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Sequential Change – Point Analysis Based on Weighted Moving Averages

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Mario Kühn
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Erster Berichterstatter:	Prof. Dr. Josef Steinebach
Zweiter Berichterstatter:	Prof. Dr. Lajos Horváth

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Zusammenfassung

Ziel der sequentiellen Change-Point Analyse ist es, geeignete Methoden zum Auffinden struktureller Brüche in stochastischen Prozessen bereitzustellen. Diese Arbeit befasst sich mit nicht-parametrischen Verfahren auf der Basis gewichteter, gleitender Durchschnitte. Es werden bekannte Prozeduren diskutiert und neue Testverfahren vorgestellt. Die Hauptresultate dieser Arbeit, der PWMA-Test (polynomially weighted moving average) und der FWMA-Test (fractionally weighted moving average), sind Erweiterungen des CUSUM-Tests. Weiterhin werden in dieser Dissertation die Grenzverteilungen der bedingten Stoppzeiten des MOSUM-Tests bestimmt und zur Konstruktion asymptotischer Konfidenzintervalle für den Zeitpunkt eines Strukturbruches eingesetzt.

Abstract

The aim of sequential change-point analysis is to provide adequate methods for detecting structural breaks in stochastic processes. This work is concerned with non-parametric procedures, which are based on weighted moving averages. We discuss known control-charts and also introduce new procedures. The main results of this work, namely the PWMA-Test (polynomially weighted moving average) and the FWMA-Test (fractionally weighted moving average), are generalizations of the CUSUM-Test. Furthermore, we derive the limiting distributions of the conditional stopping-times for the MOSUM-Test. The latter result is utilized to provide asymptotic confidence intervals for the location of a change-point.

Preface

The foundation of this work was laid at a *Travis* concert in Frankfurt, which I attended together with Alexander Aue. At this point, Alexander had already left Marburg, where we had studied together, to work on his dissertation at the University of Cologne. Since I was interested in the subject of change-point analysis, he initiated contact to my current supervisor Prof. Dr. Josef Steinebach, whom I had met briefly during his teaching time at the Philipps – University in Marburg. It was a great concert.

Acknowledgements

I am deeply grateful to my supervisor Prof. Dr. Josef Steinebach. I highly benefited from his rich experience in many fields of stochastics. He also supported my research by providing me with the opportunity to meet other mathematicians, who were working in related fields. Furthermore, I want to sincerely thank Prof. Dr. Lajos Horváth and Prof. Dr. Alexander Aue. The discussions with both of them during the coffee breaks at the University of Utah in Salt Lake City had a great influence on this work and broadened my horizon. I hope that I can return their extraordinary hospitality. Finally, I would like to thank my family, my friends, and especially Dr. Peggy Möller for their active support as well as for providing the deeper insight that there also exist other important fields of research beyond mathematics.

Conventions

We shortly state some conventions, which will be used in the sequel without further comment:

$$\text{iff}: \text{ if and only if},$$

$$\inf \emptyset = \infty,$$

$$\sum_{i=a}^{b} \ldots = 0, \quad \text{if} \quad b < a.$$

Let $\{x_t\}_{t\in T}$ and $\{y_t\}_{t\in T}$ be two real-valued sequences, where for the index set either $T = \mathbb{N}$, or $T = \mathbb{R}_+$ holds. Then, we define

$$x_t \sim y_t \Leftrightarrow \frac{x_t}{y_t} \to 1 \quad \text{as} \quad t \to \infty,$$

$$x_t \simeq y_t \Leftrightarrow \frac{x_n}{y_n} \to c \quad \text{as} \quad t \to \infty \quad \text{for some} \quad c > 0,$$

$$|x_t| \gtrsim |y_t| \Leftrightarrow \liminf_{t\to\infty} \left|\frac{x_t}{y_t}\right| \geq c > 0.$$

Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two sequences of real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$, where for the index set either $T = \mathbb{N}$, or $T = \mathbb{R}_+$ holds. Then, we define

$$X_t \overset{\mathrm{P}}{\sim} Y_t \Leftrightarrow \frac{X_t}{Y_t} \xrightarrow{\mathrm{P}} 1 \quad \text{as} \quad t \to \infty,$$

$$X_t \overset{\mathrm{P}}{\simeq} Y_t \Leftrightarrow \frac{X_n}{Y_n} \xrightarrow{\mathrm{P}} c \quad \text{as} \quad t \to \infty \quad \text{for some} \quad c > 0,$$

$$|X_t| \overset{\mathrm{P}}{\gtrsim} |Y_t| \Leftrightarrow \lim_{t\to\infty} \mathrm{P}\left(\left|\frac{X_t}{Y_t}\right| \geq c\right) = 1 \quad \text{for some} \quad c > 0.$$

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Introduction

The implementation of the Shewhart chart (see Shewhart, 1931) in quality control laid the foundation for the branch of statistics subsumed under the notion of change-point analysis.

The aim of change-point analysis is to provide and investigate methods, which allow to detect a structural break in an underlying data set. Following Brodsky and Darkhovsky (1993), we distinguish between *a-posteriori* and *sequential* procedures. A-posteriori procedures are designed to detect a change in a fixed data set. If data arrive online, and after each new observation we have to decide whether the assumption of homogeneity still holds, sequential procedures are applied. This classification can furthermore be divided in *parametric* and *non-parametric* tests. Parametric tests require *a priori* information on the stochastic model of the underlying data set, while *non-parametric* tests are based on asymptotic results for large data sets, or the investigation of a limiting process. This work is concerned with sequential, non-parametric procedures, which are furthermore partitioned into *closed-end* and *open-end* procedures. Even though monitoring will sooner or later end, open-end procedures are required, if the size of the monitoring period is not a-priori known.

Procedures for change-point detection are also named control-charts. This term is originated from the possibility of a graphical evaluation, which is provided by several tests as for example Page's CUSUM-chart (Page, 1954) and the EWMA-chart (exponentially weighted moving average) introduced by Roberts (see Roberts, 1959). Due to the fact that the boundaries in the latter charts are chosen as constant values, for large sizes of the monitoring period the null hypothesis of structural homogeneity will be rejected by both of them with high probability, even though it is true. However, the CUSUM- and the EWMA-chart are very popular tests, since their typical performance measure, the average run length (ARL), is for many applications more important than the errors of type one. If a false alarm is costly, other procedures are required and an adequate approach is given by the concept of asymptotic tests with (asymptotic) power one (see Robbins, 1970). While the CUSUM-procedure has been adapted to the concept of asymptotic tests by several authors, we refer to Chu, Stinchcombe and White (1996), Horváth, Hušková, Kokoszka and Steinebach (2004), Aue, Horváth, Hušková, Kokoszka (2006) and Aue and Kühn (2008) for more details, the only extension of the EWMA-chart in this direction, to our knowledge, has been given by Gut and Steinebach (2004). We will extend the latter result for exponential weights to a wider class of possible weight functions in Chapter 2.

If we compare the CUSUM-chart (without renewals) and the EWMA-chart, we see that both can be considered as tests, which are based on weighted moving averages of the underlying observations. The main part of this work is concerned with the question how weights, which lie in between the constant weights of the CUSUM-chart and the exponentially decreasing weights of the EWMA-chart, can be utilized to construct asymptotic tests. As result, we introduce two new control-charts, namely the PWMA-chart (polynomially weighted moving average, Chapter 3) and the FWMA-chart (fractionally weighted moving average, Chapter 4).

If a control-chart detects a change-point, it is also of interest where the change-point is located. Based on the limiting distribution for the conditional stopping times of the MOSUM-chart (moving sum, Chapter 1 and Chapter 5), we will derive asymptotic confidence intervals for the location of a change-point.

The simulations in this work are performed using the free software *R*, see `http://cran.r-project.org` for more information.

Part I

Monitoring Procedures

Chapter 1

Control charts based on moving sums

In this chapter, we consider the so called MOSUM-chart (moving sum). This chart takes action, if at time k the sum of the observations $X_{k-h+1}, \ldots, X_k$ crosses a predetermined boundary function, where h is the so called window size. MOSUM-charts have been suggested by several authors as more quickly reacting alternatives to CUSUM-type procedures in case of a late change. We refer to Bauer and Hackl (1978,1980), Chu, Hornik and Kuan (1995), Leisch, Hornik and Kuan (2000) and Gut and Steinebach (2002) for more details. Due to the sensitivity of the MOSUM-chart, one has to choose the boundary function with great care, which has been pointed out by Horváth, Kühn and Steinebach (2008).

We derive the asymptotic limit distribution of the extremes of standardized moving sums for known and estimated in-control parameters and furthermore, investigate the finite sample behavior by a simulation study.

1.1 Closed-end control charts

1.1.1 Model assumptions for known μ and σ

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of real random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$. We suppose that

$$\mathrm{E}\,\varepsilon_i = 0 \quad \text{and} \quad \mathrm{Var}\,\varepsilon_i = \sigma^2 > 0 \quad \text{for all} \quad i = 1, 2, \ldots \tag{1.1.1}$$

Furthermore, we claim that there exists a Wiener process $\{W(t), t \geq 0\}$ such that

$$\sup_{1 \leq k < \infty} \frac{1}{k^{1/\nu}} \left| \sum_{i=1}^{k} \varepsilon_i - \sigma W(k) \right| < \infty \quad \text{a.s.} \tag{1.1.2}$$

for some $\nu > 2$.

The sequence of observations is modeled as a stochastic process $\{X_i\}_{i=1,2,\ldots,h_N+N}$, following

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \le i \le h_N + k^*, \\ \mu + \triangle + \varepsilon_i & , \quad h_N + k^* < i \le h_N + N, \end{cases} \tag{1.1.3}$$

where μ is the in-control mean, $\triangle$ represents the size of a level shift, and k^* is the unknown time of a possible change. The window size h_N determines the number of observations needed to initialize the first detector. Hence, these observations can not be monitored sequentially. The assumption that a possible change occurs after this 'initialization period' is mostly made for technical reasons. According to the treatment of late changes one can show that also earlier changes may be detected.

We assume that h_N is an increasing, integer-valued function of N satisfying

$$1) \quad \lim_{N\to\infty} \left(\frac{(N+h_N)^{2/\nu}}{h_N} \log \frac{N}{h_N} \right) = 0 \tag{1.1.4}$$

and

$$2) \quad \lim_{N\to\infty} \frac{h_N}{N} = 0. \tag{1.1.5}$$

We are interested in testing either

$$H_0 : k^* \ge N \quad \text{versus} \quad H_1 : k^* < N,\ \triangle > 0 \quad \text{(one-sided alternative)}, \tag{1.1.6}$$

or

$$H_0 : k^* \ge N \quad \text{versus} \quad H_2 : k^* < N,\ \triangle \ne 0 \quad \text{(two-sided alternative)}. \tag{1.1.7}$$

1.1.2 Monitoring procedures for known μ and σ

We define the sequence of detectors $\{M_{k,N}\}_{k=1,\ldots,N}$ by

$$M_{k,N} = \sum_{i=k-h_N+1}^{k} (X_{i+h_N} - \mu) \quad \text{for all} \quad k = 1, \ldots, N. \tag{1.1.8}$$

We mention here that since the asymptotics are carried out for $N \to \infty$, we do not consider a sequence of detectors, but a triangular array. However, we only accentuate the triangular structure, if it is needed.

Depending on the alternative, we reject H_0 if

$$\tau_1 = \tau_1(\alpha, N) = \inf \left\{ 1 \le k \le N : M_{k,N} > c_1(\alpha, N)\sigma\sqrt{h_N} \right\}, \tag{1.1.9}$$

or

$$\tau_2 = \tau_2(\alpha, N) = \inf\left\{1 \le k \le N : |M_{k,N}| > c_2(\alpha, N)\sigma\sqrt{h_N}\right\} \tag{1.1.10}$$

are finite (with the usual convention that $\inf \emptyset = \infty$), where $\alpha \in]0, 1[$ is the level of significance.

The critical constants $c_1(\alpha, N)$ and $c_2(\alpha, N)$ are given by

$$c_1(\alpha, N) = \frac{q_1(1-\alpha) + b_N}{a_N} \quad \text{and} \quad c_2(\alpha, N) = \frac{q_2(1-\alpha) + b_N}{a_N}, \tag{1.1.11}$$

where a_N and b_N are defined in (1.1.15), respectively (1.1.16). The constants $q_1(1-\alpha)$ and $q_2(1-\alpha)$ are chosen as

$$q_1(1-\alpha) = -\log(-\log(1-\alpha)) \;\text{ and }\; q_2(1-\alpha) = -\log\left(-\frac{1}{2}\log(1-\alpha)\right). \tag{1.1.12}$$

Note that monitoring always ends in $\min\{\tau_i, N\},\; i = 1, 2.$

The following theorem justifies the choice of the critical constants, since it shows that the false alarm rate of the procedures converges to α as N tends to infinity.

Theorem 1.1.1 *Let the sequence* $\{M_{k,N}\}_{k=1,\dots,N}$ *be defined as in* (1.1.8). *We assume that* h_N *is an increasing, integer-valued function of* N*, satisfying* (1.1.4) *and* (1.1.5). *Then, under* H_0*, for all real* x *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} - b_N \le x\right) = \exp(-e^{-x}) \tag{1.1.13}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{|M_{k,N}|}{\sigma\sqrt{h_N}} - b_N \le x\right) = \exp(-2e^{-x}), \tag{1.1.14}$$

where

$$a_N = \sqrt{2\log\frac{N}{h_N}} \tag{1.1.15}$$

and

$$b_N = 2\log\frac{N}{h_N} + \frac{1}{2}\log\log\frac{N}{h_N} - \frac{1}{2}\log\pi. \tag{1.1.16}$$

Moreover, under slight restrictions on k^*, both procedures have asymptotic power one as can be seen by the next theorem.

Theorem 1.1.2 *Let the sequence* $\{M_{k,N}\}_{k=1,\ldots,N}$ *and the function* h_N *be defined as in Theorem 1.1.1. If* $k^* = k^*(N) \leq N - \sqrt{h_N} N^\rho$ *for some* $\rho > 0$*, then, under* H_1*, for all real* x *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\leq k\leq N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} - b_N > x\right) = 1 \tag{1.1.17}$$

and under H_2 *we have*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\leq k\leq N} \frac{|M_{k,N}|}{\sigma\sqrt{h_N}} - b_N > x\right) = 1, \tag{1.1.18}$$

where a_N *and* b_N *are defined in* (1.1.15) *and* (1.1.16)*, respectively.*

1.1.3 Model assumptions for unknown μ and σ

If the target parameters are unknown, a common approach is to utilize the observations of a training period for the estimation (see Chu, Stinchcombe and White, 1996). The data obtained within the training period is assumed to be homogeneous (noncontamination assumption). We follow this idea, which requires slight modifications of the model assumptions.

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of real random variables, satisfying (1.1.1) and (1.1.2). We assume that the observations $\{X_i\}_{i=1,2,\ldots,m_N+N}$ satisfy

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \leq i \leq m_N + k^*, \\ \mu + \triangle + \varepsilon_i & , \quad m_N + k^* < i \leq m_N + N, \end{cases} \tag{1.1.19}$$

where μ, $\triangle$ and k^* denote the same parameters as in (1.1.3) and m_N denotes the size of the training period.

The window size h_N is chosen as in case of known parameters and the interplay between the window size, the training period, and the monitoring period N is described by

$$\lim_{N\to\infty}\left(\frac{h_N}{m_N}\log\frac{N}{h_N}\right) = 0 \quad \text{and} \quad \frac{m_N}{N} \to 0 \quad \text{as} \quad N \to \infty. \tag{1.1.20}$$

We furthermore assume that the size of the training period is increasing, as the monitoring period increases. Note that if we choose for example $h_N = N^{2\phi}$ and $m_N = N^{2\varphi}$ with $1/\nu < \phi < \varphi < 1/2$, then (1.1.4), (1.1.5) and (1.1.20) are satisfied.

The procedures proposed in the next paragraph are designed to test either

$$H_0 : k^* \geq N \quad \text{versus} \quad H_1 : k^* < N,\ \triangle > 0 \quad \text{(one-sided alternative)}, \tag{1.1.21}$$

or

$$H_0 : k^* \geq N \quad \text{versus} \quad H_2 : k^* < N,\ \triangle \neq 0 \quad \text{(two-sided alternative)}. \tag{1.1.22}$$

1.1.4 Monitoring procedures for unknown μ and σ

If we estimate μ by

$$\hat{\mu}_{m_N} = \frac{1}{m_N} \sum_{i=1}^{m_N} X_i, \tag{1.1.23}$$

then, a consequence of the invariance (1.1.2) principle is that

$$\hat{\mu}_{m_N} - \mu = \boldsymbol{O}_P \left(\frac{1}{\sqrt{m_N}} \right) \quad \text{as} \quad N \to \infty.$$

We furthermore can assume that there exists an estimator $\hat{\sigma}^2_{m_N}$ for σ^2 such that

$$\hat{\sigma}^2_{m_N} - \sigma^2 = \boldsymbol{o}_P \left(\frac{1}{m_N^{\vartheta}} \right) \quad \text{as} \quad N \to \infty \quad \text{for some} \quad \vartheta > 0, \tag{1.1.24}$$

what will be shown in Lemma 1.1.3 below. Plugging in the mean estimator yields the sequence of detectors

$$\hat{M}_{k,N} = \sum_{i=k-h_N+1}^{k} \left(X_{i+m_N} - \hat{\mu}_{m_N} \right), \quad k = 1, \ldots, N \tag{1.1.25}$$

and depending on the alternative, we reject the null hypotheses if the corresponding stopping time

$$\hat{\tau}_1 = \hat{\tau}_1(N) = \inf \left\{ 1 \leq k \leq N : \hat{M}_{k,N} > c_1(\alpha, N) \hat{\sigma}_{m_N} \sqrt{h_N} \right\}, \tag{1.1.26}$$

or

$$\hat{\tau}_2 = \hat{\tau}_2(N) = \inf \left\{ 1 \leq k \leq N : |\hat{M}_{k,N}| > c_2(\alpha, N) \hat{\sigma}_{m_N} \sqrt{h_N} \right\} \tag{1.1.27}$$

is finite.

If the critical constants are chosen as in (1.1.11), the false alarm rate of the procedures converges to α as $N \to \infty$, what is implied by the counterpart of Theorem 1.1.1 for estimated in-control parameters, coming next.

Theorem 1.1.3 *Let the sequence $\{\hat{M}_{k,N}\}_{k=1,\ldots,N}$ be defined as in* (1.1.25)*. The window size h_N is assumed to satisfy* (1.1.4) *and* (1.1.5) *and the size of the training period is determined via* (1.1.20)*. Then, under H_0, for all real x it holds that*

$$\lim_{N \to \infty} \mathrm{P} \left(a_N \sup_{1 \leq k \leq N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N} \sqrt{h_N}} - b_N \leq x \right) = \exp(-e^{-x}) \tag{1.1.28}$$

and

$$\lim_{N \to \infty} \mathrm{P} \left(a_N \sup_{1 \leq k \leq N} \frac{|\hat{M}_{k,N}|}{\hat{\sigma}_{m_N} \sqrt{h_N}} - b_N \leq x \right) = \exp(-2e^{-x}), \tag{1.1.29}$$

where a_N and b_N are defined in (1.1.15) *and* (1.1.16)*, respectively.*

Like in case of known parameters, the asymptotic power of the procedures is one.

Theorem 1.1.4 *Let the sequence* $\{\hat{M}_{k,N}\}_{k=1,\ldots,N}$ *be defined as in* (1.1.25). *We suppose that* (1.1.4), (1.1.5) *and* (1.1.20) *hold. If* $k^* = k^*(N) \leq N - \sqrt{h_N} N^\rho$ *for some* $\rho > 0$, *then, under* H_1, *for all real* x *holds*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\leq k\leq N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} - b_N > x\right) = 1 \tag{1.1.30}$$

and under H_2 *we have*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\leq k\leq N} \frac{|\hat{M}_{k,N}|}{\hat{\sigma}_{m_N}\sqrt{h_N}} - b_N > x\right) = 1, \tag{1.1.31}$$

where a_N *and* b_N *are defined in* (1.1.15) *and* (1.1.16), *respectively.*

1.1.5 Simulations and discussion

In this section, we report the results of a simulation study on the finite sample properties of the MOSUM-chart.

We start with a small discussion about the framework for unknown in-control parameters. It is important to mention that in practice the size of the training period is predetermined. Based on a simulation, a test is considered to be applicable, if its empirical size for fixed m and N is smaller or equal than the prescribed nominal size α. The considered MOSUM-chart allows for an adjustment via the window size h, but in view of (1.1.20) the possibilities in our setting are limited, since the asymptotic results of Theorem 1.1.3 require $h < m$. The simulation is carried out for innovations with slightly more than five existing moments and we consider training periods of size $m = 10, 50, 100, 250, 500$. We take into account three window sizes h, which are also used in case of known in-control parameters. Under the null hypothesis the monitoring periods are chosen such that the empirical sizes keep, and also exceed the nominal size, which shows the possibilities for applications.

The tables providing the empirical power of the MOSUM-chart contain a five point summary of the empirical distribution of the stopping time. Note that the stopping time is set to N, if a change is not detected. The power is simulated for monitoring periods N that are chosen such that the empirical sizes lie close to the nominal size.

Since we are dealing with constant boundary functions, it suffices to consider a structural break immediately after the initializing, respectively training period, hence $k^* = 0$. We focus on the stopping times τ_1 and $\hat{\tau}_1$, stated in (1.1.9) and (1.1.26), where the level shift is chosen as $\triangle = 1$.

The innovations $\{\varepsilon_i\}_{i=1,2,\ldots}$ are independent identical symmetric Pareto(5.1) distributed, where we consider a random variable X to be symmetric Pareto(κ) disributed, if its density function follows

$$f_X(x) = \frac{\kappa}{2}(1+|x|)^{-(\kappa+1)} \quad \text{for all} \quad -\infty < x < \infty, \quad \text{where} \quad \kappa > 0.$$

Clearly it holds that

$$\mathrm{E}\,\varepsilon_1 = 0, \quad \mathrm{Var}\,\varepsilon_1 = 1 \quad \text{and} \quad \mathrm{E}\,|\varepsilon_1|^\nu < \infty \quad \text{for all} \quad \nu < \kappa$$

and the strong approximation according to Komlós, Major and Tusnády (1975,1976) and Major (1976) shows that (1.1.2) is satisfied. The variance of the observations is estimated by

$$\hat{\sigma}_m^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \hat{\mu}_m)^2,$$

obviously satisfying (1.1.24).

All values in the following tables are based on $5,000$ replications and the nominal size has been chosen as $\alpha = 0.05$.

Empirical sizes

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$					
h	N				
	10	20	30	40	50
9	0.0124	0.0246	0.0334	0.0346	0.0422
8	0.0138	0.0326	0.0376	0.0426	0.0428
7	0.0196	0.0312	0.0392	0.0462	0.0524
h	N				
	150	200	250	300	350
40	0.0324	0.0372	0.0382	0.0380	0.0408
35	0.0346	0.0394	0.0458	0.0490	0.0484
30	0.0400	0.0444	0.0476	0.0498	0.0512
h	N				
	500	600	700	800	900
85	0.0334	0.0346	0.0372	0.0364	0.0378
80	0.0370	0.0388	0.0402	0.0414	0.0398
75	0.0404	0.0398	0.0424	0.0450	0.0462
h	N				
	2000	2500	3000	3500	4000
210	0.0366	0.0386	0.0422	0.0436	0.0442
200	0.0396	0.0422	0.0436	0.0450	0.0446
190	0.0364	0.0392	0.0428	0.0448	0.0472
h	N				
	4000	5000	6000	7000	8000
420	0.0334	0.0344	0.0388	0.0382	0.0406
400	0.0320	0.0372	0.0392	0.0412	0.0412
380	0.0344	0.0364	0.0402	0.0390	0.0414

Table 1.1: MOSUM-charts, Empirical sizes for stopping time τ_1

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$						
m	h	N				
		10	12	14	16	18
10	9	0.0604	0.0810	0.1010	0.1106	0.1192
	8	0.0664	0.0858	0.1022	0.1172	0.1226
	7	0.0754	0.1056	0.1164	0.1166	0.1336
m	h	N				
		50	60	70	80	90
50	40	0.0392	0.0546	0.0698	0.0726	0.0844
	35	0.0500	0.0684	0.0734	0.0822	0.0860
	30	0.0604	0.0688	0.0860	0.0872	0.0902
m	h	N				
		100	120	140	160	180
100	85	0.0336	0.0462	0.0614	0.0778	0.0820
	80	0.0352	0.0474	0.0698	0.0812	0.0836
	75	0.0426	0.0536	0.0732	0.0844	0.0900
m	h	N				
		250	300	350	400	450
250	210	0.0328	0.0464	0.0588	0.0668	0.0748
	200	0.0342	0.0488	0.0594	0.0676	0.0814
	190	0.0400	0.0508	0.0614	0.0682	0.0830
m	h	N				
		500	600	700	800	900
500	420	0.0312	0.0424	0.0610	0.0648	0.0752
	400	0.0314	0.0486	0.0624	0.0698	0.0730
	380	0.0322	0.0472	0.0644	0.0716	0.0778

Table 1.2: MOSUM-charts, Empirical sizes for stopping time $\hat{\tau}_1$

Empirical power

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$							
N	h	min	$Q_{.25}$	$Q_{.5}$	$Q_{.75}$	max	power
50	9	1	8	11	19	50	0.9870
	8	1	8	13	21	50	0.9754
	7	1	8	14	26	50	0.9496
350	40	1	18	22	26	50	1
	35	1	17	21	24	54	1
	30	1	16	19	23	65	1
900	85	1	26	32	38	72	1
	80	1	26	31	37	72	1
	75	1	25	30	36	62	1
4000	210	1	43	53	63	112	1
	200	1	43	52	61	108	1
	190	2	41	51	60	101	1
8000	420	6	61	75	87	160	1
	400	1	59	73	86	144	1
	380	1	59	72	84	146	1

Table 1.3: MOSUM-charts, Empirical power for stopping time τ_1

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$								
m	N	h	min	$Q_{.25}$	$Q_{.5}$	$Q_{.75}$	max	power
10	10	9	1	5	8	10	10	0.6478
		8	1	5	7	10	10	0.6620
		7	1	5	7	10	10	0.6628
50	50	40	2	14	19	24	50	0.9852
		35	2	13	17	22	50	0.9832
		30	1	12	16	21	50	0.9806
100	100	85	3	23	28	35	100	0.9990
		80	6	22	27	33	100	0.9998
		75	6	21	26	32	100	0.9988
250	300	210	14	36	44	52	130	1
		200	7	35	42	50	108	1
		190	8	34	41	49	116	1
500	600	420	26	53	62	72	200	1
		400	24	51	61	71	226	1
		380	14	50	59	69	147	1

Table 1.4: MOSUM-charts, Empirical power for stopping time $\hat{\tau}_1$

Discussion

As we can see by Table 1.1, in case of known parameters μ and σ, the empirical size of the test can be adjusted via the window size h, such that the test is applicable for various monitoring periods N. However, if the in-control parameters are estimated (Table 1.2), acceptable values for the empirical sizes are only obtained, if the length of the monitoring period is chosen close to the length of the training period. It seems that the empirical sizes are decreasing, as the training period increases, but the rate is very slow. The results under the alternative (Tables 1.3 and 1.4) indicate that the MOSUM-chart has asymptotic power one.

We now consider three possibilities to reduce the empirical false alarm rates in case of unknown in-control parameters. Heuristically the test takes action, if the value of

$$\frac{1}{\sqrt{h}} \sum_{i=k-h+1}^{k} \varepsilon_{i+m} - \frac{\sqrt{h}}{m} \sum_{j=1}^{m} \varepsilon_j$$

exceeds a boundary function. In our setting, the application of extreme value theory requires $h/m \to 0$, however, also in case of $h/m \to c > 0$ limit distributions for the extremes can be derived and we refer to Horváth, Kühn and Steinebach (2008) for details. The results given there show that adequate chosen boundary functions depend on the number of existing moments of the innovations and it should be mentioned that the empirical sizes in case of $h/m \to 1$ in a simulation with $m = 100$ and $N = 100 \cdot m$ did not exceed the nominal size significantly.

Another possibility is to improve the rates of the estimators. This leads to recursive estimators (sequential estimators), introduced by Brown, Durbin and Evans (1975). We refer to Gut and Steinebach (2002), Horváth, Hušková, Kokoszka and Steinebach (2004) and Aue and Kühn (2008) for an overview on the usage of recursive estimators in an asymptotic framework. The basic idea is that if no action has been taken at time k, all past observations may be used to estimate the parameters. The simulations in Aue and Kühn (2008), carried out for a CUSUM-type detector, showed that recursive estimators lead to lower empirical sizes, however, also increase the reaction time. The main reason for this effect is that the underlying test reacts with some delay, hence the recursive estimators also take values after the change into account. An approach to avoid the slower reaction time may be the following. If no action has been taken at time k, the estimation is based on the observations up to time $k - d$, where d is chosen such that with high probability $k^* > k - d$. The last approach requires detailed information about the distribution of the delay time and may be realized by taking into account the results of Chapter 5.

The third method follows the idea of exponential smoothing and is elaborated in the next chapter.

1.1.6 Proofs

Proof of Theorem 1.1.1

We define the sequence $\{Q_{k,N}\}_{k=1,\ldots,N}$ as

$$Q_{k,N} = \sigma(W(k+h_N) - W(k)) \quad \text{for all} \quad k = 1, \ldots, N, \tag{1.1.32}$$

where $\{W(t), t \geq 0\}$ is the approximating Wiener-process introduced in (1.1.2).

Lemma 1.1.1 *Let the sequences* $\{M_{k,N}\}_{k=1,\ldots,N}$ *and* $\{Q_{k,N}\}_{k=1,\ldots,N}$ *be defined as in* (1.1.8) *and* (1.1.32)*, respectively. Then*

$$a_N \left(\max_{1\leq k\leq N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} - \max_{1\leq k\leq N} \frac{Q_{k,N}}{\sigma\sqrt{h_N}} \right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad N \to \infty. \tag{1.1.33}$$

Proof: We have

$$\begin{aligned} &\frac{a_N}{\sigma\sqrt{h_N}} \max_{1\leq k\leq N} |M_{k,N} - Q_{k,N}| \\ &\qquad = \frac{a_N}{\sigma\sqrt{h_N}} \max_{1\leq k\leq N} |(S(k+h_N) - S(k)) - (W(k+h_N) - W(k))| \\ &\qquad \leq 2\frac{a_N}{\sigma\sqrt{h_N}} \max_{1\leq k\leq N+h_N} |S(k) - W(k)| \\ &\qquad = \boldsymbol{O}_P\left(a_N \frac{(N+h_N)^{1/\nu}}{\sqrt{h_N}} \right) \quad \text{as} \quad N \to \infty, \end{aligned}$$

where we have used (1.1.2). Now the lemma follows by (1.1.4).

□

Let the process $\{U_N(t)\}_{t\geq 0}$ be defined as

$$U_N(t) = \sigma(W(t+h_N) - W(t)) \quad \text{for all} \quad t \geq 0, \tag{1.1.34}$$

where $\{W(t), t \geq 0\}$ again is the approximating Wiener-process.

Lemma 1.1.2 *Let the processes* $\{Q_{k,N}\}_{k=1,\ldots,N}$ *and* $\{U_N(t)\}_{t\geq 0}$ *be defined as in* (1.1.32) *and* (1.1.34)*, respectively. Then*

$$a_N \left(\sup_{1\leq t\leq N} \frac{Q_{\lfloor t\rfloor,N}}{\sigma\sqrt{h_N}} - \sup_{1\leq t\leq N} \frac{U_N(t)}{\sigma\sqrt{h_N}} \right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad N \to \infty. \tag{1.1.35}$$

Proof: By the definition of the processes we have

$$\begin{aligned} &\frac{a_N}{\sigma\sqrt{h_N}} \sup_{1\leq t\leq N} |Q_{\lfloor t\rfloor,N} - U_N(t)| \\ &\qquad \leq 2\frac{a_N}{\sigma\sqrt{h_N}} \sup_{1\leq t\leq N+h_N} |W(t) - W(\lfloor t\rfloor)| \end{aligned}$$

and the lemma follows by (1.1.4) and Theorem 1.2.1 of Csörgő and Révész (1981).

□

Now the self-similarity of the Wiener-process implies

$$\sup_{1\le t\le N}\frac{U_N(t)}{\sigma\sqrt{h_N}} \stackrel{\mathrm{D}}{=} \sup_{1/h_N\le t\le N/h_N}(W(t+1)-W(t)) \tag{1.1.36}$$

and (1.1.13) follows by Lemma 1.1.1, Lemma 1.1.2 and on combining (1.1.36) with Theorem 12.3.5 of Leadbetter, Lindgren and Rootzén (1983). Assertion (1.1.14) follows by (1.1.13) and the asymptotic independence of the maxima and minima (see Bickel and Rosenblatt, Theorem A 1, 1973).

Proof of Theorem 1.1.2

We only show (1.1.17), since (1.1.18) follows by similar arguments. If we define the sequence $\{M_{k,N}^{(0)}\}_{k=1,\ldots,N}$ as

$$M_{k,N}^{(0)} = \sum_{i=k-h_N+1}^{k}\varepsilon_{i+h_N} \quad \text{for all} \quad k=1,\ldots,N,$$

then it holds that

$$M_{k,N} = M_{k,N}^{(0)} + (k-\max\{k^*,k-h_N\})\,\triangle \quad \text{for all} \quad k=1,\ldots,N.$$

Since $k^* < N-\sqrt{h_N}N^\rho$ (w.l.o.g. $N^\rho/\sqrt{h_N}\to 0$ as $N\to\infty$) we have

$$\max_{1\le k\le N}(k-\max\{k^*,k-h_N\})\,\triangle \ge \sqrt{h_N}N^\rho\,\triangle$$

and it follows that for all real x it holds that

$$\begin{aligned} &\mathrm{P}\left(a_N\max_{1\le k\le N}\frac{M_{k,N}}{\sigma\sqrt{h_N}}-b_N\le x\right)\\ &\quad\le\ \mathrm{P}\left(a_N\max_{1\le k\le N}\frac{|M_{k,N}^{(0)}|}{\sigma\sqrt{h_N}}-b_N\le x-\frac{N^\rho}{\sigma}\,\triangle+2a_N\max_{1\le k\le N}\frac{|M_{k,N}^{(0)}|}{\sigma\sqrt{h_N}}\right). \end{aligned} \tag{1.1.37}$$

Now Theorem 1.1.1 implies

$$a_N\max_{1\le k\le N}\frac{|M_{k,N}^{(0)}|}{\sigma\sqrt{h_N}}-b_N=\boldsymbol{O}_P(1) \quad \text{as} \quad N\to\infty$$

and

$$a_N\max_{1\le k\le N}\frac{|M_{k,N}^{(0)}|}{\sigma\sqrt{h_N}}=\boldsymbol{O}_P(a_N^2) \quad \text{as} \quad N\to\infty,$$

hence $\frac{N^\rho}{\sigma}\,\triangle$ is the dominating term in (1.1.37) and (1.1.17) follows as $N\to\infty$.

Proof of Theorem 1.1.3

First, we provide the rates of the estimators, following the results in Steinebach (1995).

Lemma 1.1.3 *Let $\hat{\mu}_{m_N}$ be defined as in* (1.1.23) *and let $\hat{\sigma}^2_{m_N}$ be given by*

$$\hat{\sigma}^2_{m_N} = \frac{1}{\lfloor m_N/v_N \rfloor} \sum_{i=1}^{\lfloor m_N/v_N \rfloor} \frac{1}{v_N} \left(\sum_{j=(i-1)v_N+1}^{iv_N} X_j - v_N \check{\mu}_{m_N} \right)^2,$$

where $v_N \le m_N$ is an integer-valued function of N satisfying $v_N \sim m_N^\delta$ as $N \to \infty$ for some $2/\nu < \delta < 1$ and

$$\check{\mu}_{m_N} = \frac{1}{\lfloor m_N/v_N \rfloor} \sum_{i=1}^{\lfloor m_N/v_N \rfloor} \frac{1}{v_N} \sum_{j=(i-1)v_N+1}^{iv_N} X_j.$$

Then, it holds that

$$\hat{\mu}_{m_N} - \mu = \boldsymbol{O}_P\left(\frac{1}{\sqrt{m_N}}\right) \quad \text{as} \quad N \to \infty \tag{1.1.38}$$

and

$$\hat{\sigma}^2_{m_N} - \sigma^2 = \boldsymbol{o}_P\left(\frac{1}{m_N^\vartheta}\right) \quad \text{as} \quad N \to \infty \tag{1.1.39}$$

for all $0 < \vartheta < \min\{1/2 - \delta/2, \delta/2 - 1/\nu\}$.

Proof: The invariance principle (1.1.2) yields

$$\begin{aligned}
\hat{\mu}_N - \mu &= \frac{1}{m_N} \sum_{j=1}^{m_N} \varepsilon_j \\
&= \frac{1}{m_N} \left(\sum_{j=1}^{m_N} \varepsilon_j - \sigma W(m_N) \right) + \frac{1}{m_N} \sigma W(m_N) \\
&= \boldsymbol{O}_P\left(m_N^{1/\nu - 1}\right) + \boldsymbol{O}_P\left(\frac{1}{\sqrt{m_N}}\right) \quad \text{as} \quad N \to \infty,
\end{aligned}$$

which implies (1.1.38) since $\nu > 2$.

Next, we consider the estimator for the variance. If $\{W(t), t \ge 0\}$ denotes the approximating Wiener process and $K_N = \lfloor m_N/v_N \rfloor$, we get

$$\begin{aligned}
&\frac{1}{K_N} \sum_{i=1}^{K_N} \frac{1}{v_N} \left(\sum_{j=(i-1)v_N+1}^{iv_N} X_j - v_N \check{\mu}_{m_N} \right)^2 \\
&\quad - \frac{1}{K_N} \sum_{i=1}^{K_N} \frac{\sigma^2}{v_N} \left(W(iv_N) - W((i-1)\,v_N) - \frac{1}{K_N} W(v_N K_N) \right)^2 \\
&\quad =: \frac{1}{K_N} \sum_{i=1}^{K_N} \frac{1}{v_N} I_1(i) \cdot I_2(i),
\end{aligned} \tag{1.1.40}$$

where for all $1 \le i \le K_N$

$$I_1(i) = \left(\sum_{j=(i-1)v_N+1}^{iv_N} X_j - v_N \check{\mu}_{m_N} \right) - \sigma \left(W(iv_N) - W((i-1)v_N) - \frac{1}{K_N} W(v_N K_N) \right)$$

and

$$I_2(i) = \left(\sum_{j=(i-1)v_N+1}^{iv_N} X_j - v_N \check{\mu}_{m_N} \right) + \sigma \left(W(iv_N) - W((i-1)v_N) - \frac{1}{K_N} W(v_N K_N) \right).$$

Since

$$\begin{aligned}\max_{1\le i\le K_N} |I_1(i)| &\le 2 \max_{1\le i\le m_N} \left| \sum_{j=1}^{i} \varepsilon_j - \sigma W(j) \right| + \frac{1}{K_N} \left| \left(\sum_{j=1}^{v_N K_N} \varepsilon_j - W(v_N K_N) \right) \right| \\ &= \boldsymbol{O}_P\left(m_N^{1/\nu} \right) + \boldsymbol{O}_P\left(\frac{v_N}{m_N^{1-1/\nu}} \right) \quad \text{as} \quad N \to \infty\end{aligned}$$

and $v_N \le m_N$ it follows that

$$\max_{1\le i\le \lfloor m_N/v_N \rfloor} |I_1(i)| = \boldsymbol{O}_P\left(m_N^{1/\nu} \right) \quad \text{as} \quad N \to \infty. \tag{1.1.41}$$

Furthermore, it holds that

$$\begin{aligned}&\max_{1\le i\le K_N} |I_2(i)| \\ &\quad\le \max_{1\le i\le K_N} |I_1(i)| + 2 \max_{1\le i\le K_N} \sigma \left| W(iv_N) - W((i-1)v_N) - \frac{1}{K_N} W(v_N K_N) \right|.\end{aligned}$$

The definition of v_N and Theorem 1.2.1 of Csörgő and Révész (1981) yield

$$\max_{1\le i\le K_N} \sigma \left| W(iv_N) - W((i-1)v_N) \right| = \boldsymbol{O}\left(\sqrt{v_N \log m_N} \right) \quad \text{a.s.} \quad \text{as} \quad N \to \infty$$

and obviously

$$\left| \frac{1}{K_N} W(v_N K_N) \right| = \boldsymbol{O}_P\left(\frac{v_N}{\sqrt{m_N}} \right) \quad \text{as} \quad N \to \infty.$$

Comparing the rates we see that

$$\max_{1\le i\le K_N} |I_2(i)| = \boldsymbol{O}_P\left(\sqrt{v_N \log m_N} \right) = \boldsymbol{O}_P\left(m_N^{\delta/2} \sqrt{\log m_N} \right) \quad \text{as} \quad N \to \infty \tag{1.1.42}$$

and (1.1.40)–(1.1.42) show that

$$\frac{1}{K_N} \sum_{i=1}^{K_N} \frac{1}{v_N} I_1(i) \cdot I_2(i) = \boldsymbol{O}_P\left(\frac{\sqrt{\log m_N}}{m_N^{\delta/2-1/\nu}} \right) \quad \text{as} \quad N \to \infty.$$

Hence,

$$\frac{1}{K_N}\sum_{i=1}^{K_N}\frac{1}{v_N}I_1(i)\cdot I_2(i)=\boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta_1}}\right)\quad\text{as}\quad N\to\infty \tag{1.1.43}$$

for all $0\le\vartheta_1<\delta/2-1/\nu$.

Now

$$\frac{1}{K_N}\sum_{i=1}^{K_N}\frac{\sigma^2}{v_N}\left(W(iv_N)-W((i-1)\,v_N)-\frac{1}{K_N}W(v_NK_N)\right)^2\overset{\mathrm{D}}{=}\frac{\sigma^2}{K_N}\,\chi^2_{K_N-1},$$

where $\chi^2_{K_N-1}$ is a chi-square distributed random variable with K_N-1 degrees of freedom. The law of the iterated logarithm implies that

$$\frac{\sigma^2}{K_N}\,\chi^2_{K_N-1}-\sigma^2=\boldsymbol{o}_P\left(\frac{1}{K_N^{\theta}}\right)\quad\text{as}\quad N\to\infty$$

for all $0\le\theta<1/2$ and we see that

$$\frac{1}{K_N}\sum_{i=1}^{K_N}\frac{\sigma^2}{v_N}\left(W(iv_N)-W((i-1)\,v_N)-\frac{1}{K_N}W(v_NK_N)\right)^2=\boldsymbol{o}_P\left(\frac{1}{m^{\vartheta_2}}\right) \tag{1.1.44}$$

for all $0\le\vartheta_2\le(1-\delta)\theta$ and $0\le\theta<1/2$. Assertion (1.1.39) now follows from (1.1.43) and (1.1.44).

□

Now let the sequence $\{H_{k,N}\}_{k=1,\ldots,N}$ be defined as

$$H_{k,N}=\sum_{i=k-h_N+1}^{k}\left(X_{i+m_N}-\mu\right),\quad k=1,\ldots,N. \tag{1.1.45}$$

Condition (1.1.20) and the proof of Theorem 1.1.1 immediately imply that under the null hypothesis

$$\lim_{N\to\infty}\mathrm{P}\left(a_N\sup_{1\le k\le N}\frac{H_{k,N}}{\sigma\sqrt{h_N}}-b_N\le x\right)=\exp(-e^{-x})\quad\text{as}\quad N\to\infty \tag{1.1.46}$$

and also

$$\lim_{N\to\infty}\mathrm{P}\left(a_N\sup_{1\le k\le N}\frac{|H_{k,N}|}{\sigma\sqrt{h_N}}-b_N\le x\right)=\exp(-2e^{-x})\quad\text{as}\quad N\to\infty \tag{1.1.47}$$

hold.

