The Cone of Moving Curves on Algebraic Varieties

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Abstract

We give a new description of the closed cone $\overline{\mathrm{NM}}(X) \subset N_1(X)_{\mathbb{R}}$ of moving curves of a smooth Fano three- or fourfold X by finitely many linear equations. In particular, the cone $\overline{\mathrm{NM}}(X)$ is polyhedral. The proof in the threefold case relies on a famous result of Bucksom, Demailly, Paun and Peternell which says that the cone of moving curves is dual to the cone of pseudoeffective divisors. Additionally, the proof in the fourfold case uses a result of Kawamata which describes the exceptional locus and the flip of a small contraction on a smooth fourfold. This proof provides an inductive way to compute the cone of moving curves and gives a description of the Mori cone of the variety obtained by the the flip of a small contraction.

Kurzzusammenfassung

Wir geben eine neue Beschreibung des Kegels $\overline{\mathrm{NM}}(X) \subset N_1(X)_{\mathbb{R}}$ beweglicher Kurven auf einer glatten Fano-Drei- oder Vierfaltigkeit X durch endlich viele lineare Gleichungen. Insbesondere ist der Kegel $\overline{\mathrm{NM}}(X)$ polyhedral. Der Beweis für den Fall einer glatten Dreifaltigkeit beruht auf einem berühmten Resultat von Bucksom, Demailly, Paun und Peternell, welches besagt, dass der Kegel beweglicher Kurven dual zum Kegel der pseudoeffektiven Divisoren ist. Der Beweis für den Fall glatter Fano-Vierfaltigkeiten benutzt zusätzlich ein Resultat von Kawamata, welches den exzeptionellen Ort und den Flip einer kleinen Kontraktion auf einer glatten Vierfaltigkeit beschreibt. Der Beweis bietet zudem eine Möglichkeit den Kegel beweglicher Kurven iterativ zu berechnen und liefert eine Beschreibung des Mori Kegels einer Vierfaltigkeit, die man durch den Flip einer kleinen Kontraktion erhält.

Contents

Zusammenfassung

Eine irreduzible Kurve c auf einer projektiven Varietät X heisst *bewegliche Kurve*, falls sie Mitglied einer algebraischen, überdeckenden Familie ist.

Ein intuitives Beispiel für bewegliche Kurven sind die Geraden auf einer glatten Quadrik. Man sieht unendlich viele Geraden, aber nur zwei verschiedene

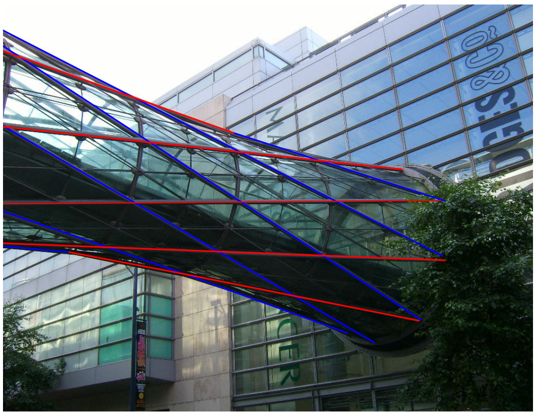

Abbildung 1. Zwei Familien von Geraden auf der glatten Quadrik.

Familien von Geraden auf dieser Fläche, nämlich die in Rot und die in Blau eingefärbten Geraden in Abbildung 1.[1] Es macht offenbar keinen Sinn zwischen zwei Geraden der selben Familie zu unterscheiden. Dies motiviert den folgenden Begriff.

Wir sagen, dass zwei Kurven c_1 und c_2 auf einer projektiven Varietät X *numerisch äquivalent* sind, wenn ihre Schnittzahlen $c_1 . D$ und $c_2 . D$ für jeden Cartier Divisor D auf X übereinstimmen, also $c_1 . D = c_2 . D$ gilt.

Man sieht unmittelbar, dass je zwei Geraden derselben Familie dieselbe numerische Äquivalenzklasse haben. Es gibt also insbesondere nur zwei verschiedene Äquivalenzklassen von Geraden auf dieser glatten Quadrik.

Numerische Äquivalenz können wir völlig analog auch auf der freien abelschen Gruppe

$$Z_1(X)_{\mathbb{R}} = \left\{ \sum\nolimits_{i=1}^{m} a_i c_i \mid a_i \in \mathbb{R}, c_i \subset X \text{ eine irreduzible Kurve} \right\}$$

von 1-Zykeln mit reellen Koeffizienten und der Gruppe der $\mathbb{R}$-Divisoren $\mathrm{Div}(X)_{\mathbb{R}}$ definieren. Die Gruppe der 1-Zykel mit reellen Koeffizienten

[1] Ich danke Patrick Litherland für die Erlaubnis das Bild aus Abbildung 1 benutzen zu dürfen, welches eine Brücke in der Corporation Street, Manchester (UK) zeigt.

modulo numerischer Äquivalenz und die Gruppe der $\mathbb{R}$-Divisoren modulo numerischer Äquivalenz sind tatsächlich sogar endlich-dimensionale $\mathbb{R}$-Vektorräume, welche wir mit

$$N_1(X)_{\mathbb{R}} := Z_1(X)_{\mathbb{R}}/\equiv_{\text{num}} \quad \text{und } N^1(X)_{\mathbb{R}} := \text{Div}(X)_{\mathbb{R}}/\equiv_{\text{num}}$$

bezeichnen. Dies ermöglicht es uns Überlegungen auf dem Niveau der Linearen Algebra anzustellen.

Der *Kegel beweglicher Kurven* $\overline{\text{NM}}(X)$ einer projektiven Varietät X ist definiert als der Abschluss des konvexen Kegels in $N_1(X)_{\mathbb{R}}$, welcher durch numerische Klassen von beweglichen Kurven aufgespannt wird.

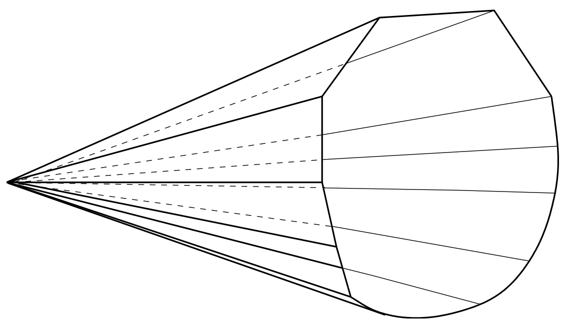

ABBILDUNG 2. Skizze eines Kegels in $N_1(X)_{\mathbb{R}}$. Eine „Ecke" des Kegels wird *extremaler Strahl* genannt.

Dieser Kegel ist für uns von Interesse, weil er in Verbindung mit dem Minimalen-Modell-Programm steht. Ein extremaler Strahl des Mori Kegels, dessen extremale Kontraktion eine Faserung ist, wird durch die Klasse einer rationalen beweglichen Kurve aufgespannt. Ein weiterer interessanter Zusammenhang ist durch den folgenden Sachverhalt gegeben.

Im Jahr 2004 haben Boucksom, Demailly, Paun und Peternell gezeigt, dass der Kegel beweglicher Kurven dual zum Kegel der pseudoeffektiven Divisorenklassen ist, welche man als Grenzwert von Folgen von $\mathbb{Q}$-Divisorenklassen erhält.

Theorem.[**BDPP04**, Theorem 2.2 und 2.4] *Sei X eine irreduzible projektive Varietät der Dimension n. Dann sind die Kegel $\overline{\text{NM}}(X)$ und $\overline{\text{Eff}}(X)$ dual zueinander und somit*

$$\overline{\text{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma \,.\, \Delta \geq 0, \text{ für alle } \Delta \in \overline{\text{Eff}}(X)\}.$$

Es ist klar, dass der Kegel beweglicher Kurven im dualen Kegel des pseudoeffektiven Kegels enthalten ist, weil eine bewegliche Kurve jeden irreduziblen Divisor nicht negativ schneidet. Die andere Inklusion war ein einschlagendes Resultat. Die Autoren konnten mit diesem Theorem eine Antwort auf eine der wichtigsten offenen Fragen der Klassifikationstheorie von projektiven bzw. kompakten Kählermannigfaltigkeiten geben. Sie haben gezeigt, dass eine projektive Mannigfaltigkeit X von rationalen Kurven überdeckt ist, wenn der kanonische Divisor K_X nicht pseudoeffektiv ist.

Mit Hilfe des obigen Theorems werden wir beweisen, dass der Kegel der beweglichen Kurven einer glatten Fano-Drei- oder Vierfaltigkeit polyhedral ist, indem wir explizit die endlich vielen linearen Gleichungen angeben, die den Kegel ausschneiden. Dies ist das Hauptresultat dieser Arbeit.

Wir beschreiben kurz die Situation. Sei X eine glatte Fano-Varietät der Dimension n, so dass jede extremale Kontraktion eines extremalen Strahls von $\overline{\mathrm{NE}}(X)$ eine Faserung oder eine divisorielle Kontraktion ist. Der exzeptionelle Divisor einer divisoriellen Kontraktion ist irreduzibel und wird deshalb von jeder beweglichen Kurve nicht negativ geschnitten.

Das folgende Theorem sagt, dass der Kegel beweglicher Kurven tatsächlich sogar genau durch diese Bedingung festgelegt ist, also durch die linearen Gleichungen ausgeschnitten wird, welche von den exzeptionellen Divisoren der divisoriellen Kontraktionen auf X induziert werden.

Theorem. *Sei X eine $\mathbb{Q}$-faktorielle Fano-n-Faltigkeit mit höchstens terminalen Singularitäten, so dass jede extremale Kontraktion eines K_X-negativen extremalen Strahls eine divisorielle oder eine Faserkontraktion ist. Seien $\varphi_i : X \to X_i$, $i = 1, ..., k$, die divisoriellen Kontraktionen mit exzeptionellen Divisoren $E_i \subset X$, die zu extremalen Strahlen $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, des Mori Kegels $\overline{\mathrm{NE}}(X)$ von X gehören. Dann ist*

$$\overline{\mathrm{NM}}(X) = \left\{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma . [E_i] \geq 0, \textit{ für alle } i = 1, ..., k\right\} \subset N_1(X)_{\mathbb{R}}.$$

Insbesondere ist $\overline{\mathrm{NM}}(X)$ ein abgeschlossener, konvexer, polyhedraler Kegel.

Als Konsequenz erhalten wir das Resultat für glatte Fano-Dreifaltigkeiten.

Korollar. *Sei X eine glatte Fano-Dreifaltigkeit und seien $\varphi_i : X \to X_i$, $i = 1, ..., k$, die divisoriellen Kontraktionen mit exzeptionellen Divisoren $E_i \subset X$, die zu extremalen Strahlen $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, des Mori Kegels $\overline{\mathrm{NE}}(X)$ von X gehören. Dann ist*

$$\overline{\mathrm{NM}}(X) = \left\{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma . [E_i] \geq 0, \textit{ für alle } i = 1, ..., k\right\} \subset N_1(X)_{\mathbb{R}}.$$

Insbesondere ist $\overline{\mathrm{NM}}(X)$ ein abgeschlossener, konvexer, polyhedraler Kegel.

Um obiges Theorem zu erhalten, werden wir lediglich das Dualitätsresultat von Bucksom, Demailly, Paun und Peternell aus [**BDPP04**] verwenden.

Alex Küronya und Endre Szabó haben, unabhängig von dieser Arbeit, in dem bisher unveröffentlichten Preprint [**KS**] bewiesen, dass der Kegel $\overline{\mathrm{Eff}}(X)$ der pseudoeffektiven Divisoren einer projektiven Varietät X ein endlicher rationaler Polyeder ist, wenn X eine glatte Fano-Dreifaltigkeit ist. Also impliziert nach [**BDPP04**] jeweils ein Resultat das andere.

Im Anschluss werden wir den Beweis für glatte Fano-Vierfaltigkeiten geben.

Theorem. *Sei X eine glatte Fano-Vierfaltigkeit. Dann ist der Kegel $\overline{\mathrm{NM}}(X)$ beweglicher Kurven auf X ein konvexer, polyhedraler Kegel in $N_1(X)_{\mathbb{R}}$, gegeben durch*

$$\overline{\mathrm{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma . \Delta \geq 0 \textit{ für alle } \Delta \in \mathrm{Eq}(X)\},$$

wobei $\mathrm{Eq}(X) \subset N^1(X)_{\mathbb{R}}$ eine explizit bekannte, endliche Menge von Klassen von Divisoren auf X ist.

Wir werden die Menge $\mathrm{Eq}(X)$ in Kapitel 4 detailliert konstruieren, an einem Beispiel zeigen, dass diese Menge auch berechenbar ist und die genaue Aussage detaillierter in Theorem 4.33 wiederholen.

Die Situation ist für Vierfaltigkeiten komplizierter, weil der exzeptionelle Ort einer extremalen Kontraktion „zu klein" sein kann, oder genauer gesagt, grössere Kodimension als eins haben kann. Der Schlüssel zum Beweis dieses Theorems ist ein Resultat von Kawamata, welches den exzeptionellen Ort und den Flip einer kleinen Kontraktion einer glatten Vierfaltigkeit beschreibt. Siehe Theorem 4.14 und Korollar 4.16 in Kapitel 4.

Wir halten fest, dass das Ziel dieser Arbeit ist, den Kegel beweglicher Kurven von einem neuen Standpunkt aus zu betrachten. Gemeint ist damit die Beschreibung des Kegels durch lineare Gleichungen.

Der zum Kegel der pseudoeffektiven Divisorenklassen duale Kegel wurde schon früher von Batyrev in [**Bat92**] untersucht. Grob gesagt, hat er den Kegel durch extremale Strahlen beschrieben, welche man durch Rückzug einer rationalen Kurve erhält, die in einer allgemeinen Faser eines Mori Faserraums liegt.

Als Konsequenz daraus erhält man, dass der Kegel beweglicher Kurven einer Fano-Dreifaltigkeit nur endlich viele extremale Strahlen hat, die alle durch rationale Kurven repräsentiert werden.

Der Leser, der mit der verwendeten Terminologie noch nicht vertraut ist, sollte die genaue Aussage, welche von Araujo in [**Ara05**] letztlich bewiesen wurde, eventuell überspringen. Siehe dazu [**Ara05**, Bemerkung 3.4].

Theorem.[**Ara05**, Theorem 3.3] *Sei X eine $\mathbb{Q}$-faktorielle Dreifaltigkeit und Δ ein Randdivisor, so dass (X, Δ) höchstens terminale Singularitäten hat. Dann gilt:*

(a) Für jedes $\varepsilon > 0$ und jeden amplen Divisor A auf X existieren endlich viele Klassen von Kurven $\varsigma_1, \ldots, \varsigma_r \in N_1(X)_{\mathbb{R}}$, so dass
 - *(1) $0 < -K_X \,.\, \varsigma_i < 6$ ist,*
 - *(2) es einen Mori Faserraum $f_i : X_i \to S_i$ gibt, welchen man durch Anwendung des $(K_X + \Delta)$-MMP erhält, so dass ς_i der Rückzug der Klasse einer rationalen Kurve ist, die in einer allgemeinen Faser von f_i liegt und*
 - *(3) $\overline{\mathrm{NE}}(X)_{([K_X]+\Delta+\varepsilon[A])\geq 0} + \overline{\mathrm{NM}}(X) = \overline{\mathrm{NE}}(X)_{([K_X]+\Delta+\varepsilon[A])\geq 0} + \sum \mathbb{R}_+\varsigma_i$.*

(b) Es existieren abzählbar viele Klassen von Kurven $\varsigma_i \in N_1(X)_{\mathbb{R}}$, so dass
 - *(1) $0 < -K_X \,.\, \varsigma_i < 6$ ist,*
 - *(2) es einen Mori Faserraum $f_i : X_i \to S_i$ gibt, welchen man durch Anwendung des $(K_X + \Delta)$-MMP erhält, so dass ς_i der Rückzug der Klasse einer rationalen Kurve ist, die in einer allgemeinen Faser von f_i liegt und*
 - *(3) $\overline{\mathrm{NE}}(X)_{([K_X]+\Delta)\geq 0} + \overline{\mathrm{NM}}(X) = \overline{\mathrm{NE}}(X)_{([K_X]+\Delta)\geq 0} + \sum \mathbb{R}_+\varsigma_i$.*

Die extremalen Strahlen $\mathbb{R}_+\varsigma_i$ werden coextremale Strahlen *genannt.* □

Für Fano-Varietäten haben Birkar, Cascini, Hacon und M^c^Kernan kürzlich eine Verallgemeinerung dieser Aussage in dem Preprint [**BCHM06**] bewiesen. Sie erhalten die analoge Aussage, dass der Kegel beweglicher Kurven einer glatten Fano-Varietät der Dimension n polyhedral ist und dass alle extremalen Strahlen durch den Rückzug einer rationalen Kurve repräsentiert werden.

Korollar.[BCHM06, Korollar 1.3.4] *Sei (X, Δ) ein $\mathbb{Q}$-faktorielles, Kawamata log terminales Paar, so dass $-(K_X + \Delta)$ ample ist. Dann ist $\overline{\mathrm{NM}}(X)$ ein rationaler Polyeder. Wenn $F = F_i$ ein extremaler Strahl des Kegels beweglicher Kurven ist, dann existiert ein Divisor Θ, so dass das Paar (X, Θ) Kawamata log terminal ist und wir ein $(K_X + \Theta)$-MMP $\pi : X \dashrightarrow Y$ durchlaufen können. Dieses endet mit einem Mori Faserraum $f : Y \to Z$, so dass F von dem Rückzug nach X der Klasse einer beliebigen Kurve $\Sigma \subset Y$ aufgespannt wird, welche durch f kontrahiert wird.*

Die Autoren erhalten dieses Korollar als Konsequenz ihres Haupttheorems, welches ein sehr tiefliegendes Resultat ist.

Zusammenfassend notieren wir, dass der hier präsentierte Beweis einen detaillierten Einblick in die Geometrie einer Fano-Drei- oder Vierfaltigkeit gibt und dass wir, anders als andere Autoren, eine Beschreibung des Kegels durch lineare Gleichungen erzielen.

Danksagung

Zuerst möchte ich der DFG und der Universität zu Köln für mein Vollzeit-Stipendium im Graduiertenkolleg „Globale Strukturen in Geometrie und Analysis" meine Dankbarkeit aussprechen. Ich danke Stefan Kebekus, Thomas Eckl, allen Korrekturlesern und allen Mitgliedern des Graduiertenkollegs, vor allem Sebastian Neumann, für zahlreiche Diskussionen, all ihre Hilfe und ihren Zuspruch. Ich möchte außerdem James McKernan, Seán Keel, Alex Küronya, Carolina Araujo, Laurent Bonavero und besonders Cinzia Casagrande für die Beantwortung meiner Fragen danken. Zudem bin ich Laurent Bonavero und Andreas Höring für einige fruchtbare Diskussionen, welche während der Sommerschule „Geometry of complex projective varieties and the minimal model program" im Jahr 2007 und in einem Workshop im Jahr 2008 am Institut Fourier in Grenoble geführt wurden, zu Dank verpflichtet.

Zu guter Letzt danke ich meinen Freunden, meiner Familie und vor allem meiner Frau Susanne für ihre Geduld, ihre Unterstützung und Liebe.

Introduction

Summary

An irreducible curve c on a projective variety X is called a *movable* or *moving curve* if c is a member of an algebraic family of curves which covers a dense subset of X.

An intuitive example of a moving curve is a line on a smooth quadric surface. Figure 3[2] illustrates many different lines on a smooth quadric surface, but

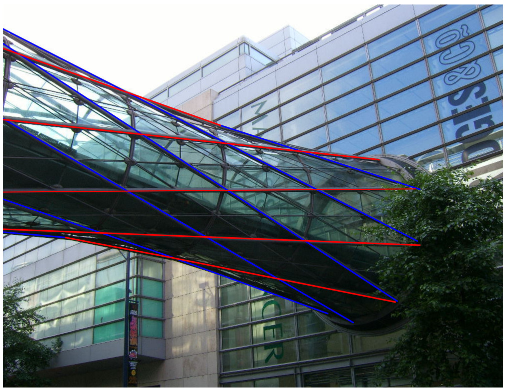

FIGURE 3. Two different rulings on the smooth quadric surface.

there are just two different families of lines, namely the red and the blue ones. Expediently, we do not want to distinguish between two lines of the same family. This motivates the following concept. Two curves c_1 and c_2 on a projective variety X are *numerically equivalent* if the intersection numbers $c_1 \,.\, D$ and $c_2 \,.\, D$ coincide for every Cartier divisor D on X.

We see that two lines of the same family have the same numerical equivalence class. In particular, there are just two numerical equivalence classes of lines in Figure 3; one is represented by a blue line, the second one by a red line.

More general, we can define numerical equivalence analogously on the free abelian group of 1-cycles

$$Z_1(X)_{\mathbb{R}} = \left\{ \sum\nolimits_{i=1}^{m} a_i c_i \mid a_i \in \mathbb{R}, c_i \subset X \text{ an irreducible curve} \right\}$$

with real coefficients and on the group $\mathrm{Div}(X)_{\mathbb{R}}$ of $\mathbb{R}$-divisors on X. In fact, both groups modulo numerical equivalence are finite dimensional real vector

[2]I would like to thank Patrick Litherland for the permission to use the picture in Figure 3, which shows a bridge in Corporation Street, Manchester (UK).

spaces; we will denote these spaces by

$$N_1(X)_{\mathbb{R}} := Z_1(X)_{\mathbb{R}}/\equiv_{\text{num}} \text{ and by } N^1(X)_{\mathbb{R}} := \text{Div}(X)_{\mathbb{R}}/\equiv_{\text{num}}.$$

This enables us to put our considerations down to the level of linear algebra.

The *cone of moving curves* $\overline{\text{NM}}(X)$ of a projective variety X is defined as the closure of the convex cone in $N_1(X)_{\mathbb{R}}$ which is spanned by numerical equivalence classes of moving curves.

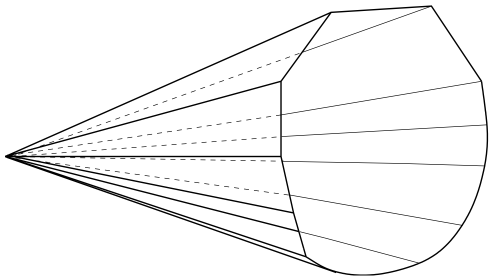

FIGURE 4. Sketch of a cone in $N_1(X)_{\mathbb{R}}$. An "edge" of the cone is called *extremal ray*.

We are interested in this cone since it is related to the minimal model program. An extremal ray of the Mori cone which has a fibration as corresponding extremal contraction is spanned by the class of a rational moving curve. Another interesting relation is given by the following result.

In 2004 Boucksom, Demailly, Paun and Peternell showed that the cone of moving curves is dual to the cone of pseudoeffective divisor classes which is spanned by classes that appear as limits of sequences of effective $\mathbb{Q}$-divisors.

Theorem 1.[**BDPP04**, Theorem 2.2 and 2.4] *Let X be an irreducible projective variety of dimension n. Then the cones $\overline{\text{NM}}(X)$ and $\overline{\text{Eff}}(X)$ are dual; in other words,*

$$\overline{\text{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma\,.\,\Delta \geq 0, \text{ for all } \Delta \in \overline{\text{Eff}}(X)\}.$$

It is clear that the moving cone is contained in the dual of the pseudoeffective cone since a movable curve has non-negative intersection with every irreducible divisor. The other inclusion was an exciting result. With this theorem, the authors were able to give an answer to one of the major open problems in the classification theory of projective and compact Kähler manifolds. They proved that a projective manifold X is covered by rational curves if the canonical divisor K_X is not pseudoeffective.

The theorem above is our key to prove that the moving cone of a smooth Fano three- or fourfold is polyhedral: we will indicate finitely many linear equations which cut out the cone. This will be the main result of this thesis.

Here is a sketch of the situation. Assume that X is a smooth Fano n-fold such that every extremal contraction of an extremal ray of $\overline{\text{NE}}(X)$ is a divisorial or a fibre contraction. It is clear that every movable curve has non-negative intersection with the exceptional divisor of a divisorial contraction since the exceptional divisor is irreducible.

The following theorem states that in this case the moving cone is completely determined by exactly this condition; more precisely: it is cut out by the linear equations induced by the exceptional divisors of the divisorial contractions on X.

Theorem 2. *Let X be a $\mathbb{Q}$-factorial Fano n-fold with only terminal singularities such that every extremal contraction of a K_X-negative extremal ray is a divisorial or a fibre contraction. Let $\varphi_i : X \to X_i$, $i = 1, ..., k$, be the divisorial contractions with exceptional divisors $E_i \subset X$, which correspond to some extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, of the Mori cone $\overline{\mathrm{NE}}(X)$ of X. Then*

$$\overline{\mathrm{NM}}(X) = \big\{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma\,.[E_i] \geq 0, \textit{ for all } i = 1, ..., k\big\}.$$

In particular, $\overline{\mathrm{NM}}(X)$ is a closed, convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$.

An immediate consequence of this theorem is the announced result for smooth Fano threefolds.

Corollary 3. *Let X be a smooth Fano threefold and let $\varphi_i : X \to X_i$, $i = 1, ..., k$, be the divisorial contractions with exceptional divisors $E_i \subset X$, which correspond to some extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, of the Mori cone $\overline{\mathrm{NE}}(X)$ of X. Then*

$$\overline{\mathrm{NM}}(X) = \big\{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma\,.[E_i] \geq 0, \textit{ for all } i = 1, ..., k\big\}.$$

In particular, $\overline{\mathrm{NM}}(X)$ is a closed, convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$.

To prove the above theorem, we will use nothing more than the duality result of Bucksom, Demailly, Paun and Peternell in [**BDPP04**].

We have learned that Alex Küronya and Endre Szabó have independently proved in the unpublished preprint [**KS**] that the cone $\overline{\mathrm{Eff}}(X)$ of pseudoeffective divisors on a projective variety X is a finite rational polyhedron if X is a smooth Fano threefold. Therefore, one result implies the other by [**BDPP04**].

Then we will give the proof for the case of a smooth Fano fourfold.

Theorem 4. *Let X be a smooth Fano fourfold. Then the moving cone $\overline{\mathrm{NM}}(X)$ of X is a convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$, given by*

$$\overline{\mathrm{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma\,.\,\Delta \geq 0 \textit{ for all } \Delta \in \mathrm{Eq}(X)\},$$

where $\mathrm{Eq}(X) \subset N^1(X)_{\mathbb{R}}$ is a well-known, finite set of divisor classes.

We will construct the set $\mathrm{Eq}(X)$ explicitely in Chapter 4, we will show by example that this set is actually computable, and we will give the precise statement with Theorem 4.33.

The situation for fourfolds is more complicated since the exceptional set of an extremal contraction can be small; in other words, the exceptional set has codimension at least two. The main ingredient for the proof is a result of Kawamata which describes the exceptional locus and the flip of a small contraction on a smooth fourfold. See Theorem 4.14 and Corollary 4.16 in chapter 4.

Note that the aspiration of the thesis is to approach the moving cone from a new point of view, meaning the description of the cone by linear equations.

The dual of the cone of pseudoeffective divisor classes has been studied earlier by Batyrev in [**Bat92**]. Roughly speaking, he described the cone in terms of extremal rays which are represented by the class of the pullback of rational curves lying in general fibres of fibrations obtained by the minimal model program. As a consequence, he deduced that the moving cone of a Fano threefold has just finitely many extremal rays, which are represented by rational curves.

The reader who is not yet familiar with the terminolgy may skip the precise statement, which was finally proved by Araujo in the preprint [**Ara05**], but see [**Ara05**, Remark 3.4].

Theorem 5.[**Ara05**, Theorem 3.3] *Let X be a $\mathbb{Q}$-factorial threefold and Δ be a boundary divisor such that (X,Δ) has only terminal singularities. Then*

(a) for any $\varepsilon > 0$ and any ample divisor A on X there are finitely many classes of curves $\varsigma_1,\dots,\varsigma_r \in N_1(X)_{\mathbb{R}}$ such that
   (1) $0 < -K_X \,.\, \varsigma_i < 6$,
   (2) there is a Mori fibre space $f_i : X_i \to S_i$, which can be obtained by running the $(K_X+\Delta)$-MMP, such that ς_i is the pullback class of a rational curve lying on a general fibre of f_i, and
   (3) $\overline{\mathrm{NE}}(X)_{([K_X]+\Delta+\varepsilon[A])\geq 0}+\overline{\mathrm{NM}}(X) = \overline{\mathrm{NE}}(X)_{([K_X]+\Delta+\varepsilon[A])\geq 0}+\sum \mathbb{R}_+\varsigma_i$.

(b) There are countably many classes of curves $\varsigma_i \in N_1(X)_{\mathbb{R}}$ such that
   (1) $0 < -K_X \,.\, \varsigma_i < 6$,
   (2) there is a Mori fibre space $f_i : X_i \to S_i$, which can be obtained by running the $(K_X+\Delta)$-MMP, such that ς_i is the pullback class of a rational curve lying on a general fibre of f_i, and
   (3) $\overline{\mathrm{NE}}(X)_{([K_X]+\Delta)\geq 0}+\overline{\mathrm{NM}}(X) = \overline{\mathrm{NE}}(X)_{([K_X]+\Delta)\geq 0}+\sum \mathbb{R}_+\varsigma_i$.

The extremal rays $\mathbb{R}_+\varsigma_i$ above are called coextremal rays. □

Very recently, Birkar, Cascini, Hacon and M^cKernan have proved a generalization for Fano varieties in the preprint [**BCHM06**]. They obtained the analogous statement that the moving cone of a Fano n-fold is polyhedral, and that the extremal rays of the cone are represented by the pullback of rational curves via usage of the minimal model program.

Corollary 6.[**BCHM06**, Corollary 1.3.4] *Let (X,Δ) be a $\mathbb{Q}$-factorial Kawamata log terminal pair such that $-(K_X+\Delta)$ is ample. Then $\overline{\mathrm{NM}}(X)$ is a rational polyhedron. If $F = F_i$ is an extremal ray of the closed cone of moving curves, then there exists a divisor Θ such that the pair (X,Θ) is Kawamata log terminal and we may run a $(K_X+\Theta)$-MMP $\pi : X \dashrightarrow Y$ which ends with a Mori fibre space $f : Y \to Z$ such that F is spanned by the pullback to X of the class of any curve Σ which is contracted by f.*

The authors achieved this corollary as a consequence of their main theorem, which is a really deep result.

Recapitulating, we can note that the proof which we will present here gives a detailed geometric insight. Moreover, we will pursue a different approach to the description of the cone by linear equations.

Outline of the thesis

In section 1.1 we will give the basic definitions and recall the introduced theorem of Bucksom, Demailly, Paun and Peternell. In section 1.2 we will give a brief review of the basic version of the minimal model program. In chapter 2 we will derive some results on the moving cone of projective surfaces from the duality statement of [**BDPP04**], and we will give a result on the extremal faces of the moving cone. In order to use the minimal model program and the related morphisms, we want to take "pullbacks" and "pushforwards" of 1-cycles on higher dimensional varieties. Therefore, these notions need to be specified, which will be done in chapter 3. Finally, we will give some examples of the moving cones of smooth Fano fourfolds, and we will start to prove the main results of the thesis in chapter 4.

Notation and conventions

In this short section we will fix the basic notation which will de used throughout the whole thesis.

- We will work with normal projective varieties defined over $\mathbb{C}$.
- With $\mathbb{N}$ we mean the set $\{0, 1, 2, \ldots\}$ of non-negative integers and $\mathbb{R}_+$ denotes the set $\{r \in \mathbb{R} \mid r \geq 0\}$.
- Let $R \in \{\mathbb{R}, \mathbb{R}_+\}$. If V is a $\mathbb{R}$-vector space and $S \subset V$ is a subset then $\langle S\rangle_R := \overline{\{\sum_{i=1}^m a_i \cdot s_i \mid a_i \in R, s_i \in S\}}$ denotes the closure of the set of finite linear combinations of elements of the set S with coefficients in R. If the set S is finite, $S = \{s_1, \ldots, s_n\}$ say, then by abuse of notation we will just write $\langle s_1, \ldots, s_n\rangle_R$.
- $\mathrm{Div}(X)$ denotes the group of Cartier divisors on X.
- $\mathrm{Div}(X)_{\mathbb{R}} = \{\sum_{i=1}^n a_i D_i \mid a_i \in \mathbb{R}, D \in \mathrm{Div}(X)\}$ denotes the set of $\mathbb{R}$-(Cartier) divisors.
- $Z_1(X)_{\mathbb{R}}$ denotes the free abelian group of 1-cycles on X with coefficients in $\mathbb{R}$.
- If $\dim(X) > 2$, we will denote curves or 1-cycles on X by small Latin letters and divisors on X by capital Latin letters.
- If two $\mathbb{R}$-divisors D_1 and D_2 are numerically equivalent, i.e., we have $D_1 . c = D_2 . c$ for all irreducible curves $c \subset X$, then we will write $D_1 \equiv_{\mathrm{num}} D_2$. We will use the analogous notation for elements of $Z_1(X)_{\mathbb{R}}$.
- If two $\mathbb{R}$-divisors D_1 and D_2 are numerically proportional, i.e we have $D_1 \equiv_{\mathrm{num}} \lambda D_2$ for a suitable $\lambda \in \mathbb{R}$, then we will write $D_1 \sim_{\mathrm{num}} D_2$. We will use the analogous notation for elements of $Z_1(X)_{\mathbb{R}}$.
- By abuse of notation, we will sometimes say that a 1-cycle or $\mathbb{R}$-divisor is numerically equivalent or numerically proportional to a numerical equivalence class of a 1-cycle or $\mathbb{R}$-divisor, respectively.
- If c is a 1-cycle on X with coefficients in $\mathbb{R}$ and D is an $\mathbb{R}$-divisor on X, then $[c]$ and $[D]$ denote the numerical equivalence classes of c in $Z_1(X)_{\mathbb{R}}/\equiv_{\mathrm{num}}$ and of D in $\mathrm{Div}(X)_{\mathbb{R}}/\equiv_{\mathrm{num}}$.
- If $\dim(X) > 2$, we will denote numerical equivalence classes of 1-cycles by small Greek letters and numerical equivalence classes of $\mathbb{R}$-divisors by capital Greek letters.

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Finally, I want to thank my friends, my family and, above all, my wife Susanne for her patience, her support and her love.

CHAPTER 1

Technical basics

In this chapter we will accomplish the basic terms which we will need for the following chapters. We will define the cones that we will use, we will recall the theorem of [**BDPP04**] and we will give a brief review of the minimal model program.

1.1. Cones in the Néron-Severi spaces

We start with the definition of our working space.

Definition 1.1 (Néron-Severi spaces)**.** The vector space $Z_1(X)_{\mathbb{R}}/\equiv_{\mathrm{num}}$ of numerical equivalence classes of 1-cycles on X with real coefficients is called the *Néron-Severi space of curves* and is denoted by $N_1(X)_{\mathbb{R}}$. The vector space $\mathrm{Div}(X)_{\mathbb{R}}/\equiv_{\mathrm{num}}$ of numerical equivalence classes of $\mathbb{R}$-divisors on X is called the *Néron-Severi space of divisors* and is denoted by $N^1(X)_{\mathbb{R}}$.

