although there are infinitely many of these model manifolds, they all have a rather nice structure.

In Chapter 7 we give a slightly modified version of a joint article with Klaus Niederkrüger. We consider the disk-bundle associated to T^*S^n as page of the open book and Dehn twists as the monodromy. This is the simplest possible case for a non-trivial monodromy. In addition, in

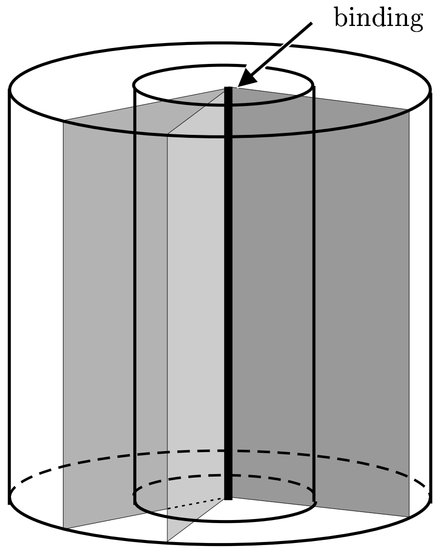

FIGURE 1.2. Open book in $\mathbb{R}^3$

the case $n = 2$ Seidel [**46**] has shown that these Dehn twists generate the group of compactly supported symplectomorphisms of T^*S^2. His results also imply that the Dehn twists have order 2 diffeomorphically (relative to the boundary), but are of infinite order symplectically. In other words, on the disk-bundle associated to T^*S^2 there are many Dehn twists that are isotopic relative to the boundary, but not symplectically so.

The second part of this thesis concerns itself with contact homology. One motivation for that theory is the following. An important problem in contact topology is to classify all contact structures on a given contact manifold. In dimension three, there are cases where this problem is completely solved, but in higher dimensions the situation is not clear at all. Let us consider the simplest compact contact manifolds, the odd-dimensional spheres, to clarify this.

In dimension three, we have the notion of overtwistedness to roughly classify contact structures. Due to a result of Gromov [**27**] it is known that overtwisted contact manifolds cannot be symplectically fillable. Results of Eliashberg [**12**] settle the situation for S^3. There is a unique tight contact structure (that is, in fact, also fillable) and in all homotopy classes of plane fields, there is precisely one overtwisted contact structure up to isomorphism.

Higher dimensional spheres are very different. Indeed, most examples are known to be fillable. If we restrict ourselves to dimension 5, the existence of a non-standard contact structure on S^5 was established by Eliashberg [**11**], who showed that the filling of the standard structure is some sense unique. He used this to exhibit an exotic contact sphere. Unfortunately, we cannot distinguish contact structures by their filling. In particular, exotic contact spheres cannot always be distinguished from one another by their filling.

The next step was made by Eliashberg and Hofer, following ideas of Floer. They developed a homology theory for contact manifolds, which can be used as an invariant of contact manifolds. Roughly speaking, this contact homology works as follows. The chain complex is generated by closed Reeb orbits and the differential is defined by counting certain pseudo-holomorphic curves connecting closed Reeb orbits. This theory is still very much being developed, since its foundations do not all have published proofs.

If we return to our example of the sphere, Ustilovsky's result comes to mind. He showed using contact homology that S^{4n+1} admits infinitely many contact structures. In itself, this does not seem to be very different from the 3-dimensional case. However, Ustilovsky's contact structures are all fillable and they all have the same classical invariants, which were used to classify the 3-dimensional case.

Our contribution can be seen as an extension of Ustilovsky's results. We present an algorithm to compute the contact homology of Brieskorn manifolds. Unlike Ustilovsky's approach we do not compute the contact homology by perturbing the contact form. The latter would make computations extremely hard, if not impossible. Instead, we use Bourgeois's Morse Bott contact homology, which enables us to compute contact homology with certain degenerate contact forms. In this way, we are able to use the nice natural contact form on the Brieskorn manifolds. All Reeb orbits of these natural contact forms are closed, and hence we get an S^1-action on the Brieskorn manifolds. Interestingly, the contact homology can be expressed in terms of the homology of the orbit spaces of this S^1-action with some degree shifts.

Application of the algorithm in various cases gives several interesting examples. We obtain many examples of index negative contact manifolds whose contact homology is only non-zero in negative degrees. We also find many other exotic contact structures on spheres that were not covered by Ustilovsky's results. We use some of these in a connected sum construction and thus we show that there are many manifolds admitting infinitely many contact structures. This covers a class of manifolds that is not restricted to spheres.

Structure of this thesis. We will introduce a lot of theory in this thesis, so we will start by describing known results before giving our own. In Chapter 2 we will describe some basic notions that are needed. This is standard material to those who work in the field of contact or symplectic geometry, but we provide it for the sake of completeness. Chapter 3 is about Stein manifolds, a special class of symplectic manifolds where the relation with contact manifolds is particularly interesting. It provides an overview on well-known properties of Stein manifolds that we need.

Chapter 4 concerns a particular class of contact manifolds. Again most results are taken from the literature, but some are not so well-known. We conclude the first part of our preparations Chapter 5. We present Giroux's construction. Most of the other results in that chapter are from the literature as well.

These preparations are used to present our results on open books for contact manifolds in Chapter 6. Results from a joint paper with Klaus Niederkrüger are presented separately in Chapter 7.

Our other results concern applications of contact homology. We present a very rough outline of that theory in Chapter 8, Chapter 9 and Chapter 10. Results from these chapters are all taken from the literature. The chapter on Maslov indices contains fairly standard results, but the other two preparatory chapters contain more advanced results. We are rather sketchy in those chapters, because of the complexity of the theory. Although we have tried to make everything as accessible as possible, it is probably helpful if the reader has some knowledge of Floer homology when reading Chapter 9 and Chapter 10. After these preparations, we present our results in Chapter 11. In the appendix we have included the code of a computer program that implements the algorithms from Chapter 4 and Chapter 11.

CHAPTER 2

Basic notions

In this chapter we will recall the basic notions that we will be using. A reader familiar with the subject can certainly skip this chapter.

2.1. Symplectic structures

DEFINITION 2.1. A **symplectic vector space** is a pair (V, ω), where V is a vector space and $\omega : V \times V \to \mathbb{R}$ is a skew-symmetric, non-degenerate bilinear form.

Given a linear subspace in a symplectic vector space (V, ω), we define the **symplectic complement** of W as

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

We say W is **isotropic** if $W \subset W^\omega$. This means that $\omega|_W$ vanishes. We say W is *symplectic* if $W \cap W^\omega = \{0\}$. There are other related notions, but we will not use them.

DEFINITION 2.2. A **symplectic manifold** is a smooth manifold M with a non-degenerate, closed differential two-form ω.

We see that symplectic manifolds are always even-dimensional, since two-forms on an odd-dimensional manifold must at least have a 1-dimensional kernel. If $\dim M = 2n$, then the non-degeneracy condition for ω can also be written as

$$\omega^n \neq 0.$$

Sometimes we will speak about the symplectic *structure* on a manifold, by which we mean the symplectic form. An isomorphism in the symplectic category is often called a symplectomorphism. More explicitly, suppose (M, ω) and (N, Ω) are symplectic manifolds, then a diffeomorphism $f : M \to N$ is called a **symplectomorphism** if $f^*\Omega = \omega$.

Since we are mostly concerned with contact manifolds, we shall not elaborate on symplectic manifolds in this section. To appreciate the strong relation between symplectic and contact manifolds, we will mention one basic result though. To that end we define the standard symplectic form on $\mathbb{R}^{2n}$, which we denote by ω_0. Let us use coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for $\mathbb{R}^{2n}$ and define

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

THEOREM 2.3 (Darboux's theorem). *Every symplectic manifold (M, ω) of dimension $2n$ is locally symplectomorphic to an open subset of $(\mathbb{R}^{2n}, \omega_0)$.*

EXAMPLE 2.4. Given a smooth manifold M, we can endow the cotangent bundle T^*M with a natural symplectic structure. There are many constructions for the symplectic structure and here we choose a construction with local coordinates. See [**34**] for another construction. Around a point $x \in M$, there are local coordinates $(q_1, \ldots, q_n)$ such that $(0, \ldots, 0)$ corresponds to x. The associated chart gives rise to a trivialization of the cotangent bundle. We write $(p_1, \ldots, p_n)$ for coordinates on the fibers of the cotangent bundle. In these local coordinates, we can define the **canonical 1-form**

$$\lambda_{can} = \sum_{i=1}^n p_i dq_i = \mathbf{p}\ d\mathbf{q}.$$

This 1-form is, in fact, independent of the choice of coordinates and induces a global 1-form. The exterior derivative of λ_{can} is denoted by ω_{can} and is called the canonical symplectic form on the cotangent bundle of M.

2.2. Contact manifolds

Let M be a $(2n+1)$-dimensional, smooth manifold.

DEFINITION 2.5. A **contact structure** on M is a hyperplane field ξ that is locally given by the kernel of a 1-form α such that

$$\alpha \wedge d\alpha^n \neq 0.$$

The pair (M, ξ) is called a **contact manifold**.

Another way of phrasing this definition is to say that the contact structure is maximally non-integrable. The identity

$$d\alpha(X, Y) = \mathcal{L}_X \alpha(Y) - \mathcal{L}_Y \alpha(X) - \alpha([X, Y])$$

explains this formulation. Namely, a distribution ξ is integrable precisely when sections of ξ are closed under the Lie-bracket. Since ξ is locally the kernel of the 1-form α, integrability of ξ means

$$\alpha([X, Y]) = 0$$

for all sections X and Y of ξ. For sections of ξ, the above identity reduces to

$$d\alpha(X, Y) = -\alpha([X, Y]).$$

We are requiring maximal rank for $d\alpha$ and therefore we are as far as we can possibly be from an integrable distribution. That is not to say that contact structures do not have integral submanifolds. However, the maximal possible dimension that an integrable submanifold of contact structure can have is less than that of other distributions. This is stated by the following proposition, see [**20**] or [**34**].

PROPOSITION 2.6. *Let (M, ξ) be a contact manifold of dimension $2n+1$. Let L be an integral submanifold of ξ. Then at each point $q \in L$ the tangent space T_qL is an isotropic subspace of $(\xi_q, d\alpha_q)$. In particular* $\dim L \leq n$.

Integrable submanifolds of maximal dimension are called **Legendrian**. Note that the above statement does not depend on the choice of local 1-form α. Indeed, if we take $\alpha' = f\alpha$, then the exterior derivative of α and of α' differ only by the non-zero function f, when restricted to the contact structure,

$$d(f\alpha)|_\xi = df \wedge \alpha|_\xi + f d\alpha|_\xi = f d\alpha|_\xi.$$

Like in symplectic geometry, contact manifolds have nice local models and have similar stability properties.

THEOREM 2.7 (Darboux's theorem). *Let (M, ξ) be a $(2n+1)$-dimensional contact manifold. Let α be a contact form on a neighborhood of a point p in M. Then there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on a possibly smaller neighborhood U of p such that*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j \, dy_j.$$

THEOREM 2.8 (Gray stability). *Let ξ_t for $t \in [0,1]$ be a smooth family of contact structures on a closed manifold M. Then there is an isotopy ψ_t for $t \in [0,1]$ of M such that*

$$T\psi_t(\xi_0) = \xi_t \text{ for each } t \in [0,1].$$

There are several theorems that emphasize the topological nature of contact manifolds. The following theorem, Theorem 2.41 from [**20**], is one of them.

THEOREM 2.9. *Let $j_t : L \to (M, \xi)$, for $t \in [0,1]$, be an isotopy of isotropic embeddings of a closed manifold L in a contact manifold (M, ξ). Then there is a compactly supported contact isotopy $\psi_t : M \to M$ with $\psi_t(j_0(L)) = j_t(L)$.*

In the special case of a Legendrian embedding, we will refer to this theorem as the Legendrian isotopy extension theorem.

An important problem in contact geometry is the classification of contact structures on a given manifold. It was discovered by Bennequin that some contact 3-manifolds carry non-isomorphic contact structures. Nowadays the contact structures that Bennequin studied are referred to as overtwisted. This comes from the following invariant, the existence of an overtwisted disk.

Definition 2.10. Let (M, ξ) be a contact 3-manifold. An **overtwisted disk** on M is a disk D, such that D is tangent to the contact structure ξ at the boundary,

$$TD|_\gamma = \xi|_\gamma.$$

If M has an overtwisted disk, then we say that M is **overtwisted**. If M does not have an overtwisted disk, then we say that M is **tight**.

Note that we see in particular that the boundary of such a disk is a Legendrian curve.

Example 2.11. We regard S^3 as a subset of $\mathbb{C}^2$. On $\mathbb{C}^2$ we choose complex coordinates (z_1, z_2). This way we can give S^3 a contact structure by considering the 1-form

$$\alpha = \frac{i}{2} \sum_{j=1}^{2} z_j d\bar{z}_j - \bar{z}_j dz_j.$$

We will denote the restriction of this 1-form to S^3 by $\tilde{\alpha}$. The kernel of $\tilde{\alpha}$ is a contact structure, which we will call the standard contact structure on S^3. This contact structure is tight, which was shown by Bennequin and in fact, it is the unique tight contact structure on S^3. It is also a holomorphically fillable contact structure. The latter notion will be defined and discussed in more detail in Chapter 3.

Note that in this particular example, the contact structure is the kernel of a globally defined form. This is a more general phenomenon; if the contact structure is cooriented, we have a global contact form α. The differential of α gives the contact structure ξ the structure of a symplectic vector bundle, which we denote by $(\xi, d\alpha)$. We also see that $d\alpha$ has a 1-dimensional kernel on TM. We use this fact to define the Reeb field.

Definition 2.12. The **Reeb field** of the contact manifold M with contact form α is the vector field defined by

$$\begin{aligned} i_R d\alpha &= 0, \\ i_R \alpha &= 1. \end{aligned}$$

Note that the Reeb field is strongly dependent on the choice of contact form. If f is a positive function, the contact form $f\alpha$ defines the same contact structure as α. The associated Reeb field $R_{f\alpha}$, however, differs in general from R_α. It can be expressed in terms of the Reeb field R_α and a perturbation Y. We have

$$R_{f\alpha} = R_\alpha + Y,$$

where the vector Y is determined by the following two equations

$$\begin{aligned} i_Y d\alpha &= \frac{df}{f^2} - \frac{R(f) + Y(f)}{f}\alpha, \\ i_Y \alpha &= \frac{1-f}{f}. \end{aligned}$$

Note that we can restrict the first equation to the contact structure ξ, which allows us to simplify the first equation to

$$i_Y d\alpha|_\xi = \frac{df}{f^2}|_\xi.$$

These formulae are often useful in order to see what happens to the Reeb field after a perturbation.

Remark 2.13. From now on, we will only consider contact manifolds with a coorientable contact structure.

2.2.1. Almost contact structures. If we are given a $(2n+1)$-dimensional contact manifold M with coorientable contact structure, then we split the tangent bundle of M into the Reeb line bundle and the contact structure, which carries the structure of a symplectic vector bundle. A symplectic vector bundle can be given the structure of a complex vector bundle.

DEFINITION 2.14. Let (E,ω) be a symplectic vector bundle over M. An endomorphism $J: E \to E$ is said to be a **complex structure** if $J^2 = -id$. We say J is **compatible** with ω if

- $\omega_p(Jv, Jw) = \omega_p(v,w)$ for all $v,w \in E_p$ and all $p \in M$.
- $\omega_p(v, Jv) > 0$ for all non-zero vectors $v \in E_p$ and all $p \in M$.

Since compatible complex structures always exist, see [**34**], we can indeed endow symplectic vector bundles with the structure of a complex vector bundle.

Thus we can reduce the structure group of M from $\mathrm{SO}(2n+1)$ to $U(n) \times \mathbb{1}$. We might be interested in doing the converse. This gives rise to the following definition,

DEFINITION 2.15. An **almost contact structure** on $2n+1$-dimensional manifold M is a reduction of the structure group of TM from $\mathrm{SO}(2n+1)$ to $U(n) \times \mathbb{1}$.

An important and unsolved question is whether any almost contact structure can deformed into an honest contact structure. In dimension 3 the answer is known to be positive.

THEOREM 2.16. *Let M be a closed, oriented 3-manifold. Then M admits a contact structure in any homotopy class of almost contact structures.*

For higher dimensional manifolds, Geiges has solved the above question in the affirmative for highly connected manifolds, see [**18**] and [**19**]. In Chapter 6 we will discuss this matter in greater detail.

Related to this question is the existence of an almost contact structure. Since we are mostly concerned with five-dimensional manifolds, the following result due to Geiges [**18**] is useful. Let M be a five-dimensional manifold. The obstruction to the existence of an almost contact structure on M is given by the third integral Stiefel-Whitney class, W_3. In other words, there exists an almost contact structure on M if and only if $W_3 = 0$. This characteristic class can be obtained by looking at the coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0,$$

which induces a long exact sequence in cohomology,

$$H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2) \xrightarrow{\delta} H^3(M;\mathbb{Z}).$$

The image of second Stiefel-Whitney class, w_2, under δ is W_3.

2.2.1.1. *Chern class of a contact structure.* Now let (M,ξ) be a contact manifold with contact form α. The associated almost contact structure is an invariant of the contact structure. In general, this invariant can be hard to compute. We can, however, also assign other topological invariants to a contact structure, which are easier to compute.

In the above discussion we already saw that the contact structure can be given the structure of a complex vector bundle. We can define the Chern class of a contact structure ξ by choosing a compatible complex structure for $(\xi, d\alpha)$ and computing the Chern class of that complex bundle. Different choices of compatible complex structure lead to the same Chern class, since the space of compatible complex structures $\mathcal{J}(E,\omega)$ on a symplectic vector bundle (E,ω) is connected. In fact, $\mathcal{J}(E,\omega)$ is contractible. See for instance [**34**].

Again, in dimension 5 we can say more due to a result of Geiges [**18**]. Indeed, the almost contact structure in dimension 5 is completely determined by the first Chern class.

CHAPTER 3

Stein manifolds, convex symplectic manifolds and symplectic cobordisms

In this chapter we want to describe a class of symplectic manifolds which will be the basic building blocks for our construction of contact manifolds. Roughly speaking, we are interested in exact symplectic manifolds whose boundaries carry an induced contact structure. These symplectic manifolds are called convex symplectic manifolds. We describe this notion following Eliashberg and Gromov, see [**10**]. Stein manifolds are a special class of complex manifolds that are convex symplectic.

We also introduce the notion of a symplectic cobordism which we will use a lot when we introduce contact homology. However, the notion of a symplectic cobordism can also be regarded as a generalization of a convex symplectic manifold.

3.1. Convex symplectic manifolds

Let (V, ω) be a symplectic manifold and let U be a domain in V bounded by a smooth hypersurface S. A vector field X is called an **expanding** field for ω if and only if

$$\mathcal{L}_X \omega = f\omega, \tag{3.1}$$

for a positive function f. The notion of a contracting field can be defined similarly. The hypersurface S is said to be **ω-convex** with respect to U if there exists an expanding field X that is transverse to S and looking outward. Of course, we could also ask for a contracting field looking inward. The notion of ω-concavity can be defined similarly. We say that a hypersurface S is **ω-concave** with respect to U if there is an expanding field X that is transverse to S and looking inward.

Remark 3.1. Note that the function f in the above definition must be constant if $\dim V \geq 4$. Indeed, if $\mathcal{L}_X \omega = f\omega$, then we get a one-parameter family of symplectic forms $f_t\omega$ by integrating along flow lines of X and pulling back. In particular $f_t\omega$ should be closed, so we have

$$df_t \wedge \omega = 0.$$

Non-degeneracy of ω then implies that $df_t = 0$ if $\dim V \geq 4$. On the other hand, if we denote the time t flow of X by Fl_t^X, we have

$$f\omega = \mathcal{L}_X \omega = \frac{d}{dt}(Fl_t^{X^*}\omega)|_{t=0} = \frac{d}{dt}(f_t\omega)|_{t=0}.$$

By taking the exterior derivative here and using $df_t \wedge \omega = 0$, we see

$$df \wedge \omega = 0.$$

By applying the same argument as for f_t, we see that $df = 0$, which means that f is constant on a connected component where X is defined. If the constant f is equal to 1, i.e.

$$\mathcal{L}_X \omega = \omega,$$

then the vector field X is also called **Liouville vector field**.

There are three possibilities for the domain where X is defined.

1. X is defined in a neighborhood of S. The hypersurface S is then said to be **locally ω-convex**.
2. X is defined in U including S. We say we have ω-convexity in U.

3. X is defined on all of V. This is referred to as global ω-convexity or ω-convexity in V.

REMARK 3.2. The notion of local ω-convexity for S corresponds to S being a hypersurface of contact type in the sense of Weinstein, since the function f in Formula (3.1) is constant by Remark 3.1.

3.2. Stein manifolds

In this section we will give several descriptions of Stein manifolds. We will also outline some relations they have with contact manifolds.

DEFINITION 3.3. A **Stein manifold** is a complex manifold that admits a proper biholomorphic embedding into $\mathbb{C}^N$ for some $N \in \mathbb{N}$.

Examples of Stein manifolds are zero-sets of polynomials in $\mathbb{C}^N$. In particular the complex affine space $\mathbb{C}^n$ is a Stein manifold. Notice that it immediately follows that Stein manifolds are Kähler manifolds, they being complex submanifolds of the Kähler manifold $\mathbb{C}^N$. Note that Stein manifolds are always exact symplectic. The symplectic form comes from the Kähler form and because the Kähler form is exact on $\mathbb{C}^N$, the induced form on a Stein manifold is exact as well. It also follows that a Stein manifold cannot be closed unless it consists of isolated points. Namely, suppose a given Stein manifold is closed. Then the coordinate functions are holomorphic functions which assume their extrema in an interior point. However, by the maximum principle the coordinate functions are then locally constant. This can only occur if the Stein manifold consists of isolated points.

To see how Stein manifolds relate to the previous section, we shall cite a few theorems. First we need to define the following notion.

DEFINITION 3.4. Let (M, J) be a complex manifold. We say a function $f : M \to \mathbb{R}$ is **(strictly) plurisubharmonic** if the two-form

$$\omega_\varphi = -d(J^* d\varphi)$$

satisfies $\omega_\varphi(v, Jv) > 0$ for all non-zero tangent vectors v.

REMARK 3.5. In the literature the term plurisubharmonic function is used for functions that satisfy $\omega_\varphi(v, Jv) \geq 0$. If there is a strict inequality, the notion of *strictly* plurisubharmonic function is used. However, since we will only consider the strict case, we will omit the word "strictly".

We say a two-form ω is **J-invariant** if $\omega(Jv, Jw) = \omega(v, w)$ for all tangent vector v and w. Note that above two-form ω_φ is J-invariant. We can associate a symmetric form g to a J-invariant two-form ω by putting

$$g(v, w) = \omega(v, Jw).$$

For a plurisubharmonic function φ we have a two-form ω_φ coming from the above definition. The associated symmetric form is then positive definite. In other words, a plurisubharmonic function gives rise to a Kähler metric and an associated two-form ω_φ. This two-form is symplectic by positive definiteness of g_φ. The notion of plurisubharmonic function can be used to characterize Stein manifolds. This is done in the following theorem due to Grauert [**26**].

THEOREM 3.6. *A complex manifold M without boundary is a Stein manifold if and only if M admits a proper Morse function $\varphi : M \to [0, \infty)$ which is strictly plurisubharmonic.*

By our previous observations, we see in particular that a plurisubharmonic function endows a Stein manifold with a symplectic structure. We might wonder whether the symplectic structure depends on the choice of plurisubharmonic function. The following theorem by Eliashberg and Gromov [**10**] shows this not to be the case.

THEOREM 3.7 (Eliashberg and Gromov). *If (M, J) is a Stein manifold, then all symplectic structures coming from plurisubharmonic functions on M are isomorphic.*

Let φ be a plurisubharmonic function on M. The following proposition shows that the level sets of φ are globally ω_φ-convex, which gives a relation between Stein manifolds and our previous section.

PROPOSITION 3.8. *Let $\nabla\varphi$ denote the gradient vector field of φ with respect to g_φ. Then $\nabla\varphi$ is an expanding vector field for ω_φ.*

Hence the level sets of a plurisubharmonic function are contact manifolds. Such contact manifolds are said to be **Stein fillable** or **holomorphically fillable**.

REMARK 3.9. We will use the term **compact Stein manifold** from time to time. This term is a bit strange, because we just pointed out that Stein manifolds cannot be compact unless they are points. This concept is to be understood in the following way. A compact complex manifold M with non-empty boundary K is called a **compact Stein manifold** if M admits a strictly plurisubharmonic Morse function f such that K is a level set of f. We get a proper Stein manifold by removing the boundary.

The following theorem of Lefschetz shows that the Stein condition puts considerable constraints on the topology of the underlying complex manifold, see for instance [**38**].

THEOREM 3.10. *Every Stein manifold of real dimension $2n$ has the homotopy type of a cell complex of dimension n.*

In handlebody language, this means that Stein manifolds admit a handlebody decomposition without handles of index higher than n. The following theorem by Eliashberg gives the converse if the real dimension of the manifold is greater than 4, see [**15**].

THEOREM 3.11. *Let X be a $2n$-dimensional smooth manifold with an almost complex structure J and $n > 2$. Let $\varphi : M \to \mathbb{R}$ be a proper Morse function such that the indices of all its critical points are $\leq n$. Then J is homotopic to a complex structure $\tilde{J}$ such that φ is $\tilde{J}$-convex. In particular, the complex manifold $(M, \tilde{J})$ is Stein.*

Note that this theorem covers both the case of proper Stein manifolds and of compact Stein manifolds. With suitable modifications, another version of this theorem can actually be made to work in real dimension 4 as well. This was also done by Eliashberg, but here we will follow a description due to Gompf. We need to introduce some notation and definitions for his theorem and we will start with that.

3.2.1. Some handlebody theory. In this section, we shall give a summary of some of the results from the book by Gompf and Stipsicz [**25**]. We start by recalling how to construct a smooth oriented manifold by adding handles to D^4. We will attach the handles by increasing index. This is, in fact, the most general case, because we can isotope the attaching maps such that handles are attached in order of increasing index, see for instance proposition 4.2.7 from [**25**]. Note that a handle of index k can be regarded as a thickened k-cell. Therefore the above theorem of Lefschetz shows that we only need to attach handles of index one and two.

Later we will use these constructions of Stein manifolds, but only in a few simple cases, where we just attach two-handles to D^4. Because of this, I opted to give a rough outline of the general case (attaching both one-handles and two-handles) and will give a more detailed description in the presence of just two-handles. During the discussion, it might be helpful to refer to Figure 3.2 to visualize what the terms mean.

The attaching of one-handles to D^4 is done as follows. Let A_i be a copy of S^0 embedded in $S^3 = \partial D^4$ for $i = 1, \ldots, k$. The embeddings should be chosen disjointly, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$. These sets A_i are called the **attaching spheres** of the one-handles. We find neighborhoods of the A_i that are disjoint and diffeomorphic to $S^0 \times D^3$. Fix such neighborhood N_i of A_i and fix a diffeomorphism φ_i to $S^0 \times D^3$ for each i. The neighborhoods N_i are referred to as **attaching regions**.

Set $H_i = D^1 \times D^3$ for $i = 1, \ldots, k$. These are the one-handles we want to glue in. The four-ball with one-handles glued in is the identification space

$$X = D^4 \cup H_1 \cup \ldots \cup H_k / \sim$$

where $D^4 \ni x \sim y \in H_i$ if and only if $x \in N_i$ and $\varphi_i(x) = y \in S^0 \times D^3 \subset \partial H_i$. Note that X is not a smooth manifold. We can, however, use a smoothing corners procedure to obtain a well-defined smooth manifold. We will denote this smooth manifold by X as well, where it is implicitly

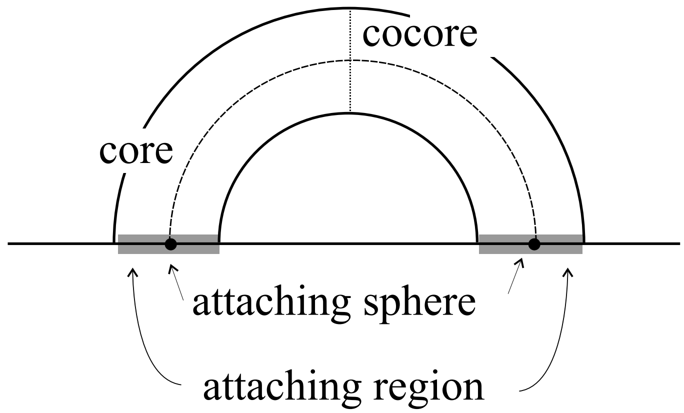

FIGURE 3.2. A handle attachment

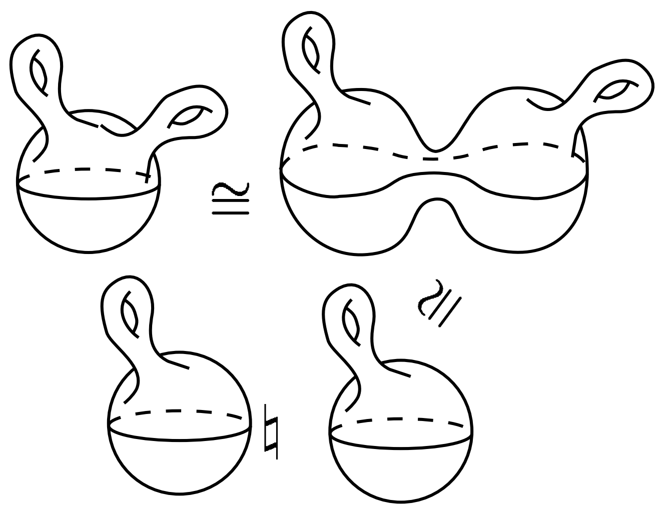

FIGURE 3.3. Boundary connected sum of two solid tori

understood that the corners are smoothed. The handlebody we obtain this way depends only on the isotopy class of the embeddings φ_i. Replacing φ_i by an embedding that is isotopic to φ_i gives a handlebody diffeomorphic to the original one. This is a general principle in handle attachment, but one-handles are special in the sense that there is no obstruction in finding an isotopy of one embedding φ_i to another one (unless the handlebody is disconnected).

Note that the boundary of X is diffeomorphic to $\#_k S^1 \times S^2$. Indeed, attaching a single one-handle to D^4 gives us $S^1 \times D^3$. Adding multiple one-handles can be interpreted as taking the boundary connected sum of copies of D^4 with a single one-handle attachment, see also Figure 3.3 for an analogous situation with three-dimensional handlebodies. Since the boundary of a boundary connected sum is a connected sum of the boundaries, our claim follows. This illustrates explicitly that attaching handles along isotopic attaching spheres gives diffeomorphic manifolds.

In general we can attach the two-handles in a similar way. We choose embeddings of S^1 into the boundary of the handlebody, which we call **attaching circles**, and glue the two-handle

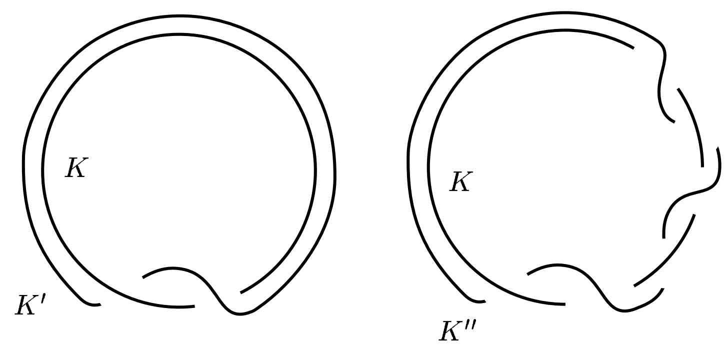

FIGURE 3.4. Framings on an unknot

$D^2 \times D^2$ to the handlebody, by identifying a neighborhood of the embedded S^1 with a subset of the boundary of the two-handle, namely with $S^1 \times D^2$. A few remarks need to be made here. The identification is not unique and needs to be indicated with additional data. In the case of one-handles this was not a problem, because all framings of the normal bundle of the embedded S^0 are isotopic. In the case of two-handles, with respect to some metric we can choose a orthogonal trivialization of the normal bundle, so the homotopy class of framings is represented by a class in $\pi_1(SO(2)) \cong \mathbb{Z}$. Note also that there are embeddings of S^1 in S^3, or in general in $\#_k S^1 \times S^2$, that are not isotopic even if they are homotopic; an embedded circle can, for instance, be knotted.

This latter point is not a problem, but only indicates that the resulting handlebody depends in a more essential way on the choice of embeddings than in the case of just attaching one-handles. The other point, the framing, needs to be taken into account, though. We will describe the issue of framing in the absence of one-handles.

3.2.1.1. *Framing of two-handles.* Let K be an embedding of S^1 into D^4 along which we want to attach a two-handle. The framing of K is a trivialization of the normal bundle of K. Hence we can indicate a framing by a parallel copy K' of K that is disjoint from K and is contained in a tubular neighborhood of K that can be identified with the normal bundle of K. In the previous section, we claimed that the homotopy class of framings of the normal bundle of K correspond to $\pi_1(SO(2)) \cong \mathbb{Z}$. In order to work with this correspondence, we need to specify a framing corresponding to $0 \in \pi_1(SO(2))$. For that, we take any framing as we just described by choosing a parallel copy K' of K and say it corresponds to $0 \in \pi_1(SO(2))$. Any other framing K'' can then be defined by taking K' and putting additional twists around K in it, see Figure 3.4.

Of course, this correspondence is not unique. In the absence of one-handles, we can single out a distinguished framing though. First orient K. If there are no one-handles, the curve K is null-homologous in S^3, and we can therefore choose an oriented Seifert surface Σ for K. If K' is another embedding of S^1 disjoint from K, then we define the **linking number** of K and K' as the algebraic number of intersections of K' with the Seifert surface of K. Note that we can always isotope K' to a curve that is transverse to Σ, in which case we can count the signed intersection of K' with Σ. Alternatively, the linking number can be defined as the intersection product of the class that K' represents in $H_1(S^3 - N(K))$ and the class that Σ represents in $H_2(S^3 - N(K), \partial N(K))$. Here we have used $N(K)$ to indicate a tubular neighborhood of K. We will denote the linking number of K and K' by $lk(K, K')$.

REMARK 3.12. We should remark at this point that the linking number is independent of the choice of Seifert surface. By definition, the linking number depends only on the homology class of K' in $H_1(S^3 - N(K))$. Also, the linking number is symmetric,

$$lk(K, K') = lk(K', K).$$

With the notion of linking number we can define a distinguished copy K' of K by requiring $lk(K, K') = 0$. The correspondence of the framing with $\pi_1(SO(2))$ can then also be made more explicit. Namely if K is a framed embedding of S^1 (with framing K''), we define the **framing coefficient** of K as $lk(K, K'')$.

In the presence of one-handles, we can in principle do something similar, but for attaching circles that run over one-handles (and might therefore not be null-homologous) more work is required. Because we will be using the above handle construction only in cases where there are no one-handles, we will not go into this.

The attaching of a two-handle can now be made unambiguous by specifying an embedding of the attaching circle in $S^3 \subset D^4$ and a framing coefficient to indicate the framing of the normal bundle used in the identification. For the attachment of a single two-handle to D^4, set $H = D^2 \times D^2$. Choose a framed embedding K of S^1 into $S^3 = \partial D^4$ such that the framing coefficient of K coincides with the given one. Denote a tubular neighborhood of K that corresponds to the normal bundle of K by $N(K)$. The framed embedding gives rise to a diffeomorphism φ from $N(K)$ to $S^1 \times D^2$. Then we define

$$X = D^4 \cup H/\sim,$$

where $D^4 \ni x \sim y \in H$ if and only if $x \in N(K)$ and $\varphi_i(x) = y \in S^1 \times D^2 \subset \partial H$. Again we need to smooth corners to ensure that X is a smooth manifold. This process can be repeated in order to attach any number of two-handles.

Remark 3.13. We will only be attaching two-handles to D^4, so the attaching circles are just curves in S^3. This allows us to use the standard convention for drawing knots in S^3, i.e. we assume that the knots miss at least one point, so we can draw them in $\mathbb{R}^3$. We choose a projection of $\mathbb{R}^3$ onto the plane such that the intersections in the projection are at most double points and transverse. This allows us to visualize the attaching circles.

Remark 3.14. Although we said we needed an embedding of a link and framing coefficients to determine the attaching of the two-handles unambiguously, the actual handlebody depends only on the isotopy type of the link and the framing coefficients. That is to say, an isotopy of an attaching circle gives rise to a diffeomorphism of the resulting handlebody.

3.2.1.2. *Topology of a handlebody.* The handlebody language is a very useful description when we want to say something about the (algebraic) topology of a manifold. For instance, the handlebody X obtained by attaching k one-handles to D^4 gets an additional generator of $\pi_1(X)$ for each one-handle, so $\pi_1(X)$ is a free group on k generators. Note that a two-handle can cancel a one-handle (see Figure 3.5) and in fact, adding two-handles to a handlebody gives relations for its fundamental group. A handle cancellation is a simple example where the relation is that the generator corresponding to the one-handle is trivial. By letting the attaching circle of a two-handle run in different ways over the one-handles (over several one-handles or multiple times over the same one-handle), one can make more complicated relations. In dimension two and three not every relation can be realized, because the attaching circles need to form an embedded link. In dimension four there is no restriction, since the boundary of a handlebody is three-dimensional. That leaves enough room to ensure attaching circles realizing any given relation can be embedded.

Our main point of interest concerns a handlebody without one-handles, so we will not prove the above statements about the fundamental group. Earlier, we made the claim that attaching a k-handle can be regarded as adding a thickened k-cell. By doing the reverse (shrinking the handlebodies to cells), we also see that adding two-handles to the zero-handle D^4 gives a handlebody that is homotopy equivalent to a bouquet of k two-spheres. So if we define X as a handlebody D^4 with k two-handles attached, we see that X is simply connected and $H_2(X) \cong \mathbb{Z}^k$.

The generators of the second homology can be explicitly represented by closed surfaces in X in the following way. Let us denote the link of oriented attaching circles in ∂D^4 by L. The oriented components of L will be denoted by $K_1, \ldots, K_k$. Each component K_i of the link admits a Seifert surface that lies in ∂D^4. We perturb the Seifert surface in such a way that its interior lies in the interior of D^4. Let us denote this surface with boundary K_i by F_i. If we attach the two-handle along K_i, we see a disk with boundary K_i lying in the two-handle, namely the core $D^2 \times \{0\}$. If we glue the core of the two-handle to F_i along K_i, we get a closed surface $\tilde{F}_i$. If

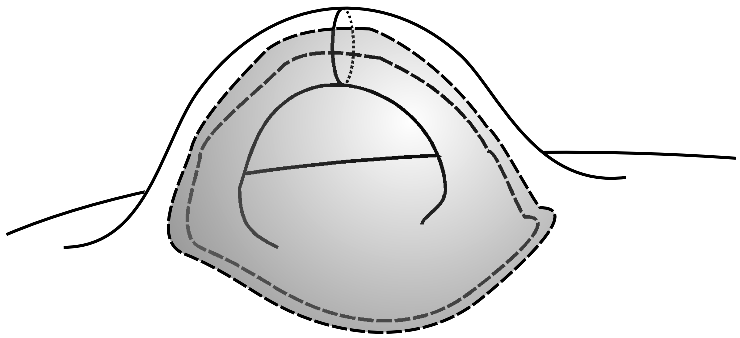

FIGURE 3.5. A two-handle cancelling a one-handle

the perturbed Seifert surface F_i is oriented such that its orientation on the boundary coincides with the orientation of K_i, we can extend the orientation to $\tilde{F}_i$. We can repeat this process for all attaching circles and get a closed, oriented surface $\tilde{F}_i$ for each K_i, for $i = 1, \ldots, k$. The classes $\alpha_i := [\tilde{F}_i] \in H_2(X)$ form a basis for $H_2(X)$.

We can describe the intersection product of X with respect to this basis. At this point, the orientation of the link L and of the associated Seifert surfaces comes into play, because arguments for the following theorem rely on the linking number. We need one additional definition before we can state the theorem.

DEFINITION 3.15. The **linking matrix** of an oriented, framed link $L = \cup_{i=1}^k K_i$, where K_i is an embedded circle in ∂D^4, is given by the following $(k \times k)$-matrix $[a_{ij}]$. The entry a_{ij} is given by $lk(K_i, K_j)$ for $i \neq j$ and by the framing coefficient of K_i if $i = j$.

Note that the linking matrix is symmetric. The intersection form of X is given by the following theorem.

THEOREM 3.16 (see [**25**]). *Let X be a connected handlebody given by attaching k two-handles to D^4 along an oriented, framed link L in ∂D^4. The matrix of the intersection form of X with respect to the basis $\alpha_1, \ldots, \alpha_k$ as defined above is given by the linking matrix of L.*

The surfaces $\tilde{F}_i$ used to represent the generators of the homology of X are not unique. We can push the interior of the Seifert surface F_i to different "depths" in D^4, see Figure 3.6. Of course, such a perturbation extends to the surface $\tilde{F}_i$ and does not affect the homology class of $\tilde{F}_i$. This can be used to prove the above theorem. Suppose we want to compute the intersection product of α_i with α_j. We can assume by pushing $\tilde{F}_j$ deeper that $\tilde{F}_i$ intersects $\tilde{F}_j$ at points where $\tilde{F}_j$ is "vertical" (meaning that $\tilde{F}_j$ looks like $I \times K_j$), see Figure 3.6. In other words, the intersection points of $\tilde{F}_i$ with $\tilde{F}_j$ correspond to the intersection points of F_i with K_j. Note also that the orientations of the intersections of $\tilde{F}_i$ with $\tilde{F}_j$ correspond to the ones of F_i with K_j. Thus we obtain

$$\alpha_i \cdot \alpha_j = \tilde{F}_i \cdot \tilde{F}_j = F_i \cdot K_j = lk(K_i, K_j).$$

A similar argument can be applied to show that $\alpha_i \cdot \alpha_i = lk(K_i, K_i')$.

We would like to finish this section with an example, where we give a handlebody decomposition for disk-bundles over S^2.

EXAMPLE 3.17. In this example we will attach an index 0 or 2 to the sets to indicate whether they belong to a zero-handle or to a two-handle, respectively. We do this to avoid confusion, since

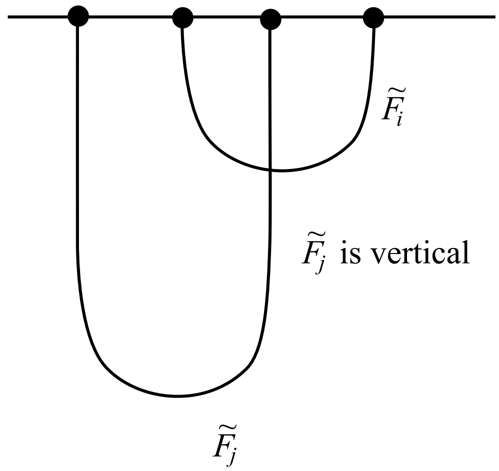

FIGURE 3.6. Pushing Seifert surfaces to different depths

we will write the zero-handle as a product $D^2 \times D^2$. Since the two-handle had the same form, it might be hard to distinguish which one is which.

Suppose we want to attach a two-handle along a framed unknot K in $S^3 \subset {D^4}_0$. We can regard the zero-handle, ${D^4}_0$, as the product $D^2 \times {D^2}_0$. By an isotopy of the unknot K we can assume that K lies in $(\partial D^2) \times {D^2}_0$. In fact, we can do the isotopy in such a way that the identity is used in the first factor to glue the two-handle $D^2 \times D^2_2$ to the zero-handle $D^2 \times D^2_0$. Moreover, the identification in the second factor can be assumed to be an element in $SO(2)$. In other words, we do the following. Let x_0 be an element in the boundary of the zero-handle $D^2 \times {D^2}_0$ and let y_2 denote an element in the $(\partial D^2 \times {D^2}_2)$-part of the two-handle. The attaching map is given by

$$\begin{array}{rrcl} f: & S^1 \times {D^2}_2 & \to & S^1 \times {D^2}_0 \\ & (p, v) & \mapsto & (p, A(p)v), \end{array}$$

for a loop A of $SO(2)$-matrices. Now we say that $x_0 \sim y_2$ if $x_0 = f(y_2)$. The handlebody X is given by the identification space

$$X = D^2 \times {D^2}_0 \cup D^2 \times {D^2}_2 / \sim .$$

It immediately follows that X is a disk-bundle over S^2. More precisely, if the framing coefficient of K is given by k, we see that we get a disk-bundle over S^2 with Euler number k. Indeed, it suffices to compute the self-intersection of a surface generating $H_2(X)$. One sphere $\tilde{F}$ that generates $H_2(X)$ can be seen by taking the core of the two-handle, $D^2 \times \{0\}$ and gluing this disk to the Seifert surface F of the unknot K, which also is a disk. For another sphere representing the same class as $\tilde{F}$, take a disk $D^2 \times \{p\}$ where $p \neq 0$. This disk intersects the zero-handle in a parallel copy K' which indicates the framing of k. Since K' is the unknot as well, the Seifert surface F' for K' is also a disk. Gluing these two disks together gives a sphere $\tilde{F}'$. Note that $\tilde{F}$ represents the same homology class as $\tilde{F}$.

From the construction, it follows that $\tilde{F}$ and $\tilde{F}'$ can only intersect each other in the interior of the zero-handle if we push the interiors of F and F' into the interior of D^4. Moreover, we can assume that F' is vertical at the intersection points. Therefore the self-intersection of the class of $\tilde{F}$ is given by

$$\tilde{F} \cdot \tilde{F}' = F \cdot F' = F \cdot K' = lk(K, K') = k.$$

3.2.2. Legendrian curves. In the previous section, we roughly outlined how to attach one- and two-handles to D^4, by attaching them along attaching spheres in $\partial D^4 = S^3$. It turns out that we get a natural Stein structure on a handlebody if we attach the two-handles along distinguished curves with a specific framing. In fact, Eliashberg has proved a theorem showing that any compact Stein manifold of dimension 4 can be represented in such a way.

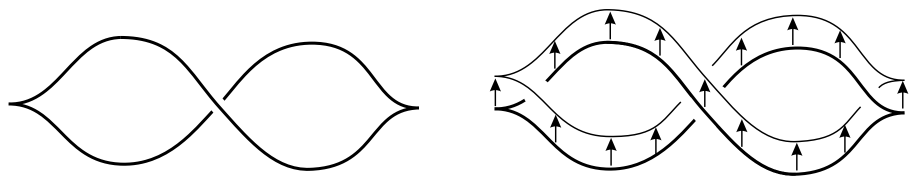

FIGURE 3.7. Canonical framing of a Legendrian curve

The first thing to note is that the zero-handle in a handle-body decomposition carries a Stein structure. In fact, the four-ball D^4 is a Stein filling for the unique tight contact structure on S^3. This observation can be regarded as a guideline, because the boundary of a compact Stein manifold is a contact manifold. In other words, constructing a Stein structure on manifold M gives the boundary of M a contact structure. In order to ensure that the boundary of a two-handlebody carries a contact structure, the attaching circles of the two-handles need to be adapted to the geometric structure. We do this by requiring the attaching circles to be Legendrian curves. There are special requirements for the framings of the two-handles.

In the following, we will talk about Legendrian curves. We only intend to apply the discussion to handlebody constructions without one-handles. Hence we will consider only $S^3 = \partial D^4$ with its unique fillable contact structure ξ. A lot of points will apply in more general situations though. In particular, everything that is done here can be made to work in the presence of one-handles. For more information, see chapter 11 of [**25**].

