Random Matrix Models for Disordered Bosons

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Zusammenfassung

Es werden drei verschiedene Zufallsmatrixmodelle untersucht, welche sich aus ungeordneten physikalischen Systemen mit bosonischen Anregungen ergeben. Das erste Modell entspringt der Betrachtung eines ungeordneten optischen Mediums und wird mit rein numerischen Methoden behandelt. Es werden zwei verschiedene Realisierungen der Unordnung betrachtet. Das zweite Modell geht vom allgemeinen bosonischen Hamiltonian aus, der bilinear in den Erzeugungs- und Vernichtungsoperatoren ist und Stabilität der Bewegung garantiert. Um die Unordnung zu modellieren, wurde eine möglichst einfache Familie von Wahrscheinlichkeitsmaßen gewählt. Diese erlaubt, alle Korrelationsfunktionen des Modells mit Hilfe bi-orthogonaler Polynome explizit zu berechnen. Das dritte Modell unterscheidet sich vom letzteren nur durch zusätzliche Zeitumkehrinvarianz. Die Anwendung bi-orthogonaler Polynome ist nun nicht mehr möglich, es wird auf die Supersymmetrie-Methode zurückgegriffen. Hier kommt die kürzlich von Littelmann, Sommers und Zirnbauer entwickelte Superbosonisierungs-Identität zum Einsatz, welche zunächst am Beispiel des bereits in der Literatur bekannten und vollständig gelösten Gauß'schen orthogonalen Ensembles demonstriert wird.

Abstract

We study three different random matrix models, which arise from disordered physical systems with bosonic excitations. The first model comes from the consideration of a disordered optical medium and is treated by means of numerical methods. Two different kinds of disorder are analyzed. The second one starts from the most general Hamiltonian which is bilinear in the creation and annihilation operators. To guarantee stability of the motion a certain condition is imposed. To treat disorder, we study a simple family of probability measures. This family allows to calculate all correlation functions by means of bi-orthogonal polynomials. The third model differs from the latter by additional time reversal symmetry. The method of bi-orthogonal polynomials cannot be applied, thus the supersymmetry method is used. We apply the superbosonization identity that was recently developed by Littelmann, Sommers and Zirnbauer. Its usage is illustrated at the example of the well-known Gaussian orthogonal ensemble.

Contents

Chapter 1

Introduction

1.1 Random matrix theory

The subject of random matrix theory is the eigenvalue statistics of large matrices of a certain symmetry class. This question arose the first time in the 1930s. However, it did not attract much attention at that time.

In the 1950s, intensive studies of random matrices in physics began with the work of Wigner [Wig51, Wig58]. The physical aim behind these studies was to understand excitations of heavy nuclei, representing a strongly interacting many-particle system. The key physical idea can be formulated as follows: In general, it is impossible to exactly calculate the energy levels of a strongly interacting many-particle system. Thus one assumes that a few fundamental properties of the system, for instance total spin and symmetries, adequately characterize the system. The statistical properties of the eigenvalue distribution are then obtained by means of the assumption that all further details of the Hamiltonian behave as random variables. This way of thinking turned out to be very successful, not only in nuclear physics, but also in many other fields of research. For instance, it can be applied to chaotic quantum systems, or even to number theory: It is a surprising and fascinating observation that the correlations of the zeros of the Riemann Zeta function correspond to those of the eigenvalues of a Hermitean random matrix.

Using the above mentioned assumptions, one is able to construct a random ensemble of $N \times N$ matrices as an idealization of a class of disordered physical systems. For the choice of the probability measure there are many possibilities. However, it is often restricted by the symmetry transformations which leave the class of physical systems invariant. In the limit of large N the statistical properties of the eigenvalues are in a wide range independent of the details of the chosen probability distribution of the matrix elements. This can be understood as a generalization of the well-known central limit theorem.

Historically, the most famous random matrix ensembles are the three Wigner-Dyson Gaussian ensembles of orthogonal, unitary, and symplectic symmetry: GOE, GUE, and GSE. In the case of GOE the domain of matrices are the symmetric real ones, the probability measure is just a Gaussian distribution of each entry. The GUE consists of Hermitean matrices, the probability measure is again Gaussian. From a physical point of view, the difference between these two is the additional time re-

versal symmetry in the GOE. The GSE consists of self-dual Hermitean quaternionic matrices, the probability measure is Gaussian in all independent degrees of freedom.

What one wishes to calculate are often ensemble averages of functions depending only on the eigenvalues of the matrices. These functions are called *radial*. Let M be the set of allowed matrices X of dimension $N \times N$, $d\mu(X)$ its probability distribution, and $F(X)$ a radial function. Then the task is to calculate the disorder integral

$$\langle F \rangle = \int_M F(X) d\mu(X)$$

in the limit of large N, or better, for any N. Usually, one is interested in n-point correlation functions of eigenvalues $\omega_1, \omega_2, \ldots, \omega_n$ given by

$$R_n(\omega_1, \ldots, \omega_n) = \left\langle \prod_{i=1}^{n} \delta(\omega_i \mathbf{1} - X) \right\rangle \ .$$

It turns out that the level correlations of heavy nuclei coincide with those of the GOE. However, it should be always kept in mind that the entire spectra are not reproduced correctly. Particularly, *Wigner's semicircle law* has no counterpart in nuclear physics. For an introduction in random matrix theory see e.g. [Meh04], [BFF$^+$81] and references therein.

Of course, there has been much progress in random matrix theory since the work of Wigner and Dyson. Random matrix ensembles occurred, which do not belong to the Wigner-Dyson symmetry classes [AZ05]. The relation of random matrix theory an symmetric spaces has been emphasized [CM04] and symmetric superspaces occur in the classification of the different models [Zir98b]. As a consequence, it is recognized that there are not only the historical Wigner-Dyson ensembles, but exactly ten symmetry classes.

1.2 Bosonic excitations of disordered systems

There are many examples of disordered systems with bosonic excitations. Let us mention excitations of Bose glasses [FWGF89], electromagnetic modes in an optical medium, normal modes of a pinned charge density wave [GS77], spin waves in a magnet, which are called *magnons*, the vibrational modes of a solid or the oscillations of the superfluid density of a Bose-Einstein condensate. In the vicinity of the ground state these systems can be linearized. What is obtained, is a bosonic Hamiltonian, which is bilinear in the creation and annihilation operators. The general question we ask is the following: How does generally disorder affect the physics of the system, especially at low energies?

To answer these questions, one may try to look at the results for problems with fermionic excitations. Here, the situation is well understood: In the case of high-energy excitations, the systems exhibit the universal behavior predicted by the Wigner-Dyson statistics [Dys62]. For low energies, the properties are also well understood. The canonical anticommutation relations of the fermionic Fock operators lead to restrictions of the Hamiltonian matrix. A classification of possible symmetries has been carried out and can be found in [HHZ05].

In the case of bosonic systems progress has been much slower. The method applied to fermionic systems cannot be copied to bosonic problems. The usual strategy for modelling disorder on the fermionic site is to choose the entries of the Hamiltonian matrix as independent and identically distributed random variables. If one tries to add disorder to a bosonic system using this scheme, the system exhibits instable dynamics rather than stable oscillations in the vicinity of the ground state, i.e. we have to look for a different strategy.

1.3 Outline

In this thesis, we investigate three different random matrix ensembles arising from physical models. It is organized as follows: In chapter 2 we establish a model for a disordered optical medium. We use a discretization of Maxwell's equation on a pair of 3-complexes (K, K'). For two different kinds of disorder, we derive random matrix ensembles from the model. The average spectrum is calculated by means of numerical methods, i.e. we use a routine for numerical diagonalization and average the spectrum of eigenvalues over many randomly chosen systems. As a result, we find for the second type of disorder an interesting behavior of the density of states at low frequencies. At approximately one fourth of the maximal frequency of the spectrum, a peak arises that goes up with increasing disorder.

The approach pursued in chapter 3 is in some sense complementary to 2, since it is more mathematically oriented. Here, we start from the most general Hamiltonian which is bilinear in the bosonic creation and annihilation operators of a Fock space with N different bosons. It is also possible to regard this Hamiltonian as the most general classical one, which is quadratic in N positions and momenta. The bilinear structure of this Hamiltonian can be build up using a $2N \times 2N$ matrix. At this stage of abstraction the difference between the classical and the quantum system cannot be seen anymore. The requirement of stability of the motion leads to a restriction for the allowed matrices, they must be elements of a positive cone $\mathcal{E}^0$ in the symplectic Lie algebra. To model the disorder, a simple family of probability measures is imposed. These measures are not of Gaussian type, since the symmetry group of the positive cone is noncompact, thus we present a different strategy to obtain natural measures. The question for the statistics of eigenfrequencies of the Hamiltonians averaged over $\mathcal{E}^0$ leads to a random matrix model, which is completely solved. For large N, the usual bulk scaling limit can be found and we obtain sine-kernel universality for the correlation functions, as known from the GUE. In addition, there exists an unusual scaling limit at the hard edge of the spectrum, a rescaling of the energy according to $\omega \to z\sqrt{N}$. The results of this chapter have been published in [LSZ06].

Chapter 4 is needed as a preparation for the model considered in chapter 5. Nevertheless, this chapter is useful on its own. We present how Efetov's supersymmetry method [Efe83] works at the example of the Gaussian orthogonal ensemble. Instead of the usually used Hubbard-Stratonovich transformation, we apply the recently developed superbosonization identity [LSZ08]. In the following, the limit of a large number of degrees of freedom can be performed. As a result, we obtain Wigner's semicircle law and a relatively simple superintegral which generates the correlation

functions. To solve this superintegral, literature helps: One has just to look for explicit forms of the GOE correlation functions. Hence, a direct evaluation of the superintegral is not necessary.

In chapter 5 we consider a random matrix ensemble very similar to that from chapter 3. As an additional symmetry, we require time reversal invariance, which leads to a restriction of the positive cone $\mathcal{E}^0$. This structure makes the ensemble much harder to handle than the one of chapter 3. In particular, the Harish-Chandra–Itzykson–Zuber integral found in chapter 3, has no analog. Thus we concentrate on the application of the supersymmetry method to obtain results. Again, it turns out that for large N two convergent scaling limits exist. In the bulk scaling, a complete solution of the ensemble is possible, i.e. we are able to calculate all correlation functions in the form of superintegrals. Not surprisingly, these superintegrals are already known from the results of chapter 4. Since the correlations of the GOE are well known in a very explicit form, we use this result to give an explicit form for the correlations of the ensemble. In the microscopic scaling at the hard edge of the spectrum the situation is much more complicated, here we find a non-analytical behavior of the density of states. An attempt to apply the methods of chapter 3 to the model is presented in appendix C.2 . However, for the integral found therein no strategy to solve it is known.

Chapter 2

Two models of a disordered optical medium

In this chapter we derive a random matrix model from the question, how the electromagnetic modes behave in a disordered medium. The optical medium is modeled by a pair of 3-complexes (K, K'), i.e. a bipartite lattice, which provides a basis for discrete versions of Maxwell's equations. Disorder is obtained by assigning each vertex a value for the magnetic permeability μ and the electric permittivity ε. In our model these quantities are isotropic, i.e. we do not use their tensor structure. After the choice of a certain probability distribution for μ and ε, we compute the average spectrum of the system by means of numerical diagonalization.

2.1 Classical electrodynamics

Maxwell's theory of classical electrodynamics is usually summarized in form of the four Maxwell equations. A comprehensive introduction to this subject is e.g. the famous textbook by J. D. Jackson [Jac75]. In this book the theory is presented, as preferred by the majority of authors, in the usual vector calculus. Here we prefer the formulation in terms of differential forms. It can be found e.g. in [Zir98a, HO03]. In this formulation Maxwell's equations read

$$\begin{aligned} dD &= \rho & dE &= -\dot{B} \\ dH &= j + \dot{D} & dB &= 0 \,. \end{aligned} \tag{2.1}$$

In addition, we need

$$D = \varepsilon \star E \quad \text{and} \quad B = \mu \star H \,. \tag{2.2}$$

As usual, E denotes the electric field, B the magnetic induction, D the electric displacement, and H the magnetic field. In this formulation of electrodynamics, E is a (even) 1-form, B is a (even) 2-form, D is an odd 2-form, and H is an odd 1-form. This classification allows a simple geometric view on these quantities. We imagine a k-form in a n-dimensional space as an $(n-k)$-chain. In the case of even differential forms, the chain carries an outer orientation, in the case of odd forms an inner one. Figure 2.1 illustrates this picture. This geometric view helps us to construct our random matrix model, since it is easy to discretize.

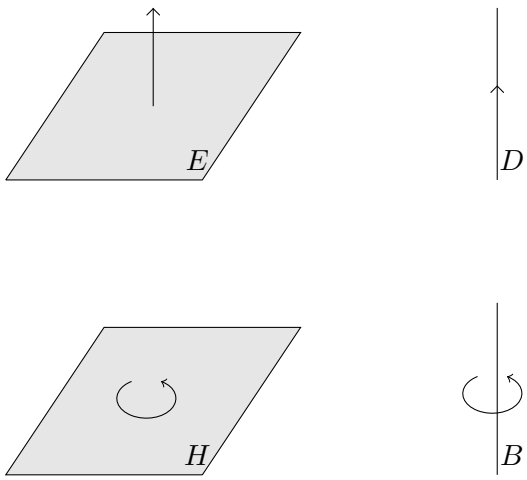


Figure 2.1: The geometric view of the electrodynamic fields as chains. E and B carry an inner orientation, D and H an outer one.

2.2 Discrete electrodynamics

We consider a dielectric medium with randomly distributed dielectric constant and magnetic permeability. Our aim is to calculate the modes of the electromagnetic field and to average over disorder. We start by constructing a discrete version of Maxwell's equations (2.1). This can be done on a pair of 3-complexes (K, K') equipped with a Hodge operator $\star : C_i(K) \to C_{3-i}(K')$. In this scheme, E and B are represented by 2- resp. 1-chains on K, and D and H by 1- resp 2-chains on K'. In our approach we use the fields B and D as the dynamical variables. Due to the absence of free charges and currents the picture of figure 2.1 can be simplified: now D is a closed 2-form, and therefore the 1-chains to represent it are always closed loops. The same holds for B.

In this picture, the dynamics can be obtained as follows. The closed loops allow us to define a degenerate commutator relation of B and D, denoted by $[B, D]$, as illustrated by figure 2.2 . The degeneracy occurs due to the gauge invariance of electrodynamics. A concrete realization of the pair (K, K') can be establish by covering space by a bipartite cubic lattice. Note that the actual geometry of the lattice does not influence the dynamics in the limit of large N, thus we are free to choose the most convenient one. The elementary degrees of freedom are now the closed loops given by the boundaries of the quadratic plaquettes. Thus, if the lattice has N vertices in each of the two components, we have $6N$ degrees of freedom. The lattice is composed of unit cells shown in figure 2.3 . A unit cell consists of three plaquettes of K and K' each. Figure 2.4 shows a part of the lattice consisting of three unit cells. The coordinates of the unit cells are given by $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{n} + \mathbf{e}_1$, and $\mathbf{n} + 2\mathbf{e}_1$. To clarify our choice of closed loops and orientations, one loop of every type is highlighted.

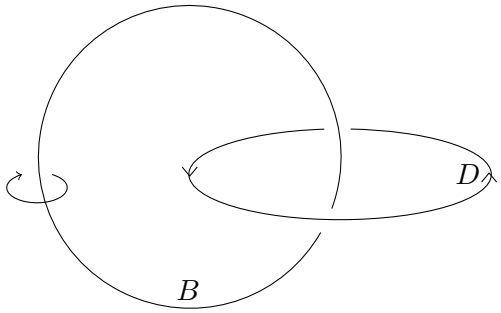


Figure 2.2: The geometric definition of the commutator. If the orientations of the loops match, we define $[B, D] = +1$, if they do not $[B, D] = -1$. If the loops do not intersect, the value is $[B, D] = 0$.

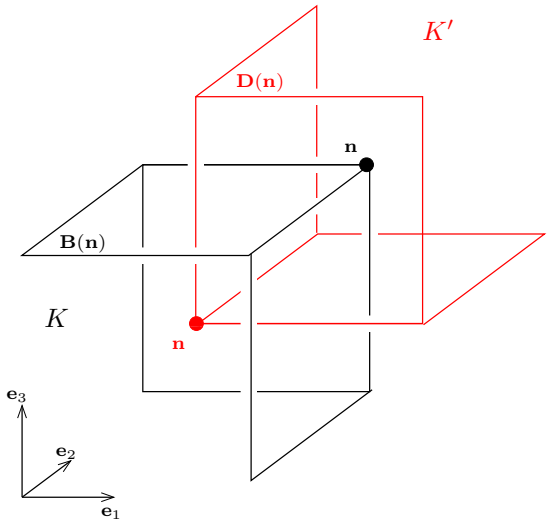


Figure 2.3: A unit cell of the pair of 3-complexes (K, K'). It consists of six plaquettes. It is denoted by the coordinates $\mathbf{n} = (n_1, n_2, n_3)$.

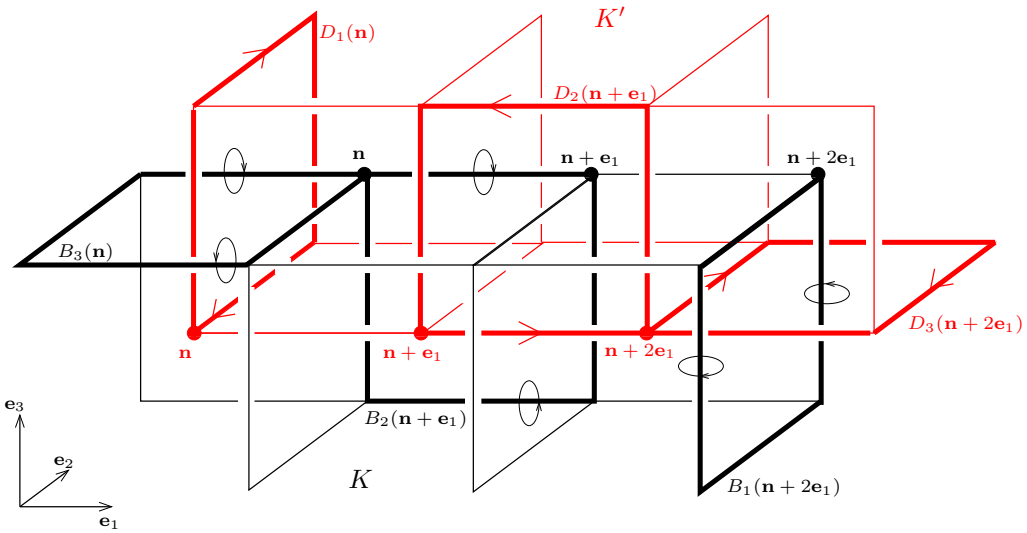


Figure 2.4: Realization of the pair of 3-complexes (K, K') by a bipartite cubic lattice. We can read off the nonvanishing commutators of the bold loops by checking the orientation: $[B_3(\mathbf{n}), D_1(\mathbf{n})] = 1$ and $[B_1(\mathbf{n}+2\mathbf{e}_1), D_3(\mathbf{n}+2\mathbf{e}_1)] = -1$.

2.3 The commutators of B and D

Using the law defined by figure 2.2, we can read off the following relations from figure 2.4:

$$\begin{aligned} [B_i(\mathbf{n}), D_j(\mathbf{n})] &= \sum_k \epsilon_{ijk} \\ [B_i(\mathbf{n}), D_j(\mathbf{n}-\mathbf{e}_k)] &= -\epsilon_{ijk} \ , \end{aligned}$$

or equivalently

$$\begin{aligned} [D_i(\mathbf{n}), B_j(\mathbf{n})] &= -\sum_k \epsilon_{ijk} \\ [D_i(\mathbf{n}), B_j(\mathbf{n}+\mathbf{e}_k)] &= \epsilon_{ijk} \, . \end{aligned} \tag{2.3}$$

The vector $\mathbf{e}_k$ denotes a displacement of one unit in the k-direction in the lattice. As usual, ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is a cyclic permutation of } (1,2,3) \ , \\ -1 & \text{if } (i,j,k) \text{ is an anticyclic permutation of } (1,2,3) \ , \\ 0 & \text{else} \ . \end{cases}$$

To each unit cell we assign a pair of values for the dielectric constant and the magnetic permeability $(\varepsilon(\mathbf{n}), \mu(\mathbf{n}))$.

2.4 Hamiltonian system

Now it is simple to obtain the equations of motion for our problem. As usual, the Hamiltonian is given by the total energy, which reads in our case

$$H = \sum_{\mathbf{n}} \left(\frac{1}{2\mu(\mathbf{n})} \mathbf{B}(\mathbf{n})^2 + \frac{1}{2\varepsilon(\mathbf{n})} \mathbf{D}(\mathbf{n})^2 \right) . \tag{2.4}$$

The sum runs over all vertices of the (K, K'), $\mathbf{B}(\mathbf{n})$ and $\mathbf{D}(\mathbf{n})$ contain the three components of B and D in each vertex. To handle all degrees of freedom in a compact form, we define the vector

$$x = (B_1(1), B_2(1), B_3(1), \ldots, B_1(N), B_2(N), B_3(N), D_1(1), \ldots, D_3(N))^t ,$$

wherein the natural numbers are placeholders for the coordinates of the lattice. The number of vertices of the lattice is given by $N = N_1 \times N_2 \times N_3$. The equations of motion of the Hamiltonian system defined above are given by

$$\dot{x} = X_H(x) = [x, H] . \tag{2.5}$$

This equation determines the Hamiltonian vector field $X_H(x)$. Since we are looking for the characteristic frequencies of the electromagnetic field, we have to solve the equation

$$\dot{x} = X_H(x) = i\omega x .$$

In other words, we want to know the eigenvalues of X_H. From the Hamiltonian (2.4) and the commutator relations we can read off the structure of this matrix. It can be written in the block form

$$X_H = \begin{pmatrix} 0 & AT_{\varepsilon^{-1}} \\ -A^t T_{\mu^{-1}} & 0 \end{pmatrix} .$$

Each of the blocks has dimension $3N \times 3N$. The matrices T_μ^{-1} and T_ε^{-1} contain the electric permittivity ε and the magnetic permeability μ. They are given by

$$T_{\mu^{-1}} = \operatorname{diag}\left(\mu^{-1}(1), \mu^{-1}(1), \mu^{-1}(1), \mu^{-1}(2), \ldots, \mu^{-1}(N)\right)$$

and

$$T_{\varepsilon^{-1}} = \operatorname{diag}\left(\varepsilon^{-1}(1), \varepsilon^{-1}(1), \varepsilon^{-1}(1), \varepsilon^{-1}(2), \ldots, \varepsilon^{-1}(N)\right) .$$

The structure of the matrix A can be extracted from our lattice and (2.5) by means of the commutator relations (2.3). Every loop of the complex K is entangled with four loops of the dual complex K' and vice versa. Thus A is sparse and has only four non-zero entries in every row and every column.

Before we start to calculate the density of states (DOS), let us state the following: Due to the gauge invariance of electrodynamics, we expect to find a spectrum, which has one third zero-modes. Hence, our calculations always yield a huge peak of zero-modes, but we do not show it in the figures. In principle, it is possible to establish a lattice model which avoids these modes in advance by reducing the number of independent degrees of freedom. However, this complicates the structure of the matrix A. The numerical advantage, which is gained by reducing the matrix size, is compensated by the much more complicated setting of the problem.

2.5 Analytic solution of the problem without disorder

First, we want to answer the rather simple question, how the DOS can be calculated in the absence of disorder, i.e. we set $\varepsilon = \mu = 1$ in every vertex of the lattice. In this case, a standard calculation of solid state physics (see e.g. [Czy00]) can be used to calculate the eigenfrequencies.

We want to solve the equation

$$\begin{pmatrix} \dot{B} \\ \dot{D} \end{pmatrix} = X_H \begin{pmatrix} B \\ D \end{pmatrix} = i\omega \begin{pmatrix} B \\ D \end{pmatrix}$$

for ω. We introduce the notation $x_{an\alpha}$ for the vector which contains all components of B and D. The index $n = (n_0, n_1, n_2)$ denotes the vertex in the lattice, B, D the field, and $\alpha \in 1, 2, 3$ the spatial component. Using this notation the matrix X_H has six indices, $X_{Han\alpha,a'n'\alpha'}$. Since the problem is invariant under translations, the matrix elements only depend on the difference of the lattice sites $\mathbf{R}_n - \mathbf{R}_{n'}$, therefore

$$X_{Ha\alpha n,a'\alpha'n'} = X_{Ha\alpha,a'\alpha'}(\mathbf{R}_n - \mathbf{R}_{n'}) \ .$$

In each vertex of the lattice, we make the ansatz

$$x_{an\alpha} = \sqrt{c_a}\, v_{a\alpha n} e^{-i\omega t} \ ,$$

to model the time dependence, where $c_B = \mu_0$ and $c_D = \varepsilon_0$. Therefore we obtain for the coefficients $v_{a\alpha n}$

$$i\omega v_{a\alpha n} = \sum_{a'\alpha' n'} \Omega_{a\alpha,a'\alpha'}(\mathbf{R}_n - \mathbf{R}'_n) v_{n'a'\alpha'} \ ,$$

where $\Omega_{a\alpha,a'\alpha'} = \frac{c_a}{\sqrt{\varepsilon\mu}} X_{Ha\alpha,a'\alpha'}$. For the dependence of the lattice site we make the ansatz $v_{na\alpha} = w_{a\alpha} e^{i\mathbf{q}\mathbf{R}_n}$. Inserting this into the previous equation yields

$$i\omega w_{a\alpha} = \sum_{n'a'\alpha'} \Omega_{a\alpha,a'\alpha'}(\mathbf{R}_n - \mathbf{R}_{n'}) e^{i\mathbf{q}(\mathbf{R}_{n'} - \mathbf{R}_n)} w_{a'\alpha'} \ .$$

With the Definition $\Omega_{a\alpha,a'\alpha'}(\mathbf{q}) = \sum_n \Omega_{a\alpha,a'\alpha'}(\mathbf{R}_n) e^{i\mathbf{q}\mathbf{R}_n}$ we get

$$i\omega w_{a\alpha} = \sum_{a\alpha,a'\alpha'} \Omega_{a\alpha,a'\alpha'}(\mathbf{R}_n) w_{a'\alpha'} \ ,$$

which has to be solved for all possible values of the wave vector $\mathbf{q}$. The explicit form of the Fourier transform of the matrix $\Omega(\mathbf{R})$ reads

$$\Omega(\mathbf{q}) = \frac{1}{\sqrt{\varepsilon\mu}} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 1-e^{iq_3} & -1+e^{iq_2} \\ 0 & 0 & 0 & -1+e^{iq_3} & 0 & 1-e^{iq_1} \\ 0 & 0 & 0 & 1-e^{iq_2} & -1+e^{iq_1} & 0 \\ 0 & 1-e^{-iq_3} & -1+e^{-iq_2} & 0 & 0 & 0 \\ -1+e^{-iq_3} & 0 & 1-e^{-iq_1} & 0 & 0 & 0 \\ 1-e^{-iq_2} & -1+e^{-iq_1} & 0 & 0 & 0 & 0 \end{pmatrix} .$$

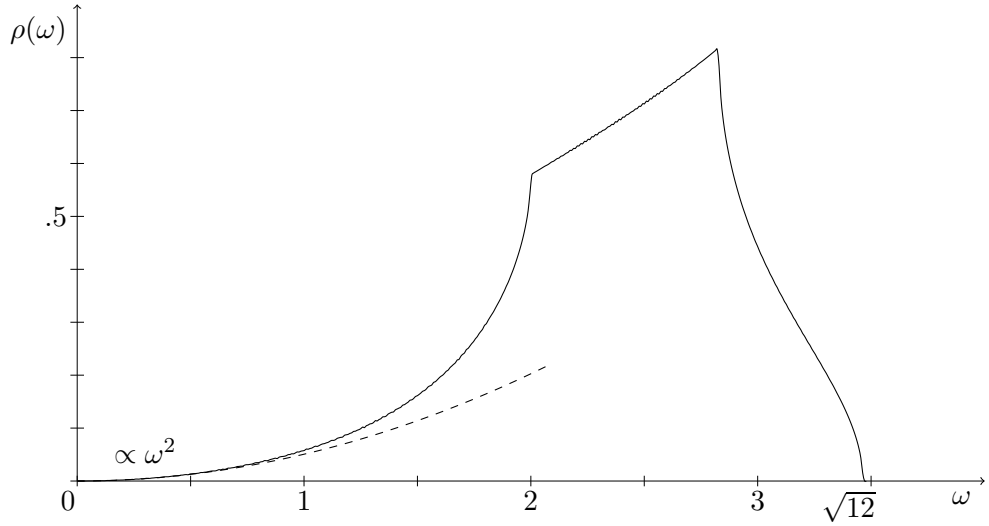


Figure 2.5: DOS of the clean system. The dashed line shows the asymptotics for small ω, which is given by $\omega^2/(2\pi^2)$. Since the DOS is symmetric with respect to $\omega = 0$ it is reasonable to show only its positive part.

The six eigenvalues of $\Omega(\mathbf{q})$ are given by 0 and $\pm\frac{2}{\sqrt{\varepsilon\mu}}\sqrt{\sin^2\frac{q_1}{2}+\sin^2\frac{q_2}{2}+\sin^2\frac{q_3}{2}}$, each of them with multiplicity two. As predicted in the previous section, one third of the eigenvalues is zero. The vector $\mathbf{q}$ must be an element of the first Brillouin zone, i.e. for a lattice with dimensions $N_1 \times N_2 \times N_3$ we have the condition

$$q_\alpha = \frac{2\pi l_\alpha}{N_\alpha} \,,$$

with $l_\alpha \in \{-\frac{N_\alpha}{2}+1, \ldots, \frac{N_\alpha}{2}\}$. To calculate the density of states we have to evaluate the integral

$$\frac{1}{8\pi^3}\int_{S(\omega)} \frac{ds}{|\nabla_{\mathbf{q}}\omega(\mathbf{q})|}$$

Calculation of the full integral in a closed analytical expression is impossible. Using numerical methods of integration leads to the density of states (DOS) shown in figure 2.5. The spectrum ends at $\omega = \frac{\sqrt{12}}{\sqrt{\varepsilon\mu}}$. The two kinks reflect the geometry of the first Brillouin zone.

It is possible to calculate the behavior near $\omega = 0$ for $\omega < 0$ and $\omega > 0$. Due to symmetry we consider only the case $\omega > 0$, i.e. $\omega(\mathbf{q}) = \sqrt{\sin^2\frac{q_1}{2}+\sin^2\frac{q_2}{2}+\sin^2\frac{q_3}{2}}$. With

$$|\nabla_{\mathbf{q}}\omega(\mathbf{q})| = \frac{1}{\omega}\sqrt{\sin^2 q_1 + \sin^2 q_2 + \sin^2 q_3}$$

we linearize the problem in the vicinity of $\omega = 0$. Since the only zero of the square root $\sqrt{\sin^2\frac{q_1}{2}+\sin^2\frac{q_2}{2}+\sin^2\frac{q_3}{2}}$ in the first Brillouin zone is $q_1 = q_2 = q_3 = 0$ we simply find

$$|\nabla_{\mathbf{q}}\omega(\mathbf{q})| \approx 1 \,.$$

Now the integral becomes very simple,

$$\frac{1}{8\pi^3}\int_{S(\omega)}\frac{ds}{|\nabla_{\mathbf{q}}\omega(\mathbf{q})|}\approx\frac{1}{8\pi^3}\int_{S(\omega)}ds=\frac{4\pi}{8\pi^3}\omega^2=\frac{1}{2\pi^2}\omega^2\ .$$

The good agreement of the DOS with this expansion in the vicinity of $\omega = 0$ can be seen in figure 2.5. This quadratic behavior corresponds to that expected from Debye's model for the excitations of a lattice.

2.6 Numerical calculation of the eigenmodes

To calculate the eigenvalues of X_H we first notice that the matrix can be transformed into an antisymmetric form:

$$\begin{aligned}\tilde{X}_H &:= \begin{pmatrix} T_{\mu^{-1/2}} & 0 \\ 0 & T_{\varepsilon^{-1/2}}\end{pmatrix} X_H \begin{pmatrix} T_{\mu^{1/2}} & 0 \\ 0 & T_{\varepsilon^{1/2}}\end{pmatrix} \\ &= \begin{pmatrix} 0 & T_{\mu^{-1/2}} A T_{\varepsilon^{-1/2}} \\ -T_{\varepsilon^{-1/2}} A^t T_{\mu^{-1/2}} & 0\end{pmatrix}\ .\end{aligned}$$

In particular, $\tilde{X}_H$ can be diagonalized by an orthogonal transformation. To simplify the numeric problem, we consider the matrix

$$\tilde{X}_H^2 = \begin{pmatrix} -T_{\mu^{-1/2}} A T_{\varepsilon^{-1}} A^t T_{\mu^{-1/2}} & 0 \\ 0 & -T_{\varepsilon^{-1/2}} A^t T_{\mu^{-1}} A T_{\varepsilon^{-1/2}}\end{pmatrix}\ .$$

Since the upper and the lower block have the same spectrum of eigenvalues, it is sufficient to consider only the upper block. Its eigenvalues are simply the eigenvalues of X_H squared. Of course, the multiplicity of the eigenvalues decreases by a factor two, which optimizes the problem for numeric calculations. Moreover, the block is symmetric. This allows us to use the routine *dsbev* of the linear algebra library *LAPACK*. This routine is optimized for the diagonalization of real symmetric matrices with double numerical accuracy.

2.7 Gaussian disorder

In the clean system the values of μ and ε are equal to 1. The first model for the disorder we consider is of Gaussian type. We consider the complex (K, K') and replace the constant values of $\varepsilon = 1$ and $\mu = 1$ by Gaussian distributions with mean 1 and variance τ.

Now the strategy is simple, it is the usual Monte-Carlo integration. For a system consisting of N vertices we randomly choose a stochastic set of N values of ε and μ each and construct the matrix $\tilde{X}_H$. Then the eigenmodes are calculated by numerical diagonalization and stored in a list. Then the procedure has to be restarted.

In principle, every eigenvalue produces a δ-peak in the spectrum. To see a curve, the peaks have to be regularized by a peak with a finite width. The choice of the value of the width is rather uncritical. However, it should be small enough in order not to cover up the structure of the curve.

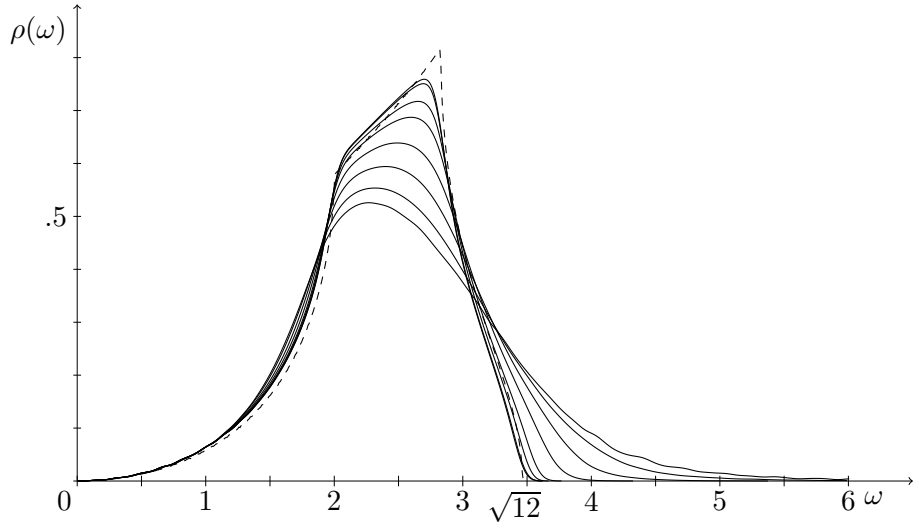


Figure 2.6: Spectrum of the system with Gaussian disorder.

For $\tau = 0$ the system is clean and we obtain the DOS shown in figure 2.5 . For increasing values of τ, the DOS is shown in figure 2.6 .

The figure shows systems with variance 0.01 up to 0.28 . The effect of the increasing value of τ is a broadening of the support; recall that the hightest eigenvalue of the clean system is $\sqrt{12}$. As an additional effect, the edges of the spectrum are rounded.

From a physical point of view it is clear that the variance must be limited: With increasing values of τ the probability to obtain negative values for ε or μ becomes bigger. This should be avoided. One possibility to avoid this problem is to cut the Gaussian distribution and to use just its positive part. However, for the numerical calculation, this makes no difference as long as τ does not exceed approximately 0.29 . If τ becomes larger, complex eigenvalues appear, i.e. the system becomes instable.

One may ask what happens if the variances of μ and ε are set independently. We find that the qualitative result does not differ from figure 2.6 .