Remark 1.2. The Néron-Severi spaces are finite-dimensional real vector spaces. This fact follows from the theorem of Néron and Severi, which states that the group of divisors modulo algebraic equivalence is a finitely generated abelian group, see [**Har77**, Chapter V, Remark 1.9.1]. By construction of the Néron-Severi spaces, taking intersection numbers of 1-cycles with Cartier divisors yields a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R},\ (\Delta, \gamma) \mapsto \Delta\,.\,\gamma \in \mathbb{R}.$$

See [**Laz04**, Definition 1.4.25]. The dimension $\dim_{\mathbb{R}}(N^1(X)_{\mathbb{R}}) =: \rho(X)$ of $N^1(X)_{\mathbb{R}}$ is called the *Picard number* of X.

Definition 1.3 (Mori Cone)**.** Let

$$\mathrm{NE}(X) := \left\{ \sum\nolimits_{i=1}^{m} a_i[c_i] \mid c_i \subset X \text{ an irreducible curve}, a_i \geq 0 \right\} \subset N_1(X)_{\mathbb{R}}$$

be the convex cone spanned by all classes of effective 1-cycles on X. Its closure $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbb{R}}$ is called the *Mori cone of* X.

Definition 1.4 (Movable and strongly movable curves)**.** An irreducible curve $c \subset X$ is called *movable* or *moving curve* if $c = c_{t_0}$ is a member of an algebraic family $\{c_t\}_{t \in T} \subset X$ of irreducible curves such that $\bigcup_{t \in T} c_t \subset X$ is dense in X. A 1-cycle class $\gamma \in N_1(X)_{\mathbb{R}}$ is called *movable* or *moving class* if $\gamma = [c]$ is the class of a movable curve $c \subset X$.

A curve $s \subset X$ is called a *strongly movable curve* if there exists a projective birational mapping $\mu : X' \dashrightarrow X$, together with ample divisors $A_1, ..., A_{n-1}$ on X', such that $s = \mu_*(A_1 \cap ... \cap A_{n-1})$. A class $\gamma \in N_1(X)_{\mathbb{R}}$ is called a *strongly movable class* if $\gamma = [s]$ is the class of a strongly movable curve $s \subset X$.

Definition 1.5 (Moving cone, strongly movable cone)**.** The closure in $N_1(X)_{\mathbb{R}}$ of the cone generated by classes of moving curves on X

$$\overline{\mathrm{NM}}(X) := \overline{\left\{\sum_{i=1}^{m} a_i\gamma_i \mid a_i \geq 0, \gamma_i \in N_1(X)_{\mathbb{R}} \text{ movable } \right\}}$$

is called the *moving* or *movable cone of* X. The closure in $N_1(X)_{\mathbb{R}}$ of the cone generated by classes of strongly movable curves on X

$$\overline{\mathrm{SNM}}(X) := \overline{\left\{\sum_{i=1}^{m} a_i\gamma_i \mid a_i \geq 0, \gamma_i \in N_1(X)_{\mathbb{R}} \text{ strongly movable}\right\}}$$

is called the *strongly movable cone of* X.

Fact 1.6 (See [**BDPP04**, §2])**.** The cones $\overline{\mathrm{NM}}(X)$ and $\overline{\mathrm{SNM}}(X)$ are closed convex cones in $N_1(X)_{\mathbb{R}}$. We have the following inclusions

$$\overline{\mathrm{SNM}}(X) \subseteq \overline{\mathrm{NM}}(X) \subseteq \overline{\mathrm{NE}}(X).$$

Note that the paper [**BDPP04**] treats projective manifolds, however, everything holds true for $\mathbb{Q}$-factorial projective varieties, as well.

Definition 1.7 (Pseudoeffective cone)**.** The *pseudoeffective cone*

$$\overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$$

is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors on X.

Definition 1.8 (Extremal face)**.** Let $K \subset V$ be a closed convex cone in a finite-dimensional real vector space. An *extremal face* $F \subset K$ is a subcone of K having the following property:

(1.a) if $v + w \in F$ for some $v, w \in K$, then necessarily $v, w \in F$.

A one-dimensional extremal face R of K is called *extremal ray*.

Remark 1.9. A subcone with the property (1.a) is often called "*geometrically extremal*".

Definition 1.10 (Extremal class)**.** Let K be closed convex cone in $N^1(X)_{\mathbb{R}}$ or in $N_1(X)_{\mathbb{R}}$. A class $\gamma \in K$ is called *extremal class* if the ray $r = \mathbb{R}_+\gamma$ spanned by γ is an extremal ray of K.

1.1.1. Two important theorems of BDPP. The next theorem is one of the main results in [**BDPP04**]. As mentioned in the introduction, this result was a great achievement. With this theorem the authors were able to give a characterization of uniruled varieties: a smooth projective variety X is uniruled if and only if K_X is not pseudoeffective.

Theorem 1.11 (see [**BDPP04**, Theorem 2.2])**.** *Let X be an irreducible projective variety of dimension n. Then the cones $\overline{\mathrm{SNM}}(X)$ and $\overline{\mathrm{Eff}}(X)$ are dual; in other words,*

$$\overline{\mathrm{SNM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma \cdot \Delta \geq 0, \text{ for all } \Delta \in \overline{\mathrm{Eff}}(X)\}.$$

□

The following theorem shows why Theorem 1.11 is useful for our purposes.

Theorem 1.12 (see [**BDPP04**, Theorem 2.4])**.** *Let X be an irreducible projective variety of dimension n. Then the cones $\overline{\mathrm{SNM}}(X)$ of strongly movable curves and $\overline{\mathrm{NM}}(X)$ of moving curves coincide; in other words,*

$$\overline{\mathrm{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma \cdot \Delta \geq 0, \text{ for all } \Delta \in \overline{\mathrm{Eff}}(X)\}.$$

Proof. We have already seen that $\overline{\mathrm{Eff}}(X)^{\vee} = \overline{\mathrm{SNM}}(X) \subseteq \overline{\mathrm{NM}}(X)$. On the other hand, every movable curve has non-negative intersection with every irreducible divisor. This is due to the fact that we can move the curve out of the support of the divisor. Moreover, every movable class appears as a limit of classes of effective 1-cycles which consist of movable curves. Since taking intersection numbers is continuous, this yields that $\overline{\mathrm{NM}}(X) \subseteq \overline{\mathrm{Eff}}(X)^{\vee}$. □

Therefore, we can use the duality-statement of Theorem 1.11 in the forthcoming computations.

1.2. A brief review of the Minimal Model Program

In this section we want to give a brief review of the minimal model program. Note that the version of the minimal model program (mmp) which is presented here is just the easiest one. There are more powerful versions as mmp with scaling and the improvement of the program is still an active field of research. Recently, there has been much development due to the paper [**BCHM06**] which appeared on the arXiv in 2006. However, we will not present anything else than the basic version since we will just need the elementary methods.

The aim of the program is to find a birational model Y of a given variety X such that K_Y is nef or such that there exists a fibration $f : Y \to Z$ with the property that K_F is nef, where F denotes a general fibre of f. The idea is to "get rid" of curves which have negative intersection with K_X.

Even if X is a smooth variety, running the minimal model program for X can produce singular varieties. Therefore, we have to introduce the notion of *terminal singularities* as a first step.

Then we will state the *Cone* and the *Contraction Theorem.* Both theorems are essential for the following.

1.2.1. The Cone and the Contraction Theorem.

Definition 1.13 (Terminal singularities)**.** A normal variety X of dimension n has only *terminal singularities* if

(i) K_X is $\mathbb{Q}$-Cartier; in other words, there exists an integer $m \in \mathbb{N}$ such that mK_X is a Cartier divisor,
(ii) there exists a projective birational morphism $f : Y \to X$ from a smooth variety Y such that all coefficients a_j of the exceptional divisors E_j of f in the ramification formula $K_Y = f^*(K_X) + \sum a_j E_j$ are strictly positive.

Definition 1.14. A normal variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor D on X is $\mathbb{Q}$-Cartier, which means that there exists in integer $m \in \mathbb{N}$ such that mD is Cartier.

Now we are able to state the Cone Theorem.

Theorem 1.15 (Cone Theorem, see [**Mat02**, Theorem 7-2-1])**.** *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities such that K_X fails to be nef. Then*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_+[r_i],$$

where $\overline{\mathrm{NE}}(X)_{K_X \geq 0} := \{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma \,.\, K_X \geq 0\}$ *and the* $[r_i]$ *are locally discrete classes of rational curves in the half-space*

$$\overline{\mathrm{NE}}(X)_{K_X < 0} := \{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma \,.\, K_X < 0\}.$$

□

Remark 1.16 ([**Deb01**, Chapter 6.1 and 7.9])**.** The proof of the Cone Theorem given in [**Deb01**] shows that for every ample $\mathbb{R}$-divisor A and $\varepsilon > 0$ there are only finitely many extremal rays in the half space $N_1(X)_{K_X+\varepsilon A<0}$. This shows that $\overline{\mathrm{NE}}(X)$ is a convex polyhedral cone if $-K_X$ is ample. See [**Laz04**, Example 1.5.34].

The following picture illustrates the statement of the Cone Theorem.

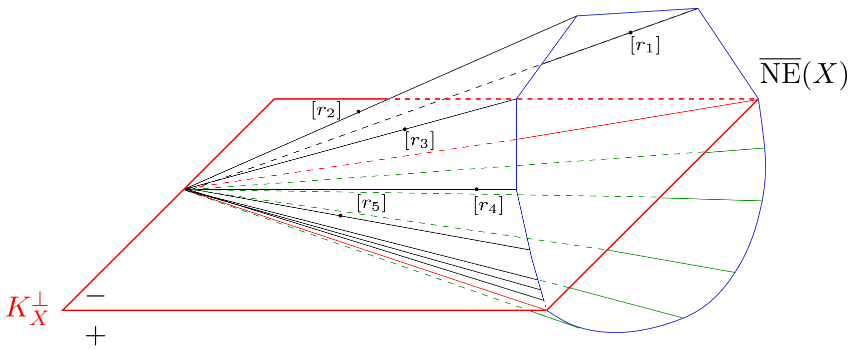


Figure 1. Illustration of the Cone Theorem. Cf. [**Deb01**, Theorem 6.1]

Definition 1.17. A normal projective variety X of dimension n is called a *Fano n-fold* if its anticanonical bundle $-K_X$ is ample.

Another important concept is the notion of an *extremal contraction*.

Definition 1.18 (Extremal contraction)**.** Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities. Then a morphism $\varphi : X \to Y$ is called an *extremal contraction* if

(i) φ is not an isomorphism,
(ii) if $c \subset X$ is a curve with $\varphi(c) = \text{pt.}$, then $c \,.\, K_X < 0$,
(iii) all the curves contracted by φ are numerically proportional; in other words, $\varphi(c) = \text{pt.} = \varphi(c') \Rightarrow [c] = \lambda[c']$, for a suitable $\lambda \in \mathbb{Q}_+$,
(iv) φ has connected fibres with Y being normal and projective.

Theorem 1.19 (Contraction Theorem, see [**Mat02**, Theorem 8-1-3])**.** *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities such that K_X is not nef. For each extremal ray $\mathbb{R}_+[r]$ of $\overline{\mathrm{NE}}(X)$ with $r \cdot K_X < 0$ there exists a morphism $\varphi_r : X \to Y$, called the* contraction of the extremal ray $\mathbb{R}_+[r]$, *such that*

(i) φ_r is an extremal contraction,
(ii) if $c \subset X$ is a curve, then $\varphi_r(C) = \mathrm{pt.} \Leftrightarrow [c] \in \mathbb{R}_+[r]$. □

Definition 1.20. Let $\varphi : X \to Y$ be a birational morphism. Then there exists a birational map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ as rational maps. There exists a maximal open subset $V \subset Y$ such that $\psi|_V : V \to X$ is a morphism. For details see [**Har77**, Chapter V.5]. The inverse image $\mathrm{Exc}_X(\varphi) := \varphi^{-1}(Y \setminus V) \subset X$ of $Y \setminus V$ under φ is called the *exceptional locus* of φ in X.

Remark 1.21 (See [**KM98**, Proposition 2.5])**.** There are three different types of extremal contractions $\varphi : X \to Y$.

(i) The morphism φ is birational. If $\mathrm{codim}_X \mathrm{Exc}_X(\varphi) = 1$, we say φ is a *divisorial contraction* or *of divisorial type.*
(ii) The morphism φ is birational. If $\mathrm{codim}_X \mathrm{Exc}_X(\varphi) \geq 2$, we say φ is a *small contraction* or *of flipping type.*
(iii) If $\dim Y < \dim X$, we say φ is *of fibre type.*

In case (i) the exceptional locus $\mathrm{Exc}_X(\varphi)$ consists of a unique irreducible divisor, and Y is again $\mathbb{Q}$-factorial with only terminal singularities. For a proof of this fact see also [**KMM87**, Proposition 5-1-6]. The morphism φ reduces the Picard number of X by one; in other words, $\rho(Y) = \rho(X) - 1$.

In case (iii) X is a *Mori fibre space*.

In case (ii) the morphism φ is birational, but the canonical divisor K_Y of Y is not $\mathbb{Q}$-Cartier anymore. This can be seen as follows.

If K_Y was $\mathbb{Q}$-Cartier, then we would have $K_X = \varphi^*(K_Y)$ since φ is an isomorphism in codimension one. If c is a curve in a fibre of φ, then we have $c\,.\,K_X < 0$ by definition of an extremal contraction. On the other hand, the equality $K_X = \varphi^*(K_Y)$ would imply that

$$0 = c\,.\,\varphi^*(K_Y) = c\,.\,K_X < 0.$$

This is a contradiction.

For more details see [**Mat02**, Chapters 3 and 8] or [**KMM87**].

Definition 1.22. Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities such that K_X is not nef. Let $\mathbb{R}_+[r]$ be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. We say

(i) $\mathbb{R}_+[r]$ is a *divisorial extremal ray* and $[r]$ is a *divisorial extremal class* if the corresponding extremal contraction is divisorial,
(ii) $\mathbb{R}_+[r]$ is a *small* or *flipping extremal ray* and $[r]$ is a *small* or *flipping extremal class* if the corresponding extremal contraction is small,
(iii) $\mathbb{R}_+[r]$ is an *extremal ray of fibre type* and $[r]$ is an *extremal class of fibre type* if the corresponding extremal contraction is of fibre type.

Because of the existence of small contractions, we have to introduce the notion of *flips*.

Definition 1.23 (Flip)**.** Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities and let $\varphi : X \to Y$ be a small contraction. A birational map

$$\phi : X \dashrightarrow X^+$$

to a $\mathbb{Q}$-factorial projective variety X^+ with only terminal singularities is called a *flip* of φ if the following conditions hold.

(i) There exists a proper birational morphism $\varphi^+ : X^+ \to Y$ such that the diagram (1.b) is commutative.

(1.b)
$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Y & \end{array}$$

(ii) The morphism φ^+ is an isomorphism in codimension one. In particular, φ^+ is small; in other words, the exceptional locus $\mathrm{Exc}_{X^+}(\varphi^+)$ of φ^+ in X^+ has codimension at least two.
(iii) K_{X^+} is φ^+-ample.

We say that the diagram (1.b) is the *flip diagram* of φ, we call φ^+ the *flipped small contraction* and X^+ the *flipped variety* obtained by ϕ.

Remark 1.24. By abuse of notation, we will sometimes call the map ϕ a *flip of an extremal ray* or a *flip of an extremal class*. Note that φ^+ has connected fibres and all the curves in fibres of φ^+ are numerically proportional. Moreover, if a flip of a small contraction exists, then it is unique. For details see [**Mat02**, Chapter 9]. The existence and termination of flips was and partially still is a major problem in Mori theory.

Conjecture 1.25 (Flip Conjecture)**.** *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities and let $\varphi : X \to Y$ be a small contraction.*

(i) The flip $\phi : X \dashrightarrow X^+$ of φ exists.
(ii) There is no infinite sequence of flips.

This conjecture has been proved in dimension three and four due to the work of Mori, Kollár, Shokurov, Kawamata, Hacon, M^c^Kernan and others.

Part (i) of the conjecture was proved in the preprint [**BCHM06**] by Birkar, Cascini, Hacon and M^c^Kernan in 2006.

Part (ii) of the conjecture is still not known in general and one of the big problems in modern algebraic geometry. As mentioned before, there has been a lot of development in the field and there are many cases where important results have been proved. A good reference for a detailed overview is the book [**Cor07**].

We will present the very basic version of the minimal model program now, and we will see why this conjecture is so important.

1.2.2. The minimal model program.

(1) We start with a $\mathbb{Q}$-factorial projective variety X with only terminal singularities.
(2) We ask if K_X is nef. If the answer is yes, we call X a *minimal model* and stop the program. If the answer is no, we perform the next step of the program.
(3) Since K_X is not nef, there exists a contraction $\varphi : X \to Y$ of an extremal ray $\mathbb{R}_+[r]$ of $\overline{\mathrm{NE}}(X)$ with $r \cdot K_X < 0$. Now, we ask if $\dim Y < \dim X$. If the answer is yes, X is a Mori fibre space and we stop the program. If the answer is no, we perform the next step of the program.
(4) We ask if $\mathrm{codim}_X \mathrm{Exc}(\varphi) \geq 2$. If the answer is no, Y is again $\mathbb{Q}$-factorial and projective with only terminal singularities and we restart the program with Y instead of X. If the answer is yes, we perform the next step of the program.
(5) By the existence of flips, we can flip the map φ, and obtain a $\mathbb{Q}$-factorial projective variety X^+ with only terminal singularities; we restart the program with X^+ instead of Y.

If the termination conjecture 1.25 (ii) for flips holds, this procedure terminates and results in a minimal model or a Mori fibre space since the Picard number decreases by one with every divisorial contraction.

Recall that there are more powerful versions of the minimal model program, but the basic version suffices for our purposes.

In Chapter 2 we will now consider the moving cone of surfaces.

CHAPTER 2

Surfaces

In this chapter we will investigate the cone of moving curves for projective varieties of dimension two. We will see that for projective surfaces the situation is quite special. This is due to the following fact.

Fact 2.1. Let X be a $\mathbb{Q}$-factorial projective surface. Then the cones $\overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{Eff}}(X)$, respectively $\overline{\mathrm{NM}}(X)$ and $\mathrm{Nef}(X)$, coincide since prime divisors are nothing but irreducible curves; as usual, we denote by $\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ the cone of nef $\mathbb{R}$-divisors on a projective variety X.

Therefore, we will obtain a structure theorem for the cone of moving curves by simply "dualising" the statement of Mori's Cone Theorem. This will be done in the following section 2.1.

In section 2.2 we will investigate how extremal faces of the moving cone of a smooth surface X can be located inside the Mori cone of X.

Finally, we will compute some examples, and we will see in Corollary 2.17 that an extremal ray of the moving cone is not necessarily represented by a rational curve. In contrast to the situation of the Mori cone, cf. Theorem 1.15, this may happen even if the ray has negative intersection with the class of the canonical divisor.

2.1. The moving cone of a projective surface

As said before, we will use the duality of the nef cone and the Mori cone in this section. Here is the idea.

If X is a $\mathbb{Q}$-factorial projective variety such that $K_X + \varepsilon A$ is not nef for a given ample divisor A on X and $\varepsilon > 0$, then we can consider the line segment in $N^1(X)_{\mathbb{R}}$ connecting $[K_X + \varepsilon A]$ and the class of an arbitrary ample divisor on X. These line segments intersect a special subset $P_\varepsilon \subset \partial\mathrm{Nef}(X)$.

Construction and Definition 2.2. Let X be a $\mathbb{Q}$-factorial projective variety such that K_X is not nef. Fix an ample divisor A on X and choose $\varepsilon > 0$ such that $K_X + \varepsilon A$ is not nef. We can connect $[K_X + \varepsilon A]$ with every point $\Gamma \in \mathrm{Amp}(X) = \mathrm{Int}(\mathrm{Nef}(X))$ by a line segment $l_\varepsilon(\Gamma) \subset N^1(X)_{\mathbb{R}}$, parametrised by

$$l_{\Gamma,\varepsilon}(t) = [K_X + \varepsilon A] + t(\Gamma - [K_X + \varepsilon A]),\ t \in [0,1],$$

which intersects $\partial\mathrm{Nef}(X)$. We set

$$P_\varepsilon := \{\Sigma \in \mathrm{Nef}(X) \mid \exists_{\Gamma \in \mathrm{Amp}(X)} : \Sigma \in l_\varepsilon(\Gamma) \cap \partial\mathrm{Nef}(X)\}.$$

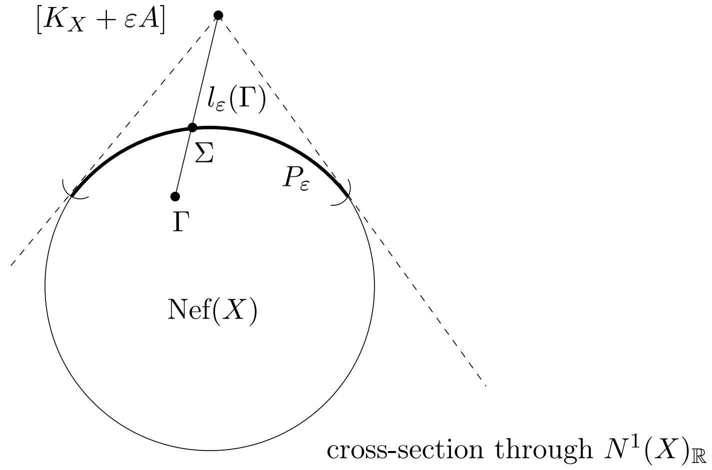


FIGURE 1. The line segment $l_\varepsilon(\Gamma)$ between $[K_X + \varepsilon A]$ and Γ meets the point $\Sigma \in P_\varepsilon \subset \partial\mathrm{Nef}(X)$.

Theorem 2.3. *Let X be a $\mathbb{Q}$-factorial projective variety such that K_X is not nef and let A be an ample divisor on X. Then for every $\varepsilon > 0$ with $[K_X + \varepsilon A] \notin \mathrm{Nef}(X)$ the set $P_\varepsilon \subset \partial\mathrm{Nef}(X)$ is an open set in $\partial\mathrm{Nef}(X)$, and it is cut out of $\mathrm{Nef}(X)$ by finitely many linear equations given by the $K_X + \varepsilon A$-negative extremal rays of $\overline{\mathrm{NE}}(X)$. Moreover, $P_\varepsilon \subset P_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon} < \varepsilon$.*

Strategy for the proof: We will divide the proof into three steps.

In step one, we will show that the following property holds: for every point $\Sigma \in P_\varepsilon$ there is a 1-cycle class $\delta \in \overline{\mathrm{NE}}(X)_{K_X + \varepsilon A < 0}$ which gives a hyperplane $\delta^\perp := \{\Lambda \in N^1(X)_\mathbb{R} \mid \delta \,.\, \Lambda = 0\}$ with $\Sigma \in \delta^\perp \cap \mathrm{Nef}(X) \subset \partial\mathrm{Nef}(X)$.

By usage of step one, we will show that P_ε is cut out of $\mathrm{Nef}(X)$ by finitely many equations which are given by $[K_X + \varepsilon A]$-negative extremal classes of $\overline{\mathrm{NE}}(X)$. This will be done in step two.

In step three we will show that $P_\varepsilon \subset P_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon} < \varepsilon$.

Proof. Let $\varepsilon > 0$ be a real number such that $K_X + \varepsilon A$ is not nef. The set P_ε is clearly open by construction. Now let $\Sigma \in P_\varepsilon$ be an arbitrary point. By definition, there is an element $\Gamma \in \mathrm{Amp}(X)$ and a scalar $t_0 \in (0, 1)$ such that

$$\Sigma = [K_X + \varepsilon A] + t_0(\Gamma - [K_X + \varepsilon A]).$$

1. Step: The nef cone $\mathrm{Nef}(X)$ is dual to the Mori cone $\overline{\mathrm{NE}}(X)$. Therefore, there is a hyperplane $\delta^\perp$ which is given by an element $\delta \in \partial\overline{\mathrm{NE}}(X)$ such that $\Sigma \in \delta^\perp \cap \mathrm{Nef}(X) \subset \partial\mathrm{Nef}(X)$. Hence we find

$$\begin{aligned} 0 &= \delta \,.\, \Sigma = \delta \,.([K_X + \varepsilon A] + t_0(\Gamma - [K_X + \varepsilon A])) \\ &= \underbrace{(1 - t_0)}_{>0} \delta \,.[K_X + \varepsilon A] + \underbrace{t_0\ \delta \,.\, \Gamma}_{>0} \\ &\Rightarrow \delta \,.[K_X + \varepsilon A] < 0. \end{aligned}$$

2. Step: According to the first step, δ is contained in the $(K_X+\varepsilon A)$-negative part of $\overline{\mathrm{NE}}(X)$. Therefore, we find an effective linear combination $\delta = \rho + a_1[r_1] + \cdots + a_k[r_k]$, where $\rho \in \overline{\mathrm{NE}}(X)_{K_X+\varepsilon A\geq 0}$ and $[r_1], ..., [r_k] \in \overline{\mathrm{NE}}(X)_{K_X+\varepsilon A<0}$ are some extremal classes in $\overline{\mathrm{NE}}(X)_{K_X+\varepsilon A<0}$. Hence

$$0 = \delta \,.\, \Sigma = \underbrace{\rho \,.\, \Sigma}_{\geq 0} + \underbrace{a_1[r_1] \,.\, \Sigma}_{\geq 0} + \cdots + \underbrace{a_k[r_k] \,.\, \Sigma}_{\geq 0}.$$

This yields that $\rho \,.\, \Sigma = [r_i] \,.\, \Sigma = 0$ for all $i = 1, \ldots, k$, but the class ρ has non-negative intersection number with $K_X + \varepsilon A$ and every class in $\mathrm{Nef}(X)$ at the same time.

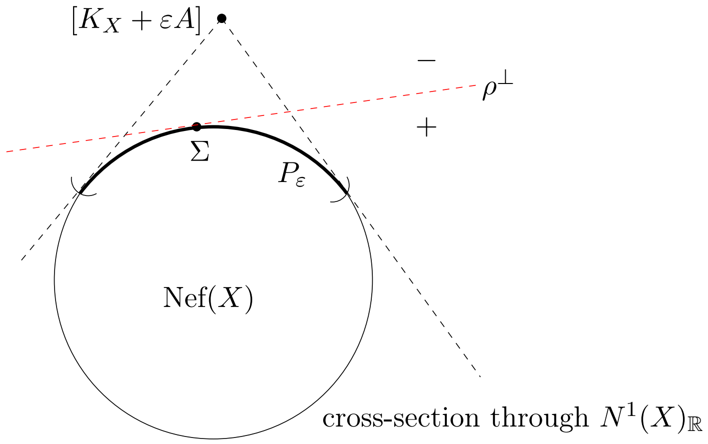


FIGURE 2. If $\rho \neq 0$, then it cannot have non-negative intersection with $K_X + \varepsilon A$ and $\mathrm{Nef}(X)$ at the same time.

Therefore, $\rho = 0 \in N_1(X)_{\mathbb{R}}$ and $\Sigma \in \bigcap_{i=1}^{k} [r_i]^{\perp}$.

Since there are just finitely many extremal rays in $\overline{\mathrm{NE}}(X)_{K_X+\varepsilon A<0}$, this implies that the set P_ε is cut out of $\mathrm{Nef}(X)$ by finitely many equations given by $(K_X + \varepsilon A)$-negative extremal rays of $\overline{\mathrm{NE}}(X)$.

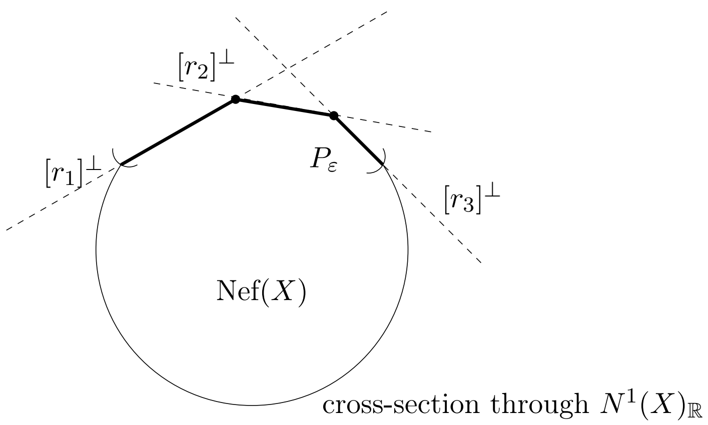


FIGURE 3. Extremal classes of $\overline{\mathrm{NE}}(X)$ cut out the set P_ε.

3. Step: Now let $0 \le \tilde{\varepsilon} < \varepsilon$ be a real number. We show that $P_\varepsilon \subset P_{\tilde{\varepsilon}}$.
We can write Σ in the following way.

$$\begin{aligned}\Sigma &= [K_X + \varepsilon A] + t_0(\Gamma - [K_X + \varepsilon A]) \\ &= ([K_X + \tilde{\varepsilon} A] + (\varepsilon - \tilde{\varepsilon})[A]) + t_0\,(\Gamma - [K_X + \tilde{\varepsilon} A] + (\varepsilon - \tilde{\varepsilon})[A])) \\ &= [K_X + \tilde{\varepsilon} A] + t_0 \left((\Gamma + \tfrac{1-t_0}{t_0}(\varepsilon - \tilde{\varepsilon})[A]) - [K_X + \tilde{\varepsilon} A] \right).\end{aligned}$$

The divisor class $(\Gamma + \frac{1-t_0}{t_0}(\varepsilon - \tilde{\varepsilon})[A])$ is ample since $0 < t_0 < 1$ and $\varepsilon > \tilde{\varepsilon}$. This yields that $\Sigma \in P_{\tilde{\varepsilon}}$.

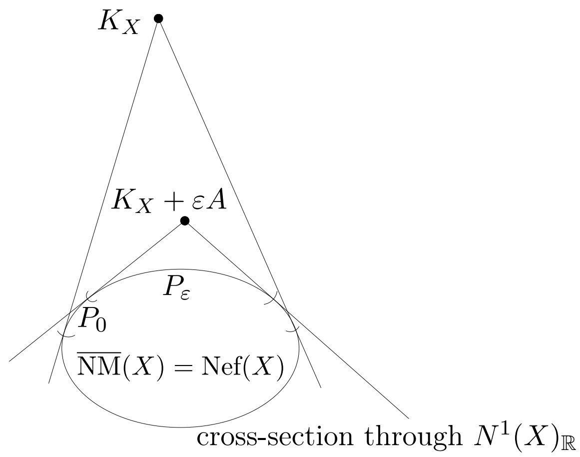


FIGURE 4. The sets P_0 and P_ε for $\varepsilon >> 0$.

□

Corollary 2.4. *Let X be a $\mathbb{Q}$-factorial projective surface such that K_X is not nef and let A be an ample divisor on X. Then for every $\varepsilon > 0$ with $[K_X + \varepsilon A] \notin \mathrm{Nef}(X)$ the set $P_\varepsilon \subset \partial\overline{\mathrm{NM}}(X)$ is an open set in $\partial\overline{\mathrm{NM}}(X)$, and it is cut out of $\overline{\mathrm{NM}}(X)$ by finitely many linear equations given by the $K_X + \varepsilon A$-negative extremal rays of $\overline{\mathrm{NE}}(X)$. Moreover, $P_\varepsilon \subset P_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon} < \varepsilon$.*

Proof. The cones $\overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{Eff}}(X)$, respectively $\overline{\mathrm{NM}}(X)$ and $\mathrm{Nef}(X)$, coincide since prime divisors are nothing but irreducible curves. □

Remark 2.5. Because of Fact 2.1, it does not make any sense to keep different notation for divisors and 1-cycles on surfaces.

Notation 2.6. If X is a projective surface, we will just use different notation for divisors or 1-cycles on X and their numerical equivalence classes in $N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$. We will denote divisors or 1-cycles on X by Latin letters and their numerical equivalence classes by Greek letters.

We have already seen that the Mori cone of a Fano surface is polyhedral. The same holds true for the moving cone of a Fano surface.

Proposition 2.7. *If X is a $\mathbb{Q}$-factorial Fano surface, then*

$$\overline{\mathrm{NM}}(X) = \{\mu \in N_1(X)_{\mathbb{R}} \mid \mu\,.[R_i] \ge 0 \textit{ for all } i = 1, \dots, n\},$$

where $R_1, \dots, R_n$ are the rational curves which span $\overline{\mathrm{NE}}(X)$. In particular, $\overline{\mathrm{NM}}(X)$ is a closed, convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$.

Proof. By definition, the anticanonical divisor $-K_X$ is ample. Thus the Cone Theorem 1.15 yields that $\overline{\mathrm{NE}}(X)$ is a convex, polyhedral cone spanned by classes of rational curves $R_1, \ldots, R_n \in N^1(X)_\mathbb{R}$. If $\mu \in N_1(X)_\mathbb{R}$ is a class with $\mu\,.[R_i] \geq 0$, for all $i = 1, \ldots, n$, then $\mu\,.\,\gamma \geq 0$ for all $\gamma \in \overline{\mathrm{NE}}(X)$. Hence $\{\mu \in N_1(X)_\mathbb{R} \mid \mu\,.[R_i] \geq 0 \text{ for all } i = 1, \ldots, n\} \subseteq \overline{\mathrm{NM}}(X)$. The inclusion $\overline{\mathrm{NM}}(X) \subseteq \{\mu \in N_1(X)_\mathbb{R} \mid \mu\,.[R_i] \geq 0 \text{ for all } i = 1, \ldots, n\}$ is obvious. □

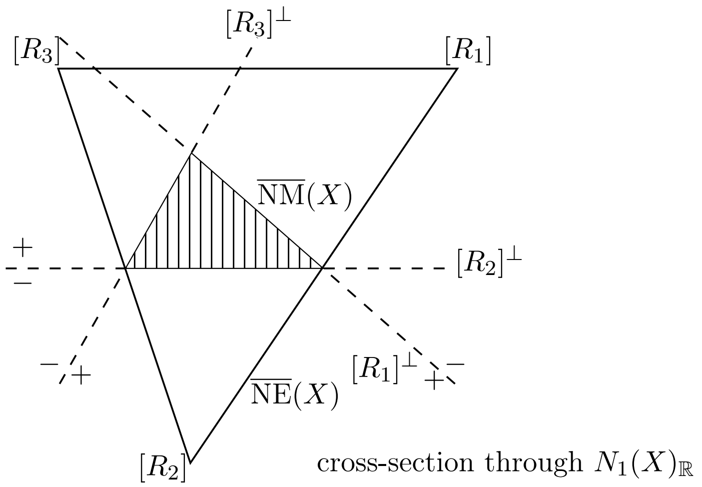


FIGURE 5. The moving cone $\overline{\mathrm{NM}}(X)$ of X is cut out by the hyperplanes $[R_i]^\perp = \{\gamma \in N^1(X)_\mathbb{R} \mid \gamma\,.[R_i] = 0\}$, $i = 1, 2, 3$.

