A Field Theory Approach to Universality in Quantum Chaos

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Abstract

We show that in $f \leq 3$ dimensions, the spectral statistical properties of a certain ensemble of quantum mechanical systems, chosen such that these systems share the same classical limit, are universal provided that the underlying classical dynamics is chaotic. These universal properties are faithful to random matrix theory up to universal corrections due to quantum interference effects, which stem from the Heisenberg uncertainty relation. In addition, we explain the formation of a universal gap in the electronic spectrum of a normal conducting chaotic system when the latter is brought in contact with a superconductor. This gap opens in the vicinity of the Fermi surface and its universal width is also set by quantum uncertainty. The method which we employ is the ballistic σ–model, a quantum field theory which is known for ten years now but whose evaluation was to date only possible with additional assumptions which we identify to be dispensable upon closer inspection. The insights gained enable us to draw novel parallels to the semiclassical approach and to the theory of dynamical systems.

Zusammenfassung

Wir zeigen, daß in $f \leq 3$ Dimensionen die Spektralstatistik eines gewissen Ensembles quantenmechanischer Systeme, die den gleichen klassischen Limes haben, universell ist, sofern die zugrundeliegende klassische Dynamik chaotisch ist. Insbesondere deckt sich dieses universelle Verhalten mit dem von Zufallsmatrixensembles bis auf universelle Korrekturen, die durch Quanteninterfenzeffekte entstehen und auf die Heisenbergsche Unschärferelation zurückzuführen sind. Darüberhinaus erklären wir die Entstehung einer universellen Lücke im elektronischen Spektrum eines normalleitenden chaotischen Systems, sobald dieses in Kontakt mit einem Supraleiter gebracht wird. Diese Lücke entsteht in der Nähe der Fermikante, und ihre universelle Breite wird ebenfalls durch Quantenunschärfe gesetzt. Als Methode verwenden wir das ballistische σ–Modell, eine Quantenfeldtheorie, die seit zehn Jahren bekannt ist, deren Auswertung bisher aber nur mithilfe zusätzlicher Annahmen möglich war, die wir durch eine sorgfältigere Betrachtung als entbehrlich herausstellen. Durch die gewonnenen Erkenntnisse sind wir in der Lage, neue Parallelen zum semiklassischen Ansatz und zur Theorie dynamischer Systeme zu ziehen.

Contents

Chapter 1

Quantum chaos and universality

This chapter provides the background needed to place this thesis into context. The first section 1.1 is dedicated to explaining the term 'quantum chaos' and some of its most prominent universal features. The subsequent two sections serve as introductions to later chapters: while section 1.2 briefly sets the historical stage for the physics of the proximity effect to be discussed in chapter 3, section 1.3 reviews the basics of the semiclassical approach to quantum chaos. The latter section is a bit more detailed than the other introductory sections as it serves as a primer for chapter 5, where we are going to elaborate on the parallels between the semiclassical approach and the field theoretical formalism presented in this thesis.

1.1 Introduction

In the last century, it has become clear that there are two radically different types of classical Hamiltonian motion: regular and chaotic dynamics.[1] Perhaps the most easily accessible difference between these two classes is the sensitivity to deviations of the initial conditions. While the final deviation of two originally close–by states only grows linearly in time in a regular system, the same error grows exponentially in chaotic systems. Accordingly, regular systems are analytically under control. In fact, the semiclassical approximation is perfectly understood and exact for regular systems, and it is the Bohr–Sommerfeld quantization scheme which led to an extrapolation of the quantum laws from the macroscopic world. The high degree of symmetry which stems from the independent integrals of motion manifests itself in the quantum spectrum through the independence of disjoint spectral series. In fact, at sufficiently large energies, the eigenvalues of a regular quantum system are completely randomized and appear to be Poisson distributed over the energy axis. Of course, the complexity of chaotic motion leaves one helpless in the first place. How would one establish a semiclassical correspondence? There is no such vehicle as the action–angle variable pairs which allows a neat classification of orbits. Quite on the contrary, the chaotic dynamics will eventually bring each orbit close to any other orbit, and these 'spaghetti' of trajectories are rather perplexing. One might then think that one just has to accept that there is no remedy but to merely take the quantum laws which one has guessed successfully from the classical world and to quantize the chaotic

[1] For a more profound introduction to the topics covered here we refer to the textbooks [1, 2, 3].

Hamiltonian at hand. No need for a semiclassical correspondence in the 'generic' case beyond this.

Interestingly, however, it turns out that the qualitative difference of chaotic dynamics leaves characteristic footprints in the quantum world: these quantum signatures of chaos are commonly termed 'quantum chaos'. Let us start our journey in the late 1950s/early 1960s, when Wigner and Dyson successfully applied statistical methods in order to describe the spectra of heavy nuclei, systems beyond the reach of standard quantum mechanical methods [4, 5, 6]. These latter methods are known to always rely on some high degree of symmetry and can at most deal with small perturbations of some primitive reference system such as a harmonic oscillator, a hydrogen atom, or some free particle or spin. However, by the experience from statistical mechanics, the complexity of chaotic dynamics represents a promising weak spot. Inspired by that observation, Wigner and Dyson modeled nuclei by statistical ensembles of random Hamiltonian matrices which respect the basic symmetries such as rotational or time reversal symmetry and are otherwise maximally entropic. This so–called random matrix theory (RMT) allows a very accurate prediction of the statistical spectral properties of sufficiently complex quantum systems. The characteristic feature of the resulting quantum spectra is a high degree of *regularity*: the eigenlevels tend to repel each other and to arrange themselves on a one–dimensional lattice with fluctuations about the equilibrium value which are universal in the sense that (upon proper rescaling of energies in units of the local mean level spacing) they only depend on the underlying symmetries of the given system.

While anyone who has had some exposition to standard statistical mechanics might say now that this universality is not very surprising at all, it is indeed very striking inasmuch as there is no proper explanation as to how this *universality* arises in an *individual* system. Quantum mechanics is a linear theory and therefore apparently ignorant of the non–linear complexity which is characteristic for classical chaos. It took twenty years to distill the criterion which is responsible for universality. Based upon empiric results, Bohigas, Giannoni, and Schmit (BGS) formulated a conjecture in 1984 [7]:

- If the underlying classical dynamics of a quantum system displays 'generic' chaos, then the statistical spectral properties of that system are universal and depend only on the symmetries of that system.

This celebrated conjecture is infamous inasmuch as it has kept a whole industry of 'quantum chaologists' busy ever since.[2] The 1980s and early 1990s brought a wealth of experimental results which substantiated the BGS conjecture and illustrated how deviations from RMT may arise [8, 9, 10]. Universal spectral statistics of the RMT type has been observed in totally different systems ranging from the energy levels of heavy nuclei, highly excited (so–called Rydberg) atoms, and mesoscopic structures such as two–dimensional electron gases, up to microwave modes in quasi two–dimensional cavities and ordinary sound modes in rigid bodies. RMT statistics even serves as a null model for the eigenvalues of stock–market price covariance matrices and helps to identify non–trivial correlations, say, among different industry sectors. As a more abstract phenomenon, also the zeroes of the Riemann zeta function display RMT statistics, and the mathematicians hope to find a physical system (the 'Riemannium') whose spectrum coincides with these zeros. In contrast to this, positive theoretical progress towards a

[2] For a review and a wealth of references cf. the monographies by Stöckmann [1] and Haake [2].

proof of the BGS conjecture went rather stagnant. As a first line of research, there is the semiclassical approach which employs the Gutzwiller trace formula in order to express spectral properties in terms of classical dynamical quantities. The semiclassical branch will be reviewed in some detail in subsection 1.3. As an orientation as to how slow progress went, the milestones date to 1985 (Berry [11]), 1996 (Bogomolny & Keating [12]), and 2001 (Sieber & Richter [13]). The work by Sieber & Richter (SR) took quantum uncertainty[3] into account and ignited a wave of publications [18, 19, 20, 21, 22, 23, 24, 25] which culminated in a fairly complete semiclassical understanding of spectral correlations.

An alternative approach dates back to 1983, a time even prior to the BGS conjecture, when Efetov employed field theoretical methods to prove that the statistical spectral properties of *disordered* systems are faithful to RMT [26]. Efetov's theory was formulated in terms of the so–called diffusive non–linear σ–model. The name alludes to the fact that a perturbative evaluation of this model is formulated in terms of a diffusion–type propagator $(D\mathbf{q}^2 - i\omega/\hbar)^{-1}$. In a system of finite extension L, this propagator displays a gap of the order of the Thouless energy $E_{\mathrm{c}} \equiv \hbar D/L^2$; that is, for energies below this threshold only the homogeneous ($\mathbf{q} = 0$) configuration remains dominant while inhomogeneous configurations become strongly suppressed. The effective low–energy theory is thus governed by the so–called 'zero–dimensional' σ–model, which is just jargon for a reduced field space which is structureless in configuration space. Efetov employed the supersymmetric formulation of the zero–dimensional σ–model and showed how to derive the RMT answer from it. While this seminal result is not directly applicable to the realm of *clean* chaotic systems, it illustrates how powerful field theoretical methods are in deriving universal properties.

Tempted by that promise and as late as 1995, Muzykantskii & Khmel'nitskii formulated the counterpart of the diffusive σ–model for clean systems [27]. In field theory jargon, clean is synonymous to 'ballistic',[4] and consequently the resulting field theory was termed the ballistic σ–model. This model is the central object of interest to this thesis. It is formulated in terms of fields which reside on classical phase space, and its perturbative evaluation is formulated in terms of the Liouville propagator of phase space densities. The ballistic σ–model therefore promises to even allow a systematic study of non–universal properties of clean systems. Perhaps yet more attractive is that a reduction of its field space to those configurations which are homogeneous on phase space again yields the zero–dimensional σ–model and hence the RMT answer. That this expectation once might prove successful is backed up by the fact that the Liouville propagator of chaotic systems is highly unstable against noise. Namely, in the presence of noise, the Liouville propagator acts on a reduced space of smooth phase space densities. The resulting object is called the Perron–Frobenius propagator and is known to display a gap $\hbar/t_{\mathrm{mix}}$ against the homogeneous configuration, just like its diffusive cousin.[5] Inter-

[3] Quantum uncertainty is reflected in a finite so–called Ehrenfest time, the time it takes to amplify quantum details to classical size by means of Lyapunov expansion. This time scale was first identified by Larkin & Ovchinnikov in the 1960s [14, 15]. It was again (Aleiner and) Larkin [16, 17] who in 1996 first identified the Ehrenfest time to play a pivotal role in quantum chaos.

[4] This terminology alludes to the ballistic (diffusive) motion of electrons in a clean (disordered) system.

[5] In fact and as a crucial point of this thesis, the propagator of the ballistic σ–model turns out to be indeed the Perron–Frobenius propagator, but may not be evaluated in the naïve sense that it becomes

estingly, taking the long–time limit first, this gap even prevails in the limit of vanishing noise. This is one of the rare (but usually fundamental) cases in physics where two limits — in the case at hand the long–time limit and the limit of vanishing noise — do not commute. Altogether this field theory was very appealing to condensed matter physicists and promised them a way to explain how universality emerges in quantum chaos.

So it took only another year until Andreev *et al.* proposed a derivation of the ballistic σ–model in their seminal works [28, 29]. Building upon the observation that it always takes some statistical ensemble to average over in order to derive an effective field theory, they took advantage of the fact that spectral correlations are invariant under translations of the spectrum. Averaging over this reference energy, Andreev *et al.* were able to derive the action by Muzykantskii & Khmel'nitskii from first principles. In the following years and even in the original works, however, a number of drawbacks were identified which plagued both the derivation and the evaluation of the effective theory [28, 29, 30, 31, 32]. Apart from the phenomenological (nevertheless very insightful) works by Larkin and collaborators [16, 17, 33, 34] and Altland & Taras–Semchuk [35] on the topic of quantum interference effects, it is safe to say that the drawbacks of the ballistic σ–model have remained unresolved to date. This thesis aims to reconsider these problems in depth and to cure most of them.

But before, let us provide some more background on universality in quantum chaos in the remainder of this introductory chapter. First, we are going to give a brief account of the history of the 'proximity effect', a term for the anomalous properties which normalconducting systems inherit from an adjacent superconductor. After that we aim to review the basic ideas entering the semiclassical approach to universal spectral correlations. These ideas rely on the phenomenon of quantum interference, hence the title of that section. We conclude with an outline of this thesis.

1.2 The proximity effect

It was as early as in the 1960s [36] that superconductors were found to tend to export some of their anomalous properties to adjacent normal conducting materials.[6] This so–called 'proximity effect' has been of great interest to mesoscopic physics ever since. Perhaps the most direct manifestation is the suppression of the density of states (DoS) in the vicinity of the Fermi surface. More importantly to us, depending on the nature of the underlying classical dynamics of the normal region, this suppression falls into one of only two qualitatively different categories [41, 42]: while the suppression is approximately linear in the integrable case, the DoS displays a finite 'Andreev' gap otherwise. This result has been experimentally tested on normalconducting films on superconducting substrates [43]. While no gap was observed in the clean case, a gap opened in the presence of disorder. Yet, for *clean* systems which are non–integrable merely due to their *chaotic* classical dynamics, the formation of a gap is to date but an expectation backed up

homogeneous for times scales in excess of t_{mix}. It is more adequate to think of it as a discretized version of the Liouville propagator, where the discretization length scales (due to quantum uncertainty) as $\hbar$; this object is protected against decay to universality up to the Ehrenfest time (cf. the discussion in section 1.3 and chapter 4), and only after this delay time has elapsed, it decays on the time scale set by t_{mix}.

[6] For extensive reviews cf. [37, 38, 39, 40].

by numerical results while an experimental verification is still lacking (cf. the review [40]). Due to technological progress experiments might nevertheless soon come into reach. For example it is possible to fabricate billiards of almost ballistic two–dimensional electron gases which are coupled to a superconductor (cf. [44] and references therein).

We already have encountered a spectral 'litmus test' for classical chaos in the introduction, namely the formation of non–trivial and universal spectral correlations. In contrast to the latter, the proximity effect is based on the DoS itself. Similar to the BGS conjecture, the Andreev gap turned out to be recalcitrant to theoretical explanation. An important step was taken by Lodder & Nazarov (LN), who built upon the Eilenberger equation to relate the DoS to the distribution of lengths of classical paths in the normal region [45]. They found that since paths of *any* length (and thus of arbitrarily long flight time) exist there are states down to even the lowest energies. In other words, they could not confirm the formation of a gap in the DoS. This is not what one expects since quantum uncertainty sets a minimal resolution in phase space below which the notion of individual trajectories loses its meaning. Indeed, in chaotic systems, the quantum uncertainty is amplified to classical dimensions after the so–called Ehrenfest time t_{E}, so trajectories longer than t_{E} should have established contact to the superconductor and not contribute to the DoS any more. Correspondingly, LN themselves already conjectured the presence of a gap of the order of $\hbar/t_{\mathsf{E}}$.[7] This conjecture was substantiated phenomenologically by Taras–Semchuk & Altland [35] and Vavilov & Larkin [33] and has been tested numerically [41, 42]. Nonetheless, the question remains how quantum uncertainty enters the 'hard–core' quasiclassics *à la* LN. Building upon the basic field theoretical framework developed in chapter 2, we will turn to this question in chapter 3.

1.3 Quantum interference: Semiclassical background

In this section we review the semiclassical results for the behavior of globally hyperbolic (chaotic[1]) quantum systems at time scales t larger than the mixing time t_{mix} yet smaller than the Heisenberg time $t_{\mathsf{H}} \equiv 2\pi\hbar/\Delta$. While the first condition implies that non–universal aspects of the classical dynamics are inessential, the second ensures that concepts of perturbation theory (in the parameter $\tau \equiv t/t_{\mathsf{H}}$) are applicable. If not mentioned explicitly otherwise, the results covered here are a strongly simplified condensate of recent publications by the Haake group [22, 23].

To describe correlations in the spectrum of the system we consider the two–point correlation function

$$R_2(\omega) \equiv \Delta^2 \big\langle \rho(E+\omega/2)\rho(E-\omega/2)\big\rangle_E - 1 \tag{1.1a}$$

and its Fourier transform

$$K(t) \equiv \frac{1}{\Delta}\int \mathrm{d}\omega\, \mathrm{e}^{-\frac{i}{\hbar}\omega t} R_2(\omega), \tag{1.1b}$$

[7] If the Ehrenfest time is finite, the gap is $\sim \min\{\hbar/t_{\mathsf{D}}, \hbar/t_{\mathsf{E}}\}$, where t_{D} is the dwell time of the normalconducting region [40].

[1] In fact, all results apply to general mixing rather than just uniformly hyperbolic systems. The point is that mixing implies ergodicity and non–integrability, and hence any mixing system will appear to have constant global Lyapunov exponents when evaluated on time scales $t \gg t_{\mathsf{mix}}$.

the spectral form factor. Here, $\rho(E)$ is the energy dependent DoS, and $\langle \cdots \rangle_E$ denotes the average over a sufficiently large portion of the spectrum centered around some reference energy E_0. In semiclassics, one makes use of the representation of the (oscillatory part of the) DoS by means of the Gutzwiller trace formula [46, 47]

$$\rho(E) - \langle \rho(E) \rangle_E \sim \frac{1}{\pi\hbar} \mathrm{Re} \sum_\gamma A_\gamma \mathrm{e}^{iS_\gamma/\hbar} \tag{1.2}$$

to express the spectral form factor as

$$K_{\mathsf{sc}}(\tau) = \Big\langle \sum_{\gamma\gamma'} A_\gamma A^*_{\gamma'} \mathrm{e}^{i(S_\gamma - S_{\gamma'})/\hbar} \delta\Big(\tau - \frac{T_\gamma + T_{\gamma'}}{2t_{\mathsf{H}}}\Big)\Big\rangle, \tag{1.3}$$

where the sums are over periodic orbits γ and γ', S_γ the classical action of the orbit γ, T_γ its revolution time, and A_γ its classical stability amplitude.

Before turning to a more detailed discussion let us briefly summarize the main results recently obtained for the semiclassical form factor: for times $\tau < 1$, K_{sc} can be expanded in a series in τ. As shown by Berry [11], the dominant contribution to this expansion, $K^{(1)}_{\mathsf{sc}} = 2\tau$, is provided by pairs of identical ($\gamma = \gamma'$) or mutually time reversed ($\gamma = \mathcal{T}\gamma'$) paths.[8] All corrections to the leading contribution $K^{(1)}_{\mathsf{sc}}$ hinge on the mechanism of quantum interference. E.g., the sub–dominant contribution $K^{(2)}_{\mathsf{sc}}$ to the form factor is provided by pairs (γ, γ') that are nearly identical except for one 'encounter region': in this region, one of the paths self–intersects[9] while its partner just so avoids the intersection (cf. figure 1.1). Alternatively, one may think of two trajectories that start out nearly identical, then split up and later recombine to form an interfering Feynman amplitude pair. The two paths are, thus, topologically distinct yet may carry almost identical classical actions [16]. Specifically, SR [13] have shown that for sufficiently shallow self intersections (crossing angle in configuration space of $\mathcal{O}(\hbar)$) the action difference $|S_\gamma - S_{\gamma'}| \lesssim \hbar$. For these angles, the duration of the encounter process is of the order of the Ehrenfest time

$$t_{\mathsf{E}} \equiv \frac{1}{\lambda} \ln \frac{c^2}{\hbar} \tag{1.4}$$

where λ is the phase space average of the dominant Lyapunov exponent of the system and c a classical reference scale (see below) whose detailed value is of secondary importance. This identifies t_{E} as the minimal time required to build up quantum corrections to the form factor (as well as to other physical observables [16]). Throughout we shall assume $t > t_{\mathsf{mix}}$ and the hierarchy $t_{\mathsf{mix}} \ll t_{\mathsf{E}} \ll t_{\mathsf{H}}$, where the condition $t_{\mathsf{mix}} \ll t_{\mathsf{E}}$ is imposed to guarantee that for time scales $t \sim t_{\mathsf{E}}$, the system already behaves universally.[10]

Summation over all SR pairs [13] leads to the universal result $K_{\mathsf{sc}} \simeq K^{(1)}_{\mathsf{sc}} + K^{(2)}_{\mathsf{sc}} = 2\tau - 2\tau^2$, which is consistent with the short time expansion of the random matrix form

[8] For the sake of simplicity, we focus on the case of $f = 2$ dimensions and of time reversal and spin rotational invariant systems (orthogonal symmetry) in this section. All results generalize to an arbitrary number f of degrees of freedom and general symmetry classes.

[9] Notice that in $f = 2$ dimensions a path of duration $t \gg t_{\mathsf{mix}}$ typically has many self–intersections in *configuration* space.

[10] For $t_{\mathsf{mix}} > t_{\mathsf{E}}$, the time window $t_{\mathsf{E}} < t < t_{\mathsf{mix}}$ is characterized by the prevalence of correlations that are non–universal yet quantum mechanical in nature.

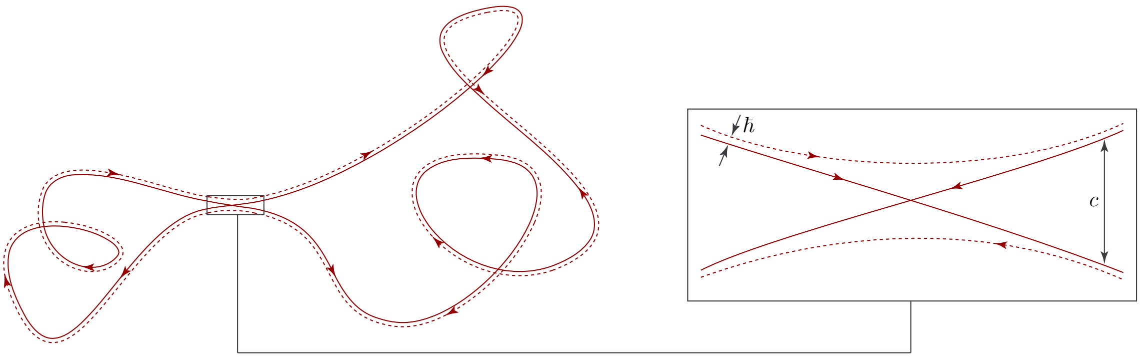

Figure 1.1: Cartoon of a pair of topologically distinct paths (γ, γ') contributing to the first quantum correction to the spectral form factor. Notice that γ and γ' differ in exactly one intersection region (crossing vs. avoided crossing). Inset: blow–up of the intersection region.

factor[11]

$$K_{\mathsf{RMT}}(\tau) \overset{\tau \leq 1}{=} 2\tau - \tau \ln(1+2\tau) = 2\tau - 2\tau^2 + \cdots . \qquad (1.5)$$

At higher orders in the τ–expansion, orbit pairs of more complex topology enter the stage. (For some families of pairs contributing to the next–to–leading correction, $K^{(3)}_{\mathsf{sc}}$, see figure 1.2.) The summation over all these pairs [23] — feasible under the presumed condition $t > t_{\mathsf{mix}}$ — obtains an infinite τ–series which equals the series expansion of the RMT result (1.5). It is also noteworthy that both the topology of the contributing orbit pairs and the combinatorial aspects of the summation are in one–to–one correspondence to the impurity–diagram expansion [48] of the spectral correlation function of disordered quantum systems.[12]

Central to our comparison of semiclassics and field theory below will be the understanding of the encounter regions where formerly pairwise aligned orbit stretches reorganize. The analysis of these objects is greatly facilitated by switching from the configuration space representation originally used by SR to one in phase space [19, 20, 21]. In the following we briefly discuss the phase space structure of the regions where periodic orbits rearrange. In chapter 5 we will compare these structures to the (somewhat

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[11] For the sake of completeness, we report the full random matrix result, which reads

$$K_{\mathsf{RMT}}(\tau) = \begin{cases} 2\tau - \tau\ln(1+2\tau), & \tau < 1, \\ 2 - \tau \ln \frac{2\tau+1}{2\tau-1}, & \tau > 1 \end{cases}$$

in the orthogonal case, while in the unitary case the form factor is given by

$$K_{\mathsf{RMT}}(\tau) = \begin{cases} \tau, & \tau < 1, \\ 1, & \tau > 1. \end{cases}$$

[12] Due to the notorious non–analyticity of $K_{\mathsf{RMT}}(\tau)$ at $\tau = 1$ [3], the form factor at times $\tau > 1$ was until very recently believed to be beyond the reach of semiclassical summation schemes. For $\tau > 1$, however, there is another expansion scheme which is organized in terms of orbit/'pseudo–orbit' pairs and yields the $\tau > 1$ expansion of the form factor [25]. These orbit/pseudo–orbit pairs have also been shown to be related term by term to a disorder diagram expansion about an additional 'non–standard' saddle–point.