Behavior at $\omega = 0$

Figure 2.7 shows the curves of figure 2.6 under a magnifying glass. It is remarkable that the curves do not differ from each other, i.e. the disorder of Gaussian type has no effect on the low-frequency excitations.

Similar results in the literature

The results obtained by modelling the Gaussian disorder are very similar to those found in [SDG98]. In this article, a cubic lattice of one-dimensional harmonic oscillators of equal masses is analyzed. The coupling constants between nearest neighbors are given by a Gaussian distribution. Disregarding the zero modes, we found if the system is clean that the DOS is exactly the same which has been found in section

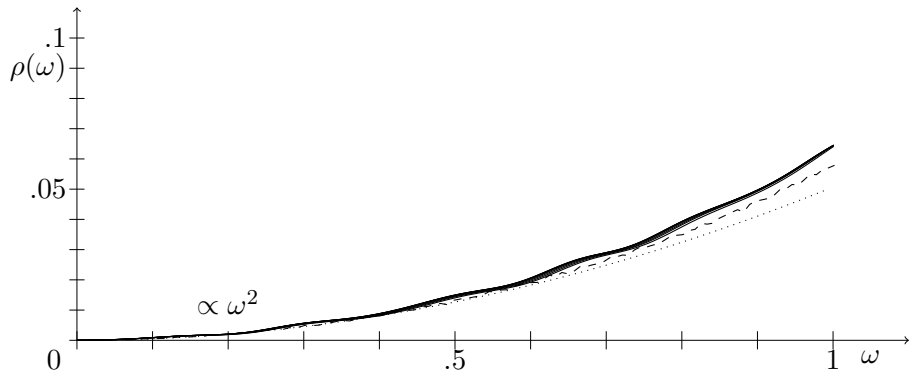


Figure 2.7: Spectrum of the system with Gaussian disorder near $\omega = 0$. The dashed line shows the analytical solution of the clean system, the dotted one the function $\omega^2/(2\pi^2)$.

2.5, see figure 2.5. Switching on the disorder, there is at least a qualitative similarity left. However, a total equivalence of the models is impossible, since the number of degrees of freedom differs by a factor 6 but the number of random variables, namely μ and ε compared with the number coupling constants of the harmonic oscillators, only by 2/3.

2.8 Non-Gaussian disorder

Our second, more interesting model of disorder is constructed as follows: We have two kinds of impurities, the one is modelled by changing the value of ε from 1 to $\varepsilon_{\text{pert}}$, the other by setting $\mu = \mu_{pert}$. We want to choose $\varepsilon_{\text{pert}} = \infty$ and $\mu_{\text{pert}} = 0$, which corresponds to a metallic or a superconducting grain in the optical medium, respectively. However, these values cause problems in the numerical calculations, thus we approximate them by $\varepsilon_{\text{pert}} = 10^{10}$ and $\mu_{\text{pert}} = 10^{-10}$. Now it remains to fix the concentration c of impurities. We analyzed systems starting from $c = 0$ up to $c = 0.5$.

Again, we use Monte-Carlo integration. For a system consisting of N vertices we choose a stochastic sets of $N \cdot c$ vertices where we set $\varepsilon = \varepsilon_{\text{pert}}$ and another stochastic set with $\mu = \mu_{pert}$. The eigenmodes are calculated by numerical diagonalization and stored in a list.

Of course, the more random matrices are diagonalized, the smoother the average DOS will be. In our case, we choose a system size up to $N_1 = N_2 = N_3 = 16$, which corresponds to a matrix of size of approximately 12200×12200. Using such large matrices, the convergence of the DOS is rather fast, it is sufficient to diagonalize a few hundred or even less matrices to obtain a smooth DOS.

Results

Figure (2.8) shows the average DOS for a series of values of c. It is clear that for small values of c the curve resembles the DOS of the clean system shown in figure

2.5. Let us remark that the total mass of all spectra is normalized to 1, in order to compare them.

In figure 2.8 a), we see some different effects of the disorder. The first one is a general shift of the DOS to lower frequencies. The right edge of the DOS moves left with increasing disorder.

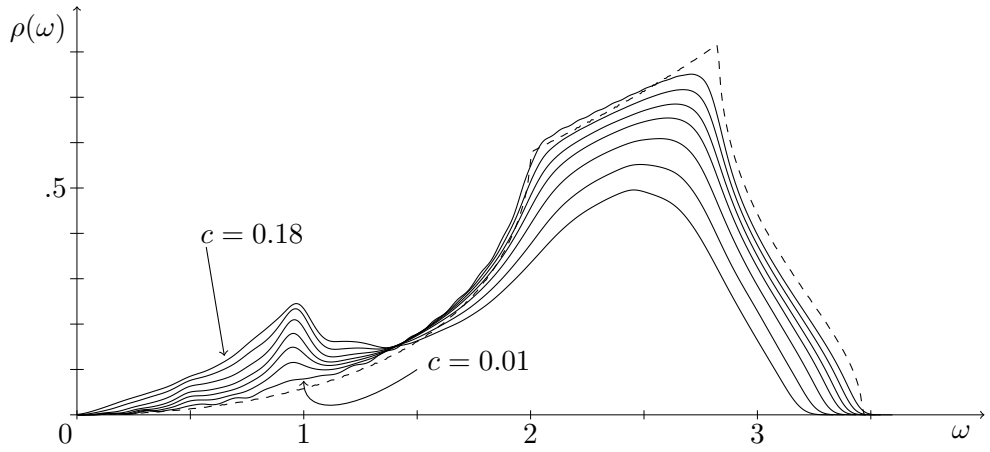
The second one is a softening of the structure of the clean DOS. The corners at $\omega \approx 2$ and $\omega \approx 3$ disappear.

The third effect is the appearance of a peak around $\omega \approx 0.9$ for almost all values of c. For higher values of c, this peak is accompanied by two more peaks. Properties of the peak around $\omega = 0.9$ are shown in figures 2.9a)-d). Using *Gnuplot*, we fitted a Gaussian curve $h_p \exp(-(\omega - \omega_p)^2/w_p^2)$ with the free parameters position ω_p, height h_p, and width ω_p in the peak. *Gnuplot* uses the nonlinear least-square Marquardt-Levenberg algorithm and produces reliable results. This has been done for many values of the impurity concentration c, namely $c = 0.01, 0.02, \ldots, 0.3, 0.32, \ldots, 0.5$. Figure 2.9a) shows that the position is almost constant. The dependence on c is only weak. Figure 2.9b) shows the relative number of modes which appear at low frequencies due to the disorder. Up to the value $c \approx 0.35$, the number increases almost linearly. The height, shown in figure 2.9b) also starts linearly for small values of c.

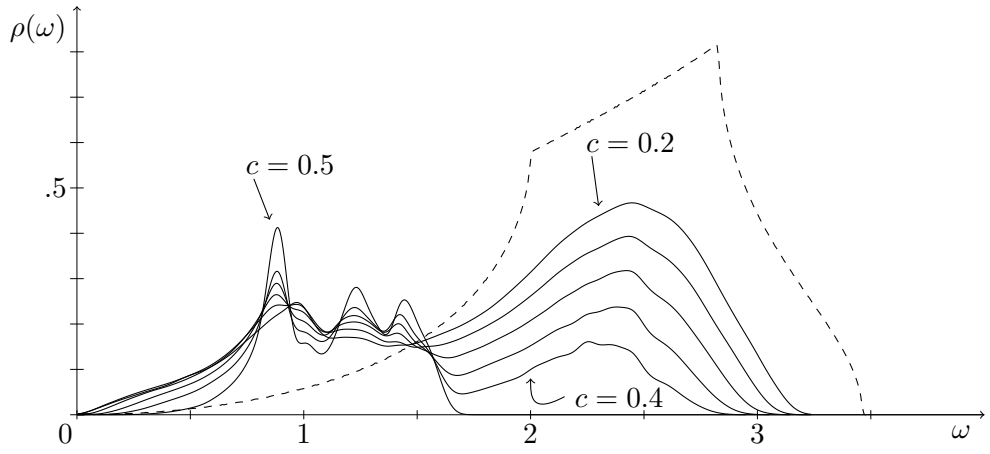
Thus, we have found an interesting feature of a disordered optical medium using this special model of disorder. The peak can be interpreted as *boson peak*, since it fits into the context in which this notion is used: It is an enhancement of the DOS for low frequency excitations of a bosonic system.

2.9 Outlook

It would be desirable to have an analytical method to compare the results of this chapter with. Here, it is promising to search for a kind of mean-field approximation. The task is to construct an effective optical medium, which has approximately the same DOS as the disorder-averaged medium under consideration. The great advantage of such a medium would be its translational invariance. This permits the application of analytical methods and to calculate the DOS of large systems. This method is known as *coherent potential approximation* [Czy00]. In [SDG98] it has been successfully applied to a lattice of harmonic oscillators with a random nearest neighbor coupling. However, a direct transfer of this method to the optical medium is impossible.

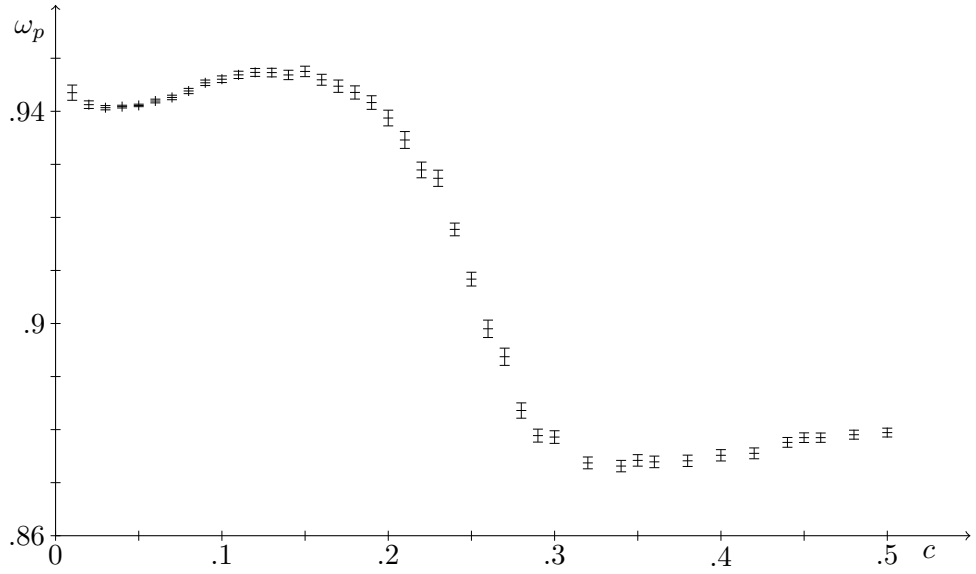


a)

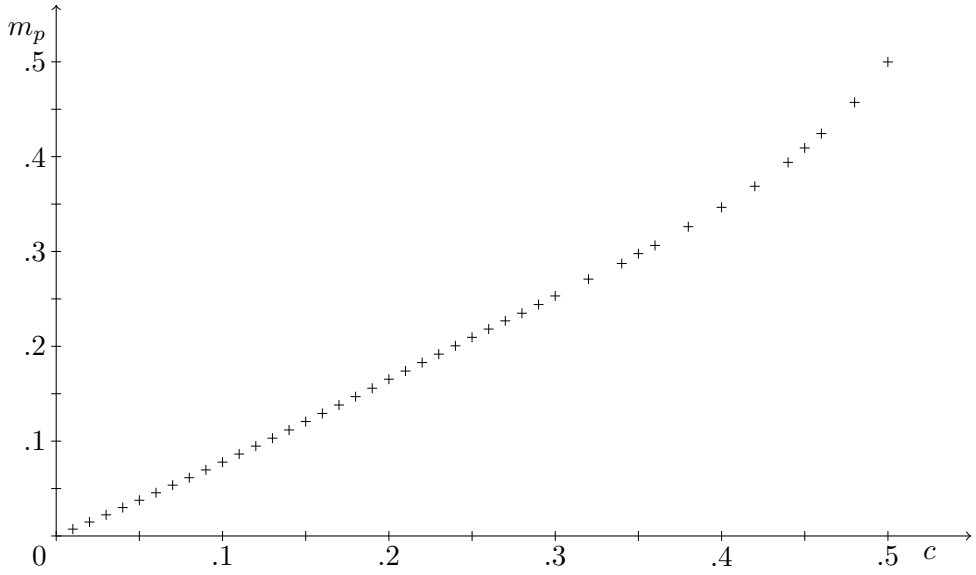


b)

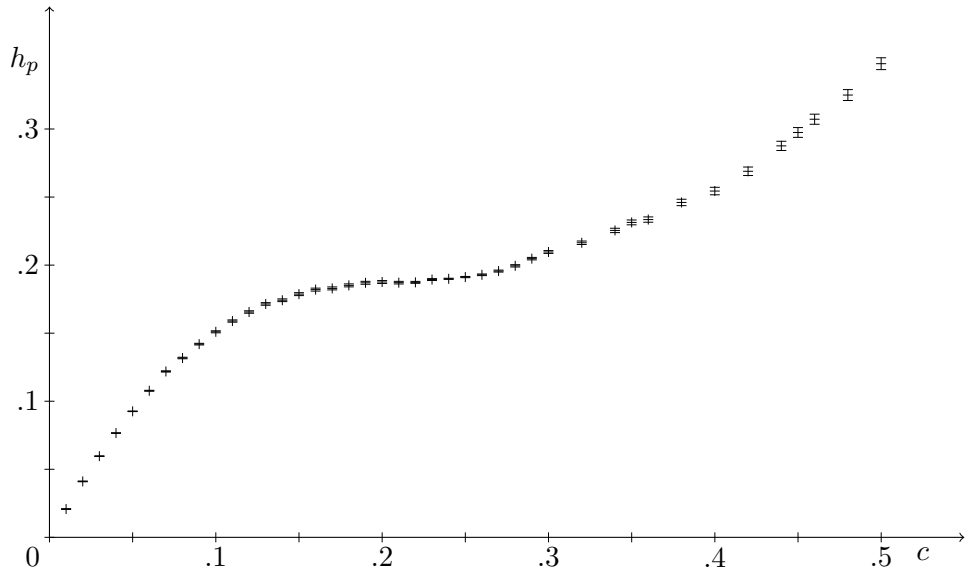
Figure 2.8: The density of states $\rho(\omega)$. Figure a) shows spectra with $c = 0.01$ up to $c = 0.15$, figure b) with $c = 0.2$ up to $c = 0.5$. The dashed line shows the DOS of the clean system.



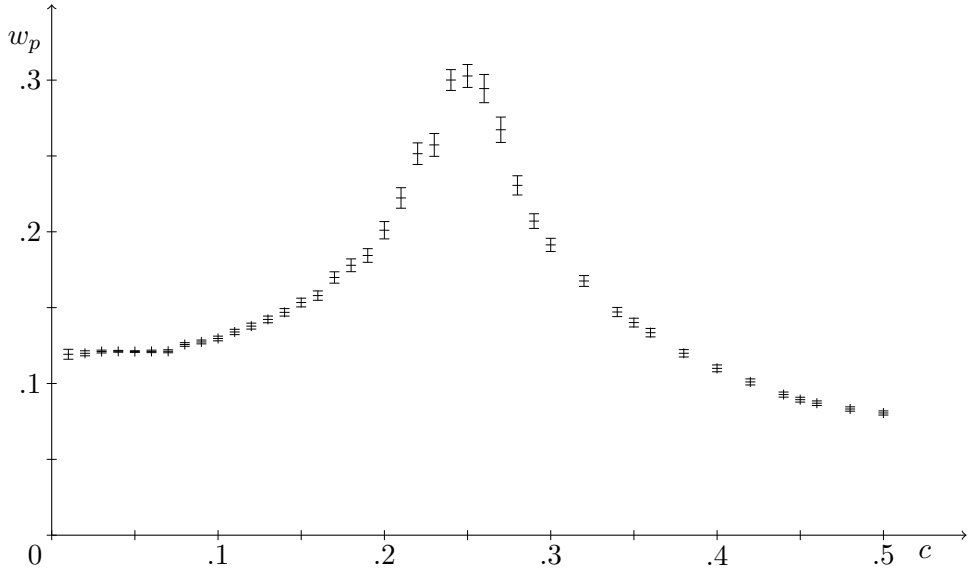
a)



b)



c)



d)

Figure 2.9: Properties of the peak near $\omega = 0$. On the previous page, diagram a) shows the position of the peak and b) the relative number of all modes which appear at low frequencies due to the disorder. In diagram c) the height of the peak is plotted and in d) the width.

Chapter 3

Bosonic random matrix ensemble

In this chapter we investigate a random matrix model for disordered bosons. The model can be understood as a system of N coupled harmonic oscillators, or alternatively, as the most general Hamiltonian which is bilinear in N different pairs of bosonic Fock operators. The random matrix in this model is given by the masses and stiffnesses of the oscillators. In order to define a stable system, this matrix has to fulfill a certain stability condition, which fixes the domain of our random matrix ensemble. We will see that it is diffeomorphic to a cone in the real symplectic Lie algebra. To construct a reasonable probability measure, we are guided by simplicity and symmetry. These principles lead to a model which can be completely solved. That is, all correlation functions can be calculated by means of bi-orthogonal polynomials for all N. A comprehensive discussion has been published in [LSZ06]. However, here we present another approach to the arising Harish-Chandra–Itzykson–Zuber type integral.

3.1 The quadratic Hamiltonian of a stable motion

Let $Q_1, \ldots, Q_N$ be canonical position variables and $P_1, \ldots, P_N$ their conjugated momenta. We start with a quadratic Hamiltonian of the form

$$H = \frac{1}{2} \sum_{i,j=1}^{N} \left(P_i B_{ij} P_j + Q_i C_{ij} Q_j + P_i A_{ij} Q_j + Q_i A_{ij}^t P_j \right), \tag{3.1}$$

where A, B, and C are real matrices satisfying the conditions $B = B^t$ and $C = C^t$. In addition, we impose

$$h := \begin{pmatrix} B & A \\ A^t & C \end{pmatrix} > 0\,, \tag{3.2}$$

i.e. all eigenvalues of this matrix should be positive. This condition ensures positivity of the characteristic frequencies of the system described by (3.1) and therefore a stable motion.

Now we rewrite the Hamiltonian in the form

$$H = \frac{1}{2} \begin{pmatrix} P & Q \end{pmatrix} X \begin{pmatrix} Q \\ -P \end{pmatrix}$$

with

$$X = \begin{pmatrix} A & -B \\ C & -A^t \end{pmatrix} = hJ \quad \text{and} \quad J = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix} .$$

To handle these kind of Hamiltonians we use the theory of Lie groups and Lie algebras [Bak04].

It can be easily seen that X fulfills the defining condition for the Lie algebra $\mathfrak{sp}(2N, \mathbb{R})$ of the real symplectic group $\mathrm{Sp}(2N, \mathbb{R})$. The latter is defined by

$$\mathrm{Sp}(2N, \mathbb{R}) = \{g \in \mathrm{GL}(2N, \mathbb{R}) | g^t J g = J\} ,$$

and the former is given by

$$\mathfrak{sp}(2N, \mathbb{R}) = \{X \in \mathrm{M}_{2N}(\mathbb{R}) | X^t J + JX = 0\} .$$

Therein, $\mathrm{M}_{2N}(\mathbb{R})$ denotes the set of real $2N \times 2N$ matrices. Via a symplectic and an additional unitary transformation X can be brought to diagonal form

$$UgXg^{-1}U^{-1} = i\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N, -\omega_1, -\omega_2, \ldots, -\omega_N) =: i\Omega \tag{3.3}$$

with some particular $g \in \mathrm{Sp}(2N, \mathbb{R})$ and

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_N & i\mathbf{1}_N \\ \mathbf{1}_N & -i\mathbf{1}_N \end{pmatrix} .$$

The eigenvalues ω_i are the characteristic frequencies of the Hamiltonian (3.1). In order to obtain a concise representation of the set containing all systems with positive frequencies, we combine the symplectic and the unitary transformation. This leads to a representation of the real symplectic group as a subgroup of $\mathrm{GL}(2N, \mathbb{C})$,

$$\tilde{\mathrm{Sp}}(2N, \mathbb{R}) = \{\tilde{g} \in \mathrm{GL}(2N, \mathbb{C}) | \tilde{g} = \Sigma_3 \tilde{g}^{-1\dagger} \Sigma_3 = \Sigma_1 \tilde{g} \Sigma_1\} .$$

To distinguish this representation from the fundamental one, we added a tilde over the symbols. The matrices Σ_1 and Σ_3 are given by

$$\Sigma_1 = \begin{pmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_N & 0 \end{pmatrix} \quad \text{and} \quad \Sigma_3 = \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix} .$$

The corresponding Lie algebra reads

$$\tilde{\mathfrak{sp}}(2N, \mathbb{R}) = \{X \in \mathrm{M}_{2N}(\mathbb{C}) | X = -\Sigma_3 X^\dagger \Sigma_3 = \Sigma_1 \bar{X} \Sigma_1\} . \tag{3.4}$$

For the Hamiltonian H the unitary transformation U corresponds to a transformation to the representation with annihilation and creation operators $\frac{1}{\sqrt{2}}(Q + iP) \mapsto \hat{a}$ and $\frac{1}{\sqrt{2}}(Q - iP) \mapsto \hat{a}^\dagger$. The transformed matrix $\tilde{X} = UXU^{-1}$ is contained in $\tilde{\mathfrak{sp}}(2N, \mathbb{R})$ and has the form

$$\tilde{X} = \begin{pmatrix} a & b \\ b^\dagger & -a^t \end{pmatrix} , \tag{3.5}$$

where $a^\dagger = -a$ and $b^t = b$. All matrices of this type with positive frequencies ω_i are given by the set

$$\mathcal{E}^0 = \{\tilde{X} \in \tilde{\mathfrak{sp}}(2N,\mathbb{R}) | X = iAd(\tilde{g})\Omega, \tilde{g} \in \tilde{\mathrm{Sp}}(2N,\mathbb{R})/T, \omega_i > 0\}\,,$$

which is a positive cone in $\tilde{\mathfrak{sp}}(2N,\mathbb{R})$. Since the diagonal matrices Ω commute with elements of the maximal torus T in $\tilde{\mathrm{Sp}}(2N,\mathbb{R})$, the representation of a particular $\tilde{X}$ by $i\mathrm{ad}(g)\Omega$ is not unique. Therefore we restrict $\tilde{g}$ to the quotient of the symplectic group and its maximal Torus.

To simplify our notation we omit the tilde from now on. The symbols $\mathrm{Sp}(2N,\mathbb{R})$, $\mathfrak{sp}(2N,\mathbb{R})$, g, X, etc. shall always refer the representations of the Lie group and its Lie algebra obtained by the unitary transformation (3.3) and (3.4), which is equivalent to the fundamental representation.

3.2 Probability measure

The definition of a random matrix ensemble requires two important components. First, one needs to define the domain of the ensemble, which has been done in the previous section. Second, a reasonable probability measure is required to fix the distribution of the ensemble.

Now we want to define a probability measure $P(X)dX$ on the set $\mathcal{E}^0$, which shall give the probability to find a system corresponding to the matrix X in the volume element dX. In the case of GOE, GUE, and GSE the situation is clear. The Gaussian measures are constructed in such a way that they are invariant under the action of the whole symmetry group. This suggests to make a similar ansatz for the desired probability measure,

$$P(X)dX \overset{?}{\propto} e^{-\mathrm{Tr}X^2}dX\,,$$

which is invariant under the symmetry group of our ensemble given by the real symplectic group $\mathrm{Sp}(2N,\mathbb{R})$. This causes a problem: since $\mathrm{Sp}(2N,\mathbb{R})$ is noncompact the integral over the domain $\mathcal{E}^0$ does not exist.

Thus the best we can do is to postulate invariance under the maximal compact subgroup of $\mathrm{Sp}(2N,\mathbb{R})$, which is given by $\mathrm{U}(N)$. In order to enforce normalizability of the probability measure we use the Lebesgue measure on $\mathcal{E}^0$ together with an exponential factor,

$$P(X)dX = c\exp(i\tau\mathrm{Tr}\Sigma_3 X/2)dX\,, \tag{3.6}$$

with a free parameter $\tau > 0$ and a normalization constant c. The $\mathrm{U}(N)$-invariance can be seen by writing

$$g = \begin{pmatrix} u & 0 \\ 0 & -u^t \end{pmatrix}$$

with a unitary u and substituting X by gXg^{-1}.

The measure (3.6) can be generalized to a whole family of measures. Since the determinant of X is always positive, we can multiply by an additional factor $\mathrm{Det}(X)^{(l-1)/2}$ with positive integer l. We get

$$P(X)dX = c\exp(i\tau\mathrm{Tr}\Sigma_3 X/2)\mathrm{Det}(X)^{(l-1)/2}dX\,. \tag{3.7}$$

This measure is still $U(N)$-invariant.

A more natural motivation to add this factor comes from the following point of view: Every positive symmetric matrices h, given by (3.2) can be obtained by adding up a sufficiently large number of rank-one projectors, according to

$$h_{ij} = \sum_{\alpha=1}^{M} v_{i\alpha} v_{j\alpha} \quad , i,j = 1,\dots,2N \ ,$$

with a set of real numbers $v_{i\alpha}$. The number of different projectors must be at least N, we introduced here $M := N + l$. We now consider the $v_{i\alpha}$ as the fundamental variables, and choose them to be independently and normally distributed with zero mean and variance τ^{-1}. Pushing forward the probability distribution of the $v_{i\alpha}$ to a probability distribution $d\mu(h)$ for h leads to

$$d\mu(h) \propto e^{-\tau \mathrm{Tr} h/2} \mathrm{Det}(h)^{(l-1)/2} \prod_{i \le j} dh_{ij} \ .$$

Using $X = hJ$, we immediately obtain the family of measures given by (3.7). We see that the parameter l arises in a natural way by the choice of M.

3.3 Disorder average of a radial function

We are interested in the ensemble average of so called radial functions f, which only depend on the characteristic frequencies $\omega_1,\dots,\omega_N$ of the system. Averaging over all allowed systems leads to the integral

$$\begin{aligned} &\int_{\mathcal{E}^0} \exp\left(i\tau \mathrm{Tr}(\Sigma_3 X)/2\right) \mathrm{Det}(X)^{(l-1)/2} f(\omega_1,\dots,\omega_N) dX \\ &= \int_{\mathbb{R}_+^N} \prod_{i=1}^{N} d\omega_i \int_{\mathrm{Sp}(2N,\mathbb{R})/T} \exp\Big(i\tau \mathrm{Tr}(\Sigma_3 g \Omega g^{-1})/2\Big) \mathrm{Det}(i\Omega)^{(l-1)/2} j f(\omega_1,\dots,\omega_N) dg_T \ . \end{aligned} \tag{3.8}$$

Therein j denotes the Jacobian of the transformation $X = ig\Omega g^{-1}$ which will be calculated in the next paragraph. $T \simeq \mathrm{U}(1)^N$ is the maximal torus in $\mathrm{Sp}(2N,\mathbb{R})$, the invariant measure on $\mathrm{Sp}(2N,\mathbb{R})/T$ is denoted by dg_T.

Derivation of the Jacobian

The Jacobian of the transformation $X = i\mathrm{ad}(h)\Omega$ is given by

$$j = \sqrt{|\mathrm{Det}\, g_{ij,kl}|}$$

where $g_{ij,kl}$ is the matrix of the metric induced by the Killing form, see e.g. [Kna05]. For a matrix $X \in \mathfrak{sp}(2N,\mathbb{R})$ with entries x_{ij} we have $\sum_{ij,kl} g_{ij,kl} dx^{ij}\, dx^{kl} = \mathrm{Tr}(dX)^2$, i.e. the metric is flat. In our case we have $X = ih\Omega h^{-1}$ and therefore

$$h^{-1} dX h = i d\Omega + [h^{-1} dh, i\Omega] \, .$$

The term $h^{-1}dh$ can be regarded as an element of the Lie algebra, and therefore it can be written as a linear combination of eigenvectors of the map $\mathrm{ad}(i\Omega)$, where $i\Omega$ lies in a suitable chosen Cartan subalgebra,

$$h^{-1}dh = \sum_{\alpha} (h^{-1}dh)_{\alpha} \,. \tag{3.9}$$

The sum runs over all roots of the symplectic Lie algebra, which we calculate now.

Roots of $\mathfrak{sp}(2N,\mathbb{R})$. In our representation of $\mathfrak{sp}(2N,\mathbb{R})$, which is isomorphic to the fundamental representation, evaluation of the conditions given by (3.4) leads to the explicit form

$$\mathfrak{sp}(2N,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ b^{\dagger} & -a^{t} \end{pmatrix} \Big| a,b \in M_N(\mathbb{C}), a = -a^{\dagger}, b = b^{t} \right\} \,.$$

with $a = -a^{\dagger}$ and $b = b^{t}$. To calculate the roots, it is necessary to complexify the Lie algebra, which means to replace the underlying field $\mathbb{R}$ by $\mathbb{C}$. In the complexified Lie algebra the condition $a = -a^{\dagger}$ is empty, i.e. a can be any complex matrix. In addition, b and $b^{\dagger}$ become independent, therefore the complexified Lie algebra reads

$$\mathfrak{sp}(2N,\mathbb{R}) \otimes \mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & -a^{t} \end{pmatrix} \Big| a,b,c \in M(N,\mathbb{C}), b = b^{t}, c = c^{t} \right\}$$

A basis for these matrices is given by

$$a \in \mathrm{span}(E_{ij}, i,j = 1 \ldots N) \,,\; b,c \in \mathrm{span}((E_{ij} + E_{ji})/2, i,j = 1 \ldots N) \,,$$

where E_{ij} is an $N \times N$-matrix which has an entry 1 in the i-th row and j-th column and zeros else. A Cartan subalgebra in our representation is given by

$$\mathfrak{a} = \{ i\Omega | \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N, -\omega_1, \cdots - \omega_N), \omega_i \in \mathbb{R} \} \,.$$

It is easy to see that for any $X \in \mathfrak{a}$ the mapping ad_X is diagonal in the chosen basis:

$$\begin{aligned}
\mathrm{ad}_X \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} &= i(\omega_i - \omega_j) \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \,, \\
\mathrm{ad}_X \begin{pmatrix} 0 & (E_{ij} + E_{ji})/2 \\ 0 & 0 \end{pmatrix} &= i(\omega_i + \omega_j) \begin{pmatrix} 0 & (E_{ij} + E_{ji})/2 \\ 0 & 0 \end{pmatrix} \,, \\
\mathrm{ad}_X \begin{pmatrix} 0 & 0 \\ (E_{ij} + E_{ji})/2 & 0 \end{pmatrix} &= -i(\omega_i + \omega_j) \begin{pmatrix} 0 & 0 \\ (E_{ij} + E_{ji})/2 & 0 \end{pmatrix} \,.
\end{aligned}$$

Hence, the roots are given by

$$\pm 2i\omega_i \,,\; \pm i(\omega_i - \omega_j) \,, \text{ and } \pm\, i(\omega_i + \omega_j) \,.$$

The indices i and j, $i \neq j$ run from 1 to N. Using relation (3.9) and the special form (3.3) of Ω we get

$$\mathrm{Tr}(dX)^2 = -2 \sum_{i=1}^{N} (d\omega_i)^2 + 2 \sum_{\alpha > 0} \mathrm{Tr}(g^{-1}dg)_{\alpha} (g^{-1}dg)_{-\alpha} \,.$$

By reading off the coefficients of the metric g^{ij} in the new coordinates we arrive at

$$j = 2^{N/2} \prod_{\alpha>0} \alpha^2 = 2^{5N/2} \left(\prod_{i=1}^{N} \omega_i \prod_{\substack{i=2 \\ j<i}}^{N} (\omega_i^2 - \omega_j^2) \right)^2 .$$

The prefactor $2^{5N/2}$ does not influence the calculations, we absorb it into the normalization constant $c_{N,l}$ which will be introduced below.

3.4 Reduction of the disorder integral

In this section we calculate the inner integral of (3.8), which is an integral of the Harish-Chandra–Itzikson–Zuber type, i.e. it can be evaluated in a closed form. There exist three different methods to treat it. One can exploit the semiclassical exactness of the integral, or, in hindsight, introduce coordinates which reduce the problem to simple Gaussian integration, as presented in [LSZ06]. Here, we show how a differential equation for the integral can be constructed. In the second step we solve this equation. This method can be found e.g. in [GW96], where it is applied to a different class of integrals and superintegrals.

We start with a slight generalization of the original problem,

$$I(S,T) = \int\limits_{\mathrm{Sp}(2N,\mathbb{R})/T} \exp(\mathrm{Tr}(TgSg^{-1}) dg_T \,, \tag{3.10}$$

with $S, T \in \mathfrak{sp}(2N,\mathbb{R})$. Recall that the desired integral in (3.8) reads

$$I(\Omega, i\tau\Sigma_3/2,) = \int\limits_{\mathrm{Sp}(2N,\mathbb{R})/T} \exp\left(i\tau \mathrm{Tr}(\Sigma_3 g \Omega g^{-1})/2\right) dg_T \,. \tag{3.11}$$

We will derive this from (3.10) by the replacements $T \to i\tau\Sigma_3/2$ and $S \to \Omega$. As we will see, this replacement contains a non-trivial limit process.