2.2. Extremal faces of the moving cone

The moving cone is a subcone of the Mori cone by definition. Therefore, one could ask how the moving cone is located inside the Mori cone. Construction 2.2 shows that the location of the set $P_0 \subset \partial\overline{\mathrm{NM}}(X)$ of a projective surface with $[K_X] \notin \overline{\mathrm{NE}}(X)$ is influenced by the selfintersection of K_X. Figure 6 and 7 sketch $\overline{\mathrm{NM}}(X)$ inside $\overline{\mathrm{NE}}(X)$ for a projective surface X with $K_X^2 < 0$ and with $K_X^2 > 0$, respectively.

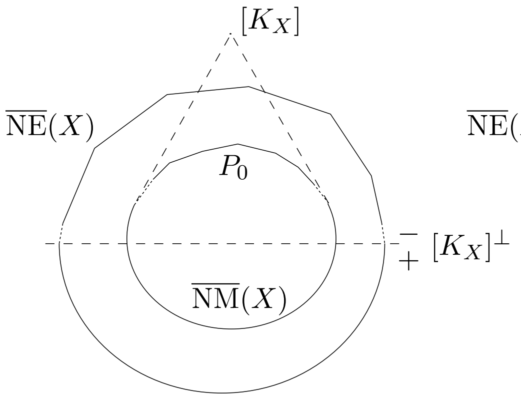


FIGURE 6. $K_X^2 < 0$.

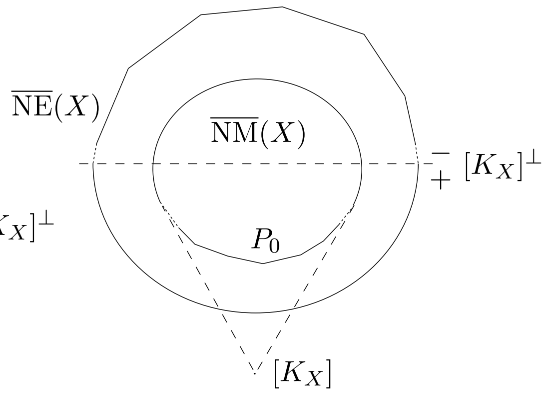


FIGURE 7. $K_X^2 > 0$.

In respect of the equations which cut out P_0 or the whole cone for Fano surfaces, one could ask how extremal faces of the moving cone sitting inside an extremal face of the Mori cone are constituted. The following theorem gives an answer to that question for smooth projective surfaces.

Theorem 2.8. *Let X be a smooth projective surface. If $M \subsetneq \overline{\mathrm{NM}}(X)$ is an extremal face of $\overline{\mathrm{NM}}(X)$ which is contained in an extremal face $F \subsetneq \overline{\mathrm{NE}}(X)$ with $M \cap \overline{\mathrm{NE}}(X)_{K_X<0} \neq \emptyset$, then M is an extremal ray $\mathbb{R}_+\mu$ of $\overline{\mathrm{NM}}(X)$.*

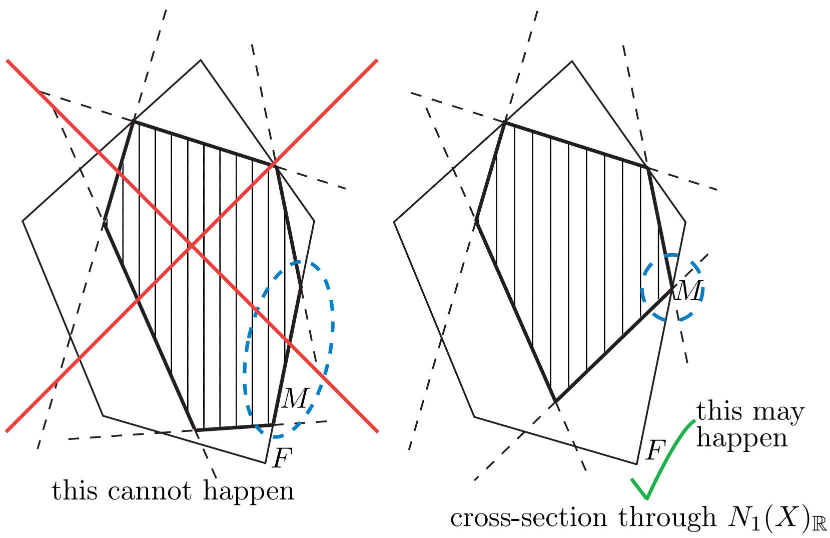


FIGURE 8. The hatched area sketches the moving cone inside the Mori cone, and the dashed lines sketch the hyperplanes which cut out the moving cone. The situation which is displayed on the left hand side cannot occur. A "moving face" M inside a "Mori face" F has to be a ray if there is a K_X-negative movable class $\mu \in M$.

The key to the proof is that if X is a smooth surface such that $\overline{\mathrm{NE}}(X)$ has an extremal face which contains a K_X-negative movable class, then X is forced to be a birational model of a ruled surface X_e over a smooth curve C:

$$X = X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{k-1}} X_k = X_e \to C.$$

Moreover, every movable class in that extremal face is the pullback of a fibre of the ruled surface X_e via the map

$$(\phi_{k-1} \circ \cdots \circ \phi_0)^* : N_1(X_e)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}.$$

2.2.1. Proof of Theorem 2.8. The following lemmas are preparatory work for the proof of the forthcoming Proposition 2.12, which will be the main argument for the proof of Theorem 2.8.

At first, we want to conclude information about a smooth projective surface X by investigating the selfintersection of the rational curves which span a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$.

Lemma 2.9 ([**Deb01**, Lemma 6.2])**.** *Let X be a smooth projective surface.*

(i) The class of an irreducible curve C on X satisfying $C^2 \leq 0$ is in $\partial\overline{\mathrm{NE}}(X)$.
(ii) The class of an irreducible curve C on X satisfying $C^2 < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$.
(iii) If the class of an irreducible curve C on X satisfying $C^2 = 0$ and $K_X . C < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$, then the surface X is ruled over a smooth curve, C is a fibre and X has Picard number $\rho(X) = 2$.
(iv) If ν spans an extremal ray of $\overline{\mathrm{NE}}(X)$, then either $\nu^2 \leq 0$ or X has Picard number $\rho(X) = 1$.
(v) If ν spans an extremal ray of $\overline{\mathrm{NE}}(X)$ and $\nu^2 < 0$, then the extremal ray is spanned by the class of an irreducible curve. □

Corollary 2.10. *Let X be a smooth surface with Picard number $\rho(X) \geq 2$ and let $(R_i)_{i\in I} \subset X$ be the K_X-negative rational curves which span the K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$. Then one of the following statements holds.*

(i) $R_i^2 = -1$ for all $i \in I$ or
(ii) $R_i^2 = 0$ for some $i \in I$, X is a ruled surface with Picard number $\rho(X) = 2$ over a smooth curve C and R_i is a fibre.

Proof. Since $\rho(X) > 1$ by assumption, Lemma 2.9 (iv) yields that $R_i^2 \leq 0$. Using the asssumption $R_i . K_X < 0$ and the adjunction formula [**Har77**, Chapter V, Exercise 1.3], we obtain

$$2g_a(R_i) - 2 = R_i .(R_i + K_X) < 0 \Rightarrow g_a(R_i) = 0, \text{ for all } i \in I.$$

Hence $-2 = R_i^2 + R_i . K_X < R_i^2 \Rightarrow -1 \leq R_i^2 \leq 0$ for all $i \in I$. If $R_i^2 = 0$ for some $i \in I$, then Lemma 2.9 (iii) says that X is a ruled surface with Picard number $\rho(X) = 2$ over a smooth curve C and that R_i is a fibre. □

This gives a nice criterion to detect if a surface X is ruled. We will now see that extremal faces of $\overline{\mathrm{NE}}(X)$ which contain a K_X -negative movable class behave very well under a divisorial contraction of a (-1)-curve of that face.

Lemma 2.11. *Let X be a smooth surface and let $F \subset \overline{\mathrm{NE}}(X)$ be an extremal face of $\overline{\mathrm{NE}}(X)$. If $\varphi : X \to X'$ is the contraction of a (-1)-curve E with $[E] \in F$ and $\mu \in (\overline{\mathrm{NM}}(X) \cap F)_{K_X<0}$ is a movable class, then the following holds:*

(i) $\varphi_*(\mu)$ is $K_{X'}$-negative,
(ii) $\varphi^*(\varphi_*(F)) \subset F$,
(iii) $F' := \varphi_*(F)$ is an extremal face of $\overline{\mathrm{NE}}(X')$,
(iv) $F \subsetneq \overline{\mathrm{NE}}(X) \Rightarrow F' \subsetneq \overline{\mathrm{NE}}(X')$.

Proof. A short computation shows that $\varphi_*(\mu) \in \varphi_*(F) = F'$ is again movable, and F' is a subcone of $\overline{\mathrm{NE}}(X')$ since $\varphi_* : N^1(X)_{\mathbb{R}} \to N^1(X')_{\mathbb{R}}$ is a homomorphism. *Let us prove (i):*

$$\varphi_*(\mu) . [K_{X'}] = \mu . \varphi^*([K_{X'}]) = \mu .([K_X] - [E]) = \underbrace{\mu . [K_X]}_{<0} - \underbrace{\mu . [E]}_{\geq 0} < 0.$$

To prove (ii), let $\delta \in F$ be an arbitrary class. We have $\varphi^*(\varphi_*(\delta)) = \delta + k[E]$ for some $k \in \mathbb{R}$. Note that $k < 0$ is possible if $[E]$ is a component of δ.

If $k \geq 0$, we deduce that $\varphi^*(\varphi_*(\delta)) \in F$ since $[E]$ and δ lie in F. If $k < 0$, we have that

$$\varphi^*(\varphi_*(\delta)) + (-k)[E] = \delta \in F.$$

However, the classes $\varphi^*(\varphi_*(\delta))$ and $(-k)[E]$ are contained in $\overline{\mathrm{NE}}(X)$. This implies that $\varphi^*(\varphi_*(\delta)) \in F$ since F is an extremal face of $\overline{\mathrm{NE}}(X)$.

We will now use (ii) to prove (iii). Let $\gamma' \in F'$ be an arbitrary class such that $\gamma' = \nu + \eta$ for some $\nu, \eta \in \overline{\mathrm{NE}}(X')$. There exists a class $\gamma \in F \subset \overline{\mathrm{NE}}(X)$ with $\gamma' = \varphi_*(\gamma)$ since φ_* is surjective, and we have

$$F \ni \varphi^*(\varphi_*(\gamma)) = \varphi^*(\gamma') = \varphi^*(\nu) + \varphi^*(\eta)$$

by (ii). Therefore, since F is an extremal face of $\overline{\mathrm{NE}}(X)$, we infer that the classes $\varphi^*(\nu)$ and $\varphi^*(\eta)$ are contained in F. This yields that the classes

$$\nu = \varphi_*(\varphi^*(\nu)) \text{ and } \eta = \varphi_*(\varphi^*(\eta)) \text{ are contained in } \varphi_*(F) = F'.$$

Let us prove (iv): Aiming for a contradiction, assume that $F \subsetneq \overline{\mathrm{NE}}(X)$ and that $F' = \overline{\mathrm{NE}}(X')$. Then, there exists an element $\xi \in F'$ such that

$$\varphi_*(\nu_1) = \xi = \varphi_*(\nu_2)$$

for some $\nu_1 \in F$ and $\nu_2 \in \overline{\mathrm{NE}}(X) \setminus F$. Therefore, $\nu_2 - \nu_1$ is a multiple of $[E]$.

Assume that $\nu_2 - \nu_1 = \lambda[E]$ with $\lambda \geq 0$. Then $\nu_2 = \lambda E + \nu_1 \in F$ since ν_1 and $[E]$ lie in F. This is a contradiction since $\nu_2 \notin F$.

Assume that $\nu_2 - \nu_1 = \lambda[E]$ with $\lambda < 0$. Then $\nu_1 - \nu_2$ is a positive multiple of $[E]$ and thus contained in F. In addition, the sum $(\nu_1 - \nu_2) + \nu_2 = \nu_1$ is contained in F since $\nu_1 \in F$ by assumption. This yields that $\nu_2 \in F$ as $\nu_2 \in \overline{\mathrm{NE}}(X)$ and F is an extremal face. As before, this is a contradiction.

This proves statement (iv) and concludes the proof. □

Proposition 2.12. *Let X be a smooth surface with Picard number $\rho(X) \geq 3$, and let $F_0 \subsetneq \overline{\mathrm{NE}}(X)$ be a proper extremal face of $\overline{\mathrm{NE}}(X)$ such that $F_0 \cap \overline{\mathrm{NE}}(X)_{K_X<0} \neq \emptyset$. If $\mu \in F_0 \cap \overline{\mathrm{NE}}(X)_{K_X<0}$ is a movable class, then there exists a finite sequence of monoidal transformations*

$$X =: X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{k-1}} X_k$$

such that X_k is a ruled surface over a smooth curve C.

Each monoidal transformation ϕ_i is the contraction of a (-1)-curve R_i with $[R_i] \in F_i := (\phi_{i-1})_*(F_{i-1}) \subsetneq \overline{\mathrm{NE}}(X_i)$, $i = 1 \dots k-1$. In particular, each movable class in F_0 will be mapped to a multiple of the class of a fibre in X_k.

Proof. Since the Picard number $\rho(X)$ of X is greater than two, Corollary 2.10 yields that all K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$ are spanned by (-1)-curves.

Since $F_0 \cap \overline{\mathrm{NE}}(X)_{K_X<0} \neq \emptyset$, there exists a K_X-negative extremal class $[R_0] \in F_0$. The Contraction Theorem 1.19 guarantees the existence of an extremal contraction $\phi_0 : X =: X_0 \to X_1$ which contracts the extremal ray $\mathbb{R}_+[R_0] \subset \overline{\mathrm{NE}}(X)$. Since R_0 is a (-1)-curve, Lemma 2.11 ensures that $F_1 := (\phi_0)_*(F_0)$ is an extremal face of $\overline{\mathrm{NE}}(X_1)$ and that $\mu_1 := (\phi_0)_*(\mu) \in F_1$ is again a K_{X_1}-negative movable class. Therefore, F_1 is not contained in $\overline{\mathrm{NE}}(X_1)_{K_{X_1}\geq 0}$ and there exists a K_{X_1}-negative extremal ray $\mathbb{R}_+[R_1] \in F_1$ of $\overline{\mathrm{NE}}(X_1)$ in F_1. Moreover, we have $\rho(X_1) = \rho(X) - 1 \geq 2$ and thus $(R_1)^2 \leq 0$ by Corollary 2.10.

Case 1: If the extremal face F_1 is a ray, then $F_1 = \mathbb{R}_+[R_1]$ and

$$0 \leq (\lambda\mu_1)^2 = [R_1]^2 \leq 0 \Rightarrow (R_1)^2 = 0, \text{ for a suitable } \lambda > 0.$$

This follows from the fact that the K_{X_1}-negative movable class μ_1 is contained in $F_1 = \mathbb{R}_+[R_1]$. Hence X_1 is a ruled surface over a smooth curve C with $\rho(X_1) = 2$ by Corollary 2.10, and every movable class in F_0 is mapped to a multiple of a fibre in X_1 via $(\phi_0)_*$.

Case 2: Conversely, if $\rho(X_1) = 2$, then the face F_1 has to be a ray and we are in the situation of case 1.

Case 3: If the Picard number $\rho(X_1)$ of X_1 is still greater than two, Corollary 2.10 implies that all K_{X_1}-negative extremal rays of $\overline{\mathrm{NE}}(X_1)$ are spanned by (-1)-curves. In particular, R_1 is a (-1)-curve, all movable classes in F_0 are mapped to movable classes in F_1 via $(\phi_0)_*$ and we can can continue in the same way as above.

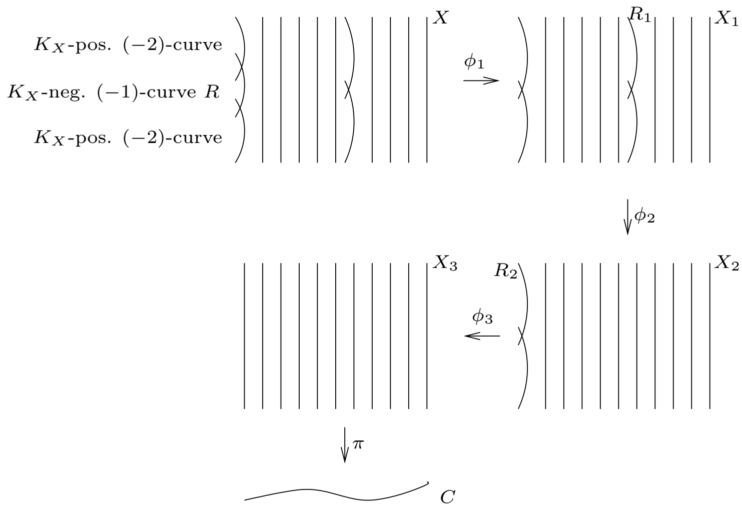


FIGURE 9. Illustration of this procedure with $k = 3$.

This procedure ends in a ruled surface X_k over a smooth curve C since the Picard number drops by one with every contraction of a (-1)-curve. ☐

Proof of Theorem 2.8. We can assume that $\rho(X) \geq 3$. Proposition 2.12 yields that there exists a finite sequence of monoidal transformations

$$X \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{k-1}} X_k$$

such that $X_k \xrightarrow{\pi} C$ is a ruled surface over a smooth curve C. Denote by ϕ^j the map $\phi_{k-1} \circ \dots \circ \phi_j : X_j \to X_k$, $0 < j < k-1$, and set $\phi := \phi^0 : X \to X_k$.

By [**Deb01**, Lemma 6.7 (c)], the extremal face $F_0 := F \subset \overline{\mathrm{NE}}(X)$ has a supporting function; in other words, there exists a non-zero $\mathbb{R}$-divisor class $\Lambda \in N^1(X)_{\mathbb{R}}$ such that

$$F_0 = \Lambda^{\perp} \cap \overline{\mathrm{NE}}(X) = \{\Delta \in \overline{\mathrm{NE}}(X) \mid \Lambda \,.\, \Delta = 0\}.$$

This implies that Λ or $-\Lambda$ is nef. W.l.o.g. assume that Λ is nef.

Claim. The class Λ is a real multiple of the class $[f_0]$ of a fibre $f_0 \subset X$ of $\pi \circ \phi$ in X.

Proof of the claim. Let f_k be a fibre of π in X_k. Set $f_i := (\phi_i)^*(f_{i+1})$, $i = k-1, \ldots, 0$, and $F_j := (\phi_{j-1})_*(F_{j-1})$, $j = 1, \ldots, k$.

By Proposition 2.12, every movable class $\mu \in M \subset F_0$ is mapped to a multiple of the class $[f_k] \in N_1(X_k)_{\mathbb{R}}$ by ϕ_*. This implies that $[f_k] \in F_k$. Therefore, Lemma 2.11 (ii) yields that $\phi^*([f_k]) \in F_0$, and, using the projection formula, we compute that

$$\phi_*(\Lambda) \,.\, [f_k] = \Lambda \,.\, \phi^*([f_k]) = 0.$$

Since X_k is a ruled surface, we see that $\phi_*(\Lambda) = k[f_k]$, for suitable $k \in \mathbb{R}$. Note that f_i is a fibre of $\pi \circ \phi^i$ in X_i and by construction

$$[f_i] = [(\phi_i)^*(f_{i+1})] = (\phi_i)^*([f_{i+1}])$$

for all $i = k-1, \ldots, 0$. Therefore, $\phi^*(\phi_*(\Lambda)) = k[f_0]$ is a multiple of the class of the fibre f_0 of $\pi \circ \phi$ in X.

It remains to show that $\phi^*(\phi_*(\Lambda)) = \Lambda$.

We have already seen that the class $[f_0] = \phi^*([f_k])$ of a general fibre of $\pi \circ \phi$ in X is contained in F_0. Moreover, the extremality condition on F_0 implies that the class of every ϕ-exceptional curve in X is also contained in F_0. In particular,

$$\Lambda \,.\, [R] = 0, \text{ for every } \phi\text{-exceptional curve } R \text{ in } X.$$

This enables us to show that $\phi^*(\phi_*(\Lambda)) = \Lambda$.

Let D be an arbitrary irreducible $\mathbb{R}$-divisor on X. The projection formula gives

$$\phi^*(\phi_*(\Lambda)) \,.\, [D] = \phi_*(\Lambda) \,.\, \phi_*([D]) = \Lambda \,.\, \phi^*(\phi_*([D])).$$

The class $\phi^*(\phi_*([D]))$ can be written in the form $[D] + m[E]$, where $m \in \mathbb{R}$ and E is a linear combination of ϕ-exceptional curves. Therefore,

$$\phi^*(\phi_*(\Lambda)) \,.\, [D] = \Lambda \,.\, ([D] + m[E]) = \Lambda \,.\, [D] + m\Lambda \,.\, [E] = \Lambda \,.\, [D]$$

and hence $k[f_0] = \phi^*(\phi_*(\Lambda)) = \Lambda$. $\square_{Claim}$

Now let $\mu \in M \subset F_0$ be an arbitrary movable class. Then necessarily $0 = \Lambda \,.\, \mu = k[f_0] \,.\, \mu$. This implies that μ itself has intersection number zero with every ϕ-exceptional curve in X and, as above, $\phi^*(\phi_*(\mu)) = \mu$.

Recall: Proposition 2.12 guarantees that the movable class $\mu \in M \subset F_0$ is mapped to a multiple of the class $[f_k] \in N_1(X_k)_{\mathbb{R}}$ by ϕ_*. Therefore, $\mu = \phi^*(\phi_*(\mu)) = t[f_0]$ for a suitable $t \in \mathbb{R}_+$, and every movable class in M lies on the ray $\mathbb{R}_+[f_0]$. This concludes the proof. $\square$

Remark 2.13. This theorem is not true for varieties of higher dimension. See Example 4.11 and Corollary 4.12 in section 4.2.

Corollary 2.14. *Let X be a smooth projective surface with Picard number $\rho(X) \geq 3$ and let $F \subset \overline{\mathrm{NE}}(X)_{K_X<0}$ be a K_X-negative extremal face, which is spanned by extremal classes $[R_1] \dots [R_k]$ of $\overline{\mathrm{NE}}(X)_{K_X<0}$. If F contains a movable class μ, then k is an even number.*

Proof. Proposition 2.12 yields that there exists a finite sequence of monoidal transformations $X \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_k} X_k$ and X_k is a ruled surface. Moreover, all extremal classes in F are (-1)-curves which are contained in fibres of X.

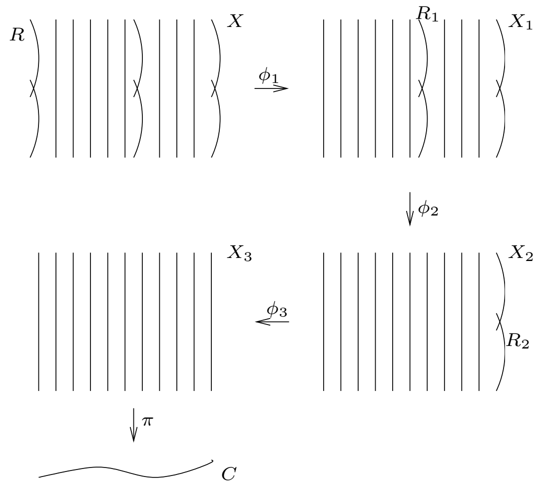


FIGURE 10. Fibres in X are either irreducible or they consist of two (-1)-curves.

Note that each fibre is either irreducible or it contains two (-1)-curves since the blow up of a smooth fibre produces exactly two (-1)-curves. Another blow up of this fibre would produce at least one (-2)-curve, but there are just (-1)-curves in F. □

2.3. Some examples

In this section we will compute a few examples and we will see in Corollary 2.17 that an extremal ray of the moving cone is not necessarily spanned by a rational curve.

Lemma 2.15. *Let X be a ruled surface with invariant $e < 0$ over a smooth curve C of genus $g \geq 0$. Let C_0 be the image of the section with selfintersection $-e$ and let f be a fibre. Then $\overline{\mathrm{NM}}(X) = \langle f, 2C_0 + ef \rangle_{\mathbb{R}_+}$.*

Proof. Since X is a surface $\overline{\mathrm{NM}}(X) = \mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ and by [**Har77**, Chapter V, Proposition 2.21] a divisor $D = aC_0 + bf$ on X is ample if and only if $a > 0$ and $b > \frac{a}{2}e$. □

Lemma 2.16. *Let X be a ruled surface with invariant $e \geq 0$ over a smooth curve C of genus $g \geq 0$. Let C_0 be the image of the section with selfintersection $-e$ and let f be a fibre. Then $\overline{\mathrm{NM}}(X) = \langle f, C_0 + ef \rangle_{\mathbb{R}_+}$.*

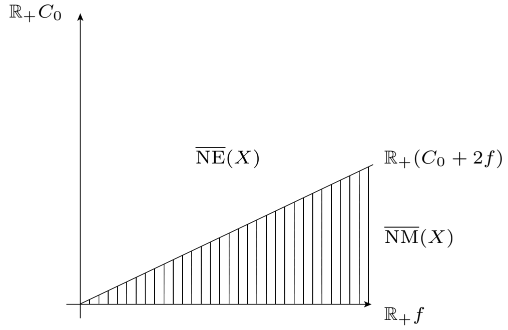


FIGURE 11. Moving Cone of X with invariant $e = 2$

Proof. Since X is a surface $\overline{\mathrm{NM}}(X) = \mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ and by [**Har77**, Chapter V, Proposition 2.20] a divisor $D = aC_0 + bf$ on X is ample if and only if $a > 0$ and $b > ae$. □

Corollary 2.17. *Let C be a smooth curve of genus $g > 0$, let $\mathscr{L}$ be a line bundle on C with $-e := \deg \mathscr{L} \leq 0$ and set $\mathcal{E} := \mathcal{O}_C \oplus \mathscr{L}$. Let X be the ruled surface $X := \mathbb{P}(\mathcal{E})$ over C, let C_0 be the image of the section with selfintersection $-e$ and let f be a fibre. Then there exists a section of X with image $C' \sim C_0 + ef$. In particular, C' is a smooth curve of genus $g > 0$ which spans an extremal ray of $\overline{\mathrm{NM}}(X)$.*

Proof. By construction, $X = \mathbb{P}(\mathcal{E})$ is a ruled surface with invariant $e \geq 0$ over a smooth curve C of genus $g > 0$. Let C' be the image of the section corresponding to the surjection

$$\mathcal{E} \to \mathcal{O}_C \to 0.$$

In particular, C' is isomorphic to C and hence a smooth curve of genus $g > 0$. By [**Har77**, Chapter V, Proposition 2.9] $C' \sim C_0 + ef$ and by Lemma 2.16 C' spans an extremal ray of $\overline{\mathrm{NM}}(X)$. □

2.3.1. Monoidal transformations of ruled surfaces. We will now illustrate a method to use Theorem 1.12 and a given divisorial contraction for the computation of the moving cone on the easy level of ruled surfaces.

Lemma 2.18. *Let $\pi : X_{e,g} \to C$ be a ruled surface with invariant $e \geq 0$ over a smooth curve C of genus g. Let C_0 be the image of the section with selfintersection $-e$ and let f be a fibre. Denote by $\mu : X \to X_{e,g}$ the blow up of $X_{e,g}$ in a point p on C_0. Then $\overline{\mathrm{NM}}(X) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{\mathbb{R}_+}$, where*

$$\gamma_1 = [f'] + [E],$$
$$\gamma_2 = [C_0'] + e[f'] + (e+1)[E],$$
$$\gamma_3 = [C_0'] + (e+1)([f'] + [E]),$$

f' denotes the strict transform of f, C_0' denotes the strict transform of C_0 and E is the exceptional divisor of the blow up. Moreover, $\gamma_1 . [K_X] = -2$, $\gamma_2 . [K_X] = 2g - (e+2)$, $\gamma_3 . [K_X] = 2g - (e+3)$.

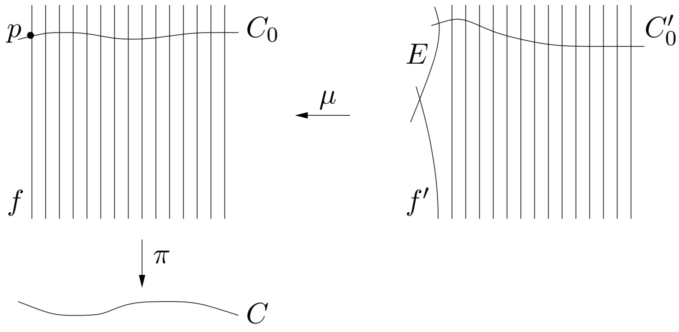


FIGURE 12. Blow up of $X_{e,g}$ in a point p on C_0

Proof. By [**Har77**, Chapter V, Proposition 2.3], $\rho(X_{e,g}) = 2$ and $N_1(X_{e,g})_{\mathbb{R}}$ is spanned by $[C_0], [f]$. The canonical divisor $K_{X_{e,g}}$ is numerically equivalent to

$$-2C_0 + (2g - 2 - e)f$$

by [**Har77**, Chapter V, Corollary 2.11]. Hence

$$N_1(X)_{\mathbb{R}} = \langle [C_0'], [f'], [E] \rangle_{\mathbb{R}}$$

by [**Har77**, Chapter V, Proposition 3.2] and

$$[K_X] = \mu^*([K_{X_{e,g}}]) + [E] = -2[C_0'] + (2g - 2 - e)[f'] + (2g - 2 - (e + 1))[E]$$

by [**Har77**, Chapter V, Proposition 3.3]. The intersection numbers on X' are given in the following table.

.	C_0'	f'	E
C_0'	$-(e+1)$	0	1
f'	0	-1	1
E	1	1	-1

Now we want to investigate how to express classes of irreducible curves on X in terms of the basis $[C_0'], [f'], [E]$. So let C' be an irreducible curve on X which is not numerically equivalent to C_0', f' or E, and set

$$a := C' \,.\, C_0', \quad b := C' \,.\, f', \quad c := C' \,.\, E.$$

Since $\mu_*(C')$ is again irreducible, [**Har77**, Chapter V, Proposition 2.20] yields that

$$\mu_*(C') = \alpha C_0 + \beta f$$

with $\alpha > 0$ and $\beta \geq \alpha e$. Therefore, we obtain

$$C' + cE = \mu^*(\mu_*(C')) = \alpha\mu^*(C_0) + \beta\mu^*(f')$$
$$\Rightarrow C' = \alpha C_0' + \beta f' + (\alpha + \beta - c)E.$$

Here the first equality is given by [**Har77**, Chapter V, Proposition 3.6]. By taking intersections of C' with C_0', respectively f', we deduce the equations

$$a = -e\alpha + \beta - c,\ b = \alpha - c \quad \Rightarrow \quad \alpha = b + c,\ \beta = a + c + e(b + c).$$

Hence $C' = (b + c)C_0' + (a + c + e(b + c))f' + (a + (e + 1)(b + c)E)$ and all coefficients are non-negative.

Now let

$$\gamma = \lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E]$$

be the class of an arbitrary movable curve on X.

Then we have $0 \le \gamma\,.[C_0'], 0 \le \gamma\,.[f'], 0 \le \gamma\,.[E]$. This yields that

$$\begin{aligned}\lambda_2 &\le \lambda_3\\ \lambda_3 &\le \lambda_1 + \lambda_2\\ \lambda_3 &\ge (e+1)\lambda_1\end{aligned}$$

and, using Theorem 1.12, the previous considerations show that these equations give $\overline{\mathrm{NM}}(X)$. The extremal rays of $\overline{\mathrm{NM}}(X)$ are given by

$$\begin{aligned}\gamma_1 &\equiv_{\mathrm{num}} [f'] + [E],\\ \gamma_2 &\equiv_{\mathrm{num}} [C_0'] + e[f'] + (e+1)[E],\\ \gamma_3 &\equiv_{\mathrm{num}} [C_0'] + (e+1)([f'] + [E]).\end{aligned}$$

Some short computations give

$$\gamma_1\,.[K_X] = -2, \gamma_2\,.[K_X] = 2g - (e+2), \gamma_3\,.[K_X] = 2g - (e+3).$$

□

Lemma 2.19. *Let $\pi : X_{e,g} \to C$ be a ruled surface with invariant $e < 0$ over a smooth curve C of genus g. Let C_0 be the image of the section with selfintersection $-e$ and let f be a fibre. Denote by $\mu : X \to X_{e,g}$ the blow up of $X_{e,g}$ in a point p on C_0. Then $\overline{\mathrm{NM}}(X) = \langle\gamma_1, \gamma_2, \gamma_3\rangle_{\mathbb{R}_+}$, where*

$$\begin{aligned}\gamma_1 &\equiv_{num} [f'] + [E],\\ \gamma_2 &\equiv_{num} [C_0'] + \tfrac{e}{2}[f'] + \tfrac{e+2}{2}[E],\\ \gamma_3 &\equiv_{num} [C_0'] + \tfrac{e+1}{2}([f'] + [E]),\end{aligned}$$

f' denotes the strict transform of f, C_0' denotes the strict transform of C_0 and E is the exceptional divisor of the blow up. See Figure 12. Moreover, $\gamma_1\,.[K_X] = -2$, $\gamma_2\,.[K_X] = 2g-2$, $\gamma_3\,.[K_X] = 2g-2$.

Proof. By [**Har77**, Chapter V, Proposition 2.3], $\rho(X_{e,g}) = 2$ and $N_1(X_{e,g})_{\mathbb{R}}$ is spanned by $[C_0], [f]$. The canonical divisor $K_{X_{e,g}}$ is numerically equivalent to

$$-2C_0 + (2g - 2 - e)f$$

by [**Har77**, Chapter V, Corollary 2.11]. Hence

$$N_1(X)_{\mathbb{R}} = \langle [C_0'], [f'], [E]\rangle_{\mathbb{R}}$$

by [**Har77**, Chapter V, Proposition 3.2] and

$$[K_X] = \mu^*([K_{X_{e,g}}]) + [E] = -2[C_0'] + (2g-2-e)[f'] + (2g-2-(e+1))[E]$$

by [**Har77**, Chapter V, Proposition 3.3]. The intersection numbers on X' are given in the following table.

.	C_0'	f'	E
C_0'	$-(e+1)$	0	1
f'	0	-1	1
E	1	1	-1

Now let C' be an irreducible curve on X' which is not numerically equivalent to C_0', f' or E and set $\gamma := C' \,.\, E$.