Lemma 1.1.4 *Let the sequences* $\{\hat{M}_{k,N}\}_{k=1,\dots,N}$ *and* $\{H_{k,N}\}_{k=1,\dots,N}$ *be defined as in* (1.1.25) *and* (1.1.45)*, respectively. Furthermore, we assume that* (1.1.4)*,* (1.1.5) *and* (1.1.20) *are satisfied. Then, under* H_0*, it holds that*

$$a_N \left(\max_{1\le k\le N} \frac{H_{k,N}}{\sigma\sqrt{h_N}} - \max_{1\le k\le N} \frac{\hat{M}_{k,N}}{\sigma\sqrt{h_N}} \right) = \boldsymbol{o}_P(1) \quad as \quad N \to \infty. \tag{1.1.48}$$

Proof: With (1.1.38) we get

$$\begin{aligned} \frac{a_N}{\sigma\sqrt{h_N}} \max_{1\le k\le N} \left| H_{k,N} - \hat{M}_{k,N} \right| &= \frac{a_N}{\sigma\sqrt{h_N}} \max_{1\le k\le N} |h_N(\mu - \hat{\mu}_{m_N})| \\ &= \frac{a_N}{\sigma} \frac{\sqrt{h_N}}{\sqrt{m_N}} \boldsymbol{O}_P(1) \quad \text{as} \quad N \to \infty \end{aligned}$$

and the lemma follows by (1.1.20). □

Lemma 1.1.5 *Let the sequence* $\{\hat{M}_{k,N}\}_{k=1,\dots,N}$ *be defined as in* (1.1.25)*. If* $\hat{\sigma}^2_{m_N}$ *satisfies* (1.1.24)*, then, under* H_0*, it holds that*

$$a_N \left(\max_{1\le k\le N} \frac{\hat{M}_{k,N}}{\sigma\sqrt{h_N}} - \max_{1\le k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \right) = \boldsymbol{o}_P(1) \quad as \quad N \to \infty. \tag{1.1.49}$$

Proof: We have

$$a_N \max_{1\le k\le N} \left| \frac{\hat{M}_{k,N}}{\sigma\sqrt{h_N}} - \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \right| = \left| 1 - \frac{\sigma}{\hat{\sigma}_{m_N}} \right| a_N \max_{1\le k\le N} \frac{|\hat{M}_{k,N}|}{\sigma\sqrt{h_N}}.$$

Now (1.1.46), (1.1.47) and Lemma 1.1.4 imply that

$$a_N \max_{1\le k\le N} \frac{|\hat{M}_{k,N}|}{\sigma\sqrt{h_N}} = \boldsymbol{O}_P\left(\log \frac{N}{h_N}\right) \quad \text{as} \quad N \to \infty$$

and since (1.1.39) implies

$$|\hat{\sigma}_{m_N} - \sigma| = \boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta}}\right) \quad \text{as} \quad N \to \infty \quad (\forall\ 0 < \vartheta < \min\{1/2 - \delta/2, \delta/2 - 1/\nu\}),$$

the lemma follows.

□

Combining (1.1.46), (1.1.47), Lemma 1.1.4 and Lemma 1.1.5 now yields Theorem 1.1.3.

Proof of Theorem 1.1.4

Since ρ can be chosen arbitrarily small, we can assume by (1.1.20) and (1.1.39) that

$$\frac{N^\rho}{\hat{\sigma}_{m_N}} \vartriangle - \frac{N^\rho}{\sigma} \vartriangle = \boldsymbol{o}_P\left(\frac{1}{N^\epsilon}\right) \quad \text{as} \quad N \to \infty \quad \text{for some} \quad \epsilon > 0.$$

Hence the proof of Theorem 1.1.2 carries over.

Chapter 2

Control charts based on weighted averages of moving sums

As we have seen in the previous chapter, the discussed MOSUM-chart is too sensitive, if the model parameters are estimated. Gut and Steinebach (2004) provided a modification of the chart, which is based on exponential smoothing methods and allows to control the sensitivity of the test via a weight parameter λ.

In this chapter we extend the approach of Gut and Steinebach (2004) to a wide class of weight functions including many convergent series and all finite sequences, which are decreasing and non-negative.

2.1 Closed-end control charts

2.1.1 Model assumptions for known μ and σ

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of real valued random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P}\,)$ with

$$\mathrm{E}\,\varepsilon_i = 0 \quad \text{and} \quad \mathrm{Var}\,\varepsilon_i = \sigma^2 > 0 \quad \text{for all} \quad i = 1, 2, \ldots. \tag{2.1.1}$$

We assume that there exists a Wiener process $\{W(t), t \geq 0\}$ such that for some $\nu > 2$

$$\sup_{1 \leq k < \infty} \frac{1}{k^{1/\nu}} \left| \sum_{j=1}^{k} \varepsilon_j - \sigma W(k) \right| < \infty \quad \text{a.s.} \tag{2.1.2}$$

The observations are modeled as a discrete-time stochastic process $\{X_i\}_{i=1,2,\ldots,h_N+N}$ satisfying

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \leq i \leq h_N + k^*, \\ \mu + \triangle + \varepsilon_i & , \quad h_N + k^* \leq h_N + N, \end{cases} \tag{2.1.3}$$

where μ and $\triangle$ are real parameters and k^* is the unknown time of a possible change in the mean.

The window size h_N is chosen such that

$$h_N \simeq N^{2\phi} \quad \text{as} \quad N \to \infty, \quad \text{where} \quad 1/\nu < \phi < 1/2. \tag{2.1.4}$$

We are interested in testing

$$H_0 : k^* = N \quad \text{versus} \quad H_1 : k^* < N, \ \triangle > 0 \quad \text{(one-sided alternative)}, \tag{2.1.5}$$

or

$$H_0 : k^* = N \quad \text{versus} \quad H_2 : k^* < N, \ \triangle \neq 0 \quad \text{(two-sided alternative)}. \tag{2.1.6}$$

2.1.2 Monitoring procedures for known μ and σ

The detectors are defined as weighted averages of standardized moving sums that is

$$A_{k,N} = \sum_{j=0}^{k-1} w_j B_{k-j,N} \quad \text{for all} \quad k = 1, 2, \ldots, N, \tag{2.1.7}$$

where $\{w_j\}_{j=0,1,\ldots}$ is a real sequence satisfying

$$1) \quad \{w_j\}_{j=0,1,\ldots} \quad \text{is non-increasing}, \tag{2.1.8}$$

$$2) \quad \sum_{j=0}^{\infty} w_j = 1, \tag{2.1.9}$$

$$3) \quad \sum_{j=0}^{k} j\, w_j = \boldsymbol{O}\left(k^{\phi}\right) \quad \text{as} \quad k \to \infty \quad \text{for some} \quad 0 < \phi < 1, \tag{2.1.10}$$

$$4) \quad \sum_{j=k}^{\infty} w_j = \boldsymbol{O}\left(\frac{1}{k^{\psi}}\right) \quad \text{as} \quad k \to \infty \quad \text{for some} \quad \psi > 0. \tag{2.1.11}$$

The sequence $\{B_{k,N}\}_{k=1,\ldots,N}$ is defined as

$$B_{k,N} = \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \left(X_{k-i+h_N} - \mu\right) \quad \text{for all} \quad k = 1, 2, \ldots, N. \tag{2.1.12}$$

Note that it follows by (2.1.8) and (2.1.9) that $w_j \geq 0$ for all $j = 0, 1, \ldots$, hence

$$j w_j = \boldsymbol{o}(1) \quad \text{as} \quad j \to \infty. \tag{2.1.13}$$

Remark 2.1.1 *Condition* (2.1.9) *is only made for technical reasons and may be replaced by the assumption that the sum of the weights is converging. Conditions* (2.1.10) *and* (2.1.11) *are satisfied by many convergent series as for example*

$$w_j = \frac{1}{(j+1)^\gamma}, \quad j = 0, 1, \ldots, \quad \gamma > 1$$

but do not hold for

$$w_j = \frac{1}{(j+1)(\log(j+2))^\gamma}, \quad j = 0, 1, \ldots, \quad \gamma > 1,$$

even though the corresponding series converges.

The influence of the weights on the boundary function of the procedure is described by the following sequence

$$\sigma_N^2 = \sigma^2 \left(\sum_{k=0}^{\infty} w_k^2 + 2 \sum_{k=0}^{\infty} w_k \left(\sum_{j=1}^{h_N - 1} \frac{h_N - j}{h_N} w_{k+j} \right) \right). \tag{2.1.14}$$

Note that (A.1.2) implies that

$$\sigma_N - \sigma \to 0 \quad \text{as} \quad N \to \infty. \tag{2.1.15}$$

If we test the null hypotheses versus the one-sided alternative, we reject H_0 if $\tau_1 < \infty$, where

$$\tau_1 = \tau_1(\alpha, N) = \inf\{1 \leq k \leq N : A_{k,N} > c_1(\alpha, N)\sigma_N\}. \tag{2.1.16}$$

As usual $\alpha \in]0, 1[$ denotes the level of significance. If the alternative is two-sided, we replace τ_1 by τ_2, where

$$\tau_2 = \tau_2(\alpha, N) = \inf\{1 \leq k \leq N : |A_{k,N}| > c_2(\alpha, N)\sigma_N\}. \tag{2.1.17}$$

The critical constants $c_1(\alpha, N)$ and $c_2(\alpha, N)$ are given by

$$c_1(\alpha, N) = \frac{q_1(1-\alpha) + b_N}{a_N} \quad \text{and} \quad c_2(\alpha, N) = \frac{q_2(1-\alpha) + b_N}{a_N}, \tag{2.1.18}$$

where a_N and b_N are defined in (2.1.22) and (2.1.23) below and

$$q_1(1-\alpha) = -\log(-\log(1-\alpha)), \quad q_2(1-\alpha) = -\log\left(-\frac{1}{2}\log(1-\alpha)\right). \tag{2.1.19}$$

The choice of critical constants is based on the next theorem.

Theorem 2.1.1 *Let the sequence* $\{A_{k,N}\}_{k=1,\dots,N}$ *be defined as in* (2.1.7) *and assume that* h_N *satisfies* (2.1.4)*. If we define* σ_N *via* (2.1.14)*, then, under* H_0*, for all real* x *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{A_{k,N}}{\sigma_N} - b_N \le x\right) = \exp(-e^{-x}) \tag{2.1.20}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|A_{k,N}|}{\sigma_N} - b_N \le x\right) = \exp(-2e^{-x}), \tag{2.1.21}$$

where

$$a_N = \sqrt{2\log\frac{N}{h_N}} \tag{2.1.22}$$

and

$$b_N = 2\log\frac{N}{h_N} + \frac{1}{2}\log\log\frac{N}{h_N} - \frac{1}{2}\log\pi. \tag{2.1.23}$$

Under weak conditions imposed on k^*, the procedures also have asymptotic power one.

Theorem 2.1.2 *Let* $\{A_{k,N}\}_{k=1,\dots,N}$*,* h_N *and* σ_N *be defined as in Theorem 2.1.1. If we assume that* $k^* = k^*(N) < N - h_N$*, then for all real* x *it holds that under* H_1

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{A_{k,N}}{\sigma_N} - b_N > x\right) = 1 \tag{2.1.24}$$

and under H_2

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{|A_{k,N}|}{\sigma_N} - b_N > x\right) = 1, \tag{2.1.25}$$

where a_N *and* b_N *are defined in* (2.1.22) *and* (2.1.23)*, respectively.*

2.1.3 Model assumptions for unknown μ and σ

We assume that the innovations $\{\varepsilon_i\}_{i=1,2,\dots}$ satisfy (2.1.1) and (2.1.2).

Again the estimators are based on a training period that is assumed to be homogeneous, hence we suppose that

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \le i \le m_N + k^*, \\ \mu + \triangle + \varepsilon_i & , \quad m_N + k^* < i \le m_N + N. \end{cases} \tag{2.1.26}$$

The conditions imposed on the size of the training period are

$$1) \quad m_N + N \sim N \quad \text{as} \quad N \to \infty \tag{2.1.27}$$

and

$$2) \quad \lim_{N\to\infty} \left(\frac{h_N}{m_N} \log \frac{N}{h_N} \right) = 0, \tag{2.1.28}$$

where the window size h_N still satisfies (2.1.4).

2.1.4 Monitoring procedures for unknown μ and σ

Replacing μ by the estimator

$$\hat{\mu}_{m_N} = \frac{1}{m_N} \sum_{j=1}^{m_N} X_j \tag{2.1.29}$$

we define

$$\hat{A}_{k,N} = \sum_{j=0}^{k-1} w_j \hat{B}_{k-j,N} \quad \text{for all} \quad k = 1, \ldots, N, \tag{2.1.30}$$

where $\{w_j\}_{j=0,1,\ldots}$ satisfies (2.1.8)–(2.1.11) and

$$\hat{B}_{k,N} = \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \left(X_{k-i+m_N} - \hat{\mu}_{m_N} \right) \quad \text{for all} \quad k = 1, \ldots, N. \tag{2.1.31}$$

With $\hat{\sigma}^2_{m_N}$ being an estimator for σ^2 satisfying

$$\hat{\sigma}^2_{m_N} - \sigma^2 = \boldsymbol{o}_P \left(\frac{1}{m_N^{\vartheta}} \right) \quad \text{as} \quad N \to \infty \quad \text{for some} \quad \vartheta > 0 \tag{2.1.32}$$

(the existence has been shown in Lemma 1.1.3), we set

$$\hat{\sigma}^2_N = \hat{\sigma}^2_{m_N} \left(\sum_{k=0}^{\infty} w_k^2 + 2 \sum_{k=0}^{\infty} w_k \left(\sum_{j=1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j} \right) \right) \tag{2.1.33}$$

which yields the modified stopping times

$$\hat{\tau}_1 = \hat{\tau}_1(\alpha, N) = \inf\{1 \le k \le N : \hat{A}_{k,N} > c_1(\alpha, N) \hat{\sigma}_N\} \tag{2.1.34}$$

and

$$\hat{\tau}_2 = \hat{\tau}_2(\alpha, N) = \inf\{1 \le k \le N : |\hat{A}_{k,N}| > c_2(\alpha, N) \hat{\sigma}_N\}. \tag{2.1.35}$$

The critical constants $c_1(\alpha, N)$ and $c_2(\alpha, N)$ are chosen as in (2.1.18) according to the next theorem.

Theorem 2.1.3 *Let the sequence* $\{\hat{A}_{k,N}\}_{k=1,\ldots,N}$ *be defined as in* (2.1.30)*. Furthermore,* $\hat{\sigma}_N$ *is chosen as in* (2.1.33) *and we assume that* (2.1.4)*,* (2.1.27) *and* (2.1.28) *are satisfied. Then for all real* x *it holds that under* H_0

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{\hat{A}_{k,N}}{\hat{\sigma}_N} - b_N \le x\right) = \exp(-e^{-x}) \tag{2.1.36}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{|\hat{A}_{k,N}|}{\hat{\sigma}_N} - b_N \le x\right) = \exp(-2e^{-x}), \tag{2.1.37}$$

where a_N *and* b_N *are chosen as in* (2.1.22) *and* (2.1.23)*, respectively.*

The asymptotic power of the test is not influenced by the estimations, so it still holds that the procedures have asymptotic power one.

Theorem 2.1.4 *Let* $\{\hat{A}_{k,N}\}_{k=1,\ldots,N}$ *and* $\hat{\sigma}_N$ *be defined as in Theorem 2.1.3 and suppose that* (2.1.4)*,* (2.1.27) *and* (2.1.28) *are satisfied. If we assume that* $k^* = k^*(N) < N - h_N$*, then for all real* x *it holds that under* H_1

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{\hat{A}_{k,N}}{\hat{\sigma}_N} - b_N > x\right) = 1 \tag{2.1.38}$$

and under H_2

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le k\le N} \frac{|\hat{A}_{k,N}|}{\hat{\sigma}_N} - b_N > x\right) = 1, \tag{2.1.39}$$

where a_N *and* b_N *are chosen as in* (2.1.22) *and* (2.1.23)*, respectively.*

2.1.5 Simulations and discussion

The aim of the simulations provided in this paragraph is to investigate the effect of smoothing for different kinds of weights. If we choose the weight function

$$w_j = \begin{cases} 1 & , \quad j = 0, \\ 0 & , \quad j > 0, \end{cases}$$

we obtain the MOSUM-chart for which we already know the empirical results stated in Tables 1.1–1.4. These tables provide guidelines for the choices of the parameters m, h and N and will be used as benchmark.

We expect that the empirical sizes decrease, if we put more weight on past observations, but we also expect that the test then will be less sensitive. Since we have seen that in case of known parameters the MOSUM-chart is well applicable for different sizes of the monitoring period, there is no need to put more weight on past observations. Hence, we focus on the weighted averages for unknown μ and σ.

The simulation is carried out for the stopping time $\hat{\tau}_1$ and the symmetrical Pareto(5.1) variables specified in the simulation part of Chapter 1. The variance of the observations is estimated by

$$\hat{\sigma}^2_{m_N} = \frac{1}{m_N - 1}\sum_{i=1}^{m_N}(X_i - \hat{\mu}_{m_N})^2.$$

The empirical power is simulated for $\triangle = 1$ and $k^* = 0$. The parameters m, h and N are chosen with regard to the results stated in Table 1.2.

The values for $\hat{\sigma}^2_{m_N}$ are approximated by

$$\hat{\sigma}^2_N \approx \hat{\sigma}^2_{m_N}\left(\sum_{k=0}^{4999} w_k^2 + 2\sum_{k=0}^{4999} w_k\left(\sum_{j=1}^{h_N-1}\frac{h_N - j}{h_N} w_{k+j}\right)\right).$$

Each result in the following tables is based on $5,000$ replications and the nominal size has been chosen as $\alpha = 0.05$.

Weight functions (WF)

- WF–1: The weights are chosen according to

$$w_j = \frac{1}{C_\gamma (j+1)^\gamma}, \quad j = 0, 1, \ldots, \quad \gamma > 1, \quad C_\gamma = \sum_{j=0}^{\infty}\frac{1}{(j+1)^\gamma}.$$

- WF–2: The weights are chosen according to

$$w_j = (1-\lambda)\lambda^j, \quad j = 0, 1, \ldots, \quad \lambda \in [0, 1[.$$

Empirical sizes

symmetrical Pareto(5.1), $\alpha = 0.05$							
m	h	N	$\gamma = 1.1$	$\gamma = 1.2$	$\gamma = 1.3$	$\gamma = 1.4$	$\gamma = 1.5$
10	8	10	0.0366	0.0420	0.0454	0.0480	0.0516
		14	0.0790	0.0796	0.0800	0.0836	0.0924
		18	0.0970	0.0978	0.0992	0.1022	0.1034
50	35	50	0.0328	0.0332	0.0346	0.0378	0.0422
		70	0.0506	0.0510	0.0552	0.0562	0.0616
		90	0.0646	0.0696	0.0702	0.0720	0.0766
100	80	100	0.0214	0.0254	0.0298	0.0312	0.0328
		140	0.0428	0.0466	0.0518	0.0510	0.0524
		180	0.0578	0.0636	0.0608	0.0628	0.0666
250	200	250	0.0204	0.0214	0.0228	0.0258	0.0276
		350	0.0394	0.0442	0.0468	0.0498	0.0546
		450	0.0536	0.0564	0.0602	0.0612	0.0656
500	400	500	0.0208	0.0240	0.0256	0.0282	0.0310
		700	0.0336	0.0474	0.0488	0.0464	0.0494
		900	0.0496	0.0574	0.0592	0.0640	0.0664

Table 2.1: WAMS-charts, Empirical sizes for stopping time $\hat{\tau}_1$ and WF–1

symmetrical Pareto(5.1), $\alpha = 0.05$							
m	h	N	$\lambda = .95$	$\lambda = .90$	$\lambda = .85$	$\lambda = .80$	$\lambda = .75$
10	8	10	0.0114	0.0220	0.0398	0.0444	0.0510
		14	0.0298	0.0550	0.0740	0.0750	0.0874
		18	0.0536	0.0834	0.0980	0.0996	0.1052
50	35	50	0.0190	0.0322	0.0352	0.0408	0.0418
		70	0.0364	0.0492	0.0496	0.0554	0.0572
		90	0.0580	0.0672	0.0706	0.0722	0.0730
100	80	100	0.0226	0.0260	0.0274	0.0290	0.0322
		140	0.0390	0.0406	0.0476	0.0558	0.0564
		180	0.0536	0.0540	0.0616	0.0626	0.0680
250	200	250	0.0188	0.0202	0.0228	0.0264	0.0272
		350	0.0390	0.0450	0.0486	0.0556	0.0566
		450	0.0502	0.0542	0.0604	0.0672	0.0686
500	400	500	0.0226	0.0254	0.0256	0.0278	0.0280
		700	0.0422	0.0426	0.0454	0.0514	0.0552
		900	0.0512	0.0610	0.0620	0.0648	0.0682

Table 2.2: WAMS-charts, Empirical sizes for stopping time $\hat{\tau}_1$ and WF–2

Empirical power

symmetrical Pareto(5.1), $\alpha = 0.05$									
m	N	h	γ	min	$Q_{.25}$	$Q_{.5}$	$Q_{.75}$	max	power
10	10	8	1.1	2	6	9	10	10	0.6054
			1.3	1	6	8	10	10	0.6080
			1.5	1	6	8	10	10	0.6246
50	70	35	1.1	5	17	21	26	70	0.9932
			1.3	6	16	20	25	70	0.9918
			1.5	6	15	19	24	70	0.9898
100	140	80	1.1	13	29	34	40	140	0.9996
			1.3	4	26	31	38	140	0.9998
			1.5	9	25	30	36	140	0.9986
250	350	200	1.1	27	49	56	64	181	1
			1.3	18	44	51	59	136	1
			1.5	18	40	47	56	119	1
500	700	400	1.1	45	72	81	91	168	1
			1.3	35	64	72	83	173	1
			1.5	24	59	68	78	187	1

Table 2.3: WAMS-charts, Empirical power for stopping time $\hat{\tau}_1$ and WF–1

symmetrical Pareto(5.1), $\alpha = 0.05$									
m	N	h	λ	min	$Q_{.25}$	$Q_{.5}$	$Q_{.75}$	max	power
10	10	8	.95	3	10	10	10	10	0.3230
			.85	2	7	10	10	10	0.5666
			.75	1	6	9	10	10	0.6250
50	70	35	.95	12	23	27	31	70	0.9920
			.85	8	17	21	26	70	0.9890
			.75	6	16	19	24	70	0.9894
100	140	80	.95	19	35	40	46	140	0.9992
			.85	11	26	31	37	140	0.9994
			.75	7	24	29	36	140	0.9992
250	350	200	.95	31	52	58	66	132	1
			.85	20	40	48	55	125	1
			.75	18	38	45	53	127	1
500	700	400	.95	37	69	77	88	155	1
			.85	26	57	66	76	145	1
			.75	20	54	63	74	167	1

Table 2.4: WAMS-charts, Empirical power for stopping time $\hat{\tau}_1$ and WF–2

Discussion

Tables 2.1 and 2.2 clearly show that the empirical sizes reduce, if we put more weight on past observations. We furthermore see that the empirical sizes also decrease, if the size of the monitoring period increases. However, for the considered values of m, even if we put rather heavy weights on the past observations the test keeps the nominal size only if $N < 2m$. Better proportions seem to be only possible for larger training periods.

Under the alternative we see that the delay times increase, if we put higher weights on past observations.

A commendation which one of the weight functions is preferable is not possible. For finite samples both functions may be adjusted via the parameters γ and λ, respectively, so that more ore less weight lies on the past observations.

We finally give a possible explanation why, in case of estimated parameters, the empirical sizes do exceed the nominal size for rather small monitoring periods (compared to the training periods). To this end we consider the detectors of the MOSUM-chart for known and unknown μ, namely

$$M_k = \sum_{i=k-h+1}^{k} \varepsilon_{i+h} \quad \text{and} \quad \hat{M}_k = \sum_{i=k-h+1}^{k} \varepsilon_{i+m} - \frac{h}{m} \sum_{j=1}^{m} \varepsilon_j .$$

If the innovations are for example independent, standard normal variables, the corresponding variances are given by

$$h \quad \text{and} \quad h + \frac{h^2}{m} = h\left(1 + \frac{h}{m}\right).$$

Remember that the application of extreme value theory required the standardization of the detectors. Since the asymptotic results for $\{\hat{M}_k\}$ have been traced back to the results for $\{M_k\}$, each $\hat{M}_k$ is standardized by $\sqrt{h}$, even though the real standard deviation is given by $(h\,(1 + h/m))^{1/2}$. Now we have a dilemma. Obviously, h/m should be small, but since in a real setting m is fixed, this goal can only be achieved by reducing the window size h. However, for fixed m the simulations for the MOSUM-chart clearly showed that the empirical sizes increase, as the window size decreases. A solution for this problem may be to consider boundary functions, which take the standard deviations of the detectors for estimated parameters into account. This will be done in the next chapter for polynomially weighted moving averages.

2.1.6 Proofs

First, we introduce a process that will be needed frequently. We extend the approximating Wiener process in the usual way to a two-sided Wiener process denoted by $\{W(t), -\infty < t < \infty\}$ and define $\{V_{t,N}, -\infty < t < \infty\}$ by

$$V_{t,N} = \frac{1}{\sqrt{h_N}} \sigma(W(t+h_N) - W(t)) \quad \text{for all} \quad -\infty < t < \infty. \tag{2.1.40}$$

It is clear that the following proofs are concerned with triangular arrays, however, this will be only pointed out if needed.

Proof of Theorem 2.1.1

We define the sequence $\{Q_{k,N}\}_{k=1,\ldots,N}$ by

$$Q_{k,N} = \sum_{j=0}^{k-1} w_j V_{k-j,N} \quad \text{for all} \quad k = 1, \ldots, N. \tag{2.1.41}$$

Lemma 2.1.1 *Let the sequences $\{A_{k,N}\}_{k=1,\ldots,N}$ and $\{Q_{k,N}\}_{k=1,\ldots,N}$ be given by* (2.1.7) *and* (2.1.41)*, respectively. σ_N is determined by* (2.1.14)*. Then it holds that*

$$a_N \left(\max_{1\le k\le N} \frac{A_{k,N}}{\sigma_N} - \max_{1\le k\le N} \frac{Q_{k,N}}{\sigma_N} \right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad N \to \infty. \tag{2.1.42}$$

PROOF: We have

$$\begin{aligned} \frac{a_N}{\sigma_N} \max_{1\le k\le N} |A_{k,N} - Q_{k,N}| &= \frac{a_N}{\sigma_N} \max_{1\le k\le N} \left| \sum_{j=0}^{k-1} w_j (B_{k-j,N} - V_{k-j,N}) \right| \\ &\le \frac{a_N}{\sigma_N} \sum_{j=0}^{\infty} w_j \max_{1\le k\le N} |B_{k,N} - V_{k,N}| \\ &\le \frac{a_N}{\sigma_N} \max_{1\le k\le N} |B_{k,N} - V_{k,N}|. \end{aligned}$$

The weak approximation (2.1.2) implies that

$$a_N \max_{1\le k\le N} |B_{k,N} - V_{k,N}| = \boldsymbol{O}_P \left(a_N \frac{(N+h_N)^{1/\nu}}{\sqrt{h_N}} \right) \quad \text{as} \quad N \to \infty$$

and since $\sigma_N \to \sigma$ as $N \to \infty$, (2.1.42) follows by (2.1.4).

□

Next, we approximate $\{Q_{k,N}\}_{k=1,\ldots,N}$ by the continuous-time process $\{Q_N(t)\}_{0\le t\le N}$, defined as

$$Q_N(t) = \sum_{j=0}^{\lfloor t\rfloor -1} w_j V_{t-j,N} \quad \text{for all} \quad 0 \le t \le N. \tag{2.1.43}$$

Lemma 2.1.2 *Let the sequence* $\{Q_{k,N}\}_{k=1,\dots,N}$ *be given by* (2.1.41) *and define the process* $\{Q_N(t)\}_{0\le t\le N}$ *as in* (2.1.43). *Then*

$$a_N \left(\sup_{1\le t\le N} \frac{Q_{\lfloor t\rfloor,N}}{\sigma_N} - \sup_{1\le t\le N} \frac{Q_N(t)}{\sigma_N} \right) = \boldsymbol{o}_P(1) \quad \text{as} \quad N\to\infty. \tag{2.1.44}$$

Proof: Obviously,

$$\begin{aligned} \frac{a_N}{\sigma_N} \sup_{1\le t\le N} \left| Q_{\lfloor t\rfloor,N} - Q_N(t) \right| &\le \frac{a_N}{\sigma_N} \sup_{1\le t\le N} \sum_{j=0}^{\lfloor t\rfloor -1} w_j \left| V_{t-j,N} - V_{\lfloor t\rfloor -j,N} \right| \\ &\le \frac{a_N}{\sigma_N} \sup_{1\le t\le N} \left| V_{t,N} - V_{\lfloor t\rfloor,N} \right| \\ &\le 2\, \frac{\sigma a_N}{\sigma_N} \sup_{1\le t\le N+h_N} \frac{|W(t)-W(\lfloor t\rfloor)|}{\sqrt{h_N}}. \end{aligned}$$

Now Theorem 1.2.1 of Csörgő and Révész (1981) shows that

$$\sup_{1\le t\le N+h_N} |W(t)-W(\lfloor t\rfloor)| = \boldsymbol{O}\left(\sqrt{\log(N+h_N)}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty$$

and since $\sigma_N \to \sigma$ as $N\to\infty$, the Lemma follows by (2.1.4).

□

The next step is to introduce a stationary extension of $\{Q_N(t)\}_{0\le t\le N}$. To this end we define the process $\{U_N(t)\}_{0\le t\le N}$ as

$$U_N(t) = \sum_{j=0}^{\infty} w_j V_{t-j,N} \quad \text{for all} \quad 0\le t\le N. \tag{2.1.45}$$

Lemma 2.1.3 *Let the processes* $\{Q_N(t)\}_{0\le t\le N}$ *and* $\{U_N(t)\}_{0\le t\le N}$ *be defined as in* (2.1.43) *and* (2.1.45), *respectively. Then*

$$a_N \left(\sup_{1\le t\le N} \frac{U_N(t)}{\sigma_N} - \sup_{1\le t\le N} \frac{Q_N(t)}{\sigma_N} \right) = \boldsymbol{o}_P(1) \quad \text{as} \quad N\to\infty. \tag{2.1.46}$$

Proof: We first show that for some suitable chosen $\tilde N$ it holds that

$$a_N \sup_{1\le t\le \tilde N} \frac{Q_N(t)}{\sigma_N} - b_N \xrightarrow{\text{P}} -\infty \quad \text{as} \quad N\to\infty. \tag{2.1.47}$$

If we set $\theta = 2\phi + (1-2\phi)\gamma$ for some $0<\gamma<1$, then we see by (2.1.4) that

$$\frac{N^\theta}{h_N} \simeq \left(\frac{N}{h_N}\right)^\gamma \quad \text{as} \quad N\to\infty. \tag{2.1.48}$$

Now

$$\sup_{1\le t\le N^\theta} Q_N(t) \le \sup_{1\le t\le N^\theta} \left| \sum_{j=0}^{\lfloor t\rfloor -1} w_j V_{t-j,N} \right| \le \sum_{j=0}^{\infty} w_j \sup_{1\le t\le N^\theta} |V_{t,N}| = \sup_{1\le t\le N^\theta} |V_{t,N}|$$

and from Theorem 1.2.1 of Csörgő and Révész (1981) it follows that

$$\limsup_{N\to\infty} \left(2 \left(\log \frac{N^\theta + h_N}{h_N} + \log\log(N^\theta + h_N) \right) \right)^{-1/2} \sup_{1\le t\le N^\theta} |V_{t,N}| \le \sigma \quad \text{a.s.}$$

Since obviously

$$\frac{N^\theta + h_N}{h_N} \simeq \left(\frac{N}{h_N}\right)^\gamma \quad \text{as} \quad N\to\infty$$

this implies that for all $\epsilon > 0$ it holds

$$\mathrm{P}\left(\sup_{1\le t\le N^\theta} |V_{k,N}| \ge \left(2\gamma \log \frac{N}{h_N}\right)^{1/2} (1+\epsilon) \right) \to 0 \quad \text{as} \quad N\to\infty,$$

which together with the definitions of a_N, b_N and σ_N shows that for all $\tilde{N} \lesssim N^\theta$ $(N\to\infty)$ assertion (2.1.47) holds. A similar statement for $\{U_N(t)\}_{0\le t\le N}$ is given in Lemma 2.1.5 below.

Now, in view of (2.1.47) and the corresponding result for $\{U_N(t)\}_{0\le t\le N}$ it suffices to show (2.1.46) on a truncated range for the suprema. Without loss of generality we assume that $\tilde{N}$ is integer-valued. We have

$$\begin{aligned} \frac{a_N}{\sigma_N} \sup_{\tilde{N}\le t\le N} |U_N(t) - Q_N(t)| &= \frac{a_N}{\sigma_N} \sup_{\tilde{N}\le t\le N} \left| \sum_{j=\lfloor t\rfloor}^{\infty} w_j V_{t-j} \right| \\ &\le \frac{a_N}{\sigma_N} \sum_{j=\tilde{N}}^{\infty} w_j \sup_{0\le s<1} \left| V_{\tilde{N}-j-s} \right|. \end{aligned} \tag{2.1.49}$$

Note that

$$\sup_{0\le s<1} \left| V_{\tilde{N}-j-s} \right| \overset{\mathrm{D}}{=} \sup_{0\le s<1} |V_{-s}| \quad \text{for all} \quad j\ge \tilde{N} \tag{2.1.50}$$

and

$$\begin{aligned} \sup_{0\le s<1} |V_{-s}| &= \sup_{0\le s<1} \frac{|W(h_N - s) - W(-s)|}{\sqrt{h_N}} \\ &\le \sup_{0\le s<1} \frac{|W(h_N - s)|}{\sqrt{h_N}} + \sup_{0\le s<1} \frac{|W(-s)|}{\sqrt{h_N}} \\ &\le \sup_{0\le t\le h_N} \frac{|W(t)|}{\sqrt{h_N}} + \sup_{0\le s<1} \frac{|W(-s)|}{\sqrt{h_N}} \\ &=: I_1(N) + I_2(N). \end{aligned}$$

For some suitable chosen $C_1 > 0$ independent of N (cf. Karatzas and Shreve, Chapter 2.8, 1988) it holds that

$$\mathrm{E}\, I_1(N) \leq C_1 \tag{2.1.51}$$

and

$$\sqrt{h_N}\, \mathrm{E}\, I_2(N) \leq C_1, \tag{2.1.52}$$

hence, on combining (2.1.50)–(2.1.52) we see that

$$\mathrm{E} \sup_{0 \leq s < 1} \left| V_{\tilde{N}-j-s} \right| \leq 2\, C_1 \quad \text{for all} \quad j \geq \tilde{N}$$

and

$$\mathrm{E} \left(\sum_{j=\tilde{N}}^{\infty} w_j \sup_{0 \leq s < 1} \left| V_{\tilde{N}-j-s} \right| \right) \leq 2\, C_1 \cdot \sum_{j=\tilde{N}}^{\infty} w_j. \tag{2.1.53}$$

Now (2.1.11) and the definition of $\tilde{N}$ yield the existence of some $\epsilon > 0$, such that

$$\sum_{j=\tilde{N}}^{\infty} w_j = \boldsymbol{O}\left(\frac{1}{N^{\epsilon}} \right) \quad \text{as} \quad N \to \infty \tag{2.1.54}$$

and the lemma follows by (2.1.22), (2.1.49), (2.1.53), (2.1.54), and the Markov-inequality.

□

Finally, we consider the extremes of $\{U_N(t)\}_{0 \leq t \leq N}$.

Lemma 2.1.4 *Let the process* $\{U_N(t)\}_{0 \leq t \leq N}$ *be given by* (2.1.45). *For all real* x *it holds that*

$$\lim_{N \to \infty} \mathrm{P} \left(a_N \sup_{1 \leq t \leq N} \frac{U_N(t)}{\sigma_N} - b_N \leq x \right) = \exp(-e^{-x}). \tag{2.1.55}$$

PROOF: We define the processes $\{\tilde{U}_N(\tilde{t}\,)\}_{\tilde{t} \geq 0}$ as

$$\tilde{U}_N(\tilde{t}\,) := U_N(\tilde{t} h_N) \quad \text{for all} \quad \tilde{t} \geq 0. \tag{2.1.56}$$

Obviously, it holds that

$$\sup_{1 \leq t \leq N} \frac{U_N(t)}{\sigma_N} = \sup_{1/h_N \leq \tilde{t} \leq N/h_N} \frac{\tilde{U}_N(\tilde{t}\,)}{\sigma_N}$$

and since Lemma A.3.1 shows that the assumptions of Theorem 5.1 in Gut and Steinebach (2004) are satisfied, the lemma follows. □

Lemma 2.1.5 *Let the process* $\{U_N(t)\}_{0\le t\le N}$ *be defined as in* (2.1.45)*. Furthermore, let* $\theta = 2\phi + (1-2\phi)\gamma$ *for some* $0<\gamma<1$*. Then for all real* x *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le t\le N^\theta} \frac{U_N(t)}{\sigma_N} - b_N \le x\right) = 1. \tag{2.1.57}$$

PROOF: In the following, we also denote N^θ by $\tilde{N}$.

Since $2\phi < \theta < 1$ we can find a constant $\tilde{\phi}$ such that $1/\nu < \tilde{\phi} < 1/2$ and

$$h_N \simeq \tilde{N}^{2\tilde{\phi}} \quad \text{as} \quad N\to\infty. \tag{2.1.58}$$

Comparing (2.1.58) with (2.1.4) we see that Lemma 2.1.4 also holds, if we replace N by $\tilde{N}$ in the domain of the supremum. Hence ($\tilde{N}\to\infty$ iff $N\to\infty$)

$$\lim_{N\to\infty} \mathrm{P}\left(a_{\tilde{N}} \max_{1\le t\le \tilde{N}} \frac{U_N(t)}{\sigma_N} - b_{\tilde{N}} \le x\right) = \exp(-e^{-x}), \tag{2.1.59}$$

where

$$a_{\tilde{N}} = \sqrt{2\log\frac{\tilde{N}}{h_N}}$$

and

$$b_{\tilde{N}} = 2\log\frac{\tilde{N}}{h_N} + \frac{1}{2}\log\log\frac{\tilde{N}}{h_N} - \frac{1}{2}\log\pi.$$

Now

$$\mathrm{P}\left(a_N \max_{1\le t\le \tilde{N}} \frac{U_N(t)}{\sigma_N} - b_N \le x\right) = \mathrm{P}\left(a_{\tilde{N}} \max_{1\le t\le \tilde{N}} \frac{U_N(t)}{\sigma_N} - b_{\tilde{N}} \le \frac{a_{\tilde{N}}}{a_N}(x+b_N) - b_{\tilde{N}}\right)$$

and since elementary calculations show that for any real x it holds

$$\frac{a_{\tilde{N}}}{a_N}(x+b_N) - b_{\tilde{N}} \to \infty \quad \text{as} \quad N\to\infty,$$

the lemma follows by (2.1.58).

□

On combining the Lemmas (2.1.1)–(2.1.5) we see that assertion (2.1.20) of Theorem 2.1.1 holds. The symmetry of $\{U_N(t)\}_{0\le t\le N}$ and the asymptotic independence of maxima and minima in the underlying extreme value asymptotic imply (2.1.21) (see also Gut and Steinebach, 2004).

Proof of Theorem 2.1.2

Under the one-sided alternative for all $1 \leq k \leq N$ we have

$$\begin{aligned}
A_{k,N} &= \sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \left(X_{k-j-i+h_N} - \mu\right) \\
&= \sum_{i=0}^{h_N-1} \frac{1}{\sqrt{h_N}} \sum_{j=0}^{k-1} w_j \left(X_{k-j-i+h_N} - \mu\right) \\
&= \sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \varepsilon_{k-j-i+h_N} + \frac{\triangle}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \sum_{j=0}^{k-i-k^*-1} w_j \\
&=: I_1(N) + I_2(N)
\end{aligned}$$

and since for $I_2(N)$ it holds that

$$\begin{aligned}
\frac{\triangle}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \sum_{j=0}^{k-i-k^*-1} w_j =& \frac{\triangle}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \left(\sum_{j=0}^{k-h_N-k^*-1} w_j + \sum_{j=k-h_N-k^*}^{k-i-k^*-1} w_j \right) \\
=& \sqrt{h_N}\, \triangle \sum_{j=0}^{k-h_N-k^*-1} w_j + \frac{\triangle}{\sqrt{h_N}} \sum_{j=0}^{h_N-1} (h_N - j) w_{k-h_N-k^*+j} \\
\geq& \sqrt{h_N}\, \triangle \sum_{j=0}^{k-h_N-k^*-1} w_j,
\end{aligned}$$

the assumption $k^* < N - h_N$ implies

$$\begin{aligned}
\max_{1\leq k\leq N} \frac{A_{k,N}}{\sigma_N} \geq& \frac{\sqrt{h_N}\, \triangle}{\sigma_N} \sum_{j=0}^{N-h_N-k^*-1} w_j - \max_{1\leq k\leq N} \left| \frac{1}{\sigma_N} \sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \varepsilon_{k-j-i+h_N} \right| \\
\geq& \frac{\sqrt{h_N}\, \triangle}{\sigma_N} w_0 - \max_{h_N\leq k\leq N} \left| \frac{1}{\sigma_N} \sum_{j=0}^{k-h_N} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \varepsilon_{k-j-i} \right|.
\end{aligned}$$

Now (2.1.21) shows that

$$\max_{1\leq k\leq N} \left| \frac{1}{\sigma_N} \sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \varepsilon_{k-j-i+h_N} \right| = \boldsymbol{O}_P\left(\sqrt{\log \frac{N}{h_N}} \right) \quad \text{as} \quad N \to \infty$$

and assertion (2.1.24) follows, since for all real x

$$\begin{aligned}
&\mathrm{P}\left(a_N \max_{1\leq k\leq N} \frac{A_{k,N}}{\sigma_N} - b_N \leq x \right) \\
&\qquad = \mathrm{P}\left(\max_{1\leq k\leq N} \frac{A_{k,N}}{\sigma_N} \leq \frac{x + b_N}{a_N} \right) \\
&\qquad \leq \mathrm{P}\left(\frac{\sqrt{h_N}\, \triangle}{\sigma_N} w_0 - \max_{1\leq k\leq N} \left| \frac{1}{\sigma_N} \sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} \varepsilon_{k-j-i} \right| \leq \frac{x + b_N}{a_N} \right)
\end{aligned}$$

and

$$\frac{x+b_N}{a_N} = \boldsymbol{O}\left(\sqrt{\log\frac{N}{h_N}}\right) \quad \text{as} \quad N\to\infty.$$

Similar arguments show (2.1.25), hence the proof of Theorem 2.1.2 is complete.

Proof of Theorem 2.1.3

Note that we already know by Lemma 1.1.3 that

$$\hat{\mu}_{m_N} - \mu = \boldsymbol{O}_P\left(\frac{1}{\sqrt{m_N}}\right) \quad \text{as} \quad N\to\infty \tag{2.1.60}$$

and

$$\hat{\sigma}^2_{m_N} - \sigma^2 = \boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta}}\right) \quad \text{as} \quad N\to\infty \tag{2.1.61}$$

for some $\vartheta > 0$.

The proof of Theorem 2.1.3 is traced back to the proof of Theorem 2.1.1, but since the training period m_N does not coincide with the window size h_N, we need some preliminaries. We define

$$\tilde{A}_{k,N} = \sum_{j=0}^{k-1} w_j \tilde{B}_{k-j,N} \quad \text{for all} \quad k=1,2,\ldots,N, \tag{2.1.62}$$

where

$$\tilde{B}_{k,N} = \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} (X_{k-i+m_N} - \mu) \quad \text{for all} \quad k=1,2,\ldots,N. \tag{2.1.63}$$

It follows immediately by (2.1.27) and (2.1.28) that the results of Theorem 2.1.1 also hold for the sequence $\{\tilde{A}_{k,N}\}_{k=1,\ldots,N}$.