The matrices S and T are diagonalizable by symplectic matrices. We can write

$$S = \mathrm{ad}(h) S_d = h S_d h^{-1} \quad \text{and} \quad T = \mathrm{ad}(g^{-1}) T_d = g^{-1} T_d g \,,$$

with $g, h \in \mathrm{Sp}(2N,\mathbb{R})$ and S_d, T_d diagonal, i.e.

$$\begin{aligned} S_d &= i\mathrm{diag}(s_1, \ldots, s_N, -s_1, \ldots, -s_N) \quad \text{and} \\ T_d &= i\mathrm{diag}(t_1, \ldots, t_N, -t_1, \ldots, -t_N) \end{aligned} \tag{3.12}$$

with real s_i and t_i . For $S, T \in \mathfrak{sp}(2N,\mathbb{R})$ we define the function

$$W(S,T) = \exp\left(\mathrm{Tr}(TS)\right) .$$

Due to the chosen representation of the symplectic group the elements of $\mathfrak{sp}(2N,\mathbb{R})$ can be written in the form

$$\begin{aligned} S &= \begin{pmatrix} a & b \\ b^\dagger & -a^t \end{pmatrix} = h S_d h^{-1} = \mathrm{ad}(h) S_d \\ T &= \begin{pmatrix} c & d \\ d^\dagger & -c^t \end{pmatrix} = g^{-1} T_d g = \mathrm{ad}(g^{-1}) T_d, \end{aligned}$$

where a, b, c and d are arbitrary complex N by N matrices, $g, h \in \mathrm{Sp}(2N,\mathbb{R})/T$. We obtain for the trace

$$\begin{aligned}
\mathrm{Tr}(TS) &= \mathrm{Tr}\begin{pmatrix} ca+db^\dagger & \dots \\ \dots & d^\dagger b + c^t a^t \end{pmatrix} \\
&= \mathrm{Tr}(ca + \bar{c}\bar{a} + bd^\dagger + \bar{d}\bar{b}^\dagger) \\
&= 2\mathrm{Tr}(\Re(ca) + \Re(b\bar{d})) \\
&= 2\sum_{i,j=1}^{N}\Big(\Re(c_{ij})\Re(a_{ji}) - \Im(c_{ij})\Im(a_{ji}) + \Re(b_{ij})\Re(d_{ji}) + \Im(b_{ij})\Im(d_{ji})\Big) \\
&= 2\sum_{i,j=1}^{N}\Big(-\Re(c_{ij})\Re(a_{ij}) - \Im(c_{ij})\Im(a_{ij}) + \Re(b_{ij})\Re(d_{ij}) + \Im(b_{ij})\Im(d_{ij})\Big)\,.
\end{aligned}$$

For the construction of the differential equation, we need the Laplace operator. In the used representation it acts by

$$\Delta_S = \sum_{i,j=1}^{N}\left(-\frac{\partial^2}{\partial\Re^2(a_{ij})} - \frac{\partial^2}{\partial\Im^2(a_{ij})} + \frac{\partial^2}{\partial\Re^2(b_{ij})} + \frac{\partial^2}{\partial\Im^2(b_{ij})}\right).$$

It is easy to check that $W(S,T)$ satisfies the relation

$$\begin{aligned}
\Delta_S W(S,T) &= \Delta_S \exp[\mathrm{Tr}(TS)] \\
&= 2\mathrm{Tr}(T^2)W(S,T) \\
&= 2\mathrm{Tr}(T_d^2)W(S,T)\,,
\end{aligned}$$

i.e. the eigenvalue of the Laplace operator only depends on the eigenvalues of T arranged in the diagonal matrix T_d. Due to the invariance of the integration measure dg_T we obtain the following equation for $I(S,T_d)$,

$$\begin{aligned}
\Delta_S I(S,T_d) &= \Delta_S \int\limits_{\mathrm{Sp}(2N,\mathbb{R})/T} W(g^{-1}T_d g, S)\,dg_T \\
&= 2\mathrm{Tr}(T_d^2)I(S,T_d)\,.
\end{aligned}$$

Since this differential equation does not depend on g and h it must be possible to decompose Δ_S according to

$$\Delta_S = \Delta_{S_d} + \Delta_{S_h}\,.$$

The part Δ_{S_d} is called the *radial part* and only acts on the eigenvalues of S. By the independence of the integral of h we immediately read off the relation

$$\Delta_{S_d} I(S,T_d) = \Delta_S I(S,T_d)\,.$$

For the radial part of the Laplace operator we have the relation

$$\Delta_{S_d} = -\sum_{n=1}^{N}\frac{1}{j(s_1,\dots,s_n)}\frac{\partial}{\partial s_n} j(s_1,\dots,s_n)\frac{\partial}{\partial s_n}\,,$$

where $j(s_1, \ldots, s_n)$ is the Jacobian of the transformation $S = hS_d h^{-1}$. The minus sign appears due to the definition (3.12). As we have seen, it is given by the product over all roots of $\mathrm{Sp}(2N, \mathbb{R})$,

$$j(s_1, \ldots, s_N) = 2^{5N/2} \left(\prod_{i=1}^{N} s_i \prod_{\substack{j=2 \\ 1 \le k < j}}^{N} (s_j^2 - s_k^2) \right)^2 .$$

For simplicity, we introduce an extra symbol for the square root of the Jacobian and neglect the constant factor,

$$j_s(S_d) \equiv j_s(s_1, \ldots, s_N) := \prod_{i=1}^{N} s_i \prod_{\substack{j=2 \\ 1 \le k < j}}^{N} (s_j^2 - s_k^2) , \tag{3.13}$$

For the symmetric factor of j we introduce

$$j_+(S_d) \equiv j_+(s_1, \ldots, s_N) := \prod_{i=1}^{N} s_i \prod_{\substack{i=2 \\ 1 \le k < j}}^{N} (s_j + s_k) . \tag{3.14}$$

For the integral we now make the ansatz

$$I(S_d, T_d) = \frac{\psi(S_d, T_d)}{j_s(S_d) j_s(T_d)}$$

and apply the radial part of the Laplace operator,

$$\begin{aligned} \Delta_{S_d} I(S,T) = \Delta_{S_d} \frac{\psi(S_d, T_d)}{j_s(S_d) j_s(T_d)} &= -\frac{1}{j_s(S_d) j_s(T_d)} \sum_{i=1}^{N} \frac{\partial^2 \psi(S,T)}{\partial s_i^2} \\ &= -\frac{1}{j_s(S_d) j_s(T_d)} \sum_{i=1}^{N} t_i^2 \psi(S,T) . \end{aligned} \tag{3.15}$$

Details of this calculations are given in appendix A.1 . We recall the relations $(S_d)_{ii} = is_i$ and $(T_d)_{ii} = it_i$ due to which a minus sign appears. The solution of this differential equation is restricted by symmetries of $I(S,T)$: obviously $I(S_d, T_d) = I(T_d, S_d)$ holds. Since the matrices which permute the eigenvalues of S are contained in the symplectic group, $I(S_d, T_d)$ is total symmetric in the two set of variables $s_1, \ldots, s_N$ and $t_1, \ldots, t_N$. By the antisymmetry of $j_s(S_d)$ we see that ψ has to be total antisymmetric in the s_i and t_j, which suggests the ansatz

$$\psi(S,T) \propto \mathrm{Det}(\exp(\pm 2 s_i t_j))_{i,j} , \tag{3.16}$$

where either in all terms the positive or the negative sign must be chosen. To show that the correct solution contains negative signs, we set $T_d = \alpha \tilde{T}_d$ with a positive parameter α and investigate how the integral depends on it for any fixed

$\tilde{T}_d$. With the definition $A = (-i\Sigma_3\tilde{T}_d)^{1/2}hg(-i\Sigma_3 S_d)^{1/2}$, where $...^{1/2}$ denotes the positive square root of the diagonal matrix in its argument, we get

$$\begin{aligned}\frac{\partial}{\partial\alpha}\exp(\mathrm{Tr}(ST)) &= \frac{\partial}{\partial\alpha}\exp(\mathrm{Tr}(g^{-1}\tilde{T}_d g h S_d h^{-1}) \\ &= \frac{\partial}{\partial\alpha}\exp(-\alpha\mathrm{Tr}(AA^\dagger) \\ &= -\alpha\mathrm{Tr}(AA^\dagger)\exp(-\alpha\mathrm{Tr}(AA^\dagger) < 0\,.\end{aligned}$$

It follows that the value of the integral cannot increase with α. The same must hold for the solution (3.16) of the differential equation, which proves the claim.

Therefore, we get

$$I(S,T) = c_N \frac{\mathrm{Det}(\exp(-2s_i t_j))_{i,j}}{j_s(S_d)j_s(T_d)}\,,$$

where c_N is a normalization constant which is not determined by the differential equation.

Now, we want to transfer this result to the original integral (3.11). That means to substitute $t_i = \tau/2$ for all i; the variables s_i are replaced by the frequencies ω_i. Therefore, the next step consist in performing the limit $t_i \to \tau/2$. As shown in the appendix, the limit is given by

$$\lim_{t_i\to\tau/2}\frac{\mathrm{Det}\big(\exp(-2t_i s_j)\big)}{j_s(t_1,\ldots,t_N)j_s(s_1,\ldots,s_N)} = \frac{(-1)^{\lfloor N/2\rfloor}\exp\left(-\tau\sum\limits_{i=1}^{N}s_i\right)}{j_+(s_1,\ldots,s_N)(\tau/2)^{N(N+1)/2}}\prod_{m=1}^{N-1}\frac{1}{m!}\,. \quad (3.17)$$

Using this result, the integration over the elliptic domain has been split into the symplectic group and the integration over the eigenvalues of X. Thus, we arrive for a reasonable radial test function $f(\Omega)$ at

$$\begin{aligned}\int\limits_{\mathcal{E}^0}\exp(i\tau\mathrm{Tr}\Sigma_3 X)f(\Omega)dX &= c_{N,l}\int\limits_{(\mathbb{R}^+)^N}\frac{f(\Omega)\exp\left(-\tau\sum\limits_{i=1}^{N}\omega_i\right)}{j_+(\omega_1,\ldots,\omega_N)(\tau/2)^{\frac{N(N+1)}{2}}}\prod_{m=1}^{N-1}\frac{1}{m!}j\prod_{i=1}^{N}d\omega_i \\ &= c_{N,l}\int\limits_{(\mathbb{R}^+)^N}d\mu_{N,l}(\omega_1,\ldots,\omega_N)f(\Omega)\,,\end{aligned}$$

where the joint probability density is given by

$$d\mu_{N,l}(\omega_1,\ldots,\omega_N) = c_{N,l}\prod_{1\le i<j\le N}(\omega_i-\omega_j)(\omega_i^2-\omega_j^2)\prod_{k=1}^{N}\omega_k^l e^{-\tau\omega_i}d\omega_i\,. \quad (3.18)$$

3.5 The asymptotic density of states in the bulk

There are many methods to obtain the asymptotic density of states (DOS) in the bulk. It can be calculated by means of superbosonization [Zir06], which is very

similar to the calculation in chapter 5, or by using the method of bi-orthogonal polynomials, or as follows [LSZ06]. This calculation has been done by Martin Zirnbauer.

The DOS $\rho(\omega)\,d\omega$ is defined as the probability density for any one of the eigenfrequencies ω_i to have the value of ω, irrespective of what the values of the other eigenfrequencies are; thus $\rho(\omega)$ is the function

$$\rho(\omega) := \int \sum_{i=1}^{N} \delta(\omega - \omega_i)\, d\mu_{N,l}(\omega_1, \ldots, \omega_N)\ , \tag{3.19}$$

which has the properties $\rho(\omega) \geq 0$ and

$$\int_0^\infty \rho(\omega)\, d\omega = N\ . \tag{3.20}$$

We are now interested in the behavior of the density function $\rho(\omega)$ in the limit of $N \to \infty$. From the expression and experience with similar problems (see e.g. [Meh04]), we expect that this limit can be obtained by maximizing the functional

$$F = \frac{1}{2}\int_0^\infty \int_0^\infty \ln\left((\omega-\omega')^2(\omega+\omega')\right)\rho(\omega)\rho(\omega')\,d\omega' d\omega + \int_0^\infty \ln(\omega^l \mathrm{e}^{-\omega\tau})\,\rho(\omega)\,d\omega \tag{3.21}$$

subject to the constraint (3.20) and the condition $\rho(\omega) \geq 0$. More precisely, the limit is expected to exist in the scaled variable $x := \omega\tau/N$; i.e., there should exist a certain non-negative function $\rho_\infty(E)$ with $\int \rho_\infty(E)\,dE = 1$ such that $\rho(\omega)$ is asymptotic to $\tau\rho_\infty(\omega\tau/N)$.

Varying F with respect to $\rho(\omega)$ we get

$$\frac{\delta F}{\delta\rho(\omega)} = l\ln\omega - \omega\tau + \int_0^\infty \left(2\ln|\omega-\omega'| + \ln(\omega+\omega')\right)\rho(\omega')\,d\omega'\ .$$

We now insert the asymptotic equality $\rho(NE/\tau) \approx \tau\rho_\infty(E)$ and pass to the limit $N \to \infty$ in the scaling variable E. Let $[0, b]$ be the region of support of ρ_∞. Then the condition $\delta F/\delta\rho(\omega) = N\lambda$, where λ is a Lagrange multiplier for the constraint (3.20), yields the equation

$$\int_0^b \left(2\ln|E-E'| + \ln(E+E')\right)\rho_\infty(E')\,dE' - E = \lambda \qquad (0 < E < b)\ . \tag{3.22}$$

It can be shown that our functional F is convex; as a result, the solution ρ_∞ of equation (3.22) exists and is unique when supplemented by the normalization condition

$$\int_0^b \rho_\infty(E)\,dE = 1\ . \tag{3.23}$$

Now, the task is to construct the solution to the mathematical problem posed by (3.22) and (3.23). It turns out to be

$$\rho_\infty(E) = \frac{1}{2\pi}(E/b)^{-1/3}\left((1+\sqrt{1-E^2/b^2})^{1/3} - (1-\sqrt{1-E^2/b^2})^{1/3}\right)\ , \tag{3.24}$$

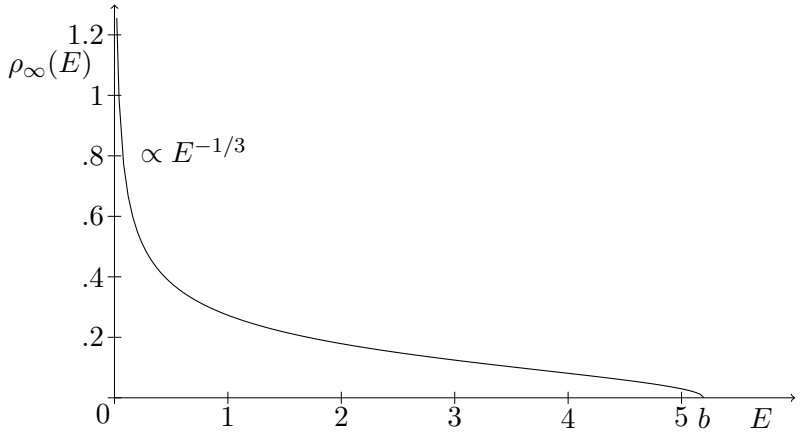


Figure 3.1: The asymptotic density of states $\rho_\infty(E)$ in the bulk.

with $0 < E \leq b = 3\sqrt{3}$. For the details how to get this solution we refer to [LSZ06].

From the solution, the behavior near the lower edge $E = 0$ is

$$\rho_\infty(E) \simeq \frac{1}{2\pi}(2b/E)^{1/3} \qquad (0 < E \ll b) \,,$$

while close to the upper edge $E = b$ one gets

$$\rho_\infty(E) \simeq \frac{1}{3\pi}(1 - E^2/b^2)^{1/2} \qquad (E < b \,,\ E \to b) \,.$$

In the vicinity of the upper and lower edges there exists crossover to a fine-scale behavior that cannot be found by the present method of maximization of the functional F. The crossover at the upper edge involves Airy functions on a scale $N^{1/3}$, which is small compared to the bulk scale N. At the lower edge, the crossover occurs on a very fine scale, $N^{-1/2}$, which is small even in comparison with the bulk mean level spacing (which is of order N^0).

3.6 The asymptotic density of states near z=0

Equation (3.19) shows, how the density of states is defined. In section 3.5, we derived the asymptotic density of states in the bulk by minimizing the functional F. Here we want to calculate the exact DOS for all N. Due to the symmetry in the variables $\omega_1, \ldots, \omega_N$ of the measure (3.18), our measure can be reduced to integrate out all variable ω_i but one, say ω_N. This can be achieved by a direct calculation. We only need the elementary integral

$$\int_{\mathbb{R}^+} \omega^k e^{-\omega} d\omega = k! \,, \ k \in \mathbb{N}_0 \,,$$

to integrate out a single variable ω_i. That is, the integrations to do are rather simple. The main challenge is to handle the combinatorics for a large number N of variables

ω_i. Since τ can be absorbed to the normalization constant $c_{N,l}$ by a simple rescaling, we set $\tau = 1$ without loss of generality. Thus with the explanations below we find for the DOS

$$\begin{aligned}\rho_{N,l}(\omega) &= \int\limits_{(\mathbb{R}^+)^N} \sum_{i=1}^{N} \delta(\omega-\omega_i) d\mu_{N,l}(\omega_1,\ldots,\omega_n) \\ &= c_N(N-1)! \sum_{i,j=1}^{N} \omega_N^{2j+i-2}(-1)^{j_1+j_2}\mathrm{Det}(\hat{A}_{ij}) \\ &= c_N(N-1)!\,\omega_N^{l-3}\mathrm{Det}\begin{pmatrix} & & & \omega_N^2 \\ & A & & \vdots \\ & & & \omega_N^{2N} \\ \omega_N & \cdots & \omega_N^N & 0 \end{pmatrix}. \end{aligned} \tag{3.25}$$

A is an N by N matrix and given by the entries $A_{ij} = (i+2j-3+l)!$, that is

$$A = \begin{pmatrix} l! & (l+2)! & \cdots & (l+2N)! \\ (l+1)! & (l+3)! & \cdots & (l+2N+1)! \\ \vdots & \vdots & \ddots & \vdots \\ (l+N)! & (l+N+2)! & \cdots & (l+3N)! \end{pmatrix}.$$

By $\hat{A}_{ij}$ we denote a matrix obtained by eliminating the i^{th} row and the j^{th} column of A. The second equality in (3.25) is obtained by expansion of the last determinant with respect to the last row and with respect to the last column. The first one can be seen by rewriting the measure in terms of Vandermonde determinants:

$$\begin{aligned}&\prod_{i<j}(\omega_i-\omega_j)(\omega_i^2-\omega_j^2)\prod_k \omega_k^l = \\ &\quad= \begin{vmatrix} 1 & \ldots & 1 \\ \omega_1 & \ldots & \omega_N \\ \vdots & \ddots & \vdots \\ \omega_1^{N-1} & \ldots & \omega_N^{N-1} \end{vmatrix} \begin{vmatrix} 1 & \ldots & 1 \\ \omega_1^2 & \ldots & \omega_N^2 \\ \vdots & \ddots & \vdots \\ \omega_1^{2(N-1)} & \ldots & \omega_N^{2(N-1)} \end{vmatrix} \prod_k \omega_k^l \qquad (3.26) \\ &\quad= \mathrm{Det}(\omega_i^{j-1})\mathrm{Det}(\omega_i^{2(j-1)})\prod_k \omega_k^l \\ &\quad= \sum_{\tau,\sigma\in S_N} \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\omega_1^{\tau(1)+2\sigma(1)-3+l}\omega_2^{\tau(2)+2\sigma(2)-3+l}\cdots\omega_N^{\tau(N)+2\sigma(N)-3+l}. \qquad (3.27)\end{aligned}$$

The double sum runs over all elements of the symmetric group S_N. Integrating out the variables $\omega_1, \omega_2, \ldots, \omega_{N-1}$ replaces their powers by factorials. To understand the equality we ask for the numerical factor in front of the power $x^{i+2j-3+l}$. It is

given by

$$\begin{aligned}\sum_{\substack{\sigma\in S_N\\ \sigma(N)=i}}\sum_{\substack{\tau\in S_N\\ \tau(N)=j}}\mathrm{sgn}(\sigma\tau)\prod_{k=1}^{N-1}(\sigma(k)+2\tau(k)-3+l)!\\ =(N-1)!\sum_{\sigma(N)=j}\prod_{k=1,k\neq i}^{N}(k+2\sigma(k)-3+l)!\\ =(N-1)!(-1)^{i+j}\mathrm{Det}\hat{A}_{ij}\ .\end{aligned}$$

By a simple calculation, the normalization constant c_N can be determined. We start with (3.27) and replace the power of every ω_i by the factorial of its exponent. Than, the sum over one of the symmetric groups can be carried out, it just produces a factor $N!$. The remaining sum has the value $\mathrm{Det}A$. This determinant can be obtained by a Gaussian elimination process, it is given by

$$\mathrm{Det}A=\prod_{k=0}^{N-1}2^k k!(2k+l)!\ .$$

Due to the condition

$$\int d\mu_{N,l}(\omega_1,\ldots,\omega_N)=N$$

we arrive at $c_N=1/\mathrm{Det}A$. The determinants of the matrices $\hat{A}_{ij}$ can be calculated by elementary methods. However, the calculations are very cumbersome. In particular, the limit of large N is hard to obtain. Thus we just state the result and show the much more elegant calculation presented in [LSZ06], which has been found by H.-J. Sommers. As we will see in the next section, in principle, he solved the same problem, i.e. he calculated the the determinants of the matrices $\hat{A}_{ij}$. But, he found a very nice representation of these determinants as contour integrals, which makes it much more easier to handle the limit of large N.

In the scaling limit

$$\tilde{\rho}_l(z)=\lim_{N\to\infty}\frac{1}{\sqrt{N}}\rho_{N,l}(\sqrt{N}z)$$

the DOS converges to finite values for every z. Explicitly, it reads

$$\tilde{\rho}_l(z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma\left(\frac{m+l+1}{2}\right)m!)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+l)!n!(2n+m+l+1)}z^{2n+m+l}\qquad(3.28)$$

Figure 3.2 shows the DOS for $l=0$ and $l=1$.

3.7 Complete solution using bi-orthogonal polynomials

The method of bi-orthogonal polynomials completely solves the random matrix model under consideration, i.e. all correlation functions can be found for all values of N. In principle, this method is a generalization of the method of orthogonal

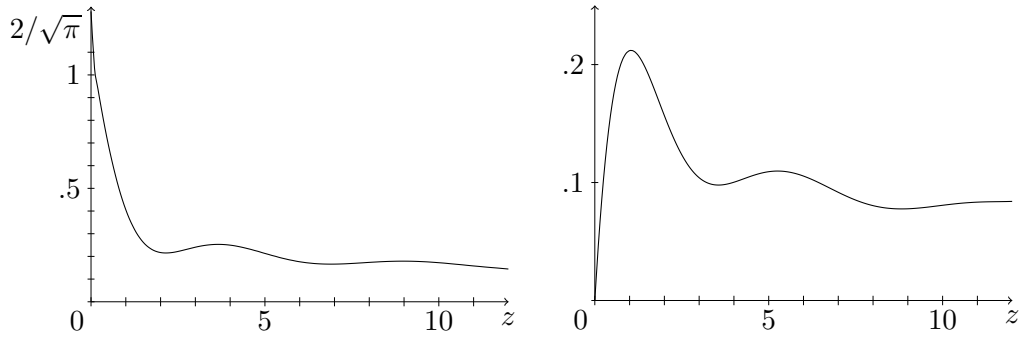


Figure 3.2: The asymptotic density of states $\tilde{\rho}_l(z)$ for $l = 0$ (left) and $l = 1$ (right)

polynomials which can be used to obtain the correlation functions for the Gaussian orthogonal or unitary ensemble (GOE/GUE). We consider again the reduced probability measure (3.18),

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau) \prod_{i<j} (\omega_i - \omega_j)(\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \mathrm{e}^{-\tau\omega_k}\, \omega_k^l \, d\omega_k \;, \tag{3.29}$$

and embark on another approach to handle it.

Again, the first step is to rewrite the measure in terms of Vandermonde determinants. According to (3.26), we obtain

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau)\, \mathrm{Det}\,(\omega_j^{i-1})\, \mathrm{Det}\,(\omega_j^{2i-2}) \prod_{k=1}^{N} \mathrm{e}^{-\tau\omega_k}\, \omega_k^l \, d\omega_k \;. \tag{3.30}$$

By standard properties of the determinant, $\mathrm{Det}\,(\omega_j^{i-1})$ changes only by a multiplicative constant when the monomials ω_j^{i-1} are replaced by any polynomials in ω_j of degree $i-1$. We have two Vandermonde determinants, $\prod_{i<j}(\omega_i - \omega_j)$ and $\prod_{i<j}(\omega_i^2 - \omega_j^2)$, so we introduce two sets of polynomials, denoting those of the first set by $P_{i-1}(\omega_j)$ and those of the second one by $Q_{i-1}(\omega_j^2)$. Our measure then becomes

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = \tilde{c}_{N,l}\, \mathrm{Det}\,\big(P_{i-1}(\omega_j)\big)\, \mathrm{Det}\,\big(Q_{i-1}(\omega_j^2)\big) \prod_{k=1}^{N} \mathrm{e}^{-\omega_k}\, \omega_k^l \, d\omega_k \;. \tag{3.31}$$

In order for the introduction of the polynomials $P_n(\omega)$ and $Q_n(\omega^2)$ to be useful we require them to be orthogonal with respect to the integration measure $\mathrm{e}^{-\omega}\omega^l \, d\omega$:

$$I_{m,n} \equiv \int_0^\infty P_m(\omega)\, Q_n(\omega^2)\, \mathrm{e}^{-\omega}\omega^l \, d\omega = h_n\, \delta_{m,n} \;, \tag{3.32}$$

where the numbers $h_n = I_{n,n}$ depend on the choice of normalization for $P_n(\omega)$ and $Q_n(\omega^2)$. Such polynomials are constructed by a variant of the Gram-Schmidt algorithm. For details, we refer again to [LSZ06], here we just state the result We obtain the following expressions for the polynomials:

$$P_n(\omega) = D_{n-1}^{-1} \begin{vmatrix} l! & \ldots & (l+2n-2)! & \omega^0 \\ (l+1)! & \ldots & (l+2n-1)! & \omega^1 \\ \vdots & \ddots & \vdots & \vdots \\ (l+n)! & \ldots & (l+3n-2)! & \omega^n \end{vmatrix} \;. \tag{3.33}$$

and

$$Q_n(\omega^2) = D_{n-1}^{-1} \begin{vmatrix} l! & (l+2)! & \dots & (l+2n)! \\ \vdots & \vdots & \ddots & \vdots \\ (l+n-1)! & (l+n+1)! & \dots & (l+3n-1)! \\ \omega^0 & \omega^2 & \dots & \omega^{2n} \end{vmatrix} , \tag{3.34}$$

which are easy to verify. These matrices are very similar to those appearing in the first approach in section 3.6. Using the relation $(n+1)! = n \cdot n!$ an easy Gauss elimination process gives the normalization constant as

$$D_n = \begin{vmatrix} l! & \dots & (l+2n)! \\ \vdots & \ddots & \vdots \\ (l+n)! & \dots & (l+3n)! \end{vmatrix} = \prod_{k=0}^{n} 2^k k!\,(l+2k)! \; . \tag{3.35}$$

From this, note the diagonal pairing matrix elements $h_0 = (l)!$ and

$$\int_0^\infty P_n(\omega)\, Q_n(\omega^2)\, \mathrm{e}^{-\omega} \omega^l d\omega = h_n = D_n/D_{n-1} = 2^n n!\,(l+2n)! \qquad (n \geq 1) \; . \tag{3.36}$$

n-level correlation functions

The n-level correlation function $R_n(\omega_1, \dots, \omega_n)$ in the present context is defined as

$$R_n(\omega_1, \dots, \omega_n) = n! \int\limits_{\mathbb{R}_+^N} \sum_{i_1 < i_2 < \dots < i_n} \delta(\omega_1 - \tilde{\omega}_{i_1}) \cdots \delta(\omega_n - \tilde{\omega}_{i_n})\, d\mu_{N,l}(\tilde{\omega}_1, \dots, \tilde{\omega}_N) \; . \tag{3.37}$$

A closed-form expression for it can be given from the bi-orthogonal polynomials $P_{n'}(\omega)$ and $Q_{n'}(\omega^2)$ for $0 \leq n' \leq N$. The result will take its most succinct form when expressed in terms of the modified functions

$$\tilde{P}_n(\omega) := (-2)^{-n} n!^{-1} \mathrm{e}^{-\omega} P_n(\omega) \; , \tag{3.38}$$

$$\tilde{Q}_n(\omega) := (-1)^n (l+2n)!^{-1} \omega^l Q_n(\omega^2) \; , \tag{3.39}$$

(the motivation for the sign $(-1)^n$ will become clear later), which from (3.32) and (3.36) obey the orthogonality relations

$$\int_0^\infty \tilde{P}_m(\omega)\, \tilde{Q}_n(\omega)\, d\omega = \delta_{m,n} \; . \tag{3.40}$$

The probability measure (3.31) expressed by these functions takes the form

$$d\mu_{N,l}(\omega_1, \dots, \omega_N) = \frac{1}{N!} \operatorname{Det}\left(\tilde{P}_{i-1}(\omega_j)\right) \operatorname{Det}\left(\tilde{Q}_{i-1}(\omega_j)\right) \prod\nolimits_k d\omega_k \; .$$

Now, by using the multiplicative property of the determinant, we can also write

$$d\mu_{N,l}(\omega_1, \dots, \omega_N) = \frac{1}{N!} \operatorname{Det}\left(K_N(\omega_i\,, \omega_j)\right)_{i,j=1,\dots,N} \prod\nolimits_k d\omega_k \; , \tag{3.41}$$

where the kernel $K(\omega_i\,,\omega_j)$ is defined by

$$K_N(\omega_i\,,\omega_j) = \sum_{n=0}^{N-1} \tilde{P}_n(\omega_i)\,\tilde{Q}_n(\omega_j)\;. \tag{3.42}$$

From the orthogonality relations (3.40) this kernel has the reproducing property

$$\int_0^\infty K_N(\omega_i\,,\omega)\,K_N(\omega,\omega_j)\,d\omega = K_N(\omega_i\,,\omega_j)\;, \tag{3.43}$$

and the trace

$$\int_0^\infty K_N(\omega,\omega)\,d\omega = N\;. \tag{3.44}$$

To proceed further, take notice of the relation

$$\int_0^\infty \begin{vmatrix} K_N(\omega_1,\omega_1) & \dots & K_N(\omega_1,\omega_n) \\ \vdots & \ddots & \vdots \\ K_N(\omega_n,\omega_1) & \dots & K_N(\omega_n,\omega_n) \end{vmatrix} d\omega_n = (N-n+1) \begin{vmatrix} K_N(\omega_1,\omega_1) & \dots & K_N(\omega_1,\omega_{n-1}) \\ \vdots & \ddots & \vdots \\ K_N(\omega_{n-1},\omega_1) & \dots & K_N(\omega_{n-1},\omega_{n-1}) \end{vmatrix}\;,$$

which is proved by expanding the determinant with respect to the last row or column and exploiting the properties (3.43) and (3.44). Using it, an inductive procedure starting from $R_N(\omega_1,\dots,\omega_N) = \mathrm{Det}\left(K_N(\omega_i\,,\omega_j)\right)_{i,j=1,\dots,N}$ gives the n-level correlation functions as

$$R_n(\omega_1,\dots,\omega_n) = \mathrm{Det}\left(K_N(\omega_i\,,\omega_j)\right)_{i,j=1,\dots,n}\;. \tag{3.45}$$

Thus the correlations are those of a determinantal process and are completely determined by the kernel $K_N(\omega_i\,,\omega_j)$. The remaining discussion therefore focuses on this kernel, but first we make another preparatory step.

Contour integral representation

We are now going to show that the functions $\tilde{P}_n(\omega)$ and $\tilde{Q}_n(\omega)$ have expressions as complex contour integrals:

$$\tilde{P}_n(\omega) \;=\; \oint_{S_\epsilon(1)} \mathrm{e}^{-\omega u}\,(1-u^{-2})^{-n-1}u^{l-2}du/\pi\mathrm{i}\;, \tag{3.46}$$

$$\tilde{Q}_n(\omega) \;=\; \oint_{S_\epsilon(0)} \mathrm{e}^{\omega v}\,(1-v^{-2})^{n}v^{-l-1}dv/2\pi\mathrm{i}\;. \tag{3.47}$$

Both integrals are over circles in the complex plane with radius ϵ and counterclockwise orientation; the first circle is centered at $u=1$ and has radius $\epsilon<2$ (to avoid the singularity at $u=-1$), the second one is centered at $v=0$.

Our proof of these expressions for $\tilde{P}_n(\omega)$ and $\tilde{Q}_n(\omega)$ will be indirect and in two steps. First, we establish some information on power series. In the case of $\tilde{Q}_n(\omega)$ we

insert the power series of the exponential function $\mathrm{e}^{\omega v}$, use the binomial expansion of $(1-v^{-2})^n$, and compute a residue to obtain

$$\tilde{Q}_n(\omega) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \omega^{2k+l}}{(2k+l)!} \,. \tag{3.48}$$

In the case of $\tilde{P}_n(\omega)$, calculating the residue at $u=1$ we have that

$$\tilde{P}_n(\omega) = \frac{2}{n!} \frac{d^n}{du^n} \left(\frac{\mathrm{e}^{-\omega u} u^{n+l-1}}{(1+u^{-1})^{n+1}} \right) \Bigg|_{u=1} \,. \tag{3.49}$$

In both cases, defining $P_n(\omega)$ and $Q_n(\omega^2)$ by the reverse of the relations (3.38,3.39), we see from (3.48,3.49) that these are polynomials of degree n in ω resp. ω^2 and that the highest-degree term (ω^n resp. ω^{2n}) has coefficient one.

Recall now that, given these properties, the polynomials $P_n(\omega)$ and $Q_n(\omega^2)$ are completely determined by the orthogonality relations (3.32) for $m \neq n$. Via (3.38,3.39) the latter are in one-to-one correspondence with the orthogonality relations (3.40) (still for $m \neq n$). Therefore, defining

$$\tilde{I}_{m,n} = \int_0^\infty \tilde{P}_m(\omega)\, \tilde{Q}_n(\omega)\, d\omega \,, \tag{3.50}$$

the second and final step of our proof is to show that $\tilde{I}_{m,n} = 0$ for $m \neq n$.

To that end, we insert the expressions (3.46, 3.47) into (3.50), replacing also $\mathrm{e}^{\omega v}$ by $\sinh(\omega v)$ since only the latter contributes to the residue at $v=0$. The ω-dependence then is $\mathrm{e}^{-\omega u}\sinh(\omega v)$ with $u \in S_\epsilon(1)$ and $v \in S_\epsilon(0)$. Taking the radius ϵ to be very small ($\epsilon \ll 1$), we have that $\mathrm{e}^{-\omega u}\sinh(\omega v)$ decreases rapidly as ω goes to $+\infty$. Therefore, the integral over ω exists, and we may interchange the order of integrations. We then do the ω-integral first, using

$$\int_0^\infty \mathrm{e}^{-\omega u} \sinh(\omega v)\, d\omega = \frac{v}{u^2 - v^2} \,.$$

The remaining contour integrals for $\tilde{I}_{m,n}$ defined by (3.50) are

$$\tilde{I}_{m,n} = \oint_{S_\epsilon(1)} \frac{u^{2l-1}}{(1-u^{-2})^{m+1}} \left(\oint_{S_\epsilon(0)} \frac{(1-v^{-2})^n\, dv}{v^l (v^2 - u^2)} \right) \frac{du}{2\pi^2} \,.$$

To simplify the inner integral over v we use the identity

$$\left(\frac{1-v^{-2}}{1-u^{-2}} \right)^n = 1 - \frac{v^2-u^2}{v^2(1-u^2)} \sum_{k=0}^{n-1} \left(\frac{1-v^{-2}}{1-u^{-2}} \right)^k \,.$$

Inserting this into the expression for $\tilde{I}_{m,n}$ we see that the terms in the k-sum do not contribute as the residue at $v=0$ vanishes for all of those terms. Doing the v-integral for the first term on the right-hand side, we get

$$\oint_{S_\epsilon(0)} v^{-l-1} (v^2 - u^2)^{-1} dv = -2\pi \mathrm{i}\, u^{-l-1} \,.$$

The remaining u-integral is

$$\tilde{I}_{m,n} = (\pi\mathrm{i})^{-1} \oint_{S_\epsilon(1)} (1-u^{-2})^{n-m-1} u^{-3} du \; .$$

This integrand is holomorphic near $u = 1$ for $m < n$, and the integral therefore vanishes in that case. For $m > n$ we use the invariance of the integration form under $u \to -u$ to write $\tilde{I}_{m,n}$ as an integral over a sum of two circles:

$$\tilde{I}_{m,n} = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{u^{2m-2n-1} du}{(u^2-1)^{m-n+1}} \; , \qquad \gamma = S_\epsilon(1) + S_\epsilon(-1) \; .$$

The integrand in this case is holomorphic near $u = 0$. In the punctured plane $\mathbb{C}\setminus\{1,-1\}$ the chain $S_\epsilon(1)+S_\epsilon(-1)$ is homologous to the circle at infinity, where the integrand vanishes. Therefore the integral again is zero. This proves that $\tilde{I}_{m,n} = 0$ for $m \neq n$, which in turn completes our proof that the contour integrals (3.46) and (3.47) are the same as the functions $\tilde{P}_n(\omega)$, $\tilde{Q}_n(\omega)$ defined from (3.33, 3.34) by (3.38, 3.39). As a final check, note that

$$\tilde{I}_{n,n} = (\pi\mathrm{i})^{-1} \oint_{S_\epsilon(1)} \frac{du}{u(u^2-1)} = 1 \; ,$$

which is how it ought to be in view of (3.40).

Now we harvest a major benefit from the contour integral representations (3.46) and (3.47): using these, we can carry out the sum in the definition (3.42) of the kernel K_N as a geometric sum. The result is a double contour integral:

$$K_N(\omega_1,\omega_2) = \oint_{S_\epsilon(1)} du \oint_{S_\epsilon(0)} dv \; F_N(u,v\,;\omega_1,\omega_2) \; , \tag{3.51}$$

$$F_N(u,v\,;\omega_1,\omega_2) = \frac{1}{2\pi^2}\, \mathrm{e}^{-\omega_1 u + \omega_2 v}\, \frac{u^l v^{-l+1}}{u^2-v^2} \left(\left(\frac{1-v^{-2}}{1-u^{-2}} \right)^N - 1 \right) \; . \tag{3.52}$$

This exact expression represents the complete solution of our problem. We will now use it to determine the large-N asymptotics in the bulk and at the hard edge $\omega = 0$.

Asymptotics in the bulk

The kernel on the diagonal $\omega_1 = \omega_2$ is the same as the 1-level function, $R_1(\omega) = K_N(\omega,\omega)$; see (3.45). We already know from section 3.5 the asymptotics of $R_1(\omega) \equiv \rho(\omega)$ in the bulk: introducing the scaling variable $E = \omega/N$, this is

$$\lim_{N\to\infty} K_N(NE,NE) = \rho_\infty(E) \; ,$$

with $\rho_\infty(x)$ given by (3.24). One can show that the scaling limit of the kernel $K_N(\omega_1,\omega_2)$ off the diagonal leads to sine-kernel universality for all level correlation functions:

$$\lim_{N\to\infty} R_n(NE+\omega_1,\ldots,NE+\omega_n) = \mathrm{Det}\left(\frac{\sin\left(\pi\rho_\infty(E)(\omega_i-\omega_j)\right)}{\pi(\omega_i-\omega_j)} \right)_{i,j=1,\ldots,n} \; , \tag{3.53}$$

as is expected for systems in the universality class of the Gaussian Unitary Ensemble. There are two methods to obtain this result: For the one we refer to [LSZ06], the main idea is a saddle-point evaluation of (3.51). We skip this calculation here. The second way uses the ideas of the supersymmetry method. In chapter 4 this method is illustratedd at the example of the Gaussian orthogonal ensemble, in chapter 5 it is applied to another random matrix model for disordered bosons.

Asymptotics near $\omega = 0$

At the lower edge ($\omega = 0$) of the spectrum, a new type of behavior is expected to emerge. This behavior, as we shall see presently, occurs on a scale $\omega \sim N^{-1/2}$.