Since $\mu_*(C')$ is again irreducible, [**Har77**, Chapter V, Proposition 2.20] yields that

$$\mu_*(C') \equiv_{\text{num}} \alpha C_0 + \beta f$$

with either $\alpha = 1, \beta \geq 0$ or $\alpha \geq 2, \beta \geq \frac{1}{2}\alpha e$. Therefore, we obtain

$$\begin{aligned} C' + \gamma E &\equiv_{\text{num}} \mu^*(\mu_*(C')) \equiv_{\text{num}} \alpha\mu^*(C_0) + \beta\mu^*(f') \\ \Rightarrow C' &\equiv_{\text{num}} \alpha C_0' + \beta f' + (\alpha + \beta - \gamma)E. \end{aligned}$$

Since f' is a (-1)-curve, there exists a monoidal transformation

$$\nu : X \to X_{(e+1),g}$$

contracting f'.

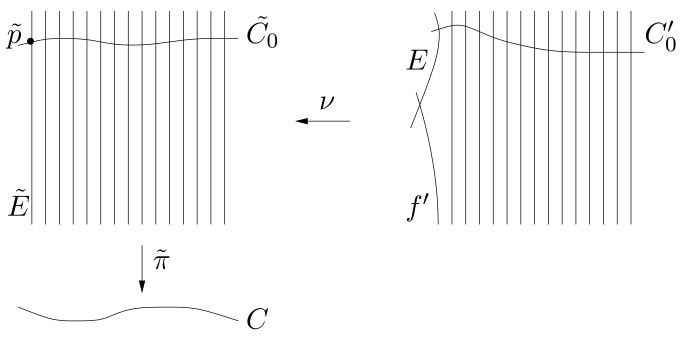


FIGURE 13. Blow up of $X_{(e+1),g}$ in a point $\tilde{p}$ on $\tilde{C}_0$

The curve $\nu_*(C_0') := \tilde{C}_0$ has selfintersection $-(e+1)$ and $\nu_*(E) := \tilde{E}$ is a fibre of $X_{(e+1),g}$. As before, we obtain a representation

$$\nu_*(C') \equiv_{\text{num}} \tilde{\alpha}\tilde{C}_0 + \tilde{\beta}\tilde{E},$$

and [**Har77**, Chapter V, Proposition 2.20] yields that either $\tilde{\alpha} = 1, \tilde{\beta} \geq 0$ or $\tilde{\alpha} \geq 2, \tilde{\beta} \geq \frac{1}{2}\tilde{\alpha}(e+1)$. Since $C' \,.\, f' = \alpha - \gamma$, we deduce

$$C' \equiv_{\text{num}} \tilde{\alpha} C_0' + (\tilde{\beta} + \gamma - \alpha) f' + \tilde{\beta} E.$$

Now let $\eta = \lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E]$ be a movable class on X'. Taking the intersection numbers of η with $[C'], [C_0'], [f']$ and $[E]$ we obtain the following inequalities.

$$\begin{aligned} &0 \leq (\lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E])\,.(\alpha[C_0'] + \beta[f'] + (\alpha+\beta-\gamma)[E]) \\ &\quad = \lambda_1(\beta - e\alpha - \gamma) + \lambda_2(\alpha-\gamma) + \lambda_3\gamma, \end{aligned} \tag{2.a}$$

$$\begin{aligned} &0 \leq (\lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E])\,.(\tilde{\alpha}C_0' + (\tilde{\beta}+\gamma-\alpha)f' + \tilde{\beta}E) \\ &\quad = \lambda_1(\tilde{\beta} - (e+1)\tilde{\alpha}) + \lambda_2(\alpha-\gamma) + \lambda_3(\gamma - \alpha + \tilde{\alpha}), \end{aligned} \tag{2.b}$$

$$\begin{aligned} &0 \leq (\lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E])\,.[C_0'] \\ &\quad = -(e+1)\lambda_1 + \lambda_3, \end{aligned} \tag{2.c}$$

$$\begin{aligned} &0 \leq (\lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E])\,.[f'] \\ &\quad = -\lambda_2 + \lambda_3, \end{aligned} \tag{2.d}$$

$$\begin{aligned} &0 \leq (\lambda_1[C_0'] + \lambda_2[f'] + \lambda_3[E])\,.[E] \\ &\quad = \lambda_1 + \lambda_2 - \lambda_3. \end{aligned} \tag{2.e}$$

We can assume that $\beta = \frac{1}{2}\alpha e$ in inequality (2.a) and $\tilde{\beta} = \frac{1}{2}\tilde{\alpha}(e+1)$. To obtain the extremal rays of the moving cone, we take equality in (2.a) to (2.e) and intersect the resulting hyperplanes pairwise.

The equations (2.a) and (2.e) give the extremal ray spanned by the class $\gamma_2 = [C_0'] + \frac{e}{2}[f'] + \frac{e+2}{2}[E]$, (2.d) and (2.e) give the extremal ray spanned by the class $\gamma_1 = [f'] + [E]$. The equations (2.b) and (2.d) give the extremal ray spanned by the class $\gamma_3 = [C_0'] + \frac{e+1}{2}([f'] + [E])$. Other admissible choices for α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ or combinations of (2.a) to (2.e) do not yield any other extremal rays of the moving cone.

Simple computations give

$$\gamma_1\,.[K_X] = -2, \gamma_2\,.[K_X] = 2g-2, \gamma_3\,.[K_X] = 2g-2.$$

□

These two examples illustrate that the presence of a divisorial contraction can be very useful. We were able to take advantage of our knowledge about a birational model of our surface by usage of the projection formula.

Close inspection reveals that we have benefited from the projection formula for divisors and divisor classes on surfaces throughout the whole chapter. This formula is a powerful computational tool and we would like to benefit from an analogous formula in higher dimensions.

Therefore, we want to establish a "numerical projection formula" for the comparison and the computation of intersection numbers on birational models obtained by running the minimal model program. This is the subject of chapter 3.

CHAPTER 3

Numerical pullback and pushforward

For computational reasons it would be very useful to take "pullbacks" and "pushforwards" of 1-cycles via morphisms or isomorphisms in codimension one. Whereas the pushforward of a 1-cycle via a proper morphism is easy to define, a definition of pushforward of 1-cycles via a birational map has to be handled carefully. This is due to the fact that a 1-cycle could be entirely contained in the indeterminacy locus of the map. The situation is even more difficult if we want to define a pullback of 1-cycles.

On the other hand, the situation is more practical for divisors. We can define pullbacks and pushforwards of divisor classes via birational maps which are surjective in codimension one without difficulties. This enables us to define a notion of pullback or pushforward of 1-cycles as dual linear maps of pushforwards or pullbacks of divisors, respectively. This idea was given and briefly explained in [**Ara05**, Section 3]. A detailed treatment of pushforwards and pullbacks of k-cycles is given in [**Ful98**].

3.1. Basic definitions and notation

Notation 3.1. In this section let X and Y be $\mathbb{Q}$-factorial varieties with only terminal singularities and $\varphi : X \dashrightarrow Y$ be a birational map which is surjective in codimension one; that is, $\operatorname{codim}_Y(Y \setminus \operatorname{im}(\varphi)) \geq 2$.

Fact 3.2. The pullback of $\mathbb{Q}$-Cartier divisors on Y via φ gives an injective linear map

$$\varphi^* : N^1(Y)_{\mathbb{R}} \hookrightarrow N^1(X)_{\mathbb{R}}$$

and the pushforward of $\mathbb{Q}$-Cartier divisors on X via φ gives a surjective linear map

$$\varphi_* : N^1(X)_{\mathbb{R}} \twoheadrightarrow N^1(Y)_{\mathbb{R}}$$

such that $\varphi_* \circ \varphi^* = \operatorname{id}_{N^1(Y)_{\mathbb{R}}}$. See [**Ara05**, Definition 3.1].

Remark 3.3. Here is an explanation of this fact. The varieties X and Y have only terminal singularities by Notation 3.1. In particular, X and Y are normal by definition. Thus the indeterminacy locus of φ in X and the singular loci in X and Y have codimension at least two. In addition, the map φ is surjective in codimension one by Notation 3.1. Therefore, we can choose algebraic sets $Z \subset X$ and $Z' \subset Y$ with $\operatorname{codim}_X(Z) \geq 2$ and $\operatorname{codim}_Y(Z') \geq 2$ such that $U := X \setminus Z$ and $V := Y \setminus Z'$ are non-singular and such that $\varphi|_U : U \to V$ is a proper morphism. Now [**Har77**, Ch. II, Proposition 6.5] yields that

$$\operatorname{Cl}(X) \cong \operatorname{Cl}(U) \cong \operatorname{CaCl}(U) \text{ and } \operatorname{Cl}(Y) \cong \operatorname{Cl}(V) \cong \operatorname{CaCl}(V),$$

where Cl(.) denotes the group of Weil divisors modulo linear equivalence, and CaCl(.) denotes the group of Cartier divisors modulo linear equivalence on X, Y, U or V, respectively. Moreover, X and Y are $\mathbb{Q}$-factorial. This yields that the maps $(\varphi|_U)^*$ and $(\varphi|_U)_*$ extend to the injective linear map φ^* and to the surjective linear map φ_* of Fact 3.2.

Remark 3.4. The condition $\operatorname{codim}_Y(Z') \geq 2$ is not necessary for the definition of the pushforward, but we lose the surjectivity of the map if we drop this requirement. Here is why. If $\operatorname{codim}_Y(Z') = 1$, then we still have a short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}(Y) \to \mathrm{Cl}(V) \to 0.$$

The second map is defined by the assignment $1 \mapsto 1 \cdot Z'$ and the third map is given by the assignment $\sum k_i \cdot W_i \mapsto \sum k_i \cdot (W_i \cap V)$, where we ignore those $W_i \cap V$ which are empty. See [**Har77**, Ch. II, Proposition 6.5]. This sequence is split since taking closure in Y and intersecting with V afterwards gives the identity map on $\mathrm{Cl}(V)$. Hence $\mathrm{Cl}(Y) \cong \mathrm{Cl}(V) \oplus (\mathbb{Z} \cdot Z')$. This shows that Z' is not contained in the image of φ_*.

Definition 3.5 (Numerical pullback and pushforward)**.** Let

$$\varphi_1^* : N_1(Y)_{\mathbb{R}} \hookrightarrow N_1(X)_{\mathbb{R}}$$

be the dual linear map of the pushforward $\varphi_* : N^1(X)_{\mathbb{R}} \twoheadrightarrow N^1(Y)_{\mathbb{R}}$ of $\mathbb{R}$-divisors on X and let

$$\varphi_{*1} : N_1(X)_{\mathbb{R}} \twoheadrightarrow N_1(Y)_{\mathbb{R}}$$

be the dual linear map of the pullback $\varphi^* : N^1(Y)_{\mathbb{R}} \hookrightarrow N^1(X)_{\mathbb{R}}$ of $\mathbb{R}$-divisors on Y. We call φ_1^* the *numerical pullback via* φ and φ_{*1} the *numerical pushforward via* φ.

Remark 3.6. We can drop the assumption that φ is surjective in codimension one if we require that φ is a proper morphism of complete varieties or projective schemes.

Remark 3.7. Every class $\gamma \in N_1(Y)_{\mathbb{R}}$ can be considered as a linear map

$$\gamma : N^1(Y)_{\mathbb{R}} \to \mathbb{R},\ \Delta \mapsto \gamma \,.\, \Delta.$$

Therefore, the class $\varphi_1^*(\gamma) \in N_1(X)_{\mathbb{R}}$ can be considered as the composition

$$\gamma \circ \varphi_* : N^1(X)_{\mathbb{R}} \to \mathbb{R},\ \Delta' \mapsto \gamma \,.\, \varphi_*(\Delta').$$

In the same way $\varphi_{*1}(\gamma') \in N_1(Y)_{\mathbb{R}}$ can be considered as the map

$$\gamma' \circ \varphi^* : N^1(Y)_{\mathbb{R}} \to \mathbb{R},\ \tilde{\Delta} \mapsto \gamma' \,.\, \varphi^*(\tilde{\Delta})$$

for every $\gamma' \in N_1(X)_{\mathbb{R}}$. This yields the following two statements.

Lemma 3.8 (Projection formula)**.** *Let $\varphi : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial projective varieties with only terminal singularities which is surjective in codimension one.*

(i) If $\gamma \in N_1(Y)_{\mathbb{R}}$ and $\Delta \in N^1(X)_{\mathbb{R}}$, then $\varphi_1^*(\gamma) \,.\, \Delta = \gamma \,.\, \varphi_*(\Delta)$.
(ii) If $\gamma \in N_1(X)_{\mathbb{R}}$ and $\Delta \in N^1(Y)_{\mathbb{R}}$, then $\gamma \,.\, \varphi^*(\Delta) = \varphi_{*1}(\gamma) \,.\, \Delta$. □

Lemma 3.9 ([**Ara05**, Remark 3.2])**.** *The composition $\varphi_{*1} \circ \varphi_1^*$ is the identity map on $N_1(Y)_{\mathbb{R}}$ and the numerical pullback via φ satisfies the following.*

(i) If $\Delta \in N^1(Y)_{\mathbb{R}}$ and $\gamma \in N_1(Y)_{\mathbb{R}}$, then $\varphi^*(\Delta) \,.\, \varphi_1^*(\gamma) = \Delta \,.\, \gamma$.
(ii) If $\Delta \in \ker \varphi_*$ and $\gamma \in \mathrm{im}\varphi_1^*$, then $\Delta \,.\, \gamma = 0$. □

Remark 3.10. The numerical pullback via a birational map is defined in an abstract way and should be handled carefully. It is possible to take the numerical pullback of a class of a curve which is entirely contained in the indeterminacy locus of the map. Thus, a priori, it is not clear what we will have to expect in such a situation. In section 4.2 we will see an example where the numerical pullback of an effective 1-cycle class is no longer effective. The same holds for the numerical pushforward via a birational map.

Now the question arises if there are other more geometrical definitions for the pushforward und pullback of cycles which coincide with these numerical definitions for 1-cycle classes.

In [**Ful98**] Fulton gives several definitions for the pullback of cycles depending on the properties of the used morphism and varieties. We will not enlarge upon this now, but we will give a nice definition of the pushforward of cycles via a proper morphism. For a detailed treatment see [**Ful98**, Chapter 1 and 2].

Proposition and Definition 3.11 (Pushforward of cycles)**.** *Let $f : X \to Y$ be a proper morphism of complete varieties or projective schemes and let V be any subvariety of X. The image $W = f(V)$ is a closed subvariety of Y and there exists an induced imbedding of the function field $K(W)$ of W in the function field $K(V)$ of V, which is a finite field extension if $\dim_{\mathbb{C}}(W) = \dim_{\mathbb{C}}(V)$. Set*

$$\deg(V/W) := \begin{cases} [K(V) : K(W)] & \text{, if } \dim_{\mathbb{C}}(W) = \dim_{\mathbb{C}}(V) \\ 0 & \text{, if } \dim_{\mathbb{C}}(W) < \dim_{\mathbb{C}}(V) \end{cases}$$

and

$$f_*([V]) := \deg(V/W) \cdot [W],$$

where $[V]$ and $[W]$ denote the rational equivalence classes of V and W. □

Corollary 3.12. *Let $f : X \to Y$ be a proper birational morphism of $\mathbb{Q}$-factorial projective varieties with only terminal singularities. Then $f_{*1}(\alpha) = f_*(\alpha)$, for all 1-cyles $\alpha \in N_1(X)_{\mathbb{R}}$.*

Proof. Let D be an arbitrary Cartier divisor on Y. Then

$$f^*(D) \,.\, \alpha = D \,.\, f_*(\alpha)$$

by [**Ful98**, Proposition 2.5]. Thus Lemma 3.8 yields $D \,.\, f_*(\alpha) = D \,.\, f_{*1}(\alpha)$ and hence the claim. □

If we are dealing with a proper morphism, we may use both concepts. The pushforward of cycles applies to geometric deliberations and the numerical pushforward is a good choice for abstract computations.

3.2. Numerical pullback of movable extremal classes

Finally, we want to propose a method to compute extremal rays of the moving cone of a $\mathbb{Q}$-factorial projective variety. It is based on ideas of Carolina Araujo. In [**Ara05**] she takes numerical pullbacks of curves lying in general fibres of Mori fibre spaces obtained by running the minimal model program.

We want to run the minimal model program and take the numerical pullback of extremal rays of the moving cones of birational models, successively. This will be very useful for the computation of some examples.

Therefore, we show a correspondence between the extremal rays of the movable cone of a $\mathbb{Q}$-factorial projective variety and extremal rays of the movable cone of a birational model obtained by a divisorial contraction or a flip.

Theorem 3.13. *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities and $\varphi : X \to Y$ a divisorial contraction of an extremal ray of $\overline{\mathrm{NE}}(X)$. Then the following holds.*

(i) A class $\gamma \in N_1(Y)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(Y)$ if and only if the class $\varphi_1^*(\gamma) \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$.

(ii) Let $\gamma \in \overline{\mathrm{NM}}(X)$ be a movable class such that

$$\varphi_1^*(\varphi_{*1}(\gamma)) = \gamma + \varepsilon\eta$$

for some $\varepsilon \in \mathbb{R}$, $\eta \in \ker \varphi_{*1}$ and such that $\gamma + \varepsilon\eta \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$. Then $\varphi_{*1}(\gamma + \varepsilon\eta) = \varphi_{*1}(\gamma) \in N_1(Y)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(Y)$.

Proof. Step 1 of (i). Let $\gamma \in N_1(Y)_{\mathbb{R}}$ be an extremal class of $\overline{\mathrm{NM}}(Y)$. We want to prove that $\varphi_1^*(\gamma) \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$.

Let $\Delta \in \overline{\mathrm{Eff}}(X)$ be a effective divisor class on X. The projection formula, Lemma 3.8, gives

$$\varphi_1^*(\gamma) \,.\, \Delta = \gamma \,.\, \varphi_*(\Delta).$$

Since $\varphi_*(\Delta)$ is again effective and $\gamma \in \overline{\mathrm{NM}}(Y)$, Theorem 1.12 implies that $\varphi_1^*(\gamma) \,.\, \Delta \geq 0$ and hence $\varphi_1^*(\gamma) \in \overline{\mathrm{NM}}(X)$.

Now let $\nu, \omega \in \overline{\mathrm{NM}}(X)$ be arbitrary classes with $\nu + \omega \in \mathbb{R}_+\varphi_1^*(\gamma)$ and let Ξ be the class of the exceptional divisor $\mathrm{Exc}_X(\varphi) \subset X$, which is contracted by φ. Thanks to Lemma 3.9 (ii), we have

$$0 = \varphi_1^*(\gamma) \,.\, \Xi = \lambda(\nu + \omega) \,.\, \Xi$$

for a suitable $\lambda \in \mathbb{R}_+$. Thus $\nu \,.\, \Xi = -\omega \,.\, \Xi$, and Theorem 1.12 yields that

$$0 \leq \nu \,.\, \Xi = -\omega \,.\, \Xi \leq 0 \Rightarrow \nu \,.\, \Xi = 0 = \omega \,.\, \Xi$$

since Ξ is effective. This implies

$$\nu = \varphi_1^*(\varphi_{*1}(\nu)) \text{ and } \omega = \varphi_1^*(\varphi_{*1}(\omega)). \tag{3.a}$$

Now let $\Delta' \in \overline{\mathrm{Eff}}(Y)$ be an effective divisor class on Y. The pullback $\varphi^*(\Delta')$ of Δ' is again effective. Together, Lemma 3.8 and Theorem 1.12 give that

$$\varphi_{*1}(\nu) \,.\, \Delta' = \nu \,.\, \varphi^*(\Delta') \geq 0 \text{ and } \varphi_{*1}(\omega) \,.\, \Delta' = \omega \,.\, \varphi^*(\Delta') \geq 0.$$

Hence $\varphi_{*1}(\nu), \varphi_{*1}(\omega) \in \overline{\mathrm{NM}}(Y)$.

If $\hat{\Delta} \in N^1(Y)_{\mathbb{R}}$ is a divisor class on Y, then we find

$$\begin{aligned}(\varphi_{*1}(\nu) + \varphi_{*1}(\omega)) \,.\, \hat{\Delta} &= \varphi_{*1}(\nu + \omega) \,.\, \hat{\Delta} && \text{by linearity} \\ &= (\nu + \omega) \,.\, \varphi^*(\hat{\Delta}) && \text{by Lemma 3.8} \\ &= \lambda\varphi_1^*(\gamma) \,.\, \varphi^*(\hat{\Delta}) && \text{by assumption} \\ &= \lambda\gamma \,.\, \hat{\Delta} && \text{by Lemma 3.9 (i).}\end{aligned}$$

Hence the class $(\varphi_{*1}(\nu) + \varphi_{*1}(\omega))$ lies on the ray $\mathbb{R}_+\gamma$. Since $\gamma \in \overline{\mathrm{NM}}(Y)$ is an extremal class, the classes $\varphi_{*1}(\nu)$ and $\varphi_{*1}(\omega)$ are forced to lie on $\mathbb{R}_+\gamma$, too. Now, for suitable $\mu, \mu' \in \mathbb{R}_+$ and $\tilde{\Delta} \in N^1(X)_{\mathbb{R}}$ an arbitrary divisor class on X, the following holds:

$$\begin{aligned}\varphi_1^*(\gamma) \,.\, \tilde{\Delta} &= \gamma \,.\, \varphi_*(\tilde{\Delta}) && \text{by Lemma 3.8} \\ &= \mu \cdot \varphi_{*1}(\nu) \,.\, \varphi_*(\tilde{\Delta}) \\ &= \mu \cdot \varphi_1^*(\varphi_{*1}(\nu)) \,.\, \tilde{\Delta} && \text{by Lemma3.8} \\ &= \mu \cdot \nu \,.\, \tilde{\Delta} && \text{by (3.a).}\end{aligned}$$

Exactly the same computation holds if we replace ν by ω and μ by μ'. This implies that ν and ω lie on $\mathbb{R}_+\varphi_1^*(\gamma)$. So, $\varphi_1^*(\gamma) \in \overline{\mathrm{NM}}(X)$ is an extremal class.

Step 2 of (i). Now let $\varphi_1^*(\gamma) \in \overline{\mathrm{NM}}(X)$ be an extremal class for some class $\gamma \in N_1(Y)_{\mathbb{R}}$. We want to prove that $\gamma \in N_1(Y)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(Y)$.

Let $\Delta \in \overline{\mathrm{Eff}}(Y)$ be an effective divisor class on Y. Together, Lemma 3.9 (ii) and Theorem 1.12 give that

$$\gamma \,.\, \Delta = \varphi_1^*(\gamma) \,.\, \varphi^*(\Delta) \geq 0$$

since $\varphi^*(\Delta) \in N^1(X)_{\mathbb{R}}$ is effective. Thus $\gamma \in \overline{\mathrm{NM}}(Y)$ by Theorem 1.12.

We choose arbitrary classes $\nu, \omega \in \overline{\mathrm{NM}}(Y)$ such that $\nu + \omega \in \mathbb{R}_+\gamma$. If $\Delta' \in \overline{\mathrm{Eff}}(X)$ is an effective divisor class on X, then $\varphi_*(\Delta') \in N^1(Y)_{\mathbb{R}}$ is effective, too. As before, the projection formula and Theorem 1.12 guarantee that

$$\varphi_1^*(\nu) \,.\, \Delta' = \nu \,.\, \varphi_*(\Delta') \geq 0 \text{ and } \varphi_1^*(\omega) \,.\, \Delta' = \omega \,.\, \varphi_*(\Delta') \geq 0.$$

This implies that $\varphi_1^*(\nu)$ and $\varphi_1^*(\omega)$ are contained in $\overline{\mathrm{NM}}(X)$.

Now choose an arbitrary divisor class $\hat{\Delta} \in N^1(X)_{\mathbb{R}}$ and a suitable $\lambda \in \mathbb{R}_+$ such that $\nu + \omega = \lambda\gamma$. Then the following holds.

$$\begin{aligned}(\varphi_1^*(\nu) + \varphi_1^*(\omega)) \,.\, \hat{\Delta} &= \varphi_1^*(\nu + \omega) \,.\, \hat{\Delta} && \text{by linearity} \\ &= (\nu + \omega) \,.\, \varphi_*(\hat{\Delta}) && \text{by Lemma 3.8} \\ &= \lambda\gamma \,.\, \varphi_*(\hat{\Delta}) && \text{by assumption} \\ &= \lambda\varphi_1^*(\gamma) \,.\, \hat{\Delta} && \text{by Lemma 3.8.}\end{aligned}$$

We find that $(\varphi_1^*(\nu) + \varphi_1^*(\omega))$ lies on the ray $\mathbb{R}_+\varphi_1^*(\gamma)$, and, by assumption, $\varphi_1^*(\gamma)$ is an extremal class of $\overline{\mathrm{NM}}(X)$. This yields that $\varphi_1^*(\nu)$ and $\varphi_1^*(\omega)$ lie on the ray $\mathbb{R}_+\varphi_1^*(\gamma)$ as well, and enables us to conclude the proof.

Let $\tilde{\Delta} \in N^1(Y)_{\mathbb{R}}$ be an arbitrary divisor class. The following computation holds for suitable $\mu, \mu' \in \mathbb{R}_+$.

$$\begin{aligned} \nu \,.\, \tilde{\Delta} &= \varphi_1^*(\nu) \,.\, \varphi^*(\tilde{\Delta}) && \text{by Lemma 3.9 (i)} \\ &= \mu \cdot \varphi_1^*(\gamma) \,.\, \varphi^*(\tilde{\Delta}) \\ &= \mu \cdot \gamma \,.\, \tilde{\Delta} && \text{by Lemma 3.9 (i).} \end{aligned}$$

This computation holds if we replace ν by ω and μ by μ'. Hence ν and ω lie on the ray $\mathbb{R}_+\gamma$, and $\gamma \in \overline{\mathrm{NM}}(Y)$ is an extremal class. $\square_{(i)}$

Proof of (ii). Let $\gamma \in \overline{\mathrm{NM}}(X)$ be a movable class such that

$$\varphi_1^*(\varphi_{*1}(\gamma)) = \gamma + \varepsilon\eta$$

for some $\varepsilon \in \mathbb{R}$, $\eta \in \ker \varphi_{*1}$ and such that $\gamma + \varepsilon\eta \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$. We want to show that $\varphi_{*1}(\gamma + \varepsilon\eta) = \varphi_{*1}(\gamma)$ is an extremal class of $\overline{\mathrm{NM}}(Y)$. This is an immediate consequence of (i). By assumption, the class $\varphi_1^*(\varphi_{*1}(\gamma))$ is an extremal class of $\overline{\mathrm{NM}}(X)$ and thus (i) yields that $\varphi_{*1}(\gamma)$ is an extremal class of $\overline{\mathrm{NM}}(Y)$. $\square$

Proposition 3.14. *Let X and X^+ be $\mathbb{Q}$-factorial projective varieties with only terminal singularities and $\phi : X \dashrightarrow X^+$ be the flip of a small contraction $\varphi : X \to Y$ of an extremal ray of $\overline{\mathrm{NE}}(X)$. Then $\phi_1^* \circ \phi_{*1} = \mathrm{id}_{N_1(X)_{\mathbb{R}}}$. In particular, ϕ_1^* is an isomorphism with inverse ϕ_{*1} and the following holds.*

(i) A class $\gamma \in N_1(X^+)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X^+)$ if and only if the class $\phi_1^*(\gamma) \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$.
(ii) A class $\gamma \in N_1(X)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X)$ if and only if the class $\phi_{*1}(\gamma) \in N_1(X^+)_{\mathbb{R}}$ is an extremal class of $\overline{\mathrm{NM}}(X^+)$.

Proof. Let $\gamma \in N_1(X)_{\mathbb{R}}$ be an arbitray class of a 1-cycle and let $\Delta \in N^1(X)_{\mathbb{R}}$ be an arbitrary divisor class. By definition, we have

$$\phi_1^*(\phi_{*1}(\gamma)) \,.\, \Delta = \phi_{*1}(\gamma) \,.\, \phi_*(\Delta) = \gamma \,.\, \phi^*(\phi_*(\Delta)).$$

Since ϕ is the flip of a small contraction, it is an isomorphism in codimension one, and we find that $\phi^*(\phi_*(\Delta)) = \Delta$. Hence $\phi_1^* \circ \phi_{*1} = \mathrm{id}_{N_1(X)_{\mathbb{R}}}$ and ϕ_1^* is an isomorphism with inverse map ϕ_{*1}.

Now one proves statement (i) and (ii) in exactly the same way as in the proof of Theorem 3.13. $\square$

Remark 3.15. Recall Remark 3.10, which states that ϕ_1^* is not an isomorphism on $\overline{\mathrm{NE}}(X^+)$. Nevertheless, it is an isomorphism on $\overline{\mathrm{NM}}(X^+)$ as Proposition 3.14 shows.

CHAPTER 4

Higher dimensions

In this chapter we will more or less adapt the methods used in chapter 2 to gain some information about the moving cone of a higher dimensional variety. The situation is more complicated than for dimension two since we lose the duality to the Mori cone. Therefore, we cannot just dualize a general structure statement since there is none for the pseudoeffective cone. In addition, it seems that there is no direct way to prove a general structure theorem for the moving cone by applying the ideas used in the proof of Mori's Cone Theorem 1.15.

Nevertheless, the situation is much better for Fano varieties. As mentioned before in the introduction and in chapter 1, C. Birkar, P. Cascini, C. Hacon and J. M^{c}Kernan have published the paper [**BCHM06**] on the arXiv in 2006. As a consequence of their main theorem they achieve in [**BCHM06**, Corollary 1.3.4] a generalization of Theorem 5 for Fano varieties. They prove that the cone of nef curves, that is, the dual cone of the pseudoeffective cone, of a $\mathbb{Q}$-factorial Fano variety with klt singularities is polyhedral, but see Corollary 6. A detailed treatment of singularities can be found in [**Cor07**].

However, there is a large machinery involved in [**BCHM06**]. We will treat the subject from a different point of view. Instead of describing the cone in terms of extremal rays, we will give a characterization by linear equations. We will explicitely indicate finitely many linear equations which cut out the moving cone of a smooth Fano variety of dimension three or four. The methods that we will use for the proof will give a detailed geometric insight.

The proof of the threefold case is the topic of the following section.

4.1. The moving cone of a smooth Fano threefold

We want to adapt the idea of the proof of Proposition 2.7 for the higher dimensional situation. There we have just used the duality of the moving cone and the Mori cone which is given for surfaces by Theorem 1.12. The result was that the moving cone is precisely the set of classes which have non-negative intersection with the extremal rays of the Mori cone. Another point of view is that the moving cone is the set of classes which have non-negative intersection with the exceptional divisors of the extremal contractions.

Unfortunately, the situation is not that easy in higher dimensions since the extremal contractions are not just divisorial, but this idea will yield the following statements.

Theorem 4.1. *Let X be a smooth Fano n-fold such that every extremal contraction of a K_X-negative extremal ray is a divisorial or a fibre contraction. Let $\varphi_i : X \to X_i$, $i = 1, ..., k$, be the divisorial contractions with exceptional divisors $E_i \subset X$, which correspond to some extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, of the Mori cone $\overline{\mathrm{NE}}(X)$ of X. Then*

$$\overline{\mathrm{NM}}(X) = \{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma\,.[E_i] \geq 0, \textit{ for all } i = 1, ..., k\}.$$

In particular, $\overline{\mathrm{NM}}(X)$ is a closed, convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$.

Corollary 4.2. *Let X be a smooth Fano threefold and let $\varphi_i : X \to X_i$, $i = 1, ..., k$, be the divisorial contractions with exceptional divisors $E_i \subset X$, which correspond to some extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, of the Mori cone $\overline{\mathrm{NE}}(X)$ of X. Then*

$$\overline{\mathrm{NM}}(X) = \{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma\,.[E_i] \geq 0, \textit{ for all } i = 1, ..., k\}.$$

In particular, $\overline{\mathrm{NM}}(X)$ is a closed, convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$.

The following statement will be the technical basis for the proof of Theorem 4.1.

Proposition 4.3. *Let X be a smooth Fano n-fold such that every extremal contraction of a K_X-negative extremal ray is a divisorial or a fibre contraction. Let $\varphi_i : X \to X_i$, $i = 1, ..., k$, be the divisorial contractions, which correspond to some K_X-negative extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., k$, of the Mori cone $\overline{\mathrm{NE}}(X)$ of X. If $\gamma \in \overline{\mathrm{NE}}(X)$ is a class with $\gamma\,.[\mathrm{Exc}_X(\varphi_i)] \geq 0$ for all $i = 1, ..., k$, then $\gamma\,.[D] \geq 0$ for all irreducible divisors $D \subset X$.*

Proof. By Theorem 1.15, $\overline{\mathrm{NE}}(X)$ is a convex, polyhedral cone spanned by finitely many extremal rays $\mathbb{R}_+[r_i]$, $i = 1, ..., m$. Theorem 1.19 guarantees the existence of an extremal contraction

$$\varphi_i : X \to X_i$$

for every extremal class $[r_i]$ of $\overline{\mathrm{NE}}(X)$, contracting exactly $\mathbb{R}_+[r_i]$; in other words, contracting all curves $c \subset X$ which are numerically proportional to r_i. By assumption, these contractions are either divisorial or of fibre type.

Assume that φ_i is divisorial for $i = 1, ..., k \leq m$ and let $E_i := \mathrm{Exc}_X(\varphi_i) \subset X$ denote the exceptional divisor which is contracted by φ_i, for $i = 1, ..., k$. The divisors E_i are irreducible by [**Mat02**, Proposition 8-2-1].

Now let $\gamma \in \overline{\mathrm{NE}}(X)$ be an arbitrary class with

$$\gamma\,.[E_i] \geq 0 \text{ for all } i = 1, ..., k \tag{4.a}$$

and $[D] \in \overline{\mathrm{Eff}}(X)$ be the class of an arbitrary irreducible divisor on X. Because of (4.a), we can assume that $D \neq E_i$ for all $i = 1, ..., k$. Since $\overline{\mathrm{NE}}(X)$ is polyhedral, we find an effective linear combination of the extremal classes $[r_i]$ such that

$$\gamma = \sum_{i=1}^{m} a_i[r_i], \ a_i \geq 0.$$

Therefore, to conclude the proof, it is sufficient to show that $D\,.\,r_i \geq 0$ for all $i = 1, ..., m$. So let $i \in \{1, ..., m\}$.

Case 1: The contraction φ_i, corresponding to the extremal class $[r_i]$, is of fibre type. The divisor D cannot contain all fibres of φ_j. Hence there has to be a fibre F which intersects D properly or $F \cap D = \emptyset$. Since all curves lying in fibres of φ_j are numerically proportional, we can take a curve $c \in F$ and obtain $0 \leq \lambda c \,.\, D = r_j \,.\, D$, for a suitable $\lambda \in \mathbb{Q}_+$.

Case 2: The contraction φ_i, corresponding to the extremal class $[r_i]$, is divisorial. If $D \cap E_i = \emptyset$, we have $r_i \,.\, D = 0$, since r_i is contained in E_i. If $D \cap E_i \neq \emptyset$, choose a curve $r' \subset E_i$ such that r' is not contained in D and $r' \equiv_{\text{num}} \lambda r_i$ for a suitable $\lambda > 0$. This is possible since $D \neq E_i$ and E_i is the unique irreducible divisor containing all curves $c \subset X$ which are numerically proportional to r_i. We obtain that $0 \leq r' \,.\, D = \lambda r_i \,.\, D$.