Lemma 2.1.6 *Let the sequences* $\{\tilde{A}_{k,N}\}_{k=1,\ldots,N}$ *and* $\{\hat{A}_{k,N}\}_{k=1,\ldots,N}$ *be defined as in* (2.1.62) *and* (2.1.30)*, respectively. Then it holds that*

$$a_N\left(\max_{1\le k\le N}\frac{\tilde{A}_{k,N}}{\sigma_N} - \max_{1\le k\le N}\frac{\hat{A}_{k,N}}{\sigma_N}\right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad N\to\infty. \tag{2.1.64}$$

Proof: By (2.1.60) it follows that

$$\begin{aligned}
\frac{a_N}{\sigma_N}\max_{1\le k\le N}\left|\tilde{A}_{k,N} - \hat{A}_{k,N}\right| &= \frac{a_N}{\sigma_N}\max_{1\le k\le N}\left|\sum_{j=0}^{k-1} w_j \frac{1}{\sqrt{h_N}}\sum_{i=0}^{h_N-1}(\hat{\mu}_{m_N}-\mu)\right| \\
&\le \frac{a_N}{\sigma_N}\sqrt{h_N}\,|\,\hat{\mu}_{m_N}-\mu| \\
&= \frac{a_N}{\sigma_N}\frac{\sqrt{h_N}}{\sqrt{m_N}}\,\boldsymbol{O}_P(1) \quad \text{as} \quad N\to\infty
\end{aligned}$$

and since $\sigma_N \to \sigma$ $(N\to\infty)$, (2.1.28) implies the lemma.

□

Next, we replace σ_N by $\hat{\sigma}_N$.

Lemma 2.1.7 *Let the sequence* $\{\hat{A}_{k,N}\}_{k=1,\ldots,N}$ *be defined as in* (2.1.30)*. Furthermore,* σ_N *and* $\hat{\sigma}_N$ *are defined via* (2.1.14) *and* (2.1.33)*, respectively. Then it holds that*

$$a_N \left(\max_{1\le k\le N} \frac{\hat{A}_{k,N}}{\sigma_N} - \max_{1\le k\le N} \frac{\hat{A}_{k,N}}{\hat{\sigma}_N} \right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad N\to\infty. \tag{2.1.65}$$

Proof: Since Theorem 2.1.1 together with Lemma 2.1.6 implies

$$\max_{1\le k\le N} \frac{\hat{A}_{k,N}}{\sigma_N} = \boldsymbol{O}_P\left(\sqrt{\log\frac{N}{h_N}}\right) \quad \text{as} \quad N\to\infty$$

and (2.1.61) gives

$$\frac{\sigma_N}{\hat{\sigma}_N} = 1 + \boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta}}\right) \quad \text{as} \quad N\to\infty, \quad \text{for some} \quad \vartheta>0,$$

we get

$$a_N\frac{\hat{A}_{k,N}}{\hat{\sigma}_N} = a_N\frac{\hat{A}_{k,N}}{\sigma_N} + \boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta}}\log\frac{N}{h_N}\right) \quad \text{as} \quad N\to\infty,$$

uniformly in $k=1,2,\ldots,N$. The lemma now follows by (2.1.4) and (2.1.28).

□

Obviously, Theorem 2.1.3 now follows on combining Lemma 2.1.6 and Lemma 2.1.7.

Proof of Theorem 2.1.4

The proof of Theorem 2.1.4 follows by the same proof steps as Theorem 2.1.2, hence we omit the details. We only mention that (2.1.61) implies

$$\frac{\sqrt{h_N}\,\triangle}{\hat{\sigma}_N} = \frac{\sqrt{h_N}\,\triangle}{\sigma_N} + \boldsymbol{o}_P\left(\frac{\sqrt{h_N}}{m_N^{\vartheta}}\right) \quad \text{as} \quad N\to\infty.$$

and

$$\frac{\sqrt{h_N}}{m_N^{\vartheta}} = \boldsymbol{O}\left(h_N^{1/2-\epsilon}\right) \quad \text{as} \quad N\to\infty \quad \text{for some} \quad \epsilon>0.$$

Chapter 3

Control charts based on polynomially weighted moving averages

Following the idea of exponential smoothing, in this chapter we introduce a control chart based on polynomially weighted moving averages (PWMA), which allows for a flexible adjustment of the weights that are assigned to recent observations.

We first consider the PWMA-chart for a closed-end setting and then extend the results to an open time horizon. To this end we derive the asymptotic boundary crossing probabilities of the Wiener process for a new class of boundary functions.

3.1 Closed-end control charts

3.1.1 Model assumptions for known μ and σ

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P}\,)$ with

$$\mathrm{E}\,\varepsilon_i = 0 \quad \text{and} \quad \mathrm{Var}\,\varepsilon_i = \sigma^2 > 0 \quad \text{for all} \quad i = 1, 2, \ldots \tag{3.1.1}$$

We assume that there exists a Wiener process $\{W(t), t \geq 0\}$ satisfying

$$\sup_{1 \leq k < \infty} \frac{1}{k^{1/\nu}} \left| \sum_{i=1}^{k} \varepsilon_i - \sigma W(k) \right| < \infty \quad \text{a.s.} \tag{3.1.2}$$

for some $\nu > 2$.

We are interested in monitoring a discrete-time stochastic process $\{X_i\}_{i=1,\ldots,N}$, which is assumed to follow the model

$$X_i = \begin{cases} \mu + \varepsilon_i & : 1 \leq i \leq k^*, \\ \mu + \Delta + \varepsilon_i & : k^* < i \leq N, \end{cases} \tag{3.1.3}$$

where μ is the in-control mean, k^* is the unknown time of a possible change and $\triangle$ is the size of the level shift.

The hypotheses which are tested sequentially can be stated as

$$H_0 : k^* = N \quad \text{versus} \quad H_1 : k^* < N,\ \triangle > 0 \quad \text{(one-sided alternative)}, \tag{3.1.4}$$

or

$$H_0 : k^* = N \quad \text{versus} \quad H_2 : k^* < N,\ \triangle \neq 0 \quad \text{(two-sided alternative)}. \tag{3.1.5}$$

3.1.2 Monitoring procedures for known μ and σ

Let q be a monomial of degree $d \in \mathbb{N}_0$. If we define

$$p_{j,k} = q\left(\frac{j}{k}\right) = \left(\frac{j}{k}\right)^d \quad \text{for all} \quad j = 0, 1, \ldots, k, \quad k = 1, 2, \ldots \tag{3.1.6}$$

and

$$p(x,t) = q\left(\frac{x}{t}\right) = \left(\frac{x}{t}\right)^d \quad \text{for all} \quad 0 \leq x \leq t, \quad t > 0, \tag{3.1.7}$$

the sequence of detectors is given by

$$P_k = \sum_{j=1}^{k} p_{j,k}(X_j - \mu), \quad k = 1 \ldots, N. \tag{3.1.8}$$

Remark 3.1.1 *The proofs show that any function $f(x) = x^\gamma$, $\gamma \geq 0$, $x \geq 0$ may be used to define the weights. Monomials are chosen with regard to an extension of the model to wider classes of polynomials.*

The variances of the detectors can be conveniently approximated by

$$\sigma_t^2 = \sigma^2 \int_0^t p^2(x,t)dx = \sigma^2 \int_0^t \left(\frac{x}{t}\right)^{2d} dx = \frac{\sigma^2}{2d+1}\, t, \quad t \geq 0. \tag{3.1.9}$$

If we test the null hypotheses versus the one-sided alternative ($\alpha \in]0,1[$), we stop monitoring at $\min\{\tau_1, N\}$, where

$$\tau_1 = \tau_1(\alpha, N) = \inf\{1 \leq k \leq N : P_k > c_1(\alpha, N)\sigma_k\} \tag{3.1.10}$$

and reject H_0, if $\tau_1 \leq N$ ($\inf \emptyset = \infty$). If the alternative is two-sided, we replace τ_1 by τ_2, where

$$\tau_2 = \tau_2(\alpha, N) = \inf\{1 \leq k \leq N : |P_k| > c_2(\alpha, N)\sigma_k\}. \tag{3.1.11}$$

The critical constants $c_1(\alpha, N)$ and $c_2(\alpha, N)$ are given by

$$c_1(\alpha, N) = \frac{q_1(1-\alpha) + b_N}{a_N} \quad \text{and} \quad c_2(\alpha, N) = \frac{q_2(1-\alpha) + b_N}{a_N}, \tag{3.1.12}$$

where a_N and b_N are defined in (3.1.16) and (3.1.17) below. Moreover,

$$q_1(1-\alpha) = -\log(-\log(1-\alpha)) \ \text{ and } \ q_2(1-\alpha) = -\log\left(-\frac{1}{2}\log(1-\alpha)\right). \tag{3.1.13}$$

The choice of the critical constants is well-founded, since the following theorem implies that the false alarm rate of the procedures converges to α as $N \to \infty$.

Theorem 3.1.1 *Let the sequence* $\{P_k\}_{k=1,\ldots,N}$ *be defined as in* (3.1.8)*. The sequence* $\{\sigma_k\}_{k=1,\ldots,N}$ *is defined via* (3.1.9)*. Then, it holds under* H_0 *and for all real* x *that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{P_k}{\sigma_k} - b_N \le x\right) = \exp\left(-e^{-x}\right) \tag{3.1.14}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|P_k|}{\sigma_k} - b_N \le x\right) = \exp\left(-2e^{-x}\right), \tag{3.1.15}$$

where

$$a_N = \left(2\log\log N^{2d+1}\right)^{1/2} \tag{3.1.16}$$

and

$$b_N = 2\log\log N^{2d+1} + \frac{1}{2}\log\log\log N^{2d+1} - \frac{1}{2}\log\pi. \tag{3.1.17}$$

Furthermore, the procedures have asymptotic power one, if the change is not located too close to the end of the monitored period N.

Theorem 3.1.2 *Let the sequence* $\{P_k\}_{k=1,\ldots,N}$ *and* $\{\sigma_k\}_{k=1,\ldots,N}$ *be defined as in Theorem 3.1.1. If* $k^* = k^*(N) \le N - N^\rho$*, where* $1/2 < \rho < 1$*, then, it holds under* H_1 *and for all real* x *that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{P_k}{\sigma_k} - b_N > x\right) = 1 \tag{3.1.18}$$

and under H_2

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|P_k|}{\sigma_k} - b_N > x\right) = 1, \tag{3.1.19}$$

where a_N *and* b_N *are chosen as in* (3.1.16) *and* (3.1.17)*, respectively.*

3.1.3 Model assumptions for unknown μ and σ

Suppose that (3.1.1) holds. As in the previous chapters, the estimators for the in-control parameters are based on a historical data set (training period) of size m, which is assumed to be homogeneous.

Monitoring starts after the training period and the length N of the monitoring period depends on m as follows

$$N \geq cm \quad \text{for some} \quad c > 1 \quad \text{and} \quad N \simeq m^{\lambda}\ (m \to \infty) \quad \text{for some} \quad \lambda \geq 1. \tag{3.1.20}$$

Instead of (3.1.2), we now claim that there exist two sequences of Wiener processes $\{W_{1,m}(t), t \geq 0\}_{m=1,2,\ldots}$ and $\{W_{2,m}(t), t \geq 0\}_{m=1,2,\ldots}$ satisfying

$$\sup_{1 \leq k < \infty} \frac{1}{k^{1/\nu}} \left| \sum_{i=1}^{k} \varepsilon_{m+i} - \sigma W_{1,m}(k) \right| = \boldsymbol{O}_P(1) \quad \text{as} \quad m \to \infty \tag{3.1.21}$$

and

$$\sum_{i=1}^{m} \varepsilon_i - \sigma W_{2,m}(m) = \boldsymbol{O}_P(m^{1/\nu}) \quad \text{as} \quad m \to \infty, \tag{3.1.22}$$

respectively. We mention that, for fixed m, it is not required that $\{W_{1,m}(t), t \geq 0\}$ and $\{W_{2,m}(t), t \geq 0\}$ are independent.

The process $\{X_i\}_{i=1,\ldots,m+N}$ is modeled as

$$X_i = \begin{cases} \mu + \varepsilon_i & : 1 \leq i \leq m + k^*, \\ \mu + \Delta + \varepsilon_i & : m + k^* < i \leq m + N \end{cases} \tag{3.1.23}$$

and we are interested in testing

$$H_0 : k^* = N \quad \text{versus} \quad H_1 : 0 \leq k^* < N,\ \Delta > 0 \quad \text{(one-sided alternative)}, \tag{3.1.24}$$

or

$$H_0 : k^* = N \quad \text{versus} \quad H_2 : 0 \leq k^* < N,\ \Delta \neq 0 \quad \text{(two-sided alternative)}. \tag{3.1.25}$$

3.1.4 Monitoring procedures for unknown μ and σ

Replacing μ by the estimator

$$\hat{\mu}_m = \frac{1}{m} \sum_{j=1}^{m} X_j, \quad m = 1, 2, \ldots \tag{3.1.26}$$

yields the sequence of detectors

$$\hat{P}_{m,k} = \sum_{j=1}^{k} p_{j,k}(X_{m+j} - \hat{\mu}_m) = \sum_{j=1}^{k} p_{j,k}\varepsilon_{m+j} - \frac{\sum_{j=1}^{k} p_{j,k}}{m} \sum_{i=1}^{m} \varepsilon_i, \quad k = 1, \ldots, N. \tag{3.1.27}$$

The proof of Lemma 1.1.3, together with (3.1.20), shows that there exists an estimator $\hat{\sigma}_m$ for σ, which is based on the observations that are obtained within the training period and satisfies

$$\hat{\sigma}_m - \sigma = \boldsymbol{o}_P\left(\frac{1}{m^{\vartheta}}\right) \quad \text{as} \quad m \to \infty \quad \text{for some} \quad \vartheta > 0. \tag{3.1.28}$$

With

$$g_c(m,k) = \sqrt{m}\sqrt{\left(\frac{1}{2d+1}\frac{k}{m}\right)\left(1 + \frac{2d+1}{(d+1)^2}\frac{k}{m}\right)} \quad k = 1, \ldots, N \tag{3.1.29}$$

(c stands for 'closed-end'),we define

$$\hat{\sigma}_{m,k} = \hat{\sigma}_m\sqrt{m}\sqrt{\left(\frac{1}{2d+1}\frac{k}{m}\right)\left(1 + \frac{2d+1}{(d+1)^2}\frac{k}{m}\right)}, \quad k = 1, \ldots, N. \tag{3.1.30}$$

Depending on the alternatives, the stopping times are

$$\hat{\tau}_1 = \hat{\tau}_1(\alpha, m) = \inf\{1 \le k \le N : \hat{P}_{m,k} > \hat{c}_1(\alpha, m)\hat{\sigma}_{m,k}\} \tag{3.1.31}$$

and

$$\hat{\tau}_2 = \hat{\tau}_2(\alpha, m) = \inf\{1 \le k \le N : |\hat{P}_{m,k}| > \hat{c}_2(\alpha, m)\hat{\sigma}_{m,k}\}, \tag{3.1.32}$$

where $\hat{c}_1(\alpha, m)$ and $\hat{c}_2(\alpha, m)$ are given by

$$\hat{c}_1(\alpha, m) = \frac{q_1(1-\alpha) + b_m}{a_m} \quad \text{and} \quad \hat{c}_2(\alpha, m) = \frac{q_2(1-\alpha) + b_m}{a_m}. \tag{3.1.33}$$

The quantiles q_1 and q_2 are chosen as in (3.1.13) and the definition of a_m and b_m is given in (3.1.36) and (3.1.37) below.

As in case of known parameters, the false alarm rate of the procedures is asymptotically α ($m \to \infty$) and the asymptotic power is 1, if the change occurs not too late. This can be seen by the following two theorems.

Theorem 3.1.3 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,\ldots,N}$ *and* $\{\hat{\sigma}_{m,k}\}_{k=1,\ldots,N}$ *be defined as in* (3.1.27) *and* (3.1.30)*, respectively. Then, it holds under* H_0 *and for all real* x *that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le N} \frac{\hat{P}_k}{\hat{\sigma}_{m,k}} - b_m \le x\right) = \exp\left(-e^{-x}\right) \tag{3.1.34}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le N} \frac{|\hat{P}_k|}{\hat{\sigma}_{m,k}} - b_m \le x\right) = \exp\left(-2e^{-x}\right), \tag{3.1.35}$$

where

$$a_m = \left(2\log\log m^{2d+1}\right)^{1/2} \tag{3.1.36}$$

and

$$b_m = 2\log\log m^{2d+1} + \frac{1}{2}\,\log\log\log m^{2d+1} - \frac{1}{2}\log\pi. \tag{3.1.37}$$

Theorem 3.1.4 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,\ldots,N}$ *and* $\{\hat{\sigma}_{m,k}\}_{k=1,\ldots,N}$ *be defined as in Theorem 3.1.3. If* $k^* = k^*(N) \le N - N^{\varrho}$*, where* $1-(1/2\lambda) < \varrho < 1$ *and* λ *is given in* (3.1.20)*, then, under* H_1 *for all real* x *holds*

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le N} \frac{\hat{P}_{m,k}}{\hat{\sigma}_{m,k}} - b_m > x\right) = 1 \tag{3.1.38}$$

and under H_2

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le N} \frac{|\hat{P}_{m,k}|}{\hat{\sigma}_{m,k}} - b_m > x\right) = 1, \tag{3.1.39}$$

where a_m *and* b_m *are chosen as in* (3.1.36) *and* (3.1.37)*, respectively.*

3.1.5 Simulations and discussion

We now investigate the finite sample properties of the PWMA-chart in a simulation study.

We focus on the one-sided stopping rule based on stopping time $\hat{\tau}_1$ and the simulation is carried out for the symmetric Pareto(5.1) variables, which are specified in the simulation part of Chapter 1. As mentioned in Remark 3.1.1 the degree d has not to be integer-valued and we take into account $d = 0.0, 0.5, 1.0, 1.5, 2.0$.

Since we are also interested in the effect of the curved boundary functions on the delay time (i.e. $\hat{\tau}_1 - k^*$), we consider structural breaks in $k^* = 0$ and $k^* = N/2$, where $N = 1000$ and $\triangle = 1$. The empirical delay times are described by the associated $.25-$, $.5-$ and $.75-$ quantiles and stated together with the empirical power.

All values in the following tables are based on $5,000$ replications and the nominal size has been chosen as $\alpha = 0.05$.

Empirical Sizes

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$						
m	N	$d = 0.0$	$d = 0.5$	$d = 1.0$	$d = 1.5$	$d = 2.0$
10	100	0.0734	0.0836	0.1068	0.1148	0.1420
	500	0.0704	0.0880	0.1056	0.1254	0.1376
	1000	0.0748	0.0874	0.1104	0.1264	0.1394
	2500	0.0710	0.0858	0.1018	0.1166	0.1424
	5000	0.0748	0.0866	0.1086	0.1274	0.1386
50	100	0.0410	0.0472	0.0538	0.0712	0.0748
	500	0.0458	0.0490	0.0574	0.0646	0.0778
	1000	0.0390	0.0510	0.0612	0.0736	0.0796
	2500	0.0426	0.0492	0.0560	0.0694	0.0816
	5000	0.0412	0.0420	0.0606	0.0686	0.0810
100	100	0.0324	0.0336	0.0484	0.0624	0.0684
	500	0.0292	0.0390	0.0492	0.0606	0.0660
	1000	0.0364	0.0382	0.0494	0.0586	0.0702
	2500	0.0324	0.0392	0.0502	0.0546	0.0666
	5000	0.0356	0.0406	0.0512	0.0592	0.0710
250	100	0.0264	0.0314	0.0400	0.0466	0.0584
	500	0.0286	0.0322	0.0414	0.0500	0.0614
	1000	0.0354	0.0308	0.0426	0.0514	0.0616
	2500	0.0358	0.0366	0.0422	0.0524	0.0628
	5000	0.0368	0.0382	0.0426	0.0482	0.0680
500	100	0.0300	0.0320	0.0404	0.0502	0.0584
	500	0.0240	0.0300	0.0412	0.0532	0.0576
	1000	0.0338	0.0360	0.0398	0.0496	0.0586
	2500	0.0314	0.0344	0.0376	0.0476	0.0566
	5000	0.0328	0.0332	0.0392	0.0492	0.0596

Table 3.1: Empirical sizes for stopping time $\hat{\tau}_1$

Empirical Power

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$						
m	N	d	$Q_{.25}$	$Q_{.50}$	$Q_{.75}$	power
10	1000	0.0	4	11	1000	0.7462
		0.5	3	15	1000	0.6656
		1.0	3	18	1000	0.6292
		1.0	2	21	1000	0.6070
		2.0	2	26	1000	0.6198
50	1000	0.0	6	10	19	0.9902
		0.5	7	14	29	0.9776
		1.0	7	18	39	0.9664
		1.5	8	22	51	0.9564
		2.0	6	23	59	0.9536
100	1000	0.0	6	10	16	0.9980
		0.5	8	15	24	0.9990
		1.0	9	19	32	0.9970
		1.5	9	21	40	0.9956
		2.0	9	25	46	0.9952
250	1000	0.0	7	11	16	1
		0.5	9	15	23	1
		1.0	11	19	29	1
		1.5	12	23	37	1
		2.0	11	26	42	1
500	1000	0.0	7	11	15	1
		0.5	10	16	22	1
		1.0	12	19	28	1
		1.5	13	24	36	1
		2.0	14	28	41	1

Table 3.2: Empirical power for stopping time $\hat{\tau}_1$, $k^* = 0$

innovations: symmetrical Pareto(5.1), $\alpha = 0.05$						
m	N	d	$Q_{.25}$	$Q_{.50}$	$Q_{.75}$	power
10	1000	0.0	458	1000	1000	0.2674
		0.5	447	1000	1000	0.2686
		1.0	359	1000	1000	0.3092
		1.0	309	1000	1000	0.3342
		2.0	240	1000	1000	0.3744
50	1000	0.0	229	370	1000	0.6674
		0.5	205	317	1000	0.7404
		1.0	167	263	428	0.8034
		1.5	147	228	371	0.8364
		2.0	127	197	335	0.8542
100	1000	0.0	166	239	342	0.9136
		0.5	145	204	281	0.9536
		1.0	123	171	237	0.9724
		1.5	109	151	206	0.9784
		2.0	94	129	180	0.9830
250	1000	0.0	113	154	202	0.9974
		0.5	99	131	166	0.9988
		1.0	87	113	142	0.9990
		1.5	76	98	124	0.9994
		2.0	68	91	113	0.9996
500	1000	0.0	91	122	153	1
		0.5	81	105	129	1
		1.0	72	91	111	1
		1.5	65	82	99	1
		2.0	59	74	90	1

Table 3.3: Empirical power for stopping time $\hat{\tau}_1$, $k^* = 500$

Discussion

In accordance with Lemma 3.1.7, Table 3.1 shows that most of the false alarms are given immediately after the training period. The empirical sizes increase with the degree d, however, for $d = 0$ and $m \geq 50$ they are always below the nominal size. Hence, we can adjust the test to a given nominal size quite accurately by the degree d. Note that for $d = 0.0$ we obtain a CUSUM-type detector.

The simulations under the alternative show that the performance of the test does not only depend on the degree d, but also on the location of the change-point k^* (for fixed d). While in case of an early change $d = 0$ yields the highest empirical power and smallest delay times, in case of a late change the degree should be chosen as high as possible. Tables 3.2 and 3.3 also show that the empirical power increases with the size of the training period.

Summarizing the results, we see that the PWMA-chart provides a flexible tool for change point-analysis, especially for the detection of late changes.

3.1.6 Proofs

In the proofs, sums of the innovations will also be denoted by $S(k) = \sum_{i=1}^{k} \varepsilon_i$.

We mention that in case of estimated parameters we are dealing with triangular schemes, however, this will be only pointed out if needed.

Proof of Theorem 3.1.1

We define the sequence $\{Q_k\}_{k=1,\dots,N}$ by

$$Q_k = \sigma \sum_{j=1}^{k} p_{j,k}(W(j) - W(j-1)) \quad \text{for all} \quad k = 1, \dots, N, \tag{3.1.40}$$

where $\{W(t), t \geq 0\}$ is the approximating Wiener process given in (3.1.2).

Lemma 3.1.1 *Let the sequences $\{P_k\}_{k=1,\dots,N}$ and $\{Q_k\}_{k=1,\dots,N}$ be defined as in* (3.1.8) *and* (3.1.40)*, respectively. If n is a non-decreasing, integer-valued function of N with $1 \leq n \leq N$, then it holds that*

$$\max_{n<k\leq N} \frac{P_k}{\sigma_k} - \max_{n<k\leq N} \frac{Q_k}{\sigma_k} = \boldsymbol{O}_P\left(\frac{1}{n^{1/2-1/\nu}}\right) \quad \textit{as} \quad N \to \infty. \tag{3.1.41}$$

Proof: For all $k = 1, 2, \dots$ we define the sequence $\{d_{j,k}, 1 \leq j \leq k\}$ by

$$d_{j,k} = p_{j,k} - p_{j-1,k}, \quad j = 1, 2, \dots$$

and reformulate the processes $\{P_k\}_{k=1,\dots,N}$ and $\{Q_k\}_{k=1,\dots,N}$ as

$$P_k = \sum_{j=1}^{k} d_{j,k}\,(S(k) - S(j-1)) \quad \text{for all} \quad k = 1, \dots, N \tag{3.1.42}$$

and

$$Q_k = \sigma \sum_{j=1}^{k} d_{j,k}\,(W(k) - W(j-1)) \quad \text{for all} \quad k = 1, \dots, N. \tag{3.1.43}$$

Then

$$\begin{aligned}\max_{n<k\le N}\frac{1}{\sigma_k}|P_k-Q_k| &\le \max_{n<k\le N}\frac{1}{\sigma_k}\sum_{j=1}^{k} d_{j,k}\left(|S(k)-\sigma W(k)|+|S(j-1)-\sigma W(j-1)|\right)\\ &\le 2\max_{n<k\le N}\sum_{j=1}^{k} d_{j,k}\max_{n\le k\le N}\frac{1}{\sigma_k}|S(k)-\sigma W(k)|\\ &= 2\max_{n\le k\le N}\frac{1}{\sigma_k}|S(k)-\sigma W(k)|\end{aligned}$$

and the Lemma follows by (3.1.9) and (3.1.2).

□

The next step is to approximate the sequence $\{Q_k\}_{k=1,\dots,N}$ by the continuous-time process $\{U(t)\}_{t\ge 1}$ defined as

$$U(t)=\sigma\int_0^t p(x,t)\,dW(x)=\sigma\int_0^t\left(\frac{x}{t}\right)^d dW(x)\quad\text{for all}\quad t\ge 1. \tag{3.1.44}$$

Lemma 3.1.2 *Let the sequence $\{Q_k\}_{k=1,\dots,N}$ and the process $\{U(t)\}_{t\ge 1}$ be defined as in* (3.1.40) *and* (3.1.44)*, respectively. If $\tilde t$ is a non-decreasing function of N with $1\le\tilde t\le N$, then*

$$\sup_{\tilde t\le t\le N}\frac{U(t)}{\sigma_t}-\sup_{\tilde t\le t\le N}\frac{Q_{\lfloor t\rfloor}}{\sigma_{\lfloor t\rfloor}}=\boldsymbol{O}_P\left(\left(\frac{\log N}{\tilde t}\right)^{1/2}\right)\quad as\quad N\to\infty. \tag{3.1.45}$$

Proof: Integration by parts yields for all $t\ge 1$

$$\begin{aligned}\frac{1}{\sigma}U(t)&=W(t)-\int_0^t p'(x,t)W(x)\,dx\\ &=W(t)-\sum_{j=1}^{\lfloor t\rfloor}\int_{j-1}^{j}p'(x,t)W(x)\,dx-\int_{\lfloor t\rfloor}^{t}p'(x,t)W(x)\,dx.\end{aligned}$$

Taking into account (3.1.43) we get for all $k=1,2,\dots$

$$\begin{aligned}\frac{1}{\sigma}Q_k&=W(k)-\sum_{j=1}^{k}d_{j,k}W(j-1)\\ &=W(k)-\sum_{j=1}^{k}(p_{j,k}-p_{j-1,k})W(j-1)\\ &=W(k)-\sum_{j=1}^{k}\int_{j-1}^{j}p'(x,k)W(j-1)\,dx.\end{aligned}$$

Hence, for all $t \geq 1$ we have

$$\frac{1}{\sigma}\left(U(t) - Q_{\lfloor t\rfloor}\right) = (W(t) - W(\lfloor t\rfloor)) - \sum_{j=1}^{\lfloor t\rfloor}\int_{j-1}^{j} p'(x,t)W(x) - p'(x,\lfloor t\rfloor)W(j-1)dx$$
$$- \int_{\lfloor t\rfloor}^{t} p'(x,t)W(x)\,dx$$

and get

$$\begin{aligned}
&\sup_{\tilde{t}\leq t\leq N}\left|\frac{U(t)}{\sigma_t} - \frac{Q_{\lfloor t\rfloor}}{\sigma_t}\right| \\
&\qquad \leq \sup_{\tilde{t}\leq t\leq N}\frac{\sqrt{2d+1}}{\sqrt{t}}\left|W(t) - W(\lfloor t\rfloor)\right| \\
&\qquad\quad + \sup_{\tilde{t}\leq t\leq N}\frac{\sqrt{2d+1}}{\sqrt{t}}\sum_{j=1}^{\lfloor t\rfloor}\int_{j-1}^{j}\left|p'(x,t)W(x) - p'(x,\lfloor t\rfloor)W(j-1)\right|dx \\
&\qquad\quad + \sup_{\tilde{t}\leq t\leq N}\frac{\sqrt{2d+1}}{\sqrt{t}}\int_{\lfloor t\rfloor}^{t}\left|p'(x,t)W(x)\right|dx \\
&\qquad =: I_1(N) + I_2(N) + I_3(N).
\end{aligned} \tag{3.1.46}$$

Theorem 1.2.1 of Csörgő and Révész (1981) immediately implies that

$$I_1(N) = \boldsymbol{O}\left(\frac{\sqrt{\log N}}{\tilde{t}^{1/2}}\right) \quad \text{a.s.} \quad \text{as} \quad N \to \infty. \tag{3.1.47}$$

For the second term we get

$$\begin{aligned}
I_2(N) &\leq \sup_{\tilde{t}\leq t\leq N}\frac{\sqrt{2d+1}}{\sqrt{t}}\sum_{j=1}^{\lfloor t\rfloor}\int_{j-1}^{j}\left|p'(x,t) - p'(x,\lfloor t\rfloor)\right|\left|W(x)\right|dx \\
&\quad + \sup_{\tilde{t}\leq t\leq N}\frac{\sqrt{2d+1}}{\sqrt{t}}\sum_{j=1}^{\lfloor t\rfloor}\int_{j-1}^{j}\left|p'(x,\lfloor t\rfloor)\right|\left|W(x) - W(j-1)\right|dx \\
&=: J_1(N) + J_2(N).
\end{aligned}$$

For $J_1(N)$ it holds that

$$
\begin{aligned}
&J_1(N) \\
&\le \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{0\le s\le t} |W(s)| \sum_{j=1}^{\lfloor t\rfloor} \int_{j-1}^{j} p'(x,\lfloor t\rfloor) - p'(x,t)dx \\
&= \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{0\le s\le t} |W(s)| \sum_{j=1}^{\lfloor t\rfloor} (p(j,\lfloor t\rfloor) - p(j-1,\lfloor t\rfloor) - (p(j,t) - p(j-1,t))) \\
&= \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{0\le s\le t} |W(s)| \left(1 - \sum_{j=1}^{\lfloor t\rfloor} (p(j,t) - p(j-1,t))\right) \\
&= \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{0\le s\le t} |W(s)| \left(1 - \left(\frac{\lfloor t\rfloor}{t}\right)^d\right) \\
&= \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{0\le s\le t} |W(s)| \frac{t^d - \lfloor t\rfloor^d}{t^d}.
\end{aligned}
$$

Since the law of the iterated logarithm implies

$$
\sup_{0\le s\le t} \frac{|W(s)|}{\sqrt{t}} = \boldsymbol{O}(\sqrt{\log\log t}) \quad \text{a.s.},
$$

while an application of the mean value theorem shows that

$$
\frac{t^d - \lfloor t\rfloor^d}{t^d} \le \frac{dt^{d-1}(t-\lfloor t\rfloor)}{t^d} \le \frac{d}{t},
$$

we obtain

$$
J_1(N) = \boldsymbol{O}\left(\frac{\sqrt{\log\log N}}{\tilde{t}}\right) \quad \text{as} \quad N\to\infty. \tag{3.1.48}
$$

For $J_2(N)$ it holds that

$$
J_2(N) \le \sup_{\tilde{t}\le t\le N} \frac{\sqrt{2d+1}}{\sqrt{t}} \sup_{1\le j\le t} \sup_{0\le s\le 1} |W(j-1+s) - W(j-1)|
$$

and again by Theorem 1.2.1 of Csörgő and Révész it follows that

$$
J_2(N) = \boldsymbol{O}\left(\frac{\sqrt{\log N}}{\tilde{t}^{1/2}}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty. \tag{3.1.49}
$$

Combining (3.1.48) and (3.1.49) we derive

$$
I_2(N) = \boldsymbol{O}\left(\frac{\sqrt{\log N}}{\tilde{t}^{1/2}}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty. \tag{3.1.50}
$$

Now since

$$\sup_{1\le t<\infty}\sup_{0\le s\le t} p'(x,t) = d,$$

Theorem 1.2.1 implies

$$I_3(N) = \boldsymbol{O}\left(\frac{\sqrt{\log N}}{\tilde{t}^{1/2}}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty. \tag{3.1.51}$$

Consequently, (3.1.46)–(3.1.51) yield the intermediate result

$$\sup_{\tilde{t}\le t\le N}\left|\frac{U(t)}{\sigma_t} - \frac{Q_{\lfloor t\rfloor}}{\sigma_t}\right| = \boldsymbol{O}\left(\frac{\sqrt{\log N}}{\tilde{t}^{1/2}}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty. \tag{3.1.52}$$

Next, we have

$$\sup_{\tilde{t}\le t\le N}\left|\frac{Q_{\lfloor t\rfloor}}{\sigma_{\lfloor t\rfloor}} - \frac{Q_{\lfloor t\rfloor}}{\sigma_t}\right| = \sup_{\tilde{t}\le t\le N}\sqrt{2d+1}\sum_{j=1}^{\lfloor t\rfloor} p_{j,\lfloor t\rfloor}|W(j)-W(j-1)|\frac{\sqrt{t}-\sqrt{\lfloor t\rfloor}}{\sqrt{t}\sqrt{\lfloor t\rfloor}}.$$

The law of the iterated logarithm for weighted sums (see Li and Tomkins, 1996) yields

$$\limsup_{\lfloor t\rfloor\to\infty}\frac{\sqrt{2d+1}\sum_{j=1}^{\lfloor t\rfloor} p_{j,\lfloor t\rfloor}|W(j)-W(j-1)|}{\sqrt{2\lfloor t\rfloor\log\log\lfloor t\rfloor}} = 1 \quad \text{a.s.} \quad \text{as} \quad N\to\infty,$$

implying that

$$\sup_{\tilde{t}\le t\le N}\left|\frac{Q_{\lfloor t\rfloor}}{\sigma_{\lfloor t\rfloor}} - \frac{Q_{\lfloor t\rfloor}}{\sigma_t}\right| = \boldsymbol{O}\left(\left(\frac{\log\log N}{\tilde{t}}\right)^{1/2}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty \tag{3.1.53}$$

and the Lemma follows by (3.1.52) and (3.1.53).

□

Finally, we consider the extremes of the process $\{U(t)/\sigma_t, 1\le t\le N\}$.

Lemma 3.1.3 *Let the process* $\{U(t)\}_{t\ge1}$ *be defined as in* (3.1.44). *If* $\{\sigma_t\}_{t\ge1}$ *is defined via* (3.1.9), *then for all real* x *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N\sup_{1\le t\le N}\frac{U(t)}{\sigma_t} - b_N \le x\right) = \exp\left(-e^{-x}\right) \quad \textit{as} \quad N\to\infty, \tag{3.1.54}$$

where a_N *and* b_N *are defined in* (3.1.16) *and* (3.1.17).

PROOF: First, note that $\{U(t)/\sigma_t, 1 \le t \le N\}$ is a standardized Gaussian process and for all $0 \le s \le t \le N$ its autocorrelation function $r(s,t)$ is given by

$$r(s,t) = \operatorname{Cov}\left(\frac{U(s)}{\sigma_s}, \frac{U(t)}{\sigma_t}\right) = \frac{2d+1}{\sqrt{st}} \int_0^s \left(\frac{x}{s}\right)^d \left(\frac{x}{t}\right)^d dx = \frac{s^{d+1/2}}{t^{d+1/2}}.$$

Comparing the autocorrelation functions, we see that

$$\left\{\frac{U(t)}{\sigma_t}, 1 \le t \le N\right\} \overset{\mathrm{D}}{=} \left\{\frac{W(t^{2d+1})}{t^{d+1/2}}, 1 \le t \le N\right\}$$

and since

$$\sup_{1 \le t \le N} \frac{W(t^{2d+1})}{t^{d+1/2}} = \sup_{1 \le t \le N^{2d+1}} \frac{W(t)}{\sqrt{t}},$$

the lemma follows by Theorem 12.3.5 of Leadbetter, Lindgren and Rootzén (1983).

□

We now put the results together. Lemma 3.1.1 shows that for any non-decreasing $1 \le n \le N$ it holds that

$$\left(\max_{1 \le k \le n} \frac{P_k}{\sigma_k} - \max_{1 \le k \le n} \frac{Q_k}{\sigma_k}\right) = \boldsymbol{O}_P(1) \quad \text{as} \quad N \to \infty.$$

If we chose $n = (\log N)^\delta$ for some $\delta > 1$, then, the law of the iterated logarithm for weighted sums (see Li and Tomkins, 1996) shows that

$$\max_{1 \le k \le (\log N)^\delta} \frac{Q_k}{\sigma_k} = \boldsymbol{O}(\log\log\log N) \quad \text{a.s.} \quad \text{as} \quad N \to \infty.$$

Hence, for any real x it follows that

$$\lim_{N \to \infty} \mathrm{P}\left(a_N \max_{1 \le k \le (\log N)^\delta} \frac{Q_k}{\sigma_k} - b_N \le x\right) = 1 \tag{3.1.55}$$

and also

$$\lim_{N \to \infty} \mathrm{P}\left(a_N \max_{1 \le k \le (\log N)^\delta} \frac{P_k}{\sigma_k} - b_N \le x\right) = 1. \tag{3.1.56}$$

Assertion (3.1.14) now follows on combining the Lemmas 3.1.1, 3.1.2 and 3.1.3. The statement in display (3.1.15) holds, since the above computations can be modified in an obvious way to consider the minimum of $\{P_k\}_{k=1,\dots,N}$ and it is well known, that the maxima and minima in the underlying extreme value asymptotic are independent (see Bickel and Rosenblatt, 1973, Theorem A1.). Note that (3.1.15) can also be shown directly by applying an extended result of Darling and Erdős (1956) to the approximations of $\{|P_k|/\sigma_k\}_{k=1,\dots,N}$ (cf. Horváth, Kokoszka and Steinebach, 2007).

Proof of Theorem 3.1.2

We only show (3.1.18), since (3.1.19) follows by the same arguments. Defining the sequence $\{P_k^{(0)}\}_{k=1,\dots,N}$ as

$$P_k^{(0)} = \sum_{j=1}^{k} p_{j,k}\varepsilon_j, \quad k = 1, \dots N, \tag{3.1.57}$$

we get the decomposition

$$P_k = P_k^{(0)} + \sum_{j=k^*+1}^{k} p_{j,k}\,\triangle, \quad k = 1, \dots N.$$

For the drift it holds that

$$\begin{aligned}\max_{1\le k\le N} \sum_{j=k^*+1}^{k} p_{j,k}\,\triangle &= \sum_{j=k^*+1}^{N} p_{j,N}\,\triangle \\ &\ge \frac{\triangle}{d+1}\frac{N^{d+1} - {k^*}^{d+1}}{N^d} \\ &\ge \frac{\triangle}{d+1}\frac{N^{d+1} - (N - N^\rho)^{d+1}}{N^d}\end{aligned}$$

and elementary calculations show that

$$\frac{N^{d+1} - (N - N^\rho)^{d+1}}{N^d} \sim N^\rho \quad \text{as} \quad N \to \infty. \tag{3.1.58}$$

For all real x we have

$$\begin{aligned}&\mathrm{P}\left(a_N \max_{1\le k\le N} \frac{P_k}{\sigma_k} - b_N \le x\right) \\ &\le \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma_k} - b_N \le x - a_N \frac{\sum_{j=k^*+1}^{N} p_{j,N}\,\triangle}{\sigma_N} + 2a_N \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma_k}\right).\end{aligned}$$

Theorem 3.1.1 implies

$$a_N \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma_k} - b_N = \boldsymbol{O}_P(1) \quad \text{as} \quad N \to \infty$$

and

$$a_N \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma_k} = \boldsymbol{O}_P(a_N^2) \quad \text{as} \quad N \to \infty,$$

hence (3.1.18) follows, since

$$\frac{\sum_{j=k^*+1}^{N} p_{j,N}\,\triangle}{\sigma_N} \gtrsim N^{\rho - \frac{1}{2}} \quad \text{as} \quad N \to \infty$$

and $\rho > 1/2$.

Proof of Theorem 3.1.3

First, we show that early time lags k do not contribute to the extreme value asymptotic. For all $k = 1, \dots, N$ we define

$$\hat{Q}_{m,k} = \sigma \sum_{j=1}^{k} p_{j,k}(W_{1,m}(j) - W_{1,m}(j-1)) - \frac{\sum_{j=1}^{k} p_{j,k}}{m} W_{2,m}(m). \tag{3.1.59}$$

Lemma 3.1.4 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,\dots,N}$ *and* $\{\hat{Q}_{m,k}\}_{k=1,\dots,N}$ *be defined as in* (3.1.27) *and* (3.1.59)*, respectively. If the sequence* $\{g_c(m,k)\}_{k=1,\dots,N}$ *is chosen according to* (3.1.29)*, then for any* $\delta > 0$ *it holds that*

$$\frac{1}{(\log\log\log m)^{1/2}} \max_{1 \le k \le (\log m)^\delta} \frac{\hat{P}_{m,k}}{\sigma g_c(m,k)} = \boldsymbol{O}_P(1) \quad as \quad m \to \infty \tag{3.1.60}$$

and

$$\frac{1}{(\log\log\log m)^{1/2}} \max_{1 \le k \le (\log m)^\delta} \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} = \boldsymbol{O}_P(1) \quad as \quad m \to \infty. \tag{3.1.61}$$

PROOF: We have

$$\begin{aligned} \max_{1 \le k \le (\log m)^\delta} \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} \le{}& \max_{1 \le k \le (\log m)^\delta} \frac{1}{g_c(m,k)} \sum_{j=1}^{k} p_{j,k}(W_{1,m}(j) - W_{1,m}(j-1)) \\ &+ \max_{1 \le k \le (\log m)^\delta} \frac{1}{g_c(m,k)} \frac{\sum_{j=1}^{k} p_{j,k}}{m} |W_{2,m}(m)| \\ ={}& I_1(m) + I_2(m). \end{aligned}$$

Obviously,

$$\sigma g_c(m,k) \ge \sigma_k \quad \text{for all} \quad k = 1, \dots, N \quad \text{and} \quad m = 1, 2, \dots,$$

hence, the final considerations to the proof of Theorem 3.1.1 carry over to $I_1(m)$ and we have

$$I_1(m) = \boldsymbol{O}\left(\log\log\log m\right) \quad \text{a.s.} \quad \text{as} \quad m \to \infty. \tag{3.1.62}$$

Now let $\{W_2(t), t \ge 0\}$ be a standard Wiener process and define D as

$$D = \frac{2d+1}{(d+1)^2}. \tag{3.1.63}$$

The distribution of $W_{2,m}(m)/\sqrt{m}$ does not depend on m, hence

$$\begin{aligned} \max_{1 \le k \le (\log m)^\delta} \frac{\sum_{j=1}^{k} p_{j,k} \frac{1}{m} |W_{2,m}(m)|}{g_c(m,k)} &\overset{\mathcal{D}}{=} \max_{1 \le k \le (\log m)^\delta} \frac{\sum_{j=1}^{k} p_{j,k} \frac{1}{m} |W_2(1)|}{\left(\frac{1}{2d+1} \frac{k}{m} \left(1 + D\frac{k}{m}\right)\right)^{1/2}} \\ &= \boldsymbol{O}_P(1) \max_{1 \le k \le (\log m)^\delta} \frac{\frac{1}{m} \sum_{j=1}^{k} p_{j,k}}{\left(\frac{1}{2d+1} \frac{k}{m} \left(1 + D\frac{k}{m}\right)\right)^{1/2}} \end{aligned}$$

as $m \to \infty$. Since

$$\frac{1}{d+1}k = \int_0^k p(x,k)\,dx \le \sum_{j=1}^{k} p_{j,k} \le \int_0^k p(x,k)\,dx + 1$$

it follows that

$$\begin{aligned}
\max_{1\le k\le(\log m)^\delta} \frac{\frac{1}{m}\sum_{j=1}^k p_{j,k}}{\left(\frac{1}{2d+1}\frac{k}{m}\left(1+D\frac{k}{m}\right)\right)^{1/2}} &\le \max_{1\le k\le(\log m)^\delta} \frac{\frac{1}{d+1}\frac{k}{m}+\frac{1}{m}}{\left(\frac{1}{2d+1}\frac{k}{m}\left(1+D\frac{k}{m}\right)\right)^{1/2}} \\
&= \max_{1\le k\le(\log m)^\delta}\left(\frac{D\frac{k}{m}}{1+D\frac{k}{m}}\right)^{1/2} + \boldsymbol{O}(1)
\end{aligned}$$

as $m \to \infty$, and we see that

$$\max_{1\le k\le(\log m)^\delta} \frac{\sum_{j=1}^k p_{j,k}\frac{1}{m}|W_{2,m}(m)|}{g_c(m,k)} = \boldsymbol{O}_P(1) \quad \text{as} \quad m\to\infty. \tag{3.1.64}$$

Assertion (3.1.61) now follows from (3.1.62) and (3.1.64). Assertion (3.1.60) is an immediate consequence of (3.1.61) and Lemma 3.1.5 below.