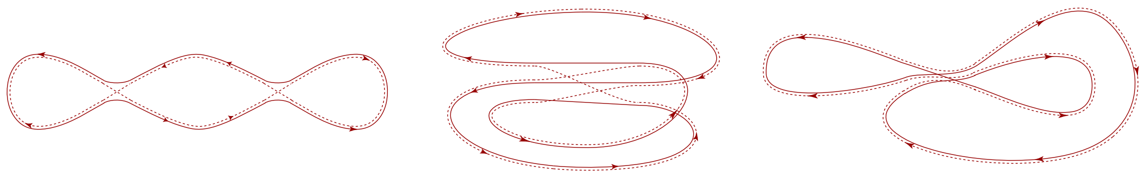

Figure 1.2: Cartoon of three classes of orbit pairs that contribute to the expansion of the form factor at order τ^3. (The triple–encounter region shown in the two figures on the right is the analog of the Hikami hexagon familiar from the impurity diagram approach to disordered systems.) The existence of the middle pair does not rely on time reversal invariance.

different) field theoretical variant of encounter processes.

Considering the correction $K^{(2)}_{\mathrm{sc}}$ as an example, we note that the encounter region contains four orbit stretches in close proximity to each other (cf. figures 1.1, 1.3): two segments $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ of the orbits γ and γ' traversing the encounter region and the (close to) *time reversed*[13] $\mathbf{x}(t_1+t_2)$ and $\mathbf{x}'(t_1+t_2)$ of these trajectory segments reentering after one of the loops adjacent to the encounter region has been traversed (t_2 is the duration of the loop traversal, and t_1 parameterizes the time during which the encounter region is passed). To describe the dynamics of these trajectory segments, it is convenient to introduce a Poincaré surface of section $\mathcal{S}$ transverse to the trajectory $\mathbf{x}(t_1)$. For the sake of simplicity, let us consider a system with two degrees of freedom (a billiard, say), in which case $\mathcal{S}$ is a two–dimensional plane slicing through the three–dimensional subspace of constant energy in phase space. We chose the origin of $\mathcal{S}$ such that it coincides with $\mathbf{x}(t_1)$. Introducing coordinate vectors $\mathbf{e}_u$ and $\mathbf{e}_s$ along the stable and unstable direction in $\mathcal{S}$, the three points $\bar{\mathbf{x}}(t_1+t_2)$, $\mathbf{x}'(t_1)$ and $\bar{\mathbf{x}}'(t_1+t_2)$ are then represented by the coordinate pairs $(u,s), (u,0)$, and $(0,s)$, respectively. (Notice that the trajectory $\gamma'/\mathcal{T}\gamma'$ traverses the encounter region on the unstable $(s=0)$/stable $(u=0)$ manifold thus deviating from/approaching the reference orbit γ.)

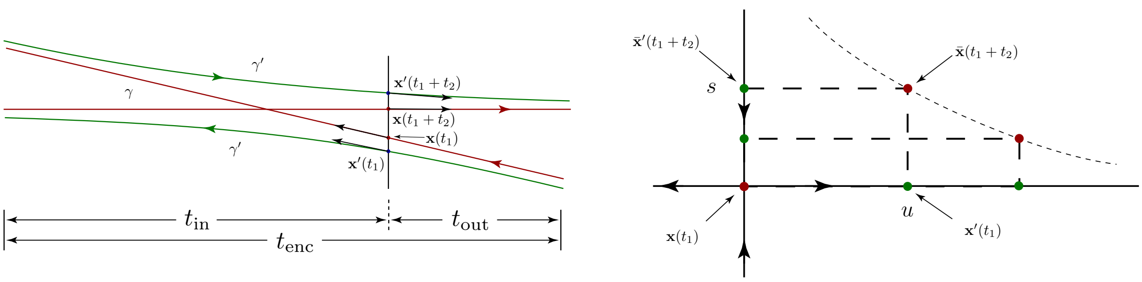

Figure 1.3: The structure of the encounter region. The picture on the right shows how the parallelogram spanned by the four points evolves in time t_1, while its symplectic area us is conserved.

The above coordinate system is optimally adjusted to a description of the two main characteristics of the encounter region: its duration t_{enc} and the action difference $S_\gamma - S_{\gamma'}$. Indeed, it is straightforward to show that the total action difference is simply given by the area of the parallelogram spanned by the four reference points in phase space,

[13] In a position–momentum representation, $\mathbf{x}=(\mathbf{q},\mathbf{p})$, time reversal is defined as $\bar{\mathbf{x}} \equiv (\mathbf{q},-\mathbf{p})$.

$S_\gamma - S_{\gamma'} = us$ [21]. As for the encounter duration, let us assume that the distance between the orbit points may grow up to a value c before they leave what we call the 'encounter region'. (It is natural to identify c with the typical phase space scale up to which the dynamics can be linearized around $\mathbf{x}(t_1)$; however, any other classical scale will be just as good.) After the trajectory γ has entered the encounter region, it takes a time $t_{\mathsf{in}} \sim \lambda^{-1}\ln(c/s)$ to reach the surface of section and then a time $t_{\mathsf{out}} \sim \lambda^{-1}\ln(c/u)$ to continue to the end of the encounter region. (Here, λ is the Lyapunov exponent of the system. Thanks to the assumption $t_{\mathsf{mix}} \ll t_{\mathsf{E}}$, λ may be assumed to be a 'self averaging quantity', constant in phase space.) The total duration of the passage is thus given by $t_{\mathsf{enc}}(u,s) \equiv t_{\mathsf{out}} + t_{\mathsf{in}} \sim \lambda^{-1}\ln(c^2/(us))$. The action difference of orbit pairs contributing significantly to the double sum must be small, $|S_\gamma - S_{\gamma'}| = us \lesssim \hbar$. Consequently, $t_{\mathsf{enc}} \gtrsim \lambda^{-1}\ln(c^2/\hbar) \equiv t_{\mathsf{E}}$, where t_{E} is the Ehrenfest time introduced above. (Notice that both $S_\gamma - S_{\gamma'}$ and t_{enc} depend only on the product us. While the individual coordinates u and s depend on the positioning of the surface of section, their product us is a canonical invariant and, therefore, independent of the choice of $\mathcal{S}$.)

Having discussed the microscopic structure of the encounter region, we next need to ask a question of statistical nature: given a long periodic orbit γ of total time t, what is the number $N(u,s,t)\,\mathrm{d}u\,\mathrm{d}s$ of encounter regions with Poincaré parameters in $[u, u+\mathrm{d}u] \times [s, s+\mathrm{d}s]$? (To each of these encounter regions there will be exactly one topologically distinct partner orbit γ' that is identical to γ in all other $(N-1)$ encounters. Thus, $N(u,s,t)\,\mathrm{d}u\,\mathrm{d}s$ is the number of SR pairs for a given parameter configuration and $\int \mathrm{d}u\,\mathrm{d}s\,N(u,s,t)$ is the total number of SR pairs.) Since the times t_1 and t_2 defining the two traversals of the encounter region are arbitrary (except for the obvious condition $|t_1 - t_2| > t_{\mathsf{enc}}$), N is proportional to the double integral $N(u,s,t)\,\mathrm{d}u\,\mathrm{d}s \propto \frac{1}{2}\int_{0,|t_2-t_1|>t_{\mathsf{enc}}}^{t} \mathrm{d}t_1\,\mathrm{d}t_2\,P_{\mathsf{ret}}(u,s,t_2)\,\mathrm{d}u\,\mathrm{d}s$. The integrand P_{ret} is the probability to propagate from point $(0,0)$ in the Poincaré section to the time reverse of (u,s) in the time t_2. Since $t_2 > t_{\mathsf{enc}} \gg t_{\mathsf{mix}}$, this probability is constant and equals the inverse of the volume $\Omega = 2\pi\hbar\, t_{\mathsf{H}}$ of the energy shell, $P_{\mathsf{ret}}(u,s,t_2) = \Omega^{-1}$. Thanks to the constancy of P_{ret}, the temporal integrals can be performed and we obtain $N(u,s,t) \propto t(t - 2t_{\mathsf{enc}})/(2\Omega)$. The normalization of N is fixed by noting that the temporal double integral weighs each encounter event with a factor t_{enc}. The appropriately normalized number of encounters thus reads $N(u,s) = \frac{t(t-2t_{\mathsf{enc}})}{2t_{\mathsf{enc}}\Omega}$. Substitution of $N(u,s,t)$ into the Gutzwiller sum obtains

$$
\begin{aligned}
K^{(2)}(\tau) &= \sum_\gamma |A_\gamma|^2 \delta\Big(\tau - \frac{T_\gamma}{t_{\mathsf{H}}}\Big) \int_{-c}^{c} \mathrm{d}u\,\mathrm{d}s\, N(u,s,t) 2\cos(us/\hbar) \\
&= \frac{\tau^2}{2\pi\hbar}\int_{-c}^{c} \mathrm{d}u\,\mathrm{d}s \Big(\frac{t}{t_{\mathsf{enc}}(u,s)} - 2\Big)\cos(us/\hbar) \overset{\hbar\to 0}{=} -2\tau^2,
\end{aligned}
\tag{1.6}
$$

where we used the sum rule $\sum_\gamma |A_\gamma|^2\delta(\tau - T_\gamma/t_{\mathsf{H}}) = \tau$ of Hannay and Ozorio de Almeida [49] and noted that in the semiclassical limit the first term in the integrand does not contribute (due to the singular dependence of t_{enc} on $\hbar$).

Before closing this section, let us discuss one last point related to the semiclassical approach: the analysis above hinges on the ansatz made for the classical transition probability $P_t(\mathbf{x},\mathbf{x}')$ between different points in phase space. Specifically, a naïve interpretation of ergodicity — $P_t(\mathbf{x},\mathbf{x}') = \Omega^{-1} = \text{const.}$ for times $t > t_{\mathsf{mix}}$ — is too crude to obtain a physically meaningful picture of weak localization. One rather has to take into account that the unstable coordinate, $u(t)$, separating two initially close ($u(0) \ll c$) points $\mathbf{x}$ and $\mathbf{x}'$ grows as $u(t) \sim u(0)\exp(\lambda t)$. For sufficiently small initial separation, the time it takes before the region of local linearizability is left,

$$\tfrac{1}{2}t_{\mathsf{E}}(\mathbf{x},\mathbf{x}') \equiv \frac{1}{\lambda}\ln\frac{c}{u(0)}, \tag{1.7}$$

may well be larger than t_{mix}. This is important because during the process of exponential divergence, the probability to propagate from $\mathbf{x}$ to the time reverse $\bar{\mathbf{x}}'$ is identically zero. (Simply because the proximity of $\mathbf{x}$ and $\mathbf{x}'$ implies that $\mathbf{x}$ and $\bar{\mathbf{x}}'$ are far away from each other.) Only after the domain of linearizable dynamics has been left, this quantity becomes finite and, in fact, constant:

$$P_t(\mathbf{x},\bar{\mathbf{x}}') = \frac{1}{\Omega}\Theta\big(t - t_{\mathsf{E}}(\mathbf{x},\mathbf{x}')\big), \text{ if } |\mathbf{x}-\mathbf{x}'| \ll c. \tag{1.8}$$

This concludes our brief survey of the semiclassical approach to quantum coherence. It will be the purpose of this thesis to discuss the corresponding field theoretical formulation. By chapter 5, we will have obtained a sufficient grip on the field theory to discuss its structural parallels to the semiclassical formalism just presented.

1.4 Outline of this thesis

In chapter 2 we start out reviewing the known facts about the field theory approach to quantum chaos. Since the field theory in question — the ballistic σ–model — is plagued by a number of problems, we will stress the delicate points which were missed to date. Doing so, we show how to solve some of these problems in chapters 3 and 4. As our knowledge how to evaluate the field theory grows, applications are discussed along the way. In particular, we present a consistent semiclassical approach to the proximity effect in chapter 3. We give an explanation of the proximity gap in chaotic SN structures and confirm the conjecture by LN that the width of this gap is set by the inverse Ehrenfest time. Chapter 4 is dedicated to developing a consistent evaluation scheme of the ballistic σ–model based upon a regularization procedure. We show how a variant of the BGS conjecture may be derived from first principles, which does not hold for *individual* systems, but rather for an *ensemble* of quantum systems which share the same classical limit. In chapter 5, we show how the quantum interference corrections to universal statistics found by the Haake group [24] are obtained from the field theory. We take this example to explain how the field theory approach presented in this thesis compares to the semiclassical approach which we discussed in section 1.3. We draw our conclusions and give an outlook in chapter 6.

Chapter 2

Field theory: The ballistic σ–model

In this chapter we review the original derivation of the ballistic σ–model by Andreev *et al.* in the seminal papers [28, 29]. We lay particular emphasis on the subtleties which one might miss when extracting a semiclassical theory from an underlying quantum theory and which were overlooked by the original authors. We close with a discussion of the principal and ostensible problems of the ballistic σ–model which kept and keep puzzling the semiclassical and condensed matter community to date.

2.1 Derivation of the effective field theory

This introductory section consists of a review of the standard derivation of the ballistic σ–model by Andreev *et al.* until the point where the authors introduce the quasiclassical approximation. Starting from rather general assumptions, we obtain an effective (still quantum) field theory.

2.1.1 Representation of observables: An example

Since it is our aim to apply field theoretical methods to examine spectral properties we have to give a representation of spectral quantities amenable to field theory. To that end we write

$$\rho(E) = \frac{1}{\pi} \operatorname{Im} \operatorname{tr} G^{-}(E) = \frac{1}{2\pi i} \operatorname{tr}\bigl(G^{-}(E) - G^{+}(E)\bigr), \tag{2.1}$$

for the DoS, where the generalized retarded $(+)$ and advanced $(-)$ Green functions are defined as

$$G^{\pm}(E) \equiv \bigl(E - H \pm i0\bigr)^{-1}. \tag{2.2}$$

In order to obtain a field theoretical representation we employ the generating functional[1]

$$Z(E,\omega) \equiv \int \mathcal{D}[\bar{\psi},\psi]\, \mathrm{e}^{i\bar{\psi}G^{-1}(E)\psi}, \quad G^{-1}(E) \equiv E - \tfrac{1}{2}\omega^{+}\sigma_3^{\mathsf{ar}} - H, \tag{2.3}$$

where the ψ–fields are Grassmann–valued vectors in advanced/retarded (ar), N–dimensional Hilbert space, and R–dimensional replica space; since observables are represented

[1] As a side remark, we mention that for discrete time quantum maps it is possible to formulate an analogous field theory by means of the color–flavor transformation [50].

by *logarithmic* derivates of the generating functional, the auxiliary R–fold replication of the theory serves to ensure proper normalization by means of the identity

$$\ln z = \lim_{R\to 0} \frac{z^R - 1}{R}.$$

As an example of relevance to us, we want to consider the spectral two–point correlation function, R_2, which was defined in (1.1a). In terms of Green functions, R_2 can be represented as

$$R_2(\omega) = \frac{\Delta^2}{2\pi^2} \operatorname{Re} \big\langle \operatorname{tr} G^+(E+\omega/2) \operatorname{tr} G^-(E-\omega/2) \big\rangle_{E,\mathsf{c}},$$

where $\langle AB \rangle_{\mathsf{c}} \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$ denotes the connected average. To derive this representation from (2.1) we used that $\langle G^+(E_1)G^+(E_2)\rangle_E = \langle G^+(E_1)\rangle_E \langle G^+(E_2)\rangle_E$ (and $+ \rightleftharpoons -$). R_2 is then obtained by two–fold differentiation of the averaged generating functional, $Z(\omega) \equiv \langle Z(E,\omega)\rangle_E$, according to

$$R_2(\omega) = -\frac{\Delta^2}{2\pi^2} \lim_{R\to 0} \frac{1}{R^2} \operatorname{Re} \partial_\omega^2 Z(\omega).$$

This follows from the fact that (by construction)

$$Z(\omega_1 - \omega_2) = \big\langle \det[iG^+(E+\omega_1)]^R \det[iG^-(E+\omega_2)]^R \big\rangle_E.$$

It is then straightforward to verify that

$$\begin{aligned}
\lim_{R\to 0} \frac{1}{R^2} \operatorname{Re} \partial^2_{\omega_1-\omega_2} Z(\omega_1-\omega_2) &= -\operatorname{Re} \big\langle \operatorname{tr} G^+(E+\omega_1) \operatorname{tr} G^-(E+\omega_2) \big\rangle_{E,\mathsf{c}} \\
&= -\frac{2\pi^2}{\Delta^2} R_2(\omega_1-\omega_2).
\end{aligned}$$

In terms of the dimensionless variables $s \equiv \pi\omega/\Delta$ and $\tau \equiv t/t_{\mathsf{H}}$, the field theoretical representations of R_2 and the form factor read

$$R_2(s) = -\lim_{R\to 0} \frac{1}{R^2} \operatorname{Re} \partial_s^2 Z(s), \tag{2.4a}$$

$$K(\tau) = (2\tau)^2 \lim_{R\to 0} \frac{1}{R^2} \operatorname{Re} Z(\tau). \tag{2.4b}$$

2.1.2 Energy average

Given the conditions that the width E_{av} of the energy window is much larger than Δ yet much smaller than the width of the spectrum, R_2 is translationally invariant under shifts of the center–of–mass energy E within this window, and therefore the result does not depend on the precise form of the distribution which is used to perform the energy average in (1.1a). Assuming these conditions to be given we are thus free to employ a Gaussian average,

$$\langle \cdots \rangle_E = (2\pi E_{\mathsf{av}}^2)^{-\frac{1}{2}} \int \mathrm{d}E\, \mathrm{e}^{-\frac{1}{2}\left(\frac{E-E_0}{E_{\mathsf{av}}}\right)^2} (\cdots), \tag{2.5}$$

which induces an interaction term $\propto (\bar{\psi}\psi)^2$,

$$Z(\omega) = \int \mathcal{D}[\bar{\psi}, \psi]\, \mathrm{e}^{i\bar{\psi}G^{-1}(E_0)\psi - \frac{E_{\mathsf{av}}^2}{2}(\bar{\psi}\psi)^2}.$$

The interaction term is invariant under unitary transformations

$$\psi \mapsto \mathcal{T}\psi, \quad \bar{\psi} \mapsto \bar{\psi}\mathcal{T}^\dagger, \quad \mathcal{T} \in \mathrm{U}(2RN^2), \tag{2.6}$$

a symmetry which is broken by the other terms in the action, $\omega \cdot \bar{\psi}\sigma_3^{\mathsf{ar}}\psi$ and $\bar{\psi}H\psi$. We may decouple the interaction by means of a Hubbard–Stratonovich transformation[2]

$$\mathrm{e}^{-\frac{E_{\mathsf{av}}^2}{2}(\bar{\psi}\psi)^2} = \int \mathcal{D}\tilde{\mathcal{Q}} \mathrm{e}^{-\frac{1}{2}\operatorname{tr}\tilde{\mathcal{Q}}^2 + E_{\mathsf{av}}\bar{\psi}\tilde{\mathcal{Q}}\psi}.$$

The invariance of the interaction term has to be respected by the term $\bar{\psi}\tilde{\mathcal{Q}}\psi$. Accordingly, the symmetry transformation (2.6) induces the transformation

$$\tilde{\mathcal{Q}} \mapsto \mathcal{T}\tilde{\mathcal{Q}}\mathcal{T}^{-1}$$

on the Hubbard–Stratonovich field $\tilde{\mathcal{Q}}$. Performing the (now Gaussian) integral over the ψ–fields we find that the partition function is given by

$$Z(\omega) = \int \mathcal{D}\tilde{\mathcal{Q}}\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}\tilde{\mathcal{Q}}^2 + \operatorname{tr}\ln(\mathcal{G}^{-1}[\tilde{\mathcal{Q}}] - \frac{1}{2}\omega^+\sigma_3^{\mathsf{ar}})}, \quad \mathcal{G}^{-1}[\tilde{\mathcal{Q}}] \equiv E_0 - H - iE_{\mathsf{av}}\tilde{\mathcal{Q}}. \tag{2.7}$$

2.1.3 Saddle–point equation

Varying the action of the effective generating functional (2.7) w.r.t. $\tilde{\mathcal{Q}}$ and neglecting terms of order ω, one obtains the saddle–point equation

$$\tilde{\mathcal{Q}}_0 = -iE_{\mathsf{av}}\mathcal{G}[\tilde{\mathcal{Q}}_0]. \tag{2.8}$$

Applying an ansatz for $\tilde{\mathcal{Q}}_0$ which is diagonal in the 'internal' (that is, all but the Hilbert space) indices, one finds the solutions

$$\tilde{\mathcal{Q}}_0(H) = -i\frac{E_0 - H}{2E_{\mathsf{av}}} + \Lambda\sqrt{1 - \Big(\frac{E_0 - H}{2E_{\mathsf{av}}}\Big)^2},$$

where Λ is an arbitrary traceless diagonal matrix with entries ± 1.