This concludes the proof. □

Proof of Theorem 4.1 and Corollary 4.2. We want to prove that

$$\overline{\text{NM}}(X) = \left\{\gamma \in \overline{\text{NE}}(X) \mid \gamma \,.[E_i] \geq 0, \text{ for all } i = 1, ..., k\right\} =: M.$$

Let $\gamma \in \overline{\text{NM}}(X)$. We have already seen in the proof of Theorem 1.12 that $\gamma \,.[E_i] \geq 0$ for all $i = 1, \ldots, m$ since the exceptional divisors E_i are irreducible. Thus $\overline{\text{NM}}(X) \subseteq M$. Proposition 4.3 and Theorem 1.12 give the inclusion $\overline{\text{NM}}(X) \supseteq M$ and hence $\overline{\text{NM}}(X) = M$.

Note that the assumptions of Proposition 4.3 are fulfilled if X is a smooth Fano threefold. For this fact see [**Mat02**, Example 8-3-8]. □

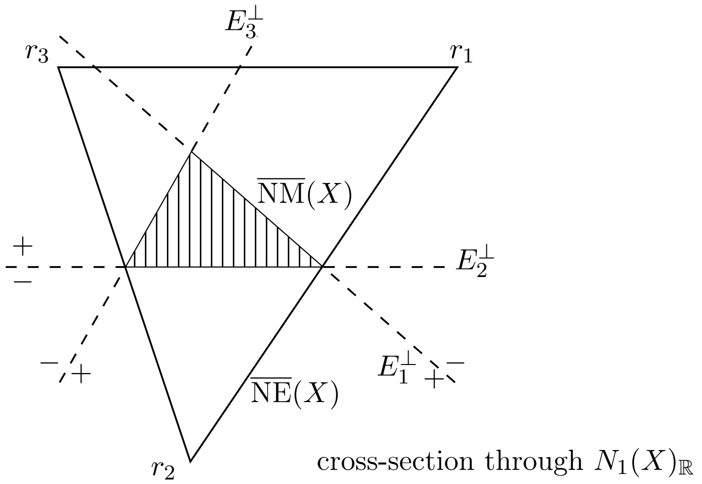


FIGURE 1. The moving cone $\overline{\text{NM}}(X)$ is cut out by the hyperplanes $E_i^\perp = \{\gamma \in N_1(X)_\mathbb{R} \mid \gamma \cdot E_i = 0\}$, $i = 1, 2, 3$.

The existence of small contractions makes the situation much more complicated. Fortunately, the circumstances are not that bad for smooth fourfolds. We will illustrate this now by some very useful examples. Later we will give some precise statements, due to Yujiro Kawamata, which will enable us to prove that the moving cone of a smooth fourfold is polyhedral.

4.2. Smooth fourfolds

4.2.1. Some useful examples. The next lemma describes the behaviour of the exceptional divisor and the canonical divisor of a smooth blow up; it will be useful for the following examples.

Lemma 4.4 (See [**Deb01**, 6.15])**.** *Let Y be a smooth projective variety, let Z be a smooth subvariety of Y of codimension c and let $\pi : X \to Y$ be the blow up of Y in Z with exceptional divisor E. Then we have*

(i) $K_X = \pi^*(K_Y) + (c-1)E$,
(ii) any fibre F of $\pi|_E : E \to Z$ is isomorphic to $\mathbb{P}^{c-1}$ and $\mathcal{O}_X(E)|_F$ is isomorphic to $\mathcal{O}_F(-1)$,
(iii) if η is the class of a line contained in F, then $K_X \,.\, \eta = -(c-1)$ and η spans a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$ whose extremal contraction is π. □

Example 4.5. Let $\mu : X \to \mathbb{P}^n$, $n \geq 3$, be the blow up of $\mathbb{P}^n$ in a line l. The exceptional divisor E of the blow up is a $\mathbb{P}^{n-2}$-bundle $\mu|_E : E \to l$ over l. Choosing local coordinates, one computes easily that the normal bundle $N_{l/X}$ of l in X is isomorphic to $\mathcal{O}_l(1)^{\oplus(n-1)}$. Thus $E = \mathbb{P}(N_{l/X})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$.

The variety X has Picard number $\rho(X) = 2$ and Lemma 4.4 yields that

$$[K_X] = -(n+1)\Gamma + (n-2)[E],$$

where $\Gamma := [\mu^*(H)]$ is the class of the pullback of a hyperplane H in $\mathbb{P}^n$.

Denote by $\gamma := \mu_1^*([g])$ the numerical pullback of the class of a line g in $\mathbb{P}^n$ and let $\eta := [f]$ be the class of a line f which is contained in a fibre F of $\mu|_E$.

The Néron-Severi spaces are spanned by these classes:

$$N^1(X)_{\mathbb{R}} = \langle \Gamma, [E] \rangle_{\mathbb{R}} \text{ and } N_1(X)_{\mathbb{R}} = \langle \gamma, \eta \rangle_{\mathbb{R}}.$$

The intersection numbers on X are given in the following table.

.	γ	η
Γ	1	0
$[E]$	0	-1

Since $\rho(X) = 2$, the cones $\overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{NM}}(X)$ are both spanned by two extremal rays.

We will show that $\overline{\mathrm{NM}}(X) = \langle \gamma, (\gamma-\eta) \rangle_{\mathbb{R}_+}$ and that $\overline{\mathrm{NE}}(X) = \langle (\gamma-\eta), \eta \rangle_{\mathbb{R}_+}$.

Let us start with $\overline{\mathrm{NM}}(X)$ and let $\nu = s\gamma + t\eta$, $s, t \in \mathbb{R}$, be a movable class. The class of the strict transform $\bar{H}$ of a hyperplane H' in $\mathbb{P}^n$ which contains the line l is given by $[\bar{H}] = \Gamma - [E]$.

Theorem 1.12 yields that $0 \leq [\bar{H}] \,.\, \nu = s + t$, $0 \leq [E] \,.\, \nu = -t$. Hence $\overline{\mathrm{NM}}(X) \subset \langle \gamma, (\gamma - \eta) \rangle_{\mathbb{R}_+}$. The class $\gamma - \eta$ is given by the strict transform of a line in $\mathbb{P}^n$ which meets the line l in one point; thus $\gamma - \eta$ is movable. We obtain that $\overline{\mathrm{NM}}(X) = \langle \gamma, (\gamma - \eta) \rangle_{\mathbb{R}_+}$.

Now let λ be the class of an irreducible curve c in X. We have $\lambda = a\gamma + b\eta$ for suitable $a, b \in \mathbb{R}$. Since $\mu^*(H)$ is nef, we have $a = \Gamma . \lambda \geq 0$. If c does not meet the exceptional divisor E, then clearly $\lambda \sim_{\text{num}} \gamma$ and $b = 0$.

Now assume that c meets E and that c is not contained E. Let $\widetilde{H}$ be a hyperplane in $\mathbb{P}^n$ which contains l and meets $\mu_*(c)$ properly. This is possible since l is the complete intersection of $n-1$ hyperplanes in $\mathbb{P}^n$ and $\mu_*(c) \neq l$. The class of the strict transform of $\widetilde{H}$ under μ is given by $\Gamma - [E]$. Note that the curve c is not contained in the strict transform of $\widetilde{H}$. Therefore,

$$0 \leq \lambda . (\Gamma - [E]) = (a\gamma + b\eta) . (\Gamma - [E]) = a + b \Rightarrow -b \leq a.$$

Finally, assume that c is contained in E. If c is contained in a fibre F of $\mu|_E$, then Lemma 4.4 yields that $\lambda \sim_{\text{num}} \eta$.

Otherwise c is numerically proportional to a line l' in $E \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ which is not contracted by μ.

Let H'' be a hyperplane in $\mathbb{P}^n$ which meets the line l transversaly in one point. The pullback of H'' intersects E in the fibre of $\mu|_E$ which lies over the intersection point of H'' and l. Therefore, $[l'] . \Gamma = 1$ since l' intersects each fibre F of $\mu|_E$ transversaly in one point.

The only thing left is to compute the number $l' . E$. A short computation in local coordinates shows that the restriction of the line bundle $\mathcal{O}_X(E)$ to l' is the trivial bundle on l'. Hence $l' . E = \deg(\mathcal{O}_X(E)|_{l'}) = 0$ and $[l'] = \gamma$.

All this implies that

$$\overline{\text{NE}}(X) = \langle (\gamma - \eta), \eta \rangle_{\mathbb{R}_+}.$$

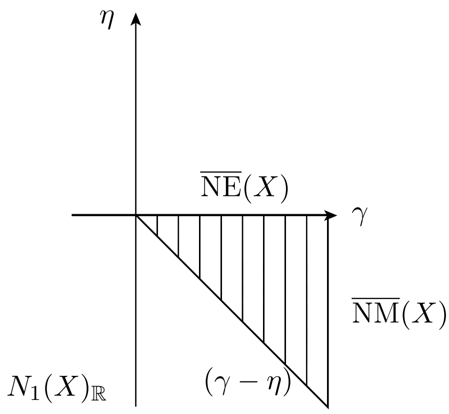


FIGURE 2. $\overline{\text{NE}}(X)$ is given by the area inbetween η and $\gamma - \eta$. The hatched area is a sketch of $\overline{\text{NM}}(X)$.

Construction 4.6 (See [**Deb01**, 6.19])**.** Let $g'' \subset \mathbb{P}^4$ be a smooth line and let $S'' \subset \mathbb{P}^4$ be a smooth surface in $\mathbb{P}^4$ such that g'' meets S'' transversaly in a point $p = g'' \cap S''$. Let

$$\mu_1 : X' \to \mathbb{P}^4$$

be the blow up of $\mathbb{P}^4$ in g'' with exceptional divisor E_1'.

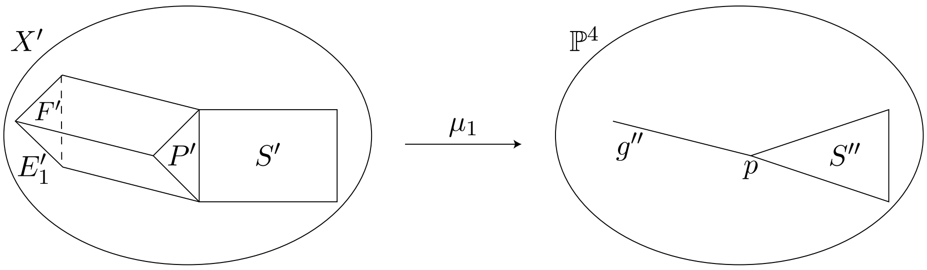


FIGURE 3. Sketch of the blow up $\mu_1 : X' \to \mathbb{P}^4$ of $\mathbb{P}^4$ in the line g''.

By the universal property of the blow up, the strict transform S' of S'' is isomorphic to the blow up $\mathrm{Bl}_p(S'')$ of S'' in the point p. The divisor E_1' is a $\mathbb{P}^2$-bundle $\mu_1|_{E_1'} : E_1' \to g''$ over g''. Denote by $P' \subset E_1'$ the fibre of $\mu_1|'_{E_1}$ over the point p and let e_1' be a line in a fibre $F' \neq P'$ of $\mu_1|_{E_1'}$. Now let

$$\mu_2 : X \to X'$$

be the blow up of X' in S'.

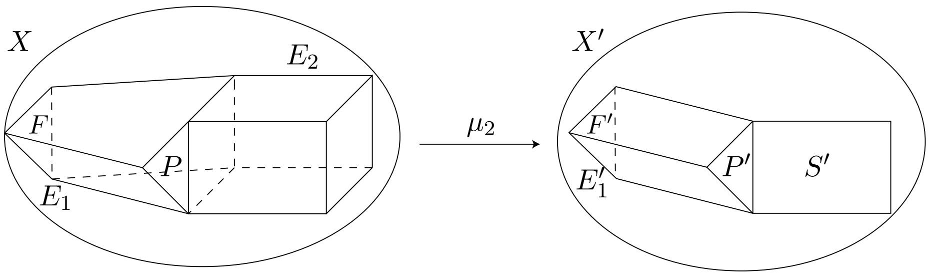


FIGURE 4. Sketch of the blow up $\mu_2 : X \to X'$ of X' in the surface S'.

The exceptional divisor E_2 of the blow up is a $\mathbb{P}^1$-bundle $\mu_2|_{E_2} : E_2 \to S'$ over S'. We fix the following notation. Denote by P the strict transform of P', by $E_1 := \mu_2^*(E_1')$ the pullback of E_1' and by $\Gamma := [\mu_2^*(\mu_1^*(H''))]$ the class of the pullback of the hyperplane divisor H'' of $\mathbb{P}^4$.

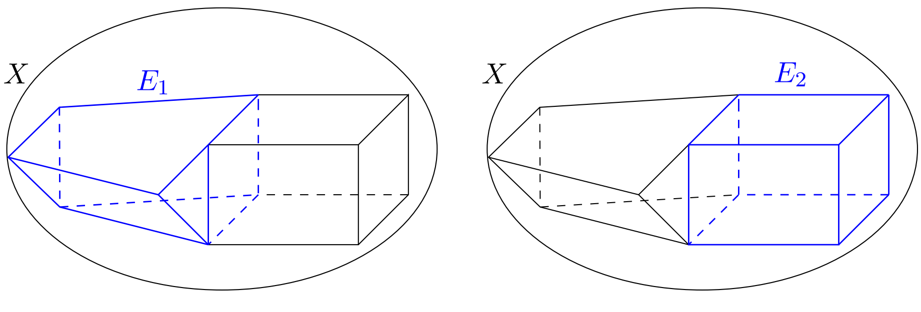


FIGURE 5. Sketch of the classes $[E_1]$ and $[E_2]$.

Fact: Note that P is isomorphic to $\mathbb{P}^2$ and that $N_{P/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, where $N_{P/X}$ denotes the normal bundle of P in X.

Furthermore, let $\gamma := (\mu_2)_1^*((\mu_1)_1^*([l'']))$ be the numerical pullback of the class of a line l'' in $\mathbb{P}^4$, let $\eta_1 := (\mu_2)_1^*([e_1'])$ be the numerical pullback of the class of e_1' and let η_2 be the class of a fibre of $\mu_2|_{E_2}$.

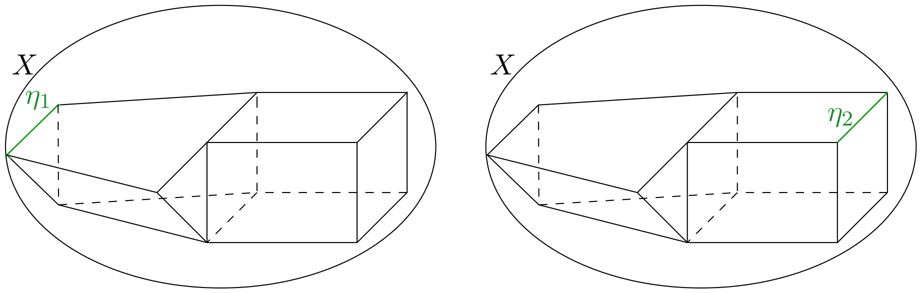


FIGURE 6. Sketch of the classes η_1 and η_2.

The variety X has Picard number $\rho(X) = 3$ and $N^1(X)_{\mathbb{R}} = \langle \Gamma, [E_1], [E_2] \rangle_{\mathbb{R}}$, $N_1(X)_{\mathbb{R}} = \langle \gamma, \eta_1, \eta_2 \rangle_{\mathbb{R}}$.

By Lemma 4.4, we have

$$[K_X] = -5\Gamma + 2[E_1] + [E_2].$$

The intersection numbers on X are given in the following table.

.	Γ	$[E_1]$	$[E_2]$
γ	1	0	0
η_1	0	-1	0
η_2	0	0	-1

Construction 4.7. Let X be as in Construction 4.6 and let

$$\mu_3 : \widetilde{X} \to X$$

be the blow up of X in P with exceptional divisor $\widetilde{E}_3$.

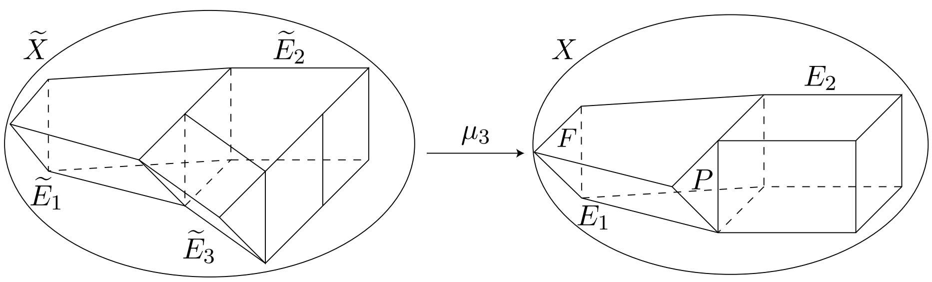


FIGURE 7. Sketch of the blow up $\mu_3 : \widetilde{X} \to X$ of X in the surface P.

The exceptional divisor $\widetilde{E}_3$ is a $\mathbb{P}^1$-bundle $\mu_3|_{\widetilde{E}_3} : \widetilde{E}_3 \to P$ over P and actually isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$. Moreover, the normal bundle $N_{\widetilde{E}_3/\widetilde{X}}$ of $\widetilde{E}_3$ in $\widetilde{X}$ is isomorphic to $pr_1^*(\mathcal{O}_{\mathbb{P}^2}(-1)) \oplus pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1))$, where pr_1 and pr_2 denote the projections onto the first and the second factor of $\mathbb{P}^2 \times \mathbb{P}^1$, respectively.

Denote by $\widetilde{\Gamma} := \mu_3^*(\Gamma)$, $\widetilde{E}_1 := \mu_3^*(E_1)$ and $\widetilde{E}_2 := \mu^*(E_2)$ the pullbacks of Γ, E_1 and E_2, respectively.

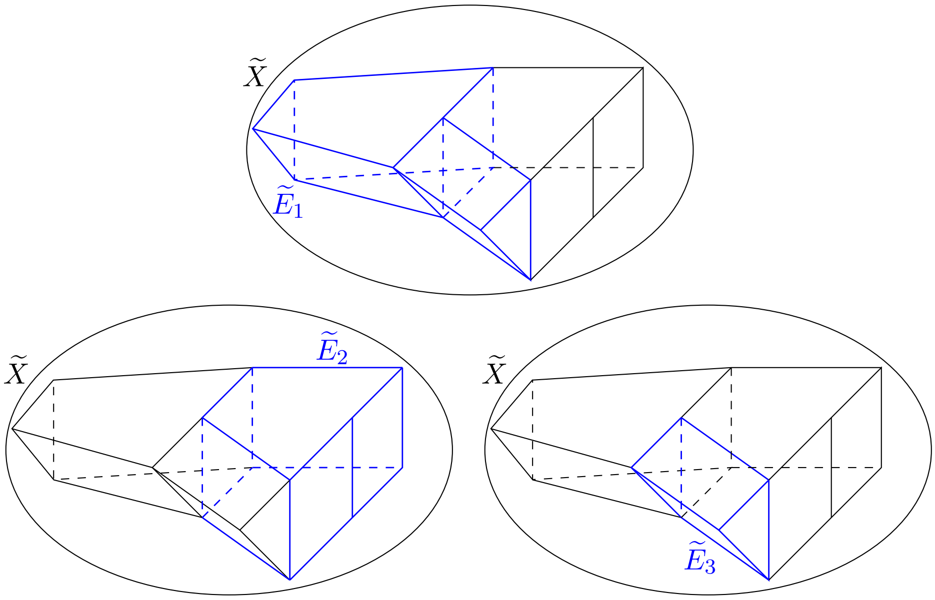


FIGURE 8. Sketch of the classes $[\widetilde{E}_1]$, $[\widetilde{E}_2]$ and $[\widetilde{E}_3]$.

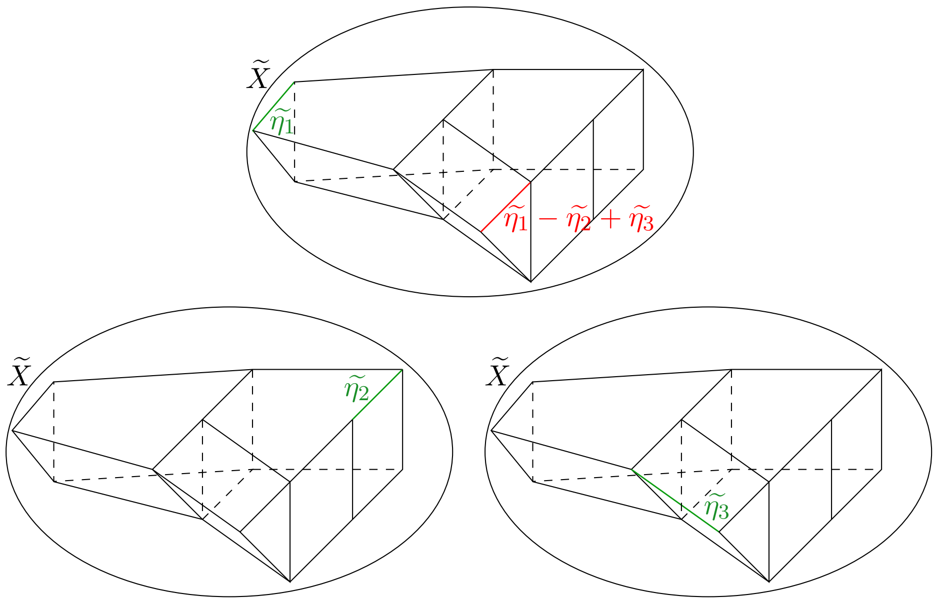


FIGURE 9. Sketch of the classes $\widetilde{\eta}_1$, $\widetilde{\eta}_2$, $\widetilde{\eta}_3$ and $\widetilde{\eta}_1 - \widetilde{\eta}_2 + \widetilde{\eta}_3$.

Besides, let $\widetilde{\gamma} := (\mu_3)_1^*(\gamma)$, $\widetilde{\eta_1} := (\mu_3)_1^*(\eta_1)$ and $\widetilde{\eta_2} := (\mu_3)_1^*(\eta_2)$ be the numerical pullbacks of γ, η_1 and η_2, respectively. Finally, let $\widetilde{\eta_3}$ be the class of a fibre of $\mu_3|_{\widetilde{E}_3}$.

The variety $\widetilde{X}$ has Picard number $\rho(\widetilde{X}) = 4$ and

$$N^1(\widetilde{X})_{\mathbb{R}} = \langle \widetilde{\Gamma}, [\widetilde{E}_1], [\widetilde{E}_2], [\widetilde{E}_3] \rangle_{\mathbb{R}}, \quad N_1(\widetilde{X})_{\mathbb{R}} = \langle \widetilde{\gamma}, \widetilde{\eta_1}, \widetilde{\eta_2}, \widetilde{\eta_3} \rangle_{\mathbb{R}}.$$

Lemma 4.4 yields that

$$[K_{\widetilde{X}}] = -5\widetilde{\Gamma} + 2[\widetilde{E}_1] + [\widetilde{E}_2] + [\widetilde{E}_3].$$

The intersection numbers on $\widetilde{X}$ are given in the following table.

.	$\widetilde{\Gamma}$	$[\widetilde{E}_1]$	$[\widetilde{E}_2]$	$[\widetilde{E}_3]$
$\widetilde{\gamma}$	1	0	0	0
$\widetilde{\eta_1}$	0	-1	0	0
$\widetilde{\eta_2}$	0	0	-1	0
$\widetilde{\eta_3}$	0	0	0	-1

In the following examples we will compute the cones

$$\overline{\mathrm{NE}}(X),\ \overline{\mathrm{NE}}(\widetilde{X}),\ \overline{\mathrm{Eff}}(X),\ \overline{\mathrm{Eff}}(\widetilde{X}),\ \overline{\mathrm{NM}}(X) \text{ and } \overline{\mathrm{NM}}(\widetilde{X}).$$

Example 4.8 (See Construction 4.6 and 4.7)**.** We will show that

$$\overline{\mathrm{NE}}(X) = \langle (\gamma - \eta_1 - \eta_2), (\eta_1 - \eta_2), \eta_2 \rangle_{\mathbb{R}_+} \text{ and that}$$

$$\overline{\mathrm{NE}}(\widetilde{X}) = \langle (\widetilde{\gamma} - \widetilde{\eta_1} - \widetilde{\eta_2}), (\widetilde{\eta_1} - \widetilde{\eta_2} + \widetilde{\eta_3}), (\widetilde{\eta_2} - \widetilde{\eta_3}), \widetilde{\eta_3} \rangle_{\mathbb{R}_+} \text{ first.}$$

Let c be an irreducible curve in X which is not contained in E_1 or E_2. Furthermore, let $\widetilde{c}$ be an irreducible curve in $\widetilde{X}$ which is not contained in $\widetilde{E}_1, \widetilde{E}_2$ or $\widetilde{E}_3$.

We set $\alpha := \Gamma\,.[c]$, $\widetilde{\alpha} := \widetilde{\Gamma}\,.[\widetilde{c}]$, $\beta := [E_1]\,.[c]$, $\widetilde{\beta} := [\widetilde{E}_1]\,.[\widetilde{c}]$, $\gamma := [E_2]\,.[c]$, $\widetilde{\gamma} := [\widetilde{E}_2]\,.[\widetilde{c}]$ and $\widetilde{\delta} := [\widetilde{E}_3]\,.[\widetilde{c}]$.

A short computation shows that

$$[c] = \alpha(\gamma - \eta_1 - \eta_2) + (\alpha - \beta)(\eta_1 - \eta_2) + (2\alpha - \beta - \gamma)\eta_2, \tag{4.b}$$

$$\begin{aligned}[\widetilde{c}] = {} & \widetilde{\alpha}(\widetilde{\gamma} - \widetilde{\eta_1} - \widetilde{\eta_2}) + (\widetilde{\alpha} - \widetilde{\beta})(\widetilde{\eta_1} - \widetilde{\eta_2} + \widetilde{\eta_3}) \\ & + (2\widetilde{\alpha} - \widetilde{\beta} - \widetilde{\gamma})(\widetilde{\eta_2} - \widetilde{\eta_3}) + (\widetilde{\alpha} - \widetilde{\gamma} - \widetilde{\delta})\widetilde{\eta_3}.\end{aligned} \tag{4.c}$$

To prove our claim, we have to show that all coefficients in (4.b) and (4.c) are non-negative and that all classes in (4.b) and (4.c) are represented by rational curves. Then we will investigate curves which are contained in one of the exceptional divisors.

The intersection numbers α and $\widetilde{\alpha}$ are non-negative since Γ and $\widetilde{\Gamma}$ are nef. The other intersection numbers are non-negative since c is not contained in any E_i and $\widetilde{c}$ is not contained in any $\widetilde{E}_i$ by assumption. Denote by c'' the image $(\mu_1)_*((\mu_2)_*(c))$ of c in $\mathbb{P}^4$. The line g'' is the complete intersection of three hyperplanes in $\mathbb{P}^4$. Since $c'' \neq g''$, there exists a hyperplane $\hat{H}$ containing g'' which meets c'' properly. Note that the curve c is not contained in the strict transform of $\hat{H}$ under $\mu_1 \circ \mu_2$, which has the class $\Gamma - [E_1]$. Thus

$$0 \leq [c]\,.(\Gamma - [E_1]) = \alpha - \beta \Rightarrow \beta \leq \alpha.$$

An analogous argumentation shows that $\alpha \geq \gamma$ and that $\widetilde{\alpha} \geq \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\gamma} + \widetilde{\delta}$. Hence all the coefficients in (4.b) and (4.c) are non-negative.

Now assume that c is contained in E_1. If c is contracted by $\mu_1 \circ \mu_2$, then $c' := (\mu_2)_*(c)$ is contained in a fibre F of $\mu_1|_{E_1}$. If $F \neq P'$ then $c \sim_{\text{num}} \eta_1$, otherwise $c \sim_{\text{num}} (\eta_1 - \eta_2)$.

If c is not contracted by $\mu_1 \circ \mu_2$, then c' is an effective curve in the blow up of $\mathbb{P}^4$ in the line g''. Therefore, $[c'] = \alpha'(\mu_1)_1^*([l'']) - \beta'[e_1']$ with $\alpha' \geq \beta'$, by Example 4.5.

This implies that $[c] = \alpha'\gamma - \beta'\eta_1 - \gamma'\eta_2$ for some $\gamma' \in \mathbb{R}$. Since c is contained in E_1, it is not contained in E_2. Now we take a hyperplane $\bar{H}$ in $\mathbb{P}^4$ which containes S'' and intersects $c'' := (\mu_1)_*((\mu_2)_*(c))$ properly. Then the curve c is not contained in the strict transform of $\bar{H}$ under $\mu_1 \circ \mu_2$, which has class $\Gamma - [E_2]$. Thus

$$0 \leq [c] . (\Gamma - [E_2]) = (\alpha'\gamma - \beta'\eta_1 - \gamma'\eta_2) . (\Gamma - [E_2]) = \alpha' - \gamma' \Rightarrow \gamma' \leq \alpha'.$$

If c is contained in E_2, then exactly the same argumentation shows that $[c'] = \alpha'\gamma - \beta'\eta_1 - \gamma'\eta_2$, where $\alpha' \geq \beta', \gamma'$.

Finally, the class $\gamma - \eta_1 - \eta_2$ is represented by the strict transform of a line in $\mathbb{P}^4$ which meets g'' in a point p' and S'' in a point p'' such that $p \neq p'' \neq p' \neq p$. The class $\eta_1 - \eta_2$ is represented by a line in P. In particular, the extremal contraction, φ say, of the ray $\mathbb{R}_+(\eta_1 - \eta_2)$ is small.

A completely analogous argumentation applies to the computation of $\overline{\text{NE}}(\widetilde{X})$. It remains to compute the class of a curve $\widetilde{c}$ which is contained in $\widetilde{E}_3$. If $\widetilde{c}$ is contracted by μ_3, then $\widetilde{c} \sim_{\text{num}} \widetilde{\eta}_3$. Otherwise, $(\mu_3)_*(\widetilde{c})$ is a curve in P and hence $[\widetilde{c}] = \alpha''\widetilde{\eta}_1 - \alpha''\widetilde{\eta}_2 + \beta''\widetilde{\eta}_3$ for a suitable $\beta'' \in \mathbb{R}$ and $\alpha'' \geq 0$. The strict transform of E_1 under μ_3 is given by the class $\widetilde{E}_1 - \widetilde{E}_3$. We can assume that $\widetilde{c}$ is not contained in $\widetilde{E}_1 - \widetilde{E}_3$ and hence $\widetilde{c} . (\widetilde{E}_1 - \widetilde{E}_3) \geq 0$. This implies that

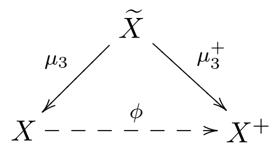
$$0 \leq \widetilde{c} . (\widetilde{E}_1 - \widetilde{E}_3) = -\alpha'' + \beta'' \Rightarrow \beta'' \geq \alpha'' \geq 0.$$

The class $(\widetilde{\gamma} - \widetilde{\eta}_1 - \widetilde{\eta}_2)$ is given by the strict transform of a curve with class $(\gamma - \eta_1 - \eta_2) \in \overline{\text{NE}}(X)$.

The class $(\widetilde{\eta}_2 - \widetilde{\eta}_3)$ is represented by a line $\widetilde{e}_3$ in $\widetilde{E}_2$ such that $\mu_3(\widetilde{e}_3)$ is a fibre of μ_2 which lies over a point in P'.

Denote by π_1 the projection $\widetilde{E}_3 \xrightarrow{\sim} \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^2$ and by π_2 the projection $\widetilde{E}_3 \xrightarrow{\sim} \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{pr_2} \mathbb{P}^1$. The class $(\widetilde{\eta}_1 - \widetilde{\eta}_2 + \widetilde{\eta}_3)$ is given by a line in $\widetilde{E}_3$ which is mapped to a line in $\mathbb{P}^2$ by π_1 and which is mapped to a point by π_2.

The extremal contraction μ_3^+ of the ray $\mathbb{R}_+(\widetilde{\eta}_1 - \widetilde{\eta}_2 + \widetilde{\eta}_3)$ is divisorial with exceptional divisor $\widetilde{E}_3$. Therefore, we have a commutative diagram



and the birational map $\phi = \mu_3^+ \circ \mu_3^{-1}$ is the flip of the small contraction φ.

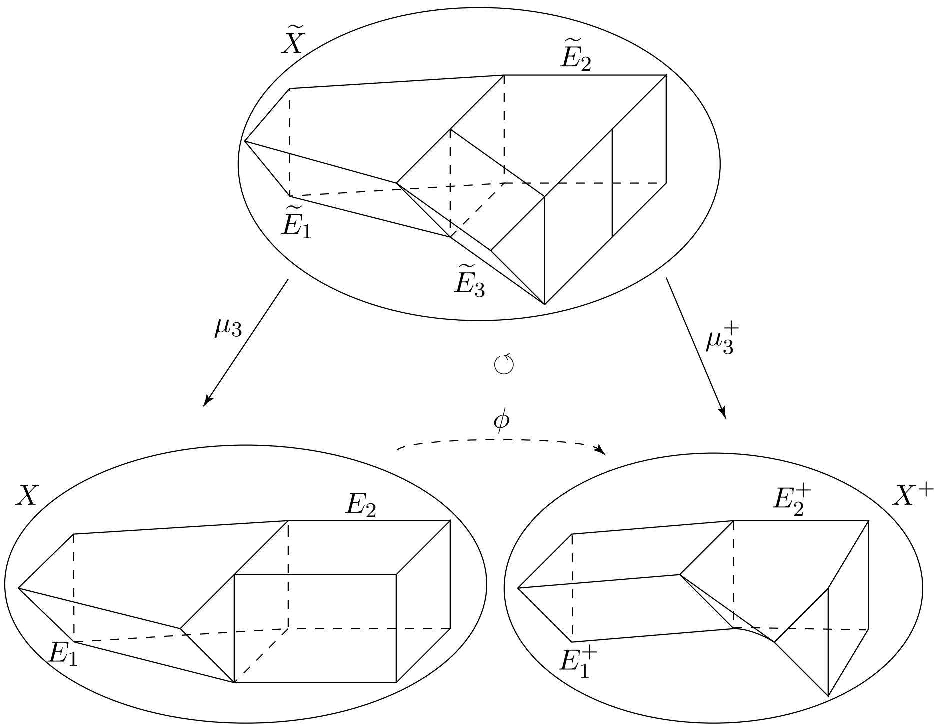


FIGURE 10. Sketch of the flip ϕ.