For practical purposes, it is useful to draw the Legendrian curves in a diagram. As is common in knot diagrams, we take out a point of S^3 to be able to draw curves in $\mathbb{R}^3$. If we restrict the tight contact structure on S^3 to $\mathbb{R}^3$ we get a contact structure that is contactomorphic to the standard contact structure on $\mathbb{R}^3$, see for instance Proposition 2.13 of [**20**]. We will use the convention of Gompf and Stipsicz, so we consider $\mathbb{R}^3$ with contact form $\alpha = dz + x\,dy$. We will use the front projection, where a curve is projected to the yz-plane. This is done for the following reason. Consider the curve

$$\gamma(t) = (x(t), y(t), z(t)) \text{ in } \mathbb{R}^3.$$

If γ is a Legendre curve, then $\frac{d\gamma}{dt} \subset \ker\alpha$, which means that $x = -\frac{dz}{dy}$. In other words, the slope of γ in the yz-projection will determine the x-coordinate of a Legendrian curve. Moreover, for a Legendrian curve, the projection cannot have vertical tangencies. Instead a Legendrian curve can have cusps in its front projection. We also see from the condition $x = -\frac{dz}{dy}$ that an arc with a larger slope will cross underneath an arc with a smaller slope, since larger slope implies a smaller x-coordinate.

We will consider Legendrian isotopies, i.e. isotopies through Legendrian curves. By the Legendrian isotopy extension theorem in contact geometry, a Legendrian isotopy extends to an isotopy of the ambient contact manifold.

Another feature of (oriented) Legendrian curves is that they come with a canonical framing. This framing is induced by any vector field transverse to the contact structure, for instance the Reeb field; we get a copy L' of a Legendrian knot L by pushing L along such a transverse vector field, see Figure 3.7. This framing is preserved by Legendrian isotopies. We get an invariant of L by computing the framing coefficient of the canonical framing. This invariant is called the **Thurston-Bennequin invariant**, $tb(L) \in \mathbb{Z}$, and can be easily computed from the front projection of L. The canonical framing is given by a copy L' of L, obtained by pushing L upward. Here, we use the fact that with our conventions the Reeb field points up in the front projection, i.e. $R = \frac{\partial}{\partial z}$. The framing coefficient of the canonical framing of L is then given by $tb(L) = lk(L, L')$. This linking number can be computed from the front projection of L by counting the crossings of L underneath L' with orientation. Alternatively, we can count the writhe of L (signed crossings of

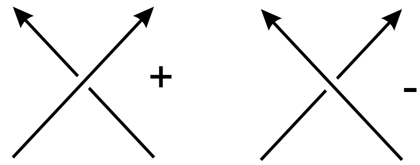

FIGURE 3.8. Orientation of a crossing

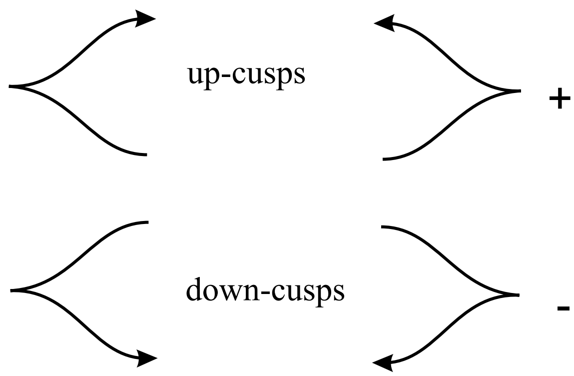

FIGURE 3.9. How to count cusps for the rotation number

L with itself) and the number of cusps, since every cusp corresponds to a crossing of L and L'. In Figure 3.8, we have indicated the usual convention for calling a crossing positive or negative. We obtain the following result for the Thurston-Bennequin invariant of L,

$$tb(L) = writhe(L) - \frac{1}{2}\#cusps,$$

where the $writhe(L)$ is given by the positive crossings minus the negative crossings. See also [**25**].

In a similar way we can introduce another simple invariant of Legendrian curves. Let L be an oriented Legendrian curve. Since we are mainly interested in Stein manifolds without one-handles, we assume that L is null-homotopic and hence we can find a Seifert surface F for L. The restriction of ξ to the Seifert surface F is symplectically trivial, since F is a surface with boundary. In particular, we get a trivialization of $\xi|_L$. Since L is oriented, we can choose a tangent vector v to L that defines the orientation of L. Because L is Legendrian, the vector v lies in $\xi|_L$. Hence we may define the **rotation number** $r(L)$ to be the winding number of v with respect to the trivialization given by the Seifert surface F.

In case L is represented by a diagram in the front projection, we can use a global trivialization of the contact structure on $\mathbb{R}^3$. Namely, the vector $\frac{\partial}{\partial x}$ lies in the contact structure. The winding number with respect to this trivialization can be determined by counting with sign how often L crosses $\pm\frac{\partial}{\partial x}$. This can only happen in cusps. We introduce the following notation to compute $r(L)$ more explicitly. In Figure 3.9, we have indicated which cusps we refer to as up-cusps and to which ones we refer as down-cusps. We claim the rotation number of L is given by

$$r(L) = \frac{1}{2}(\#\{\text{up-cusps}\} - \#\{\text{down-cusps}\}).$$

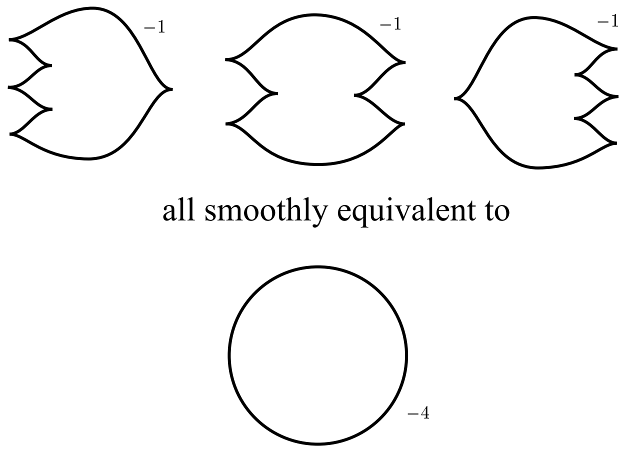

FIGURE 3.10. Different Stein structures on Σ_4

This can also be found in [**25**]. See Figure 3.10 for an example of Legendrian curves distinguished by these invariants. We should point out that these invariants do not always suffice to distinguish Legendrian knots from each other and that other, more subtle invariants exist, for instance invariants coming from contact homology, see [**8**].

3.2.2.1. *Two-handle attachment along Legendrian links.* We have now introduced enough notation to state Eliashberg's theorem on Stein manifolds.

THEOREM 3.18 (Eliashberg). *A smooth, oriented open four-manifold X admits a Stein structure if and only if it is the interior of a handlebody such that the following conditions hold:*

(a) *Each handle has index ≤ 2.*
(b) *Each two-handle h_i is attached along a Legendrian curve L_i in the contact structure induced on the boundary of the underlying handle body consisting of zero- and one-handles.*
(c) *The framing for attaching each h_i is obtained from the canonical framing on L_i by adding a single negative twist.*

A smooth, oriented compact four-manifold X admits a Stein structure if and only if it has a handlebody decomposition satisfying (a), (b) and (c). Both in case of a compact Stein manifold and in the case of a proper Stein manifold, such a handle decomposition comes from a strictly plurisubharmonic function (with, in the compact case, the boundary of X as a level set).

In case there are no one-handles, condition (c) from the theorem can be restated by requiring that the framing coefficient for the attaching handle h_i be given by $tb(L_i) - 1$. This can actually be done in general, but we have not defined framing coefficients in the presence of one-handles. See for instance [**25**] for more details.

We will use the theorem to construct compact Stein manifolds. A compact Stein manifold can be represented by an oriented Legendrian link in $\#_k S^1 \times S^2$. Since we will not use one-handles, we consider only oriented Legendrian links in S^3. These will always represent simply connected Stein manifolds. Note that the converse does not hold true. A handlebody decomposition with one-handles can still give rise to a simply connected Stein manifold, because of handle cancellation. Since all data that is needed to construct a Stein manifold can be obtained from the front projection, we will represent Stein manifolds by drawing Legendrian links in the front projection. This gives rise to a **Kirby diagram**. We simply draw the oriented, Legendrian attaching circles of the two-handles. The framing coefficient for each component L_i of the Legendrian link is given by $tb(L_i) - 1$.

The first Chern class of a Stein manifold provides an invariant that shows that a four-manifold can carry non-isomorphic Stein structures. In absence of one-handles, the first Chern class can be expressed as follows. Suppose X is a Stein manifold that is represented by a Kirby diagram. If

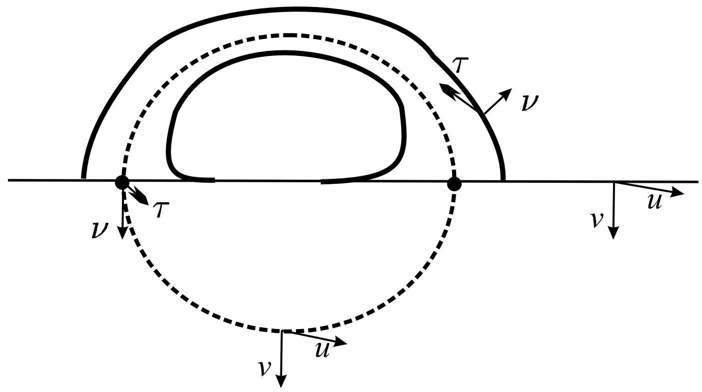

FIGURE 3.11. Trivializations on the core of the two-handle and on the Seifert surface of L_i in ∂D^4

there are no one-handles, then $H_1(X) = 0$ and hence we can think of $H^2(X)$ as $\mathrm{Hom}(H_2(X), \mathbb{Z})$. In particular, the first Chern class would be such a homomorphism from $H_2(X)$ to $\mathbb{Z}$. Recall that an oriented link used for the attaching circles gives a canonical basis for $H_2(X)$. Each link component L_i gives a surface F_i that represents a generator $[F_i]$ of $H_2(X)$. Then we have the following proposition (see proposition 2.3 from [**24**]).

PROPOSITION 3.19. *With the above notation, the first Chern class of X is characterized by*

$$\langle c_1(X), [F_i] \rangle = r(L_i).$$

PROOF. In Figure 3.11 we have drawn a picture of the situation. This might help to clarify the data involved. We start by giving suitable trivializations on the zero-handle and on the two-handles. Since the two-handles are glued along S^1's to the zero-handle, we can express the Chern class in terms of the winding number of the trivializations on the two-handles with respect to the trivialization on the zero-handle. More precisely, on the boundary of the zero-handle D^4 we choose a trivialization of the complex tangent bundle. We take $u = \frac{\partial}{\partial x}$ (coming from a trivialization of the fillable contact structure on S^3) and v, which denotes the inward normal to S^3 in D^4. For the two-handles h_i, $i = 1, \ldots, k$, we note that a two-handle h_i can be regarded as a neighborhood of $D^2 \times \{0\} \subset i\mathbb{R}^2 \times \mathbb{R}^2$. The circle given by $\partial D^2 \times \{0\}$ denotes the attaching circle of the two-handle h_i. We write S_i for this circle. We denote the tangent vector to S_i by τ and the outward pointing normal vector (in $D^2 \times \{0\}$) by ν. Note that (τ, ν) do not just trivialize $TD^2|_{S_i}$ but they give a complex trivialization of $Th_i|_{S_i}$ as well. With respect to the product trivialization on D^2, the trivialization (τ, ν) represents a (non-trivial) element in $\pi_1(SO(2))$. As a complex trivialization, (τ, ν) also represents an element in $\pi_1(U(2))$, again with respect to the product framing. However, $SO(2) \subset SU(2) \subset U(2)$ and $SU(2)$ is simply connected, so the complex trivialization (τ, ν) is homotopic to the product trivialization. Hence (τ, ν) extends to a complex trivialization of Th_i over all of h_i. When we now glue a two-handle h_i to the zero-handle, gluing S_i to L_i, we identify the tangent τ to the two-handle with the tangent field to the attaching circle L_i. Also the vector field ν becomes an inward normal to ∂D^4 in D^4. Let us call the resulting handlebody X. We can see the surface F_i formed by the Seifert surface of L_i in D^4 glued to the core of the two-handle h_i.

The bundle $TX|_{F_i}$ is a rank 2 complex vector bundle over F_i and hence it splits into complex line bundles. Actually, we can see that one of these line bundles is a trivial bundle, because the vector fields v and ν match on L_i and hence give rise to a nowhere vanishing section of $TX|_{F_i}$. If we call this trivial bundle L, then we see that its complement, which we shall call Q, is trivialized by u and τ on the two disks making up F_i. Since L is trivial, we have

$$\langle c_1(TX), [F_i]\rangle = \langle c_1(Q), [F_i]\rangle.$$

We finish by showing that the Chern class of the line bundle Q is indeed given by $r(L_i)$. As we saw before, the surface F_i can be split into two surfaces with boundary, S_1 and S_2, the first one lying in D^4 and the second one being the core of the two handle h_i. The line bundle Q is trivialized on S_1 by the vector field u. On S_2 the vector field τ trivializes Q. Hence we get two trivializations of Q on the intersection L_i of the two surfaces S_1 and S_2, one coming from u and one coming from τ. The Chern class of $Q|_{F_i}$ can then be computed as the winding number of τ with respect to u. This winding number is equal to the rotation number $r(L_i)$. □

See Figure 3.10 for an example of three different Stein structures on Σ_4, the disk-bundle over S^2 with Euler number -4. In particular we see that disk-bundles over S^2 can carry many Stein structures. In fact, Gompf showed in [**24**] that many four manifolds admit infinitely many Stein structures. We only consider a few simple cases of disk-bundles over S^2 to construct contact structures on five-manifolds, but even so we get infinitely many contact structures on some them, distinguished (not surprisingly) by their Chern class. The exotic Stein structures that Gompf provides might be used to construct more interesting examples of contact five manifolds.

3.3. Symplectic cobordisms

In this section we define symplectic cobordisms following [**14**]. The notation introduced here will only be used in Chapter 9 and later chapters, but, as we shall see, we may regard Stein manifolds as special symplectic cobordisms.

Definition 3.20. Let (W, ω) be a symplectic manifold. If (W, ω) has ends of the form

$$E^- = V^- \times (-\infty, 0] \text{ and } E^+ = V^+ \times [0, \infty),$$

such that $V^\pm$ are compact manifolds and $\omega|_{V^\pm} = d(e^t\alpha^\pm)$ with $\alpha^\pm$ contact forms on $V^\pm$, then we say that (W, ω) is a symplectic manifold with **cylindrical ends**. If we denote the contact structure given by $\ker\alpha^\pm$ on $V^\pm$ by $\xi^\pm$, then we also say that such a symplectic manifold (M, ω) is a **directed symplectic cobordism** between the contact manifolds (V^+, ξ^+) and (V^-, ξ^-). We write $\overrightarrow{V^-V^+}$ for the symplectic cobordism and call the contact manifolds (V^+, ξ^+) and (V^-, ξ^-) symplectically cobordant.

If we use our previously introduced notation, we see that the contact manifolds V^- and V^+ are concave and convex, respectively. Namely, the expanding vector field $\frac{\partial}{\partial t}$ points inward for the end E^- and outward for E^+ if we use the coordinates from the definition. We say that the end E^+ is convex and the end E^- is concave.

Remark 3.21. This definition of a cobordism is different from what one usually encounters in the literature, since in general cobordisms do have a boundary and the ends are usually given in another order. This way of ordering the ends is the usual one in symplectic field theory though, see [**14**]. We can relate a symplectic cobordism to a cobordism in the usual sense in the following way. Let (W, ω) be a symplectic cobordism. Since the ends of a symplectic cobordism have the form $V^- \times (-\infty, 0]$ and $V^+ \times [0, \infty)$, we can remove the interiors of the ends and obtain a compact cobordism

$$W^0 = W - (\mathrm{Int}E^+ \cup \mathrm{Int}E^-).$$

This cobordism W^0 is referred to as a compact symplectic cobordism. The symplectic cobordism W itself is then sometimes called **completed** symplectic cobordism.

To relate this definition to the previous sections, note that Stein and convex symplectic manifolds can be regarded as a symplectic cobordism between a certain contact manifold V and the empty set.

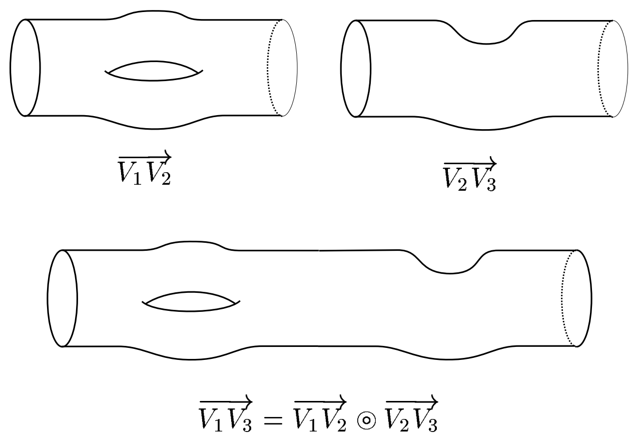

FIGURE 3.12. Gluing symplectic cobordisms

Suppose now that we are given a contact manifold (V, ξ) with contact form α. Then we define the **symplectization** of V as the symplectic manifold

$$V \times \mathbb{R} \text{ with symplectic form } \omega = d(e^t\alpha),$$

where t denotes the coordinate on $\mathbb{R}$. We can also regard the symplectization as a trivial symplectic cobordism. This point of view can sometimes be useful in dealing with symplectic cobordisms, see Chapter 9. Note also that the ends of any symplectic cobordism are isomorphic to the upper or lower half of a symplectization.

Due to the structure at the ends of a symplectic cobordism, we see that two symplectic cobordisms $\overrightarrow{V_0V_1}$ and $\overrightarrow{V_1V_2}$ can be be glued to give a symplectic cobordism $\overrightarrow{V_0V_2}$. This is illustrated in Figure 3.12. More precisely, we have the following construction. Let $W^- = \overrightarrow{V_1V_2}$ and $W^+ = \overrightarrow{V_2V_3}$ be symplectic cobordisms. The contact form on V_2 is denoted by α. Then the convex end E^+ of W^- looks like $(V_2 \times [1, \infty), d(e^t\alpha))$ and the concave end E^- of W^+ looks like $(V_2 \times (-\infty, -1], d(e^t\alpha))$. If we remove the interior of the ends $E^\mp$ from the cobordisms $W^\pm$, we get two cobordisms, each with boundary V_2. In fact, we can arrange that closed collar neighborhoods of the boundary of $W^\pm$ are symplectomorphic to $([-1, 1] \times V_2, d(e^t\alpha))$. In this identification, the boundary of $W^- - \text{Int } (E^-)$ is given by $V_2 \times \{1\}$, and the boundary of $W^+ - \text{Int } (E^+)$ is given by $V_2 \times \{-1\}$. Hence we can glue $W^- - \text{Int } (E^-)$ and $W^+ - \text{Int } (E^+)$ along the above collar neighborhood and we obtain a glued cobordism $W = W^- \circledcirc W^+$ (see also [**14**]).

CHAPTER 4

Some facts on Brieskorn manifolds

4.1. Brieskorn manifolds

In this section we would like to introduce Brieskorn manifolds. These are obtained by intersecting the zero-set of a polynomial of the form $z_0^{a_0}+\ldots+z_n^{a_n}$ in $\mathbb{C}^{n+1}$ with a sphere S^{2n+1}. It was shown that these sets are contact manifolds by Lutz and Meckert [**30**] if the radius of the sphere is sufficiently small. Although it has probably been known for a much longer time that the radius of the sphere used in defining Brieskorn manifolds is unimportant, I have not been able to find a reference, so we give a reasonably short proof here of that claim.

We should also mention that the obvious filling of a Brieskorn manifold by the zero-set of the polynomial $z_0^{a_0}+\ldots+z_n^{a_n}$ is not a manifold, since it has a singularity at 0. If we replace the defining polynomial by $z_0^{a_0}+\ldots+z_n^{a_n}-\varepsilon$, we find a Stein filling for a Brieskorn manifold. These give simple examples of Stein manifolds and we will use them in the coming chapters. This will be discussed in more detail in Chapter 6. For now, let us define the following.

Definition 4.1. The **Brieskorn manifold** $\Sigma_R(a_0,\ldots,a_n)\subset\mathbb{C}^{n+1}$ (with $a_0,\ldots,a_n\in\mathbb{N}$) is defined as the intersection of the sphere $S^{2n+1}=\{(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}|\ |z_0|^2+\ldots+|z_n|^2=R\}$ with the zero set of the polynomial $f(z_0,\ldots,z_n)=z_0^{a_0}+\ldots+z_n^{a_n}$.

Theorem 4.2. *The sets $\Sigma_R(a_0,\ldots,a_n)\subset\mathbb{C}^{n+1}$ as defined in Definition 4.1 are smooth manifolds. The form*

$$\alpha=\frac{i}{8}\sum_{j=0}^{n}a_j(z_jd\bar z_j-\bar z_jdz_j)$$

restricts to a contact form on $\Sigma_R(a_0,\ldots,a_n)$, which we will refer to as **standard contact form**. *We have that $\Sigma_R(a_0,\ldots,a_n)$ is contactomorphic to $\Sigma_{R'}(a_0,\ldots,a_n)$ with their standard contact forms for all $R,R'>0$. In particular we have that $\Sigma_R(a_0,\ldots,a_n)$ is diffeomorphic to $\Sigma_{R'}(a_0,\ldots,a_n)$ for all $R,R'>0$.*

Remark 4.3. The theorem justifies the term *manifold*. Since Brieskorn manifolds with the same exponents and standard contact structure are always contactomorphic, we will write $\Sigma(a_0,\ldots,a_n)$ and omit the radius of the sphere which is used to define the manifold.

Proof. Consider $\mathbb{C}^{n+1}-\{0\}$ with symplectic form

$$\omega=d\alpha=\frac{i}{4}\sum_{j=0}^{n}a_jdz_j\wedge d\bar z_j.$$

We identify the tangent bundle of $\mathbb{C}^{n+1}-\{0\}$ with $\mathbb{C}^{n+1}$ and consider the subbundle given by $\xi^\omega=\mathrm{span}(X_1,Y_1,X_2,Y_2)$, where

$$X_1=(\bar z_0^{a_0-1},\ldots,\bar z_n^{a_n-1}),\qquad Y_1=iX_1,$$
$$X_2=-2i(\frac{z_0}{a_0},\ldots,\frac{z_n}{a_n}),\qquad Y_2=(z_0,\ldots,z_n).$$

From this definition it is not clear whether these vectors are linearly independent. This will be shown in the following though. In fact, we will show that ξ^ω is a trivial symplectic vector bundle of rank 4. First of all, let us use the Gram-Schmidt process to turn the above vectors into a symplectic standard basis. The Gram-Schmidt process yields

$$\tilde{X}_1 = \frac{X_1}{\sqrt{\omega(X_1,Y_1)}}, \qquad \tilde{Y}_1 = \frac{Y_1}{\sqrt{\omega(X_1,Y_1)}}$$
$$\tilde{X}_2 = X_2, \qquad \tilde{Y}_2 = Y_2 - \frac{\sum a_i z_i^{a_i}}{2\omega(X_1,Y_1)} X_1,$$

where we have used that $\omega(X_1,Y_1) = \frac{1}{2}\sum_j a_j |z_j|^{2(a_j-1)} > 0$. Putting these vectors into ω shows that $\tilde{X}_1$, $\tilde{Y}_1$, $\tilde{X}_2$ and $\tilde{Y}_2$ form a symplectic standard basis. Hence ξ^ω is a trivial symplectic vector bundle over $\mathbb{C}^{n+1} - \{0\}$.

In addition, the computation shows that X_1, Y_1, X_2 and Y_2 are linearly independent. This can be used to show in an elementary way that the Brieskorn manifolds as defined above are independent of the radius used. First of all, we need to show that the set $\Sigma_R(a_0,\dots,a_n)$ is indeed a manifold for each $R>0$. First write $\Sigma_R = \Sigma_R(a_0,\dots,a_n)$ to simplify our notation.

Define the function

$$\begin{array}{rccl} h: & \mathbb{C}^{n+1} & \to & \mathbb{R}^3 \\ & z & \mapsto & (f(z)+\bar{f}(z), (f(z)-\bar{f}(z))/i, |z|^2). \end{array}$$

The first component of h is, of course, twice the real part of f and the second component of h is twice the imaginary part of f. Hence $\Sigma_R = h^{-1}(0,0,R^2)$. The differential of h is given by

$$Th = \begin{pmatrix} a_0 z_0^{a_0-1} & \dots & a_n z_n^{a_n-1} & a_0 \bar{z}_0^{a_0-1} & \dots & a_n \bar{z}_n^{a_n-1} \\ -ia_0 z_0^{a_0-1} & \dots & -ia_n z_n^{a_n-1} & ia_0 \bar{z}_0^{a_0-1} & \dots & ia_n \bar{z}_n^{a_n-1} \\ \bar{z}_0 & \dots & \bar{z}_n & z_0 & \dots & z_n \end{pmatrix}$$

We use the vectors X_1, Y_1 and Y_2 to show that the Th has full rank on the set $h^{-1}(0,0,R^2)$. More explicitly, we find that on Σ_R the following holds

$$Th(X_1) = \begin{pmatrix} 2\sum_{j=0}^n a_j |z_j|^{2(a_j-1)} \\ 0 \\ 0 \end{pmatrix}, \qquad Th(Y_1) = \begin{pmatrix} 0 \\ 2\sum_{j=0}^n a_j |z_j|^{2(a_j-1)} \\ 0 \end{pmatrix},$$
$$Th(Y_2) = \begin{pmatrix} 2\mathrm{Re}(\sum_{j=0}^n a_j z_j^{a_j}) \\ 2\mathrm{Im}(\sum_{j=0}^n a_j z_j^{a_j}) \\ 2R^2 \end{pmatrix}.$$

Note that at each point of Σ_R, the vectors $Th(X_1)$, $Th(Y_1)$ and $Th(Y_2)$ span $\mathbb{R}^3$, so Th has full rank on Σ_R. This shows that Σ_R is a smooth submanifold of $\mathbb{C}^{n+1}$ for each $R>0$.

Next, we show that different radii R used to define Σ_R give rise to diffeomorphic manifolds. This is done by observing that Σ_R is a codimension 1 submanifold of the complex manifold

$$M = \{z \in \mathbb{C}^{n+1} \mid f(z) = 0 \text{ and } z \neq 0\},$$

by noting that Σ_R is given by the preimage of R^2 under the map $|\dots|^2 : \mathbb{C}^{n+1} \to \mathbb{R}_{\geq 0}$. By taking this point of view, we see that Σ_{R_1} and Σ_{R_2} are diffeomorphic for each positive R_1 and R_2. Indeed, $|\dots|^2$ is a function on M without critical points, so our claim follows (see for instance theorem 3.1 from [**38**]).

We also see that Σ_R is a contact manifold for each R. This can be done in the following way. Since Σ_R is a submanifold of $\mathbb{C}^{n+1} - \{0\}$, we can restrict $T(\mathbb{C}^{n+1} - \{0\})$ to Σ_R and obtain a trivial complex vector bundle of rank $n+1$, which we will denote by V_R. The symplectic complement of $\xi^\omega|_{\Sigma_R}$ in V_R will be denoted by ξ. Note that ξ is a subbundle of the tangent bundle of Σ_R. Indeed, if we plug in X_1 into ω, we obtain

$$i_{X_1}\omega = \frac{i}{4}\sum_{j=0}^n (a_j \bar{z}_j^{a_j-1} d\bar{z}_j - a_j z_j^{a_j-1} dz_j),$$

which is a linear combination the differential of f and of $\bar{f}$, which were used to define Σ_R. Plugging Y_1 into ω gives a similar result, and for X_2 we have

$$i_{X_2}\omega = \frac{1}{2}\sum_{j=0}^{n}(z_j d\bar{z}_j + \bar{z}_j dz_j).$$

This is one half times the differential of the function $z \mapsto |z|^2$. In other words, the condition that a vector $w \in V_R$ lie in the symplectic complement of X_1, Y_1 and X_2 implies that w is tangent to Σ_R. We can also see that $\xi = \ker\alpha$. Namely, if we plug Y_2 into ω, we get

$$i_{Y_2}\omega = \frac{i}{4}\sum_{j=0}^{n} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j) = 2\alpha,$$

which shows that if $w \in \xi$, then w is both in the kernel of α and tangent to Σ_R.

To complete the argument that Σ_R is a contact manifold, note that X_2 is tangent to M_R and $i_{X_2}\alpha = R^2 \neq 0$. This shows that the form $\alpha \wedge d\alpha^{n-1}$ is nowhere zero, because $d\alpha$ is a symplectic form on ξ and $i_{X_2}\alpha \neq 0$.

Finally, we observe that Σ_{R_1} is in fact contactomorphic to Σ_{R_2}. Indeed, we get a smooth path of contact forms on Σ_{R_1} by pulling back the contact form on each Σ_{R_t} for R_t between R_1 and R_2. This path starts at the standard contact form described above on Σ_{R_1} and ends at the contact form obtained by pulling back the standard form on Σ_{R_2}. By Gray stability, these forms are contactomorphic.

□

Remark 4.4. Note that this proof also shows that the Chern class of the contact structure ξ is trivial. Indeed, the symplectic complement of ξ in $\mathbb{C}^{n+1}$, given by ξ^ω, is trivial. We have $c(\xi \oplus \xi^\omega) = c(\xi)c(\xi^\omega) = c(\mathbb{C}^{n+1}) = 1$. Since $c(\xi^\omega) = 1$, the conclusion follows.

4.1.1. Reeb flow on Brieskorn manifolds. The Reeb vector field of the contact form

$$\alpha = \frac{i}{8}\sum_{j=0}^{n} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j)$$

has the particularly simple shape

$$R = 4i(z_0/a_0, \dots, z_n/a_n),$$

where we regard $T\Sigma(a_0, \dots, a_n)$ as a subset of $T\mathbb{C}^{n+1}$. The Reeb flow is then given by

$$\varphi_t(z) = (e^{4it/a_0}z_0, \dots, e^{4it/a_n}z_n), \tag{4.1}$$

which means that all Reeb orbits are closed. Hence we obtain an S^1-action on the Brieskorn manifold $\Sigma(a_0, \dots, a_n) \subset \mathbb{C}^{n+1}$.

4.2. Topology of Brieskorn manifolds

The above description of Brieskorn manifolds lends itself to computations, since we can use coordinates. It can sometimes be difficult to see the topology though. Brieskorn manifolds have, however, been studied extensively in the past and a lot is known about their topology. In particular, many exotic spheres (spheres that are homeomorphic, but not diffeomorphic to the standard sphere) can be realized as Brieskorn manifolds. We describe a few of these facts before we give Randell's algorithm to compute the homology of a Brieskorn manifold.

In [**28**] it is shown that the fundamental group of a Brieskorn manifold is abelian if its dimension is larger than 3. This can be used to show that Brieskorn manifolds are highly connected, i.e. $\Sigma(a_0, \dots, a_n)$ is $(n-2)$-connected. In other words, provided the middle homology group $H_{n-1}(\Sigma(a_0, \dots, a_n); \mathbb{Z})$ vanishes, the Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ is a homotopy sphere. There is a nice criterion for this, which is much easier to check than using Randell's algorithm to compute the homology. We have taken this claim from [**28**].

Let $\Gamma(a)$ denote the graph, whose vertices are the exponents $a = (a_0, \dots, a_n)$. We connect two vertices a_i and a_j (with $i \neq j$) if $\gcd(a_i, a_j) > 1$.

THEOREM 4.5. *Assume $n > 2$. The Brieskorn manifold $\Sigma(a_0, \ldots, a_n)$ is a homotopy sphere if either of the following two conditions is satisfied:*

(1) *$\Gamma(a)$ has two isolated points.*
(2) *$\Gamma(a)$ has an isolated point and a connected component C with an odd number of points such that if $a_i, a_j \in C$ with $i \neq j$ then $\gcd(a_i, a_j) = 2$.*

4.2.1. Homology of Brieskorn manifolds. In [**42**] Randell proves an algorithm which can be used to compute the homology of a Brieskorn manifold. We give a short summary of his results in the case of Brieskorn manifolds. The algorithm in Randell can also be used to compute the homology of so-called generalized Brieskorn manifolds. Generalized Brieskorn manifolds are defined as the common zero set of several complex polynomials, each one having a shape similar to the polynomials used in defining Brieskorn manifolds, intersected with a sphere. See [**42**] for more details.

As shown in the previous section, the Brieskorn manifolds admit the S^1-action

$$t(z_0, \ldots, z_n) = (t^{q_0} z_0, \ldots, t^{q_n} z_n)$$

for $t \in S^1$ and with $q_i = \frac{\mathrm{lcm}_j\, a_j}{a_i}$. Then define $M^* := M/S^1$, which, in general, will be an orbifold. In his proof, Randell also shows how to compute the homology of the orbifold M^*, which will in fact turn out to be useful in Chapter 11.

We introduce the following notation for the algorithm of [**42**]. Let I denote the set $\{0, \ldots, n\}$. A subset of I with s elements will by denoted by I_s. If $I_s = \{i_1, \ldots, i_s\}$, then let $K(I_s)$ denote the Brieskorn manifold $\Sigma(a_{i_1}, \ldots, a_{i_s})$ of dimension $2s-3$. Note that $M = K(I)$ contains all manifolds $K(I_s)$ for all $1 < s \leq n+1$ in a natural way by restricting to suitable coordinate hyperplanes. We define, following Randell,

$$\kappa(K(I_s)) = \sum_{I_t \subset I_s} (-1)^{s-t} \frac{\prod_{i \in I_t} a_i}{\underset{j \in I_t}{\mathrm{lcm}}\ a_j}.$$

Then we have for the free part of the homology

$$\mathrm{rk}\ \tilde{H}_{n-1}(M, \mathbb{Z}) = \kappa(K(I)).$$

For the torsion part, a few additional definitions are required. We set

$$k(K(I_s)) = \begin{cases} \kappa(K(I_s)), & \text{if } n+1-s \text{ is odd}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C(\emptyset) = \underset{i \in I}{\gcd}(a_i)$ and set

$$C(I_s) = \frac{\underset{i \in (I - I_s)}{\gcd}\ a_i}{\prod_{I_t \subsetneq I_s} C(I_t)}.$$

Now set $d_j = \prod_{k(K(I_s)) \geq j} C(I_s)$ and $r = \max\{k(K(I_s)) | I_s \subset I\}$. Then we have

$$\mathrm{Tor} H_{n-1}(K(I), \mathbb{Z}) = Z_{d_1} \oplus \ldots \oplus Z_{d_r}.$$

There is another interesting result from Randell's paper that we shall use. He computes the rational (as well as its torsion part) homology of M^*. We shall only need the rational homology of M^*. Randell's results is as follows:

$$H_q(M^*, \mathbb{Q}) \cong \left\{ \begin{array}{l} \mathbb{Q},\ q \text{ even}, 0 \leq q \leq \dim M^* \\ 0, \text{ otherwise} \end{array} \right\} \oplus \left\{ \begin{array}{l} \mathbb{Q}^{\kappa},\ q = \frac{1}{2}\dim M^* \\ 0 \text{ otherwise} \end{array} \right\}, \tag{4.2}$$

where $\kappa = \kappa(K(I))$.

A computer program that performs the steps of Randell's algorithm can be found in Appendix A.

CHAPTER 5

Construction of contact open books

5.1. Open books

DEFINITION 5.1. An **open book** on a closed manifold M is a pair (K, ϑ), where

(a) K is a codimension 2 submanifold of V with trivial normal bundle and

(b) ϑ is a fibration $\vartheta: \ M - K \to S^1$ that in a neighborhood $K \times D^2$ of K is the angular coordinate on D^2.

The set K is called the **binding** of the open book and the closure of the fibers of ϑ are called **pages**.

Let M be a closed manifold with open book (K, ϑ). Let F be a page of the open book. By pushing a page F once around in the S^1-direction, we get a diffeomorphism of F, the monodromy of the open book. This can be made precise in the following way, see [**31**]. The vector field $\frac{\partial}{\partial \vartheta}$ on S^1 can be pulled back to M outside the binding K. The time t-flow of this vector field gives a diffeomorphism Ψ_t of $M - K$ which can be extended to M by putting $\Psi_t|_K = id|_K$. The map Ψ_1 is determined uniquely up to isotopy by the fibration ϑ and is called the **monodromy** of the open book.

Now let F be a $2n$-dimensional manifold with non-empty boundary K. Then we can use a diffeomorphism of F to play the role of monodromy and hence we can construct an open book. Namely, let Ψ be a diffeomorphism of F that is the identity near the boundary of F. The mapping torus V of Ψ is defined as

$$V = F \times I/(x, 1) \sim (\Psi(x), 0).$$

Since the Ψ is the identity near K, the boundary of V can be naturally identified with $K \times S^1$. Hence we can glue in $K \times D^2$, which has the same boundary. We obtain the **relative mapping torus** $\tilde{V}$,

$$\tilde{V} = V \cup_{K \times S^1} K \times D^2.$$

The binding of the relative mapping torus is given by $K \times \{0\} \subset K \times D^2$. The projection to S^1 which we denote by ϑ is given by the natural projection to S^1 on V (note that by definition V is an F-bundle over S^1). Near the boundary, V looks like $K \times I \times S^1$. This allows us to extend the projection ϑ to $K \times D^2 - K \times \{0\}$ by putting

$$\begin{array}{rcl} \vartheta: \ K \times D^2 - K \times \{0\} & \to & S^1 \\ (x, r, \varphi) & \mapsto & \varphi, \end{array}$$

where r, φ are polar coordinates on D^2. Hence $\tilde{V}$ admits an open book with monodromy given by Ψ. Observe that every open book can be obtained from the above construction. We will call an open book obtained from such a construction an abstract open book formed from (F, Ψ) and we write this open book as $\tilde{V}_{(F,\Psi)}$.

5.1.1. Open book connected sum. If we are given two manifolds M_1 and M_2 with open book structures, then their connected sum $M_1 \# M_2$ admits an open book as well. This is called the **open book connected sum**. It is performed as follows. Let K_i be the binding of the open book on M_i for $i = 1, 2$ and let ϑ_i denote the fibration $M_i - K_i \to S^1$. Assume that the bindings are connected. Let D_i be a neighborhood of a point in K_i. By choosing this neighborhood sufficiently small, we can assume, that $D_i \cong D^{2n+1}$ and that $D_i \cap K_i \cong D^{2n-1}$. The set $D_i - (D_i \cap K_i)$ fibers over S^1 and corresponds to the trivial fibration with fiber diffeomorphic to D^{2n-1}, see Figure 5.1. Take the connected sum of M_1 and M_2 in the usual way, by removing a smaller ball $\tilde{D}_i \subset D_i$ from

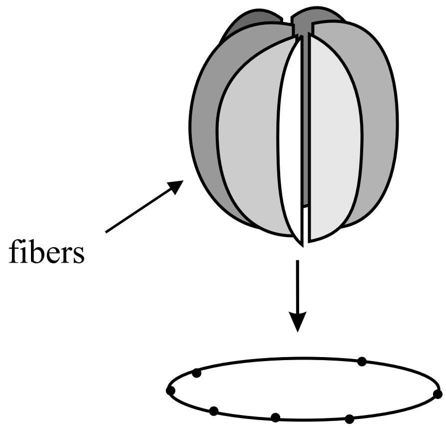

FIGURE 5.1. Fibration of a ball minus binding over S^1

M_i and identifying $D_1 - \tilde{D}_1$ with $D_2 - \tilde{D}_2$ via an orientation reversing diffeomorphism. We can arrange this in such a way that the fibration over S^1 is respected. When we perform the connected sum in this way, we connect sum the bindings to form the set $K_1 \# K_2$. Note that outside this set $K_1 \# K_2 \subset M_1 \# M_2$, we have a fibration over S^1, which coincides with the given fibration $M_i - K_i$ for $i = 1, 2$. Since the normal bundle of $K_1 \# K_2$ in $M_1 \# M_2$ is trivial, we get an induced open book on $M_1 \# M_2$. If F_i denotes the page of the open book on M_i, then the page of the induced open book on $M_1 \# M_2$ is given by $F_1 \natural F_2$, the boundary connected sum of F_1 and F_2. If the bindings K_i are connected, the book connected sum is well-defined. The construction itself will work just as well in cases where the binding is disconnected, but the results will then of course depend on the choice of D_i.

The book connected sum can also be described using abstract open books. Let M_i be the abstract open book formed from (F_i, φ_i) for $i = 1, 2$. Since the monodromy φ_i is required to be equal to the identity near the boundary of F_i, we get a diffeomorphism $\varphi_1 \natural \varphi_2$ of $F_1 \natural F_2$ which coincides with φ_1 on F_1 and with φ_2 on F_2. This allows us to form the abstract open book from $(F_1 \natural F_2, \varphi_1 \natural \varphi_2)$, which is isomorphic to the book connected sum of M_1 and M_2.

5.1.2. A special stabilization. In this section we will describe a special case of a so-called stabilization of an open book. The notion of stabilization belongs to contact open books, which we have not defined yet. However, the special case which we describe here can be applied to all open books. We present this example in order to show that open books are not unique, and as a preparation for later applications.

We need a few definitions before we can describe the construction. We denote the unit disk bundle associated to T^*S^n by $T^*_{|v|\leq 1}S^n$ (v stands for a vector in a fiber of T^*S^n).

DEFINITION 5.2. A **(right-handed) Dehn twist** τ in $T^*_{|v|\leq 1}S^n$ is the composition of the time π map of the geodesic flow on S^n and the differential of the antipodal map.

In particular, a Dehn twist is the identity at the boundary, and by an isotopy we can ensure that it is the identity in a neighborhood of the boundary. We will call this isotoped map a Dehn twist as well.

REMARK 5.3. For a left-handed Dehn twist, we simply take the time $-\pi$ map of the geodesic flow on S^n. For now, the difference is not important, but it will play a role if we talk about open books and contact structures. Note also that if we restrict a Dehn twist to the zero section of T^*S^n, where $v = 0$, then the Dehn twist is just the antipodal map. If n is even, we quickly see that a Dehn twist is not isotopic to the identity, because the induced homomorphism

$$\tau_* : \ H_n(T^*_{|v|\leq 1}S^n; \mathbb{Z}) \quad \rightarrow \quad H_n(T^*_{|v|\leq 1}S^n; \mathbb{Z})$$

equals minus the identity map on $\mathbb{Z}$. For n odd this argument does not hold, but such Dehn twists are not isotopic to the identity either. The latter can, for instance, be seen by constructing the abstract open book (T^*S^n, τ^k) for n odd. We will show in Chapter 7 that this gives a manifold that is diffeomorphic to $\Sigma(k, 2, \dots, 2)$. For n odd these are all non-diffeomorphic manifolds distinguished by $H_n(\Sigma(k, 2, \dots, 2)) \cong Z_k$. It follows that the maps τ^k are not isotopic for different k and hence τ cannot be isotopic to the identity.

Note that for n even, the composition of an even number of Dehn twists can be isotopic to the identity, e.g. Dehn twists on T^*S^2 have this property, as was shown by Seidel [**46**]. In general, the situation is more complicated though, which can be seen by considering the abstract open book (T^*S^n, τ^k). These are diffeomorphic to $\Sigma(k, 2, \dots, 2)$ and can for instance be exotic spheres.

Finally, we observe that $T^*_{|v|\leq 1}S^n$ carries a canonical symplectic structure and that the Dehn twist can be chosen to be a symplectomorphism. We will say more on this in Section 5.2.2.1.

The abstract open book obtained from $(T^*_{|v|\leq 1}S^n, \tau)$ is diffeomorphic to S^{2n+1}, which we will explain in Chapter 7. Let us denote this abstract open book by $V_{(T^*_{|v|\leq 1}S^n, \tau)}$. Now let M be a manifold with open book (K, ϑ). We can alter the given open book structure on M by book connect summing with $V_{(T^*_{|v|\leq 1}S^n, \tau)}$. The new open book on M is the **stabilization** of (K, ϑ). We will give a few more details on general stabilizations in the section on contact open books.

In general there are many distinct open book structures on a given manifold (provided that at least one open book structure exists). However, it turns out that some open books carry a natural contact structure on them, and the process of stabilization (of which we just described a special case) does not change the contact structure.

5.2. A basic construction of contact open books

In this section, we want to explain a basic construction for contact structures in terms of open books due to Giroux [**23**]. Roughly speaking, we take a symplectic manifold with convex boundary K and a symplectomorphism of that manifold. The mapping torus of this symplectomorphism can be arranged in such a way that the resulting manifold admits a contact structure. Because of convexity, the boundary K inherits a contact structure and a thickened K (i.e. $K \times D^2$) can be glued onto the boundary of the mapping torus to give a closed contact manifold. Originally, a similar construction was used by Thurston and Winkelnkemper to give a short proof that all oriented three manifolds admit a contact structure [**49**]. Recently, Giroux proposed to use this construction for contact manifolds in all dimensions. In fact, he even showed that the converse holds true. Any closed contact manifold admits an open book such that the given contact structure and the contact structure coming from the above construction are isomorphic.

Let (Σ, ω) be a symplectic manifold with non-empty ω-convex boundary K. Let φ be a symplectomorphism of Σ that is the identity near the boundary of Σ. We start by constructing a contact form on the mapping torus of (Σ, φ). The convexity condition gives a vector field X such that $\mathcal{L}_X\omega = \omega$ near the boundary. Let β be the primitive of ω given by $i_X\omega$ so that $\beta|_K$ is a contact form on K. We have $\varphi^*\omega = \omega$, but φ might not preserve β. However, the symplectomorphism condition shows that $\varphi^*\beta - \beta$ is closed. In fact, we can perturb φ such that the difference is even exact, as the following lemma of Giroux shows.

Lemma 5.4 (Giroux). *The symplectomorphism φ can be isotoped to a symplectomorphism φ_1 such that $\varphi_1^*\beta - \beta$ is exact.*

Proof. Let us denote the one-form $\varphi^*\beta - \beta$ by μ. Since ω is non-degenerate, we find a unique solution Y to the equation $i_Y\omega = -\mu$. The flow of the vector field Y preserves ω, because μ is closed,

$$0 = -d\mu = di_Y\omega = \mathcal{L}_Y\omega.$$

Since φ is the identity near the boundary, μ and hence Y vanish near the boundary. If we denote the time t flow of Y by ψ_t, then we see that $\varphi_1 = \varphi \circ \psi_1$ is a symplectomorphism that is the identity near the boundary. Note that $\mathcal{L}_Y\mu = 0$, so $\psi_t^*\mu = \mu$ for all t. We check that the difference

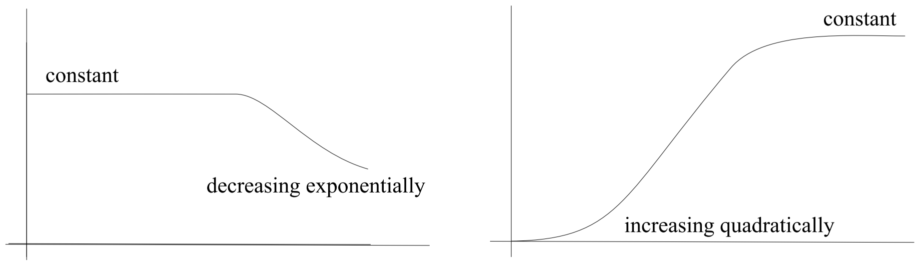

FIGURE 5.2. A sketch of k_1 and k_2, respectively

of the pullback of β and β is indeed exact. We have

$$(\varphi \circ \psi_1)^* \beta - \beta = \psi_1^*(\mu + \beta) - \beta = \mu + \psi_1^* \beta - \beta.$$

On the other hand, we can express the difference $\psi_1^* \beta - \beta$ as

$$\psi_1^* \beta - \beta = \int_0^1 \frac{d}{dt} \psi_t^* \beta = \int_0^1 \psi_t^* \mathcal{L}_Y \beta = \int_0^1 \psi_t^* \left(i_Y \omega + d(i_Y \beta) \right) = -\mu + \int_0^1 d\psi_t^* (i_Y \beta).$$

Moving μ to the left-hand-side, we see that $\mu + \psi_1^* \beta - \beta$ is exact, which shows the claim of the lemma. □

Using this lemma, we can assume that $\varphi^* \beta - \beta = -dh$. Now, we construct the mapping torus of (Σ, φ), but in a way that differs from the standard way. First choose a primitive h of the above equation that is positive. This can always be done by adding a constant to h. We define

$$V = \Sigma \times \mathbb{R}/H,$$

where $H(x,t) = (\varphi(x), t + h(x))$. By choosing h positive we see directly that V is diffeomorphic to the standard definition of a mapping torus, since we can deform V by replacing h by h_t, where h_t interpolates h to 1. Note that H preserves $\alpha = \beta + dt$. This means that the mapping torus V inherits a contact form from the contact form α on $\Sigma \times \mathbb{R}$. We denote this contact form on V by $\tilde{\alpha}$.

