International Real Business Cycles in the Developed and Emerging Economies of NAFTA and the EU

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Berlin, 02. Mai. 2008

Esther Conrad

To my parents, to Eva and Charlotte
and to my dog, Loki

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Abstract

This study empirically and theoretically evaluates economic interdependence of emerging and developed economies in terms of business cycles. In addition to evaluating Mexico’s business cycles relative to the developed NAFTA economies, it considers the business cycles of some of the Eastern European emerging economies in the EU relative to developed EU economies. By evaluating intra- and cross-country statistics, the study finds that are empirical regularities (stylized facts) for emerging economies just as there are for developed ones. A key empirical finding is that developed economies belonging to the same trade agreement tend to have highly synchronized business cycles and hence positive output and consumption correlations, but that this relationship does not necessarily hold with respect to emerging economies. In fact, the correlations are virtually absent and sometimes even negative when comparing the emerging economies’ business cycles with those of their developed trading partners. It is shown that the intra-country statistics for both types of economies can successfully be reproduced using a one-country international real business cycle model with an endogenous interest rate. In addition, the non-existent or negative output and consumption correlations between the two economy types can be captured by a two-country international real business cycle model using portfolio adjustment costs and applying negative spillover effects in the productivity process of the emerging economy. The negative spillover effect also allows for a reversal of the usual theoretical implication of theses model types that there should be more consumption- than output smoothing (while data shows the opposite to be true). The study additionally gives a comprehensive overview of contemporary solution mechanisms used to solve this class of models.

I n d e x

Index I

List of Tables IV

List of Figures VI

Chapter 1: Introduction 1

1.1 Literature and Summary for Chapter 2 3

1.2 Literature and Summary for Chapter 3 5

1.3 Literature and Summary for Chapter 4 7

Chapter 2: Measuring Business Cycles in Developed and Emerging Economies 9

2.1 Empirical Results 11

2.1.1 Intra-Country Findings 13

2.1.2 Cross-Country Findings 21

2.2 Concluding Remarks Chapter 2 35

Chapter 3: Modeling Real Business Cycles of Developed and Emerging Economies 37

3.1 The Debt Elastic Interest Rate Model 39

3.2 The Steady State 45

3.3 Model 1: Calibration and Results for a Developed Economy 48

3.3.1 Impulse Response Analysis for Model 1 51

3.3.2 Empirical vs. Theoretical Business Cycle Moments for Model 1 55

3.4 Model 2: Calibration and Results for an Emerging Economy 58

3.4.1 Impulse Response Analysis for Model 2 62

3.4.2 Empirical vs. Theoretical Business Cycle Moments for Model 2 65
3.5 Model 3: Calibration and Results for a Two-Country Model of a Developed and Emerging Economy 67
3.5.1 Impulse Response Analysis for Model 3 75
3.5.2 Empirical vs. Theoretical Business Cycle Moments for Model 3 83
3.6 Concluding Remarks Chapter 3 85

Chapter 4: A Primer on Solving Open Economy SDGE Models 87

4.1 Log-Linearizing the Debt Elastic Interest Rate Model 91
4.2 Derivation of Model 4 94
4.2.1 Solution Stability and Uniqueness of Model 4 97
4.2.2 Characterizing the Solution of Model 4 100
4.3 The Method of Undetermined Coefficients 102
4.3.1 Interpreting the Undetermined Coefficients 106
4.4 The Eigenvalue Decomposition 109
4.4.1 Applying the Eigenvalue Decomposition to Model 4 109
4.5 The Schur (QZ)-Decomposition for Higher Dimensional Models 113

Chapter 5: Conclusion 122

Chapter 6: Appendices 126

6.1 Data Appendix A: Chapter 2 126
6.2 Technical Appendix B: Chapter 3 128
6.2.1 Solving the Optimization Problem 128
6.2.2 Differentiation Needed for Steady State Determination 129
6.2.3 A Theoretical Discussion of Impulse Response Functions 131
6.2.4 Model 3: Remaining Business Cycle Statistics 133
6.3 Technical Appendix C: Chapter 4 134
6.3.1 Linearizing the Trade Balance and Current Account 134
Ratios 134
6.3.2 The Eigenvalue Decomposition in the n-Variable Case 135

6.3.3 The Eigenvalue Decomposition: On the Role of the Stable Eigenvalue in Transition Functions in Model 4 144
6.3.4 The Schur-Decomposition: Reducing the Coefficient Matrix W........ 145
6.4 Computer Algorithms........ 147
6.4.1 Uhlig's Toolbox 149
6.4.2 Oviedo's Toolbox........ 151

References 154

List of Tables

Chapter 2

Table 2.1: Standard Deviations 14
Table 2.2: Selected Relative Standard Deviations 15
Table 2.3: Contemporaneous Correlation with Output 17
Table 2.4: Absolute and Relative Saving and Investment Correlations 19
Table 2.5: First-Order Autocorrelations 20
Table 2.6: Cross-Country Output Correlations 33
Table 2.7: Cross Country Output Correlations of European Countries Only 34
Table 2.8: Cross-Country Consumption Correlations 34
Table 2.9: Cross Country Consumption Correlations of European Countries Only 35

Chapter 3

Table 3.1: Averages of the Great Ratios 47
Table 3.2: Parameter Values for Model 1 (Developed Economy) 50
Table 3.3: Business Cycle Summary Stats: Model 1 vs. Average of Developed SMOPECs.. 56
Table 3.4: Parameter Values for Model 2 (Emerging Economy) 61
Table 3.5: Business Cycle Summary Stats: Model 2 vs. Average of Emerging Economies.... 65
Table 3.6: Parameter Values for Model 3 (the Two-Country Model) 74
Table 3.7: Cross Country Business Cycle Summary Statistics : Model 3 vs. Data 84
Table 3.8: Intra-Country Business Cycle Summary Statistics: Model 3 vs. Data 84

Chapter 4

Table 4.1: Variable Classification and Notation 90
Table 4.2: Policy and Transition Function Coefficients for Productivity in Model 4 (in %) 109

Chapter 6

Table 6.1: Additional Business Cycle Statistics of Model 3 133
Table 6.2: Decomposing Uhlig's Toolbox 151
Table 6.3: Decomposing Oviedo's Toolbox 153

List of Figures

Chapter 2

Figure 2.1: Synchronized Canadian and U.S. Business Cycles 23
Figure 2.2: Business Cycles in Countries Belonging to a Common Trade Agreement 24
Figure 2.3: The Eastern European Countries' Output Cycle vs. the EU LOPECs................... 26
Figure 2.4: The Eastern European Countries' Output Cycle vs. the EU SMOPECs 28
Figure 2.5: The Eastern European Countries' Consumption Cycle vs. the EU LOPECs 29
Figure 2.6: The Eastern European Countries' Consumption Cycle vs. the EU SMOPECs..... 31

Chapter 3

Figure 3.1: Impulse Response Functions of the Developed Economy (Model 1) 52
Figure 3.2: Impulse Response Functions of the Emerging Economy (Model 2)..................... 62
Figure 3.3: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of the Developed Economy Only.. 77
Figure 3.4: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of the Emerging Economy Only 80
Figure 3.5: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of Both Economies... 81

Chapter 4

Figure 4.1: The Characteristic Polynomial of the Matrix *A* for Model 4................................ 99

Chapter 1

Introduction

As globalization pushes countries into ever increasing, supranational interdependence, the nature of economic ties between developed and emerging economies is catapulted to the forefront of public debates. The word interdependence hosts ‘dependence’, a word that in light of the recent housing, banking, oil and food shortage crises that have propagated through developed and emerging economies alike, gains even more of a negative connotation. Conversely, the ability of some economies to realize unprecedented growth through globalization, specifically via trade and more specifically via their ability to exploit their comparative advantages (think China), elicits more positive emotions. In short, economic interdependence is a topic inquisitive minds should grapple with.

Three issues (that certainly do not exhaust the potential spectrum of issues) come to mind: The first is a simple matter of measurement: How interdependent are economies of similar or different development types, how has this changed over time and how do regional aspects factor into interdependence? The second issue is how real business cycle analysis, which has proven vastly successful in modelling the business cycles of developed economies, can help us understand the behaviour of emerging economies and the interaction between emerging and developed economies. The last issue is qualitative: Do the benefits of interdependence outweigh the potential pitfalls? In other words, is the interdependence of business cycles dangerous (can one country’s recession drag down another leading to a “domino effect”?) or does it lead to greater prosperity for all involved?

This study is mainly concerned with addressing the first two issues from a quantitative perspective and only treats the qualitative debate on the desirability of interdependence on the periphery. The five main findings are:

(1) There are empirical regularities for emerging economies similar to the ‘stylized facts’ that have been found for developed economies in the real business cycle literature.

(2) Two and half decades of data show that developed economies within close geographical proximity to one another or belonging to the same trade agreement tend to have highly synchronized business cycles, while the emerging economies' business cycles with respect to their developed neighbors or trading partners display little synchronization.
(3) Many empirical features of both the developed as well as the emerging economies can successfully be reproduced with an international real business cycle model based on an endogenous interest rate.
(4) It is also theoretically possible to capture the lack of business cycle synchronization across the two economy types in a two-country model using portfolio adjustment costs and negative spillover effects in the productivity process of the emerging economy.[1]
(5) Even though the desirability of linked business cycles has lost some of its shine due to the recent events mentioned above, it remains a fact that countries with highly synchronized business cycles are the more prosperous ones and that such a harmonization is therefore likely to be more advantageous than not.

The following chapters first establish intra-country and cross-country business cycle statistics for developed and emerging economies in North America and Europe. The countries were chosen depending on the reliability and availability of data and because the issues surrounding economic interdependence have been particularly poignant for them in recent years as they entered comprehensive trade agreements. Second, the question as to what kind of international real business cycle model can be used to reproduce the empirical intra- and cross-country data findings for both types of economies is taken up. These topics are treated in chapters 2 and 3 respectively. Chapter 4 gives an in-depth treatment of contemporary solution mechanisms for stochastic dynamic equilibrium (SDGE) models, with an application to one of the models introduced in chapter 3. Chapter 5 offers concluding remarks, while chapter 6 provides data and technical appendices to chapters 2 – 4. The remainder of chapter 1's introduction provides a brief overview of the related literature for each chapter followed by a chapter summary.

[1] This theoretical mechanism translates into allowing emerging economies to exploit their comparative advantage.

1.1 Literature and Summary for Chapter 2 (Measuring Business Cycles in Developed and Emerging Economies)

Studies on business cycles in developed economies are abundant, while analyses of business cycles in emerging economies are, as of yet, more of an ‘up-and-coming’ research trend. After being dormant for several decades, modern real business cycle (RBC) research was reignited by the work of Kydland and Prescott (1982). Initially most RBC studies focused on the United States (U.S.) using closed economy models. Starting in the early nineties, however, the models received more of an international flair: Mendoza (1991) estimated a small open economy model for Canada. Baxter and Crucini (1993) delivered a comprehensive overview of international business cycle frequencies for a selection of small and large economies and developed a model that could account for the saving-investment correlation puzzle (which finds a home-bias in saving). Backus, et al. (1992) were among the first to discover that a two-country real business cycle model (calibrated for the United States versus a European aggregate) generates higher consumption than output correlations – a finding that is at odds with the data. Stockmann and Tesar (1990) and Zimmermann (1995) are additional useful references on business cycle statistics for developed economies. More recent studies focus on emerging economies as well: Uribe and Yue (2006), for example, study interest rate premia for a set of emerging economies using a real business cycle model with habits, a working-capital constraint and debt adjustment costs. García-Cicco, et al. (2006) develop a real business cycle model with growth, an endogenous interest rate and a combination of transitory and permanent shocks, which they then compare to Argentine data. They find that the model is not able to account for Argentine business cycles. Neumeyer and Perri (2005) present a range of business cycle statistics for emerging economies and then develop a model with working-capital and an endogenous interest rate subject to interest rate shocks. Also calibrated to Argentina, they find (contrary to García-Cicco, et al. (2006)) that their model can generate results that are coherent with the data. Aguiar and Gopinath (2007) examine an even broader country set and create a model that successfully mimics the behavior of both a developed (Canada) and an emerging economy (Mexico). In addition they provide a precise decomposition of transitory and permanent shock components to productivity.

The emerging and developed economies of chapter 2 were chosen based on their geographical proximity to one another and their membership in a common preferential trade agreement (with two exceptions noted below). The first group is given by the countries belonging to the North American Free Trade Agreement (NAFTA): The developed ones are Canada and the U.S., the emerging one is Mexico. The second group is given by European countries, which, with one exception, belong to the European Union (EU): Here the developed countries are given by Belgium, France, Germany, the Netherlands and Switzerland (not an EU member), while the emerging ones are given by the Czech Republic, Poland and the Slovak Republic. Australia is included additionally as a 'wild card'.

Chapter 2, in line with other research, confirms several empirical regularities (stylized facts) regarding the business cycles of the developed economies considered in this study: Most prominently, consumption tends to be less volatile than output, labor input is about as volatile as output, capital is about half as volatile as output and investment is about three times more volatile than output (all measured by standard deviations), while net exports and the current account tend to be acyclical. In addition, cross-country comparisons of the developed economies generate positive consumption and output correlations, although there appears to be more output than consumption smoothing (a puzzling and robust feature of the data, which Backus, et al. (1992) were among the first to discover). For the emerging economies of chapter 2, the variables are generally more volatile than in developed economies, although the ranking of the variables' volatility remains approximately the same. The main exceptions are that the relative consumption to output volatility is larger than one (versus less than one in developed economies) and that the international variables tend to be much more countercyclical.

Chapter 2 also provides some insight on a topic that seems to have received little attention in the literature so far: The cross-country correlations for countries in the same vicinity but of a different development level. Even though consumption and output correlations across the developed economies in both the North American and European group indeed tend to be positive, the same correlations tend to be non-existent or even negative when comparing a developed and an emerging economy. Paradoxically, this is sometimes even more pronounced for neighboring countries belonging to the same regional preferential trade agreement – a fact that initially seems counterintuitive. Theoretically one ought to expect that positive productivity shocks translate into greater spillover effects (and hence positive output and consumption correlations) for countries with little geographical distance between them or for countries sharing membership in a regional trade agreement or both.

Although the business cycles of the developed economies in each of the two geographical groups exhibit precisely this behavior, emerging economies vis-à-vis their developed neighbors and preferential trade partners do not. As mentioned, the answer to this puzzle may be that spillover effects do not disseminate equally in both directions. Rather a positive spillover of productivity shocks going from developed to emerging economies but a negative spillover effect going from emerging to developed economies may be the source of the discrepancy. A negative spillover effect, as explained in chapter 3, could be interpreted as a mechanism that allows the emerging economy to exploit its comparative advantage at the expense of the developed economy. With negative spillover effects, a positive productivity shock in the emerging country would, ceteris paribus, eventually lead to a decline in the productivity of the developed country and could therefore explain the lack of business cycle synchronization

1.2 Literature and Summary for Chapter 3 (Modeling Real Business Cycles of Developed and Emerging Economies)

Some of the literature related to chapter 2 relates to chapter 3 as well. Modifications to the standard, non-stationary open economy model are presented by Mendoza (1991), Schmitt–Grohé and Uribe (2001, 2003) and Kim and Kose (2003), who all examine the role of an endogenous discount factor[2] as a stationarity–inducing mechanism in an open economy SDGE model. Schmitt–Grohé and Uribe (2003) give a comprehensive overview on small open economy real business cycle models by comparing and contrasting five modeling specifications: (1) a model based on an endogenous discount factor which either depends on the average per capita level of consumption or on a representative agent's consumption level, (2) a model featuring a debt elastic interest rate premium, (3) a model including portfolio adjustment costs to debt holdings, (4) a complete asset market model and (5) the standard non-stationary open economy model. They conclude that each of the stationarity–inducing

[2] Also known as Uzawa type preferences, where individuals become more impatient the more they consume. See Uzawa (1968)

instruments just mentioned predict similar dynamics and that the computationally more involved endogenous discount factor model is therefore the least parsimonious –if the researcher's aim is to simplify numerical approximations. Kim, et al. (2001) compare the welfare implications of complete versus incomplete asset markets in the context of an endowment economy. Two country models have been examined by Kollmann (1996, 1998) using an incomplete asset market structure while Backus, et al. (1992) and Baxter and Crucini (1993) are among the standard works for two-country models with a complete asset market structure.

Chapter 3 is based on two of the small open economy models discussed in Schmitt–Grohé and Uribe (2003): The first incorporates an interest rate premium (i.e. an endogenous interest rate) and the second uses portfolio adjustment costs, both of which are a function of deviations of debt from the steady state. The interest rate premium model is used twice, once to match selected data features for an average of the developed economies introduced in chapter 2 (model 1) and once to match the same features for an average of the emerging economies (model 2).[3] The portfolio adjustment cost approach is used in a two-country model (model 3), which is primarily calibrated to match the characteristics of cross-country consumption and output correlations between averages of the emerging and developed economies. It turns out that an interest rate premium model (models 1 and 2) is not just able to capture statistical features of the developed economies but of the emerging economies as well (a contended issue in the literature). Additionally, using some potentially unconventional parameterization for the exogenous productivity process (i.e. by allowing for the possibility of negative spillover effects), the two-country portfolio adjustment cost model is able to match the virtually non-existent consumption and output correlations between the two types of economies and reverses the 'usual' modeling implication that there should be more consumption than output smoothing.

The use of either an endogenous interest rate premium or a portfolio adjustment cost function in a small open economy model is a way to ensure that the steady state is well defined and the solution stationary. A standard small open economy model without such mechanisms can exhibit infinite second moments because either steady state consumption or debt are not well defined. According to Schmitt-Grohé and Uribe (2003), this can imply that "endogenous variables … wonder around an infinitely large region in response to bounded

[3] An alternative to taking averages for each type of economy would be to choose one representative emerging and developed economy as in Aguiar and Gopinath (2007) who chose Mexico and Canada. The reason this was not the course of action chosen here is because the Eastern European countries and Mexico at times display very different results.

shocks. This introduces serious computational difficulties because all available techniques are valid locally around a given stationary path" (p.164). It should be intuitive that the interest rate premium and the portfolio adjustment costs increase in a country's debt obligations (or decrease in a country's assets holdings).