Let us fix some notation. We set $\Gamma^+ := (\mu_3^+)_*(\widetilde{\Gamma})$, $E_1^+ := (\mu_3^+)_*(\widetilde{E}_1)$, $E_2^+ := (\mu_3^+)_*(\widetilde{E}_2)$, $\gamma^+ := (\mu_3^+)_{*1}(\widetilde{\gamma})$, $\eta_1^+ := (\mu_3^+)_{*1}(\widetilde{\eta}_1)$, $\eta_2^+ := (\mu_3^+)_{*1}(\widetilde{\eta}_2)$ and $\eta_3^+ := (\mu_3^+)_{*1}(\widetilde{\eta}_3)$. Since the class $\widetilde{\eta}_1 - \widetilde{\eta}_2 + \widetilde{\eta}_3$ is contracted by μ_3^+, we find the relation

$$\eta_3^+ = \eta_2^+ - \eta_1^+.$$

The variety X^+ has Picard number $\rho(X) = 3$ and

$$N^1(X^+)_{\mathbb{R}} = \langle \Gamma^+, [E_1^+], [E_2^+] \rangle_{\mathbb{R}}, \quad N_1(X)_{\mathbb{R}} = \langle \gamma^+, \eta_1^+, \eta_2^+ \rangle_{\mathbb{R}}.$$

By Lemma 4.4 we have

$$[K_{X^+}] = -5\Gamma^+ + 2[E_1^+] + [E_2^+].$$

The intersection numbers on X are given in the following table.

.	Γ^+	$[E_1^+]$	$[E_2^+]$
γ^+	1	0	0
η_1^+	0	-1	0
η_2^+	0	0	-1

The surjectivity of $(\mu_3^+)_{*1}$ yields that

$$\overline{\mathrm{NE}}(X^+) = \langle (\gamma^+ - \eta_1^+ - \eta_2^+), \eta_1^+, (\eta_2^+ - \eta_1^+) \rangle_{\mathbb{R}_+}$$

and a short computation shows that

$$\phi_{*1}(\gamma) = \gamma^+, \phi_{*1}(\eta_1) = \eta_1^+, \phi_{*1}(\eta_2) = \eta_2^+.$$

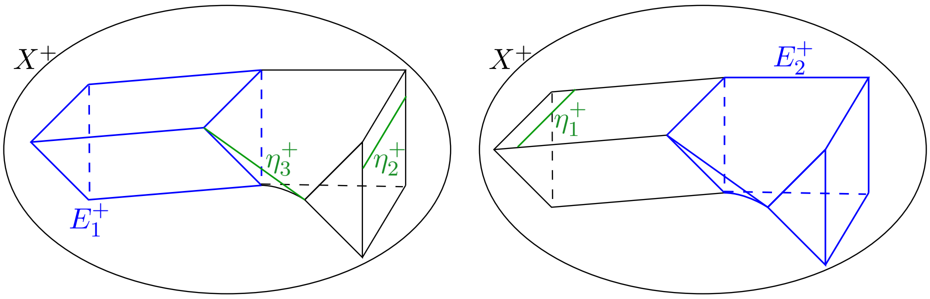


FIGURE 11. Sketch of the classes E_1^+, E_2^+, η_1^+, η_2^+ and $\eta_3^+ = \eta_2^+ - \eta_1^+$.

The class $\eta_2^+ - \eta_1^+$ is K_{X^+}-positive. The other extremal rays of $\overline{\mathrm{NE}}(X^+)$ are K_{X^+}-negative.

Example 4.9 (See Construction 4.6 and 4.7)**.** We will show that

$$\overline{\mathrm{Eff}}(X) = \langle(\Gamma - [E_1] - [E_2]), [E_1], [E_2]\rangle_{\mathbb{R}_+} \text{ and that}$$
$$\overline{\mathrm{Eff}}(\widetilde{X}) = \langle(\widetilde{\Gamma} - [\widetilde{E_1}] - [\widetilde{E_2}] - [\widetilde{E_3}]), ([\widetilde{E_1}] - [\widetilde{E_3}]), [\widetilde{E_2}], [\widetilde{E_3}]\rangle_{\mathbb{R}_+}.$$

With this knowledge Theorem 1.12 enables us to compute

$$\overline{\mathrm{NM}}(X) = \langle\gamma, (\gamma - \eta_1), (\gamma - \eta_2)\rangle_{\mathbb{R}_+} \text{ and}$$
$$\overline{\mathrm{NM}}(\widetilde{X}) = \langle\widetilde{\gamma}, (\widetilde{\gamma} - \widetilde{\eta}_1), (\widetilde{\gamma} - \widetilde{\eta}_2), (2\widetilde{\gamma} - \widetilde{\eta}_1 - \widetilde{\eta}_3)\rangle_{\mathbb{R}_+}.$$

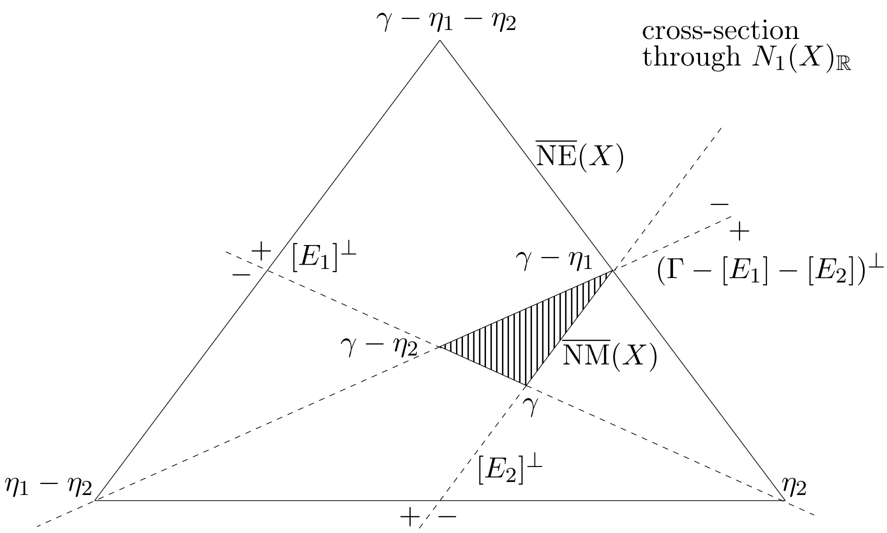


FIGURE 12. The hatched area is a sketch of a cross-section of $\overline{\mathrm{NM}}(X)$ inside $\overline{\mathrm{NE}}(X)$.

Thus Proposition 3.14 and the previous example show that

$$\overline{\mathrm{NM}}(X^+) = \langle\gamma^+, (\gamma^+ - \eta_1^+), (\gamma^+ - \eta_2^+)\rangle_{\mathbb{R}_+}.$$

Note that a cross-section through $\overline{\mathrm{NE}}(X^+)$ looks very similar to Figure 12, but the hyperplanes $[E_1^+]^\perp$ and $[E_2^+]^\perp$ are swapped.

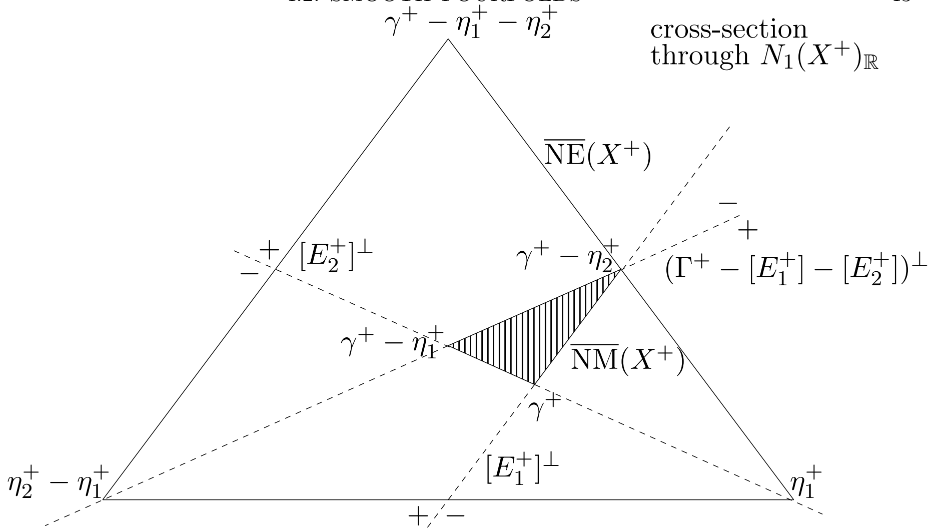


FIGURE 13. The hatched area is a sketch of a cross-section of $\overline{\mathrm{NM}}(X^+)$ inside $\overline{\mathrm{NE}}(X^+)$.

We will now start with the computation of $\overline{\mathrm{Eff}}(X)$ and $\overline{\mathrm{Eff}}(\widetilde{X})$.

Let $D \neq E_1, E_2$ be an irreducible divisor on X. We have

$$[D] = \alpha\Gamma - \beta[E_1] - \gamma[E_2]$$

with $\alpha := [D]\,.\,\gamma$, $\beta := [D]\,.\,\eta_1$, $\gamma := [D]\,.\,\eta_2 \in \mathbb{R}_+$. We have seen in Example 4.5 that the class $[(\mu_1)_*(l'')] - [e_1'] \in N_1(X')_\mathbb{R}$ is movable. Therefore, Theorem 3.13 gives

$$\alpha - \beta = (\gamma - \eta_1)\,.\,[D] = ([(\mu_1)_*(l'')] - [e_1'])\,.\,[\mu_{2*}(D)] \geq 0$$

since $\mu_{2*}(D)$ is effective. The class $\gamma - \eta_2$ is given by the strict transform $\hat{l}$ of a curve in $\mathbb{P}^4$ which meets S'' in one point and which does not meet g''. It is impossible that all of these strict transforms are contained in the support of D. Hence

$$\alpha - \gamma = (\gamma - \eta_2)\,.\,[D] = [\hat{l}]\,.\,[D] \geq 0.$$

A short computation shows that the class $\Gamma - [E_1] - [E_2]$ is given by the strict transform of a hyperplane in $\mathbb{P}^4$ which contains g'' and S''. The class $[E_1]$ is given by the strict transform of E_1', hence by an irreducible divisor.

All this together yields that

$$\overline{\mathrm{Eff}}(X) = \langle(\Gamma - [E_1] - [E_2]), [E_1], [E_2]\rangle_{\mathbb{R}_+}.$$

Now let $\widetilde{D}$ be an irreducible divisor on $\widetilde{X}$ which is not equal to $(\widetilde{E_1} - \widetilde{E_3}), \widetilde{E_2}$ or $\widetilde{E_3}$. The divisor $(\mu_3)_*(\widetilde{D})$ is again irreducible and hence

$$[(\mu_3)_*(\widetilde{D})] = \alpha'\Gamma - \beta'[E_1] - \gamma'[E_2]$$

for suitable α', β', $\gamma' \in \mathbb{R}_+$ with $\alpha' \geq \beta', \gamma'$. Therefore,

$$[\widetilde{D}] = (\mu_3)^*((\mu_3)_*(\widetilde{D})) - \delta'[E_3] = \alpha'\Gamma - \beta'[E_1] - \gamma'[E_2] - \delta'[E_3],$$

for a suitable $\delta' \in \mathbb{R}_+$. The class $\widetilde{\eta_2} - \widetilde{\eta_3}$ is given by a line $\widetilde{e_3}$ in $\widetilde{E_2}$ such that $(\mu_3)_*(\widetilde{e_3})$ is a fibre of μ_2 over a point in P'. Since $\widetilde{D} \neq \widetilde{E_2}$, we find that

$$\gamma' - \delta' = (\widetilde{\eta_2} - \widetilde{\eta_3})\,.\,[\widetilde{D}] = [\widetilde{e_3}]\,.\,[\widetilde{D}] \geq 0.$$

Hence $\alpha' \geq \gamma' \geq \delta'$. A short computation shows that the class $\widetilde{\Gamma} - [\widetilde{E_1}] - [\widetilde{E_2}] - [\widetilde{E_3}]$ is given by the strict transform of a hyperplane in $\mathbb{P}^4$ which contains g'' and S''. The class $[\widetilde{E_1}] - [\widetilde{E_3}]$ is given by the strict transform of E_1 and the class $[\widetilde{E_2}]$ is given by the strict transform of E_2.

Combining these results, we achieve

$$\overline{\mathrm{Eff}}(\widetilde{X}) = \langle(\widetilde{\Gamma} - [\widetilde{E_1}] - [\widetilde{E_2}] - [\widetilde{E_3}]), ([\widetilde{E_1}] - [\widetilde{E_3}]), [\widetilde{E_2}], [\widetilde{E_3}]\rangle_{\mathbb{R}_+}.$$

Construction 4.10 (see [**Deb01**, 1.36])**.** Let $k, j > 0$ be positive integers and let $\mathscr{E}_{k,j}$ be the vector bundle $\mathcal{O}_{\mathbb{P}^k} \oplus (\mathcal{O}_{\mathbb{P}^k}(1)^{\oplus(j+1)})$ over $\mathbb{P}^k$. The variety

$$Y_{k,j} := \mathbb{P}(\mathscr{E}_{k,j})$$

is smooth and $\dim_{\mathbb{C}}(Y_{k,j}) = k + j + 1$. Here are some more facts about $Y_{k,j}$:

(i) $\rho(Y_{k,j}) = \dim_{\mathbb{R}} N^1(Y_{k,j})_{\mathbb{R}} = 2$.
(ii) The projection map $\pi_{k,j} : Y_{k,j} \to \mathbb{P}^k$ has a section corresponding to the quotient $\mathscr{E} \to \mathcal{O}_{\mathbb{P}^k} \to 0$. Let $S_{k,j} \subset Y_{k,j}$ be the image of this section.
(iii) The normal bundle $\mathrm{N}_{S_{k,j}/Y_{k,j}}$ of $S_{k,j}$ in $Y_{k,j}$ is isomorphic to the vector bundle $\mathcal{O}_{S_{k,j}}(-1)^{\oplus(j+1)}$.

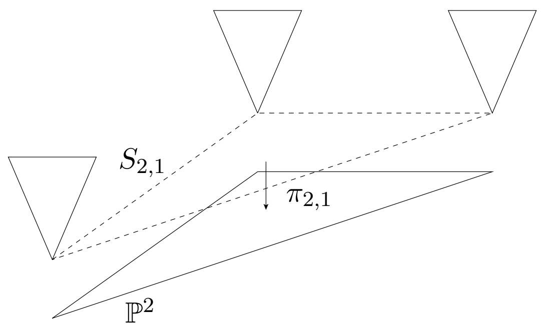


FIGURE 14. Sketch of the variety $Y_{2,1}$. Fibres of $\pi_{2,1}$ are isomorphic to $\mathbb{P}^2$.

The Néron-Severi space of divisors $N^1(Y_{k,j})_{\mathbb{R}}$ is generated by the class Λ' of the line bundle $\mathcal{O}_{Y_{k,j}}(1)$ and the class Γ' of the inverse image of the hyper-

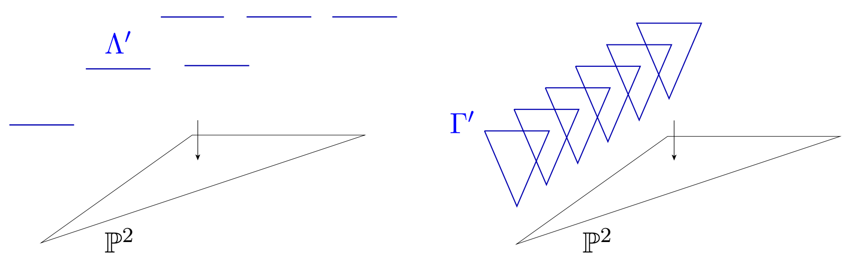


FIGURE 15. Sketch of the divisor classes Λ' and Γ' in $Y_{2,1}$.

plane class in $\mathbb{P}^k$. According to [**Deb01**, Example 3.16(2)], the class of the canonical divisor is given by

$$[K_{Y_{k,j}}] = -(j + 2)\Lambda' + (j - k)\Gamma'.$$

Now let

$$\mu_{k,j} : X_{k,j} \to Y_{k,j}$$

be the blow up of $S_{k,j}$ and let E be the exceptional divisor of the blow up. We have

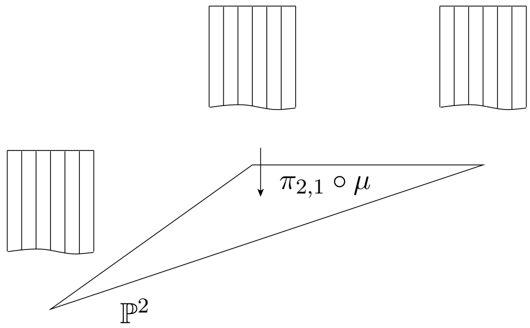


FIGURE 16. Sketch of the variety $X_{2,1}$. Fibres of $\pi_{2,1}$ are isomorphic to the blow up of $\mathbb{P}^2$ in one point.

$$E \cong \mathbb{P}^j \times \mathbb{P}^k \text{ and } \mathcal{O}_E(E) \cong pr_1^*(\mathcal{O}_{\mathbb{P}^j}(-1)) \otimes pr_2^*(\mathcal{O}_{\mathbb{P}^k}(-1))$$

since $\mathrm{N}_{S_{k,j}/Y_{k,j}} \cong \mathcal{O}_{S_{k,j}}(-1)^{\oplus(j+1)}$. The Néron-Severi space of divisors $N^1(X_{k,j})_{\mathbb{R}}$ has dimension three and is generated by $\Lambda := \mu^*(\Lambda')$, $\Gamma := \mu^*(\Gamma')$ and $[E]$.

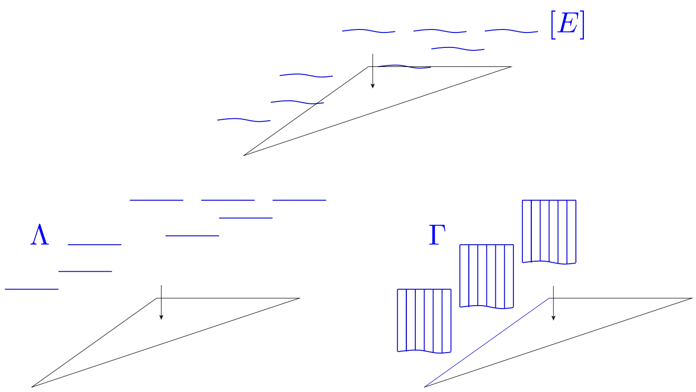


FIGURE 17. Sketch of the divisor classes Λ, Γ and $[E]$ in $X_{2,1}$.

Following [**Deb01**, Example 3.16(2)], the class of the canonical divisor is given by

$$[K_{X_{k,j}}] = -(j+2)\Lambda + (j-k)\Gamma + j[E].$$

Example 4.11 (see Construction 4.10)**.** We want to compute $\overline{\mathrm{NM}}(Y_{k,j})$ and $\overline{\mathrm{NM}}(X_{k,j})$ by usage of Theorem 4.1 if possible. Therefore, we we will compute $\overline{\mathrm{NE}}(Y_{k,j})$ and $\overline{\mathrm{NE}}(X_{k,j})$ first. Let us start with $\overline{\mathrm{NE}}(Y_{k,j})$.

Let λ' be the class of a line in a fibre of $\pi_{k,j} : Y_{k,j} \to \mathbb{P}^k$ and let γ' be the class of a line which is contained in $S_{k,j}$. The Néron-Severi space of cycles

$N_1(Y_{k,j})_{\mathbb{R}}$ is spanned by these two classes and the intersection numbers on $Y_{k,j}$ are given in the following table.

.	γ'	λ'
Γ'	1	0
Λ'	0	1

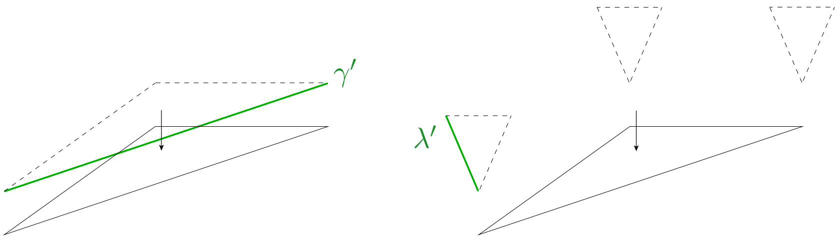


FIGURE 18. Sketch of the classes γ' and λ' in $Y_{2,1}$.

This yields that

$$\overline{\mathrm{NE}}(Y_{k,j}) = \langle \gamma', \lambda' \rangle_{\mathbb{R}_+},$$

since Γ' and Λ' are nef. The extremal contraction of the $K_{Y_{k,j}}$-negative extremal ray $\mathbb{R}_+\lambda'$ is a fibre contraction. The extremal ray $\mathbb{R}_+\gamma'$ is $K_{Y_{k,j}}$-negative for $j < k$. In this case there exists an extremal contraction $\varphi_{\gamma'}$ of $\mathbb{R}_+\gamma'$. However, this contraction is small. Its exceptional locus is given by $S_{k,j}$ and has codimension $j + 1 \geq 2$. So we can't use Theorem 4.1.

We continue with $\overline{\mathrm{NE}}(X_{k,j})$. Let λ be the class of the strict transform of a line in a fibre of $\pi_{k,j} : Y_{k,j} \to \mathbb{P}^k$. Furthermore, let γ be the class of a line in E which is mapped to a line in $S_{k,j}$ by μ and let η be the class of a line in E which is contracted by μ. The Néron-Severi space of cycles $N_1(X_{k,j})_{\mathbb{R}}$

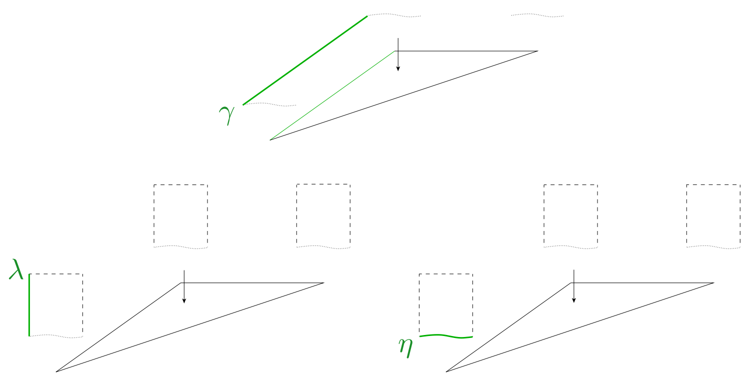


FIGURE 19. Sketch of the classes γ, λ and η in $X_{2,1}$.

is spanned by these three classes and the intersection numbers on $X_{k,j}$ are given in the following table.

.	γ	λ	η
Γ	1	0	0
Λ	0	1	0
$[E]$	-1	1	-1

Now let $\nu = a_1\gamma + a_2\lambda + a_3\eta$ be the class of an irreducible curve c on $X_{k,j}$. The classes Γ and Λ are the pullbacks of nef classes and hence nef. Therefore,

$$0 \leq \nu\,.\,\Gamma = a_1 \text{ and } 0 \leq \nu\,.\,\Lambda = a_2.$$

It remains to show that $a_3 = \nu\,.(\Lambda - \Gamma - [E]) \geq 0$. A divisor with class $\Lambda - \Gamma - [E]$ is given by the strict transform of a divisor with class $\Lambda' - \Gamma'$ since $S_{k,j}$ is the complete intersection of $j+1$ divisors with class $\Lambda' - \Gamma'$.

Now assume that c is not contained in E as a first step. Then it cannot be contained in the support of all divisors with class $\Lambda - \Gamma - [E]$. Otherwise $\mu_*(c)$ would be contained in the support of all divisors with class $\Lambda' - \Gamma'$ and hence in $S_{k,j}$, but this would yield that $c \subset \mu^{-1}(\mu(c)) \subset E$. This is a contradiction. Therefore,

$$0 \leq \nu\,.(\Lambda - \Gamma - [E]) = a_3.$$

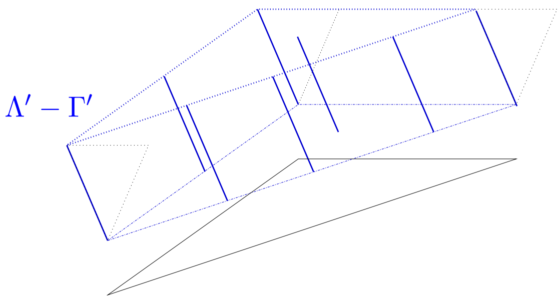


FIGURE 20. Sketch of the divisor class $\Lambda' - \Gamma'$ in $Y_{2,1}$.

If c is contained in E, then we have two cases. If c is contained in the support of one and hence all divisors with class $\Lambda - \Gamma - [E]$, then necessarily $c \sim_{\text{num}} \gamma$; in other words, $a_1 > 0$ and $a_2 = a_3 = 0$. If c is not contained in the support of all divisors with class $\Lambda - \Gamma - [E]$, then

$$0 \leq \nu\,.(\Lambda - \Gamma - [E]) = a_3.$$

This implies that

$$\overline{\text{NE}}(X_{k,j}) = \langle \gamma, \lambda, \eta \rangle_{\mathbb{R}_+}$$

and $X_{k,j}$ is Fano for all $k, j > 0$. The extremal contractions

$$\mu^+ : X_{k,j} \to Y^+,$$

$$\mu : X_{k,j} \to Y_{k,j}$$

of the extremal rays $R_1 = \mathbb{R}_+\gamma$ and $R_3 = \mathbb{R}_+\eta$ are divisorial with exceptional divisor $\text{Exc}_{X_{k,j}}(\mu^+) = \text{Exc}_{X_{k,j}}(\mu) = E$. The birational map

$$\phi := \mu^+ \circ \mu^{-1} : Y_{k,j} \dashrightarrow Y^+$$

is the flip of the small contraction $\varphi_{\gamma'}$ and Y^+ is isomorphic to $Y_{j,k}$. The extremal contraction of the extremal ray $R_2 = \mathbb{R}_+\lambda$ is a fibre contraction onto $\mathbb{P}^j \times \mathbb{P}^k$. Cf. [**Deb01**, Example 1.36].

Theorem 4.1 yields that

$$\begin{aligned}\overline{\mathrm{NM}}(X_{k,j}) &= \{\nu \in \langle \gamma, \lambda, \eta\rangle_{\mathbb{R}_+} \mid \nu\,.[E] \geq 0\} \\ &= \langle (\gamma+\lambda), \lambda, (\lambda+\eta)\rangle_{\mathbb{R}_+}.\end{aligned}$$

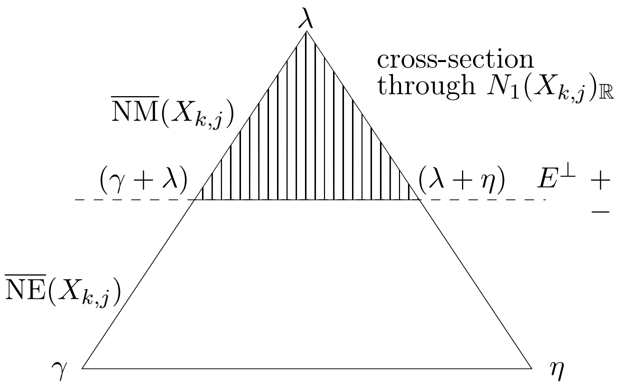


FIGURE 21. The hyperplane $E^\perp$ cuts out $\overline{\mathrm{NM}}(X_{k,j})$.

A simple computation shows that $\mu_1^*(\gamma') = \gamma - \eta$ and $\mu_1^*(\lambda') = \lambda + \eta$. Hence, Theorem 3.13 yields that $\mathbb{R}_+\lambda'$ and $\mathbb{R}_+(\gamma'+\lambda')$ are extremal rays of $\overline{\mathrm{NM}}(Y_{k,j})$. Therefore,

$$\overline{\mathrm{NM}}(Y_{k,j}) = \langle (\gamma'+\lambda'), \lambda'\rangle_{\mathbb{R}_+}$$

since $\rho(Y_{k,j}) = 2$.

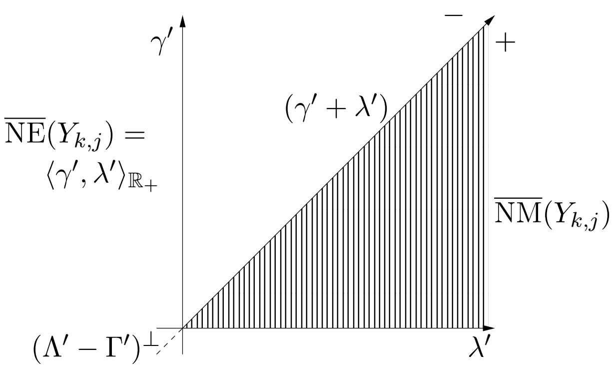


FIGURE 22. The hyperplane $(\Lambda' - \Gamma')^\perp$ cuts out $\overline{\mathrm{NM}}(Y_{2,1})$.

We have $\overline{\mathrm{NM}}(Y_{k,j}) = \{\nu \in \langle\gamma', \lambda'\rangle_{\mathbb{R}_+} \mid \nu\,.(\Lambda' - \Gamma') \geq 0\}$. We will see in Example 4.34 that the class $\Lambda' - \Gamma'$ is the pullback of a nef divisor class on Y^+ via the flip ϕ.

Corollary 4.12. *Theorem 2.8 is wrong for varieties of dimension greater than two.* □

These examples illustrate the ideas presented in the following section which describe the behaviour of effective cycles under a flip.

4.2.2. Flips for smooth fourfolds.

Lemma 4.13. *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities and let $\varphi : X \to Y$ be a small contraction with the following flip diagram.*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Y & \end{array}$$

Moreover, let γ be the class of a curve which lies in a fibre of φ and let γ^+ be the class of a curve which lies in a fibre of φ^+. Then

$$\phi_1^*(\gamma^+) = \tfrac{r^+}{r}\gamma,$$

where $r^+ := [K_{X^+}] \,.\, \gamma^+, r := [K_X] \,.\, \gamma$ and $\frac{r^+}{r} < 0$.

Proof. We have $r^+ := [K_{X^+}] \,.\, \gamma^+ > 0$ and $r := [K_X] \,.\, \gamma < 0$ by definition of an extremal contraction and the flip-diagram. Now, using Lemma 3.9 (i), we find

(4.d) $$0 < r^+ = [K_{X^+}] \,.\, \gamma^+ = \phi^*([K_{X^+}]) \,.\, \phi_1^*(\gamma^+) = [K_X] \,.\, \phi_1^*(\gamma^+).$$

It remains to show that $\phi_1^*(\gamma^+)$ is contracted by φ^+. For this purpose let A be an arbitrary ample Cartier divisor on Y. Then, using the commutativity of the flip diagram and Lemma 3.9 (i), we compute

$$0 = \gamma^+ \,.(\varphi^+)^*([A]) = \phi_1^*(\gamma^+) \,.\, \phi^*((\varphi^+)^*([A])) = \phi_1^*(\gamma^+) \,.\, \varphi^*([A]).$$

Therefore, the class $\phi_1^*(\gamma^+)$ is contracted by φ and thus $\phi_1^*(\gamma^+) = \lambda\gamma$ for a suitable $\lambda \in \mathbb{R}$. Now equation (4.d) implies that $\lambda = \frac{r^+}{r} < 0$. □

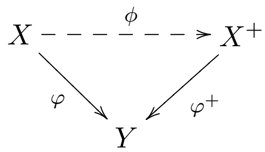
As mentioned before, the examples of section 4.2.1 are representative for all smooth fourfolds. This is because every small contraction on a smooth fourfold is locally like that. We will cite the exact statement which was proved by Yujiro Kawamata in [**Kaw89**].

Theorem 4.14 ([**Kaw89**, Theorem 1.1])**.** *Let X be a non-singular projective variety of dimension four and let $\varphi : X \to Y$ be a small contraction. Then the exceptional locus S of φ is a disjoint union of its irreducible components S_i, $i = 1 \ldots n$, such that $S_i \cong \mathbb{P}^2$ and $N_{S_i/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, where $N_{S_i/X}$ denotes the normal bundle of S_i in X.* □

Remark 4.15. The exceptional locus may be reducible. An example can be given by modifying Construction 4.6. If we replace the line g'' by a smooth curve c'' which intersects the surface S'' transversaly in n points $p_1, \ldots, p_n$, then the analogous construction leads to a variety with a small contraction with reducible exceptional locus.

Moreover, Kawamata proves the following corollary.

Corollary 4.16 ([**Kaw89**, Corollary 1.2])**.** *Let $\varphi : X \to Y$ be as in Theorem 4.14. Then there exists a flip*



where X^+ is a non-singular projective variety. □

Remark 4.17. The flip is constructed in the same way as in the previous examples. If we blow up the exceptional locus S in X, then the exceptional divisor of the blow-up is a disjoint union of irreducible components $E_i \cong \mathbb{P}^2 \times \mathbb{P}^1$. Furthermore, the normal bundle of each E_i is isomorphic to $pr_1^*(\mathcal{O}_{\mathbb{P}^2}(-1)) \oplus pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1))$, where pr_1, resp. pr_2, denotes the projection on the first, resp. second, factor of $\mathbb{P}^2 \times \mathbb{P}^1$. By contracting the exceptional divisor in the other direction, we obtain a smooth projective variety X^+ and the commutative flip-diagram (4.e), but see [**Kaw89**].

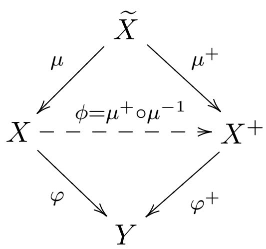
Remark 4.18. Let X be as in Theorem 4.14 with flip diagram (4.e). Moreover, let γ be the class of a line g which lies in a fibre of φ and let γ^+ be the class of a curve g^+ which lies in a fibre of φ^+.
Fact: With the aid of the normal bundle sequence, one computes that

$$N_{g/X} \cong \mathcal{O}_g(1) \oplus (\mathcal{O}_g(-1)^{\oplus 2}) \text{ and } N_{g^+/X^+} \cong \mathcal{O}_{g^+}(-1)^{\oplus 3}.$$

Taking the degree of the first Chern classes in the normal bundle sequences for $N_{g/X}$ and N_{g^+/X^+} yields that $\gamma^+ .[K_{X^+}] = 1 = -\gamma .[K_X]$. Thus we can rephrase Lemma 4.13 in our situation as follows. Let γ^+ be the class of a curve which lies in a fibre of φ^+ and let γ be the class of a line which lies in a fibre of φ. Then $\phi_1^*(\gamma^+) = -\gamma$.