2.1.4 Effective field theory

These solutions are in fact but some of a whole manifold of solutions, which is understood as follows: as we have seen above, the $\tilde{\mathcal{Q}}$-field transforms according to $\tilde{\mathcal{Q}} \mapsto \mathcal{T}\tilde{\mathcal{Q}}\mathcal{T}^{-1}$. We see that for $\omega = 0$, the subgroup of transformations for which $[H, \mathcal{T}] = 0$ leaves the action invariant. Hence, for $\omega = 0$, these transformations give rise to a whole manifold of saddle–points. If ω or $[H, \mathcal{T}]$ are non–zero, the $\mathcal{T}$-fields generate the low–energy

[2] The missing i in the decoupled term is due to the anticommutativity of Grassmann fields, viz. $-\operatorname{tr}(\bar{\psi}\psi)^2 = +\operatorname{tr}(\psi\bar{\psi})^2$, which is decoupled as $-\operatorname{tr}(\tilde{\mathcal{Q}}\psi\bar{\psi}) = \bar{\psi}\tilde{\mathcal{Q}}\psi$.

configurations $\tilde{\mathcal{Q}} = \mathcal{T}\tilde{\mathcal{Q}}_0\mathcal{T}^{-1}$. The transformations for which $[\mathcal{T}, \tilde{\mathcal{Q}}_0] = 0$ have to be factored out since these do not give rise to new configurations $\tilde{\mathcal{Q}}$. We note that the first term of $\tilde{\mathcal{Q}}_0$ is diagonal in the internal indices and therefore contributes only a constant to the soft mode action and that configurations with matrix elements $\mathcal{T}_{\alpha\alpha'}$ for which $|E_0 - E_\alpha| > 2E_{\mathsf{av}}$ result in a purely imaginary contribution which is exponentially suppressed. It is therefore allowed to effectively replace the saddle–point by

$$\tilde{\mathcal{Q}}_0(H) = \pi E_{\mathsf{av}}\Lambda\delta_{E_{\mathsf{av}}}\big(E_0 - H\big), \quad \delta_{E_{\mathsf{av}}}(E) \equiv \frac{1}{\pi E_{\mathsf{av}}}\,\mathrm{Re}\sqrt{1-\Big(\frac{E}{2E_{\mathsf{av}}}\Big)^2}. \tag{2.9}$$

The 'delta function' is in fact a normalized semicircle distribution and E_{av} denotes (a quarter of) its width. Keeping only the configurations which are soft and do not leave $\tilde{\mathcal{Q}}_0$ invariant, the effective action reads (modulo inessential additive constants)

$$\begin{aligned} S[\mathcal{T}] &= -\frac{\beta}{2}\,\mathrm{tr}\ln\big(\mathcal{G}^{-1}[\tilde{\mathcal{Q}}] - \tfrac{1}{2}\omega^+\sigma_3^{\mathsf{ar}}\big) \\ &= -\frac{\beta}{2}\,\mathrm{tr}\ln\big(\mathcal{G}^{-1}[\tilde{\mathcal{Q}}_0] - \tfrac{1}{2}\omega^+\mathcal{T}^{-1}\sigma_3^{\mathsf{ar}}\mathcal{T} - \mathcal{T}^{-1}[H,\mathcal{T}]\big). \end{aligned} \tag{2.10}$$

For the explanation of the factor of $\beta/2$, note that we so far only considered the case of no discrete symmetries whatsoever (unitary symmetry class, $\beta = 2$). In appendix B we review the necessary modifications to take time reversal invariance into account (orthogonal symmetry class, $\beta = 1$), which roughly speaking leads to another doubling of field space by a 'time reversal' (tr) sector and an additional symmetry obeyed by the fields.[3] Further evaluation of this effective quantum field theory is possible only with additional assumptions as will become clear momentarily.

2.2 Semiclassical representation

This rather formal section serves to introduce a semiclassical expansion scheme of quantum theories which will later be employed to semiclassically evaluate the quantum action (2.10). The insights gained are nevertheless crucial for this work since they lay the ground for an adequate treatment of quantum uncertainty and other subtleties which tended to be obfuscated in the conventional field theory approach to quantum chaos.

2.2.1 Stratonovich–Weyl correspondence

A Stratonovich–Weyl correspondence is a family (parameterized by $s \in \mathbb{R}$) of one–to–one linear maps associating to each Hilbert space operator A a function $f_A^{(s)}$ on phase space Γ — called the symbol of the operator A — satisfying

(i) $f_{A^\dagger}^{(s)} = \big[f_A^{(s)}\big]^*$ (reality)

(ii) $\int \mathrm{d}\mathbf{x}\, f_A^{(s)}(\mathbf{x}) = \mathrm{tr}\, A$ (standardization)

(iii) $\int \mathrm{d}\mathbf{x}\, f_A^{(s)}(\mathbf{x}) f_B^{(-s)}(\mathbf{x}) = \mathrm{tr}(AB)$ (traciality)

[3] For more complicated symmetry classes we refer the reader to [51].

along with covariance properties ensuring that the symbols transform properly when a quantum mechanical system is rotated, boosted, etc. [52, 53]. These conditions ensure that quantum mechanical expectation values — which are of the form $\mathrm{tr}(\rho A)$ — may be interpreted as statistical expectation values of phase space observables. Together with traciality, standardization ensures that the unit operator is mapped to the constant function 1.[4] The operator product maps to the star product

$$\big(f_A^{(s)} \star f_B^{(s)}\big)(\mathbf{x}) \equiv f_{AB}^{(s)}(\mathbf{x}).$$

Being the symbol of an operator product, the star product inherits associativity and non–commutativity.

2.2.2 Standard phase space and the Wigner symbol

In the remainder of this work we want to restrict ourselves to standard phase space $\Gamma = \mathbb{R}^f \times \mathbb{R}^f$. The general case is discussed in [53] and applies, e.g., to the case $\Gamma = S^2$ of quantum mechanical spin [54]. We want to parameterize Γ using rescaled dimensionless phase space variables,

$$\mathbf{x} = \begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{q}/q_0\\ \mathbf{p}/p_0\end{pmatrix},$$

which we will again denote by $\mathbf{x}$. Here, q_0 (p_0) is some classical and otherwise inessential constant of dimension length (momentum), say the system size (Fermi momentum). Whenever we talk about $\hbar$ in the following we actually mean $\hbar_{\mathsf{eff}} \equiv \hbar/(q_0 p_0)$. This non–canonical rescaling allows a notion of Euclidean distance on phase space,[5]

$$|\mathbf{x} - \mathbf{x}'| \equiv \sqrt{(\mathbf{q}-\mathbf{q}')^2 + (\mathbf{p}-\mathbf{p}')^2}.$$

While the $s \neq 0$ members of the Stratonovich–Weyl correspondence for standard phase space are of importance in different contexts as well,[6] we will only make use of the $s = 0$ member, the so–called Wigner symbol

$$A(\mathbf{x}) = \int \mathrm{d}\Delta\mathbf{q}\, \mathrm{e}^{-\frac{i}{\hbar}\mathbf{p}\cdot\Delta\mathbf{q}} \langle \mathbf{q} + \tfrac{1}{2}\Delta\mathbf{q}|A|\mathbf{q} - \tfrac{1}{2}\Delta\mathbf{q}\rangle. \tag{2.11}$$

The corresponding star product is called Moyal product [56] and affords the two alternative representations[7]

$$(AB)(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f}\frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f} \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2} A(\mathbf{x}+\mathbf{x}_1)B(\mathbf{x}+\mathbf{x}_2) \tag{2.12a}$$

$$= A(\mathbf{x})\mathrm{e}^{\frac{i\hbar}{2}\overleftarrow{\partial}_{\mathbf{x}}^{\mathrm{T}} I \overrightarrow{\partial}_{\mathbf{x}}} B(\mathbf{x}), \tag{2.12b}$$

[4] The phase space integral is normalized to unity.

[5] Due to the Theorem of Darboux [55], there exist coordinates such that the symplectic form reads $\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$, which induces a Euclidean metric.

[6] In particular, the $(s = \pm 1)$–pair of Husimi and Cahill–Glauber symbols.

[7] These and other properties of the Wigner symbol are verified in appendix C.

where $I \equiv \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix} \right)$ is the symplectic unit operator. To obtain a convenient representation of the product of more than two operators, we iteratively apply the prototype formula equation (2.12a). A straightforward calculation then yields the general product formula

$$(A_1 \cdots A_{2n})(\mathbf{x}) = \int \prod_{i=1}^{2n} \frac{\mathrm{d}\mathbf{x}_i}{(\pi\hbar)^f} \, \mathrm{e}^{\frac{i}{\hbar} S(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})} A_1(\mathbf{x}+\mathbf{x}_1) \cdots A_{2n}(\mathbf{x}+\mathbf{x}_{2n}), \qquad (2.13)$$

where the multilinear form $S(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) \equiv 2 \sum_{i<j} (-)^{i+j+1} \mathbf{x}_i^{\mathrm{T}} I \mathbf{x}_j$. A view at (2.12a) reveals that the oscillatory term kills all contributions outside a box where $\mathbf{x}_i^{\mathrm{T}} I \mathbf{x}_j \lesssim \hbar$, i.e. it nails the phase space arguments onto each other only in the semiclassical limit $\hbar \to 0$. One of the most important messages to take home from this work may be formulated at this early stage already: the fuzziness of the coordinates in the Moyal product formula (2.12a) at finite values of $\hbar$ stems from the non–commutativity of quantum operator products and is therefore a direct manifestation of the uncertainty principle. Finally, the Moyal commutator is given by

$$\begin{aligned}\big([A,B]\big)(\mathbf{x}) &= 2iA(\mathbf{x}) \sin\Big(\frac{\hbar}{2} \overleftarrow{\partial}_{\mathbf{x}}^{\mathrm{T}} I \overrightarrow{\partial}_{\mathbf{x}}\Big) B(\mathbf{x}) \\ &= i\hbar \{A(\mathbf{x}), B(\mathbf{x})\} + \ldots, \end{aligned} \qquad (2.14)$$

where $\{f, g\}_{\mathbf{x}}$ is the Poisson bracket. *A priori*, the omitted terms are by no means small. This apparently trivial point and the quantum uncertainty inherent to the Moyal product have already been understood in several different contexts [57, 58, 59, 60]. They have nevertheless so far been overlooked in the semiclassical treatment of the ballistic σ–model; yet, they are crucial for its adequate derivation and evaluation. In that sense it is fair to call them novel achievements of this thesis, a predicate which also applies to the findings in the following subsections which deal with the regularity properties of the field degrees of freedom.

2.2.3 Off–shell structure of the fields

With this background we are in shape to construct a semiclassical representation of the action (2.10). To that end let us decompose phase space into one energy variable $E = H(\mathbf{x})$ and $(2f-1)$ variables which parameterize the shells $\Gamma(E)$ of constant energy E. According to the Theorem of Darboux, it is locally possible to decompose these variables into one variable t canonically conjugate to E which represents the travel time along the flow and $2(f-1)$ canonical pairs denoted by $\mathbf{y} = (\mathbf{u}, \mathbf{s})$ (standing for unstable/stable) which parameterize the cross–section transversal to the flow. These coordinates are taylor–made to reveal the features of the Hamiltonian flow.

Let us now turn to the phase space structure of the fields $\mathcal{T}$ rotating the reference point $\tilde{\mathcal{Q}}_0$ in (2.9). We start out representing these fields as $\mathcal{T} = \mathbb{1} + \mathcal{W}$, where the generators $\mathcal{W}$ obey the condition $[\mathcal{W}, \tilde{\mathcal{Q}}_0]_+ = 0$ (since a component commuting with $\tilde{\mathcal{Q}}_0$ would not effect a rotation). Choosing $[\mathcal{W}, \Lambda]_+ = 0$,[8] it follows that $\mathcal{W}$ has to commute with the Hilbert space content of $\tilde{\mathcal{Q}}_0$, $[\mathcal{W}, \delta_{E_{\mathsf{av}}}(E_0 - H)] = 0$. Using the Moyal product formula (2.12a) plus the decomposition $\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2 = \mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2 + E_1 t_2 - E_2 t_1$ of the symplectic

[8] That is, $\mathcal{W}$ is block off–diagonal w.r.t. an ordering $(+1, \ldots, +1, -1, \ldots, -1)$ of the entries of Λ.

product, and transforming to a frequency representation, $f(\epsilon) \equiv (2\pi\hbar)^{-1} \int \mathrm{d}t\, \mathrm{e}^{\frac{i}{\hbar}\epsilon t} f(t)$, it is straightforward to verify that

$$\begin{aligned}\big[\delta_{E_{\mathsf{av}}}(E_0 - H)\mathcal{W}\big](E,\epsilon,\mathbf{y}) &= \mathcal{W}(E,\epsilon,\mathbf{y})\delta_{E_{\mathsf{av}}}\Big(\frac{\epsilon}{2} - (E - E_0)\Big)\\ \big[\mathcal{W}\delta_{E_{\mathsf{av}}}(E_0 - H)\big](E,\epsilon,\mathbf{y}) &= \mathcal{W}(E,\epsilon,\mathbf{y})\delta_{E_{\mathsf{av}}}\Big(\frac{\epsilon}{2} + (E - E_0)\Big).\end{aligned}$$

Obviously, only those generators span up the field manifold which do not annihilate the saddle–point, implying that neither of these terms must vanish. On the other hand, the commutator $[\delta_{E_{\mathsf{av}}}(E_0 - H), \mathcal{W}]$ does have to vanish, that is

$$0 = \mathcal{W}(E,\epsilon,\mathbf{y})\Big\{\delta_{E_{\mathsf{av}}}\Big(\frac{\epsilon}{2} - (E - E_0)\Big) - \delta_{E_{\mathsf{av}}}\Big(\frac{\epsilon}{2} + (E - E_0)\Big)\Big\}.$$

Taken together, these requirements amount to the restrictions

$$|E - E_0| \le E_{\mathsf{av}}, \quad |\epsilon| \le E_{\mathsf{av}} - |(E - E_0)| \le E_{\mathsf{av}},$$

which are easily verified to carry over to arbitrary powers of $\mathcal{W}$. Altogether, we see that a given width E_{av} of the energy window does not only nail the support of the fields down to a window of width of order E_{av} about the energy shell; it also induces an uncertainty of the $\mathcal{T}$–fields in the t–direction on the scale $t_{\mathsf{av}} \equiv 2\pi\hbar/E_{\mathsf{av}}$. In order to resolve all details, t_{av} necessarily has to be smaller than any other relevant time scale of the system. Since the smallest such time scale is certainly classical (typically, it is set by the Lyapunov time t_{L}), it suffices to take t_{av} much smaller yet still classical. We then find that averaging over an energy window as narrow as $E_{\mathsf{av}} \sim \hbar$ suffices to resolve all details of interest, and we will stick to this choice throughout the remainder of this work.[9]

2.2.4 Regularity of the field space

Having understood that all Wigner symbols appearing in the action (2.10) are confined to a small neighborhood of the energy shell, the following important observation applies: let the support of both $A(\mathbf{x})$ and $(AB)(\mathbf{x})$ be classically finite. Then, the Moyal product effects a smoothing of scales scaling smaller than $\hbar$. To see this, define the convolution of an operator A with a Gaussian,

$$\big\langle A\big\rangle_{\hbar^\alpha}(\mathbf{x}) = \int \mathrm{d}\mathbf{x}'\, A(\mathbf{x}')g_{\hbar^\alpha}(\mathbf{x} - \mathbf{x}'),$$

where $g_{\hbar^\alpha}$ stands for a unit normalized isotropic Gaussian with width $\sigma \sim \hbar^\alpha$. It is then easy to see that the Moyal product is invariant under smoothing in the sense that

$$AB = A\big\langle B\big\rangle_{\hbar^\alpha} = \big\langle AB\big\rangle_{\hbar^\alpha}, \quad \alpha > 1. \tag{2.15}$$

[9] In contrast to our findings, Efetov *et al.* [61] obtain an infinitely thin energy shell. It is unclear to us how the authors can do so without losing the resolution in direction of the flow.

This is shown as follows: an application of the integral formula (2.12a) gives

$$
\begin{aligned}
\big(A\langle B\rangle_{\hbar^\alpha}\big)(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f}\frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f}\int \mathrm{d}\mathbf{x}'\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2} A(\mathbf{x}+\mathbf{x}_1)B(\mathbf{x}')g_{\hbar^\alpha}(\mathbf{x}+\mathbf{x}_2-\mathbf{x}') \\
&= \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f}\frac{\mathrm{d}\mathbf{x}'}{(\pi\hbar)^f}\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I (\mathbf{x}'-\mathbf{x})} A(\mathbf{x}+\mathbf{x}_1)B(\mathbf{x}')\int \mathrm{d}\mathbf{x}_2\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2} g_{\hbar^\alpha}(\mathbf{x}_2) \\
&= \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f}\frac{\mathrm{d}\mathbf{x}'}{(\pi\hbar)^f}\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}'} A(\mathbf{x}+\mathbf{x}_1)B(\mathbf{x}+\mathbf{x}')\mathrm{e}^{-(\mathbf{x}_1\hbar^{\alpha-1})^2} = (AB)(\mathbf{x}).
\end{aligned}
$$

These equations should be read bottom up. In the crucial last line, finiteness of $\mathbf{x}$ and $\mathbf{x}+\mathbf{x}_1$ implies that $\mathbf{x}_1$ is also finite, which in turn means (since $\alpha > 1$) that the Gaussian is effectively unity. Summarizing, we may *a priori* restrict ourselves to a field space consisting only of those Wigner symbols which display no features smaller than $\hbar$.

2.3 The ballistic σ–model

Having understood about the subtleties of the Wigner representation we now continue our review of the derivation by Andreev *et al.* We start out with a *bona fide* review of their last steps towards the ballistic σ–model (2.18); this serves to put the reader into the position of the perplexed reader of the original works. After a discussion of the problems of which the ballistic σ–model suffers if taken at face value, we outline how a regularization scheme would resolve one major drawback.

2.3.1 The original derivation of the ballistic σ–model

Let us assume that an (*a priori* formal) expansion of the effective action (2.10) in the terms $\omega\,\mathrm{tr}(\sigma_3^{\mathsf{ar}}\tilde{\mathcal{Q}})$ and $[H,\mathcal{T}]$ that break the symmetry of the action is justified. We are then led to

$$S_0[\mathcal{T}] = \frac{i\beta\pi}{2\Delta}\int \frac{\mathrm{d}\mathbf{x}}{\Omega}\,\mathrm{tr}\big(\tfrac{1}{2}\omega^+\sigma_3^{\mathsf{ar}}\mathcal{Q} - \mathcal{T}\mathcal{Q}_0[H,\mathcal{T}^{-1}]\big),$$

where use of Weyl's law $\Omega\Delta = (2\pi\hbar)^f$ was made and all products are understood as Moyal products. We further made use of the saddle–point equation (2.8), and spurious normalization factors were absorbed by letting

$$\mathcal{Q} \equiv (\pi E_{\mathsf{av}})^{-1}\tilde{\mathcal{Q}}. \tag{2.16}$$

Let us further assume that the commutator can be replaced by the Poison bracket according to

$$[H,\mathcal{T}^{-1}] \mapsto i\hbar\{H,\mathcal{T}^{-1}\}. \tag{2.17}$$

This replacement results from a (once again just formal) truncation of the expansion (2.14) at first order. In addition — and, as will turn out shortly, closely related to the 'quasiclassical' truncation (2.17) — let us assume 'mode locking', i.e. that the fields $\mathcal{T}(\mathbf{x})$ do not depend on the energy variable $E = H(\mathbf{x})$ on their support (which is concentrated to the energy shell of classically vanishing width $E_{\mathsf{av}} \sim \hbar$). We then arrive at the celebrated ballistic σ–model action

$$S[T] = \frac{\beta\pi\hbar}{2\Delta}\int(\mathrm{d}\mathbf{x})\,\mathrm{tr}\Big(\frac{i\omega^+}{2\hbar}\sigma_3^{\mathsf{ar}}Q + T\Lambda\{H,T^{-1}\}\Big), \tag{2.18}$$

where $\int(\mathrm{d}\mathbf{x}) \equiv \Omega^{-1}\int \mathrm{d}\mathbf{x}\,\delta(E_0 - H(\mathbf{x}))$ stands for the unit normalized integral over the energy shell, $Q = T\Lambda T^{-1}$ denotes the reduced[10] field, the products are ordinary local products in the variables (E, t) and — a novel point of this work — Moyal products in the transversal directions $\mathbf{y}$.

2.3.2 Problems of the original model

In order to put ourselves in the perspective of the discussion which followed up the publications by Andreev *et al.* we want to emphasize the oversimplifications of their semiclassical expansion scheme. The authors falsely alleged the validity of the quasiclassical replacement (2.17). Apart of that, all products were taken to be ordinary local products, and the presence of the intrinsic UV cut–off on phase space scales of order $\hbar$ which we discussed in the preceding section 2.2 was missed. Taken at face value, the ballistic σ–model (2.18) then apparently suffers from a number of problems:

(i) A look at the quantum action (2.10) at $\omega = 0$ reveals that it possesses a host of N exact zero modes, where $N \equiv E_{\mathsf{av}}/\Delta$ denotes the (effective) dimension of Hilbert space. These are given by the field configurations which are diagonal in an eigenbasis of the Hamiltonian and therefore commute with the latter. In the semiclassical limit ($N \to \infty$) this instability foils any low–energy expansion [31]. Relatedly, the mode locking assumption and the quasiclassical replacement (2.17) of the von Neumann commutator by the Liouville bracket remain uncontrolled without regularization. We want to stress that these drawbacks are *inescapable and principal* in any (not only in the present field theoretical) approach without some additional regularization procedure. This point was first mentioned by Zirnbauer [31], while a partial aspect of this problem was already identified in the original work by Andreev *et al.* [29]: since the latter authors were not aware of the UV cut–off they were confronted with the pathological feature of the Liouville operator that it does not couple at all among different trajectories and disjoint energy shells.[11]

(ii) Due do the fact that different trajectories do not seem to couple and the fields seem to resolve infinitely small details it is not obvious how Ehrenfest time effects related to quantum uncertainty emerge from the theory. A phenomenological answer to this problem was proposed by Larkin and collaborators [16, 17, 34]. They introduced a regulator term in order to phenomenologically model the quantum uncertainty which underlies the Ehrenfest time physics. How quantum uncertainty is *intrinsic* to the ballistic σ–model remained nevertheless obscure.

(iii) The theory fails to reproduce the leading 'diagonal approximation' to the form factor due to an overcounting of periodic orbits. For a discussion of this so–called 'repetition problem' cf. the review by Mirlin [32].

In this work, we propose affirmative answers on how to solve problems (i) and (ii), while we have no solution to offer for the repetition problem (iii). We will not approach

[10] In the sense that the field structure in the E–direction has become trivial.

[11] However, Andreev *et al.* did not recognize the fundamental need for regularization.

the regularization issue (i) straight ahead, but we find it more instructive to take a detour in chapter 3 to address problem (ii) before. The merit of doing so is that we understand how the Ehrenfest time is intrinsic to the field theory, simply because of the principal regularity of the fields on linear scales $\hbar$ which constitutes an important modification w.r.t. the original work by Andreev. Problems (i) and (ii) therefore turn out to be largely independent inasmuch as the Ehrenfest time physics well relies on the validity of the quasiclassical replacement[12] (2.17) but not on the strength of the regulator which ensures this validity. The concrete implementation of a regularization procedure and its physical interpretation will be postponed to chapter 4.

2.3.3 Effects of regularization

Let us nevertheless briefly discuss the consequences of regularity of the fields below some cut–off scale larger than $\hbar$:

- Since the window about the energy shell is narrower than this cut–off scale, mode locking would be ensured. In addition, the Moyal products would become local (up to a sufficiently fine resolution t_{av}) in the time direction.
- Turning to the directions $\mathbf{y}$ transversal to the flow, a look at (2.12b) reveals that the expansion of the Moyal product could be terminated at lowest non–vanishing order. In particular, the quasiclassical truncation (2.17) of the von Neumann operator to the Liouville operator would become well–controlled.
- As a 'byproduct', the Liouville operator would only act on functions which are coarse–grained over scales $\hbar$. Since for a chaotic system, the resulting propagator eventually decays to the homogeneous mode (a point to be discussed in chapter 4), universality would come into reach.