To exhibit the scaling limit near $\omega = 0$, it is best to send the integration variables u, v to their reciprocals, $u \to u^{-1}$ and $v \to v^{-1}$. Then $du \to -u^{-2}du$, $dv \to -v^{-2}dv$, and the integration contour for v has its radius inverted and orientation reversed, $S_\epsilon(0) \to -S_{1/\epsilon}(0)$. However, since the integrand is holomorphic in v on $\mathbb{C} \setminus \{0\}$ we may return to the original radius ϵ (or any other radius, for that matter). In the case of u we take the radius ϵ of $S_\epsilon(1)$ to be very small. Then inversion $u \to u^{-1}$ sends $S_\epsilon(1)$ to itself (or, in any case, to the same homology class on $\mathbb{C}\setminus\{1\}$), with no change of orientation. Altogether, then, carrying out the transformation $(u, v) \to (u^{-1}, v^{-1})$ the integral representation (3.51) continues to hold true if we make the replacement

$$\begin{aligned} F_N(u, v\,; \omega_1, \omega_2) \to &- u^{-2}v^{-2}F_N(u^{-1}, v^{-1}; \omega_1, \omega_2) \\ &= \frac{1}{2\pi^2}\, \mathrm{e}^{-\omega_1/u+\omega_2/v}\, \frac{u^{-l}v^{l-1}}{u^2 - v^2} \left(\left(\frac{1 - v^2}{1 - u^2} \right)^N - 1 \right) . \end{aligned}$$

Next, as another preparation for taking the limit $N \to \infty$, we deform the u-contour $S_\epsilon(1)$ to some axis parallel to the imaginary axis. The deformed contour crosses the real axis between $uF = 0$ and $u = 1$ and is directed from $u = +\mathrm{i}\infty$ to $u = -\mathrm{i}\infty$. We also reverse the direction of integration for u and change the overall sign of the integral.

Then we set $\omega_j = N^{-1/2}z_j$ and rescale $u \to N^{-1/2}u$ and $v \to N^{-1/2}v$ accordingly. Again, in view of the analytic properties of the integrand we can keep the integration contours fixed while rescaling. Because the u-integral converges at infinity we have a good limit

$$\lim_{N\to\infty} (1 - u^2/N)^{-N} = \exp(u^2) .$$

In total, we thus obtain the following scaling limit for our kernel K_N :

$$\begin{aligned} k(z_1, z_2) &:= \lim_{N\to\infty} N^{-1/2}K_N(N^{-1/2}z_1, N^{-1/2}z_2) \\ &= \frac{1}{2\pi^2} \int\limits_{\mathrm{i}\mathbb{R}+\epsilon} du \oint\limits_{\mathrm{U}_1} \frac{dv}{v}\, \mathrm{e}^{-z_1/u+z_2/v}\, (v/u)^l\, \frac{\mathrm{e}^{u^2-v^2} - 1}{u^2 - v^2} , \end{aligned} \tag{3.54}$$

where $\mathrm{U}_1 \equiv S_1(0)$ means the unitary numbers, and $\mathrm{i}\mathbb{R} + \epsilon$ is the imaginary axis translated by $\epsilon > 0$ into the right half of the complex plane.

Taking the same scaling limit for the functions $\tilde{P}_N(\omega)$ and $\tilde{Q}_N(\omega)$ in (3.46) and (3.47) one gets

$$p(z) = \lim_{N\to\infty} N^{-l}\tilde{P}_N(N^{-1/2}y) = \frac{1}{\pi\mathrm{i}} \int\limits_{\mathrm{i}\mathbb{R}+\epsilon} \mathrm{e}^{u^2-z/u}\, u^{-l} du \,, \tag{3.55}$$

$$q(z) = \lim_{N\to\infty} N^{l+1/2}\tilde{Q}_N(N^{-1/2}z) = \frac{1}{2\pi\mathrm{i}} \oint\limits_{\mathrm{U}_1} \mathrm{e}^{-v^2+z/v}\, v^{l-1} dv \,. \tag{3.56}$$

Both functions have convergent series expansions:

$$p(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma((l+n+1)/2)} \,, \qquad q(z) = z^l \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!\,(2n+l)!} \,. \tag{3.57}$$

The expansion for $q(z)$ can be obtained either directly from (3.56), or by taking the limit $N \to \infty$ in (3.48). In the case of $p(z)$, the earlier formula (3.49) is not suitable; rather, in order to verify (3.57) for $p(z)$ one expands the integrand of (3.55) in powers of z, makes use of $\mathfrak{Re}\, u = \epsilon > 0$ to write

$$u^{-l-n} = (l+n-1)!^{-1} \int_0^{\infty} \mathrm{e}^{-tu}\, t^{l+n-1}\, dt \,.$$

Then, one does the Gaussian u-integral by completing the square, and finally uses the duplication formula for the Gamma function.

The expansions (3.57) lead to the asymptotic DOS (3.28) as follows. Clearly, we have

$$\tilde{\rho}_l(z) = k(z,z) \,.$$

By setting $z_1 = z_2 = z$ in (3.54) and the rescaling $(u,v) \to (uz, vz)$ and in (3.54) and (3.55) we obtain the relation

$$\frac{d}{dz}(zk(z,z)) = p(z)q(z) \,.$$

This suggests

$$\begin{aligned} k(z,z) &= \frac{1}{z}\int_0^z dz' p(z')q(z') \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma\left(\frac{m+l+1}{2}\right) m!)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+l)!\, n!\, (2n+m+l+1)} z^{2n+m+l} \,. \end{aligned} \tag{3.58}$$

The integration does not produce any constants independent of z, except for the case $l = 0$. That this is correct can be seen by evaluating the integral (3.54) in $z_1 = z_2 = 0$, leading to

$$k(0,0) = \begin{cases} \frac{2}{\sqrt{\pi}} & l = 0 \\ 0 & l > 0 \end{cases} \,,$$

which coincides with the constant terms in (3.58).

It is clear from the discussion of the asymptotics in the bulk in section 3.5 that the DOS in the present scaling limit behaves as $\tilde{\rho}_l(z) \propto z^{-1/3}$, since the behavior at $x = 0$ in the bulk scaling must be recovered for large z in the microscopic scaling. Indeed, this expectation can be confirmed by using the method of saddle-point evaluation for the integral (3.54).

Chapter 4

The Gaussian orthogonal ensemble

In this chapter, we pursue two different goals. First, we want to illustrate how the supersymmetry method and the superbosonization works looking at the relatively simple example of the well known Gaussian orthogonal ensemble (GOE). In chapter 5 this will serve as a template to solve the more complicated ensemble considered there. Second, following [Zir98c], we want to obtain representations for the correlation functions as superintegrals over Riemannian symmetric superspaces, which will help us to understand similar superintegrals obtained in chapter 5. Although the GOE has been considered comprehensively in the literature, see e.g. [Meh04],[PM83], there is no reference wherein the following calculation can be found, thus the present chapter is interesting on its own.

The GOE is defined on $N \times N$ real symmetric matrices,

$$\mathrm{Sym}_N(\mathbb{R}) := \{H \in \mathrm{Mat}_{N,N}(\mathbb{R}) | H = H^t\} \; .$$

The measure on this space is, with a Euclidean measure dH,

$$d\mu(H) = \exp(-\mathrm{Tr}H^2/(2v^2))dH \;\; , \;\; \int\limits_{\mathrm{Sym}_N(\mathbb{R})} d\mu(H) = 1 \; ,$$

with a positive constant v. It is invariant under orthogonal transformations,

$$H \mapsto gHg^t \;\; , \;\; g \in \mathrm{O}(N) \; .$$

The ensemble average of an appropriate function $f(H)$ is defined as

$$\langle f \rangle = \int\limits_{\mathrm{Sym}_N(\mathbb{R})} f(H) d\mu(H) \; . \tag{4.1}$$

To calculate the n-point correlation function R_n of the ensemble means to evaluate

$$R_n(\omega_1, \ldots, \omega_n) = \lim_{\varepsilon \to 0} \left\langle \prod_{i=1}^{n} \mathfrak{I} \left(\frac{-1}{\pi} \mathrm{Tr}(H - \omega_i + i\varepsilon)^{-1} \right) \right\rangle \; , \tag{4.2}$$

where $\omega_1, \ldots, \omega_n$ are energy variables. To simplify the situation for our purposes we observe

$$\mathfrak{I} \left(\mathrm{Tr}(H - \omega_i + i\varepsilon)^{-1} \right) = -\frac{i}{2} \left(\mathrm{Tr}(H - \omega_i + i\varepsilon)^{-1} + \mathrm{Tr}(H - \omega_i - i\varepsilon)^{-1} \right) \; .$$

Inserting this into (4.2) and expanding the product, we see that 2^n terms occur, which differ by the signs of $i\varepsilon$. This seems to cause much work. However, we will see that in the limit of large N only the number of positive and negative signs affects the result. Due to symmetry, the order of the signs does not matter. Hence, it is convenient to fix the number of positive and negative signs n_R and $n_A = n - n_R$, respectively, and define

$$R_{n_A,n_R}(\omega_1,\ldots,\omega_n) = \lim_{\varepsilon\to 0}\left\langle \prod_{i=1}^{n_A}\left(\frac{i}{\pi}\mathrm{Tr}(H-\omega_i-i\varepsilon)^{-1}\right)\prod_{i=n_A+1}^{n}\left(\frac{i}{\pi}\mathrm{Tr}(H-\omega_i+i\varepsilon)^{-1}\right)\right\rangle . \quad (4.3)$$

In the following sections we compute this quantity in the limit of large N. We will show how this can be achieved by applying the superbosonization identity [LSZ08]. If desired, the correlation function R_n can be recovered by adding all contributions,

$$R_n(\omega_1,\ldots,\omega_n) = \sum_{\substack{n_A=0\\ n_R=n-n_A}}^{n}\binom{n}{n_A}R_{n_A,n_R}(\omega_1,\ldots,\omega_n) .$$

4.1 The supersymmetry method

The supersymmetry method [Efe83, Efe99] always starts by rewriting the resolvent of the Hamiltonian (or of the matrices) in terms of determinants. In the next step, these determinants are written as a superintegral, which allows us to calculate the disorder average. Then, one can use a Hubbard-Stratonovich transformation or the recently developed superbosonization formula [LSZ08] to reduce the number of integration variables to a constant independent of N. The dimension N becomes a (large) parameter, which can be used to apply the method of steepest descent to the resulting superintegral.

Generating function.

Due to the formula $\mathrm{Det}A = \exp(\mathrm{Tr}\log A)$ for a regular matrix A, it is clear that we can rewrite the resolvent of H at the energy ω according to

$$\mathrm{Tr}(H-\omega)^{-1} = \frac{d}{d\beta}\mathrm{Det}\left(\frac{H-\beta_i}{H-\alpha_i}\right)\Big|_{\alpha=\beta=\omega} , \quad (4.4)$$

where ω must have a small imaginary part. To obtain the n-point correlation function we start considering the generating function

$$Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) = \left\langle \prod_{i=1}^{n}\mathrm{Det}\left(\frac{H-\beta_i}{H-\alpha_i}\right)\right\rangle , \quad (4.5)$$

where the numbers n_A and n_R determine how many of the pairs (α_i,β_i) have negative or positive imaginary parts in both entries. These signs have already occurred in eq. (4.3).

According to eq. (4.4), the correlation function R_{n_A,n_R} can be calculated by differentiating Z_{n_A,n_R},

$$R_{n_A,n_R}(\omega_1,\ldots,\omega_n) = \lim_{\varepsilon\to 0}\left(\frac{-\varepsilon}{\pi}\right)^n \prod_{i=1}^{n_A}\frac{\partial}{\partial\alpha_i}\Big|_{\alpha_i=\omega_i-i\varepsilon} \prod_{i=n_A+1}^{n}\frac{\partial}{\partial\alpha_i}\Big|_{\alpha_i=\omega_i+i\varepsilon}$$
$$\times Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\omega_1-i\varepsilon,\ldots,\omega_{n_A}-i\varepsilon,\omega_{n_A+1}+i\varepsilon,\ldots,\omega_n+i\varepsilon)\ . \quad (4.6)$$

We see from (4.6) that it is sufficient to perform all calculations using the generating function (4.5). The differentiations with respect to the energy parameters can be postponed to the end of our calculations, provided the different limit processes and integrations can be interchanged without causing problems. Indeed, for all of the following calculations this is the case.

Rewriting the determinants as Gaussian superintegrals

As already mentioned, the essential structure of the supersymmetry method is to rewrite the determinants occurring in the resolvent as Gaussian superintegrals. For that purpose we define the supermatrix

$$\nu = E_B\otimes\mathbf{1}_2\otimes\sum_{i=1}^{n}E_{ii}\alpha_i + E_F\otimes\mathbf{1}_2\otimes\sum_{i=1}^{n}E_{ii}\beta_i\ .$$

The matrices $E_B = \mathrm{diag}(1,0)$ and $E_F = \mathrm{diag}(0,1)$ are elementary 2×2 supermatrices, E_{ij} is an $n\times n$ matrix which has only one entry 1 in the i-th row and j-th column.

A direct calculation gives

$$\prod_{i=1}^{n}\mathrm{Det}\left(\frac{H-\beta_i}{H-\alpha_i}\right) = \mathrm{SDet}(\mathbf{1}_{4n}\otimes H-\nu\otimes\mathbf{1}_N)\ .$$

It is clear that the symmetry of H must be taken into account in the auxiliary Gaussian superintegrals. We need the following matrices: Let u be a $N\times n$-matrix of independent complex variables and $\tilde{u}=\bar{u}$ its complex conjugate. As anticommuting variable we take the independent $N\times n$-matrices ξ and $\tilde{\xi}$. We put these components in large matrices,

$$\Psi = \begin{pmatrix} u & \tilde{u} & \xi & \tilde{\xi}\end{pmatrix}\quad,\quad \tilde{\Psi} = \begin{pmatrix}\tilde{u} & u & \tilde{\xi} & -\xi\end{pmatrix}^t = \mathcal{C}\Psi^t\ , \quad (4.7)$$

with

$$\mathcal{C} = (E_{BB}\otimes\sigma_1 + E_{FF}\otimes i\sigma_2)\otimes\mathbf{1}_n\ .$$

As usual, σ_1 and σ_2 are Pauli's matrices. The appropriate Berezin-measure reads

$$D(\Psi,\tilde{\Psi}) = \pi^{-nN}\prod_{i=1}^{N}\prod_{j=1}^{n}du_{ij}d\bar{u}_{ij}\frac{\partial}{\partial\xi_{ij}}\frac{\partial}{\partial\tilde{\xi}_{ij}}\ .$$

Following [Zir98c], we find

$$\mathrm{SDet}(\mathbf{1}_{4n}\otimes H-\nu\otimes\mathbf{1}_N) = \int D(\Psi,\tilde{\Psi})\exp(-i\mathrm{Tr}H\Psi\Lambda\tilde{\Psi}+i\mathrm{STr}\Lambda\nu\tilde{\Psi}\Psi)\ . \quad (4.8)$$

Convergence of this superintegral is enforced by the imaginary part of the entries of the supermatrix ν. Due to the different signs the matrix Λ must be present in the exponent. We define

$$\tilde{\lambda} = \mathrm{diag}(-\mathbf{1}_{n_A}, \mathbf{1}_{n_R}) \quad , \quad \lambda = \mathbf{1}_2 \otimes \tilde{\lambda} \quad , \quad \Lambda = E_{BB} \otimes \lambda + E_{FF} \otimes \lambda \, .$$

Ensemble average

Now, we are able to use (4.8) to carry out the ensemble average in the generating function (4.5) according to (4.1). This means doing the outer integral of the expression

$$\left\langle \prod_{i=1}^{n} \mathrm{Det}\left(\frac{H-\beta_i}{H-\alpha_i}\right) \right\rangle = \int\limits_{\mathrm{Sym}_N(\mathbb{R})} d\mu(H) \int D(\Psi, \tilde{\Psi}) \exp(-i\mathrm{Tr} H\Psi\Lambda\tilde{\Psi} + i\mathrm{STr}\Lambda\nu\tilde{\Psi}\Psi) \, .$$

Now, we need to interchange the inner and outer integration. To simplify notation for the next step, we omit the integration over the auxiliary space and the factor containing ν.

$$\begin{aligned} \int\limits_{\mathrm{Sym}_N(\mathbb{R})} d\mu(H) \exp(-i\mathrm{Tr} H\Psi\Lambda\tilde{\Psi}) &= \int\limits_{\mathrm{Sym}_N(\mathbb{R})} dH \exp(-i\mathrm{Tr} H\Psi\Lambda\tilde{\Psi} - \mathrm{Tr} H^2/(2v^2)) \\ &= \exp\big(-v^2 \mathrm{Tr}(\Psi\Lambda\tilde{\Psi})^2/2\big) \end{aligned}$$

Here the general strategy of the supersymmetry method ends. The ensemble average has been performed successfully and we are left with integrations over the auxiliary space. The usual strategy now would be to follow [Zir98c] and to apply a kind of Hubbard–Stratonovich transformation. However, here we want to use the recently developed superbosonization identity [LSZ08].

4.2 Application of the superbosonization identity

The identity

$$\mathrm{Tr}(\Psi\Lambda\tilde{\Psi})^2 = \mathrm{STr}(\Lambda\tilde{\Psi}\Psi)^2$$

helps us to apply the superbosonization formula. By means of (4.7) we can write down $\tilde{\Psi}\Psi$ explicitly,

$$\tilde{\Psi}\Psi = \begin{pmatrix} \tilde{u}^t u & \tilde{u}^t \tilde{u} & \tilde{u}^t \xi & \tilde{u}^t \tilde{\Psi} \\ u^t u & u^t \tilde{u} & u^t \xi & u^t \tilde{\xi} \\ \tilde{\xi}^t u & \tilde{\xi}^t \tilde{u} & \tilde{\xi}^t \xi & 0 \\ -\xi^t u & -\xi^t \tilde{u} & 0 & -\xi^t \tilde{\xi} \end{pmatrix} ,$$

which is exactly the form needed to apply the superbosonization formula for orthogonal symmetry. In the notation of [LSZ08], we have $p = q = n$, i.e. the number of bosonic and fermionic replicas is equal. The number of orbitals is N. Putting things

together, we find

$$\begin{aligned} Z_{n_A,n_R}(\nu) &\equiv Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) \\ &= \left\langle \prod_{i=1}^{n} \mathrm{Det}\left(\frac{H-\beta_i}{H-\alpha_i}\right)\right\rangle \\ &= \int D(\Psi,\tilde{\Psi}) \exp\big(-\mathrm{STr}(\frac{v^2}{2}(\Lambda\tilde{\Psi}\Psi)^2 - i\nu\Lambda\tilde{\Psi}\Psi)\big) \\ &= \int\limits_{D_n^0\times D_n^1} DQ \exp\big(-\mathrm{STr}(\frac{v^2}{2}(\Lambda Q)^2 - i\nu\Lambda Q)\big)\mathrm{SDet}^{N/2}(Q)\ . \end{aligned} \tag{4.9}$$

In the second step of (4.9) the superbosonization identity is used. The integration over the $2N$ bosonic and $2N$ fermionic variables is replaced by an integration over the $4n\times 4n$ supermatrix Q, which is its crucial feature. The advantage is clear: instead of the large number of integration variables we are dealing with a relatively simple superintegral, whose dimension $4n\times 4n$ is independent of N. This replacement produces the N^{th} power of the superdeterminant. Now, let us consider the supermatrix Q. In block form, we write

$$Q = \begin{pmatrix} X & \sigma \\ \tau & Y \end{pmatrix}\ .$$

From [LSZ08] we know how to choose the supermatrix Q, the measure DQ and the domain of integration $D_n^0\times D_n^1$.

The following conditions hold: The supermatrix Q is subject to the symmetry

$$Q = TQ^{\mathrm{st}}T^{-1}\ ,\quad T = E_{BB}\otimes t_s + E_{FF}\otimes t_a\ ,\quad t_s = \sigma_1\otimes\mathbf{1}_n\ ,\ t_a = (-i)\sigma_2\otimes\mathbf{1}_n\ . \tag{4.10}$$

Therefore, there are only $(2n)^2$ independent Grassmann variables in Q, explicitly we have $\sigma = t_s\tau^t t_a^{-1}$. For the boson-boson block X, the domain of integration is

$$D_n^0 := \{X\in \mathrm{Mat}_{2n,2n}(\mathbb{C}) | X = X^\dagger > 0, X = t_s X t_s^{-1}\}\ .$$

The dimension is $\dim_{\mathbb{R}}(D_n^0) = 2n^2+n$. For the fermi-fermi block Y we have

$$D_n^1 := \{Y\in U(2n) | Y = t_a Y^t t_a^{-1}\}\ .$$

Its dimension is $\dim_{\mathbb{R}}(D_n^1) = 2n^2-n$. To give eq. (4.9) a meaning, the definition of the Berezin measure DQ is missing. It can be written as

$$DQ = d\mu_{D_n^0}(X)d\mu_{D_n^1}(Y)\prod_{i=1}^{2n}\prod_{j=1}^{2n}\frac{\partial}{\partial\sigma_{ij}}\frac{\mathrm{Det}^n(X-\sigma Y^{-1}\tau)\mathrm{Det}^n(Y-\tau X^{-1}\sigma)}{(2\pi)^{2n^2}\mathrm{Det}^{1/2}(1-X^{-1}\sigma Y^{-1}\tau)}\ .$$

Up to a constant factor, the measures $d\mu_{D_n^0}(X)$ and $d\mu_{D_n^1}(Y)$ are defined by invariance under the transformations

$$\begin{aligned} X &\mapsto gXt_s g^t t_s \quad ,\ g^\dagger = t_s g^t t_s\ ,\quad g\in \mathrm{GL}(n,\mathbb{C}) \quad \text{and} \\ Y &\mapsto gYt_a g^t t_a^{-1} \quad ,\ g\in \mathrm{U}(2n)\ . \end{aligned}$$

To uniquely determine the measures, normalization constants are needed. They can be fixed by Gaussian integrals, see [LSZ08].

4.3 Correlation functions in the limit of large N

The last superintegral in (4.9) is still exact, no approximation has been made so far. Now we are interested in the limit of large N. To this aim it is beneficial to use the method of steepest descent, N shall serve as a large parameter. Starting at (4.9), we rescale by $Q \mapsto \sqrt{N}Q$. The superdeterminant $\mathrm{SDet}(Q)$ and the measure DQ are left unchanged, also the domains of integration D_n^0 and D_n^1. However, the energy scale must be changed by $1/\sqrt{N}$. For this purpose, we replace ν according to

$$\frac{\nu}{\sqrt{N}} = E \cdot \mathbf{1}_{4n} + \frac{\hat{\omega}}{N} ,$$

i.e. we calculate the correlations in the vicinity of the energy $\sqrt{N}\,E$. The diagonal matrix $\hat{\omega}/N$ is regarded as a small perturbation of lower order in N. We use

$$\hat{\omega} := E_B \otimes \mathbf{1}_2 \otimes \sum_i E_{ii}\hat{\alpha}_i + E_F \otimes \mathbf{1}_2 \otimes \sum_i E_{ii}\hat{\beta}_i .$$

Hence,

$$\int\limits_{D_n^0 \times D_n^1} DQ \exp\left(-\mathrm{STr}\left[\frac{v^2}{2}(\Lambda Q)^2 - i\nu\Lambda Q\right]\right) \mathrm{SDet}^{N/2}(Q) = \int\limits_{D_n^0 \times D_n^1} DQ \exp\left(NF(Q) + \hat{\omega}\right) \tag{4.11}$$

with

$$F(Q) = \frac{1}{2}\mathrm{STr}\left(\log Q - v^2(Q\Lambda)^2 + 2iE\Lambda Q\right) . \tag{4.12}$$

The contribution $\hat{\omega}$ is of order 1 and does not affect the saddle-point. To start the saddle-point analysis, we set $Q = Q_0 + \delta Q$ with a small variation δQ and demand that the term linear in δQ vanishes in the expansion of $F(Q_0 + \delta Q)$. We obtain

$$Q^{-1} - 2v^2\Lambda Q\Lambda + 2iE\Lambda = 0 ,$$

which is equivalent to

$$(Q\Lambda)^2 - \frac{iE}{v^2}Q\Lambda - \frac{1}{2v^2} = 0 . \tag{4.13}$$

Following Berezin [Ber87], eq. (4.13) defines a supermanifold. Our goal is to factor superintegral (4.11) in an superintegral over this supermanifold and the remaining Gaussian superintegral in the limit of large N.

The strategy reads as follows: First, we have to determine the supermanifold defined by (4.13). We start by setting all Grassmann variables to zero and construct the manifold emerging from the boson-boson sector. Then, we calculate its analogue in the fermi-fermi sector. In the third step, we put these manifolds together and obtain the whole saddle-point supermanifold. In the last step, we must determine the remaining Gaussian superintegral and the resulting superintegral over the supermanifold.

The boson-boson block. Setting all Grassmann variables to zero, in the BB-component the restriction of eq. (4.13) reads

$$(X\lambda)^2 - \frac{iE}{v^2} X\lambda - \frac{1}{2v^2} = 0 \; . \tag{4.14}$$

We may also restrict F to the BB-block,

$$F_B(X) = \frac{1}{2}\mathrm{Tr}\big(\log X - v^2 (X\lambda)^2 + 2iE\lambda X \big) \; .$$

First we observe that the domain of integration, D_n^0, can be embedded in a larger space of matrices by discarding the hermiticity and positivity,

$$D_n^0 \subset D_{\mathbb{C},n}^0 = \{ X \in \mathrm{Mat}_{n,n}(\mathbb{C} | X = t_s X^t t_s \} \; .$$

Since F_B is holomorphic up to a branch cut, D_n^0 can be deformed into the larger space $D_{\mathbb{C},n}^0$ to reach possible saddle-points.

It is clear that the different solution spaces of (4.14) in $D_{\mathbb{C},n}^0$ can be distinguished by the eigenvalues of $X\lambda$. Thus we search a solution of (4.14) in the diagonal matrices and construct the whole saddle-point manifold in the BB-sector, M_B, using a group action. Since on the diagonal matrices $F_B(X)$ is a sum of functions of the diagonal elements,

$$F_B(X) = \sum_{i=1}^{n} F_{B,i}(X_{ii}) \; , \; F_{B,i}(X_{ii}) = (\log X_{ii} - v^2 X_{ii}^2 + 2iE\lambda_{ii} X_{ii}) \; ,$$

we can reduce (4.14) to just one diagonal element of X,

$$X_{ii}^2 - \tilde{\lambda}_{ii} \frac{iE}{v^2} X_{ii} - \frac{1}{2v^2} = 0 \; .$$

This simple quadratic equation has the solutions

$$X_{ii} = \tilde{\lambda}_{ii} \frac{iE}{2v^2} \pm \frac{1}{2v^2} \sqrt{2v^2 - E^2} \; . \tag{4.15}$$

This equation has solutions with non-zero real part if $|E| < \sqrt{2}\, v$. We denote the solutions for $\lambda_{ii} = \pm 1$ by $s_{1\pm}$ and $s_{2\pm}$ for positive and negative real part, respectively. Note that for $i = 1, 2$ the relation $s_{i+} = \bar{s}_{i-}$ holds. Now we need a criterion telling us which saddle-point contributes. It is clear that D_n^0 contains diagonal matrices of the form $X = \mathrm{diag}(X_{11}, \ldots, X_{nn}, X_{11}, \ldots, X_{nn})$, $X_{ii} \in \mathbb{R}^{\geq 0}$. The original integration domain, $\mathbb{R}^{\geq 0}$, of every X_{ii} may be distorted separately into the complex plane. Therefore, it is possible to consider only one contribution of F_B, say $F_{B,i}(X_{ii})$ and to answer the question whether it is possible to reach the saddle-point via an allowed path. Running along such a path, $\Re(F_{B,i}(X_{ii}))$ must assume its absolute maximum in the saddle-point. Figure 4.3 shows the situation for $\lambda_{ii} = +1$ and fixed E; it does not change qualitatively for $\lambda_{ii} = -1$ and different values of E. Due to $\Re(F_B(s_1)) = \Re(F_B(s_2))$, both saddle-points s_1 and s_2 can be shown in one figure. Hence, it is clear that the only reachable saddle-point in the

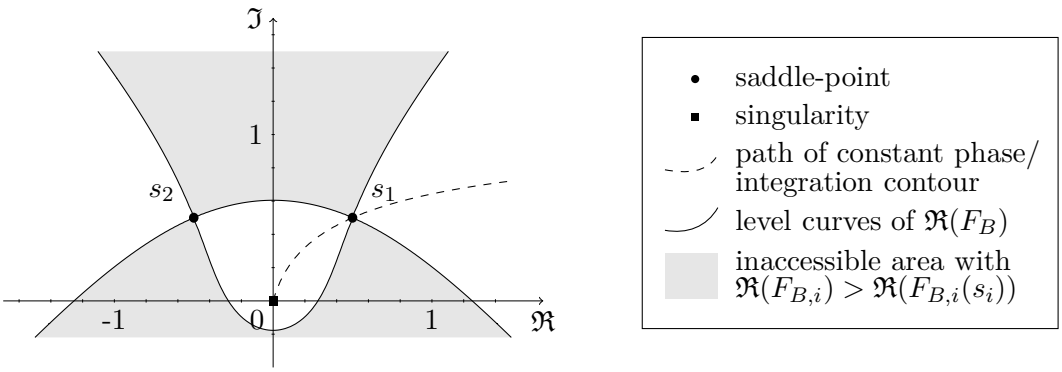


Figure 4.1: Saddle-points s_1 and s_2 of F_B

diagonal matrices reads

$$X_0 = \Re(s_1)\mathbf{1}_{2n} + \Im(s_1)\lambda \ .$$

To construct the whole saddle-point manifold M_B starting from X_0, we use the structure of D_n^0. We know from [LSZ08] that a symmetry group $G_B \simeq \mathrm{GL(n,\mathbb{R})}$ acts transitively on D_n^0 by

$$X \mapsto gX^t t_s g^t t_s \ , t_s g^t t_s = g^\dagger \ . \tag{4.16}$$

This symmetry must still be present in the distorted domain of integration, at least in a local sense.

To handle the situation, we introduce two commuting involutory automorphisms on the group G_B,

$$\Phi_B(g) = t_s (g^{-1})^t t_s^{-1} \ \text{ and } \quad \Theta_B(g) = t_s \lambda (g^{-1})^t (t_s\lambda)^{-1} \ . \tag{4.17}$$

The fixed point set $\mathrm{Fix}(\Phi)$ of Φ_B is the isotropy group of the action (4.16) in $X = \mathbf{1}_{2n}$. The set $\mathrm{Fix}(\Theta)$ is also a group and generates M_B by

$$X_0 \mapsto g X_0^t t_s g^t t_s \ , \tag{4.18}$$

as can be checked easily. Since Φ_B and Θ_B commute, the intersection $H = \mathrm{Fix}(\Phi) \cap \mathrm{Fix}(\Theta)$ is the isotropy group of (4.18). Thus we find $M_B \simeq \mathrm{Fix}(\Theta_B)/H$. Hence, our task is now to determine this space. For this purpose, it is convenient to switch to the real group $\mathrm{GL}(2n,\mathbb{R})$ by the unitary transformation

$$g \mapsto h = pgp^\dagger \quad , \quad p = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_n & \mathbf{1}_n \\ i\mathbf{1}_n & -i\mathbf{1}_n \end{pmatrix} \ , \ p^t p = t_s \ , \ p\lambda p^\dagger = \lambda \ .$$

Using this transformation, the condition $t_s g^t t_s = g^\dagger$ just turns into $h = \bar{h}$. The further conditions become

$$\begin{aligned} &p^\dagger h p = \Theta(p^\dagger h p) = t_s \lambda p^t (h^{-1})^t \bar{p} (t_s\lambda)^{-1} \quad \text{and} \\ &p^\dagger h p = \Phi(p^\dagger h p) t_s p^t (h^{-1})^t \bar{p} (t_s)^{-1} \ . \end{aligned}$$

Due to $pt_s\lambda p^t = \lambda$ the and $pt_s p^t = \mathbf{1}_{2n}$ they simplify to

$$h\lambda h^t = \lambda \quad \text{and} \quad hh^t = \mathbf{1}_{2n} \ ,$$

i.e. we have $\mathrm{Fix}(\Theta_B) \simeq \mathrm{O}(2n_A, 2n_R)$ and $\mathrm{Fix}(\Phi_B) \simeq \mathrm{O}(2n)$, and consequently $\mathrm{Fix}(\Theta_B) \cap \mathrm{Fix}(\Phi_B) \simeq \mathrm{O}(2n_A) \times \mathrm{O}(2n_R)$, which means

$$M_B \simeq \mathrm{O}(2n_A, 2n_R)/\mathrm{O}(2n_A) \times \mathrm{O}(2n_R) = \mathrm{SO}(2n_A, 2n_R)/\mathrm{S}\big(\mathrm{O}(2n_A) \times \mathrm{O}(2n_R)\big) \ .$$

This is a Riemannian symmetric space of noncompact type in class BDI in Cartan's list [Hel78], in particular we have $\dim_{\mathbb{R}} M_B = 4n_A n_R$. We denote the deformed domain of integration which contains M_B by $\tilde{D}_n^0$.

The Fermi-Fermi block. The restriction of $F(Q)$ to the FF-sector reads

$$F_F(Y) = \frac{1}{2}\mathrm{Tr}\big(\log Y - v^2(Y\lambda)^2 + 2iE\lambda Y\big) \ .$$

The saddle-point equation (4.13) becomes

$$(Y\lambda)^2 - \frac{iE}{v^2}Y\lambda - \frac{1}{2v^2} = 0 \ . \tag{4.19}$$

We look for saddle-points in the diagonal matrices which can be reached by distortion of D_n^1. However, in principle all saddle-points are reachable by the deformed integration domain. Thus we need a new criterion to decide which saddle-point is the most important one. Here supersymmetry helps as follows: It is an elementary observation that each Gaussian integration around the saddle-point supermanifold leads to a factor N^{-1} and N^{+1} in the case of bosonic and fermionic directions, respectively. Thus we have to minimize the transverse superdimension $d_B - d_F$ of the saddle-point supermanifold in order to obtain the leading order in N. Since the total number of fermionic generators is independent of our analysis ($4n^2$), this is achieved by maximizing the number of fermionic generators which do not conserve eq. (4.13). This can be realized by fulfilling the equation $X_{ii} - Y_{jj} = 0$ for as many pairs (i, j) as possible. Thus the good choice for Y_0 is to take a copy of X_0, $Y_0 = X_0$.

On D_n^1 we have a group action,

$$Y \mapsto gYt_a g^t t_a^{-1} \quad , \quad g \in \mathrm{U}(2n) \ , \tag{4.20}$$

which allows us to construct the saddle-point manifold M_F. Again, we introduce two commuting involutory automorphisms,

$$\Phi_F(g) = t_a(g^{-1})^{\mathrm{st}}t_a^{-1} \ \text{ and } \ \Theta_F(g) = t_a\lambda(g^{-1})^{\mathrm{st}}(t_a\lambda)^{-1} \ . \tag{4.21}$$

The isotropy group (as a subgroup of $\mathrm{U}(2n)$) of the action (4.20) at $Y = \mathbf{1}_{2n}$ is $\mathrm{Fix}(\Phi_F)$. The solutions of eq. (4.19) are obtained by $Y = gY_0t_a g^t t_a^{-1}$, $g \in \mathrm{Fix}(\Theta_F)$. Since $(\lambda t_a)^2 = -\mathbf{1}_{2n}$, we have $\mathrm{Fix}(\Theta_F) = \mathrm{USp}(2n)$.[1] Clearly, the isotropy group of Y_0 in $\mathrm{Fix}(\Theta_F)$ is the intersection $\mathrm{Fix}(\Theta_F) \cap \mathrm{Fix}(\Phi_F)$, which is isomorphic to

[1] Some authors use different notations. Here, we mean $\mathrm{USp(2n)} = \mathrm{Sp(2n)} = \mathrm{U(2n)} \cap \mathrm{Sp}(2n, \mathbb{C})$.