Lemma 4.19. *Let X be as in Theorem 4.14. By Remark 4.17, we have a commutative diagram*

(4.e)



where $\mu : \widetilde{X} \to X$ is the blow-up of S in X and $\mu^+ : \widetilde{X} \to X^+$ is the blow-down of the exceptional divisor E of μ in the other direction. Let $\alpha \in N_1(X)_{\mathbb{R}}$ and $\alpha^+ \in N_1(X^+)_{\mathbb{R}}$ be arbitrary classes. Then

$$\phi_1^*(\alpha^+) = \alpha \Leftrightarrow \mu_1^*(\alpha) = (\mu^+)_1^*(\alpha^+).$$

Proof. Let us assume first that $\phi_1^*(\alpha^+) = \alpha$ and let D' be an arbitrary divisor on $\widetilde{X}$. We have to show that $[D'] . \mu_1^*(\alpha) = [D'] . (\mu^+)_1^*(\alpha^+)$. For $D' = E$

both intersection numbers are zero. So let us assume $D' \neq E$. Then we have

$$\begin{aligned}
[D'] \,.\, \mu_1^*(\alpha) &= \mu_*([D']) \,.\, \alpha && \text{by Lemma 3.8} \\
&= \mu_*([D']) \,.\, \phi_1^*(\alpha^+) && \text{by assumtion} \\
&= \phi_*(\mu_*([D'])) \,.\, \alpha^+ && \text{by Lemma 3.8} \\
&= (\mu^+ \circ \mu^{-1})_*(\mu_*([D'])) \,.\, \alpha^+ && \text{by diagram (4.e)} \\
&= \mu_*^+((\mu^{-1} \circ \mu)_*([D'])) \,.\, \alpha^+ && \text{by Remark 3.4} \\
&= \mu_*^+(\mathrm{id}_*([D'])) \,.\, \alpha^+ && \mathrm{id} : \widetilde{X} \to \widetilde{X}, \mathrm{Exc}_{\widetilde{X}}(\mathrm{id}) = E \\
&= \mu_*^+([D']) \,.\, \alpha^+ && \text{since } D' \neq E \\
&= [D'] \,.\, (\mu^+)_1^*(\alpha^+) && \text{by Lemma 3.8.}
\end{aligned}$$

Now assume that $\mu_1^*(\alpha) = (\mu^+)_1^*(\alpha^+)$ and let D be an arbitrary irreducible divisor on X. We have to show that $\phi_1^*(\alpha^+) = \alpha$. We find

$$\begin{aligned}
[D] \,.\, \phi_1^*(\alpha^+) &= \phi_*([D]) \,.\, \alpha^+ && \text{by Lemma 3.8} \\
&= (\mu^+)_1^*(\phi_*([D])) \,.\, (\mu^+)_1^*(\alpha^+) && \text{by Lemma 3.9} \\
&= (\mu^+)^*(\phi_*([D])) \,.\, \mu_1^*(\alpha) && \text{by assumption} \\
&= \mu_*((\mu^+)^*(\phi_*([D]))) \,.\, \alpha && \text{by Lemma 3.8} \\
&= [D] \,.\, \alpha
\end{aligned}$$

The last equality follows from Remark 3.4 together with diagram (4.e) since its commutativity yields that $\mu_* \circ (\mu^+)^* \circ (\mu^+)_* = \mu_*$. □

Corollary 4.20. *Let X be as in Theorem 4.14 with flip-diagram (4.e) and let c^+ be an irreducible curve on X^+. Then $\phi_1^*([c^+])$ is an effective class if and only if c^+ is not contained in the exceptional locus S^+ of the flipped small contraction φ^+.*

Proof. Let c^+ be an irreducible curve on X^+. Using Lemma 4.13, we see that $\phi_1^*([c^+])$ is not effective if c^+ is contained in S^+.

Let us assume that c^+ is not contained in S^+. Lemma 4.19 yields that

$$\mu_1^*(\phi_1^*([c^+])) = (\mu^+)_1^*([c^+])$$

and hence Lemma 3.8 gives $\phi_1^*([c^+]) = \mu_{*1}((\mu^+)_1^*([c^+]))$. Thus it suffices to show that $\mu_{*1}((\mu^+)_1^*([c^+]))$ is effective. This will result from the following Lemma 4.21. □

Lemma 4.21. *Let X and $S \subset X$ be as in Theorem 4.14 and let $\mu : \widetilde{X} \to X$ be the blow-up of S in X.*

(i) Let $\gamma \in N_1(X)_{\mathbb{R}}$ be the class of an effective 1-cycle such that no component of γ is contained in S. Then the class $\mu_1^*(\gamma) \in N_1(\widetilde{X})_{\mathbb{R}}$ is effective.

(ii) Let $\alpha \in N_1(\widetilde{X})_{\mathbb{R}}$ be an effective class. Then the class $\mu_{*1}(\alpha) \in N_1(X)_{\mathbb{R}}$ is effective.

Proof of (i). Let $\gamma \in N_1(X)_{\mathbb{R}}$ be the class of an arbitrary effective 1-cycle such that no component of γ is contained in S. We can assume that γ is given by the class $[c]$ of an irreducible curve c on X. Then the proof applies naturally to an arbitrary effective 1-cycle by regarding its irreducible components together with linearity.

Recall that S is the disjoint union of its irreducible components S_i, $i = 1 \dots n$, such that $S_i \cong \mathbb{P}^2$ and $N_{S_i/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, where $N_{S_i/X}$ denotes the normal bundle of S_i in X. Furthermore, by Remark 4.17 the exceptional divisor E of μ is a disjoint union of irreducible components $E_i \cong \mathbb{P}^2 \times \mathbb{P}^1$.

We will now show that the numerical pullback of $[c]$ coincides with the class

$$[c''] := [c'] + \sum_{i=1}^{n} k_i [e_i], \tag{4.f}$$

where

- c' denotes the strict transform of c,
- $e_i \cong \mathbb{P}^1$ denotes a fibre of μ in the irreducible component E_i of the exceptional divisor $E = \sum_{i=1}^{n} E_i$ and
- $k_i := c' . E_i$ is the intersection number of c' with the irreducible component E_i of E.

We know that c' is not contained in E since c is not contained in S. Thus the intersection number k_i is non-negative for all $i = 1, \dots, n$. By Lemma 4.4 (ii), we have $e_i . E = \deg(\mathcal{O}_{\widetilde{X}}(E)|_{e_i}) = \deg(\mathcal{O}_{\mathbb{P}^1}(-1)) = -1$ for all $i = 1, \dots, n$. Therefore, $e_i . E_j = -\delta_{ij}$ for $i, j \in \{1, \dots, n\}$, where δ_{ij} denotes the Kronecker-Delta. This yields that

$$[c''] . [E_i] = 0 = \mu_1^*([c]) . [E_i] \text{ for all } i = 1, \dots, n. \tag{4.g}$$

Now let D' be an arbitrary irreducible divisor on $\widetilde{X}$. We can write D' in the form

$$D' = \mu^*(\mu_*(D')) - \sum_{i=1}^{n} m_i E_i,$$

for suitable $m_i \in \mathbb{N}$. Together with (4.g) this yields that

$$[c''] . [D'] = [c''] . [\mu^*(\mu_*(D'))].$$

On the other hand, we have $\mu_1^*([c]) . [D'] = [c] . [\mu_*(D')]$ by Lemma 3.8.

Hence it is sufficient to show that

$$[c''] . \mu^*([D]) = [c] . [D]$$

for an arbitrary divisor D on X, and this is a short computation. We have

$$\begin{aligned}
[c''] . \mu^*([D]) &= ([c'] + \sum_{i=1}^{n} k_i [e_i]) . \mu^*([D]) \\
&= [c'] . \mu^*([D]) + \sum_{i=1}^{n} k_i [e_i] . \mu^*([D]) \\
&= \mu_{*1}([c']) . [D] + \sum_{i=1}^{n} k_i \mu_{*1}([e_i]) . [D] && \text{by Lemma 3.8} \\
&= \mu_*([c']) . [D] + \sum_{i=1}^{n} k_i \mu_*([e_i]) . [D] && \text{by Corollary 3.12} \\
&= [c] . [D] && \text{by Definition 3.11}
\end{aligned}$$

since c' is the strict transform of c and the e_i are contracted by μ, in other words, $\mu_*([e_i]) = 0$. Thus $\mu_1^*([c]) = [c'] + \sum_{i=1}^{n} k_i [e_i]$ is an effective class.

Proof of (ii). This follows immediately from Corollary 3.12 since the image of an effective 1-cycle under μ is again an effective 1-cycle. □

Remark 4.22. Let X be as in Theorem 4.14 with flip-diagram (4.e) and let c^+ be an irreducible curve on X^+ which is not contracted by φ^+. Lemma 4.19 and the proof of Lemma 4.21 show that

$$\phi_1^*([c^+]) = [c] + k[r_s],$$

where c is the strict transform of c^+, k is a non-negative integer and r_s is an irreducible curve which is contained in a fibre of the small contraction φ. The analogous statement holds for the numerical pushforward of a curve which is not contracted by φ.

4.2.3. The moving cone of a smooth fourfold. The following proposition can be viewed as an induction step which will enable us to prove that the moving cone of a smooth fourfold is polyhedral. The proof of the proposition relies on Kawamata's results. The essential arguments that we will use are Remark 4.22 and the fact that the exceptional locus of the flipped small contraction is good natured in the sense that it is a disjoint union of finitely many rational curves on the flipped variety.

Proposition 4.23. *Let X be a smooth projective fourfold such that K_X fails to be nef and such that $\overline{\mathrm{NE}}(X)$ is a convex, polyhedral cone in $N_1(X)_{\mathbb{R}}$, $\overline{\mathrm{NE}}(X) = \langle [p_1], \dots, [p_k], [n_1], \dots, [n_m] \rangle_{\mathbb{R}_+}$ say, where each p_i is an irreducible curve on X such that $[p_i] . [K_X] > 0$. Assume that*

(i) there exists an ample divisor A on X and a real number $\varepsilon > 0$ such that every irreducible curve $c \neq p_1, \dots, p_k$ on X is $(K_X + \varepsilon A)$-negative and that

(ii) each n_j is a rational curve on X, in particular $[n_j] . ([K_X] + \varepsilon[A]) < 0$ by (i).

Moreover, assume that $[n_m]$ is a small extremal class with extremal contraction $\varphi : X \to Y$ and let

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Y & \end{array}$$

be the flip of φ. Denote by $S := \mathrm{Exc}_X(\varphi)$ the exceptional set of φ in X and by $S^+ := \mathrm{Exc}_{X^+}(\varphi^+)$ the exceptional set of φ^+ in X^+.

Then $\overline{\mathrm{NE}}(X^+)$ is a convex, polyhedral cone in $N_1(X^+)_{\mathbb{R}}$. For every ample divisor A^+ on X^+ there exists an $\varepsilon^+ > 0$ such that every irreducible curve $c^+ \neq p_1^+, \dots, p_k^+$ which is not contained in S^+ is $(K_{X^+} + \varepsilon^+ A^+)$-negative, where p_i^+ denotes the strict transform of p_i under ϕ, for $i = 1, \dots, k$.

Strategy for the proof. Since $\overline{\mathrm{NE}}(X^+)$ contains no lines, it is the span of its extremal rays by [**Deb01**, Lemma 6.7 (b)]. Therefore, to show that $\overline{\mathrm{NE}}(X^+)$ is polyhedral, it is sufficient to show that $\overline{\mathrm{NE}}(X^+)$ has only finitely many extremal rays. To achieve this, we will show two statements.

The first one is that

$$\overline{\mathrm{NE}}(X^+) = \overline{\mathrm{NI}}(X^+) + \mathbb{R}_+[n_m^+] + \mathbb{R}_+[p_1^+] + \cdots + \mathbb{R}_+[p_k^+],$$

where $\overline{\mathrm{NI}}(X^+)$ is a certain subcone of $\overline{\mathrm{NE}}(X^+)$ that we will define later.

This yields that every extremal ray R of $\overline{\mathrm{NE}}(X)$ which is not equal to $\mathbb{R}_+[n_m^+]$ or $\mathbb{R}_+[p_i^+]$, $i = 1, \ldots, k$, is an extremal ray of $\overline{\mathrm{NI}}(X^+)$. The existence of such an extremal ray R is guaranteed by the existence of $[K_{X^+}]$-negative classes in $\overline{\mathrm{NE}}(X^+)$. Otherwise $\overline{\mathrm{NE}}(X^+)$ would be the span of $[n_m^+], [p_1^+], \ldots, [p_k^+]$ and hence entirely $[K_{X^+}]$-positive.

The second statement is that for every ample divisor A^+ on X^+ there exists an $\varepsilon^+ \in \mathbb{R}_{>0}$ such that $\overline{\mathrm{NI}}(X^+) \setminus \{0\}$ is entirely $[K_{X^+} + \varepsilon^+ A^+]$-negative.

Once we have this, statement one yields that every extremal ray R of $\overline{\mathrm{NE}}(X)$ which is not equal to $\mathbb{R}_+[n_m^+]$ or $\mathbb{R}_+[p_1^+], \ldots, \mathbb{R}_+[p_k^+]$ is $[K_{X^+} + \varepsilon^+ A^+]$-negative. This will conclude the proof since Mori's Cone Theorem says that $\overline{\mathrm{NE}}(X^+)$ has only finitely many $[K_{X^+} + \varepsilon^+ A^+]$-negative extremal rays.

Proof. First of all, note that the existence of the map ϕ is guaranteed by Corollary 4.16, X^+ is a smooth fourfold and

$$\phi\,|_{X \setminus S} \colon X \setminus S \xrightarrow{\sim} X^+ \setminus S^+$$

is an isomorphism. Moreover, we know that S is the disjoint union of its irreducible components, which are isomorphic to $\mathbb{P}^2$, and that S^+ is a disjoint union of rational curves.

Without loss of generality, we can assume that S and S^+ are irreducible. For conformity, we will denote the irreducible curve S^+ by n_m^+. Furthermore, we can assume that

$$\phi_1^*([n_m^+]) = -[n_m]$$

by Remark 4.18. Recall that for every irreducible curve $r^+ \neq n_m^+$ on X^+ the class $\phi_1^*([r^+])$ is effective by Corollary 4.20. Let p_i^+ be the strict transform of p_i under ϕ for all $i = 1, \ldots, k$, define $P := \{n_m^+, p_1^+, \ldots, p_k^+\}$,

$$\mathrm{NI}(X^+) := \Big\{\sum_{i=1}^{s} a_i[c_i^+] \mid a_i \in \mathbb{R}_+, c_i^+ \notin P \text{ is an irreducible curve on } X^+\Big\}.$$

and denote by $\overline{\mathrm{NI}}(X^+)$ the closure of $\mathrm{NI}(X^+)$ in $N_1(X^+)_{\mathbb{R}}$. We say $\overline{\mathrm{NI}}(X^+)$ is spanned by *effectively flipping curves*.

Claim 1. $\overline{\mathrm{NE}}(X^+) = \underbrace{\overline{\mathrm{NI}}(X^+) + \mathbb{R}_+[n_m^+] + \mathbb{R}_+[p_1^+] + \cdots + \mathbb{R}_+[p_k^+]}_{=:W}$

Proof of Claim 1. We will prove $\overline{\mathrm{NE}}(X^+) \supseteq W$ first. Let $\alpha \in W$. By definition of W, we find $\lambda_1, \ldots, \lambda_k, \lambda_m \geq 0$ and $\alpha' \in \overline{\mathrm{NI}}(X^+)$ such that

$$\alpha = \alpha' + \lambda_m[n_m^+] + \lambda_1[p_1^+] + \ldots + \lambda_k[p_k^+].$$

By definition of $\overline{\mathrm{NI}}(X^+)$, there exists a sequence $(\alpha_l)_{l\in\mathbb{N}} \subset \mathrm{NI}(X^+)$ which converges to α'. Thus the series $(\alpha_l')_{l\in\mathbb{N}}$ with

$$\alpha_l' := \alpha_l + \lambda_m[n_m^+] + \lambda_1[p_1^+] + \ldots + \lambda_k[p_k^+] \in \mathrm{NE}(X^+), \text{ for all } l \in \mathbb{N},$$

converges to α and hence $\alpha \in \overline{\mathrm{NE}}(X^+)$.

Now we will prove $\overline{\mathrm{NE}}(X^+) \subseteq W$. Let $\beta \in \overline{\mathrm{NE}}(X^+)$. By definition of the Mori cone, there exists a sequence $(\beta_l)_{l\in\mathbb{N}} \subset \mathrm{NE}(X^+)$ such that $\beta_l \xrightarrow{l\to\infty} \beta$. By construction of $\overline{\mathrm{NI}}(X^+)$, we can decompose β_l into the sum

$$\beta_l = \beta_l' + \beta_l^1 + \dots \beta_l^k + \beta_l^m,$$

for all $l \in \mathbb{N}$, where $\beta_l' \in \mathrm{NI}(X^+) \subset \overline{\mathrm{NI}}(X^+)$, $\beta_l^1 \in \mathbb{R}_+[p_1^+]$, … ,$\beta_l^k \in \mathbb{R}_+[p_k^+]$ and $\beta_l^m \in \mathbb{R}_+[n_m^+]$. This yields that $\beta \in W$ since each $\beta_l \in W$ and W is a closed set in $N_1(X^+)_\mathbb{R}$. $\square_{\text{Claim 1}}$

By assumption (i), we know that all irreducible curves on X except $p_1, \dots, p_k$ are $(K_X + \varepsilon A)$-negative for a certain ample divisor A on X and a certain real number $\varepsilon > 0$.

Now let A^+ be an arbitrary ample divisor on X^+ and set

$$\varepsilon' := \min\left\{ \tfrac{\varepsilon[A]\,.\,[p_i]}{\phi^*([A^+])\,.\,[p_i]}, \tfrac{\varepsilon[A]\,.\,[n_j]}{\phi^*([A^+])\,.\,[n_j]} \mid i = 1, \dots, k,\ j = 1, \dots, m-1 \right\}.$$

We have to check that this is well-defined. Since none of the p_i, $1 \le i \le k$, or n_j, $1 \le j < m$, are contained in the exceptional locus S of φ, all the classes $\phi_{*1}([p_i])$ and $\phi_{*1}([n_j])$ are effective by Remark 4.22. Thus $0 < [A^+]\,.\,\phi_{*1}([p_i]) = \phi^*([A^+])\,.\,[p_i]$ and $0 < [A^+]\,.\,\phi_{*1}([n_j]) = \phi^*([A^+])\,.\,[n_j]$ for all $i = 1, \dots, k$ and $j = 1, \dots, m-1$. In particular, $\varepsilon' > 0$ by the previous consideration. With this definition, assumption (i) yields that

$$\begin{aligned} ([K_X] + \varepsilon'\phi^*([A^+]))\,.\,[p_i] &= [K_X]\,.\,[p_i] + \varepsilon'\phi^*([A^+])\,.\,[p_i] \\ &\le [K_X]\,.\,[p_i] + \tfrac{\varepsilon[A]\,.\,[p_i]}{\phi^*([A^+])\,.\,[p_i]}\phi^*([A^+])\,.\,[p_i] \\ &= ([K_X] + \varepsilon[A])\,.\,[p_i], \text{ for } 1 \le i \le k, \end{aligned} \tag{4.h}$$

$$\begin{aligned} ([K_X] + \varepsilon'\phi^*([A^+]))\,.\,[n_j] &= [K_X]\,.\,[n_j] + \varepsilon'\phi^*([A^+])\,.\,[n_j] \\ &\le [K_X]\,.\,[n_j] + \tfrac{\varepsilon[A]\,.\,[n_j]}{\phi^*([A^+])\,.\,[n_j]}\phi^*([A^+])\,.\,[n_j] \\ &= ([K_X] + \varepsilon[A])\,.\,[n_j] < 0, \text{ for } 1 \le j < m, \end{aligned} \tag{4.i}$$

and that

$$\begin{aligned} ([K_X] + \varepsilon'\phi^*([A^+]))\,.\,[n_m] &= [K_X]\,.\,[n_m] + \varepsilon'\phi^*([A^+])\,.\,[n_m] \\ &= [K_X]\,.\,[n_m] + \varepsilon'[A^+]\,.\,\phi_{*1}([n_m]) \text{ by 3.8} \\ &= [K_X]\,.\,[n_m] - \varepsilon'[A^+]\,.\,[n_m^+] \text{ by 4.18} \\ &< [K_X]\,.\,[n_m] \\ &< ([K_X] + \varepsilon[A])\,.\,[n_m] < 0 \text{ by (ii)}. \end{aligned} \tag{4.j}$$

Since $\overline{\mathrm{NE}}(X) = \langle [p_1], \dots, [p_k], [n_1], \dots, [n_m] \rangle_{\mathbb{R}_+}$, these equations show that

$$([K_X] + \varepsilon'\phi^*([A^+]))\,.\,\eta \le ([K_X] + \varepsilon[A])\,.\,\eta \text{ for all } \eta \in \overline{\mathrm{NE}}(X). \tag{4.k}$$

Now set

$$\varepsilon^+ := \tfrac{\varepsilon'}{2} = \tfrac{1}{2}\min\left\{ \tfrac{\varepsilon[A]\,.\,[p_i]}{\phi^*([A^+])\,.\,[p_i]}, \tfrac{\varepsilon[A]\,.\,[n_j]}{\phi^*([A^+])\,.\,[n_j]} \mid i = 1, \dots, k,\ j = 1, \dots, m-1 \right\}.$$

Claim 2. Every class $\gamma \in \overline{\mathrm{NI}}(X^+) \setminus \{0\}$ is $([K_{X^+}] + \varepsilon^+[A^+])$-negative.

Proof of Claim 2. Let $0 \neq \gamma \in \overline{\mathrm{NI}}(X^+)$. By definition of $\overline{\mathrm{NI}}(X^+)$, there exists a sequence of effective cycles $(\gamma_l)_{l\in\mathbb{N}} \subset \mathrm{NI}(X^+)$ such that $\gamma_l \xrightarrow{l\to\infty} \gamma$. Let $l \in \mathbb{N}$ be an arbitrary integer. The class γ_l is given by a finite sum

$$\sum_{i=1}^{s} a_i[c_i^+],$$

where $a_i \geq 0$ and c_i^+ is an irreducible curve on X^+ which is not contained in the set $P = \{n_m^+, p_1^+, \ldots, p_k^+\}$ for all $i = 1, \ldots, s$. Remark 4.22 yields that $\phi_1^*([c_i^+]) = [c_i] + k_i[n_m]$, where c_i denotes the strict transform of c_i^+ and k_i is a suitable non-negative integer, for all $i = 1, \ldots, s$.

We have that each $c_i \neq p_j$, for all $j = 1, \ldots, k$. This follows from the fact that none of the curves c_i^+ is equal to one of the p_j^+, $j = 1, \ldots, k$.

Therefore, $([K_X] + \varepsilon[A])\,.\,([c_i] + k_i[n_m]) < 0$ for all $i = 1, \ldots, s$ and

(4.l) $\phi_1^*(\gamma_l)$ is an effective $([K_X] + \varepsilon[A])$-negative class.

Thus

$$\begin{aligned}([K_{X^+}] + \varepsilon'[A^+])\,.\,\gamma_l &= \phi^*([K_{X^+}] + \varepsilon'[A^+])\,.\,\phi_1^*(\gamma_l) && \text{by Lemma 3.9}\\ &= ([K_X] + \varepsilon'\phi^*([A^+]))\,.\,\phi_1^*(\gamma_l) &&\\ &\leq ([K_X] + \varepsilon[A])\,.\,\phi_1^*(\gamma_l) < 0 && \text{by (4.k) and (4.l)}\end{aligned}$$

This yields that $([K_{X^+}] + \varepsilon'[A^+])\,.\,\gamma = \lim\limits_{l\to\infty}([K_{X^+}] + \varepsilon'[A^+])\,.\,\gamma_l \leq 0$. Since $\gamma \in \overline{\mathrm{NI}}(X^+) \subset \overline{\mathrm{NE}}(X^+)$ and A^+ is ample, we attain

$$[K_{X^+}] + \varepsilon^+[A^+])\,.\,\gamma = \underbrace{([K_{X^+}] + \varepsilon'[A^+])\,.\,\gamma}_{\leq 0} - \underbrace{\tfrac{\varepsilon'}{2}[A^+]\,.\,\gamma}_{>0} < 0. \quad \Box_{\text{Claim 2}}$$

By definition of the cone $\overline{\mathrm{NI}}(X^+)$, this yields that every irreducible curve $c^+ \neq p_1^+, \ldots, p_k^+$ which is not contained in S^+ is $(K_{X^+} + \varepsilon^+A^+)$-negative.

Now let $R := \mathbb{R}_+\nu$ be an extremal ray of $\overline{\mathrm{NE}}(X^+)$ such that ν is not numerically proportional to $[p_1^+], \ldots, [p_k^+]$ or $[n_m^+]$. The existence of such an extremal ray R is guaranteed by the following fact. The cone $\overline{\mathrm{NE}}(X^+)$ contains no lines and is thus the span of its extremal rays by [**Deb01**, Lemma 6.7 (b)]. If there is no extremal ray $R \neq \mathbb{R}_+[n_m^+], \mathbb{R}_+[p_1^+], \ldots, \mathbb{R}_+[p_k^+]$, then $\overline{\mathrm{NE}}(X^+)$ is spanned by the classes $[n_m^+], [p_1^+], \ldots, [p_k^+]$ and thus $[K_{X^+}]$-positive. This is impossible since the classes $\phi_{*1}(n_i)$, $i = 1, \ldots, m-1$, are $[K_{X^+}]$-negative and effective by Corollary 4.20.

Now claim 1 yields that $\nu = \nu' + a_m[n_m^+] + a_1[p_1^+] + \ldots a_k[p_k^+]$ for some suitable $a_m, a_1, \ldots, a_k \in \mathbb{R}_+$ and a class $\nu' \in \overline{\mathrm{NI}}(X^+)$. We will now show that $a_m = a_1 = \ldots = a_k = 0$.
Assume that one of these coefficients is non-zero; w.l.o.g. assume that $a_m > 0$. Then $(\nu' + a_1[p_1^+] + \ldots a_k[p_k^+]) + a_m[n_m^+] \in R$ and both terms are effective. Since R is an extremal ray of $\overline{\mathrm{NE}}(X^+)$ this yields that $a_m[n_m^+] \in R$ and hence $R = \mathbb{R}_+[n_m^+]$, a contradiction. Thus $a_m = a_1 = \ldots = a_k = 0$ and $R = \mathbb{R}_+\nu = \mathbb{R}_+\nu'$ is an extremal ray of $\overline{\mathrm{NI}}(X^+)$. In particular, R is $([K_{X^+}] + \varepsilon^+[A^+])$-negative by claim 2.

Since there are only finitely many $([K_{X^+}] + \varepsilon^+[A^+])$-negative extremal rays in $\overline{\mathrm{NE}}(X^+)$ by Remark 1.16, this shows that $\overline{\mathrm{NE}}(X^+)$ has only finitely many extremal rays and concludes the proof. □

Remark 4.24. Note that it is not true that the strict transforms of curves which span extremal rays of $\overline{\mathrm{NE}}(X)$ will always span extremal rays of $\overline{\mathrm{NE}}(X^+)$. Even if they do, the types of extremal contractions can change.

Definition 4.25. A smooth projective fourfold which satisfies the requirements of Proposition 4.23 is called a *pmc-fourfold*. Let X_0 be a pmc-fourfold. A finite sequence

$$X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} X_n$$

of birational maps is called a *pmc-flip sequence for* X_0 if

(i) the map $\phi_i : X_{i-1} \dashrightarrow X_i$ is the flip of a small contraction which contracts a $K_{X_{i-1}}$-negative extremal ray $\mathbb{R}_+[s_{i-1}]$, for all $i = 1, \ldots, n$, and
(ii) the Mori cone $\overline{\mathrm{NE}}(X_n)$ of the fourfold X_n has no small extremal rays.

We will call a pmc-flip sequence for X_0 a *pmc-flip sequence for* $\mathbb{R}_+[s_0]$ if the first map $\phi_1 : X_0 \dashrightarrow X_1$ in the sequence is the flip of the small contraction which contracts the K_{X_0}-negative extremal ray $\mathbb{R}_+[s_0]$.

The number of flips in a pmc-flip sequence is called the *length of the pmc-flip sequence*.

Remark 4.26. Note the following two apparent statements.

(i) Each variety X_i, for $i = 0, \ldots, n-1$, in such a pmc-flip sequence is a pmc-fourfold by Proposition 4.23. In particular, $\overline{\mathrm{NE}}(X_n)$ is polyhedral.
(ii) If X is a smooth Fano fourfold such that $\overline{\mathrm{NE}}(X)$ has a small extremal ray, then X is a pmc-fourfold. This is obvious, as we can take $A = -K_X$ and $\varepsilon = \frac{1}{2}$, for example. Therefore, we have already seen two pmc-flip sequences of length one in Example 4.8 and Example 4.11.

We will now show that pmc-flip sequences exist and that there are just finitely many pmc-flip sequences for each pmc-fourfold. This is an immediate consequence of the following theorem due to Y. Kawamata, K. Matsuda and K. Matsuki.

Theorem 4.27 (See [**KMM87**, Theorem 5-1-15])**.** *The Termination Conjecture 1.25 (ii) holds for threefolds and fourfolds.* □

Thanks to this result we are able to prove the afore-noted statement.

Lemma 4.28. *Let X_0 be a pmc-fourfold. Then there exists a pmc-flip sequence for every small extremal ray of $\overline{\mathrm{NE}}(X_0)$. Moreover, there exist only finitely many pmc-flip sequences for X_0.*

Proof. Let R be a small extremal ray of $\overline{\mathrm{NE}}(X_0)$. By Corollary 4.16, there exists a flip $\phi_1 : X_0 \dashrightarrow X_1$ of the corresponding small contraction and $\overline{\mathrm{NE}}(X_1)$ is polyhedral by Proposition 4.23. If $\overline{\mathrm{NE}}(X_1)$ has no small extremal ray, $\phi_1 : X_0 \dashrightarrow X_1$ is a pmc-flip sequence for X_0 of length one. Otherwise,

X_1 is a pmc-fourfold and we can iterate this procedure. Since there is no infinite sequence of flips by Theorem 4.27, this procedure will stop with a smooth fourfold X_n such that $\overline{\mathrm{NE}}(X_n)$ has no small extremal rays. Moreover, Proposition 4.23 guarantees that $\overline{\mathrm{NE}}(X_n)$ is polyhedral.

We will prove the second statement of the lemma by contradiction. Suppose there are infinitely many pmc-flip sequences for X_0. Since $\overline{\mathrm{NE}}(X_0)$ has only finitely many extremal rays, there has to be a small extremal ray $\mathbb{R}_+[s_0]$ with flip $\phi_1 : X_0 \dashrightarrow X_1$ such that infinitely many pmc-flip sequences start with ϕ_1. Proposition 4.23 yields that $\overline{\mathrm{NE}}(X_1)$ is polyhedral, too. Therefore, there has to be a small extremal ray $\mathbb{R}_+[s_1]$ of $\overline{\mathrm{NE}}(X_1)$ with flip $\phi_2 : X_1 \dashrightarrow X_2$ such that infinitely many pmc-flip sequences start with the map $\phi_1 \circ \phi_2$. In this manner we can construct an infinite sequence of flips successively which, however, is a contradiction to Theorem 4.27. □

The following statement gives a more detailed description of K_{X_i}-positive curves on a pmc-fourfold X_i which affects in a pmc-flip sequence for a smooth Fano fourfold.

Lemma 4.29. *Let X_0 be a smooth Fano fourfold such that $\overline{\mathrm{NE}}(X_0)$ has at least one small extremal ray and let*

$$X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} X_n$$

be an arbitrary pmc-flip sequence for X_0. Then one of the following statements holds for all $i = 1, \ldots, n$ and every K_{X_i}-positive irreducible curve c_i on X_i. Either

- *(i) there exists an index $j < i-1$ and a rational curve s_j which spans a small extremal ray of $\overline{\mathrm{NE}}(X_j)$ such that c_i is the strict transform s_j^i of a curve on X_{j+1} with class $-(\phi_{j+1})_{*1}([s_j])$ under the map $\phi_i \circ \cdots \circ \phi_{j+2}$, or*
- *(ii) c_i is contained in the exceptional locus of the flipped small contraction $\varphi_{i-1}^+ : X_i \to Y_i$ obtained by ϕ_i.*

In particular, if R is a K_{X_i}-positive extremal ray of $\overline{\mathrm{NE}}(X_i)$, then R is spanned by the class of one of these finitely many K_{X_i}-positive irreducible curves on X_i.

Proof. This is an immediate consequence of Proposition 4.23 and can be proved inductively. □

This lemma finishes the preparatory work which is necessary for the proof of Theorem 4 of the introduction and we will construct the set $\mathrm{Eq}(X) \subset N^1(X)_{\mathbb{R}}$ which was introduced in this statement.

Construction and Definition 4.30. Let X be a pmc-fourfold and let $\mathbb{R}_+[s]$ be a small extremal ray of $\overline{\mathrm{NE}}(X)$. Let X_k be a smooth projective fourfold which affects in a pmc-flip sequence for $\mathbb{R}_+[s]$ and denote by $\Phi_k : X \dashrightarrow X_k$ the induced birational map. The Mori cone of X_k is polyhedral,

$$\overline{\mathrm{NE}}(X_k) = \{\alpha \in N_1(X_k)_{\mathbb{R}} \mid \alpha\,.[N_i] \geq 0, i = 1, \ldots, m\} \text{ say,}$$

where $N_1, \ldots, N_m$ are nef divisors on X_k which span $\mathrm{Nef}(X_k)$.

We define

$$\mathrm{Eq}(X_k)_{nef} := \{(\Phi_k)^*([N_i]) \mid i = 1\ldots, m\}$$

as the set of pullbacks via the map Φ_k of nef divisors on X_k which span $\mathrm{Nef}(X_k)$. We set

$$\mathrm{Eq}(X_k)_{div} := \{(\Phi_k)^*([E_i]) \mid i = 1, \ldots, l\},$$

where $E_1, \ldots, E_l$ are the exceptional divisors which correspond to the divisorial extremal rays of $\overline{\mathrm{NE}}(X_k)$.

Furthermore, let $\mathrm{Poly}(\mathbb{R}_+[s])$ be the set of varieties which affect in a pmc-flip sequence for $\mathbb{R}_+[s]$.

We call the set

$$\mathrm{Eq}(\mathbb{R}_+[s]) := \bigcup_{X_k \in \mathrm{Poly}(\mathbb{R}_+[s])} (\mathrm{Eq}(X_k)_{nef} \cup \mathrm{Eq}(X_k)_{div})$$

the *set of equations for* $\mathbb{R}_+[s]$.

Remark 4.31. Note that the set $\mathrm{Eq}(\mathbb{R}_+[s])$ is a finite set of classes of divisors on X for every small extremal ray of $\overline{\mathrm{NE}}(X)$.