We still need to glue in the binding to get a closed contact manifold. The restriction of β to K is a contact form and will be denoted by γ. This form is extended to a contact form on the product $K \times D^2$. We choose the contact form

$$\delta = k_1(r)\gamma + k_2(r) d\vartheta,$$

where (r, ϑ) are polar coordinates on D^2 and r represents the radial direction. For $k_1(r) = 1$ and $k_2 = r^2$, the form δ is a contact form, but its Reeb dynamics do not match those on the mapping torus V. This is corrected by perturbing k_1 and k_2. This idea is inspired by the Lutz twist, see for instance [**20**]. The condition for δ to be a contact form amounts to

$$\delta \wedge (d\delta)^n = n k_1^{n-1} (k_1 k_2' - k_2 k_1') \gamma \wedge d\gamma^{n-1} \wedge dr \wedge d\vartheta \neq 0,$$

so δ is contact provided that $k_1 k_2' - k_2 k_1' \neq 0$ and $k_1 \neq 0$. Following Lutz, we can define k_1 and k_2 such that δ is contact and that δ coincides near the boundary of $K \times D^2$ with $\tilde{\alpha}$ near the boundary of V. See Figure 5.2 for a sketch of how k_1 and k_2 could look like.

These choices ensure that we get a contact form on the relative mapping torus $V \cup_\partial K \times D^2$. This can be seen as follows. First of all, let us introduce some notation for the gluing procedure. We will denote the set $\{(x,p) \in K \times D^2 | \, |p| = r\}$ by $K \times S^1_r$. By our choice of k_1 and k_2, the contact form looks like

$$\delta = e^{-t}\gamma + d\vartheta$$

in a small enough neighborhood of the boundary. We show that the contact form has the same form in a neighborhood of the boundary of the mapping torus.

Recall that we introduced the vector field X with $\mathcal{L}_X\omega = \omega$ near the boundary of Σ. This vector field can be used to define a suitable collar neighborhood of the boundary in V. We get the following identification of a collar neighborhood of $\partial\Sigma$,

$$\begin{aligned}\Psi : [-\varepsilon, 0] \times K &\to \Sigma \\ (s, x) &\mapsto Fl_s^X(x).\end{aligned}$$

Here we use Fl_s^X to denote the time s flow of the vector field X. We use the form $i_X\omega = \beta$ to construct the contact form on the mapping torus V. We claim that the pull-back of β under Ψ is

$$(\Psi^*\beta)(s, x) = e^s\beta|_{K,x} = e^s\gamma_x.$$

We see that this holds, because $i_X\beta = 0$ and $\frac{d}{ds}\Psi^*|_{s=0}\beta = (\mathcal{L}_X\beta)|_K = \beta|_K = \gamma$. This shows that we can find a neighborhood of the boundary of V, where the contact form looks like

$$\alpha = e^s\gamma + d\vartheta,$$

where we have used Ψ to identify a neighborhood of K in Σ with $K \times [-\varepsilon, 0]$. Since the s-direction corresponds to the negative r-direction in $K \times D^2$, we see that the contact forms δ and α match in a collar neighborhood of the $K \times S^1$. Hence we can glue V and $K \times D^2$ along a collar neighborhood of the boundary and obtain a closed contact manifold.

We summarize the results of the above construction and introduce some new notation. If an abstract open book is formed from a convex symplectic page F with symplectic monodromy φ, we get a contact structure on the abstract open book. Let us call the contact manifold (M, ξ) obtained from such a construction an **(abstract) contact open book**. We write $(M_{(F,\varphi)}, \xi_{(F,\varphi)})$ for such a contact open book.

5.2.1. Compatible open books. In the previous section we saw that Giroux's construction gave us an open book with a contact structure. Not every manifold with an open book structure admits a contact structure though. On the other hand, many contact manifolds come with natural open books, so it makes sense to have a notion which describes when an open book is compatible with its contact structure.

DEFINITION 5.5. A contact structure ξ on a manifold V is **supported** by an open book (K, ϑ) if ξ is the kernel of a 1-form α that satisfies

(1) α induces a positive contact form on K, and
(2) $d\alpha$ induces a positive symplectic form on each fiber F of ϑ.

Such a 1-form α is said to be **adapted** to (K, ϑ).

REMARK 5.6. We should stress that $d\alpha$ is required to induce a positive symplectic form on each fiber F of ϑ and **not** on every page, i.e. closure of F. In fact, the latter does not hold.

REMARK 5.7. We use the vector pointing out of the page to orient the binding as the boundary of the page. This convention ensures that the standard contact structure on $\mathbb{R}^3$ is supported by the natural open book (K, ϑ), where K is the z-axis and ϑ the standard angular coordinate on the xy-plane. For practical purposes, positivity of the contact form can be checked by finding a copy of the binding in the interior of a page, obtained by pushing the binding along a collar into the page.

EXAMPLE 5.8. The standard contact structure on S^3 is supported by (H^+, π_+), where

$$H^+ = \{(z_1, z_2) \in S^3 |\ z_1 z_2 = 0\},$$

$$\begin{aligned}\pi_+ :\ S^3 - H^+ &\to S^1 \\ (z_1, z_2) &\mapsto \frac{z_1 z_2}{|z_1 z_2|}.\end{aligned}$$

Here we regard S^3 as a subset of $\mathbb{C}^2$, given by $\{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\}$. The set H^+ is an embedded Hopf link in S^3. This is an interesting example, since the standard contact structure on S^3 is not supported by (H^-, π_-), where

$$H^- = \{(z_1, z_2) \in S^3 |\ z_1 \bar{z}_2 = 0\},$$

$$\begin{array}{rcl} \pi_- : \; S^3 - H^- & \to & S^1 \\ (z_1, z_2) & \mapsto & \dfrac{z_1 \bar{z}_2}{|z_1 z_2|}. \end{array}$$

Note the bindings of the open books (H^+, π_+) and (H^-, π_-) are the same sets in S^3, but their monodromy is different. In fact, (H^-, π_-) supports an overtwisted contact structure on S^3.

5.2.2. Stabilization. Suppose (M, ξ) is a contact manifold that is supported by the open book (K, ϑ). As we have seen in the example from Section 5.1.2, we can construct other open books for M. The construction outlined there can be made compatible with the contact structure and it can be generalized. We shall now describe this procedure using the abstract open book picture. Just like in the special case from Section 5.1.2, we need a Dehn twist.

5.2.2.1. *Symplectic Dehn twist.* Let $T^*_{|p|\leq 1}S^n$ denote the unit cotangent space of S^n. If we use coordinates $(q,p) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with $|q| = 1$ and $p \cdot q = 0$ for $T^*_{|p|<1}S^n$, then we can write the canonical 1-form as $\lambda_{can} = p \cdot dq$. The differential of λ_{can} defines the canonical symplectic form ω_{can} on $T^*_{|p|\leq 1}S^n$.

As before, we define a **right-handed symplectic Dehn twist** τ by the composition of the time π map of the geodesic flow on S^n and the differential of the antipodal map. Since both maps are symplectic, their composition τ is symplectic as well. Note that on the boundary $T^*_{|p|=1}S^n$ the map τ is the identity. Similarly, a left-handed Dehn twist can be defined as the composition of the time $-\pi$ map of the geodesic flow on S^n and the differential of the antipodal map. For a more explicit description of Dehn twists, see [**40**].

REMARK 5.9. The choice of canonical 1-form is important in this case, because taking $-\lambda_{can}$ instead of λ_{can} will reverse the roles of right-handed Dehn twist and left handed Dehn twists in the following.

We end this section by introducing some notation. Let (F, ω) be a symplectic manifold of dimension $2n$ and let

$$\psi : S^n \to F$$

be a Lagrangian embedding, i.e. $\psi^*\omega = 0$. By the Lagrangian neighborhood theorem [**34**], it follows that a neighborhood of $\psi(S^n)$ in F is symplectomorphic to T^*S^n. Hence we can extend ψ to a symplectic embedding of $T^*|_{|p|\leq 1}S^n$. We get a symplectic embedding

$$\tilde{\psi} : (T^*_{|p|\leq 1}S^n, \omega_{can}) \to (F, \omega).$$

If we perform a Dehn twist along this embedded cotangent bundle, we get a symplectomorphism of F, which we will denote by τ_ψ. Note that this symplectomorphism has support only on $\tilde{\psi}(T^*|_{|p|\leq 1}S^n)$.

5.2.2.2. *Constructing a new contact open book.* Let F be a compact Stein manifold of real dimension $2n$ and let

$$\psi : D^n \to F$$

be a proper Lagrangian embedding. Note that the embedded boundary, which we denote by K, of the disk is a Legendrian sphere in ∂F. If $\dim F = 4$, then we have seen in Chapter 3 that we can attach a two-handle along K in a canonical way in order to obtain a new Stein manifold (by using the canonical framing of K). This procedure works in all dimensions as shown by Eliashberg [**15**]: there is a canonical way to attach an n-handle along K to obtain a Stein manifold F'. Moreover, we obtain an embedded Lagrangian sphere in F' by gluing the core of the n-handle along K to the Lagrangian disk $\psi(D^n)$. By abuse of notation, we write ψ for the Lagrangian embedding

$$\psi : S^n \to F'.$$

Now let $\varphi : F \to F$ be a symplectomorphism that is the identity near the boundary. We can extend φ to a symplectomorphism (also denoted by φ) of F' by setting φ to be the identity on the attached n-handle. Let $\varphi' = \tau_\psi \circ \varphi$. Giroux calls this extension of the pair (F, φ) to (F', φ') a **positive Lagrangian plumbing**. He has proved the following proposition about these plumbings.

PROPOSITION 5.10 (Giroux). *The contact manifolds* $(M_{F,\varphi}, \xi_{F,\varphi})$ *and* $(M_{F',\varphi'}, \xi_{F',\varphi'})$ *are contactomorphic.*

With this proposition it makes sense to give the following notion of stabilization.

DEFINITION 5.11. The contact open book $(M_{F',\varphi'}, \xi_{F',\varphi'})$ is said to be a **stabilization** of $(M_{F,\varphi}, \xi_{F,\varphi})$ if (F', φ') can be obtained from (F, φ) by a sequence of positive Lagrangian plumbings.

In the special case of a stabilization presented in Section 5.1.2 we see that this proposition holds by applying Proposition 5.13 from the next section. By that proposition, the contact open book $(M_{F',\varphi'}, \xi_{F',\varphi'})$ is contactomorphic to the connected sum of $(M_{F,\varphi}, \xi_{F,\varphi})$ and $(M_{T^*S^n,\tau}, \xi_{T^*S^n,\tau})$. Since the contact open book $(M_{T^*S^n,\tau}, \xi_{T^*S^n,\tau})$ is contactomorphic to S^5 with its standard contact structure, the conclusion follows.

REMARK 5.12. Giroux's proposition does not hold true if we replace a positive Lagrangian plumbing by a negative one, i.e. if we replace right-handed Dehn twists by left-handed ones. This is where the contact category differs significantly from differentiable category, because the underlying manifold can not see the difference between a left- or right-handed Dehn twist.

5.2.3. Connect summing open books. Suppose now we are given two abstract contact open books, $(M_{(F_1,\varphi_1)}, \xi_{(F_1,\varphi_1)})$ and $(M_{(F_2,\varphi_2)}, \xi_{(F_2,\varphi_2)})$. We can take their book connected sums to obtain a new abstract contact open book, $(M_{(F_1\natural F_2,\varphi_1\natural\varphi_2)}, \xi_{(F_1\natural F_2,\varphi_1\natural\varphi_2)})$. This is seen by noting that the boundary of $F_1\natural F_2$ remains convex and that $\varphi_1\natural\varphi_2$ is symplectic. We have the following proposition.

PROPOSITION 5.13. *Let* $(M_{(F_i,\varphi_i)}, \xi_{(F_i,\varphi_i)})$ *be abstract contact open books for* $i = 1, 2$. *Then*

$$(M_{(F_1,\varphi_1)}, \xi_{(F_1,\varphi_1)})\#_{book}(M_{(F_2,\varphi_2)}, \xi_{(F_2,\varphi_2)}) \cong (M_{(F_1\natural F_2,\varphi_1\natural\varphi_2)}, \xi_{(F_1\natural F_2,\varphi_1\natural\varphi_2)}).$$

Here is a rather informal argument for this claim.

PROOF. Let α_i denote the contact form obtained from Giroux's construction of the abstract contact open book on $M_i = (M_{(F_i,\varphi_i)}, \xi_{(F_i,\varphi_i)})$. We use the point of view of Weinstein [**53**] to connect sum M_1 and M_2, see Figure 5.3, which we adapt for open books. Let D_i be a neighborhood of a point in the binding of M_i. This neighborhood can be chosen such that

$$\alpha_i|_{D_i} = (\pm dz) + (\pm \sum_{j=1}^{n-1} y_j dx_j) + r^2 d\vartheta = (\pm dz) + (\pm \sum_{j=1}^{n-1} y_j dx_j) + x_n dy_n - y_n dx_n$$

in local coordinates. Here z and $(x_1, y_1, \dots, x_{n-1}, y_{n-1})$ are local coordinates for the binding of M_i. We take a $+$ sign for α_2 and a $-$ sign for α_1. The coordinates (r, ϑ) are polar coordinates for a disk neighborhood of the binding such that the fibration coming from the open book is given by the ϑ coordinate. The polar coordinates (r, ϑ) correspond to the cartesian coordinates (x_n, y_n) in the usual way. We see the contact form can be brought in this form by applying Darboux's theorem to the binding and using the special form of the α_i near the binding.

We embed both neighborhoods D_1 and D_2 into the symplectic manifold $(\mathbb{R}^{2n+2}, \omega)$. If we use coordinates $(u, z, x_1, y_1, \dots, x_n, y_n)$, then we embed D_1 at $u = -1$ and D_2 at $u = 1$. The symplectic form ω is the standard form, so that we have a Liouville vector field on $\mathbb{R}^{2n+2}$ that gives us the contact forms on D_1 and D_2. Define $R^2 = z^2 + \sum_{j=1}^n x_j^2 + y_j^2$ and choose a function f such that the zero set of $u^2 - f(R^2)$ looks as in Figure 5.4. Let us denote the zero set of $u^2 - f(R^2)$ by T.

We require $f(R^2) = 1$ for large R^2 and we specify f further as we go. In particular f can be chosen such that the vector field

$$X = 2u\frac{\partial}{\partial u} - z\frac{\partial}{\partial z} + \frac{1}{2}(\sum_{j=1}^n x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j})$$

is a Liouville vector field transverse to T. We will use this "tube" to form the connected sum of D_1 and D_2. By removing smaller balls in D_1 and D_2 and replacing them with the zero set of $u^2 - f(R^2)$, we obtain the connected sum of D_1 and D_2. Since X is a Liouville vector field and

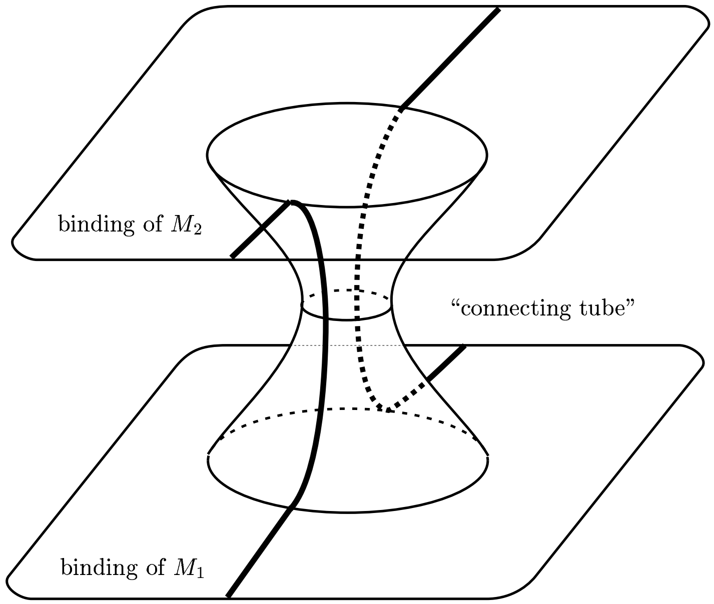

FIGURE 5.3. Weinstein's picture for a connected sum of contact manifolds

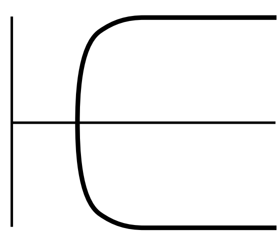

FIGURE 5.4. Profile of the connecting tube

transverse to T, the connected sum $D_1 \# D_2$ is a contact manifold. Note that the fibration extends to the connecting tube, so this construction is compatible with the book connected sum. Since we have not altered anything outside D_1 and D_2, we get the book connected sum of M_1 and M_2. Because the fibration "tube" $\to S^1$ is trivial, we see that we can take the monodromy to be the identity on the connected sum region (the "tube"). Hence we see that the monodromy of the open book is equal to φ_1 on $M_1 - D_1$ and to φ_2 on $M_2 - D_2$. It is the identity on the "tube", so we get an induced contact open book on the connected sum of M_1 and M_2, which is isomorphic to $M_{(F_1 \natural F_2, \varphi_1 \natural \varphi_2)}, \xi_{(F_1 \natural F_2, \varphi_1 \natural \varphi_2)}$. $\square$

REMARK 5.14. Another way to formulate this proposition is to say that contact open books for (M_1, ξ_1) and (M_2, ξ_2) give a contact open book for $(M_1 \# M_2, \xi_1 \# \xi_2)$.

CHAPTER 6

Open books on five-manifolds

6.1. Open book decompositions for simply connected contact five-manifolds

In this chapter we will give examples of Giroux's theorem in dimension 5. We will give a new proof of the following theorem.

THEOREM 6.1 (Geiges). *Let M be a simply-connected 5-manifold. Then M admits a contact structure in every homotopy class of almost contact structures.*

Along the way we find some other interesting results as well, which can be used to construct more contact structures.

We begin by recalling Barden's theorem on closed, simply connected five-manifolds [**2**]. Let M_1 and M_2 be closed, simply connected five-manifolds. Note that by simple connectedness, elements in $H^2(M_i;\mathbb{Z}_2)$ can be regarded as maps from $H_2(M_i;\mathbb{Z})$ to $\mathbb{Z}_2$ for $i=1,2$. For this reason, the second Stiefel-Whitney class $w_2(M_i)$ may be viewed as such a map from $H_2(M_i;\mathbb{Z})$ to $\mathbb{Z}_2$ $(i=1,2)$. In fact, we may find a minimal generating set of $H_2(M_i;\mathbb{Z})$ such that w_2 is only non-zero on at most one of these generators. This generator has order 2^j for some $j\in\mathbb{N}_0$. The number j is called the linking number.

THEOREM 6.2. *Two simply connected five-manifolds M_1 and M_2 are diffeomorphic if and only if $w_2(M_1)$ and $w_2(M_2)$ are isomorphic, i.e. there exists an isomorphism $A: H_2(M_1;\mathbb{Z})\to H_2(M_2;\mathbb{Z})$ such that $w_2(M_1)=w_2(M_2)\circ A$.*

Note that $w_2(M_1)$ and $w_2(M_2)$ can only be isomorphic if the linking numbers of M_1 and M_2 are the same.

For practical purposes, another description of this theorem is useful. By the above theorem there are unique (up to diffeomorphism) simply connected five-manifolds M in the following cases. By unique we mean that the manifold is determined uniquely up to diffeomorphism.

1. For p prime and $k\in\mathbb{N}$ there exists M with $H_2(M;\mathbb{Z})\cong\mathbb{Z}_{p^k}\oplus\mathbb{Z}_{p^k}$ and $w_2(M)=0$. We will denote these manifolds by B_{p^k}. These manifolds admit an almost contact structure as the obstruction to such a structure, the third integral Stiefel-Whitney class $W_3(M)$, vanishes. By Geiges' theorem [**18**] they admit a contact structure as well.
2. For $k\in\mathbb{N}$ there exists M with $H_2(M;\mathbb{Z})\cong\mathbb{Z}_2^k\oplus\mathbb{Z}_2^k$ and $w_2(M)\neq 0$. Here $W_3(M)\neq 0$, so M does not carry an almost contact structure.
3. There exists a unique M with $H_2(M;\mathbb{Z})\cong\mathbb{Z}_2$ and $w_2(M)\neq 0$. This manifold is sometimes referred to as the Wu-manifold and does not carry an almost contact structure, because $W_3(M)\neq 0$.
4. There is a unique manifold M with $H_2(M;\mathbb{Z})\cong\mathbb{Z}$ and $w_2(M)=0$. This manifold admits an almost contact structure and therefore by Geiges' theorem a contact structure.
5. There is a unique manifold M with $H_2(M;\mathbb{Z})\cong\mathbb{Z}$ and $w_2(M)\neq 0$. Its third integral Stiefel-Whitney class is 0 and hence M admits an almost contact structure. Again it is contact as well.

The manifolds in case 4 and 5 have well-known models, namely $S^2\times S^3$ and $S^2\tilde{\times}S^3$ (the non-trivial S^3-bundle over S^2), respectively. In [**2**] there are explicit models for all other cases as well. For our purposes the existence of these models is enough, and in fact we will make our own models for those manifolds in this list that admit a contact structure.

The manifolds of type 1 and 4 are said to be prime manifolds. Our notion of prime manifold differs from the usual one and is to be understood in the following sense. Any simply connected

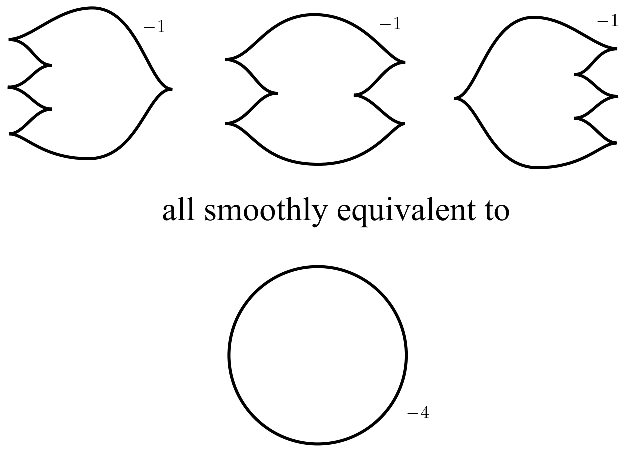

FIGURE 6.1. Different Stein structures on Σ_5

five-manifold M may be decomposed uniquely as the connected sum of prime manifolds with possibly one additional summand of type 2, 3 or 5. This additional summand is only needed if $w_2(M) \neq 0$. Of course, the empty connected sum is diffeomorphic to S^5.

Note that this implies in particular that M can only be contact if it can be written as a connected sum of manifolds of type 1, 4 and 5. Because the connected sum is an operation in the contact category, it suffices that we produce contact open books for S^5 and for the manifolds of type 1, 4 and 5 (recall the book connected sum used for connect summing open books can be made compatible with the contact structures carried by the open books, see Proposition 5.13). We know in view of Giroux's theorem that this is possible and we are even provided a starting point by A'Campo, see [**1**]. He has shown by explicit construction that simply connected five-manifolds always admit open book decompositions. Unfortunately, the pages he used in his construction do not always carry an exact symplectic form. In some other cases of his open books it is hard to see whether the monodromy can actually be realized symplectically.

However, in the case $M \cong S^5$ it can obviously be done. The page is D^4 with its standard symplectic structure, the monodromy is the identity, so the mapping torus will carry a contact structure following Giroux's construction. Note that D^4 provides a Stein filling for the binding S^3, so we are exactly in Giroux's model situation. This open book is compatible with the standard contact structure on S^5.

6.1.1. Open books for $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$. In this section, we will construct open books for manifolds of type 4 and 5 in the above notation. Our construction starts by taking a simple Stein manifold Σ_k, the 2-disk-bundle over S^2 with Euler number $-k$ with $k \geq 2$. We remark that these manifolds carry often more than one Stein structure as can be seen in Figure 6.1.

First we will show that we get contact open books for $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$, then we will show that the different realizations from Figure 6.1 can give rise to different contact structures on $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$. Namely it will turn out that they have different Chern classes.

Let S_k denote the contact boundary of Σ_k. The space S_k is a lens space and can be identified with circle bundle over S^2 with Euler number $-k$ due to Example 3.17. We will use the identity as a monodromy, so the mapping torus of the pair (Σ_k, id) is diffeomorphic to $A := \Sigma_k \times S^1$. A neighborhood of the binding will be written as $B := S_k \times D^2$. The relative mapping torus is then given by $X = A \cup_\partial B$. Here we mean that A and B are identified in X in a collar neighborhood of their boundary.

Note that X is simply connected by the Seifert-van Kampen theorem. Indeed, the generator of $\pi_1(A)$ (the S^1 direction in the product) gets killed in B and vice versa, as Σ_k is a simply connected filling for S_k. We compute the homology of X using the Mayer-Vietoris sequence for the pair (A, B) in X and then apply Barden's theorem to show that we get $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$.

We already know that $H_1(X) \cong 0$ by simple connectedness. Note that the intersection of A and B is diffeomorphic to $S_k \times S^1 \times (-1,1)$. This set is homotopy equivalent to $S_k \times S^1$. The homology groups of $A \cap B$ can therefore be computed by using the Künneth formula. These remarks show that we know all terms in the Mayer-Vietoris sequence except for $H_i(X)$. The homology sequence looks as follows.

$$
\begin{aligned}
&0 \to H_5(X) \to \underset{\cong\mathbb{Z}}{H_4(A \cap B)} \to 0 \to H_4(X) \to \underset{\cong\mathbb{Z}}{H_3(A \cap B)} \to \underset{\cong\mathbb{Z}\oplus\mathbb{Z}}{H_3(A) \oplus H_3(B)} \\
&\to H_3(X) \to \underset{\cong\mathbb{Z}_k}{H_2(A \cap B)} \to \underset{\cong\mathbb{Z}}{H_2(A) \oplus H_2(B)} \xrightarrow{g} H_2(X) \to \underset{\cong\mathbb{Z}\oplus\mathbb{Z}_k}{H_1(A \cap B)} \\
&\xrightarrow{f} \underset{\cong\mathbb{Z}\oplus\mathbb{Z}_k}{H_1(A) \oplus H_1(B)} \to \underset{\cong 0}{H_1(X)}
\end{aligned}
$$

The known terms are indicated. We may argue as follows to get the rest. The first part of the sequence shows us that $H_5(X) \cong \mathbb{Z}$. Notice that X is orientable and so we can apply Poincaré duality to see that $H^4(X) \cong H_1(X) \cong 0$. This implies here that $H_4(X) \cong 0$ as well. We may now tensor (over $\mathbb{Z}$) the sequence with $\mathbb{Q}$ to obtain an exact sequence of vector spaces over $\mathbb{Q}$. We can then easily see that $\operatorname{rk} H_3(X) = \operatorname{rk} H_2(X) = 1$. This means that $H_3(X) \cong \mathbb{Z}$, because simply connected five-manifolds cannot have torsion in $H_3(X)$.

The map f in the sequence is surjective. We now want to show that it is injective as well to be able to split off a part of the sequence. First observe that f must have the shape $f(a,b) = (g(a), h(a,b))$, because $\operatorname{Hom}(\mathbb{Z}_k, \mathbb{Z}) = 0$. Also note that surjectivity of f implies that $g(a) = \pm a$, so g is injective.

Let $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}_k$ and suppose $f(a,b) = 0$. Since g is injective, this means that $a = 0$. By applying the same argument to the inequality

$$f(a', b') \in \mathbb{Z}_k$$

we see that $h: \mathbb{Z}_k \to \mathbb{Z}_k$ is surjective. This can only hold true if h is injective, so b is 0 as well.

Note that because $\operatorname{Hom}(\mathbb{Z}_k, \mathbb{Z}) \cong 0$, the map from $H_2(A \cap B)$ to $H_2(A) \oplus H_2(B)$ is the zero homomorphism. Because we just showed that f is injective, we can split off the following part of the Mayer-Vietoris sequence. We have

$$0 \to \underset{\cong\mathbb{Z}}{H_2(A) \oplus H_2(B)} \xrightarrow{g} H_2(X) \to 0,$$

so $H_2(X) \cong \mathbb{Z}$. Using Barden's theorem we see that X is either $S^2 \times S^3$ or $S^2 \tilde{\times} S^3$. To decide which one we get, we consider the second Stiefel-Whitney class and look at the Mayer-Vietoris cohomology sequence with $\mathbb{Z}_2$ coefficients. Note that $\mathbb{Z}_2$ is a field and hence the sequence is an exact sequence of vector spaces. This means that the following part of the Mayer-Vietoris sequence,

$$\to \underset{\cong 0}{H^1(X; \mathbb{Z}_2)} \to H^1(A; \mathbb{Z}_2) \oplus H^1(B; \mathbb{Z}_2) \to H^1(A \cap B; \mathbb{Z}_2),$$

can be separated from the long exact sequence. Indeed, we observe that the ranks of $H^1(A; \mathbb{Z}_2) \oplus H^1(B; \mathbb{Z}_2)$ and $H^1(A \cap B; \mathbb{Z}_2)$ are the same. Since the map in the sequence is injective, it is an isomorphism between vector spaces. This simplifies the rest of the Mayer-Vietoris sequence. The part that is relevant to us looks like

$$0 \to H^2(X, \mathbb{Z}_2) \xrightarrow{(i^*, j^*)} H^2(A, \mathbb{Z}_2) \oplus H^2(B, \mathbb{Z}_2).$$

Here i and j denote the inclusions of A and B in X, respectively. Observe that the map (i^*, j^*) is injective, so it suffices to see whether $(i^*, j^*) w_2(X)$ is zero or not.

Because i and j are inclusions, we have that $i^* w_2(X) = w_2(A)$ and $j^* w_2(X) = w_2(B)$. The tangent bundle of B is trivial because it is the product of a 3-dimensional oriented manifold and D^2, so $w_2(B) = 0$. We now consider the Stiefel-Whitney class of A. We denote the projection from $A = \Sigma_k \times S^1$ to Σ_k by p_1 and the projection from A to S^1 by p_2. We have

$$TA \cong p_1^* T\Sigma_k \oplus p_2^* TS^1 \cong p_1^*(T\Sigma_k \oplus \varepsilon^1),$$

where ε^1 denotes the trivial real vector bundle of rank 1. This means that the Stiefel-Whitney class of A is determined by the Stiefel-Whitney class of Σ_k. Now let p denote the projection of Σ_k to S^2. We have $T\Sigma_k \cong p^*TS^2 \oplus p^*\mathcal{O}(k)$, where $\mathcal{O}(k)$ is the rank 2 vector bundle over S^2 with Euler number k. The class $w_2(TS^2)$ is zero and $w_2(\mathcal{O}(k)) = k \bmod 2$. Therefore $w_2(\Sigma_k) \in H^2(\Sigma_k; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is zero if k is even and non-zero if k is odd.

This shows that $w_2(X) = k \bmod 2$, where we use $H^2(X; \mathbb{Z}_2) = \mathbb{Z}_2$. By the classification result for five-manifolds, X is diffeomorphic to $S^2 \times S^3$ if k is even and it is diffeomorphic to $S^2 \tilde{\times} S^3$ if k is odd. We have obtained contact open book decompositions for $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$. In particular, this shows that both manifolds admit contact structures. We now turn our attention to the Chern class of the obtained contact structures. This, as it turns out, depends on the chosen Stein structure on Σ_k.

6.1.1.1. *Chern classes of contact structures.* Let us take a look at Figure 6.1. Legendrian unknots representing Σ_k have rotation numbers going from $-k+2, -k+4, \cdots, k-2$. Fix a Legendrian unknot representing Σ_k and denote its rotation number by r. Proposition 3.19 tells us that the Chern class of Σ_k is then given by $r \in \mathbb{Z} \cong H^2(\Sigma_k)$.

We now want to establish the relation between the Chern class of the contact structure corresponding to the open book decomposition we described and the Chern class of Σ_k, the page of the open book. We may regard the pull-back $p_1^*T\Sigma_k$ as a subbundle of TA. If we denote the symplectic form on Σ_k by ω, then we may write the contact form on A as $\alpha = dt + \beta$, where t is the local coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$, and β satisfies $d\beta = p_1^*\omega$. We obtain a complex structure J for $p_1^*T\Sigma_k$ by pulling back the (almost) complex structure on Σ_k that is compatible with ω.

Next, we construct a vector bundle isomorphism from $p_1^*T\Sigma_k$ to the contact structure $\xi = \ker \alpha$. Define

$$\begin{array}{rccl} \varphi: & p_1^*T\Sigma_k & \to & \xi \\ & v & \mapsto & v - \beta(v)\dfrac{\partial}{\partial t}. \end{array}$$

In the definition of this map, we regard both $p_1^*T\Sigma_k$ and ξ as subbundles of the tangent bundle. The vector field $\frac{\partial}{\partial t}$ generates the standard rotation in the S^1-direction.

The inverse of φ can be obtained as follows,

$$\varphi^{-1}(v) = H(Tp_1(v)),$$

where we use H to denote the obvious lift from $T\Sigma_k$ to TA. In other words, the inverse of φ projects out the $\frac{\partial}{\partial t}$-component of an element in $\xi \subset TA$. This map φ can be used to give ξ a complex structure. We have the following diagram.

$$\begin{tikzcd} p_1^*T\Sigma_k \arrow[r, "\varphi"] \arrow[d, "J"'] & \xi \arrow[d, "\tilde{J} = \varphi \circ J \circ \varphi^{-1}"] \\ p_1^*T\Sigma_k \arrow[r, "\varphi"] & \xi \end{tikzcd}$$

This makes φ into complex vector bundle isomorphism from $(p_1^*T\Sigma_k, J)$ to $(\xi, \tilde{J})$, because by construction $\tilde{J} \circ \varphi = \varphi \circ J$. We check now that the $\tilde{J}$ is a complex structure for ξ compatible with $d\alpha = d\beta$. We set $\tilde{v} = \varphi(v)$ and $\tilde{w} = \varphi(w)$. Then

$$d\beta(\tilde{J}\tilde{v}, \tilde{J}\tilde{w}) = d\beta(\varphi(Jv), \varphi(Jw)) = d\beta(Jv, Jw) = d\beta(v, w) = d\beta(\varphi(v), \varphi(w)) = d\beta(\tilde{v}, \tilde{w})$$

These steps hold true, because φ adds an S^1-component and $d\beta$ does not contain any dt part, so $d\beta(\varphi(\ldots), \varphi(\ldots)) = d\beta(\ldots, \ldots)$. Also, J is a complex structure on $(p_1^*T\Sigma_k, J)$ compatible with $d\beta$. For the same reasons, the following holds:

$$d\beta(\tilde{v}, \tilde{J}\tilde{v}) = d\beta(\varphi(v), \varphi(Jv)) = d\beta(v, Jv) > 0 \text{ if } \tilde{v} \neq 0.$$

This proves that $\tilde{J}$ is a complex structure compatible with the contact structure ξ. Since $(p_1^*T\Sigma_k, J)$ and $(\xi, \tilde{J})$ are isomorphic as complex vector bundles by φ (which covers the identity), their Chern classes are the same. We had already computed the Chern class of Σ_k, so we have proved that $c_1(\xi) = r \in \mathbb{Z} \cong H^2(A)$.

We resort again to a Mayer-Vietoris argument to complete our computation of the Chern class of X. Consider the Mayer-Vietoris sequence for cohomology with integer coefficients. The part that is relevant to us looks like

$$0 \to \underbrace{H^1(A)}_{\cong \mathbb{Z}} \oplus \underbrace{H^1(B)}_{\cong 0} \xrightarrow{\alpha} \underbrace{H^1(A \cap B)}_{\cong \mathbb{Z}} \xrightarrow{f} \underbrace{H^2(X)}_{\cong \mathbb{Z}} \xrightarrow{(i^*, j^*)} \underbrace{H^2(A) \oplus H^2(B)}_{\cong \mathbb{Z} \oplus \mathbb{Z}_k}.$$

Since the map α is injective, it has to map 1 to some non-zero integer, say m. If m is not equal to ± 1, then we see that $f(m) = 0$, but $f(1) \neq 0$ by exactness. However $H^2(X)$ has no torsion, so we see that $m = \pm 1$ and thus the map α is an isomorphism. Again, by exactness the map f has to be the zero homomorphism. So we see that the map (i^*, j^*) is injective. We can say a bit more, namely that i^* is injective. This can be seen by noting that $H^2(B)$ is torsion. We show that it is an isomorphism by looking at the sequence of the pair (X, A). The piece of the sequence that interests us, looks like

$$H^2(X) \xrightarrow{i^*} H^2(A) \to H^3(X, A).$$

By excision, we have $H^3(X, A) \cong H^3(B, \partial B)$. The latter group is seen to be isomorphic to $H_2(B) = 0$ by Poincaré duality. This shows that i^* is surjective.

The restriction of the first Chern class of the contact structure ξ_X on X to A is given by $c_1(\xi) = r$. Since we just checked i^* to be an isomorphism, it follows that $c_1(\xi_X) = r \in \mathbb{Z} \cong H^2(X)$. There is an ambiguity in this notation, namely it depends on which generator of $H^2(X)$ we take (actually this ambiguity is of course also present in our discussion of $c_1(\xi)$).

These ambiguities do not matter for the point we want to make, which is showing that all possible Chern classes of ξ_X can be realized by our open books. For $(X \cong S^2 \times S^3, \xi_X)$qcycle we can realize all even Chern classes and for $(X \cong S^2 \tilde{\times} S^3, \xi_X)$ we can realize all odd Chern classes. Namely, observe that the rotation number r of the diagram in Figure 6.1 can attain any even value, provided that we have chosen k even and large enough for that purpose. The same argument works for odd rotation numbers.

6.1.2. Open books for prime manifolds. In this section we will construct open book decompositions of the remaining prime manifolds, i.e. those simply connected five-manifolds with torsion H_2 and trivial Stiefel-Whitney class. We start with a few remarks establishing some notation and general arguments.

6.1.2.1. *Some general arguments for computing the homology of open books.* We will use some arguments that are quite similar to the ones we used in the previous section. We start with a compact Stein manifold Σ with boundary K and build the mapping torus

$$A := \Sigma \times I / \sim, \text{ where } (x, 0) \sim (\varphi(x), 1),$$

where φ is a symplectomorphism of Σ which is the identity near the boundary K. Define $B = K \times D^2$. Then the relative mapping torus X is formed by

$$X = A \cup_\partial B,$$

by which we mean that we identify collar neighborhoods of the boundary of A and B. As we mentioned in Section 5.2, this gives X an open book decomposition. The fundamental group of X can be computed by applying the Seifert-van Kampen theorem. Since we are interested in simply connected manifolds, we will always use a simply connected page Σ. This means that the fundamental group of A is isomorphic to $\mathbb{Z}$ (note that A always fibers over S^1), whereas $\pi_1(B) \cong \pi_1(K)$. Now note that any generator of the fundamental group of B gets killed in A, because we can first homotope a curve in B to lie in $K \times \{point\}$. Then we see that any such curve is contractible in $\Sigma \times \{point\} \subset A$. Similarly, we see that the generator of $\pi_1(A)$ is contractible in B. This is done by first homotoping any curve in A to lie in $\{point\} \times S^1$. The resulting curve is contractible in $\{point\} \times D^2$. Application of Seifert-van Kampen shows that $\pi_1(X) = 0$.

For the computation of the homology we rely on the Mayer-Vietoris sequence of the pair of the pair of subspaces (A, B) in X. Note that $A \cap B$ is homotopy equivalent to $K \times S^1$, because φ is the identity near the boundary. The homology of $K \times S^1$ can be determined by the Künneth formula assuming that we know the homology of A and K. Since A is a fiber bundle over S^1 with fiber Σ, we may apply the Wang sequence to get the homology of A (see [**32**] and also [**39**] for our particular case). Note that for a simply connected page we can split off a part of the Mayer-Vietoris sequence. We do this generalizing the argument from Section 6.1.1. A piece of the Mayer-Vietoris sequence looks like

$$H_1(A\cap B) \xrightarrow{f} H_1(A)\oplus H_1(B) \to \underbrace{H_1(X)}_{\cong 0}.$$

We follow the same line of reasoning as in Section 6.1.1 to show that f is injective. First of all, the Künneth formula shows that $H_1(A\cap B)\cong \mathbb{Z}\oplus H_1(K)\cong H_1(A)\oplus H_1(B)$, since $H_1(A)\cong\mathbb{Z}$ (by the homotopy exact sequence for $A\to S^1$) and $H_1(B)\cong H_1(K)$. Since homology groups are finitely generated abelian groups, we can write

$$H_1(A\cap B)\cong H_1(A)\oplus H_1(B)\cong \mathbb{Z}^k\oplus T.$$

In this formula $\mathbb{Z}^k$ denotes the free part and T the torsion part which is a finite group. Since the map f is a homomorphism and $\mathrm{Hom}(T,\mathbb{Z}^k)=0$, we may write f as

$$\begin{array}{rccc} f: & \mathbb{Z}^k\oplus T & \to & \mathbb{Z}^k\oplus T \\ & (a,b) & \mapsto & (g(a),h(a,b)). \end{array}$$

Because f is surjective by the above sequence, g is a surjective homomorphism from $\mathbb{Z}^k$ to $\mathbb{Z}^k$. We can, for instance by tensoring with $\mathbb{Q}$, extend this map to a linear surjection from $\mathbb{Q}^k$ to $\mathbb{Q}^k$. Since the two vector spaces have the same dimension, the extended map must be injective as well, so g is injective. In other words, if $(a,b)\in\ker f$, then a is zero, so to find an element in the kernel of f, we can restrict h to T and we obtain a surjective homomorphism from T to T. A surjective map between two finite sets with the same number of elements is injective, so the kernel is just the neutral element. Hence f is injective.

6.1.2.2. *Brieskorn varieties.* Giroux's theorem on open book decompositions of contact manifolds says that we can always assume the page to be a Stein manifold. We therefore turn attention to a particularly simple class of Stein manifolds, which we will call Brieskorn varieties. Take the polynomial

$$P_t(z)=\sum_{i=0}^{n} z_i^{a_i}-t$$

for $z=(z_0,\dots,z_n)\in\mathbb{C}^{n+1}$ and $t\in\mathbb{C}$. The zero set of this polynomial is a Stein manifold if $t\neq 0$. If $t=0$, the zero set of P_t has a singularity at 0 if one of the exponents is larger than 1. We will denote the zero set of the polynomial P_t by Σ_a, where a indicates that this set depends on the exponents $a=(a_0,\dots,a_n)$. The set Σ_a is called a Brieskorn variety. There is a group action of $\mathbb{Z}_{a_i}$ on Σ_a obtained by multiplying the i^{th} coordinate by a_i^{th} roots of unity for each $i=0,\dots,n$. These Stein manifolds can be made into compact Stein manifolds by restricting Σ_a to a ball $B_R=\{z\in\mathbb{C}^{n+1}\mid |z|\le R\}$ in $\mathbb{C}^{n+1}$ with sufficiently large radius. By abuse of notation, we will also denote this set by Σ_a. The boundary of this compact Stein manifold is a Brieskorn manifold with exponents a, provided that t is small enough. This property of Brieskorn manifolds can for instance be found in theorem 14.3 of [**28**], but we will give another argument in Section 6.1.2.3.

We would like to use Brieskorn varieties as pages with the corresponding Brieskorn manifolds as binding in open books. We may of course use the identity for the monodromy of the pages, but this will not give us any interesting open books. A Mayer-Vietoris argument similar to the one used in the previous section and some additional arguments show that such open books will give a connected sum of copies of $S^2\times S^3$ in dimension 5. Instead, we use the action of the generator of $\mathbb{Z}_{a_0}$ on Σ_a as monodromy, i.e. we use the "rotation" map

$$\begin{array}{rcl} \varphi:\Sigma_a & \to & \Sigma_a \\ (z_0,\dots,z_n) & \to & (\zeta_{a_0}z_0,z_1,\dots,z_n), \end{array}$$

where ζ_{a_0} is the a_0^{th} root of unity $e^{2\pi i/a_0}$. Since this is a biholomorphism, we get a symplectomorphism of the page, but we still need to show that we can isotope this map symplectically to the identity near the boundary of the page. We will describe this in the following interlude.

6.1.2.3. *The rotation maps φ are symplectically isotopic to the identity.* Instead of considering the polynomial P, we take the function

$$g = \sum_{i=0}^{n} z_i^{a_i} - f(r),$$

where $r = \sqrt{\sum_{i=0}^{n+1} |z_i|^2}$ and the function f is a real valued function to be specified later. We denote the zero set of g by $\tilde{\Sigma}_a$ of g. Note that this set is in general not be a Stein manifold. We will, however, show that it is symplectic for suitable f. Take a vector $X \in T\mathbb{C}^{n+1}|_{g^{-1}(0)}$. The condition that X be tangent to $\tilde{V}_a$ is

$$i_X dg = i_X \left(\sum_{i=0}^{n} a_i z_i^{a_i - 1} dz_i - \frac{1}{2}\frac{\partial f}{\partial r} \sum_{i=0}^{n} (\frac{\bar{z}_i}{r} dz_i + \frac{z_i}{r} d\bar{z}_i) \right) = 0.$$

Let now ω_0 denote the standard symplectic form on $\mathbb{C}^{n+1}$ and suppose that $\omega_0|_{\tilde{\Sigma}_a}$ is degenerate for the vector X at some point of V. Then we have

$$i_X \omega_0 = (\lambda dg + \bar{\lambda} d\bar{g})$$

for some $\lambda \in \mathbb{C}$, because we know ω_0 is non-degenerate on $\mathbb{C}^{n+1}$. Using this relation, we deduce that

$$i_X dz_j = \frac{2}{i} \left(- \left(\frac{\partial f}{\partial r} \frac{z_j}{2r} \right) (\lambda + \bar{\lambda}) + a_j \bar{z}_j^{a_j - 1} \bar{\lambda} \right).$$

Now we return to check the tangency condition of X. The previous relations now give us

$$0 = i_X dg = \frac{2}{i} \bar{\lambda} \left(\sum_j a_j^2 |z_j|^{2(a_j - 1)} - \frac{\partial f}{\partial r} \sum_j \frac{a_j}{2r} (z_j^{a_j} + \bar{z}_j^{a_j}) \right)$$

The coefficient of $\bar{\lambda}$ has a term involving $a_i^2 |z_i|^{2(a_i-1)}$ in it. Now assume that the exponents are larger than 1 and that the derivative $\frac{\partial f}{\partial r} < 1 - \varepsilon$ for some positive ε. This means that the term with $a_i^2 |z_i|^{2(a_i-1)}$ will dominate for large r, i.e. the coefficient of $\bar{\lambda}$ will be non-zero and therefore $\bar{\lambda} = 0$. Since $|\bar{\lambda}| = |\lambda|$, it follows that λ must be zero, which in turn implies that X is zero. This last step shows that V can be made symplectic for suitable f. To be more precise we choose f with the following properties.

1. The function f is constant 1 for $r \leq R_0$, where R_0 is chosen in such a way that the above mentioned term will indeed dominate.
2. For $r \geq R_1 > R_0 + 1$, the function f is constant 0. Note that this condition is not necessary for symplecticity. It will, however, be useful to make the rotation maps isotopic to the identity for large radii.
3. Between R_0 and R_1, the function f goes smoothly from 1 to 0, connecting smoothly to the already described parts of f. We will choose f such that its derivative is smaller than $1 - \varepsilon$.