$\mathrm{USp}(n_A) \times \mathrm{USp}(n_R)$. This can be seen by writing $g = \exp(C)$ and switching to the Lie-Algebra $\mathfrak{g}_F = \mathfrak{su}(2n)$. The linearized automorphisms (4.21) read

$$d\Phi_F(C) = -t_a C^t t_a^{-1} \quad \text{and} \quad d\Theta_F(C) = -t_a \lambda C^t (t_a \lambda)^{-1} \; . \tag{4.22}$$

They define two Cartan decompositions of $\mathfrak{g}_F$, given by

$$\mathfrak{g}_F = \mathfrak{p}_{\Theta_F} + \mathfrak{k}_{\Theta_F} \quad \text{and} \quad \mathfrak{g}_F = \mathfrak{p}_{\Phi_F} + \mathfrak{k}_{\Phi_F} \; ,$$

with the definitions

$$\begin{aligned} \mathfrak{p}_{\Theta_F} &= \{F \in \mathfrak{g}_F | d\Theta(F) = -F\} \; , & \mathfrak{k}_{\Theta_F} &= \{F \in \mathfrak{g}_F | d\Theta(F) = F\} \; , \\ \mathfrak{p}_{\Phi_F} &= \{F \in \mathfrak{g}_F | d\Phi(F) = -F\} \; , & \mathfrak{k}_{\Phi_F} &= \{F \in \mathfrak{g}_F | d\Phi(F) = F\} \; , \end{aligned}$$

As usual, the Lie algebra of $\mathrm{Fix}(\Theta_F) \cap \mathrm{Fix}(\Phi_F)$ is obtained by differentiating the conditions for $\mathrm{Fix}(\Theta_F)$ and $\mathrm{Fix}(\Phi_F)$ along a curve $\exp(tC)$ in $t = 0$, which leads to $C \in \mathfrak{p}_{\Phi_F} \cap \mathfrak{k}_{\Theta_F}$. This is equivalent to

$$C = -C^\dagger = -t_a C^t t_a = \lambda C \lambda \; .$$

First, we observe that the first two equations enforce an obvious block structure of C. This can be inserted in the third equation, which then reads

$$C = \begin{pmatrix} a & b \\ b^\dagger & a^t \end{pmatrix} \; , \; a = a^\dagger \; , \; b = -b^t \; .$$

We see that $b = \tilde{\lambda} b \tilde{\lambda}$ and $a = \tilde{\lambda} a \tilde{\lambda}$ must hold. Therefore, only the diagonal blocks in a and b of size $n_A \times n_A$ and $n_R \times n_R$ survive. Since the blocks of different size do not talk to each other under the Lie bracket, we obtain two independent Lie Algebras $\mathfrak{usp}(2n_A)$ and $\mathfrak{usp}(2n_R)$. Altogether, we find

$$M_F \simeq \mathrm{USp(n)}/\mathrm{USp}(n_A) \times \mathrm{USp}(n_R) \; ,$$

which is a compact Riemannian symmetric space of class CII in Cartan's list. In particular, we have $\dim(M_F) = \dim(M_B) = 4n_A n_R$.

Full supersymmetric situation. Now we are able to discuss the full supersymmetric problem. We introduce two commuting involutory automorphisms as a generalization of (4.17) and (4.21),

$$\Phi(g) = T(g^{-1})^t T^{-1} \quad \text{and} \quad \Theta(g) = T\Lambda(g^{-1})^t (T\Lambda)^{-1} \; , \; g \in \mathrm{GL}(2n|2n) \; . \tag{4.23}$$

The supermatrix Q is subject to the symmetry (4.10), thus we can generalize the actions (4.16) and (4.20) to

$$Q \mapsto g Q T g^{\mathrm{st}} T^{-1} \; . \tag{4.24}$$

The restrictions of the previous paragraphs, namely the reality and unitarity condition, must be transferred to the part of Grassmann degree zero in the BB-block and the FF-block of g, which defines a supergroup G in $\mathrm{GL}(2n|2n)$.

Analogously to the previous paragraphs, the saddle-point supermanifold is generated by those elements of G, which, acting via (4.24) on $Q_0 = E_B \otimes X_0 + E_F \otimes Y_0$, fulfill the saddle-point equation (4.13). These elements lie in the fixed point set $\mathrm{Fix}(\Theta)$. The isotropy group of (4.24) at $Q = c\mathbf{1}_{4n}$ is given by $\mathrm{Fix}(\Phi)$, thus we arrive at $M \simeq \mathrm{Fix}(\Theta)/\mathrm{Fix}(\Theta) \cap \mathrm{Fix}(\Phi)$. From the previous paragraphs we already know its structure in the BB- and FF-sector, hence we have

$$M \simeq \mathrm{OSp}(2n|2n)/\mathrm{OSp}(2n_A|2n_A) \times \mathrm{OSp}(2n_R|2n_R) \ ,$$

which is a Riemannian symmetric superspace of class BDI|CII [Zir98c].

To complete the method of steepest descent, it remains to discuss the Gaussian integrations around the saddle-point supermanifold M. To this aim, we observe that in the limit of large N it is sufficient to know the tangential space at Q_0. To understand its structure, we consider first the tangential space at a point $Q_p \propto \mathbf{1}_{4n}$ in the original supermanifold S. It can be obtained by setting $g = \exp(tA), A \in \mathfrak{g}$ and differentiating the action (4.24) at $t = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tA) Q_p T \exp(tA^{\mathrm{st}}) T^{-1} = AQ_p + Q_p T A^{\mathrm{st}} T^{-1} \ . \tag{4.25}$$

Of course, if we allow A to lie in the whole Lie algebra $\mathfrak{g}$, we obtain more than the tangential space at Q_p. Since the isotropy group of (4.24) at Q_p is $\mathrm{Fix}(\Phi)$, the restrictions for A are obtained by using the Φ-induced automorphism of $\mathfrak{g}$, given by

$$d\Phi(A) = \left.\frac{d}{dt}\right|_{t=0} \Phi\big(\exp(tA)\big) = -TA^{\mathrm{st}}T^{-1} \ . \tag{4.26}$$

This automorphism produces a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{p}_\Phi + \mathfrak{k}_\Phi \ , \ \mathfrak{p}_\Phi = \{A \in \mathfrak{g} | d\Phi(A) = -A\} \ , \ \mathfrak{k}_\Phi = \{A \in \mathfrak{g} | d\Phi(A) = A\} \ .$$

Since $\mathfrak{k}_\Phi$ is the Lie algebra of $\mathrm{Fix}(\Phi)$, we only need $\mathfrak{p}_\Phi$. It generates the tangential space at Q_p by relation (4.25). Since the deformation of $D_n^0 \times D_n^1$ to $\tilde{D}_n^0 \times \tilde{D}_n^1$ does not deform the tangential space, it is also a good candidate to generate the tangential space at Q_0. However, some rotations by phase factors can occur. Nevertheless, it is useful to start with

$$T_{Q_0} S = \{Q_0 A + A Q_0 | A \in \mathfrak{p}_\Phi\} \ .$$

Now, we need to know, which of the directions in $T_{Q_0}S$ are tangential and which are normal to M. Note that a notion of orthogonality comes from the quadratic form $\mathcal{B}(A_1, A_2) = \mathrm{STr}(A_1 A_2)$ on $\mathfrak{g} \times \mathfrak{g}$, which can be naturally transferred to $T_{Q_0}S$.

Since M is generated by the group $\mathrm{Fix}(\Theta)$ via (4.24) the automorphism $d\Theta$ on $\mathfrak{g}$, given by

$$d\Theta(A) = \left.\frac{d}{dt}\right|_{t=0} \Theta\big(\exp(tA)\big) = -T\Lambda A^{\mathrm{st}} (T\Lambda)^{-1} \tag{4.27}$$

helps us to distinguish normal and tangential directions.

Similar to $d\Phi$, it induces a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{p}_\Theta + \mathfrak{k}_\Theta \ , \ \mathfrak{p}_\Theta = \{A \in \mathfrak{g} | d\Theta(A) = -A\} \ , \ \mathfrak{k}_\Theta = \{A \in \mathfrak{g} | d\Theta(A) = A\} \ ,$$

which is orthogonal with respect to the quadratic form $\mathcal{B}$. Moreover, it induces a decomposition of $T_{Q_0}S$ into a tangential and normal part,

$$\begin{aligned} T_{Q_0}S &= T_{Q_0}M + \nu_{Q_0}M \quad \text{with} \\ T_{Q_0}M &= \{AQ_0 + Q_0A | A \in \mathfrak{p}_\Phi \cap \mathfrak{k}_\Theta\} \ , \\ \nu_{Q_0}M &= \{AQ_0 + Q_0A | A \in \mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta\} \ . \end{aligned} \tag{4.28}$$

Thus $\nu_{Q_0}M$ is a good candidate for the Gaussian integrations. To proceed, we need the quadratic expansion of F in Q_0, which reads

$$F(Q_0 + \delta Q) = F(Q_0) - \frac{1}{2}\mathrm{STr}((Q_0^{-1}\delta Q)^2 + v^2(\delta Q\Lambda)^2) + \mathcal{O}(\delta Q)^3 \ . \tag{4.29}$$

It can be easily seen that decomposition (4.28) is orthogonal with respect to the quadratic form in (4.29), as it should be. Thus the superintegral (4.11) factorizes in an superintegral over M and Gaussian integrations over $\nu_{Q_0}M$. However, inserting a tangential vector corresponding to an even generator in $\mathfrak{g}$, i.e. $\delta Q \in \nu_{Q_0,0} = \{AQ_0 + Q_0A | A \in (\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta)_0\}$, in the quadratic form of (4.29), we see that an imaginary part occurs. That means that the chosen directions are not those of steepest descent. Fortunately, this can be repaired choosing an orthogonal basis of $(\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta)_0$ and multiplying each basis vector by a phase factor obtained by inserting the corresponding tangential vector in (4.29). Moreover, we do not explicitly need these factors, since they also occur in the Jacobian of the measure, which compensates them exactly. Thus we can simply denote the space corrected by the phase factors by $(\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta)_0$. Of course in the odd part given by $(\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta)_1$ no phase factors occur.

To conclude the discussion, we must discuss the measure. The measure DQ in (4.11) is invariant under $Q \mapsto hQTh^{\mathrm{st}}T^{-1}$. During our calculations we deformed the domain of integration and chose new local coordinates in the vicinity of M, $Q = gQ_0Tg^{\mathrm{st}}T^{-1}$. The new invariant measure is simply dg, the Berezinian is just 1. It decomposes into a tangential and a transverse part. For the purpose of Gaussian integration, it can be realized locally by dA, which is the flat measure on $\mathfrak{g}$. On the saddle-point supermanifold M the measure is just the restriction of dg.

Let us consider the Gaussian superintegral. For every $A \in \mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta$ we have $AQ_0 = Q_0A$, this symmetry is not influenced by the fact that we rotate the domain of integration to $\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta$ by phase factors. Thus the Gaussian superintegral reads

$$\int\limits_{(\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta)_0} DA \exp\left(-2N\mathrm{STr}(A^2 + v^2(Q_0A)^2)\right) = c_n \ ,$$

It can be seen by counting the dimension of $A \in \mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta$ that the number of fermionic and bosonic degrees of freedom is equal, namely $4(n_A^2 + n_R^2)$. Due to the diagonal structure of Q_0 we conclude that this superintegral is independent of N and yields just a constant c_n, which we do not calculate explicitly. Due to the invariance of the measure DQ under the action (4.24), it is clear that the Gaussian superintegral is the same for the whole saddle-point supermanifold M. Thus it remains to write down the superintegral over the latter, no different Gaussian superintegrals occur. We calculate for $Q = gQ_0Tg^{\mathrm{st}}T^{-1} \in M$

$$F(Q) = \frac{1}{2}\mathrm{STr}\big(\log gQ_0Tg^{\mathrm{st}}T^{-1} - v^2(gQ_0Tg^{\mathrm{st}}T^{-1}\Lambda)^2 + 2iE\Lambda Q\big) = F(Q_0) = 0 \ .$$

Thus the leading contribution to the saddle-point superintegral must come from the term independent of N in the exponent of (4.11) and we are left with

$$\lim_{N\to\infty} Z_{n_A,n_R}(\sqrt{N}E+\hat{\omega}/\sqrt{N}) = c_n \int\limits_{M_B\times M_F} dg\exp(i\mathcal{B}(\hat{\omega},\mathrm{Ad}(g)Q_0\Lambda)) \ , \qquad (4.30)$$

This is the central result of this chapter. Before we discuss this result, it is necessary to calculate the density of states, i.e. $R_1(E)$. However, as we will see in the following section, this result is contained in the previous calculations.

4.4 Density of states

Referring to the previous calculations, it is easy to compute the density of states in the large N-limit under consideration. We just need to set $n_R = n = 1$ and $n_A = 0$, and use (4.6) and (4.30). In this case, the superintegral (4.30) is degenerate since M contains only the point Q_0. Nevertheless, we may differentiate the integrand with respect to $\hat{\alpha}$:

$$\lim_{N\to\infty} R_{0,1}(\sqrt{N}E) \propto \lim_{\varepsilon\to 0} \frac{\partial}{\partial\hat{\alpha}}\bigg|_{\hat{\alpha}=i\varepsilon} \exp(i\mathcal{B}(\hat{\omega},Q_0)) = i\mathrm{Tr}_0 Q_0 \ ,$$

with $Tr_0Q_0 := \mathrm{STr}(E_BQ_0)$. In a last step, we must take the imaginary part of $R_{0,1}$, which corresponds to the calculation of $R_{0,1} - R_{1,0}$. We obtain the density of states,

$$\rho_\infty(E) \propto \sqrt{2v^2 - E^2} \ .$$

It is also possible to do this calculation without the degenerate superintegral in (4.30), using (4.6) and (4.9), as follows,

$$\begin{aligned}\rho_N(\omega) &= \Im\left(\lim_{\varepsilon\to 0}\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\omega+i\varepsilon} Z_{0,1}(\nu)\right) \\ &= \Im\left(\lim_{\varepsilon\to 0}\int\limits_{D_n^0\times D_n^1} DQ\,\mathrm{Tr}_0 Q\exp\big(-\mathrm{STr}(\frac{v^2}{2}Q^2 - i(\omega+i\varepsilon)Q)\big)\mathrm{SDet}^{N/2}(Q)\right) \ .\end{aligned}$$

Now the same computations as in the previous section can be done, however, they are much simpler since the saddle-point manifold M is zero-dimensional, i.e. it is only a point. Thus the only thing to do is to evaluate a Gaussian superintegral at $Q = Q_0$. Since the Gaussian integration just yields a constant, Tr_0Q_0 is the important factor. Thus we find again, using (4.15)

$$\rho_\infty(E) \propto \mathrm{Tr}_0 Q_0 = \sqrt{2v^2 - E^2} \ .$$

The proportionality constant can be obtained from the Gaussian superintegral, however, it is easier to restore it from the normalization condition as follows,

$$1 = c\int\limits_{-v\sqrt{2}}^{v\sqrt{2}} dE\sqrt{2v^2 - E^2} = c\pi v^2 \ .$$

Hence, we have

$$\rho_\infty(E) = \frac{1}{\pi v^2}\sqrt{2v^2 - E^2}\ .$$

This density of states is well known as Wigner's semi-circle law. Taking into account that we have rescaled the energy by $E = \omega/\sqrt{N}$, for $v = 1$ this result assumes exactly the form found in [Meh04].

4.5 Explicit correlation functions of GOE from the literature

From the literature we already know the result of the superintegral (4.30), if we insert it into (4.6) with $\omega_1 = \omega_2 = \cdots = \omega_n = NE$ and add up all contributions for different values of n_A and n_R. For example, in [PM83] or [Meh04] the following result for the n-point correlation function in the limit of large N can be found,[2]

$$\lim_{N\to\infty}(R_n(E+\alpha_1,\ldots,E+\alpha_n)) = \mathrm{QDet}(K_E(\alpha_i,\alpha_j))_{ij}\ . \tag{4.31}$$

The symbol QDet denotes the quaternionic determinant. The kernel K_E is defined by

$$K_E(\alpha_i,\alpha_j) = \begin{pmatrix} \frac{\sin\pi\rho_\infty(E)(\alpha_i-\alpha_j)}{\alpha_i-\alpha_j} & D(\rho_\infty(E)(\alpha_i-\alpha_j)) \\ J((\rho_\infty(E)(\alpha_i-\alpha_j)) & \frac{\sin\pi\rho_\infty(E)(\alpha_i-\alpha_j)}{\alpha_i-\alpha_j} \end{pmatrix}\ ,$$

with the notation

$$D(r) = -\frac{1}{\pi}\int_0^\pi dk\, k\sin kr \quad \text{and} \quad J(r) = -\frac{1}{\pi}\int_\pi^\infty dk\,\frac{\sin kr}{k}\ .$$

For further discussion of this result we refer to the literature, see the references mentioned above.

[2] Here we use a slightly different notation due to the energy variables introduced above. Usually, the kernel K does not depend on the energy E, however, in our context this is convenient.

Chapter 5

Bosonic random matrix ensemble invariant under time reversal

Again we consider a bosonic system with N degrees of freedom. Similar to the setting in chapter 3 we start with a quadratic Hamiltonian of the form

$$H = \frac{1}{2} \sum_{i,j=1}^{N} \left(P_i B_{ij} P_j + Q_i C_{ij} Q_j \right) , \tag{5.1}$$

where $B = B^t$ and $C = C^t$ are real and positive $N \times N$ matrices. The absence of off-diagonal terms, which would mix up momenta and positions, ensures the invariance under time reversal. As usual, we are interested in the spectral correlations in the thermodynamical limit, i.e. for large N. However, a solution of the model using techniques like bi-orthogonal polynomials, similar to chapter 3 seems to be impossible. The attempt leads to problems discussed in appendix C.2. Thus here we use the supersymmetry method to derive the correlation functions in terms of superintegrals.

In analogy to the situation in chapter 3 we introduce the matrix $X = hJ$, with

$$h = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} , \quad h = h^t > 0 \, ;$$

and J the symplectic unit matrix. The characteristic frequencies of (5.1) are given by the eigenvalues of iX. The Hamiltonian assumes the form

$$H = \frac{1}{2} \begin{pmatrix} P & Q \end{pmatrix} X \begin{pmatrix} Q \\ -P \end{pmatrix} .$$

The measure on the space of all allowed Hamiltonians is again

$$d\mu_l(h) = c_l e^{-\tau \mathrm{Tr} h/2} \mathrm{Det}^{(l-1)/2}(h) dh \, , \tag{5.2}$$

with a free parameter $l \in \mathbb{N}$ and $\tau > 0$. Since τ can be scaled out, we immediately choose $\tau = 1$ without loss of generality. The normalization constant c_l is fixed by the condition

$$1 = \int\limits_{h=h^t>0} d\mu_l(h) \, .$$

Similar to the measure in chapter 3, this measure can be constructed by adding up a sufficient number of rank-one projectors. We write

$$B_{ij} = \sum_{\alpha=1}^{M} v_{i\alpha} v_{j\alpha} \quad , \quad C_{ij} = \sum_{\alpha=1}^{M} w_{i\alpha} w_{j\alpha} \quad , \quad i,j = 1,2,\ldots,N \; , \tag{5.3}$$

with $M := N + l$. The numbers $v_{i\alpha}$ and $w_{i\alpha}$ are independent Gaussian distributed with zero mean and variance 1. Pushing forward this distribution to a distribution of h leads immediately to (5.2). The disorder average for any appropriate function $f(h)$ is defined, as usual, as

$$\langle f \rangle = \int\limits_{h=h^t>0} f(h) d\mu_l(h) \; . \tag{5.4}$$

5.1 Using the supersymmetry method

For this section, we use section 4.1 as a template. However, the situation is more complicated. The first question is, how the space of matrices $h = h^t > 0$ can be realized. Since B and C are positive and symmetric, all matrices h can be reached by setting

$$h = A^t A \quad , \quad A = \begin{pmatrix} \tilde{B} & 0 \\ 0 & \tilde{C} \end{pmatrix} \quad , \quad B = \tilde{B}^t \tilde{B} \quad , \quad C = \tilde{C}^t \tilde{C} \; .$$

The matrices $\tilde{B}$ and $\tilde{C}$ are arbitrary real and rectangular of dimension $M \times N$. The advantage of this choice is the disappearance of the determinantal factor in the measure (5.2) after transforming the disorder integral. We find, using the measure $d\tilde{B}d\tilde{C}$ proportional to the flat one,

$$\langle f \rangle = \int\limits_{h=h^t>0} f(h) d\mu_l(h) = \int\limits_{\mathrm{Mat}_{M,N}(\mathbb{R})\times\mathrm{Mat}_{M,N}(\mathbb{R})} e^{-\frac{1}{2}\mathrm{Tr}A^tA} f(A^tA) d\tilde{B}d\tilde{C} \; . \tag{5.5}$$

By an invariance argument one can see that this identity is indeed true. We understand the integrals in (5.5) as functionals which map a function f to a real number and consider their behavior under the transformation $f \mapsto f \circ l_g \circ r_{g^t}$. The mappings l_g and r_{g^t} are the left and right multiplication with a matrix g and g^t, respectively, $g = \mathrm{diag}(g_1, g_2) > 0$, $g_1, g_2 \in \mathrm{GL}(N, \mathbb{R})$ and we obtain

$$\int\limits_{h=h^t>0} (f \circ l_g \circ r_{g^t})(h)(h) d\mu_l(h) = \mathrm{Det}^{l+N}(g) \int\limits_{h=h^t>0} f(h)(h) d\mu_l(h) \quad \text{and}$$

$$\int e^{-\frac{1}{2}\mathrm{Tr}A^tA} (f \circ l_g \circ r_{g^t})(A^tA) d\tilde{B}d\tilde{C} = \mathrm{Det}^{l+N}(g) \int e^{-\frac{1}{2}\mathrm{Tr}A^tA} f(A^tA) d\tilde{B}d\tilde{C} \; .$$

Hence, the transformation behavior of both is the same which tells us that the functionals are proportional to each other. The constant of proportionality can be obtained by setting $f \equiv 1$ and is absorbed into the measure $d\tilde{B}d\tilde{C}$.

We are interested in the averaged n-level correlation functions given by

$$R_n(\omega_1,\ldots,\omega_n) := \lim_{\varepsilon\to 0}\left\langle \prod_{i=1}^{n}\frac{1}{\pi}\mathfrak{R}(\mathrm{Tr}(X-i\omega_i+\varepsilon)^{-1})\right\rangle \quad , \quad X = hJ \; .$$

We start with a reformulation of the real part of the resolvent in order to use the right hand side of (5.5) for the calculation of the ensemble average. By means of

$$\begin{aligned}\mathrm{Tr}(iE - JA^tA)^{-1} &= \frac{1}{iE}\sum_{i=0}^{\infty}\mathrm{Tr}\left(\frac{1}{iE}JA^tA\right)^i \\ &= -\frac{2l}{iE} + \frac{1}{iE}\sum_{i=0}^{\infty}\mathrm{Tr}\left(\frac{1}{iE}AJA^t\right)^i \\ &= -\frac{2l}{iE} + \mathrm{Tr}(iE - AJA^t)\end{aligned}$$

we obtain

$$\begin{aligned}\mathfrak{R}(\mathrm{Tr}(X-i\omega_i+\varepsilon)^{-1}) &= \mathfrak{R}\left(-\frac{2l}{i\omega_i+\varepsilon} + \mathrm{Tr}(X-i\omega_i+\varepsilon)^{-1}\right) \\ &= \mathfrak{R}\left(-\frac{2l}{i\omega_i+\varepsilon}\right) + \mathfrak{R}\left(i\frac{\partial}{\partial\beta}\mathrm{Det}\left(\frac{AJA^t-i\beta}{AJA^t-i\alpha}\right)\Big|_{\alpha=\beta=\omega_i-i\varepsilon}\right) \; .\end{aligned}$$

The first term corrects the resolvent at the energy $\omega = 0$. It occurred since the matrix AJA^t has $2l$ eigenvalues 0, which are artefacts of the replacement $X \to JA^tA \to AJA^t$. Since this term does not affect the resolvent at finite energies, which we are interested in, we neglect it in the following calculations for the sake of simplicity. Using the fact that eigenvalues of real antisymmetric matrices occur in conjugated pairs, the second term can be split into two parts according to

$$\begin{aligned}&\mathfrak{R}\left(i\frac{\partial}{\partial\beta}\mathrm{Det}\left(\frac{AJA^t-i\beta}{AJA^t-i\alpha}\right)\Big|_{\alpha=\beta=\omega_i-i\varepsilon}\right) = \\ &\qquad i\frac{\partial}{\partial\beta}\mathrm{Det}\left(\frac{AJA^t-i\beta}{AJA^t-i\alpha}\right)\Big|_{\alpha=\beta=\omega_i-i\varepsilon} - i\frac{\partial}{\partial\beta}\mathrm{Det}\left(\frac{AJA^t-i\beta}{AJA^t-i\alpha}\right)\Big|_{\alpha=\beta=\omega_i+i\varepsilon} \; .\end{aligned}$$

Thus in a product of n resolvents 2^n terms occur. These terms differ by the sign of $i\varepsilon$, we call the number of positive and negative signs n_R and n_A, respectively. Disregarding the bahavior at $\omega_i = 0$, we can write for the n-level correlation functions

$$R_n(\omega_1,\ldots,\omega_n) = \sum_{\substack{n_A=0\\ n_R=n-n_A}}^{n}\binom{n}{n_A}R_{n_A,n_R}(\omega_1,\ldots,\omega_n) \; , \tag{5.6}$$

with

$$\begin{aligned}R_{n_A,n_R}(\omega_1,\ldots,\omega_n) = \lim_{\varepsilon\to 0}\left(\frac{i\varepsilon}{\pi}\right)^{n_A}\prod_{i=1}^{n_A}-\frac{\partial}{\partial\alpha_i}\Big|_{\alpha_i=\omega_i-i\varepsilon}\prod_{i=n_A+1}^{n}\frac{\partial}{\partial\alpha_i}\Big|_{\alpha_i=\omega_i+i\varepsilon} \\ \times Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\omega_1-i\varepsilon,\ldots,\omega_{n_A}-i\varepsilon,\omega_{n_A+1}+i\varepsilon,\ldots,\omega_n+i\varepsilon) \; .\end{aligned} \tag{5.7}$$

The generating function Z_{n_A,n_R} is given by

$$Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) = \left\langle \prod_{i=1}^{n} \mathrm{Det}\left(\frac{AJA^t - i\beta_i}{AJA^t - i\alpha_i}\right)\right\rangle \ . \tag{5.8}$$

We see from (5.7) that it is sufficient to perform all calculations using the generating function (5.8). The differentiations with respect to the energy parameters can be postponed to the end of our calculations, provided the different limit processes and integrations can be interchanged without causing problems.

Rewriting the generating function as Gaussian superintegral

As before, the fundamental structure of the supersymmetry method is to rewrite the determinants occurring in the resolvent as Gaussian superintegrals. For that purpose we define the supermatrix

$$\mu = E_B \otimes \sum_{i=1}^{n} E_{ii}\alpha_i + E_F \otimes \sum_{i=1}^{n} E_{ii}\beta_i \ .$$

The matrices $E_B = \mathrm{diag}(1,0)$ and $E_F = \mathrm{diag}(0,1)$ are elementary 2×2 supermatrices, E_{ij} is an $n\times n$ matrix which has only one entry 1 in the i-th row and j-th column.

A direct calculation gives

$$\prod_{i=1}^{n} \mathrm{Det}\left(\frac{AJA^t - i\beta_i}{AJA^t - i\alpha_i}\right) = \mathrm{SDet}(\mathbf{1}_{2n}\otimes AJA^t - i\mu\otimes \mathbf{1}_M) \ .$$

The matrix $AJA^t - i\omega$ has a special structure, it is symmetric in the diagonal blocks and antisymmetric in the off-diagonal ones. This structure must be copied to the auxiliary Gaussian superintegrals, which can be done as follows. Let v and w be $M\times n$-matrices of independent complex variables, $\tilde v = \bar v$ and $\tilde w = \bar w$ its complex conjugate. As anticommuting variables we take the independent $M\times n$-matrices η, ξ, $\tilde\eta$, and $\tilde\xi$. We put these components in large matrices,

$$\begin{aligned}
\Psi &= \begin{pmatrix} v & \tilde v & \eta & \tilde\eta \\ w & -\tilde w & \xi & -\tilde\xi \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad , \\
\tilde\Psi &= \begin{pmatrix} \tilde v & v & \tilde\eta & -\eta \\ \tilde w & -w & \tilde\xi & \xi \end{pmatrix}^t = \begin{pmatrix} \tilde\psi_1 & \tilde\psi_2 \end{pmatrix} = \mathcal{C}\Psi^t \ , \\
\mathcal{C} &= (E_B\otimes\sigma_1 + E_F\otimes i\sigma_2)\otimes \mathbf{1}_n \ .
\end{aligned} \tag{5.9}$$

The appropriate Berezin measure reads

$$D(\Psi,\tilde\Psi) = \pi^{-2Mn}\prod_{i=1}^{M}\prod_{j=1}^{n} dv_{ij}d\bar v_{ij}dw_{ij}d\bar w_{ij}\frac{\partial}{\partial\eta_{ij}}\frac{\partial}{\partial\tilde\eta_{ij}}\frac{\partial}{\partial\xi_{ij}}\frac{\partial}{\partial\tilde\xi_{ij}} \ .$$

Using the definition

$$\nu = E_B\otimes\mathbf{1}_2\otimes\sum_{i=1}^{n} E_{ii}\alpha_i + E_F\otimes\mathbf{1}_2\otimes\sum_{i=1}^{n} E_{ii}\beta_i \ , \tag{5.10}$$

we find the relation

$$\mathrm{SDet}(\mathbf{1}_{2n}\otimes AJA^t - i\mu\otimes\mathbf{1}_M) = \int D(\Psi,\tilde{\Psi})\exp(-\mathrm{Tr}AJA^t\Psi\Lambda\tilde{\Psi} + i\mathrm{STr}\Lambda\nu\tilde{\Psi}\Psi)\ . \quad (5.11)$$

As in chapter 4, convergence of this superintegral is enforced by the imaginary part of the elements of ν. Due to the different signs of them the matrix Λ must be present in the exponent. We define

$$\tilde{\lambda} = \mathrm{diag}(-\mathbf{1}_{n_A}, \mathbf{1}_{n_R}) \quad , \quad \lambda = \mathbf{1}_2 \otimes \tilde{\lambda} \quad , \quad \Lambda = E_B \otimes \lambda + E_F \otimes \lambda\ .$$

Ensemble average

Now, we are able to use (5.11) to carry out the ensemble average in the generating function (5.8) according to (5.5). This means evaluating the outer integral of

$$\left\langle \prod_{i=1}^{n} \mathrm{Det}\left(\frac{AJA^t - \beta_i}{AJA^t - \alpha_i}\right)\right\rangle = \int d\tilde{B}d\tilde{C}e^{-\frac{1}{2}\mathrm{Tr}A^2}\int D(\Psi,\tilde{\Psi})\exp(-\frac{1}{2}\mathrm{Tr}AJA^t\Psi\Lambda\tilde{\Psi} + \frac{i}{2}\mathrm{STr}\Lambda\nu\tilde{\Psi}\Psi)\ .$$

Now, we interchange the inner and the outer integration. To simplify notation for the next step, we omit the integration over the auxiliary fields and the factor containing ν,

$$\begin{aligned}
&\int d\tilde{B}d\tilde{C}\exp -\frac{1}{2}\mathrm{Tr}\left(\tilde{B}^2 + \tilde{C}^2 + AJA^t\Psi\Lambda\tilde{\Psi}\right)\\
&\quad = \int d\tilde{B}d\tilde{C}\exp -\frac{1}{2}\mathrm{Tr}\left(\tilde{B}^2 + \tilde{C}^2 + \tilde{C}^t\psi_2\Lambda\tilde{\psi}_1\tilde{B} - \tilde{B}^t\psi_1\Lambda\tilde{\psi}_2\tilde{C}\right)\\
&\quad = \int d\tilde{B}d\tilde{C}\exp -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}\tilde{B}^t & \tilde{C}^t\end{pmatrix}\begin{pmatrix}\mathbf{1} & \psi_1\Lambda\tilde{\psi}_2\\ -\psi_2\Lambda\tilde{\psi}_1 & \mathbf{1}\end{pmatrix}\begin{pmatrix}\tilde{B}\\ \tilde{C}\end{pmatrix}\right)\\
&\quad = \mathrm{Det}(1 + \psi_1\Lambda\tilde{\psi}_2\psi_2\Lambda\tilde{\psi}_1)^{-N/2}\ .
\end{aligned}$$

The integration in the last step is a simple Gaussian integral since $(\psi_1\Lambda\tilde{\psi}_2)^t = -\psi_2\Lambda\tilde{\psi}_1$, which makes the quadratic form in the exponent symmetric. Note that this symmetry occurs due to our suitable choice of ψ_1 and ψ_2, which is crucial for the model under consideration.

Thus, we calculated successfully the disorder average. However, this has been paid with the introduction of the auxiliary space. Now, our task is to integrate out these auxiliary fields.

5.2 Application of the superbosonization identity

For the following calculation, section 4.2 serves as template.

The identity

$$\begin{aligned}
\mathrm{Det}(1 + \psi_1\Lambda\tilde{\psi}_2\psi_2\Lambda\tilde{\psi}_1) &= \exp(\mathrm{Tr}\log(1 + \psi_1\Lambda\tilde{\psi}_2\psi_2\Lambda\tilde{\psi}_1))\\
&= \exp(\mathrm{STr}\log(1 + \tilde{\psi}_1\psi_1\Lambda\tilde{\psi}_2\psi_2\Lambda))\\
&= \mathrm{SDet}(1 + \tilde{\psi}_1\psi_1\Lambda\tilde{\psi}_2\psi_2\Lambda)
\end{aligned}$$

helps us to apply the superbosonization formula. By means of (5.9) we can write down $\tilde{\psi}_1\psi_1$ and $\tilde{\psi}_2\psi_2$ explicitly,

$$\tilde{\psi}_1\psi_1 = \begin{pmatrix} \tilde{v}^t v & \tilde{v}^t\tilde{v} & \tilde{v}^t\eta & \tilde{v}^t\tilde{\eta} \\ v^t v & v^t\tilde{v} & v^t\eta & v^t\tilde{\eta} \\ \tilde{\eta}^t v & \tilde{\eta}^t\tilde{v} & \tilde{\eta}^t\eta & 0 \\ -\eta^t v & -\eta^t\tilde{v} & 0 & -\eta^t\tilde{\eta} \end{pmatrix} ,$$

$$\tilde{\psi}_2\psi_2 = \begin{pmatrix} \tilde{w}^t w & -\tilde{w}^t\tilde{w} & \tilde{w}^t\xi & -\tilde{w}^t\tilde{\xi} \\ -w^t w & w^t\tilde{w} & -w^t\xi & w^t\tilde{\xi} \\ \tilde{\xi}^t w & -\tilde{\xi}^t\tilde{w} & \tilde{\xi}^t\xi & 0 \\ \xi^t w & -\xi^t\tilde{w} & 0 & -\xi^t\tilde{\xi} \end{pmatrix} .$$

The matrix $\tilde{\psi}_1\psi_1$ has exactly the form needed to apply the superbosonization formula for orthogonal symmetry, but in $\tilde{\psi}_2\psi_2$ some signs have to be changed. This can be achieved by multiplying with $\Sigma_3 = E_B \otimes \sigma_3 \otimes \mathbf{1}_n + E_F \otimes \sigma_3 \otimes \mathbf{1}_n$ from the left and from the right. Let us state that in the notation of [LSZ08], we have $p = q = n$, i.e. the number of bosonic and fermionic replicas is equal, the number of orbitals is N. Thus, we are ready to apply the superbosonization identity and obtain

$$\begin{aligned} &Z_{n_A,n_R}(\nu) \equiv Z_{n_A,n_R}(\alpha_1 \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \\ &= \left\langle \prod_{i=1}^{n} \mathrm{Det}\left(\frac{AJA^t - \beta_i}{AJA^t - \alpha_i}\right) \right\rangle \\ &= \int D(\Psi,\tilde{\Psi}) \exp\left(\mathrm{STr}\frac{1}{2} i\nu\Lambda\tilde{\Psi}\Psi\right) \mathrm{SDet}\big(1 + \tilde{\psi}_1\psi_1(\Lambda\Sigma_3)\Sigma_3\tilde{\psi}_2\psi_2\Sigma_3(\Lambda\Sigma_3)\big)^{-N/2} \\ &= \int DQ_1 DQ_2 e^{\frac{i}{2}\mathrm{STr}(\nu\Lambda(Q_1+Q_2))} \frac{\mathrm{SDet}^{M/2}(Q_1Q_2)}{\mathrm{SDet}^{N/2}(\mathbf{1} + Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)} . \end{aligned} \tag{5.12}$$

In the second step of the superbosonization identity is used. The integration over the two times $2N$ bosonic and two times $2N$ fermionic variables is reduced to an integration over two $4n \times 4n$ supermatrices Q_1 and Q_2, which produces the power of the superdeterminant. In block form, we write, using $i \in \{1, 2\}$,

$$Q_i = \begin{pmatrix} X_i & \sigma_i \\ \tau_i & Y_i \end{pmatrix} .$$

According to [LSZ08], Q_i is subject to the symmetry

$$Q_i = TQ_i^{\mathrm{st}}T^{-1} , \quad T = E_B \otimes t_s + E_F \otimes t_a , \quad t_s = \sigma_1 \otimes \mathbf{1}_n , \; t_a = (-i)\sigma_2 \otimes \mathbf{1}_n . \tag{5.13}$$

Therefore, there are only $(2n)^2$ independent Grassmann variables in each Q_1 and Q_2, explicitly we have $\sigma_i = t_s\tau_i^t t_a^{-1}$. The domain of integration for X_i reads

$$D_n^0 := \{X \in \mathrm{Mat}_{2n,2n}(\mathbb{C}) | X = X^\dagger > 0, X = t_s X t_s^{-1}\} . \tag{5.14}$$

The dimension is $\dim_{\mathbb{R}}(D_n^0) = 2n^2 + n$. For Y we have

$$D_n^1 := \{Y \in U(2n) | Y = t_a Y^t t_a^{-1}\} . \tag{5.15}$$

Its dimension is $\dim_{\mathbb{R}}(D_n^1) = 2n^2 - n$. It remains to define the measure,

$$DQ_i = d\mu_{D_n^0}(X_i) d\mu_{D_n^1}(Y_i) \prod_{i=1}^{2n} \prod_{j=1}^{2n} \frac{\partial}{\partial \sigma_{ij}} \frac{\mathrm{Det}^n(X_i - \sigma Y_i^{-1}\tau)\mathrm{Det}^n(Y_i - \tau X^{-1}\sigma)}{(2\pi)^{2n^2}\mathrm{Det}^{1/2}(1 - X_i^{-1}\sigma Y_i^{-1}\tau)} \ . \tag{5.16}$$

Up to a constant factor, the measures $d\mu_{D_n^0}(X)$ and $d\mu_{D_n^1}(Y)$ are defined by invariance under the transformations

$$\begin{aligned} X \mapsto gXt_s g^t t_s \quad &, \quad g^\dagger = t_s g^t t_s \ , \quad g \in \mathrm{GL}(n,\mathbb{C}) \quad \text{and} \\ Y \mapsto gYt_a g^t t_a^{-1} \quad &, \quad g \in \mathrm{U}(2n) \ . \end{aligned} \tag{5.17}$$

In [LSZ08], the normalization constants are fixed by Gaussian integrals.