Both model types feature incomplete asset markets. The use of an incomplete rather than a complete asset market is necessitated by the statistical findings of chapter 2, which reveal a low and sometimes even negative consumption (as well as output) correlation among developed and emerging economies. It is a well known fact, that complete asset market models, in which agents have access to a state contingent array of financial instruments, allow idiosyncratic risks to be pooled. As a result, high positive consumption correlations are created in two-country models, which would obviously constitute an incorrect modeling specification for the task at hand. In an incomplete asset market agents have access to a single risk free asset, which prevents excessive consumption and output smoothing and thereby generates lower correlations.

1.3 Literature and Summary for Chapter 4 (A Primer on Solving Open Economy SDGE Models)

The theoretical solution mechanisms underlying the kinds of international real business cycle models just mentioned are rooted in the work of Blanchard and Kahn (1980) and more recently in papers by Klein (2000) and Sims (2002). These authors show how to exploit a linear representation of a stochastic model with (potentially multiple) leads and lags of variables. The linearized model paves the way for the derivation of impulse response functions and the determination of business cycle summary statistics, which tend to be the ultimate goal of most analyses on this topic. A standard references on how to linearize these types of models is given by the contribution of King, et al. (2002). Uhlig (1997) and Oviedo (2005) also explain the linearization process in greater detail and elaborate and build on the theoretical solution methods by introducing intelligible computer toolboxes to solve this class of models. These are briefly discussed in the technical appendix to chapter 4 (chapter 6).

Second-order approximations are analyzed by Schmitt–Grohé and Uribe (2004) and will not be explored here, especially because first-order approximations usually render very accurate results. Kim and Kim (1999) take on this latter issue by examining why standard, first-order linear approximation methods sometimes imply higher welfare levels for models with incomplete versus complete asset markets. They propose a bias-correction which generates results as accurately as a second-order approximation.

Chapter 4 elucidates the underlying solution mechanisms for the models in chapter 3 by providing a primer (introductory treatment) on solving linearized small open economy SDGE models with an application to the interest rate premium model of chapter 3. In addition, chapter 4 provides a comprehensive overview of the methodological tools and the links between them. As it turns out, some key features of the interest rate premium model (models 1 and 2) can be retained even if domestic investment and the capital stock are kept constant, i.e. are always at their steady state level. The forced constancy of these two variables represents the simplification (this constitutes model 4) relative to models 1 and 2 presented in chapter 3. In effect, this amounts to eliminating one state variable (capital) from the linearized model, which can greatly facilitate the analysis if a researcher is interested in solving this kind of model via ‘back of the envelope’ calculations.

Two ‘traditional’ solution mechanisms, the method of undetermined coefficients and the eigenvalue decomposition, will be explored in the context of the simplified model. In the case of more complex models with more than two state variables, such as models 1 - 3 introduced in chapter 3, these types of calculations become virtually impossible and the Schur-or *QZ*-decomposition (the current standard of real business cycle solution algorithms) is required. The discussion on the eigenvalue decomposition will be particularly useful for understanding this more complex algorithm, since it is a special case of the Schur-decomposition.

Chapter 2

Measuring Business Cycles in Developed and Emerging Economies

Studies of business cycles in emerging and developing economies can approximately be grouped into four sets of stylized facts that form the pillars of this chapter's empirical analysis and are a proxy by which the accuracy of any model can be measured. These stylized facts can, for example, be found in the work of Backus and Kehoe (1991), Baxter and Crucini (1993) and Mendoza (1991) for developed economies. Additionally, more recent works of Uribe and Yue (2006), García-Cicco, et al. (2006), Neumeyer and Perri (2005) and Aguiar and Gopinath (2007) also focus on business cycles in emerging economies.

For developed economies the stylized facts are given by:

- consumption is pro-cyclical and tends to be less volatile than output,[4]
- labor input is pro-cyclical and tends to be as volatile as output,
- capital is acyclical and about half as volatile as output,
- saving and investment are strongly pro-cyclical and about two to three times as volatile as output,
- net exports are countercyclical and less volatile for large than for small countries,
- all of the above variables have strong positive first-order autocorrelations,
- the real interest rate tends to be acyclical and lags the business cycle.

For emerging economies the stylized facts are similar to those of developed economies, with the following additions and modifications:

- on average, variables are more volatile,

[4] A pro-cyclical variable exhibits a positive contemporaneous correlation with output. An acyclical variable exhibits almost no contemporaneous correlation with output in either direction. A counter-cyclical variable exhibits a negative contemporaneous correlation with output. The volatility measure is given by a variable's percentage standard deviation.

- consumption tends to be more volatile than output rather than less,
- saving and investment are much more volatile relative to developed economies,
- net exports are strongly countercyclical,
- the real interest rate is counter-cyclical and leads the business cycle.

In terms of cross-country correlations for developed economies (these will be extended to emerging economies below) the main stylized facts are:

- output and consumption are positively correlated across countries,
- cross-country output correlation tends to be higher than cross-country consumption correlations.

An additional and frequently cited stylized fact in both closed and open economy real business cycle literature is the high correlation between saving and investment, a ‘puzzle’ that was initially discovered by Feldstein and Horioka (1980), who demonstrated that domestic saving and investment rates for sixteen OECD countries from 1960–1974 were highly correlated. By regressing investment on saving rates, the coefficient on the latter was found to be near unity in the sample average. This was interpreted as empirical evidence *against* international capital mobility. If capital were indeed mobile, then we should find no correlation between saving and investment since, in theory, higher domestic saving should be invested where returns are highest and not necessarily remain in the domestic market. This anomaly, especially for developed, large open economies (LOPECs) such as the U.S., has remained a robust empirical finding across two and a half decades of research.

This chapter examines a set of countries, including both developed and emerging economies, which also display the stylized facts mentioned above. In addition, the data sheds some light on new evidence regarding the cross-country correlations between developed and emerging economies. The most prominent finding is that even though consumption and output correlations across *developed economies* tend to be positive, the same correlations tend to be surprisingly non-existent or negative when comparing a *developed* and an *emerging economy.* This is somewhat surprising, because a positive productivity shock in a non-autarkic country (which describes most countries today) should theoretically ‘spill over’ into other countries, at least to some extent. Perfect historical examples are the industrial revolution that began in Britain and then spread to the U.S. and Europe or the more recent personal computing revolution that emanated from the U.S. into the farthest corners of the world. It should be intuitive that positive productivity shocks can be expected to have greater spillover effects and hence positive output and consumption correlations for countries with little geographical distance between them or for countries sharing membership in a regional trade agreement. The countries in this chapter’s sample meet these criteria, but nevertheless display stark differences in output and consumption correlations depending on development level. Although the ability of emerging economies to quickly internalize positive productivity shocks stemming from other countries may be impeded by factors such as an untrained workforce or by a lack of infrastructure, the preponderance and intensity of the negative correlations still seems striking. An explanation for this phenomenon is given in chapter 3, which considers the possibility that there may be negative spillover effects for the developed economies when the emerging economies experience positive productivity shocks.

2.1 Empirical Results

The set of developed economies examined is given by some of the typical small open economies found in the literature –Australia (AUS), Belgium (BEL), Canada (CAN), the Netherlands (NEL) and Switzerland (SWI) – as well as some larger open economies: France (FRA), Germany (GER) and the U.S. These developed economies have been the focus of numerous real business cycle studies and therefore present limited opportunities for new results, except in the sense that the data is as recent as 2006. The set of emerging economies is given by the Czech Republic (CZR), Mexico (MEX), Poland (POL), and the Slovak Republic (SLR). Although Mexico, and more generally various South American and Asian countries, have been the subject of several real business cycle studies focusing on emerging markets (for example, Aguiar and Gopinath (2007), Neumeyer and Perri (2005), García-Cicco, et al. (2006), Uribe and Yue (2006) and Kydland and Zarazaga (2002)), the set of Eastern European countries has not.[5] Because of the proximity of two developed economies, Canada and the U.S., to an emerging economy, Mexico, it is a logical experiment to compare these North American countries with the developed and emerging economies of Europe. Moreover, the North American countries share membership in the North American Free Trade Agreement

[5] Notably, the Slovak Republic is actually included in the Aguiar and Gopinath (2005) country set, but Poland and the Czech Republic are not.

(NAFTA, established 1994), while the European countries share membership in the European Union (EU, established in 1992). Even though the Eastern European countries in question did not fully join the EU until 2004, they had significant access to the EU market via preferential trade agreements prior to their own accession. In particular, the Czech Republic and Poland began formal accession negotiations in 1998, while the Slovak Republic did so in 2000 (Beichelt (2004)). Australia and Switzerland obviously neither belong to the EU or NAFTA and in the case of Australia, there is also no geographical closeness relative to the rest of the sample. These two countries therefore serve as crosschecks for the obtained results.

The following applies to all data presented: All volatility measures (standard deviations) are based on the longest available data span for each series within each country, measured on a quarterly basis (see data appendix for more information). All correlations are based on the longest common sample of any two series within one country (these are intra-country correlations such as a country's saving-investment correlation) or on the longest common sample of a series across two countries (these are the cross-country correlations such as two countries' consumption correlation). Each series x_{it} is measured in constant prices (real terms), then divided by the working age population to obtain per capita terms and lastly transformed into logarithms. The exceptions are the trade-balance to output ratio, the current-account to output ratio and the real interest rate, which are all in percentage terms. All series are detrended using the Hodrick-Prescott filter, setting the smoothing parameter λ=1600.

The (per capita) series considered are: real GDP (y_t), real consumption (c_t), excluding government consumption (g_t), labor input (h_t) given by hours worked per employee in the total economy multiplied by total employment, the capital stock (k_t) given by the volume of the total economy's capital stock, saving (s_t) given by $y_t - c_t - g_t$,[6] real investment (i_t), the trade-balance to output ratio (tby_t) obtained by dividing the current value of net exports by the value of current GDP, the current-account to output ratio (cay_t) given by the current account as a percentage of current GDP and lastly the real interest rate series (r_t), which was taken from Neumeyer and Perri (2005) for Australia, Canada, the Netherlands and Mexico. Because of the difficulty to obtain comparable labor input and capital stock series, these variables were omitted for the emerging economies. For the developed countries, two average measures are considered: The first (All) takes the mean for all countries, the second eliminates the three largest economies France, Germany and the U.S. (the large open

[6] This follows the definition used by Baxter and Crucini (1993). They also discuss why true saving are difficult to measure and why this definition might be the most parsimonious.

economies abbreviated as LOPECs) from the sample, leaving the small open economies (SMOPECs).

2.1.1 Intra-Country Findings

Table (2.1) and table (2.2) display absolute and relative (to output) standard deviations of each variable, confirming the stylized facts. In terms of table (2.2), consumption is less volatile than output in developed economies, resulting in a relative standard deviation of 0.85 for the average of all developed economies.[7] This trend is reversed for the emerging economies, where consumption tends to be more volatile than output, yielding an average relative standard deviation of 1.13. For the set of all developed economies, labor-input displays similar volatility as output (yielding an average relative standard deviation of 1.03), while the capital stock is only half as variable as output (with an average relative standard deviation of 0.48). Saving and investment are approximately three to four times as volatile as output in developed economies (3.67 and 3.21 respectively), and are about four to five times as volatile as output in emerging economies (4.43 and 4.06 respectively). Lastly, table (2.1) shows that, in terms of absolute standard deviations, the trade balance and current account ratios are more than twice as volatile in emerging than in developed economies (for the trade balance ratio this is 2.41 in emerging versus 0.78 in developed economies and for the current account this is 2.39 in emerging versus 0.98 in developed).

[7] Unless there is a big discrepancy, the averages across all developed economies will be emphasized rather than always distinguishing between SMOPECs and LOPECs.

Table 2.1: Standard Deviations $\sigma(x_t)$ [a]

	$\sigma(y_t)$	$\sigma(c_t)$	$\sigma(h_t)$	$\sigma(k_t)$	$\sigma(s_t)$	$\sigma(i_t)$	$\sigma(tby_t)$	$\sigma(cay_t)$	$\sigma(r_t)$
AUS	1.43	0.96	1.66	0.35	5.93	5.51	0.98	1.03	0.50
BEL	0.99	0.97	0.79	0.54	3.61	4.11	0.73	1.12	-
CAN	1.57	1.23	1.45	0.88	5.21	4.25	0.97	1.08	0.45
CZR	1.50	1.62	-	-	5.70	5.15	1.58	2.12	-
FRA	0.82	0.77	0.72	0.38[c]	3.19	2.74	0.61	0.58	-
GER	0.88	0.91	0.74[b]	0.91	3.18	2.89	0.68	0.70	-
MEX	2.42	2.98	-	-	5.16	9.47	2.04	1.91	0.68
NEL	1.18	1.17	2.28[b]	0.33[c]	3.75	3.56	1.02	1.41	0.22
POL	2.21	1.55	-	-	14.51	6.55	1.77	1.26	-
SLR	1.53	2.31	-	-	7.96	9.06	4.27	4.28	-
SWI	1.25	0.74	0.80	0.29[c]	3.76	3.11	0.82	1.42	-
U.S.	1.31	1.02	1.40[b]	0.64	5.96	3.68	0.40	0.48	-
Averages for Developed Economies[d]									
All	1.18	0.97	1.23	0.54	4.32	3.73	0.78	0.98	0.39
SMOPEC	1.28	1.02	1.40	0.48	4.45	4.11	0.90	1.21	0.39
Averages for Emerging Economies									
	1.92	2.12	-	-	8.33	7.56	2.41	2.39	0.68

Notes: (a) All standard deviations are in percentage points per quarter and based on the maximum number of available observations. See the data appendix for additional information. (b) Total hours worked in the business sector rather than the total economy. (c) The capital stock of the business sector rather than the capital stock of the total economy. (d) Average "All" refers to the average across all developed economies, 'SMOPEC' is the average obtained by excluding France, Germany, and the U.S. (the large open economies = 'LOPEC'), leaving the small open developed economies (SMOPECs).

Table 2.2: Selected Relative Standard Deviations $\sigma(x_t/y_t)$ [a]

	$\frac{\sigma(c_t)}{\sigma(y_t)}$	$\frac{\sigma(h_t)}{\sigma(y_t)}$	$\frac{\sigma(k_t)}{\sigma(y_t)}$	$\frac{\sigma(s_t)}{\sigma(y_t)}$	$\frac{\sigma(i_t)}{\sigma(y_t)}$
AUS	0.67	1.16	0.24	4.15	3.85
BEL	0.98	0.80	0.55	3.65	4.15
CAN	0.78	0.92	0.56	3.32	2.71
CZR	1.08	-	-	3.80	3.43
FRA	0.94	0.88	0.46[c]	3.89	3.34
GER	1.03	0.84[b]	1.03	3.61	3.28
MEX	1.23	-	-	2.13	3.91
NEL	0.99	1.93[b]	0.27[c]	3.18	3.02
POL	0.70	-	-	6.57	2.96
SLR	1.51	-	-	5.20	5.92
SWI	0.59	0.64	0.23[c]	3.01	2.49
U.S.	0.78	1.07[b]	0.48	4.55	2.81
Averages for Developed Economies[d]					
All	0.85	1.03	0.48	3.67	3.21
SMOPEC	0.80	1.09	0.37	3.46	3.24
Averages for Emerging Economies					
	1.13	-	-	4.43	4.06

Notes: See table (2.1)

Table (2.3) shows that in all of the developed economies, consumption, labor input, saving and investment have positive contemporaneous correlations with output (0.71, 0.67, 0.81 and 0.79). These do not change much when considering the SMOPECs only. Capital is acyclical, yielding an average contemporaneous correlation with output of 0.17, while the correlation between the real interest rate and output for the three developed countries with available data can also be considered weakly pro-cyclical (0.51).[8] In addition, both the trade balance and current account ratios are counter-cyclical (with correlation coefficients around -0.30).

The contemporaneous output correlations in the emerging markets exhibit less similarities across the four countries. The consumption-output correlation, for instance, is much higher in Mexico (0.92) than in the Eastern European countries, where it is below 0.60 in all cases, which, with the exception of Australia (0.40), is also lower than for all the developed countries. Additionally, the saving-output correlation in the emerging economies is not as conform as in the case of the developed economies (ranging from 0.25 for the Slovak Republic to 0.71 for the Czech Republic). This variability may stem from the measurement difficulties associated with the saving definition (see second to last footnote). The investment-output correlation is not as variable as the saving-output correlation and, as is the case for developed economies, tends to be positive (0.72 for the emerging economies versus 0.79 for all developed economies). The trade-balance and current-account to output ratios are more counter-cyclical in emerging markets than in developed countries, with average correlation coefficients of around -0.40 (versus around -0.30 in the developed countries). Although the stylized fact that finds a stronger counter-cyclicality of the trade balance and current account ratios in emerging economies (relative to developed economies) is therefore confirmed, it is not as pronounced as in other studies (see for example, Aguiar and Gopinath (2007)). Lastly, although no generalization can be made due to the small sample on real interest rate series, it is confirmed that the interest rate is pro-cyclical in the three developed economies, while it is countercyclical (with a correlation coefficient of -0.47) for Mexico.

[8] Neumeyer and Perri (2005) find an acyclical interest rate for their set of developed economies with an average output correlation coefficient of 0.20.