Now that we know that $\tilde{\Sigma}_a$ is symplectic, we want to see the corresponding rotation map can be isotoped to the identity. First define the map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, sending $(z_0, \dots, z_n) \mapsto (\zeta_{a_0} z_0, z_1, \dots, z_n)$. Now choose the following Hamiltonian function on $\mathbb{C}^{n+1}$:

$$H = \sum_{i=0}^{n+1} \frac{\pi}{a_i} |z_i|^2.$$

The time t flow of the Hamiltonian vector field associated to H induces the map

$$\psi_t : (z_0, \dots, z_n) \mapsto (e^{2\pi i \frac{t}{a_0}} z_0, \dots, e^{2\pi i \frac{t}{a_n}} z_n)$$

Note that this map sends $\tilde{\Sigma}_a$ to $\tilde{\Sigma}_a$ for $r > R_1$. Choose a function h which is constant 0 for $0 \leq r \leq R_1$ and which increases to 1 at $r = R_2 > R_1$, after which it is constant 1. Let $\tilde{\psi}_t$ denote the time t flow of the Hamiltonian vector field associated to $\tilde{H} = hH$. The map $\tilde{\psi}_t$ sends $\tilde{\Sigma}_a$ to $\tilde{\Sigma}_a$ for all radii. By choosing $t_0 \in \mathbb{Z}$ such that $t_0 = -1 \bmod a_0$ and $t_0 = 0 \bmod a_i$ for $i = 1, \ldots, n$, we undo the rotation in the first coordinate for large radii and hence we see that it is the identity near the boundary. Note this choice is not always possible, but if a_0 is relatively prime to a_i for $i = 1, \ldots, n$, it is. Altogether, we have the map

$$\tilde{\varphi} = \tilde{\psi}_{t_0} \circ \varphi : \ \tilde{\Sigma}_a \to \tilde{\Sigma}_a,$$

which is the identity near the boundary of $\tilde{\Sigma}_a$. Also note that the choice of t_0 is not unique.

6.1.2.4. *Homomorphism on homology induced by the rotation map.* We would like to know what the relative mapping torus of the rotation map on $\tilde{\Sigma}_a$ is. Since the classification of simply connected five-manifolds is mainly controlled by homology, it turns out that it suffices to know what map the monodromy induces on the homology. First, we observe that φ and $\tilde{\varphi}$ are isotopic, so they induce the same maps on homology. And we may, in fact, work with the non-deformed Stein manifold Σ_a and the rotation map defined there (which we will also refer to as φ), because $\tilde{\Sigma}_a$ and Σ_a coincide in ball of radius R_0 around the origin as subsets of $\mathbb{C}^{n+1}$.

These Stein manifolds Σ_a have been studied carefully in the past (see for instance [**28**]) and many results about their properties, including their homology, are known. We will give a short summary of some of the results that we will use. The results that we are listing are from Hirzebruch-Mayer, [**28**], but date back to Pham, see [**41**].

Stein manifolds are well known to have the homotopy type of a cell-complex of half their real dimension, see Theorem 3.10. This fact is reflected in the following theorem,

THEOREM 6.3 (Pham, see [**28**] and [**41**]). *The set $U_a = \{z \in \Sigma_a \mid z_j^{a_j} \geq 0$ for all $j\}$ is a deformation retract of Σ_a. This deformation is compatible with the group action mentioned above.*

We will use the group action a lot in the following, so let us introduce some notation. The group of a_j^{th}-roots of unity will be written as $G_{a_j} \cong \mathbb{Z}_{a_j}$ when we consider it as an abstract group, and we will denote a generator of G_{a_j} by w_j. As a subgroup of $\mathbb{C}^*$, we shall write $\tilde{G}_{a_j}$. The roots of unity will be indicated by ζ_j. We will write $G_a = G_{a_1} \oplus G_{a_2} \oplus \cdots \oplus G_{a_n}$.

The set U_a can be identified with the join $G_{a_0} * \cdots * G_{a_n}$ in the following way. First a simple observation. Suppose $z \in \mathbb{C}^*$, then note that the condition $z^{a_j} \in \mathbb{R}_{\geq 0}$ is equivalent to $z = \zeta_j |z_j|$ with ζ_j an a_j^{th} root of unity. This gives another description of the set U_a,

$$U_a = \{(\zeta_0 t_0, \ldots, \zeta_n t_n) \in \mathbb{C}^{n+1} | \ \zeta_j \in \tilde{G}_{a_j}, \ t_j \geq 0 \text{ and } \sum_{i=0}^{n} t_j^{a_j} = 1\}.$$

On the other hand, the join $G_{a_0} * \cdots * G_{a_n}$ may be written as

$$\tilde{G}_{a_0} * \cdots * \tilde{G}_{a_n} = \{(\zeta_0 t_0, \ldots, \zeta_n t_n) \in \mathbb{C}^{n+1} \mid \zeta_j \in \tilde{G}_{a_j}, \ t_j \geq 0 \text{ and } \sum_{i=0}^{n} t_j = 1\}.$$

These sets can be identified if we rescale the t_j's. Notice that this identification is compatible with the group action, because G_a acts only on the roots of unity.

The join $G_{a_0} * \cdots * G_{a_n}$ is an n-dimensional simplicial complex with an n-simplex for each element in G_a. This is again compatible with the group action in the following sense. Let e denote the simplex corresponding to $1 \in G_a$. The other simplices are obtained by letting G_a act. In other words, the simplicial chain complex in degree n can be written as

$$C_n(U_a) = \mathbb{Z}(G_a)e,$$

where $\mathbb{Z}(G_a)$ denotes the group ring of G_a.

Another related way to see that we have an n-dimensional simplicial complex is by noting that we can map an n-simplex into Σ_a,

$$\begin{array}{rcl} \{(x_0,\dots,x_n) \mid x_i \geq 0 \text{ for all } i \text{ and } \sum_{j=0}^n x_j = 1\} & \to & \Sigma_a \\ (x_0,\dots,x_n) & \mapsto & (x_0^{1/a_0},\dots,x_n^{1/a_n}). \end{array}$$

Note that this is the rescaling of the t_j's we mentioned earlier. By letting G act, we give the entire space Σ_a the structure of a simplicial complex. Using this picture we can see that the differential of simplicial homology commutes with the group action.

For this, recall that in simplicial homology the differential for a single simplex $v = \langle v_0,\dots,v_q\rangle$ can be written as

$$\partial\langle v_0,\dots,v_q\rangle = \sum_{j=0}^q (-1)^j \langle v_0,\dots,\hat{v_j},\dots,v_q\rangle.$$

The notation $\hat{v_j}$ means that we omit the term v_j. This map can be extended linearly to give the boundary operator $\partial : C_n(U_a) \to C_{n-1}(U_a)$. We will denote the j^{th} term in the sum by ∂_j. So we may write

$$\partial v = \sum_{j=0}^q (-1)^j \partial_j v.$$

Since omitting a term from a simplex commutes with multiplication with group elements, we see $w\partial = \partial w$. In particular, this relation simplifies to $\partial_j \circ w_i = \partial_j \circ 1$ for the generators of the group w_j, since ∂_j forgets the j^{th} component of a simplex, whereas w_j acts only on the j^{th} component.

Hence we see that

$$h = (1-w_0)(1-w_1)\dots(1-w_n)e$$

is a cycle. In fact, this may be used to to establish the following isomorphism

$$\tilde{H}_n(U_a) \cong \mathbb{Z}(G_a)h.$$

We will give a more detailed explanation of this fact. We have already seen that the simplicial chain complex in degree n satisfies $C_n(U_a) \cong \mathbb{Z}(G_a)$. The n^{th} reduced homology group can be regarded as a subgroup of $C_n(U_a)$ given by the kernel of ∂, because there are no simplices of higher degree. Therefore we consider the following map

$$\begin{array}{rcl} C_n(U_a) \cong \mathbb{Z}(G_a) & \to & \mathbb{Z}(G_a)h \\ w & \mapsto & wh. \end{array}$$

Note that the image of this map is always representing a cycle. Let I_a denote the kernel of the map. The ideal I_a is generated by

$$1 + w_j + w_j^2 + \dots + w_j^{a_j-1} \text{ for } j = 0,\dots,n.$$

Note that I_a is a direct summand of $\mathbb{Z}(G_a)$, so $\mathbb{Z}(G_a)h$ is a free $\mathbb{Z}$-module of rank $\prod_{j=0}^n (a_j - 1)$. By a theorem of Milnor, $\mathbb{Z}(G_a)h$ is isomorphic to the homology of the join. Indeed, we have the following theorem from [**37**].

Theorem 6.4 (Milnor). *Let A, B be topological spaces, then*

$$\tilde{H}_{r+1}(A * B) = \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B) + \sum_{i+j=r-1} \operatorname{Tor}(\tilde{H}_i(A), \tilde{H}_j(B)).$$

Note that each of the groups G_{a_j} is a space with a_j points, so we have $\tilde{H}_0(G_{a_j}) \cong \mathbb{Z}^{a_j-1}$ and the other homology groups are zero. Applying Milnor's theorem to $G_{a_0} * \dots * G_{a_n}$ we find that

$$\tilde{H}_n(G_{a_0} * \dots * G_{a_n}) \cong \mathbb{Z}^{(a_0-1)(a_1-1)\cdot\ldots\cdot(a_n-1)}.$$

Hence it follows that our claim $\tilde{H}_n(U_a) \cong \mathbb{Z}(G_a)h$ holds true. We also get a convenient basis for this group. Namely, note that the elements of the form

$$w_0^{k_0} w_1^{k_1} \cdot \ldots \cdot w_n^{k_n} \text{ with } 0 \leq k_j \leq a_j - 2 \text{ for } j = 0,\dots,n$$

form a basis for $\mathbb{Z}(G_a)h$.

We will be interested in what map the group action induces on homology. That is why we reviewed the description of Hirzebruch and Mayer instead of using Milnor's theorems at an earlier stage. We introduced several spaces for this, but their homology is the same,

$$\tilde{H}_n(\Sigma_a) \cong \tilde{H}_n(U_a) \cong \tilde{H}_n(G_{a_0} * \cdots * G_{a_n}).$$

For the following, it is important to note that all identifications we made are compatible with the group action. We consider the rotation map φ induced by multiplication of the first coordinate by the first a_0^{th} root of unity. The map φ induces multiplication by w_0 in $C_n(U_a) \cong \mathbb{Z}(G_a)$. To get the map in homology, take a representative in $C_n(U_a) \cong \mathbb{Z}(G_a)$ of $A \in \tilde{H}_n(U_a)$, and multiply by w_0. The class this element represents is the image of A. It is well-defined by compatibility of the boundary operator ∂ with the group action. This induced map on homology will be called $\varphi_\#$ and can be represented with respect to a basis of $\tilde{H}_n(U_a)$, giving a matrix representation for $\varphi_\#$. We take the above basis of $\tilde{H}_n(U_a)$, given by classes represented by elements of the form

$$w_0^{k_0} w_1^{k_1} \cdot \ldots \cdot w_n^{k_n} \text{ with } 0 \leq k_j \leq a_j - 2 \text{ for } j = 0, \ldots, n.$$

The matrix representation of $\varphi_\#$ consists of $(a_1 - 1) \cdot \ldots \cdot (a_n - 1)$ blocks on the diagonal that look like

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

if we order the basis by its degree in w_1, then by its degree in w_2 and so on.

The above representation of $\varphi_\#$ can be used to compute the homology of the mapping torus

$$A' := \Sigma_a \times I/\sim, \text{ where } (x, 0) \sim (\varphi(x), 1).$$

This is done most easily using the Wang sequence. We use the facts that $H_3(\Sigma_a) = 0$ and that $\pi_1(\Sigma_a) = 0$ (and hence also $H_1(\Sigma_a) = 0$). The piece that is relevant to us looks like

$$0 \to H_3(A') \to H_2(\Sigma_a) \stackrel{\varphi_\# - id}{\to} H_2(\Sigma_a) \to H_2(A') \to 0.$$

Using the above matrix representation of $\varphi_\#$ we see that $\varphi_\# - id$ is injective, because the determinant of the associated matrix is non-zero. Hence we conclude that $H_3(A') = 0$ and that $H_2(A') \cong \operatorname{coker}(\varphi_\# - id)$. We have

$$H_2(A') \cong \operatorname{coker}(\varphi_\# - id) \cong \underbrace{\mathbb{Z}_{a_0} \oplus \cdots \oplus \mathbb{Z}_{a_0}}_{(a_1-1)\cdot \ldots \cdot (a_n-1)\text{times}}.$$

Indeed, each block of the matrix representation of $\varphi_\# - id$ corresponding to the above block has a cokernel isomorphic to $\mathbb{Z}_{a_0}$, which can be seen by performing Gauss elimination. Together with the discussion at the beginning of this section this gives us the homology of the mapping torus of $\tilde{\Sigma}_a$ with monodromy $\tilde{\varphi}$. Let A denote this mapping torus,

$$A = \tilde{\Sigma}_a \times I/\sim, \text{ where } (x, 0) \sim (\tilde{\varphi}(x), 1).$$

Then we have

$$H_2(A) \cong \underbrace{\mathbb{Z}_{a_0} \oplus \cdots \oplus \mathbb{Z}_{a_0}}_{(a_1-1)\cdot \ldots \cdot (a_n-1)\text{times}}.$$

The homotopy exact sequence of the fibration $A \to S^1$ shows that $\pi_1(A) \cong \mathbb{Z}$, so we see that $H_1(A) \cong \mathbb{Z}$ as well. All higher homology groups (grade larger than two) are zero.

6.1.2.5. *Homology of the relative mapping torus.* In this section we will take a few specific examples of Brieskorn varieties and use them to construct open book decompositions for five manifolds. We use the basic ingredients obtained in the previous sections and a Mayer-Vietoris argument similar to the one from section 6.1.1.

First of all, we consider the Brieskorn variety $\tilde{\Sigma}_a$ with exponents $a_0 = p^k$, $a_1 = 3$ and $a_2 = 2$, where p is a prime different from 2 and 3, and k some positive integer. Notice that the associated Brieskorn manifold K is then a homology sphere, i.e. $H_1(K) = 0$. The set A denotes the mapping torus of $\tilde{\Sigma}_a$ with monodromy $\tilde{\varphi}$ as in the previous section. We define $B = K \times D^2$ and set $X = A \cup_\partial B$.

We see that X is simply connected by the Seifert-van Kampen theorem. Indeed, we can apply the arguments from our general discussion in Section 6.1.2.1. By Poincaré duality we see that $H_4(X) = 0$, and since K is a homology sphere we also have $H_2(A \cap B) = 0$. Consider the following piece of the Mayer-Vietoris sequence,

$$0 \to \underbrace{H_2(A)}_{\cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}} \oplus \underbrace{H_2(B)}_{\cong 0} \to H_2(X) \to 0.$$

Here we have used the argument from section 6.1.2.1 to split off a part of the sequence. We see directly that $H_2(X) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$. In particular, the rank of $H_2(X)$ is zero, so $H_3(X) = 0$ as well by Poincaré duality and the universal coefficient theorem. This shows that the prime manifolds M with $H_2(M) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ admit contact open books for $p \neq 2, 3$. The binding is a Brieskorn homology sphere of the form $\Sigma(p^k, 3, 2)$, and the page is the Brieskorn variety $\tilde{\Sigma}_a$. Together with our earlier results, this covers all prime manifolds except those with 2- or 3-torsion in their second homology. To get them, we consider Brieskorn varieties with different exponents.

We stick to the notation we introduced before in this section, so A denotes the mapping torus of Σ_a defined in section 6.1.2.4 and B denotes a neighborhood of the binding, given by K, the boundary of Σ_a.

First we shall tackle the case of 2-torsion in homology. Consider the Brieskorn varieties $\tilde{\Sigma}_a$ with exponents $a_0 = 2^k$, $a_1 = 3$ and $a_2 = 3$. Since the exponents are not relatively prime, we cannot conclude that K is a homology sphere. We can, however, compute the homology of K by using the algorithm of Randell [**42**], which we described in Section 4.2.1. So we get $H_1(K) \cong \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$. As in the beginning of this section, the set B denotes the product neighborhood of the binding, $K \times D^2$. The relative mapping torus X is given by $X = A \cup_\partial B$. We see that $\pi_1(X) = 0$ by arguing in the same way as in Section 6.1.2.1. We apply the Mayer-Vietoris sequence to get the homology of X. The fact that $\pi_1(X) = 0$ shows that $H_1(X) \cong H_4(X) = 0$. If we consider the Mayer-Vietoris sequence with rational coefficients, we see quickly that the rank of $H_3(X)$ is zero. Together with the arguments from Section 6.1.2.1 this reduces the remaining part of the Mayer-Vietoris sequence with integer coefficients to

$$0 \to \underbrace{H_2(A \cap B)}_{\cong \mathbb{Z}^2_{2^k}} \stackrel{i \oplus j}{\to} \underbrace{H_2(A)}_{\cong Z^4_{2^k}} \oplus \underbrace{H_2(B)}_{=0} \to H_2(X) \to 0.$$

We have used to Künneth formula to determine $H_2(A \cap B)$. The rank of $H_1(K)$ is zero, so by Poincaré duality $H_2(K) = 0$ and hence we also have $H_2(B) = 0$. Formula (6.1.2.4) gives the homology of A. We may then argue for the remaining term $H_2(X)$ as follows. First tensor the above short exact sequence with $\mathbb{Z}_2$. This gives a sequence of $\mathbb{Z}_2$-vector spaces. Since tensoring is a right exact functor, we only need to check that $(i \oplus j) \otimes id_{\mathbb{Z}_2}$ is injective to show that we get a short exact sequence. Note that the map $i \oplus j$ can be represented by a (4×2) matrix with coefficients in $\mathbb{Z}_{2^k}$. Since $i \oplus j$ is injective there is a (2×2) subdeterminant that is invertible in the ring $\mathbb{Z}_{2^k}$ (i.e. a class represented by an odd number). If we then tensor with $\mathbb{Z}_2$, the corresponding subdeterminant of the matrix representation of $(i \oplus j) \otimes \mathbb{Z}_2$ is non-zero and hence $(i \oplus j) \otimes \mathbb{Z}_2$ is injective. So we have short exact sequence

$$0 \to \underbrace{H_2(A \cap B)}_{\cong \mathbb{Z}^2_2} \otimes \mathbb{Z}_2 \stackrel{i \oplus j}{\to} \underbrace{H_2(A)}_{\cong Z^4_2} \otimes Z_2 \oplus \underbrace{H_2(B)}_{=0} \otimes Z_2 \to H_2(X) \otimes Z_2 \to 0.$$

We know that the rank of the first two terms is 2 and 4, respectively. So we see that the rank of $H_2(X) \otimes \mathbb{Z}_2$ is equal to 2. On the other hand, the number of elements of $H_2(X)$ must be equal to 2^{2k} by the above short exact sequence (with integer coefficients). Combining this information we find that $H_2(X) \cong G_1 \oplus G_2$, where $G_i \cong \mathbb{Z}_{2^{k_i}}$ for $i = 1, 2$ and $k_1 + k_2 = 2k$. For simply connected five-manifolds these two groups G_1 and G_2 must be isomorphic, since the torsion parts of the second homology group come in pairs (except for the Wu manifold, but that manifold does not carry a contact structure), see Section 6.1. Therefore we get $H_2(X) \cong \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$.

The arguments for the 3-torsion case are almost completely the same. The exponents for Σ_a are different, of course. We take $a_0 = 3^k$, $a_1 = 4$ and $a_2 = 2$. As before we use the algorithm of Randell [**42**] to compute the homology of the Brieskorn manifold K. This time we get $H_1(K) \cong \mathbb{Z}_{3^k}$. Formula (6.1.2.4) shows that $H_2(A) = \mathbb{Z}^3_{3^k}$. Again using the arguments from 6.1.2.1 we can split off a part of the Mayer-Vietoris sequence. By tensoring with $\mathbb{Q}$ we see that the rank of $H_2(X)$ is zero, and hence $H_3(X) = 0$. This reduces the sequence to

$$0 \to \underset{\cong \mathbb{Z}_{3^k}}{H_2(A \cap B)} \to \underset{\cong \mathbb{Z}^3_{3^k}}{H_2(A)} \oplus \underset{\cong 0}{H_2(B)} \to H_2(X) \to 0.$$

We argue as before, by first tensoring this short exact sequence with $\mathbb{Z}_3$. We see that the rank of $H_2(X) \otimes \mathbb{Z}_3$ is two. The order of $H_2(X)$ is equal to 3^{2k}. We use the structure of simply connected five-manifolds again and see that $H_2(X) \cong \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k}$.

6.1.3. Additional applications. In the previous sections we have constructed explicit open books for all simply connected prime five-manifolds and for $S^2 \tilde{\times} S^3$ and S^5 as well. Using the book connected sum, this gives all simply connected five manifolds that can admit a contact structure, a contact open book. Moreover, we have shown that we can realize every compatible Chern class by choosing an appropriate open book. This can be seen by observing that a non-trivial Chern class in a simply connected contact five-manifold can only come from an $S^2 \times S^3$ or an $S^2 \tilde{\times} S^3$ factor. Since we have shown in those particular cases that we can obtain every compatible Chern class, the conclusion follows.

Actually, we can even give another simple proof of the statement that each simply connected almost contact five manifold admits a contact structure. This proof is based on a paper by Thomas [**48**]. He gave constructions for contact structures on five-manifolds. Thomas used Brieskorn manifolds of the form $\Sigma(p^k, 3, 3, 3)$ for p not divisible by 3 to show that prime manifolds with torsion in H_2 admit contact structures. However, we use Randell's algorithm to show that Brieskorn manifolds of the form $\Sigma(3^k, 4, 4, 2)$ realize the missing cases, i.e. prime manifolds with 3-torsion. In fact, the following table might be useful.

Brieskorn manifold	H_2	condition
$\Sigma(p, 3, 3, 3)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	p is not divisible by 3
$\Sigma(p, 4, 4, 2)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	p is not divisible by 2
$\Sigma(2^k, 6, 3^l, p)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	p is not divisible by 2 and 3; $k, l \in \mathbb{N}$
$\Sigma(2, 2^k 3^l, 3, p)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	p is not divisible by 2 and 3; $k, l \in \mathbb{N}$
$\Sigma(2k, 2, 2, 2)$	$\mathbb{Z}$	

Our computations in Chapter 4 show that Brieskorn manifolds have always trivial second Stiefel-Whitney class. In particular, we see that simply connected spin five-manifolds, i.e. $w_2 = 0$, can be written as a connected sum of Brieskorn manifolds.

We continue with our alternative proof. The above table shows that all simply connected prime manifolds (recall that our notion of prime manifold differs from the usual one) can be realized by Brieskorn manifolds and hence they carry a contact structure. To show that all almost contact manifolds carry a contact structure we still need to verify that $S^2 \tilde{\times} S^3$ admits a contact structure. This is most easily done using the open book decompositions of $S^2 \tilde{\times} S^3$ we described earlier.

6.1.3.1. *More open book constructions of contact manifolds.* In Section 6.1.2.4 we constructed the mapping torus of a Brieskorn variety, where the monodromy was obtained by multiplying the first coordinate with a suitable root of unity. We can, of course, modify the monodromy a bit, for example by multiplying several coordinates with suitable roots of unity. Then we may apply the same procedure to obtain other open books of five-manifolds. We would like to say here that this construction is of course possible and in fact easy to do.

Repeating the construction gives other open books and shows that we get many different open books for the same five-manifold. This is an interesting phenomenon, and might give rise to many non-isomorphic contact structures on contact five-manifolds. Unfortunately, I was unable to compute their contact homology, so these contact structures could still be isomorphic. In fact, an example by Giroux [**21**] shows that the standard structure on S^5 admits many different open book decompositions. If we take a closer look at his examples, we also see that they are not always related by stabilizations. This also shows that the equivalence relation we should use for open books is not clear at the moment.

Also note that in Section 6.1.2.3, we isotoped the rotation map to the identity for large radii, see formula 6.1.2.3. The t_0 in that formula was chosen such that φ was isotoped to the identity far away from the origin. The choice of t_0 is, however, not unique. We can add $\mathrm{lcm}(a_0, a_1, a_2)$ to t_0 and we obtain another symplectic isotopy from φ to the identity for large radii. Note that this resembles a Dehn twist, because of the form of the isotopy we defined in Section 6.1.2.3.

6.1.3.2. *Dehn twists.* We would like to explore this feature in a bit more detail recalling our previous discussion in 5.2.2.1. Let $\Sigma_a \subset \mathbb{C}^{n+1}$ be the Brieskorn variety with exponents $a_i = 2$ for $i = 0, \dots, n$. The Brieskorn variety Σ_a can in that case be identified with T^*S^n, see for instance 6.20 from [**34**]. Actually, it turns out that the symplectic manifolds

$$(\Sigma_a, \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j) \text{ and } (T^*S^n, \omega_{can})$$

are isomorphic via the symplectomorphism

$$\begin{array}{rcl} \psi : \Sigma_a & \to & T^*S^n \\ z = x + iy & \mapsto & (|x|^{-1}x, |x|y), \end{array}$$

where $x, y \in \mathbb{R}^{n+1}$ such that $x + iy \in \Sigma_a$. Now consider the map

$$\begin{array}{rcl} \varphi : \Sigma_a & \to & \Sigma_a \\ z & \mapsto & -z. \end{array}$$

This is a "rotation" map with a form slightly different from the type we described in Section 6.1.2.3. The exponents are now not relatively prime, but because they are the same, we can still use the same trick to isotope the map to the identity for large $|z|$, after perturbing the manifold Σ_a to the manifold $\tilde{\Sigma}_a$ described in Section 6.1.2.3. Let us denote this isotoped map by $\tilde{\varphi}$.

In other words, this map is an odd Dehn twist, and by choosing different values of t_0 as mentioned in the previous section we get different Dehn twists. If we apply the arguments from Section 6.1.2.3 to the case where we take $\tilde{\varphi}$ to be the identity, we get the even Dehn twists.

Open books with page T^*S^n and monodromy given by a Dehn twist are described in [**40**] and endow spheres in dimensions $4k + 1$ with infinitely many non-isomorphic contact structures. The fact that the contact structures are indeed non-isomorphic was first shown by Ustilovsky [**50**]. In other words, this might indicate that varying the value of t_0 in Section 6.1.2.3 could give rise to different contact structures on the same manifold in more cases, such as the open books we gave for five-manifolds.

I should add at this point though that I did not verify that either way (taking different Brieskorn varieties or isotoping the symplectomorphism in different ways to the identity) actually gives different contact structures. Only in case that all exponents of the Brieskorn variety are 2 we know that that is true by [**40**] and [**50**].

6.1.3.3. *Symplectomorphisms that are isotopic to the identity near the boundary.* In order to construct contact open books, we need an exact symplectic manifold Σ and a symplectomorphism that is the identity near the boundary of Σ. In this section we would like to give some remarks concerning these kinds of symplectomorphisms. The easiest non-trivial example would be the following one. Let Σ be a symplectic manifold of dimension $2n$ with boundary and let L be a Lagrangian embedding of S^n. Then we can do a symplectic Dehn twist on a neighborhood of L. Such a neighborhood is symplectomorphic to T^*S^n by a theorem of Weinstein that is now known as the Lagrangian neighborhood theorem [**34**]. We identify the neighborhood of L with T^*S^n and note that Dehn twists are isotopic to the identity away from the zero section. In other words, the symplectomorphism defined on a neighborhood of L can be extended to be the identity outside this neighborhood of L. We obtain a symplectomorphism that is suitable for a contact open book. This is of course related to the Lagrangian plumbings that we defined earlier.

An interesting example is that open book decompositions can sometimes be used to make a statement about the symplectomorphism group of a certain symplectic manifold. For instance, take any Legendrian knot with *tb* either 0 or 2. If we attach a two-handle as in Theorem 3.18 to such a knot, then we obtain a Stein manifold Σ, whose boundary is a homology sphere, say K. We want to argue that Σ does not admit any symplectomorphism isotopic to the identity near the boundary that induces $-id$ on homology (note $H_2(\Sigma) \cong \mathbb{Z}$). Assume that there is such a map, which we denote by φ. Then we may define the mapping torus

$$A := \Sigma \times I / \sim, \text{ where } (x, 0) \sim (\varphi(x), 1).$$

The Wang sequence for A, a Σ-bundle over S^1, shows that $H_2(A) \cong \mathbb{Z}_2$, cf. the sequence 6.1.2.4. Let $B = K \times D^2$ be a neighborhood of the binding. The arguments from Section 6.1.2.1 apply and show that $X := A \cup_\partial B$ is simply connected. A Mayer-Vietoris argument then shows that $H_2(X) \cong \mathbb{Z}_2$. This means that X is diffeomorphic to the Wu manifold, which does not admit a contact structure. Hence the assumption that there is such a symplectomorphism is false. Of course, there might be situations where it can be shown that there is not even a diffeomorphism with these properties.

CHAPTER 7

Open book decompositions for contact structures on Brieskorn manifolds

This chapter is based on a joint article with Klaus Niederkrüger, see [**40**]. We construct abstract contact open books using a simple Stein manifold, namely T^*S^n with its canonical symplectic form. For the monodromy we use Dehn twists. We show that these contact open books are certain Brieskorn manifolds, namely $\Sigma(k, 2, \dots, 2)$, where k is the order of the Dehn twist.

7.0.4. Dehn twists. Here we will give an explicit form for a Dehn twist in order to make computations. See also Sections 5.1.2 and 5.2.2.1 for a more geometric description of a Dehn twist. We write points in T^*S^{n-1} as $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n}$ with $|\mathbf{q}| = 1$ and $\mathbf{q} \perp \mathbf{p}$.

We write a k-fold right-handed Dehn twist as

$$\tau_k(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \cos g_k(\mathbf{p}) & |\mathbf{p}|^{-1} \sin g_k(\mathbf{p}) \\ -|\mathbf{p}| \sin g_k(\mathbf{p}) & \cos g_k(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}.$$

Here $g_k(\mathbf{p}) = \pi k + f_k(|\mathbf{p}|)$, and f_k is a smooth function that increases monotonically from 0 to πk for $k > 0$ on an interval that will be specified later. Outside this interval, f_k will be identically equal to 0 or πk, see Figure 7.1. Though the details do not matter for the Dehn twist itself, our computations will turn out to put some constraints on f_k. We can, of course, also write a left-handed Dehn twist in this way by choosing a negative k and by requiring f_k to decrease monotonically to πk. In the following we will restrict ourselves to right-handed Dehn twists though. Note that for small $|\mathbf{p}|$ a k-fold Dehn twist is $(-1)^k\,\mathrm{id}$, while for large $|\mathbf{p}|$ it equals the identity map.

We will now construct a mapping torus of T^*S^{n-1} using right-handed Dehn twists following the construction we described in Section 5.2. The canonical 1-form $\lambda_{\mathrm{can}} = \mathbf{p}\, d\mathbf{q}$ on T^*S^{n-1} transforms like

$$\tau_k^* \lambda_{\mathrm{can}} = \lambda_{\mathrm{can}} + |\mathbf{p}|\, d\big(f_k(|\mathbf{p}|)\big).$$

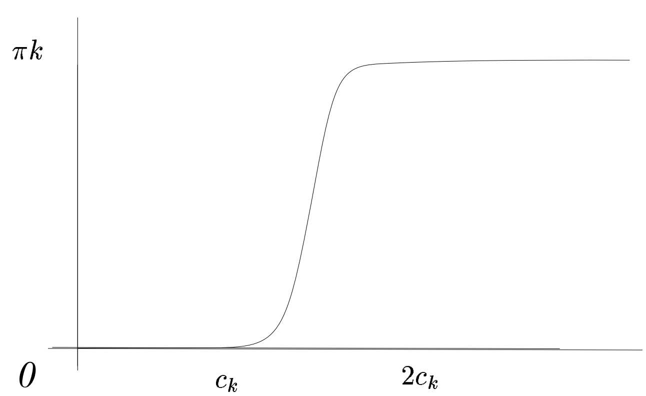

FIGURE 7.1. Sketch of f_k

Note that the difference $\lambda_{\mathrm{can}} - \tau_k^* \lambda_{\mathrm{can}}$ is exact. This implies in particular that the Dehn twists are symplectomorphisms of $(T^*S^{n-1}, d\lambda_{\mathrm{can}})$. As a primitive of this difference $\lambda_{\mathrm{can}} - \tau_k^* \lambda_{\mathrm{can}}$ we take

$$h_k(|\mathbf{p}|) := 1 - \int_0^{|\mathbf{p}|} s f_k'(s) ds.$$

Note that h_k can assumed to be positive by choosing a suitable interval where f_k increases. To be more explicit, choose a smooth function f that is identically 0 on the interval $[0,1]$, on the interval $[1,2]$ it increases monotonically from 0 to 1 and f is identically 1 on the interval $[2,\infty)$. Furthermore, we may assume that the derivative f' is bounded by 2. Then we can take $f_k(x) := k\pi f(c_k x)$ with $c_k > 3k\pi$. We have

$$\int_0^{|p|} s f_k'(s) ds \leq \int_0^{\infty} k\pi c_k s f'(c_k s) ds \leq k\pi \int_0^{\infty} y f'(y) dy / c_k \leq \frac{k\pi}{c_k} \int_1^2 y 2 dy = \frac{3k\pi}{c_k},$$

where we have substituted $y = c_k s$ and used that $f'(y) = 0$ outside the interval $[1,2]$ and that f' is bounded by 2. Our choice of c_k ensures that this integral is indeed smaller than 1, so h_k is positive. Consider the map

$$\begin{array}{rrcl} \varphi_k : & \mathbb{R} \times T^*S^{n-1} & \to & \mathbb{R} \times T^*S^{n-1}, \\ & (t; \mathbf{q}, \mathbf{p}) & \mapsto & (t + h_k(|\mathbf{p}|); \tau_k(\mathbf{q}, \mathbf{p})). \end{array}$$

This map preserves the contact form $dt + \lambda_{\mathrm{can}}$ on $\mathbb{R} \times T^*S^{n-1}$, so we obtain an induced contact structure on $\mathbb{R} \times T^*S^{n-1}/\varphi_k$.

To make computations more convenient, we construct an additional intermediate mapping torus. Let $\mathbb{R} \times T^*S^{n-1}/\sim_k$ be the mapping torus obtained by identifying

$$(t; \mathbf{q}, \mathbf{p}) \sim_k (t+1; \tau_k(\mathbf{q}, \mathbf{p})).$$

We can define a diffeomorphism

$$\mathbb{R} \times T^*S^{n-1}/\sim_k \to \mathbb{R} \times T^*S^{n-1}/\varphi_k$$

by sending $(t; \mathbf{q}, \mathbf{p})$ to $(h_k(|\mathbf{p}|)t; \mathbf{q}, \mathbf{p})$. The pull-back β_k of the described contact form under this diffeomorphism is given by

$$\beta_k = h_k(|\mathbf{p}|)dt - t|\mathbf{p}|\, d\big(f_k(|\mathbf{p}|)\big) + \lambda_{\mathrm{can}}.$$

During our computations a few more mapping tori arise, and one of the main goals is to construct contactomorphism between them. For the convenience of the reader we have included a diagram in Section 7.1.2 that contains all mapping tori and the maps between them.

7.1. Open books for the Brieskorn manifolds W_k^{2n-1}

We will consider a special class of Brieskorn manifolds (see Chapter 4 for some general facts on these manifolds), namely Brieskorn manifolds of the form $\Sigma(k, 2, \dots, 2) \subset \mathbb{C}^{n+1}$. We will denote them as W_k^{2n-1}. Some of the computations will be a little easier if we take the radius of the sphere to be $\sqrt{2}$.

Because of these special exponents, we have a large group acting on W_k^{2n-1}; the orthogonal group $\mathrm{SO}(n)$ acts linearly on $\mathbb{C}^{n+1}$ by multiplying the last n coordinates with $\mathrm{SO}(n)$ in its standard matrix representation, i.e. $A \cdot (z_0, z_1, \dots, z_n) := (z_0, A \cdot (z_1, \dots, z_n))$. This action restricts to W_k^{2n-1}, because the polynomial $f(z_0, \dots, z_n) = z_0^k + z_1^2 + \ldots + z_n^2$ can be written as $z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i\langle \mathbf{x} | \mathbf{y} \rangle$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$.

The contact form we defined in Chapter 4 is also $\mathrm{SO}(n)$-invariant. We write $z_j = x_j + iy_j$. In these coordinates the contact form looks like

$$\alpha_k := k \cdot (x_0\, dy_0 - y_0\, dx_0) + 2 \sum_{j=1}^n (x_j\, dy_j - y_j\, dx_j)\,,$$

so we see immediately that it is indeed invariant under the group action.

Although the group action will not appear explicitly in the following computations, it serves as a guide. Often it is only necessary to describe a map on a thin slice of the manifold. The group action can then be used to define the map everywhere.

Next, we note that we have an open book structure on W_k^{2n-1}: we define the binding B of the open book by the set in W_k^{2n-1} with $z_0 = 0$, and we have the fibration $\vartheta : (W_k^{2n-1} - B) \to S^1$, given by $(z_0, z_1, \dots, z_n) \mapsto z_0/|z_0|$. First we show that this does indeed define an open book, and then we show that the monodromy can be regarded as a k-fold right-handed Dehn twist.

7.1.1. The binding. Note that the binding B is naturally contactomorphic to W_2^{2n-3}. Hence the binding is a contact manifold. As a manifold, we can identify W_2^{2n-3} with the unit sphere bundle $S(T^*S^n)$. The latter fact can be seen as follows. We have an $\mathrm{SO}(n)$ action on W_2^{2n-3}. Indeed, the orbit of $(1, i, 0, \dots, 0)$ is W_2^{2n-3} and the stabilizer of $(1, i, 0 \dots, 0)$ can be seen to be $SO(n-2)$. Hence we can identify $W_2^{2n-3} = \mathrm{SO}(n)/\mathrm{SO}(n-2) \cong S(T^*S^n)$

The symplectic normal bundle of the binding is trivial, because for $k \neq 1$ we have a symplectic basis

$$\frac{1}{\sqrt{2k}}(1, 0, \dots, 0),\ \frac{1}{\sqrt{2k}}(i, 0, \dots, 0),$$

and for $k = 1$ we have the basis

$$\sqrt{\frac{2}{5}}(1, -\frac{\bar{z}_1}{4}, \dots, -\frac{\bar{z}_n}{4}),\ \sqrt{\frac{2}{5}}(i, -\frac{i\bar{z}_1}{4}, \dots, -\frac{i\bar{z}_n}{4}).$$

In particular we have that the normal bundle is trivial, which is necessary for B to be the binding of an open book.

7.1.2. The pages. In this section, we want to prove that $W_k^{2n-1} - B$ is contactomorphic to $\mathbb{R} \times T^*S^{n-1}/\sim_k$, the mapping torus of a k-fold Dehn twist.

We have an $\mathbb{R}$-action on $W_k^{2n-1} - B$, given by

$$e^{it}(z_0, z_1, \dots, z_n) = (e^{it}z_0, e^{\frac{ki}{2}t}z_1, \dots, e^{\frac{ki}{2}t}z_n).$$

This induces a diffeomorphism between the pages $\vartheta^{-1}(1)$ and $\vartheta^{-1}(e^{it})$. Note that this action is the flow of the Reeb field.

The following auxiliary mapping torus makes the computations more convenient. Define

$$M_k := \mathbb{R} \times T^*S^{n-1}/\sigma_k,$$

where

$$\sigma_k(t, \mathbf{q}, \mathbf{p}) = (t + 1, (-1)^k\mathbf{q}, (-1)^k\mathbf{p}).$$

We will now give an explicit map to show that $P = \vartheta^{-1}(1)$ is diffeomorphic to $T^*_{|\mathbf{p}|<1}S^{n-1}$. Here $T^*_{|\mathbf{p}|<1}S^{n-1}$ denotes the open unit disk bundle associated with the cotangent bundle of S^{n-1}. A point $(\mathbf{q}, \mathbf{p}) \in T^*S^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$ with $|\mathbf{q}| = 1$, $|\mathbf{p}| \leq 1$, and $\mathbf{q} \perp \mathbf{p}$ is mapped to

$$(\mathbf{q}, \mathbf{p}) \mapsto \Big(1 - |\mathbf{p}|^2, F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q}\Big),$$

with

$$F(r) = \sqrt{\frac{2 - (1 - r^2)^2 - (1 - r^2)^k}{2r^2}}$$

and

$$G(r) = \sqrt{\frac{2 - (1 - r^2)^2 + (1 - r^2)^k}{2}}.$$

Together with the $\mathbb{R}$-action this gives a map

$$\begin{aligned} \Phi_k : \mathbb{R} \times T^*_{|\mathbf{p}|<1}S^{n-1} &\to W_k^{2n-1} \\ (t, \mathbf{q}, \mathbf{p}) &\mapsto \Big(e^{2\pi it}(1 - |\mathbf{p}|^2), e^{\pi kit}(F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q})\Big). \end{aligned}$$

This descends to a diffeomorphism of the subset of M_k with $|\mathbf{p}| < 1$ to $W_k^{2n-1} - B$. For k even, one obtains $\Phi_k(t + 1, \mathbf{q}, \mathbf{p}) = \Phi_k(t, \mathbf{q}, \mathbf{p})$, so that $W_k^{2n-1} - B \cong S^1 \times T^*_{|\mathbf{p}|<1}S^{n-1}$, and for k odd, one obtains $\Phi_k(t+1,\mathbf{q},\mathbf{p})=\Phi_k(t,-\mathbf{q},-\mathbf{p})$, so that $W_k^{2n-1}-B$ is a non-trivial $T^*_{|\mathbf{p}|<1}S^{n-1}$-bundle over S^1.

The pull-back of the contact form α_k to M_k under Φ_k gives

$$\Phi_k^*\alpha_k = 2\pi k\big((1-|\mathbf{p}|^2)^2+|\mathbf{p}|^2F^2+G^2\big)\,dt+4FG\,\lambda_{\mathrm{can}} = 4\pi k\,dt+4FG\,\lambda_{\mathrm{can}}.$$

Next, we construct a diffeomorphism Ψ_k from M_k to the mapping torus $\mathbb{R}\times T^*S^{n-1}/\sim_k$ by defining

$$\begin{aligned}\Psi_k(t;\mathbf{q},\mathbf{p}) \quad = \quad & \Big[t;\mathbf{q}\cdot\cos\big(tf_k(|\mathbf{p}|)\big)+\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\sin\big(tf_k(|\mathbf{p}|)\big),\\ & \mathbf{p}\cdot\cos\big(tf_k(|\mathbf{p}|)\big)-|\mathbf{p}|\mathbf{q}\cdot\sin\big(tf_k(|\mathbf{p}|)\big)\Big].\end{aligned}$$

The map is well-defined, because $\Psi_k\circ\sigma_k(t;\mathbf{q},\mathbf{p})$ is identified with $\Psi_k(t;\mathbf{q},\mathbf{p})$ in the mapping torus $\mathbb{R}\times T^*S^{n-1}/\sim_k$. In order to show that $(W_k^{2n-1}-B,\alpha_k)$ and $(\mathbb{R}\times T^*S^{n-1}/\sim_k,\beta_k)$ are contactomorphic, we will show that the pull-back of α_k under Φ_k is contactomorphic to the pull-back of β_k under Ψ_k.

We now compute the pull-back of β_k under Ψ_k, noting that the norm of $\mathbf{p}$ is invariant under Ψ_k (we do not write the dependence of h_k and f_k on $|\mathbf{p}|$):

$$\begin{aligned}\Psi_k^*\beta_k = h_k dt - t|\mathbf{p}|df_k + (\mathbf{p}\cos(tf_k)-|\mathbf{p}|\mathbf{q}\sin(tf_k))\cdot\Big(d\mathbf{q}\cos(tf_k)-\\ -\mathbf{q}\sin(tf_k)(f_kdt+tdf_k)+(\frac{d\mathbf{p}}{|\mathbf{p}|}-\frac{\mathbf{p}\,d|\mathbf{p}|}{|\mathbf{p}|^2})\sin(tf_k)+\frac{\mathbf{p}}{|\mathbf{p}|}\cos(tf_k)(f_kdt+tdf_k)\Big).\end{aligned}$$

Since we have $\mathbf{p}\cdot\mathbf{q}=0$ and $|\mathbf{q}|^2=1$, it follows that $\mathbf{p}\,d\mathbf{q}=-\mathbf{q}\,d\mathbf{p}$ (recall that $\mathbf{p}\,d\mathbf{q}=\lambda_{\mathrm{can}}$) and $\mathbf{q}\,d\mathbf{q}=0$. We now use the standard trigonometric equalities and the fact that $h_k(y)=1-yf_k(y)+\int_0^y f_k(s)ds$ to find

$$\Psi_k^*\beta_k=\left(1+\int_0^{|\mathbf{p}|}f_k(s)ds\right)dt+\lambda_{\mathrm{can}}.$$

Note that $\Phi_k^*\alpha_k$ has a very similar form. We make the following ansatz for a contactomorphism of $(M_k|_{|\mathbf{p}|<1},\Phi_k^*\alpha_k)$ to $(M_k,\Psi_k^*\beta_k)$:

$$S_k:(t,\mathbf{q},\mathbf{p}) \quad\mapsto\quad (t,\mathbf{q},\frac{g(|\mathbf{p}|)}{|\mathbf{p}|}\mathbf{p}).$$

With this ansatz we find what $\mathbf{p}$ should map to in order to be a contactomorphism. Note that the map S_k just rescales $\mathbf{p}$. The pull-back under S_k of $\Psi_k^*\beta_k$ is given by

$$\left(1+\int_0^{g(|\mathbf{p}|)}f_k(s)ds\right)dt+\frac{g(|\mathbf{p}|)}{|\mathbf{p}|}\lambda_{\mathrm{can}}.$$

Since we want this to be a multiple of $\Phi_k^*\alpha_k$, we need to solve the following equation,

$$\frac{g(|\mathbf{p}|)}{1+\int_0^{g(|\mathbf{p}|)}f_k(s)ds}=\frac{|\mathbf{p}|FG}{k\pi}.$$

With the auxiliary function

$$h(y):=\frac{y}{1+\int_0^y f_k(s)ds},$$

the above equation becomes

$$h\left(g(|\mathbf{p}|)\right)=\frac{|\mathbf{p}|FG}{k\pi}. \tag{7.1}$$

Hence we can solve for $g(|\mathbf{p}|)$ by inverting h. The following arguments show that h can indeed be inverted. The derivative of h is given by

$$h'(y)=\frac{1-\int_0^y sf_k'(s)ds}{\left(1+\int_0^y f_k(s)ds\right)^2}=\frac{h_k(y)}{\left(1+\int_0^y f_k(s)ds\right)^2};$$

this is positive by our choice of h_k in Section 7.0.4. This shows that h is strictly increasing. Since $f_k(s) = k\pi$ for s sufficiently large, we see that $h(y)$ converges to $1/k\pi$ if y tends to infinity. Combining these two observations shows that the function h maps $[0, \infty)$ to $[0, 1/k\pi)$. Hence h can be inverted when restricted to a suitable range. A short computation shows that the right-hand side of the Equation (7.1), the term $|\mathbf{p}|FG/k\pi$, has positive derivative and is therefore strictly increasing on the interval $[0, 1)$. Moreover, it has the same range as h, namely $[0, 1/k\pi)$. Therefore we can find a smooth solution to $g(|\mathbf{p}|)$ by applying the inverse of h to $\frac{|\mathbf{p}|FG}{k\pi}$.