□

Next, we replace the observations by the corresponding increments of the approximating Wiener processes.

Lemma 3.1.5 *Let the sequences $\{\hat{P}_{m,k}\}_{k=1,\dots,N}$ and $\{\hat{Q}_{m,k}\}_{k=1,\dots,N}$ be defined as in* (3.1.27) *and* (3.1.59)*, respectively. If $\{g_c(m,k)\}_{k=1,\dots,N}$ is chosen according to* (3.1.29) *and n is a non-decreasing, integer-valued function of m with $1 \le n \le N$, then*

$$\max_{n<k\le N}\frac{\hat{P}_{m,k}}{\sigma g_c(m,k)} - \max_{n<k\le N}\frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} = \boldsymbol{O}_P\left(\frac{1}{n^{1/2-1/\nu}}+\frac{1}{m^{1/2-1/\nu}}\right) \quad \textit{as} \quad m\to\infty. \tag{3.1.65}$$

Proof: We have

$$\begin{aligned}
&\max_{n<k\le N}\left|\frac{\hat{P}_{m,k}}{\sigma g_c(m,k)} - \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)}\right| \\
&\quad\le \max_{n<k\le N}\frac{1}{\sigma g_c(m,k)}\left|\sum_{j=1}^k p_{j,k}(\varepsilon_{m+j} - \sigma(W_{1,m}(j)-W_{1,m}(j-1)))\right| \\
&\quad\quad + \max_{n<k\le N}\frac{1}{\sigma g_c(m,k)}\left|\frac{\sum_{j=1}^k p_{j,k}}{m}\left(\sum_{i=1}^m \varepsilon_i - \sigma W_{2,m}(m)\right)\right| \\
&\quad =: I_1(m) + I_2(m).
\end{aligned}$$

Since

$$\sigma g_c(m,k) \geq \sigma_k \quad \text{for all} \quad k=1,\ldots,N \quad \text{and} \quad m=1,2,\ldots,$$

the same steps as in the proof of Lemma 3.1.1 show that

$$I_1(m) = \boldsymbol{O}_P\left(\frac{1}{n^{1/2-1/\nu}}\right) \quad \text{as} \quad m\to\infty. \tag{3.1.66}$$

Furthermore, (3.1.22) implies

$$\frac{1}{\sqrt{m}}\left|\sum_{i=1}^{m}\varepsilon_i - \sigma W_{2,m}(m)\right| = \boldsymbol{O}_P\left(\frac{1}{m^{1/2-1/\nu}}\right) \quad \text{as} \quad m\to\infty$$

and since the proof of Lemma 3.1.4 shows that

$$\max_{n<k\leq N} \frac{\sum_{j=1}^{k} p_{j,k}}{m} \Big/ \sigma\sqrt{\left(\frac{1}{2d+1}\frac{k}{m}\right)\left(1+\frac{2d+1}{(d+1)^2}\frac{k}{m}\right)} = \boldsymbol{O}(1) \quad \text{as} \quad m\to\infty,$$

we conclude that

$$I_2(m) = \boldsymbol{O}_P\left(\frac{1}{m^{1/2-1/\nu}}\right) \quad \text{as} \quad m\to\infty. \tag{3.1.67}$$

The lemma now follows by (3.1.66) and (3.1.67).

□

The next result is the counterpart of Lemma 3.1.2.

Lemma 3.1.6 *Let $\tilde{n}$ and $\tilde{N}$ be non-decreasing functions of m with $1\leq\tilde{n}\leq\tilde{N}\leq N$. If $\{g_c(m,k)\}_{k=1,\ldots,N}$ is chosen according to* (3.1.29)*, then*

$$\max_{\tilde{n}\leq\lfloor t\rfloor\leq\tilde{N}} \frac{\sum_{j=1}^{\lfloor t\rfloor} p_{j,\lfloor t\rfloor}(W_{1,m}(j)-W_{1,m}(j-1))}{g_c(m,\lfloor t\rfloor)} - \max_{\tilde{n}\leq t\leq\tilde{N}} \frac{\int_0^t p(x,t)\,dW_{1,m}(x)}{g_c(m,t)} = \boldsymbol{O}_P\left(\sqrt{\frac{\log\tilde{N}}{\tilde{n}}}\right) \quad \textit{as} \quad m\to\infty \tag{3.1.68}$$

and

$$\max_{\tilde{n}\leq t\leq\tilde{N}} \frac{\int_0^t p(x,t)\,dW_{1,m}(x)}{g_c(m,t)} \stackrel{D}{=} \max_{\tilde{n}\leq t\leq\tilde{N}} \frac{1}{\sqrt{2d+1}} \frac{\frac{1}{t^d}W_1(t^{2d+1})}{g_c(m,t)}, \tag{3.1.69}$$

where $\{W_1(t), t\geq 0\}$ denotes a standard Wiener process.

Proof: Assertion (3.1.68) follows similar to the proof of Lemma 3.1.2, since

$$\sigma g_c(m,t) \geq \sigma_t \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad m = 1, 2, \ldots$$

and

$$\frac{1}{g_c(m, \lfloor t \rfloor)} - \frac{1}{g_c(m,t)} = \boldsymbol{O}\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty,$$

uniformly in $m = 1, 2, \ldots,$ where the last equation is implied by

$$\frac{1}{g_c(m, \lfloor t \rfloor)} - \frac{1}{g_c(m,t)} = \sqrt{2d+1}\frac{\sqrt{t\left(1 + D\frac{t}{m}\right)} - \sqrt{\lfloor t \rfloor \left(1 + D\frac{\lfloor t \rfloor}{m}\right)}}{\sqrt{t\left(1 + D\frac{t}{m}\right)}\sqrt{\lfloor t \rfloor \left(1 + D\frac{\lfloor t \rfloor}{m}\right)}}$$

and the fact that the derivative of the function

$$f(t) = \sqrt{t\left(1 + D\frac{t}{m}\right)}, \quad t \geq 1$$

is uniformly bounded for all $m = 1, 2, \ldots$ Note that we are dealing with a triangular array, however, the result of Lemma 3.1.2 carries over since Theorem 1.2.1 of Csörgő and Révész (1981) holds almost surely.

Assertion (3.1.69) can be seen by comparing the covariances of

$$\left\{\int_0^t p(x,t)\, dW_{1,m}(x), t \geq 1\right\} \quad \text{and} \quad \left\{\frac{\sqrt{2d+1}}{t^d} W_1(t^{2d+1}), t \geq 1\right\}.$$

□

Lemma 3.1.7 *Let the sequence* $\{\hat{Q}_{m,k}\}_{k=1,\ldots,N}$ *be defined as in* (3.1.59)*. If we choose* $\{g_c(m,k)\}_{k=1,\ldots,N}$ *according to* (3.1.29)*, then for any constant* $c > 0$ *with* $cm \leq N$ *it holds that*

$$\frac{1}{(2\log\log\log m)^{1/2}} \max_{cm/\log m \leq k \leq N} \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} = \boldsymbol{O}_P(1) \quad \textit{as} \quad m \to \infty. \tag{3.1.70}$$

Proof: We have

$$\begin{aligned}\max_{cm/\log m \leq k \leq N} \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} \leq & \max_{cm/\log m \leq k \leq N} \frac{1}{g_c(m,k)} \sum_{j=1}^{k} p_{j,k}(W_{1,m}(j) - W_{1,m}(j-1)) \\ & + \max_{cm/\log m \leq k \leq N} \frac{1}{g_c(m,k)} \frac{\sum_{j=1}^{k} p_{j,k}}{m} |W_{2,m}(m)| \\ & =: I_1(m) + I_2(m).\end{aligned}$$

Lemma 3.1.6 and (3.1.20) show that instead of $I_1(m)$ we can consider

$$\max_{cm/\log m\le t\le N} \frac{\frac{1}{t^d}W_1(t^{2d+1})}{m^{1/2}\left(\frac{t}{m}\right)^{1/2}\left(1+D\frac{t}{m}\right)^{1/2}}.$$

To this end, we split up the domain into $cm/\log m \le k \le cm$ and $cm \le k \le N$.

The rescaling property of the Wiener process yields

$$\max_{cm/\log m\le t\le cm} \frac{\frac{1}{t^d}W_1(t^{2d+1})}{m^{1/2}\left(\frac{t}{m}\right)^{1/2}\left(1+D\frac{t}{m}\right)^{1/2}} \stackrel{\mathrm{D}}{=} \max_{c/\log m\le s\le c} \frac{W_1(s^{2d+1})}{s^{d+1/2}\left(1+Ds\right)^{1/2}}$$

and by the law of the iterated logarithm for $s\to 0$ we see that

$$\begin{aligned}\max_{c/\log m\le s\le c} \frac{W_1(s^{2d+1})}{s^{d+1/2}\left(1+Ds\right)^{1/2}} &\le \max_{c/\log m\le s\le c} \frac{W_1(s^{2d+1})}{s^{d+1/2}} \\ &= \boldsymbol{O}\left((\log\log\log m)^{1/2}\right) \quad \text{a.s.} \quad \text{as} \quad m\to\infty. \qquad (3.1.71)\end{aligned}$$

For the upper part of the domain the rescaling property and the law of the iterated logarithm for $s\to\infty$ yield

$$\max_{c\le s\le N/m} \frac{W_1(s^{2d+1})}{s^{d+1/2}\left(1+Ds\right)^{1/2}} = \boldsymbol{O}\left(1\right) \quad \text{a.s.} \quad \text{as} \quad m\to\infty. \qquad (3.1.72)$$

Next, we consider $I_2(m)$. Let $\{W_2(t), 0\le t\}$ be a standard Wiener process. The distribution of $I_2(m)$ does not depend on m, hence, the rescaling property of the Wiener process gives

$$\begin{aligned}\max_{cm/\log m\le k\le N} \frac{\sum_{j=1}^k p_{j,k}\frac{1}{m}|W_{2,m}(m)|}{g(m,k)} &\stackrel{\mathrm{D}}{=} \max_{cm/\log m\le k\le N} \frac{\sum_{j=1}^k p_{j,k}\frac{1}{m}|W_2(1)|}{\left(\frac{1}{2d+1}\frac{k}{m}\left(1+D\frac{k}{m}\right)\right)^{1/2}} \\ &= \boldsymbol{O}_P(1) \max_{cm/\log m\le k\le N} \frac{\frac{1}{m}\sum_{j=1}^k p_{j,k}}{\left(\frac{1}{2d+1}\frac{k}{m}\left(1+D\frac{k}{m}\right)\right)^{1/2}}\end{aligned}$$

as $m\to\infty$. Since the proof of Lemma 3.1.4 implies that

$$\max_{cm/\log m\le k\le N} \frac{\frac{1}{m}\sum_{j=1}^k p_{j,k}}{\left(\frac{1}{2d+1}\frac{k}{m}\left(1+D\frac{k}{m}\right)\right)^{1/2}} = \boldsymbol{O}(1) \quad \text{as} \quad m\to\infty,$$

we get

$$I_2(m) = \boldsymbol{O}_P(1) \quad \text{as} \quad m\to\infty \qquad (3.1.73)$$

and the Lemma follows by (3.1.71), (3.1.72) and (3.1.73).

□

Lemma 3.1.8 *Let the sequence* $\{\hat{Q}_{m,k}\}_{k=1,\ldots,N}$ *be defined as in* (3.1.59)*. If we choose* $\{g_c(m,k)\}_{k=1,\ldots,N}$ *according to* (3.1.29)*, then for all real* x *it holds that*

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le N} \frac{\hat{Q}_{m,k}}{\sigma g_c(m,k)} - b_m \le x\right) = \exp\left(-e^{-x}\right), \tag{3.1.74}$$

where a_m *and* b_m *are defined in* (3.1.36) *and* (3.1.37)*, respectively.*

PROOF: Lemma 3.1.4 and Lemma 3.1.7 imply that (3.1.74) follows, if it holds that

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le k\le cm/\log m} \frac{\hat{Q}_{m,k}}{\sigma g(m,k)} - b_m \le x\right) = \exp\left(-e^{-x}\right), \tag{3.1.75}$$

for some $\delta > 0$, and $c > 0$ chosen so that $cm \le N$.

Since

$$a_m \max_{(\log m)^\delta \le k\le cm/\log m} \frac{\sum_{j=1}^k p_{j,k}\frac{1}{m}|W_{2,m}(m)|}{g(m,k)} = \boldsymbol{O}_P(1)\, a_m \max_{(\log m)^\delta \le k\le cm/\log m} \left(\frac{D\frac{k}{m}}{1+D\frac{k}{m}}\right)^{1/2}$$

as $m\to\infty$ and

$$\begin{aligned} a_m \max_{(\log m)^\delta \le k\le cm/\log m} \left(\frac{D\frac{k}{m}}{1+D\frac{k}{m}}\right)^{1/2} &\le a_m \max_{(\log m)^\delta \le k\le cm/\log m} \left(D\frac{k}{m}\right)^{1/2} \\ &= \boldsymbol{O}\left(\left(\frac{\log\log m}{\log m}\right)^{1/2}\right) \quad \text{as} \quad m\to\infty, \end{aligned}$$

it is sufficient to show that

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le k\le cm/\log m} \frac{\sum_{j=1}^k p_{j,k}(W_{1,m}(j)-W_{1,m}(j-1))}{\sigma g(m,k)} - b_m \le x\right) = \exp\left(-e^{-x}\right). \tag{3.1.76}$$

Using (3.1.68), it follows that

$$\begin{aligned} &a_m \max_{(\log m)^\delta \le k\le cm/\log m} \frac{\sum_{j=1}^k p_{j,k}(W_{1,m}(j)-W_{1,m}(j-1))}{\sigma g(m,k)} \\ &\quad - a_m \max_{(\log m)^\delta \le t\le cm/\log m} \frac{\int_0^t p(x,t)dW_{1,m}(x)}{\sigma g(m,t)} = \boldsymbol{O}_P\left(\left(\frac{\log m}{(\log m)^\delta}\right)^{1/2}\right) \end{aligned}$$

as $m\to\infty$. Now if we choose $\delta > 1$ and apply (3.1.69), we see that (3.1.76) is satisfied, if

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le t\le cm/\log m} \frac{1}{\sqrt{2d+1}} \frac{\frac{1}{t^d}W_1(t^{2d+1})}{g(m,t)} - b_m \le x\right) = \exp\left(-e^{-x}\right) \tag{3.1.77}$$

holds. To this end, we first show that

$$a_m \max_{(\log m)^\delta \le t \le cm/\log m} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{m^{1/2}\left(\frac{t}{m}\right)^{1/2}\left(1+D\frac{t}{m}\right)^{1/2}} \\ - a_m \max_{(\log m)^\delta \le t \le cm/\log m} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{t^{1/2}} = \boldsymbol{o}_P(1) \quad \text{as} \quad m\to\infty. \quad (3.1.78)$$

We have

$$a_m \max_{(\log m)^\delta \le t \le cm/\log m} \frac{|W_1(t^{2d+1}) - \left(1+D\frac{t}{m}\right)^{1/2} W_1(t^{2d+1})|}{t^{d+1/2}\left(1+D\frac{t}{m}\right)^{1/2}} \\ = a_m \max_{(\log m)^\delta \le t \le cm/\log m} \frac{|W_1(t^{2d+1}) - \left(1+\frac{D}{2\sqrt{1+D}}\frac{t}{m}+\boldsymbol{o}\left(\frac{t}{m}\right)\right) W_1(t^{2d+1})|}{t^{d+1/2}\left(1+D\frac{t}{m}\right)^{1/2}}$$

as $m\to\infty$. Since the law of the iterated logarithm implies

$$\max_{(\log m)^\delta \le t \le cm/\log m} \frac{|\frac{t}{m} W_1(t^{2d+1})|}{t^{d+1/2}\left(1+D\frac{t}{m}\right)^{1/2}} \le \max_{(\log m)^\delta \le t \le cm/\log m} \frac{|\frac{t}{m} W_1(t^{2d+1})|}{t^{d+1/2}} \\ = \boldsymbol{O}\left(\frac{(\log\log m)^{1/2}}{\log m}\right) \quad \text{a.s.} \quad \text{as} \quad m\to\infty,$$

we see that (3.1.78) holds.

Finally,

$$\max_{(\log m)^\delta \le t \le cm/\log m} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{t^{1/2}} = \sup_{\delta(2d+1)\log\log m \le t \le (2d+1)\log(cm/\log m)} \frac{W_1(e^t)}{e^{t/2}} \\ \stackrel{\mathcal{D}}{=} \sup_{0 \le t \le (2d+1)\log(cm/(\log m)^{1+\delta})} \frac{W_1(e^t)}{e^{t/2}}$$

so that Theorem 12.3.5 of Leadbetter, Lindgren and Rootzén (1983) yields

$$\lim_{m\to\infty} \mathrm{P}\left(\tilde{a}_m \sup_{0\le t\le (2d+1)\log(cm/(\log m)^{1+\delta})} \frac{W_1(e^{2t})}{e^t} - \tilde{b}_m \le x\right) = \exp\left(-e^{-x}\right), \quad (3.1.79)$$

where

$$\tilde{a}_m = \left(2\log\log\left(\frac{cm}{(\log m)^{1+\delta}}\right)^{2d+1}\right)^{1/2}$$

and

$$\tilde{b}_m = 2\log\log\left(\frac{cm}{(\log m)^{1+\delta}}\right)^{2d+1} + \frac{1}{2}\log\log\log\left(\frac{cm}{(\log m)^{1+\delta}}\right)^{2d+1} - \frac{1}{2}\log\pi.$$

Since elementary calculations show that

$$\tilde{a}_m(\tilde{a}_m - a_m) = \boldsymbol{o}(1) \quad \text{as} \quad m \to \infty$$

and

$$\tilde{b}_m - b_m = \boldsymbol{o}(1) \quad \text{as} \quad m \to \infty,$$

the proof of the lemma is complete.

□

The first assertion of Theorem 3.1.3 now follows on combining the Lemmas 3.1.4–3.1.8 and the fact that σ can be replaced by $\hat{\sigma}_m$ (see Lemma 1.1.3). The second follows as in case of known parameters.

Proof of Theorem 3.1.4

We only show (3.1.18). We define the sequence $\{\hat{P}^{(0)}_{m,k}\}_{k=1,\dots,N}$ by

$$\hat{P}^{(0)}_{m,k} = \sum_{j=1}^{k} p_{j,k}\varepsilon_j - \frac{\sum_{j=1}^{k} p_{j,k}}{m} \sum_{i=1}^{m} \varepsilon_i, \quad k = 1, \dots, N. \tag{3.1.80}$$

Obviously, it holds that

$$\hat{P}_{m,k} = \hat{P}^{(0)}_{m,k} + \sum_{j=k^*+1}^{k} p_{j,k}\, \triangle, \quad k = 1, \dots, N$$

and we already know by the proof of Theorem 3.1.2 that

$$\max_{1 \le k \le N} \sum_{j=k^*+1}^{k} p_{j,k}\, \triangle = \sum_{k^*+1}^{N} p_{j,N}\, \triangle \simeq N^{\varrho} = m^{\lambda\varrho} \quad \text{as} \quad m \to \infty.$$

Furthermore,

$$g_c(m,N) \simeq \frac{N}{m^{1/2}} = m^{\lambda - 1/2} \quad \text{as} \quad m \to \infty$$

and since

$$\frac{1}{\hat{\sigma}_m} - \frac{1}{\sigma} = \boldsymbol{o}_P\left(\frac{1}{m^{\vartheta}}\right) \quad \text{for some} \quad \vartheta > 0,$$

it follows that

$$\frac{\sum_{k^*+1}^{N} p_{j,N}\, \triangle}{\hat{\sigma}_m\, g_c(m,N)} - \frac{\sum_{k^*+1}^{N} p_{j,N}\, \triangle}{\sigma\, g_c(m,N)} = \boldsymbol{o}_P\left(m^{\lambda(\rho-1)+1/2-\vartheta}\right) \quad \text{as} \quad m \to \infty.$$

Now

$$\frac{\sum_{k^*+1}^{N} p_{j,N}\ \Delta}{\sigma\, g_c(m,N)} \simeq m^{\lambda(\rho-1)+1/2} \quad \text{as} \quad m \to \infty$$

and since we have assumed that

$$\lambda(\varrho - 1) + 1/2 > 0,$$

assertion (3.1.18) follows by Theorem 3.1.3 and the same considerations as in the proof of Theorem 3.1.2.

3.2 Open-end control charts

Dealing now with infinite time horizons, a delicate point is the choice of adequate boundary functions. Since we already know that the detectors $\{P_k\}_{k=1,2,\ldots}$ follow a law of the iterated logarithm for weighted sums (see Li and Tomkins, 1996), it is clear that in case of known parameters any positive function G satisfying

$$\limsup_{t\to\infty} \frac{\sqrt{t \log\log t}}{G(t)} = 0 \tag{3.2.1}$$

yields non-trivial crossing probabilities. We have seen how adequate boundary functions for the closed-end PWMA-chart can be derived by transforming boundary functions for the Wiener process, hence, we first collect some results on the boundary crossing probabilities of the Wiener process.

First, we mention that there are only a few examples of boundary functions for the Wiener process, where the crossing probabilities are explicitly known and refer to Lerche (1986) for an overview on this subject. The most popular examples are given by Robbins and Siegmund (1970) who provided the following result

Example (Robbins, Siegmund, 1970) *Denote by $S(k)$ the partial sum of k i.i.d. random variables having mean 0 and variance 1. Then*

$$\begin{aligned}
&\lim_{m\to\infty} \mathrm{P}\left(\sup_{1\le k<\infty} \frac{|S(k)|}{\sqrt{(k+m)(a^2+\log(1+k/m))}} \ge 1 \right) \\
&= \mathrm{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{\sqrt{(1+t)(a^2+\log(1+t))}} \ge 1 \right) \\
&= \exp\left(-\frac{1}{2}\,a^2\right) \quad (a>0)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{m\to\infty} \mathrm{P}\left(\sup_{1\le k<\infty} \frac{S(k)}{\sqrt{(k+m)}\;\Theta^{-1}(2\log(a/2)+\log(1+k/m))} \ge 1 \right) \\
&= \mathrm{P}\left(\sup_{0<t<\infty} \frac{W(t)}{\sqrt{(1+t)}\;\Theta^{-1}(2\log(a/2)+\log(1+t))} \ge 1 \right) \\
&= \frac{1}{2a} \quad (a>1/2),
\end{aligned}$$

where for all real x

$$\Theta(x) = x^2 + 2\log\Phi(x)$$

and Φ denotes the standard normal distribution function.

It is possible to derive boundary functions for the PWMA-detectors based on the latter example (see Appendix C), however, the derived boundary function for the one-sided alternative can only be computed by numerical methods. Hence, we consider in this chapter an alternative approach that is based on the family of boundary functions for the Wiener process, which is introduced in Appendix B.

3.2.1 Model assumptions for known μ and σ

We assume that (3.1.1)–(3.1.5) hold with $N = \infty$.

3.2.2 Monitoring procedures for known μ and σ

We define the sequence of detectors as

$$P_k = \sum_{j=1}^{k} p_{j,k}(X_j - \mu), \quad k = 1, 2, \dots, \tag{3.2.2}$$

where the weights are chosen as in (3.1.6).

For all $t_0 \geq 1$ and $t > 0$ we choose $g(t_0, t)$ according to

$$g(t_0, t) = \frac{1}{\sqrt{2d+1}} \sqrt{t \log\left(\frac{t^{2d+1}}{t_0} + e\right)}. \tag{3.2.3}$$

If we test the null hypotheses versus the one-sided alternative ($\alpha \in]0, 1[$), we reject H_0 if $\tau_1 < \infty$, where

$$\tau_1 = \tau_1(\alpha, t_0) = \inf\{1 \leq k < \infty : P_k > \sigma c_1(\alpha, t_0) g(t_0, k)\}. \tag{3.2.4}$$

If the alternative is two-sided we replace τ_1 by τ_2, where

$$\tau_2 = \tau_2(\alpha, t_0) = \inf\{1 \leq k < \infty : |P_k| > \sigma c_2(\alpha, t_0) g(t_0, k)\}. \tag{3.2.5}$$

The critical constants $c_1(\alpha, t_0)$ and $c_2(\alpha, t_0)$ are defined as

$$c_1(\alpha, t_0) = \frac{q_1(1-\alpha) + b_{t_0}}{a_{t_0}} \quad \text{and} \quad c_2(\alpha, t_0) = \frac{q_2(1-\alpha) + b_{t_0}}{a_{t_0}}, \tag{3.2.6}$$

where a_{t_0} and b_{t_0} are given in (3.2.9) and (3.2.10) below. The quantiles q_1 and q_2 are chosen as in (3.1.13), according to the following theorem.

Theorem 3.2.1 *Let the sequence* $\{P_k\}_{k=1,2,\ldots}$ *be defined as in* (3.2.2)*. Furthermore, let the sequence* $\{g(t_0,k)\}_{k=1,2,\ldots}$ *be defined via* (3.2.3) *for all* $t_0 \geq 1$*. Then, it holds under the null hypothesis and for all real* x *that*

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0} \sup_{1\leq k<\infty} \frac{P_k}{\sigma g(t_0,k)} - b_{t_0} \leq x\right) = \exp\left(-e^{-x}\right) \tag{3.2.7}$$

and

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0} \sup_{1\leq k<\infty} \frac{|P_k|}{\sigma g(t_0,k)} - b_{t_0} \leq x\right) = \exp\left(-2e^{-x}\right), \tag{3.2.8}$$

where

$$a_{t_0} = (2\log\log t_0)^{1/2} \tag{3.2.9}$$

and

$$b_{t_0} = 2\log\log t_0 + \frac{1}{2}\log\log\log t_0 - \frac{1}{2}\log\pi. \tag{3.2.10}$$

The procedures have asymptotic power one, as can be seen by the following theorem.

Theorem 3.2.2 *Let the sequences* $\{P_k\}_{k=1,2,\ldots}$ *and* $\{g(t_0,k)\}_{k=1,2,\ldots}$ *be defined as in Theorem 3.2.1. Then, under* H_1 *it holds that*

$$\lim_{t_0\to\infty} \mathrm{P}\left(\sup_{1\leq k<\infty} \frac{P_k}{\sigma g(t_0,k)} > c_1(\alpha,t_0)\right) = 1 \tag{3.2.11}$$

and under H_2 *we have*

$$\lim_{t_0\to\infty} \mathrm{P}\left(\sup_{1\leq k<\infty} \frac{|P_k|}{\sigma g(t_0,k)} > c_2(\alpha,t_0)\right) = 1, \tag{3.2.12}$$

where a_{t_0} *and* b_{t_0} *are defined in* (3.2.9) *and* (3.2.10)*, respectively.*

3.2.3 Model assumptions for unknown μ and σ

We assume that (3.1.1), (3.1.21) and (3.1.22) hold. The monitoring of the process $\{X_i\}_{i=1,2,\ldots}$ starts after a training period of size m and we suppose that

$$X_i = \begin{cases} \mu + \varepsilon_i & : 1 \leq i \leq m + k^*, \\ \mu + \Delta + \varepsilon_i & : m + k^* < i < \infty, \end{cases} \tag{3.2.13}$$

where the parameters are denoted as in the closed end setting.

We want to test either

$$H_0 : k^* = \infty \quad \text{versus} \quad H_1 : 0 \leq k^* < \infty,\ \Delta > 0 \quad \text{(one-sided alternative)}, \tag{3.2.14}$$

or

$$H_0 : k^* = \infty \quad \text{versus} \quad H_2 : 0 \leq k^* < \infty,\ \Delta \neq 0 \quad \text{(two-sided alternative)}. \tag{3.2.15}$$

3.2.4 Monitoring procedures for unknown μ and σ

For all $m = 1, 2, \ldots$ we define the sequence $\{\hat{P}_{m,k}\}_{k=1,2,\ldots}$ as in (3.1.27) with $N = \infty$.

With

$$D = \frac{2d+1}{(d+1)^2} \tag{3.2.16}$$

we define for all $m = 1, 2, \ldots$ and $t > 0$

$$g_o(m,t) = \frac{\sqrt{m}}{\sqrt{2d+1}} \sqrt{\frac{t}{m}\left(1 + D\frac{t}{m}\right) \log\left(\frac{t^{2d+1}}{m} + e\right)}. \tag{3.2.17}$$

The stopping times are

$$\hat{\tau}_1 = \hat{\tau}_1(\alpha, m) = \inf\{1 \leq k < \infty : \hat{P}_{m,k} > \hat{c}_1(\alpha, m)\hat{\sigma}_m\, g_o(m,k)\} \tag{3.2.18}$$

and

$$\hat{\tau}_2 = \hat{\tau}_2(\alpha, m) = \inf\{1 \leq k < \infty : |\hat{P}_{m,k}| > \hat{c}_2(\alpha, m)\hat{\sigma}_m\, g_o(m,k)\}, \tag{3.2.19}$$

where we assume that $\hat{\sigma}_m$ satisfies (3.1.28) and the critical constants $\hat{c}_1(\alpha, m)$ and $\hat{c}_2(\alpha, m)$ are given by

$$\hat{c}_1(\alpha, m) = \frac{q_1(1-\alpha) + b_m}{a_m} \quad \text{and} \quad \hat{c}_2(\alpha, m) = \frac{q_2(1-\alpha) + b_m}{a_m}, \tag{3.2.20}$$

with a_m as in (3.2.23) and b_m as in (3.2.24) below. The quantiles q_1 and q_2 are chosen as in (3.1.13), justified by the following theorem.

Theorem 3.2.3 *Let the sequence $\{\hat{P}_{m,k}\}_{k=1,2,\ldots}$ be defined as in (3.1.27) with $N = \infty$. The sequence $\{g_o(m,k)\}_{k=1,2,\ldots}$ is chosen as in (3.2.17) for all $m = 1, 2, \ldots$ Then, it holds under the null hypothesis and for all real x that*

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\leq k<\infty} \frac{\hat{P}_{m,k}}{\hat{\sigma}_m\, g_o(m,k)} - b_m \leq x\right) = \exp\left(-e^{-x}\right) \tag{3.2.21}$$

and

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\leq k<\infty} \frac{|\hat{P}_{m,k}|}{\hat{\sigma}_m\, g_o(m,k)} - b_m \leq x\right) = \exp\left(-2e^{-x}\right), \tag{3.2.22}$$

where

$$a_m = (2\log\log m)^{1/2} \tag{3.2.23}$$

and

$$b_m = 2\log\log m + \frac{1}{2}\log\log\log m - \frac{1}{2}\log\pi. \tag{3.2.24}$$

As in case of known parameters the procedures have asymptotic power one.

Theorem 3.2.4 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,2,\ldots}$ *and* $\{g_o(m,k)\}_{k=1,2,\ldots}$ *be defined as in Theorem 3.2.3. Then, under* H_1 *it holds that*

$$\lim_{m\to\infty} \mathrm{P}\left(\max_{1\le k<\infty} \frac{P_k}{\hat{\sigma}(m)g_o(m,k)} > \hat{c}_1(\alpha,m)\right) = 1 \tag{3.2.25}$$

and under H_2 *we have*

$$\lim_{m\to\infty} \mathrm{P}\left(\max_{1\le k<\infty} \frac{|P_k|}{\hat{\sigma}(m)g_o(m,k)} > \hat{c}_2(\alpha,m)\right) = 1, \tag{3.2.26}$$

where a_m *and* b_m *are defined as in* (3.2.23) *and* (3.2.24), *respectively.*

3.2.5 Proofs

Since

$$\sigma g_o(m,t) \ge \sigma g_c(m,t) \quad \text{for all} \quad t\ge 1 \quad \text{and} \quad m=1,2,\ldots,$$

the most parts of the following proofs are basically identical with the proofs for the closed-end setting.

Proof of Theorem 3.2.1

We define the sequence $\{Q_k\}_{k=1,2,\ldots}$ as

$$Q_k = \sigma\sum_{j=1}^{k} p_{j,k}(W(j)-W(j-1)) \quad \text{for all} \quad k=1,2,\ldots, \tag{3.2.27}$$

where $\{W(t), t\ge 0\}$ is the approximating Wiener-process.

Lemma 3.2.1 *Let the sequences* $\{P_k\}_{k=1,2,\ldots}$ *and* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (3.2.2) *and* (3.2.27), *respectively. Then for any* $\delta\ge 0$ *and all real* x *it holds that*

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0}\max_{1\le k\le(\log t_0)^\delta} \frac{P_k}{\sigma g(t_0,k)} - b_{t_0} \le x\right) = 1 \tag{3.2.28}$$

and

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0}\max_{1\le k\le(\log t_0)^\delta} \frac{Q_k}{\sigma g(t_0,k)} - b_{t_0} \le x\right) = 1. \tag{3.2.29}$$

Proof: The lemma follows by (3.1.55) and (3.1.56), since

$$\sigma_k \le \sigma g(t_0,k) \quad \text{for all} \quad k=1,2,\ldots \quad \text{and} \quad t_0\ge 1.$$

□

Lemma 3.2.2 *Let the sequences* $\{P_k\}_{k=1,2,\ldots}$ *and* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (3.2.2) *and* (3.2.27), *respectively. Then for any* $\delta > 0$ *it holds that*

$$\sup_{(\log t_0)^\delta \le k < \infty} \frac{P_k}{\sigma g(t_0, k)} - \sup_{(\log t_0)^\delta \le k < \infty} \frac{Q_k}{\sigma g(t_0, k)} = \boldsymbol{O}_P\left(\frac{1}{(\log t_0)^{\delta(1/2 - 1/\nu)}}\right) \qquad (3.2.30)$$

as $t_0 \to \infty$.

Proof: As in the proof of Lemma 3.1.1 we derive

$$\sup_{(\log t_0)^\delta \le k < \infty} \frac{|P_k - Q_k|}{\sigma g(t_0, k)} \le 2 \sup_{(\log t_0)^\delta \le k < \infty} \frac{|S(k) - \sigma W(k)|}{\sigma g(t_0, k)},$$

where $S(k), k = 1, 2, \ldots$ denote the partial sums of the innovations. Now (3.1.2) implies

$$\sup_{(\log t_0)^\delta \le k < \infty} \frac{|S(k) - \sigma W(k)|}{\sigma g(t_0, k)} = \boldsymbol{O}_P(1) \sup_{(\log t_0)^\delta \le k < \infty} \frac{k^{1/\nu}}{g(t_0, k)} \quad \text{as} \quad t_0 \to \infty$$

and since for all $t_0 \ge 1$ we have

$$\sup_{(\log t_0)^\delta \le k < \infty} \frac{k^{1/\nu}}{g(t_0, k)} \le \sup_{(\log t_0)^\delta \le k < \infty} \frac{1}{\sqrt{2d+1}} \frac{k^{1/\nu}}{k^{1/2}},$$

the lemma follows as $t_0 \to \infty$.

□

Lemma 3.2.3 *Let the sequence* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (3.2.27). *Then*

$$\sup_{t_0 \le k < \infty} \frac{Q_k}{\sigma g(t_0, k)} = \boldsymbol{o}_P(1) \quad \textit{as} \quad t_0 \to \infty. \qquad (3.2.31)$$

Proof: The law of the iterated logarithm for weighted sums (Li and Tomkins, 1996) yields

$$\sup_{t_0 \le k < \infty} \frac{Q_k}{\sigma g(t_0, k)} = \boldsymbol{O}(1) \sup_{t_0 \le k < \infty} \frac{\sqrt{k \log \log k}}{\sqrt{k \log\left(\frac{k^{2d+1}}{t_0} + e\right)}} \quad \text{a.s.} \quad \text{as} \quad t_0 \to \infty.$$

and elementary calculations show that

$$\max_{t_0 \le k < \infty} \frac{\sqrt{\log \log k}}{\sqrt{\log\left(\frac{k^{2d+1}}{t_0} + e\right)}} = \frac{\sqrt{\log \log t_0}}{\sqrt{\log\left(t_0^{2d} + e\right)}},$$

implying the lemma as $t_0 \to \infty$.

□

Next, we replace $\{Q_k\}_{k=1,2,\ldots}$ by the continuous-time process $\{U(t)\}_{t\geq 1}$ defined as

$$U(t) = \sigma \int_0^t p\,(x,t)\; dW(x) = \sigma \int_0^t \left(\frac{x}{t}\right)^d dW(x), \quad t \geq 1. \tag{3.2.32}$$

Lemma 3.2.4 *Let the processes $\{Q_k\}_{k=1,2,\ldots}$ and $\{U(t)\}_{1\leq t<\infty}$ be defined as in* (3.2.27) *and* (3.2.32)*, respectively. Then for all $\epsilon > 0$ it holds that*

$$\sup_{(\log t_0)^{1+\epsilon}\leq t\leq t_0} \frac{U(t)}{\sigma g(t_0,t)} - \sup_{(\log t_0)^{1+\epsilon}\leq t\leq t_0} \frac{Q_{\lfloor t\rfloor}}{\sigma g(t_0,\lfloor t\rfloor)} = \boldsymbol{O}_P\left(\frac{1}{(\log t_0)^{\epsilon/2}}\right) \tag{3.2.33}$$

as $t_0 \to \infty$.

Proof: Since

$$\sigma_t \leq \sigma g(t_0,t) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad t_0 \geq 1$$

and elementary calculations show that

$$\frac{1}{g(t_0,\lfloor t\rfloor)} - \frac{1}{g(t_0,t)} = \boldsymbol{O}\left(\frac{1}{t}\right) \quad \text{as} \quad t\to\infty, \quad \text{uniformly in} \quad t_0 \geq 1,$$

the proof is just a repetition of the arguments used to show Lemma (3.1.2).

□

Lemma 3.2.5 *Let the process $\{U(t)\}_{t\geq 1}$ be defined as in* (3.2.32)*. Then for all $\delta \geq 0$ it holds that*

$$\sup_{1\leq t\leq (\log t_0)^\delta} \frac{U(t)}{\sigma g(t_0,t)} = \boldsymbol{O}_P\left(\sqrt{\log\log\log t_0}\right) \quad \textit{as} \quad t_0 \to \infty. \tag{3.2.34}$$

Proof: We already know that

$$\left\{\frac{\sqrt{2d+1}}{\sigma\sqrt{t}}\,U(t), 1 \leq t < \infty\right\} \stackrel{\mathrm{D}}{=} \left\{\frac{W(t^{2d+1})}{t^{d+1/2}}, 1 \leq t < \infty\right\}, \tag{3.2.35}$$

hence, we consider

$$\sup_{1\leq t\leq (\log t_0)^\delta} \frac{W(t^{2d+1})}{\sqrt{t^{2d+1}\log\left(\frac{t^{2d+1}}{t_0}+e\right)}} = \sup_{1\leq t\leq (\log t_0)^{\delta(2d+1)}} \frac{W(t)}{\sqrt{t\log\left(\frac{t}{t_0}+e\right)}}$$

and the lemma follows by the law of the iterated logarithm as $t_0 \to \infty$.

□

Lemma 3.2.6 *Let the process* $\{U(t)\}_{t\geq 1}$ *be defined as in* (3.2.32). *Then*

$$\sup_{t_0^{1/(2d+1)}\leq t<\infty} \frac{U(t)}{\sigma g(t_0,t)} = \boldsymbol{O}_P(1) \quad as \quad t_0\to\infty. \tag{3.2.36}$$

Proof: In view of (3.2.35) we can consider

$$\sup_{t_0^{1/(2d+1)}\leq t<\infty} \frac{W(t^{2d+1})}{\sqrt{t^{2d+1}\log\left(\frac{t^{2d+1}}{t_0}+e\right)}} = \sup_{t_0\leq t<\infty} \frac{W(t)}{\sqrt{t\log\left(\frac{t}{t_0}+e\right)}}$$

and Lemma B.1.1 completes the proof.

□

Lemma 3.2.7 *If we define the process* $\{U(t)\}_{t\geq 1}$ *as in* (3.2.32), *then it holds that*

$$\sup_{(t_0/\log t_0)^{1/(2d+1)}\leq t\leq t_0^{1/(2d+1)}} \frac{U(t)}{\sigma g(t_0,t)} = \boldsymbol{O}_P\left(\sqrt{\log\log\log t_0}\right) \quad as \quad t_0\to\infty. \tag{3.2.37}$$

Proof: The same steps as in the preceding lemma yield

$$\sup_{(t_0/\log t_0)^{1/(2d+1)}\leq t\leq t_0^{1/(2d+1)}} \frac{U(t)}{\sigma g(t_0,t)} \stackrel{\mathrm{D}}{=} \sup_{t_0/\log t_0\leq t\leq t_0} \frac{W(t)}{\sqrt{t\log\left(\frac{t}{t_0}+e\right)}}$$

and Lemma B.1.2 shows that (3.2.37) holds.

□

In the next lemma we derive the limiting distribution of the extremes for the process $\{U(t)/\sigma g(t_0,t)\}_{t\geq 1}$.

Lemma 3.2.8 *Let the process* $\{U(t)\}_{t\geq 1}$ *be defined as in* (3.2.32). *Then, for any* $\delta\geq 0$ *and all real* x, *it holds that*

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0} \sup_{(\log t_0)^{\delta}\leq t\leq (t_0/\log t_0)^{1/(2d+1)}} \frac{U(t)}{\sigma g(t_0,t)} - b_{t_0}\leq x\right) = \exp\left(-e^{-x}\right), \tag{3.2.38}$$

where a_{t_0} *and* b_{t_0} *are defined in* (3.2.9) *and* (3.2.10), *respectively.*

Proof: Taking into account Lemma 3.2.5, it suffices to show the case $\delta=0$. Applying (3.2.35) we see that

$$\sup_{1\leq t\leq (t_0/\log t_0)^{1/(2d+1)}} \frac{U(t)}{\sigma g(t_0,t)} \stackrel{\mathrm{D}}{=} \sup_{1\leq t\leq t_0/\log t_0} \frac{W(t)}{\sqrt{t\log\left(\frac{t}{t_0}+e\right)}}$$

and Lemma B.1.3 together with Lemma B.1.4 yield (3.2.38).