Table 2.3: Contemporaneous Correlation (ρ) of x_t with Output y_t [(a)]

	$\rho(c_t,y_t)$	$\rho(h_t,y_t)$	$\rho(k_t,y_t)$	$\rho(s_t,y_t)$	$\rho(i_t,y_t)$	$\rho(tby_t,y_t)$	$\rho(cay_t,y_t)$	$\rho(r_t,y_t)$
AUS	0.40	0.66	-0.19	0.90	0.84	-0.43	-0.47	0.43
BEL	0.79	0.66	0.23	0.75	0.79	-0.51	-0.37	-
CAN	0.88	0.88	-0.07	0.93	0.74	-0.06	-0.18	0.50
CZR	0.59	-	-	0.71	0.61	-0.43	-0.27	-
FRA	0.78	0.74	0.26[(c)]	0.84	0.87	-0.38	-0.26	-
GER	0.63	0.70[(b)]	0.23	0.72	0.76	-0.29	-0.39	-
MEX	0.92	-	-	0.32	0.91	-0.62	-0.63	-0.47
NEL	0.70	0.56[(b)]	0.33[(c)]	0.83	0.71	-0.20	-0.27	0.59
POL	0.33	-	-	0.68	0.67	-0.29	-0.33	-
SLR	0.58	-	-	0.25	0.69	-0.39	-0.31	-
SWI	0.68	0.29	0.59[(c)]	0.65	0.70	-0.36	-0.12	-
U.S.	0.82	0.86[(b)]	-0.01	0.86	0.94	-0.47	-0.49	-
Averages for Developed Economies[(d)]								
All	0.71	0.67	0.17	0.81	0.79	-0.34	-0.32	0.51
SMOPEC	0.69	0.61	0.18	0.81	0.76	-0.31	-0.28	0.51
Averages for Emerging Economies								
	0.61	-	-	0.49	0.72	-0.43	-0.39	-0.47

Notes: (a) All correlations are based on the maximum number of *common* observations. See the data appendix for additional information. (b) Total hours worked in the business sector rather than the total economy. (c) The capital stock of the business sector rather than the capital stock of the total economy. (d) "All" refers to the average across all developed economies, 'SMOPEC' refers to the average obtained by excluding France, Germany, and the U.S. (the large open economies = 'LOPEC'), leaving the small open developed economies (SMOPECs).

Table (2.4) presents a stylized fact that is rooted in more recent research: Both in terms of absolute saving and investment and in terms of relative saving and investment rates (i.e. ratios with respect to output), contemporaneous correlations between these two variables tend to be lower for emerging economies than for developed economies. Although a recent consensus has emerged that the Feldstein-Horioka puzzle has not been as pronounced in developed economies in recent years, it does manifest itself in individual cases (most notably the U.S. where the absolute (relative) saving-investment correlation equals 0.79 (0.65) or in Australia and Belgium, where the same correlations are 0.72 (0.53) and 0.73 (0.62) respectively). One reason that the saving-investment correlations have declined in recent years is likely to be due to greater capital mobility that emerged as a consequence of globalization beginning in the 1990s. In the emerging economies, however, the Feldstein-Horioka puzzle can hardly be said to exist. Here, the absolute (relative) saving-investment correlations range from -0.29 (-0.50) for the Slovak Republic to 0.56 (0.29) for Poland. When considering these values, the caveat regarding the potentially inaccurate measurement of saving should be kept in mind. On average, the absolute saving-investment correlation of 0.64 for all developed economies decreases to 0.54 once the LOPECs are omitted and decreases even further to 0.26 once only emerging markets are considered. For the relative saving-investment rate correlation, the average taken across all developed economies is 0.43, 0.40 for the SMOPECs alone, and -0.01 for the emerging economies. Thus, it is safe to conclude that in emerging economies absolute and relative saving and investment exhibit less contemporaneous saving and investment correlations relative to developed economies. The latter seem to exhibit increasingly less of the puzzling characteristics described by Feldstein and Horioka, even though the two variables are still positively correlated throughout the sample

Table 2.4: Absolute Saving and Investment Correlations[(a)]

$\rho(s_t, i_t)$							
AUS	BEL	CAN	CZR	FRA	GER	MEX	NET
0.72	0.73	0.64	0.52	0.69	0.53	0.26	0.49
POL	SVR	SWI	U.S.				
0.56	-0.29	0.56	0.79				
Averages Developed Economies[(b)]							
All	SMOPEC						
0.64	0.54						
Averages Emerging Economies							
0.26							

Notes: See table (2.3)

Table 2.4 (continued): Relative Saving and Investment Correlations[(a)]

$\rho\left(\frac{s_t}{y_t}, \frac{i_t}{y_t}\right)$							
AUS	BEL	CAN	CZR	FRA	GER	MEX	NET
0.53	0.62	0.29	0.34	0.51	0.28	-0.16	0.20
POL	SVR	SWI	U.S.				
0.29	-0.50	0.36	0.65				
Averages Developed Economies[(b)]							
All	SMOPEC						
0.43	0.40						
Averages Emerging Economies							
-0.01							

Notes: See table (2.3)

Lastly, table (2.5) shows that most of the series are very persistent across country types. The least persistent series is the current account to output ratio, while 'sluggish' variables such as the capital stock are highly persistent. It appears that variables in developed economies are slightly more persistent than those of their emerging counterparts, but this difference is negligible.

Table 2.5: First-Order Autocorrelations $\rho(x_t, x_{t-1})$ [(a)]

	y_t	c_t	h_t	k_t	s_t	i_t	tby_t	cay_t	r_t
AUS	0.85	0.78	0.92	0.93	0.80	0.83	0.75	0.76	0.82
BEL	0.89	0.88	0.94	0.97	0.88	0.89	0.48	0.04	-
CAN	0.90	0.86	0.93	0.97	0.82	0.88	0.71	0.65	0.78
CZR	0.83	0.85	-	-	0.83	0.67	0.58	0.42	-
FRA	0.87	0.76	0.91	0.97[(c)]	0.84	0.92	0.80	0.49	-
GER	0.70	0.43	0.86[(b)]	0.74	0.67	0.54	0.56	0.51	-
MEX	0.79	0.80	-	-	0.34	0.85	0.88	0.78	0.55
NET	0.70	0.76	0.94[(b)]	0.96[(c)]	0.53	0.46	0.55	0.34	0.87
POL	0.71	0.45	-	-	0.53	0.71	0.75	0.70	-
SVR	0.72	0.62	-	-	0.57	0.74	0.70	0.61	-
SWI	0.86	0.73	0.95	0.96[(c)]	0.54	0.76	0.47	0.45	-
U.S.	0.85	0.85	0.92[(b)]	0.96	0.79	0.88	0.77	0.72	-
Averages for Developed Economies[(d)]									
All	0.83	0.76	0.92	0.93	0.73	0.77	0.63	0.50	0.82
SMOPEC	0.84	0.80	0.93	0.96	0.71	0.77	0.59	0.45	0.82
Averages for Emerging Economies									
	0.76	0.68	-	-	0.57	0.74	0.73	0.63	0.55

Notes: See table (2.3)

2.1.2 Cross-Country Findings

Zimmermann (1995) develops a three country model consisting of a small open economy, a large neighbor and the rest of the world. Although the small open economies are developed ones (Switzerland and Canada) some of his findings naturally extend to the present analysis comparing business cycles of developed and emerging economies. The pertinent findings are (p.1):

- "[S]ize and distance can… account for the diversity in the observed business cycles…."
- "[S]mall countries are indeed more sensitive to foreign technological innovations."
- "[T]he business cycle is transmitted mostly through innovation spillovers rather than through trade."
- "[T]he volatility of innovations is higher in small countries."
- "[T]he predicted cross-correlations of output levels are much lower than those of consumption. Indeed, data exhibit more often output smoothing than consumption smoothing."

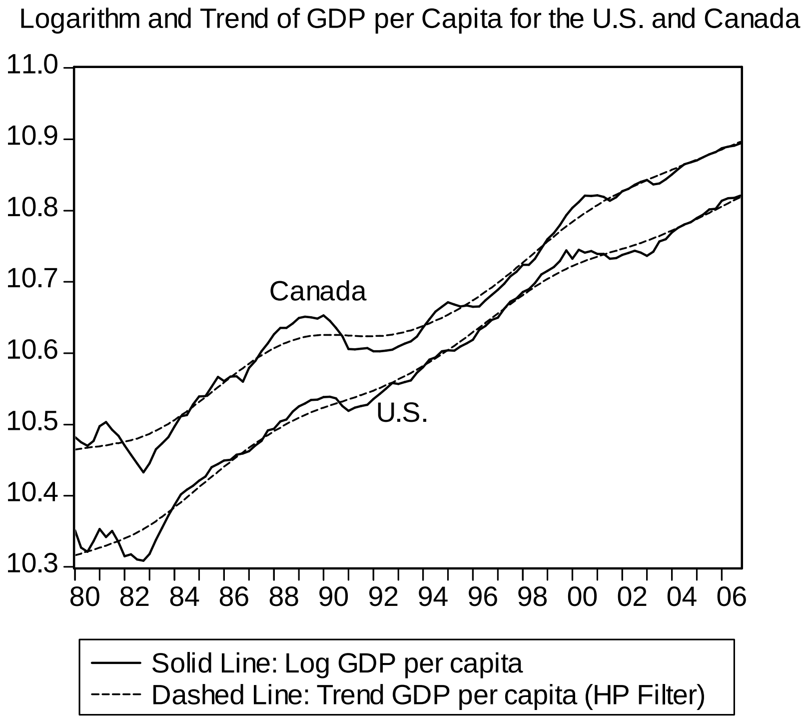
For the sample at hand, countries in the same vicinity should therefore be more likely to experience similar business cycles than two countries on opposite ends of the globe. Although according to Zimmermann, trade channels play a secondary role, it seems intuitive that agreements such as the EU, which eliminates most barriers to trade and decreases the barriers to capital and labor mobility, should have a definite positive influence on the synchronization of business cycles. The same is true for NAFTA, although the elimination of barriers is not as far-reaching. As will be shown, there are indeed very synchronized business cycles among the developed economies in either the EU or NAFTA set, but not with respect to the emerging economies, despite geographical proximity and common membership in a trade agreement. The lack of business cycle synchronization for most data points is surprisingly high, especially because the emerging economies of the sample were always neighboring the developed economies even when they did not have full access to the preferential trade agreement.

Applying this to Zimmermann's first finding, it appears that economic size (here size is used in the sense of economic size and therefore approximately interchangeable with development level) may be more important than distance for successfully transmitting spillovers. This extends naturally to his second finding (that small economies are more sensitive to spillovers), which only seems to hold for countries of the same development level. An alternative interpretation is presented below. The fact that there appears to be some convergence in business cycles across the two economy types in recent years may imply that the trade channel has begun to play a more important role than geographical proximity, which would be contrary to Zimmermann's third finding. His fourth finding regarding greater volatility of business cycles in smaller economies trivially extends to greater volatility in emerging economies. His fifth finding is validated at the end of this chapter.

As has been mentioned, emerging economies may simply lack the necessary infrastructure (technology, skilled workforce, etc.) to internalize spillovers completely or quickly. But, there may be other or additional mechanisms at work: The working hypothesis featured in the two-country model of chapter 3 is that the lack of business cycle synchronization between two countries of dissimilar development levels but meeting the geographical proximity and trade channel criteria is due to negative spillover effects from the emerging onto the developed country. According to Zimmermann, this can be interpreted as exploitation of a comparative advantage 'at the expense' of another country. In the current context, it could be postulated that the emerging economies are exploiting their comparative advantages in agriculture, (unskilled) labor intensive production or in manufacturing.

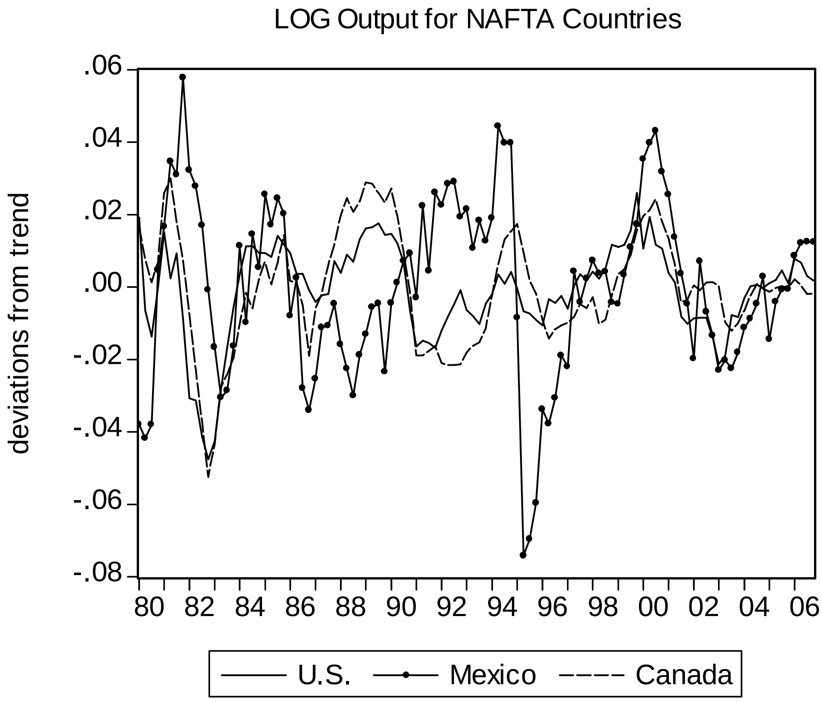
To illustrate synchronization of business cycles, consider figure (2.1), depicting the logarithm and trend of output per capita for the U.S. and Canada. Clearly both countries experience very similar business cycles, even prior to the NAFTA period (1994). Thus, factors such as geographical proximity (for Canada and the U.S. other aspects, such as a shared language or 'value' system, could also come to mind) definitely seem to play an important role in synchronizing business cycles.

Figure 2.1: Synchronized Canadian and U.S. Business Cycles



To what extend the interactions between membership in a preferential trade agreement, development level, and geographical proximity play a role in generating similar business cycles can be inferred from figure (2.2), which shows the cyclical component (i.e. the detrended series using the HP filter) of per capita log output and consumption for all NAFTA countries and selected developed EU member countries. While the developed EU countries follow the U.S.-Canada pattern in the sense that output and consumption closely track one another (particularly after 1992), it is striking how little the Mexican cycle is in sync with either of the two developed NAFTA member countries. Figures (2.3) – (2.6) depict selected Eastern European countries' detrended output and consumption series with respect to selected developed European countries. These graphs verify the observation that business cycles among developed and emerging countries in the EU have, until recently, not been very synchronized.

Figure 2.2: Business Cycles in Countries Belonging to a Common Trade Agreement



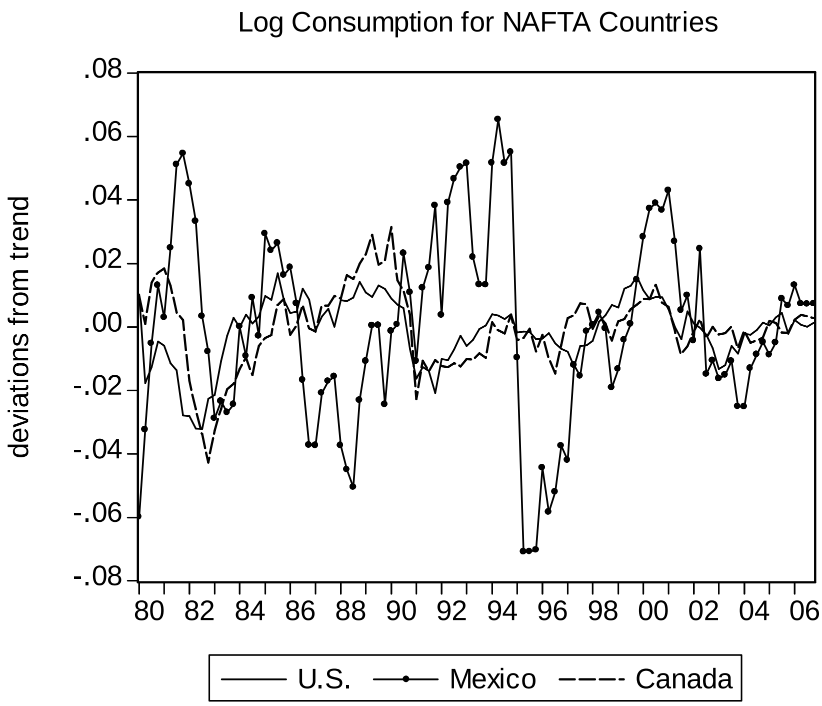
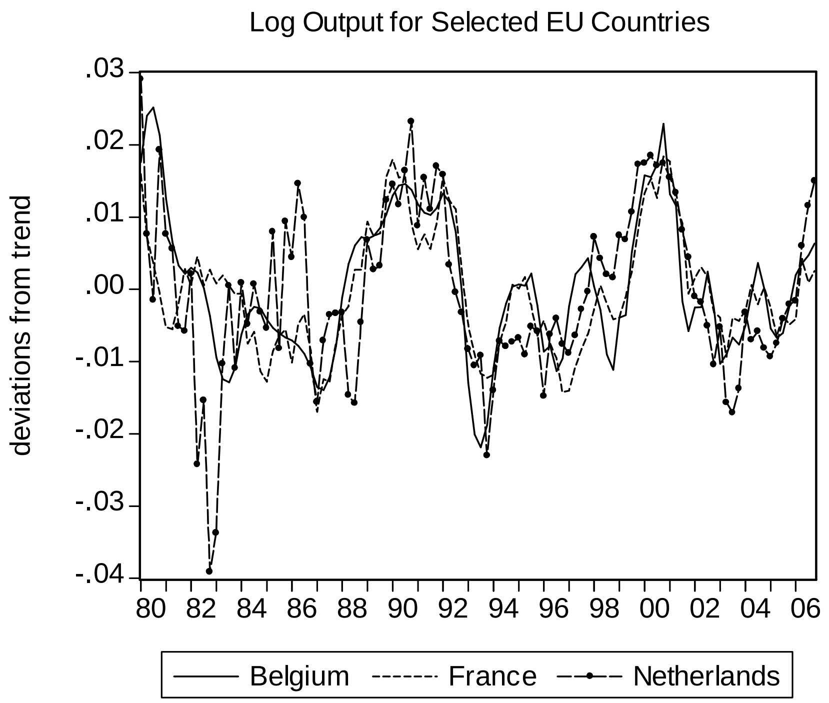
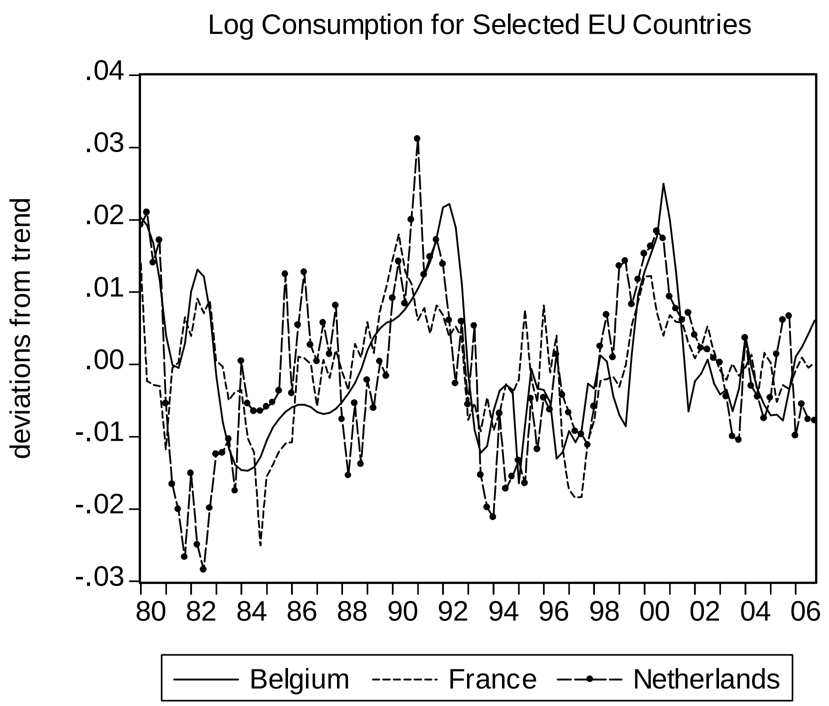


Figure 2.2 (continued)**:**





Overall, output cycles in Eastern European countries differ, and sometimes differ starkly, from those of the Western European countries (figures (2.3) and (2.5)). Notably, this pattern seems to subside somewhat in the most recent years since the Eastern European countries acceded to the EU. Thus it appears that geographical proximity facilitates synchronized business cycles among European developed and emerging economies to a lesser extent than the trade channels that were opened by joining the EU. As shown in figures (2.4) and (2.6) these observations are even more pronounced when considering the cyclical components of consumption. Most times of upswings in the EU LOPEC's consumption cycle are accompanied by times of downswings in Eastern European countries and vice versa. This is most evident for the Czech and Slovak Republic and to a lesser extent in Poland as well.