This shows that the open book (B, ϑ) on W_k^{2n-1} has page T^*S^{n-1} with monodromy given by a k-fold Dehn twist. The contactomorphism that achieves this is

$$C_k := \Phi_k \circ S_k^{-1} \circ \Psi_k^{-1} : (\mathbb{R} \times T^*S^{n-1}/\sim_k, \beta_k) \to (W_k^{2n-1} - B, \alpha_k).$$

Note that this contactomorphism also respects the projection to S^1, because the S^1-coordinate is invariant under C_k. We summarize our results in the following diagram

$$\begin{array}{l} (\mathbb{R} \times T^*S^{n-1}/\varphi_k, dt + \lambda_{\mathrm{can}}) \\ \qquad\quad \wr\!\| \\ \left(\mathbb{R} \times T^*S^{n-1}/\sim_k, \beta_k\right) \qquad \xleftarrow{\psi_k} M_k \xleftarrow{S_k} M_k|_{|\mathbf{p}|<1} \xrightarrow{\Phi_k} \left(W_k^{2n-1}, \alpha_k\right). \end{array}$$

7.1.3. The contact structure on W_k^{2n-1} is supported by the open book. We now want to show that the open book we gave on W_k^{2n-1} is adapted to the contact form α_k.

In Section 7.1.1 we showed that α_k induces a contact form on the binding B. Next we note that the Reeb field R_{α_k} is transverse to the pages, as its flow even provides a diffeomorphism from one page to another. If we denote an open page (i.e. the page without the binding) by P, this implies in particular that the rank of $d\alpha_k|_P$ is maximal, or in other words that $d\alpha_k$ is a symplectic form when restricted to P. This shows that $d\alpha_k$ induces a symplectic form on each open page.

This verifies a part of Definition 5.5, but we still need the positivity condition. For that, we use the following observation due to Giroux,

$$\int_{\partial P} \alpha \wedge (d\alpha_k)^{n-2} = \int_P (d\alpha_k)^{n-1} > 0.$$

The latter expression is positive because $(d\alpha_k)$ is symplectic, so it gives a volume form on P. We give P the orientation coming from this volume form. Hence we see that the orientation on the binding coming from the contact form matches the orientation as the boundary of a page, provided that the binding is connected. If the boundary is not connected, the form $\alpha \wedge (d\alpha_k)^{n-2}$ could still be negative on one component. We first observe that the binding is connected if $n > 2$. In case the Brieskorn manifold is 3-dimensional ($n = 2$), we see that the binding given by $z_0 = 0$ has two components, namely the subsets of the Brieskorn manifold satisfying $z_1^2 + z_2^2 = 0$, or

$$\{z_1 = iz_2\} \text{ and } \{z_1 = -iz_2\}.$$

If we send z_2 to $-z_2$, we map one component to the other without changing the form α_k. Hence their integrals must be the same, and by the above computation they are both positive.

CHAPTER 8

Maslov index and closed Reeb orbits

8.1. Maslov index for loops of symplectic matrices

In this chapter we would like to define the Conley-Zehnder index which plays the role of degree in contact homology. This index is a special kind of Maslov index. Because there are many different Maslov indices, we will introduce a few of them and then point out relations between them. Most of the statements here can be found in [**43**] and [**44**].

Since the fundamental group of $\mathrm{Sp}(2n)$ (the group of matrices that preserve the standard symplectic form in $\mathbb{R}^{2n}$) is isomorphic to the integers, one might be interested in an explicit isomorphism $\pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$. This can be given by the so-called Maslov index. By requiring some properties, we can define a distinguished isomorphism. We have the following theorem from [**34**], Theorem 2.35,

THEOREM 8.1. *There exists a unique functor μ_l, called the* **Maslov index for symplectic loops***, which assigns an integer $\mu_l(\psi)$ to every loop $\psi : S^1 \to \mathrm{Sp}(2n)$ of symplectic matrices and satisfies the following axioms:*

- *Homotopy: Two loops in $\mathrm{Sp}(2n)$ are homotopic if and only if they have the same Maslov index.*
- *Product: For any two loops $\psi_1,\ \psi_2 : S^1 \to \mathrm{Sp}(2n)$ we have $\mu_l(\psi_1\psi_2) = \mu_l(\psi_1) + \mu_l(\psi_2)$. In particular, the constant loop $\psi(t) = 1$ has Maslov index 0.*
- *Direct sum: If $n = n' + n''$, we may regard $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ as a subgroup of $\mathrm{Sp}(2n)$. The Maslov index is additive with respect to this operation: $\mu_l(\psi \oplus \psi') = \mu_l(\psi) + \mu_l(\psi')$*
- *Normalization: The loop $\psi : S^1 \to \mathrm{U}(1) \subset \mathrm{Sp}(2)$ defined by $\psi(t) = e^{2\pi i t}$ has Maslov index 1.*

Indeed, we can give an explicit description of this functor. This is done by constructing a map from $\mathrm{Sp}(2n)$ to S^1, which can be used to compose with a path of symplectic matrices. Thus we obtain a map from S^1 to S^1, whose degree we can consider. More precisely, we do the following. An element A from $\mathrm{Sp}(2n)$ can be retracted onto $U(n)$ by using

$$(AA^T)^{-1/2}A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

Here we decomposed A into a symmetric, positive definite matrix and an orthogonal matrix, and consider only its orthogonal part. We use this decomposition to define

$$\begin{array}{rccc} \rho : & \mathrm{Sp}(2n) & \to & S^1 \\ & A & \mapsto & \det(X + iY), \end{array}$$

where X, Y are defined the equation above. For a loop of symplectic matrices $\psi :\ S^1 \to \mathrm{Sp}(2n)$, we define the Maslov index as

$$\mu_l(\psi) = \deg(\rho \circ \psi).$$

This definition is convenient for many computations, but it can sometimes be helpful to have another interpretation of the Maslov index. We define $\mathrm{Sp}^*(2n)$ to be the set of symplectic matrices without an eigenvalue equal to 1. This set has two components, which can be distinguished by the sign of $\det(1-\psi)$ for $\psi \in \mathrm{Sp}^*(2n)$. The complement of $\mathrm{Sp}^*(2n)$ in $\mathrm{Sp}(2n)$ is called the Maslov cycle and is an algebraic variety of codimension 1 with a natural coorientation. For any loop $\varphi :\ S^1 \to \mathrm{Sp}(2n)$ the intersection number with the Maslov cycle turns out to be even. The Maslov index $\mu(\varphi)$ is half this number.

8.2. Conley-Zehnder index

The map ρ can also be used to define the so-called **Conley-Zehnder index**. This index assigns to every path $\psi: [0,1] \to \mathrm{Sp}(2n)$ with $\psi(0) = \mathbb{1}$ and $\psi(1) \in \mathrm{Sp}^*(2n)$ an integer. One definition goes as follows. We extend the path ψ to a map $\psi: [0,2] \to \mathrm{Sp}(2n)$ such that $\psi(s) \in \mathrm{Sp}^*(2n)$ for $s \geq 1$ and $\psi(2)$ is either $W^+ = -\mathbb{1}$ or $W^- = \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$. Such an extension is unique up to homotopy. Note here that W^+ and W^- lie in different components of $\mathrm{Sp}^*(2n)$. Since $\rho(W^\pm) = \pm 1$, we find that $\rho^2 \circ \psi: [0,2] \to S^1$ is a loop. As the degree of this map is independent of the chosen extension, the Conley-Zehnder index

$$\mu_{CZ}(\psi) = \deg(\rho^2 \circ \psi) \tag{8.1}$$

is well-defined. Following [**44**] we list the following properties of the Conley-Zehnder index, which are similar to the ones that define the Maslov index for symplectic loops. Indeed, [**45**] shows that the homotopy, loop and signature property determine the Conley-Zehnder index uniquely

- Naturality: For any path $\varphi: [0,1] \to \mathrm{Sp}(2n)$, $\mu_{CZ}(\varphi\psi\varphi^{-1}) = \mu_{CZ}(\psi)$.
- Homotopy: The Conley-Zehnder index of ψ is invariant under homotopies of ψ with fixed endpoints.
- Zero: If $\psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then $\mu_{CZ}(\psi) = 0$.
- Direct sum: If $n = n_1 + n_2$, we may regard $\mathrm{Sp}(2n_1) \oplus \mathrm{Sp}(2n_2)$ as a subgroup of $\mathrm{Sp}(2n)$. The Conley-Zehnder index is additive with respect to this operation: $\mu(\psi_1 \oplus \psi_2) = \mu(\psi_1) + \mu(\psi_2)$. The paths are both supposed to have the proper form, i.e. $\psi_i: [0,1] \to \mathrm{Sp}(2n_i)$ with $\psi_i(0) = 1$ and $\psi_i(1) \in \mathrm{Sp}^*(2n_i)$ for $i = 1, 2$.
- Loop: If $\varphi: [0,1] \to \mathrm{Sp}(2n)$ is a loop with $\varphi(0) = \varphi(1) = 1$, then
$$\mu_{CZ}(\varphi\psi) = \mu_{CZ}(\psi) + 2\mu_l(\varphi).$$
- Signature: Let S be a symmetric, non-degenerate $(2n \times 2n)$-matrix with $\|S\| < 2\pi$. Define $\psi(t) = \exp(J_0 S t)$. Then we have
$$\mu_{CZ}(\psi) = \frac{1}{2}\,\mathrm{sign}(S).$$
Here $\mathrm{sign}(S)$ is the signature of the matrix S, i.e. the number of positive minus the number of negative eigenvalues. The norm on matrices is taken to be
$$\|S\| = \max_{|x|=1} |Sx|,$$
where we have used the standard Euclidean norm on $\mathbb{R}^{2n}$.

For a lot of computations another definition of the Conley-Zehnder index is often convenient. Let ψ be a path $\psi: [0,1] \to \mathrm{Sp}(2n)$ with $\psi(0) = 1$ and $\psi(1) \in \mathrm{Sp}^*(2n)$. A number $t \in [0,1]$ is called a **crossing** if $\det(1 - \psi(t)) = 0$. For a crossing t we may define the quadratic form $\Gamma(\psi, t): \ker(1 - \psi(t)) \to \mathbb{R}$ by

$$\Gamma(\psi, t)v := \omega_0(v, \dot{\psi}(t)v)$$

for $v \in \ker(1 - \psi(t))$ and ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. This quadratic form is called the **crossing form**. A crossing t is said to be **regular** if its corresponding crossing form is non-degenerate. Note that regular crossings are always isolated. If the path ψ has only regular crossings, its Conley-Zehnder index may also be computed as follows:

$$\mu_{CZ}(\psi) = \frac{1}{2}\,\mathrm{sign}\,\Gamma(\psi, 0) + \sum_{t \text{ is a crossing}, t>0} \mathrm{sign}\,\Gamma(\psi, t). \tag{8.2}$$

Because the Conley-Zehnder index is invariant under homotopies with fixed endpoints, we may compute the Conley-Zehnder index for general paths in the same way after making a small perturbation (fixing the endpoints) to ensure that all crossings are regular. It is proved in [**43**] that the definitions in Formula (8.1) and Formula (8.2) of the Conley-Zehnder index agree.

8.2.1. Maslov index for paths. Robbin and Salamon [**43**] considered more general paths of symplectic matrices. They defined a more general Maslov index, which can be regarded as a generalization of the Conley-Zehnder index. It can be defined as follows. Let $\psi:\ [a,b]\to \mathrm{Sp}(2n)$ be any path of symplectic matrices. If this path has only regular crossings, its Maslov index may be defined by

$$\mu(\psi) := \frac{1}{2}\operatorname{sign}\Gamma(\psi,a) + \sum_{t \text{ is a crossing}, t\in(a,b)} \operatorname{sign}\Gamma(\psi,t) + \frac{1}{2}\operatorname{sign}\Gamma(\psi,b). \tag{8.3}$$

As before, $\Gamma(\psi,t)$ denotes the crossing form of ψ at t. If there is no crossing in a, the first term $(\frac{1}{2}\operatorname{sign}\Gamma(\psi,a))$ should be read as 0. The last term $(\frac{1}{2}\operatorname{sign}\Gamma(\psi,b))$ should be dropped in case there is no crossing in b. If the considered path ψ has non-regular crossings, we first make a small perturbation $\tilde\psi$ of ψ such that $\tilde\psi$ has only regular crossings. As before, we obtain this perturbation using a homotopy of ψ that fixes the endpoints. We put $\mu(\psi)=\mu(\tilde\psi)$. This general Maslov index is half-integer in general and has the following properties.

- Naturality: For $\varphi\in \mathrm{Sp}(2n)$ we have $\mu(\varphi\psi\varphi^{-1})=\mu(\psi)$.
- Homotopy: The Maslov index of ψ is invariant under homotopies of ψ with fixed endpoints.
- Catenation: Let $\psi:\ [a,b]\to\mathrm{Sp}(2n)$. Then we have for $a<c<b$
$$\mu(\psi)=\mu(\psi|_{[a,c]})+\mu(\psi|_{[c,b]})$$
- Direct sum: If $n=n_1+n_2$, we may regard $\mathrm{Sp}(2n_1)\oplus\mathrm{Sp}(2n_2)$ as a subgroup of $\mathrm{Sp}(2n)$. The Maslov index is additive with respect to this operation: $\mu(\psi_1\oplus\psi_2)=\mu(\psi_1)+\mu(\psi_2)$, where $\psi_i:\ [0,1]\to\mathrm{Sp}(2n_1)$ for $i=1,2$.

We note that the loop and signature properties as we found them for the Conley-Zehnder index will also hold for this more general index.

8.2.2. Example. Here we will compute the Maslov index of a certain path of symplectic matrices. In itself, this is not very interesting since we simply use Formula (8.3), but we will get a convenient formula that we shall use in Chapter 11.

Consider the path of symplectic matrices given by

$$\begin{array}{rcl} \psi:\ [0,T] & \to & \mathrm{Sp}(2) \\ t & \mapsto & e^{it} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}. \end{array}$$

We see that there is always a crossing at $t=0$. Depending on what T is, there can be additional crossing at integer multiples of 2π smaller than T. The derivative of ψ is given

$$\dot\psi(t) = \begin{pmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{pmatrix},$$

so at a crossing t, the derivative $\dot\psi(t)$ is the standard (almost) complex structure on $\mathbb{R}^2$. By plugging this into the second slot of the standard symplectic form, we get the standard metric on $\mathbb{R}^2$. Hence we see that the signature at each crossing is 2. In case T is not divisible by 2π, the only crossing at the boundary of $[0,T]$ is the one at 0; this contributes 1 to the Maslov index. There are $\lfloor T/2\pi\rfloor$ interior crossings, each of which increases the Maslov index by 2. On the other hand, if T is divisible by 2π, we have two crossings at the boundary, both contributing a 1 to the Maslov index, and $T/2\pi-1$ interior crossings. So we obtain

$$\mu(\psi) = \begin{cases} 2\frac{T}{2\pi} & \text{if } T \text{ is divisible by } 2\pi, \\ 2\lfloor\frac{T}{2\pi}\rfloor+1 & \text{otherwise.} \end{cases} \tag{8.4}$$

8.3. Indices for closed Reeb orbits

In this section, (M, α) will denote a closed contact manifold of dimension $2n-1$ with contact form α. Moreover, we will assume that the contact form satisfies a certain genericity condition which we shall define now. The Reeb field of α is denoted by R and the contact structure by $\xi = \ker \alpha$.

Let γ be a closed Reeb orbit of period T, and take $p \in \gamma$. The time t flow of the Reeb field, which we shall write as

$$Fl_t^R : \ M \to M,$$

preserves the contact structure ξ. In fact, the Reeb flow preserves $d\alpha$ because of $\mathcal{L}_R \alpha = d i_R \alpha + i_R d\alpha = 0$. Hence we get a symplectic map of symplectic vector spaces

$$\psi_\gamma : \ (\xi, d\alpha)|_p \to (\xi, d\alpha)|_p,$$

defined by the restriction of $T_p Fl_T^R$ to $\xi|_p$.

Definition 8.2. The map ψ_γ is called the **linearized return map**. The closed Reeb orbit γ is said to be **non-degenerate** if the linearized return map has no eigenvalue equal to 1.

For now we require closed Reeb orbits to be non-degenerate. This imposes some conditions on the contact form, but they turn out to be rather mild, as we see in the following proposition.

Proposition 8.3. *There exists a C^∞-small perturbation α' of α such that all closed Reeb orbits of α' are non-degenerate.*

A proof of this proposition can for instance be found in [**5**]. In general, a perturbation, such as the one from this proposition, destroys a lot of symmetry that could be present in the contact form. We illustrate this in the following example, whose numerics will be used again later on. We consider the $(2n-1)$-dimensional sphere as a subset of $\mathbb{C}^n$,

$$S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | \ \sum_{i=1}^n |z_i|^2 = 1\}.$$

We can take the following contact form

$$\alpha = \frac{i}{2} \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j)|_{TS^{2n-1}}.$$

We can easily verify that the Reeb field and flow are given by

$$R_\alpha = i(z_0, \ldots, z_n) \text{ with flow } Fl_t^{R_\alpha}(z_1, \ldots, z_n) = (e^{it} z_1, \ldots, e^{it} z_n).$$

We see that all Reeb orbits are closed. Indeed, the time 2π flow of the Reeb field is the identity map on S^{2n-1}. In particular, all closed Reeb orbits are degenerate.

We can also take a perturbed contact form

$$\tilde{\alpha} = \frac{i}{2} \sum_{j=1}^n a_j (z_j d\bar{z}_j - \bar{z}_j dz_j)|_{TS^{2n-1}},$$

with the a_j linearly independent over $\mathbb{Q}$. The Reeb field has a similar form to the one before, and the flow is given by

$$Fl_t^{R_{\tilde{\alpha}}}(z_1, \ldots, z_n) = (e^{it/a_1} z_1, \ldots, e^{it/a_n} z_n).$$

In this case, all but one of the the coordinates of closed Reeb orbits must be zero, i.e. we have n closed Reeb orbits (if we do not count multiple covers), and the j^{th} one is given by

$$t \mapsto (0, \ldots, 0, \underbrace{e^{it/a_j}}_{\text{at } j^{\text{th}} \text{ position}}, 0, \ldots, 0).$$

This Reeb orbit has period $2\pi a_j$. If we take $t = 0$ as a starting point for each closed Reeb orbit, then we can compute the linearized return map. At these starting points, which are given by $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, the contact structure coincides with the complex tangent space to the sphere. Hence we see that the return map of the j^{th} closed Reeb orbit is given by the diagonal $((n-1)\times(n-1))$-matrix

$$\operatorname{diag}(e^{i2\pi a_j/a_1},\dots,\widehat{e^{i2\pi a_j/a_j}},\dots,e^{i2\pi a_j/a_n}),$$

where $\hat{}$ denotes an omitted term. All closed Reeb orbits of the perturbed contact form $\tilde\alpha$ are non-degenerate, but the contact form has now a different shape for different coordinates.

We will now assume that we have a **generic contact form** on M, i.e. a contact form whose closed Reeb orbits are non-degenerate.

8.3.1. Indices for homologically trivial Reeb orbits. Let γ be a closed Reeb orbit that is homologically trivial. We define its Conley-Zehnder index in the following way. First choose a Seifert surface S_1 for γ along with a trivialization of ξ on S_1. Next, we choose a framing of ξ on S_1,

$$\varphi_1:\ S_1\times\mathbb{R}^{2n-2}\to\xi|_{S_1}$$

(of course linear in the second slot). The path of symplectic matrices we are going to consider is the differential of the Reeb flow restricted to ξ,

$$\psi_1:=\varphi_1^{-1}\circ TFl_t^R|_\xi\circ\varphi_1(\gamma(0),\dots)$$

Now we put $\mu_{CZ}(\gamma;S_1)=\mu_{CZ}(\psi_1):=\mu_{CZ}(pr_2\circ\psi_1)$, where pr_2 denotes the natural projection to $\mathbb{R}^{2n-2}$. By naturality of the Conley-Zehnder index this is independent of the framing chosen on S_1. It is, however, not independent of the chosen Seifert surface. The dependence can be expressed relatively easily though. Choose another Seifert surface S_2 and denote the framing of ξ on S_2 by $\varphi_2:\ S_2\times\mathbb{R}^{2n-2}\to\xi|_{S_2}$. We define, as above,

$$\psi_2:=\varphi_2^{-1}\circ TFl_t^R|_\xi\circ\varphi_2(\gamma(0),\dots)=\varphi_2^{-1}\circ\varphi_1\circ\psi_1\circ\varphi_1^{-1}\circ\varphi_2(\gamma(0),\dots).$$

We want to relate ψ_1 and ψ_2 via a symplectic base change. Note that the symplectic matrix on the right, $\varphi_1^{-1}\circ\varphi_2$, is evaluated in $\gamma(0)$, whereas the one on the left, $\varphi_2^{-1}\circ\varphi_1$, is evaluated in $\gamma(t)$. In other words, the naturality property does not apply. In order to make use of our list of properties, we want to relate ψ_1 and ψ_2 through a loop of symplectic matrices starting at the identity. This can be guaranteed by choosing a different framing on S_2, obtained from the old framing by multiplying with the constant matrix $(\varphi_1^{-1}\circ\varphi_2)^{-1}$ in $\gamma(0)$. Let us denote this new framing by φ_2'. We have

$$\varphi_2'(x,v)=\varphi_2(x,(\varphi_1^{-1}\circ\varphi_2)^{-1}_{\gamma(0)}v).$$

Note that the Conley-Zehnder index of γ with respect to the framings φ_2 and φ_2' are the same by naturality of this index.

We write ψ_2' for the symplectic path associated to γ measured with respect to the trivialization φ_2'. As before, we have

$$\psi_2'={\varphi_2'}^{-1}\circ\varphi_1\circ\psi_1\circ\varphi_1^{-1}\circ\varphi_2'(\gamma(0),\dots).$$

Note that the symplectic matrix on the right, $\varphi_1^{-1}\circ\varphi_2'$, is now the identity in γ by virtue of our change of framing. Therefore we have $\psi_2'={\varphi_2'}^{-1}\circ\varphi_1\circ\psi_1$. Note that ${\varphi_2'}^{-1}\circ\varphi_1$ can be regarded as a loop of symplectic matrices that starts at the identity. By the loop property of the Conley-Zehnder index we can then write $\mu_{CZ}(\psi_2')=\mu_{CZ}(\psi_1)+2\mu_l({\varphi_2'}^{-1}\circ\varphi_1)$.

The term $\mu_L({\varphi_2'}^{-1}\circ\varphi_1)$ is the winding number of the trivialization of $\xi|_{S_2}$ with respect to the trivialization of $\xi|_{S_1}$ and hence it represents $\langle c_1(\xi),[S_1\cup S_2]\rangle$. The relation between the Conley-Zehnder indices of the two trivializations φ_1 and φ_2 can then be written as

$$\mu_{CZ}(\gamma,S_2)=\mu_{CZ}(\gamma,S_1)+2\langle c_1(\xi),[S_1\cup S_2]\rangle. \tag{8.5}$$

Once we have computed the Conley-Zehnder index of a Reeb orbit, we can define the degree of that Reeb orbit as

$$\deg(\gamma)=\mu_{CZ}(\gamma,S_1)+n-3.$$

REMARK 8.4. If we compute the Conley-Zehnder index (and hence the degree) in this way, we see that this index is not well-defined unless $c_1(\xi) = 0$ (see Formula (8.5)). In our applications this will actually be the case, but in general this is a problem that has to be taken into account. We will say more on this in Chapter 10. See also Remark 8.6. In principle, we do not need the term $n-3$, since we will only be using cylindrical contact homology, but the $n-3$ is needed for generalizations and we include it to stick with the usual conventions.

8.3.1.1. *A practical computation of the Conley-Zehnder index.* Let (M, ξ) be a contact manifold of dimension $2n-1$ with contact form α. Suppose the symplectic vector bundle $(\xi, d\alpha)$ is symplectically stably trivial, i.e.

$$(\xi \oplus \varepsilon^{2k}, d\alpha \oplus \omega_{2k}) \cong (\varepsilon^{2n+2k-2}, \omega_{2n+2k-2}) \text{ for some } k \in \mathbb{N},$$

where $(\varepsilon^{2k}, \omega_{2k})$ denotes the trivial symplectic vector bundle of rank $2k$. Then we might use the trivialization of $(\varepsilon^{2n+2k-2}, \omega_{2n+2k-2})$ instead of choosing a disk.

Consider a contractible closed Reeb orbit γ. We extend the linearized Reeb flow along γ to $(\xi \oplus \varepsilon^{2k}, d\alpha \oplus \omega_{2k})$. Let us denote the extended flow by Φ, the restriction of this extension to $(\varepsilon^{2k}, \omega_{2k})$ by Φ_{ξ^ω}, and the linearized Reeb flow by Φ_ξ.

Then we compute the Maslov index of the extended Reeb flow (note that we allow the extension to have degenerate endpoints). By the direct sum property of the Maslov index, we expect

$$\mu(\Phi) = \mu(\Phi_\xi \oplus \Phi_{\xi^\omega}) = \mu(\Phi_\xi) + \mu(\Phi_{\xi^\omega}).$$

Note that the Maslov index does not depend on the choice of trivialization of the symplectic vector bundle $(\varepsilon^{2n+2k-2}, \omega_{2n+2k-2})$. Indeed, if Φ' is the extended linear flow with respect to another trivialization, then there is a basis transformation along γ from one trivialization to the other, which can be extended to M. We write $\psi : S^1 = \mathbb{R}/\mathbb{Z} \to (\varepsilon^{2n+2k-2}, \omega_{2n+2k-2})$ for this basis transformation. The two paths of symplectic matrices are related by

$$\Phi'(t) = \psi(t)\Phi(t)\psi(0)^{-1}.$$

By the loop property (which is also valid for the Maslov index), we see that the Maslov indices of the two paths are related by

$$\mu(\Phi') = \mu(\Phi) + 2\mu_l(\psi) = \mu(\Phi).$$

The latter equality holds because the trivializations extend to the whole manifold M. In particular, they extend to a disk bounding the Reeb orbit γ, so the map ψ must be homotopic to the constant map. The same argument applies to the path Φ_{ξ^ω}. In particular, we can consider a trivialization of $(\xi, d\alpha)$ on a disk and extend this trivialization to $(\varepsilon^{2n+2k-2}, \omega_{2n+2k-2})$. Hence we can compute the Conley-Zehnder index of the Reeb orbit γ as

$$\mu(\Phi_\xi) = \mu(\Phi) - \mu(\Phi_{\xi^\omega}).$$

REMARK 8.5. Note that a contact structure ξ is symplectically stably trivial if and only if $\xi \oplus \varepsilon^2$ is trivial. See for instance Ustilovsky's thesis [**51**], Chapter 2, for this claim. Also observe that the total Chern class of a symplectically stably trivial contact structure is trivial.

8.3.2. Indices for homologically non-trivial Reeb orbits. Although we will only consider contractible Reeb orbits in our applications of contact homology (so we can actually take the Seifert surfaces to be disks in dimensions greater than 4), it is still interesting to define the index of a Reeb orbit in more general cases. For the sake of convenience we will restrict ourselves to the case that $H_1(M; \mathbb{Z})$ is torsion-free. We can choose curves $\delta_1, \ldots, \delta_r$ in M representing a *basis* of $H_1(M; \mathbb{Z})$. Here the torsion-free condition is important, because we can otherwise only assume the curves to represent a generating set of $H_1(M; \mathbb{Z})$. We also choose trivializations of ξ on each of the curves δ_i. We will call these curves δ_i **reference arcs**.

Any closed Reeb orbit γ will be either null-homologous or homologic to a (non-trivial) linear combination of the reference arcs $\delta_1, \ldots, \delta_r$. In the first case we proceed as in the previous section. In the second case we choose a surface S_γ that represents the homology between γ and a linear combination of the reference arcs. The trivializations $\xi|_{\delta_i}$ extend to the surface S_γ. Hence we get a surface with a trivialization of ξ. Since γ is in the boundary of the surface S_γ, we can then compute the Conley-Zehnder index in a way similar to the one from the previous section. The degree of a closed Reeb orbit is then similarly defined as

$$\deg(\gamma) = \mu_{CZ}(\gamma, S_\gamma) + n - 3.$$

REMARK 8.6. It is important to note that the Conley-Zehnder we compute this way is in general not well-defined, but depends on the choice of reference arcs and the homology between the closed Reeb orbits and the reference arcs. Therefore it is important to keep track of the choices made. When we introduce contact homology in Chapter 10 these choices will be part of the data and hence we can "compensate" in some sense for the fact that the Conley-Zehnder index is not well-defined. In other words, data like the surface realizing the homology between γ, the reference arcs and the homology between Reeb orbits and reference arcs shall be used later on.

In case $H_1(M;\mathbb{Z})$ is not torsion free, we can actually follow a similar strategy by choosing curves representing a basis of the free part and curves that represent a minimal generating set of the torsion part. The standard way to make this work (see for instance [**14**], section 2.9.1, or the lecture notes of Bourgeois, [**5**]) assigns a rational degree to a closed Reeb orbit.

8.3.3. Closing remarks on degrees. In our application of contact homology, the contact form has degenerate Reeb orbits, so computations as above cannot be performed directly. Since we have introduced the Maslov index as well, we can however compute that index following the same procedure as for non-degenerate orbits by choosing a Seifert surface or a homology to reference arcs if we replace the Conley-Zehnder index by the Maslov index. The degree is defined differently though, but we will say more on this in Chapter 10. Note that the Maslov index for non-degenerate Reeb orbits is the same as the Conley-Zehnder index.

From this point on, we will only consider manifolds without torsion in H_1. At some point, results could be made more general by restricting to contractible Reeb orbits instead of requiring torsion-free homology.

CHAPTER 9

Pseudo-holomorphic curves in symplectic cobordisms

In this chapter we will introduce pseudo-holomorphic curves in symplectic cobordisms. This is a rather large theory and we will not provide proofs of the statements we make. Some notions and theorems basically go back to Gromov's article on pseudo-holomorphic curves [**27**], but we will also use some of the more recently developed notions coming from symplectic field theory. In the end we are interested in a small part of symplectic field theory, namely contact homology, which, among other things, provides invariants of contact manifolds. We will describe that theory in the next chapter, but again it is a rather large theory and we will only provide sketch proofs.

Since our application of pseudo-holomorphic curves takes place in contact homology, we can think of holomorphic curves in symplectizations most of the time. However, the proof of invariance of contact homology requires holomorphic curves in more general cobordisms, so we will not restrict ourselves to symplectizations. The discussion here follows mostly the article of Eliashberg, Givental and Hofer [**14**], and some elements come from [**6**] and [**3**]. Other good references are [**44**] and [**33**].

In the following we will only consider manifolds with coorientable contact structure and from now on we will often only specify a contact manifold by a pair (V, α), where α is a contact form. Throughout the discussion we will often mention Morse homology. Morse homology can be regarded as a "prototype" for contact homology (and also Floer homology). In finite-dimensional Morse homology the technical aspects are much simpler than the technicalities of contact homology. Moreover, some of the main ideas of contact homology are already contained in Morse homology. We refer to the book of Schwarz [**47**] to cover the needed ideas.

9.1. Almost complex structures

In order to speak about pseudo-holomorphic curves in a symplectic manifold (M, ω), we need an appropriate complex structure on the tangent space of M. We will consider the following notion, but there are weaker notions which allow the study of pseudo-holomorphic curves.

Definition 9.1. (cf. Definition 2.14) An almost complex structure J on M is said to be **compatible** with ω if

1. $\omega(v, Jv) > 0$ for all $v \in TM$
2. $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in TM$.

Remark 9.2. If J is an almost complex structure compatible with ω, then $\omega(\dots, J\dots)$ is a Riemannian metric.

Definition 9.3. Let (Σ, j) be a Riemann surface with complex structure j and let (M, ω, J) be a symplectic manifold with compatible almost complex structure J. We say a map $f : (\Sigma, j) \to (M, J)$ is **pseudo-holomorphic** or (by slight abuse of notation) holomorphic if

$$J \circ Tf = Tf \circ j. \tag{9.1}$$

We see that this is a generalization of the Cauchy-Riemann equations. For instance, if we take (Σ, j) to be the standard complex plane and we take $M = \mathbb{R}^2$ with is standard symplectic and (almost) complex structure, then Equation (9.1) is just the standard Cauchy-Riemann equation. Sometimes we will use the term J-holomorphic map or (j, J)-holomorphic map instead of just pseudo-holomorphic. We shall do this in cases where we vary the almost complex structure on W or on S and W, respectively.

We rewrite Equation (9.1) and simultaneously introduce the delbar operator $\bar\partial$

$$\bar\partial f = \frac{1}{2}(Tf + J \circ Tf \circ j) = 0.$$

Like in complex analysis this is often a convenient way of writing the Cauchy-Riemann equations.

9.1.1. Symplectic cobordisms. Since we will consider curves in symplectic cobordisms, we want to use almost complex structures that respect the symplectic structure of a symplectic cobordism. Let $W = \overrightarrow{V^-V^+}$ be a symplectic cobordism between the contact manifolds (V^-, α^-) and (V^+, α^+). By definition, the symplectic form on W looks like $d(e^t\alpha^-)$ near the negative end $V^- \times (-\infty, 0]$ and like $d(e^t\alpha_+)$ near the positive end $V^+ \times [0, \infty)$. As in Section 3.3 the number t denotes the coordinate on the interval. We will consider almost complex structures J on W that are translation-invariant on the ends. By this we mean that J is invariant under the map $t \mapsto t \pm c$ ($-$ sign for the negative end and $+$ sign for the positive end) for large $|t|$ and $c > 0$. In the special case of a symplectization we require J to be globally translation invariant (not just near the ends).

If we restrict ourselves to a slice $V^\pm \times \{t\}$, we want in addition the contact structure on that slice, given by $\xi_\pm = \ker \alpha_\pm|_{V^\pm \times \{t\}}$, to be invariant under J. Furthermore, we will put

$$J\frac{\partial}{\partial t} = R_{\alpha_\pm}, \tag{9.2}$$

where $R_{\alpha_\pm}$ is the Reeb field of the contact form $\alpha_\pm$.

Note that one way to obtain such an almost complex structure on the symplectization of (V, α) is by starting with a complex structure $\tilde J$ on $\xi = \ker \alpha$ that is compatible with $d\alpha$. This complex structure can be extended to $T(V \times \mathbb{R})$ by requiring

$$J\frac{\partial}{\partial t} = R_\alpha.$$

Thus we can get an almost complex structure on $T(V \times \mathbb{R})$ that is compatible with $d(e^t\alpha)$.

9.2. Holomorphic curves

Let (V, α) be a contact manifold and let (W, ω) be its symplectization. We choose a suitable almost complex structure J for W following the discussion from the previous section. Let γ be a closed Reeb orbit in V. The vertical cylinder $\gamma \times \mathbb{R}$ is then a J-holomorphic curve in W, because we imposed the requirement (9.2) on J. In this example we see that the curve behaves nicely near the ends. We want to formalize this behavior in the following definition.

Definition 9.4. Let $f: D^2 - \{0\} \to W$ be a J-holomorphic map. We can write $f(r, \vartheta) = (f_V(r, \vartheta), f_{\mathbb{R}}(r, \vartheta)) \in V \times \mathbb{R}$ with polar coordinates on $D^2 - \{0\}$. We say f is **asymptotically cylindrical** over the closed Reeb orbit γ at $\pm\infty$ if

$$\lim_{r \to 0} f_{\mathbb{R}}(r, \vartheta) = \pm\infty$$

and

$$\lim_{r \to 0} f_V(r, \vartheta) = \tilde\gamma(\vartheta)$$

for a parametrization $\tilde\gamma$ of the Reeb orbit γ.

With this definition, we can also indicate what class of holomorphic curves we are mainly interested in. We consider rational curves that are asymptotically cylindrical over Reeb orbits, i.e. maps from the 2-sphere minus a number of punctures, where the behavior near the punctures is similar to the example of the vertical cylinder. We will also use the notation from this definition more often. The subscripts V and $\mathbb{R}$ indicate the projections from a curve to the V- and $\mathbb{R}$-component.

Note also that we can speak about asymptotically cylindrical holomorphic curves in general symplectic cobordisms by virtue of the structure of the ends (they look like part of a symplectization). In the following we will often talk about **positive** and **negative** punctures. This notion is used to indicate whether the Reeb orbit over which a holomorphic curve is asymptotically cylindrical near a puncture, is either at the positive end or the negative end of a symplectic cobordism.

The energy is an important quantity we can assign to a holomorphic curve. For the symplectization $W = V \times \mathbb{R}$ of the contact manifold (V, α) this notion is easiest to define. We choose the almost complex structure J on W as before. We denote the symplectic form on W by $\omega = d(e^t\alpha)$ and have the inclusion

$$\begin{array}{rcl} i: \; V & \to & V \times \mathbb{R} \\ x & \mapsto & (x, 0). \end{array}$$

Definition 9.5. Let $f: \; S \to W$ be a pseudo-holomorphic curve. We write $f = (u, a)$. Then the ω**-energy** of f is defined by

$$E_\omega(f) = \int_S a^* i^* \omega.$$

This energy are defined for more general two-forms ω as well. If we put $\omega = d(e^t\alpha)$, where α is a contact form on M, the ω-energy is also called **contact area**. Under suitable assumptions, the energy is finite and can be computed using Stokes' theorem. Indeed if $f: \; (S, j) \to (W, J)$ is a pseudo-holomorphic curve that is asymptotically cylindrical over the Reeb orbits Γ^- at the negative end and asymptotically cylindrical over the Reeb orbits Γ^+ at the positive end of W, then the contact area of f is equal to

$$E_\omega(f) = \int_{\Gamma^+} \alpha - \int_{\Gamma^-} \alpha.$$

Note that this expressions involves the integral of α along a curve. This quantity is known as the action and will play a role in Chapter 10. Note that the definition of contact area is very similar to the area of a holomorphic curve in Gromov's theory. However, the actual area of a holomorphic curve in a symplectization cannot be bounded. Hence the definition is modified to pick out only the "contact part" of the area and not the component in the t-direction.

Remark 9.6. There are also other notions of energy. The most important condition in later theorems will be boundedness of the energy. In [**3**] another, not equivalent notion of energy is used for these bounds. However, if we require the holomorphic maps to be proper, then boundedness of the notion we defined here and boundedness of energy in the sense of [**3**] are equivalent. See Lemma 5.15 of [**3**] for a proof of this claim. For this reason we need to include the condition of properness in some statements, which is often absent in the literature where the other notion of energy is used.

For more general symplectic cobordisms, we do the following. Let W be a symplectic cobordism, which we may write as

$$W = W^- \cup \bar{W} \cup W^+,$$

where $\bar{W}$ is the compact symplectic cobordism obtained from W by removing the ends, and $W^\pm$ are positive and negative ends of symplectizations, i.e.

$$V^- = V^- \times (-\infty \times 0], \text{ and } W^+ = V^+ \times [0, \infty)$$

The manifolds $V^\pm$ are contact manifolds. We have the inclusions

$$\begin{array}{rcl} i^\pm: \; V^\pm & \to & V^\pm \\ x & \mapsto & (x, 0) \end{array}$$

Definition 9.7. Let $f: \; S \to W$ be a pseudo-holomorphic map. We write its restriction to the ends of W as $F|_{V^\pm} = (u^\pm, a^\pm)$. The ω**-energy** of f is defined as

$$E_\omega(f) = \int_{f^{-1}(\bar{W})} f^*\omega + \int_{f^{-1}(W^-)} {u^-}^* {i^-}^* \omega + \int_{f^{-1}(W^+)} {u^+}^* {i^+}^* \omega.$$

All the energies we defined here can be shown to be non-negative, see Lemma 6.1 of [**3**]. On the other hand, in a symplectization a holomorphic curve can have contact area 0 if and only if that curve is either a vertical cylinder or a constant map.

Note that the energy of a holomorphic curve that is asymptotically cylindrical over closed Reeb orbits near all ends is always finite. In fact, there is a kind of converse for symplectizations, see [**14**].

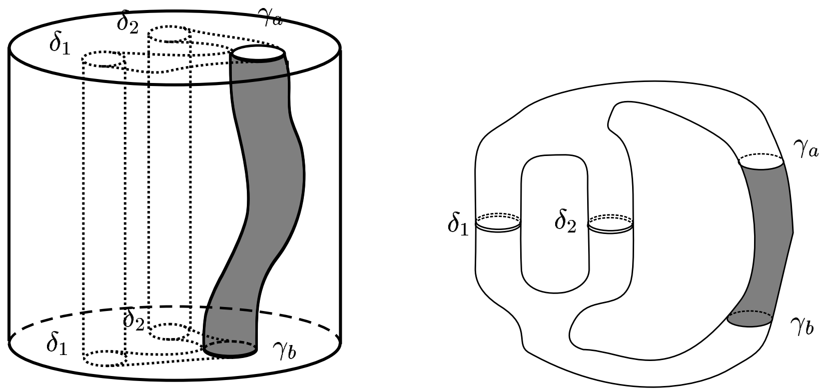

FIGURE 9.1. Homology class for a holomorphic curve

THEOREM 9.8. *Suppose that α is a generic contact form on M, i.e. all closed Reeb orbits are non-degenerate. Let $W = symp(M)$ denote the symplectization of (M, α). Let C be a non-compact Riemann surface without boundary and let $f : C \to W$ be a proper pseudo-holomorphic curve. Suppose the contact area of f is bounded. Then C is conformally equivalent to a punctured Riemann surface obtained from the closed Riemann surface S_g of genus g; we have $s^+ + s^-$ punctures, which can be divided in s^+ positive punctures and s^- negative punctures, such that the map f is asymptotically cylindrical over closed Reeb orbits near the positive punctures at $+\infty$ and that near the negative punctures the map f is asymptotically cylindrical over closed Reeb orbits at $-\infty$.*

9.2.1. Homology class of a holomorphic curve. In this section we assign a homology class to a holomorphic curve. At first this might seem a matter of book-keeping, but in later sections it will be vital to have this information.

Let $W = \overrightarrow{V^-V^+}$ be a symplectic cobordism between the contact manifolds (V^-, α_-) and (V^+, α_+). Let us assume that W is topologically trivial, i.e. $W \cong V^- \times \mathbb{R} \cong V^+ \times \mathbb{R}$. We will write V for both V^- and V^+ if we only need the topological structure of the underlying manifold. Choose a compatible almost complex structure as in Section 9.1. Let S be the 2-sphere with s^+ positive punctures, written $x^+ = \{x_1^+, \ldots, x_{s^+}^+\}$, and s^- negative punctures, denoted by $x^- = \{x_1^-, \ldots, x_{s^-}^-\}$. Let $f : S \to W$ be a holomorphic curve that is asymptotically cylindrical over Reeb orbits at all punctures.

In order to associate a homology class to f we need a closed surface. To that end we attach the surfaces we used to compute the Conley-Zehnder index of the Reeb orbits (see Section 8.3) to the holomorphic curve. We can cap off those punctures that are asymptotically cylindrical over homologically trivial Reeb orbits with their Seifert surfaces. For each of the other punctures the holomorphic curve is asymptotically cylindrical over a closed Reeb orbit γ that is not homologically trivial. In Section 8.3 we chose a surface S_γ that realized a homology between γ and a linear combination of the reference loops. We attach these surfaces to the holomorphic curve as well and hence we obtain a surface A in W with boundary equal to a linear combination of the reference loops. Note that A represents a homology between the reference loops at the positive end and the reference loops at the negative end. Hence we see that the projection of A to V is a closed surface, which represents a homology class. See Figure 9.1 for a graphical representation of this construction in case f is a holomorphic cylinder (a sphere with one positive and one negative puncture).

9.2.2. Moduli spaces of holomorphic curves. Let W be a symplectic cobordism and S a closed Riemann surface with set of "distinguished points" $\mathbf{p}$. These points can be punctures, which we denote by $\mathbf{x}^+ \cup \mathbf{x}^-$, but also **marked** points, which we denote by $\mathbf{m}$. At each of the punctures we also specify unit tangent vectors, which we will call **directions**. We use these to fix the parametrization of the Reeb orbits at the ends.

DEFINITION 9.9. We say a smooth holomorphic map $f : (S - (\mathbf{x}^+ \cup \mathbf{x}^-)) \to W$ is **stable** if

- Every component of S at which f is constant must have at least three distinguished points, punctures or marked points.
- f is not a vertical cylinder on at least one component of $S - \mathbf{p}$.

The first condition means that the Euler characteristic of such a component is negative. The reason to put this in is to ensure that the automorphism group of the component is finite. To put this into perspective, one should keep in mind that the group of biholomorphisms of S^2 is triple-transitive. The stability condition might seem a bit strange to impose at first. It will, however, play an important role, as the stability will guarantee finiteness in a later construction. Originally, this notion (only the first) was introduced by Kontsevich for closed holomorphic curves. In that case, it is in fact necessary to ensure that the moduli spaces are Hausdorff.

Let W be a symplectic cobordism with almost complex structure J. By Γ^- we mean a finite set of closed Reeb orbits on the negative end of W and by Γ^+ we mean a finite set of closed Reeb orbits on the positive end of W. Let us denote by $\mathcal{M}^A_m(\Gamma^+, \Gamma^-; W, J)$ classes of maps f that satisfy the following.

- f is a stable (j, J)-holomorphic map that sends $S - (\mathbf{x}^+ \cup \mathbf{x}^-)$ to W. Here (S, j) is a closed Riemann surface. The sets $\mathbf{x}^+$ and $\mathbf{x}^-$ are positive and negative punctures.
- f is asymptotically cylindrical over the Reeb orbits $\gamma^+ \in \Gamma^+$ at the positive end at the punctures $\mathbf{x}^+$. Similarly we require f to be asymptotically cylindrical over the Reeb orbits of Γ^- at the negative end near the punctures $\mathbf{x}^-$.
- The homology class of f in the sense of the previous section is given by A.
- Given maps $f : (S - (\mathbf{x}^+ \cup \mathbf{x}^-), j) \to (W, J)$ and $f' : (S' - (\mathbf{x}'^+ \cup \mathbf{x}'^-), j') \to (W, J)$ that satisfy the above, we say f and f' are equivalent if there is a biholomorphism $\varphi : S \to S'$ such that $f = f' \circ \varphi$.

At each of the punctures we can specify directions which fix the parametrization of the Reeb orbits in the following sense. At a positive puncture p that is asymptotically cylindrical over the Reeb orbit γ with direction v we require that

$$\lim_{t \downarrow 0} f \circ \delta(t) = \gamma(0) \times \{\infty\}$$

for paths δ with $\delta(0) = p$ and $\frac{d\delta}{dt}(0) = v$. Note that for an m-fold covered Reeb orbit γ_m there are m distinct parametrizations; we can rotate by m^{th} roots of unity around a puncture and get the same image in W. In other words, we get an action by $\mathbb{Z}_m$ on the Reeb orbits and also on the moduli space whose elements are asymptotic over γ_m at one of their punctures. For now, this action does not carry any additional information, but it will be of importance when we consider orientations of the moduli spaces.

Later on, we will be mostly interested in curves in symplectizations W. In that case we have a global $\mathbb{R}$-action on W which respects the additional structure like the almost complex structure J. Hence the moduli space inherits this $\mathbb{R}$-action and we will consider $\mathcal{M}^A(\Gamma^+, \Gamma^-; W, J)/\mathbb{R}$ instead.

We would also like to remark on the marked points. The main purpose here is to illustrate the stability condition. Note that for higher genus curves the stability condition can be modified. One needs fewer marked points to get a negative Euler characteristic (and a finite automorphism group). Besides this illustration, we shall not use the marked points directly in this chapter, but they can be used to make holomorphic curves more rigid. For instance, one can eliminate families of equivalent curves by adding marked points. Since the equivalence relation we introduced required marked points to be mapped to marked marked points under biholomorphisms, this translates to "smaller" equivalence classes. Besides being a theoretical tool, marked points can also be used to

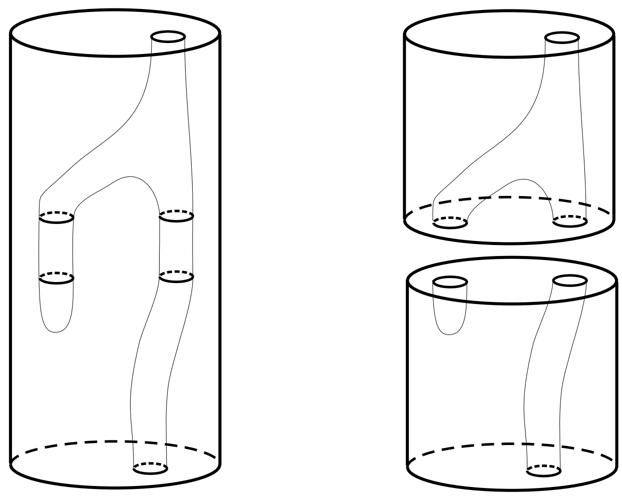

FIGURE 9.2. Breaking of a cylinder

define more sophisticated versions of contact homology, leading to finer invariants than the ones that we consider.