Lemma 4.32. *Let X_0 be a smooth Fano fourfold and let $\mathbb{R}_+[s_0]$ be small extremal ray of $\overline{\mathrm{NE}}(X_0)$. If D is an irreducible divisor on X_0 such that $[D]$ is not contained in the closed cone $\langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$ spanned by classes in $\mathrm{Eq}(\mathbb{R}_+[s_0])$, then $[D]\,.[s_0] \geq 0$.*

Proof. We will prove the lemma by contradiction. Let D be an irreducible divisor on X_0 such that $[D]$ is not contained in the closed cone $\langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$ spanned by classes in $\mathrm{Eq}(\mathbb{R}_+[s_0])$ and assume that $[D]\,.[s_0] < 0$. Furthermore, let

$$\begin{array}{ccc} X_0 & \overset{\phi_1}{\dashrightarrow} & X_1 \\ & {\scriptstyle \varphi_0}\searrow \quad \swarrow{\scriptstyle \varphi_0^+} & \\ & Y_0 & \end{array}$$

be the flip diagram for $\mathbb{R}_+[s_0]$. Let $[s_0^1]$ be the class of an irreducible curve s_0^1 in a fibre of φ_0^+ and let D_1 be the strict transform of D under ϕ_1. We know that $[D_1] = (\phi_1)_*([D])$ and that $(\phi_1)_*([D])\,.[s_0^1] > 0$. Moreover, Remark 4.17 shows that the curve s_0^1 is not contained in the support of D_1.

By Lemma 4.28 there exists a pmc-flip sequence for $\mathbb{R}_+[s_0]$ and, of course, X_1 affects in every pmc-flip sequence for $\mathbb{R}_+[s_0]$. Thus $[D_1]$ cannot be a nef class since $[D] \notin \langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$ by assumption. Hence there has to be an extremal ray R_1 of $\overline{\mathrm{NE}}(X_1)$ such that $[D_1]$ is negative on R_1. Lemma 4.29 yields that R_1 is K_{X_1}-negative since $[D_1]\,.[s_0^1] > 0$. The contraction of the ray R_1 cannot be a fibre contraction since $[D_1]$ is effective and it cannot be a divisorial contraction since $[D] \notin \langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$. Therefore, $R_1 = \mathbb{R}_+[s_1]$ is a small extremal ray.

Let

$$\begin{array}{ccc} X_1 & \overset{\phi_2}{\dashrightarrow} & X_2 \\ & {\scriptstyle\varphi_1}\searrow \quad \swarrow{\scriptstyle\varphi_1^+} & \\ & Y_1 & \end{array}$$

be the flip diagram for $\mathbb{R}_+[s_1]$. Let $[s_1^2]$ be the class of an irreducible curve s_1^2 in a fibre of φ_1^+, let D_2 be the strict transform of D_1 and let s_0^2 be the strict transform of s_0^1 under ϕ_2.

We know that $[D_2] = (\phi_2)_*([D_1])$ and that $[D_2]\,.\,[s_1^2] > 0$. Moreover, the curves s_0^2 and s_1^2 are not contained in the support of D_2. In particular, $[D_2]\,.\,[s_0^2] \geq 0$.

The variety X_2 affects in at least one pmc-flip sequence for $\mathbb{R}_+[s_0]$. Thus $[D_2]$ cannot be a nef class since $[D] \notin \langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$ by assumption. Hence there has to be an extremal ray R_2 of $\overline{\mathrm{NE}}(X_2)$ such that $[D_2]$ is negative on R_2. Lemma 4.29 yields that R_2 is K_{X_2}-negative since $[D_2]\,.\,[s_0^2] \geq 0$ and $[D_2]\,.\,[s_1^2] > 0$. The contraction of the ray R_2 cannot be a fibre contraction since $[D_2]$ is effective and it cannot be a divisorial contraction since $[D] \notin \langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$. Therefore, $R_2 = \mathbb{R}_+[s_2]$ is a small extremal ray.

Theorem 4.27 yields that this process has to end with a variety X_n and a divisor D_n on X_n. By induction, we know that D_n is not nef and non-negative on all irreducible K_{X_n}-positive curves of X_n. Thus there has to be an extremal ray R_n of $\overline{\mathrm{NE}}(X_n)$ such that D_n is negative on R_n. Lemma 4.29 yields that R_n is K_{X_n}-negative, but there are no small extremal rays in $\overline{\mathrm{NE}}(X_n)$. This yields that D_n is non-negative on R_n since $D \notin \langle \mathrm{Eq}(\mathbb{R}_+[s_0])\rangle_{\mathbb{R}_+}$, which is a contradiction. □

Theorem 4.33. *Let X be a smooth Fano fourfold. Denote by $\mathrm{Eq}(X)$ the set*

$$\mathrm{Eq}(X) := \big(\bigcup_{i=1}^{k} \mathrm{Eq}(\mathbb{R}_+[s_i])\big) \bigcup \mathrm{Eq}(X)_{nef} \bigcup \mathrm{Eq}(X)_{div},$$

where $\mathbb{R}_+[s_1], \ldots, \mathbb{R}_+[s_k]$ are the small extremal rays of $\overline{\mathrm{NE}}(X)$. Then the moving cone $\overline{\mathrm{NM}}(X)$ of X is a convex polyhedral cone in $N_1(X)_{\mathbb{R}}$, given by

$$\overline{\mathrm{NM}}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma\,.\,\Delta \geq 0 \textit{ for all } \Delta \in \mathrm{Eq}(X)\}.$$

Proof. The inclusion

$$\overline{\mathrm{NM}}(X) \subseteq \{\gamma \in N_1(X)_{\mathbb{R}} \mid \gamma\,.\,\Delta \geq 0 \text{ for all } \Delta \in \mathrm{Eq}(X)\} =: M$$

follows from the fact that the numerical pushforward of a movable class by a flip is again a movable class. Let us prove that $\overline{\mathrm{NM}}(X) \supseteq M$.

Let $\gamma \in N_1(X)_{\mathbb{R}}$ such that $\gamma\,.\,\Delta \geq 0$ for all $\Delta \in \mathrm{Eq}(X)$ and let D' be an arbitrary irreducible divisor on X. We need to show that $\gamma\,.\,[D'] \geq 0$ by Theorem 1.12. If $[D']$ is contained in the set $\langle \mathrm{Eq}(X)\rangle_{\mathbb{R}_+}$ of effective linear combinations of classes in $\mathrm{Eq}(X)$, then there's nothing to show. Thus we can assume that $[D']$ is not contained in $\langle \mathrm{Eq}(X)\rangle_{\mathbb{R}_+}$.

The inclusion $\mathrm{Eq}(X)_{nef} \subset \mathrm{Eq}(X)$ yields that $\gamma \in \overline{\mathrm{NE}}(X)$. Since X is Fano, we know that

$$\overline{\mathrm{NE}}(X) = \langle [f_1], \ldots, [f_m], [d_1], \ldots, [d_n], [s_1], \ldots, [s_k] \rangle_{\mathbb{R}_+},$$

where f_i, d_j, s_l are rational curves on X, $\mathbb{R}_+[f_i]$ is an extremal ray of fibre type, $\mathbb{R}_+[d_j]$ is a divisorial extremal ray and $\mathbb{R}_+[s_l]$ is a small extremal ray, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $l = 1, \ldots, k$.

Thus it is sufficient to show that $[f_i] . [D'] \geq 0$, $[d_j] . [D'] \geq 0$ and $[s_l] . [D'] \geq 0$, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $l = 1, \ldots, k$.

We choose arbitrary indices $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq l \leq k$.

By assumption, $\mathbb{R}_+[f_i]$ is an extremal ray of fibre type. Therefore, clearly $[f_i] . [D'] \geq 0$ since D' cannot contain every fibre of the corresponding contraction.

Let E_j denote the exceptional divisor of the extremal contraction corresponding to $\mathbb{R}_+[d_j]$. Since $[D'] \notin \mathrm{Eq}(X)_{div}$, we have $[D'] \neq [E_j]$. Therefore, we can find a curve $c \subset E_j$ such that $c \sim_{\mathrm{num}} d_j$ and $c \not\subseteq D'$. Hence $[d_j] . [D'] \geq 0$.

The last inequality $[s_l] . [D'] \geq 0$ is given by Lemma 4.32. □

A priori, the set $\mathrm{Eq}(X)$ defined in Theorem 4.33 seems to be very large and cumbersome. We will compute an example now which shows that the situation is not that bad and which illustrates the algorithmic procedure to compute the set $\mathrm{Eq}(X)$.

Example 4.34 (See Construction 4.10)**.** Let Y be the smooth fourfold $Y_{2,1}$ of Construction 4.10, let F' be a fibre of the projection

$$\pi = \pi_{2,1} : Y \to \mathbb{P}^2$$

onto $\mathbb{P}^2$ and let l be a line in F' which does not intersect the plane $S' = S_{2,1}$. As in Construction 4.10 we denote by Γ' the class of the pullback of a hyperplane in $\mathbb{P}^2$ and by Λ' the class of the line bundle $\mathcal{O}_Y(1)$. As in Example 4.11 we denote by γ' the class of a line in S' and by λ' the class of a line in a fibre of π.

Now let $\mu : X \to Y$ be the blow up of Y in l and denote by E the exceptonal divisor of the blow up and denote by η the class of a curve in a fibre of $\mu|_E : E \to l$. We fix some more notation.

Let F and S denote the strict transforms under μ of F' and S', respectively. The class of the strict transform under μ of a divisor with class Γ' which does not contain F' is given by $\mu^*(\Gamma') =: \Gamma$, of a divisor with class Γ' which contains the fibre F' by $\Gamma - [E]$, and of a divisor with class Λ' by $\mu^*(\Lambda') - [E] =: \Lambda$. The strict transform of a divisor with class $\Lambda' - \Gamma'$ is given by $\Lambda - \Gamma + [E]$, see Figure 20.

The class of the strict transform under μ of a curve with class γ' is given by $\mu_1^*(\gamma') =: \gamma$, of a curve in F' by $\mu_1^*(\lambda') - \eta =: \nu$, and of a general curve with class λ' by $\mu_1^*(\lambda') =: \lambda = \nu + \eta$.

Fact: The variety X is a smooth Fano fourfold and one can check that

$$\overline{\mathrm{NE}}(X) = \langle \gamma, \nu, \eta \rangle_{\mathbb{R}_+}.$$

The intersection product on X is given in the following table.

.	γ	ν	η	λ
Γ	1	0	0	0
Λ	0	0	1	1
$[E]$	0	1	-1	0

The extremal contractions of the extremal rays $\mathbb{R}_+\nu$ and $\mathbb{R}_+\gamma$ are small with exceptional loci F and S, respectively. Moreover, some short computations show that $\overline{\mathrm{NE}}(X)$ is cut out of $N_1(X)_\mathbb{R}$ by the nef divisor classes Γ, Λ and $\Lambda + [E] = \mu^*(\Lambda')$. Thus we have

$$\mathrm{Eq}(X)_{nef} = \{\Gamma, \Lambda, (\Lambda + [E])\} \text{ and } \mathrm{Eq}(X)_{div} = \{[E]\}.$$

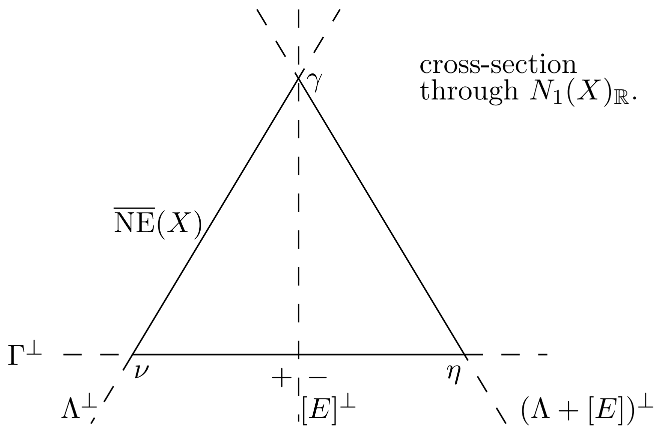


FIGURE 23. Sketch of $\overline{\mathrm{NE}}(X)$ and the hyperplanes which are spanned by elements of $\mathrm{Eq}(X)_{nef}$ and $\mathrm{Eq}(X)_{div}$.

Now we will compute the pmc-flip sequences for $\mathbb{R}_+\nu$ and $\mathbb{R}_+\gamma$, and we will start with the sequence for $\mathbb{R}_+\nu$. Let

$$\phi_1 : X \dashrightarrow X_1$$

be the flip of the small extremal ray $\mathbb{R}_+\nu$. We set

$$\Gamma_1 := (\phi_1)_*(\Gamma),$$
$$\Lambda_1 := (\phi_1)_*(\Lambda) \text{ and}$$
$$[E_1] := (\phi_1)_*([E]).$$

Let ν_1 denote the class $-(\phi_1)_{*1}(\nu)$ and let F_1 be the indeterminacy locus of ϕ_1 in X_1.

The class of the strict transform under ϕ_1 of a curve with class γ is given by $(\phi_1)_{*1}(\gamma) - \nu_1 =: \gamma_1$ if the curve intersects F, and by $\gamma_1 + \nu_1$ otherwise.

The class of the strict transform under ϕ_1 of a curve with class η is given by $(\phi_1)_{*1}(\eta) - \nu_1 =: \eta_1$ if the curve intersects F, and by $\eta_1 + \nu_1$ otherwise.

The class of the strict transform of a curve with class λ is given by $(\phi_1)_{*1}(\lambda) =: \lambda_1$. The intersection product on X_1 is given in the following table.

.	γ_1	ν_1	η_1	λ_1
Γ_1	1	0	0	0
Λ_1	0	0	1	1
$[E_1]$	1	-1	0	0

We will prove that

$$\overline{\mathrm{NE}}(X_1) = \langle \gamma_1, \nu_1, \lambda_1 \rangle_{\mathbb{R}_+}$$

and that the extremal contractions of $\mathbb{R}_+\gamma_1$ and $\mathbb{R}_+\lambda_1$ are of fibre type.

The proof of Proposition 4.23 shows that $\overline{\mathrm{NE}}(X_1)$ is the span of ν_1 and the closed cone $\overline{\mathrm{NI}}(X_1)$ spanned by effectively flipping curves on X_1, that is, irreducible curves which are not contained in F_1. We have already seen that the classes γ_1, $(\gamma_1+\nu_1)$, $\lambda_1 = \eta_1$ and $(\lambda_1+\nu_1)$ are contained in $\overline{\mathrm{NI}}(X_1)$. We will show now that actually

$$\overline{\mathrm{NI}}(X_1) = \langle \gamma_1, (\gamma_1+\nu_1), \lambda_1, (\lambda_1+\nu_1) \rangle_{\mathbb{R}}. \tag{4.m}$$

For this purpose let c_1 be an irreducible curve on X_1 such that $[c_1] \in \overline{\mathrm{NI}}(X_1)$. We know that $(\phi_1)_1^*([c_1]) = [c] + k_1\nu$ by Remark 4.22, where c denotes the strict transform of c_1 and $k_1 \geq 0$. Thus we find an effective linear combination $(\phi_1)_1^*([c_1]) = a_1\gamma + b_1\eta + (d_1+k_1)\nu$ with $d_1 \geq 0$ and hence

$$[c_1] = (\phi_1)_{*1}((\phi_1)_1^*([c_1])) = a_1\gamma_1 + b_1\lambda_1 + (a_1+b_1-d_1-k_1)\nu_1.$$

Since c_1 is not contained in F_1, we can find an irreducible divisor with class $\Gamma_1 - [E_1]$ on X_1 such that

$$0 \leq (\Gamma_1 - [E_1])\,.[c_1] = a_1 + b_1 - (d_1+k_1).$$

If $a_1 \geq a_1+b_1-d_1-k_1 \Rightarrow d_1+k_1-b_1 \geq 0$, then

$$[c_1] = (a_1+b_1-d_1-k_1)(\gamma_1+\nu_1) + (d_1+k_1-b_1)\gamma_1 + b_1\lambda_1.$$

Otherwise $a_1 < a_1+b_1-d_1-k_1 \Rightarrow b_1-d_1-k_1 > 0$, and

$$[c_1] = a_1(\gamma_1+\nu_1) + (b_1-d_1-k_1)(\lambda_1+\nu_1) + (d_1+k_1)\lambda_1.$$

This shows (4.m) and thus $\overline{\mathrm{NE}}(X_1) = \langle \gamma_1, \nu_1, \lambda_1 \rangle_{\mathbb{R}_+}$.

The extremal contraction of $\mathbb{R}_+\lambda_1$ is obviously a fibre contraction. The extremal contraction of $\mathbb{R}_+\gamma_1$ cannot be small by Theorem 4.14. Therefore, it has to be either a divisorial or a fibre contraction. However, it cannot be a divisorial contraction. This is a consequence of the following fact. If the contraction was divisorial, then the class of the exceptional divisor of the contraction would have negative intersection with γ_1 and the exceptional divisor would be the strict transform of an irreducible divisor on X which contains the set S. Therefore, the exceptional divisor would be numerically proportional to $\Lambda_1 - \Gamma_1 + [E_1]$, but $\gamma_1\,.(\Lambda_1 - \Gamma_1 + [E_1]) = 0$.

Thus there is only one pmc-flip sequence for $\mathbb{R}_+\nu$, which has length one. Furthermore, $\overline{\mathrm{NE}}(X_1)$ is cut out of $N_1(X_1)_{\mathbb{R}}$ by the hyperplanes

$$\Gamma_1^{\perp}, \Lambda_1^{\perp} \text{ and } (\Gamma_1 - [E_1])^{\perp}.$$

We obtain that

$$\mathrm{Eq}(X_1)_{nef} = \{\Gamma, \Lambda, (\Gamma - [E])\} \text{ and } \mathrm{Eq}(X_1)_{div} = \emptyset.$$

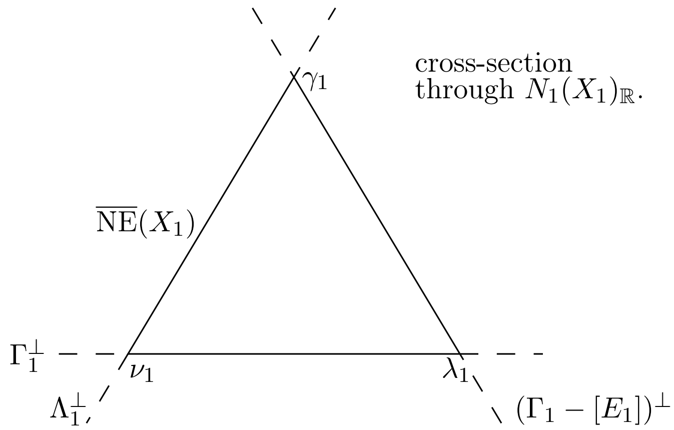


FIGURE 24. Sketch of $\overline{\mathrm{NE}}(X_1)$ and the hyperplanes which are spanned by elements of $\mathrm{Eq}(X_1)_{nef}$. The set $\mathrm{Eq}(X_1)_{div}$ is empty.

We go on with the pmc-flip sequences for γ. Let

$$\phi_2 : X \dashrightarrow X_2$$

be the flip of the small extremal ray $\mathbb{R}_+\gamma$. We set

$$\Gamma_2 := (\phi_2)_*(\Gamma),$$
$$\Lambda_2 := (\phi_2)_*(\Lambda) \text{ and}$$
$$[E_2] := (\phi_2)_*([E]).$$

Set $\gamma_2 := -(\phi_2)_{*1}(\gamma)$ and let S_2 be the indeterminacy locus of ϕ_2 in X_2.

The class of the strict transform under ϕ_2 of a curve with class ν is given by $(\phi_2)_{*1}(\nu) - \gamma_2 =: \nu_2$ if the curve intersects S, and by $\nu_2 + \gamma_2$ otherwise.

The class of the strict transform of a curve with class λ is given by $(\phi_2)_{*1}(\lambda) - \gamma_2 =: \lambda_2$ if the curve intersects S, and by $\lambda_2 + \gamma_2$ otherwise.

The class of the strict transform of a curve with class η is given by $(\phi_2)_{*1}(\eta) =: \eta_2$. The intersection product on X_2 is given in the following table.

$\cdot$	γ_2	ν_2	η_2	λ_2
Γ_2	-1	1	0	1
Λ_2	0	0	1	1
$[E_2]$	0	1	-1	0

We will show that

$$\overline{\mathrm{NE}}(X_2) = \langle \gamma_2, \nu_2, \eta_2 \rangle_{\mathbb{R}_+},$$

that the extremal contraction of $\mathbb{R}_+\nu_2$ is of fibre type and that the extremal contraction of $\mathbb{R}_+\eta_2$ is divisorial with exceptional divisor E_2.

As before, the proof of Proposition 4.23 yields that it is sufficient to compute $\overline{\mathrm{NI}}(X_2)$. By the previous considerations, the classes ν_2, $(\nu_2 + \gamma_2)$ and η_2 are contained in $\overline{\mathrm{NI}}(X_2)$. Now let c_2 be an irreducible curve on X_2 such that $[c_2] \in \overline{\mathrm{NI}}(X_2)$. By Remark 4.22, we have $(\phi_2)_1^*([c_2]) = [c'] + k_2\gamma$ for a suitable number $k_2 \geq 0$, where c' denotes the strict transform of c_2. Hence $(\phi_2)_1^*([c_2]) = a_2\nu + b_2\eta + (d_2 + k_2)\gamma$ for suitable $a_2, b_2, d_2 \geq 0$ and

$$[c_2] = (\phi_2)_{*1}((\phi_2)_1^*([c_2])) = a_2\nu_2 + b_2\eta_2 + (a_2 - d_2 - k_2)\gamma_2.$$

By assumption, we have that c_2 is not contained in S_2. Therefore, we can find a divisor with class $\Lambda_2 - \Gamma_2 + [E_2]$ on X_2 such that

$$0 \leq (\Lambda_2 - \Gamma_2 + [E_2]) \,.[c_2] = a_2 - d_2 - k_2.$$

Thus we find that

$$[c_2] = (a_2 - d_2 - k_2)(\nu_2 + \gamma_2) + (d_2 + k_2)\nu_2 + b_2\eta_2$$

is an effective linear combination of the classes $(\nu_2 + \gamma_2)$, ν_2 and η_2. This shows that $\overline{\mathrm{NI}}(X_2) = \langle(\gamma_2 + \nu_2), \nu_2, \eta_2\rangle_{\mathbb{R}_+}$ and thus $\overline{\mathrm{NE}}(X_2) = \langle\gamma_2, \nu_2, \eta_2\rangle_{\mathbb{R}_+}$.

The extremal contraction of $\mathbb{R}_+\eta_2$ is clearly divisorial with exceptional divisor E_2. The extremal contraction of $\mathbb{R}_+\nu_2$ is a fibre contraction. This can be checked by the following considerations. The contraction cannot be divisorial since the exceptional divisor would be numerically proportional to $\Gamma_2 - [E_2]$, but $\nu_2 \,.(\Gamma_2 - [E_2]) > 0$. The contraction cannot be small by Theorem 4.14.

Thus there is only one pmc-flip sequence for $\mathbb{R}_+\gamma$, which has length one, as well. Furthermore, $\overline{\mathrm{NE}}(X_2)$ is cut out of $N_1(X_2)_\mathbb{R}$ by the hyperplanes

$$\Lambda_2^\perp, (\Lambda_2 + [E_2])^\perp \text{ and } (\Lambda_2 - \Gamma_2 + [E_2])^\perp.$$

This yields that

$$\mathrm{Eq}(X_2)_{nef} = \{\Lambda, (\Lambda + [E]), (\Lambda - \Gamma + [E])\} \text{ and } \mathrm{Eq}(X_2)_{div} = \{[E]\}.$$

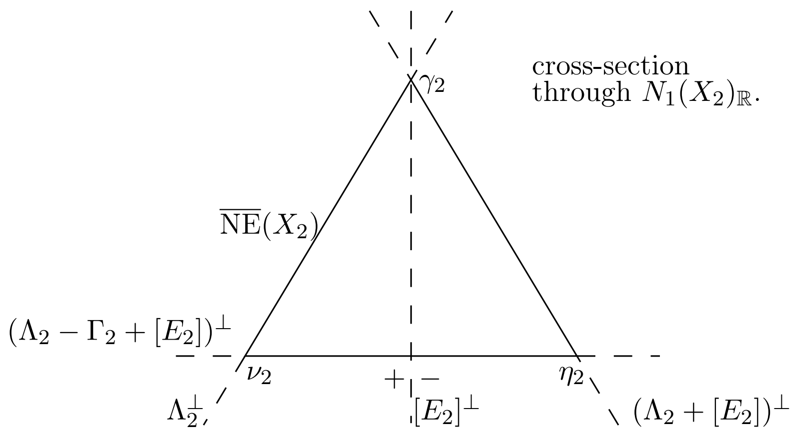


FIGURE 25. Sketch of $\overline{\mathrm{NE}}(X_2)$ and the hyperplanes which are spanned by elements of $\mathrm{Eq}(X_2)_{nef}$ and $\mathrm{Eq}(X_2)_{div}$.

Combining all this, we have

$$\mathrm{Eq}(X) = \{\Lambda, \Gamma, [E], (\Gamma - [E]), (\Lambda + [E]), (\Lambda - \Gamma + [E])\}$$

and a short computation gives

$$\begin{aligned}\overline{\mathrm{NM}}(X) &= \{\varsigma \in N_1(X)_{\mathbb{R}} \mid \varsigma \,.\, \Delta \geq 0, \text{ for all } \Delta \in \mathrm{Eq}(X)\} \\ &= \langle \lambda, (\lambda+\gamma), (\gamma+\nu) \rangle_{\mathbb{R}_+}.\end{aligned}$$

The complete situation is sketched in the following picture, where the hatched areas inside the Mori cones illustrate the moving cones of X, X_1 and X_2.

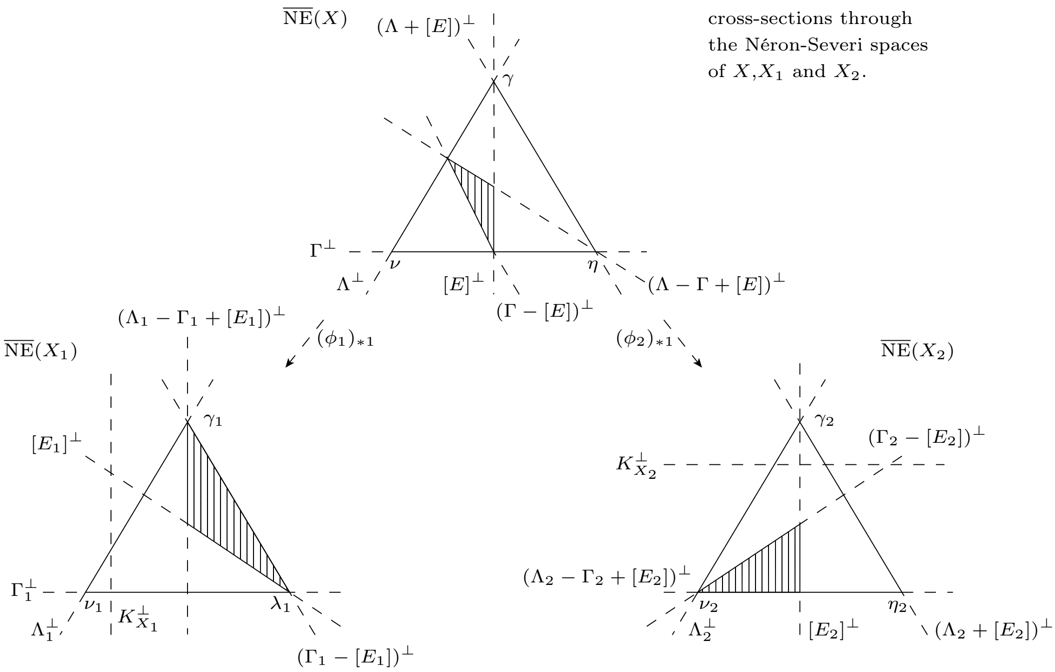


FIGURE 26. The hatched areas sketch the moving cones inside the Mori cones.

4.3. Prospects and questions

4.3.1. Prospects. It is needless to say that we would like to apply the methods used up to now in other settings, too. It seems that our argumentation works for toric varieties. This is because the Mori cone of a toric variety is polyhedral and flips are well-understood.

However, this is already covered by the work of Yi Hu and Seán Keel. They proved that the pseudoeffective cone of a toric variety is polyhedral, but see [**HK00**, Proposition 1.11 and Corollary 2.4].

What is the problem with Fano varieties of dimension greater than four or singular Fano varieties?

We have explicitly used that the numerical pushforward of an irreducible curve via the flip of a small contraction is effective if the curve is not contained in the exceptional locus of the small contraction. Thanks to Kawamata, this was easy to prove in the smooth fourfold case. However, this should also be true for the general case.

Conjecture 4.35. *Let X be a $\mathbb{Q}$-factorial projective variety with only terminal singularities and let $\varphi : X \to Y$ be a small contraction of an extremal ray of $\overline{\mathrm{NE}}(X)$ with the following flip diagram.*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Y & \end{array}$$

If c is an irreducible curve on X which is not contained in the exceptional locus $\mathrm{Exc}_X(\varphi)$ of φ, then $\phi_{*1}([c])$ is effective.

If Conjecture 4.35 holds for Fano threefolds, then our argumetation applies to singular Fano threefolds immediately since the exceptional locus of a small contraction on a threefold is just the union of finitely many curves.

This brings us to the second difficulty.

Problem 4.36. Assume that we have a curve c which is entirely contained in the exceptional locus of a small contraction φ but not contracted by φ. Then it is not clear at all whether the numerical pushforward of c via the flip of φ is still effective!

This issue seems to be quite hard to fix since we have only few knowledge about flips in general. However, if one could give a positive answer to both problems, then the methods used in chapter 4 will apply to all Fano varieties.

4.3.2. Questions. The description of the moving cone by equations is nice because it provides information about the pseudoeffective cone. The disadvantage of this description is that it does not deliver visible information about the extremal rays of the moving cone. Thus, in respect of Mori's Cone Theorem, it is reasonable to ask if extremal classes of the moving cone are represented by rational curves. However, we have already seen in Corollary 2.17 that this is in general not true, whereat this example was not a Fano variety. Bircar, Cascini, Hacon and McKernan obtain in [**BCHM06**, Corollary 1.3.4] that every extremal class of the moving cone of a Fano variety is represented by the pullback of a rational curve.

Therefore, this motivates the following question.

Question 4.37. *Is every extremal class of the moving cone represented by an effective 1-cycle?*

In addition, it can be can ask if the moving cone yields information about extremal contractions of extremal faces of the Mori cone. More specific, it can be asked if the contraction of an extremal face of the Mori cone which contains a movable class is a fibre contraction. Because of Question 4.37, this is not obvious, but it seems that the answer to this question is positive.

Conjecture 4.38. *Let X be a $\mathbb{Q}$-factorial projective variety and let F be a K_X-negative extremal face of $\overline{\mathrm{NE}}(X)$ which contains a movable class. Then the extremal contraction of F is a fibre contraction.*

This expectation is independent of Question 4.37 and we finish this chapter and the thesis with the above annotation.

Bibliography

[Ara05] Carolina Araujo. *The cone of effective divisors of log varieties after Batyrev.* math.AG/0502174, 2005. Preprint. (document), 5, 3, 3.2, 3.9, 3.2

[Bat92] V.V. Batyrev. *The cone of effective divisors of threefolds.* Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989). Contemp. Math., vol. 131, AMS, 1992. pp. 337-352. (document)

[BCHM06] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type, 2006. (document), 6, 1.2, 1.2.1, 4, 4.3.2

[BDPP04] Sebastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* 2004. (document), 1, 1, 1.6, 1.1.1, 1.11, 1.12

[Cor07] Alessio Corti, editor. *Flips for 3-folds and 4-folds*, volume 35 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2007. 1.2.1, 4

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001. 1.16, 1, 2.9, 2.2.1, 4.4, 4.6, 4.10, 4.10, 4.10, 4.11, 4.2.3, 4.2.3

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998. 3, 3.10, 3.1

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 1.2, 1.20, 2.2.1, 2.3, 2.3, 2.3, 2.3.1, 2.3.1, 2.3.1, 3.3, 3.4

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday. 4.3.1

[Kaw89] Yujiro Kawamata. Small contractions of four-dimensional algebraic manifolds. *Math. Ann.*, 284(4):595–600, 1989. 4.2.2, 4.14, 4.16, 4.17

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. 1.21

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987. 1.21, 4.27

[KS] Alex Küronya and Endre Szabó. *The locally polyhedral structure of the effective cone.* Unpublished Preprint. (document)

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. 1.2, 1.16

[Mat02] Kenji Matsuki. *Introduction to the Mori program.* Universitext. Springer-Verlag, New York, 2002. 1.15, 1.19, 1.21, 1.24, 4.1, 4.1

Glossary

$\mathrm{Amp}(X)$	ample cone of X, 9
$\mathrm{CaCl}(X)$	Cartier divisor class group of X, 25, 26
$\overline{\mathrm{NI}}(X^+)$	closed convex cone spanned by effectively flipping curves, 54
$\langle S\rangle_{\mathbb{R}_+}$	closed convex cone spanned by S, xi
$\mathrm{Cl}(X)$	divisor class group of X, 25, 26
$l_\varepsilon(\Gamma)$	line segment between $\Gamma \in \mathrm{Amp}(X)$ and $K_X + \varepsilon A$ in $N^1(X)_\mathbb{R}$, 9
$\mathrm{Nef}(X)$	nef cone of X, 9
$N_1(X)_\mathbb{R}$	Néron-Severi space of 1-cycles on X, 1
$N^1(X)_\mathbb{R}$	Néron-Severi space of divisors on X, 1
$\mathbb{N}$	non-negative integers, xi
$\mathbb{R}_+$	non-negative real numbers, xi
$\equiv_{\mathrm{num}}$	numerical equivalence, xi
$[c]$	numerical equivalence class of the 1-cycle c, xi
$[D]$	numerical equivalence class of the divisor D, xi
$\sim_{\mathrm{num}}$	numerical proportionality, xi
φ_1^*	numerical pullback via φ, 26
φ_{*1}	numerical pushforward via φ, 26
$\overline{\mathrm{NE}}(X)$	Mori cone of X, 1
$\overline{\mathrm{NM}}(X)$	moving cone of X, 2
$\rho(X)$	Picard number of X, 1
$\overline{\mathrm{Eff}}(X)$	pseudoeffective cone of X, 2
φ^*	pullback of divisors and divisor classes via φ, 25, 26
$f_*([V])$	pushforward of the cycle class $[V]$ via f, 27
φ_*	pushforward of divisors and divisor classes via φ, 25, 26
$\mathrm{Eq}(X_k)_{div}$	set of divisorial equations of X_k, 59
$\mathrm{Eq}(\mathbb{R}_+[s])$	set of equations for $\mathbb{R}_+[s]$, 59
$\mathrm{Eq}(X)$	set of equations for X, 60
$\mathrm{Eq}(X_k)_{nef}$	set of nef equations of X_k, 59
$\mathrm{Poly}(\mathbb{R}_+[s])$	set of varieties which affect in a pmc-flip sequence for $\mathbb{R}_+[s]$, 59
$\overline{\mathrm{SNM}}(X)$	strongly movable cone of X, 2
P_ε	subset of $\partial\mathrm{Nef}(X)$, 9

Index

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde.

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Die in Kapitel 3 und in Kapitel 4, Abschnitt 4.1 dargestellten Resultate wurden von mir zum Teil bereits auf dem Preprint-server *arXiv.org e-print archive* (`http://arxiv.org/`) in dem Preprint math/0703025v3 veröffentlicht.

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