□

Assertion (3.2.7) now follows on combining the Lemmas 3.2.1–3.2.8 and since the asymptotic independence of maxima and minima carries over from the underlying extreme value asymptotic (see Theorem B.1.1), we also obtain (3.2.8).

Proof of Theorem 3.2.2

It suffices to show (3.2.11), since (3.2.12) follows by similar considerations.

If $\{P_k^{(0)}\}_{k=1,2,\ldots}$ denotes the extension of (3.1.57) to an open time horizon, we get the decomposition

$$P_k = P_k^{(0)} + \sum_{j=k^*+1}^{k} p_{j,k}\, \triangle, \quad k = 1, 2, \ldots$$

For any finite $N > k^*$ it holds that

$$\begin{aligned} \mathrm{P}&\left(\max_{1\le k<\infty} \frac{P_k}{\sigma g(t_0,k)} > c_1(\alpha, t_0) \right) \\ &\ge \mathrm{P}\left(\max_{1\le k\le N} \frac{P_k}{\sigma g(t_0,k)} > c_1(\alpha, t_0) \right) \\ &\ge \mathrm{P}\left(\max_{1\le k\le N} \frac{\sum_{j=k^*+1}^{k} p_{j,k}\, \triangle}{\sigma g(t_0,k)} - \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma g(t_0,k)} > c_1(\alpha, t_0) \right) \\ &\ge \mathrm{P}\left(\frac{\sum_{j=k^*+1}^{N} p_{j,N}\, \triangle}{\sigma g(t_0,N)} - \max_{1\le k\le N} \frac{|P_k^{(0)}|}{\sigma g(t_0,k)} > c_1(\alpha, t_0) \right) \end{aligned} \tag{3.2.39}$$

and we have

$$\sum_{j=k^*+1}^{N} p_{j,N}\, \triangle \ge \frac{\triangle}{d+1} \frac{N^{d+1} - {k^*}^{d+1}}{N^d}.$$

If we choose $N = \lfloor t_0 \rfloor$, then it follows by the definition of g that

$$\frac{\sum_{j=k^*+1}^{\lfloor t_0 \rfloor} p_{j,}\, \triangle}{\sigma g(t_0, \lfloor t_0 \rfloor)} \gtrsim t_0^{\rho} \quad \text{for all} \quad \rho < \frac{1}{2} \quad \text{as} \quad t_0 \to \infty. \tag{3.2.40}$$

Now Theorem 3.2.1 implies

$$\max_{1\le k<\infty} \frac{|P_k^{(0)}|}{\sigma g(t_0,k)} = \boldsymbol{O}_P\left(\sqrt{\log\log t_0}\right) \quad \text{as} \quad t_0 \to \infty \tag{3.2.41}$$

and from the definition of a_{t_0} and b_{t_0} follows

$$c_1(\alpha, t_0) \simeq \sqrt{\log\log t_0} \quad \text{as} \quad t_0 \to \infty. \tag{3.2.42}$$

Hence, (3.2.11) follows by putting together (3.2.39)–(3.2.42).

Proof of Theorem 3.2.3

Lemma 3.2.9 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,2,\ldots}$ *and* $\{\hat{Q}_{m,k}\}_{k=1,2,\ldots}$ *be defined as in* (3.1.27) *and* (3.1.59) *with* $N = \infty$. *The sequence* $\{g_o(m,k)\}_{k=1,2,\ldots}$ *is given by* (3.2.17). *If* $\tilde{n}$ *and is a non-decreasing functions of* m *with* $1 \le \tilde{n} \le \tilde{N}$, *where* $\tilde{N}$ *is either a non-decreasing function of* N, *or* $\tilde{N} = \infty$. *Then*

$$\max_{\tilde{n}<k\le\tilde{N}} \frac{\hat{P}_{m,k}}{\sigma g_o(m,k)} - \max_{\tilde{n}<k\le\tilde{N}} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} = \boldsymbol{O}_P\left(\frac{1}{\tilde{n}^{1/2-1/\nu}} + \frac{1}{m^{1/2-1/\nu}}\right) \quad \text{as} \quad m \to \infty. \tag{3.2.43}$$

Proof: We have

$$\begin{aligned}
&\max_{\tilde{n}<k\le\tilde{N}} \frac{\hat{P}_{m,k}}{\sigma g_o(m,k)} - \max_{\tilde{n}<k\le\tilde{N}} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} \\
&\qquad \le \max_{\tilde{n}<k\le\tilde{N}} \frac{1}{\sigma g_o(m,k)} \left| \sum_{j=1}^{k} p_{j,k}(\varepsilon_{m+j} - \sigma(W_{1,m}(j) - W_{1,m}(j-1))) \right| \\
&\qquad\quad + \max_{\tilde{n}<k\le\tilde{N}} \frac{1}{\sigma g_o(m,k)} \left| \frac{\sum_{j=1}^{k} p_{j,k}}{m} \left(\sum_{i=1}^{m} \varepsilon_i - \sigma W_{2,m}(m) \right) \right| \\
&\qquad =: I_1(m) + I_2(m).
\end{aligned}$$

Recalling the proofs of Lemma (3.1.1) and Lemma 3.1.5, we see that the results did not depend on the upper bound of the domain for the maximum. Since furthermore

$$g_o(m,k) \ge g_c(m,k) \quad \text{for all} \quad k = 1,2,\ldots, \quad m = 1,2,\ldots,$$

the proofs can be modified in an obvious way to show the lemma.

□

Lemma 3.2.10 *Let the sequences* $\{\hat{P}_{m,k}\}_{k=1,2,\ldots}$, $\{\hat{Q}_{m,k}\}_{k=1,2,\ldots}$ *and* $\{g_o(m,k)\}_{k=1,2,\ldots}$ *be defined as in Lemma 3.2.9. Then for all real* x *and* $\delta > 0$ *it holds that*

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le(\log m)^\delta} \frac{\hat{P}_{m,k}}{\sigma g_o(m,k)} - b_m \le x\right) = 1 \tag{3.2.44}$$

and

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1\le k\le(\log m)^\delta} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} - b_m \le x\right) = 1. \tag{3.2.45}$$

Proof: The lemma obviously follows by the proof of Lemma 3.1.4, since

$$g_o(m,k) \ge g_c(m,k) \quad \text{for all} \quad k = 1,2,\ldots \quad \text{and} \quad m = 1,2,\ldots$$

□

Lemma 3.2.11 *Let the sequences* $\{\hat{Q}_{m,k}\}_{k=1,2,\ldots}$ *and* $\{g_o(m,k)\}_{k=1,2,\ldots}$ *be defined as in Lemma 3.2.9. Then*

$$\max_{m\le k<\infty}\frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)}=\boldsymbol{O}_P(1)\quad \textit{as}\quad m\to\infty. \tag{3.2.46}$$

Proof: We have

$$\begin{aligned}
&\max_{m\le k<\infty}\frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)}\\
&\qquad\le \max_{m\le k<\infty}\frac{1}{\sigma g_o(m,k)}\left|\sum_{j=1}^{k}p_{j,k}(\sigma(W_{1,m}(j)-W_{1,m}(j-1)))\right|\\
&\qquad\quad+\max_{m\le k<\infty}\frac{1}{\sigma g_o(m,k)}\left|\frac{\sum_{j=1}^{k}p_{j,k}}{m}\,\sigma W_{2,m}(m)\right|\\
&\qquad=: I_1(m)+I_2(m).
\end{aligned} \tag{3.2.47}$$

Since

$$g_o(m,k)\ge g(m,k)\quad\text{for all}\quad k=1,2,\ldots,\quad m=1,2,\ldots,$$

where $g(m,k)$ is defined in (3.2.3), Lemma 3.2.3 implies that

$$I_1(m)=\boldsymbol{o}_P(1)\quad\text{as}\quad m\to\infty. \tag{3.2.48}$$

For the second term we get

$$I_2(m)=\boldsymbol{O}_P(1)\max_{m\le k<\infty}\frac{\frac{1}{\sqrt{m}}\sum_{j=1}^{k}p_{j,k}}{g_o(m,k)}\quad\text{as}\quad m\to\infty$$

and, since for all $k=1,2,\ldots$

$$\sum_{j=1}^{k}p_{j,k}\le\frac{1}{d+1}k+1,$$

we have

$$\max_{m\le k<\infty}\frac{\frac{1}{\sqrt{m}}\sum_{j=1}^{k}p_{j,k}}{g_o(m,k)}\le\max_{m\le k<\infty}\frac{\sqrt{D}\frac{k}{m}+\frac{\sqrt{2d+1}}{m}}{\sqrt{\frac{k}{m}\left(1+D\frac{k}{m}\right)\log\left(\frac{k^{2d+1}}{m}+e\right)}}.$$

Now

$$\max_{m\le k<\infty}\frac{\sqrt{D}\frac{k}{m}}{\sqrt{\frac{k}{m}\left(1+D\frac{k}{m}\right)\log\left(\frac{k^{2d+1}}{m}+e\right)}}\le\max_{m\le k<\infty}\frac{1}{\sqrt{\log\left(\frac{k^{2d+1}}{m}+e\right)}}$$
$$=\frac{1}{\sqrt{\log\left(m^{2d}+e\right)}}$$
$$=\boldsymbol{o}(1)\quad\text{as}\quad m\to\infty,$$

hence, also

$$\max_{m\le k<\infty}\frac{\frac{\sqrt{2d+1}}{m}}{\sqrt{\frac{k}{m}\left(1+D\frac{k}{m}\right)\log\left(\frac{k^{2d+1}}{m}+e\right)}}=\boldsymbol{o}(1)\quad\text{as}\quad m\to\infty,$$

so that

$$I_2(m)=\boldsymbol{o}_P(1)\quad\text{as}\quad m\to\infty \tag{3.2.49}$$

and the lemma follows by (3.2.47), (3.2.48) and (3.2.49).

□

Lemma 3.2.12 *Let $\tilde{n}$ and $\tilde{N}$ be non-decreasing functions of m with $1\le\tilde{n}\le\tilde{N}<\infty$. Then*

$$\max_{\tilde{n}\le\lfloor t\rfloor\le\tilde{N}}\frac{\sum_{j=1}^{\lfloor t\rfloor}p_{j,\lfloor t\rfloor}(W_{1,m}(j)-W_{1,m}(j-1))}{g_o(m,\lfloor t\rfloor)}$$
$$-\max_{\tilde{n}\le t\le\tilde{N}}\frac{\int_0^t p\,(x,t)\;dW_{1,m}(x)}{g_o(m,t)}=\boldsymbol{O}_P\left(\sqrt{\frac{\log\tilde{N}}{\tilde{n}}}\right)\quad\textit{as}\quad m\to\infty \tag{3.2.50}$$

and

$$\max_{\tilde{n}\le t\le\tilde{N}}\frac{\int_0^t p(x,t)dW_{1,m}(x)}{g_o(m,t)}\stackrel{\mathcal{D}}{=}\max_{\tilde{n}\le t\le\tilde{N}}\frac{1}{\sqrt{2d+1}}\frac{\frac{1}{t^d}W_1(t^{2d+1})}{g_o(m,t)}, \tag{3.2.51}$$

where $\{W_1(t),0\le t\}$ is a standard Wiener-process.

Proof: Assertion (3.2.50) follows by the same computations as Lemma 3.1.2, since

$$\sigma g_o(m,t)\ge\sigma_t\quad\text{for all}\quad t\ge1\quad\text{and}\quad m=1,2,\ldots$$

and elementary calculations show that

$$\frac{1}{g_o(m,\lfloor t\rfloor)}-\frac{1}{g_o(m,t)}=\boldsymbol{O}\left(\frac{1}{t}\right)\quad\text{as}\quad t\to\infty,\quad\text{uniformly in}\quad m\in\mathbb{N}.$$

Assertion (3.2.51) follows by comparing the covariances of

$$\left\{\int_0^t p\,(x,t)\;dW_{1,m}(x),t\ge1\right\}\quad\text{and}\quad\left\{\frac{W_1(t^{2d+1})}{t^d\sqrt{2d+1}},t\ge1\right\}.$$

□

Lemma 3.2.13 *Let the sequences* $\{\hat{Q}_{m,k}\}_{k=1,2,\dots}$ *and* $\{g_o(m,k)\}_{k=1,2,\dots}$ *be defined as in Lemma 3.2.9. Then*

$$\max_{(m/\log m)^{1/(2d+1)} \le k \le m} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} = \boldsymbol{O}_P\left(\sqrt{2\log\log\log m}\right) \quad as \quad m \to \infty \tag{3.2.52}$$

Proof: We have

$$\begin{aligned}
&\max_{(m/\log m)^{1/(2d+1)} \le k \le m} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} \\
&\qquad \le \max_{(m/\log m)^{1/(2d+1)} \le k \le m} \frac{1}{\sigma g_o(m,k)} \left| \sum_{j=1}^{k} p_{j,k}(\sigma(W_{1,m}(j) - W_{1,m}(j-1))) \right| \\
&\qquad\quad + \max_{(m/\log m)^{1/(2d+1)} \le k \le m} \frac{1}{\sigma g_o(m,k)} \left| \frac{\sum_{j=1}^{k} p_{j,k}}{m} \sigma W_{2,m}(m) \right| \\
&\qquad =: I_1(m) + I_2(m).
\end{aligned} \tag{3.2.53}$$

Since

$$g_o(m,k) \ge g(m,k) \quad \text{for all} \quad k = 1,2,\dots, \quad m = 1,2,\dots,$$

the Lemmas 3.2.6, 3.2.7 and 3.2.12 imply

$$I_1(m) = \boldsymbol{O}_P\left(\sqrt{2\log\log\log m}\right) \quad \text{as} \quad m \to \infty. \tag{3.2.54}$$

Furthermore, similar computations as in the proof of Lemma 3.2.11 yield

$$I_2(m) = \boldsymbol{O}_P(1) \frac{1}{\sqrt{\log\left(\frac{1}{\log m} + e\right)}} \quad \text{as} \quad m \to \infty. \tag{3.2.55}$$

and the lemma follows by (3.2.53), (3.2.54) and (3.2.55).

□

Lemma 3.2.14 *Let the sequences* $\{\hat{Q}_{m,k}\}_{k=1,2,\dots}$ *and* $\{g_o(m,k)\}_{k=1,2,\dots}$ *be defined as in Lemma 3.2.9. Then*

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{1 \le k < \infty} \frac{\hat{Q}_{m,k}}{\sigma g_o(m,k)} - b_m \le x\right) = \exp\left(-e^{-x}\right) \tag{3.2.56}$$

PROOF: In view of the Lemmas 3.2.9– 3.2.13 it suffices to show that for all real x

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le k \le (m/\log m)^{1/(2d+1)}} \frac{\hat{Q}_{m,k}}{\sigma g(m,k)} - b_m \le x\right) = \exp\left(-e^{-x}\right), \quad (3.2.57)$$

for some $\delta > 1$.

Since $g_o(m,k) \ge g_c(m,k)$ for all $k = 1, 2, \dots$ and $m = 1, 2, \dots$, it follows as in the proof of Lemma 3.1.8 that

$$a_m \max_{(\log m)^\delta \le k \le (m/\log m)^{1/(2d+1)}} \frac{\sum_{j=1}^{k} p_{j,k} \frac{1}{m} |W_{2,m}(m)|}{g_o(m,k)} = \boldsymbol{o}_P(1) \quad \text{as} \quad m \to \infty$$

and we see that (3.2.57) is satisfied, if for all real x and some $\delta > 1$

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le k \le (m/\log m)^{1/(2d+1)}} \frac{\sum_{j=1}^{k} p_{j,k}(W_{1,m}(j) - W_{1,m}(j-1))}{\sigma g_o(m,k)} - b_m \le x\right) = e^{-e^{-x}}. \quad (3.2.58)$$

Lemma 3.2.12 implies that (3.2.58) holds, if

$$\lim_{m\to\infty} \mathrm{P}\left(a_m \max_{(\log m)^\delta \le t \le (m/\log m)^{1/(2d+1)}} \frac{1}{\sqrt{2d+1}} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{g_o(m,t)} - b_m \le x\right) = \exp\left(-e^{-x}\right) \quad (3.2.59)$$

is satisfied and as in the proof of Lemma 3.1.8, it follows that

$$\begin{aligned} &a_m \max_{(\log m)^\delta \le t \le (m/\log m)^{1/(2d+1)}} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{g_o(m,k)} \\ &\quad - a_m \max_{(\log m)^\delta \le t \le (m/\log m)^{1/(2d+1)}} \frac{\frac{1}{t^d} W_1(t^{2d+1})}{\sqrt{t \log\left(\frac{t^{2d+1}}{m} + e\right)}} = \boldsymbol{o}_P(1) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Now without loss of generality we assume that $\delta = \tilde{\delta}/(2d+1)$. Then

$$\begin{aligned} &\max_{(\log m)^{\tilde{\delta}/(2d+1)} \le t \le (m/\log m)^{1/(2d+1)}} \frac{W_1(t^{2d+1})}{\sqrt{t^{2d+1} \log\left(\frac{t^{2d+1}}{m} + e\right)}} \\ &\qquad = \max_{(\log m)^{\tilde{\delta}} \le t \le m/\log m} \frac{W_1(t)}{\sqrt{t \log\left(\frac{t}{m} + e\right)}} \end{aligned}$$

and by Lemma 3.2.5 and Lemma 3.2.8 we see that the proof is complete.

□

Combining the Lemmas 3.2.9– 3.2.14 and since σ can be replaced by $\hat{\sigma}_m$, we obtain the first assertion of Theorem 3.2.3. The second assertion is an immediate consequence of the first assertion and the asymptotic independence of maxima and minima.

Proof of Theorem 3.2.4

We define the sequence $\{\hat{P}^{(0)}_{m,k}\}_{k=1,2,\ldots}$ as

$$\hat{P}^{(0)}_{m,k} = \sum_{j=1}^{k} p_{j,k}\varepsilon_j - \frac{\sum_{j=1}^{k} p_{j,k}}{m}\sum_{i=1}^{m}\varepsilon_i \quad \text{for all} \quad k = 1, 2, \ldots \tag{3.2.60}$$

Under the alternative it holds that

$$\hat{P}_{m,k} = \hat{P}^{(0)}_{m,k} + \sum_{j=k^*+1}^{k} p_{j,k}\ \triangle, \quad k = 1, 2, \ldots$$

and as in the proof of Theorem 3.2.2 we get for any finite $N > k^*$

$$\begin{aligned} &\mathrm{P}\left(\max_{1\le k<\infty} \frac{\hat{P}_{m,k}}{\hat{\sigma}(m)g_0(m,k)} > \hat{c}_1(\alpha, m)\right) \\ &\qquad \ge \mathrm{P}\left(\frac{\sum_{j=k^*+1}^{N} p_{j,N}\ \triangle}{\hat{\sigma}(m)g_o(m,N)} - \max_{1\le k\le N}\frac{|\hat{P}^{(0)}_{m,k}|}{\hat{\sigma}(m)g_o(m,k)} > \hat{c}_1(\alpha, m)\right). \end{aligned} \tag{3.2.61}$$

For $N = m^{3/2}$ it holds that

$$\sum_{j=k^*+1}^{m^{3/2}} p_{j,m^{3/2}}\ \triangle \simeq m^{3/2} \quad \text{as} \quad m \to \infty$$

and since the definition of g_o yields

$$\sigma g_o(m,k) \simeq m\sqrt{\log m} \quad \text{as} \quad m \to \infty,$$

we see that

$$\frac{\sum_{j=k^*+1}^{m^{3/2}} p_{j,m^{3/2}}\ \triangle}{\sigma g_o(m)} \simeq \sqrt{\frac{m}{\log m}} \quad \text{as} \quad m \to \infty.$$

Now Theorem 3.2.3 implies

$$\max_{1\le k<\infty} \frac{|\hat{P}^{(0)}_{m,k}|}{\hat{\sigma}_m g_o(m,k)} = \boldsymbol{O}_P\left(\sqrt{\log\log m}\right) \quad \text{as} \quad m \to \infty$$

and it holds that

$$\hat{c}_1(\alpha, m) \simeq \sqrt{\log\log m} \quad \text{as} \quad m \to \infty,$$

hence, since

$$\frac{1}{\hat{\sigma}_m} - \frac{1}{\sigma} = \boldsymbol{o}_P\left(\frac{1}{m^{\vartheta}}\right) \quad \text{for some} \quad \vartheta > 0,$$

it follows that the drift term is dominating in (3.2.61), which gives (3.2.25). Assertion (3.2.26) follows by similar arguments.

Chapter 4

Control charts based on fractionally weighted moving averages

A carefully study of the proofs for the PWMA-chart shows that the application of the invariance principle would only require a boundary function, which is slightly stronger increasing than the rate of the approximation. Hence, in case of independent observations, we see that the number of existing moments of the innovations yields a lower bound for possible weight functions. This observation laid the foundation for a control-chart, which is based on fractionally weighted moving averages.

The chart is only provided for known parameters. An extension to estimated parameters, which follows the lines of the proof for the PWMA-chart, will be part of future work.

4.1 Closed-end control charts

4.1.1 Model assumptions for known μ and σ

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$, with

$$\mathrm{E}\,\varepsilon_i = 0 \quad \text{and} \quad \operatorname{Var}\varepsilon_i = \sigma^2 > 0 \quad \text{for all} \quad i = 1, 2, \ldots. \tag{4.1.1}$$

We assume the there exists a Wiener process $\{W(t), t \geq 0\}$, satisfying

$$\sup_{1 \leq k < \infty} \frac{1}{k^{1/\nu}} \left(\sum_{i=1}^{k} \varepsilon_i - \sigma W(k) \right) < \infty \quad \text{a.s.} \tag{4.1.2}$$

for some $\nu > 2$.

The sequence of observations $\{X_i\}_{i=1,\dots,N}$ follows the model

$$X_i = \begin{cases} \mu + \varepsilon_i & : 1 \le i \le k^*, \\ \mu + \triangle + \varepsilon_i & : k^* < i \le N, \end{cases} \tag{4.1.3}$$

where μ, k^* and $\triangle$ denote the usual parameters.

We are interested in testing either

$$H_0 : k^* = N \quad \text{versus} \quad H_1 : k^* < N,\ \triangle > 0 \quad \text{(one-sided alternative)}, \tag{4.1.4}$$

or

$$H_0 : k^* = N \quad \text{versus} \quad H_2 : k^* < N,\ \triangle \neq 0 \quad \text{(two-sided alternative)}. \tag{4.1.5}$$

4.1.2 Monitoring procedures for known μ and σ

Let $f_\phi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ be defined as

$$f_\phi(x) = \begin{cases} \dfrac{1}{1/2+\phi} & , \quad x = 0, \\ \dfrac{1}{x^{1/2-\phi}} & , \quad x > 0, \end{cases} \tag{4.1.6}$$

where $1/\nu < \phi \le 1/2$. In the sequel we omit the index ϕ and for integer-valued x we also use the notation $f(x) = f_x$.

The detectors are given by

$$F_k = \sum_{j=0}^{k-1} f_j (X_{k-j} - \mu) \quad \text{for all} \quad k = 1, \dots, N \tag{4.1.7}$$

and we define

$$\sigma_t^2 = \sigma^2 \int_0^t f^2(x)dx = \frac{\sigma^2}{2\phi} t^{2\phi} \quad \text{for all} \quad t \ge 0. \tag{4.1.8}$$

The stopping times are given by

$$\tau_1 = \tau_1(\alpha, N) = \inf\{1 \le k \le N : F_k > c_1(\alpha, N)\sigma_k\} \tag{4.1.9}$$

and

$$\tau_2 = \tau_2(\alpha, N) = \inf\{1 \le k \le N : |F_k| > c_2(\alpha, N)\sigma_k\}, \tag{4.1.10}$$

where $\alpha \in]0, 1[$, if $\tau_i < \infty$, $i = 1, 2$.

The critical constants $c_1(\alpha, N)$ and $c_2(\alpha, N)$ are chosen as

$$c_1(\alpha, N) = \frac{q_1(1-\alpha) + b_N}{a_N} \quad \text{and} \quad c_2(\alpha, N) = \frac{q_2(1-\alpha) + b_N}{a_N}, \tag{4.1.11}$$

where a_N and b_N are defined in (4.1.15) and (4.1.15) below and

$$q_1(1-\alpha) = -\log(-\log(1-\alpha)), \quad q_2(1-\alpha) = -\log\left(-\frac{1}{2}\log(1-\alpha)\right). \tag{4.1.12}$$

The following theorems show that the false alarm rate of the procedures is asymptotically α and both have asymptotic power one.

Theorem 4.1.1 *Let the sequence* $\{F_k\}_{k=1,2,\ldots}$ *be defined as in* (4.1.7)*. If* $\{\sigma_k\}_{k=1,\ldots,N}$ *is given by* (4.1.8)*, then, it holds under the null hypothesis and for all real* x *that*

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{F_k}{\sigma_k} - b_N \le x\right) = \exp\left(-e^{-x}\right) \tag{4.1.13}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|F_k|}{\sigma_k} - b_N \le x\right) = \exp\left(-2e^{-x}\right), \tag{4.1.14}$$

where

$$a_N = \sqrt{2\log\log N}, \tag{4.1.15}$$

$$b_N = 2\log\log N + \frac{1-\phi}{2\phi}\log\log\log N + \log\left(C^{1/(2\phi)} H_{2\phi} \frac{2^{(1-\phi)/(2\phi)}}{\sqrt{2\pi}}\right) \tag{4.1.16}$$

and

$$C = 2\phi \int_0^\infty z^{-1+2\phi}\left(1 - \left(1 + \frac{1}{z}\right)^{-1/2+\phi}\right) dz. \tag{4.1.17}$$

The definition of the constant $H_{2\phi}$ *is given, for example, in Leadbetter, Lindgren and Rootzén (1983).*

Theorem 4.1.2 *Let the sequences* $\{F_k\}_{k=1,\ldots,N}$ *and* $\{\sigma_k\}_{1\le k\le N}$ *be defined as in* (4.1.7) *and* (4.1.8)*. If* $k^* = k^*(N) < N - N^\rho$*, where* $\rho > \phi/(1/2+\phi)$*, then, it holds under* H_1 *that for all real* x

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{F_k}{\sigma_k} - b_N > x\right) = 1 \tag{4.1.18}$$

and under H_2

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|F_k|}{\sigma_k} - b_N > x\right) = 1. \tag{4.1.19}$$

4.1.3 Discussion

Even though a simulation is not provided, we discuss briefly the related problems. Only two values of the constant H are explicitly known, namely $H_1 = 1$ and $H_2 = 1/\pi$. Shao (1996) provided boundaries for H, so that a more conservative adjustment of the boundary function is possible. However, detailed study is required and will be carried out, when the proof for the FWMA-chart with estimated parameters is complete.

4.1.4 Proofs

In the sequel, sums of the innovations will also be denoted by $S(k) = \sum_{i=1}^{k} \varepsilon_i$.

Intermediate results in the following proofs are derived via integration by parts, but since the weight function has a singularity in zero, we need the following considerations.

Preliminaries

Let $\{W(t), -\infty < t < \infty\}$ be the two-sided extension of the Wiener-process, claimed in (4.1.2). We define the process $\{H(t)\}_{t\geq 0}$ as

$$H_t = \int_{t-1}^{t} f(t-x)\, dW(x), \quad t \geq 0. \tag{4.1.20}$$

Note that the process is well defined, since f is square-integrable.

Obviously $\operatorname{Cov}(H(s), H(t)) = 0$, if $|s-t| > 1$ and for $|s-t| \leq 1$ we get $(s \leq t)$

$$\begin{aligned}\operatorname{Cov}(H(s), H(t)) &= \int_{t-1}^{s} (s-x)^{-1/2+\phi}(t-x)^{-1/2+\phi} dx \\ &= \int_{0}^{1-(t-s)} y^{-1/2+\phi}(y+(t-s))^{-1/2+\phi} dy,\end{aligned}$$

implying that $\{H(t)\}_{t\geq 0}$ is a stationary process with

$$\operatorname{Var} H(t) = \int_0^1 y^{-1+2\phi} dy = \frac{1}{2\phi}. \tag{4.1.21}$$

The autocorrelation function of $\{H(t)\}_{t\geq 0}$ is given by

$$r(h) = \begin{cases} 0 & , \quad |h| > 1, \\ 2\phi \displaystyle\int_0^{1-|h|} y^{-1/2+\phi}(y+|h|)^{-1/2+\phi} dy & , \quad |h| \leq 1,\ 0 < \phi < \dfrac{1}{2}, \\ 1 - |h| & , \quad |h| \leq 1,\ \phi = \dfrac{1}{2}. \end{cases} \tag{4.1.22}$$

The next lemma provides the limiting distribution for the extremes of the process $\{H(t)\}_{t\geq 0}$.

Lemma 4.1.1 *Let the Gaussian process $\{H(t)\}_{t\geq 0}$ be defined as in (4.1.20). Then, for all real x it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(A_N \max_{0\leq t\leq N} \sqrt{2\phi}H(t) - B_N \leq x\right) = \exp\left(-e^{-x}\right) \tag{4.1.23}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(A_N \max_{0\leq t\leq N} |\sqrt{2\phi}H(t)| - B_N \leq x\right) = \exp\left(-2e^{-x}\right), \tag{4.1.24}$$

where

$$A_N = \sqrt{2\log N}, \tag{4.1.25}$$

$$B_N = 2\log N + \frac{1-\phi}{2\phi}\log\log N + \log\left(C^{1/2\phi} H_{2\phi} \frac{2^{(1-\phi)/2\phi}}{\sqrt{2\pi}}\right) \tag{4.1.26}$$

and

$$C = 2\phi \int_0^\infty z^{-1+2\phi}\left(1 - \left(1 + \frac{1}{z}\right)^{-1/2+\phi}\right) dz. \tag{4.1.27}$$

$H_{2\phi}$ is defined as in Theorem 4.1.1.

Proof: In view of Theorem 12.3.5 in Leadbetter, Lindgren and Rootzén (1983), we have to show that

$$r(h)\log h \to 0 \quad \text{as} \quad h \to \infty \tag{4.1.28}$$

and

$$r(h) = 1 - C|h|^\alpha + \boldsymbol{o}\left(|h|^\alpha\right) \quad \text{as} \quad h \to 0 \quad \text{for some} \quad 0 < \alpha \leq 2 \quad \text{and} \quad C > 0. \tag{4.1.29}$$

In case of $\phi = 1/2$ it is easy to see that these conditions hold, hence, we only consider the case $0 < \phi < 1/2$. Obviously (4.1.28) is satisfied, so it remains to show (4.1.29). Note that ($h \geq 0$)

$$2\phi \int_0^{1-h} (y+h)^{-1+2\phi} dy \leq r(h) \leq 2\phi \left(\int_0^{1-h} y^{-1+2\phi} dy \int_0^{1-h} (y+h)^{-1+2\phi} dy\right)^{1/2},$$

where the right-hand side of the inequality follows by the Cauchy-Schwarz inequality. Elementary calculations show that

$$1 - h^{2\phi} \leq r(h) \leq (1-h)^\phi \left(1 - h^{2\phi}\right)^{1/2} \leq \left(1 - h^{2\phi}\right)^{1/2},$$

so that for $h \to 0$

$$1 - h^{2\phi} \leq r(h) \leq 1 - \frac{1}{2}h^{2\phi} + \boldsymbol{o}\left(h^{2\phi}\right).$$

Now the last inequality justifies the assumption

$$r(h) = 1 - C|h|^{2\phi} + \boldsymbol{o}\left(|h|^{2\phi}\right) \quad \text{for some} \quad C > 0 \quad (h \to 0), \tag{4.1.30}$$

which will be shown next.

Since

$$1 = 2\phi \int_0^{1-h} y^{-1+2\phi} dy + 2\phi \int_{1-h}^1 y^{-1+2\phi} dy$$

we have

$$\begin{aligned}
\frac{1 - r(h)}{h^{2\phi}} &= \frac{2\phi}{h^{2\phi}} \left(\int_0^{1-h} y^{-1+2\phi} \left(1 - \left(1 + \frac{h}{y}\right)^{-1/2+\phi} \right) dy + \int_{1-h}^1 y^{-1+2\phi} dy \right) \\
&= \frac{2\phi}{h} \int_0^{1-h} \left(\frac{y}{h}\right)^{-1+2\phi} \left(1 - \left(1 + \frac{h}{y}\right)^{-1/2+\phi} \right) dy + \frac{1}{h^{2\phi}} \left(1 - (1-h)^{2\phi}\right) \\
&=: I_1(h) + I_2(h).
\end{aligned}$$

First, we consider $I_1(h)$. We have

$$I_1(h) = 2\phi \int_0^{1/h-1} z^{-1+2\phi} \left(1 - \left(1 + \frac{1}{z}\right)^{-1/2+\phi} \right) dz.$$

Note that for any fixed h the integral is finite, since

$$\int_0^1 z^{-1+2\phi} dz < \infty.$$

Now

$$1 - \left(1 + \frac{1}{z}\right)^{-1/2+\phi} = \left(\frac{1}{2} - \phi\right) \frac{1}{z} + \boldsymbol{o}\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty$$

and we see that ($\phi < 1/2$)

$$\lim_{h \to 0} I_1(h) < \infty. \tag{4.1.31}$$

Next, the Taylor expansion of $\left(1 - (1-h)^{2\phi}\right)$ yields

$$\left(1 - (1-h)^{2\phi}\right) = 2\phi h + \boldsymbol{o}(h) \quad \text{as} \quad h \to 0$$

and it follows that

$$\lim_{h \to 0} I_2(h) = 0. \tag{4.1.32}$$

(4.1.31) and (4.1.32) now show that (4.1.30) holds with $C = \lim_{h\to 0} I_1(h)$ and Theorem 12.3.5 of Leadbetter, Lindgren and Rootzén et al. completes the proof of (4.1.23). The statement given in display (4.1.24) follows by (4.1.23) and the asymptotic independence of maxima and minima.

□

Remark 4.1.1 *An immediate consequence of Lemma 4.1.1 is that*

$$\max_{0\le t\le N} |H_t| = \boldsymbol{O}_P\left(\sqrt{\log N}\right) \quad \textit{as} \quad N \to \infty. \tag{4.1.33}$$

Proof of Theorem 4.1.1

We define

$$Q_k = \sigma \sum_{j=0}^{k-1} f_j(W(k-j) - W(k-j-1)) \quad \text{for all} \quad 1 \le k \le N, \tag{4.1.34}$$

where $\{W(t), t \ge 0\}$ is the Wiener-process, introduced in (4.1.2).

Lemma 4.1.2 *Let the sequences* $\{F_k\}_{k=1,\dots,N}$ *and* $\{Q_k\}_{k=1,\dots,N}$ *be defined as in* (4.1.7) *and* (4.1.34)*, respectively. If* n *is a non-decreasing, integer-valued function of* N*, with* $1 \le n \le N$*, then*

$$\max_{1\le k\le n} \frac{F_k}{\sigma_k} = \boldsymbol{O}_P\left(\sqrt{\log\log n}\right) \quad \textit{as} \quad N \to \infty \tag{4.1.35}$$

and

$$\max_{1\le k\le n} \frac{Q_k}{\sigma_k} = \boldsymbol{O}\left(\sqrt{\log\log n}\right) \quad \textit{a.s.} \quad \textit{as} \quad N \to \infty. \tag{4.1.36}$$

Proof: Since Q_k is a weighted sum of normal random variables and

$$\sigma \sum_{i=0}^{k-1} f_i^2 \sim \sigma_k^2 \simeq k^{2\phi} \quad \text{as} \quad k \to \infty,$$

the law of the iterated logarithm for weighted sums according to Stadtmüller (Corollary 1, 1984) holds and we get

$$\limsup_{k\to\infty} \frac{Q_k}{\sqrt{2\sigma_k^2 \log\log \sigma_k^2}} = 1 \qquad \text{a.s.},$$

implying (4.1.36). Assertion (4.1.35) follows on combining (4.1.36) with Lemma 4.1.3 below.

□

Lemma 4.1.3 *Let the sequences* $\{F_k\}_{k=1,\dots,N}$ *and* $\{Q_k\}_{k=1,\dots,N}$ *be defined as in* (4.1.7) *and* (4.1.34)*, respectively. If* n *is a non-decreasing, integer-valued function of* N*, with* $1 \le n \le N$*, then*

$$\max_{n\le k\le N} \frac{F_k}{\sigma_k} - \max_{n\le k\le N} \frac{Q_k}{\sigma_k} = \boldsymbol{O}_P\left(\frac{1}{n^{\phi-1/\nu}}\right) \quad \textit{as} \quad N \to \infty. \tag{4.1.37}$$

Proof: With

$$d_i = f_i - f_{i+1} \quad \text{for all} \quad i = 0, 1, \ldots,$$

we get

$$F_k = \sum_{i=0}^{k-1} d_i \left(\sum_{j=1}^{k} \varepsilon_j - \sum_{j=1}^{k-i-1} \varepsilon_j \right) + f_k \sum_{j=1}^{k} \varepsilon_j \quad \text{for all} \quad 1 \le k \le N \tag{4.1.38}$$

and

$$Q_k = \sum_{i=0}^{k-1} d_i \left(W(k) - W(k-i-1) \right) + f_k W(k) \quad \text{for all} \quad 1 \le k \le N. \tag{4.1.39}$$

Then

$$\begin{aligned}
\max_{n \le k \le N} \frac{1}{\sigma_k} |F_k - Q_k| &= \max_{n \le k \le N} \frac{1}{\sigma_k} \Bigg| \sum_{i=0}^{k-1} d_i \left(S(k) - W(k) \right) \\
&\quad + \sum_{i=0}^{k-1} d_i \left(W(k-i-1) - S(k-i-1) \right) + f_k \left(S(k) - W(k) \right) \Bigg| \\
&\le p_0 \sqrt{2\phi} \frac{1}{n^{\phi - 1/\nu}} \max_{n \le k \le N} \left| \frac{S(k) - W(k)}{k^{1/\nu}} \right| \\
&\quad + p_0 \sqrt{2\phi} \frac{1}{n^{\phi - 1/\nu}} \max_{n \le k \le N} \max_{1 \le i \le k-1} \left| \frac{S(i) - W(i)}{k^{1/\nu}} \right| \\
&\quad + p_0 \sqrt{2\phi} \frac{1}{n^{\phi - 1/\nu}} \max_{n \le k \le N} \left| \frac{S(k) - W(k)}{k^{1/\nu}} \right|
\end{aligned}$$

and (4.1.37) follows by (4.1.2).

□

Next, we approximate $\{Q_k\}_{1 \le k \le N}$ by the continuous-time process

$$U(t) = \sigma \int_0^t f(t-x) dW(x), \quad t \ge 0. \tag{4.1.40}$$

Lemma 4.1.4 *Let the processes $\{Q_k\}_{k=1,\ldots,N}$ and $\{U(t)\}_{t \ge 0}$ be defined as in (4.1.34) and (4.1.40), respectively. $\{\sigma_t\}_{t \ge 0}$ is given by (4.1.8). If $\tilde{t}$ is a non-decreasing function of N with $1 \le \tilde{t} \le N$, then*

$$\left(\sup_{\tilde{t} \le t \le N} \frac{Q_{\lfloor t \rfloor}}{\sigma_{\lfloor t \rfloor}} - \sup_{\tilde{t} \le t \le N} \frac{U(t)}{\sigma_t} \right) = \boldsymbol{O}_P \left(\frac{\sqrt{\log N}}{\tilde{t}^{\phi}} \right) \quad \textit{as} \quad N \to \infty. \tag{4.1.41}$$

PROOF: Integration by parts yields

$$
\begin{aligned}
\frac{U(t)}{\sigma} &= \int_0^{t-1} f(t-x)dW(x) + \int_{t-1}^{t} f(t-x)dW(x) \\
&= f(t-x)W(x)\Big|_0^{t-1} + \int_0^{t-1} f'(t-x)W(x)dx + \int_{t-1}^{t} f(t-x)dW(x) \\
&= f(1)W(t-1) + \int_1^{t} f'(x)W(t-x)dx + \int_{t-1}^{t} f(t-x)dW(x) \\
&= f(1)W(t-1) + \sum_{i=1}^{\lfloor t\rfloor -1} \int_i^{i+1} f'(x)W(t-x)dx + \int_{\lfloor t\rfloor}^{t} f'(x)W(t-x)dx \\
&\quad + \int_{t-1}^{t} f(t-x)dW(x). \qquad (4.1.42)
\end{aligned}
$$

Furthermore, using (4.1.39) we get

$$
\begin{aligned}
\frac{Q_{\lfloor t\rfloor}}{\sigma} &= \sum_{i=0}^{\lfloor t\rfloor -1} (f_i - f_{i+1})\left(W(\lfloor t\rfloor) - W(\lfloor t\rfloor - i - 1)\right) + f_{\lfloor t\rfloor} W(\lfloor t\rfloor) \\
&= f(0)W(\lfloor t\rfloor) - \sum_{i=0}^{\lfloor t\rfloor -1} (f_i - f_{i+1}) W(\lfloor t\rfloor - i - 1) \\
&= f(0)\left(W(\lfloor t\rfloor) - W(\lfloor t\rfloor - 1)\right) + f(1)W(\lfloor t\rfloor - 1) - \sum_{i=1}^{\lfloor t\rfloor -1} (f_i - f_{i+1}) W(\lfloor t\rfloor - i - 1) \\
&= f(1)W(\lfloor t\rfloor - 1) + \sum_{i=1}^{\lfloor t\rfloor -1} \int_i^{i+1} f'(x)W(\lfloor t\rfloor - i - 1)dx \\
&\quad + f(0)\left(W(\lfloor t\rfloor) - W(\lfloor t\rfloor - 1)\right) \qquad (4.1.43)
\end{aligned}
$$

and (4.1.42), together with (4.1.43) shows that

$$
\begin{aligned}
\frac{1}{\sigma}\sup_{1\le t\le N}\left|U(t) - Q_{\lfloor t\rfloor}\right| \quad &\le \quad \sup_{1\le t\le N} f(1)\left|W(t-1) - W(\lfloor t\rfloor - 1)\right| \\
&\quad + \sup_{1\le t\le N} \sum_{i=1}^{\lfloor t\rfloor -1} \int_i^{i+1} |f'(x)||W(t-x) - W(\lfloor t\rfloor - i - 1)|dx \\
&\quad + \sup_{1\le t\le N} \int_{\lfloor t\rfloor}^{t} |f'(x)||W(t-x)|dx \\
&\quad + \sup_{1\le t\le N} \left|\int_{t-1}^{t} f(t-x)dW(x)\right| \\
&\quad + \sup_{1\le t\le N} f(0)\left|W(\lfloor t\rfloor) - W(\lfloor t\rfloor - 1)\right| \\
&=: \quad I_1(N) + I_2(N) + I_3(N) + I_4(N) + I_5(N). \qquad (4.1.44)
\end{aligned}
$$

First, note that Theorem 1.2.1 of Csörgő and Révész (1981) implies

$$
I_1 = \boldsymbol{O}\left(\sqrt{\log N}\right) \quad \text{a.s.,} \quad I_5 = \boldsymbol{O}\left(\sqrt{\log N}\right) \quad \text{a.s.} \qquad (4.1.45)
$$

and also

$$I_2 = \boldsymbol{O}\left(\sqrt{\log N}\right) \quad \text{a.s.} \quad \text{as} \quad N \to \infty, \tag{4.1.46}$$

where the last equation follows since $|(\lfloor t \rfloor - i - 1) - (t - x)| \leq 2$ for all $x \in [i, i+1]$.