9.2.3. Convergence of holomorphic curves. The moduli spaces we introduced in the previous section are not compact, so we will describe the compactification here. The behavior of a sequence of stable maps can be described in a few steps. The picture we can keep in mind is that besides the phenomena we can expect from Gromov compactness, curves can "break" in a way that is similar to the broken gradient flow in Morse homology, see Figure 9.2.

To be more precise, we recall the definition of "cusp" or nodal curves which appear in Gromov compactness. Let (S, j) be a Riemann surface with an even number of distinct marked points, which we shall call **special points**. We denote these special points by D and we require them to come in pairs. Hence we may write $D = \{(c_1, d_1), \dots, (c_k, d_k)\}$. We define the **nodal surface coming from** (S, j) **and** D as

$$S_D = S/\{c_i \sim d_i \text{ for } i = 1, \dots, k\}.$$

So the nodal surface S_D is formed from the components of S by gluing them along the pairs in the set D. Because special points are all distinct, the nodal surface S_D is a possibly singular surface with at most double points as singularities. The standard example of a nodal curve would be the case $S = S^2 \cup S^2$ with a special point on each S^2. The nodal curve consists of two spheres glued together at one point, see Figure 9.3. Alternatively, we can describe a nodal surface as a Riemann surface $(\tilde{S}, \tilde{j})$ with a finite set of disjoint circles $\gamma_1, \dots, \gamma_k$. By collapsing the circles to points we obtain the nodal surface (the collapsed circle corresponds of course to the double point singularity in the previous picture. See for instance [**29**] for the latter description. We will call the singular points of a nodal curve **nodes**.

We restrict ourselves to topologically trivial cobordisms, i.e. symplectic cobordisms that are diffeomorphic (not necessarily symplectomorphic) to a symplectization of the contact manifold V. The following definitions come from [**6**], with some elements from [**3**]. Since we consider topologically trivial cobordisms $V \times \mathbb{R}$, we can always write a map f from a Riemann surface S to $V \times \mathbb{R}$ as a pair $(u, a) \in V \times \mathbb{R}$. We use this notation in the following where we will always denote holomorphic maps by f, possibly with an index. The pair (u, a) inherits this index.

DEFINITION 9.10. Let (S, j) be a nodal curve. A holomorphic map of **height (or level)** k in the symplectization of V is a triple (S, j, f) consists of the following data.

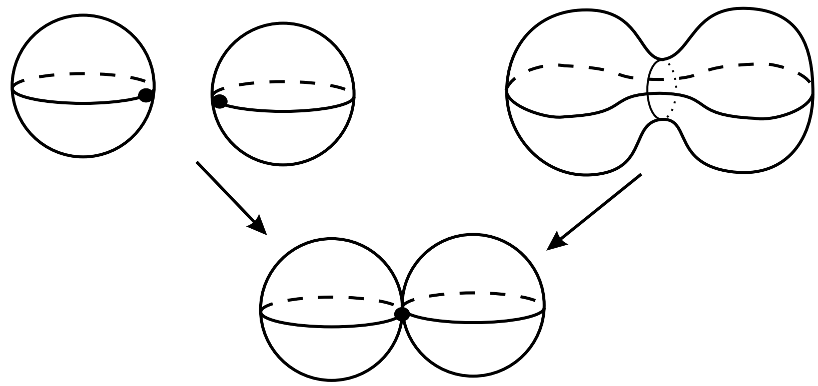

FIGURE 9.3. Getting a nodal curve by identifying points and by collapsing circles

- The connected components of $\tilde{S} = S - \{\text{singular points}\}$ are labelled by $\{1, \dots, k\}$. We refer to these labels as levels and require that levels of two components of $\tilde{S}$ differ by at most 1 if the closure of the two components share a node. We use $S^{(i)}$ to denote the union of connected components of $\tilde{S}$ with level i.
- We have proper holomorphic maps $f^{(i)} : (S^{(i)}, j) \to (V \times \mathbb{R}, J)$ with bounded energy for $i = 1, \dots, k$ making up f. In addition we require that each node shared by $S^{(i)}$ and $S^{(i+1)}$ is a positive puncture for $f^{(i)}$ at which $f^{(i)}$ is asymptotic to a closed Reeb orbit γ and a negative puncture for $f^{(i+1)}$ at which $f^{(i+1)}$ is asymptotic to the same Reeb orbit γ such that $f^{(i)}$ extends continuously across the nodes from $S^{(i)}$.

Height k curves are also referred to as **holomorphic buildings**. To stick with this terminology, we will sometimes call $f^{(i)} : S^{(i)} \to W$ the **floors** of the holomorphic building. The definition of height k curves in more general symplectic cobordisms W is obtained by splitting W into cobordisms $W_1, \dots, W_k$ such that $W = W_1 \circledcirc \cdots \circledcirc W_k$. The map $f^{(i)}$ in the above definition should be interpreted as a map to W_i. Note that we can perform this splitting procedure for a symplectization W in such a way that the split components $W_1, \dots, W_k$ are all symplectizations. Of course, this definition can also be used for symplectic cobordisms that are not topologically trivial.

The notion of stability for height k curves is similar to the notion we defined previously, basically imposing the stability condition on each level.

DEFINITION 9.11. We say a height k holomorphic map (S, j, f) to $(V \times \mathbb{R}, \omega)$ is **stable** if the connected components of $S^{(i)}$ on which $f^{(i)}$ is constant have negative Euler characteristic (after removing marked points) or $E_\omega(f^{(i)}) > 0$.

Negative Euler characteristic implies that the automorphism group of such a component is finite. This is important for the moduli space to be well-behaved. We already mentioned that $E_\omega(f) \geq 0$ with equality only holding if f is a vertical cylinder or a constant map.

We now come to the definition of convergence of stable curves of height k. We can keep convergence in the sense of Gromov in mind, since for height 1 the two notions coincide. Here is the definition for a symplectization.

DEFINITION 9.12. A sequence of stable curves of height k given by (S_n, j_n, f_n), $n \in \mathbb{N}$, is said to converge to a stable curve (S, j, f) of height $k' \geq k$ if there is a sequence of maps $\varphi_n : S_n \to S$ and sequences $t_n^{(i)} \in \mathbb{R}$ for $i = 1, \dots, k'$, such that

- the maps φ_n are diffeomorphisms outside a (possibly empty) set of circles C_n. Distinct circles from C_n are mapped to distinct nodes in S. Away from these nodes of S, the sequence $(\varphi_n)_* j_n$ of complex structures converges to j.
- The sequences $(u_n \circ \varphi_n^{-1}, t_n^{(i)} + a_n \circ \varphi_n^{-1}) : S^{(i)} \to V \times \mathbb{R}$ converge in the C^∞-topology to $f^{(i)} : S^{(i)} \to V \times \mathbb{R}$ on every compact subset of $S^{(i)}$ for $i = 1, \dots, k'$.
- For each node p of S between adjacent levels, we can consider a sequence of curves $\gamma_n : (-\varepsilon, \varepsilon) \to S_n$ intersecting $\varphi_n^{-1}(p)$ transversely at $t = 0$ and satisfying $\varphi_n \circ \gamma_n = \gamma$ for all n. Then $\lim_{t\downarrow 0} u(\gamma(t)) = \lim_{t\uparrow 0} u(\gamma(t))$.

In case we are considering general symplectic cobordisms, we can generalize the definition by using the splitting of a cobordism we mentioned earlier. We also need to interpret the second point in the definition in another way. In general symplectic cobordisms there is no translation invariance, but we can still use the structure at the ends of a symplectic cobordism.

We briefly mention how these notions will appear in contact homology. We might think of contact homology as a Morse homology where the critical points are given by closed Reeb orbits and the gradient flow is given by holomorphic curves that are asymptotic to closed Reeb orbits. Index 2 gradient flow lines in Morse homology form a one dimensional manifold whose ends can be described by broken gradient flow lines. Stable curves of height 2 are the analogue of this in contact homology.

In a similar spirit we shall sometimes speak about broken holomorphic curves. By that we mean stable curves of height $k > 1$ (actually $k = 2$ for our purposes).

9.2.4. Compactness of the moduli space. If we want to work with compact moduli spaces, we should, of course, at least add the stable curves of height k for all k to the moduli space. Let W be a symplectic cobordism that is topologically trivial and choose a suitable almost complex structure J. We write $\bar{\mathcal{M}}(W)$ for the moduli spaces of pseudo-holomorphic curves of any level. In case we consider pseudo-holomorphic curves with specified asymptotics we will write as before $\bar{\mathcal{M}}(\Gamma^+, \Gamma^-, W)$, where Γ^+ and Γ^- denote the Reeb orbits at the positive and negative end, respectively. The bar stands for adding stable curves of all heights k.

That these moduli spaces are indeed compact under certain assumptions, has been established in [**3**]. We have

THEOREM 9.13 (Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder). *For every $E > 0$, the space $\{f \in \bar{\mathcal{M}}(W) \mid E(f) \leq E\}$ is compact.*

There are much more general versions of this theorem, but this suffices for our needs. We rephrase the compactness result to stress its importance. Suppose that we are given a sequence of stable height k curves f_n such that $E(f_n) \leq E$. Then there is a subsequence f_{n_i} which converges to a stable curve of height $k' \geq k$ in the sense of Definition 9.12. In other words, the notion of stable curves suffices to describe all elements in the compactified moduli space.

9.2.5. The structure of the moduli space. In this section we will discuss the elements which form the heart of the matter together with the compactness results. In order to keep things brief, we will be sketchy.

We start by giving another description of the moduli space of maps from a Riemann surface S to an almost complex manifold (W, J). We first give a rough idea, whose origins lie in Floer homology, and then indicate what complications appear and what modifications of the setup must be made. In particular, we should inform that reader that the description that we will give at first serves to illustrate the main ideas. As given, the actual description only works in a few situations.

Let $\mathcal{B}(S, W, \Gamma)$ be the space of stable maps from S to W asymptotic over the Reeb orbits Γ. There is a vector bundle $\mathcal{E}$ over $\mathcal{B}(S, W, \Gamma)$, such that the fiber at $[u] \in \mathcal{B}(S, W, \Gamma)$ is isomorphic to the space of $(0,1)$-forms on S with values in (u^*TW), i.e. the space of sections $\Gamma(S, \Omega^{0,1}(u^*TW))$. Since the delbar operator $\bar{\partial}_J$ acting on a map u gives such a $(0,1)$-form with values in u^*TW, we can regard $\bar{\partial}_J : \mathcal{B}(S, W, \Gamma) \to \mathcal{E}$ as a section of the bundle $\mathcal{E}$. The zero set of this section describes the J-holomorphic maps and therefore the moduli space we are interested in.

REMARK 9.14. Some comment is needed on a proper description of the above setup. Since we require that maps converge to closed Reeb orbits near the punctures, this needs to be included in the data in a suitable way. One can, for instance similar to the setup of Floer homology in the monotone case, try to consider the Banach manifold of maps with prescribed asymptotics. In order to still have a norm without compactness of W, we need to include asymptotic weights. The bundle $\mathcal{E}$ is a Banach bundle in this setup. Doing this properly complicates the discussion and we will not go into this. Furthermore, there are additional complications which we want to show before indicating how to tackle these.

Using the implicit function theorem we can try and show that this zero set is a smooth manifold. To apply that theorem, we need that the linearization of $\bar{\partial}_J$ at a holomorphic curve u is a surjective Fredholm operator. We use that $T_{(u,0)}\mathcal{E} \cong T_u\mathcal{B}(S,W,\Gamma) \oplus \mathcal{E}_u$ and consider the projection of the linearized Cauchy-Riemann operator to $\mathcal{E}_u$, which we denote by

$$\bar{\partial}_u : \ T_u\mathcal{B}(S,W,\Gamma) \to \mathcal{E}_u.$$

From now on we will also call this projection the linearized Cauchy-Riemann operator, since it contains all information we need. From [**9**], Proposition 2.10, we have the following expression for $\bar{\partial}_u$,

$$\bar{\partial}_u \xi = \nabla_s \xi + J(u)\nabla_t \xi + (\nabla_\xi \tilde{J}(u))u_t.$$

Here ∇ denotes the covariant derivative with respect to the natural metric on the symplectization,

$$g_{sym} = dt \otimes dt + \alpha \otimes \alpha + d\alpha(\ldots, J\ldots).$$

One still needs to verify the Fredholm property of $\bar{\partial}_u$. A proof of this property can be found in [**9**], Theorem 3.6.

9.2.5.1. *Index of $\bar{\partial}_u$ and Conley-Zehnder indices of Reeb orbits.* The expression for the linearized operator can be written in a nice form near the punctures. Let us take the case of a positive puncture where the solution u of the Cauchy-Riemann equation is asymptotic to the closed Reeb orbit γ. We can choose a unitary trivialization of TW on u near the positive puncture,

$$\Phi : \ \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n} \quad \to \quad TW|_u.$$

We require this map to be linear on the $\mathbb{R}^{2n}$-factor. By a unitary trivialization of $TW|_u$ we mean a trivialization which maps the symplectic structure ω and almost complex structure J on $TW|_u$ to the standard structures ω_0 and J_0 on $\mathbb{R}^{2n}$. With respect to a unitary trivialization, the linearized operator can be written as

$$\partial_s + J_0 \partial_t + S(t). \tag{9.3}$$

Here we can assume that S is represented by symmetric matrices. This expression is also useful for establishing a relation between the Conley-Zehnder indices of Reeb orbits and the index of the linearized operator. We denote the limit $\lim_{s\to\infty} \Phi(s,t)$ by $\Phi^+(t)$. Since we know that u converges to a closed orbit at infinity, we may consider

$$\Psi(t) = \Phi^+(t)^{-1}(TFl^R_t(x_0))\Phi^+(0).$$

This is a path of symplectic matrices. We can compute its derivative and find (cf. [**45**], Section 5)

$$\dot{\Psi} = J_0 S \Psi. \tag{9.4}$$

This relation is needed to establish the connection between the Conley-Zehnder index of the Reeb orbits and the index of (9.3). We will use this relation later on to derive some information on the Cauchy-Riemann operator in a particular case.

REMARK 9.15. Note that S is determined by the Reeb orbit γ. In other words, the asymptotic form of the linearized Cauchy-Riemann operator is determined by the “boundary” conditions.

REMARK 9.16. We see again that the Conley-Zehnder index of a Reeb orbit need not be well-defined. The choice of trivialization Φ might give another index than the trivialization we have chosen before in Section 8.3.

9.2.5.2. *Dimension formula.* First we define some notions that are needed in our discussion.

DEFINITION 9.17. Let (S, j) be a Riemann surface and (W, J) a symplectic cobordism with compatible almost complex structure J. We say a pseudo-holomorphic curve $u: S \to W$ is **multiply covered** if there exists a Riemann surface (S', j'), a pseudo-holomorphic curve $u': S' \to W$ and a holomorphic branched covering $\varphi: S \to S'$ such that

$$u = u' \circ \varphi \text{ with } \deg(\varphi) > 1.$$

A pseudo-holomorphic map is said to be **simple** if it is not multiply covered. We say a pseudo-holomorphic map $u: S \to W$ is **somewhere injective** if there exists a point $z \in S$ such that

$$u^{-1}(u(z)) = \{z\} \text{ and } T_z u \neq 0.$$

If $\bar{\partial}_u$ surjective, then the $\bar{\partial}_J^{-1}(0)$ is a smooth submanifold of dimension given by the Fredholm index of $\bar{\partial}_u$. This set can still be empty though.

A nice argument to compute the Fredholm index is presented in [**6**]. From there we have the following dimension formula for the moduli space (if we have transversality)

$$\dim \mathcal{M}^A(\Gamma^+, \Gamma^-; W, J) = \sum_{j=1}^{s^+} \mu_{CZ}(\gamma_j^+) - \sum_{i=1}^{s^-} (\mu_{CZ}(\gamma_i^-) + n - 3) + (n-3)(2 - s^+) + 2\langle c_1(J), A\rangle. \tag{9.5}$$

We will often refer to this index as the virtual dimension of the moduli space. Note that for one positive puncture, one negative puncture and $c_1(J) = 0$, this formula has the form

(degree at the positive puncture) – (degree at the negative puncture),

which is very similar to the dimension of the space of gradient trajectories in Morse homology.

Up to this point, the construction of the moduli spaces is similar to the one in Floer homology in the monotone case. The next step in that case would be perturbation of the almost complex structure and/or the Hamilton function. This way we can always achieve transversality in Floer homology; in our situation there are some cases where this works as well. Indeed, after a suitable perturbation of J, the moduli space of simple holomorphic maps in a symplectization is a smooth manifold whose dimension is given by the above formula. This is due to the following theorem of Dragnev [**9**].

THEOREM 9.18. *Let $\mathcal{B}_s^A(\Gamma^+, \Gamma^-)$ denote the set of somewhere injective, finite energy curves asymptotic to Γ^+ near the positive punctures and asymptotic to Γ^- near the negative punctures. Then $\mathcal{B}_s^A(\Gamma^+, \Gamma^-)$ carries the structure of a separable Banach manifold.*

The moduli space of somewhere injective holomorphic curves can be seen as a subset of $B_s^A(\Gamma^+, \Gamma^-)$. The Sard-Smale theorem shows that regular almost complex structures J are generic, and hence we can obtain transversality for such a generic choice of J. As a result, the moduli space of somewhere injective curves carries the structure of a smooth manifold with dimension given by the above formula. A proof of a similar statement for closed holomorphic curves can be found in Section 3.2 of [**33**]. Both proofs rely on the assumption that curves are somewhere injective.

Unfortunately, if we drop the latter assumption, we cannot, in general, achieve transversality by perturbing the almost complex structure. We now give a simple example of this fact. Let us consider the case of a holomorphic cylinder, i.e. $s_+ = s_- = 1$ in the above dimension formula. Suppose for instance that $\mu_{CZ}(\gamma^+) - \mu_{CZ}(\gamma^-)$ is large enough so that

$$\dim \mathcal{M}^A(\gamma^+, \gamma^-; W, J) = \mu_{CZ}(\gamma^+) - \mu_{CZ}(\gamma^-) + 2\langle c_1(J), A\rangle \geq 0$$

for a certain fixed homology class A. Assume that this moduli space is non-empty, so we can take $u \in \mathcal{M}^A(\gamma^+, \gamma^-; W, J)$. Now we might consider positive multiples of the homology class A, which we will denote by NA with $N \in \mathbb{N}$. For $N > 1$, the expected dimension of $\mathcal{M}^{NA}(\gamma_N^+, \gamma_N^-; W, J)$ could be different from the dimension of $\mathcal{M}^A(\gamma^+, \gamma^-; W, J)$ (this depends on the precise data). We use γ_N to denote the N-fold covering of the Reeb orbit γ.

However, there exists a covering of the cylinder of degree N, for instance $z \mapsto z^N$ if we use $\mathbb{C} - \{0\}$ as a model for a cylinder. Let us take such a map φ_N. Then the composition $u \circ \varphi_N$ represents an element in $\mathcal{M}^{NA}(\gamma^+, \gamma^-; W, J)$, and we see that all elements of $\mathcal{M}^A(\gamma^+, \gamma^-; W, J)$ give rise to an element of $\mathcal{M}^{NA}(\gamma_N^+, \gamma_N^-; W, J)$.

In particular, if the dimension of $\mathcal{M}^A(\gamma^+, \gamma^-; W, J)$ does not coincide with the dimension of $\mathcal{M}^{NA}(\gamma_N^+, \gamma_N^-; W, J)$ (more generally, we should take the dimension of the space of covering maps of degree N into account), we cannot expect $\mathcal{M}^{NA}(\gamma_N^+, \gamma_N^-; W, J)$ to carry the structure of a smooth manifold. Even in cases where the moduli space is smooth, the virtual dimension might not coincide with the actual dimension. These arguments show that we cannot always obtain transversality by perturbing J. Indeed, as we mentioned earlier, our description sofar does not work in all cases.

9.2.5.3. *Some remarks about multivalued perturbations.* In the previous section we mentioned that transversality cannot always be obtained by perturbing the almost complex structure. Therefore we need another kind of perturbation. In principle we could try and perturb the complex structure on the Riemann surface as well. In the literature, it is more usual though to consider inhomogeneous perturbations, i.e. perturbations by sections ν, so that we consider the inhomogeneous Cauchy-Riemann equation

$$\bar{\partial}_J u = \nu.$$

This allows us in fact to keep some symmetry. For instance, this way it is not necessary to perturb a symmetric almost complex structure J, whereas a symmetric almost complex structures would often not be admissible in our previous description. We have so far neglected to mention one of the features of the setup explicitly, namely equivariance of $\bar{\partial}_J$ under the automorphism group of the Riemann surface. Because of the stability condition, this automorphism group is always finite. If $u \in \mathcal{B}(S, W, \Gamma)$ and $\varphi \in \mathrm{Aut}'(S)$, then

$$\bar{\partial}_J(u \circ \varphi) = \varphi^* \bar{\partial}_J(u).$$

We require the perturbation to respect this structure. Now we really need to specify the space of maps, about which we have been rather unspecific up to this point. In the setup for Floer homology in the monotone case and also in Dragnev's case of somewhere injective maps, the space $\mathcal{B}$ of maps carries the structure of a smooth Banach manifold.

This is no longer true in our case. We consider the space of stable maps into a symplectic cobordism with prescribed asymptotics. This space (or at least a neighborhood of the moduli space of holomorphic curves in this space) can be endowed with the structure of a Banach orbifold, similar to [**35**]. That paper gives some details for closed holomorphic curves in symplectic manifolds. A similar problem arises there if one does not restrict to somewhere injective curves.

Roughly speaking, the orbifold structure comes from the group of automorphisms of a holomorphic curve. This group is finite due to the stability condition. Indeed, some curves might have a non-trivial automorphism group, whereas other nearby curves lack that symmetry and have only a trivial group of automorphisms. Hence the uniformization charts for the orbifold can be thought of as follows. We can choose a parametrization $\tilde{f}$ for a stable map $f \in \mathcal{B}$. We can define $\tilde{U}$ as a neighborhood of $\tilde{f}$ consisting of parametrized stable maps that are close to $\tilde{f}$ in a suitable metric. Let us denote the automorphism group of the curve f by G. The action of G on $\tilde{f}$ can be extended to a possibly shrunken $\tilde{U}$ such that $\tilde{U}/G$ can be identified with a neighborhood of f in the space of stable maps $\mathcal{B}$. If we call this identification π_G, then the uniformization chart around f is given by $(\tilde{U}, G, \pi_G)$.

Similarly, the bundle $\mathcal{E}$ mentioned earlier can also be defined, but it is an orbibundle. For these bundles there is also a notion of a Fredholm section. Indeed, the Cauchy-Riemann operator can be regarded as such a section. The moduli space of holomorphic curves is still the zero set of this section, but it will in general be a singular space, since the ambient space $\mathcal{B}$ is no longer smooth. We should remark here that the moduli space does not in general carry the structure of an orbifold, but rather that of a branched "manifold". A more precise description of the structure of the moduli space requires a more thorough discussion about the virtual cycle language from [**35**], for instance.

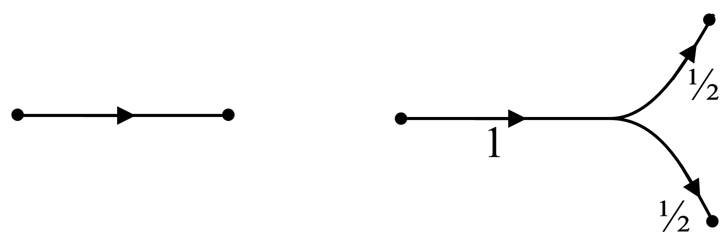

FIGURE 9.4. Examples of 1-dimensional moduli spaces; one that carries a smooth structure and one that does not.

Also note that we still need some argument to obtain transversality (indeed, outside the singular points of $\mathcal{B}$ the situation is as before). Since we are now dealing with orbibundles, the perturbations in general have to be multivalued.

In Morse homology (or early versions of Floer homology), one of the arguments to show that the theory is a homology theory boils down to the following. The boundary of smooth, compact 1-manifolds is zero with orientation. We would like to mimic that argument, but since the moduli spaces are no longer smooth, there can be an odd number of boundary points. On the other hand, we need to take the automorphism group of the curves into account.

One can assemble the data of the moduli spaces needed for later applications in contact homology into the so-called virtual moduli cycle. This is a rational homology cycle relative to the boundary representing the moduli space, i.e. its fundamental class. The elements in this cycle are weighted by $1/|G|$, where $|G|$ is the order of the automorphism group. The idea is that by taking these weights into account, we can still mimic the argument from Morse homology, see Figure 9.4. The arrows in that figure indicate an orientation that we will discuss in the next section. Also notice that we have added weights in the non-smooth case. The idea is, that if we count the oriented boundary with these weights, we get zero just as in the smooth case. As a result, the differential in contact homology will involve rational coefficients.

9.2.6. Gluing and orientation of the moduli space. In this section we will describe two important ingredients for later constructions. In order to simplify the discussion we consider the case of one positive and one negative puncture. That particular case was done by Floer and Hofer [**16**] and suffices for our needs, since we will consider cylindrical contact homology, where only curves with one positive and one negative puncture are considered.

9.2.6.1. *Gluing.* Let us describe the gluing procedure, which we use to glue two holomorphic curves with matching asymptotics. This gluing procedure can also be performed at the level of Cauchy-Riemann operators. We consider maps from a cylinder into a symplectic cobordism W that satisfy the Cauchy-Riemann equation. We require convergence of the map to a Reeb orbit at both the positive and the negative puncture. We already saw in Formula (9.3) that the Cauchy-Riemann operator can be given a specific form near the punctures, and we will consider operators that have such a form. We write $\mathcal{O}(\gamma^+,\gamma^-)$ for the space of Fredholm operators that have the form of Formula (9.3), where the behavior near the positive puncture and negative puncture is determined by γ^+ and γ^-, respectively, in the sense of Remark 9.15. We will use $\mathcal{O}$ to denote the space of all such operators,

$$\mathcal{O} = \bigcup_{\gamma^+,\gamma^-} \mathcal{O}(\gamma^+, \gamma^-).$$

The space of operators $\mathcal{O}(\gamma^+, \gamma^-)$ will play an important role for the orientation which we describe below, but we will first continue our discussion of the gluing procedure, which is due to Floer. For simplicity, let us consider the case of a symplectization of the contact manifold (V, α) with $c_1 = 0$. Then there is a well-defined grading and we need not concern ourselves with the homology class of the holomorphic curves. Suppose we are given holomorphic cylinders $u \in \mathcal{M}(\gamma_a, \gamma_b)/\mathbb{R}$ and $v \in \mathcal{M}(\gamma_b, \gamma_c)/\mathbb{R}$ and that both moduli spaces are 0-dimensional, so

$$\mu_{CZ}(\gamma_a) - 1 = \mu_{CZ}(\gamma_b) = \mu_{CZ}(\gamma_c) + 1.$$

In the sense of Definition 9.10, the curves u and v are the levels of a broken cylinder of height 2. We want to show that this broken cylinder lies in the boundary of the 1-dimensional moduli space $\mathcal{M}(\gamma_a, \gamma_c)/\mathbb{R}$. With some abuse of notation, we will denote the parametrizations of the holomorphic curves u and v by the same letters. Moreover, we can parametrize them such that their directions "match", i.e.

$$\lim_{t\to-\infty} u_V(t, \varphi) = \lim_{t\to\infty} v_V(t, \varphi) = \gamma_b(\varphi).$$

The next step is define an approximation of the gluing of the holomorphic cylinders for every $R > 0$. It turns out that for large R this approximation can be deformed into a genuine holomorphic cylinder. Before we give the approximation, let us first define the auxiliary functions $\zeta(t, \varphi)$ and $\eta(t, \varphi)$ by the conditions

$$\begin{aligned} u_V(t, \varphi) &= exp_{\gamma_b(\varphi)}\zeta(t, \varphi), \\ v_V(t, \varphi) &= exp_{\gamma_b(\varphi)}\eta(t, \varphi). \end{aligned}$$

We also need to choose a smooth cutoff-function $\beta(t)$ that is identically 0 for $t \le -1$, and identically equal to 1 for $t \ge 1$. Let us write Floer's approximation to the glued cylinders as $\tilde{w}_R = (\tilde{w}_V, \tilde{w}_\mathbb{R})$, where

$$\tilde{w}_V = \begin{cases} v_V(t + R, \varphi) & \text{for } t \le -1, \\ exp_{\gamma_b(\varphi)}(\beta(t)\zeta(t - R, \varphi) + (1 - \beta(t))\eta(s + R, \varphi)) & \text{for } t \in [-1, 1], \\ u_V(t - R, \varphi) & \text{for } t \ge 1, \end{cases}$$

and

$$\tilde{w}_\mathbb{R} = \begin{cases} v_R(t, \varphi) & t \le -1 \\ \beta(t)u_R(t, \varphi) + (1 - \beta(t))v_R(t, \varphi) & t \in [-1, 1] \\ u_R(t, \varphi) & t \ge 1 \end{cases}$$

with the abbreviations $v_R(t, \varphi) = v_\mathbb{R}(t+R, \varphi) - v_\mathbb{R}(R, 0)$ and $u_R(t, \varphi) = u_\mathbb{R}(t-R, \varphi) - u_\mathbb{R}(-R, 0)$. An argument due to Floer allows us to deform the approximation to a genuine solution. One needs the condition that the operators $\bar{\partial}_u$ and $\bar{\partial}_v$ are surjective. Then the linearization of ∂_J at an approximate solution $\tilde{w}_R$ can be shown to be surjective for large R, and to admit a right inverse. In addition, for large R we have that $\bar{\partial}_J \tilde{w}_R$ tends to 0. Using the mentioned right inverse, it follows from the implicit function theorem that near $\tilde{w}_R$ there exists a holomorphic cylinder w_R. Hence we get a family of holomorphic cylinders w_R for $R > R_0$.

In the limit $R \to \infty$, this family of cylinders converges to a broken cylinder of height 2 in the sense of Definition 9.12. Indeed, for the sequence of approximate solutions $\tilde{w}_n$ we can insert $t_n^1 = \pm n$ in Definition 9.12 to obtain a sequence that converges to u respectively v on all compact sets $[-N, N]$. Hence we see that $\tilde{w}_n$ converges to a broken curve consisting of u and v.

Hence we see that broken cylinders are in the boundary of the 1-dimensional moduli space $\mathcal{M}(\gamma_a, \gamma_c)/\mathbb{R}$,

$$\bigcup_{\gamma_b} \mathcal{M}(\gamma_a, \gamma_b)/\mathbb{R} \times \mathcal{M}(\gamma_b, \gamma_c)/\mathbb{R} \subset \partial\mathcal{M}(\gamma_a, \gamma_c)/\mathbb{R}.$$

The boundary may consist of more than just holomorphic cylinders, even if we consider only the boundary of a 1-dimensional moduli space $\mathcal{M}(\gamma_a, \gamma_c)/\mathbb{R}$. Since the moduli space is not always a smooth manifold, we should keep in mind that these boundary points might be different from what one expects from the analogous situation in Floer homology. At any rate, note the similarity with Morse homology, where broken trajectories make up the boundary of a 1-dimensional moduli space. The above construction also works without the assumption that $c_1 = 0$. In that case, we need to take the homology class of the holomorphic curve into account. We have

$$\bigcup_{\gamma_b, A^+ + A^- = A} \mathcal{M}^{A^+}(\gamma_a, \gamma_b)/\mathbb{R} \times \mathcal{M}^{A^-}(\gamma_b, \gamma_c)/\mathbb{R} \subset \partial \mathcal{M}^A(\gamma_a, \gamma_c)/\mathbb{R}.$$

9.2.6.2. *Orientation on the space of operators* $\mathcal{O}(\gamma^+, \gamma^-)$. Suppose that we are given two operators with matching boundary conditions, i.e. $L \in \mathcal{O}(\gamma_1, \gamma_2)$ and $K \in \mathcal{O}(\gamma_2, \gamma_3)$. Then we can construct a new operator $M \in \mathcal{O}(\gamma_1, \gamma_3)$ following Floer and Hofer [**16**]. The new operator M is often denoted by $L\#K$ and is called the glued operator. For later applications, we need some relations of index bundle and this gluing procedure. The index bundle $\mathcal{L}$ at an operator L is defined as the top exterior power of the kernel of L tensored with the top exterior power of the cokernel of L,

$$\operatorname{Det}(L) = \bigwedge^{top} \ker(L) \otimes \left(\bigwedge^{top} \operatorname{coker}(L) \right)^* .$$

This is in fact a line bundle over $\mathcal{O}$. Actually, since we consider operators whose asymptotics are fixed, this is a trivial line bundle. Floer and Hofer show this by proving that for fixed asymptotics, the space $\mathcal{O}$ is contractible, and hence any vector bundle over $\mathcal{O}$ is trivial. Because the bundle $\mathcal{L}$ is trivial, we can choose non-vanishing sections for $\mathcal{L}$ over $\mathcal{O}(\gamma_1, \gamma_2)$ and $\mathcal{O}(\gamma_2, \gamma_3)$. In particular, we get orientations for $\mathcal{L}_L$ and $\mathcal{L}_K$. There exists a natural isomorphism

$$\psi : \operatorname{Det}(L) \otimes \operatorname{Det}(K) \to \operatorname{Det}(M),$$

which is defined in [**16**] and also in [**4**], Corollary 7. Suppose $\sigma(L)$ is an orientation of $\operatorname{Det}(L)$ (a non-vanishing vector), and $\sigma(K)$ is an orientation of $\operatorname{Det}(K)$. We get an orientation on $\operatorname{Det}(M)$ induced by ψ, which we denote by $\sigma(L)\#\sigma(K)$. We will use this discussion to orient the moduli space.

In the literature the moduli space is not oriented directly, in order to avoid having to deal with transversality issues at this stage. Instead we find an orientation of the index bundle $\mathcal{L}$ of the space $\mathcal{O}(\gamma^+, \gamma^-)$. We have a map from the moduli $\mathcal{M}(\gamma^+, \gamma^-)$ to $\mathcal{O}(\gamma^+, \gamma^-)$ given by

$$\begin{array}{rccc} \pi : & \mathcal{M}(\gamma^+, \gamma^-) & \to & \mathcal{O}(\gamma^+, \gamma^-) \\ & u & \mapsto & \bar{\partial}_u \end{array}$$

where $\bar{\partial}_u$ is the linearization of the Cauchy Riemann operator $\bar{\partial}_J$ near u, which we also saw in Section 9.2.5.

It turns out that when transversality is satisfied, then the top exterior power of $T\mathcal{M}$ is canonically isomorphic to $\pi^*\mathcal{L}$, so that an orientation of $\mathcal{L}$ induces one on the moduli space. Indeed, in case the Cauchy-Riemann operator is surjective, the tangent space to the moduli space is given by

$$T\mathcal{M} = \bigcup_{u \in \mathcal{M}} \{u\} \times \ker(\bar{\partial}_u),$$

so the highest exterior power of $T\mathcal{M}$ can be written as (again using surjectivity of $\bar{\partial}_u$)

$$\bigwedge^{top} T_u\mathcal{M} = \bigwedge^{top} \ker(\bar{\partial}_u) \otimes \mathbb{R}^* = \operatorname{Det}(\bar{\partial}_u).$$

We now want to choose coherent orientations of the index bundle, i.e. a map $\sigma : \mathcal{O} \to \mathcal{L}$ such that if L and K are operators with matching asymptotics, then

$$\sigma(L)\#\sigma(K) = \sigma(L\#K).$$

The choice of such an orientation is possible, see theorem 1 of [**4**].

Remark 9.19. In some situations there are natural orientations for the index bundle. For this, we consider the asymptotic form of the operators,

$$\partial_s + J_0\partial_t + S(t).$$

Note that the first part $\partial_s + J_0\partial_t$ is always complex linear, i.e. commutes with J_0. The second part, $S(t)$, is in general only real linear. If the second part is also complex linear, the whole operator is (asymptotically) complex linear and hence its kernel is endowed with a natural orientation, which we can use for the coherent orientation procedure. We will use this argument in a slightly different setup later on.

Now we remind ourselves of the discussion of directions in Section 9.2.2. There we saw that for a Reeb orbit γ_m that is an m-fold covering of a simple closed Reeb orbit, we have a $\mathbb{Z}_m$-action on that orbit and in fact on moduli spaces whose curves are asymptotic to γ_m at some puncture. By itself, this action does not seem to be of any interest, but we need to see whether the coherent orientations respect this action. To that end we define the following.

DEFINITION 9.20. A closed Reeb orbit γ_m is said to be **bad** if γ_m is the m-fold covering of some Reeb orbit γ and if the difference

$$\mu_{CZ}(\gamma_m) - \mu_{CZ}(\gamma)$$

is odd. This can only happen is m is even. Closed Reeb orbits that are not bad are called **good**.

We have the following theorem (theorem 3 from [**4**])

THEOREM 9.21 (Bourgeois, Mohnke). *Let γ_m be an m-fold cover of a simple closed Reeb orbit γ. Suppose γ_m is one of the Reeb orbits at $\pm\infty$. Then the above $\mathbb{Z}_m$ action on $\mathcal{M}^A(\Gamma^+, \Gamma^-; W, J)$ is orientation preserving if and only if γ_m is good.*

This theorem will be the main reason for excluding the bad Reeb orbits from the chain complex of contact homology. We see that in the presence of bad Reeb orbits, different parametrizations of closed Reeb orbits can give the moduli spaces different orientations. Hence we can only orient the moduli spaces in a meaningful way if we do not consider bad orbits.

CHAPTER 10

Cylindrical contact homology

10.1. Introduction

In this chapter we present the basic setup of contact homology. We will restrict ourselves to cylindrical contact homology, though we will indicate roughly how to generalize this.

10.2. Contact homology for generic contact forms

Let (M,ξ) be a contact manifold that admits a contact form α such that $\xi=\ker\alpha$. As usual we define the Reeb field R_α of α by

$$i_{R_\alpha}d\alpha=0 \text{ and } i_{R_\alpha}\alpha=1.$$

Keep in mind that the Reeb field depends on the choice of contact form. Let us now consider the action functional

$$\begin{array}{rcl}\mathcal{A}:C^\infty(S^1,M) & \to & \mathbb{R}\\ \gamma & \mapsto & \displaystyle\int_\gamma \alpha.\end{array}$$

This is the starting point for contact homology. Roughly speaking, we want to do a kind of Morse theory for this functional and use the ideas from Morse theory as a guide, similar to Floer homology. This brings us to the first issue, the critical points of $\mathcal{A}$.

We find the critical points by inserting 1-parameter families of loops into $\mathcal{A}$. If $\gamma_t:\ S^1=\mathbb{R}/\mathbb{Z}\to M$ for $t\in[0,1]$ is such a family, we get

$$\frac{d}{dt}|_{t=0}\int_{S^1}\gamma_t^*\alpha=\int_{S^1}\gamma_0^*\mathcal{L}_X\alpha=\int_{\gamma_0(S^1)}i_X d\alpha=\int_0^1 d\alpha(X(s),\dot\gamma_0(s))ds. \tag{10.1}$$

Here X is the vector field that generates the family up to first order, $X=\frac{d}{dt}\gamma_t|_{t=0}$. We used Stokes' theorem to eliminate one of the terms that arises in Cartan's formula for the Lie derivative. The final expression is only 0 for all possible families of paths and hence for all X if $i_{\dot\gamma_0}d\alpha=0$, where $\dot\gamma_0$ is the tangent vector to the curve γ_0. This means that tangent vectors to γ_0 are parallel to the Reeb field. By reparametrizing γ_0 we see that γ_0 is a closed Reeb orbit.

Like in Morse homology, we want to consider critical points that are in some sense non-degenerate, i.e. the Hessian of $\mathcal{A}$ at critical points is non-degenerate. This amounts to the same notion of non-degenerate Reeb orbits as in Section 8.3.

Let us introduce the chain complex for contact homology in a more precise fashion. Suppose the dimension of M is $2n-1$. Choose a generic contact form for ξ, which can be done according to Proposition 8.3. Next, let γ a closed non-degenerate Reeb orbit and assume that γ is homologically trivial. Then we define the degree of γ by (see also Section 8.3)

$$\deg\gamma=\mu_{CZ}(\gamma)+(n-3).$$

Note that the Conley-Zehnder index depends on the choice of trivialization. Since we assumed γ to be homologically trivial, we can find a Seifert surface S for γ and use the trivialization of ξ over S to find the Conley-Zehnder index. Now let A be a surface in M with homology class $[A]$. This gives us another trivialization of ξ coming from $S\#A$. Their Conley-Zehnder indices are related in the following way (this formula is of course related to Formula (8.5))

$$\mu_{CZ}(\gamma,S\#A)=\mu_{CZ}(\gamma,S)+2\langle c_1(\xi),[A]\rangle. \tag{10.2}$$

In other words, even if we always compute the Conley-Zehnder index by using trivializations coming from Seifert surfaces, the degree is still not well-defined, unless $c_1(\xi) = 0$. In our applications we actually have that the first Chern class is trivial, but we want to indicate in the following what to do if the Chern class of ξ is not trivial. We already mentioned this in Remark 8.6

Remark 10.1. In Section 8.3 we also defined the degree for Reeb orbits that were not homologically trivial (we described that case where $H_1(M)$ is torsion free). This was done using reference arcs where we fixed a trivialization of the contact structure. The contact homology we define in the next sections can also be defined in a similar way for homologically non-trivial Reeb orbits. Indeed the associated contact homology can still be used as an invariant of the contact structure, but it will in general depend on the choice of reference arcs.

To put this into perspective, we recall that in the Morse homology case, the dimension of the space of gradient trajectories between p and q is given by

$$\dim \mathcal{M}_{Morse}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q).$$

In contact homology we work with the moduli space of holomorphic curves. If we just consider holomorphic cylinders between Reeb orbits γ_+ and γ_-, the (virtual) dimension is given by

$$\dim \mathcal{M}^A(\gamma_+,\gamma_-) = \deg(\gamma_+) - \deg(\gamma_-) + 2\langle c_1(\xi), A\rangle.$$

In particular, the dimension does not just depend on the critical points, but on additional information, namely the homology class of a holomorphic curve as well. If we take that point of view, we see that the closed Reeb orbits by themselves do not carry enough information to make a statement about the dimension of the moduli space. Hence we need to put additional information in the chain complex, which is done using the coefficient ring for the chain complex.

The coefficient ring. Let us continue with our discussion of the Conley-Zehnder index of a homologically trivial Reeb orbit. In case the first Chern class of ξ is not trivial, the Conley-Zehnder index depends on the chosen Seifert surface. The Conley-Zehnder index with respect to a different Seifert surface is given by (10.2). In order to account for the dependence on a homology class, we give the homology classes a grading and include them in the coefficient ring in the following way. The grading on homology is given by

$$|A| = -2\langle c_1(\xi), A\rangle \text{ for } A \in H_2(M;\mathbb{Z}). \tag{10.3}$$

Now let $\mathcal{R}$ be a submodule of $H_2(M;\mathbb{Z})$ with zero grading, i.e. $A \in \mathcal{R}$ implies $|A| = 0$. This choice is, of course, not unique. In fact, in a few of our applications later on, where $c_1(\xi) = 0$, we actually choose $\mathcal{R} = H_2(M;\mathbb{Z})$. In the discussion of the differential in contact homology in Section 10.2.2, we will clarify the choice of $\mathcal{R}$. This choice can vary from application to application, but sometimes it is necessary to choose $\mathcal{R} = 0$ even in cases where $c_1(\xi) = 0$.

By choosing a submodule $\mathcal{R}$ in the above way, we see that the quotient $H_2(M;\mathbb{Z})/\mathcal{R}$ has a well-defined grading. We then choose our coefficient ring to be the graded ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$. By using such coefficient rings we are able to provide additional information in the chain complex for contact homology, and we will see that this is enough to get the dimension of the moduli space. We will sometimes refer to the graded ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$ as the Novikov ring of $H_2(M;\mathbb{Z})/\mathcal{R}$.

10.2.1. Chain complex of contact homology. Roughly speaking, the chain complex of contact homology is generated by closed Reeb orbits. Some closed Reeb orbits need to be excluded though, namely the bad Reeb orbits which we defined in Section 9.2.6. Without these, the moduli space can be oriented using the coherent orientations.

Definition 10.2. The chain complex $C_*(M,\alpha)$ of cylindrical contact homology is the graded module freely generated by the good Reeb orbits over the graded ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$.

We will write elements of the coefficient ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$ as finite sums of the form

$$\sum_{i=1}^k q_i e^{A_i}, \text{ where } q_i \in \mathbb{Q} \text{ and } A_i \in H_2(M;\mathbb{Z})/\mathcal{R}.$$

That way we have encoded the multiplication structure of the coefficient ring in a convenient way. We split the module C_* by the homotopy class of a Reeb orbit. Hence we can write C_* as a sum over free homotopy classes a,

$$C_* = \oplus_a C_*^a.$$

The differential we define later respects this splitting.

10.2.2. Differential for cylindrical contact homology. First we would like to give some motivation for the differential, see also [**14**]. We would like to pursue the ideas of Morse homology here, which means that we need the gradient flow of $\mathcal{A}$. To that end we need to have a metric on the free loop space

$$\Lambda(M) = C^\infty(S^1, M).$$

We choose a complex structure J on ξ compatible with $d\alpha$. This gives rise to a metric on ξ,

$$g(v, w) = d\alpha(v, Jw),$$

which we extend to TM by saying that the Reeb field R_α is a unit normal field to ξ. The tangent bundle of $\Lambda(M)$ at curve γ can be expressed as

$$T_\gamma\Lambda(M) \cong C^\infty(S^1 = \mathbb{R}/\mathbb{Z}, \gamma^*TM),$$

so we get an induced metric on $\Lambda(M)$ given by

$$g_\Lambda(X, Y) = \int_0^1 g_{\gamma(t)}(X(t), Y(t))dt$$

for vectors $X, Y \in C^\infty(\mathbb{R}/\mathbb{Z}, \gamma^*TM)$. This metric allows us to pursue the ideas of Morse homology with the action functional. The idea to extend Morse homology to such a setting was first put forward by Floer, see for instance [**17**], and was used to establish various kinds of Floer homology theories. The reader should keep this in mind, since many ideas of contact homology come from Floer's observations.

We now consider the gradient flow of the action functional $\mathcal{A}$ with respect to the metric we just defined. Let γ be a curve in M. From the computation in Equation (10.1) we see that the differential of the action in the direction of $X \in C^\infty(\mathbb{R}/\mathbb{Z}, \gamma^*TM)$ at γ is given by

$$d\mathcal{A}(\gamma)[X] = \int_0^1 d\alpha_{\gamma(t)}(X(t), \dot{\gamma}(t))dt,$$

where $\dot{\gamma}(t)$ is the the tangent vector to γ at $\gamma(t)$. If we use π to denote the projection from TM to ξ along the Reeb field, we see that

$$\nabla\mathcal{A}(\gamma) = -J\pi(\dot{\gamma})$$

solves the equation

$$\int_0^1 d\alpha_{\gamma(t)}(X(t), \dot{\gamma}(t))dt = \int_0^1 d\alpha(\nabla\mathcal{A}(\gamma), JX)dt = g_\Lambda|_\xi(\nabla\mathcal{A}(\gamma), X)$$

for all $X \in C^\infty(\mathbb{R}/\mathbb{Z}, \gamma^*TM)$. Therefore it also solves $d\mathcal{A}(\gamma)[X] = g_\Lambda(\nabla\mathcal{A}(\gamma), X)$ for all X, so the gradient of $\mathcal{A}$ with respect to g_Λ is given by $\nabla\mathcal{A}(\gamma) = -J\pi(\dot{\gamma})$. If we write a gradient flow line as a map

$$u: \ S^1 \times \mathbb{R} \to M,$$

then u has to satisfy the following equation that has the flavor of a Cauchy-Riemann equation (we use the convention for the gradient flow that $\dot{u} = -\nabla u$),

$$\frac{\partial u}{\partial s}(t, s) - J\pi(\frac{\partial u}{\partial t}(t, s)) = 0.$$

Note that due to our convention for the gradient flow this would actually correspond to an "anti-holomorphic" curve. This, however, makes no difference for the subsequent arguments involved. We will call the curves involved later on, although incorrectly, holomorphic curves. Our choice for the negative gradient flow is mostly in order to stick with the conventions of Morse homology.