Figure 2.3: The Eastern European Countries' Output Cycle vs. the EU LOPECs

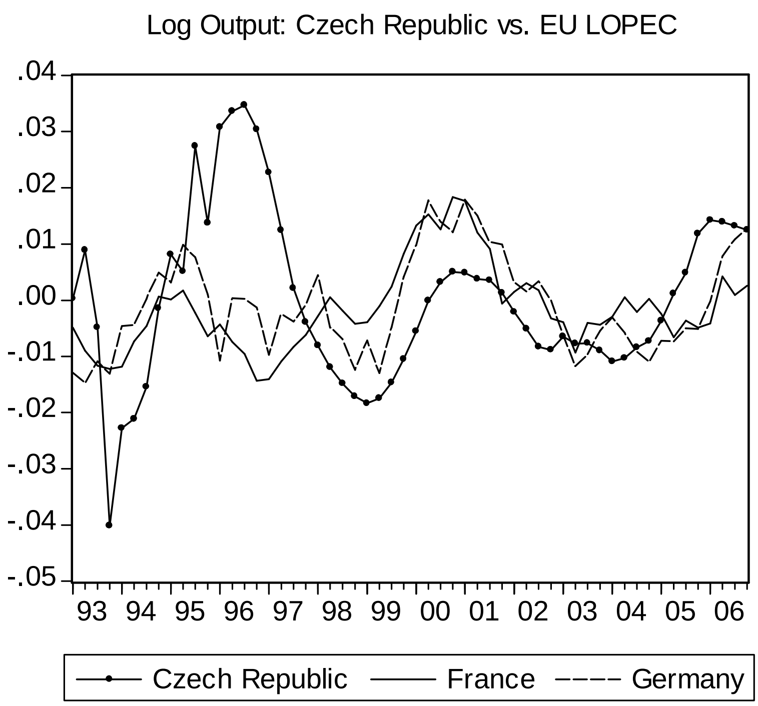
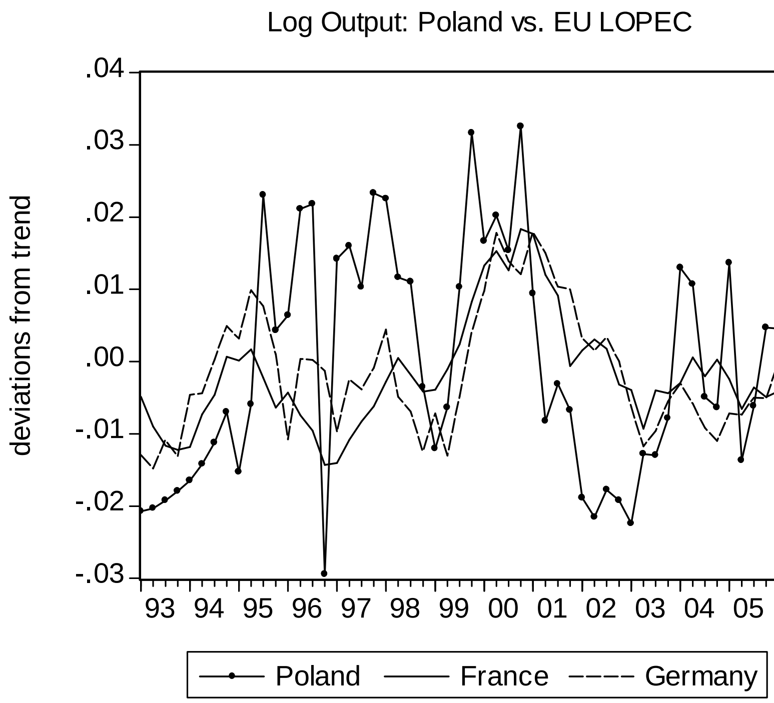


Figure 2.3 (continued)



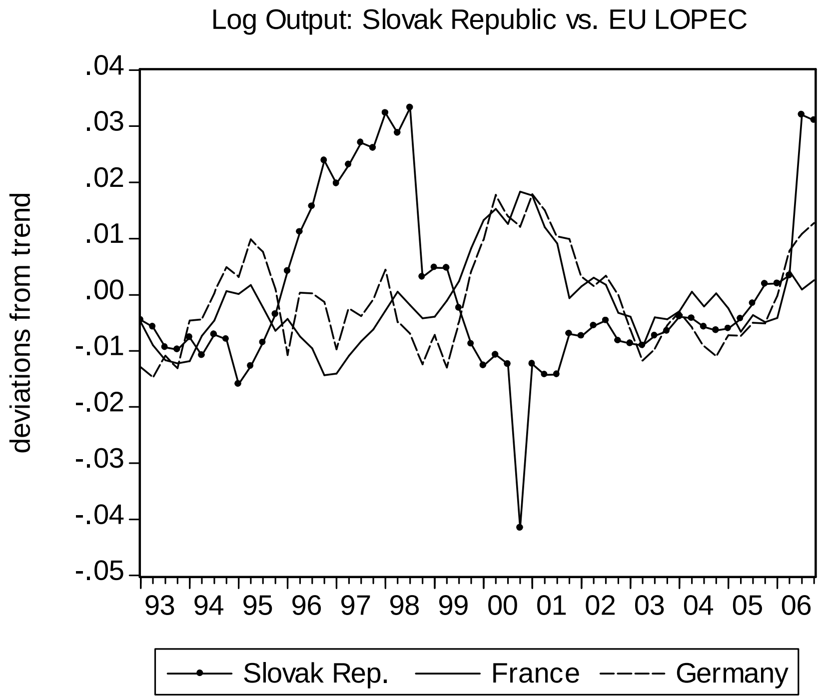
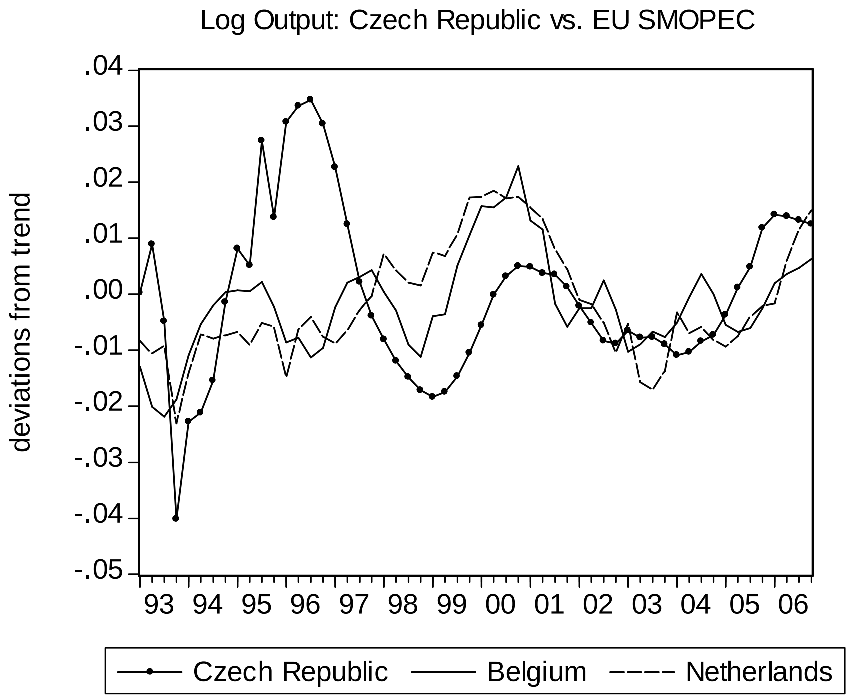


Figure 2.4: The Eastern European Countries' Output Cycle vs. the EU SMOPECs



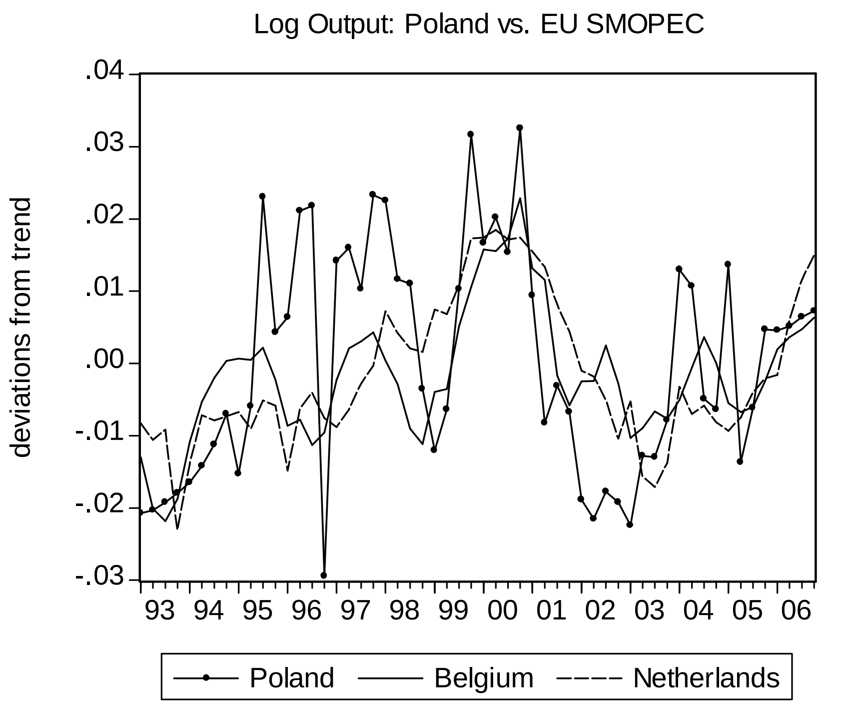


Figure 2.4 (continued)**:**

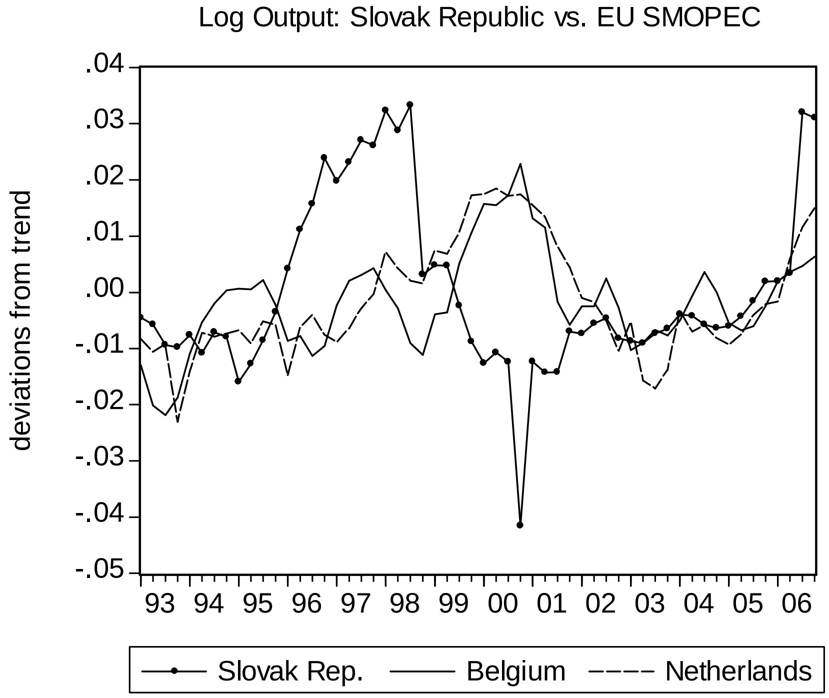


Figure 2.5: The Eastern European Countries' Consumption Cycle vs. the EU LOPECs

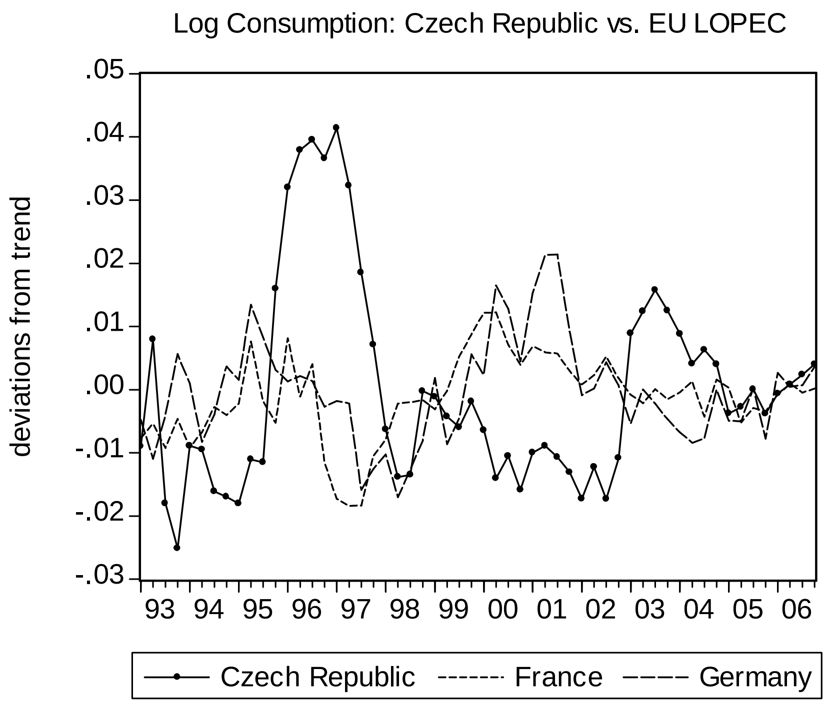
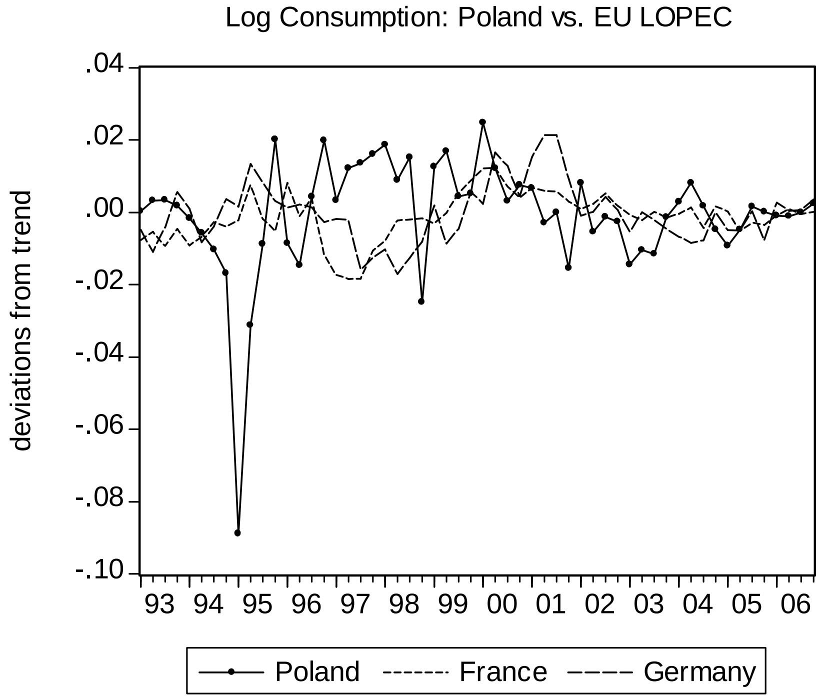


Figure 2.5 (continued)