Obviously

$$I_3 = \boldsymbol{O}(1) \quad \text{a.s.} \quad \text{as} \quad N \to \infty \tag{4.1.47}$$

and since we already know by Remark 4.1.1 that

$$I_4 = \boldsymbol{O}_P\left(\sqrt{\log N}\right) \quad \text{as} \quad N \to \infty, \tag{4.1.48}$$

we conclude by (4.1.44)–(4.1.48) that

$$\frac{1}{\sigma} \sup_{1 \leq t \leq N} \left| U(t) - Q_{\lfloor t \rfloor} \right| = \boldsymbol{O}_P\left(\sqrt{\log N}\right) \quad \text{as} \quad N \to \infty. \tag{4.1.49}$$

Now

$$\begin{aligned} \sup_{\tilde{t} \leq t \leq N} \left| \frac{Q_{\lfloor t \rfloor}}{\sigma_{\lfloor t \rfloor}} - \frac{U(t)}{\sigma_t} \right| &\leq \sup_{\tilde{t} \leq t \leq N} \frac{1}{\sigma_{\lfloor t \rfloor}} \left| Q_{\lfloor t \rfloor} - U(t) \right| + \sup_{\tilde{t} \leq t \leq N} |U(t)| \left| \frac{\sigma_t - \sigma_{\lfloor t \rfloor}}{\sigma_t \sigma_{\lfloor t \rfloor}} \right| \\ &=: J_1(N) + J_2(N). \end{aligned}$$

By (4.1.49) and (4.1.8) we see that

$$J_1(N) = \boldsymbol{O}_P\left(\frac{\sqrt{\log N}}{\tilde{t}^{\phi}}\right) \quad \text{as} \quad N \to \infty$$

and Remark 4.1.2 below, together with (4.1.8), implies

$$J_2(N) = \boldsymbol{O}_P\left(\frac{\sqrt{\log \log N}}{\tilde{t}^{2\phi}}\right) \quad \text{as} \quad N \to \infty,$$

completing the proof of the Lemma.

$\square$

We now consider the extremes of $\{U(t), t \geq 0\}$ on $[1, N]$.

Lemma 4.1.5 *Let the process* $\{U(t)\}_{t \geq 0}$ *be defined as in* (4.1.40). *With* $\{\sigma_t\}_{t \geq 0}$ *being defined via* (4.1.8) *for all real* x *holds*

$$\lim_{N \to \infty} \mathrm{P}\left(a_N \max_{1 \leq t \leq N} \frac{U(t)}{\sigma_t} - b_N \leq x\right) = \exp\left(-e^{-x}\right) \quad \textit{as} \quad N \to \infty, \tag{4.1.50}$$

where a_N *and* b_N *are defined in* (4.1.15) *and* (4.1.16), *respectively.*

PROOF: First, note that Lemma (3.1.3) yields the result for $\phi = 1/2$. Hence, in the following we assume that $1/\nu < \phi < 1/2$.

If we define the process $\{\tilde{U}(t)\}_{t\geq 0}$ by

$$\tilde{U}(t) = U(e^t) \quad \text{for all} \quad t \geq 0, \tag{4.1.51}$$

then obviously

$$\max_{1\leq t\leq N} \frac{U(t)}{\sigma_t} = \max_{0\leq t\leq \log N} \frac{\tilde{U}(t)}{\sigma_{e^t}}.$$

The covariance of $\{\tilde{U}(t)\}_{t\geq 0}$ is given by $(0 \leq s \leq t)$

$$\begin{aligned}
\text{Cov}\left(\tilde{U}(s), \tilde{U}(t)\right) &= \sigma^2 \int_0^{e^s} (e^s - x)^{-1/2+\phi} \left(e^t - x\right)^{-1/2+\phi} dx \\
&= \sigma^2 e^{2\phi t} \int_0^{1-(1-e^{s-t})} y^{-1/2+\phi} \left(y + 1 - e^{s-t}\right)^{-1/2+\phi} dy,
\end{aligned}$$

hence

$$\text{Var}\left(\tilde{U}(t)\right) = \frac{\sigma^2 e^{2\phi t}}{2\phi} \tag{4.1.52}$$

and for the autocorrelation function $\tilde{r}$ of $\{\tilde{U}(t)\}_{t\geq 0}$ holds $(\phi < 1/2)$

$$\tilde{r}(h) = 2\phi\ e^{\phi|h|} \int_0^{1-(1-e^{-|h|})} y^{-1/2+\phi} \left(y + 1 - e^{-|h|}\right)^{-1/2+\phi} dy, \quad h \in \mathbb{R}, \tag{4.1.53}$$

showing that $\{\tilde{U}(t)\}_{t\geq 0}$ is a stationary process.

Now, as in the proof of Lemma 4.1.1, we show that

$$\tilde{r}(h) \log h \to 0 \quad \text{as} \quad h \to \infty \tag{4.1.54}$$

and

$$\tilde{r}(h) = 1 - C|h|^\alpha + \boldsymbol{o}\left(|h|^\alpha\right) \quad \text{as} \quad h \to 0 \quad \text{for some} \quad 0 < \alpha \leq 2 \quad \text{and} \quad C > 0. \tag{4.1.55}$$

Since

$$\tilde{r}(h) \leq 2\phi\ e^{\phi|h|} \int_0^{e^{-|h|}} y^{-1+2\phi} dy = e^{-\phi|h|},$$

we see that (4.1.54) is satisfied.

With $1 - e^{-|h|} = \tilde{h}$ and r as in (4.1.22), we get

$$\begin{aligned}
\frac{1 - \tilde{r}(h)}{|h|^{2\phi}} &= \frac{1 - r(\tilde{h}) + r(\tilde{h})(1 - e^{\phi|h|})}{|h|^{2\phi}} \\
&= \frac{1 - r(\tilde{h})}{\tilde{h}^{2\phi}} \cdot \frac{\tilde{h}^{2\phi}}{|h|^{2\phi}} + r(\tilde{h}) \frac{1 - e^{\phi|h|}}{|h|^{2\phi}} \\
&=: I_1(h) + I_2(h).
\end{aligned}$$

The Taylor expansion of $\tilde{h}$, together with the Taylor expansion of $x \mapsto x^{2\phi}$ in 0, yields

$$\lim_{h\to 0} \frac{\tilde{h}^{2\phi}}{|h|^{2\phi}} = 1$$

and the proof of Lemma 4.1.1 shows that

$$\lim_{h\to 0} \frac{1-r(\tilde{h})}{\tilde{h}^{2\phi}} = 2\phi \int_0^\infty z^{-1+2\phi} \left(1 - \left(1+\frac{1}{z}\right)^{-1/2+\phi}\right) dz,$$

hence

$$\lim_{h\to 0} I_1(h) = C, \tag{4.1.56}$$

with C being defined as in (4.1.17).

Since $r(\tilde{h}) \to 1$ as $h \to 0$ and the Taylor expansion of $1 - e^{\phi|h|}$ shows that

$$\frac{1-e^{\phi|h|}}{|h|^{2\phi}} = \boldsymbol{O}\left(h^{1-2\phi}\right) \quad \text{as} \quad h \to 0,$$

it follows that

$$I_2(h) = \boldsymbol{o}(1) \quad \text{as} \quad h \to 0. \tag{4.1.57}$$

Now (4.1.56) and (4.1.57) show that (4.1.55) also holds with $\alpha = 2\phi$ and C chosen as in (4.1.17), hence the assumptions of Theorem 12.3.5 in Leadbetter, Lindgren and Rootzén (1983) are satisfied and the proof of the lemma is complete.

□

Remark 4.1.2 *Lemma 4.1.5, the asymptotic independence of the maximum and minimum and the symmetry of $\{U(t)\}_{t\geq 0}$ immediately imply*

$$\sup_{1\leq t\leq N} \frac{|U(t)|}{\sigma_t} = \boldsymbol{O}_P\left(\sqrt{\log\log N}\right) \quad \textit{as} \quad N \to \infty. \tag{4.1.58}$$

In view of Lemma 4.1.2 and Lemma 4.1.3 for any $\delta > 0$ it holds that

$$\begin{aligned}\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{1\leq k\leq N} \frac{F_k}{\sigma_k} - b_N \leq x\right) &= \lim_{N\to\infty} \mathrm{P}\left(a_N \max_{(\log N)^\delta\leq k\leq N} \frac{F_k}{\sigma_k} - b_N \leq x\right) \\ &= \lim_{N\to\infty} \mathrm{P}\left(a_N \max_{(\log N)^\delta\leq k\leq N} \frac{Q_k}{\sigma_k} - b_N \leq x\right).\end{aligned}$$

If δ is chosen so that $\delta\phi > 1/2$ holds, Lemma 4.1.4 implies

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \max_{(\log N)^\delta\leq k\leq N} \frac{Q_k}{\sigma_k} - b_N \leq x\right) = \lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{(\log N)^\delta\leq t\leq N} \frac{U(t)}{\sigma_t} - b_N \leq x\right)$$

and since an immediate consequence of Remark 4.1.2 is that

$$\lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{(\log N)^\delta \le t\le N} \frac{U(t)}{\sigma_t} - b_N \le x\right) = \lim_{N\to\infty} \mathrm{P}\left(a_N \sup_{1\le t\le N} \frac{U(t)}{\sigma_t} - b_N \le x\right),$$

assertion (4.1.13) of Theorem 4.1.1 follows by Lemma 4.1.5. Assertion (4.1.14) is derived by the usual arguments.

Proof of Theorem 4.1.2

We only show (4.1.18), since (4.1.19) follows by the same arguments. We define the sequence $\{F_k^{(0)}, 1 \le k \le N\}$ as

$$F_k^{(0)} = \sum_{j=0}^{k-1} f_j \varepsilon_{k-j}, \quad 1 \le k \le N, \tag{4.1.59}$$

and get the decomposition

$$F_k = F_k^{(0)} + \sum_{j=0}^{k-k^*-1} f_j \,\triangle, \quad 1 \le k \le N.$$

For the drift holds

$$\begin{aligned}\max_{1\le k\le N} \sum_{j=0}^{k-k^*-1} f_j \,\triangle &= \sum_{j=0}^{N-k^*-1} f_j \,\triangle \\ &\ge \sum_{j=0}^{N^\rho} f_j \,\triangle \\ &\ge \frac{\triangle}{1/2+\phi} N^{\rho(1/2+\phi)}.\end{aligned}$$

and for all real x we get

$$\begin{aligned}&\mathrm{P}\left(a_N \max_{1\le k\le N} \frac{F_k}{\sigma_k} - b_N \le x\right) \\ &\quad\le \mathrm{P}\left(a_N \max_{1\le k\le N} \frac{|F_k^{(0)}|}{\sigma_k} - b_N \le x - a_N \frac{\sum_{j=0}^{N-k^*-1} f_j \,\triangle}{\sigma_N} + 2a_N \max_{1\le k\le N} \frac{|F_k^{(0)}|}{\sigma_k}\right).\end{aligned}$$

By Theorem 4.1.1 we see that

$$a_N \max_{1\le k\le N} \frac{|F_k^{(0)}|}{\sigma_k} - b_N = \boldsymbol{O}_P(1) \quad \text{as} \quad N \to \infty$$

and

$$a_N \max_{1\le k\le N} \frac{|F_k^{(0)}|}{\sigma_k} = \boldsymbol{O}_P(a_N^2) \quad \text{as} \quad N \to \infty,$$

hence (4.1.18) follows since

$$\frac{\sum_{j=0}^{N-k^*-1} f_j \ \triangle}{\sigma_N} \gtrsim N^{\rho(1/2+\phi)-\phi} \quad \text{as} \quad N \to \infty$$

and $\rho(1/2+\phi)-\phi > 0$.

Part II

Conditional Stopping Times

Chapter 5

The conditional stopping time of the MOSUM-chart

A first result on the asymptotic normality for stopping times, which are based on curved boundary functions has been given by Siegmund (1968), followed by the result of Teicher (1973). The asymptotic normality of the stopping time for CUSUM-procedures has been investigated by Aue (2003) and Aue, Horváth, Kokoszka and Steinebach (2007). In this chapter we investigate how the limiting distribution of a conditional stopping time can be used to construct an asymptotic confidence interval for the location of a detected change-point.

5.1 One-sided alternatives

Throughout this section, we adopt the nomenclature of Section 1.1. For the convenience of the reader and since we have to modify the model assumptions, we first repeat the framework and some definitions, which are frequently used in the sequel. Note that all computations are carried out under the alternative H_1.

5.1.1 Model assumptions for known μ and σ

Let $\{\varepsilon_i\}_{i=1,2,\ldots}$ be a sequence of independent, identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathrm{P}\,)$. We assume that

$$\mathrm{E}\,\varepsilon_1 = 0, \quad \mathrm{Var}\,\varepsilon_1 = \sigma^2 > 0 \quad \text{and} \quad \mathrm{E}\,|\,\varepsilon_1|^\nu < \infty \quad \text{for some} \quad \nu > 2. \tag{5.1.1}$$

The sequence $\{X_i\}_{i=1,2,\ldots}$ is assumed to satisfy

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \le i \le h_N + k^*, \\ \mu + \triangle + \varepsilon_i & , \quad h_N + k^* < i \le h_N + N, \end{cases}$$

where $\triangle > 0$. Instead of (1.1.4) and (1.1.5) we now impose a more restrictive condition on the window size h_N. Namely, it is required that

$$\sqrt{h_N} \simeq N^{\phi} \quad \text{as} \quad N \to \infty, \quad \text{where} \quad 1/\nu < \phi < 1/2. \tag{5.1.2}$$

The change-point k^* is assumed to depend on N as follows

$$k^* = k^*(N) = \lfloor \kappa N \rfloor \quad \text{for some} \quad 0 < \kappa < 1. \tag{5.1.3}$$

We are interested in the limit distribution of the stopping time

$$\tau_1 = \tau_1(\alpha, N) = \inf\left\{1 \le k \le N : M_{k,N} > c_1(\alpha, N)\sigma\sqrt{h_N}\right\}, \tag{5.1.4}$$

where

$$M_{k,N} = \sum_{i=k-h_N+1}^{k} (X_{i+h_N} - \mu) \quad \text{for all} \quad k = 1, \ldots, N \tag{5.1.5}$$

and the critical constant $c_1(\alpha, N)$ is defined in (1.1.11).

5.1.2 The conditional limit distribution of τ_1

The first result is obtained for known in-control parameters and provides the conditional limit distribution of τ_1.

Theorem 5.1.1 *Let τ_1 be defined as in* (5.1.4) *and assume that k^* satisfies* (5.1.3). *Then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{(\tau_1 - k^*) - \alpha_N}{\beta_N} \le x \,\middle|\, \tau_1 > k^*\right) = \Phi(x), \tag{5.1.6}$$

where

$$\beta_N = \frac{\sigma\sqrt{h_N}}{\triangle} \quad \textit{and} \quad \alpha_N = c_1(\alpha, N)\beta_N. \tag{5.1.7}$$

Remark 5.1.1 *It can be seen from the proof of Theorem 1.1.2 that the parametrization of k^* does not affect the asymptotic power of the underlying test, hence it still holds that* $\lim_{N\to\infty} \mathrm{P}(\tau_1 \le N) = 1$.

If $\triangle$ is known, which is of course unrealistic, Theorem 5.1.1 allows to define two kinds of asymptotic confidence intervals for k^*.

Corollary 5.1.1 *Let τ_1 be defined as in* (5.1.4) *and assume that k^* satisfies* (5.1.3). *Then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\tau_1 - (c_1(\alpha, N) + x)\frac{\sigma\sqrt{h_N}}{\triangle} \le k^* < \tau_1 \,\middle|\, \tau_1 > k^*\right) = \Phi(x) \tag{5.1.8}$$

and

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 - (c_1(\alpha, N) + x)\frac{\sigma\sqrt{h_N}}{\triangle} \le k^* < \tau_1\right) \ge \Phi(x)(1-\alpha). \tag{5.1.9}$$

Proof: (5.1.8) is a direct consequence of (5.1.6), and (5.1.9) follows, since

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 > k^*\right) \ge 1 - \limsup_{N\to\infty} \mathrm{P}\left(\tau_1 \le k^*\right) \ge 1 - \alpha.$$

□

Next, we treat the case of an unknown level shift $\triangle$. The following lemma provides an adequate estimator for $\triangle$ under the assumption that the level shift is bounded.

Lemma 5.1.1 *Let τ_1 be defined as in* (5.1.4). *We assume that k^* satisfies* (5.1.3) *and furthermore, that $0 < \triangle \le \triangle_{\max}$. If we define*

$$\hat{\triangle}_N = \frac{1}{\lambda_N} \sum_{i=\tau_1-\lambda_N+1}^{\tau_1} X_i - \mu, \tag{5.1.10}$$

where

$$\lambda_N = \left\lfloor \frac{\sigma\sqrt{h_N}}{\triangle_{\max}} \right\rfloor, \tag{5.1.11}$$

then, for any $0 \le \gamma < \phi - 1/\nu$ and $\epsilon > 0$, it holds that

$$\lim_{N\to\infty} \mathrm{P}\left(|\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma} \,\middle|\, \tau_1 > k^*\right) = 0. \tag{5.1.12}$$

We are now prepared to extend Theorem 5.1.1.

Theorem 5.1.2 *Let τ_1 be defined as in* (5.1.4). *We assume that k^* satisfies* (5.1.3) *and that $0 < \triangle \le \triangle_{\max}$. If $\hat{\triangle}_N$ is defined as in* (5.1.10), *then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{(\tau_1 - k^*) - \hat{\alpha}_N}{\hat{\beta}_N} \le x \,\middle|\, \tau_1 > k^*\right) = \Phi(x), \tag{5.1.13}$$

where

$$\hat{\beta}_N = \frac{\sigma\sqrt{h_N}}{\hat{\triangle}_N} \quad \textit{and} \quad \hat{\alpha}_N = c_1(\alpha, N)\hat{\beta}_N. \tag{5.1.14}$$

The corresponding asymptotic confidence intervals are given in the next corollary.

Corollary 5.1.2 *Let* τ_1 *be defined as in* (5.1.4). *We assume that* k^* *satisfies* (5.1.3) *and that* $0 < \Delta \leq \Delta_{\max}$. *If* $\hat{\Delta}_N$ *is defined as in* (5.1.10), *then, for all real* x, *it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\tau_1 - (c_1(\alpha, N) + x)\frac{\sigma\sqrt{h_N}}{\hat{\Delta}_N} \leq k^* < \tau_1 \;\middle|\; \tau_1 > k^*\right) = \Phi(x) \tag{5.1.15}$$

and

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 - (c_1(\alpha, N) + x)\frac{\sigma\sqrt{h_N}}{\hat{\Delta}_N} \leq k^* < \tau_1\right) \geq \Phi(x)(1-\alpha). \tag{5.1.16}$$

5.1.3 Model assumptions for unknown μ and σ

We assume that the sequence of innovations $\{\varepsilon_i\}_{i=1,2,\dots}$ satisfies (5.1.1) and that the observations $\{X_i\}_{i=1,2,\dots}$ follow the model

$$X_i = \begin{cases} \mu + \varepsilon_i & , \quad 1 \leq i \leq m_N + k^*, \\ \mu + \Delta + \varepsilon_i & , \quad m_N + k^* < i \leq m_N + N, \end{cases}$$

where

$$\lim_{N\to\infty}\left(\frac{h_N}{m_N}\log\frac{N}{h_N}\right) = 0 \quad \text{and} \quad \frac{m_N}{N} \to 0 \quad \text{as} \quad N \to \infty. \tag{5.1.17}$$

The window size h_N is chosen as in (5.1.2)and the change-point k^* is assumed to depend on N as in (5.1.3).

We are interested in the limit distribution of the stopping time

$$\hat{\tau}_1 = \hat{\tau}_1(\alpha, N) = \inf\left\{1 \leq k \leq N : \hat{M}_{k,N} > c_1(\alpha, N)\hat{\sigma}_{m_N}\sqrt{h_N}\right\}, \tag{5.1.18}$$

where

$$\hat{M}_{k,N} = \sum_{i=k-h_N+1}^{k} (X_{i+m_N} - \hat{\mu}_{m_N}) \quad \text{for all} \quad k = 1, \dots, N \tag{5.1.19}$$

and the critical constant $c_1(\alpha, N)$ is defined in (1.1.11). Taking into account the results of Lemma 1.1.3 we can assume that the estimators $\hat{\mu}_{m_N}$ and $\hat{\sigma}_{m_N}$ satisfy

$$\hat{\mu}_{m_N} - \mu = \boldsymbol{o}_P\left(\frac{1}{\sqrt{m_N}}\right) \quad \text{as} \quad N \to \infty \tag{5.1.20}$$

and

$$\hat{\sigma}^2_{m_N} - \sigma^2 = \boldsymbol{o}_P\left(\frac{1}{m_N^{\vartheta}}\right) \quad \text{as} \quad N \to \infty \tag{5.1.21}$$

for some $\vartheta > 0$.

5.1.4 The conditional limit distribution of $\hat{\tau}_1$

The next Theorem shows that the conditional limit distribution of $\hat{\tau}_1$ basically coincides with the conditional limit distribution of τ_1. Note that each of the following accents has a different meaning: $\hat{\alpha}$, $\widehat{\alpha}$, $\widehat{\hat{\alpha}}$.

Theorem 5.1.3 *Let $\hat{\tau}_1$ be defined as in (5.1.18) and assume that k^* satisfies (5.1.3). Then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{(\hat{\tau}_1 - k^*) - \widehat{\alpha}_N}{\widehat{\beta}_N} \leq x \;\middle|\; \hat{\tau}_1 > k^*\right) = \Phi(x), \tag{5.1.22}$$

where

$$\widehat{\beta}_N = \frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\triangle} \quad \textit{and} \quad \widehat{\alpha}_N = c_1(\alpha, N)\widehat{\beta}_N. \tag{5.1.23}$$

If $\triangle$ is known, we get the following asymptotic confidence intervals for k^*.

Corollary 5.1.3 *Let $\hat{\tau}_1$ be defined as in (5.1.18) and assume that k^* satisfies (5.1.3). Then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - (c_1(\alpha, N) + x)\frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\triangle} \leq k^* < \hat{\tau}_1 \;\middle|\; \hat{\tau}_1 > k^*\right) = \Phi(x) \tag{5.1.24}$$

and

$$\liminf_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - (c_1(\alpha, N) + x)\frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\triangle} \leq k^* < \hat{\tau}_1\right) \geq \Phi(x)(1-\alpha). \tag{5.1.25}$$

To make use of the theoretical results above, we need an estimator for $\triangle$.

Lemma 5.1.2 *Let $\hat{\tau}_1$ be defined as in (5.1.18). We assume that k^* satisfies (5.1.3) and furthermore, that $0 < \triangle \leq \triangle_{\max}$. If we define*

$$\widehat{\triangle}_N = \frac{1}{\hat{\lambda}_N} \sum_{i=\hat{\tau}_1 - \hat{\lambda}_N + 1}^{\hat{\tau}_1} X_i - \hat{\mu}_{m_N}, \tag{5.1.26}$$

where

$$\hat{\lambda}_N = \left\lfloor \frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\triangle_{\max}} \right\rfloor, \tag{5.1.27}$$

then, for any $0 \leq \zeta < \phi - 1/\nu$ and $\epsilon > 0$, it holds that

$$\lim_{N\to\infty} \mathrm{P}\left(|\widehat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\zeta} \;\middle|\; \tau_1 > k^*\right) = 0. \tag{5.1.28}$$

The following theorem and the corresponding corollary may be considered as the main result of this chapter.

Theorem 5.1.4 *Let $\hat{\tau}_1$ be defined as in* (5.1.18). *We assume that k^* satisfies* (5.1.3) *and that $0 < \triangle \leq \triangle_{\max}$. If $\widehat{\triangle}_N$ is defined as in* (5.1.26), *then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{(\hat{\tau}_1 - k^*) - \widehat{\alpha}_N}{\widehat{\beta}_N} \leq x \,\middle|\, \hat{\tau}_1 > k^* \right) = \Phi(x), \tag{5.1.29}$$

where

$$\widehat{\beta}_N = \frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\widehat{\triangle}_N} \quad \text{and} \quad \widehat{\alpha}_N = c_1(\alpha, N)\widehat{\beta}_N. \tag{5.1.30}$$

Corollary 5.1.4 *Let $\hat{\tau}_1$ be defined as in* (5.1.18). *We assume that k^* satisfies* (5.1.3) *and that $0 < \triangle \leq \triangle_{\max}$. If $\widehat{\triangle}_N$ is defined as in* (5.1.26), *then, for all real x, it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - (c_1(\alpha, N) + x)\frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\widehat{\triangle}_N} \leq k^* < \hat{\tau}_1 \,\middle|\, \hat{\tau}_1 > k^* \right) = \Phi(x) \tag{5.1.31}$$

and

$$\liminf_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - (c_1(\alpha, N) + x)\frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\widehat{\triangle}_N} \leq k^* < \hat{\tau}_1 \right) \geq \Phi(x)(1-\alpha). \tag{5.1.32}$$

5.1.5 Proofs

Proof of Theorem 5.1.1

For any real x we define

$$\delta = \delta(x, \alpha, N) = \frac{(c_1(\alpha, N) + x)\sigma\sqrt{h_N}}{\triangle\, k^*} \tag{5.1.33}$$

and

$$n = n(x, \alpha, N) = k^* + \lfloor \delta k^* \rfloor. \tag{5.1.34}$$

Note that, for any x, it holds that δ is positive if N is large enough and furthermore,

$$\lim_{N\to\infty} \left(c_1(\alpha, N) - \frac{\triangle\, (n - k^*)}{\sigma\sqrt{h_N}} \right) = -x. \tag{5.1.35}$$

Lemma 5.1.3 *Let τ_1 be defined as in* (5.1.4) *and assume that k^* satisfies* (5.1.3). *With $n(x, \alpha, N)$ being defined as in* (5.1.34), *it holds that*

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 \leq n \,|\, \tau_1 > k^*\right) \geq \Phi(x). \tag{5.1.36}$$

PROOF: First, note that

$$\begin{aligned}
P(\tau_1 \le n \,|\, \tau_1 > k^*) &= \frac{P\left(\max_{1\le k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \max_{k^*<k\le n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1\right)}{P(\tau_1 > k^*)} \\
&\ge \frac{P\left(\max_{1\le k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right)}{P(\tau_1 > k^*)}
\end{aligned} \tag{5.1.37}$$

and for the numerator of the last term it holds

$$\begin{aligned}
&P\left(\max_{1\le k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right) \\
&= P\left(\max_{1\le k\le k^*-h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \max_{k^*-h_N<k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right).
\end{aligned}$$

By the stationarity of the innovations, we conclude that

$$P\left(\max_{k^*-h_N<k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1\right) = P\left(\max_{1\le k\le h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1\right)$$

and the law of the iterated logarithm yields (extend the partial sums)

$$\max_{1\le k\le h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} = \boldsymbol{O}\left(\sqrt{\log\log h_N}\right) \quad \text{a.s.} \quad \text{as} \quad N\to\infty.$$

Hence, since $c_1(\alpha, N) \simeq \sqrt{\log(N/h_N)}$ as $N\to\infty$, we have

$$\lim_{N\to\infty} P\left(\max_{1\le k\le h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1\right) = 1 \quad \text{as} \quad N\to\infty,$$

showing that

$$\begin{aligned}
\liminf_{N\to\infty} P&\left(\max_{1\le k\le k^*-h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \max_{k^*-h_N<k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right) \\
&= \liminf_{N\to\infty} P\left(\max_{1\le k\le k^*-h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right) \\
&= \liminf_{N\to\infty} P\left(\max_{1\le k\le k^*-h_N} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1\right) P\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right) \\
&= \liminf_{N\to\infty} P\left(\max_{1\le k\le k^*} \frac{M_{k,N}}{\sigma\sqrt{h_N}} \le c_1\right) P\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right).
\end{aligned}$$

Combined with (5.1.37), the latter result yields

$$\liminf_{N\to\infty} P(\tau_1 \le n \,|\, \tau_1 > k^*) \ge \liminf_{N\to\infty} P\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right).$$

Now

$$\liminf_{N\to\infty} P\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} > c_1\right) = \liminf_{N\to\infty} \mathrm{P}\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} > c_1 - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}}\right)$$

and the central limit theorem together with (5.1.35) yield the Lemma.

□

Lemma 5.1.4 *Let* τ_1 *be defined as in* (5.1.4) *and assume that* k^* *satisfies* (5.1.3)*. If* $n(x,\alpha,N)$ *is defined as in* (5.1.34)*, then, it holds that*

$$\limsup_{N\to\infty} \mathrm{P}\left(\tau_1 \le n \,|\, \tau_1 > k^*\right) \le \Phi(x). \tag{5.1.38}$$

PROOF: For any $\xi > 0$ we get the decomposition

$$\begin{aligned}
\mathrm{P}\left(\tau_1 \le n \,|\, \tau_1 > k^*\right) &= \frac{\mathrm{P}\left(k^* < \tau_1 \le n, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle(n-k^*)}{\sigma\sqrt{h_N}} > -x(1+\xi)\right)}{\mathrm{P}\left(\tau_1 > k^*\right)} \\
&\quad + \frac{\mathrm{P}\left(k^* < \tau_1 \le n, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle(n-k^*)}{\sigma\sqrt{h_N}} \le -x(1+\xi)\right)}{\mathrm{P}\left(\tau_1 > k^*\right)} \\
&=: I_1(N) + I_2(N).
\end{aligned}$$

First, note that the same arguments as in the proof of Lemma 5.1.3 show that

$$\begin{aligned}
&\limsup_{N\to\infty} I_1(N) \\
&\quad = \limsup_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\le n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} > -x(1+\xi)\right) \\
&\quad \le \limsup_{N\to\infty} \mathrm{P}\left(\frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} > -x(1+\xi)\right)
\end{aligned}$$

and therefore the central limit theorem implies

$$\limsup_{N\to\infty} I_1(N) \le \Phi(x(1+\xi)). \tag{5.1.39}$$

Next, we show that $I_2(N)$ tends to zero. It holds that

$$I_2(N) = \frac{\sum_{k=k^*+1}^{n} \mathrm{P}\left(\tau_1 = k, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle(n-k^*)}{\sigma\sqrt{h_N}} \le -x(1+\xi)\right)}{\mathrm{P}\left(\tau_1 > k^*\right)}$$

and since

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 > k^*\right) \ge 1-\alpha,$$

it suffices to consider the numerator of $I_2(N)$. We get

$$\begin{aligned}
&\sum_{k=k^*+1}^{n} \mathrm{P}\left(\tau_1 = k, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} \leq -x(1+\xi)\right)\\
&\quad= \sum_{k=k^*+1}^{n} \mathrm{P}\left(\tau_1 = k, \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} \leq -x(1+\xi)\right)\\
&\quad\leq \sum_{k=k^*+1}^{n} \mathrm{P}\left(\frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1, \frac{M_{n,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}} \leq -x(1+\xi)\right)\\
&\quad\leq \sum_{k=k^*+1}^{n} \mathrm{P}\left(\frac{M_{n,N}-M_{k,N}}{\sigma\sqrt{h_N}} - \frac{\triangle\,(n-k)}{\sigma\sqrt{h_N}} \leq -x(1+\xi) - c_1 + \frac{\triangle\,(k-k^*)}{\sigma\sqrt{h_N}}\right)\\
&\quad\leq \sum_{k=k^*+1}^{n} \mathrm{P}\left(\frac{\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N}}{\sigma\sqrt{h_N}} \leq -x\xi - \frac{\triangle\,(n-k)}{\sigma\sqrt{h_N}}\right)\\
&\quad= \sum_{k=k^*+1}^{n} \mathrm{P}\left(\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N} \leq -x\xi\sigma\sqrt{h_N} - \triangle\,(n-k)\right),
\end{aligned} \tag{5.1.40}$$

where we have used (N large)

$$c_1 \geq -x + \frac{\triangle\,(n-k^*)}{\sigma\sqrt{h_N}}.$$

Now assume $x > 0$. The Markov inequality yields

$$\begin{aligned}
&\mathrm{P}\left(\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N} \leq -x\xi\sigma\sqrt{h_N} - \triangle\,(n-k)\right)\\
&\qquad\leq \mathrm{P}\left(\left|\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N}\right| \geq x\xi\sigma\sqrt{h_N} + \triangle\,(n-k)\right)\\
&\qquad\leq \frac{\mathrm{E}\left|\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N}\right|^{\nu}}{\left(x\xi\sigma\sqrt{h_N} + \triangle\,(n-k)\right)^{\nu}}.
\end{aligned}$$

By Rosenthal's inequality (see Rosenthal, Theorem 3, 1970) there exists a constant $C_1 > 0$, only depending on ν, such that

$$\mathrm{E}\left|\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N}\right|^{\nu} \leq C_1 \max\left\{\left(2(n-k)\sigma^2\right)^{\nu/2},\ 2(n-k)\,\mathrm{E}\left|\varepsilon_1\right|^{\nu}\right\}.$$

Hence

$$\mathrm{E}\left|\sum_{j=k+1}^{n}\varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N}\varepsilon_{j+h_N}\right|^{\nu} \leq C_2(n-k)^{\nu/2}$$

for some suitable $C_2 > 0$, independent of n and k, and for the last term in (5.1.40) it holds

$$\begin{aligned}
&\sum_{k=k^*+1}^{n} \mathrm{P}\left(\sum_{j=k+1}^{n} \varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N} \varepsilon_{j+h_N} \le -x\xi\sigma\sqrt{h_N} - \triangle\,(n-k) \right) \\
&\qquad \le C_2 \sum_{k=k^*+1}^{n} \frac{(n-k)^{\nu/2}}{\left(x\xi\sigma\sqrt{h_N} + \triangle\,(n-k)\right)^{\nu}} \\
&\qquad = C_2 \sum_{k=1}^{\lfloor \delta k^* \rfloor} \frac{(\lfloor \delta k^* \rfloor - k)^{\nu/2}}{\left(x\xi\sigma\sqrt{h_N} + \triangle\,(\lfloor \delta k^* \rfloor - k)\right)^{\nu}} \\
&\qquad = C_2 \sum_{k=1}^{\lfloor \delta k^* \rfloor - 1} \frac{k^{\nu/2}}{\left(x\xi\sigma\sqrt{h_N} + \triangle\, k\right)^{\nu}} \\
&\qquad = \frac{C_2}{\triangle^{\nu}} \sum_{k=1}^{\lfloor \delta k^* \rfloor - 1} \frac{k^{\nu/2}}{\left(C_3\sqrt{h_N} + k\right)^{\nu}},
\end{aligned}$$

where

$$C_3 = \frac{x\xi\sigma}{\triangle}.$$

Now

$$\begin{aligned}
\sum_{k=1}^{\lfloor \delta k^* \rfloor - 1} \frac{k^{\nu/2}}{\left(C_3\sqrt{h_N} + k\right)^{\nu}} &\le \sum_{k=\lfloor C_3\sqrt{h_N} \rfloor + 1}^{\lfloor \delta k^* \rfloor + \lfloor C_3\sqrt{h_N} \rfloor - 1} \frac{(k - \lfloor C_3\sqrt{h_N} \rfloor)^{\nu/2}}{k^{\nu}} \\
&\le \sum_{k=\lfloor C_3\sqrt{h_N} \rfloor + 1}^{\infty} \frac{1}{k^{\nu/2}} \to 0 \quad \text{as} \quad N \to \infty
\end{aligned}$$

and we see that in case of $x > 0$

$$\lim_{N\to\infty} I_2(N) = 0 \quad \text{as} \quad N \to \infty. \tag{5.1.41}$$

Next, we assume $x < 0$. Since $\delta k^* \simeq \sqrt{h_N \log(N/h_N)}$ as $N \to \infty$, it follows that for sufficiently large N it holds that

$$\triangle\,(n-k) \ge 2|x|\xi\sigma\sqrt{h_N}.$$

Hence (N large)

$$\begin{aligned}
&\mathrm{P}\left(\sum_{j=k+1}^{n} \varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N} \varepsilon_{j+h_N} \le -x\xi\sigma\sqrt{h_N} - \triangle\,(n-k) \right) \\
&\qquad \le \mathrm{P}\left(\sum_{j=k+1}^{n} \varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N} \varepsilon_{j+h_N} \le x\xi\sigma\sqrt{h_N} \right) \\
&\qquad \le \mathrm{P}\left(\left| \sum_{j=k+1}^{n} \varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N} \varepsilon_{j+h_N} \right| \ge |x|\xi\sigma\sqrt{h_N} \right).
\end{aligned}$$

Similar computations as before show that for (5.1.40) now it holds

$$\sum_{k=k^*+1}^{n} \mathrm{P}\left(\sum_{j=k+1}^{n} \varepsilon_{j+h_N} - \sum_{j=k-h_N+1}^{n-h_N} \varepsilon_{j+h_N} \le -x\xi\sigma\sqrt{h_N} - \triangle\,(n-k)\right)$$
$$\le \frac{C_2}{\left(x\xi\sigma\sqrt{h_N}\right)^{\nu}} \sum_{k=1}^{\lfloor \delta k^* \rfloor - 1} k^{\nu/2} \to 0 \quad \text{as} \quad N \to \infty,$$

where the convergence is a consequence of

$$\sum_{k=1}^{\lfloor \delta k^* \rfloor - 1} k^{\nu/2} \simeq \left(\sqrt{h_N \log \frac{N}{h_N}}\right)^{1+\nu/2} \qquad \text{as} \quad N \to \infty$$

and $\nu > 1 + \nu/2$. So, for $x < 0$ also holds

$$\lim_{N\to\infty} I_2(N) = 0 \quad \text{as} \quad N \to \infty \tag{5.1.42}$$

and the lemma follows for all $x \neq 0$ by (5.1.39), (5.1.41) and (5.1.42) as $\xi \to 0$. The continuity of the standard normal distribution function now implies that the lemma also holds for $x = 0$.

□

Theorem 5.1.1 now follows obviously from Lemma 5.1.3, Lemma 5.1.4 and the definition of n.

Proof of Lemma 5.1.1

Theorem 5.1.1 implies that for any sequence $\{x_N\}_{N=1,2,\ldots}$, with $x_N \to -\infty$ as $N \to \infty$, it holds that

$$\lim_{N\to\infty} \mathrm{P}\left(\frac{(\tau_1 - k^*) - \alpha_N}{\beta_N} \le x_N \,\middle|\, \tau_1 > k^*\right) = 0,$$

hence

$$\lim_{N\to\infty} \mathrm{P}\left(\tau_1 - (c_1(\alpha, N) + x_N)\beta_N \le k^* \,\middle|\, \tau_1 > k^*\right) = 0.$$

The choice $x_N = 1 - c_1(\alpha, N)$ yields

$$\lim_{N\to\infty} \mathrm{P}\left(\tau_1 - \beta_N \le k^* \,\middle|\, \tau_1 > k^*\right) = 0$$

and since

$$\beta_N \geq \frac{\sigma\sqrt{h_N}}{\triangle_{\max}} = \lambda_N,$$

it follows that

$$\lim_{N\to\infty} \mathrm{P}\left(\tau_1 - \lambda_N \leq k^* \,\middle|\, \tau_1 > k^*\right) = 0. \tag{5.1.43}$$

Now

$$\begin{aligned}\mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma} \,\middle|\, \tau_1 > k^*\right) &= \mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, \tau_1 - \lambda_N \leq k^* \,\middle|\, \tau_1 > k^*\right)\\ &\quad + \mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, \tau_1 - \lambda_N > k^* \,\middle|\, \tau_1 > k^*\right)\\ &:= I_1(N) + I_2(N)\end{aligned}$$

Obviously, (5.1.43) implies

$$\lim_{N\to\infty} I_1(N) = 0. \tag{5.1.44}$$

For the second term we have

$$I_2(N) = \frac{\mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, \tau_1 - \lambda_N > k^*\right)}{\mathrm{P}\left(\tau_1 > k^*\right)}$$

and since

$$\liminf_{N\to\infty} \mathrm{P}\left(\tau_1 > k^*\right) \geq 1 - \alpha$$

we only consider the numerator. Note that

$$\begin{aligned}&\limsup_{N\to\infty} \mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, k^* + \lambda_N < \tau_1\right)\\ &\qquad = \limsup_{N\to\infty} \mathrm{P}\left(|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, k^* + \lambda_N < \tau_1 \leq N\right),\end{aligned}$$

since the underlying test has asymptotic power one.

It holds that

$$\left\{|\,\hat{\triangle}_N - \triangle\,| > \frac{\epsilon}{N^\gamma},\, k^* + \lambda_N < \tau_1 \leq N\right\} \subset \left\{\max_{k^*+\lambda_N<k\leq N} \frac{N^\gamma}{\lambda_N}\left|\sum_{i=k-\lambda_N+1}^{k} \varepsilon_{i+h_N}\right| > \epsilon\right\}.$$

Applying the strong invariance principle according to Komlós, Major and Tusnády (1975,1976) and Major (1976) we can find a Wiener process $\{W(t), t \geq 0\}$, such that

$$\begin{aligned} &\max_{k^*+\lambda_N<k\leq N} \frac{N^\gamma}{\lambda_N} \left| \sum_{i=k-\lambda_N+1}^{k} \varepsilon_{i+h_N} \right| \\ &\quad - \max_{k^*+\lambda_N<k\leq N} \frac{N^\gamma}{\lambda_N} |W(k+h_N) - W(k+h_N-\lambda_N)| = \boldsymbol{o}\left(\frac{N^\gamma (N+h_N)^{1/\nu}}{\lambda_N}\right) \quad \text{a.s.} \end{aligned}$$

as $N \to \infty$ and since $\lambda_N \simeq \sqrt{h_N}$ as $N \to \infty$, we see that

$$\frac{N^\gamma (N+h_N)^{1/\nu}}{\lambda_N} \to 0 \quad \text{as} \quad N \to \infty.$$

Next, Theorem 1.2.1 of Csörgő and Révész (1981) implies

$$\max_{k^*+\lambda_N<k\leq N} \frac{N^\gamma}{\lambda_N} |W(k+h_N) - W(k+h_N-\lambda_N)| = \boldsymbol{O}\left(\frac{N^\gamma \sqrt{\log N}}{\sqrt{\lambda_N}}\right) \quad \text{a.s.}$$

as $N \to \infty$ and since $\gamma < \phi - 1/\nu$, it follows that

$$\frac{N^\gamma \sqrt{\log N}}{\sqrt{\lambda_N}} \to 0 \quad \text{as} \quad N \to \infty.$$

But then

$$\lim_{N\to\infty} I_2(N) = 0. \tag{5.1.45}$$

and Lemma 5.1.1 follows from (5.1.44) and (5.1.45).