Summarizing, regularization would provide a cure for problem (i) and a justification for the assumptions that entered the derivation of the ballistic σ–model (2.18) as presented in subsection 2.3.1.

2.4 Summary

In this section we have not only reviewed the original work by Andreev *et al.* but also gained a number of additional insights. Specifically, we gave an account on the semiclassical representation of quantum theories sufficiently detailed to reveal a number of aspects which have so far been neglected in the literature of the ballistic σ–model, namely

- a precise statement on the minimal requirement to stabilize the quasiclassical truncation (2.17) which, as a byproduct, led to the insight that the mode locking mechanism is but another side of the same coin,
- the identification of a principal lower bound of order $\hbar$ on fluctuations of the field configurations, and

[12] Which allows to talk about classical dynamics in the first place.

- of the non–local nature of the transversal Moyal products appearing in the ballistic σ–model action (2.18).

The first point is the key for identifying a regularization procedure which is minimally invasive[13] and ensures sufficient regularity to justify the ballistic σ–model action (2.18); this will be the topic of chapter 4. The last two points are related to quantum uncertainty and allow an understanding of the Ehrenfest time physics, the issue of point (ii) to which we will now turn.

[13] At least it is the minimal intrusion which allows for the quasiclassical truncation (2.17).

Chapter 3

The proximity effect in SN systems

In this chapter we show that *any* quasiclassical approach — be it field theoretical or not — to SN systems principally leads to contradictions. We show that these inconsistencies can be overcome by the observation that quantum uncertainty effectively *averages* the kinetic term in the Eilenberger equation. With this modification, the quasiclassical ansatz leads to a solution which turns out to be both self–consistent and unique. As a result, we find that the DoS of the chaotic normalconducting component displays a gap which is of the order of the inverse Ehrenfest time. Moreover, this chapter serves to gain some intuition for section 4.2 where we will formalize some of the ideas developed here.

3.1 Field theoretical formulation

We follow Taras–Semchuk & Altland [35] and represent the DoS as a derivative of the generating functional $Z(E) = \int \mathcal{D}\Psi \, \mathrm{e}^{i\bar{\Psi}G_E^{-1}(\epsilon^+)\Psi}$ for the retarded Gor'kov Green function $G_E^{-1}(\epsilon^+) = E - (\epsilon^+ + \hat{\Delta})\sigma_3^{\mathsf{ph}} - H$. Outsourcing notations and conventions to appendix D, we want to say here no more than that the superscript 'ph' refers to the particle–hole doubling of the field space which stems from the Nambu spinor representation of the Hamiltonian of the composite SN system, H is the Hamiltonian describing the dynamics of the normal conducting component, $\hat{\Delta}$ is the matrix representation of the superconducting order parameter, and E stands for the Fermi energy. Using the freedom to choose a gauge we take the superconducting order parameter to be real and, for simplicity but without loss of generality, to be homogeneous in the superconductor,

$$\hat{\Delta}(\mathbf{q}) = \begin{cases} \Delta\sigma_1^{\mathsf{ph}}, & \mathbf{q} \in \mathsf{S}, \\ 0, & \mathbf{q} \in \mathsf{N}. \end{cases}$$

Since the spectrum, when measured w.r.t. the Fermi surface, is invariant under translations of the latter, we may invoke the energy average which is essential for the construction of an effective field theory; we merely have to replace E_0 in (2.5) by E_{F}, the latter being some arbitrary but classically large energy which will again be called the Fermi energy. Notice that in contrast to the standard ballistic σ–model as discussed in chapter 2, the energy variable ϵ has a different meaning; while in the context of spectral correlations of normal systems it measured the distance between energy levels, it stands here for the offset w.r.t. the Fermi surface. The derivation of the effective field theory then proceeds entirely analogous along the lines of chapter 2 — the only differences being that E_0, $\omega/2$, and the ar–sector are replaced by E_{F}, $(\epsilon+\hat{\Delta})$, and the ph–sector, respectively. Assuming mode locking we obtain the effective action[1]

$$S_0[T] = i\pi\nu \int (\mathrm{d}\mathbf{x})\,\mathrm{tr}\Big((\epsilon^+ + \hat{\Delta})\sigma_3^{\mathsf{ph}} Q + T\Lambda[H, T^{-1}]\Big), \tag{3.1}$$

which is identical to the standard ballistic σ–model action (2.18) up to the minor modifications mentioned above.

We stress that we do not make explicit use of a regulator (apart from the mode locking property) in this chapter, as it is our aim to show that the UV cut–off of order $\hbar$ alone suffices to smooth the quasiclassical solution, which will turn out to be singular otherwise. For clarity of this argument, we did not yet replace the von Neumann commutator by the Liouville bracket as in (2.17). As reference point of the non–linear field Q, we choose $\Lambda = \sigma_3^{\mathsf{ph}}$. The observable of interest, namely the DoS at energy ϵ w.r.t. the Fermi surface, is obtained from the Gor'kov Green function as

$$\nu(\epsilon) = -\frac{1}{2\pi}\,\mathrm{Im}\,\big\langle \mathrm{tr}\big(G(E,\epsilon^+)\sigma_3^{\mathsf{ph}}\big)\big\rangle_E = -\lim_{R\to 0}\frac{\nu}{2R}\int (\mathrm{d}\mathbf{x})\,\mathrm{Re}\,\big\langle \mathrm{tr}\big(Q\sigma_3^{\mathsf{ph}}\big)\big\rangle_Q, \tag{3.2}$$

where we made use of (D.1) in the first equality and of the saddle–point equation (2.8) in the second. Having thus constructed an effective field theory for SN systems, we finally vary the effective action (3.1) w.r.t. the soft–mode fields T to obtain the mean field equation

$$\Big[(\epsilon^+ + \hat{\Delta})\sigma_3^{\mathsf{ph}} - H, \bar{Q}\Big] = 0, \quad \bar{Q}^2 = 1. \tag{3.3}$$

3.2 Solution of the mean field equation

In this section, we reiterate the quasiclassical approach to the mean field equation (3.3) due to Lodder & Nazarov (LN) in order to demonstrate how *any* quasiclassical approach is doomed to fail. The reason is found to lie in the breakdown of the quasiclassical approximation at the Ehrenfest time and is traced back to the fact that, as it stands, the quasiclassical scheme is ignorant of quantum uncertainty. We then invoke the UV cut–off of chapter 2 to construct a unique and self–consistent 'semiclassical'[2] solution which basically consists of a quasiclassical solution to the *coarse–grained* mean field equation. This solution is found to imply a gap in the DoS which is of the order of the inverse Ehrenfest time.

[1] Since the letter Δ is reserved to the order parameter, we use ν for the inverse level spacing: the average DoS. Since we are working on phase space, ν deviates from the standard notation in fermionic theories inasmuch as it is not divided by the configuration space volume.

[2] In this chapter, we mean by 'semiclassical' (in parentheses) 'quasiclassical plus quantum uncertainty'. This has to be distinguished from the common terminology where the word semiclassical (which we also use, but without parentheses) indicates the use of a stationary phase approximation to the Feynman propagator in order to obtain a description in terms of, say, the Gutzwiller trace formula. An example is the semiclassical approach to spectral statistics of section 1.3.

3.2.1 The quasiclassical approach

LN analyze equation (3.3) in the quasiclassical approximation which relies on the assumptions that

(A) the quantum von Neumann commutator can be substituted by its quasiclassical limit, the Poisson bracket, $[H, \bar{Q}] \mapsto i\hbar\{H, \bar{Q}\}$, and that

(B) the condition $\bar{Q}^2(\mathbf{x}) = 1$ is taken to be local, $[\bar{Q}(\mathbf{x})]^2 = 1$.

The resulting equation is called the Eilenberger equation [62]. These assumptions were found in chapter 2 to be justified only if $\bar{Q}(\mathbf{x})$ is sufficiently a smooth function on phase space Γ; but for the time being let us take them for granted. Restricting our attention to the normal region, the quasiclassical solution is constructed as follows:

(i) To a given phase space point $\mathbf{x} \in \Gamma$ (of the N region) one associates the classical trajectory γ through $\mathbf{x}$. Each orbit γ starts and ends at the SN interface Σ and has a length[3] $T(\mathbf{x}) \equiv T_\gamma$. Since the classical dynamics is assumed to be chaotic and hence ergodic, the set of all orbits γ constitutes the entire phase space of the N component.

(ii) Following LN one solves the mean field equation on each orbit γ. The solutions are given by

$$\bar{Q}_\gamma(\tau) = \sin\theta_\gamma \left(\sin\frac{\epsilon(t-t_\gamma)}{\hbar}\,\sigma_1^{\mathsf{ph}} + \cos\frac{\epsilon(t-t_\gamma)}{\hbar}\,\sigma_2^{\mathsf{ph}} \right) + \cos\theta_\gamma\,\sigma_3^{\mathsf{ph}}, \qquad (3.4)$$

where t is a time variable parameterizing the orbit. In the regime $\epsilon/\Delta \ll 1$, continuity of $\bar{Q}_\gamma(t)$ at the interfaces $(t = \pm T_\gamma/2)$ fixes the parameters according to

$$t_\gamma = 0, \quad \cos\theta_\gamma = -i\tan\frac{\epsilon^+ T_\gamma}{\hbar}.$$

Evaluating (3.2) in a saddle–point approximation at the quasiclassical solution $\bar{Q}$, and using that $\mathrm{Im}\,\tan x^+ = -\pi\sum_{n\in\mathbb{Z}}\delta\big(x-(n+\frac{1}{2})\pi\big)$, one obtains the 'Bohr–Sommerfeld' result

$$\bar{\nu}(\epsilon) = \pi\nu\int(\mathrm{d}\mathbf{x})\sum_{n\in\mathbb{Z}}\delta\Big(\frac{\epsilon T(\mathbf{x})}{\hbar} - (n+\tfrac{1}{2})\pi\Big). \qquad (3.5)$$

Thus, in order to compute the DoS one needs to determine the distribution of path lengths for the given system. For a chaotic system this distribution decays exponentially on the scale of the dwell time of the N component [45, 63], which in turn implies a small but finite (gapless!) contribution at arbitrarily small energies.

3.2.2 Breakdown of quasiclassics

So far, we have merely reiterated the quasiclassical treatment, which to date represented the only viable approach to the DoS of an SN system. In this subsection we point out the

[3] We identify lengths and times by means of the Fermi velocity, $L = v_{\mathsf{F}} T$.

principal inconsistency of this approach. To be specific, let us check whether the quasiclassical solution (3.4) constitutes a self–consistent solution of the actual quantum equation (3.3), i.e., whether it is sufficiently regular to justify the assumptions (A) and (B) of the quasiclassical approximation which we mentioned in subsection 3.2.1. From chapter 2 we know that condition (A) is satisfied whenever $\bar{Q}$ possesses no details as fine as $\hbar$,[4] while condition (B) asks for $\bar{Q}$ to be approximately constant on any surface of area $\delta u \cdot \delta s \sim \hbar$ transversal to the flow.[5]

Our strategy to check these conditions is the following: given the solution (3.4) of the quasiclassical version of the mean field equation (3.3) on all trajectories, we state that for conditions (A) and (B) to be applicable, the solution (3.4) should be constant over all sections transversal to the classical flow which are (A) as wide as $\hbar$ in any transversal direction and (B) of the canonically invariant transversal area $\delta u \cdot \delta s = \hbar$. To check whether this consistency condition is obeyed, we choose an (arbitrary but fixed) orbit γ and form a bundle of all trajectories γ' which end at the interface $\Sigma_\pm$ together with γ in a continuous fashion. As the solution for one trajectory is uniquely determined by the trajectory length and the trajectories within each bundle are by construction of approximately the same length, the quasiclassical solution (3.4) does not fluctuate over transversal sections of these bundles; hence, we will call the bundles 'protected regions', namely, protected against field fluctuations transversal to the flow. By means of the criterion for applicability of the quasiclassical approximation, we understand how the Ehrenfest time enters the stage here: the protected region of an orbit γ of length $T_\gamma \gtrsim t_{\mathsf{E}}$ necessarily acquires a diameter smaller than $\hbar$, cf. Fig. 3.1. This implies that any protected region around a long[6] orbit gets 'squeezed' within the background of short orbits (which make up the overwhelming fraction of phase space). Therefore, the orbit lengths (and hence the quasiclassical solution (3.4)) vary significantly on scales of order $\hbar$, which implies that condition (A) is violated. The main conclusion is thus that the field configuration (3.4) obtained by LN is not consistent with the quasiclassical approximation and therefore does not solve the original quantum equation (3.3). In fact, we also see that in *any* quasiclassical treatment of the quantum equation, talking of orbits longer than the Ehrenfest time is meaningless. In the next subsection we discuss to what extent the classical dynamics can nonetheless be employed to construct a self–consistent solution to (3.3).

3.2.3 Self–consistent solution of the quantum equation

Having understood that any quasiclassical treatment inevitably leads to contradictions, we now employ our findings about the smoothness properties of the $\bar{Q}$–field to try and find a solution which is 'semiclassical' in the weaker sense that the ansatz of the quasiclassical replacement (A) only holds on average over the minimal grain size,

$$[H, \bar{Q}] \stackrel{(2.15)}{=} \big\langle [H, \bar{Q}] \big\rangle_{\hbar^\alpha} \stackrel{(\mathsf{A})}{=} i\hbar \big\langle \{H, \bar{Q}\} \big\rangle_{\hbar^\alpha}, \quad \alpha > 1. \tag{A'}$$

Note that the inconsistency with the quasiclassical assumptions only arose for long orbits. Motivated by this observation we define the set $\tilde{\Gamma} \equiv \{\gamma | T_\gamma \gtrsim t_{\mathsf{E}}\}$ of exceptionally long

[4] Recall the differential version of the Moyal product (2.12)

[5] In this chapter, we restrict ourselves to the case $f = 2$.

[6] 'Long'/'short' refers to longer/shorter than t_{E}.

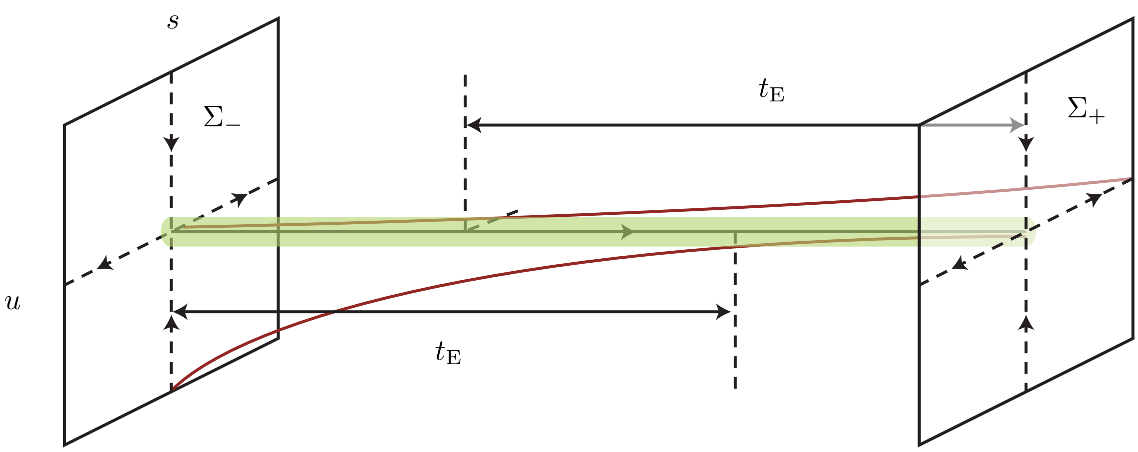

Figure 3.1: Construction of the protected region about a long orbit. The shaded region marks the $\hbar$–threshold. The segment within the dashed lines is protected (in the sense that its $\hbar$–neighborhood contains only long orbits) while the complement has established contact to the background of short orbits.

orbits and state that for $\mathbf{x} \in \Gamma \setminus \tilde{\Gamma}$, Eq. (3.4) constitutes the unique solution to the mean field equation (3.3, A') and is consistent with conditions (A) and (B). Thus, on $\Gamma \setminus \tilde{\Gamma}$ a 'semiclassical' solution has to be identical to the solution by LN. On the long orbits, $\mathbf{x} \in \gamma \in \tilde{\Gamma}$, choose $\mathbf{x}$ outside the dashed lines of Fig. 3.1 and consider an $\hbar$–neighborhood U of $\mathbf{x}$.[7] The smallest transversal extension of the protected component $U \cap \tilde{\Gamma}$ of U scales as $\mathrm{e}^{-\lambda(T_\gamma/2+|t|)}$, where t denotes the time–like coordinate of $\mathbf{x} = \gamma(t)$ (cf. Fig. 3.1). This means that for $|t| > t_{\mathsf{E}} - T_\gamma/2$, this diameter is smaller than the diameter $\hbar$ of U by a factor which scales as a positive power of $\hbar$. Thus, the protected component of the $\hbar$–neighborhood U makes up a negligible fraction.[8] With this knowledge, we may now rewrite the mean field equation (3.3, A') at the point $\mathbf{x}$ according to

$$[\epsilon^+\sigma_3^{\mathsf{ph}}, \bar{Q}] \overset{(3.3)}{=} [H, \bar{Q}] \overset{(\mathrm{A'})}{=} i\hbar \big\langle \{H, \bar{Q}\} \big\rangle_U \approx i\hbar \big\langle \{H, \bar{Q}\} \big\rangle_{U \setminus \tilde{\Gamma}} \overset{(3.3)}{=} \big\langle [\epsilon^+\sigma_3^{\mathsf{ph}}, \bar{Q}] \big\rangle_{U \setminus \tilde{\Gamma}}.$$

The first equality is the mean field equation. Secondly, we applied the modified 'semiclassical' ansatz (A'). The third step is valid because $U \cap \tilde{\Gamma}$ by construction makes up a vanishingly small fraction of U. In the last step we used that the quasiclassical solution $\bar{Q}|_{\Gamma \setminus \tilde{\Gamma}}$ is self–consistent and therefore also a solution of the full mean field equation.

We are now able to formulate the central result of this chapter: wherever the protected region has become negligibly thin, the 'semiclassical' solution is uniquely obtained as the mean value (over an $\hbar$–neighborhood) of its quasiclassical value on the background of short orbits,

$$\bar{Q} = \big\langle \bar{Q}|_{\Gamma \setminus \tilde{\Gamma}} \big\rangle_\hbar. \tag{3.6}$$

[7] In the sense that U is of linear dimensions scaling as $\hbar^\alpha$, $\alpha \searrow 1$.

[8] Think of U as of a ball and of its protected component $U \cap \tilde{\Gamma}$ as of a pancake within this ball which is very thin in comparison to the ball's diameter.

3.2.4 The Ehrenfest gap

For energies $\epsilon \leq \pi\hbar/(2t_\mathsf{E})$, the short orbits stay clear of the resonance condition (3.5), which implies that no contributions to the DoS are formed. Furthermore, the mean field assumes the approximate value σ_2^ph on these orbits which coincides with the bulk value of $\bar{Q}$ deep inside the superconductor. Due to equation (3.6), the same holds for those protected regions which have become negligibly thin. Note that for orbits with length $t_\mathsf{E} < T_\gamma < 2t_\mathsf{E}$, there remains a protected region of short length $2t_\mathsf{E} - T_\gamma < t_\mathsf{E}$ (cf. figure 3.1); yet, this region is indistinguishable of a *short* protected region which has its ends attached to the background σ_2^ph of generic trajectories, that is, to the superconducting bulk. Thus, also on $\Gamma \setminus \tilde{\Gamma}$, and hence everywhere

$$\bar{Q} \approx \sigma_2^\mathsf{ph}. \tag{3.7}$$

We stress once more that (3.7) is the unique low–energy solution which is consistent with the 'semiclassical' approximation scheme. If no other solutions exist, the DoS in an SN system will display a gap of width $\pi\hbar/(2t_\mathsf{E})$. Orbits of length in excess of the Ehrenfest time are always dominated by the background of short orbits; the contact to the shorter neighbors is established on the Lyapunov time scale $t_\mathsf{L} = \lambda^{-1}$, so we conclude that at finite $\hbar$ the gap is not hard but the DoS exponentially decays at $\hbar/t_\mathsf{E}$ on a scale $\hbar t_\mathsf{L}/t_\mathsf{E}^2$. However, the quotient of this scale and the gap width is given by $t_\mathsf{L}/t_\mathsf{E}$ and goes to zero in the semiclassical limit.

In principle there are solutions to the mean field equation (3.3) which are *quantum* in the sense that they cannot be identified by 'semiclassical' means. These solutions can only be discarded with an additional regulator which ensures the quasiclassical truncation (A), but we stress that quantum uncertainty alone sets the lower bound for the size of the gap, and not the precise form or strength of the regulator.

3.3 Summary

In this chapter we have shown how the quasiclassical approximation leads to contradictions at times in excess of the Ehrenfest time. This failure was traced back to the missing account for quantum uncertainty in the quasiclassical approach. We then invoked the smoothness properties of the Wigner symbol which in chapter 2 were found to stem from the non–commutativity of quantum mechanics. This way we were able to cure the inconsistencies of the quasiclassical approach and to obtain a unique and self–consistent 'semiclassical' solution. For energies below the inverse Ehrenfest time, this solution was found to be given by the superconducting bulk value and thus responsible for the formation of a gap in the DoS. This gap was seen to rely upon the presence of a regulator only implicitly, in the sense that the gap *size* is set by quantum uncertainty alone.

Chapter 4

Perturbation theory I: Regularization and universality

Building upon the field theoretical framework of chapter 2 we construct a perturbation theory. As seen with the Andreev billard in the preceding chapter, any quasiclassical field theory of chaotic systems will generate arbitrarily small details due to Lyapunov contraction. We propose a regularization scheme which consists of introducing a small amount of diffusion into the dynamics which counteracts the Lyapunov contraction, and which we argue to be physically reasonable in $f \leq 3$ dimensions since it does not force an integrable system into RMT statistics. We point out that we do not succeed to justify a regularization scheme which would leave the spectrum of an *individual* system intact; but we are able to derive a regulator from an averaging procedure over an *ensemble* of quantum systems all of which share the same classical limit. An analogous argument was earlier given by Zirnbauer [31], yet we are able to weaken his assumptions significantly; in particular, the influence of the regulator is so weak that quantum uncertainty is still intrinsically manifest.

In the last and rather formal section, we invoke classic and mathematically rigorous results from ergodic theory whose precise statements in our opinion are too little known at least in the condensed matter community. In order to apply these results appropriately we present a small dictionary which translates them to the field theoretical context. As one byproduct, it turns out that a proper understanding of these results resolves the apparent paradox that the propagator of the field theory decays on the classical time scale t_{mix} on one hand, which is subject to a delay until the quantum Ehrenfest time t_{E} (associated to the coarse graining due to quantum uncertainty) has elapsed, on the other. As another byproduct, we cite a concrete notion of what is meant by 'generic' chaos in mathematics nowadays. We finally argue that establishing the universal regime at energies below the inverse Ehrenfest time amounts to an explanation of universal spectral correlations and hence of BGS spectral statistics for our ensemble.