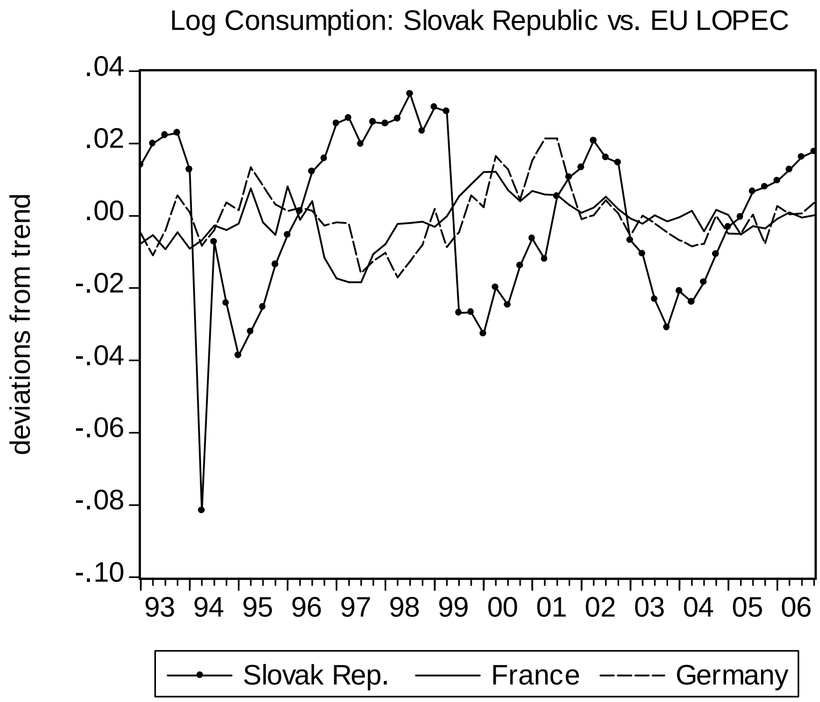
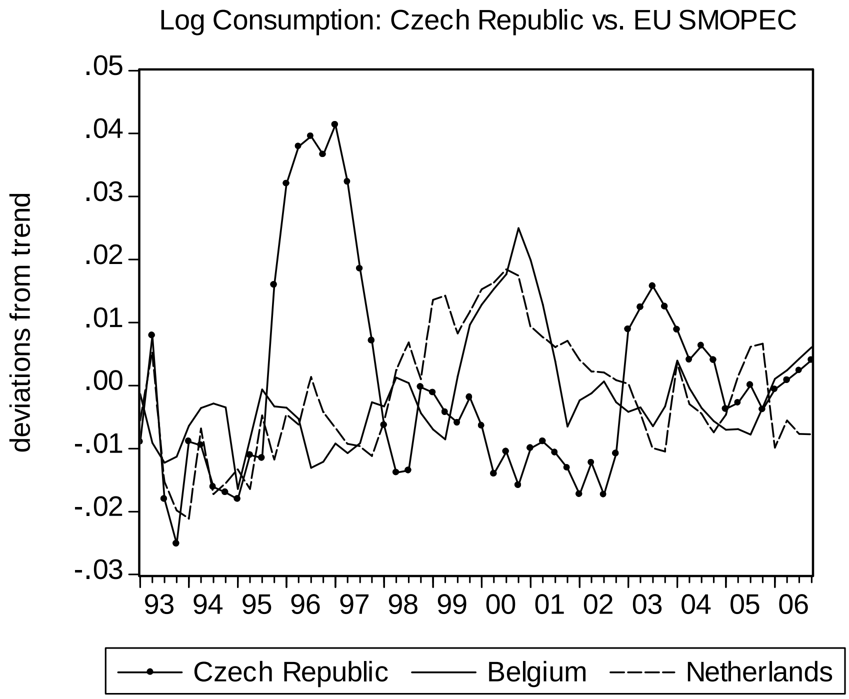


Figure 2.6: The Eastern European Countries' Consumption Cycle vs. the EU SMOPECs



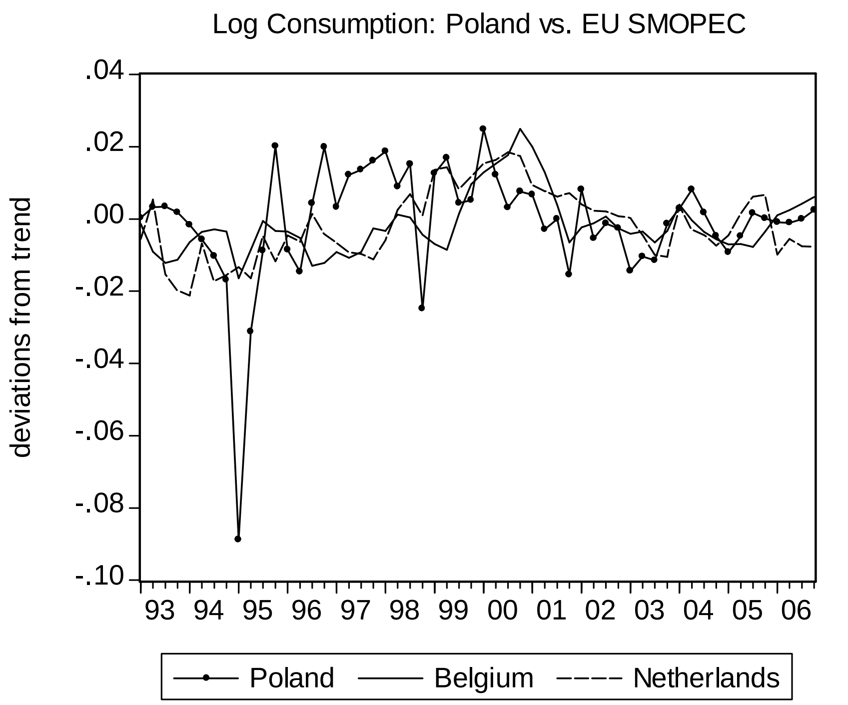
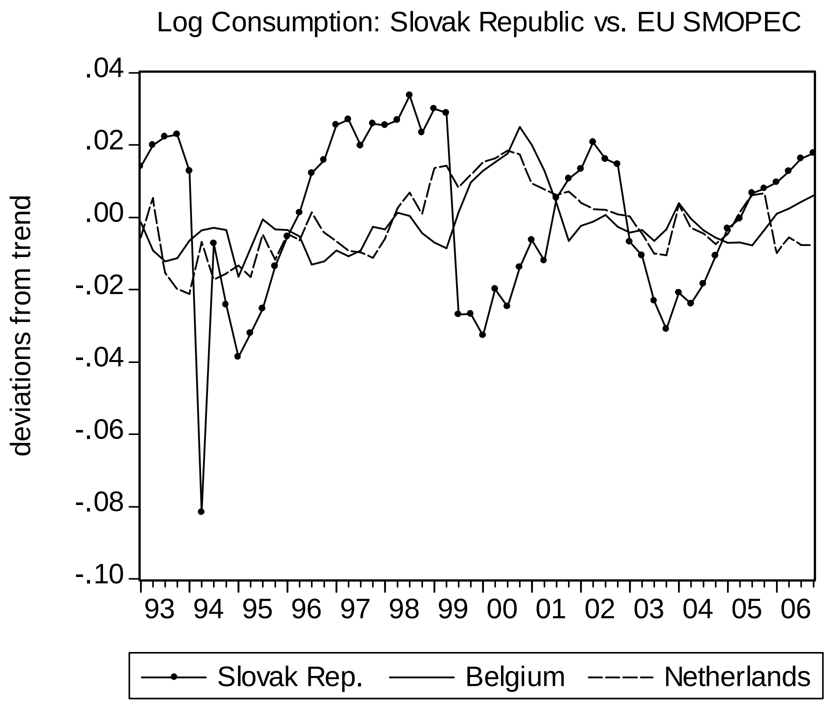


Figure 2.6 (continued)



In sum, it appears that for the NAFTA countries, business cycles are fairly synchronized in the developed economies (Canada and the U.S.), but not with respect to the emerging country (Mexico), with the exception of recent data points. Similarly, among the developed countries of the EU there is a considerably harmonious business cycle. Once each of the Eastern European countries is considered, however, this correspondence is almost entirely absent (again with some convergence after the Eastern European countries' EU accession). The picture that therefore emerges, is one in which developed countries in the same vicinity and sharing membership in the same trade agreement tend to exhibit synchronized business cycles, while emerging economies vis-à-vis their developed economy counterparts do not. These observations are more pronounced over the entire sample period but are less substantiated in recent years. This leads to the tentative conclusion that geographical proximity appears to play less of a role in synchronizing cycles than common membership in a trade agreement.

Tables (2.6) – (2.9) confirm the previous observations. Tables (2.6) and (2.8) display output and consumption correlations for countries with data available from 1980:Q1 – 2006:Q4 respectively. Tables (2.7) and (2.9) display the same correlations for the Eastern

European countries in comparison to Western European countries for the period 1993:Q1 – 2006:Q4.

In table (2.6), Mexico, as the only emerging economy, has lower output correlations vis-à-vis the developed economies than the developed economies have vis-à-vis one another. Moreover, its output correlations with respect to its NAFTA partners are among the three lowest (0.08 and 0.14 for Canada and the U.S. respectively). This result is even more pronounced when considering the cross-country consumption correlations of the NAFTA countries in table (2.8), which fall to -0.01 and -0.07 for the pairs Mexico-Canada and Mexico-U.S. Thus, the picture conveyed by the above graphs is confirmed. Mexico has some of its lowest, virtually non-existent contemporaneous output or consumption correlation with its fellow NAFTA member countries. Similar and sometimes even more pronounced observations hold for the Eastern European countries in tables (2.7) – (2.9). Strikingly, all developed EU countries exhibit considerable positive cross-output and consumption correlations, while the three emerging countries exhibit very low and at times negative cross-output and consumption correlations vis-à-vis the developed EU economies. An exception is Poland’s cross-country output correlation of around 0.50 with respect to Belgium and the Netherlands.

Table 2.6: Cross-Country Output Correlations [(a)]

	AUS	BEL	CAN	FRA	MEX	NLD	SWZ	U.S.
AUS	1.00	0.25	0.78	0.02	0.09	0.24	0.47	0.63
BEL	0.25	1.00	0.43	0.76	0.19	0.63	0.68	0.24
CAN	0.78	0.43	1.00	0.23	0.08	0.45	0.56	0.77
FRA	0.02	0.76	0.23	1.00	0.20	0.57	0.60	0.10
MEX	0.09	0.19	0.08	0.20	1.00	0.25	0.40	0.14
NLD	0.24	0.63	0.45	0.57	0.25	1.00	0.71	0.51
SWZ	0.47	0.68	0.56	0.60	0.40	0.71	1.00	0.47
U.S.	0.63	0.24	0.77	0.10	0.14	0.51	0.47	1.00
Averages[(b)]								
All	0.35	0.45	0.47	0.36	0.19	0.48	0.56	0.41
SMOPEC	0.37	0.43	0.46	-	0.20	0.46	0.56	-
NAFTA	-	-	0.43	-	0.11	-	-	0.46

Notes: (a) The table presents correlations for those countries with data from 1980:Q1-2006:Q4. (b) Averages “All” includes the U.S. and France, “SMOPEC” excludes these and NAFTA computes the average correlations among the three NAFTA countries, Canada, Mexico and the U.S.

Table 2.7: Cross Country Output Correlations of European Countries Only[a]

	European SMOPECs			European LOPECs		Eastern European Countries		
	BEL	NLD	SWI	FRA	GER	CZR	POL	SVR
BEL	1.00	0.75	0.73	0.80	0.75	0.11	0.57	-0.17
NLD	0.75	1.00	0.80	0.78	0.69	0.05	0.51	0.03
SWI	0.73	0.80	1.00	0.80	0.73	-0.08	0.44	-0.12
FRA	0.80	0.78	0.80	1.00	0.73	-0.05	0.36	-0.42
GER	0.75	0.69	0.73	0.73	1.00	0.29	0.37	-0.15
CZR	0.11	0.05	-0.08	-0.05	0.29	1.00	0.29	0.23
POL	0.57	0.51	0.44	0.36	0.37	0.29	1.00	0.23
SVR	-0.17	0.03	-0.12	-0.42	-0.15	0.23	0.23	1.00
Averages[b]								
All	0.51	0.52	0.47	0.43	0.49	0.12	0.40	-0.05
EU West	0.77	0.74	-	0.77	0.72	0.10	0.45	-0.18
SMOPEC	-	-	-	-	-	0.08	0.54	-0.07

(a) The table presents correlations for those countries with data from 1993:Q1-2006:Q4. (b) Averages "EU West" computes the average correlation of each country with respect to Belgium, the Netherlands, France and Germany. Average "SMOPEC" excludes France and Germany.

Table 2.8: Cross-Country Consumption Correlations [a]

	AUS	BEL	CAN	FRA	MEX	NLD	SWZ	U.S.
AUS	1.00	0.23	0.18	0.42	0.11	0.01	0.26	0.05
BEL	0.23	1.00	0.09	0.63	0.33	0.45	0.66	-0.22
CAN	0.18	0.09	1.00	0.07	-0.01	0.26	0.34	0.61
FRA	0.42	0.63	0.07	1.00	0.05	0.34	0.61	-0.04
MEX	0.11	0.33	-0.01	0.05	1.00	0.02	0.26	-0.07
NLD	0.01	0.45	0.26	0.34	0.02	1.00	0.53	0.28
SWZ	0.26	0.66	0.34	0.61	0.26	0.53	1.00	0.12
U.S.	0.05	-0.22	0.61	-0.04	-0.07	0.28	0.12	1.00
Averages[b]								
All	0.18	0.31	0.22	0.30	0.10	0.27	0.40	0.10
SMOPEC	0.16	0.35	0.17	-	0.14	0.25	0.41	-
NAFTA	-	-	0.30	-	-0.04	-	-	0.27

Notes: (a) The table presents correlations for those countries with data from 1980:Q1-2006:Q4. (b) Averages "All" includes the U.S. and France, "SMOPEC" excludes these and NAFTA computes the average correlations among the three NAFTA countries, Canada, Mexico and the U.S.

Table 2.9: Cross Country Consumption Correlations of European Countries Only[(a)]

	European SMOPECs			European LOPECs		Eastern European Countries		
	BEL	NLD	SWI	FRA	GER	CZR	POL	SVR
BEL	1.00	0.58	0.73	0.59	0.44	-0.32	0.30	-0.27
NLD	0.58	1.00	0.61	0.54	0.21	-0.15	0.33	-0.01
SWI	0.73	0.61	1.00	0.62	0.49	-0.43	0.09	-0.06
FRA	0.59	0.54	0.62	1.00	0.57	-0.35	-0.11	-0.39
GER	0.44	0.21	0.49	0.57	1.00	-0.22	-0.17	-0.26
CZR	-0.32	-0.15	-0.43	-0.35	-0.22	1.00	0.17	0.13
POL	0.30	0.33	0.09	-0.11	-0.17	0.17	1.00	0.34
SVR	-0.27	-0.01	-0.06	-0.39	-0.26	0.13	0.34	1.00
Averages[(b)]								
All	0.29	0.30	0.29	0.21	0.15	-0.17	0.14	-0.07
EU West	0.54	0.44	-	0.57	0.41	-0.26	0.09	-0.23
SMOPEC	-	-	-	-	-	-0.23	0.31	-0.14

(a) The table presents correlations for those countries with data from 1993:Q1-2006:Q4. (b) Averages "EU West" computes the average correlation of each country with respect to Belgium, the Netherlands, France and Germany. Average "SMOPEC" excludes France and Germany.

2.2 Concluding Remarks Chapter 2

Chapter 2 has shown that the inclusion of a sample of Eastern European countries corroborates previous findings on emerging markets: These countries exhibit similar stylized facts as other, more commonly studied, emerging markets in South America and Asia. The data shows that among *developed* economies, factors such as geographical proximity and membership in a common trade agreement results in synchronized business cycles. This result is to be expected since theoretically one expects the spillover effects of productivity shocks to manifest themselves in precisely this way for countries in precisely this type of constellation. A surprising result is how little this holds for emerging economies in close geographical proximity to developed economies. Using several years of data, both Mexico as well as the set of Eastern European countries display little synchronization of business cycles with respect to their developed counterparts, which results in small and sometimes even negative consumption and output correlations. It is possible that these results are a mere product of the sample period being considered, since there appears to be some convergence in recent years, therefore lending support to the theory that the trade channel is an important mechanism for synchronizing business cycles. It is also possible that the assumption of positive productivity shocks is simply a fallacy. If there are negative spillover effects from the emerging onto the developed economies, which allows the emerging economies to exploit their comparative advantage while ‘catching up’, then the lack of correlations in business cycles could be explained. This issue is the focus of the next chapter.

Chapter 3

Modeling Real Business Cycles of Developed and Emerging Economies

In order to match the intra- and cross-country stylized facts of chapter 2, three open economy real business cycle models based on Schmitt-Grohé and Uribe (2003) will be introduced in this chapter. Model 1 is a one-country, small open economy model with an interest rate premium calibrated to match key long run statistical features of developed economies. In an effort to contribute to the ongoing discussion on whether real business cycle models are an appropriate tool for modeling the business cycles of emerging economies (see, for example, Aguipar and Gopinath (2007), Neumeyer and Perri (2005), García-Cicco, et al. (2006) and Kydland and Zarazaga (2002)), model 2 also uses an endogenous interest rate as in model 1 but will be calibrated to match the key long run statistical features of emerging economies. Model 3 constructs a two-country model for an emerging and a developed economy but features portfolio adjustment costs to debt rather than an interest rate premium. This two-country model attempts to replicate certain cross-country 'stylized facts' such as the low or negative output and consumption correlations between developed and emerging economies that were found in chapter 2. The mechanisms that connect the two hypothetical economies of model 3 are linked innovations to the productivity processes and the international market for financial claims. Schmitt–Grohé and Uribe (2003) prove that both model types –the endogenous interest rate and the portfolio adjustment cost model– generate very similar results.