We can try and turn the above equation into a genuine Cauchy-Riemann equation by extending the equation to the symplectization of M. By extending the complex structure on ξ in a suitable way to $T(M \times \mathbb{R})$, for instance using Equation (9.2), we get the following equation for $U(t,s) = (u(t,s), a(t,s)) \in M \times \mathbb{R}$,

$$\frac{\partial U}{\partial s}(t,s) - J\frac{\partial U}{\partial t}(t,s) = 0. \tag{10.4}$$

With some abuse of notation, we use J to denote also the almost complex structure on $M \times \mathbb{R}$ obtained by extension from ξ. As in the previous chapter we can consider the natural metric on the symplectization, which is given by

$$\tilde{g}(\ldots,\ldots) = dt \otimes dt + \alpha \otimes \alpha + \omega(\ldots, J\ldots).$$

Now we write this Cauchy-Riemann equation as a system,

$$\begin{aligned} \frac{\partial u}{\partial s}(t,s) - J\pi(\frac{\partial u}{\partial t}(t,s)) - \frac{\partial a}{\partial t}(t,s)R(u(t,s)) &= 0, \\ \frac{\partial a}{\partial s}(s,t) + \tilde{g}\left(\frac{\partial u}{\partial t}(t,s), R(u(t,s))\right) &= 0. \end{aligned}$$

Since $d\mathcal{A}(\gamma)[fR] = 0$ for any function f, and $-J\pi(\frac{\partial u}{\partial t}(t,s))$ is the gradient of $\mathcal{A}$, we see that the first equation of the above system can be seen as the flow of the gradient-like vector field

$$-J\pi(\frac{\partial u}{\partial t}(t,s)) - \frac{\partial a}{\partial t}(t,s)R(u(t,s)).$$

Hence our extension to the symplectization of the original gradient flow equations does not change our interpretation of the Cauchy-Riemann equation too much. Indeed, trajectories between critical points (closed Reeb orbits) described by Equation (10.4) can now be interpreted as pseudo-holomorphic curves that are asymptotically cylindrical over closed Reeb orbits. These holomorphic curves are precisely those elements of the moduli spaces we described in Chapter 9. We try to make a homology theory out of this setup by counting these holomorphic curves between critical points in a suitable way, similar to Morse homology.

Since the dimension of the moduli space does not just depend on the degree of the Reeb orbits, but also on the homology class of a holomorphic curve, we count elements of all zero-dimensional moduli spaces instead of just considering connecting trajectories between Reeb orbits of index difference 1, which we would do in Morse homology. Note here that we consider pseudo-holomorphic curves in symplectizations. In that special case the moduli space of holomorphic curves carries an $\mathbb{R}$-action induced by the translation in the $\mathbb{R}$-direction on the symplectization. Hence we can count the number of components in the zero-dimensional moduli space $\mathcal{M}/\mathbb{R}$. Also note that the $\mathbb{R}$-action gives the 1-dimensional components of $\mathcal{M}$ an orientation, which we will call the flow orientation. On the other hand, the components have also been oriented by the coherent orientation procedure. By comparing the orientations on the 1-dimensional components of $\mathcal{M}$ we get signs which we use to count the elements of $\mathcal{M}/\mathbb{R}$. We define the differential, using these counting rules, on the generators of the chain complex and extend linearly. Let γ_a be a closed Reeb orbit and define

$$\partial\gamma_a = \kappa_{\gamma_a} \sum_{\substack{\gamma_b \in \mathrm{Crit}\mathcal{A}, \\ A \in H_2(M)}} \left(\sum_{\substack{C \in \mathcal{M}^A(\gamma_a,\gamma_b)/\mathbb{R} \\ \dim\ \mathcal{M}^A(\gamma_a,\gamma_b)=1}} \frac{\pm 1}{\kappa_C} \right) e^{\pi(A)}\gamma_b.$$

The sign in this formula is determined by the procedure we just described. Note that this is the same procedure that can be used to produce the sign in Floer homology, which is in turn analogous to determining the sign in Morse homology. We have used κ_{γ_a} to denote the multiplicity of the Reeb orbit γ_a, and κ_C to denote the order of the covering group of the cylinder C. Recall our brief discussion where we mentioned the virtual cycle for a motivation of those weights. The projection π maps elements of $H_2(M;\mathbb{Z})$ to $H_2(M;\mathbb{Z})/\mathcal{R}$. Note that in case γ_a is simple and the cylinder C is simple, this formula is the same as in the original formulation of Floer homology with $\mathbb{Z}$-coefficients. Also note that by defining the differential this way, the overall degree of $e^{\pi(A)}\gamma_b$ is 1 less than the degree of γ_a. We also see that in favorable situations, for instance if $c_1(\xi) = 0$, the dimension of the moduli space depends only on the difference in degree between Reeb orbits, and hence the differential counts only Reeb orbits of degree difference 1. The Reeb orbit γ_a gives us an energy bound for holomorphic curves involved in the differential of γ_a. This is due to the fact that $\mathcal{A}(\gamma_a)$ is finite, giving us an upper bound for the contact area or $d\alpha$-energy. Then the compactness theorems we discussed in Section 9.2.4 guarantee that the expression used for the differential only contains finite sums of non-zero elements.

With this differential we have completed the definition of the chain complex (C_*, ∂). Although we will not use that fact, it is important to note that this chain complex comes with an additional filtration given by the action of a Reeb orbit. The differential respects this filtration. Indeed, since the $d\alpha$-energy of a holomorphic cylinder going from γ_a to γ_b is always non-negative, the action of the Reeb orbit of γ_b satisfies

$$\mathcal{A}(\gamma_b) \leq \mathcal{A}(\gamma_a).$$

Under suitable circumstances, the square of ∂ vanishes and the associated homology is an invariant of the contact manifold (M, ξ).

THEOREM 10.3. *Let con denote the free homotopy class of contractible curves. If $C_k^{con} = 0$ for $k = -1, 0, 1$, then $\partial^2 = 0$ and the homology $H_*(C_*, \partial)$ is independent of choice of the contact form α, the choice of complex structure J, and the choice of perturbation ν as described in Section 9.2.5.3.*

If the above theorem applies, then we define the **cylindrical contact homology** $HC_*(M, \xi)$ as the homology of the above chain complex. Note that cylindrical contact homology comes with an additional grading given by free homotopy classes of curves. This works since we are counting cylinders with our differential and those can not change the homotopy class of a Reeb orbit. We write $HC_*^a(M, \xi)$ for the part of the homology that corresponds to the homotopy class a.

Note that it is not true that contact homology is independent of all choices made. For instance, if $c_1(\xi)$ is not trivial, then we might change the degree of a Reeb orbit by taking a different surface for the trivialization of the contact structure, needed to compute the degree, according to Formula (8.5). In case there are homologically non-trivial Reeb orbits, we have to choose reference arcs, and the contact homology will depend on those choices. We can still use it as an invariant though by fixing these choices when comparing different contact structures. Also note that in case the above theorem applies and $c_1(\xi) = 0$, the homology $HC_*^{con}(M, \xi)$ can always be used as an invariant without specifying these choices.

10.2.3. Sketch of proof for cylindrical contact homology. We will now give a rough sketch of the proof of this theorem following Bourgeois's lecture notes [**5**]. An important guiding idea for the proof is the proof of the corresponding statement in Morse homology. This in itself was also used for Floer homology, and a fairly detailed proof in that case can be found in [**44**]. At several places we define maps by counting certain holomorphic curves, but we will do this in cases where $c_1(\xi) = 0$. To obtain the general case we have to use our coefficient ring; we do not just consider a single "target" Reeb orbit, but also products with elements in the coefficient ring. In other words, replace γ_b by $e^A \gamma_b$ and consider those γ_b and A such that the moduli space has the proper dimension, i.e. like the differential, the maps we consider have the form

$$\Phi \gamma_a = \kappa_{\gamma_a} \sum_{\substack{\gamma_b \\ A \in H_2(M)}} \left(\sum_{\substack{C \in \mathcal{M}^A(\gamma_a, \gamma_b) \\ \deg(\gamma_a) - \deg(\gamma_b) + 2\langle c_1, A \rangle = k}} \frac{\pm 1}{\kappa_C} \right) e^{\pi(A)} \gamma_b,$$

where k is the index difference, which is $1, 0$ or -1 in the cases we consider. Of course, the moduli space in this formula needs to be replaced by the particular moduli space we consider at that point.

Moreover, we will detail the additional assumptions imposed on the non-existence of certain Reeb orbits. Also, we assume that transversality is satisfied. This can be done in some cases by perturbation of the complex structure, but in most cases we need to use more general perturbations,

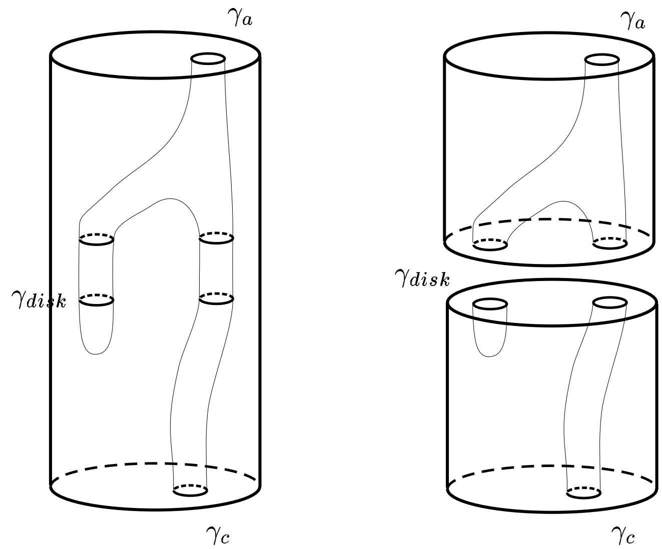

FIGURE 10.1. Breaking of a cylinder

such as the ones provided by virtual cycle techniques. Keep in mind that this is the point where contact homology differs significantly from the original formulation of Floer homology.

First we consider ∂^2. We can write that map out and see that it counts broken trajectories between γ_a and γ_c with intermediate critical point γ_b. These trajectories represent elements of $\mathcal{M}(\gamma_a, \gamma_c)$.

The argument from Morse homology we want to mimic is that the boundary of compact 1-dimensional manifolds vanishes if we count with orientation. We cannot expect this argument to work straight away since the moduli spaces might not be smooth manifolds. On the other hand, we count curves with weights, so the situation is slightly different. The moduli space involved, $\mathcal{M}(\gamma_a, \gamma_c)$, is compact and has dimension 1. By Floer's gluing construction we already know that

$$\partial\mathcal{M}(\gamma_a, \gamma_c) \supset \cup_b \mathcal{M}(\gamma_a, \gamma_b) \times \mathcal{M}(\gamma_b, \gamma_c).$$

Hence we see that all curves counted by ∂^2 lie in $\partial\mathcal{M}(\gamma_a, \gamma_c)$. We are in business if

$$\partial\mathcal{M}(\gamma_a, \gamma_c) = \cup_b \mathcal{M}(\gamma_a, \gamma_b) \times \mathcal{M}(\gamma_b, \gamma_c).$$

For the other inclusion, i.e. "$\subset$ ", we need to use one of our additional assumptions. Consider a family of cylinders that breaks into a cylinder, a tree like curve and same disks capping off some of the punctures. See Figure 10.1 for an example with one such capping disk.

We use Formula (9.5) to see what the dimension of the moduli space of broken components is, assuming that transversality is satisfied. Since every component should have a non-negative dimension for its moduli space, we see that the degree of γ_{disk} should be at least 1. However, degree 2 or higher is not possible, since the whole curve should be on the boundary of a moduli space of dimension 1 (the whole curve has index 2 and a tree-like curve cannot have index 0 in a symplectization). Hence we see that generically the breaking of a cylinder into similar curves with more such capping disks cannot occur.

The remaining case of one capping disk can happen, but since we assumed that there were no degree 1 contractible Reeb orbits, there will be no such curves at the boundary of the moduli space, and hence we see that the boundary of the 1-dimensional moduli space $\mathcal{M}(\gamma_a, \gamma_c)/\mathbb{R}$ has the structure we just mentioned.

If all Reeb orbits involved are simple, then the moduli spaces are smooth manifolds, and the identity $\partial^2 = 0$ follows from the fact that one-dimensional manifolds have vanishing boundary if we count with orientation. For the general case, this argument might be taken as inspiration. Namely, in general the fundamental cycle of $\mathcal{M}/\mathbb{R}$ is a rational cycle, instead of an integer cycle in the smooth case. The boundary of this cycle can be identified with ∂^2. Geometrically, this corresponds to the following. The count by ∂^2 consists of products of weights of the height 2 curves and the factor κ_{γ_b}. This factor indicates different ways in which the cylinders can be glued. Indeed, κ_{γ_b} is the multiplicity of the Reeb orbit γ_b where we glue. Hence a rotation around the punctures corresponding to γ_b by a κ_{γ_b}-th root of unity does not affect the image curve, giving rise to κ_{γ_b} inequivalent ways of gluing the cylinder. Thus one can show that the count by ∂^2 equals κ_{γ_a} times the weight $1/\kappa_C$, where C is a height 2 cylinder at the boundary of a 1-dimensional moduli space $\mathcal{M}/\mathbb{R}$. These cylinders multiplied by their weight represent the boundary of the rational fundamental cycle.

This shows that $\partial^2 = 0$, so we obtain indeed a homology theory. Now we want to show that the resulting homology is independent of the choice of contact form and almost complex structure. Let α_1 and α_2 be contact forms describing the contact structure ξ on M and let J_1 and J_2 be compatible almost complex structures on the symplectization of (M, α_1) and (M, α_2), respectively. Instead of considering curves in the symplectization, we look at another symplectic cobordism. Namely, we can choose a homotopy of the contact forms, β_s, such that $\beta_0 = \alpha_1$ and $\beta_1 = \alpha_2$. Next, choose a function $f: \mathbb{R} \to [0,1]$ such $\omega = d(e^t \beta_{f(t)})$ is a symplectic form on $M \times \mathbb{R}$. In addition, we require $f(t)$ to be identically 1 for large negative t and to be identically 0 for large positive t. Thus we get a symplectic cobordism W between the contact manifolds (M, α_2) and (M, α_1). We choose an almost complex structure on J on $W = (M \times \mathbb{R}, \omega)$ that is equal to J_2 for large negative r, and we require it to be equal to J_1 for large positive r. Here r is the coordinate on $\mathbb{R}$. This symplectic cobordism allows us to define a map Φ_W from the chain complex $C_*(\alpha_1)$ to the chain complex $C_*(\alpha_2)$ by counting cylinders connecting Reeb orbits. The definition is very similar to the definition of the differential, but we count elements of zero-dimensional components of $\mathcal{M}$ instead of counting elements of $\mathcal{M}/\mathbb{R}$, since we do not have global translation invariance on a general symplectic cobordism.

To see that Φ_W is a chain map, we then argue using similar arguments as in the case of $\partial^2 = 0$. If we write the differentials for the chain complexes $C_*(\alpha_1)$ and $C_*(\alpha_1)$ as ∂_1 and ∂_2, respectively, the chain map property can be written as

$$\Phi_W \circ \partial_1 - \partial_2 \circ \Phi_W = 0.$$

Note that the left hand side of this equation amounts in our case to counting certain broken holomorphic curves; height 2 curves that first follow a cylinder in the symplectization of (M, α_1) and then a cylinder in W, or height 2 curves that consist of a cylinder in W followed by a cylinder in the symplectization of (M, α_2). We want to argue by showing that all these curves lie in the boundary of the one-dimensional moduli space $\mathcal{M}(\gamma_1, \gamma_2)$, where $\gamma_1 \in C_*(\alpha_1)$ and $\gamma_2 \in C_*(\alpha_2)$.

Similar to the situation of $\partial^2 = 0$, a cylinder can break in different ways. We refer again to Figure 10.1. Cylinders can break into height 2 curves that consist of a tree-like curve, several disks and a cylinder. An index counting argument similar to the one given before shows that the only situation that could happen is a tree-like curve, one disk bounding a Reeb orbit of degree 0 and a cylinder (we can copy the argument from before if we note that we are now counting curves in general symplectic cobordisms). Since we assumed that there are no contractible degree 0 orbits, we see that all curves on the boundary are indeed counted by

$$\Phi_W \circ \partial_1 - \partial_2 \circ \Phi_W.$$

To be more precise, this argument has similar complications as the argument for $\partial^2 = 0$.

In other words, a homotopy of the contact forms and other data involved allows us to define a chain map between the associated chain complexes, and hence we obtain a map on the level of homology, which we denote by $\bar{\Phi}_W$. Finally, we show that this map is an isomorphism. Towards that end, we construct an inverse of $\bar{\Phi}_W$. On the level of chains, the first choice is the map associated to the "reversed" symplectic cobordism W. We take the symplectic cobordism $\bar{W} = (M \times \mathbb{R}, d(e^t \beta_{1-f(t)}))$ and follow the same procedure to find a suitable almost complex structure J as in the case of W.

The composition $\Phi_{\bar{W}} \circ \Phi_W$ can also be described by considering curves in the glued symplectic cobordism $\bar{W} \circledcirc W$. Instead of counting index 0 curves in this cobordism directly, we first want to deform the symplectic structure on $\bar{W} \circledcirc W$ to the structure of the symplectization of (M, α_1). The map $\Phi_{sympl(M,\alpha_1)}$ corresponding to this deformed cobordism is the identity, because the only index 0 curves in a symplectization are the vertical cylinders. Moreover, these vertical cylinder preserve the orientation of the Reeb orbits, i.e. $\gamma \mapsto +\gamma$.

We now want to show that such a deformation of the symplectic structure on a cobordism gives rise to chain homotopy $A: \ C_*(M, \alpha_1) \to C_{*+1}(M, \alpha_1)$. That way we see that the original map $\Phi_{\bar{W}} \circ \Phi_W$ also gives the identity on homology.

Let us do this construction in a more precise manner. Let ω_s be a path of symplectic forms for $s \in [0, 1]$ such that ω_0 corresponds to the symplectic form on the glued cobordism $\bar{W} \circledcirc W$ and such that ω_1 is the symplectic form of the symplectization of (M, α_1). Hence we get a family of cobordisms W_s, and we choose compatible almost complex structures J_s. We construct the chain homotopy by counting J_s-holomorphic curves in the family of cobordisms W_s. Note that we cannot use the dimension formula since we cannot obtain transversality for a 1-parameter family of cobordisms by perturbations. For a generic k-parameter family of cobordisms W_s, $s \in [0, 1]^k$, we have the following dimension formula:

$$\begin{aligned} \dim \mathcal{M}^A_{k-par}(\Gamma^+, \Gamma^-; W, J) = \mu_{CZ}(\gamma^+) - \sum_{i=1}^{s^-} (\mu_{CZ}(\gamma_i^-) + n - 3) \\ + (n-3)(2 - s^+) + 2\langle c_1(J), A\rangle + k. \end{aligned}$$

By counting cylinders connecting γ_a to γ_b such that $\deg(\gamma_a) - \deg(\gamma_b) = -1$, we obtain a map

$$A: \ C_*(M, \alpha_1) \quad \to \quad C_{*+1}(M, \alpha_1).$$

Note that the curves counted by A lie always in W_s with $s \in (0, 1)$, since transversality is satisfied at the boundary.

Once again we consider the boundary of the one-dimensional moduli space $\mathcal{M}(\gamma_a, \gamma_c)$, where $\deg(\gamma_a) - \deg(\gamma_c) = 0$. The boundary of this moduli space consists of several components. We have components coming from the boundary of the parameter space $[0, 1]$, which are counted by $id - \Phi_{\bar{W}} \circ \Phi_W$ (recall that the counting of index 0 curves in the symplectization gives the identity at the level of chains). On the other hand, there can be components which consist of height 2 holomorphic curves. Since the height 2 curve is the limit of a sequence of cylinders, we are in a similar situation as before. Ideally, the first floor would consist of a cylinder counted by either A or ∂_1 and the second floor would have a cylinder counted by ∂_1 or A (such that the whole curve has "index 0"). However, as before a cylinder might break into a tree-like curve, some disks and a cylinder. We can once again apply an index counting argument to show that only the case of a tree-like curve with one capping disk and a cylinder can occur. If this happens, the Reeb orbit bounded by the capping disk has degree -1, so in our situation all curves in the boundary of $\mathcal{M}(\gamma_a, \gamma_c)$ are counted by

$$id - \Phi_{\bar{W}} \circ \Phi_W - \partial_1 \circ A - A \circ \partial_1.$$

By the same argument as in the previous two cases, we see that A does indeed give a chain homotopy between id and $\Phi_{\bar{W} \circledcirc W} = \Phi_{\bar{W}} \circ \Phi_W$, so it follows that $\Phi_{\bar{W} \circledcirc W}$ induces the identity on homology. We repeat this construction for $\Phi_W \circ \Phi_{\bar{W}}$ and find that $\bar{\Phi}_W$ is an isomorphism, as claimed.

Remark 10.4. We put in additional assumptions to deal with the splitting of a cylinder into a tree-like curve and a capping disk. We can define a more general contact homology that does not require these additional assumptions. This is done by not considering cylinders, but tree-like curves instead, i.e. curves with an arbitrary number of negative punctures. If we consider tree-like curves all along the splitting would just be another curve counted by ∂ or one of the other maps. It turns out, that the chain complex needs an appropriate modification and is, in fact, an algebra generated by Reeb orbits.

This generalization also explains the need for the term $n-3$ in the definition of degree. By making this choice, one can ensure that the unit element in the algebra has degree 0 as well as making the form of the differential more elegant, cf. Formula (9.5).

REMARK 10.5. Different coefficient rings give wildly different results for the contact homology, see [**5**] for an example with contact structures on T^3. Even if $c_1(\xi)=0$ it can sometimes be useful to use more complicated coefficients such as $\mathbb{Q}[H_2(M)]$, because that allows us to extract more information from contact homology, such as the particular homology class of the holomorphic curves.

10.2.4. Example. In Section 8.3 we gave an example for non-degenerate Reeb orbits on the sphere with standard contact structure if we deformed the contact form. This deformed contact form is admissible for contact homology and allows us to compute the cylindrical homology of the standard contact structure on spheres.

We use the notation introduced in Section 8.3, where we found that the closed Reeb orbits all had the form

$$(0,\dots,0,\underset{\text{at } j^{\text{th}} \text{ position}}{e^{it/a_j}},0,\dots,0).$$

We call this orbit the j^{th} closed Reeb orbit and write γ_j^N for an N-fold covering of this orbit. In order to compute the Conley-Zehnder index, we observe that $S^{2n-1}\subset\mathbb{C}^n$. We have the symplectic form (we extended $\tilde{\alpha}$ to $\mathbb{C}^n$)

$$\omega=d\tilde{\alpha}=i\sum_{j=1}^{n}a_j dz_j\wedge d\bar{z}_j.$$

With respect to this symplectic form, the symplectic complement ξ^ω of the contact structure ξ is trivial. We have the following basis for the symplectic complement

$$X_1=-i(\frac{z_1}{a_1},\dots,\frac{z_n}{a_n}),\qquad Y_1=(z_1,\dots,z_n).$$

Note that we can extend the Reeb flow in obvious way to $\mathbb{C}^n$. The differential of this extended Reeb flow is a path of symplectic matrices given by the diagonal matrices

$$diag(e^{it/a_1},\dots,e^{it/a_n}).$$

We write $\Phi_{\mathbb{C}^n}$, Φ_ξ and Φ_{ξ^ω} for the path of symplectic matrices obtained by restricting the extended Reeb flow to $\mathbb{C}^n$, ξ and ξ^ω, respectively. Note that the extended Reeb flow acts trivially on X_1 and Y_1 and thus on ξ^ω. Hence we see that the following holds

$$\mu_{CZ}(\Phi_{\mathbb{C}^n})=\mu_{CZ}(\Phi_\xi\oplus\Phi_{\xi^\omega})=\mu_{CZ}(\Phi_\xi).$$

We can compute the first Maslov index easily by using the direct sum property of the Maslov index and Formula 8.4. The period of the closed Reeb orbit above is given by $2\pi a_j$.

We see that the Conley-Zehnder index of the j^{th} closed Reeb orbit and its multiples is given by

$$\mu_{CZ}(\gamma_j^N)=2N+2\sum_{i\neq j}\lfloor\frac{a_j}{a_i}\rfloor+n-1.$$

For the degree of these Reeb orbits we just need to add $n-3$. Note that the resulting degree is always even, so the differential is the zero map. To see what the resulting contact homology is up to degree $2n-2+D$ for any $D\in\mathbb{Z}_{\geq 0}$, we use invariance of contact homology. Namely, we choose the coefficients a_i such that $a_{i+1}/a_i>D$. By Gray stability the resulting contact structure is contactomorphic to the standard contact structure on S^{2n-1}. By the above formula, the degree of a closed Reeb orbit will always be larger than $2D$ unless we consider the multiples of the first closed Reeb orbit γ_1^N. The latter give precisely one generator in each even degree between degrees $2n-2$ and D. By invariance of contact homology, we can repeat this argument for all D and hence we see that

$$CH_k(S^{2n-1},\xi_{std})\cong\begin{cases}\mathbb{Q} & \text{if } k \text{ is even and } k\geq 2n-2\\ 0 & \text{otherwise.}\end{cases}$$

10.3. Morse-Bott contact homology

In many cases a contact manifold comes with a natural contact form that has degenerate closed Reeb orbits. The symmetry often also means degenerateness of Reeb orbits (see for instance the example of S^{2n-1} with its standard contact structure in Section 8.3), so we need to perturb the contact form if we want to use the description of contact homology we gave in the last section. We can again use Morse homology as a guide. There it is possible to construct a homology with functions having degenerate critical points if they are of a suitable type, namely we can consider Morse-Bott functions. These ideas can be adapted to contact homology [**6**]. There is, however, quite a bit of effort involved in making these ideas work. We will give a recipe-like description of the theory of Bourgeois. We will give a bit more detail at some places when we use Bourgeois's ideas in another form later on. In some of his theorems We shall restrict ourselves to contact forms for which all Reeb orbits are closed. This is not a requirement in Bourgeois's work and only takes care of a small technical detail. We will also assume that the first Chern class of the contact structure is trivial. This was a requirement in Bourgeois's earlier work, [**6**] but this restriction was lifted later on.

Let $\sigma(\alpha)$ denote the action spectrum of α, i.e. the critical values of the action functional $\mathcal{A}$. Instead of generic contact forms, we allow contact forms with degenerate Reeb orbits. We still need some regularity conditions as is shown in the following definition.

Definition 10.6. A contact form α is said to be of **Morse-Bott type** if the action spectrum $\sigma(\alpha)$ is discrete and if, for every $T \in \sigma(\alpha)$, $M_T = \{p \in M | \varphi_T(p) = p\}$ is a closed, smooth submanifold of M, such that the rank $d\alpha|_{M_T}$ is locally constant and $T_pM_T = \ker(T\varphi_T - id)_p$.

The Reeb flow induces an S^1-action on $M_{T'}$. Using this action we define the orbit space $S_{T'} := M_{T'}/S^1$. Note that these orbit spaces are orbifolds in general.

The chains of the Morse-Bott chain complex will correspond to the critical points of suitable Morse functions on the orbit spaces. Bourgeois constructs these Morse functions by induction (note that he needs Morse functions on orbifolds for which he introduces a suitable notion in [**6**]). We will now describe his construction.

For the smallest $T \in \sigma(\alpha)$, the orbit space S_T is a smooth manifold. Take any Morse function f_T on it. For larger $T \in \sigma(\alpha)$, S_T is an orbifold where the singularities are the orbit spaces $S_{T'}$ with T' dividing T. The previously defined Morse functions $f_{T'}$ on the orbit spaces $S_{T'}$ are extended to a function f_T on S_T by requiring that the Hessian of f_T restricted to the normal bundle of $S_{T'}$ is positive definite. These Morse functions are then lifted to M_T and extended to a function $\bar{f}_T$ on M such that they have support only in a tubular neighborhood of M_T.

For $T \in \sigma(\alpha)$, Bourgeois considers the following family of contact forms $\alpha_\lambda = (1+\lambda \bar{f}_T)\alpha$. We have

Lemma 10.7 (Bourgeois). *For all T, we can choose $\lambda > 0$ small enough such that the periodic orbits of R_{α_λ} in M of action $T' \leq T$ are non-degenerate and correspond to the critical points of $f_{T'}$.*

Let $p \in S'_T$ be a critical point of f'_T and denote its corresponding closed Reeb orbit (and its multiple covers) by $\gamma^p_{kT'}$ for $k = 1, 2, \ldots$. As Bourgeois's construction is explicit enough, he is able to compute the Conley-Zehnder indices of these Reeb orbits provided λ has been chosen so small such that Lemma 10.7 applies:

$$\mu_{CZ}(\gamma^p_{kT'}) = \mu(S_{kT'}) - \frac{1}{2}\dim S_{kT'} + \mathrm{index}_p(f_{kT'}). \tag{10.5}$$

This determines the degree of the Reeb orbits associated with the perturbed contact form with small period. The following notion is helpful in dealing with orbits of larger period.

Definition 10.8. The orbit spaces S_T are said to have **index positivity** if there exist constants $c > 0$ and c' such that $\mu(S_T) > cT + c'$ for all $T \in \sigma(\alpha)$.

Similarly, we define **index negativity** of the orbit spaces S_T if there are constants $c < 0$ and c' such that $\mu(S_T) < cT + c'$. In order to control the behavior of orbits with larger period, Bourgeois has the following result.

LEMMA 10.9 (Bourgeois). *Assume that the orbit spaces S_T have index positivity, that $c_1(\xi) = 0$ and that all Reeb orbits are closed. Then there exists a $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$, all period orbits γ_λ of R_{α_λ} of action greater than T satisfy $\mu_{CZ}(\gamma_\lambda) > cT/2$, where c is the positive constant from the index positivity of S_T.*

This lemma makes sure that other closed Reeb orbits that do not correspond to any critical point of the Morse functions on the orbit spaces have large Maslov indices. A similar result holds in case of index negativity of the orbit spaces. Hence we push those orbits that do not correspond to critical points of the chosen Morse functions away to larger degree in the case of index positivity, so that those orbits do not affect the differential of contact homology in lower degree. In the case of index negativity, orbits are, of course, pushed to large negative degree

The chains of the Morse-Bott complex are the critical points p of the Morse functions f_T for $T \in \sigma(\alpha)$, with degree given by

$$\deg(p) = \mu(S_{kT'}) - \frac{1}{2}\mathrm{dim} S_{kT'} + \mathrm{index}_p(f_{kT'}) + n - 3. \tag{10.6}$$

Before we give the definition of the differential, first recall that the fibered product of A and B over C for maps $f : A \to C$, $g : B \to C$ is given by

$$A \times_C B = \{(a, b) \in A \times B |\ f(a) = g(b)\}.$$

Let $\mathcal{M}(S^+, S^-)$ denote the moduli space of holomorphic curves with degenerate asymptotics, with orbit spaces S^+ and S^- for the positive and negative punctures, respectively. We will use the moduli space of generalized holomorphic cylinders, which is given by

$$\mathcal{M}^{f_T}(S^+, S^-) = \mathcal{M}(S^+, S^-) \cup \left(\mathcal{M}(S^+, S') \times_{S'} \left(\mathbb{R}^+ \times \mathcal{M}(S', S^-)\right)\right) \cup \ldots,$$

where the union runs over successive fibered products (cf. height k stable curves we introduced before). Note that the union is finite, because a holomorphic curve has to have positive energy (the energy is equal the action of the top Reeb orbit minus the action of the bottom Reeb orbit) and the action spectrum is discrete. The projection maps for these fibered products are ev^- and $\varphi^{f_T} \circ ev^+$. The maps ev^- and ev^+ are the evaluation maps at the negative and positive punctures, respectively, i.e. $ev^\pm : \mathcal{M}(S^+, S^-) \to S^\pm$, and here φ^{f_T} is the gradient flow of f_T on the orbit space. In other words $\varphi^{f_T} \circ ev^+ :\ (t, u) \mapsto \varphi^{f_T}(t)\,(ev^+(u))$, where $t \in \mathbb{R}^+$ and $u \in \mathcal{M}(S', S'')$.

10.3.1. Orientation of the moduli space. The moduli space of holomorphic curves needs to be oriented. Bourgeois does this using the coherent orientation procedure, in a similar way as in contact homology with generic contact forms, discussed in Section 9.2.6. There are a few obstructions to the existence of coherent orientations. First of all, there is the notion of a bad Reeb orbit. Since we are working here with Morse-Bott instead of generic contact forms, the notion is slightly different from the notion we gave before.

DEFINITION 10.10. A Reeb orbit γ is said to be **bad** if it is the $2m$-fold cover of a simple Reeb orbit $\gamma' \in S_T$ and if

$$(\mu(S_{2T}) \pm \frac{1}{2}\mathrm{dim} S_{2T}) - (\mu(S_T) \pm \frac{1}{2}\mathrm{dim} S_T)$$

is odd. If a Reeb orbit is not bad, it is said to be **good**.

In Morse-Bott contact homology, there can be another obstruction to the existence of coherent orientations. For this, we remind ourselves of the discussion in Section 9.2.6, where we indicated how to orient the moduli space in case the asymptotics are non-degenerate. We have a map π from the moduli space of holomorphic curves $\mathcal{M}$ to the space of Fredholm operators $\mathcal{O}(\gamma^+, \gamma^-)$ defined by sending a holomorphic map to its linearized Cauchy-Riemann operator. Let $\mathcal{L}$ denote the determinant bundle over $\mathcal{O}(\gamma^+, \gamma^-)$. Then $\pi^*\mathcal{L}$ is naturally isomorphic to the top exterior power of $T\mathcal{M}$ in case of transversality. In case the asymptotics for $\mathcal{O}(\gamma^+, \gamma^-)$ are fixed, $\mathcal{O}(\gamma^+, \gamma^-)$ is contractible and hence $\mathcal{L}$ is trivial. In this case there are no obstructions to orient $\mathcal{M}$.

On the other hand, if the asymptotics are allowed to vary, as is the case for the Morse-Bott formalism, then we may get a non-contractible space of Fredholm operators. This can happen, because the space of operators fibers over submanifolds of the form M_T. If that space M_T is not simply connected, it may contain a loop of operators such that the determinant bundle over that loop is not trivial. Such a loop is called a **disorienting loop**. We should remark here that if the projection of this loop to the orbit space S_T is contractible, the loop in M_T is homotopic to a bad Reeb orbit.

These phenomena can be present in general, because the linearized Cauchy-Riemann operator is usually only real linear and not complex linear. In favorable cases, the linearized Cauchy-Riemann operator is complex linear as well and an orientation on the determinant line bundle can be obtained directly from the induced complex structure on the kernel and cokernel of the linearized Cauchy-Riemann operator. This removes the need to see whether there are disorienting loops or bad orbits, because they cannot occur in that case.

10.3.2. Differential for Morse-Bott contact homology. With these remarks in mind, the differential can be defined. The differential of the chain complex is given by

$$dp = \partial p + \sum_q n_q q, \tag{10.7}$$

where $p \in S_T$, ∂ is the Morse-Witten differential of the Morse function f_T on $S_{T'}$, $q \in S^{T'}$ and n_q is the algebraic number of elements in the fibered product

$$(W^u(p) \times_{S_T} \mathcal{M}^{f_T}(S_T; S_{T'}) \times_{S_{T'}} W^s(q))/\mathbb{R}$$

if this product is zero-dimensional, and 0 otherwise ($q \in S_{T'}$). In this product, W^s and W^u denote the stable and unstable manifolds of a critical point of f_T on an orbit space, respectively.

Theorem 10.11 (Bourgeois). *Let α be a contact form of Morse-Bott type for a contact structure ξ on M that satisfies $c_1(\xi) = 0$. Assume that all Reeb orbits are closed. Assume that, for all $T \in \sigma(\alpha)$, M_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all Reeb orbits in S_T are good. Assume that the almost complex structure J is invariant under the Reeb flow on all submanifolds M_T. Assume that the cylindrical homology is well defined: the Morse-Bott chain complex has no contractible orbits of index $-1, 0$ or $+1$. Assume furthermore that all orbit spaces S_T of contractible periodic orbits have index positivity or index negativity. Then the homology $H_*(C_*^{\bar{a}})$ of the Morse-Bott chain complex $(C_*^{\bar{a}}, d)$ of (M, α) is isomorphic to the cylindrical homology $HF_*^{\bar{a}}(M, \xi)$ of (M, ξ) with coefficients in the Novikov ring of $H_2(M, \mathbb{Z})/\mathcal{R}$.*

There are other (improved) versions of this theorem, but this one suffices for our needs. In addition we will take the ring $\mathcal{R}$ in the above theorem to be $H_2(M, \mathbb{Z})$, or in other words we will use $\mathbb{Q}$-coefficients for the Morse-Bott chain complex.

CHAPTER 11

Applications of contact homology

Most of the results in this chapter are from my paper [**52**]. In short, we use Morse-Bott contact homology to study the cylindrical homology of Brieskorn manifolds. We obtain an algorithm which can be used to compute the contact homology. In addition, we show how some Brieskorn manifolds give us "nice" contact manifolds, where nice means that the contact manifold admits a contact form for which the cylindrical contact homology is well-defined. By using connected sums, many of these nice contact manifolds can be shown to admit infinitely contact structures. That way we show that a large class of contact manifolds admit infinitely many contact structures.

11.0.3. Algorithm for the computation of the cylindrical homology of Brieskorn manifolds. Consider the Brieskorn manifold $M=\Sigma(a_0,\dots,a_n)\subset\mathbb{C}^{n+1}$ with contact form induced by $\alpha=\frac{i}{8}\sum a_j(z_jd\bar z_j-\bar z_jdz_j)$ and assume $n\geq 3$, which means that M is at least 5-dimensional. The Reeb flow of the contact form α is given by $\varphi_t(z)=(e^{4it/a_0}z_0,\dots,e^{4it/a_n}z_n)$. In Remark 4.4 we observed that the first Chern class of the contact structure induced by α is zero. Moreover, all Reeb orbits are closed. We will use the notation we introduced in Chapter 4, in particular Section 4.2.1. The algorithm works as follows. It might be helpful to look at the examples in Section 11.0.5.1 to clarify the statements of the algorithm.

(1) Compute the homology of M using the algorithm of Randell [**42**] which we described in Section 4.2.1. This information can be used to determine more precisely what manifold M is (if the dimension of M is five, this step provides enough information to use the classification of Barden, see [**2**]). This step also involves some numerics that are used in Step (4).
(2) Identify all orbit types that can occur for the Reeb flow. This is done as follows. For all subsets $I_s\subset I=\{0,\dots,n\}$, $s>1$, the minimal positive time T such that $2T/\pi$ is divisible by all elements of a_i, $i\in I_s$, is $\frac{\pi}{2}\operatorname{lcm}_{i\in I_s}(a_i)$. The same minimal time T can occur for several sets I_s. Let J_T denote the largest such set. We obtain a collection of sets $J_{T_1},\dots,J_{T_k}$ for different $T_1,\dots,T_k$. The corresponding submanifolds $M_{T_i}:=K(J_{T_i})$ indicate submanifolds that are invariant under the time T_i Reeb flow. As in Section 4.2.1, $K(J_{T_i})$ denotes the Brieskorn manifold with exponents $a_{j_1},\dots,a_{j_s}$ where we have written $J_{T_i}=\{j_1,\dots,j_s\}$.
(3) Compute the Maslov indices of an orbit in M_{T_i} with time NT_i ($N\in\mathbb{N}$). In order to ensure that we do not consider orbits of another orbit space, we have to choose N such that for $i\neq j$ multiples NT_i are not divisible by T_j whenever $J_{T_i}\subset J_{T_j}$. Note that the Maslov index will only depend on the orbit type and not on the orbit itself. We may use the following formula if we choose N as above
$$\mu(S_{NT_i})=2\sum_{j\in J_{T_i}}\frac{2NT_i}{\pi a_j}+2\sum_{j\in I-J_{T_i}}\lfloor\frac{2NT_i}{\pi a_j}\rfloor+\#(I-J_{T_i})-4N\frac{T_i}{\pi}.$$
The algorithm fails if one does not obtain index positivity or negativity for the orbit spaces at this point. The conditions for these are given by
$$\sum_{j=0}^n\frac{1}{a_j}>1 \text{ for index positivity, and } \sum_{j=0}^n\frac{1}{a_j}<1 \text{ for index negativity.}$$
(4) The dimension of the orbit space $S_{T_i}=M_{T_i}/S^1$ is given by $2\#J_{T_i}-4$. Compute the rational homology of the orbit spaces S_{T_i} in the following way. First of all, compute the

rank of the $H_{\#J_{T_i}-2}(M_{T_i})$, given by

$$\kappa = \mathrm{rk}\tilde{H}_{\#J_{T_i}-2}(M_{T_i}) = \sum_{I_s \subset J_{T_i}} (-1)^{\#J_{T_i}-s} \frac{\prod_{j' \in I_s} a_{j'}}{\operatorname{lcm}_{j \in I_s} a_j}.$$

With Formula (4.2) we can compute the rational homology of the orbit spaces.

$$H_q(S_{T_i}, \mathbb{Q}) \cong \left\{ \begin{array}{l} \mathbb{Q},\ q \text{ even},\ 0 \le q \le \dim S_{T_i} \\ 0, \text{ otherwise} \end{array} \right\} \oplus \left\{ \begin{array}{l} \mathbb{Q}^\kappa,\ q = \frac{1}{2}\dim S_{T_i} \\ 0 \text{ otherwise} \end{array} \right\}.$$

(5) The cylindrical contact homology with $\mathbb{Q}$-coefficients of M with induced contact structure is a $\mathbb{Q}$-vector space, where the number of generators in each degree can be determined as follows. For each T_i we get rk $H_j(S_{T_i}, \mathbb{Q})$ generators in degree $\mu(S_{NT_i})+n-3+j-\frac{1}{2}\dim S_{T_i}$ for $j = 0, \dots, \dim S_{T_i}$ and $N \in \mathbb{N}$ such that for $j \neq i$ the multiples NT_i are not divisible by T_j whenever $J_{T_i} \subset J_{T_j}$ (using the Maslov-index that has been computed in Step 3).

For the cylindrical contact homology to be well-defined and an invariant of the contact structure there should be no generators in degree $-1, 0$ or 1. To check this, one needs to define Morse functions f_T on the orbit spaces S_T following Bourgeois's construction described in Section 10.3. The critical points of these Morse functions form the Morse-Bott chain complex with grading given by Formula (10.6). If there are no critical points with degree $-1, 0$ or 1 then the algorithm yields the cylindrical contact homology for contractible Reeb orbits. Note that these computations can depend on the choice of Morse functions.

Remark 11.1. We should emphasize at this point that cylindrical contact homology of Brieskorn manifolds is a periodic repetition of certain $\mathbb{Q}$-vector spaces with a degree shift. This can be seen as follows. Let us consider the orbit space S_{T_i} and multiple coverings S_{NT_i} where N is chosen such that for $i \neq j$ the time NT_i is not divisible by T_j whenever $J_{T_i} \subset J_{T_j}$. If we add

$$s_i := \frac{\pi \operatorname{lcm}_{j' \in I} a_{j'}}{2T_i}$$

to N, the corresponding orbit space remains the same, since $\frac{\pi \operatorname{lcm}_{j' \in I} a_{j'}}{2}$ is divisible by all T_j. We see that the Maslov index changes as follows,

$$\mu(S_{(N+s_i)T_i}) = \mu(S_{NT_i}) + 2 \operatorname{lcm}_{j \in I} a_j \left(\sum_{j'=0}^{n} \frac{1}{a_{j'}} - 1 \right).$$

This shift of the Maslov index is independent of the orbit space S_{T_i} and hence the terms in the contact homology corresponding to the orbit space S_{T_i} are repeated with period at most

$$2 \operatorname{lcm}_{j \in I} a_j \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1 \right).$$

The periodicity of contact homology allows us to stop the algorithm after a finite number of steps.

Remark 11.2. Note that the requirement that $n \geq 3$, is not strictly necessary. For $n = 2$ though, we are looking at 3-dimensional Brieskorn manifolds, which are in general not simply connected. We can of course deal with these cases in an easy way by considering only contractible Reeb orbits, but then one needs to investigate which Reeb orbits are contractible, which I have not done.

Remark 11.3. If we consider Brieskorn manifolds with large exponents, we can ensure that we have index negativity. In addition, large exponents ensure that the grading is strictly less than -1, guaranteeing that cylindrical contact homology is well-defined. Indeed if

$$\min_{i \in I} a_i \geq \frac{5n}{2},$$

then the algorithm will always give the cylindrical contact homology of the Brieskorn manifold $\Sigma(a_0,\dots,a_n)$. This estimate is rather rough and can be obtained from the formula for the Maslov index from Step (3).

Remark 11.4. Note that the contact homology is completely determined by the degree shifts coming from the Maslov index and the homology of the orbit spaces, which can be encoded in the homology of the orbifold $\Sigma(a_0,\dots,a_n)/S^1$ and its singular subspaces. See [**52**] for a more extensive discussion on a possible relation with orbifold cohomology and some open problems.

We have implemented this algorithm in a simple computer program whose code can be found in the appendix.

11.0.4. Proof of the algorithm. Consider the Brieskorn manifold $M=\Sigma(a_0,\dots,a_n)\subset\mathbb{C}^{n+1}$ with contact form induced by $\alpha=\frac{i}{8}\sum a_j(z_jd\bar z_j-\bar z_jdz_j)$. The differential $\omega:=d\alpha=\frac{i}{4}\sum a_jdz_j\wedge d\bar z_j$ is a symplectic form on $\mathbb{C}^{n+1}$. Let ξ be the contact structure given by $\ker\alpha|_{TM}$. Note that the contact form is of Morse-Bott type (Definition 10.6). This is seen as follows. Discreteness of the action spectrum is guaranteed by Step (2) of the algorithm. The sets M_T in Definition 10.6 are Brieskorn manifolds with their standard contact form. In particular they are closed submanifolds of M. Note that this verifies the rank condition on $d\alpha$ as well. The last condition for α being of Morse-Bott type can be checked by observing that the differential of the Reeb flow φ is diagonal. Namely, the differential of the Reeb flow described in Formula (4.1) is the diagonal matrix

$$T\varphi=\operatorname{diag}(e^{it/a_0},\dots,e^{it/a_n}).$$

The rank condition on $T\varphi-id$ which we require for a Morse-Bott contact form is then also verified.

We verify that Step (3) gives the correct Maslov indices. For each time T_i that we found in Step (2), consider the N-fold covering of M_{T_i} with N such that NT_i is not divisible by T_j's of a larger orbit space. Now let $p\in M_{T_i}$ and consider its orbit under the Reeb flow $\gamma(t):=\varphi_t(p)$ for $t\in[0,NT_i]$. To compute its Maslov index, we use the obvious extension of the Reeb flow to $\mathbb{C}^{n+1}$, and we denote this extension by φ_t as well. The symplectic action of the extended Reeb flow φ_t on $\mathbb{C}^{n+1}$ is given by the differential $T\varphi_t=\operatorname{diag}(e^{4it/a_0},\dots,e^{4it/a_n})$ for $t\in[0,NT_i]$. We denote the path of symplectic matrices induced by the extended flow by $\Phi_{\mathbb{C}^{n+1}}$. We can now use the additivity of the Maslov index and Formula (8.4) to get the index of this path. This gives us

$$\mu(\Phi_{\mathbb{C}^{n+1}})=2\sum_{a_j\in J_{T_i}}\frac{2NT_i}{\pi a_j}+2\sum_{a_j\in I-J_{T_i}}\lfloor\frac{2NT_i}{\pi a_j}\rfloor+\#(I-J_{T_i}).$$

In order to get the Maslov index of the Reeb flow, we use the additivity property of the Maslov index to note that we may subtract the Maslov index of the restriction of $\Phi_{\mathbb{C}^{n+1}}$ to the symplectic complement ξ^ω of the contact structure ξ in $\mathbb{C}^{n+1}$.