5.3 Asymptotic correlation functions in the bulk

In (5.12), we found for the generating function the expression

$$Z_{n_A,n_R}(\nu) = \int DQ_1 DQ_2 e^{\frac{i}{2}\mathrm{STr}(\nu\Lambda(Q_1+Q_2))} \frac{\mathrm{SDet}^{M/2}(Q_1Q_2)}{\mathrm{SDet}^{N/2}(\mathbf{1} + Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)} \ , \tag{5.18}$$

with $\Sigma_3 = (E_B + E_F) \otimes \sigma_3 \otimes \mathbf{1}$. We want to calculate (5.18) in the limit of large N in the bulk scaling. Therefore, we rescale the energy according to $\nu = NE + \hat{\omega}$ with fixed E and

$$\hat{\omega} = E_B \otimes \mathbf{1}_2 \otimes \mathrm{diag}(\hat{\alpha_1}, \hat{\alpha_2}, \ldots, \hat{\alpha_n}) + E_F \otimes \mathbf{1}_2 \otimes \mathrm{diag}(\hat{\beta_1}, \hat{\beta_2}, \ldots, \hat{\beta_n}) \ .$$

In order to apply the method of steepest descent, we set

$$\begin{aligned} F(Q_1,Q_2) = N\mathrm{STr}\big(&iE\Lambda(Q_1+Q_2) + \log Q_1 + \log Q_2 \\ &- \log(\mathbf{1} + Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)\big) \ . \end{aligned} \tag{5.19}$$

The integrand of (5.18) can be written in the form

$$\exp\left(\frac{1}{2}(F(Q_1,Q_2) + i\mathrm{STr}(\hat{\omega}\Lambda(Q_1+Q_2) + (M-N)(\log Q_1 + \log Q_2)\right) \ . \tag{5.20}$$

The terms not contained in $F(Q_1,Q_2)$ are of order 1; we will see that the difference $M-N$ does not affect the result in the limit of large N. The matrix $\hat{\omega}$ can be regarded as a small perturbation . To start the saddle-point analysis, we set $Q_i = \tilde{Q}_i + \delta Q_i$ for $i \in \{1,2\}$. Demanding that the first order terms in $\delta Q_{1/2}$ vanish, we obtain the two equations

$$\begin{aligned} i\Lambda E + Q_1^{-1} - \Sigma_3\Lambda Q_2\Sigma_3\Lambda(\mathbf{1} + Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)^{-1} &= 0 \ , \\ i\Lambda E + Q_2^{-1} - \Sigma_3\Lambda Q_1\Sigma_3\Lambda(\mathbf{1} + Q_2\Sigma_3\Lambda Q_1\Sigma_3\Lambda)^{-1} &= 0 \ . \end{aligned}$$

Multiplying these equations by $(1 + Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)$ and $(1 + Q_2\Sigma_3\Lambda Q_1\Sigma_3\Lambda)$ from the right, respectively, and combining them, we obtain the saddle-point equations

$$(Q_1\Lambda)^3 + Q_1\Lambda = -iE^{-1} \quad \text{and} \tag{5.21a}$$

$$\Sigma_3 Q_2 \Sigma_3 = Q_1 \ . \tag{5.21b}$$

Let us remark that the situation is symmetric under exchange of Q_1 and Q_2, as can be e.g. seen by means of superintegral (5.18). Equations (5.21) merely do not look symmetric.

For the discussion of eq. (5.21a) the strategy of the discussion of eq. (4.13) can be copied. However, due to condition (5.21b) things are a bit more complicated.

Boson-Boson block. The saddle-point equations restricted to the BB-block reads

$$(X_1\lambda)^3 + X_1\lambda = -iE^{-1} \quad \text{and} \tag{5.22a}$$

$$\tilde{\Sigma}_3 X_2 \tilde{\Sigma}_3 = X_1 \ . \tag{5.22b}$$

The construction of the boson-boson saddle-point manifold M_B works as follows: First we embed the original space of integration, $D_n^0 \times D_n^0$ in a larger space of complex matrices $D_{\mathbb{C},n}^0 \times D_{\mathbb{C},n}^0$, with $D_{\mathbb{C},n}^0 = \{X \in \mathrm{Mat}_{2n,\mathbb{C}} | X = t_s X^t t_s\}$. As usual, since F_B is holomorphic, it is possible to deform $D_n^0 \times D_n^0$ in this larger space by dropping the conditions $X_i = X_i^\dagger > 0$, $(i \in \{1,2\})$ without changing the value of the integral. We just have to take care that the boundary of the domain does not move and the deformed domain converges to the original one at infinity. In particular, it is possible to reach saddle-points outside of $D_n^0 \times D_n^0$.

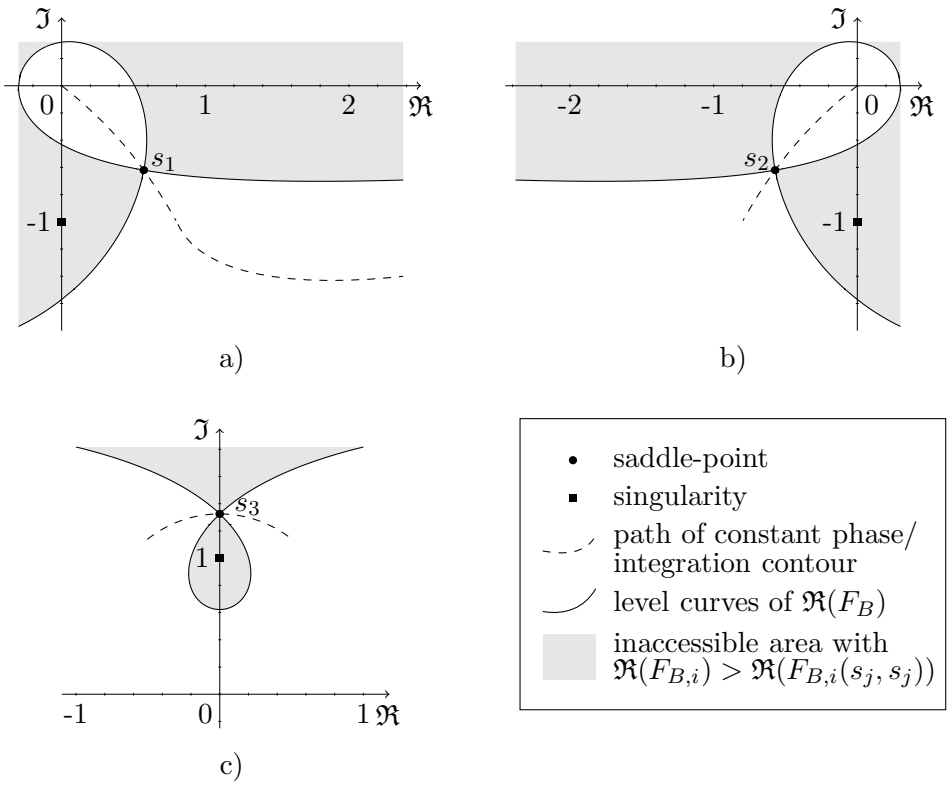


Figure 5.1: Saddle-points in the boson-boson block for $E = 1 + i\varepsilon$

Clearly, only (5.22a) is crucial for the structure of M_B, (5.22b) just copies its solution to X_2, up to the conjugation by $\tilde{\Sigma}_3$.

To start the discussion of (5.22a) we set $X_1 \equiv X$ and observe that its can be distinguished by the eigenvalues of $X\lambda$. Thus the first steps are to search for solutions in the space of diagonal matrices and fix the ones which are reachable by deformations of D_n^0. Then, we use a group action on D_n^0 to generate the whole solution space and copy it to X_2. In the last step the group action serves to find the remaining directions of steepest descent.

Obviously, (5.22a) has 3^n different solutions in the diagonal matrices for real and positive E, since every element x_{ii} fulfills

$$(\pm x_{ii})^3 \pm x_{ii} = i/E \tag{5.23}$$

independently. For each sign, there are three solutions, one with positive real part, denoted by s_1, one with negative real part, s_2; the third one, s_3, is purely imaginary. To see, which solution contributes to the integral (5.18), consider, as in chapter 4, the diagonal matrices $X = \mathrm{diag}(X_{11}, \ldots, X_{nn}, X_{11}, \ldots, X_{nn})$ in D_n^0 and deform the domain of integration of each entry, $\mathbb{R}^{\geq 0}$, in the complex plane. The remaining $2n^2$ directions which are necessary to cover the whole domain of integration, are irrelevant for this thought. In any case, there must be a path from 0 to the solution X_0 lying completely in the diagonal matrices. The criterion to choose one is that the saddle-point given by the pair $(X_0, \tilde{\Sigma}_3 X_0 \tilde{\Sigma}_3)$ must maximize $\Re(F_B(X_1, X_2))$ globally. Unfortunately, this is still a high-dimensional problem. However, it can be reduced to one complex dimension: In the diagonal matrices, F_B decomposes into n contributions, one for each pair $((X_1)_{ii}, (X_2)_{ii})$, i.e.

$$F_B(X_1, X_2) = \sum_{i=1} f_B((X_1)_{ii}, (X_2)_{ii}),$$

$$f_B(x_1, x_2) = -2N(\pm iE(x_1 + x_2) - \log x_1 - \log x_2 + \log(1 + x_1 x_2),$$

the sign depends on the related entry of λ. Hence, the condition to globally maximize $\Re(F_B(X_1, X_2))$ in the saddle-point has been reduced to the maximization of. $\Re(F_{B,i}((X_1)_{ii}, (X_2)_{ii}))$. Finally, we obtain a one dimensional complex problem if we fix one variable, say $(X_2)_{ii}$, in the saddle-point s_j and observe what happens if $(X_2)_{ii}$ runs from zero to s_j. Figure 5.3 shows the situation for a fixed value of z and a fixed sign in (5.23). However, the pictures do not change qualitatively for different values of z; changing the sign just leads to complex conjugation. As mentioned in the description of figure 5.3, the gray parts are inaccessible since the real part of $F_{B,i}$ exceeds its value in the saddle-point. In the case of s_1 it is obviously possible to distort the integration contour through it, see figure 5.3 a). In the case of s_2 and s_3 (fig. 5.3 b)-c)) it is impossible to start with the contour in 0, run along the path of constant phase through the saddle-point and escape to $+\infty$, since there is a singularity on the imaginary axis: We see from (5.19) that the pair of values with $x_1 x_2 = -1$ is singular, i.e. if we deform both contours in the same manner, a singularity in $\pm i$ occurs. In any case, we are not able to deform both contours in a permitted manner. Furthermore, it is problematic to run to values of x_1 and x_2

with negative real part since $F_{B,i}$ must have a branch cut, however, this does not affect the real part.

Consequently, the only reachable saddle-point in the set of diagonal matrices by deforming the original domain of integration is the pair (X_0, X_0), with $X_0 = \mathfrak{R}(s_1)\lambda + \mathfrak{I}(s_1)\mathbf{1}$.

The symmetry group of $D_n^0 \times D_n^0$ is $G_B \times G_B \simeq \mathrm{GL(n,\mathbb{R})} \times \mathrm{GL(n,\mathbb{R})}$. It acts transitively on $D_n^0 \times D_n^0$ by

$$(X_1, X_2) \mapsto (g_1 X_1 t_s g_1^t, g_2 X_2 t_s g_2^t) \; . \tag{5.24}$$

We call its Lie algebra $\mathfrak{g}_B \oplus \mathfrak{g}_B$.

As mentioned before, (5.22b) can be neglected to determine the structure of M_B. Thus, since we found the reachable solution of (5.22a) in the diagonal matrices, we can handle it exactly like (4.14), and restrict action (5.24) to the first element of the pair (X_1, X_2). Thus we know

$$M_B \simeq \mathrm{O}(2n_A, 2n_R)/\mathrm{O}(2n_A) \times \mathrm{O}(2n_R) = \mathrm{SO}(2n_A, 2n_R)/\mathrm{S}\big(\mathrm{O}(2n_A) \times \mathrm{O}(2n_R)\big) \; ,$$

which is again is a Riemannian symmetric space of noncompact type in class BDI in Cartan's list [Hel78], in particular we have $\dim_{\mathbb{R}} M_B = 4n_A n_R$.

For the further discussion, we denote the deformed domain of integration which contains M_B by $\tilde{D}_n^0 \times \tilde{D}_n^0$.

The Fermi-Fermi block. The restriction of $F(Q)$ to the FF-sector reads

$$F_F(X) = \frac{1}{2}\mathrm{Tr}\big(\log Y - v^2(Y\lambda)^2 + 2iE\lambda Y\big) \; .$$

The saddle-point equation become

$$(Y_1\lambda)^3 + Y_1\lambda = -iE^{-1} \quad \text{and} \tag{5.25a}$$

$$\tilde{\Sigma}_3 Y_2 \tilde{\Sigma}_3 = Y_1 \; . \tag{5.25b}$$

Searching a saddle-point in the diagonal matrices leads, using the same supersymmetric argument as in chapter 4 , to $(Y_0, Y_0) = (X_0, X_0)$. Again, on $D_n^1 \times D_n^1$ we have a group action,

$$(Y_1, Y_2) \mapsto (g_1 Y_1 t_a g_1^t t_a^{-1}, g_2 Y_2 t_a g_2^t t_a^{-1}) \quad , \quad g_1, g_2 \in \mathrm{U}(2n) \; , \tag{5.26}$$

which allows us to construct the saddle-point manifold M_F. It consists of pairs of the form $(Y, \tilde{\Sigma}_3 Y \tilde{\Sigma}_3) \in D^1_{\mathbb{C}n} \times D^1_{\mathbb{C}n}$, $D^1_{\mathbb{C}n} = \{Y \in \mathrm{Mat}_{2n,2n}(\mathbb{C}) | Y = t_a Y^t t_a^{-1}\}$. As in the BB-sector, only the first saddle-point equation (5.25a) is important. Thus we pick the first element of the pair and see immediately by means of the construction in chapter 4

$$M_F \simeq \mathrm{USp(n)}/\mathrm{USp}(n_A) \times \mathrm{USp}(n_R) \; ,$$

which is a compact Riemannian symmetric space of class CII in Cartan's list. In particular, we have $\dim(M_F) = \dim(M_B) = 4n_A n_R$.

Full supersymmetric situation. Now we are able to discuss the full supersymmetric problem. Starting with the saddle-point in the diagonal matrices, (Q_0, Q_0) with $Q_0 = E_B \otimes X_0 + E_F \otimes Y_0$, we can construct the whole saddle-point supermanifold M. The actions (5.24) and (5.26) generalize to

$$(Q_1, Q_2) \to (g_1 Q_1 T g_1^{\mathrm{st}}, g_2 Q_2 T g_2^{\mathrm{st}}) \; . \tag{5.27}$$

The restrictions of the previous paragraphs, namely the reality condition $g_B^\dagger = t_s g_B^t t_s$ in the BB-sector and the unitarity condition $g_F \in \mathrm{U}(2n)$ in the FF-sector, must be transfered to the part of Grassmann degree zero in the BB-block g_B and the FF-block g_F of g, which defines a supergroup $G \times G$ in $\mathrm{GL}(2n|2n) \times \mathrm{GL}(2n|2n)$.

Analogously to the previous paragraphs, the saddle-point supermanifold is generated by those elements of G, which, acting via (5.27) restricted to Q_1 on Q_0, fulfill the saddle-point equation (5.21a), (5.21b) may be neglected. These elements lie in the fixed point set $\mathrm{Fix}(\Theta)$. The isotropy group of the restriction of (5.27) to Q_1 at $Q_1 = c\mathbf{1}_{4n}$ is given by $\mathrm{Fix}(\Phi)$, thus we arrive at $M \simeq \mathrm{Fix}(\Theta)/\mathrm{Fix}(\Theta) \cap \mathrm{Fix}(\Phi)$. From the previous paragraphs we already know its structure in the BB- and FF-sector, hence we have

$$M \simeq \mathrm{OSp}(2n|2n)/\mathrm{OSp}(2n_A|2n_A) \times \mathrm{OSp}(2n_R|2n_R) \; ,$$

which is a Riemannian symmetric superspace of class BDI|CII [Zir98c].

It remains to construct the Gaussian superintegrals around M. The strategy is very similar to that in chapter 4. We can even use the same definitions and decompositions of $\mathfrak{g}$. Recall the automorphisms (4.26) and (4.27),

$$d\Phi(A) = -TA^{\mathrm{st}}T^{-1} \quad \text{and} \quad d\Theta(A) = -T\Lambda A^{\mathrm{st}}(T\Lambda)^{-1} \; ,$$

which induce decompositions of $\mathfrak{g}$ according to

$$\begin{aligned} \mathfrak{g} &= \mathfrak{p}_\Phi + \mathfrak{k}_\Phi \; , \quad \mathfrak{p}_\Phi = \{A \in \mathfrak{g} | d\Phi(A) = -A\} \; , \quad \mathfrak{k}_\Phi = \{A \in \mathfrak{g} | d\Phi(A) = A\} \; , \\ \mathfrak{g} &= \mathfrak{p}_\Theta + \mathfrak{k}_\Theta \; , \quad \mathfrak{p}_\Theta = \{A \in \mathfrak{g} | d\Theta(A) = -A\} \; , \quad \mathfrak{k}_\Theta = \{A \in \mathfrak{g} | d\Theta(A) = A\} \; . \end{aligned} \tag{5.28}$$

Since the isotropy supergroup of (5.27) is $\mathrm{Fix}(\Phi) \times \mathrm{Fix}(\Phi)$, a candidate for the tangential space at (Q_0, Q_0) in the original supermanifold,S, is

$$T_{(Q_0,Q_0)}S = \{(A_1 Q_0 + Q_0 A_1, A_2 Q_0 + Q_0 A_2 | A_1, A_2 \in \mathfrak{p}_\Phi\} \; .$$

The notion of orthogonality on $T_{(Q_0,Q_0)}S$ comes from the quadratic form $\mathrm{STr}(A_1 A_2 + A_3 A_4)$ on $\mathfrak{g} \oplus \mathfrak{g}$. Thus we can decompose $T_{(Q_0,Q_0)}S$. The saddle-point supermanifold is generated by the supergroup $\mathrm{Fix}(\Theta)$ by $(Q_0, Q_0) \mapsto (gQ_0TgT^{-1}, \Sigma_3 gQ_0TgT^{-1}\Sigma_3)$, thus we can use the decomposition (5.28) to separate the normal part of $T_{(Q_0,Q_0)}S$ from the part tangential to M. We obtain

$$T_{(Q_0,Q_0)}S = T_{(Q_0,Q_0)}M + \nu_{(Q_0,Q_0)}M \; ,$$

$$T_{(Q_0,Q_0)}M = \{(Q_0 A + A Q_0, \Sigma_3(Q_0 A + A Q_0)\Sigma_3 | A \in \mathfrak{p}_\Phi \cap \mathfrak{k}_\Theta\} \; , \tag{5.29a}$$

$$\nu_{(Q_0,Q_0)}M = \nu_{\parallel(Q_0,Q_0)}M + \nu_{\perp(Q_0,Q_0)}M \; , \tag{5.29b}$$

$$\nu_{\parallel(Q_0,Q_0)}M = \{(Q_0 A + A Q_0, -\Sigma_3(Q_0 A + A Q_0)\Sigma_3 | A \in \mathfrak{p}_\Phi \cap \mathfrak{k}_\Theta\} \; , \tag{5.29c}$$

$$\nu_{\perp(Q_0,Q_0)}M = \{(Q_0 A_1 + A_1 Q_0, Q_0 A_2 + A_2 Q_0 | A_1, A_2 \in \mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta\} \; . \tag{5.29d}$$

Let us remark that the three spaces are all orthogonal to each other. The further decomposition (5.29b) is useful for the Gaussian integration.

We proceed by expanding $F(Q_1, Q_2)$ to quadratic order in the point (Q_0, Q_0),

$$\begin{aligned} F(Q_0+\delta Q_1, Q_0+\delta Q_2) = N\mathrm{Tr}\big[&(Q_0^{-1}\delta Q_1)^2 + (Q_0^{-1}\delta Q_2)^2 \\ &+ \delta Q_1 \Lambda \delta Q_2 \Lambda (1+Q_0^2)^{-1} \\ &- \left(Q_0\Lambda\delta Q_2\Lambda(1+Q_0^2)^{-1}\right)^2 - \left(\delta Q_1 Q_0 (1+Q_0^2)^{-1}\right)^2 \\ &- Q_0\Lambda\delta Q_2\Lambda(1+Q_0^2)^{-1}\delta Q_1 Q_0 (1+Q_0^2)^{-1}\big] + \mathcal{O}(\delta Q_{1/2}^3) \\ &= H(\delta Q_1, \delta Q_2) + \mathcal{O}(Q_{1/2}^3)\ . \end{aligned} \tag{5.30}$$

It is clear that the three parts of $T_{(Q_0,Q_0)}S$ are orthogonal with respect to the quadratic form H, thus the superintegral (5.18) factors into three parts. However, inserting even elements of $\nu_{\perp(Q_0,Q_0)}M$ in the quadratic form H, we obtain an imaginary part, hence, similar to the situation in chapter 4, some phase factors occur. We just rotate the space $(\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta)_0$ by them and obtain the domain of integration $(\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta)_0$. For $\nu_{\parallel(Q_0,Q_0)}M$ no phase factors occur.

To write down the superintegrals, we have to consider the Berezin measure. The measure DQ_1DQ_2, in (5.18) is invariant under $(Q_1, Q_2) \mapsto (h_1 Q_1 T h_1^{\mathrm{st}} T^{-1}, h_1 Q_1 T h_1^{\mathrm{st}} T^{-1})$. For our calculations we deformed the domain of integration and chose new local coordinates in the vicinity of M, given by $(Q_1, Q_2) = (g_1 Q_0 T g_1^{\mathrm{st}} T^{-1}, g_2 Q_0 T g_2^{\mathrm{st}} T^{-1})$. The new invariant measure is simply $dg_1 dg_2$, the Berezinian is just 1. On the saddle-point supermanifold M the measure is just the restriction of $dg_1 dg_2$. For $\nu_{\parallel(Q_0,Q_0)}M$ the local realization is the restricted flat measure on $\mathfrak{g}$,dA, and for $\tilde{\nu}_{\perp(Q_0,Q_0)}M$ it is realized by the restricted flat measure on $\mathfrak{g} \oplus \mathfrak{g}$, $dA_1 dA_2$.

Let us consider the Gaussian superintegral. As already mentioned, it factorizes into two parts, according to the decomposition (5.29b). The first factor reads

$$\int\limits_{(\mathfrak{p}_\Phi \cap \mathfrak{k}_\Theta)_0} dA \exp\left(H(AQ_0 + Q_0 A, \Sigma_3(AQ_0 + Q_0 A)\Sigma_3)\right)\ .$$

The second one reads

$$\int\limits_{(\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta)_0 \oplus (\tilde{\mathfrak{p}}_\Phi \cap \tilde{\mathfrak{p}}_\Theta)_0} dA_1 dA_2 \exp\left(H(A_1 Q_0 + Q_0 A_2, A_1 Q_0 + Q_0 A_2)\right)\ .$$

It can be seen by counting the dimension of $\mathfrak{p}_\Phi \cap \mathfrak{k}_\Theta$ and $(\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta) \oplus (\mathfrak{p}_\Phi \cap \mathfrak{p}_\Theta)$ that the number of bosonic and fermionic degrees of freedom is equal in each of the two Gaussian superintegrals, namely $8n_A n_R$ and $8(n_A^2 + n_R^2)$, respectively. Due to the diagonal structure of Q_0 we conclude that these superintegrals are independent of N and just yield constants, which we do not calculate explicitly.

Due to the invariance of the measure DQ_1DQ_2 under the action (5.27), it is clear that the Gaussian superintegrals are the same for the whole saddle-point supermanifold M, thus it remains only to write down the superintegral over the latter. By invariance of the supertrace under cyclic permutations we calculate for $(Q_1, Q_2) = (gQ_0 T g^{\mathrm{st}} T^{-1}, \Sigma_3 g Q_0 T g^{\mathrm{st}} T^{-1} \Sigma_3) \in M$

$$F(Q_1, Q_2) = 0\ .$$

Thus the leading contribution to the saddle-point superintegral must come from the term independent of N in (5.20) and we obtain the result

$$\lim_{N\to\infty} Z_{n,N}(NE+\hat{\omega}) = c_n \int\limits_{M_B\times M_F} dg \exp(i\mathcal{B}(\hat{\omega}, \mathrm{Ad}(g)Q_0\Lambda)) \ , \tag{5.31}$$

with $\mathcal{B}(A_1, A_2) = \mathrm{STr}(A_1 A_2)$. We see that this superintegral is the same as obtained by the calculation in 4. Thus the correlation functions exhibit GOE universality. By means of the results of chapter 2 one would expect that: The main difference between GUE and GOE is the additional time reversal invariance. Thus if time reversal invariance is added to a model that leads to GUE-universality, this should result in GOE-universality. To obtain a more explicit expression, one has just to replace the density of states of the GOE in 4.31 by the one of the ensemble considered here.

5.4 Density of states in the bulk

Now it is easy to read off the density of states (DOS) of the ensemble under consideration. Analogously to the GOE it can be obtained by evaluating the derivative with respect to $\hat{\alpha_1}$ of (5.31) for $(n_A, n_R) = (1, 0)$ and $(n_A, n_R) = (0, 1)$ and adding up the results. Since the integral is zero-dimensional, this is achieved by searching the solutions with positive real part of (5.23) for positive and negative sign, corresponding to $(n_A, n_R) = (0, 1)$ and $(n_A, n_R) = (1, 0)$. We denote these solutions by x_+ and x_-, respectively. We find, setting $b = 3\sqrt{3}/2$,

$$x_\pm = -(2E)^{-1/3} e^{\mp\pi i/6} \left(1 - \sqrt{1 - \frac{E^2}{b^2}}\right)^{1/3} + \frac{1}{3}(2E)^{1/3} e^{\pm\pi i/6} \left(1 - \sqrt{1 - \frac{E^2}{b^2}}\right)^{-1/3}$$

and hence

$$\rho(E) = c(x_+ + x_-) = c\left(\frac{E}{b}\right)^{-1/3} \left(\left(1 + \sqrt{1 - \frac{E^2}{b^2}}\right)^{-1/3} - \left(1 - \sqrt{1 - \frac{E^2}{b^2}}\right)^{-1/3} \right) . \tag{5.32}$$

The normalization constant c can be determined by demanding

$$1 = \int_0^b \rho(E) dE \ ,$$

which leads to $c = 1/\pi$.

This density of states is exactly the same calculated in chapter 3. For a discussion we refer to section 3.5 and figure 3.1.

5.5 Correlation functions near the lower edge

For the superintegral (5.12) a second convergent scaling limit exists, as can be seen by the following calculation.

$$\begin{aligned} Z_{n_A,n_R}(\nu) &\equiv Z_{n_A,n_R}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) \\ &= \int DQ_1DQ_2 e^{-\frac{i}{2}\mathrm{STr}(\nu\Lambda(Q_1+Q_2))} \frac{\mathrm{SDet}^{M/2}(Q_1Q_2)}{\mathrm{SDet}^{N/2}(\mathbf{1}+Q_1\Sigma_3\Lambda Q_2\Sigma_3\Lambda)} \\ &= \int DQ_1DQ_2 e^{-\frac{i}{2}\mathrm{STr}(\nu\Lambda(Q_1+Q_2))} \frac{\mathrm{SDet}^{l/2}(Q_1Q_2)}{\mathrm{SDet}^{N/2}(\mathbf{1}+Q_1^{-1}\Sigma_3\Lambda Q_2^{-1}\Sigma_3\Lambda)} \\ &\approx \int DQ_1DQ_2 e^{-\frac{i}{2}\mathrm{STr}(\nu\Lambda(Q_1+Q_2))} \mathrm{SDet}^{l/2}(Q_1Q_2) \\ &\qquad \times \exp\left(-\frac{N}{2}\mathrm{STr}Q_1^{-1}\Sigma_3\Lambda Q_2^{-1}\Sigma_3\Lambda\right) . \end{aligned} \tag{5.33}$$

In the last equality we assumed N to be large and used the fact

$$\lim_{N\to\infty}(1-X/N)^N = \exp(X) ,$$

which is also true if X is a supermatrix. The diagonal matrix ν contains the parameters α_i and β_i, cf. (5.10).

Scaling limit $\tilde{\omega} = \nu\sqrt{N/2}$. The last superintegral (5.33) yields a finite result in the limit of large N if we rescale by

$$\tilde{\omega} = \nu\sqrt{N/2} \quad \text{and} \quad Q_i \to Q_i\sqrt{N/2} .$$

In the limit of large N, this leads to the following expression for the scaled generating function $\tilde{Z}_{n_A,n_R}$:

$$\begin{aligned} \tilde{Z}_{n_A,n_R}(\tilde{\omega}) &= \\ &= \lim_{N\to\infty} Z_{n_A,n_R}(\tilde{\omega}/\sqrt{N/2}) \\ &= \int DQ_1DQ_2 e^{-\frac{i}{2}\mathrm{STr}(\tilde{\omega}\Lambda(Q_1+Q_2))} \mathrm{SDet}^{l/2}(Q_1Q_2) \\ &\qquad \times \exp\left(-\mathrm{STr}Q_1^{-1}\Sigma_3\Lambda Q_2^{-1}\Sigma_3\Lambda\right) . \end{aligned}$$

Starting from this superintegral, we restrict ourselves to the simplest case $(n_A, n_R) = (0, 1)$, which is similar to the case $(n_A, n_R) = (1, 0)$. The much more complicated case with general (n_A, n_R) has not been solved within this project and requires further investigation.

5.6 Density of states at the lower edge

To simplify our notation we set $\tilde{\omega} = z_1E_B \otimes \mathbf{1}_2 + z_2E_F \otimes \mathbf{1}_2 = \mathrm{diag}(z_1, z_1, z_2, z_2)$ and $Z(z_1, z_2) := \tilde{Z}_{0,1}(\tilde{\omega})$. Equation (5.6) tells us that we have to add the contributions of

$R_{0,1}$ and $R_{1,0}$. This corresponds to adding the contributions of $Z_{0,1}$ and $Z_{1,0}$, which is equivalent to taking two times the imaginary part of $Z_{0,1}$ due to $Z_{0,1} = -\bar{Z}_{1,0}$. For the limit of the density of states $\tilde{\rho}_l$ this means

$$\tilde{\rho}_l(z) = \frac{2i}{\pi}\frac{\partial}{\partial z_2}\Im(Z(z_1,z_2))\Big|_{z_1=z_2=z} = \frac{2i}{\pi}\frac{\partial}{\partial z_1}\Im(Z(z_1,z_2))\Big|_{z_1=z_2=z} . \tag{5.34}$$

The second of these relations seems to be trivial, looking at eqs. (5.8) and (5.7). Nevertheless, we will see that it is a very useful identity.

Now, our task is to calculate the superintegral

$$Z(z_1,z_2) = \int DQ_1 DQ_2 e^{-\frac{i}{2}\mathrm{STr}(\tilde{\omega}(Q_1+Q_2))}\mathrm{SDet}^{l/2}(Q_1Q_2)\exp\left(-\mathrm{STr}Q_1^{-1}\Sigma_3 Q_2^{-1}\Sigma_3\right) . \tag{5.35}$$

For each of the supermatrices Q_1 and Q_2 symmetries, domains of integration, and the Berezin measure is fixed by the relations (5.13)-(5.17) with $n = 1$.

Superintegral (5.35) is hard to handle. Usual transformations and parametrizations for the supermatrices Q_1 and Q_2 do not lead to a useful simplification of it. Thus we tackled this superintegral by writing a package for *Mathematica* which is able to deal with Grassmann variables, see appendix B. Using this package, we can get rid of all Grassmann variables and obtain the integral

$$\begin{aligned} Z(z_1,z_2) &= \int \prod_{i=1,2} \mathrm{Det}\, X_i^{3/2}\mathrm{Det}\, Y_i^{1/2} d\mu(X_i)d\mu(Y_i)\Omega_i e^{-\frac{iz}{2}\mathrm{Tr}(Q_1+Q_2)}\times \\ &\qquad \times \mathrm{SDet}^{l/2}(Q_1Q_2)\exp\left(-\mathrm{STr}\Sigma_3 Q_1^{-1}\Sigma_3 Q_2^{-1}\right) \\ &= \int \prod_{i=1,2} \mathrm{Det}\, X_i^{3/2}\mathrm{Det}\, Y_i^{1/2} d\mu(X_i)d\mu(Y_i) F_l(X_1,X_2,Y_1,Y_2) . \end{aligned} \tag{5.36}$$

Recall that the domain of integration is $(D_1^0 \times D_1^1) \times (D_1^0 \times D_1^1)$. By evaluating the definitions (5.14) and (5.15) we find for X_i and Y_i the explicit expressions

$$\begin{aligned} X_i \in D_1^0 &= \left\{ \begin{pmatrix} x & v \\ \bar{v} & x \end{pmatrix} \Big| x > 0, v\bar{v} < x^2 \right\} , \\ Y_i \in D_1^1 &= \{\mathrm{diag}(e^{i\varphi}, e^{i\varphi}) | 0 \le \varphi < 2\pi\} , \end{aligned}$$

which are useful to handle the remaining integral. The function F_l is cumbersome

and given by

$$\begin{aligned} F_l(X_1,X_2,Y_1,Y_2) =& \frac{A^{\frac{l-3}{2}}}{2\pi^4 (y_1y_2)^{l+1}} \exp\left(-\frac{i}{2}\mathrm{Tr}(z_1(X_1+X_2)-z_2(Y_1+Y_2))\right) \\ & \times \exp\left(\frac{2}{(y_1y_2)}-\frac{C}{A}\right)\left[\frac{4}{(y_1y_2)^4}+\frac{4}{(y_1y_2)^3}\left(\frac{C}{A}+2l+2\right)\right. \\ & + \frac{2}{(y_1y_2)^2}\left(\frac{2C^2}{A^2}+\frac{C(4l-1)-2}{A}+3l^2+3l+1\right) \\ & + \frac{1}{(y_1y_2)}\left(\frac{2C^2(2l+1)+4C}{A^2}+\frac{C(5l^2-4l-3)-12}{A}+2l^3\right) \\ & + \frac{C^2l^2+4C(l-1)+4}{A^2}+\frac{C(l-2)l^2+2(l^2-5l+3)}{A} \\ & \left. + \frac{(l-1)^2l^2}{4}\right]. \end{aligned} \tag{5.37}$$

Of course, the correspondig output of *Mathematica* is much more complicated. But, recognizing that the important variables are A and C, we were able to express F_l in the present form. The check that this result is equivalent to the output of *Mathematica* is easy and very fast. The variables A and C are defined by

$$\begin{aligned} A &= \frac{1}{4}\mathrm{Tr}(\sigma_3X_1\sigma_3X_1)\mathrm{Tr}(\sigma_3X_2\sigma_3X_2) = \mathrm{Det}\, X_1X_2 \\ C &= \mathrm{Tr}\,\sigma_3X_1\sigma_3X_2 = \mathrm{Det}(X_1+X_2)-\mathrm{Det}\,X_1-\mathrm{Det}\,X_2\,. \end{aligned}$$

Furthermore, we used $Y_i = \mathrm{diag}(y_i, y_i)$, $(i = 1, 2)$. Let us start to attack the integrations.