4.1 Regularization

In order to describe spectral statistics in chaotic systems, we tie in with the field theory developed in chapter 2 and construct a perturbation theory. In the subsequent chapter 5, the resulting perturbative expansion will turn out to be equivalent to the semiclassical short time expansion of the form factor we learned about in section 1.3. We propose a regularization scheme, discuss the resulting propagator of the perturbation theory, and give a prescription for its evaluation. Finally, we argue that we are unable to justify the presence of the regulator for an *individual* system, but rather for an *ensemble* of classically equal quantum systems.

4.1.1 Perturbative action

Let us evaluate the field integral with effective action (2.18). We focus here on the orthogonal ($\beta = 1$) variant of the field theory. The evaluation of the field theory relies on a perturbative expansion of the effective field which is organized about the reference point[1] $\Lambda = \sigma_3^{\mathsf{ar}}$ and is most conveniently performed using the so–called 'rational' parameterization of the coset–valued field T. This parameterization is defined by $T = \mathbf{1} + W$, where

$$W = \begin{pmatrix} & B \\ -B^\dagger & \end{pmatrix}_{\mathsf{ar}} \tag{4.1}$$

is a matrix which anti–commutes with the saddle–point Λ introduced above. Its off–diagonal blocks satisfy the constraint $B^\dagger = B^\tau$, where B^τ denotes the generalized transposition associated to time reversal defined in appendix B. The principal advantage of the rational parameterization is that the Jacobian of the transformation from the T–matrices to the linear space of B–matrices is unity: $\int \mathcal{D}T = \int \mathcal{D}B$. Substituting this representation into the action (2.18) we obtain a series expansion $S[B] = \sum_{n=1}^{\infty} S^{(2n)}[B]$, where[2]

$$S^{(2n)}[B] = \frac{t_{\mathsf{H}}}{2} \int (\mathrm{d}\mathbf{x})\ \mathrm{tr}\big[(-B^\dagger B)^{n-1} B^\dagger L_\omega B\big] \tag{4.2}$$

is of $2n$–th order in B. Here, $L_\omega \equiv -i\omega/\hbar - [H, \ ]$ stands for the von Neumann operator which generates the time evolution of quantum density operators. In order to ensure regularity of the field space, we introduce a small amount of diffusion $\sim \mathrm{tr}(\partial_{\mathbf{x}} Q)^2$, to be discussed in detail momentarily. For now let it suffice to say that it effects both mode locking and the truncation (2.17) of the von Neumann operator to the Liouville operator $\mathcal{L}_\omega \equiv -i\omega/\hbar - \{H, \ \}$ which generates the time evolution of classical phase space densities. Anticipating this result, the quadratic action reduces to

$$S^{(2)}[B] = \frac{t_{\mathsf{H}}}{2} \int (\mathrm{d}\mathbf{x})\ \mathrm{tr}\big[B^\dagger \mathcal{L}_{\omega,\mathsf{reg}} B\big], \tag{4.3}$$

where the subscript 'reg' indicates that we added a small diffusion term $\sim \partial_{\mathbf{x}}^2$ to the Liouville operator.

[1] In fact, the perturbative evaluation method may be performed (with minor modifications which do not affect the arguments of this chapter) about *any* reference point $\Lambda = R\sigma_3^{\mathsf{ar}} R^{-1}$ which is obtained from σ_3^{ar} by means of a homogeneous rotation R. In fact, we may *define* the field space by an integral over all uniform rotations R plus fluctuations W. Since the latter form a linear matrix space, their Wigner symbols are well–defined.

[2] Here, we already tacitly assume that mode locking will be ensured self–consistently.

4.1.2 Perturbative terms and contraction rules

To compute the perturbative expansion we need to consider the non–linear contributions $S^{(2n>2)}$ to the perturbative action, as given in (4.2). We recall that all products of B–matrices have to be understood as Moyal products in the transversal directions $\mathbf{y}$, while they are local in the (E,t)–sector, so we have[3]

$$(B^\dagger B)^{2n}(\mathbf{x}) = \int \prod_{j=1}^{2n} \frac{\mathrm{d}\mathbf{y}_j}{(\pi\hbar)^{f-1}} \mathrm{e}^{\frac{i}{\hbar}S(\mathbf{y}_1,\ldots\mathbf{y}_{2n})} B^\dagger(\mathbf{x}+\mathbf{y}_1)\cdots B(\mathbf{x}+\mathbf{y}_{2n}). \tag{4.4}$$

In view of the quadratic action (4.3), the contraction rules [29] employed in calculating integrals over products of B matrices read

$$\begin{aligned}
\left\langle \mathrm{tr}(B(\mathbf{x})A)\,\mathrm{tr}(B^\dagger(\mathbf{x}')A')\right\rangle_B &= \frac{\Omega}{t_{\mathsf{H}}} P_\omega(\mathbf{x},\mathbf{x}')\ \mathrm{tr}(AA') \\
\left\langle \mathrm{tr}(B(\mathbf{x})AB^\dagger(\mathbf{x}')A')\right\rangle_B &= \frac{\Omega}{t_{\mathsf{H}}} P_\omega(\mathbf{x},\mathbf{x}')\ \mathrm{tr}(A)\,\mathrm{tr}(A') \\
\left\langle \mathrm{tr}(B(\mathbf{x})A)\,\mathrm{tr}(B(\mathbf{x}')A')\right\rangle_B &= \frac{\Omega}{t_{\mathsf{H}}} P_\omega(\mathbf{x},\bar{\mathbf{x}}')\ \mathrm{tr}(AA'^{\tau}) \\
-\left\langle \mathrm{tr}(B(\mathbf{x})AB(\mathbf{x}')A')\right\rangle_B &= \frac{\Omega}{t_{\mathsf{H}}} P_\omega(\mathbf{x},\bar{\mathbf{x}}')\ \mathrm{tr}(AA'^{\tau}),
\end{aligned} \tag{4.5}$$

where $P_\omega \equiv \mathcal{L}_{\omega,\mathsf{reg}}^{-1}$ denotes the regularized propagator, and A and A' are arbitrary fixed matrices. To compute the integral over an arbitrary product of traces of B–matrices, one first forms all possible total pairings B—$B^\dagger$, B—B, and $B^\dagger$—$B^\dagger$, and then computes individual pairings by means of (4.5). Each contraction reduces the number of matrices by two. Eventually, one obtains an expression $\sim (\mathrm{tr}\ \mathbb{1})^k = (2R)^{k\geq 2}$, where all contributions with $k>2$ vanish in the replica limit. By elementary power counting, each matrix B scales as (symbolic notation) $\sim (\mathcal{L}_\omega)^{-\frac{1}{2}} \sim \omega^{-\frac{1}{2}} \sim s^{-\frac{1}{2}}$. Therefore, each vertex $S^{(2n)}$ contributes a factor $\sim (B^\dagger B)^{n-1}B^\dagger \mathcal{L}_\omega B \sim s^{-n+1}$ to the functional integral.

From the traciality property of the Wigner representation[4] we see that the quadratic action is local in phase space. On the other hand, a view at the integral representation (4.4) of the Moyal product reveals that the field coordinates in the perturbative $(n>1)$ terms of the action (4.6) are fuzzy on the Planck cell scale $\mathbf{x}_i^{\mathrm{T}} I \mathbf{x}_j \lesssim \hbar$. The latter feature is a direct manifestation of quantum uncertainty and will be of high importance in the following. As a side (nevertheless important) remark, we want to mention that at this stage it becomes most transparent why the Wigner representation is suited best for the questions at hand: its self–duality[5] allows to treat all fields on equal footing and clearly separates

(i) the *linear* scales $\hbar$ related to the breakdown of the quasiclassical approximation (2.17) — and therefore to the regulator — on one hand

[3] Here, we are a bit sloppy adding $2f$–dimensional objects $\mathbf{x}$ and $2(f-1)$–dimensional objects $\mathbf{y}$, but the notation should be obvious.

[4] Cf. section 2.2.

[5] Namely, $+s=-s$ in the traciality property (iii) in subsection 2.2.1; this property is exclusive to the $s=0$ member of a Stratonovich–Weyl correspondence, which in the case of the standard representation is just the Wigner symbol.

(ii) from the aspects of quantum uncertainty to be associated to the symplectic *area* $\hbar$, which are manifest in the perturbative terms ($n > 1$) on the other hand.

The linear scales associated to the latter are typically much larger than the former (since $\hbar^{\delta} \cdot \hbar^{\delta'} = \hbar$ typically implies $\delta, \delta' < 1$). The separation of the aspects of regularization (i) and of quantum uncertainty (ii) has already been observed in the discussion of the Andreev billard in chapter 3 and tends to be concealed in, say, the Husimi/Cahill–Glauber representation ($s = \pm 1$) due to the fact that Husimi symbols are smooth over much larger linear scales $\hbar^{\frac{1}{2}}$. We see that in the perturbative terms, the derivatives of the regulator act on propagators whose arguments are averaged over Planck cells and hence smooth enough to render these derivatives negligible, so we find that the perturbative terms of the action (4.2) are simplified according to

$$S^{(2n>2)}[B] = \frac{t_{\mathsf{H}}}{2} \int (\mathrm{d}\mathbf{x}) \ \mathrm{tr}\big[(B^\dagger B)^{n-1} B^\dagger \mathcal{L}_\omega B\big], \tag{4.6}$$

keeping in mind, of course, that the product is given by (4.4).

4.1.3 Regularization in the chaotic case

Naïvely, one might hope that in order to achieve the reduction (2.17) it suffices that in the present context the *initial* distributions in phase space are sufficiently smooth;[6] however, what complicates the problem in the case of chaotic dynamics is that the generator $\{H, \ \}$ of classical evolution by itself leads to the dynamical buildup of singularities due to Lyapunov contraction, no matter how smooth the initial distribution was. Eventually the quasiclassical approximation will break down, a phenomenon we already came across in the context of the proximity effect in chapter 3. Namely, linearizing the flow around a given reference trajectory, the equations of motion controlling the evolution of a phase space distribution ρ assume the form $\dot{\rho} = \{H, \rho\} = \lambda s \partial_s \rho + \ldots$, where s is the coordinate which contracts strongest, λ the corresponding Lyapunov exponent, and the ellipses indicate derivatives in other coordinate directions. After a time $\lambda^{-1} \ln(\delta x_0/\hbar)$, where δx_0 denotes the characteristic initial extension of the distribution, structures in the s–direction fluctuating on scales $\lesssim \hbar$ will have formed implying that the higher–order derivatives acting in s–direction can no longer be neglected. This complication may be removed by adding to the generator of classical time evolution an elliptic operator $\sim D\partial_{\mathbf{x}}^2$, where D is a constant [58]. Indeed, it is straightforward to show (by dimensional analysis or by explicit calculation) that for the regularized operator $\lambda s \partial_s + D\partial_s^2$ the initial contraction halts at a characteristic scale $s \sim (D/\lambda)^{\frac{1}{2}}$. Choosing $D \sim \hbar^{2\alpha}$, where $\alpha \in (0, 1)$,[7] it is guaranteed that the distribution will not build up structure on scales $\hbar$, i.e. that the quantum corrections to classical dynamics remain negligible. In fact, we will stick to the weakest possible choice of $\alpha \nearrow 1$, since this will help us to separate the effects which stem from quantum uncertainty (coarse graining over Planck cells of area $\hbar$, but of linear dimensions which are much larger than $\hbar$) from the effects of a regulator cutting off details at 'some quantum scale $\hbar^{\alpha}$'. To understand the implications of the

[6]Namely, owing to the averaging of the propagator arguments, smooth over Planck cells of area $\hbar$, and thus over linear scales larger than $\hbar$.

[7]The condition $\alpha > 0$ ensures that the regulator term is classically negligible.

presence of such a regulator, let us discuss the propagator $P_\omega \equiv \mathcal{L}^{-1}_{\omega,\text{reg}}$. Importantly, P_ω is not strictly inverse to the bare Liouville operator (i.e. the Liouville operator acting in the space of unregularized functions), $\mathcal{L}_\omega P_\omega(\mathbf{x},\mathbf{x}') \neq \delta(\mathbf{x}-\mathbf{x}')$, but rather acts on a space of functions which are smooth below some cut–off scale $\hbar$. Accordingly, its time Fourier transform, $P_t(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}(t))$, can resolve the definite dynamical evolution generated by the Liouville operator only up to times

$$\tilde{t}_{\mathsf{E}} \equiv \frac{2}{\lambda}\ln\frac{c}{\hbar} \sim 2t_{\mathsf{E}},$$

where the factor of 2 stems from the fact that the propagator possesses two legs, each leg departing from a patch of linear size $\hbar$. Thereafter, the limited resolution of the boundary conditions (the effect of smoothing) renders the dynamics unpredictable and rapid mixing settles in.[8] Taken together, we find

$$P_t(\mathbf{x},\mathbf{x}') = \begin{cases} \tilde{\delta}(\mathbf{x}-\mathbf{x}'(t)) & , \; t < \tilde{t}_{\mathsf{E}}, \\ \Omega^{-1} & , \; t > \tilde{t}_{\mathsf{E}}, \end{cases}. \tag{4.7}$$

where the 'delta function' is not a mathematical delta function but a wave packet of classically negligible extension. The crossover between the two regimes takes place over time scales $\sim \max\{\delta\tilde{t}_{\mathsf{E}}, t_{\mathsf{mix}}\}$, where $\delta\tilde{t}_{\mathsf{E}} \ll \tilde{t}_{\mathsf{E}}$ is the uncertainty in $\tilde{t}_{\mathsf{E}}$ caused by an eventual non–uniformity of the Lyapunov expansion.[9] Notice that in previous discussions of the ballistic σ–model the propagator P_ω was (correctly) identified with the Perron–Frobenius operator, which was mostly (and too naïvely, it turns out) considered as an object which describes relaxation into a uniform configuration, $P_t(\mathbf{x},\mathbf{x}') \overset{t>t_{\mathsf{mix}}}{=}$ const. over classically short times. However, while the former identification is correct, it is impossible to reconcile the latter behavior with the indispensable condition that[10]

$$P_t(\mathbf{x},\bar{\mathbf{x}}') \overset{t<t_{\mathsf{E}}(\mathbf{x},\mathbf{x}')}{=} 0 \text{ for } |\mathbf{x}-\mathbf{x}'| \ll c, \tag{4.8}$$

implying that the propagator must be able to resolve fine structures in phase space over times parametrically larger than the relaxation time of the Perron–Frobenius operator. (In view of the phase space structure of the vertices (4.4), however, we see that the

[8] For a more rigorous argument, cf. the discussion of section 4.2.

[9] The results above apply to uniformly hyperbolic systems. In the case of non–uniformly hyperbolic systems, local fluctuations in the Lyapunov expansion rate $\lambda(\mathbf{x})$ need to be taken into account. The logarithmic mismatch $y(\mathbf{x},t) = \ln(u(t)/u(0))$ between two trajectories starting at $\mathbf{x}$ and $\mathbf{x}+u(0)\mathbf{e}_u$, respectively, grows as $\dot{y} = \lambda(\mathbf{x}(t))$. ($\mathbf{e}_u$ is the locally most unstable direction in phase space.) Due to inhomogeneities in the expansion rate, $y(\mathbf{x},t)$ is a fluctuating quantity with mean $y(t)$ and a certain width $\delta y(t)$. Importantly, an upper bound on fluctuations in y is imposed by Oseledec's theorem [64] which states that the phase space average λ of the Lyapunov expansion rate equals the long–time expansion rate of individual trajectories almost everywhere: $y(\mathbf{x},t)/t \to \lambda$ for $t \to \infty$ for almost all $\mathbf{x}$. Consequently, $\delta y(t) \sim t^\eta$ grows at a rate $\eta < 1$. (E.g., the model of statistically independent Gaussian fluctuations of the local expansion rate employed by AL [16] leads to $\eta = \frac{1}{2}$.) By definition of $\tilde{t}_{\mathsf{E}}$, a phase space distribution of initial extension $\sim \hbar^{2f}$ has expanded to classical dimensions when $y(t) = \lambda\tilde{t}_{\mathsf{E}}$. Defining $\delta\tilde{t}_{\mathsf{E}}$ as the time uncertainty in $\tilde{t}_{\mathsf{E}}$ (due to fluctuations in the local expansion rate), we obtain the estimate $\delta\tilde{t}_{\mathsf{E}} \sim \delta y(\tilde{t}_{\mathsf{E}})/\lambda \sim \tilde{t}^\eta_{\mathsf{E}}$. This means that $\delta\tilde{t}_{\mathsf{E}}/\tilde{t}_{\mathsf{E}} \sim \tilde{t}^{\eta-1}_{\mathsf{E}}$ vanishes in the semiclassical limit. For finite $\hbar$, the effective relaxation rate of the system is set by $\max\{\delta\tilde{t}_{\mathsf{E}}, t_{\mathsf{mix}}\}$.

[10] Recall our discussion of the semiclassical treatment of weak localization in section 1.3, in particular the result (1.8) for the classical transition amplitude between almost time reversed phase space points.

center–of–mass argument $\mathbf{x}$ of the involved propagators always appears to be averaged such that the situation (4.8) is the only case where the deviation from the uniform configuration plays a role.) Referring to section 4.2 for a more substantial resolution to this paradox, we here only state that equation (4.7) is motivated by the structure of the action and *does* resolve the classical phase space dynamics (and therefore preserves the condition (4.8)) up to $\tilde{t}_{\mathsf{E}}$. Due to quantum uncertainty, however, the initial conditions of the fields always appear to be averaged over Planck cells (with typical linear dimensions in excess of $\hbar$), so relaxation settles in already long before the threshold time $\tilde{t}_{\mathsf{E}}$ associated to the UV cut–off of the fields.[11] In any case, the delay time will still be of the order of t_{E}, so the separation of the (small, of order t_{mix}) time scale over which relaxation takes place and the (large, of order t_{E}) delay time is still intact, and the relaxation can be considered to be completed instantaneously once it has begun. Thus, the theory has long become quantum–unpredictable at a time $\tilde{t}_{\mathsf{E}}$ where the artificially introduced smearing would become virulent, and we conclude that the strength of the regulator does not explicitly enter the results of the theory.

4.1.4 Regularization in the integrable case

It is necessary to investigate the impact of the proposed regularization scheme on an integrable system. We aim to show now that in integrable systems, the regulator enforces the universal regime only at energies which scale as $\hbar^3$, which in $f \leq 3$ dimensions is of the order of (or even smaller than) the level spacing $\Delta \sim \hbar^f$, the barrier below which perturbative evaluation schemes fail anyway.[12]

Recall that an integrable system possesses f independent constants of motion, the action variables $\mathbf{I}$, along with canonically conjugate angle variables $\boldsymbol{\phi}$ which move with constant velocities $\boldsymbol{\omega}(\mathbf{I}) = \partial_{\mathbf{I}} H$. The Hamiltonian $H = H(\mathbf{I})$ is a function of the constants of motion alone. The question is now how large is the gap of the integral kernel of the quadratic action. To that end, consider the eigenvalue equation

$$\frac{-i\omega}{\hbar}\rho_\omega = \left[-\boldsymbol{\omega}(\mathbf{I}) \cdot \partial_{\boldsymbol{\phi}} + D\partial_{\mathbf{x}}^2\right]\rho_\omega.$$

which governs the propagator of the regularized action. For a finite and classically large energy shell, the diffusion operator possesses eigenvalues whose spacing scales as $D/L^2 \sim \hbar^2$. Low–lying modes with eigenvalues of this size are, for instance, given by configurations which are independent of the angle variables and depend on the action variables only on classical length scales. We conclude that the gap in ω scales no larger than $\hbar^3$, which in $f \leq 3$ dimensions is at most of the order of the level spacing.

4.1.5 Microscopic justification for the regulator

There are several ways to justify a regularization as introduced above. They usually consist of introducing an ensemble of random perturbations, $H \mapsto H+V$, to the original Hamiltonian. Examples are stochastic disorder potentials or a random Aharonov–Bohm

[11] Cf. the discussion of section 5.1.2, where we substantiate this heuristic argument by a calculation.

[12] Recall the non–analyticity of the form factor at $\tau = 1$ (cf. section 1.3) which is associated to a breakdown of perturbation theory at $\omega \sim \Delta$.

flux [65, 66, 67, 68], or a sum of Hamiltonians which generate motion along all $2f$ directions of phase space [31]. Generically, an average over the corresponding ensemble yields a correction[13]

$$\begin{aligned}\delta S_{\text{reg}} &\sim \big\langle \mathrm{tr}(\mathcal{T}\mathcal{Q}_0[V,\mathcal{T}^{-1}]\mathcal{T}\mathcal{Q}_0[V,\mathcal{T}^{-1}]\big\rangle_V \\ &\to c\int(\mathrm{d}\mathbf{x})\int(\mathrm{d}\mathbf{x}')\ \mathrm{tr}\big(Q(\mathbf{x})Q(\mathbf{x}')\big)\tilde{f}(\mathbf{x}-\mathbf{x}') \\ &\to c\int(\mathrm{d}\mathbf{x})\left(\frac{\hbar}{\xi}\right)^2(\partial_{\mathbf{x}}Q)^2.\end{aligned}$$

to the action. In the second line of this symbolic calculation, all integrals and the correlation function f have been normalized to unity and all numerical factors have been absorbed in the constant c. In the third line we assumed that a gradient expansion is valid self–consistently.[14] We want to keep the effect of the regulator term as weak as possible and therefore restrict ξ to classical values. Note that the unperturbed action (2.18) is of order ω/Δ. Thus, in order not to significantly affect the spectrum of the Hamiltonian H, we would have to demand that the theory is stabilized against the appearance of quantum corrections to the dynamics if $c \lesssim 1$. On the other hand, we read off (4.3) that $c \sim \hbar/\Delta \gg 1$.[15] In other words, an ensemble which leaves the spectrum of an individual system intact is not sufficient to derive a regulator for the ballistic σ–model in the way described above,[16] and we see no other way than to introduce a larger ensemble, which certainly implies that *all statements to be derived in this work hold only on average over this ensemble and not for individual systems*. A convenient[17] ensemble was proposed by Zirnbauer [31], who considered perturbations

$$H \mapsto H + V, \quad V = \sum_{j=1}^{2f} \xi_j X_j,$$

to the original Hamiltonian H, where the ξ_j are independent Gaussian distributed random variables with variance ϵ, and the X_j are quantizations of classical Hamiltonian functions

[13]A term $[\mathrm{tr}(QV)]^2$ also arises, but since Q has no preferred basis, all matrix elements are of the same order, and the term $\mathrm{tr}(QVQV)$ which we kept here consists of much more summands. In any case, the latter term does the job, and the two terms do not cancel each other.

[14]A straightforward calculation reveals that $\tilde{f}$ is the (normalized) symplectic Fourier transform of $f(\mathbf{x}-\mathbf{x}') = \big\langle V(\mathbf{x})V(\mathbf{x}')\big\rangle_V$, viz.

$$\begin{aligned}S_{\text{reg}}[T] &= -\frac{\beta\pi^2}{4}\big\langle \mathrm{tr}(\mathcal{Q}V)^2\big\rangle_V \\ &= -\frac{\beta\pi^2}{4}\int\frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f}\int\frac{\mathrm{d}\mathbf{x}_1}{(2\pi\hbar)^f}\,\mathrm{tr}\big(\mathcal{Q}(\mathbf{x})\mathcal{Q}(\mathbf{x}+\mathbf{x}_1)\big)\int\frac{\mathrm{d}\mathbf{x}_2}{(2\pi\hbar)^f}\mathrm{e}^{\frac{i}{\hbar}\mathbf{x}_1^{\mathrm{T}}I\mathbf{x}_2}f\Big(\frac{\mathbf{x}_2}{\xi}\Big),\end{aligned}$$

hence the notion $\hbar/\xi$ for the correlation length.