In the proof of Theorem 4.2 we found a symplectic basis of ξ^ω, which in terms of

$$X_1=(\bar z_0^{a_0-1},\dots,\bar z_n^{a_n-1}),\qquad Y_1=iX_1,$$
$$X_2=-2i(\frac{z_0}{a_0},\dots,\frac{z_n}{a_n}),\qquad Y_2=(z_0,\dots,z_n)$$

is given by

$$\tilde X_1=\frac{X_1}{\sqrt{\omega(X_1,Y_1)}},\qquad \tilde Y_1=\frac{Y_1}{\sqrt{\omega(X_1,Y_1)}}$$
$$\tilde X_2=X_2,\qquad \tilde Y_2=Y_2-\frac{\sum a_iz_i^{a_i}}{2\omega(X_1,Y_1)}X_1.$$

With respect to this basis, the action induced on ξ^ω by the extended Reeb flow is given by

$$T\varphi_t(\tilde X_1(x))=e^{4it}\tilde X_1(\varphi_t(x)),\qquad T\varphi_t(\tilde Y_1(x))=e^{4it}\tilde Y_1(\varphi_t(x))$$
$$T\varphi_t(\tilde X_2(x))=\tilde X_2(\varphi_t(x)),\qquad T\varphi_t(\tilde Y_2(x))=\tilde Y_2(\varphi_t(x)).$$

We use again Formula (8.4) with T substituted by NT_i to see to see that the Maslov index of this path of symplectic matrices is equal to

$$\mu(\Phi_{\xi^\omega}) = 4N\frac{T_i}{\pi}.$$

Taking the difference of the Maslov indices we found yields the desired result from Step (3). The conditions for index positivity and negativity can be found by observing that $\lfloor t \rfloor \geq t-1$ and $\lfloor t \rfloor \leq t$.

We show that the determinant bundle of the linearized Cauchy-Riemann operator can be oriented directly, i.e. we shall show that the linearized Cauchy-Riemann operator is asymptotic to a complex linear operator. First of all, we note that near the punctures this operator can be given the form,

$$\frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t} + S(s,t),$$

(see Formula 9.3) where S are symmetric matrices, J_0 is the standard complex structure, and (s,t) are cylindrical coordinates near the puncture, t for the S^1-coordinate. We have (see Formula (9.4))

$$\lim_{s\to\infty} S(s,t) = \tilde{S}(t),$$

with

$$\tilde{S}(t) = -J_0\frac{d\psi(t)}{dt}\psi^{-1}(t),$$

where ψ are the symplectic matrices obtained from the linearized Reeb flow in the symplectization $M\times\mathbb{R}$. Note that if these matrices $\psi(t)$ are unitary with respect to the above trivialization, the matrix $\tilde{S}$ will commute with J_0. We will verify that we can choose a trivialization such that the matrices ψ are unitary.

Note that the linearized extended Reeb flow on $\mathbb{C}^{n+1}$ is represented by a path of unitary matrices. Keep in mind that the metric is given by $\omega(\ldots, J\ldots)$, where J is the standard complex structure on $\mathbb{C}^{n+1}$. For the above purposes we need a trivialization that comes from the symplectization of M, so this one is not suitable. To that end, recall that a contact structure ξ is symplectically stably trivial if and only if $\xi\oplus\mathbb{C}$ is symplectically trivial (see also Remark 8.5). This means that we can split off a complex line bundle from our trivialization on $\mathbb{C}^{n+1}$. Define $V := \operatorname{span}_{\mathbb{R}}(\tilde{X}_1,\tilde{Y}_1)$ and let W be the orthogonal complement of V in $\mathbb{C}^{n+1}$. Because the linearized extended Reeb flow maps V to V, we can see that the linearized extended Reeb flow induces a map from W to W which is unitary with respect to the induced metric. Note also that ξ is a subbundle of W, and W contains the Reeb line bundle. Hence W can be identified with the tangent bundle of the symplectization of M restricted to $M\times\{\text{point}\}$. Let $\psi(t)$ be the path of unitary matrices given by the linearized extended Reeb flow on W with respect to an orthonormal basis of W. Then we define $\tilde{S}(t)$ as in the above.

It follows that the Cauchy-Riemann operator

$$\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \tilde{S}(t)$$

is complex linear. By the above discussion, the given linearized Cauchy-Riemann operator on the symplectization is asymptotic to this operator. As mentioned in Section 10.3, this gives us an orientation of the determinant bundle. In particular, no bad orbits or disorienting loops can occur.

Index positivity/negativity is verified in Step (3) of the algorithm. If there are no generators in degree $-1, 0$ or 1, then the homology of the Morse-Bott complex is isomorphic to cylindrical contact homology according to Theorem 10.11. Consider the Morse-Bott complex with generators the critical points p of the chosen Morse functions with grading given by $\mu(S_T) - \frac{1}{2}\dim S_T + \operatorname{index}_p f_T + n - 3$. The differential of the Morse-Bott complex is given by Formula (10.7). The differential acting on $p\in S_T$ is given by

$$dp = \partial p + \sum n_q q.$$

The first term is the Morse-Witten differential for the critical points of the Morse functions f_T. The second term counts the number of elements in the zero-dimensional part of the fibered product

$$(W^u(p) \times_S \mathcal{M}^{f_T}(S; S') \times_{S'} W^s(q))/\mathbb{R}.$$

Now note that there is an S^1-action on the symplectization of M induced by the Reeb flow. A holomorphic curve asymptotic to closed Reeb orbits comes therefore in at least an S^1-family (by letting the Reeb flow act) except in the case that the curve is a vertical cylinder. This means the above fibered product is at least 1-dimensional, so it will not contribute to the differential. This argument is due to Bourgeois, see for instance Section 9.3 of [**6**].

This implies that the only non-zero contribution to the differential comes from ∂p, which means that the cylindrical contact homology is isomorphic to the Morse-Witten homology of the orbit spaces S_T with degree shifts of $\mu(S_T) - \frac{1}{2}\dim S_T + n - 3$. As the Morse-Witten homology of the orbit spaces is equal to the rational homology of the orbit spaces which is computed in Step (4), we find that the contact homology is given by our algorithm.

11.0.5. Examples.

11.0.5.1. *Cylindrical homology of some contact structures on $S^2 \times S^3$.* In this section we consider the family of Brieskorn manifolds of the form $M = \Sigma(2l, 2, 2, 2)$ for $l > 1$. Using our algorithm, it turns out that these manifolds are diffeomorphic to $S^2 \times S^3$ and that their cylindrical contact homologies are all isomorphic. This is a bit exceptional, since typically we get very different homologies for different exponents. In the Section 11.1.1 we find some new exotic contact structures on spheres which illustrates the latter point. Let us now turn our attention to the application of the algorithm to $M = \Sigma(2l, 2, 2, 2)$. The numbering is as in the algorithm.

(1) We find $\operatorname{rk} H_2(M) = 1$. Computation of the homology torsion by Randell's algorithm shows that there is none. By the classification of simply connected five manifolds [**2**] we see that $M \cong S^2 \times S^3$, as the second Stiefel-Whitney class of M is zero.

(2) The time $T_1 = \frac{\pi}{2}2$ is the minimal time for the sets $I_s \subset \{0, \dots, n\} = I$ which do not include 0. Hence we see $J_{T_1} = \{1, 2, 3\}$.
The time $T_2 = \frac{\pi}{2}2l$ appears as minimal time for the set $I = \{0, 1, 2, 3\}$. So $J_{T_2} = \{0, 1, 2, 3\}$.

(3) We get, for $NT_1\frac{2}{\pi}$ not divisible by l,
$$\mu(S_{NT_1}) = 2N + 1 + 2\lfloor \frac{N}{l} \rfloor.$$
The principal orbits have Maslov index
$$\mu(S_{NT_2}) = 2lN + 2N.$$

(4) We find $\dim S_{T_1} = 2$ with $H_0(S_{T_1}, \mathbb{Q}) = \mathbb{Q}$ and $H_2(S_{T_1}, \mathbb{Q}) = \mathbb{Q}$. All other homology groups are 0. The orbit space S_{T_2} has dimension 4, with the homology $H_0(S_{T_2}, \mathbb{Q}) = \mathbb{Q}$, $H_2(S_{T_2}, \mathbb{Q}) = \mathbb{Q}^2$ and $H_4(S_{T_3}, \mathbb{Q}) = \mathbb{Q}$. The other homology groups are zero.

(5) The period we defined in (11.1) is equal to $2 + 2l$. This allows us to compute fewer terms. For the first period, the contributions from S_{NT_1} lie in degree
$$2N + 2\lfloor \frac{N}{l} \rfloor + k$$
for $N = 1, \dots, l-1$ (since we are considering a single period) and $k = 0, 2$. The contribution due to S_{NT_2} are in degree
$$2l + k,$$
for $k = 0, 2, 4$. For the first period, we get one generator in degree 2, two generators in degree $4, 6, \dots, 2l+2$ and one generator in degree $2l+4$. Hence the cylindrical contact homology has one generator in degree 2 and two generators in degree $2k$ for $k > 1$. We note that the cylindrical homology is well-defined, as there are no generators in degree $-1, 0$ or 1 (lowest degree is higher than 1).

We can also apply the algorithm to $\Sigma(2,2,2,2)$, which has just a single orbit space. This yields the same contact homology.

11.0.5.2. *Some contact structures with index negativity.* Let us consider Brieskorn manifolds with large exponents such that we have index negativity and that the degree is strictly less than -1. In case all exponents are equal, contact homology is particularly easy to compute. For simplicity, we consider examples coming from Brieskorn manifolds of the form $M = \Sigma(k,k,k,k)$ with $k \geq 6$. If we are just interested in the contact homology of M, then we may skip Step (1) (which would allow us to identify M). Since M has only a single orbit space, the computations are simple.

The minimal return time is $T = \frac{\pi}{2}k$. Hence we get the Maslov index

$$\mu(S_{NT}) = 2 \cdot 4 \cdot N - 2 \cdot N \cdot k = 2N(4-k).$$

Step (4) of the algorithm shows that $H_0(S_T;\mathbb{Q}) \cong \mathbb{Q}$, $H_2(S_T;\mathbb{Q}) \cong \mathbb{Q}^d$, where $d = \kappa + 1 = \frac{(k-1)^4-1}{k} + 2$. The last homology group is $H_4(S_T;\mathbb{Q}) \cong \mathbb{Q}$. This gives the information needed for a single period of contact homology. Taking the Maslov index into account we get the following for the contact homology of M.

We have one generator in degree $2N(4-k)-2$ for $N = 1,2,\ldots$. In degree $2N(4-k)$ we have $\frac{(k-1)^4-1}{k} + 2$ generators and in degree $2N(4-k)+2$ we have again one generator ($N = 1,2,\ldots$).

Another way to phrase the result is the following. First notice that since all exponents of the Brieskorn manifold are equal, the contact manifold $\Sigma(k,k,k,k)$ can be identified with an S^1-bundle over a symplectic manifold. The contact homology of $\Sigma(k,k,k,k)$ is given by the singular homology of this underlying symplectic manifold repeated with degree shifts of $8-2k$.

11.1. Exotic contact structures

Our aim in this section is to describe a certain class of contact manifolds that admits infinitely many non-isomorphic contact structures.

Given two contact manifolds (M_1,ξ_1) and (M_2,ξ_2), we can build a new contact manifold by forming their connected sum, see [**36**] and [**53**]. If we think of a connected sum as first removing a disk from both M_1 and M_2 and then gluing them via a "connecting" tube, then the contact structure on $M_1 \# M_2$ can be made to coincide with the contact structure on M_1 with a disk removed and M_2 with a disk removed.

In order to say something about the contact homology of the connected sum, we find generic contact forms α_1, α_2 for the contact structures ξ_1 and ξ_2, i.e. contact forms whose closed Reeb orbits are non-degenerate.

First we need another, but similar notion of index positivity, which we will refer to as Ustilovsky index positivity. Suppose that a contact structure ξ is symplectically stably trivial and let F be a corresponding trivialization. We may then compute the Maslov index with respect to the trivialization F. The index does depend on the trivialization and we indicate this by making the Maslov index visibly dependent on the trivialization F by writing $\mu(\ldots,F)$. We use ε to indicate the trivial bundle.

Definition 11.5. Let (M,α) be a contact manifold. Assume that $\pi_1(M) = 0$ and that the bundle $(\xi, d\alpha)$ is symplectically stably trivial. Let F be a symplectic trivialization of $\xi \oplus \varepsilon^2$. The contact form α is called **Ustilovsky index-positive** if there exist constants $c > 0$ and d such that for any Reeb orbit γ we have

$$\mu(\gamma;F) \geq c\mathcal{A}(\gamma) + d.$$

Ustilovsky has shown in his thesis that this notion does not depend on the choice of trivialization.

Example 11.6. Brieskorn manifolds are symplectically stably trivial (for instance, see our proof of Theorem 4.2). Hence we may consider Ustilovsky index positivity of a Brieskorn manifold $\Sigma(a_0,\ldots,a_n)$. Here we really need $n \geq 3$, because many Brieskorn manifolds in dimension 3 are not simply-connected. Note that a Brieskorn manifold has Ustilovsky index positivity if all orbit types have index positivity. Namely, our computation of the Maslov indices for closed orbits used the trivialization coming from $\mathbb{C}^{n+1}$. If we use that trivialization to verify Ustilovsky index positivity, we see that the formula from Step (3) of the algorithm can be modified as follows:

$$\mu(\gamma(t)|_{t\in[0,T]}) = \sum_{j=0}^{n} \mu(e^{4it/a_j}|_{t\in[0,T]}) - \mu(e^{4it}|_{t\in[0,T]}),$$

where the Reeb orbit γ is given by

$$\gamma(t) = (e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n),$$

for $(z_0, \dots, z_n) \in \Sigma(a_0, \dots, a_n) \subset \mathbb{C}^{n+1}$. Then apply Formula (8.4).

We have the following theorem from Ustilovsky ([**51**], Theorem 5.2.1):

THEOREM 11.7 (Ustilovsky). *Let (M_1, α_1), (M_2, α_2) be two simply connected contact manifolds of dimension $2n-1$ that have Ustilovsky index positivity. Assume all periodic orbits of R_{α_1} and R_{α_2} are non-degenerate. Then for any integer N there exists a contact form α on $M = M_1 \# M_2$ so that*

(1) *(M, α) is Ustilovsky index-positive.*
(2) *All periodic orbits of R_α are non-degenerate.*
(3) *If c_j^1, c_j^2 and c_j denote the numbers of periodic Reeb orbits of degree j in M_1, M_2 and M_3, respectively, then for $j \leq N$, we have $c_j = c_j^1 + c_j^2 + \beta_j$, where $\beta_j = 1$ for $j = 2n-3, 2n-1, \dots$, and $\beta_j = 0$ otherwise.*

The β_j's in this theorem are the degrees of the periodic Reeb orbits in the "connecting tube". We will take for M_1 any contact manifold satisfying the conditions of the above theorem of Ustilovsky and for M_2 we will take a special contact sphere. This sphere will have the property that its contact homology contains generators with degree lower than the lowest degree of a generator in the "connecting tube". After taking the connected sum with M_1 and using Ustilovsky's theorem, the resulting contact manifold $M_1 \# M_2$ will be diffeomorphic to M_1, but its cylindrical contact homology will have more generators in low degrees.

11.1.1. Construction of a special contact sphere. Let us consider the Brieskorn manifold $M = \Sigma(p_1, \dots, p_{n-1}, 2, 2)$ where $p_1, \dots, p_{n-1}$ are odd primes which will be specified later (they need to be chosen large enough). Notice that we immediately see that this manifold has the homology of a sphere by Theorem 4.5. Then use the fact that Brieskorn manifolds of dimension at least 5 are always simply connected and conclude that M is homeomorphic to a sphere with the generalized Poincaré conjecture as proved by Smale. We apply our algorithm to compute the first terms of the contact homology.

11.1.1.1. *Contact homology of $\Sigma(p_1, \dots, p_{n-1}, 2, 2)$.* We have the following orbit types and Maslov indices.

- $I_2 = \{2, 2\}$. The minimal return time for orbits of this type is $T = \pi$. The Maslov index of the corresponding orbits is given by

$$\mu(S_{NT}) = 2\sum_{i=1}^{n-1} \lfloor \frac{2N}{p_i} \rfloor + n - 1 \geq n - 1.$$

  If the p_i's are odd primes, the first term will vanish for at least $N = 1$. Now we turn our attention to the homology of the orbit space. It has dimension 0 and Formula (4.2) shows that $H_0(S_T, \mathbb{Q}) = \mathbb{Q}^2$. This shows that the contact homology of M has at least two generators in degree $2n-4$. Since the first term is always even, multiple covers of this orbit will have either the same degree or a higher even degree. Note that this orbit type will not give any generators in degree $2n-3$.

- Sets of the form $\{p_{i_1}, \dots, p_{i_k}\}$ with k at least 2. The minimal return time is now $T = p_{i_1} \cdot \dots \cdot p_{i_k} \frac{\pi}{2}$. The Maslov indices of the orbit spaces are given by

$$\begin{aligned} \mu(S_{NT}) &= 2\sum_j p_{i_1} \cdot \dots \cdot \hat{p}_{i_j} \cdot \dots \cdot p_{i_k} N + 4\lfloor \frac{N p_{i_1} \cdot \dots \cdot p_{i_k}}{2} \rfloor \\ &+ 2\sum_{l \neq i_j} \lfloor \frac{N p_{i_1} \cdot \dots \cdot p_{i_k}}{p_l} \rfloor + n - 1 - k + 2 - 2N p_{i_1} \cdot \dots \cdot p_{i_k}. \end{aligned}$$

  Note that $4\lfloor \frac{N p_{i_1} \cdot \dots \cdot p_{i_k}}{2} \rfloor \geq 2N p_{i_1} \cdot \dots \cdot p_{i_k} - 2$ and that $\dim S_T = 2k - 4$. Using this estimate, we find that the degree of the associated generators can be estimated from below as

$$\text{degree} \geq 2n - 2 - 2k + 2\sum_j p_{i_1} \cdot \dots \cdot \hat{p}_{i_j} \cdot \dots \cdot p_{i_k} N.$$

  Since the sum contains at least two terms, we can make the Maslov index arbitrarily large by choosing big primes. In particular, the degree of these orbit types can be assumed to be larger than $2n - 3$.
- Sets of the form $\{p_{i_1}, \dots, p_{i_k}, 2, 2\}$. The minimal return time is $T = p_{i_1} \cdot \dots \cdot p_{i_k} \pi$. The associated Maslov indices are

$$\mu(S_{NT}) = 4\sum_j p_{i_1} \dots \hat{p}_{i_j} \cdot \dots \cdot p_{i_k} N + 2\sum_{l \neq i_j} \lfloor \frac{2N p_{i_1} \cdot \dots \cdot p_{i_k}}{p_l} \rfloor + n - 1 - k.$$

  For $k > 1$ the first term will contain at least one p_i-term. This means that the degree (note that the dimension of the orbit space is now $2k$) will become as large as we like by choosing the p's accordingly. For $k = 1$ the first term is $4N$ and we see that the degree is at least $4N + n - 3 + n - 3 \geq 2n - 2$.

Summarizing these estimates, we see that by choosing suitable primes we may assume that the contact homology contains at least two generators in degree $2n - 4$ and no generators in degree $2n-3$. Note that we have (Ustilovsky) index positivity since two exponents are 2, see the condition from Step (3) of the algorithm.

In order to be able to apply Theorem 11.7, we need to have a generic contact form on M that has at least two closed Reeb orbits in degree $2n - 4$ and no generators in degree $2n - 3$. We will do this, along with a more general statement in the following interlude.

11.1.1.2. *Generic contact forms.* In this section, we want to associate a generic contact form, i.e. a contact form whose closed Reeb orbits are non-degenerate, to the Morse-Bott contact forms used in our algorithm.

Let M be a contact manifold of dimension $2n - 1$ and let α be a contact form on M that satisfies the Morse-Bott condition. We use Bourgeois's construction of Morse functions on the orbit spaces described in Section 10.3. We use his ideas to perturb α into a generic contact form such that we still have some information on the indices of the closed Reeb orbits. The following observations by Ustilovsky [**51**] will play a key role.

Lemma 11.8. *If (M, α) is Ustilovsky index-positive, then a C^∞-small perturbation α' of α the manifold (M, α') is also Ustilovsky index-positive. Moreover, if $\mu(\gamma; F) \geq c\mathcal{A}(\gamma) + d$ for orbits γ of R_α then, for α' close enough to α, $\mu(\gamma'; F') \geq c'\mathcal{A}(\gamma') + d'$ for orbits γ' of $R_{\alpha'}$, where $c' = c/2$ and $d' = -|d| - 2n$.*

Remark 11.9. Note that for a small perturbation, there is a one-to-one correspondence between non-degenerate Reeb orbits of the contact form and of the perturbed contact form up to some period. The Conley-Zehnder indices of the corresponding orbits are the same. This can be seen with the following argument.

In our definition of the Conley-Zehnder index we start by choosing an extension of the path of symplectic matrices to either W^+ or W^-. For a small perturbation, the endpoint of the path of symplectic matrices will be in the same component of $\mathrm{Sp}^*(2n)$, so we choose an extension to the same point $W^\pm$ for both the unperturbed and the perturbed path. This means that the extended paths will be homotopic with fixed endpoints, so the perturbed path will have the same Conley-Zehnder index as the unperturbed path.

LEMMA 11.10. *Let α be the standard contact form on the Brieskorn manifold $\Sigma(a_0,\dots,a_n)$, $n \geq 3$. Assume that the exponents are such that we have Ustilovsky index positivity (cf. Example 11.6). Then for all $N \in \mathbb{Z}$ there exists a generic contact form α', such that all generators of the chain complex of α' coincide with the generators of the Morse-Bott chain complex of α up to degree N.*

PROOF. Let us denote the constants from the Ustilovsky index positivity by $c > 0$ and d such that $\mu(\gamma) \geq c\mathcal{A}(\gamma) + d$ for a part γ of a Reeb orbit. Choose $T \geq \max\{N, \frac{4}{c}(N + |d| + 4n)\}$. By Lemma 10.7 we find a perturbation α'' of α such that its periodic Reeb orbits up to action T are non-degenerate and correspond to critical points of the chosen Morse functions on the orbit spaces. Since we have Ustilovsky index positivity for α, we will have the same for α'' by Lemma 11.8, where the constants are now $c/2$ and $-|d| - 2n$.

We perturb α'' further to make all Reeb orbits non-degenerate and call the perturbation α'. Once again, α' is Ustilovsky index positive with constants $c/4$ and $-|d| - 4n$. By Remark 11.9 this perturbation will not change the Conley-Zehnder indices of orbits with period up to T (and in particular up to N). The perturbation will introduce new periodic Reeb orbits. However, the newly created ones can be made to have period larger than T (by making a small enough perturbation). As a result of Lemma 11.8, their Conley-Zehnder indices will be larger than $Tc/4 - |d| - 4n \geq N$. This proves our lemma. □

11.1.1.3. *A generic contact form for $\Sigma(p_1,\dots,p_{n-1},2,2)$.* We continue with our construction of a contact sphere from Section 11.1.1. As before, we write our Brieskorn sphere as $M = \Sigma(p_1,\dots,p_{n-1},2,2)$. We get the Morse-Bott chain complex for contact homology by following Bourgeois's construction of Morse functions for the orbit spaces from Section 10.3. Note that the computation of the contact homology of M in Section 11.1.1.1 shows that the Morse-Bott complex will have at least two generators in degree $2n-4$ and no generators in degree $2n-3$. This holds true because the lowest-dimensional orbit spaces have dimension 0, so the number and degree of generators associated to those orbit spaces do not depend on the choice of Morse functions. Note that these data do depend on the choice of Morse functions for the other orbit spaces.

Now we apply Lemma 11.10 to obtain a generic contact form where the the number and degree of generators of the chain complex coincide with those of the Morse-Bott complex up to degree $2n-2$. This gives M a generic contact form and allows us to use Theorem 11.7.

11.1.2. Constructing new contact structures.

THEOREM 11.11. *Let (M,ξ) be a simply-connected contact manifold. Assume furthermore that M admits a nice contact form (a contact form without any closed Reeb orbits of degree $-1, 0$ or 1) that has Ustilovsky index positivity (in particular $c_1(\xi) = 0$). Then M admits infinitely many non-isomorphic contact structures.*

PROOF. Let $N' = \Sigma(p_1,\dots,p_{n-1},2,2)$. The above discussion shows that N' admits a generic contact form that is Ustilovsky index positive with at least two generators in degree $2n-4$, no generators in degree $2n-3$ and no generators in lower degrees. As remarked before, we know that N' is homeomorphic to a sphere. Since the differentiable structures on a sphere form a finite group, we can find an r such that $N := \underbrace{N'\#\dots\#N'}_{r}$ is diffeomorphic to the standard sphere. By Theorem 11.7, N admits a generic contact form whose cylindrical contact homology has at least $2r$ generators in degree $2n-4$, precisely $r-1$ generators in degree $2n-3$ and no generators in lower degrees.

Now apply Theorem 11.7 to M and N. The connected sum $M\#N$ will be diffeomorphic to M, since N is diffeomorphic to S^{2n-1}. The theorem shows that the connected sum still admits a nice contact form (because no generators are added in degrees $-1, 0$ or 1). The number of generators of the chain complex of the cylindrical contact homology is increased by $2r$ in degree $2n-4$ and by r in degree $2n-3$. Because of Ustilovsky index positivity there are only a finite number of generators in each degree. Since the number of generators in degree $2n-5$ is unchanged by taking the connected sum with N, taking repeated connected sums with N will ensure that we get a contact structure on M with a contact homology different from the original one, namely with more generators in degree $2n-4$. By taking more connected sums with N we get infinitely many contact structures on M, all with different cylindrical contact homology distinguished by the rank of the homology in degree $2n-4$. □

REMARK 11.12. Although Ustilovsky never mentions this theorem, I am sure he knew this result already, at least in case the contact manifold has dimension $4k+1$, since we can also use the exotic contact spheres of Ustilovsky in those cases. The theorem in itself is not so useful if we do not have a way to generate "nice" contact forms which we require in the theorem. In fact the only cases I could find in the literature are spheres. Because of Lemma 11.10 I thought this theorem was worth mentioning, since Lemma 11.10 gives a way to construct a large family of "nice" contact manifolds suitable for use in the theorem. This family of Brieskorn manifolds does not consist of spheres only.

Therefore we can combine this theorem with our algorithm to construct infinitely many contact structures on a large class of manifolds. We will focus here on contact manifolds of dimension 5. The methods do work in all dimensions, but we should mention that in case the dimension of the contact manifold is $4k-1$, there are infinitely many homotopy classes of almost contact structures on the sphere. This means that invariants from algebraic topology could then already suffice to prove the existence of infinitely many contact structures on a given contact manifold, see for instance [**19**]. However, in dimension $4k+1$ there are only finitely many homotopy classes of almost contact structures on the sphere.

From the example we considered earlier we see that the Brieskorn manifold $\Sigma(2,\dots,2)$ has an index positive contact structure whose cylindrical contact homology is well-defined. Note that we can identify $\Sigma(2,\dots,2)$ with (ST^*S^n,λ_{can}). We can use Lemma 11.10 to perturb the contact form to a nice contact form. Thus Theorem 11.11 implies that ST^*S^n admits infinitely many contact structures for $n>2$ with trivial first Chern class. Repeated application of the theorem shows the following corollary.

COROLLARY 11.13. *Let $n>2$ and $k>0$. The manifold*

$$\underbrace{ST^*S^n\#\dots\#ST^*S^n}_{k\ times}$$

admits infinitely many contact structures with trivial first Chern class.

11.1.3. Exotic contact structures in dimension 5. We want to show that a large class of simply connected five-manifolds admit infinitely many contact structures in a single homotopy class of almost contact structures. In the last section we have already shown that $ST^*S^3\cong S^2\times S^3$ satisfies this. Since connected sums of nice index positive contact manifolds are again nice index positive contact manifolds, we try to find such contact structures on some of the prime manifolds, see Section 6.1. Application of our algorithm shows that the Brieskorn manifolds $\Sigma(2,3,3,3)$, $\Sigma(2,3,4,4)$ and $\Sigma(5,2,3,6)$ are both nice index positive contact manifolds. Their homologies can be computed using Randell's algorithm and are isomorphic to $\mathbb{Z}_2\oplus\mathbb{Z}_2$, $\mathbb{Z}_3\oplus\mathbb{Z}_3$ and $\mathbb{Z}_5\oplus\mathbb{Z}_5$, respectively. This means that we have covered a few of the prime manifolds mentioned in Section 6.1.

We will show that many other prime manifolds admit infinitely many contact structures as well. I have not found a nice family of index positive contact structures on them, so we cannot invoke Theorem 11.11 to use them in connected sums. Let us consider the Brieskorn manifolds of the form $\Sigma(2^k,6,9,p)$, where $k\geq 2$ and p is relatively prime with 2 and 3. We use Randell's algorithm to compute the homology and find that

$$H_2(\Sigma(2^k,6,9,p);\mathbb{Z})\cong\mathbb{Z}_p\oplus\mathbb{Z}_p.$$

Hence we have found a one-parameter family of Brieskorn manifolds with the same homology. Let us put $a_0=2^k$, $a_1=6$, $a_2=9$ and $a_3=p$ in order to use the same notation as in our algorithm. Since $\sum_{i=0}^3\frac{1}{a_i}<1$, the contact structure on these Brieskorn manifolds are index negative. Although it is rather straightforward to compute the contact homology now, we still need to show that the contact homologies for different k are not isomorphic. To that end, we introduce a kind of weighted Euler characteristic. Define

$$\chi_W((M,\alpha)) = \lim_{N\to\infty} \frac{\sum_{i=-N}^{N}(-1)^i \operatorname{rk}(CH_i(M,\alpha))}{N}.$$

Since the cylindrical contact homology of Brieskorn manifolds is, if defined, a periodic repetition of $\mathbb{Q}$-vector spaces, this limit exists. Indeed, we see that this limit is equal to the Euler characteristic of a single period as computed in our algorithm of the contact homology divided by the period

$$2 \operatorname*{lcm}_{j\in I} a_j \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1 \right),$$

since this period is always even. Note that odd periods would give a 0 "weighted" Euler characteristic.

Note that the above definition for this "weighted" Euler characteristic is obviously an invariant of the homology. We compute this invariant by computing the Euler characteristic of single period of contact homology and dividing by the period. We will now give some details for this computation. These details give almost enough information to get the full contact homology as well, so the reader might ask why one could not use this directly. To distinguish contact structures by the ranks of the contact homology, we could try to follow Ustilovsky [**50**] and find that there are "degree jumps" between the homologies of different Brieskorn manifolds. However, we would have many more orbit types involved in such an argument. Therefore I preferred the present approach.

The above formula for the period gives

$$\text{period} = 2\left(3^2 p + 2^k(\frac{3}{2}p + p + 3 - 3^2 p)\right).$$

As mentioned before, we see that this is a negative number for the values of p and k that we are considering. To complete the argument, we list a table containing the data we need. We list the orbit types, the minimal closure time of the Reeb orbits in that orbit type, how often that orbit type contributes within a single period of contact homology, the homology of that orbit space and its contribution to the Euler characteristic. I determined how often a orbit type appears within the first period by dividing the time of a full period by the time of that orbit type. That way one overcounts because of the larger orbit spaces that appear. We subtract how often larger orbit types appear. In other words, we made the table starting at the largest orbit type, which can only appear once, and worked our way back.

Orbit type	Minimal time	How often	H_*	χ
$(2^k,6)$	$2^k\cdot 3$	$3p-(p-1)-(3-1)-1$	$\mathbb{Q}^2$	2
$(2^k,p)$	$2^k p$	$3^2-(3-1)-1$	$\mathbb{Q}$	1
$(6,9)$	$2\cdot 3^2$	$2^{k-1}p-(p-1)-(2^{k-1}-1)-1$	$\mathbb{Q}^3$	3
$(6,p)$	$2\cdot 3p$	$2^{k-1}\cdot 3-(2^{k-1}-1)-(3-1)-1$	$\mathbb{Q}$	1
$(9,p)$	$3^2 p$	$2^k-(2^{k-1}-1)-1$	$\mathbb{Q}$	1
$(2^k,6,9)$	$2^k\cdot 3^2$	$p-1$	$\mathbb{Q},\mathbb{Q}^2,\mathbb{Q}$	0
$(2^k,6,p)$	$2^k\cdot 3p$	$3-1$	$\mathbb{Q},0,\mathbb{Q}$	2
$(6,9,p)$	$2\cdot 3^2 p$	$2^{k-1}-1$	$\mathbb{Q},0,\mathbb{Q}$	2
$(2^k,6,9,p)$	$2^k\cdot 3^2 p$	1	$\mathbb{Q},0,\mathbb{Q},0,\mathbb{Q}$	3

We get the Euler characteristic of the first period of contact homology by multiplying the Euler characteristic of each orbit type with the number of times that orbit type appears. We obtain the following formula for the "weighted" Euler characteristic.

$$\chi_W(\Sigma(2^k,6,9,p)) = -\frac{p+8+(\frac{3p}{2}+1)2^k}{-18p+(13p-18)2^k}.$$

Note that functions of the form $\frac{a+bx}{c+dx}$ are injective if denominator and numerator are not multiples of one another. For fixed p the above expression has such a form and hence we see that the contact manifolds $\Sigma(2^k, 6, 9, p)$ are not contactomorphic for different k, although they are diffeomorphic for fixed p.

We summarize a somewhat weakened version of the above discussion in the following corollary.

COROLLARY 11.14. *The prime manifolds B_{p^k} admit infinitely many contact structures for primes $p > 3$ and all $k \geq 1$. The prime manifolds B_2, B_3 and $B_\infty = S^2 \times S^3$ also admit infinitely many contact structures. In addition, we have that connected sums of the form*

$$S^2 \times S^3 \underbrace{\# \dots \#}_{k \ times} S^2 \times S^3 \# B_2 \underbrace{\# \dots \#}_{l \ times} B_2 \# B_3 \underbrace{\# \dots \#}_{m \ times} B_3 \# B_5 \underbrace{\# \dots \#}_{n \ times} B_5$$

admit infinitely many contact structures. All contact structures constructed in this corollary have the same formal homotopy class of almost contact structures.

Appendix A: Brieskorn algorithm in C

This program performs the steps of Randell's algorithm and of the algorithm to compute contact homology. It will always show a bit more than a single period of contact homology. If more is needed, then one needs to modify the "mindeg" and "maxdeg" variables in an appropriate way.

```
#include <iostream.h>

typedef int *wijs;
typedef wijs *wijzer;

int n=4;

class list{
public:
  list *first;
  list *next;
  int k,nummer;
  int lcm;
  list()
  {
    next=NULL;
  }
  void nieuw(int nieuw_lcm,int k1,int n1);
  void check(int nieuw_lcm,int k1,int n1);
  void laatzien();
};

void list::laatzien()
{
  cout << k << " " << nummer << " met lcm " << lcm << endl;
  if( next!=NULL )
    next->laatzien();
}

void list::nieuw(int nieuw_lcm,int k1,int n1)
{
  next=new list;
  next->first=first;
  next->k=k1;
  next->nummer=n1;
  next->lcm=nieuw_lcm;
}

void list::check(int nieuw_lcm,int k1,int n1)
{
```

```
  if( nieuw_lcm==lcm )
    {
      k=k1; nummer=n1;
    }
  else
    {
      if( next!=NULL )
        {
          next->check(nieuw_lcm,k1,n1);
        }
      else
        {
          // create a new member in the list
          nieuw(nieuw_lcm,k1,n1);
        }
    }
}

int fact(int k)
{
  if( k==0 )
    return 1;
  else
    return k*fact(k-1);
}

int binom(int n, int k)
{
  return fact(n)/fact(k)/fact(n-k);
}

int gcd(int k, int l)
{
  int hulp;

  if( l>k )
    {
      hulp=l; l=k; k=hulp;
      return gcd(k,l);
    }
  if( k%l==0 )
    return l;
  else
    return gcd(l,k%l);
}

int gcdset(int *exponent,wijzer *b, int k, int tel)
{
  int i;
  int currentgcd=1;
  for( i=0; i<=k; i++)
    {
      currentgcd*=exponent[b[i][k][tel]];
    }
```

```
  for( i=0; i<=k; i++)
    {
      currentgcd=gcd(exponent[b[i][k][tel]],currentgcd);
    }
  return currentgcd;
}

int gcdsetcomplement(int *exponent,wijzer *b, int k, int tel)
{
  int i,j,found;
  int currentgcd=1;

  for( i=0; i<n; i++)
    {
      currentgcd*=exponent[b[i][n-1][0]];
    }
  for( i=0; i<n; i++)
    {
      found=0; j=0;
      while( !found && j<=k )
        {
          if( b[j][k][tel]==b[i][n-1][0] )
            found=1;
          j++;
        }
      if( !found )
        {
          currentgcd=gcd(exponent[b[i][n-1][0]],currentgcd);
        }
    }
  return currentgcd;
}

int lcm(int k, int l)
{
  return k*l/gcd(k,l);
}

int lcmset(int *exponent,wijzer *b, int k, int tel)
{
  int i;
  int currentlcm=1;
  for( i=0; i<=k; i++)
    {
      currentlcm=lcm(exponent[b[i][k][tel]],currentlcm);
    }
  return currentlcm;
}

int prodset(int *exponent,wijzer *b, int k, int tel)
{
  int i;
  int prod=1;
  for( i=0; i<=k; i++)
```

```
    {
      prod*=exponent[b[i][k][tel]];
    }
  return prod;
}

int setcheck(wijzer *b,int k,int tel,int i,int l)
{
  int j1,j2,found;

  for( j2=0; j2<=i; j2++ )
    {
      j1=0; found=0;
      while( !found && j1<=k )
        {
          if( b[j2][i][l]==b[j1][k][tel] )
            {
              found=1;
            }
          j1++;
        }
      if( !found )
        return 0;
    }
  return 1;
}


main()
{
  cout << "Number of exponents: ";
  cin >> n;

  int i,tel,k,l;
  int loop;
  int *a,*last;
  wijs *kappa,*lcms,*tkappa,*C;
  wijzer *b;
  int *exponent;
  int sign,rmax,d;
  int found,maxlcm;
  list *orbits,*first;
  int *chrank;
  int mindeg,mindeg2,maxdeg,maxdeg2,period,degree;

  //allocate memory
  a=new int[n];
  last=new int[n];
  kappa=new wijs[n];
  tkappa=new wijs[n];
  lcms=new wijs[n];
  C=new wijs[n];

  b=new wijzer[n];
```

```
exponent=new int[n];
orbits=new list;

for( i=0; i<n; i++ )
  {
    cout << "exponent a_" << i << ": ";
    cin >> exponent[i];
  }

for( l=0; l<n; l++ )
  {
    b[l]=new wijs[n];
    for( i=0; i<n; i++ )
      {
        (b[l])[i]=new int[binom(n,i+1)];
      }
    kappa[l]=new int[binom(n,l+1)];
    tkappa[l]=new int[binom(n,l+1)];
    lcms[l]=new int[binom(n,l+1)];
    C[l]=new int[binom(n,l+1)];
  }

//initialize arrays to count properly; the entries are b[k][i][j]
//we count sets with i elements as follows
// (0,..,i-1),(0,..,i-2,i),..,(0,..,i-2,i),(0,..,i-3,i-1,i)
// i indicates the number of elements, j which i-element set
//(as counted above) and k the k-th entry of that set
//eg b[i-1][i-1][0]=i-1, b[i-1][i-1][1]=i, b[0][i-1][1]=0
for( i=0; i<n; i++ )
  {
    last[i]=binom(n,i+1)-1;
    for( k=0; k<=i; k++ )
      {
        ((b[k])[i])[last[i]]=(n-1)-i+k;
      }
    for( k=0; k<=i; k++ )
      {
        ((b[k])[i])[0]=k;
      }
  }

for( i=0; i<n; i++ )
  {
    for( tel=1; tel<binom(n,i+1)-1; tel++ )
      {
        loop=1;
        k=i;
        for( l=0; l<i; l++ )
          {
            ((b[l])[i])[tel]=((b[l])[i])[tel-1];
          }
        ((b[k])[i])[tel]=((b[k])[i])[tel-1]+1;
        while( loop )
          {
```

```
            if( ((b[k])[i])[tel]==((b[k])[i])[last[i]]+1 && k!=0 )
              {
                ((b[k-1])[i])[tel]=((b[k-1])[i])[tel-1]+1;
                k--;
              }
            else
              loop=0;
          }
        //reset
        k++;
        while( k<=i )
          {
            ((b[k])[i])[tel]=((b[k-1])[i])[tel]+1;
            k++;
          }
      }
  }

//compute lcm
for( k=0; k<n; k++ )
  {
    for( tel=0; tel<binom(n,k+1); tel++ )
      {
        lcms[k][tel]=lcmset(exponent,b,k,tel);
      }
  }

//compute kappa as in Randell's algorithm
rmax=0;
for( k=0; k<n; k++ )
  {
    for( tel=0; tel<binom(n,k+1); tel++ )
      {
        sign=(k+1)%2;
       if( sign==0 )
         sign=+1;
       else
         sign=-1;
        kappa[k][tel]=sign;

        // sum over subsets with i+1 elements
        for(i=0; i<k; i++)
          {
            // sets with i+1 elements
            for( l=0; l<binom(n,i+1); l++ )
              {
                //check whether set is a subset of b[-][k][tel]
                if( setcheck(b,k,tel,i,l) )
                  {
                    sign=(k-i)%2;
                    if( sign==0 )
                      sign=+1;
                    else
                      sign=-1;
```

```
                    kappa[k][tel]+=sign*prodset(exponent,b,i,l)/lcmset(exponent,b,i,l);
                  }
              }
          }
        // set b[-][k][tel] always occurs as a subset of itself
        kappa[k][tel]+=prodset(exponent,b,k,tel)/lcmset(exponent,b,k,tel);

        sign=(n+1-k)%2;
        if( sign==0 )
          tkappa[k][tel]=0;
        else
          tkappa[k][tel]=kappa[k][tel];

        if( tkappa[k][tel]>rmax )
          rmax=tkappa[k][tel];
      }
  }

//n odd implies rmax is at least 1 (empty set contributes in that case)
if( n%2!=0 )
  {
    if( rmax==0 )
      rmax=1;
  }

for( k=0; k<n; k++ )
  {
    for( tel=0; tel<binom(n,k+1); tel++ )
      {
        C[k][tel]=gcdsetcomplement(exponent,b,k,tel)/gcdset(exponent,b,n-1,0);
        // subsets with i+1 elements
        for(i=0; i<k; i++)
          {
            // sets with i+1 elements
            for( l=0; l<binom(n,i+1); l++ )
              {
                //check whether set is a subset of b[-][k][tel]
                if( setcheck(b,k,tel,i,l) )
                  {
                    C[k][tel]/=C[i][l];
                  }
              }
          }
      }
  }

cout << "Rang H_" << n-2 << " is equal to " << kappa[n-1][0] << endl;
cout << "The torsion components of H_" << n-2 << " are" << endl;
loop=0;
for( i=0; i<rmax; i++ )
  {
    d=1;
    for( k=0; k<n; k++ )
      {
```

```
          for( tel=0; tel<binom(n,k+1); tel++ )
          {
            if( tkappa[k][tel]>=i+1 )
              {
                d*=C[k][tel];
              }
          }
      }
    //empty set counts once if n is odd, tkappa=0
    if( n%2!=0 )
      d*=gcdset(exponent,b,n-1,0);
    if( d!=1 )
      {
        cout << "Z_d_" << i << "=" << d << endl;
        loop=1;
      }
  }
if( loop==0 )
  cout << "No torsion" << endl;
//Begin computing CCH; lcm determines orbit type
orbits->lcm=lcms[1][0];
orbits->k=1; orbits->nummer=0;
for( k=1; k<n; k++ )
  {
    for( tel=0; tel<binom(n,k+1); tel++ )
      {
        // checks whether this lcm has already been found and updates structure
        orbits->check(lcms[k][tel],k,tel);
      }
  }
//orbits->laatzien();

first=orbits;
//determine minimal and maximal degree in the first period of ch
//(rough estimates to get bounds for a table)
// get  period as well;
d=1;
mindeg=-2-2*d-10; maxdeg=4*n-6-2*d;
period=-2*lcms[n-1][0];
for( i=0; i<n; i++ )
  {
    mindeg+=2*((int) d/exponent[i]); maxdeg+=2*((int) d/exponent[i]);
    period+=2*(lcms[n-1][0]/exponent[i]);
  }
for( d=2; d<=lcms[n-1][0]; d++ )
  {
    mindeg2=2*n-2-2*d; maxdeg2=4*n-6-2*d;
    for( i=0; i<n; i++ )
      {
        mindeg2+=2*((int) d/exponent[i]); maxdeg2+=2*((int) d/exponent[i]);
      }
    if( mindeg2<mindeg )
      mindeg=mindeg2;
```

```
    if( maxdeg2>maxdeg )
      maxdeg=maxdeg2;
  }
mindeg-=10;
maxdeg+=10;
//normalize maxdeg
maxdeg=maxdeg-mindeg;
chrank=new int[maxdeg+1];
cout << "mindeg/maxdeg: " << mindeg << " " << maxdeg << endl;
for( i=0; i<=maxdeg; i++ )
  {
    chrank[i]=0;
  }
// consider time pi/2*d
for( d=1; d<=lcms[n-1][0]; d++ )
  {
    //find largest orbit space whose period divides d*pi/2
    k=0; tel=0; found=0; maxlcm=1;

    while( orbits!=NULL )
      {
        if( d%orbits->lcm==0 )
          {
            if( found )
              {
                if( orbits->lcm>maxlcm )
                  {
                    k=orbits->k;
                    tel=orbits->nummer;
                    maxlcm=orbits->lcm;
                  }
              }
            else
              {
                found=1;
                k=orbits->k;
                tel=orbits->nummer;
                maxlcm=orbits->lcm;
              }
          }
        orbits=orbits->next;
      }

    //if d is divisible by one of the lcms,
    //we get a contribution for homology
    if( found )
      {
        //increase the rank of the appropriate chgroups
        degree=0;
        for( i=0; i<n; i++ )
          {
            if( d%exponent[i]==0 )
              degree+=2*d/exponent[i];
            else
```

```
                degree+=2*((int) d/exponent[i])+1;
              }
            //k+1 is the number of exponents: 2(k+1)-4 is dim orbit space
            degree=degree-2*d+n-4-(k+1)+2;
            for( i=0; i<=2*(k+1)-4; i+=2)
              {
                chrank[degree+i-mindeg]+=1;
              }
            chrank[degree+(k+1)-2-mindeg]+=kappa[k][tel];
          }
        orbits=first;
      }
  // now list some ranks, simple version which is not always correct
  cout << "The following is only correct if the period is not equal to 0, ";
  cout << endl << "and if there are no generators in degree -1,0 or 1";
  cout << endl << "The period is " << period << endl;
  for( i=0; i<=maxdeg; i++ )
    {
      tel=chrank[i];
      k=i-period;
      while( k<=maxdeg && k>=0 && period!=0 )
        {
          tel+=chrank[k];
          k-=period;
        }
      cout << "In degree " << i+mindeg << " ch has rank " << tel << endl;
    }
  //program does not free memory; here should be some delete's
}
```

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