Integration in the FF-sector

In the FF-sector, i.e. for Y_1 and Y_2 the situation is simple. The integrations can be performed analytically as follows:

For $Y_i = \mathrm{diag}(y_i, y_i)$ and a test function f we find

$$\int_{D_1^0} \mathrm{Det}\, Y_i^{1/2} d\mu(Y_i) f(y_i) = \int_{U_1} dy_i f(y_i)\,,$$

since the measure $d\mu(Y)$ is invariant under the transformation $Y \mapsto gY^t t_a g^t t_a^{-1}$. This invariance is equivalent to invariance under $y \mapsto \alpha y$, $\alpha \equiv \mathrm{Det}\, g \in U_1$, since we have for any 2×2-matrix g the simple formula $gt_ag^tt_a^{-1} = \mathrm{diag}(\mathrm{Det}\, g, \mathrm{Det}\, g)$. Therefore, the Y-integrations can be performed by calculating the residual:

$$\begin{aligned} \int\limits_{U_1\times U_1} dy_1dy_2 \exp\left(iz_2(y_1+y_2)+\frac{2}{y_1y_2}\right)\frac{1}{(y_1y_2)^k} = \\ -\frac{8\pi^2}{2^k}\sum_{i=\mathrm{Max}(k-1,0)}^{\infty}\frac{(-2)^i z_2^{2i}}{i!^2(i-k+1)!}\,. \end{aligned} \tag{5.38}$$

Note that these integrals are real for real z_2. The derivative with respect to z_2, which is necessary to obtain $\tilde{\rho}_l(z)$ from $Z(z_1, z_2)$, can be easily done. Now we analyze the remaining integral by means of standard complex analysis.

Integration in the BB-sector

In the BB-sector the situation is more complicated. We have to integrate six real degrees of freedom, only one of these integrals is trivial.

Extracted from eqs. (5.37) and (5.36), the integrations have the structure

$$I_{l,i,j}(z_1) = \Im\left(\int d\mu(X_1, X_2) \exp\left(-(iz_1+\varepsilon)\mathrm{Tr}(X_1+X_2) - \frac{C}{A}\right) A^{\frac{l}{2}-i} C^j\right) . \quad (5.39)$$

Since the y-integrations yield purely real terms and we are interested in the imaginary part of $Z(z_1, z_2)$, cf. (5.34), here we need the imaginary part.

For each value of l there are six cases for the pair of parameters (i, j): $(0, 0)$, $(1, 0)$, $(2, 0)$, $(1, 1)$, $(2, 1)$ and $(2, 2)$. In the following, we treat all six cases together.

A parametrization of the matrices X_1 and X_2 is given by

$$X_i = x_i \begin{pmatrix} 1 & u_i e^{i\varphi_i} \\ u_i e^{-i\varphi_i} & 1 \end{pmatrix} ,$$

with $x_i \in \mathbb{R}^+$, $\varphi_i \in [0, 2\pi)$, and $u_i \in [0, 1]$. Now we want to reduce the Integral (5.39) as far as possible. First one observes that one of the six remaining integrations is trivial. Defining $\varphi = \varphi_1 - \varphi_2$ we obtain by means of the above parametrization $A = \mathrm{Det} X_1 X_2 = (x_1 x_2)^2 (1-u_1^2)(1-u_2^2)$ and $C = \mathrm{Tr}\, \sigma_3 X_1 \sigma_3 X_2 = 2(1 - u_1 u_2 \cos\varphi)$, i.e. the integrand only depends on the difference $\varphi_1 - \varphi_2$. Hence, we are confronted with five non-trivial integrations. For two of them there exist analytical tools, but for the remaining three we will need numerical integration.

Meijer's G-function. We therefore wish to evaluate integrals of the following type:

$$\lim_{\varepsilon\to 0} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^1 du_1 \int_0^1 du_2 \int_0^{2\pi} d\varphi e^{-(iz+\varepsilon)(x_1+x_2) - \frac{2(1-u_1u_2\cos\varphi)}{x_1x_2(1-u_1^2)(1-u_2^2)}} \times \frac{(2(1-u_1u_2\cos\varphi))^j}{((1-u_1^2)(1-u_2^2))^{(3-l)/2+i}} (x_1x_2)^n u_1 u_2 , \quad (5.40)$$

with $n = l - 1 - 2i + j$. The crucial steps to perform the limit $\varepsilon \to 0$ are contained in the following integrations with a positive constant v. The first integration is

standard, the following steps can be found in [Erd53] and [GR65],

$$\begin{aligned}
&\int\limits_{(\mathbb{R}^+)^2} dx_1 dx_2 (x_1 x_2)^n e^{-(iz+\varepsilon)(x_1+x_2)-\frac{v}{x_1 x_2}} = \\
&\quad = 2\left(\frac{v}{iz+\varepsilon}\right)^{\frac{n+1}{2}} \int\limits_{\mathbb{R}^+} dx\, e^{-x(iz+\varepsilon)} K_{n+1}\left(2\sqrt{\frac{v(iz+\varepsilon)}{x}}\right) x^{\frac{n-1}{2}} \\
&\quad = \frac{v^n}{iz+\varepsilon} \int\limits_{\mathbb{R}^+} dx\, e^{-x(iz+\varepsilon)} G^{2\,0}_{0\,2}\left(\frac{v(iz+\varepsilon)}{x}\Big| 1, -n\right) \\
&\quad = \frac{v^n}{iz+\varepsilon} \int\limits_{\mathbb{R}^+} dx\, e^{-x(iz+\varepsilon)} G^{0\,2}_{2\,0}\left(\frac{x}{v(iz+\varepsilon)}\Big| n+1, 0\right) \\
&\quad = \frac{v^n}{(iz+\varepsilon)^2} G^{0\,3}_{3\,0}\left(\frac{1}{v^2(iz+\varepsilon)^2}\Big| 0, 0, n+1\right) \\
&\quad = \frac{v^n}{(iz+\varepsilon)^2} G^{3\,0}_{0\,3}\left(v^2(iz+\varepsilon)^2\Big| 1, 1, -n\right) \,. \qquad (5.41)
\end{aligned}$$

Meijer's G-function $G^{3\,0}_{0\,3}\left(v^2(iz+\varepsilon)^2\Big|1,1,-n\right)$ is meromorphic with a singularity in 0 and has a branch cut on the negative real axis. In particular, it can be evaluated – despite the branch cut – on the negative real axis, i.e. for $\varepsilon = 0$. For the remaining three integrations which are not shown here (but still present), no analytical method exists. Thus, we continue by means of numerical methods. Hence, we need a fast method to calculate values of Meijer's G-function up to a prescribed accuracy. The function can be written as a sum of residuals, which are increasing powers of the argument. This leads to rapidly converging series for small arguments: From the definition of the G-function we find

$$\begin{aligned}
G^{3\,0}_{0\,3}(x|1,1,-n) &= \int\limits_{i\mathbb{R}+\max(n,-1)+\varepsilon} ds\, \Gamma^2(1+s)\Gamma(-n+s)x^{-s} \\
&= 2\pi i \sum_{i=\max(n,-1)}^{\infty} \mathrm{res}_{s=-i}\Gamma^2(1+s)\Gamma(-n+s)x^{-s}\,. \qquad (5.42)
\end{aligned}$$

The explicit expressions for the residuals are rather complicated, but it is no problem to calculate them using e.g. *Mathematica*. For large arguments we use the differential equation of the G-function [Erd53], which reads in our case

$$\left[x + \left(x\frac{d}{dx} - 1\right)^2 \left(x\frac{d}{dx} + n\right)\right] G^{3\,0}_{0\,3}(x|1,1,-n) = 0\,. \qquad (5.43)$$

Also from [Erd53] we know the G-function vanishes exponentially for large arguments, which fixes the correct solution of (5.43) in the three-dimensional space of solutions. As an ansatz, we assume that an asymptotic expansion of the form

$$G^{3\,0}_{0\,3}(x|1,1,-n) \approx e^{-3x^{1/3}} \sum_{j=n-1}^{j_{\max}} f_j x^{-j/3} \qquad (5.44)$$

exists. Then we use the differential equation to fix the coefficients f_i, which leads to the recurrence relation

$$f_j(-j+n-1)+f_{j-1}\frac{1}{9}(4-15n-3(j-1)(2n-j-2))+f_{j-2}\frac{1}{27}(j+1)^2(3n-j+2)=0\,.$$

In particular, we see from this recurrence that from a certain limit $i_{\max}$ on the coefficients f_j will increase at least as $j!$. That means, we only have an asymptotic expansion which diverges for any given x if j becomes too large. Nevertheless, if x is large enough, we obtain (numerically) arbitrarily good values from this approximation. Moreover, this method is much faster (orders of magnitude, in particular for large x!) than the built-in algorithms in *Mathematica* or *Maple*, which evaluate the G-function. It is also much faster than the evaluation of the integral (5.42). Therefore, to evaluate the remaining three integrations, we use the following approximation of the G-function:

$$G^{3\,0}_{0\,3}(x|1,1,-n)\approx\begin{cases}2\pi i\sum\limits_{j=\max(n,-1)}^{j_{\max}}\operatorname{res}_{s=-j}\Gamma^2(1+s)\Gamma(-n+s)x^{-s} & \text{if } x<x_0\\ e^{-3x^{1/3}}\sum\limits_{i=n-1}^{i_{\max}}f_i x^{-i/3} & \text{if } x\geq x_0\,.\end{cases}$$

This approximation can be driven to arbitrary accuracy. One just has to increase the values of $j_{\max}$, x_0, and $i_{\max}$. In our numerically calculations, we worked with a relative accuracy of 10^{-7}, according to values around $j_{\max}=13$, $x_0\approx 25$, and $i_{\max}=13$. However, the value of x_0 should be tuned separately for each value of n. Using the above techniques and (5.41), we are able to evaluate integrals of the type (5.40), which reduce to

$$\int du_1du_2d\varphi G^{3\,0}_{0\,3}\left(\frac{z^2v_1}{v_2}\Big|1,1,-n\right)v_2^{-l/2-1/2+i-j}v_1^{l-1-2i+2j}u_1u_2 \tag{5.45}$$

with

$$n=l-1-2i+j\quad,\quad v_2=(1-u_1^2)(1-u_2^2)\quad\text{, and}\quad v_1=2(1-u_1u_2\cos\varphi)\,. \tag{5.46}$$

To obtain all terms in (5.37), for each l the six cases $(i,j)=(0,0)$, $(1,0)$, $(2,0)$, $(1,1)$, $(2,1)$, and $(2,2)$ must be evaluated. The result can be seen in figure 5.2 for different values of l.

Asymptotic expansion for large z

In section 5.4 we calculated the asymptotic density of states in the bulk scaling limit. We know from chapter 3 that $\rho(E)$ given by (5.32) behaves as $E^{-1/3}$ for small E. Clearly, the asymptotic behavior in $z=0$ in the scaling used there must be recovered in the asymptotics of the microscopic scaling for large z. Therefore, we have

$$\tilde{\rho}_l(z)\propto z^{-1/3}$$

for large z, independent of l, which is illustrated by figure 5.2.

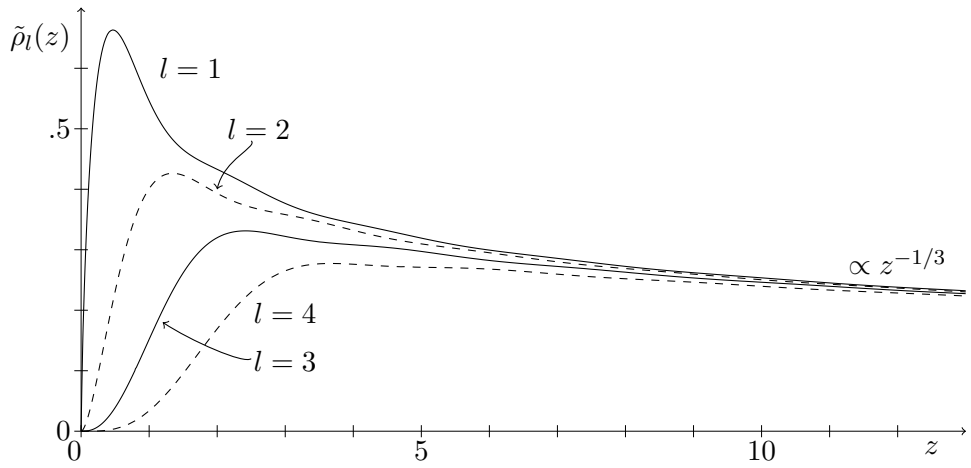


Figure 5.2: The asymptotic density of states $\tilde{\rho}_l(z)$

Asymptotic expansion in $z = 0$

The expansion in $z = 0$ is more problematic, the behavior there is non-analytic. The fact that we need the imaginary part of integral (5.39) causes problems. Indeed, for big enough values of l an expansion in powers of z up to a certain order is possible, but unfortunately the terms obtained by this procedure are real. The lowest order in the expansion producing an imaginary part diverges. Nevertheless, we are able to calculate the leading terms of the DOS for all odd values of the parameter l, we obtain $\tilde{\rho}_l(z) = (a + b \ln z) z^l + \mathcal{O}(\ln z \, z^{l+1})$, $a, b \in \mathbb{R}$. We suspect that this formula is also true for even l, however, we are not able to proof it. The calculation for odd l is rather complicated: Since $v = v_1/v_2$ is the only variable which appears in the argument of Meijer's G-function, we choose v as new variable and transform integral (5.45) to an one-dimensional integral,

$$\begin{aligned} \int du_1 du_2 d\varphi G^{3\,0}_{0\,3}\left(\frac{z^2 v_1}{v_2}\Big|1,1,-n\right) v_2^{-l/2-1/2+i-j} v_1^{l-1-2i+2j} u_1 u_2 \\ = \int_2^\infty G^{3\,0}_{0\,3}\left(z^2 v\Big|1,1,-n\right) \mathrm{vol}_k(v) v^{n+j} dv \; . \end{aligned} \tag{5.47}$$

Of course the volumefunction $\mathrm{vol}_k(c)$ depends on the exponents of v_1 and v_2, we define $k = l/2 - 3/2 - i + j$. For the definition of $\mathrm{vol}_k(c)$, see (C.3) in the appendix. Since the imaginary part of the G-function, which we are interested in, is 0 at $z^2 c = 0$ and decays exponentially for large arguments, it is reasonable to assume that only large values of v contribute in the limit of small z. This assumption turns out to be true for odd l, for even l it fails. Therefore, we need an expansion for $\mathrm{vol}(v)$ in $v = \infty$. Using the estimates in section C.1, we only need to evaluate the relatively simple integral

$$\int_0^\pi d\varphi \int_0^{u_0} du \frac{u(1 - u\cos\varphi)^{k+1}}{1 - u^2} \quad , \quad u_0 = 1 - \sqrt{\frac{1 - \cos\varphi}{2v}} \; .$$

This can be done using the integrals (C.6)-(C.7). The result is

$$\mathrm{vol}_k(v) = \frac{a_k \ln v - b_k}{v^{k+2}} + \mathcal{O}(\ln v / v^{k+3})$$

with

$$\begin{aligned} a_k &= 4^{k-1/2}\sqrt{\pi}\Gamma((2k+3)/2)/\Gamma(k+2) \quad \text{and} \\ b_k &= 2^{k-1}\sqrt{\pi}\Bigg(\frac{2^{k+2}\Gamma((2k+3)/2)}{\Gamma(k+2)} \sum_{i=1}^{2k+2} \frac{(-1)^i}{i} \\ &+ \sum_{i=0}^{[(k+1)/2]} \sum_{j=1}^{2i+1} \binom{2i+1}{j} \binom{k+1}{2i} \frac{\Gamma((2i+1)/2)}{j\Gamma((2i+2)/2)} (-1)^j (2-2^j)\Bigg) . \end{aligned}$$

Using (5.42) we can explicitly do the v-integration in (5.47), calculate the residuals and obtain approximations for our integrals of the form (5.39).

$$\mathfrak{I}\left(\int d\mu(X_1, X_2) \exp\left(-(z+\varepsilon)\mathrm{Tr}(X_1+X_2) - \frac{C}{A}\right) A^{\frac{l}{2}-i} C^j\right) = \int_2^\infty \mathfrak{I}\left(G^{3\,0}_{0\,3}(vz^2)\right) \mathrm{vol}_k(v) v^{n+j} dv \, .$$

However, inserting the results in (5.37) and (5.36) we see that the leading order in z vanishes. Therefore we also need the second order, which can be obtained rather simply. In equation (5.8) we defined Z_{n_A,n_R}. Since the scaling limit does not change things, we must have, $\mathfrak{I}(Z(z,z)) \equiv 0$, since this comes from the average over 1. Moreover, we find

$$\left[\mathfrak{I}(\partial_{z_1} Z(z_1,z_2)) + \mathfrak{I}(\partial_{z_2} Z(z_1,z_2))\right]\Big|_{z_1=z_2=z} = \pi(\tilde{\rho}_l(z) - \tilde{\rho}_l(z)) \equiv 0 \, .$$

This property allows us to calculate the desired second order for the integrals (5.39), the final result now is

$$\tilde{\rho}_l(z) = \frac{2^{(l-3)/2} l!!\pi}{l!^2} \left(\sum_{i=0}^{(l-1)/2} \frac{1}{2i+1} - \frac{3}{2}\gamma - \ln z \right) z^l + \mathcal{O}(z^{l+2}) \, . \tag{5.48}$$

For details of the calculation, see appendix C.1. For $l = 1$ and $l = 3$ we obtain

$$\tilde{\rho}_1(z) = \frac{\pi}{4}(2 - 3\gamma - 2\ln z)z + \mathcal{O}(\ln z \, z^3) \tag{5.49a}$$

$$\tilde{\rho}_3(z) = \frac{\pi}{72}(8 - 9\gamma - 6\ln z)z^3 + \mathcal{O}(\ln z \, z^5) \, . \tag{5.49b}$$

Unfortunately, analytical continuation of (5.48) to even l using derivatives of the Γ-function leads to wrong results. However, if we assume the same form of the expansion, it is possible to calculate the coefficients by means of numerical methods,

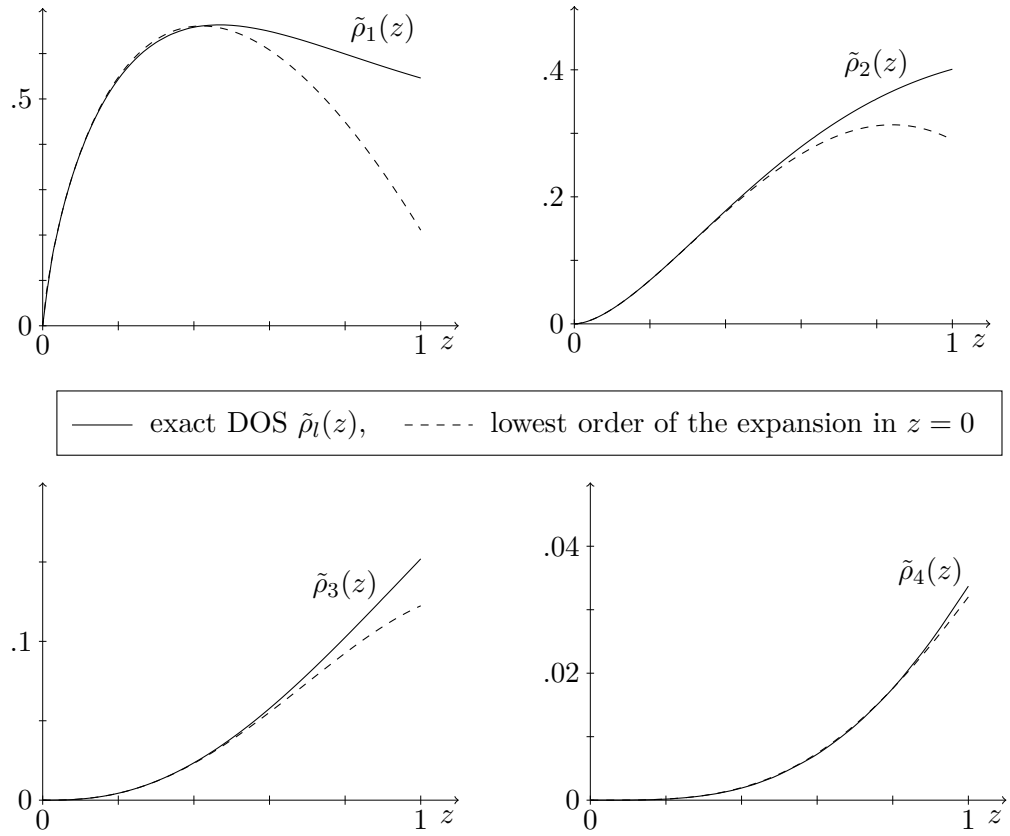


Figure 5.3: The asymptotic DOS in $z = 0$. The dashed line shows the first-order expansion, the smooth line is the exact curve.

e.g. a least square fit. Using this method in the vicinity of $z = 0$ for the curves of $\tilde\rho_2(z)$ and $\tilde\rho_4(z)$, see figure 5.2, we obtain

$$\tilde\rho_2(z) = \big((0.290 \pm 0.003) - (0.886 \pm 0.001)\ln z\big) z^2 \,, \tag{5.49c}$$

$$\tilde\rho_4(z) = \big((0.032 \pm 0.001) - (0.049 \pm 0.001)\ln z\big) z^4 \,. \tag{5.49d}$$

The numerical values of the coefficients of the logarithmic terms are very close to the values obtains by analytical continuation of (5.48), namely we use

$$-\frac{2^{(l-3)/2} l!!\pi}{l!^2} = -\frac{\pi}{2\Gamma(l+1)\Gamma((l+1)/2)} \,.$$

This leads to $-\frac{\pi}{2\Gamma(2+1)\Gamma((2+1)/2)} \approx -0.0862$ and $-\frac{\pi}{2\Gamma(4+1)\Gamma((4+1)/2)} \approx -0.04923$ for the coefficients of the logarithm in (5.49c), and (5.49d). This is an indicator for the correctness of our assumption. A plot of (5.49a-d) together with a part of the curves of figure 5.2 is shown in figure 5.3. For a better understanding, we show an extra picture for each value of l.

5.7 Numerical confirmation of the result

It is also possible to obtain the asymptotic density of states by Monte-Carlo integration, see e.g. [PFTV88]. In analogy to chapter 2, we programmed a routine which generates random matrices subject to the given symmetry. The correct distribution can be realized by the fact that every matrix can be written in terms of a sum of projectors, which underlie a Gaussian distribution each, see (5.3). Let us mention that –as expected– the result for large matrices is in a wide range independent of the nature of the probability distribution of the original variables $v_{i\alpha}$ and $w_{i\alpha}$. For example, instead of a Gaussian distribution one can take a rectangular distribution with zero mean and variance 1, which is very easy to generate by means of numerics.

After the diagonalization of a large number of matrices, a smooth curve can be obtained by smearing out every eigenvalue by a finite width. This can be achieved, e.g., by replacing every peak by a Gaussian curve of area 1 and adding up all values. However, if one increases the number of random matrices, the actual form of the curve does not affect the result, only the width does. The choice of the width is governed by two different aspects: On the one hand, the width must be broad enough to obtain a smooth result. On the other hand, it must be smaller than the size of the smallest structures we want to see.

Using such a routine, it is very easy to obtain the DOS of the bulk scaling limit, which has been calculated in section 5.4. The curve converges very rapidly, and it is sufficient to choose $N = 100$. Within the limits of numerical accuracy, we obtained a result which cannot be distinguished from the exact one.

Reproducing the results of section 5.6 is much more difficult. The main obstacle is the following: Without any rescaling, a $N \times N$-random matrix of our model has a spectrum with a largest eigenvalue proportional to N, i.e. the density of eigenvalues is independent of the matrix size. However, we want to expand the spectrum by a factor $\sqrt{N}$. Thus in a given interval the number of eigenvalues of a matrix decreases by $1/\sqrt{N}$. That is, the larger N is the more matrices have to be diagonalized to obtain a smooth curve. Furthermore, in this rescaling the convergence is much slower with respect to N, which forces us to make N much larger than in the bulk scaling limit. For fig. 5.4 we chose $N = 500$ and averaged over the spectra of approximately $150,000$ matrices each. For the smearing, we used a Gaussian curve of width 0.05, which is much smaller than the width of the maximum. For a comparison, we show again the results of section. 5.6 in the figures.

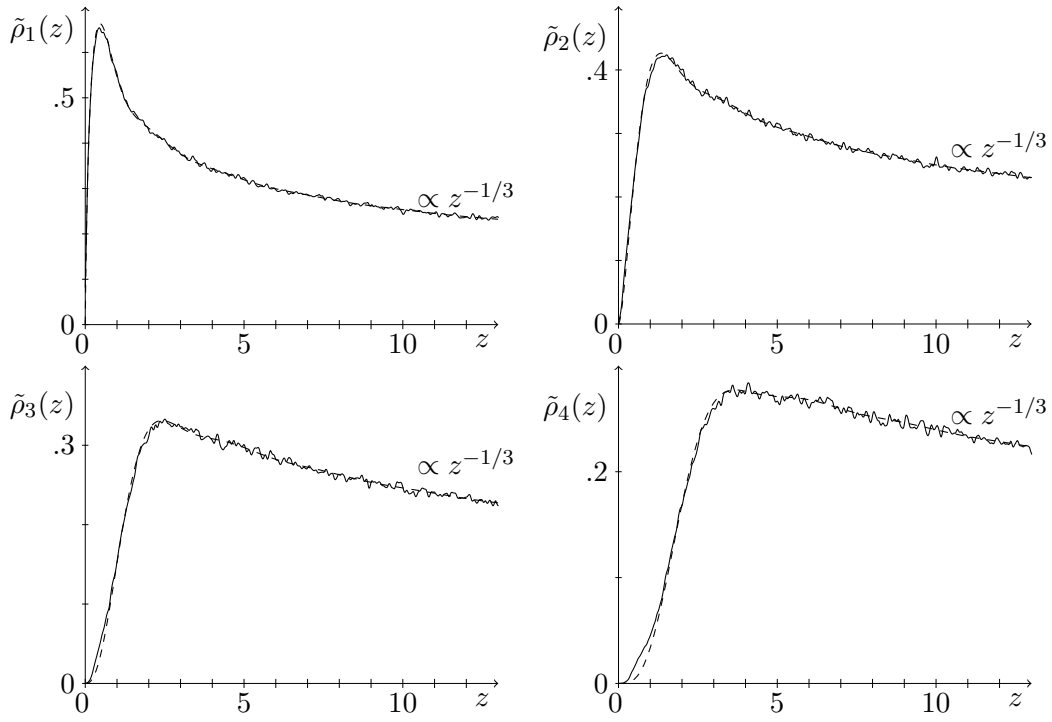


Figure 5.4: The smooth lines show the asymptotic density of states $\tilde{\rho}_l(z)$, obtained by Monte Carlo integration for $l = 1, 2, 3$, and 4. For comparison, the exact results are given by the dashed lines.

Chapter 6

Conclusion

In chapter 2 we used numerical methods to calculate the density of states (DOS) of an optical medium. In our model of discrete electrodynamics we were able to derive it for a clean system by means of analytical methods. In the case of a Gaussian probability distribution for the magnetic permeability μ and the dielectric permittivity ε we used numerical methods to calculate the DOS. We found that the disorder does not influence at low energies. It just softens the edges of the DOS and increases the highest energy levels. In the case of non-Gaussian disorder, we also used numerics and found evidence for the existence of the *boson peak* in an optical medium.

In chapter 3 we introduced a random matrix model for a system with bosonic excitations. We were able to compute the joint probability density of the characteristic frequencies of the system, which is equivalent to a reduction of the integrations to an integral over those. Rewriting the joint probability density by means of Vandermonde determinants enabled us to apply the method of bi-orthogonal polynomials. This lead to explicit expressions for all correlation functions. Near the hard edge of the spectrum a new scaling limit was found. The energy had to be rescaled by the factor $\sqrt{N}$. Of course, also the usual bulk scaling limit exists, here the model exhibits sine-kernel universality, i.e. the correlation functions are those of the GUE.

The model of chapter 5 differs from that of chapter 3 by additional time reversal invariance. In appendix C we tried to copy the strategy of chapter 3 and found that for the calculation of the joint probability density of the characteristic frequencies it is necessary to solve an integral, which is not of the Harish-Chandra–Itzykson–Zuber type. To our knowledge, there is no method available to solve it. Thus we applied the supersymmetry method, which is more powerful. In particular, we applied the superbosonization identity, which lead to a formulation of the correlation functions in terms of superintegrals. Of course, the additional time reversal symmetry did not change the existing scaling limits of the model considered in chapter 3. Near the hard edge of the spectrum evaluation of the superintegrals turned out to be very laborious. We solved it for the simplest case, which is the density of states. We calculated the leading order of its asymptotic expansion in $z = 0$ and found a logarithmic singularity.

Evaluation of the superintegrals in the bulk scaling limit was the main motivation to consider the GOE in 4. There, we also used the supersymmetry method and obtained exactly the same superintegrals. Since more explicit expressions for the

correlation function of the GOE can be found in literature, we immediately obtained the solution for these. This result is not surprising: Since in chapter 3 in the bulk scaling GUE-universality was found and the GOE differs from GUE by additional time reversal symmetry, we expected to recover the GOE correlation functions.

The second motivation of chapter 4 was to illustrate the supersymmetry method by a relatively simple example. The methods and strategies applied there were also used in chapter 5, however, in a more complicated manner.

Appendix A

Calculations to chapter 3

This part of the appendix contains calculations which are needed to understand the steps performed in chapter 3. We explicitly apply the radial part of the Laplace operator to the function at hand. In the second part, we compute a limit, which is needed to obtain the desired integral (3.11) from (3.10).

A.1 Radial part of the Laplace operator

Applied to the function $\psi(S_d, T_d)/j_s(S_d)$ the radial part of the Laplace operator acts according to

$$\begin{aligned} -\Delta_{S_d}\frac{\psi(S_d,T_d)}{j_s(S_d)} &= \sum_{i=1}^{N}\frac{2}{j_s^2}\frac{\partial}{\partial s_i}j_s^2\frac{\partial}{\partial s_i}\frac{\psi(x,y)}{j_s} \\ &= \frac{2}{j_s^2}\sum_{i=1}^{N}\frac{\partial}{\partial s_i}\left(j_s\frac{\partial\psi}{\partial s_i}-\psi\frac{\partial j_s}{\partial s_i}\right) \\ &= \frac{2}{j_s}\sum_{i=1}^{N}\left(\frac{\partial^2\psi}{\partial s_i^2}-\frac{\psi}{j_s}\frac{\partial^2 j_s}{\partial s_i^2}\right) \\ &= \frac{2}{j_s}\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial s_i^2}\,. \end{aligned} \tag{A.1}$$

The second term in the sum before the last equality sign vanishes:

$$\begin{aligned} \sum_{i=1}^{N}\frac{\partial^2 j_s(S_d)}{\partial s_i^2} &= \sum_{i=1}^{N}\frac{\partial}{\partial s_i}\left(\frac{1}{s_i}\sum_{j\neq i}\frac{2s_i}{s_i^2-s_j^2}\right) \\ &= \sum_{i=1}^{N}j_s\left[\left(\frac{1}{s_i}\sum_{j\neq i}\frac{2s_i}{s_i^2-s_j^2}\right)^2-\frac{1}{s_i^2}+\sum_{j\neq i}\left(\frac{2}{s_i^2-s_j^2}-\frac{4}{(s_i^2-s_j^2)^2}\right)\right] \\ &= j_s\left(6\sum_{j\neq i}^{N}\frac{1}{s_i^2-s_j^2}+8\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}}\frac{s_i^2}{(s_i^2-s_j^2)(s_i^2-s_k^2)}\right)\,. \end{aligned} \tag{A.2}$$

Since every term in the last line occurs two times with different signs, this sum vanishes. The second sum vanishes due to its symmetry:

$$\begin{aligned}
&\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \frac{s_i^2}{(s_i^2-s_j^2)(s_i^2-s_j^2)} \\
&= \frac{1}{3}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \left(\frac{s_i^2}{(s_i^2-s_j^2)(s_i^2-s_k^2)} + \frac{s_j^2}{(s_j^2-s_k^2)(s_j^2-s_i^2)} + \frac{s_k^2}{(s_k^2-s_i^2)(s_k^2-s_j^2)}\right) \\
&= \frac{1}{3}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \frac{s_i^2(s_k^2-s_j^2)+s_j^2(s_i^2-s_k^2)+s_k^2(s_j^2-s_i^2)}{(s_i^2-s_j^2)(s_j^2-s_k^2)(s_k^2-s_i^2)} \\
&= 0\,.
\end{aligned}$$

A.2 The limit $t_i \to \tau/2$

In eq. (3.17) we claimed that the following limit holds:

$$\lim_{t_i\to(\tau/2)} \frac{\mathrm{Det}\big(\exp(-2t_is_j)\big)}{j_s(t_1,\ldots,t_N)j_s(s_1,\ldots,s_N)} = \frac{(-1)^{\lfloor N/2\rfloor}\exp\left(-\tau\sum_{i=1}^N s_i\right)}{j_+(s_1,\ldots,s_N)(\tau/2)^{N(N+1)/2}}\prod_{m=1}^{N-1}\frac{1}{m!}\,.$$

This can be shown by induction. For $N=1$ the claim is obviously correct. The step $N-1\to N$ is a bit tedious. To simplify the notation, we introduce the following convention: The hat, for example in $j_+(s_1,.,\hat{s}_j,.,s_N)$, always denotes that s_j in the argument is skipped. Of course, the number of variables in the argument of the occurring functions increases with each step of the induction. However, since the meaning of the functions is clear in their context, we do not introduce a new symbol to distinguish functions with a different number of arguments. In addition to the definitions of j_s and j_+, which are given by (3.13) and (3.14), respectively, we need the Vandermonde determinant

$$\Delta(s_1,\ldots,s_N) := \prod_{i<j}^{N}(s_i-s_j)\,.$$

The induction step reads as follows:

$$\begin{aligned}
&\lim_{t_i\to\tau/2}\frac{\mathrm{Det}\big(\exp(-2t_is_j)\big)}{j_s(t_1,\ldots,t_N)j_s(s_1,\ldots,s_N)} \\
&\overset{(i)}{=} \lim_{t_i\to\tau/2}\sum_{j=1}^{N}(-1)^{j+N}\frac{\mathrm{Det}(\exp(-2s_it_k))_{i\neq j,k\neq N}}{j_s(s_1,.,\hat{s}_j,.,s_N)j_s(t_1,\ldots,t_{N-1})}\frac{(-1)^{N-j}\exp(-2s_jt_N)}{s_j\prod_{i\neq j}(s_j^2-s_i^2)t_N\prod_{i\neq N}(t_N^2-t_i^2)}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(ii)}{=} \lim_{t_N\to\tau/2}\sum_{j=1}^{N}\frac{(-1)^{j+N+\lfloor(N-1)/2\rfloor}\exp\left(-\sum\limits_{i\neq j}s_i\tau\right)\prod\limits_{m=1}^{N-2}\frac{1}{m!}}{j_+(s_1,.,\hat{s}_j,.,s_N)(\tau/2)^{N(N-1)/2}}\times\\
&\qquad\qquad\times\frac{(-1)^{N-j}\exp(-2s_jt_N)}{t_N(t_N-\tau/2)^{N-1}s_j\prod\limits_{i\neq j}(s_j^2-s_i^2)}\\
&\overset{(iii)}{=}\frac{(-1)^{\lfloor(N-1)/2\rfloor}\exp\left(-\sum\limits_{i=1}^{N}s_i\tau\right)\prod\limits_{m=1}^{N-2}\frac{1}{m!}}{j_+(s_1,\ldots,s_N)\tau^{N-1}(\tau/2)^{N(N-1)/2}}\times\\
&\qquad\qquad\times\lim_{d\to0}\sum_{j=1}^{N}\frac{(-1)^{N+j+N-j}e^{-2s_jd}}{(d(\tau+d))^{N-1}(\tau/2+d)\prod\limits_{i\neq j}(s_j-s_i)}\\
&\overset{(iv)}{=}\frac{(-1)^{\lfloor(N-1)/2\rfloor}\exp\left(-\sum\limits_{i=1}^{N}s_i\tau\right)}{j_+(s_1,.,s_N)\tau^{N-1}(\tau/2)^{N(N-1)/2}}\frac{\prod\limits_{m=1}^{N-2}\frac{1}{m!}}{\Delta(s_1,.,s_N)}\times\\
&\qquad\qquad\times\lim_{d\to0}\sum_{j=1}^{N}\frac{(-1)^{N+j}\Delta(s_1,.,\hat{s}_j,.,s_N)e^{-2s_jd}}{(d(\tau+d))^{N-1}(\tau/2+d)}\\
&\overset{(v)}{=}\frac{(-1)^{\lfloor N/2\rfloor}\exp\left(-\tau\sum\limits_{i=1}^{N}s_i\right)}{j_+(s_1,\ldots,s_N)(\tau/2)^{N(N+1)/2}}\prod_{m=1}^{N-1}\frac{1}{m!}\,.
\end{aligned}
\tag{A.3}
$$

In step (i) the determinant is expanded with respect to the last column. For (ii) the induction hypothesis for $N-1$ is used, in step (iii) we set $t_N=\tau/2+d$. In the last step, (v), the limit is performed by using the relations

$$(d(\tau+d))^{N-1}\left(\frac{\tau}{2}+d\right)=\frac{\tau}{2}(\tau)^{N-1}d^{N-1}+\mathcal{O}(d^N)$$

and

$$
\begin{aligned}
\sum_{j=1}^{N}(-1)^{N+j}\Delta(s_1,.,\hat{s}_j,.,s_N)e^{-2s_jd}&=\mathrm{Det}\begin{pmatrix}1&\ldots&1\\ s_1&\ldots&s_N\\ \vdots&\ddots&\vdots\\ s_1^{N-2}&\ldots&s_N^{N-2}\\ e^{-2s_1d}&\ldots&e^{-2s_Nd}\end{pmatrix}\\
&=\frac{(-2d)^{N-1}\Delta(s_1,\ldots,s_N)}{(N-1)!}+\mathcal{O}(d^N).
\end{aligned}
\tag{A.4}
$$

Here we use that in the determinant rows can be added to each other without changing its value. Therefore in the power series of the exponential function all terms up to d^{N-2} can be eliminated. In the highest order a Vandermonde-Determinant is obtained. If N is even, a sign appears.