[15]From our discussion of the regularized dynamics in subsection 4.1.3 we also know that this choice of c ensures the validity of the gradient expansion: indeed, the regulator smoothes the dynamics at scales $\hbar^\alpha$, $\alpha \nearrow 1$, before the dangerous scales $\hbar/\xi$ (which for classical ξ are of $\mathcal{O}(\hbar)$) build up where the gradient expansion would break down.

[16]This does not exclude more sophisticated regularization schemes which systematically keep track of both the classical dynamics *and* quantum corrections. A promising direction of research seems to be the recent work by Dittrich *et al.* on the Wigner representation of the von Neumann propagator [69].

[17]The reader can convince himself that Gaussian distributed stochastic disorder with classically large (phase space) correlation length ξ will do equally well.

with associated linear independent Hamiltonian vector fields Ξ_j. In fact, we are free to choose the phase space coordinates $X_j = x_j$ as perturbations. These have the virtue to be linear, implying that the expansion (2.14) always terminates at leading order, $[x_j, \,] = i\hbar\{x_j, \,\}$, which is just a partial derivative w.r.t. the coordinate conjugate to x_i. Expanding the action (2.10) to second order in the perturbation and averaging, we find a correction

$$\begin{aligned} S_{\text{reg}} &= -\frac{\beta\pi^2}{4}\big\langle \mathrm{tr}\big(\mathcal{T}\mathcal{Q}_0[V,\mathcal{T}^{-1}]\mathcal{T}\mathcal{Q}_0[V,\mathcal{T}^{-1}]\big)\big\rangle_V \\ &= \frac{\beta(\pi\hbar)^2}{4}\int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f}\sum_{j=1}^{2f}\epsilon\,\mathrm{tr}\big(\mathcal{T}\mathcal{Q}_0\{\Xi_j,\mathcal{T}^{-1}\}\mathcal{T}\mathcal{Q}_0\{\Xi_j,\mathcal{T}^{-1}\}\big) \\ &= \frac{\beta(\pi\hbar)^2}{4E_{\text{av}}\Delta}\int (\mathrm{d}\mathbf{x})\sum_{j=1}^{2f}\epsilon\,\mathrm{tr}(TQ_0\partial_{\mathbf{x}}T^{-1})^2, \end{aligned}$$

where we carried out the Gaussian average to obtain the second line, and we assumed mode–locking to be self–consistently given in the third line.[18] There are two factors of $E_{\text{av}}^{-1} \sim \hbar^{-1}$ coming from the saddle points $\mathcal{Q}_0$; one got absorbed into the integral measure, while the other one remains uncompensated (cf. the discussion of section 2.3). Comparing with the unperturbed action (2.18) and counting powers of $\hbar$, we find that the variance ϵ can be identified (up to classical constants) with the diffusion constant D of the regulator. Zirnbauer was led to the same identification, but note that he postulated $\epsilon \sim \hbar^\alpha$, while we find that a much smaller variance $\epsilon \sim \hbar^{2\alpha}$ suffices,[19] since all we require is the suppression of the quantum corrections to the expansion (2.14) of the von Neumann commutator. Similarly to Zirnbauer, Aleiner & Larkin [16, 17] postulated a phenomenological 'quantum scattering potential' of strength $\epsilon \sim \hbar$ much larger than ours. We finally perform a straightforward expansion in the generators B to readily obtain the postulated Gaussian action (4.3).

It remains to investigate the effect of the regulator upon those modes B which commute with the Hamiltonian. To that end, consider the expansion of S_{reg} to quadratic order in the B–fields and represent the action w.r.t. an eigenbasis $H|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$ of the Hamiltonian, which reads (up to constants of order unity)

$$\begin{aligned} \big\langle \mathrm{tr}(QVQV)\big\rangle_V \to & \frac{1}{E_{\text{av}}^2}\Big[\big\langle V_{\alpha\gamma}V_{\gamma\alpha}\big\rangle_V\,\mathrm{tr}(\bar{B}_{\alpha\beta}B_{\beta\alpha}) - \big\langle V_{\beta\gamma}V_{\delta\alpha}\big\rangle_V\,\mathrm{tr}(\bar{B}_{\alpha\beta}B_{\gamma\delta})\Big] \\ \to & \frac{1}{E_{\text{av}}^2}\big(\big\langle V_{\alpha\gamma}V_{\gamma\alpha}\big\rangle_V\delta_{\alpha\beta} - \big\langle V_{\alpha\beta}V_{\beta\alpha}\big\rangle_V\big)\,\mathrm{tr}(\bar{B}_{\alpha\alpha}B_{\beta\beta}). \end{aligned}$$

Recall that the saddle–point (2.9) projects onto the energy window $[E_0 - E_{\text{av}}, E_0 + E_{\text{av}}]$, so the summations are restricted to those states whose energy lies within this range. In the second line we restricted ourselves to the modes $B_{\alpha\alpha}$ which commute with the Hamiltonian. The matrix $M_{\alpha\beta}$ coupling these modes has the following properties:

(i) M is symmetric.

[18] In fact, the expansion relies on the validity of a gradient expansion to omit higher order terms. As already explained, our final tuning of the strength parameter ϵ will ensure this.

[19] In both cases, $\alpha \in (0, 1)$, and we always take $\alpha \nearrow 1$.

(ii) $\sum_\alpha M_{\alpha\beta} = 0$.

(iii) Since H is chaotic, the eigenbases of V and H are generically uncorrelated It follows that the diagonal entries of M are approximately equal and of size $M_{\alpha\alpha} \sim E_{\mathsf{av}}/\lambda \cdot \left\langle (V^2)_{\alpha\alpha} \right\rangle_V / E_{\mathsf{av}}^2 \sim \hbar$,[20] while its off–diagonal entries are random numbers which are smaller than the diagonal entries by a factor of $1/N$.

Matrices of this type are known [70] to possess one non–degenerate eigenvalue zero and $(N-1)$ eigenvalues of order $M_{\alpha\alpha}(1-N^{-\frac{1}{2}})$. Comparing with the magnitude ω/E_{av} of the unperturbed action, we conclude that the modes which commute with the Hamiltonian are no longer zero modes of the regularized theory but acquire a mass $\sim M_{\alpha\alpha}E_{\mathsf{av}} \sim \hbar^2$ — except for the single universal mode $B \propto \mathbb{1}$ which is an exact zero mode of the action. This ensures the suppression of all modes $B_{\alpha\alpha}$ but the universal mode at energies of the order of the level spacing[21] even in the lowest possible dimension $f = 2$.

Let us finally comment on the effect of the ensemble average to the semiclassical approach to spectral correlations as reviewed in section 1.3. As we have just seen, this ensemble consists of systems whose Hamiltonians differ from the reference Hamiltonian H by a correction $V = \hbar v$, where v is a classical operator. The effect of such a correction is that the action of the Gutzwiller trace formula (1.2) has to be modified according to

$$S_\gamma \mapsto S_\gamma + \delta S_\gamma, \quad \delta S_\gamma = \int \mathrm{d}t\, V(\gamma(t)),$$

while the sum is still over the periodic orbits of the *unperturbed* Hamiltonian flow [47]. For a single DoS, a disorder average leads to an exponential suppression of contributions of orbits whose length T_γ exceeds some classical threshold time T^*,[22] which just means that the resolution of single levels is limited to scales $\hbar/T^*$. In contrast to this, the semiclassical evaluation of the form factor is *insensitive* to any such perturbation, as we will now show: as we have learned in section 1.3, the partner orbits γ, γ' are identical outside encounter regions, so it remains to check that $\delta S_\gamma - \delta S_{\gamma'} \ll \hbar$ also inside an encounter region. But even inside an encounter region, there is always a stretch of γ' to each stretch of γ such that their mutual distance $\delta \mathbf{x} \lesssim \hbar^{\frac{1}{2}}$ (cf. Fig.1.3), and thus

$$\delta S_\gamma - \delta S_{\gamma'} \lesssim t_{\mathsf{E}}\, \delta\mathbf{x} \cdot \partial_{\mathbf{x}} V \lesssim |\ln \hbar|\, \hbar^{\frac{3}{2}} \ll \hbar,$$

which completes our argument that perturbations $V = \hbar v$ leave the semiclassical form factor invariant. The semiclassical approach selects terms which *do* contribute coherently to the form factor without justifying systematically why the omitted terms *do not* contribute. This forbids to call the semiclassical result in favor of BGS statistics of individual chaotic systems more than a 'physicist's proof'. In particular, it will turn out in the following chapter that the semiclassical and the field theoretical formalism are by

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[20]The factor E_{av}/λ, where λ denotes the (classical) width of the spectrum of H, accounts for the restriction of the summation over β in $\sum_\beta |V_{\alpha\beta}|^2 \sim E_{\mathsf{av}}/\lambda \cdot (V^2)_{\alpha\alpha}$ to the energy window.

[21]Which is the relevant energy regime for these modes; indeed, the $B_{\alpha\alpha} \sim |\alpha\rangle\langle\alpha|$ project onto the shell of energy ϵ_α, thus they parameterize those modes which are homogeneous over individual energy shells but still display transversal fluctuations and are therefore are interesting only for the question of mode–locking in the universal regime.

[22]For classically smooth Gaussian disorder in a fermionic billard, the condition for suppression is $\left\langle (\delta S_\gamma/\hbar)^2 \right\rangle_V \sim v^2 T_\gamma \xi / v_{\mathsf{F}} \gg 1$ [66].

and large equivalent. Since equivalent approaches should usually face similar problems, the question remains open whether the field theorists are just too ignorant to find a more sophisticated regularization scheme or if semiclassics is overlooking something.

4.2 Universality: A step towards mathematical rigor

In this section we aim to resolve the paradox of subsection 4.1.3, namely how to reconcile the classical gap of the Perron–Frobenius propagator with the existence of Ehrenfest time effects. To that end, we need to somewhat formalize our language in order to be able to reconsider the original works by Bowen and Ruelle which date back to the 1970s. It turns out that these classic works already bear the key to resolving this puzzle. As a byproduct, we build a bridge to the current state of debate in ergodic theory and report what mathematicians mean today by the (still controversial) term 'generic' in the BGS conjecture. Relying on this definition, we derive universal spectral correlations from the regularized ballistic σ–model.

4.2.1 Setup: What is 'generic' chaos?

As before, we shall consider closed quantum systems whose classical dynamics displays 'generic' chaos. Certainly, the flows under consideration shall be exponentially mixing,

$$\lim_{t\to\infty}\left|\int(\mathrm{d}\mathbf{x})\,f(\mathbf{x})g(\mathbf{x}(t))-\int(\mathrm{d}\mathbf{x})\,f(\mathbf{x})\cdot\int(\mathrm{d}\mathbf{x})\,g(\mathbf{x})\right|\leq C_{f,g}\cdot\mathrm{e}^{-t/t_{\mathsf{mix}}}. \tag{4.9}$$

The identification of a 'generic' class of flows satisfying the property of exponential decay of correlations is a rather difficult and controversial topic. Usually, people resort to the 'harmonic oscillators' of chaos, maps which satisfy the so–called Axiom A.[23] Let us mention that while the qualitative mixing property is well–established for Axiom A flows and maps, and exponential decay of correlations in Axiom A *maps* was established in the 1970s in a series of papers by Sinai, Bowen and Ruelle, the question regarding the speed of mixing of *flows* is subject to fruitful research to date. The main difficulty for flows in contrast to maps is the neutral direction along the flow. Dolgopyat [72] proposed a notion of 'generic' suspension flows over a subshift of finite type for which he proved exponential decay of correlations. In order to give the term 'generic' a well–defined meaning, we want to stick to Dolgopyat's definition. Roughly speaking, we are dealing with flows admitting a Poincaré section $\mathcal{S}$ (with sufficiently regular first return time function τ whose expectation value τ_0 sets the discrete time scale) such that the discrete time dynamics induced by the map $\Phi:\mathcal{S}\to\mathcal{S}$ of first return to $\mathcal{S}$ allows a representation in terms of a symbolic dynamics (Σ_A,σ) with finite grammar.[24] The less ambitioned reader might think of a member of this certainly non–empty class as of a 'generic' chaotic billiard and of the (usually short and certainly classical) time τ_0 as of the flight time across that billiard.

[23]That is, uniformly hyperbolic. For recent results on partial or non–uniformly hyperbolic systems, see the survey [71] and references therein.

[24]For details consult the pedagogic survey [73], which we follow — at times verbatim — in notation and argumentation.

4.2.2 Ehrenfest time and universality: A mathematics dictionary for physicists

We discussed in section 4.1 that the appearance of Moyal products in the perturbative action implies an average of the field coordinates over Planck cells.[25] Thus, the perturbative expansion consists of the basic building block

$$\left\langle B(\mathbf{x})B^\dagger(\mathbf{x}')\right\rangle_B = \frac{\Omega}{t_\mathsf{H}} P_\omega(\mathbf{x},\mathbf{x}'),$$

where $P_\omega(\mathbf{x},\mathbf{x}')$ is the Fourier transform of the propagator $P_t(\mathbf{x},\mathbf{x}')$ of the classical Hamiltonian flow, and the coordinates $\mathbf{x}$ and $\mathbf{x}'$ are averaged implicitly over Planck cells. We therefore proposed the prescription (4.7). We experienced that parts of the physics community (including us) found it hard to reconcile the spectral gap of the Perron–Frobenius operator which implies exponential decay of correlations in chaotic systems on a *classical* time scale t_mix intrinsic to the flow with the existence of the Ehrenfest time, up to which the relaxation of (4.7) is prohibited, and which may become *arbitrarily large* in the semiclassical limit $\hbar \to 0$. The resolution to this paradox is found in a classical theorem by Ruelle [74]. Since this theorem is formulated in the language of symbolic dynamics and the 'thermodynamic formalism' for ergodic systems, we try and establish a rudimentary dictionary translating between our field theoretical language and ergodic theory.

The crucial tool we need is symbolic dynamics: let us consider the Poincaré map $\Phi : \mathcal{S} \to \mathcal{S}$ associated to our 'generic' flow. By virtue of the assumptions made above it is possible to find a suitable partition $\mathcal{R} = \{R_1, \dots, R_m\}$ of $\mathcal{S}$ into cells, corresponding to the letters $\{1, \dots, m\}$ of an alphabet, together with a transition matrix A with entries

$$A_{ij} = \begin{cases} 1, & \text{if } R_i \cap \Phi^{-1}(\mathrm{int}\ R_j) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}.$$

The latter defines a so–called 'topological Markov chain' (TMC)

$$\Sigma_A = \left\{\omega = (\cdots \omega_{-1}\omega_0\omega_1 \cdots),\, \omega_i \in \{1, \dots, m\},\, \forall i \in \mathbb{Z} : A_{\omega_i\omega_{i+1}} = 1\right\}$$

together with a left shift $\sigma : \Sigma_A \to \Sigma_A$ defined by $\sigma(\omega)_i = \omega_{i+1}$. (A finite word is understood to be periodically continued.) Our setup implies that this TMC (Σ_A, σ) is mixing, i.e., that some positive power A^{n_mix} of the transition matrix contains no zeroes. The words $\omega \in \Sigma_A$ correspond to points $\mathbf{x} \in \mathcal{S}$ by means of the identification

$$\mathbf{x} \equiv \pi(\omega) = \bigcap_{n\in\mathbb{Z}} \Phi^{-n}(R_{\omega_n}),$$

and one has $\pi \circ \sigma = \Phi \circ \pi$. Let $-\infty < m < n < \infty$ and $\omega_{[m,n]} = \omega_m\omega_{m+1}\cdots\omega_n$ be an admissible word. The set

$$C(\omega_{[m,n]}) = \left\{\omega' \in \Sigma_A \middle|\, \forall m \le i \le n : \omega'_i = \omega_i\right\}$$

[25] In a more conservative discussion, we should replace Planck cells by coarse grains of linear dimension $\hbar$ and the Ehrenfest time t_E by $\tilde{t}_\mathsf{E} > t_\mathsf{E}$, respectively. This difference is inessential for the present discussion, since the relative size of these times is of order unity, and we are ultimately interested in the universal regime of energies $\omega \ll \hbar/t_\mathsf{E}$.

is called an (m,n)–cylinder. It can be shown that cylinders are equivalent to balls in phase space, and that the length of a cylinder needed to specify a ball of diameter ϵ is proportional to $n_\epsilon \sim \frac{2}{S}\ln(c/\epsilon)$, where S is the so–called entropy of the map, which by Pesin's formula is just the sum of the (positive) Lyapunov exponents.[26] The logarithmic dependence is intuitively clear: if one imagines the energy shell of volume c^{2f-1} to be discretized by a mesh of size ϵ, then one will need $\sim (c/\epsilon)^{2f-1}$ cells which may be numbered by $\sim \ln(c/\epsilon)$ digits; in symbolic dynamics, these digits are realized as a substring specifying a cylinder of length n_{E}. A special case of importance to us is the time needed to specify a Planck cell, that is $\epsilon \sim \hbar^{\frac{1}{2}}$, which naturally is equal to the Ehrenfest time

$$t_{\mathsf{E}} = \frac{1}{S}\ln\frac{c^2}{\hbar}. \tag{4.10}$$

Note that in the case $f=2$, we have $S=\lambda$, and we find back the estimate (1.4).

Summarizing, it is sometimes useful to think of a point $\mathbf{x}\in\mathcal{S}$ as of a generalized word or decimal fraction (the 'rationals' being periodic orbits) and to identify a phase space cell with a cylinder or finite substring.

4.2.3 Delay of mixing and universality

In the present context, a theorem due to Ruelle [74] applies, according to which there are constants $c>0$, $d\in(0,1)$ such that for any two cylinders $C=C(\omega_{[0,r]})$ and $D=D(\omega_{[0,s]})$ we have

$$|\rho(C\cap\sigma^{-n}D)-\rho(C)\rho(D)| \le c\rho(C)\rho(D)d^{n-s}, \quad n\ge 0.$$

Here, ρ is the lift of the Liouville measure to (Σ_A,σ). Note that $(n-s)$ is the gap between the intervals on which the cylinders C and $\sigma^{-n}D$ are based. Thanks to our dictionary, we may translate this rather abstract statement into 'everyday' language. Namely, comparing this result with the definition (4.9), we see that our 'generic' flows enjoy an exponential mixing property. We now recall that Planck cells are equivalent to cylinders. Since the time needed to specify the Planck cells is given by the Ehrenfest time (4.10), the corresponding cylinders have length $n_{\mathsf{E}} = t_{\mathsf{E}}/\tau_0$, and we conclude that mixing settles in only after the Ehrenfest time. In other words, we have proven the validity of our prescription (4.7). The paradox which was mentioned above is now resolved as follows: while the physicist associates the term 'exponential decay with a rate t_{mix}^{-1}' to an immediate decay of any initial distribution, the mathematician means an exponential upper bound to correlations — the prefactor in (4.9) may well be large. Indeed, in the case of f,g being characteristic functions of Planck cells, $C_{f,g}\sim \mathrm{e}^{t_{\mathsf{E}}/t_{\mathsf{mix}}}$, so mixing remains invisible before the Ehrenfest time has elapsed.

Let us now recall[27] that all perturbative terms consisted of propagators with the same energy ω. Thus, we only have to transform (4.7) back to frequency space to obtain

$$P_\omega(\mathbf{x},\mathbf{x}') \sim \frac{i\hbar}{\omega^+}\big(1+\mathcal{O}(\omega t_{\mathsf{E}}/\hbar)\big),$$

[26] For a more precise discussion of this result, consult [75]. The factor of two stems from the time needed to approach a given trajectory and to leave it again.

[27] Cf. the discussion of perturbation theory in section 4.1.

where the error accounts for contributions from the non–universal integration interval $(0, t_{\mathsf{E}})$. We conclude that for energies $\omega \ll \hbar/t_{\mathsf{E}}$, the error becomes small, and only the zero–mode survives. In this low–energy regime, the ballistic σ–model action (2.18) thus collapses to the zero–mode action

$$S_0[Q] = \frac{i\beta\pi s^+}{4}\,\mathrm{tr}(Q\sigma_3^{\mathsf{ar}}), \quad Q = T\sigma_3^{\mathsf{ar}}T^{-1}, \tag{4.11}$$

which is known to reproduce universal spectral statistics [76, 77]. One might object the validity of our argument, since the replica treatment is restricted to perturbative evaluations. Yet, note that the replica structure is inessential to the arguments above, so the supersymmetry result [26] applies equally well. We have therefore shown how to obtain universal spectral statistics from the regularized ballistic σ–model.

4.3 Summary

In this chapter we developed a consistent evaluation scheme for the ballistic σ–model of 'generic' chaotic systems. We have seen that any regularization scheme which keeps track of the classical dynamics by cutting off quantum corrections as in (2.17) strongly distinguishes among chaotic and integrable dynamics; more precisely, we found that in chaotic systems, the universal RMT limit is enforced at comparatively high energies $\omega \sim \hbar/t_{\mathsf{E}} \sim \hbar/\ln\hbar$, while in integrable systems the corresponding energy scales as $\hbar^3$, which in $f \leq 3$ dimensions is at most of the order of the level spacing. We argued that the field theory may at present (i.e., without a better semiclassical understanding of the evolution of quantum density matrices, which might hint at a more sophisticated regularization scheme) not justify BGS for individual systems, but rather for an ensemble of quantum systems which all share the same classical limit; such an ensemble was proposed before by Zirnbauer [31], yet we were able to significantly reduce the strength of the perturbations, which allowed us to separate aspects of regularization from those of intrinsic quantum uncertainty. We further demonstrated that a variation of the Hamiltonian within the ensemble leaves the semiclassical form factor invariant. Finally, we translated well–established mathematical results about mixing properties to the present context. As a result, we understood how universality emerges (only) after the Ehrenfest time, and we were able to derive universal spectral correlations for the mentioned ensemble. As a byproduct, we gave a proper definition to the term 'generic' in the BGS conjecture.

Summarizing, we have proposed an ensemble average which is *sufficient* to ensure regularization of the ballistic σ–model and to yield RMT spectral statistics. We cannot prove, however, that an average over such a large ensemble is *necessary*, i.e. we cannot exclude that the von Neumann propagator could be destabilized by other means — maybe even by an intrusion which is small enough not to affect the spectrum of an *individual* system.

Chapter 5

Perturbation theory II: Quantum interference and parallels to semiclassics

In this chapter we shall explain how corrections to the universal behavior of spectral correlations of chaotic systems arise and how these compare to the semiclassical approach. Building upon the field theoretical framework of chapters 2 and 4 we find how quantum interference may be explained, i.e. effects which are related to the recombination of classical paths due to quantum uncertainty. Within the ballistic σ–model this question — termed 'problem (ii)' in chapter 2 — was so far only answered phenomenologically by Larkin and collaborators [16, 17, 34]. In accord with our experience from the preceding chapters it is once again the Moyal product which makes the quantum uncertainty manifest. We demonstrate the ease of the field theoretical description on the example of the leading (in a short time expansion) quantum interference correction to the form factor $K(\tau)$ — the famous Sieber–Richter term. We benefit from the insights we gained from our discussion of semiclassical methods in section 1.3 and point out how far the analogies between field theory and semiclassics reach for the Sieber–Richter term and beyond.