Proof of Theorem 5.1.2

In view of Theorem 5.1.1 it suffices to show that for any $\epsilon > 0$ it holds that

$$\lim_{N\to\infty} \mathrm{P}\left(\left| \frac{(\tau_1 - k^*) - \alpha_N}{\beta_N} - \frac{(\tau_1 - k^*) - \hat{\alpha}_N}{\hat{\beta}_N} \right| > \epsilon \;\middle|\; \tau_1 > k^* \right) = 0. \tag{5.1.46}$$

We have

$$
\begin{aligned}
\mathrm{P}\left(\left|\frac{(\tau_1-k^*)-\alpha_N}{\beta_N}-\frac{(\tau_1-k^*)-\hat{\alpha}_N}{\hat{\beta}_N}\right|>\epsilon\,\middle|\,\tau_1>k^*\right)& \\
\leq \quad & \mathrm{P}\left(\left|\frac{(\hat{\beta}_N-\beta_N)(\tau_1-k^*)}{\beta_N\hat{\beta}_N}\right|>\epsilon\,\middle|\,\tau_1>k^*\right) \\
& +\mathrm{P}\left(\left|\frac{\hat{\alpha}_N-\alpha_N}{\hat{\beta}_N}\right|>\epsilon\,\middle|\,\tau_1>k^*\right) \\
& +\mathrm{P}\left(\left|\frac{\alpha_N(\hat{\beta}_N-\beta_N)}{\beta_N\hat{\beta}_N}\right|>\epsilon\,\middle|\,\tau_1>k^*\right) \\
=: \; & I_1(N)+I_2(N)+I_3(N).
\end{aligned}
$$

Before we can consider $I_1(N)$, we need some preliminaries. For any real x we choose n according to (5.1.34). Theorem 5.1.1 implies that

$$
\begin{aligned}
\Phi(x) &= \lim_{N\to\infty} \mathrm{P}\left(\tau_1\leq n\,|\,\tau_1>k^*\right) \\
&= \lim_{N\to\infty} \mathrm{P}\left(\frac{\tau_1-k^*}{\alpha_N}\leq\frac{n-k^*}{\alpha_N}\,\middle|\,\tau_1>k^*\right) \\
&= \lim_{N\to\infty} \mathrm{P}\left(\frac{\tau_1-k^*}{\alpha_N}\leq 1+\frac{x}{c_1}\,\middle|\,\tau_1>k^*\right),
\end{aligned}
$$

hence it holds, for all $\epsilon>0$

$$
\lim_{N\to\infty} \mathrm{P}\left(\left|\frac{\tau_1-k^*}{\alpha_N}-1\right|\leq\epsilon\,\middle|\,\tau_1>k^*\right)=1. \tag{5.1.47}
$$

Now, taking into account (5.1.47), we get

$$
\begin{aligned}
\limsup_{N\to\infty} I_1(N) &= \limsup_{N\to\infty} \mathrm{P}\left(\left|\left(1-\frac{\beta_N}{\hat{\beta}_N}\right)\frac{\tau_1-k^*}{\beta_N}\right|>\epsilon\,\middle|\,\tau_1>k^*\right) \\
&= \limsup_{N\to\infty} \mathrm{P}\left(\left|1-\frac{\hat{\Delta}_N}{\Delta}\right|\left|\frac{\tau_1-k^*}{\alpha_N}\right|c_1>\epsilon\,\middle|\,\tau_1>k^*\right) \\
&\leq \limsup_{N\to\infty} \mathrm{P}\left(\left|\Delta-\hat{\Delta}_N\right|\frac{c_1}{\Delta}>\frac{\epsilon}{2}\,\middle|\,\tau_1>k^*\right).
\end{aligned}
$$

Since

$$
c_1(\alpha,N)=\boldsymbol{O}\left(\sqrt{\log N}\right) \quad \text{as} \quad N\to\infty,
$$

display (5.1.12) shows that

$$\lim_{N\to\infty} I_1(N) = 0. \tag{5.1.48}$$

Moreover, (5.1.12) also implies

$$\begin{aligned}\lim_{N\to\infty} I_2(N) &= \lim_{N\to\infty} \mathrm{P}\left(\left|1-\frac{\beta_N}{\hat\beta_N}\right| c_1 > \epsilon \,\middle|\, \tau_1 > k^*\right)\\ &= \lim_{N\to\infty} \mathrm{P}\left(\left|\triangle - \hat\triangle_N\right| \frac{c_1}{\triangle} > \epsilon \,\middle|\, \tau_1 > k^*\right)\\ &= 0 \end{aligned}\tag{5.1.49}$$

and

$$\lim_{N\to\infty} I_3(N) = 0. \tag{5.1.50}$$

Hence (5.1.46) holds and the theorem follows.

Proof of Theorem 5.1.3

We start with some technical preliminaries. Let $\epsilon > 0$. We define the stoping times η_l and η_u as

$$\eta_l = \eta_l(\epsilon,\alpha,N) = \inf\left\{1 \le k \le N : M_{k,N} > (c_1(\alpha,N)-\epsilon)\,\sigma\,\sqrt{h_N}\right\} \tag{5.1.51}$$

and

$$\eta_u = \eta_u(\epsilon,\alpha,N) = \inf\left\{1 \le k \le N : M_{k,N} > (c_1(\alpha,N)+\epsilon)\,\sigma\,\sqrt{h_N}\right\}. \tag{5.1.52}$$

Additionally, for any real x we set

$$\delta_l = \delta_l(\epsilon,x,\alpha,N) = \frac{((c_1(\alpha,N)-\epsilon)+(x-\epsilon))\sigma\sqrt{h_N}}{\triangle\ k^*}, \tag{5.1.53}$$

$$\delta_u = \delta_u(\epsilon,x,\alpha,N) = \frac{((c_1(\alpha,N)+\epsilon)+(x+\epsilon))\sigma\sqrt{h_N}}{\triangle\ k^*}, \tag{5.1.54}$$

$$n_l = n_l(\epsilon,x,\alpha,N) = k^* + \lfloor \delta_l k^* \rfloor, \tag{5.1.55}$$

$$n_u = n_u(\epsilon,x,\alpha,N) = k^* + \lfloor \delta_u k^* \rfloor. \tag{5.1.56}$$

Lemma 5.1.5 *With the notation of* (5.1.51)–(5.1.56) *for any real x and $\epsilon > 0$ it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\eta_l \le n_l \,|\, \eta_l > k^*\right) = \Phi(x-\epsilon) \tag{5.1.57}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(\eta_u \le n_u \,|\, \eta_u > k^*\right) = \Phi(x+\epsilon). \tag{5.1.58}$$

Furthermore, the limits are completely determined by the alarms after the change, i.e.

$$\lim_{N\to\infty} \mathrm{P}\left(k^* < \eta_l \le n_l\right) = \Phi(x-\epsilon) \tag{5.1.59}$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(k^* < \eta_u \le n_u\right) = \Phi(x+\epsilon). \tag{5.1.60}$$

PROOF: The proofs of (5.1.57) and (5.1.58) are just a repetition of the arguments, used to show Theorem 5.1.1.

(5.1.59) and (5.1.60) follow by (5.1.57) and (5.1.58), since the extremes of the the detectors before and after the change are asymptotically independent (cf. the proof of Lemma 5.1.3) and it holds that

$$\lim_{N\to\infty} \mathrm{P}\left(\eta_l > k^*\right) = 1-\alpha_l \quad \text{for some} \quad 0<\alpha_l<1$$

and

$$\lim_{N\to\infty} \mathrm{P}\left(\eta_u > k^*\right) = 1-\alpha_u \quad \text{for some} \quad 0<\alpha_u<1,$$

which can be seen utilizing the same arguments as those used in the proof of Lemma 5.1.6 below.

□

Lemma 5.1.6 *Let τ_1 and $\hat{\tau}_1$ be defined as in* (5.1.4) *and* (5.1.18)*, respectively. Furthermore, assume that k^* satisfies* (5.1.3)*. Then it holds that*

$$\lim_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right) = \lim_{N\to\infty} \mathrm{P}\left(\tau_1 > k^*\right) = 1-\tilde{\alpha} \quad \textit{for some} \quad 0<\tilde{\alpha}<1. \tag{5.1.61}$$

PROOF: First, note that

$$\liminf_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right) = \liminf_{N\to\infty} \mathrm{P}\left(N \ge \hat{\tau}_1 > k^*\right)$$

and

$$\limsup_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right) = \limsup_{N\to\infty} \mathrm{P}\left(N \ge \hat{\tau}_1 > k^*\right),$$

since the underlying test has asymptotic power one.

Similar arguments as in the proof of Lemma 5.1.3 show that

$$\begin{aligned}
&\liminf_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right)\\
&\qquad = \liminf_{N\to\infty} \mathrm{P}\left(\max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \le c_1, \max_{k^*<k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right)\\
&\qquad = \liminf_{N\to\infty} \mathrm{P}\left(\max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \le c_1\right) \mathrm{P}\left(\max_{k^*<k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right)
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right)\\
&\qquad = \limsup_{N\to\infty} \mathrm{P}\left(\max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \le c_1, \max_{k^*<k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right)\\
&\qquad = \limsup_{N\to\infty} \mathrm{P}\left(\max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \le c_1\right) \mathrm{P}\left(\max_{k^*<k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right).
\end{aligned}$$

Now the proof of Theorem 1.1.4 implies

$$\lim_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\le N} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right) = 1,$$

hence it suffices to show that

$$\lim_{N\to\infty} \mathrm{P}\left(\max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} \le c_1\right) = 1 - \tilde{\alpha}. \tag{5.1.62}$$

The parametrization of k^* and the proof of Theorem 1.1.3 show that

$$a_{k^*} \max_{1\le k\le k^*} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} - b_{k^*} \xrightarrow{\mathrm{D}} G \quad \text{as} \quad N\to\infty,$$

where

$$a_{k^*} = \sqrt{2\log\frac{k^*}{h_N}}, \tag{5.1.63}$$

$$b_{k^*} = 2\log\frac{k^*}{h_N} + \frac{1}{2}\log\log\frac{k^*}{h_N} - \frac{1}{2}\log\pi \tag{5.1.64}$$

and G denotes a Gumbel-distributed random variable.

Finally, elementary calculations yield

$$a_{k^*}c_1 - b_{k^*} \to \tilde{q} \quad \text{as} \quad N\to\infty$$

and we see that (5.1.62) holds.

It is obvious that the same calculations also hold for τ_1, hence the proof of the lemma is complete.

$\square$

Lemma 5.1.7 *Let τ_1 and $\hat{\tau}_1$ be defined as in* (5.1.4) *and* (5.1.18)*, respectively. Furthermore, assume that k^* satisfies* (5.1.3)*. For any real x we define $n = n(x, \alpha, N)$ according to* (5.1.34)*. Then*

$$\lim_{N\to\infty} \mathrm{P}\left(k^* < \hat{\tau}_1 \leq n\right) = \lim_{N\to\infty} \mathrm{P}\left(k^* < \tau_1 \leq n\right) \tag{5.1.65}$$

Proof: Assume for the moment that

$$\lim_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right) = \lim_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1\right) \tag{5.1.66}$$

holds. Then the asymptotic independence of the maxima of the detectors before and after the change (see the proof of Lemma 5.1.3), together with the proof of Lemma 5.1.6 yields

$$\begin{aligned}
\lim_{N\to\infty} \mathrm{P}\left(k^* < \hat{\tau}_1 \leq n\right) &= (1-\tilde{\alpha}) \lim_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right) \\
&= (1-\tilde{\alpha}) \lim_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1\right) \\
&= \lim_{N\to\infty} \mathrm{P}\left(k^* < \tau_1 \leq n\right).
\end{aligned}$$

To see that (5.1.66) holds, first note that for sufficiently large N we have

$$\begin{aligned}
&\mathrm{P}\left(\max_{k^*<k\leq n} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right) \\
&\quad = \mathrm{P}\left(\frac{\sigma}{\hat{\sigma}_{m_N}} \max_{k^*<k\leq n} \frac{\sum_{i=k-h_N+1}^{k}(X_{i+m_N}-\mu)}{\sigma\sqrt{h_N}} - \sqrt{h_N}\,\frac{\hat{\mu}_{m_N}-\mu}{\hat{\sigma}_{m_N}} > c_1\right) \\
&\quad = \mathrm{P}\left(\frac{\sigma}{\hat{\sigma}_{m_N}} \max_{k^*<k\leq n} \frac{\sum_{i=k-h_N+1}^{k}(X_{i+h_N}-\mu)}{\sigma\sqrt{h_N}} - \sqrt{h_N}\,\frac{\hat{\mu}_{m_N}-\mu}{\hat{\sigma}_{m_N}} > c_1\right),
\end{aligned}$$

where the last equation follows by the stationarity of the innovations and the fact that $(k^* - h_N)/m_N \to \infty$ as $N \to \infty$. Hence

$$\mathrm{P}\left(\max_{k^*<k\leq n} \frac{\hat{M}_{k,N}}{\hat{\sigma}_{m_N}\sqrt{h_N}} > c_1\right) = \mathrm{P}\left(\max_{k^*<k\leq n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1\frac{\hat{\sigma}_{m_N}}{\sigma} + \sqrt{h_N}\,\frac{\hat{\mu}_{m_N}-\mu}{\sigma}\right) \tag{5.1.67}$$

For any fixed $\epsilon > 0$, Lemma 5.1.5 and the definition of n_l, n_u imply

$$\begin{aligned}
\liminf_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1-\epsilon\right) &\geq \liminf_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\leq n_l} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1-\epsilon\right) \\
&= \Phi(x-\epsilon)
\end{aligned} \tag{5.1.68}$$

and

$$\limsup_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\le n} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1 - \epsilon\right) \le \limsup_{N\to\infty} \mathrm{P}\left(\max_{k^*<k\le n_u} \frac{M_{k,N}}{\sigma\sqrt{h_N}} > c_1 - \epsilon\right)$$
$$= \Phi(x+\epsilon). \tag{5.1.69}$$

Now (5.1.17), (5.1.20) and (5.1.21) show that

$$\left(c_1 \frac{\hat{\sigma}_{m_N}}{\sigma} + \sqrt{h_N}\,\frac{\hat{\mu}_{m_N} - \mu}{\sigma}\right) - c_1 = \boldsymbol{o}_P(1) \quad \text{as} \quad N \to \infty \tag{5.1.70}$$

and we see that (5.1.66) follows by (5.1.67)–(5.1.70), if we let $\epsilon \to 0$.

□

Theorem 5.1.3 now follows by Lemma 5.1.6 and Lemma 5.1.7.

Proof of Lemma 5.1.2

Theorem 5.1.3 implies that (cf. the proof of Lemma 5.1.1)

$$\lim_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - \widehat{\beta}_N \le k^* \;\middle|\; \hat{\tau}_1 > k^*\right) = 0$$

and since

$$\widehat{\beta}_N \ge \frac{\hat{\sigma}_{m_N}\sqrt{h_N}}{\Delta_{\max}} \ge \hat{\lambda}_N,$$

it follows that

$$\lim_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 - \hat{\lambda}_N \le k^* \;\middle|\; \hat{\tau}_1 > k^*\right) = 0.$$

Hence, in the decomposition

$$\mathrm{P}\left(|\,\widehat{\Delta}_N - \Delta\,| > \frac{\epsilon}{N^\zeta} \;\middle|\; \hat{\tau}_1 > k^*\right) = \mathrm{P}\left(|\,\widehat{\Delta}_N - \Delta\,| > \frac{\epsilon}{N^\zeta},\, \hat{\tau}_1 - \hat{\lambda}_N \le k^* \;\middle|\; \hat{\tau}_1 > k^*\right)$$
$$+\, \mathrm{P}\left(|\,\widehat{\Delta}_N - \Delta\,| > \frac{\epsilon}{N^\zeta},\, \hat{\tau}_1 - \hat{\lambda}_N > k^* \;\middle|\; \hat{\tau}_1 > k^*\right)$$
$$:= I_1(N) + I_2(N)$$

only $I_2(N)$ has to be considered.

Since furthermore

$$\liminf_{N\to\infty} \mathrm{P}\left(\hat{\tau}_1 > k^*\right) \ge 1 - \alpha$$

and the underlying test has asymptotic power one, we can focus on

$$\limsup_{N\to\infty} \mathrm{P}\left(|\, \widehat{\triangle}_N - \triangle \,| > \frac{\epsilon}{N^{\zeta}},\, k^* + \hat{\lambda}_N < \hat{\tau}_1 \le N \right).$$

Now note that

$$\left\{ |\, \widehat{\triangle}_N - \triangle \,| > \frac{\epsilon}{N^{\zeta}},\, k^* + \hat{\lambda}_N < \hat{\tau}_1 \le N \right\} \subset \left\{ \max_{k^*+\hat{\lambda}_N < k \le N} \frac{N^{\zeta}}{\hat{\lambda}_N} \left| \sum_{i=k-\hat{\lambda}_N+1}^{k} \varepsilon_{i+m_N} \right| > \epsilon \right\}.$$

It follows by (5.1.2), (5.1.17) and (5.1.21) that

$$\lim_{N\to\infty} \mathrm{P}\left(\lambda_N - N^{\phi(1-\vartheta)} \le \hat{\lambda}_N \le \lambda_N + N^{\phi(1-\vartheta)} \right) = 1,$$

which implies

$$\begin{aligned}
&\limsup_{N\to\infty} \mathrm{P}\left(\max_{k^*+\hat{\lambda}_N < k \le N} \frac{N^{\zeta}}{\hat{\lambda}_N} \left| \sum_{i=k-\hat{\lambda}_N+1}^{k} \varepsilon_{i+m_N} \right| > \epsilon \right) \\
&\quad = \limsup_{N\to\infty} \mathrm{P}\left(\max_{k^*+\hat{\lambda}_N < k \le N} \frac{N^{\zeta}}{\hat{\lambda}_N} \left| \sum_{i=k-\hat{\lambda}_N+1}^{k} \varepsilon_{i+m_N} \right| > \epsilon,\, |\, \hat{\lambda}_N - \lambda_N| \le N^{\phi(1-\vartheta)} \right).
\end{aligned}$$

Obviously

$$\begin{aligned}
&\left\{ \max_{k^*+\hat{\lambda}_N < k \le N} \frac{N^{\zeta}}{\hat{\lambda}_N} \left| \sum_{i=k-\hat{\lambda}_N+1}^{k} \varepsilon_{i+m_N} \right| > \epsilon,\, |\, \hat{\lambda}_N - \lambda_N| \le N^{\phi(1-\vartheta)} \right\} \\
&\qquad\qquad \subset \left\{ \frac{N^{\zeta}}{\lambda_N - N^{\phi(1-\vartheta)}} \max_{k^* < k \le N} \max_{|\lambda-\lambda_N| \le N^{\phi(1-\vartheta)}} \left| \sum_{i=k-\lambda+1}^{k} \varepsilon_{i+m_N} \right| > \epsilon \right\}.
\end{aligned}$$

The strong invariance principle according to Komlós, Major and Tusnády (1975,1976) and Major (1976) shows that there exists a Wiener-process $\{W(t), t \ge 0\}$, such that

$$\sum_{i=1}^{l} \varepsilon_i - W(l) = \boldsymbol{o}\left(l^{1/\nu}\right) \quad \text{a.s.} \quad \text{as} \quad l \to \infty,$$

hence

$$\begin{aligned}
&\frac{N^{\zeta}}{\lambda_N - N^{\phi(1-\vartheta)}} \max_{k^* < k \le N} \max_{|\lambda-\lambda_N| \le N^{\phi(1-\vartheta)}} \left| \sum_{i=k-\lambda+1}^{k} \varepsilon_{i+m_N} \right| \\
&\quad - \frac{N^{\zeta}}{\lambda_N - N^{\phi(1-\vartheta)}} \max_{k^* < k \le N} \max_{|\lambda-\lambda_N| \le N^{\phi(1-\vartheta)}} |W(k+m_N) - W(k+m_N-\lambda)| \\
&\qquad = \boldsymbol{o}\left(\frac{N^{\zeta}(N+m_N)^{1/\nu}}{\lambda_N - N^{\phi(1-\vartheta)}} \right) \quad \text{a.s.} \quad \text{as} \quad N \to \infty.
\end{aligned}$$

Since

$$(N + m_N)^{1/\nu} \sim N^{1/\nu} \quad \text{as} \quad N \to \infty,$$

$$\lambda_N - N^{\phi(1-\vartheta)} \sim N^{\phi} \quad \text{as} \quad N \to \infty$$

and $\zeta < \phi - 1/\nu$ we see that

$$\frac{N^{\zeta}(N + m_N)^{1/\nu}}{\lambda_N - N^{\phi(1-\vartheta)}} \sim \frac{N^{\zeta} N^{1/\nu}}{N^{\phi}} \to 0 \quad \text{as} \quad N \to \infty.$$

Finally, Theorem 1.2.1 of Csörgő and Révész (1981) implies

$$\frac{N^{\zeta}}{\lambda_N - N^{\phi(1-\vartheta)}} \max_{k^* < k \le N} \max_{|\lambda - \lambda_N| \le N^{\phi(1-\vartheta)}} |W(k + m_N) - W(k + m_N - \lambda)|$$
$$= \boldsymbol{O}\left(\frac{N^{\zeta}\sqrt{\log N}}{\sqrt{\lambda_N}}\right) \quad \text{a.s.} \quad \text{as} \quad N \to \infty$$

and since $\zeta < \phi - 1/\nu$, it follows by the definition of λ_N that

$$\frac{N^{\zeta}\sqrt{\log N}}{\sqrt{\lambda_N}} \to 0 \quad \text{as} \quad N \to \infty.$$

But then also

$$\lim_{N \to \infty} I_2(N) = 0$$

and Lemma 5.1.2 follows.

Proof of Theorem 5.1.4

If we show that, for any $\epsilon > 0$, it holds that

$$\lim_{N \to \infty} \mathrm{P}\left(\left|\frac{(\hat{\tau}_1 - k^*) - \widehat{\alpha}_N}{\widehat{\beta}_N} - \frac{(\hat{\tau}_1 - k^*) - \widehat{\alpha}_N}{\widehat{\beta}_N}\right| > \epsilon \;\middle|\; \hat{\tau}_1 > k^*\right) = 0, \tag{5.1.71}$$

then Theorem 5.1.4 follows by Theorem 5.1.3.

We have

$$
\begin{aligned}
\mathrm{P}\left(\left|\frac{(\hat{\tau}_1-k^*)-\widehat{\alpha}_N}{\widehat{\beta}_N}-\frac{(\hat{\tau}_1-k^*)-\widehat{\alpha}_N}{\widehat{\beta}_N}\right|>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)&\\
\leq\quad &\mathrm{P}\left(\left|\frac{(\widehat{\beta}_N-\widehat{\beta}_N)(\hat{\tau}_1-k^*)}{\widehat{\beta}_N\widehat{\beta}_N}\right|>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)\\
&+\mathrm{P}\left(\left|\frac{\widehat{\alpha}_N-\widehat{\alpha}_N}{\widehat{\beta}_N}\right|>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)\\
&+\mathrm{P}\left(\left|\frac{\widehat{\alpha}_N(\widehat{\beta}_N-\widehat{\beta}_N)}{\widehat{\beta}_N\widehat{\beta}_N}\right|>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)\\
=:\;&I_1(N)+I_2(N)+I_3(N).
\end{aligned}
$$

We only consider $I_1(N)$ in detail, since all three terms finally depend on the same expression.

Theorem 5.1.3 implies that for any $\epsilon>0$ we have

$$
\lim_{N\to\infty}\mathrm{P}\left(\left|\frac{\hat{\tau}_1-k^*}{\widehat{\alpha}_N}-1\right|\leq\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)=1. \tag{5.1.72}
$$

Now, with (5.1.72) we get

$$
\begin{aligned}
\limsup_{N\to\infty}I_1(N)&=\limsup_{N\to\infty}\mathrm{P}\left(\left|\left(1-\frac{\widehat{\beta}_N}{\widehat{\beta}_N}\right)\frac{\hat{\tau}_1-k^*}{\widehat{\beta}_N}\right|>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)\\
&=\limsup_{N\to\infty}\mathrm{P}\left(\left|1-\frac{\widehat{\Delta}_N}{\Delta}\right|\left|\frac{\hat{\tau}_1-k^*}{\widehat{\alpha}_N}\right|c_1>\epsilon\;\middle|\;\hat{\tau}_1>k^*\right)\\
&\leq\limsup_{N\to\infty}\mathrm{P}\left(\left|\Delta-\hat{\Delta}_N\right|\frac{c_1}{\Delta}>\frac{\epsilon}{2}\;\middle|\;\tau_1>k^*\right)
\end{aligned}
$$

and since

$$
c_1(\alpha,N)=\boldsymbol{O}\left(\sqrt{\log N}\right)\quad\text{as}\quad N\to\infty,
$$

it follows by (5.1.28) that

$$
\lim_{N\to\infty}I_1(N)=0.
$$

By similar computations we obtain $\lim_{N\to\infty}I_2(N)=0$ and also $\lim_{N\to\infty}I_3(N)=0$, hence the proof of Theorem 5.1.4 is complete.

Appendix A

Covariances

In this chapter we consider the covariance function of the process $\{U_N(t)\}_{t\geq 0}$, laid down in (2.1.45).

Before we start, we provide a simple but useful lemma.

Lemma A.0.1 *Let $\{p_n\}_{n=0,1,\ldots}$ be a sequence of real numbers. Then for all $q = 0, 1, \ldots$ and $n = 1, 2, \ldots$ it holds that*

$$\sum_{i,j=0}^{n-1} p_{|i-j+q|} = \sum_{j=0}^{n-1}(n-j)p_{q+j} + \sum_{j=1}^{\min\{n-1,q\}} (n-j)p_{q-j} + \sum_{j=q+1}^{n-1}(n-j)p_{j-q} \tag{A.0.1}$$

Proof: The proof carried out via induction over n. Obviously assertion (A.0.1) holds for $n = 1$. Assuming that it also holds for $n-1$ we get

$$\begin{aligned}
\sum_{i,j=0}^{n} p_{|i-j+q|} &= \sum_{i,j=0}^{n-1} p_{|i-j+q|} + \sum_{j=0}^{n} p_{|n-j+q|} + \sum_{i=0}^{n-1} p_{|i-n+q|} \\
&= \sum_{j=0}^{n-1}(n-j)p_{q+j} + \sum_{j=1}^{\min\{n-1,q\}} (n-j)p_{q-j} + \sum_{j=q+1}^{n-1}(n-j)p_{j-q} \\
&\qquad + \sum_{j=0}^{n} p_{q+j} + \sum_{i=\max\{0,n-q\}}^{n-1} p_{i-n+q} + \sum_{i=0}^{n-q-1} p_{n-i-q} \\
&= \sum_{j=0}^{n-1}(n-j)p_{q+j} + \sum_{j=1}^{\min\{n-1,q\}} (n-j)p_{q-j} + \sum_{j=q+1}^{n-1}(n-j)p_{j-q} \\
&\qquad + \sum_{j=0}^{n} p_{q+j} + \sum_{j=1}^{\min\{n,q\}} p_{q-j} + \sum_{j=q+1}^{n} p_{j-q} \\
&= \sum_{j=0}^{n}(n+1-j)p_{q+j} + \sum_{j=1}^{\min\{n,q\}} (n+1-j)p_{q-j} + \sum_{j=q+1}^{n}(n+1-j)p_{j-q},
\end{aligned}$$

showing (A.0.1). □

Remark A.0.2 *In case of $q = 0$ assertion* (A.0.1) *can be stated as*

$$\sum_{i,j=0}^{n-1} p_{|i-j|} = np_0 + 2\sum_{j=1}^{n-1}(n-j)p_j. \tag{A.0.2}$$

We are now prepared to carry out the so called elementary calculations.

Since obviously $\{U_N(t)\}_{t\geq 0}$ is a strictly stationary process, we can assume that

$$U_N(t) = \sum_{j=0}^{\infty} w_j V_{t-j,N} \quad \text{for all} \quad 0 \leq t < \infty, \tag{A.0.3}$$

where

$$V_{t,N} = \frac{1}{\sqrt{h_N}}\sigma(W(t) - W(t-h_N)) \quad \text{for all} \quad -\infty < t < \infty \tag{A.0.4}$$

and $\{W(t), -\infty < t < \infty\}$ is a two-sided Wiener process.

A.1 The variance of $\{U_N(t)\}$

Lemma A.1.1 *For all natural numbers N the variance of $\{U_N(t)\}_{t\geq 0}$ is given by*

$$\operatorname{Var} U_N(t) = \sigma_N^2 = \sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k \left(\sum_{j=1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j}\right). \tag{A.1.1}$$

Furthermore, it holds that

$$1 - \sigma_N^2 = \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.1.2}$$

where $\gamma = \min\{1-\phi, \psi\}$ with ϕ and ψ being defined as in (2.1.10) *and* (2.1.11), *respectively .*

Proof: Let $\{N_t\}_{-\infty<t<\infty}$ and $\{P_t\}_{-\infty<t<\infty}$ be defined as

$$N_t = W(t) - W(t-1) \quad \text{for all} \quad -\infty < t < \infty \tag{A.1.3}$$

and

$$P_t = \sum_{j=0}^{\infty} w_j N_{t-j} \quad \text{for all} \quad -\infty < t < \infty. \tag{A.1.4}$$

Then

$$\begin{aligned}
U_N(t) =& \sum_{j=0}^{\infty} w_j \frac{1}{\sqrt{h_N}} \sum_{i=0}^{h_N-1} N_{t-i-j} \\
=& \sum_{i=0}^{h_N-1} \frac{1}{\sqrt{h_N}} \sum_{j=0}^{\infty} w_j N_{t-i-j} \\
=& \sum_{i=0}^{h_N-1} \frac{1}{\sqrt{h_N}} P_{t-i} \quad \text{for all} \quad -\infty < t < \infty,
\end{aligned}$$

and for all real t it holds that

$$\begin{aligned}
\mathrm{E}\,(U_N(t)U_N(t)) =& \mathrm{E}\,(U_N(0)U_N(0)) \\
=& \mathrm{E}\left(\sum_{i=0}^{h_N-1} \frac{1}{\sqrt{h_N}} P_{-i} \sum_{j=0}^{h_N-1} \frac{1}{\sqrt{h_N}} P_{-j} \right) \\
=& \frac{1}{h_N} \sum_{i=0}^{h_N-1} \sum_{j=0}^{h_N-1} \mathrm{E}\,(P_{-i}P_{-j}).
\end{aligned}$$

The definition of $\{P_t, -\infty < t < \infty\}$ yields

$$\mathrm{E}\,(P_{-i}P_{-j}) = \mathrm{E}\,(P_0 P_{|i-j|}) = \sum_{k=0}^{\infty} w_k w_{k+|i-j|} \quad \text{for all} \quad i, j = 0, 1, \ldots,$$

hence

$$\mathrm{E}\,(U_N(t)U_N(t)) = \frac{1}{h_N} \sum_{k=0}^{\infty} w_k \sum_{i=0}^{h_N-1} \sum_{j=0}^{h_N-1} w_{k+|i-j|}$$

and applying (A.0.2) we get

$$\begin{aligned}
\mathrm{E}\,(U_N(t)U_N(t)) =& \frac{1}{h_N} \sum_{k=0}^{\infty} w_k \left(h_N w_k + 2 \sum_{j=0}^{h_N-1} (h_N - j) w_{k+j} \right) \\
=& \sum_{k=0}^{\infty} w_k^2 + 2 \sum_{k=0}^{\infty} w_k \sum_{j=1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j},
\end{aligned}$$

what is (A.1.1).

If $N \to \infty$, (A.1.2) follows immediately by (2.1.8)–(2.1.11), since

$$\begin{aligned}
1 - \sigma_N^2 =& \left(\sum_{k=0}^{\infty} w_k \right)^2 - \sigma_N^2 \\
=& \sum_{k=0}^{\infty} w_k^2 + 2 \sum_{k=0}^{\infty} w_k \sum_{j=1}^{\infty} w_{k+j} - \left(\sum_{k=0}^{\infty} w_k^2 + 2 \sum_{k=0}^{\infty} w_k \sum_{j=1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j} \right) \\
=& 2 \sum_{k=0}^{\infty} w_k \left(\sum_{j=1}^{h_N-1} \frac{j}{h_N} w_{k+j} + \sum_{j=h_N}^{\infty} w_{k+j} \right).
\end{aligned}$$

□

A.2 The covariance function of $\{U_N(t)\}$

In this section we provide the covariance function of $\{U_N(t)\}_{t\geq 0}$. In view of Lemma 2.1.4 we are finally interested in the covariance function of the time-transformed process $\{\tilde{U}_N(\tilde{t}\,)\}_{t\geq 0}$, defined as

$$\tilde{U}_N(\tilde{t}\,) := U_N(\tilde{t}h_N) \quad \text{for all} \quad -\infty < \tilde{t} < \infty, \tag{A.2.1}$$

hence we assume in the following that the time parameters s and t are chosen as

$$s = \tilde{s}h_N \quad \text{and} \quad t = \tilde{t}h_N \quad \text{for some} \quad -\infty < \tilde{s}, \tilde{t} < \infty. \tag{A.2.2}$$

Furthermore, we use the following notation

$$q = \lfloor |s| \rfloor \quad \text{and} \quad \zeta = |s| - q. \tag{A.2.3}$$

First, we consider the covariance function of $\{P_t\}_{-\infty<t<\infty}$, defined in (A.1.4). Since

$$\mathrm{E}\,(P_t P_{t+s}) = \mathrm{E}\,(P_0 P_{|s|}) \quad \text{for all} \quad -\infty < s, t < \infty,$$

it suffices to consider $\mathrm{E}\,(P_0 P_s)$ for some $s \geq 0$. We have

$$\mathrm{E}\,(P_0 P_s) = \mathrm{E}\left(\sum_{j=0}^{\infty} w_j N_{-j} \sum_{i=0}^{\infty} w_i N_{s-i}\right) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_j w_i\, \mathrm{E}\,(N_{-j} N_{s-i})$$

and since for all $i, j = 0, 1, \ldots$

$$\mathrm{E}\,(N_{-j} N_{s-i}) = \begin{cases} 1-\zeta & , \quad i = j+q, \\ \zeta & , \quad i = j+q+1, \\ 0 & , \quad \text{else}, \end{cases}$$

we get

$$\begin{aligned} \mathrm{E}\,(P_0 P_s) &= \sum_{j=0}^{\infty} w_j (w_{j+q} E(N_{-j} N_{s-(j+q)}) + w_{j+q+1} E(N_{-j} N_{s-(j+q+1)})) \\ &= \sum_{j=0}^{\infty} w_j (w_{j+q}(1-\zeta) + w_{j+q+1}\zeta) \\ &= \sum_{j=0}^{\infty} w_j (w_{j+q} + \zeta(w_{j+q+1} - w_{j+q})). \end{aligned} \tag{A.2.4}$$

Next, note that ($0 \leq s = q + \zeta$)

$$\begin{aligned}
\mathrm{E}\left(U_N(t)U_N(t+s)\right) =& \mathrm{E}\left(U_N(0)U_N(s)\right)\\
=& \frac{1}{h_N}\sum_{i=0}^{h_N-1}\sum_{j=0}^{h_N-1} \mathrm{E}\left(P_{-i}P_{s-j}\right)\\
=& \frac{1}{h_N}\sum_{i=0}^{h_N-1}\sum_{j=0}^{h_N-1} E(P_0P_{i-j+s})\\
=& \frac{1}{h_N}\sum_{i=0}^{h_N-1}\sum_{j=0}^{h_N-1} E(P_0P_{|i-j+q+\zeta|})
\end{aligned}$$

and since

$$|i-j+q+\zeta| = \begin{cases} |i-j+q| + \zeta & , \quad j \leq q+i, \\ |i-j+q+1| + (1-\zeta) & , \quad j > q+i, \end{cases}$$

(A.2.4) shows that

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\frac{1}{h_N}\sum_{i=0}^{h_N-1}\left(\sum_{j=0}^{\min\{q+i,h_N-1\}} E(P_0P_{|i-j+q|+\zeta}) + \sum_{j=q+i+1}^{h_N-1} E(P_0P_{|i-j+q+1|+(1-\zeta)})\right)\\
&=\frac{1}{h_N}\sum_{i=0}^{h_N-1}\left(\sum_{j=0}^{\min\{q+i,h_N-1\}}\sum_{k=0}^{\infty} w_k(w_{k+|i-j+q|} + \zeta(w_{k+|i-j+q|+1} - w_{k+|i-j+q|}))\right.\\
&\quad\left. + \sum_{j=q+i+1}^{h_N-1}\sum_{k=0}^{\infty} w_k(w_{k+|i-j+q+1|} + (1-\zeta)(w_{k+|i-j+q+1|+1} - w_{k+|i-j+q+1|}))\right)\\
&=\frac{1}{h_N}\sum_{i=0}^{h_N-1}\left(\sum_{j=0}^{\min\{q+i,h_N-1\}}\sum_{k=0}^{\infty} w_k(w_{k+|i-j+q|} + \zeta(w_{k+|i-j+q|+1} - w_{k+|i-j+q|}))\right.\\
&\quad\left. + \sum_{j=q+i+1}^{h_N-1}\sum_{k=0}^{\infty} w_k(w_{k+|i-j+q|} + \zeta(w_{k+|i-j+q+1|} + w_{k+|i-j+q|}))\right)\\
&=\frac{1}{h_N}\sum_{i=0}^{h_N-1}\sum_{j=0}^{h_N-1}\sum_{k=0}^{\infty} w_k(w_{k+|i-j+q|} + \zeta(w_{k+|i-j+q+1|} + w_{k+|i-j+q|})). \qquad \text{(A.2.5)}
\end{aligned}$$

Now applying (A.0.1) on (A.2.5) yields the intermediate result

$$
\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right) \\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \Bigg(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{\min\{h_N-1,q\}}(h_N-j)w_{k+q-j} + \sum_{j=q+1}^{h_N-1}(h_N-j)w_{k-q+j} \\
&+\zeta\Bigg(\Bigg(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+(q+1)+j} + \sum_{j=1}^{\min\{h_N-1,q+1\}}(h_N-j)w_{k+(q+1)-j} + \sum_{j=(q+1)+1}^{h_N-1}(h_N-j)w_{k-(q+1)+j}\Bigg) \\
&-\Bigg(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{\min\{h_N-1,q\}}(h_N-j)w_{k+q-j} + \sum_{j=q+1}^{h_N-1}(h_N-j)w_{k-q+j}\Bigg)\Bigg)\Bigg).
\end{aligned}
\tag{A.2.6}
$$

A further simplification of this expression depends on q. We have to consider the cases $q = 0$, $0 < q < h_N - 1$ and $h_N - 1 \leq q$.

First, we assume $q = 0$. Note that then $0 \leq s = \zeta < 1$, hence $0 \leq \tilde{s} < 1/h_N$. Reformulating (A.2.6) yields

$$
\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right) \\
&\quad=\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \Bigg(h_N w_k + 2\sum_{j=1}^{h_N-1}(h_N-j)w_{k+j} \\
&\qquad+\zeta\Bigg(\Bigg(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+j+1} + (h_N-1)w_k + \sum_{j=2}^{h_N-1}(h_N-j)w_{k+j-1}\Bigg) \\
&\qquad-\Bigg(h_N w_k + 2\sum_{j=1}^{h_N-1}(h_N-j)w_{k+j}\Bigg)\Bigg)\Bigg) \\
&\quad=\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \Bigg(h_N w_k + 2\sum_{j=1}^{h_N-1}(h_N-j)w_{k+j} + \zeta\sum_{j=0}^{h_N-1}(w_{k+j+1}-w_{k+j})\Bigg) \\
&\quad=\sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k\Bigg(\sum_{j=1}^{h_N-1}\frac{h_N-j}{h_N}w_{k+j}\Bigg) - \zeta\frac{1}{h_N}\Bigg(\sum_{k=0}^{\infty} w_k^2 - \sum_{k=0}^{\infty} w_k w_{k+h_N}\Bigg).
\end{aligned}
\tag{A.2.7}
$$

Next, we treat the case $0 < q < h_N - 1$, equivalent to $1/h_N \leq \tilde{s} \leq (h_N - 1)/h_N$.

Then (A.2.6) can be stated as

$$
\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{q}(h_N-j)w_{k+q-j} + \sum_{j=q+1}^{h_N-1}(h_N-j)w_{k-q+j} \right.\\
&+\zeta\left(\left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+(q+1)+j} + \sum_{j=1}^{q+1}(h_N-j)w_{k+(q+1)-j} + \sum_{j=(q+1)+1}^{h_N-1}(h_N-j)w_{k-(q+1)+j}\right)\right.\\
&\left.\left.-\left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{q}(h_N-j)w_{k+q-j} + \sum_{j=q+1}^{h_N-1}(h_N-j)w_{k-q+j}\right)\right)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=q}^{h_N+q-1}(h_N+q-j)w_{k+j} + \sum_{j=0}^{q-1}(h_N-q+j)w_{k+j} + \sum_{j=1}^{h_N-q-1}(h_N-q-j)w_{k+j} \right.\\
&+\zeta\left(\left(\sum_{j=1}^{h_N}(h_N-j+1)w_{k+q+j} + \sum_{j=0}^{q}(h_N-j-1)w_{k+q-j} + \sum_{j=q+1}^{h_N-2}(h_N-j-1)w_{k-q+j}\right)\right.\\
&\left.\left.-\left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{q}(h_N-j)w_{k+q-j} + \sum_{j=q+1}^{h_N-1}(h_N-j)w_{k-q+j}\right)\right)\right)
\end{aligned}
$$

and summing up matching terms we obtain

$$
\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=q}^{h_N+q-1}(h_N+q-j)w_{k+j} + \sum_{j=0}^{q-1}(h_N-q+j)w_{k+j} + \sum_{j=1}^{h_N-q-1}(h_N-q-j)w_{k+j} \right.\\
&\qquad\left.+\zeta\left(\sum_{j=1}^{h_N} w_{k+q+j} - \sum_{j=0}^{q} w_{k+q-j} - \sum_{j=q+1}^{h_N-1} w_{k-q+j}\right)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=q}^{h_N+q-1}(h_N+q-j)w_{k+j} + \sum_{j=0}^{q-1}(h_N-q+j)w_{k+j} + \sum_{j=1}^{h_N-q-1}(h_N-q-j)w_{k+j} \right.\\
&\qquad\left.+\zeta\left(\sum_{j=q+1}^{h_N+q} w_{k+j} - \sum_{j=0}^{q} w_{k+j} - \sum_{j=1}^{h_N-q-1} w_{k+j}\right)\right). \qquad \text{(A.2.8)}
\end{aligned}
$$

Considering the upper bounds of the sums, we see that further splitting is required. We first suppose that $0 < q < h_N/2$ and get

$$
\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\frac{1}{h_N}\sum_{k=0}^{\infty}w_k\Bigg((h_N-q)w_k+2\sum_{j=1}^{q-1}(h_N-q)w_{k+j}+2\sum_{j=q}^{h_N-q-1}(h_N-j)w_{k+j}+\sum_{j=h_N-q}^{h_N+q-1}(h_N+q-j)w_{k+j}\\
&\qquad -\zeta\left(w_k+2\sum_{j=1}^{q}w_{k+j}-\sum_{j=h_N-q}^{h_N+q}w_{k+j}\right)\Bigg)\\
&=\frac{1}{h_N}\sum_{k=0}^{\infty}w_k\left(\left(h_Nw_k+2\sum_{j=1}^{q-1}h_Nw_{k+j}+2\sum_{j=q}^{h_N-q-1}(h_N-j)w_{k+j}\right)\right.\\
&\qquad -(q+\zeta)\left(w_k+2\sum_{j=1}^{q}w_{k+j}\right)\\
&\qquad\quad \left.+\left(\sum_{j=h_N-q}^{h_N+q-1}(h_N+q-j)w_{k+j}+2qw_{k+q}+\zeta\sum_{j=h_N-q}^{h_N+q}w_{k+j}\right)\right)\\
&=:\frac{1}{h_N}\sum_{k=0}^{\infty}w_k\left(I_1(N)-(q+\zeta)I_2(N)+I_3(N)\right)
\end{aligned}
\tag{A.2.9}
$$

We now cosider $\frac{1}{h_N}\sum_{k=0}^{\infty}w_kI_3(N)$ in more detail. Since $q < h_N/2$, we get by (2.1.8) and (2.1.11)

$$
\frac{1}{h_N}\sum_{j=h_N-q}^{h_N+q-1}(h_N+q-j)w_{k+j}\leq\frac{1}{h_N}\sum_{j=\lfloor h_N/2\rfloor}^{\infty}2qw_j\leq\sum_{j=\lfloor h_N/2\rfloor}^{\infty}w_j=\boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right)
$$

as $N\to\infty$.

Furthermore, by (2.1.8), (2.1.11) and $0\leq\zeta<1$, it follows that

$$
\frac{1}{h_N}\zeta\sum_{j=h_N-q}^{h_N+q}w_{k+j}\leq\frac{1}{h_N}\sum_{j=\lfloor h_N/2\rfloor}^{\infty}w_j=\boldsymbol{O}\left(\frac{1}{h_N^{1+\psi}}\right)\quad\text{as}\quad N\to\infty
$$

and for the last summand we get by (2.1.8) and (2.1.13)

$$
\frac{1}{h_N}2qw_{k+q}\leq\frac{2}{h_N}qw_q=\boldsymbol{O}\left(\frac{1}{h_N}\right)\quad\text{as}\quad N\to\infty.
$$

Since the asymptotics above hold uniformly on the underlying domain, we get

$$
\frac{1}{h_N}\sum_{k=0}^{\infty}w_kI_3=\boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right)\quad\text{as}\quad N\to\infty,
\tag{A.2.10}
$$

uniformly in $0 < q < h_N/2$.

Combining (A.2.9) with (A.2.10) we see that for all $-\infty < s, t < \infty$

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\sum_{k=0}^{\infty} w_k^2 + \sum_{k=0}^{\infty} w_k \left(2\sum_{j=1}^{q-1} w_{k+j} + 2\sum_{j=q}^{h_N-q-1} \frac{h_N - j}{h_N} w_{k+j}\right)\\
&\qquad - |s|\frac{1}{h_N}\left(\sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k \sum_{j=1}^{q} w_{k+j}\right) + \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N \to \infty, \quad \text{(A.2.11)}
\end{aligned}$$

uniformly in $0 < \lfloor |s| \rfloor < h_N/2$.