If the results generated by models 1-3 reflect the findings of chapter 2, it can be concluded that the use of stochastic international real business cycle models is appropriate not just for developed economies but also for studying emerging economies and/or characterizing the statistical interactions between developed and emerging economies. The empirical findings to which the models' results will be compared are the averages that were obtained for the developed small open economies (i.e. excluding France, Germany and the U.S.) and the averages for the emerging economies of chapter 2. Alternatively, a representative country could have been chosen from each economy type. But since the averages also tend to reflect the stylized facts for an arbitrary developed or emerging market, this approach was chosen.

Because it is the easiest to calibrate, it is not surprising that model 1 of a developed economy does well relative to the data: It correctly predicts the ranking of the volatilities for developed economies, for example, that output is more variable than consumption and that investment is more variable than output. It also correctly predicts the acyclical behavior of the trade balance and current account ratios, but understates their volatility. The contemporaneous output and first order auto-correlations are all well matched with the exception that the contemporaneous correlation of capital and consumption with respect to output is overstated. Additionally, the saving and investment correlation is overstated.

Model 2 of an emerging economy also correctly predicts the volatility rankings. It is noteworthy that it can reproduce the fact that consumption is more volatile than output in an emerging economy (rather than the opposite as in the case of developed economies) and that investment and the international variables are more volatile than in the developed economy. The volatility of the trade balance and current account ratios, however, is again understated. In contrast to model 1, model 2 accurately predicts the relatively low contemporaneous correlation of consumption and output. Unfortunately, the stylized fact that the trade balance and current account ratios are even more acyclical in the emerging than in the developed economy can not be replicated. Instead, model 2 understates the acyclical behavior of these variables. Lastly, the saving and investment correlation in the data is successfully mirrored by model 2.

The key success of the two-country model 3 is that it is able to mimic two important cross-country findings of chapter 2: The first is that there is less consumption than output smoothing among countries in general and the second is that the two hypothetical economies representing a developed and emerging country exhibit small, negative cross-country consumption and output correlations. Although the predicted difference in the cross-country contemporaneous consumption and output correlations is not as large as in the data, it seems that the parameterization is pointing in the right direction. The intra-country statistics of model 3 approximately compare to those of models 1 and 2: For the developed economy, the consumption and output correlation is again overstated. The trade balance and current account ratios perform relatively worse in comparison to model 1, because they are again not volatile enough while their acyclical nature is now also understated. For the emerging economy, model 3 can no longer capture the fact that consumption is more volatile than output (a fact that is likely a result of the lower steady state interest rate used in model 3). Model 3 correctly retains the prediction that the trade balance and current account ratios are more volatile in the emerging than in the developed economy. In addition, it now accurately replicates the fact that they are more acyclical in the emerging economy even though it understates this correlation. Lastly, it overstates the saving-investment correlation in both economy types.

Each of the following sections introduces one of the three models, followed by an impulse response analysis and a comparison of the models' forecasted business cycle statistics with relevant empirical averages obtained in chapter 2 for the developed SMOPECs and emerging economies. The impulse response analysis tests the effects of productivity shocks in individual countries (models 1 and 2) and the effects across countries (model 3)

3.1 The Debt Elastic Interest Rate Model

This section describes the debt elastic interest rate model[9] in general. Because the ultimate goal of the two-country model with portfolio adjustment costs (model 3) is to relate a developed economy to an emerging one, where there exist, on average, virtually no output and consumption correlations (as in the data presented in chapter 2), the use of an incomplete asset market structure is the right choice. In incomplete asset markets, the market for financial claims is usually characterized by a single instrument (in our case debt or assets) that costs/pays a risk-free rate of return. In the first two models, this rate of return will depend on the world interest rate plus a country specific interest rate premium, which in turn pins down the steady state level of debt or assets (in this way 'closing' the model, see Schmitt–Grohé and Uribe (2003)). In complete asset markets, conversely, the market for financial claims is characterized by an array of instruments, whose rate of return is contingent upon the state of the economy. Kollmann (1996) describes the trade-off between each type of market structure as: "The elimination of trade in state contingent assets limits international risk sharing. A country affected by an idiosyncratic (country-specific) income shock can mitigate the effect

[9] Endogenous or debt elastic interest rate model are two labels that may be used interchangeably.

of this shock on its consumption by trading in bonds. One may expect, however, that countries are less able to offset the effects of idiosyncratic income shocks when markets are incomplete than when markets are complete (p. 3).” In an Arrow-Debreu (complete asset) market setting, consumption is usually highly correlated, because a combined optimization problem is solved by a benevolent social planner that results in proportionate consumption levels. Therefore, if countries are “less able to offset the effects” of productivity shocks, then consumption should be less correlated. Given the findings of chapter 2, this is clearly the more appropriate way to model the market for financial claims.

Model 1, the endogenous interest rate model for a developed economy, can be described as follows: A single good is produced, which can be used for consumption and investment. The economy is populated by an infinite number of identical consumers and there exists a representative agent, who solves the following optimization problem with respect to per capita variables:

$$\max_{\{c_t, h_t, d_{t+1}, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \tag{3.1}$$

subject to:

$$d_{t+1} = (1+r_t)d_t - y_t + c_t + i_t + \Phi(k_{t+1} - k_t) \tag{3.2}$$

$$y_t = A_t F(h_t, k_t) \tag{3.3}$$

$$k_{t+1} = i_t + (1-\delta)k_t \tag{3.4}$$

$$r_t = r + \rho\left(\tilde{d}_{t+1}\right) \tag{3.5}$$

The exogenous productivity process, which the agent takes as given, follows a first order autoregressive process in logarithms:

$$\ln\left(A_{t+1}\right) = \rho \ln\left(A_t\right) + \varepsilon_{t+1};\ \forall \varepsilon_{t+1} \sim NIID(0, \sigma_\varepsilon^2),\ t \geq 0 \tag{3.6}$$

The first order conditions after maximizing (3.1) subject to (3.2) – (3.5) are:

$$U_c(c_t, h_t) = \lambda_t \tag{3.7}$$

$$-U_h(c_t,h_t)=\lambda_t A_t F_h(k_t,h_t) \tag{3.8}$$

$$\lambda_t=\beta(1+r_t)E_t\lambda_{t+1} \tag{3.9}$$

$$\lambda_t\left[1+\Phi'(k_{t+1}-k_t)\right]=\beta E_t\lambda_{t+1}\left[A_{t+1}F_k(h_{t+1},k_{t+1})+1-\delta+\Phi'(k_{t+2}-k_{t+1})\right] \tag{3.10}$$

where $U_c=\partial U/\partial c$, $F_h=\partial F/\partial h$, $F_k=\partial F/\partial k$ and the constraints (3.2) – (3.5) must hold with equality. In addition the following transversality condition must be satisfied:[10]

$$\lim_{t\to\infty}\beta^t\lambda_t\left(k_{t+1}-d_{t+1}\right)=0 \tag{3.11}$$

Utility is an increasing function of consumption (c_t) and a decreasing function of labor input (h_t). Output (y_t) is produced with labor input and domestic capital (k_t), which is subject to a convex capital adjustment cost function $\Phi(\cdot)$ satisfying $\Phi(0)=0,\ \Phi'(0)=0$. The law of motion for the exogenous productivity series (A_t) is subject to independent and identically distributed (i.i.d.) shocks ε_t with mean zero and variance σ_ε^2. The parameter ρ measures the persistence of the productivity process and is also referred to as the autocorrelation coefficient of productivity.

This is an open economy model because saving (s_t, defined below) can either be invested (i_t) in the domestic market for physical capital (k_t) or on the international market for financial claims. The market for financial claims can simultaneously be characterized by the stock of per capita assets (b_t) or the stock of per capita foreign debt (d_t), which, in short, will be referred to as 'assets' or 'debt'. The interest premium discussed below, however, initially is a function of aggregate per capita foreign debt ($\tilde{d}_t$), which will be abbreviated as 'aggregate debt' (similarly aggregate per capita assets are given by ($\tilde{b}_t$)).[11]

The current account is defined as the change in net foreign assets over one period. Let $b_t=-d_t=$ per capita foreign assets. Then the (per capita) current account can be defined as

[10] The transversality condition arises with state variables to ensure that their evolution over time does not become explosive in the infinite horizon. In model 1, the two means by which output can be invested, i.e. the domestic asset (the capital stock) and the international asset, can be grouped together. The transversality condition rules out that the agent can accumulate or borrow assets forever.

[11] To avoid confusion between aggregate per capita debt and per capita debt, consider the following example: Individual A has 10 units of debt and individual B has 20 units of debt which equal their per capita debt respectively. Their aggregate debt is therefore 10+20=30 units, while their aggregate per capita debt is (10+20)/2 = 15 units. The assumption that individuals are identical in equilibrium implies that both per capita debt and aggregate per capita debt are 15 units.

$ca_t = b_{t+1} - b_t = \Delta b_{t+1}$ in terms of net foreign assets and $ca_t = -\left(d_{t+1} - d_t\right) = -\Delta d_{t+1}$ in terms of net foreign debt. The budget constraint (3.2) could therefore also be written as $ca_t = -r_t d_t + y_t - c_t - i_t - \Phi\left(k_{t+1} - k_t\right)$. If debt increases from period t to $t+1$, the country is a net borrower ($\Delta d_{t+1} > 0$) and its current account deficit is increasing (or its current account surplus is decreasing). If debt falls from period t to $t+1$ the country is a net lender ($\Delta d_{t+1} < 0$) and its current account surplus is increasing (or its current account deficit is decreasing). In the steady state it must be the case that $\Delta d_{t+1} = 0 \Rightarrow ca_t = 0$.

Other useful current account relationships are $ca_t = tb_t + r_t b_t = tb_t - r_t d_t = s_t - i_t$, where $tb_t = y_t - c_t - i_t - \Phi\left(k_{t+1} - k_t\right)$ denotes the trade balance. In the steady state the following identities must therefore hold: $tb = -rb = rd$. If the steady state trade balance is negative, it must be the case that the steady state value of debt is negative (the steady state value of assets is positive) and if the steady state trade balance is positive, it must be the case that the steady state value of debt is positive (the steady state value of assets is negative). In the sense that the steady state implies an empirical long run average, the interpretation is that a country that is an average borrower must have a positive average trade balance while a country that is an average lender must have a negative average trade balance. From here on out, the concept 'debt' rather than 'assets' is used, with the understanding that a negative debt level implies an asset that pays interest.

Each of the debt measures (d_t and $\tilde{d}_t$) costs the country specific interest rate r_t. This interest rate is decomposed into the world interest rate r (which will differ across models 1 and 2) and an interest rate premium function $\rho(\cdot)$ that is an increasing function of aggregate debt relative to the steady state. Note that the first Euler equation (3.9) is as of yet independent of the interest rate rule involving *aggregate* per capita foreign debt, because the agent initially takes this variable as given. This implies that the interest rate premium function given by (3.5) so far is not explicitly part of the agent's first order conditions except as a binding constraint. However, because the representative agent is representative in equilibrium and agents are assumed to be identical, aggregate per capita debt must equal individual per capita debt in any equilibrium, that is: $\tilde{d}_t = d_t$ (see previous footnote). Knowing this, any equilibrium must be characterized by replacing all equations involving aggregate per capita debt ($\tilde{d}_t$) by per capita debt (d_t).

The model could, using appropriate substitution, be entirely written in terms of the exogenous state variable A_t (the productivity process), the two endogenous state variables d_t and k_t (debt and capital) and the two control variables c_t and h_t (consumption and labor input). All other variables will be referred to as flow variables (for the uninitiated, this classification is explained in greater detail in chapter 4). In other words, flow variables are those that could potentially be eliminated from the model, but can be backed out later if necessary. If they can be eliminated from the model, they must therefore logically either be a function of a control or a state variable or a combination thereof, potentially at different dates. These include output (y_t), investment (i_t) and the interest rate (r_t) given by equations (3.3), (3.4) and (3.5) respectively. There are additional flow variables, which are implicitly part of the model: These include saving, the trade balance and the current account. Since it is common practice to work with the trade balance and current account ratios with respect to output rather than with their absolute values, the following three flow variables are now added to the model:

Saving is formally given by: [12]

$$s_t = y_t - \Phi\left(k_{t+1} - k_t\right) - c_t \tag{3.12}$$

The trade balance to output ratio is given by:

$$tby_t = \frac{tb_t}{y_t} = \left(\frac{y_t - c_t - i_t - \Phi\left(k_{t+1} - k_t\right)}{y_t}\right) = 1 - \left(\frac{c_t + i_t + \Phi\left(k_{t+1} - k_t\right)}{y_t}\right) \tag{3.13}$$

And the current account to output ratio is given by:

$$cay_t = \frac{ca_t}{y_t} = tby_t - \frac{r_t d_t}{y_t} \tag{3.14}$$

In addition, there is one minor technicality associated with the capital adjustment costs in equation (3.10). For many algorithms (see appendix) used to solve these types of models, it is

[12] Note that this definition of saving differs slightly from the one used in chapter 2 where saving equaled output minus household consumption minus government consumption. Since there is no government sector in this model, it is implicitly assumed that the government sector is usurped by the households.

pivotal to reduce the range of variable dates to two: t and $t+1$. As of now, the model is written in three time periods with variables dated at t, $t+1$ and $t+2$ in equation (3.10). To reduce the time span to t and $t+1$ requires the introduction of yet another variable, the auxiliary capital stock, given by $k_t^a = k_{t+1} \Rightarrow k_{t+1}^a = k_{t+2}$. Formally, its equation is:

$$k_t^a - k_{t+1} = 0 \tag{3.15}$$

This allows equation (3.10) to be written as: $\lambda_t \left[1+\Phi'(k_{t+1}-k_t)\right] = \beta E_t \lambda_{t+1}\left[A_{t+1}F_k(h_{t+1},k_{t+1})+1-\delta+\Phi'(k_{t+1}^a - k_{t+1})\right]$. Since k_t is considered an endogenous state variable, while k_{t+1} is considered a control variable, k_{t-1}^a must be a state variable while k_t^a is a control variable. Given the auxiliary capital stock, model 1 is now described by variables belonging only to t and $t+1$.

A rational expectations equilibrium can now be defined as a set of processes $\left\{\lambda_t, d_{t+1},\ \tilde{d}_{t+1},\ k_{t+1}, c_t,\ h_t\right\}_{t=0}^{\infty}$ that satisfy the first order conditions (3.7) – (3.11), the constraints (3.2) – (3.5) and the flow variable equations (3.12) – (3.15) given the exogenous productivity process (3.6) as well as the starting values for the state variables A_0, $\tilde{d}_0(d_0)$ and k_0 (which usually take on the values of the steady state). All other variables are flow variables that are functions of the variables listed in the rational expectations equilibrium and therefore need not be mentioned separately. In other words, once the endogenous control and state variables satisfy the rational expectations equilibrium, so do the remaining variables.

Following Schmitt–Grohé and Uribe (2003), the following functional forms are assigned to utility, the production function, the adjustment cost function and the interest rate premium:

$$U(c,h) = \frac{(c-\omega^{-1}h^{\omega})^{1-\gamma}-1}{1-\gamma}$$

$$F(k,h) = k^{\alpha}h^{1-\alpha}$$

$$\Phi(x) = \frac{\phi}{2}x^2$$

$$\rho(\tilde{d}) = \psi(\exp^{\tilde{d}-\bar{d}}\text{-}1)$$

The functional form for utility was first introduced by Greenwood, et al. (1988) and has frequently been employed in the small open economy real business cycle literature because it implies a static labor supply curve, independent of consumption decisions and only dependent on the real wage. According to Mendoza (1991), "this simplification facilitates the numerical simulations and allows the model to focus expressly on the interaction of foreign assets and domestic capital as alternative vehicles of saving, at the cost of eliminating the wealth effect on labor supply" (p.801). In addition, recall that it must be the case, that in any equilibrium $\rho\left(\tilde{d}_t - \overline{d}\right) = \rho\left(d_t - \overline{d}\right)$. Therefore this country specific interest rate premium is strictly increasing if debt exceeds the parameter $\overline{d}$ and vice versa.

3.2 The Steady State

This section characterizes the steady state of the model described above and develops some useful relationships among the variables. The steady state simply describes the resting point of our system, that is, the long run equilibrium of the model economy if there are no stochastic disturbances, i.e. $\varepsilon_t = 0,\ \forall t$.[13] . Let variables without time subscripts denote steady state values and note that the subjective discount factor must satisfy $\beta = 1/(1+r)$.[14] Substituting the functional forms and their respective derivatives (see appendix) into the first order conditions (3.7) – (3.10), leads to the following steady state relationships. Equation (3.9) in the steady state implies that:

$$\beta^{-1} = 1 + r + \rho(d) \;\; \Rightarrow \;\; \psi(\exp^{d-\overline{d}} - 1) = 0 \quad \Rightarrow \; d = \overline{d} \tag{3.16}$$

[13] In a model where all variables but labor are allowed to grow at the same rate as technological progress, e.g. that found in King and Rebelo (2000), the steady state is instead referred to as a balanced growth path.
[14] This is a standard assumption in small open economy models and ensures that the model has well defined dynamics. See Mendoza (p. 799) for an explanation.