5.1 Application to quantum interference

Having derived a perturbation theory and understood how to evaluate it, we now turn to a discussion of the results which readily follow. As long as we restrict ourselves to perturbative operations, i.e. an expansion of the two–level correlation function (1.1a) in a series

$$R_2(s) \stackrel{s\geq 1}{=} \mathrm{Re} \sum_{n=2}^{\infty} c_n (is^+)^{-n} \tag{5.1}$$

of powers of the dimensionless energy variable $s = \pi\omega/\Delta$, the replica limit $R \to 0$ is well–defined. A straightforward Fourier transformation, $K(\tau) = \pi^{-1} \int \mathrm{d}s\, \mathrm{e}^{-2is\tau} R_2(s)$, shows that the coefficients c_n are related to the coefficients d_n of the spectral form factor $K(\tau) \equiv \sum_{n=1}^{\infty} d_n \tau^n$ through

$$d_n = -\frac{(-2)^n}{n!} c_{n+1}. \tag{5.2}$$

In fact, however, there are much further–reaching analogies between the temporal and the frequency representation of spectral correlations: at every given order n, various topologically distinct families of orbit/partner orbit pairs ('diagrams') contribute to the coefficient d_n. Likewise, the expansion coefficients c_n obtain as sums of Wick contractions of the generating functional $Z(\omega)$. We shall see that there is an exact correspondence between field theoretical and semiclassical diagrams (both in topological structure and numerical value) which simply means that the two approaches describe spectral correlations in terms of the same semiclassical interference processes. We begin with the leading 'Berry' term (of order τ) and a brief discussion of the problem of repetitions. We continue to elaborate the perturbation theory on the paradigmatic example of the 'Sieber–Richter' (SR) term (of order τ^2) and then proceed to the terms of order τ^3. Building upon that experience we claim that this procedure finds a generalization to arbitrarily high orders.

5.1.1 The Berry term and problem of repetitions

Let us now turn to the perturbative expansion of the functional integral. The dominant contribution to the series (5.1) obtains by integration over the quadratic action:

$$R_2^{(2)}(s) = -\tfrac{1}{2} \lim_{R\to 0} \frac{1}{R^2} \mathrm{Re}\; \partial_s^2 \int \mathcal{D}B \mathrm{e}^{-S^{(2)}[B]} = -\tfrac{1}{2}\; \mathrm{Re} \lim_{R\to 0} \frac{1}{R^2} \partial_s^2 (\det P_\omega)^{2R^2} =$$
$$= \mathrm{Re}\; \partial_s^2 \ln \det(P_\omega^{-1}) \overset{\omega \ll \hbar/t_{\mathsf{mix}}}{\simeq} \mathrm{Re}\, (is^+)^{-2}. \quad (5.3)$$

This result implies (cf. equations (5.1, 5.2)) $d_1 = 2$ in accord with the semiclassical analysis, where the corresponding term is called the 'diagonal approximation' or 'Berry term' [11]. It is worthwhile to notice that the agreement between semiclassics and field theory does not pertain to times $t < t_{\mathsf{mix}}$: for these times short periodic orbits traversed more than once influence the behavior of the form factor. For reasons that are only partly understood the σ–model fails to correctly count the integer statistical weight associated to the repetitive traversal of periodic orbits. The essence of the problem [32] is that the degrees of freedom of the σ–model (the B's) describe the joint propagation of amplitudes locally paired in phase space. However, an n–fold repetitive process is governed by the local correlation of $2n$ Feynman amplitudes. Perturbative approaches to the problem fail to correctly describe these correlations, which is termed the 'problem of repetitions' and was mentioned in chapter 2 as problem (iii). Interestingly, a non–perturbative evaluation of the functional integral — feasible in the artificial case of the harmonic oscillator — leads to the correct result [78].

5.1.2 The Sieber–Richter term

The dominant correction ($\sim s^{-3}$) to the leading contribution (5.3) obtains by first order expansion in the vertex $S^{(4)}$:

$$R_2^{(3)}(s) = -\,\mathrm{Re} \lim_{R\to 0} \frac{t_{\mathsf{H}}}{(2R)^2} \partial_s^2 \int (\mathrm{d}\mathbf{x}) \big\langle \mathrm{tr}(B^\dagger B B^\dagger \mathcal{L}_\omega B) \big\rangle_B. \quad (5.4)$$

We first notice that the trace of the product (2.13) of four operators reduces to the expression[1]

$$\operatorname{tr}(A_1A_2A_3A_4) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f} \int \frac{\mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2}{(2\pi\hbar)^{2f}}\,\mathrm{e}^{\frac{i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2}\times \\ \times A_1\big(\mathbf{x}+\tfrac{1}{2}\mathbf{x}_1)A_2(\mathbf{x}+\tfrac{1}{2}\mathbf{x}_2)A_3(\mathbf{x}-\tfrac{1}{2}\mathbf{x}_1)A_4(\mathbf{x}-\tfrac{1}{2}\mathbf{x}_2).$$

Inserting this result into (4.4), applying the contraction rules (4.5), and taking the replica limit we obtain

$$R_2^{(3)}(s) = \operatorname{Re}\frac{\Omega^2}{t_{\mathsf{H}}}\partial_s^2 \int (\mathrm{d}\mathbf{x}) \int \frac{\mathrm{d}\mathbf{y}_1\,\mathrm{d}\mathbf{y}_2}{(\pi\hbar)^{2(f-1)}}\mathrm{e}^{\frac{i}{\hbar}\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2}\times \\ \times P_\omega(\overline{\mathbf{x}+\mathbf{y}_1/2}, \mathbf{x}-\mathbf{y}_1/2)\mathcal{L}_{\omega,\mathbf{x}-\mathbf{y}_2/2}P_\omega(\mathbf{x}-\mathbf{y}_2/2, \overline{\mathbf{x}+\mathbf{y}_2/2}),$$

where the coordinate subscript in $\mathcal{L}_{\omega,\mathbf{x}}$ indicates the argument on which the Liouvillian acts. The physical meaning of this expression is best revealed by switching to the Fourier conjugate picture. Inserting the definition (1.1b) of the form factor, we obtain

$$K^{(2)}(\tau) = -2\tau^2\Omega^2 \operatorname{Re}\int (\mathrm{d}\mathbf{x}) \int \frac{\mathrm{d}\mathbf{y}_1\,\mathrm{d}\mathbf{y}_2}{(\pi\hbar)^{2(f-1)}}\mathrm{e}^{\frac{i}{\hbar}\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2}\times \\ \times \int_0^t \mathrm{d}t'\, P_{t-t'}(\overline{\mathbf{x}+\mathbf{y}_1/2}, \mathbf{x}-\mathbf{y}_1/2)\mathcal{L}_{t',\mathbf{x}-\mathbf{y}_2/2}P_{t'}(\mathbf{x}-\mathbf{y}_2/2, \overline{\mathbf{x}+\mathbf{y}_2/2}), \quad (5.5)$$

where $\mathcal{L}_{t,\mathbf{x}} \equiv \partial_t - \{H,\ \}$. The result obtained for $K^{(2)}(\tau)$ critically depends on the behavior of the propagator P_t at times $t \sim t_{\mathsf{E}}$, cf. equations (1.7, 1.8). Namely, the total time which has to elapse before a non–vanishing contribution is possible is given by[2]

$$t_{\mathsf{E}}(\tfrac{1}{2}\mathbf{y}_1, -\tfrac{1}{2}\mathbf{y}_1) + t_{\mathsf{E}}(\tfrac{1}{2}\mathbf{y}_2, -\tfrac{1}{2}\mathbf{y}_2) = \frac{2}{\lambda}\ln\frac{c}{u_1} + \frac{2}{\lambda}\ln\frac{c}{s_2} \simeq \frac{2}{\lambda}\ln\frac{c^2}{\hbar} = 2t_{\mathsf{E}},$$

where we restricted ourselves to the case of $f = 2$ dimensions to facilitate comparison with the semiclassical results of section 1.3. To understand the meaning of the approximation of the delay time $t_{\mathsf{E},2}$ of the second propagator, notice that once the distance between the stretches which form the legs of the propagators has acquired classical dimensions, the fraction of the Planck cell $\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2 \lesssim \hbar$ which has not yet been transported out of the linearization regime[3] about the orbit $\mathbf{x}(t)$ shrinks exponentially on the classical Lyapunov scale $\lambda^{-1} \ll t_{\mathsf{E}}$. This means that the approximation $t_{\mathsf{E},2} \simeq 2t_{\mathsf{E}} - t_{\mathsf{E},1}$ holds up to an insignificant uncertainty of $\mathcal{O}(\lambda^{-1})$. We may hence use

[1] This expression holds for arbitrary operators, so the (E,t)–sector is still present in this identity.

[2] At any rate, the individual propagators relax at times $2\delta t_{\mathsf{E}}$ (or $2(1-\delta)t_{\mathsf{E}}$), $0 < \delta < 1$, which is typically long before the time $\tilde{t}_{\mathsf{E}} \sim 2t_{\mathsf{E}}$ associated to the regulator has elapsed, cf. the discussion of subsection 4.1.3.

[3] Notice that a shift $\mathbf{x} \to \mathbf{x} + \mathbf{y}_2/2$ of integration variables in (5.5) eliminates the dependence of the first argument of the second propagator on $\mathbf{y}_2$. At the same time, the first propagator $P_{t-t'}(\overline{\mathbf{x} + (\mathbf{y}_1+\mathbf{y}_2)/2}, \mathbf{x} - (\mathbf{y}_1-\mathbf{y}_2)/2) \simeq \Theta(t-t'-t_{\mathsf{E}}(\mathbf{y}_1/2, -\mathbf{y}_1/2))$ remains invariant under this shift, and the integral over $\mathbf{y}_2$ may be carried out (which otherwise would be hindered by the presence of the Poisson bracket), revealing the second argument of the second propagator to be effectively averaged over scales $(u_2, s_2) \lesssim (\hbar/u_1, \hbar/s_1)$ (which span the Planck cell $\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2 \lesssim \hbar$) and hence to effectively depend only on $t_{\mathsf{E},1}$.

that $\partial_t P_t(\mathbf{x},\bar{\mathbf{x}}') = \Omega^{-1}\delta(t-(2t_{\mathsf{E}}-t_{\mathsf{E},1}))$, where $\delta(t)$ is some smeared δ–function whose detailed functional structure is not of much importance.[4] Turning to the Poisson bracket component of the Liouville operator, we note that the second propagator depends only on the delay time $t_{\mathsf{E},2} \simeq 2t_{\mathsf{E}} - t_{\mathsf{E},1}$, and hence only on the coordinate $\mathbf{y}_1$. We conclude that the action of the Poisson bracket on the second propagator gives only a negligible contribution. Inserting these results into (5.5), as well as the normalization relations $\int(\mathrm{d}\mathbf{x}) = 1$ and $\int \mathrm{d}\mathbf{y}_1\,\mathrm{d}\mathbf{y}_2 \mathrm{e}^{\frac{i}{\hbar}\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2} = (2\pi\hbar)^{2(f-1)}$, we obtain

$$K^{(2)}(\tau) \simeq -2\tau^2 \int_0^t \mathrm{d}t'\,\Theta(t-t'-t_{\mathsf{E},1})\delta\big(t'-(2t_{\mathsf{E}}-t_{\mathsf{E},1})\big) = -2\tau^2\Theta(t-2t_{\mathsf{E}})$$

in agreement with the result of the semiclassical analysis. In fact, equation (5.5) makes the analogies (as well as a number of differences) between the semiclassical and the field theoretical description of quantum corrections explicit: central to both approaches are two semi–loops shown schematically in figure 1.1 on page 13. In either case, the proximity of these loops is controlled by phase factors which contain the coordinates of the end points (in a canonically invariant manner) as their arguments. However, unlike with semiclassics, equation (5.5) does not relate the unification of the two semiloops to specific periodic orbits. Rather, the two halves are treated as independent entities, each described in terms of its own probability factor P. It is nevertheless straightforward to identify periodic orbit partners γ, γ' with revolution times $T_\gamma = T_{\gamma'} = t$ which asymptotically travel along the semiloops and switch in the encounter region at $\mathbf{x}$. These orbits pierce the Poincaré surface of section through $\mathbf{x}$ at appropriately chosen intersections of the stable and unstable manifolds of $\mathbf{x} \pm \mathbf{y}_i/2$, cf. figure 5.1 and the similar construction which was applied in section 1.3 in order to identify the partner orbit γ' of γ. The action difference $S_\gamma - S_{\gamma'} = u_1 s_2$ is equal to the shaded area in figure 5.1. This result violates an apparent symmetry among $\mathbf{y}_1$ and $\mathbf{y}_2$, yet note that this symmetry is broken by causality:[5] indeed, if both retarded propagators are traversed in the causal direction, the corresponding semi–loops shadow the orbit γ, and the order of traversal of the loop is $(-1 \to \overline{1} \to \overline{2} \to -2 \to -1)$, whereas if one of the semi–loops (the second, say) is time reversed in order to shadow the partner orbit γ', the order of traversal is $(-1 \to \overline{1} \to \overline{-2} \to 2 \to -1)$. At any rate, the propagators in (5.5) depend only on u_1 and s_2. Consequently, the integral over the transversal coordinates in (5.5) is in fact reduced according to

$$\int \frac{\mathrm{d}\mathbf{y}_1\,\mathrm{d}\mathbf{y}_2}{(2\pi\hbar)^{2(f-1)}} \mathrm{e}^{\frac{i}{\hbar}\mathbf{y}_1^{\mathrm{T}} I \mathbf{y}_2} \ldots \to \int \frac{\mathrm{d}u_1\,\mathrm{d}s_2}{(2\pi\hbar)^{f-1}} \mathrm{e}^{\frac{i}{\hbar}u_1 s_2} \ldots$$

Relabeling $(u_1, s_2) \mapsto (u, s)$, the action difference of the corresponding orbit partners is equal to us and manifests itself in a phase factor $\exp(ius/\hbar)$, just as in the semiclassical expression (1.6) in section 1.3.

[4] All we shall rely upon is $\int \mathrm{d}t'\, f(t')\delta(t-t') \simeq f(t)$ for functions which vary slowly on the scales where $\delta(t)$ varies.

[5] In fact, we have come across this asymmetry already when we calculated the delay times $t_{\mathsf{E},1}$ and $t_{\mathsf{E},2}$.

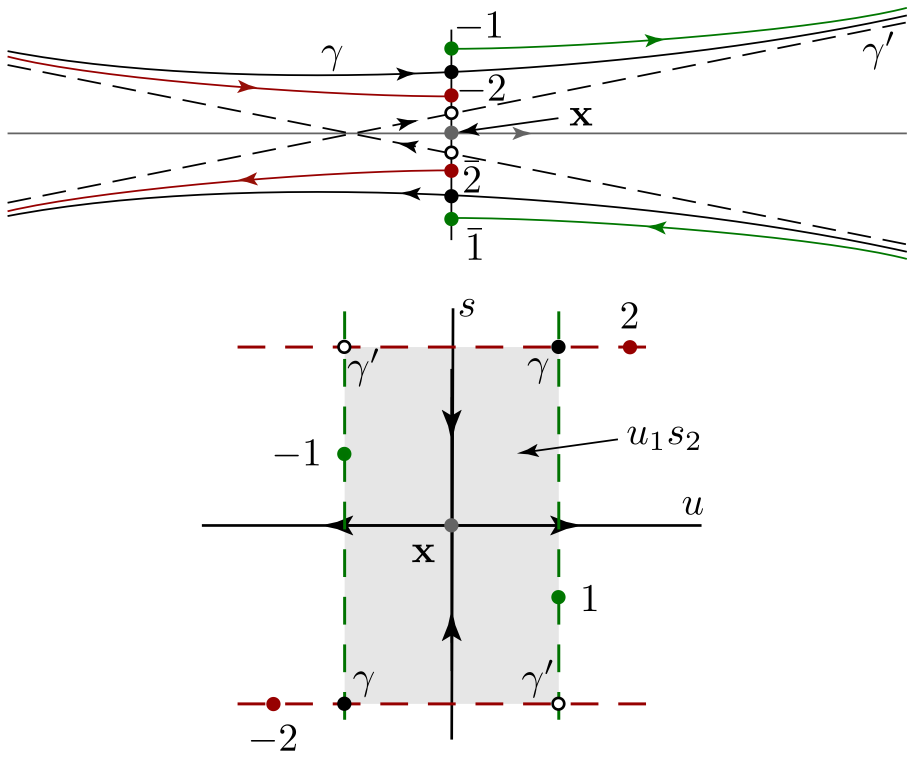

Figure 5.1: Identification of the periodic orbit partners γ, γ' pertaining to the semiloops 1 (green) and 2 (red). As in figure 1.3 on page 14, the upper sketch is drawn in configuration space, while the lower sketch depicts the (phase space) Poincaré surface of section through $\mathbf{x}$. The latter corresponds to the vertical line in the upper picture, and the points in the two pictures are identified by colours and labels, where -1 is a shorthand for $\mathbf{x} - \mathbf{y}_1/2$, $\bar{1}$ for $\overline{\mathbf{x} + \mathbf{y}_1/2}$, and similarly for $1 \rightleftharpoons 2$.

5.1.3 Higher orders of perturbation theory

What happens at higher orders in perturbation theory in the parameter s^{-1}? Before turning to the problem in full, it is instructive to have a look at the zero mode approximation to the model. The action of the zero mode configuration — formally obtained by setting $T(\mathbf{x}) \equiv T =$ const. — is given by (4.11). Parameterizing the matrix $T = 1 + W$ as in (4.1), an expansion in the generators B obtains the expression

$$S_0[B] = \sum_{n=1}^{\infty} S_0^{(2n)}[B], \qquad S_0^{(2n)}[B] = -is^+ \operatorname{tr}(-B^\dagger B)^n.$$

It is known [26] that, term by term in an expansion in s^{-1}, the zero mode functional reproduces the RMT approximation to the correlation function $R_2(s)$. Second, there exists a far–reaching structural connection between the perturbative expansion of the zero mode theory on one hand and the Gutzwiller double sum on the other.[6] More specifically, to each term contributing to the Wick contraction of

$$\left\langle (S_0^{(4)}[B])^{m_2} \, (S_0^{(6)}[B])^{m_3} \ldots \right\rangle_0 \tag{5.6}$$

[6]In fact, the correspondence Gutzwiller sum $\leftrightarrow$ zero dimensional σ–model $\leftrightarrow$ RMT played a pivotal role in the proof that the semiclassical expansion coincides with the RMT result [23, 24, 25].

there corresponds precisely one semiclassical orbit/partner orbit pair (or 'diagram'). Counting powers one finds that this diagram contributes to the correlation function at order $s^{-2-\sum_n m_n(n-1)}$. For every value of $n = 2, 3, \ldots$, it contains m_n encounter regions where n orbit segments meet and $\sum_n nm_n$ inter–encounter orbit stretches. The topology of the diagram is fixed by the way in which the B matrices are contracted.[7] Importantly, the minimum time required for the buildup of a diagram (i.e., the time required to traverse the $\sum_n m_n$ encounter regions) is given by $t_{\mathsf{E}} \times \sum_n nm_n$.

Turning back to the full problem, let us consider the analog of the zero dimensional expression (5.6),

$$\langle (S^{(4)}[B])^{m_2} \, (S^{(6)}[B])^{m_3} \ldots \rangle, \tag{5.7}$$

where $S^{(2n)}$ is given by (4.6) and the average is over the full quadratic action. It is natural to expect that the unique correspondence between Wick contractions and semiclassical diagrams carries over to the full model. If so, individual contractions should vanish/reduce to the universal RMT result for times shorter/much larger than $t < t_{\mathsf{E}} \times \sum_n nm_n$. In subsection 5.1.2 this correspondence was exemplified for the simplest non–trivial example, the SR diagram $\langle S^{(4)}[B] \rangle$.

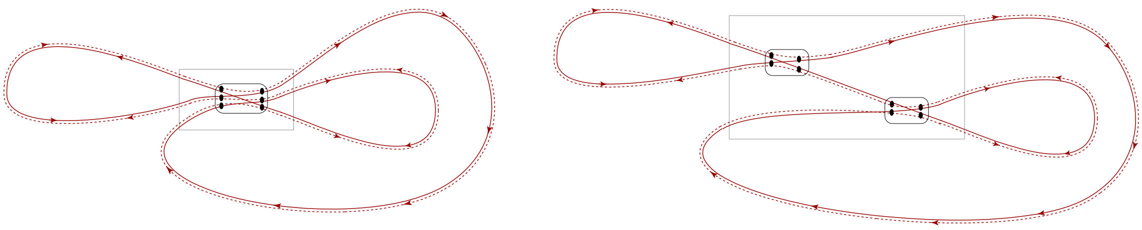

Figure 5.2: Two representatives of the 'clover leaf' diagram class contributing to the form factor at $\mathcal{O}(\tau^3)$. Discussion, see text.

Perhaps unexpectedly, the straightforward one–to–one correspondence outlined above does not pertain to higher orders in perturbation theory. To anticipate our main findings, it turns out that at order $(s^{-4} \leftrightarrow \tau^3)$ in the series expansion, propagators of short duration $P_{t<t_{\mathsf{E}}}$ — absent in the $(s^{-3} \leftrightarrow \tau^2)$ term considered above — begin to play a role. This implies that individual contractions may relate to more than one semiclassical diagram class. Nonetheless, integration over all time parameters obtains a universal result.

By way of example, let us consider the $(1-3, 2-6, 4-8, 5-7)$ contraction of $\langle \mathrm{tr}(B^\dagger B B^\dagger B) \, \mathrm{tr}(B^\dagger B B^\dagger B) \rangle$. For generic values ($t_i \sim t_{\mathsf{H}} \gg t_{\mathsf{E}}$) of the time arguments carried by the four resulting propagators the contraction corresponds to the orbit pair shown in figure 1.2 (left) on page 14. However, the integration over times t_i also extends over exceptional values where one of the two propagators connecting the two encounter regions ($(2-6)$ or $(4-8)$) is of short duration $< t_{\mathsf{E}}$. Such a short time propagator connects two *distinct* vertices.[8] This results in a structure as shown in figure 5.2 right, where the two clusters of dots indicate the eight phase space arguments of the B–fields,

[7]For example, the first of the diagrams shown in figure 1.2 on page 14 corresponds to the contraction $(1-3, 2-6, 4-8, 5-7)$ of $\mathrm{tr}(B^\dagger B B^\dagger B) \, \mathrm{tr}(B^\dagger B B^\dagger B)$, the second diagram to the contraction $(1-4, 2-5, 3-6)$ of $\mathrm{tr}(B^\dagger B B^\dagger B B^\dagger B)$, etc.

[8]While, in principle, the theory also permits the formation of short time propagators connecting two phase space points of a single vertex, these contributions are practically negligible: imagine a propagator

the straight line–pair represents the short propagator, and the box indicates that all phase space points lie in a *single* encounter region. Evidently, this structure corresponds to a pair of orbits visiting a single encounter region twice. Diagrams of this structure are canonically obtained by contraction of a 'Hikami hexagon' $\mathrm{tr}(B^\dagger B B^\dagger B B^\dagger B)$, as indicated in figure 5.2 left. Fortunately, the absence of a unique assignment to semiclassical orbit families, does not significantly complicate the actual computation of the diagrams: closer inspection shows that taking the Liouville operators involved in the definition of the Hikami boxes into account and integrating by parts,[9] we again obtain the universal zero–mode result.