Appendix B

Using Mathematica with Grassmann variables

In section 5.6 we were confronted with the problem to calculate the superintegral (5.35). Since other techniques to solve this problem failed, an elementary package was written, which enables the computer algebra system *Mathematica* to deal with Grassmann variables.

The basic idea is simple: If the Grassmann algebra has n different generators $\xi_1, \xi_2, \ldots, \xi_n$ its dimension is 2^n. That suggests a representation of the generators or its product by a binary number with n digits. If the product of generators contains ξ_i, the $n-i+1$-th digit is set to 1, if not, it is zero. Of course, we must take into account the order of the generators in the product. Thus we have to fix a standard order, which is no problem. In our convention a product $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_j}$ is in standard order, if and only if $i_1 > i_2 > \cdots > i_j$. Thus we choose a basis of the 2^n-dimensional Grassmann algebra as

$$\{1, \xi_1, \xi_2, \xi_2\xi_1, \ldots, \xi_n\xi_{n-1}\cdots\xi_1\} .$$

As easy to check, the above conventions are chosen in such a way that the position of a basis vector in the basis, indicated by an integer $0, 1, \ldots, n-1$ gives the involved generators, if regarded as a binary number.

If two basis vectors, η_1 and η_2, are multiplied, the desired computational routine must do several things: First, it has to be checked if the result is simply zero. This can be achieved by usage of the build-in function *BitAnd* of *Mathematica*. Just feed in it the numbers of the two basis vectors, if the result is not zero, the multiplication is trivial and nothing remains to do. If the result is zero, the multiplication is non-trivial and the (binary) index of the resulting vector in the basis is simply the sum of the indices of η_1 and η_2. Furthermore, the routine must produce the correct sign. The elementary code to be fed in *Mathematica* may look as follows:

```
(* Grassman - multiplication *)
Clear[GMMult];
GMMult[xi_, eta_] := Module[
  (* Initialization of the necessary variables *)
  {
    result      = 0,
```

```
    check       = 0,
    table1      = 0,
    sign = 1
  },
  (* Check, if the result is simply 0 *)
  check  = 1 - BitAnd[xi[[2]], eta[[2]]];
  (* If the result is not 0, start its calculation *)
  If[check == 1,
    If[And[xi[[2]]>0,eta[[2]]>0],
      table1 = Table[
        If[
          BitAnd[eta[[2]],2^s] > 0,
          Table[
            If[BitAnd[xi[[2]],2^k] > 0, -1, 1],
            {k,0,s-1}
          ],
          List[1]
        ],
        {s, 1, Floor[Log[2,eta[[2]]]]}
      ];
      (* Now the sign of the result must be fixed *)
      sign = Product[
        Product[
          table1[[s]][[k]],
          {k,1,Length[table1[[s]]]}
        ],
        {s,1,Length[table1]}
      ]
    ]
  ];
  (* Output of the result as a list with two entries.
    The first one is simply the product of the coefficients
    of the input lists, multiplied by a sign *)
  List[If[
        check == 1, (xi[[1]] * eta[[1]]) * check * sign, 0
        ],
        If[
        check == 1, xi[[2]]+eta[[2]], 0
        ]
  ]
];
```

This routine has two arguments, xi and eta, which are lists of length two. The second element of both must be an integer, which denotes the basis vector in the Grassmann algebra, and the first one serves as coefficient. It may be any complex number or algebraic expression understood by *Mathematica*. The output is again a list with two entries with the same meaning as in the input.

However, an element of the Grassmann algebra may be a linear combination of all basis vectors. Thus we represent it by a list of length of at most $2 \times 2^n + 1$, as explained in the context of the following code.

```
(* Grassmann-multiplication of arbitrary
```

```
elements of the Grassmann algebra*)
GLMult[xi_, eta_] := Module[
  (* Initialization of the necessary variables *)
  {
    table1 = 0,
    table2 = 0,
    result = {},
    n = Length[xi]-1,
    m = Length[eta]-1
  },
  (* Build a table of the maximal possible length for the
    internal representation of the result *)
  table1 = Table[
    If[Mod[i,2]==1,0,i/2-1],{i,1,2*(xi[[n]]+eta[[m]]+2)}
  ];
  (* Pairwise multiplication of all entries of the input lists.
    These are n*m/4 multiplications. Write the result in the
    correct place in table1. *)
  Do[
    Do[
      {table2=GMMult[
        {xi[[2i-1]],xi[[2i]]},{eta[[2j-1]],eta[[2j]]}
        ],
      table1[[2*table2[[2]]+1]]+=table2[[1]]
      {j,m/2}
    ],
    {i,n/2}
  ];
  (* Simplify the result *)
  Do[{
      If[
        table1[[2i-1]]==0,0,
        If[
          result == {},
          result = {table1[[2i-1]],table1[[2i]]},
          result = Append[
            Append[result,table1[[2i-1]]],table1[[2i]]
          ]
        ],
        If[
          result == {},
          result = {table1[[2i-1]],table1[[2i]]},
          result = Append[
            Append[result,table1[[2i-1]]],table1[[2i]]
          ]
        ]
      ]
    {i,Length[table1]/2}
  ];
  Append[result,"G"]
];
```

```
GLMult[A1_,A2_,A3_] := GLMult[A1,GLMult[A2,A3]]
```

In this routine, the input variable xi and eta are lists, too. If xi is a linear combination of j basis vectors of the Grassmann algebra, it is a list of length $2j+1$. In the slots with even indices it contains the integers denoting the basis vectors, in the slots with odd indices the coefficient of belonging to the following basis vector. The last slot with index $2j+1$ is just a tag to distinguish the lists used for the Grassmann algebra from ordinary lists, it must contain the character "G". The output is also a list of this type. The last line of the code generalizes the multiplication to three factors.

Now it is straightforward to define the usual operations like the inverse of a Grassmann expression or the sum of two expressions. It is also possible to define supermatrices and all desired operations with them, like the superdeterminant, the matrixproduct, the exponential of a supermatrix or whatever is needed. It is convenient that some built-in functions of *Mathematica* can be applied to lists without further definitions. For instance, the very important command "Simplify" just simplifies the algebraic expressions in the list. Of course, the integers indicating the basis vectors are left unchanged.

However, note that the effort to carry out such operations grows exponentially with the number n of involved Grassmann variables. In chapter 5 we dealt with 8 generators, hence the dimension of the Grassmann algebra is 256. Using the fact that a reasonable expression contains only odd or only even linear combinations, reduces the effective dimension by the factor two. Nevertheless, *Mathematica* needs to run several minutes to produce the result 5.37.

Appendix C

Calculations and remarks to chapter 5

C.1 Calculation of the expansion in $z = 0$

This section contains calculations which are needed to obtain the expansion of the density of states $\tilde{\rho}_l(z)$ in section 5.6 . We want to calculate the small-z-limit of the integral

$$\begin{aligned}
&\int_0^1 du_1 \int_0^1 du_2 \int_0^\pi d\varphi f(z^2 v) v_1^{n+j} v_2^{k-n-j} u_1 u_2 \\
&\qquad = \int_0^1 du_1 \int_0^1 du_2 \int_0^\pi d\varphi f(z^2 v)(1-u_1^2)^k (1-u_2^2)^k v^{n+j} u_1 u_2 \\
&\qquad = \int_2^\infty dv f(z^2 v) \mathrm{vol}_k(v) v^{n+j} \ , \qquad \text{(C.1)}
\end{aligned}$$

for an appropriate function f. For our purpose, f is the imaginary part of Meijer's G-function, whose asymptotic series expansion is known in $z^2 v = 0$ and $z^2 v = \infty$. The variable v is defined by

$$v = \frac{v_1}{v_2} = \frac{2(1 - u_1 u_2 \cos\varphi)}{(1-u_1^2)(1-u_2^2)} \ , \qquad \text{(C.2)}$$

see also (5.46). Since $f(0) = 0$, we start with the reasonable assumption that for small z the main contribution of the integral C.1 comes from the large values of v. Thus our task is to find a useful representation for the factor $\mathrm{vol}_k(v)$ for large v.

First, we use the symmetry in u_1 and u_2 and integrate only over $u_1 > u_2$. Then, we choose u_1, φ and v as new coordinates and need to compute

$$\mathrm{vol}_k(v) = 2 \int_0^\pi d\varphi \int_0^{\tilde{u}_0} du_1 j u_1 u_2 (1-u_1^2)^k (1-u_2^2)^k \ , \qquad \text{(C.3)}$$

with the Jacobian $j = \partial u_2 / \partial v$. The old coordinate u_2 is now considered as a function of u_1, v, and φ. The value $\tilde{u}_0$ is determined by the condition

$$v = \frac{2(1 - \tilde{u}_0^2 \cos\varphi)}{(1-\tilde{u}_0^2)^2},$$

we obtain the expansion

$$\tilde{u}_0 = 1 - \frac{\cos\varphi}{2v} - \sqrt{\frac{\cos^2\varphi}{4v^2} + \frac{1-\cos\varphi}{2v}} = 1 - \sqrt{\frac{1-\cos\varphi}{2v}} + \mathcal{O}(1/v)\,. \tag{C.4}$$

The minimal value of u_2 is given by $u_2 = \tilde{u}_0$, we set $u_2 = 1 - \varepsilon$, which implies $\varepsilon < 1/\sqrt{2v}$. In the first order approximation we find

$$\varepsilon = \frac{1 - u_1\cos\varphi}{v(1-u_1^2) - u_1\cos\varphi}\,.$$

Using a computer-algebra system, it can be seen that the main contribution to (C.3) is given by

$$\int_0^\pi d\varphi \int_0^{u_0} du_1 \frac{u_1(1-u_1\cos\varphi)^{k+1}}{1-u_1^2}\,, \quad u_0 = 1 - \sqrt{(1-\cos\varphi)/2v}\,. \tag{C.5}$$

This can also be seen doing a tedious calculations by hand, see next section. This integral can be evaluated by expanding the $(k+1)$th power and applying the following identities, they can be found by means of [GR65].

$$\begin{aligned}\int_0^{u_0} \frac{u_1^n du_1}{1-u_1^2} &= -\frac{1}{4}\ln(1-\cos\varphi) + \frac{1}{4}\ln 2v + \frac{(-1)^n \ln 2}{2} \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}\binom{n}{j}\frac{1}{j}\left[(-1)^j + (-1)^{n-j}(2^i-1)\right] + \mathcal{O}(\ln v/v)\end{aligned} \tag{C.6}$$

$$\int_0^\pi d\varphi (1-\cos\varphi)^{k+1}\ln(1-\cos\varphi) = 2^{k+2}\sqrt{\pi}\,\frac{\Gamma((2k+3)/2)}{\Gamma(k+2)}\left(\sum_{i=1}^{2k+2}\frac{(-1)^{i+1}}{i} - \frac{\ln 2}{2}\right)$$

$$\int_0^\pi d\varphi (1-\cos\varphi)^{k+1} = 2^{k+1}\sqrt{\pi}\,\frac{\Gamma((2k+3)/2)}{\Gamma(k+2)}\,.$$

If k is even, we have

$$\int_0^\pi d\varphi \cos^k\varphi = \sqrt{\pi}\,\frac{\Gamma((k+1)/2)}{\Gamma((k+2)/2)}\,. \tag{C.7}$$

Putting these integrals together, we find

$$\mathrm{vol}_k(v) = \frac{a_k \ln c - b_k}{v^{k+2}} + \mathcal{O}\left(\frac{\ln v}{v^{k+3}}\right)\,,$$

which is the desired expansion of $\mathrm{vol}_k(v)$ for large v. The coefficient a_k and b_k are given by

$$\begin{aligned} a_k &= 4^{k-1/2}\sqrt{\pi}\Gamma((2k+3)/2)/\Gamma(k+2) \quad \text{and} \\ b_k &= 2^{k-1}\sqrt{\pi}\Bigg(\frac{2^{k+2}\Gamma((2k+3)/2)}{\Gamma(k+2)}\sum_{i=1}^{2k+2}\frac{(-1)^i}{i} \\ &\quad + \sum_{i=0}^{[(k+1)/2]}\sum_{j=1}^{2i+1}\binom{2i+1}{j}\binom{k+1}{2i}\frac{\Gamma[(2i+1)/2}{j\Gamma((2i+2)/2)}(-1)^j(2-2^j)\Bigg)\,. \end{aligned} \tag{C.8}$$

Now we are able to calculate the hightest order of the integrals (5.39).

$$\begin{aligned} I_{l,i,j}(z) &= \Im\left(\int d\mu(X_1,X_2)\exp\left(-(z+\varepsilon)\mathrm{Tr}(X_1+X_2)-\frac{C}{A}\right)A^{\frac{l}{2}-i}C^j\right) \\ &= \int_2^\infty \Im\left(G^{3\,0}_{0\,3}\left(z^2v\Big|11,-n\right)\right)v^{n+j}\mathrm{vol}_k(v)dv \\ &= \int\limits_{i\mathbb{R}+\max(n,-1)+\varepsilon} ds \int_2^\infty dv\,\Im(\Gamma^2(1+s)\Gamma(-n+s)(vz^2)^{-s})v^{n+j}\mathrm{vol}_k(v)\ . \end{aligned} \tag{C.9}$$

We defined $n=l-1-2i+j$ and $k=l/2-3/2-i+j$. For odd $l\geq 1$ the integral can be evaluated in the leading order of z, for even l the exponent $n+j$ is not an integer, due to which problems occur. We use the approximation for $\mathrm{vol}_k(v)$ to perform the v-integration and then the calculus of residuals. We find

$$\begin{aligned} I_{l,0,0}(z) &= \pi\Gamma\left(\frac{m+1}{2}\right)\left[3\Big(\gamma-\sum_{i=1}^{(m-1)/2}\frac{1}{i}\Big)a_{(m-3)/2}\right. \\ &\quad \left.+(2\ln z-1)b_{(l-1)/2}\right]\frac{1}{z^{l+1}}+\mathcal{O}\left(\frac{\ln z}{z^{l-1}}\right)\ . \end{aligned}$$

The integrals with $j=1,2$ can be obtained by differentiating with respect to z, explicitly we have $I_{l,i,j}=I_{l+2i,0,j}$ and $I_{l,1,j+1}=-1/2z^{-2l-2j+1}\frac{\partial}{\partial z}z^{2l+2j}I_{l,0,j}$. Inserting all these integrals in $Z(z_1,z_2)$, given by (5.36) we obtain, using computer algebra, the stated result for the leading term in $\tilde{\rho}_l(z)$.

Main contribution of the integral

We want to compute the main contribution of integral (C.3) by hand. Inserting $u_2=1-\varepsilon$ in (C.2) and expanding in ε yields $\varepsilon=\frac{1-a\cos\varphi}{v(1-u_1^2)-u_1\cos\varphi}+\mathcal{O}(1/v^2)$ and, using $j=\partial u_2/\partial v$,

$$\begin{aligned} &\int_0^{u_0}du_1\int_0^\pi d\varphi j u_1u_2(1-u_1^2)^k(1-u_2^2)^k= \\ &\quad\int_0^{u_0}du_1\int_0^\pi d\varphi\frac{(1-u_1^2)^{k+1}(1-u_1\cos\varphi)^{k+1}(v(1-u_1^2)-1)u_1}{(v(1-u_1^2)-u_1\cos\varphi)^{k+3}}\ . \end{aligned} \tag{C.10}$$

To proceed, we expand the integrand in large v,

$$\begin{aligned} &v^{-k-2}\frac{(1-u_1^2)^{k+1}(1-u_1\cos\varphi)^{k+1}(v(1-u_1^2)-1)u_1}{(v(1-u_1^2)-u_1\cos\varphi)^{k+3}}= \\ &\quad\left[\frac{u_1(1-u_1\cos\varphi)^{k+1}}{1-u_1^2}-\frac{u_1(1-u_1\cos\varphi)^{k+1}}{v(1-u_1^2)^2}\right]\sum_{i=0}^\infty\binom{i+k+2}{i}\left(\frac{u_1\cos\varphi}{v(1-u_1^2)}\right)^i\ . \end{aligned} \tag{C.11}$$

Now we prove that only the $(i = 0)$-term in the sum combined with the first term in the squared brackets contributes in the leading order in v. Consider

$$\left| \int_0^\pi d\varphi \int_0^{u_0} du_1 \frac{u_1^j \cos\varphi^i (1 - u_1 \cos\varphi)^{k+1}}{(1 - u_1^2)^{i+1}} \right| < \int_0^\pi d\varphi \int_0^{u_0} \frac{du_1}{(1 - u_1)^{i+1}} \,. \tag{C.12}$$

For our purposes it is sufficient to replace the numerator (C.12) by 1 and ignore the factor $(1 + u_1)^{i+1}$ in the denominator, which surely increases the absolute value of the integral. We split the u_1-integration into to parts, $0 < u_1 < 1 - v^{-s}$ and $1 - v^{-s} < u_1 < u_0$, $0 < s < 1/2$. The first part does not cause any problems. Since $0 < u_1 < 1$ for $i > 0$ we have

$$\int_0^\pi d\varphi \int_0^{1-v^{-s}} \frac{du_1}{(1 - u_1)^{i+1}} < \frac{\pi v^{si}}{i} \,.$$

To understand the rest of the u_1-integration, we split the φ-integration according to $0 < \varphi < v^{-s}$ and $v^{-s} < \varphi < \pi$. For the first part we have $u_0 < 1 - 1/v$ if v is large enough. This leads to

$$\int_0^{v^{-s}} d\varphi \int_{1-v^{-s}}^{u_0} du_1 \frac{1}{(1 - u_1)^{i+1}} < \frac{v^{i-s}}{i} \,.$$

For the second part we have, using $\cos\varphi = 1 - \varphi^2/2 + \mathcal{O}(\varphi^2)$ and $u_0 < 1 - v^{-s-1/2}/2$,

$$\int_{v^{-s}}^\pi d\varphi \int_{1-v^{-s}}^{u_0} du_1 \frac{1}{(1 - u_1)^{i+1}} < \frac{\pi v^{i(s+1/2)}}{i} \,.$$

Using these estimates we see, taking into account $0 < s < 1/2$ as well as the factors v^{-i} and v^{-i-1}, respectively, that all contributions except for the first term in the squared bracket of (C.11) combined with the $(i = 0)$-part, vanish at least as $v^{-\delta}$ for some positive δ. It still remains to show that for the $(i = 0)$-contribution $u_0 = 1 - \sqrt{(1 - \cos\varphi)/(2v)}$ is a good choice for the upper boundary of the integral (C.10). To do so, we set $u_0' = 1 - \sqrt{(1 - \cos\varphi)/(2v)} - \cos\varphi/v$, which is the next order of the expansion (C.4), and show that the term $\cos\varphi/v$ does not contribute in the leading order.

$$\begin{aligned} \int_0^\pi d\varphi \int_0^{u_0'} \frac{du_1}{1 - u_1} &= \int_0^\pi d\varphi \ln(1 - u_0) \\ &= \int_0^\pi d\varphi \ln\left(\frac{\sin(\varphi/2)}{\sqrt{2v}}\right) + \int_0^\pi d\varphi \ln\left(1 + \frac{\cos\varphi}{\sin(\varphi/2)\sqrt{2v}}\right) \end{aligned}$$

The first integral equals $\pi \ln 2v$ and comes from the integration up to u_0, the second integral is comes from the integration between u_0 and u_0'. We want to show that

the second contribution can be omitted. To see that it vanishes if v diverges, we consider three cases, (i) $0 < \varphi < v^{-s}$, (ii) $v^{-s} < \varphi < \pi/2$, and (iii) $\pi/2 < \varphi < \pi$. For (i) we use $\cos\varphi/\sin\varphi/2 = 2/t + \mathcal{O}(t)$,

$$\begin{aligned}\left|\int_0^{v^{-s}} d\varphi \ln\left(1+\frac{\cos\varphi}{\sin(\varphi/2)\sqrt{2v}}\right)\right| &< \left|\int_0^{v^{-s}} d\varphi \ln\left(1+\frac{2}{\varphi\sqrt{2v}}\right)\right| \\ &= \frac{2}{\sqrt{v}}\ln\left(1+\frac{1}{2}v^{1/2-s}\right)+v^{-s}\ln(1+2v^{-1/2+s}) \\ &= \frac{\sqrt{2}(1/2-s)\ln v}{\sqrt{v}}+\mathcal{O}(1/\sqrt{v})\,.\end{aligned}$$

For (ii), we find

$$\begin{aligned}\left|\int_{v^{-s}}^{\pi/2} d\varphi \ln\left(1+\frac{\cos\varphi}{\sin(\varphi/2)\sqrt{2v}}\right)\right| &< \left|\int_{v^{-s}}^{\pi/2} d\varphi \ln(1+\sqrt{2}v^{s-1/2})\right| \\ &= \frac{\pi}{\sqrt{2}}v^{s-1/2}+\mathcal{O}(v^{2s-1})\,.\end{aligned}$$

Part (iii) yields

$$\begin{aligned}\left|\int_{\pi/2}^{\pi} d\varphi \ln\left(1+\frac{\cos\varphi}{\sin(\varphi/2)\sqrt{2v}}\right)\right| &< \left|\int_{\pi/2}^{\pi} d\varphi \ln\left(1-\frac{1}{2\sqrt{v}}\right)\right| \\ &= \frac{\pi}{4\sqrt{v}}+\mathcal{O}(1/v)\,.\end{aligned}$$

Thus all three parts vanish if v becomes large.

C.2 Attempt of complete solution

As stated in chapter 1, the strategy applied to the random matrix ensemble considered in chapter 3 does not work in the case of additional time reversal invariance. In particular, the reduction of the integrations to an integral over the eigenfrequencies of the system cannot be achieved by an integral of Harish-Chandra–Itzykson–Zuber type. Here we show which integral has to be solved in order to reduce the integration.

Recall that we consider Hamiltonians of the form

$$H = \frac{1}{2}\sum_{i,j=1}^{N}(P_iB_{ij}P_j + Q_iC_{ij}Q_j) = \begin{pmatrix}\mathbf{P} & \mathbf{Q}\end{pmatrix}\begin{pmatrix}B & 0\\ 0 & C\end{pmatrix}\begin{pmatrix}\mathbf{P}\\ \mathbf{Q}\end{pmatrix}$$

with $B = B^t > 0$, $C = C^t > 0$. We want to transform H into the standard form

$$H = \frac{1}{2}\sum_{i=1}^{N}(p_i^2+\omega_i^2q_i^2)\,,$$

where ω_i are the positive characteristic frequencies of the system. To find the linear canonical transformation, we first transform B into its standard form, the identity matrix in N dimensions. The existence of the standard form is guaranteed by Sylvester's theorem. Thus we can write $B' = g^t B g = \mathbf{1}_N$, which corresponds to the transformation

$$\begin{array}{rcl} \Psi_1 : \mathbb{R}^{2N} & \to & \mathbb{R}^{2N} \\ \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} & \mapsto & \begin{pmatrix} \mathbf{p}' \\ \mathbf{q}' \end{pmatrix} = \begin{pmatrix} g^{-1} & 0 \\ 0 & g^t \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} , \end{array}$$

with $g \in \mathrm{GL}(N, \mathbb{R})$. The block structure of the transformation arises due to the requirement that is must be a canonical transformation. Since Ψ_1 is linear, this is ensured by

$$\begin{pmatrix} g^{-1} & 0 \\ 0 & g^t \end{pmatrix} \in \mathrm{Sp}(N, \mathbb{R}) = \{X \in \mathrm{GL}(N, \mathbb{R}) | X^t J X = J\} ,$$

which is easy to see:

$$\begin{array}{rcl} \begin{pmatrix} (g^{-1})^t & 0 \\ 0 & g \end{pmatrix} J \begin{pmatrix} g^{-1} & 0 \\ 0 & g^t \end{pmatrix} & = & \begin{pmatrix} (g^{-1})^t & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g^t \end{pmatrix} \\ & = & \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \end{array}$$

This transformation leads to $C' = g^{-1} C (g^t)^{-1}$. To obtain a diagonal form of C we now apply another canonical transformation,

$$\begin{array}{rcl} \Psi_2 : \mathbb{R}^{2N} & \to & \mathbb{R}^{2N} \\ \begin{pmatrix} \mathbf{p}' \\ \mathbf{q}' \end{pmatrix} & \mapsto & \begin{pmatrix} \mathbf{P}' \\ \mathbf{Q}' \end{pmatrix} = \begin{pmatrix} h^{-1} & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{p}' \\ \mathbf{q}' \end{pmatrix} . \end{array} \tag{C.13}$$

Since C' is symmetric h is orthogonal. We obtain $B'' = B' = \mathbf{1}_N$ and $C'' \equiv \Omega^2 = h^{-1} g^{-1} C (g^{-1})^t h$, with $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_2), \omega_N \geq \omega_{N-1} \geq \cdots \geq \omega_1 > 0$. The ordering is necessary since the permutation matrices are orthogonal.

Conversely, the Hamiltonian can be written as

$$H = \frac{1}{2} \begin{pmatrix} \mathbf{p} & \mathbf{q} \end{pmatrix} \begin{pmatrix} (g^{-1})^t g^{-1} & 0 \\ 0 & g h \Omega^2 (g h)^t \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} .$$

In order to obtain a bijection between the spaces of all possible Hamiltonians and the spaces which contain the matrices g, h and ω^2 we must factor out the subgroups which leave (C.14) invariant. If we denote the space of positive diagonal ordered $N \times N$ matrices by t_+ and the space of real symmetric and positive matrices by S_+, we obtain a bijection by choosing g as a member of the equivalence classes of $\mathrm{GL}(N, \mathbb{R})/\mathrm{O(N)}$ and h as an element of the classes in $\mathrm{O(N)}/T$. $T \cong \mathbb{Z}_{\not 2}{}^N$ is the maximal subgroup in O(N) which commutes with all matrices Ω^2. The bijection reads

$$\begin{array}{rcl} \Phi : \ \mathrm{GL}(N, \mathbb{R})/\mathrm{T} \times t_+ & \to & S_+ \times S_+ \\ (gT, \Omega) & \mapsto & (g \Omega^2 g^t, (g g^t)^{-1}) . \end{array} \tag{C.14}$$

Probability measure

The probability measure was discussed in chapter 5, we have

$$P(h)dh = c\, e^{-\frac{1}{2}\mathrm{Tr}(B+C)} \prod_{i\leq j=1}^{N} dB_{ij} dC_{ij} \ . \tag{C.15}$$

with

$$h = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

and a normalization constant c.

Integral

We want to calculate the expectation of some function $F(h)$, which only depends on the characteristic frequencies Ω,

$$\int\limits_{\mathrm{S}_+\times\mathrm{S}_+} F(\Omega)P(h)dh = c \int\limits_{t_+} \left(\int\limits_{\mathrm{GL}(N,\mathbb{R})} e^{-\frac{1}{2}\mathrm{Tr}(g\Omega^2 g^t + (gg^t)^{-1})} j(g,\Omega)dg \right) F(\Omega)d\Omega \ . \tag{C.16}$$

The measure is the flat one on $\mathrm{GL}(N,\mathbb{R})$, we have $dg = \prod\limits_{i,j=1}^{N} dg_{ij}$.

The system in one dimension

The one-dimensional case is easy to solve. The general Hamiltonian is given by $H = \frac{1}{2}(bp + cq)$, $b, c \in \mathbb{R}_+$. We use the transformation $b = x^2$ and $c = \omega^2/x^2$ to calculate the integral,

$$\begin{aligned} \int\limits_{S_+\times S_+} F(\Omega)P(X)dX &= c \int\limits_0^\infty d\omega F(\omega) \int\limits_0^\infty dx\, e^{-\frac{1}{2}\left(x^2 + \frac{\omega^2}{x^2}\right)} j(x,\omega) \\ &= c \int\limits_0^\infty d\omega F(\omega) \int\limits_0^\infty dx\, e^{-\frac{1}{2}\left(x^2 + \frac{\omega^2}{x^2}\right)} \frac{4\omega}{x} \\ &= 4c \int\limits_0^\infty d\omega F(\omega)\omega K_0(\omega) \ . \end{aligned}$$

The Jacobian is simply given by $j(x,\omega) = 4\omega/x$, K_0 is a modified Bessel function (also called MacDonald function). The constant c can be calculated by integration with $F(\omega) \equiv 1$, one obtains $c = 1/4$.

The benefit from this calculation is the insight that the density of states is not analytic in $\omega = 0$, the expansion of K_0 reads

$$K_0(\omega) = \ln 2 - \gamma - \ln\omega + \mathcal{O}(\omega^2 \ln\omega) \ .$$

This is a hint that we must not expect to find analytic behavior near $\omega = 0$ for any N or in the limit of large N. Rather we suppose to find an expansion similar to that of K_0. Indeed, this is verified by the calculations in chapter 5 and in the previous section of this appendix.

Integration in N dimensions

It is possible to explicitly calculate the Jacobian in (C.16). It is given by

$$j(g,\Omega) = \left| \frac{4^N}{\mathrm{Det}(g)^N} \prod_{i<j} (\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \omega_k \right| .$$

To calculate the Jacobian we understand the spaces of interest as subsets of the $\mathbb{R}^{N^2+N}$, $\mathrm{GL}(n,\mathbb{R})\times\Omega \subset \mathbb{R}^{N^2+N}$ and $S_+\times S_+ \subset \mathbb{R}^{N^2+N}$. For the first space we choose the basis consisting of all pairs $(E_{ij},\mathbf{0})$, $i,j = 1\dots N$ and $(\mathbf{0},E_{kk})$, $k = 1\dots N$. For the space $S_+ \times S_+$ the basis is already fixed by the choice of the measure (C.15), it consists of the vectors $(E_{ij}+E_{ji},\mathbf{0})$, $(\mathbf{0},(E_{ij}+E_{ji}))$, $j = 1\dots N$, $i<j$, $(E_{ii},\mathbf{0})$, and $(\mathbf{0},E_{ii})$, $i = 1\dots N$. The Jacobian is the determinant of the differential of the mapping Φ, given by (C.14).

$$\begin{aligned} d\Phi_{(g,\Omega)}(A,B) &= \frac{d}{dt}\Big|_{t=0} \Phi(g+tA,\Omega+tB) \\ &= \frac{d}{dt}\Big|_{t=0} ((g+tA)(\Omega+tB)^2(g+tA)^t, ((g+tA)^{-1})^t(g+tA)^{-1}) \\ &= (g(g^{-1}A\Omega^2 + (g^{-1}A\Omega^2)^t + 2\Omega B)g^t, -(g^{-1})^t(A+A^t)g^{-1}) \\ &= (\tilde\Psi_g \circ \tilde\Phi \circ (l_{g^{-1}} \times \mathrm{Id}))(A,B) \end{aligned} \tag{C.17}$$

In the last step, the differential has been decomposed into three linear mappings, $\tilde\Psi_g$,$\tilde\Phi$, and $l_{g^{-1}} \times \mathrm{Id}$. They are defined by

$$\begin{aligned} \tilde\Psi_g : S\times S &\to S\times S \\ (X,Y) &\mapsto (gXg^t, (g^{-1})^t Y g^{-1}) , \end{aligned}$$

$$\begin{aligned} \tilde\Phi : \mathfrak{gl}(N,\mathbb{R})\times\mathbb{R}^N &\to S\times S \\ (A,B) &\mapsto \Big(A\Omega^2 + (A\Omega^2)^t + 2\Omega B, -(A^t+A)\Big) , \end{aligned}$$

and

$$\begin{aligned} l_{g^{-1}} \times \mathrm{Id} : \mathfrak{gl}(N,\mathbb{R})\times\mathbb{R}^N &\to \mathfrak{gl}(N,\mathbb{R})\times\mathbb{R}^N \\ (A,B) &\mapsto (g^{-1}A, B) . \end{aligned}$$

The determinant of the left multiplication is given by

$$\mathrm{Det}(l_{g^{-1}} \times \mathrm{Id}) = \frac{1}{\mathrm{Det}(g)^N} ,$$

the determinant of $\tilde\Psi$ is simply

$$\mathrm{Det}\Psi_g = \mathrm{Det}(l_g)\mathrm{Det}(r_{g^t})\mathrm{Det}(l_{(g^{-1})^t}\mathrm{Det}(r_{g^{-1}}) = 1 ,$$

since it can be decomposed in right and left multiplications which act on one of the subspaces S of $S \times S$. To calculate the determinant of $\tilde{\Phi}$ we consider its matrix representation with respect to the basises of $\mathbb{R}^{N^2+N}$ chosen above. Since the spaces are linear we identify the tangential space in each point with the space itself. Applying $\tilde{\Phi}$ to a vector of the basis yields

$$\tilde{\Phi}(E_{ji}, \mathbf{0}) = \Big(\omega_i^2(E_{ij} + E_{ji}), -(E_{ij} + E_{ji})\Big)$$

and

$$\tilde{\Phi}(\mathbf{0}, E_{ii}) = \Big(\omega_i E_{ii}, \mathbf{0}\Big) \; .$$

Thus the matrix of $\tilde{\Phi}$ assumes a simple block form. This induces a decomposition of both spaces in $N(N+1)/2$ two-dimensional subspaces. $N(N-1)/2$ of these subspaces are given by the pairs

$$\Big(\mathrm{span}((E_{ij}, \mathbf{0}), (E_{ji}, \mathbf{0})), \mathrm{span}((\frac{1}{2}(E_{ij} + E_{ji}), \mathbf{0}), (\mathbf{0}, \frac{1}{2}(E_{ij} + E_{ji}))\Big) \; . \tag{C.18}$$

The remaining N subspaces are spanned by the diagonal basis matrices, they are given by the pairs

$$\Big(\mathrm{span}((E_{ii}, \mathbf{0}), (\mathbf{0}, E_{ii})), \mathrm{span}((E_{ii}, E_{ii}), (E_{ii}, \mathbf{0}))\Big) \; . \tag{C.19}$$

On each of the pairs (C.18) the restricted mapping $\tilde{\Phi}$ is well-defined and has the matrix

$$\begin{pmatrix} \omega_i^2 & \omega_j^2 \\ -1 & -1 \end{pmatrix} \; ,$$

on each of the pairs (C.18) it assumes the form

$$2 \begin{pmatrix} \omega_i^2 & \omega_i \\ -1 & 0 \end{pmatrix} \; .$$

Now it is easy to read off the determinant of $\tilde{\Phi}$, it is the product of the determinants of these two by two matrices:

$$\mathrm{Det}\tilde{\Phi} = 4^N \prod_{i<j} (\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \omega_k \; .$$

Thus we arrive at the Jacobian

$$j(g, \Omega) = \left| \frac{2^{2N}}{\mathrm{Det} g^N} \prod_{i<j} (\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \omega_k \right| \; .$$

At this point, our attempt to solve the model in analogy to that in chapter 3 ends. Up to now, for the remaining integral in (C.16) no solution is known.

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