Thus, the steady state value of debt d equals the parameter $\bar{d}$. The preceding logic also implies that there is no interest rate premium in the steady state, that is $r_t = r$. Turning to equation (3.10):

$$1 = \beta\left[AF_k(h,k) + 1 - \delta\right] \;\Rightarrow\; \beta^{-1} = \alpha k^{\alpha-1} h^{1-\alpha} + (1-\delta) \;\Rightarrow \frac{h}{k} = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \tag{3.17}$$

To arrive at this last expression, two steady state properties are used: The first is based on $\ln A = \rho \ln A$. Since $\rho \neq 0$, i.e. shocks are *not* assumed to be purely transitory in nature, it must therefore be the case that $A = 1$. The second property simply trades the subjective discount factor for the interest rate definition. Thus, equation (3.17) provides an expression for the steady state labor input to capital ratio as a function of parameters only. Turning to equation (3.8) after substituting the appropriate derivatives:

$$(c - \omega^{-1}h^{\omega})^{-\gamma}(h^{\omega-1}) = (c - \omega^{-1}h^{\omega})^{-\gamma} A(1-\alpha)k^{\alpha}h^{-\alpha} \Rightarrow (h^{\omega-1}) = (1-\alpha)\left(\frac{k}{h}\right)^{\alpha}$$

$$\Rightarrow h = \left[(1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1}{\omega-1}} \tag{3.18}$$

Equation (3.18) shows that the labor input choice in the steady state is independent of consumption decisions. Given parameters, the steady state level of labor input can be calculated and equations (3.17) and (3.18) can then be used to find the steady state level of capital k (by dividing (3.17) by (3.18)). This yields steady state values for investment $i = \delta k$ and output $y = F(h,k)$. In the steady state it must be the case that the current account is zero: $ca = tb - rd = 0 \Rightarrow tb = rd$. With these results, the parameter $\bar{d}$ and hence the steady state level of debt can be pinned down by applying the trade balance ratio definition:

$$\frac{tb}{y} = tby = \frac{rd}{y} = \frac{r\bar{d}}{y}$$

Given an *empirical* average of the trade balance to output ratio (left hand side), and given the calculated steady state value for output and the assumed world interest rate (right hand side), the level of steady state debt that replicates the average trade balance to output ratio can be determined (also see the section on calibration). Given values for $\bar{d} = d$ and $tb = rd$, consumption can be found via the steady state budget constraint $c = y - tb - i$. Lastly, saving is simply given by $s = y - c$ because there are no adjustment costs in the steady state: $\Phi(k - k) = \Phi(0) = 0$.

At this point, it is instructive to include what are referred to as the 'great ratios' in the real business cycle literature, which are pivotal for calibrating the parameters of the next sections. The great ratios are simply empirical variable ratios vis-à-vis output, which have proven fairly immune to time. The ratios are calculated for the countries presented in chapter 2, where C/Y refers to the consumption to output ratio, I/Y refers to the investment to output ratio and TB/Y refers to the trade balance to output ratio.

Table 3.1: Averages of the Great Ratios[(a)]

	C/Y	I/Y	TB/Y
AUS	0.764	0.225	-0.015
BEL	0.791	0.183	0.024
CAN	0.785	0.195	0.025
CZR	0.733	0.283	-0.019
FRA	0.812	0.187	-0.001
GER	0.780	0.204	0.019
MEX	0.806	0.184	-0.006
NEL	0.749	0.205	0.049
POL	0.826	0.190	-0.014
SLR	0.765	0.281	-0.046
SWI	0.717	0.217	0.037
USA	0.848	0.174	-0.024
Averages for Developed Economies[(b)]			
All	0.781	0.199	0.014
SMOPEC	0.761	0.205	0.024
Averages for Emerging Economies			
	0.783	0.235	-0.021

(a) All data is in real terms and quarterly; consumption includes government consumption. See data appendix for more details.(b) Average 'All' for developed countries refers to the averages including the largest three economies the US, Germany, and France, average 'SMOPEC' excludes these.

Table 3.1 shows that the main difference in the great ratios for developed versus emerging economies arises in the net exports category. All emerging economies exhibit an average trade balance deficit while most of the developed economies have a trade balance surplus. Excluding the LOPECs from the set of developed economies therefore implies an average trade balance ratio of 0.024 across space and time and an average of -0.021 for the set of emerging economies. These ratios will be used to calibrate the steady level of debt as described above. In addition, the average investment ratio is higher in the emerging countries than in the developed countries. This ratio will come into play for the calibration of the depreciation rate of capital. Once the trade balance and investment rate ratios have been used for calibration of parameters in the steady state, the steady state consumption ratio must naturally follow.

3.3 Model 1: Calibration and Results for a Developed Economy

In order to calibrate the model presented in sections 3.1 – 3.2 to a hypothetical developed economy, the *averages* obtained in chapter 2 for the developed SMOPECs will be used rather than focusing on a single country. Parameters are assumed to fit into one of three categories: fixed, assumed and free. Fixed parameters are based on prior studies (sometimes these have a fixed range). Assumed parameters reflect certain data or steady state properties. Free parameters make up the remainder and can be set to generate a 'better fit' of the model:

Fixed Parameters:

- Capital's share of output: α=0.32. This is a standard assumption.
- Coefficient of relative risk aversion: γ=2. In the real business cycle literature, a consensus has emerged that $1 \leq \gamma \leq 5$.
- Intertemporal elasticity of labor supply: ω=1.4. The range that is commonly found in the literature is $1.4 \leq \omega \leq 1.7$ (see Neumeyer and Perri (2005) or Mendoza (1991)).

Assumed Parameters:

- World interest rate: $r = 0.015$. This implies an average annual interest rate of 6 percent. Some studies set this as low as 4 percent per annum (i.e. a quarterly rate of 1 percent), but given the higher averages found for SMOPECs in Neumeyer and Perri (2005), 6 percent per annum is within reason.
- Subjective discount factor: β =0.985. This is a forced value based on the steady state assumption that $\beta = 1/(1+r)$.
- The depreciation rate: δ =0.0266. This is in line with the standard value of δ = 0.025 commonly used in the RBC literature and, in the steady state, matches the average investment to output ratio $i/y = \delta k/y = 0.205$ of the developed SMOPECs.
- Steady state level of per capita debt: $\bar{d}$ =17.5703. This matches the average trade-balance to output ratio of the developed SMOPECS using the steady state condition $tby = r\bar{d}/y = 0.024$ where steady state output can be calculated to equal 10.981.

Free Parameters:

The free parameters are (ab)used to generate the 'best fit' of the model, given all other parameter values. They are, however, expected to be in line with theoretical expectations.

- The capital adjustment cost parameter ϕ that moderates investment's response to productivity shocks, is set to match the average standard deviation of investment in the data for SMOPECs, given all other parameters. Mendoza (1991) obtains a range of $0.023 \leq \phi \leq 0.028$ for his annual model of Canada. Incidentally, the best results for the volatility of investment in the quarterly model at hand are obtained by setting ϕ =0.024/4=0.006.
- The interest rate rule parameter ψ, "measuring the sensitivity of the country interest-rate premium to deviations of external debt from trend…[should be]…assigned[ed] a small value… with the sole purpose of ensuring independence of the deterministic steady state from initial conditions, without affecting the short-run dynamics of the model." (García-Cicco, et al. (2006), p. 8). Following this recommendation ψ is set to 0.0001.
- Lastly, and perhaps most controversially, the productivity process's persistence parameter ρ and the associated shock variance σ_{ε}^{2} are set with the sole purpose of replicating the empirical average volatility and first-order autocorrelation of output for the developed SMOPECs, given the other assigned parameter values. A formal calculation of Solow

residuals would have been an alternative, but given that this method has, at times proven problematic, the present study follows Mendoza (1991) and uses these as free parameters. The 'best fit', given all other parameters is found for $\rho = 0.715$ and $\sigma_\varepsilon = 0.00383$, which are both plausible values.

Table 3.2 summarizes the above discussion:

Table 3.2: Parameter Values for Model 1 (Developed Economy)

Variable	Description	Value
α	capital share in output	0.32
β	rate of time preference/discount factor, set to $1/(1+r)$	0.985
δ	capital's depreciation rate	0.0266
$\bar{d}$	steady state foreign debt level, matches SMOPEC average *tby*	17.5703
γ	coefficient of relative risk aversion	2
ϕ	capital adjustment cost parameter	0.006
ψ	interest rate rule parameter	0.0001
r	steady state world interest rate	0.015
ρ	productivity process persistence parameter	0.715
σ_ε	standard deviation of technological innovation	0.00383
ω	1 + inverse of intertemporal elasticity of substitution in labor	1.4

The first set of model 1's results examined in this section are impulse response functions, which use each variable's policy or transition function to generate a response curve to a productivity shock (see section 6.2.3 for an explanation of policy or transition functions and their link to impulse response functions). This is a theoretical exercise since impulse response functions are not observed empirically. However, they ought to coincide with theoretical expectations (e.g. productivity increases should translate into output gains) and can therefore serve as a rough mechanism with which to judge the performance of the model for a hypothetical developed economy. A more precise mechanism is given by the second set of results in table 3.3 which compares the empirical business cycle statistics obtained in chapter

2 with the theoretical moments generated by the model and thus allows the model's predictions to be weighed against real world occurrences.

3.3.1 Impulse Response Analysis for Model 1

Figure 3.1 depicts how each of the variables responds to a one percent increase in the productivity process at $t=1$, i.e. $\hat{\varepsilon}_1 = 1$ over a twenty-five year period. The initial response to a positive productivity shock calls for a distinction between the behavior of 'sluggish' and 'jump' variables: Sluggish variables can not adjust contemporaneously to the shock. In the present model these are represented by the endogenous state variables debt and capital, which are pre-determined in each period, including $t=1$, and hence do not adjust until one period after the shock has occurred. Investment, even though it is solely a function of the capital stock, can adjust immediately, because it is defined by the regular and auxiliary capital stock. Recall that the latter can instantaneously respond to shocks because it is defined by next period's regular capital stock. Similarly, the debt elastic interest rate adjusts immediately because it is defined in terms of next period's debt. The remaining control and flow variables are determined at the beginning of each period, allowing them to adjust contemporaneously to shocks. These represent the set of jump variables.

Figure 3.1: Impulse Response Functions of the Developed Economy (Model 1) after a One Percent Increase in the Technological Innovation

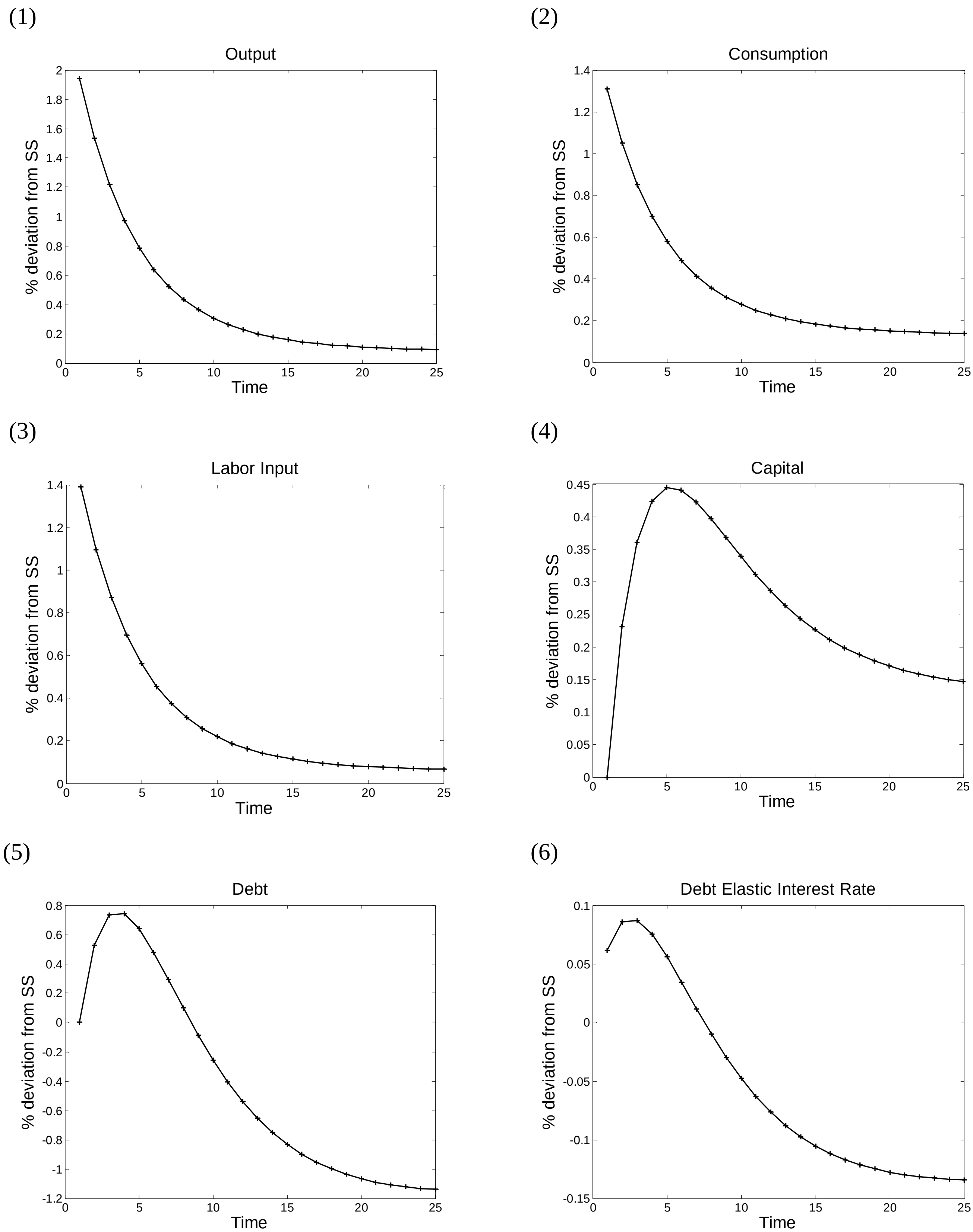
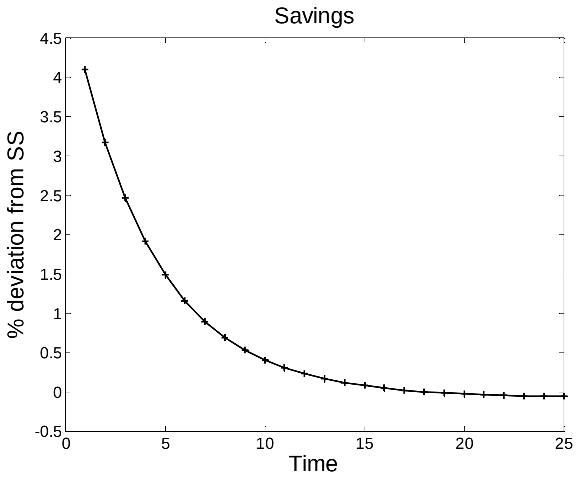
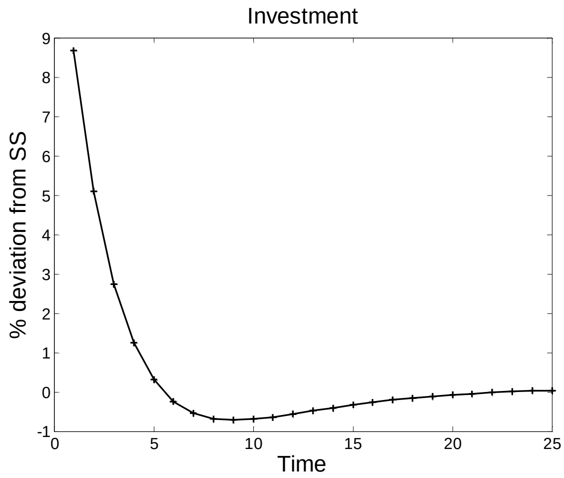


Figure 3.1 (continued)

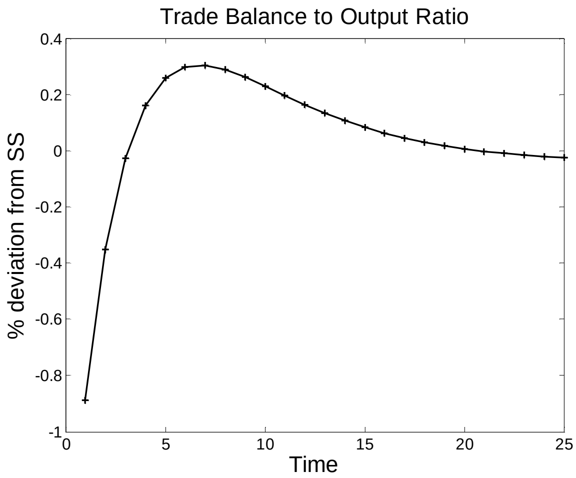
(7)



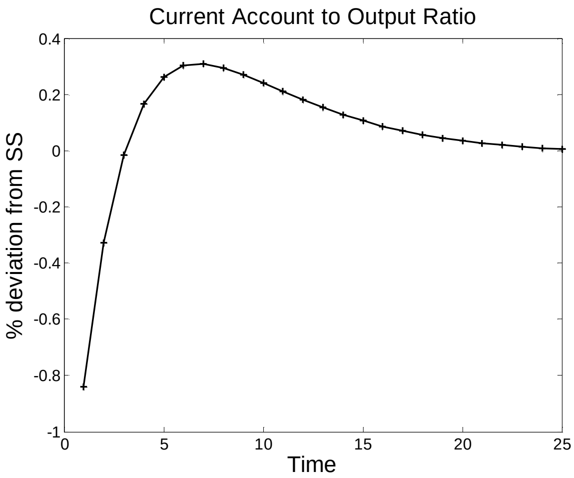
(8)



(9)



(10)



In panels (1), (2) and (3), the positive effect of the technological innovation is reflected in the law of motion for consumption, labor input and output respectively. Most obviously, the shock translates into increases in output via the production function, raising output by 1.94 percent in the short and 0.92 percent in the long run. The long run effect is attributable to the fact that capital and labor input remain permanently higher, allowing production to remain perpetually above its initial steady state level even though the productivity process effect dissipates. Because of the increases in output, consumption naturally also increases – a well known feature of any basic consumption function: Because the marginal propensity to consume is less than one and because the marginal utility of consumption is decreasing, consumption increases by less than output (initially by 1.31 percent and in the long run by

0.14 percent above its steady state value). The remaining output is saved for a rainy day, a course of action known as ‘consumption smoothing’.

Labor input increases by 1.39 percent initially and remains above its steady state value by a factor of 0.07 percent. The fact that labor input increases in response to a productivity shock is similar to the capital stock effect. Because the marginal product of labor schedule $A_t F_h(k_t, h_t)$ dictating the wage rate is positively affected by increases in productivity, it raises the opportunity cost of taking leisure. This stimulating effect on labor input also declines as the effect on the productivity process wears out, causing the wage at the beginning of the productivity propagation cycle to be higher relative to future expected wages. Usually the wealth effect on labor supply describing a positive correlation between higher income levels and taking leisure would also have to be considered. The choice for the utility function, however, ensures that this wealth effect on labor supply is eliminated.