Summarizing, we have seen that at next to leading order in perturbation theory short time propagators begin to play a role. While this complication prevents the assignment of Wick contractions to orbit pairs of definite topology, the results obtained after integration over all temporal configurations remain universal (agree with the RMT prediction). We trust that the structures discussed above are exemplary for the behavior of the ballistic σ–model at arbitrary orders of perturbation theory, i.e. that after integration over all intermediate times, each contraction contributing to (5.7) produces the universal result otherwise obtained by its zero dimensional analog equation (5.6).

5.2 Summary

In this chapter we took a closer look at the perturbative expansion of the ballistic σ–model of chaotic systems. We have shown how the results about universal spectral correlations and their quantum interference corrections for $\tau < 1$ compare to the semiclassical approach. We found quantitative agreement between field theory and semiclassics, and we pointed out the structural similarities and differences.

$P_t(\mathbf{x}, \mathbf{x}')$ returning after a short time to its point of departure ($|\mathbf{x} - \mathbf{x}'| \sim \hbar^\delta$). Since t is much shorter than the Ehrenfest time, all other propagators departing from the concerned Hikami box will essentially follow the trajectory traced out by the return propagator, and, after a time t, also return to the departure region. In semiclassical language, we are dealing with an orbit that traverses a loop structure in phase space repeatedly. It is known, however, that for large time scales, the probability to find repetitive orbits is exponentially small (in the parameter $\exp(-\lambda t)$), i.e. short self–retracing contractions are negligible.

[9]'Integrating by parts' is the classical counterpart of the identity $\mathrm{tr}(A[H,B]) = -\,\mathrm{tr}([H,A]B)$.

Chapter 6

Conclusions and remarks

In this thesis we treated the ballistic σ–model in order to reconsider some of the problems that prevent its proper understanding and evaluation.

In particular, we proposed a regularization scheme which strongly destabilizes the quantum theory of an ensemble of classically equal chaotic systems towards the universal random–matrix regime, which implies that the BGS conjecture holds on average over this ensemble. The destabilization towards RMT takes place already at comparatively high energies below the inverse Ehrenfest time, $\omega \lesssim \hbar/t_{\mathsf{E}} \sim \hbar/\ln\hbar$, while for integrable systems universality is enforced only at parametrically much lower energies $\sim \hbar^3$, which in $f \leq 3$ dimensions is at most of the order of the level spacing. We have demonstrated that the semiclassical form factor is invariant under perturbations by members of our ensemble, and since the semiclassical approach relies on an uncontrolled choice of relevant terms, the question remains open whether the BGS conjecture is true for individual systems. While we found that BGS for individual systems cannot be proven within the field theory approach by quasiclassical means (i.e. by simply cutting off quantum corrections according to (2.17)), there is still scope for a more sophisticated approach to the evaluation of the fundamental building block of the field theory, the Wigner propagator of density operators. There is a promising novel approach by Dittrich *et al.* to this object which allows to separate classical dynamics from quantum coherence effects [69]. Maybe this approach leads to a better understanding of both universal and non–universal features of the theory.

Nevertheless, our careful analysis of the semiclassical correspondence allowed us to reveal the mechanisms responsible for the manifestation of quantum uncertainty. This put us in the position to provide a solution to an open problem of quantum chaos, namely the proximity gap in ballistic SN structures. Second, we presented a field theoretical explanation of the short time quantum interference corrections to universal spectral correlations and the parallels to the semiclassical approach. We stress that our conclusions are drawn from first principles. We only rely on the validity of the effective quantum field theory.

We have thus partly rehabilitated the ballistic σ–model as an important and promising tool to face all kinds of problems related to quantum chaos. It is also possible to extend the results presented here to non–standard symmetry classes and discrete time systems. Due to the far–reaching parallels to semiclassics it should be straightforward to translate all universal results obtained by semiclassical methods to the field theory and *vice versa*

— at least if they are perturbative in nature: as an example, a semiclassical treatment of the proximity effect is missing so far.

An open issue remains the non–universal short time regime. Plagued by the problem of repetitions, the ballistic σ–model can so far not be considered reliable to describe non–universal corrections. As an additional problem in the short time regime, we want to mention a discrepancy among the results of Tian & Larkin [34] and ours (which agree with the findings of the Haake group [24]). This discrepancy is explained easiest on the τ^3–correction to the unitary form factor. As we laid out in subsection 5.1.3, we find an exact cancellation of this correction due to the fact that two 4–vertices merge in the presence of a short time propagator to form a 6–vertex, cf. figure 5.2 on page 54. Tian & Larkin disagree on this inasmuch as their theory does not allow for short time propagators at all. Brouwer *et al.* very recently added a new perspective to the semiclassical side of the coin by their treatment of universal short time corrections to the form factor, taking into account also repetitions. While the trust in the semiclassical result makes me believe that our result is correct as long as repetitions are neglected, the problems of the ballistic σ–model concerning the correct counting of repetitions prevent me from taking up the cudgels for the field theory when it comes to short time effects. This is a grave drawback as the attractiveness of the ballistic σ–model stems not only from its effortless description of universal effects, but also from its potential to be a 'theory of everything'. I conjecture that in the non–universal regime of the short time expansion, the reference point $\Lambda = \sigma_3^{\mathrm{ar}}$ is ill–chosen, and that instead mean fields (similar to those of the Andreev billard in chapter 3) which reflect strong fluctuations on individual orbits should be investigated.

Appendix A

Classical chaos

In this appendix we present some rudimentary vocabulary about classical chaotic mechanics. While every physics student learns in her or his first course in theoretical mechanics what an integrable system is, it is not so clear what a chaotic system should be. 'Chaos' is stronger a property than 'non–integrability'. In order to provide the reader with some intuition we present two properties which express a certain degree of stochasticity, ergodicity and mixing. We further explain hyperbolicity, another feature charasteristic to non–integrable systems. For supplementary reading we refer to the monograph by Gaspard [79].

A Hamiltonian system consists of a phase space Γ and some flow $\Phi_t(\mathbf{x}) \equiv \mathbf{x}(t)$ which preserves energy and the Liouville measure $\mu(\mathbf{x}) = \Omega^{-1}\delta(E_0 - H(\mathbf{x}))$, where $H(\mathbf{x})$ is the Hamiltonian and Ω is the volume of the shell of constant energy E_0. Probably the first property one would instinctively associate with chaos is ergodicity. A system is called ergodic if for any every observable f, the average w.r.t. the Liouville measure and the long time average along the trajectory departing from μ–almost every $\mathbf{x}_0$ coincide,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, f(\mathbf{x}_0(t)) = \int (\mathrm{d}\mathbf{x})\, f(\mathbf{x}) \text{ for } \mu\text{–a.e. } \mathbf{x}_0,$$

where $(\mathrm{d}\mathbf{x}) \equiv \mathrm{d}\mu(\mathbf{x})$. Note, however, that ergodicity does not even imply non–integrability as the following counterexample shows: take the (integrable) flow describing the free motion of a particle with initial velocity $\mathbf{v} = (v_1, v_2)$ on the unit 2–torus. If the quotient v_1/v_2 is rational, the orbit will eventually close and the corresponding motion is of course not ergodic. If, on the other hand, that quotient is irrational, any orbit fills the torus densely, and the motion is ergodic. Having said that, we certainly need a stronger property to characterize chaos. The criterion we are looking for is the mixing property, which says that the correlations of any two observables[1] f, g decay,

$$\lim_{t\to\infty} \left| \int (\mathrm{d}\mathbf{x})\, f(\mathbf{x}) g(\mathbf{x}(t)) - \int (\mathrm{d}\mathbf{x})\, f(\mathbf{x}) \cdot \int (\mathrm{d}\mathbf{x})\, g(\mathbf{x}) \right| \to 0. \tag{A.1}$$

We say that a system is exponentially mixing if this decay happens to take place exponentially fast. When we talk about a chaotic system, we will always assume it to be exponentially mixing.

[1] The function space from which the observables are chosen has to be specified in order to give this definition a well–defined meaning. Usually, this space is strictly smaller than $L^2(\Gamma)$.

Another property which is attributed to chaotic dynamics is the strong sensitivity to a variation of the initial conditions. Linearizing the flow about some general reference trajectory and fixing a time t, the differential $\mathrm{d}\Phi_t(\mathbf{x})$ is said to be hyperbolic if all its eigenvalues have a modulus different from unity. Since the differential of a Hamiltonian flow is symplectic, its eigenvalues come in conjugate pairs (Λ, Λ^{-1}).[2] The information about exponential instability may be filtered from the dynamics as follows: locally, one may define stretching rates $\lambda_i(\mathbf{x}(t))$, which are accompanied by squeezing rates $-\lambda_i(\mathbf{x}(t))$ and, taken together, form the spectrum of $\mathrm{d}_t \ln \mathrm{d}\Phi_t(\mathbf{x})$. The long time averages are called the local Lyapunov exponents $\lambda_i(\mathbf{x})$. According to the theorem of Oseledec [64], their long time averages are μ–almost everywhere well–defined in an ergodic system and are called the Lyapunov exponents. In a uniformly hyperbolic system, all local stretching rates are uniformly bounded away from zero. There are also several kinds of weaker hyperbolicity properties, such as partial or non–uniform hyperbolicity, which will play no role in this thesis.

[2] If Λ is complex, also $(\bar{\Lambda}, \bar{\Lambda}^{-1})$ are eigenvalues.

Appendix B

Time reversal invariant systems

In this appendix we want to summarize the modifications which are necessary for time reversal invariant systems. For the understanding of the derivation of the ballistic σ–model, these modifications are immaterial, and this subsection can be skipped at first reading. Time reversal invariance implies a symmetry $H = H^{\mathrm{T}}$ of the Hamiltonian. To safely encapsulate this feature in the field theory, one introduces a further doubling of the field space: to that end, we write

$$\bar{\psi} H \psi = \tfrac{1}{2}(\bar{\psi} H \psi - \psi^{\mathrm{T}} H \bar{\psi}^{\mathrm{T}}) \equiv \bar{\Psi} H \Psi,$$

where we introduced the fields

$$\bar{\Psi} \equiv \frac{1}{\sqrt{2}}(\bar{\psi}, -\psi^{\mathrm{T}})_{\mathsf{tr}}, \quad \Psi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \bar{\psi}^{\mathrm{T}} \end{pmatrix}_{\mathsf{tr}},$$

which obey the time reversal symmetry

$$\bar{\Psi} = (C\Psi)^{\mathrm{T}}, \quad C \equiv i\sigma_2^{\mathsf{tr}}. \tag{B.1}$$

Replacing $\psi \to \Psi$, the calculations are analogous to the unitary case. The energy average (2.5) results in an interaction with a different symmetry than in the unitary case. Transforming $\Psi \mapsto \mathcal{T}\Psi$, the time reversal symmetry relation (B.1) fixes the action of $\mathcal{T}$ on $\bar{\Psi}$. Now, the transformations leaving the interaction invariant fulfill the condition

$$\mathcal{T}^{\mathrm{T}} C \mathcal{T} = C \text{ or } \mathcal{T} \in \mathrm{Sp}(4RN^2).$$

The decoupling of the interaction is somewhat more subtle. Consider the position representation $\langle \mathbf{q} + \frac{1}{2}\Delta\mathbf{q} | \tilde{\mathcal{Q}} | \mathbf{q} - \frac{1}{2}\Delta\mathbf{q} \rangle$ of the Hilbert space operator $\tilde{\mathcal{Q}}$. In chapter 2 we lay out that the low–energy configurations are varying slowly — in the sense that they do not display details on length scales smaller than $\hbar$ — w.r.t. the center–of–mass coordinate $\mathbf{q}$, so one has to explicitly separate such contributions. This is achieved turning to the momentum representation,

$$S_{\mathsf{int}}[\bar{\Psi}, \Psi] = \frac{E_{\mathsf{av}}^2}{2}(\bar{\Psi}\Psi)^2 \approx \frac{E_{\mathsf{av}}^2}{2} \sum_{\mathbf{P},\mathbf{p}} \big(\bar{\Psi}_{\mathbf{P}} \Psi_{-\mathbf{P}} \bar{\Psi}_{\mathbf{P}+\mathbf{p}} \Psi_{-\mathbf{P}-\mathbf{p}} + \bar{\Psi}_{\mathbf{P}} \Psi_{-\mathbf{P}} \bar{\Psi}_{-\mathbf{P}-\mathbf{p}} \Psi_{\mathbf{P}+\mathbf{p}} \big),$$

where $\mathbf{p}$ is restricted by some classical bound. Using the time reversal symmetry relation one sees that these terms are equal, which leads to a factor of two and thus a modified decoupling

$$\mathrm{e}^{-S_{\mathsf{int}}[\bar{\Psi},\Psi]} = \int \mathcal{D}\tilde{\mathcal{Q}}\mathrm{e}^{-\frac{1}{4}\operatorname{tr}\tilde{\mathcal{Q}}^2 + E_{\mathsf{av}}\bar{\Psi}\tilde{\mathcal{Q}}\Psi}.$$

where $\tilde{\mathcal{Q}}$ is now of dimension $4RN^2$ and restricted to be slowly varying w.r.t. to its center–of–mass coordinate $\mathbf{q}$. Again, the invariance of $\operatorname{tr}(\bar{\Psi}\tilde{\mathcal{Q}}\Psi)$ requires that $\tilde{\mathcal{Q}}$ transforms as $\tilde{\mathcal{Q}} \mapsto \mathcal{T}\tilde{\mathcal{Q}}\mathcal{T}^{-1}$ and satisfies the symmetry constraint

$$\tilde{\mathcal{Q}} = C\tilde{\mathcal{Q}}^{\mathrm{T}}C^{\mathrm{T}} \equiv \tilde{\mathcal{Q}}^{\tau}.$$

Carrying out the Gaussian integral over the Ψ–fields we find the effective partition function

$$Z(\omega) = \int \mathcal{D}\tilde{\mathcal{Q}}\,\mathrm{e}^{-\frac{1}{4}\operatorname{tr}\tilde{\mathcal{Q}}^2 + \frac{1}{2}\operatorname{tr}\ln(\mathcal{G}^{-1}(E_0) - \frac{1}{2}\Omega^+\sigma_3^{\mathsf{ar}})}, \quad \mathcal{G}^{-1}(E_0) \equiv E_0 - H - iE_{\mathsf{av}}\tilde{\mathcal{Q}}.$$

Note the factor of $\frac{1}{2}$ in front of the logarithm in the action reflecting the fact that due to time reversal symmetry, the number of integration variables is reduced by one half. One may summarize the modifications as follows: the action differs from its unitary counterpart by a global factor $\frac{1}{2}$ which compensates the doubled rank of the matrix fields in the action. In the sequel we will thus explicitly notate a factor $\beta/2$, where $\beta = 1$ (2) in the orthogonal (unitary) case and implicitly keep in mind that the field space is characterized by an auxiliary tr–sector and a symplectic symmetry in the orthogonal case.

Appendix C

Wigner representation

The aim of this appendix is to summarize and check some properties of the Wigner symbol (2.11) to which we refer repeatedly in the main text. Here, the matrix I is defined through $\mathbf{x}^T I \mathbf{x}' \equiv \mathbf{q} \cdot \mathbf{p}' - \mathbf{p} \cdot \mathbf{q}'$. The key relation for the proof of all these identities is the completeness relation of Fourier transformation,

$$\int \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^f} \mathrm{e}^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{q}} = \delta(\mathbf{q}). \tag{C.1}$$

Let us begin with the characteristic properties

(i) Reality:

$$\big(A(\mathbf{x})\big)^* = \int \mathrm{d}\Delta\mathbf{q}\, \mathrm{e}^{+\frac{i}{\hbar}\mathbf{p}\cdot\Delta\mathbf{q}} \langle \mathbf{q} - \tfrac{1}{2}\Delta\mathbf{q} | A^\dagger | \mathbf{q} + \tfrac{1}{2}\Delta\mathbf{q} \rangle = (A^\dagger)(\mathbf{x}) \quad \square$$

(ii) Normalization:

$$\int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f} A(\mathbf{x}) = \int \mathrm{d}\mathbf{q}\, \langle \mathbf{q} | A | \mathbf{q} \rangle = \mathrm{tr}\, A \quad \square$$

(iii) Traciality:

$$\begin{aligned} \int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f} A(\mathbf{x}) B(\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^f} \int \mathrm{d}\Delta\mathbf{q} \int \mathrm{d}\Delta\mathbf{q}'\, \mathrm{e}^{\frac{i}{\hbar}\mathbf{p}\cdot(\Delta\mathbf{q}+\Delta\mathbf{q}')} \times \\ &\qquad \times \langle \mathbf{q} + \tfrac{1}{2}\Delta\mathbf{q} | A | \mathbf{q} - \tfrac{1}{2}\Delta\mathbf{q} \rangle \langle \mathbf{q} + \tfrac{1}{2}\Delta\mathbf{q}' | B | \mathbf{q} - \tfrac{1}{2}\Delta\mathbf{q}' \rangle \\ &= \int \mathrm{d}\mathbf{q}_+ \int \mathrm{d}\mathbf{q}_-\, \langle \mathbf{q}_+ | A | \mathbf{q}_- \rangle \langle \mathbf{q}_- | B | \mathbf{q}_+ \rangle = \mathrm{tr}(AB) \quad \square \end{aligned}$$

These properties are characteristic inasmuch they define — together with a covariance condition which relates a transformation on phase space to a transformation on Hilbert space, examples being the well–known Hilbert space representations of the Galilei group — uniquely define the Weyl symbol [53, 54]. The Moyal product formula (2.12b) for the case $n = 1$ is proven as follows: a straightforward application of the defining relation (2.11) gives

$$\begin{aligned}
\int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f} \int \frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f}\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2} A(\mathbf{x}+\mathbf{x}_1) B(\mathbf{x}+\mathbf{x}_2) \\
&= \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f} \int \frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f}\, \mathrm{e}^{\frac{2i}{\hbar}(\mathbf{x}_1-\mathbf{x})^{\mathrm{T}} I (\mathbf{x}_2-\mathbf{x})} A(\mathbf{x}_1) B(\mathbf{x}_2) \\
&= \int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f} \int \frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f} \int \mathrm{d}\Delta\mathbf{q}_1 \int \mathrm{d}\Delta\mathbf{q}_2\, \mathrm{e}^{-\frac{i}{\hbar}(\mathbf{p}_1\cdot\Delta\mathbf{q}_1+\mathbf{p}_2\cdot\Delta\mathbf{q}_2)} \mathrm{e}^{\frac{2i}{\hbar}(\mathbf{x}_1-\mathbf{x})^{\mathrm{T}} I (\mathbf{x}_2-\mathbf{x})} \times \\
&\quad \times \langle \mathbf{q}_1 + \tfrac{1}{2}\Delta\mathbf{q}_1 | A | \mathbf{q}_1 - \tfrac{1}{2}\Delta\mathbf{q}_1\rangle \langle \mathbf{q}_2 + \tfrac{1}{2}\Delta\mathbf{q}_2 | A | \mathbf{q}_2 - \tfrac{1}{2}\Delta\mathbf{q}_2\rangle,
\end{aligned}$$

where the first step consists of a shift $\mathbf{x}_i \to \mathbf{x}_i - \mathbf{x}$ of the integration variables. The integrals over $\mathbf{p}_1$, $\mathbf{p}_2$ can be performed yielding

$$\tfrac{1}{2}\Delta\mathbf{q}_1 = \mathbf{q} - \mathbf{q}_2, \quad \tfrac{1}{2}\Delta\mathbf{q}_2 = -\mathbf{q} + \mathbf{q}_1.$$

Carrying out the integrals over $\mathbf{q}$ and $\Delta\mathbf{q}$, and writing $\mathbf{q}_\pm \equiv \mathbf{q}_1 \pm \frac{1}{2}\Delta\mathbf{q}_1$, we find that

$$\begin{aligned}
\int \frac{\mathrm{d}\mathbf{x}_1}{(\pi\hbar)^f} \int \frac{\mathrm{d}\mathbf{x}_2}{(\pi\hbar)^f}\, \mathrm{e}^{\frac{2i}{\hbar}\mathbf{x}_1^{\mathrm{T}} I \mathbf{x}_2} A(\mathbf{x}+\mathbf{x}_1) B(\mathbf{x}+\mathbf{x}_2) = \\
&= \int \mathrm{d}\mathbf{q}_+ \int \mathrm{d}\mathbf{q}_-\, \mathrm{e}^{-\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{q}_+-\mathbf{q})} \langle \mathbf{q}_+ | A | \mathbf{q}_-\rangle \langle \mathbf{q}_- | B | 2\mathbf{q} - \mathbf{q}_+\rangle = (AB)(\mathbf{x}). \quad \square
\end{aligned}$$

The differential formula (2.12b) follows — after another application of the completeness relation (C.1) — from the Taylor formula $f(\mathbf{x}+\Delta\mathbf{x}) = \mathrm{e}^{(\Delta\mathbf{x})^{\mathrm{T}}\partial_{\mathbf{x}}} f(\mathbf{x})$. The integral formula (2.13) for general n is obtained straightforwardly by induction over n.

Appendix D

Gor'kov Hamiltonian

The dynamics of a composite superconducting/normalconducting (SN)–system which is also known as Andreev billiard is described by the so–called Gor'kov Hamiltonian. Deferring for a detailed discussion of this topic to the literature [80, 81, 82] we here just state its Nambu representation,

$$\hat{H}_{\text{Gor'kov}} = (c_\uparrow^\dagger, c_\downarrow)_{\mathsf{ph}} \begin{pmatrix} H - E_{\mathsf{F}} & \Delta \\ \Delta^\dagger & -(H - E_{\mathsf{F}}) \end{pmatrix}_{\mathsf{ph}} \begin{pmatrix} c_\uparrow \\ c_\downarrow^\dagger \end{pmatrix}_{\mathsf{ph}},$$

where the decomposition is into a particle–hole sector[1] (ph) and E_{F} denotes the Fermi energy. The spectral properties are as usually encoded in the Green functions (2.2). Inserting the Gor'kov Hamiltonian, one finds by standard methods of fermionic many–particle physics that the DoS at energy ϵ w.r.t. the Fermi energy is given by

$$\nu(\epsilon) = \mp \frac{1}{2\pi} \operatorname{Im} \operatorname{tr}\big(G_{\text{Gor'kov}}(\epsilon^\pm)\big), \tag{D.1}$$

where the factor of $\frac{1}{2}$ is traced back to the doubling of field space and the Gor'kov Green function is defined by

$$G_{\text{Gor'kov}}(\epsilon) \equiv \begin{pmatrix} -\epsilon - (H - E_{\mathsf{F}}) & -\Delta \\ -\Delta^\dagger & -\epsilon + H - E_{\mathsf{F}} \end{pmatrix}^{-1}.$$

Both, writing down a generating functional for the Gor'kov Green function and the replacements w.r.t. normal systems is straightforward and described in chapter 3 of the main text. Note that in the main text we modify our definition of the Gor'kov Green function by multiplication by σ_3^{ph} from the left which the reader finds reflected in the emergence of an additional σ_3^{ph} in the formula (3.2) for the DoS.

[1] Order of indices in this additional space is $(1,2) = (\mathsf{p}, \mathsf{h})$.

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