Next, we assume that $h_N/2 \le q < h_N - 1$. (A.2.8) can be reformulated as

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\frac{1}{h_N}\sum_{k=0}^{\infty} w_k\Bigg((h_N - q)w_k + 2\sum_{j=1}^{h_N-q-1}(h_N - q)w_{k+j} + \sum_{j=h_N-q}^{q-1}(h_N - q + j)w_{k+j} + \sum_{j=q}^{h_N+q-1}(h_N + q - j)w_{k+j}\\
&\qquad - \zeta\left(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j} + \sum_{j=h_N-q}^{q} w_{k+j} - \sum_{j=q+1}^{h_N+q} w_{k+j}\right)\Bigg)\\
&=\frac{1}{h_N}\sum_{k=0}^{\infty} w_k\Bigg(h_N w_k + 2\sum_{j=1}^{h_N-q-1} h_N w_{k+j} + \sum_{j=h_N-q}^{q-1}(h_N + j)w_{k+j} + \sum_{j=q}^{h_N+q-1}(h_N - j)w_{k+j}\\
&\qquad - q\left(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j} + \sum_{j=h_N-q}^{q-1} w_{k+j} - \sum_{j=q}^{h_N+q-1} w_{k+j}\right)\\
&\qquad\quad - \zeta\left(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j} + \sum_{j=h_N-q}^{q} w_{k+j} - \sum_{j=q+1}^{h_N+q} w_{k+j}\right)\Bigg).
\end{aligned}$$

Replacing $q + \zeta$ by s and grouping matching terms we get

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\frac{1}{h_N}\sum_{k=0}^{\infty} w_k\Bigg(\left(h_N w_k + 2\sum_{j=1}^{h_N-q-1} h_N w_{k+j}\right) - s\left(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j}\right)\\
&\quad - s\left(\sum_{j=h_N-q}^{q-1} w_{k+j} - \sum_{j=q}^{h_N+q-1} w_{k+j}\right)\\
&\quad + \left(\sum_{j=h_N-q}^{q-1}(h_N + j)w_{k+j} + \sum_{j=q}^{h_N+q-1}(h_N - j)w_{k+j} - \zeta\left(2w_{k+q} - w_{k+h_N+q}\right)\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \Bigg(\Bigg(h_N w_k + 2\sum_{j=1}^{h_N-q-1} h_N w_{k+j}\Bigg) - s\Bigg(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j}\Bigg)\\
&\quad + \Bigg(\sum_{j=h_N-q}^{q-1}(h_N - s)w_{k+j} + \sum_{j=q}^{h_N+q-1}(h_N+s)w_{k+j}\Bigg)\\
&\qquad + \Bigg(\sum_{j=h_N-q}^{q-1} j w_{k+j} - \sum_{j=q}^{h_N+q-1} j w_{k+j} - \zeta\left(2w_{k+q} - w_{k+h_N+q}\right)\Bigg)\Bigg)
\end{aligned}
$$

$$
=:\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \Bigg(\Bigg(h_N w_k + 2\sum_{j=1}^{h_N-q-1} h_N w_{k+j}\Bigg) \tag{A.2.12}
$$

$$
- s\Bigg(w_k + 2\sum_{j=1}^{h_N-q-1} w_{k+j}\Bigg) + I_1(N) + I_2(N)\Bigg) \tag{A.2.13}
$$

We now consider $\frac{1}{h_N} I_1(N)$ under the assumption $h_N/2 \leq q < h_N - 1$, which is equivalent to $h_N/2 \leq \lfloor \tilde{s} h_N \rfloor < h_N - 1$. By (2.1.8) and (2.1.11) we see that

$$
\begin{aligned}
\frac{1}{h_N}\sum_{j=h_N-q}^{q-1}(h_N - s)w_{k+j} &\leq \sum_{j=\lfloor h_N(1-\tilde{s})\rfloor}^{\infty} (1-\tilde{s}) w_j\\
&= \boldsymbol{O}\left(\frac{1-\tilde{s}}{(\lfloor h_N(1-\tilde{s})\rfloor)^{\psi}}\right) = \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N \to \infty
\end{aligned}
$$

and

$$
\frac{1}{h_N}\sum_{j=q}^{h_N+q-1}(h_N+s)w_{k+j} \leq \frac{1}{h_N}\sum_{j=\lfloor h_N/2\rfloor}^{\infty} 2h_N w_j = \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N \to \infty.
$$

Since both asymptotics hold uniformly in $h_N/2 \leq \lfloor \tilde{s} h_N \rfloor < h_N - 1$ we obtain

$$
\frac{1}{h_N} I_1(N) = \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N \to \infty, \tag{A.2.14}
$$

uniformly in $h_N/2 \leq \lfloor \tilde{s} h_N \rfloor < h_N - 1$.

Next, we investigate $\frac{1}{h_N} I_2(N)$. (2.1.8) and (2.1.10) yield

$$
\frac{1}{h_N}\sum_{j=h_N-q}^{q-1} j w_{k+j} \leq \frac{1}{h_N}\sum_{j=1}^{h_N} j w_j = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N \to \infty
$$

and

$$
\frac{1}{h_N}\sum_{j=q}^{h_N+q-1} j w_{k+j} \leq \frac{1}{h_N}\sum_{j=1}^{2h_N} j w_j = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N \to \infty.
$$

Since furthermore

$$\frac{1}{h_N}\,\zeta(2w_{k+q}-w_{k+h_N+q}) = \boldsymbol{O}\left(\frac{1}{h_N}\right) \quad \text{as} \quad N\to\infty,$$

we see that

$$\frac{1}{h_N}\,I_2(N) = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N\to\infty, \tag{A.2.15}$$

also uniformly in $h_N/2 \le \lfloor \tilde{s}h_N \rfloor < h_N - 1$. Combining (A.2.12), (A.2.14) and (A.2.15) we achieve the following result

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
&=\sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k \sum_{j=1}^{h_N-q-1} w_{k+j} - \frac{|s|}{h_N}\left(\sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k \sum_{j=1}^{h_N-q-1} w_{k+j}\right) + \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right)
\end{aligned} \tag{A.2.16}$$

as $N\to\infty$, uniformly in $h_N/2 \le \lfloor \tilde{s}h_N \rfloor < h_N - 1$, where $\gamma = \min\{1-\phi, \psi\}$.

The outstanding case is $h_N - 1 \le q$. Under this assumption (A.2.6) can be reformulated as

$$\begin{aligned}
&\mathrm{E}\left(U_N(t)U_N(t+s)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{h_N-1}(h_N-j)w_{k+q-j}\right.\\
&\quad + \zeta\left(\left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+(q+1)+j} + \sum_{j=1}^{h_N-1}(h_N-j)w_{k+(q+1)-j}\right)\right.\\
&\qquad \left.\left. - \left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{h_N-1}(h_N-j)w_{k+q-j}\right)\right)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{h_N-1}(h_N-j)w_{k+q-j}\right.\\
&\quad + \zeta\left(\left(\sum_{j=q+1}^{h_N+q}(h_N+q+1-j)w_{k+j} + \sum_{j=q-h_N+2}^{q}(h_N-q-1+j)w_{k+j}\right)\right.\\
&\qquad \left.\left. - \left(\sum_{j=q}^{h_N+q-1}(h_N+q-j)w_{k+j} + \sum_{j=q-h_N+1}^{q-1}(h_N-q+j)w_{k+j}\right)\right)\right)\\
=&\frac{1}{h_N}\sum_{k=0}^{\infty} w_k \left(\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} + \sum_{j=1}^{h_N-1}(h_N-j)w_{k+q-j}\right.\\
&\quad \left. + \zeta\left(\sum_{j=q+1}^{h_N+q} w_{k+j} - \sum_{j=q-h_N+1}^{q} w_{k+j}\right)\right).
\end{aligned} \tag{A.2.17}$$

It follows by (2.1.8)–(2.1.11) that

$$
\begin{aligned}
\frac{1}{h_N}\sum_{j=0}^{h_N-1}(h_N-j)w_{k+q+j} \leq & \frac{1}{h_N}\sum_{j=0}^{h_N-1}(h_N-j)w_{(h_N-1)+j} \\
\leq & \sum_{j=h_N-1}^{\infty} w_j = \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N\to\infty,
\end{aligned} \tag{A.2.18}
$$

$$
\begin{aligned}
\frac{1}{h_N}\sum_{j=1}^{h_N-1}(h_N-j)w_{k+q-j} \leq & \frac{1}{h_N}\sum_{j=1}^{h_N-1}(h_N-j)w_{(h_N-1)-j} \\
\leq & \frac{1}{h_N}\sum_{j=0}^{h_N-2}(j+1)w_j = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N\to\infty,
\end{aligned} \tag{A.2.19}
$$

$$
\frac{1}{h_N}\sum_{j=q+1}^{h_N+q} w_{k+j} \leq \frac{1}{h_N}\sum_{j=h_N}^{\infty} w_j = \boldsymbol{O}\left(\frac{1}{h_N^{1+\psi}}\right) \quad \text{as} \quad N\to\infty, \tag{A.2.20}
$$

$$
\frac{1}{h_N}\sum_{j=q-h_N+1}^{q} w_{k+j} \leq \frac{1}{h_N}\sum_{j=0}^{\infty} w_j = \boldsymbol{O}\left(\frac{1}{h_N}\right) \quad \text{as} \quad N\to\infty \tag{A.2.21}
$$

and since the results in (A.2.18)–(A.2.21) hold uniformly in $h_N-1\leq q$, we see that for all $h_N-1\leq |s| = |\tilde{s}h_N|$ it holds uniformly that

$$
\operatorname{Cov}\left(U_N(t),U_N(t+s)\right) = \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N\to\infty, \tag{A.2.22}
$$

where $\gamma=\min\{1-\phi,\psi\}$.

A.3 The autocorrelation functions of $\{U_N(t)\}$ and $\{\tilde{U}_N(\tilde{t}\,)\}$

With the results of the preceding sections, we finally obtain the autocorrelation functions of $\{U_N(t)\}_{t\geq 0}$ and $\{\tilde{U}_N(\tilde{t}\,)\}_{\tilde{t}\geq 0}$, denoted by r_N and $\tilde{r}_N$, respectively.

Lemma A.3.1 *The autocorrelation functions r_N and $\tilde{r}_N$ satisfy*

$$
r_N(s) = \begin{cases}
1-\dfrac{|s|}{h_N}\displaystyle\sum_{k=0}^{\infty} w_k^2 + \boldsymbol{O}\left(\dfrac{1}{h_N^{1+\gamma}}\right) & , \quad \textit{uniformly in} \quad |s|<1, \\
1-\dfrac{|s|}{h_N} + \boldsymbol{O}\left(\dfrac{1}{h_N^{\gamma}}\right) & , \quad \textit{uniformly in} \quad 1\leq |s| < h_N-1, \\
\boldsymbol{O}\left(\dfrac{1}{h_N^{\gamma}}\right) & , \quad \textit{uniformly in} \quad h_N-1\leq |s|,
\end{cases} \tag{A.3.1}
$$

as $N \to \infty$ *and*

$$\tilde{r}_N(\tilde{s}\,) = \max\{0, (1 - |\tilde{s}|)\} + \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right), \quad \textit{uniformly}, \tag{A.3.2}$$

as $N \to \infty$, *where* $\gamma = \min\{1-\phi, \psi\}$. *The parameters* ϕ *and* ψ *are laid down in* (2.1.10) *and* (2.1.11), *respectively.*

PROOF: We first consider the case $|s\,| = |\tilde{s}h_N| < 1$. With (A.1.1) and (A.2.7) we get

$$\begin{aligned} r_N(s) =& 1 - \frac{|s|}{h_N}\frac{\sum_{k=0}^{\infty} w_k^2 - \sum_{k=0}^{\infty} w_k w_{k+h_N}}{\sigma_N^2} \\ =& 1 - \frac{|s|}{h_N}\sum_{k=0}^{\infty} w_k^2 + \frac{|s|}{h_N}\frac{(\sigma_N^2 - 1)\sum_{k=0}^{\infty} w_k^2 + \sum_{k=0}^{\infty} w_k w_{k+h_N}}{\sigma_N^2}. \end{aligned}$$

Hence, by (2.1.8), (2.1.11) and (A.1.2), we see that

$$r_N(s) = 1 - \frac{|s|}{h_N}\sum_{k=0}^{\infty} w_k^2 + \frac{|s|}{h_N}\,\boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.3.3}$$

uniformly in $|s\,| = |\tilde{s}h_N| < 1$, where $\gamma = \min\{1-\phi, \psi\}$.

Next, we assume that $1 \le \lfloor |s\,| \rfloor = \lfloor |\tilde{s}h_N| \rfloor < h_N/2$. Then (A.2.11), (A.1.1) and (A.1.2) imply

$$\begin{aligned} r_N(s) =& \left(\frac{1}{\sigma_N^2}\left(\sum_{k=0}^{\infty} w_k^2 + \sum_{k=0}^{\infty} 2w_k\left(\sum_{j=1}^{q-1} w_{k+j} + \sum_{j=q}^{h_N-q-1}\frac{h_N - j}{h_N} w_{k+j}\right)\right)\right) \\ & - \frac{|s\,|}{h_N}\left(\frac{1}{\sigma_N^2}\sum_{k=0}^{\infty} w_k^2 + 2\sum_{k=0}^{\infty} w_k \sum_{j=1}^{q} w_{k+j}\right) + \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \\ =&: I_1(N) - \frac{|s\,|}{h_N} I_2(N) + \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \end{aligned}$$

as $N \to \infty$, uniformly in $1 \le \lfloor |s\,| \rfloor = \lfloor |\tilde{s}h_N| \rfloor < h_N/2$.

First, note that

$$1 - I_1(N) = \frac{1}{\sigma_N^2}\left(2\sum_{k=0}^{\infty} w_k\left(\sum_{j=h_N-q}^{h_N-1}\frac{h_N - j}{h_N} w_{k+j} - \sum_{j=1}^{q-1}\frac{j}{h_N} w_{k+j}\right)\right).$$

Since (2.1.8) and (2.1.11) imply

$$\sum_{j=h_N-q}^{h_N-1}\frac{h_N - j}{h_N} w_{k+j} \le \sum_{j=h_N-q}^{h_N-1}\frac{q}{h_N} w_{k+j} \le \frac{1}{2}\sum_{j=\lfloor h_N/2 \rfloor}^{h_N-1} w_j = \boldsymbol{O}\left(\frac{1}{h_N^{\psi}}\right) \quad \text{as} \quad N \to \infty$$

and (2.1.8), together with (2.1.10), yields

$$\sum_{j=1}^{q-1} \frac{j}{h_N} w_{k+j} \leq \frac{1}{h_N} \sum_{j=1}^{\lfloor h_N/2 \rfloor} j w_j = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N \to \infty,$$

we conclude that

$$I_1 = 1 + \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.3.4}$$

uniformly in $1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor < h_N/2$, where $\gamma = \min\{1-\phi, \psi\}$.

Carrying out similar computations for $I_2(N)$, we start with

$$1 - I_2(N) = \frac{1}{\sigma_N^2} \left(2 \sum_{k=0}^{\infty} w_k \left(\sum_{j=q+1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j} - \sum_{j=1}^{q} \frac{j}{h_N} w_{k+j} \right) \right).$$

The first inner sum is not converging uniformly to zero, but by (2.1.8) and (2.1.11) we conclude that

$$\sum_{j=q+1}^{h_N-1} \frac{h_N - j}{h_N} w_{k+j} \leq \sum_{j=\lfloor |s| \rfloor + 1}^{h_N-1} w_j = \boldsymbol{O}\left(\frac{1}{|s|^{\psi}}\right) \quad \text{as} \quad N \to \infty$$

and in the same way we get

$$\sum_{j=1}^{q} \frac{j}{h_N} w_{k+j} = \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N \to \infty,$$

uniformly in $1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor < h_N/2$.

Now $\frac{|s|}{h_N} I_2(N)$ can be stated as

$$\frac{|s|}{h_N} I_2 = \frac{|s|}{h_N} + \frac{1}{h_N} \boldsymbol{O}\left(|s|^{1-\psi}\right) + \frac{|s|}{h_N} \boldsymbol{O}\left(\frac{1}{h_N^{1-\phi}}\right) \quad \text{as} \quad N \to \infty$$

and since $\lfloor |s| \rfloor < h_N/2$, it follows that

$$\frac{|s|}{h_N} I_2 = \frac{|s|}{h_N} + \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.3.5}$$

uniformly in $1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor < h_N/2$.

Finally, (A.3.4) and (A.3.5) yield

$$r_N(s) = 1 - \frac{|s|}{h_N} + \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.3.6}$$

uniformly in $1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor < h_N/2$, where $\gamma = \min\{1-\phi, \psi\}$.

Note that (A.3.6) also holds for the case $h_N/2 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| < h_N - 1$. This follows by (A.1.1), (A.2.16) and basically the same computations as above, hence we omit the details.

For the last case, $h_N - 1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor$, we get by (A.2.22) and (A.1.2)

$$r_N(s) = \boldsymbol{O}\left(\frac{1}{h_N^{\gamma}}\right) \quad \text{as} \quad N \to \infty, \tag{A.3.7}$$

uniformly in $h_N - 1 \leq \lfloor |s| \rfloor = \lfloor |\tilde{s} h_N| \rfloor$, where $\gamma = \min\{1 - \phi, \psi\}$ and the proof of (A.3.1) is complete.

Assertion (A.3.2) is an immediate consequence of (A.3.1).

$\square$

Appendix B

A class of boundary functions for the Wiener process

B.1 Boundary functions which are $\simeq \sqrt{t \log t}$

Theorem B.1.1 *Let the family of functions* $\{g(t_0,t)\}$ *be defined as*

$$g(t_0,t) = \sqrt{t \log\left(\frac{t}{t_0} + e\right)} \quad \textit{for all} \quad t > 0, \quad t_0 \geq 1 \tag{B.1.1}$$

and let $\{W(t), t \geq 0\}$ *be a standard Wiener process. Then, for all real* x *it holds that*

$$\lim_{t_0 \to \infty} \mathrm{P}\left(a_{t_0} \sup_{1 \leq t < \infty} \frac{W(t)}{g(t_0,t)} - b_{t_0} \leq x\right) = \exp\left(-e^{-x}\right) \tag{B.1.2}$$

and

$$\lim_{t_0 \to \infty} \mathrm{P}\left(a_{t_0} \sup_{1 \leq t < \infty} \frac{|W(t)|}{g(t_0,t)} - b_{t_0} \leq x\right) = \exp\left(-2e^{-x}\right), \tag{B.1.3}$$

where

$$a_{t_0} = \sqrt{2 \log\log t_0} \tag{B.1.4}$$

and

$$b_{t_0} = 2 \log\log t_0 + \frac{1}{2}\log\log\log t_0 - \frac{1}{2}\log \pi. \tag{B.1.5}$$

Proof of Theorem B.1.1

Lemma B.1.1 *Let the family of functions* $\{g(t_0,t)\}$ *be defined as in* (B.1.1)*. Then, it holds that*

$$\sup_{t_0 \leq t < \infty} \frac{W(t)}{g(t_0,t)} = \boldsymbol{O}_P(1) \quad \textit{as} \quad t_0 \to \infty. \tag{B.1.6}$$

PROOF: We have

$$\sup_{t_0 \le t < \infty} \frac{W(t)}{\sqrt{t \log\left(\frac{t}{t_0} + e\right)}} \stackrel{\mathrm{D}}{=} \sup_{t_0 \le t < \infty} \frac{W\left(\frac{t}{t_0}\right)}{\sqrt{\frac{t}{t_0} \log\left(\frac{t}{t_0} + e\right)}} = \sup_{1 \le t < \infty} \frac{W(t)}{\sqrt{t \log\left(t + e\right)}}$$

and the lemma follows by the law of the iterated logarithm.

□

Lemma B.1.2 *Let the family of functions* $\{g(t_0, t)\}$ *be defined as in* (B.1.1). *Then, it holds that*

$$\sup_{t_0/\log t_0 \le t < t_0} \frac{W(t)}{g(t_0, t)} = \boldsymbol{O}_P(\sqrt{\log\log\log t_0}) \quad \textit{as} \quad t_0 \to \infty. \tag{B.1.7}$$

PROOF: Since

$$\sup_{t_0/\log t_0 \le t \le t_0} \frac{W(t)}{\sqrt{t \log\left(\frac{t}{t_0} + e\right)}} \le \sup_{t_0/\log t_0 \le t \le t_0} \frac{W(t)}{\sqrt{t}} \stackrel{\mathrm{D}}{=} \sup_{1/\log t_0 \le t \le 1} \frac{W(t)}{\sqrt{t}},$$

the law of the iterated logarithm in zero yields the lemma.

□

Lemma B.1.3 *Let the family of functions* $\{g(t_0, t)\}$ *be defined as in* (B.1.1). *Then, it holds that*

$$a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{\sqrt{t}} - \sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{g(t_0, t)}\right) = \boldsymbol{o}_P(1) \quad \textit{as} \quad t_0 \to \infty. \tag{B.1.8}$$

PROOF: First, note that

$$a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{\left|\sqrt{\log\left(\frac{t}{t_0} + e\right)} W(t) - W(t)\right|}{\sqrt{t \log\left(\frac{t}{t_0} + e\right)}}\right)$$
$$= a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{\left(\frac{1}{2e}\left(\frac{t}{t_0}\right) + \boldsymbol{o}\left(\frac{t}{t_0}\right) |W(t)|\right)}{\sqrt{t \log\left(\frac{t}{t_0} + e\right)}}\right) \quad \text{as} \quad t_0 \to \infty,$$

uniformly in $t \in [1, t_0/\log t_0]$.

Now

$$a_{t_0} \sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}|W(t)|}{\sqrt{t \log\left(\frac{t}{t_0} + e\right)}} = \boldsymbol{O}(1) \sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}\sqrt{\log\log t}}{\sqrt{\log\left(\frac{t}{t_0} + e\right)}} \quad \text{a.s.} \quad \text{as} \quad t_0 \to \infty$$

and

$$\sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}\sqrt{\log\log t}}{\sqrt{\log\left(\frac{t}{t_0} + e\right)}} \le \sup_{1 \le t \le t_0/\log t_0} \frac{t}{t_0}\sqrt{\log\log t} \le \frac{1}{\log t_0}\sqrt{\log\log\left(\frac{t_0}{\log t_0}\right)},$$

implying the lemma as $t_0 \to \infty$.

□

Lemma B.1.4 *Let a_{t_0} and b_{t_0} be defined as in* (B.1.4) *and* (B.1.5)*, respectively. Then,*

$$\lim_{t_0 \to \infty} \mathrm{P}\left(a_{t_0} \sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{\sqrt{t}} - b_{t_0} \le x \right) = \exp\left(-e^{-x}\right). \tag{B.1.9}$$

Proof: Since

$$\left\{ \frac{W(e^t)}{\sqrt{e^t}}, 0 \le t \le \log\left(t_0/\log t_0\right) \right\}$$

is a standardized, stationary Gaussian process, we obtain from Theorem 12.3.5 in Leadbetter, Lindgren and Rootzén (1983) that

$$\lim_{t_0 \to \infty} \mathrm{P}\left(a_{t_0/\log t_0} \sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{\sqrt{t}} - b_{t_0/\log t_0} \le x \right) = \exp\left(-e^{-x}\right), \tag{B.1.10}$$

where

$$a_{t_0/\log t_0} = \sqrt{2 \log\log\left(\frac{t_0}{\log t_0}\right)}$$

and

$$b_{t_0/\log t_0} = 2\log\log\left(\frac{t_0}{\log t_0}\right) + \frac{1}{2}\log\log\log\left(\frac{t_0}{\log t_0}\right) - \frac{1}{2}\log \pi.$$

Now elementary calculations show that

$$a_{t_0/\log t_0}\left(a_{t_0} - a_{t_0/\log t_0}\right) \to 0 \quad \text{and} \quad b_{t_0} - b_{t_0/\log t_0} \to 0 \quad \text{as} \quad t_0 \to \infty,$$

which implies the lemma.

□

Combining the Lemmas B.1.1–B.1.4, we see that assertion (B.1.2) holds. The statement given in display (B.1.3) is a consequence of (B.1.2) and the asymptotic independence of maxima and minima in the underlying extreme value asymptotic (see Bickel and Rosenblatt, 1973).

B.2 Boundary functions which are $\simeq \sqrt{t \log\log t}$

Theorem B.2.1 *Let the family of functions $\{h(t_0,t)\}$ be defined as*

$$h(t_0,t) = \sqrt{t \log\log\left(\frac{t}{t_0} + e^e\right)} \quad \text{for all} \quad t > 0, \quad t_0 \geq 1 \tag{B.2.1}$$

and let $\{W(t), t \geq 0\}$ be a standard Wiener process. Then, for all real x it holds that

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0} \sup_{1\leq t<\infty} \frac{W(t)}{h(t_0,t)} - b_{t_0} \leq x\right) = \exp\left(-e^{-x}\right) \tag{B.2.2}$$

and

$$\lim_{t_0\to\infty} \mathrm{P}\left(a_{t_0} \sup_{1\leq t<\infty} \frac{|W(t)|}{h(t_0,t)} - b_{t_0} \leq x\right) = \exp\left(-2e^{-x}\right), \tag{B.2.3}$$

where

$$a_{t_0} = \sqrt{2\log\log t_0} \tag{B.2.4}$$

and

$$b_{t_0} = 2\log\log t_0 + \frac{1}{2}\log\log\log t_0 - \frac{1}{2}\log \pi. \tag{B.2.5}$$

Proof of Theorem B.2.1

Lemma B.2.1 *Let the family of functions $\{h(t_0,t)\}$ be defined as in* (B.2.1). *Then, it holds that*

$$\sup_{t_0\leq t<\infty} \frac{W(t)}{h(t_0,t)} = \boldsymbol{O}_P(1) \quad \text{as} \quad t_0 \to \infty. \tag{B.2.6}$$

Proof: We have

$$\sup_{t_0\leq t<\infty} \frac{W(t)}{\sqrt{t\log\log\left(\frac{t}{t_0}+e^e\right)}} \overset{\mathrm{D}}{=} \sup_{t_0\leq t<\infty} \frac{W\left(\frac{t}{t_0}\right)}{\sqrt{\frac{t}{t_0}\log\log\left(\frac{t}{t_0}+e^e\right)}} = \sup_{1\leq t<\infty} \frac{W(t)}{\sqrt{t\log\log\left(t+e^e\right)}}$$

and the law of the iterated logarithm gives the lemma.

□

Lemma B.2.2 *Let the family of functions $\{h(t_0,t)\}$ be defined as in* (B.2.1). *Then, it holds that*

$$\sup_{t_0/\log t_0\leq t<t_0} \frac{W(t)}{h(t_0,t)} = \boldsymbol{O}_P(\sqrt{\log\log\log t_0}) \quad \text{as} \quad t_0 \to \infty. \tag{B.2.7}$$

Proof: Since

$$\sup_{t_0/\log t_0 \le t \le t_0} \frac{W(t)}{\sqrt{t \log\log\left(\frac{t}{t_0} + e^e\right)}} \le \sup_{t_0/\log t_0 \le t \le t_0} \frac{W(t)}{\sqrt{t}} \overset{\mathrm{D}}{=} \sup_{1/\log t_0 \le t \le 1} \frac{W(t)}{\sqrt{t}},$$

the lemma follows from the law of the iterated logarithm in zero.

□

Lemma B.2.3 *Let the family of functions* $\{h(t_0, t)\}$ *be defined as in* (B.2.1)*. Then, it holds that*

$$a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{\sqrt{t}} - \sup_{1 \le t \le t_0/\log t_0} \frac{W(t)}{h(t_0, t)}\right) = \boldsymbol{o}_P(1) \quad as \quad t_0 \to \infty. \tag{B.2.8}$$

Proof: Elementary calculations show that

$$a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{\left|\sqrt{\log\log\left(\frac{t}{t_0} + e^e\right)}W(t) - W(t)\right|}{\sqrt{t \log\log\left(\frac{t}{t_0} + e^e\right)}}\right)$$
$$= a_{t_0}\left(\sup_{1 \le t \le t_0/\log t_0} \frac{\left(\frac{1}{2\exp(e+1)}\left(\frac{t}{t_0}\right) + \boldsymbol{o}\left(\frac{t}{t_0}\right)|W(t)|\right)}{\sqrt{t \log\log\left(\frac{t}{t_0} + e^e\right)}}\right) \quad \text{as} \quad t_0 \to \infty,$$

uniformly in $t \in [1, t_0/\log t_0]$.

Since

$$a_{t_0} \sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}|W(t)|}{\sqrt{t \log\log\left(\frac{t}{t_0} + e^e\right)}}$$
$$= \boldsymbol{O}(1) \sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}\sqrt{\log\log t}}{\sqrt{\log\log\left(\frac{t}{t_0} + e^e\right)}} \quad \text{a.s.} \quad \text{as} \quad t_0 \to \infty$$

and

$$\sup_{1 \le t \le t_0/\log t_0} \frac{\frac{t}{t_0}\sqrt{\log\log t}}{\sqrt{\log\log\left(\frac{t}{t_0} + e^e\right)}} \le \sup_{1 \le t \le t_0/\log t_0} \frac{t}{t_0}\sqrt{\log\log t} \le \frac{1}{\log t_0}\sqrt{\log\log\left(\frac{t_0}{\log t_0}\right)},$$

the lemma follows as $t_0 \to \infty$.

□

Theorem (B.2.1) now follows by the same arguments as Theorem B.1.1.

Appendix C

PWMA control-charts II

C.1 Open-end control charts

In this chapter, we provide an alternative boundary function for the open-ended PWMA-chart in the case of two-sided alternatives and known parameters, which yields computable critical values. The approach is based on the following result from Robbins and Siegmund (1970).

Example *Let $S(k)$ denote the partial sum of k i.i.d. random variables having mean 0 and variance 1. Then,*

$$\begin{aligned}
\lim_{m\to\infty} \mathrm{P}&\left(\sup_{1\le k<\infty} \frac{|S(k)|}{\sqrt{(k+m)(a^2+\log(1+k/m))}} \ge 1\right) \\
&= \mathrm{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{\sqrt{(1+t)(a^2+\log(1+t))}} \ge 1\right) \\
&= \exp\left(-\frac{1}{2}\,a^2\right) \qquad (a>0).
\end{aligned}$$

C.1.1 Model assumptions for known μ and σ

We assume that (3.1.1)–(3.1.3) and (3.1.5) hold with $N = \infty$.

C.1.2 Monitoring procedure for known μ and σ

Let the sequence of detectors be defined as

$$P_k = \sum_{j=1}^{k} p_{j,k}(X_j - \mu), \quad k = 1, 2, \ldots \tag{C.1.1}$$

where the weights $\{p_{j,k}, 1 \le j \le k, k = 1, 2, \dots\}$ are given in (3.1.6).

Furthermore, we define for all $m = 1, 2, \dots$, $s > 0$ and $a > 0$

$$h(m,s,a) = \frac{\sqrt{m}}{\sqrt{2d+1}} \sqrt{\left(\frac{s}{m} + \left(\frac{m}{s}\right)^{2d}\right)\left(a^2 + \log\left(\left(\frac{s}{m}\right)^{2d+1} + 1\right)\right)}. \tag{C.1.2}$$

We test the null hypotheses H_0 versus the two-sided alternative H_2. The stopping time is defined as

$$\eta = \eta(\alpha, m) = \inf\{1 \le k < \infty : |P_k| > \sigma h(m, k, (-2\log\alpha)^{1/2})\}, \tag{C.1.3}$$

where $\alpha \in]0, 1[$.

The boundary function is justified by the following theorem.

Theorem C.1.1 *Let the sequences $\{P_k\}_{k=1,2,\dots}$ and $\{h(m,k,a)\}_{k=1,2,\dots}$ be defined as in*(C.1.1) *and* (C.1.2)*, respectively. Then, it holds under the null hypothesis that*

$$\lim_{m\to\infty} \mathrm{P}\left(\sup_{1\le k<\infty} \frac{|P_k|}{\sigma h(m,k,a)} \ge 1\right) = \exp\left(-\frac{1}{2}\,a^2\right) \tag{C.1.4}$$

Furthermore, the procedure has asymptotic power one.

Theorem C.1.2 *Let the sequences $\{P_k\}_{k=1,2,\dots}$ and $\{h(m,k,a)\}_{k=1,2,\dots}$ be defined as in*(C.1.1) *and* (C.1.2)*, respectively. Then, it holds under the alternative that*

$$\lim_{m\to\infty} \mathrm{P}\left(\sup_{1\le k<\infty} \frac{|P_k|}{\sigma h(m,k,a)} \ge 1\right) = 1. \tag{C.1.5}$$

C.1.3 Proofs

Proof of Theorem C.1.1

We define

$$Q_k = \sigma \sum_{j=1}^{k} p_{j,k}(W(j) - W(j-1)) \quad \text{for all} \quad k = 1, 2, \dots, \tag{C.1.6}$$

where $\{W(t), t \ge 0\}$ is the approximating Wiener process given in (3.1.2).

Lemma C.1.1 *Let the sequences $\{P_k\}_{k=1,2,\dots}$ and $\{Q_k\}_{k=1,2,\dots}$ be defined as in* (C.1.1) *and* (C.1.6)*, respectively. Then, it holds that*

$$\sup_{1\le k<\infty} \frac{|P_k|}{\sigma h(m,k,a)} - \sup_{1\le k<\infty} \frac{|Q_k|}{\sigma h(m,k,a)} = \boldsymbol{O}_P\left(\frac{1}{m^{1/2-1/\nu}}\right) \quad \textit{as} \quad m \to \infty. \tag{C.1.7}$$

PROOF: As in the proof of Lemma 3.1.1 we get

$$\sup_{1\le k<\infty} \frac{|P_k - Q_k|}{\sigma h(m,k,a)} \le 2 \sup_{1\le k<\infty} \frac{|S(k) - W(k)|}{\sigma h(m,k,a)}$$

and by (3.1.2) we obtain

$$2 \max_{1\le k<\infty} \frac{|S(k) - \sigma W(k)|}{\sigma h(m,k,a)} = \boldsymbol{O}_P(1) \sup_{1\le k<\infty} \frac{k^{1/\nu}}{h(m,k,a)} \quad \text{as} \quad m \to \infty.$$

Since for $k \le m$ and some suitable constant $C > 0$ it holds that

$$\frac{k^{1/\nu}}{h(m,k,a)} \le C \frac{k^{1/\nu}}{\sqrt{m}\sqrt{\frac{k}{m} + \left(\frac{m}{k}\right)^{2d}}} \le \frac{k^{1/\nu}}{m^{1/2}} \le m^{1/\nu - 1/2}$$

and for $k \ge m$ we have

$$\frac{k^{1/\nu}}{h(m,k,a)} \le C \frac{k^{1/\nu}}{\sqrt{m}\sqrt{\frac{k}{m} + \left(\frac{m}{k}\right)^{2d}}} \le \frac{k^{1/\nu}}{k^{1/2}} \le m^{1/\nu - 1/2},$$

the Lemma follows as $m \to \infty$.

□

Lemma C.1.2 *Let the sequence* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (C.1.6). *Then, it holds that*

$$\max_{1\le k\le m/\log m} \frac{Q_k}{\sigma h(m,k,a)} = \boldsymbol{o}(1) \quad a.s. \quad as \quad m \to \infty. \tag{C.1.8}$$

PROOF: Since the sequence $\{Q_k\}_{k=1,2,\ldots}$ obeys the law of the iterated logarithm for weighted sums (see Li and Tomkins, 1996), it suffices to consider

$$\max_{1\le k\le m/\log m} \frac{\sqrt{k \log\log k}}{h(m,k,a)}.$$

For some suitable chosen constant $C > 0$ and $1 \le k \le m/\log m$ it holds that

$$\frac{\sqrt{k\log\log k}}{h(m,k,a)} \le \frac{\sqrt{k\log\log k}}{\sqrt{m}\sqrt{\frac{k}{m} + \left(\frac{m}{k}\right)^{2d}}} = \frac{\sqrt{\log\log k}}{\sqrt{1 + \left(\frac{m}{k}\right)^{2d+1}}} \le \frac{\sqrt{\log\log m}}{\sqrt{1 + (\log m)^{2d+1}}}$$

and the Lemma follows as $m \to \infty$.

□

Lemma C.1.3 *Let the sequence* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (C.1.6). *Then, for all* $\delta > 0$ *it holds that*

$$\sup_{m^{1+\delta} \leq k < \infty} \frac{Q_k}{\sigma h(m,k,a)} = \boldsymbol{o}(1) \quad a.s. \quad as \quad m \to \infty. \tag{C.1.9}$$

Proof: Since for all $k \geq m^{1+\delta}$

$$\begin{aligned}
\frac{\sqrt{k \log\log k}}{h(m,k,a)} &= \frac{\sqrt{2d+1}\sqrt{k\log\log k}}{\sqrt{m}\sqrt{\left(\frac{k}{m} + \left(\frac{m}{k}\right)^{2d}\right)\left(a^2 + \log\left(\left(\frac{k}{m}\right)^{2d+1} + 1\right)\right)}} \\
&= \frac{\sqrt{2d+1}\sqrt{\log\log k}}{\sqrt{\left(1 + \left(\frac{m}{k}\right)^{2d+1}\right)\left(a^2 + \log\left(\left(\frac{k}{m}\right)^{2d+1} + 1\right)\right)}} \\
&\leq \frac{\sqrt{2d+1}\sqrt{\log\log k}}{\sqrt{\left(\log\left(\left(\frac{k}{m}\right)^{2d+1} + 1\right)\right)}} \\
&\leq \frac{\sqrt{2d+1}\sqrt{\log\log(m^{1+\delta})}}{\sqrt{\left(\log\left((m^{\delta})^{2d+1} + 1\right)\right)}},
\end{aligned}$$

the Lemma follows by the law of the iterated logarithm for weighted sums.

□

We define the process

$$U(t) = \sigma \int_0^t p\,(x,t)\, dW(x) = \sigma \int_0^t \left(\frac{x}{t}\right)^d dW(x) \quad \text{for all} \quad t > 0. \tag{C.1.10}$$

Lemma C.1.4 *Let the sequence* $\{Q_k\}_{k=1,2,\ldots}$ *be defined as in* (C.1.6). *Then, for all* $T > 0$ *it holds that*

$$\max_{m/T \leq k \leq mT} \frac{Q_k}{\sigma h(m,k,a)} \xrightarrow{D} \sup_{1/T \leq t \leq T} \frac{W(t^{2d+1})}{\tilde{h}(t^{2d+1},a)} \quad as \quad m \to \infty, \tag{C.1.11}$$

where

$$\tilde{h}(x,a) = \sqrt{(1+x)(a^2 + \log(1+x))} \quad for\ all \quad x > 0, \quad a > 0. \tag{C.1.12}$$

Proof: First, note that for all $t \geq m/T$ we have

$$\begin{aligned}
\sigma h(m,t,a) &\geq \frac{\sigma\sqrt{t}}{\sqrt{2d+1}}\left(a^2 + \log\left(\left(\frac{1}{T}\right)^{2d+1} + 1\right)\right)^{1/2} \\
&= \sigma_t \left(a^2 + \log\left(\left(\frac{1}{T}\right)^{2d+1} + 1\right)\right)^{1/2},
\end{aligned}$$

where σ_t is given in (3.1.9). Hence, Lemma 3.1.2 yields

$$\left| \max_{m/T \le k \le mT} \frac{Q_k}{\sigma h(m,k,a)} - \sup_{m/T \le t \le mT} \frac{U(t)}{\sigma h(m,t,a)} \right| = \boldsymbol{O}_P\left(\left(\frac{T \log mT}{m} \right)^{1/2} \right) = \boldsymbol{o}_P(1)$$

as $m \to \infty$.

Comparing the covariances, we see that

$$\left\{ U(t) \frac{\sqrt{2d+1}}{\sigma}, \frac{m}{T} \le t \le mT \right\} \stackrel{\mathrm{D}}{=} \left\{ \frac{W(t^{2d+1})}{t^d}, \frac{m}{T} \le t \le mT \right\},$$

which implies

$$\sup_{m/T \le t \le mT} \frac{U(t)}{\sigma h(m,t,a)} \stackrel{\mathrm{D}}{=} \sup_{m/T \le t \le mT} \frac{W(t^{2d+1})}{m^{d+1/2} \tilde{h}\left(\left(\frac{t}{m} \right)^{2d+1} \right)}.$$

Finally, the rescaling property of the Wiener process yields

$$\sup_{m/T \le t \le mT} \frac{W(t^{2d+1})}{m^{d+1/2} \tilde{h}\left(\left(\frac{t}{m} \right)^{2d+1} \right)} \stackrel{\mathrm{D}}{=} \sup_{1/T \le t \le T} \frac{W(t^{2d+1})}{\tilde{h}\left(t^{2d+1} \right)},$$

which completes the proof.

□

Since the continuity of the Wiener process implies

$$\mathrm{P}\left(\sup_{1/T \le t \le T} \frac{W(t^{2d+1})}{\tilde{h}\left(t^{2d+1} \right)} \right) = \mathrm{P}\left(\sup_{(1/T)^{2d+1} \le t \le T^{2d+1}} \frac{W(t)}{\tilde{h}\left(t \right)} \right) \to \mathrm{P}\left(\sup_{0 < t < \infty} \frac{W(t)}{\tilde{h}\left(t \right)} \right)$$

as $T \to \infty$, combining the Lemmas C.1.1–C.1.4 with Example 3 of Robbins and Siegmund (1970) yields Theorem C.1.1.

Proof of Theorem C.1.2

We define the sequence $\{P_k^{(0)}\}_{k=1,2,\ldots}$ as in (3.1.57) with $N = \infty$.

Then, it holds under the alternative that

$$P_k = P_k^{(0)} + \sum_{j=k^*+1}^{k} p_{j,k} \, \triangle \quad \text{for all} \quad k = 1, 2, \ldots$$

For any finite $N > k^*$ we have

$$\begin{aligned} \mathrm{P}\left(\sup_{1\le k<\infty}\frac{P_k}{\sigma h(m,k,a)}\ge 1\right) &\ge \mathrm{P}\left(\max_{1\le k\le N}\frac{P_k}{\sigma h(m,k,a)}\ge 1\right) \\ &\ge \mathrm{P}\left(\max_{1\le k\le N}\frac{\sum_{j=k^*+1}^{k} p_{j,k}\ \triangle}{\sigma h(m,k,a)} - \max_{1\le k\le N}\frac{|P_k^{(0)}|}{\sigma h(m,k,a)}\ge 1\right) \\ &\ge \mathrm{P}\left(\frac{\sum_{j=k^*+1}^{N} p_{j,N}\ \triangle}{\sigma h(m,N,a)} - \max_{1\le k\le N}\frac{|P_k^{(0)}|}{\sigma h(m,k,a)}\ge 1\right). \end{aligned} \tag{C.1.13}$$

Since k^* is fixed, we have

$$\max_{1\le k\le N}\sum_{j=k^*+1}^{k} p_{j,k}\ \triangle \ge \frac{\triangle}{d+1}\frac{N^{d+1}-{k^*}^{d+1}}{N^d}.$$

Hence, if we choose $N = m$, it follows by(C.1.2) that

$$\frac{\sum_{j=k^*+1}^{m} p_{j,m}\ \triangle}{\sigma h(m,m,a)} \simeq m^{1/2} \quad \text{as} \quad m\to\infty. \tag{C.1.14}$$

Now the law of the iterated logarithm for weighted sums yields

$$\max_{1\le k\le m}\frac{|P_k^{(0)}|}{\sigma h(m,m,a)} = \boldsymbol{O}\left(\sqrt{\log\log m}\right) \quad \text{a.s.} \quad \text{as} \quad m\to\infty \tag{C.1.15}$$

and Theorem C.1.2 follows by (C.1.13)–(C.1.15).

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Köln, im Dezember 2007

Mario Kühn