As can be inferred from panels (4) and (5), the variables capital and debt are sluggish variables that do not adjust contemporaneously to the shock (because their graphs do not ‘jump’ at t=1). Note that the interest rate in panel (6) follows the impulse response function of debt due to its functional form but on a smaller scale, because its sensitivity to changes in debt from trend is moderated by the parameter ψ. Initially both debt and the interest rate respond positively and reach their peak at t=3 by approximately 0.8 and 0.1 percent above their steady state values respectively. In the long run, however, a positive productivity shock should have a negative effect on an initially (steady state) positive debt level and hence the interest rate, because higher domestic productivity increases output and lessens the need to borrow output from abroad. This is reflected in the impulse response functions for both variables after $t \approx 8$.

In panel (4), the capital stock’s initial as well as long run response to increases in the productivity process is positive throughout. This is due to the stimulating effect of the technological innovation on domestic investment, which in turn determines the level of the domestic capital stock. Also note that the rental price of capital schedule given by $A_t F_k(k_t, h_t)$ is shifted upward because productivity and labor input (panel 5) immediately increase. This mechanism makes investment in domestic capital profitable and hence works to increase the capital stock. As capital is build up and the productivity and labor input effects dissipate, the marginal product of capital begins to fall again.

Panels (7) – (8) show that upon impact of the shock, saving increases by 4.08 and investment increases by 8.68. Saving slightly dips below its steady state as the consumption smoothing effect dissipates and the consumption increases and capital adjustment processes begin to outweigh the output increases. Investment, on the other hand, goes through a period where it is below its steady state value but then bounces back above its steady state. To account for the intermediate dynamics, King and Rebelo (2000) observe that, “[i]nvestment… drops below the steady state, as the economy runs down the capital that was accumulated during the initial expansion” (p.38). In the long run, however, domestic investment re-approaches and then remains above its steady state.

Recall that the current account is defined as the trade balance minus interest payments (the interest rate times the debt level), or as saving minus investment. Given that investment increases by more than saving, implies that the current account must first incur a higher deficit (or a lower surplus). This is financed by the trade balance. Since saving remains below its steady state value while investment remains above and since debt and the interest rate simultaneously remain below their respective steady state values, it must be the case that the trade balance incurs a higher deficit (or a reduced surplus). This is reflected in panel (9) – (10), where the long run trade balance remains slightly below its steady state value by a factor of -0.02 percent, while the current account matches the behavior of the trade balance but remains above its steady state value by a factor of 0.01 percent.

3.3.2 Empirical vs. Theoretical Business Cycle Moments for Model 1

Table 3.3 presents business cycle summary statistics generated by model 1 for a developed economy and compares these to the relevant average statistics obtained for developed SMOPECs in chapter 2.

Table 3.3: Business Cycle Summary Statistics: Model 1 vs. Average of Developed SMOPECs[(a)]

Variable (x)	Source	$\sigma(x_t)$	$\frac{\sigma(x_t)}{\sigma(y_t)}$	$\rho(x_t, y_t)$	$\rho(x_t, x_{t-1})$
Output (y)	Model 1	1.28	1.00	1.00	0.82
	Data	1.28	1.00	1.00	0.84
Consumption (c)	Model 1	1.04	0.81	0.96	0.87
	Data	1.02	0.80	0.69	0.80
Labor Input (h)	Model 1	0.92	0.71	1.00	0.82
	Data	1.40	1.09	0.61	0.93
Capital stock (k)	Model 1	0.76	0.59	0.62	0.99
	Data	0.48	0.37	0.18	0.96
Saving (s)	Model 1	2.50	1.94	0.92	0.78
	Data	4.45	3.46	0.81	0.71
Investment (i)	Model 1	4.11	3.18	0.79	0.58
	Data	4.11	3.24	0.76	0.77
Trade Bal. Ratio (tby)	Model 1	0.50	0.39	-0.35	0.63
	Data	0.90	-	-0.31	0.59
Current Axt. Ratio (cay)	Model 1	0.47	0.36	-0.25	0.62
	Data	1.21	-	-0.28	0.45
Correlation (s, i)	Model 1	0.83	(a) Values reported in the data rows correspond to the values calculated in chapter 2. All standard deviations are in percent per quarter		
	Data	0.54			

Obviously output performs very well with respect to its standard deviation (1.28 percent) or autocorrelation (0.82 versus. 0.84 in the data), because its parameters were set in such a way to fit the empirical findings. The same is true for investment since parameters were also set to match its standard deviation (4.11 percent). Independently, its contemporaneous correlation with output and its first order autocorrelation also match the data fairly well, although the latter statistic is somewhat understated by the model (0.79 versus. 0.76 and 0.58 versus. 0.77). Consumption’s variability, however, is accurately and independently predicted by the model even though no parameters were set in order to reproduce the data findings (1.04 versus 1.02 percent). As is common in real business cycle models in open economies, the contemporaneous correlation between consumption and output is exaggerated by the model (0.96 versus 0.69 in the data). All other statistics of consumption are well matched.

Although the model understates the *average* variability of labor input in SMOPECs (0.92 versus 1.40 percent), it is in line with certain individual countries' standard and relative deviations (i.e. Belgium, France, Germany and Switzerland; see table 2.1). An inevitable blemish of the model is that it predicts a perfect correlation between output and labor input, which is due to the specification of the utility function. Conversely, the model overstates the variability of capital (0.76 versus 0.48) but again is not too far fetched for some individual country observations (i.e. Canada and Germany). The model predicts the serial autocorrelation remarkably well (0.99 versus 0.96 in the data) but highly overstates the contemporaneous correlation with output (0.62 versus 0.18 in the data).

The theoretical predictions regarding the trade balance and current account ratios fare relatively poorly when held up against the data. In both cases the volatility is clearly understated. However, the model does a good job in replicating the negative contemporaneous correlation relative to output. For the trade balance, for example, the empirical correlation is -0.31, in the model it is -0.35. In addition, the lower serial correlation relative to the remaining model variables is also captured by the model. Lastly, the volatility of saving is clearly understated by the model, which may have to do with the difference in definitions used in the model versus the data (i.e. if government consumption is the main contributor to the empirical volatility in saving, this would not be reflected by the model since no government sector is included). This may also be the reason why the contemporaneous correlation between saving and investment in the last row is higher in the model (0.83) versus the data (0.54).

3.4 Model 2: Calibration and Results for an Emerging Economy

This section mirrors section 3.3, but now describes the emerging economy where all variables and parameters that differ across the two model types are denoted with an asterisk. Since it is a mirror economy, all explanations are the same as in section 3.3 and only the formal description is provided.

The representative agent in the emerging economy solves the following optimization problem:

$$\max_{\left\{c_t^*, h_t^*, d_{t+1}^*, k_{t+1}^*\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \left(\beta^*\right)^t U(c_t^*, h_t^*) \tag{3.19}$$

subject to:

$$d_{t+1}^* = (1+r_t^*)d_t^* - y_t^* + c_t^* + i_t^* + \Phi(k_{t+1}^* - k_t^*) \tag{3.20}$$

$$y_t^* = A_t^* F(h_t^*, k_t^*) \tag{3.21}$$

$$k_{t+1}^* = i_t^* + (1-\delta^*)k_t^* \tag{3.22}$$

$$r_t^* = r^* + \rho^*\left(\tilde{d}_t^*\right) \tag{3.23}$$

$$\ln\left(A_{t+1}^*\right) = \rho \ln\left(A_t^*\right) + \varepsilon_{t+1}^*;\ \forall \varepsilon_{t+1}^* \sim NIID(0, \sigma_\varepsilon^{*2}),\ t \geq 0 \tag{3.24}$$

The first order conditions are:

$$U_c^*(c_t^*, h_t^*) = \lambda_t^* \tag{3.25}$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \tag{3.26}$$

$$\lambda_t^* = \beta(1+r_t^*)E_t \lambda_{t+1}^* \tag{3.27}$$

$$\lambda_t^*\left[1+\Phi'(k_{t+1}^* - k_t^*)\right] = \beta E_t \lambda_{t+1}^*\left[A_{t+1}^* F_k(h_{t+1}^*, k_{t+1}^*) + 1 - \delta + \Phi'(k_{t+2}^* - k_{t+1}^*)\right] \tag{3.28}$$

$$\lim_{t\to\infty} \left(\beta^*\right)^t \lambda_t^* \left(k_{t+1}^* - d_{t+1}^*\right) = 0 \tag{3.29}$$

The additional flow variables and the auxiliary capital stock are given by:

$$s_t^* = y_t^* - \Phi^*\left(k_{t+1}^* - k_t^*\right) - c_t^* \tag{3.30}$$

$$tby_t^* = 1 - \left(\frac{c_t^* + i_t^* + \Phi\left(k_{t+1}^* - k_t^*\right)}{y_t^*}\right) \tag{3.31}$$

$$cay_t^* = tby_t^* - \frac{r_t^* d_t^*}{y_t^*} \tag{3.32}$$

$$k_t^{a*} - k_{t+1}^* = 0 \tag{3.33}$$

A rational expectations equilibrium for the emerging county can be defined as a set of processes $\left\{\lambda_t^*, d_{t+1}^*,\ \tilde{d}_{t+1}^*,\ k_{t+1}^*, c_t^*,\ h_t^*\right\}_{t=0}^{\infty}$ that satisfy the first order conditions (3.25) – (3.28), the constraints (3.20) – (3.23) and the flow variable equations (3.30) – (3.32) given the exogenous productivity process (3.24) as well as the starting values for the state variables $A_0^*, \tilde{d}_0^*\,(d_0^*)$ and k_0^*. The emerging economy is given the same functional forms for utility, the production function, the adjustment cost function and the interest rate premium as the developed economy. Note, however, that the steady state values and some of the parameters differ.

$$U(c^*, h^*) = \frac{(c^* - \omega^{-1} h^{*\omega})^{1-\gamma} - 1}{1-\gamma}$$

$$F(k,h) = \left(k^*\right)^{\alpha}\left(h^*\right)^{1-\alpha}$$

$$\Phi(x) = \frac{\phi^*}{2} x^2$$

$$\rho(\tilde{d}^*) = \psi^* (\exp^{\tilde{d}^* - \bar{d}^*}\text{-}1)$$

Again, let variables without time subscripts denote the steady state and assume that $\beta^* = 1/(1+r^*)$. Equation (3.27) yields:

$$\rho(d^*) = 0 \;\Rightarrow\; d^* = \bar{d}^* \tag{3.34}$$

The labor input to capital stock ratio is obtained from equation (3.28):

$$\frac{h^*}{k^*}=\left(\frac{r^*+\delta^*}{\alpha}\right)^{\frac{1}{1-\alpha}} \tag{3.35}$$

Lastly, labor input can be calculated using equation (3.26):

$$h^*=\left[(1-\alpha)\left(\frac{\alpha}{r^*+\delta^*}\right)^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1}{\omega-1}} \tag{3.36}$$

The remaining steady state variables can be found using the procedure described in section 3.2. Recall that the great ratios for the emerging economies were given by a consumption to output ratio of 0.783, an investment to output ratio of 0.235 and a trade-balance to output ratio of -0.021. The parameters that are assigned different values relative to the developed economy are:

- World interest rate: $r^*=0.0325$. This implies an annual interest rate of 13 percent, which is an intermediate value between the average Mexican real interest rate of 10.4 percent and the real interest rate value used by Neumeyer and Perri (for the Argentine economy) equaling 14.8 percent. Whether this represents the set of Eastern European emerging economies is up for debate. However, the value is in line with the fact that an emerging economy experiences higher real interest rates than a developing economy.
- The above implies that $\beta^*=1/(1+r^*)=0.969$.
- To replicate the average empirical investment to output ratio of 0.235, implies that the quarterly depreciation rate must be $\delta^*=0.0895$. This results in an annual depreciation rate of 36 percent, which stands in stark contrast to the usual assumption that capital depreciates at an annual rate of 10 percent. However, perhaps capital in emerging economies behaves or is used differently than capital in developed economies. To set the depreciation rate equal to the value assumed for developed economies (0.0266), given the above parameters, amounts to generating a very low steady state investment ratio of 0.144 in the emerging countries.

- Using the previous parameter values and noting that the steady state trade balance ratio should be equal to -0.021, implies that $\bar{d}* = \text{-}1.2061$.
- Given that $\delta^* \neq \delta$, it is also the case that the capital adjustment cost parameter differs across models 1 and 2 ($\phi^* \neq \phi = 0.006$). To match the empirical volatility of investment for emerging economies implies that $\phi^* = 0.064/4 = 0.016$. Again, this may be contentious, but if capital depreciates at a higher rate in emerging economies, it also makes sense that the capital adjustment cost parameter is higher.
- For the above parameters, the best results are obtained when $\rho^* = 0.540$ (to match output's average first order correlation of 0.761) and $\sigma_\varepsilon^* = 0.00633$.

In sum:

Table 3.4: Parameter Values for Model 2 (Emerging Economy)

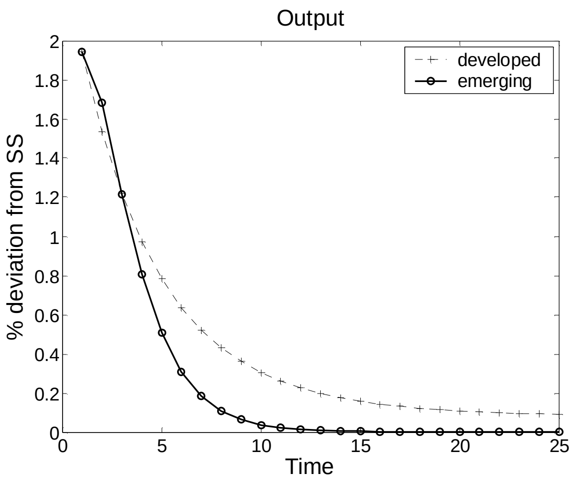
Variable	Description	Value
$\alpha^* = \alpha$	capital share in output	0.32
β^*	rate of time preference/discount factor, set to $1/(1+r)$	0.969
δ^*	capital's depreciation rate	0.0895
$\bar{d}^*$	steady state foreign debt level, matches SMOPEC average *tby*	-1.2061
$\gamma^* = \gamma$	coefficient of relative risk aversion	2
ϕ^*	capital adjustment cost parameter	0.0161
$\psi^* = \psi$	interest rate rule parameter	0.0001
r^*	steady state world interest rate	0.0325
ρ^*	productivity process persistence parameter	0.540
σ_ε^*	standard deviation of technological innovation	0.00633
$\omega^* = \omega$	1 + inverse of intertemporal elasticity of substitution in labor	1.4

3.4.1 Impulse Response Analysis for Model 2

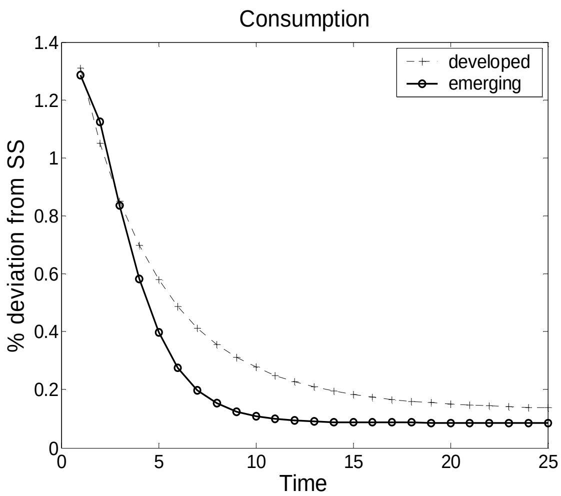
Figure 3.2 is analogous to figure 3.1 in section 3.3 and shows how the emerging economy responds to a one percent increase in the productivity process at $t = 1$, i.e. $\hat{\varepsilon}_1^* = 1$ over a twenty-five year period. For comparison the impulse responses of the developed economy are also plotted with a dashed line except for the interest rate in panel (6), where the scale on the vertical axis is much smaller in model 2 than in model 1.

Figure 3.2: Impulse Response Functions of the Emerging Economy (Model 2) after a One Percent Increase in the Technological Innovation

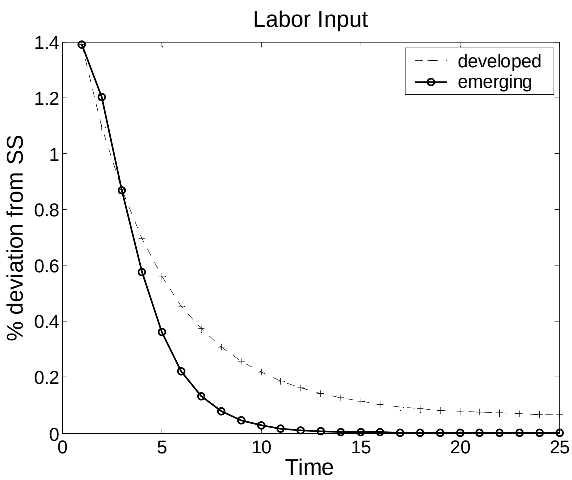
(1)



(2)



(3)



(4)

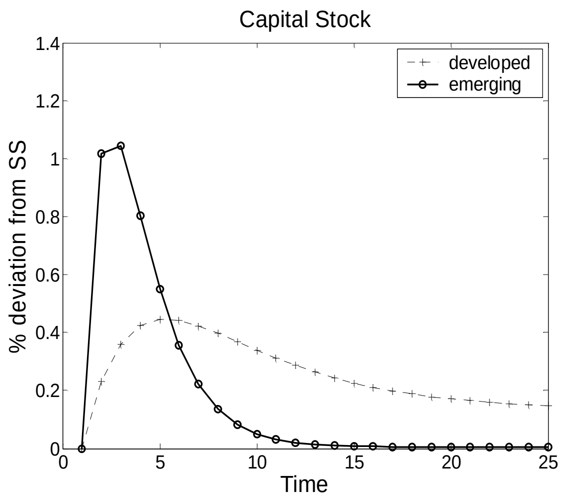
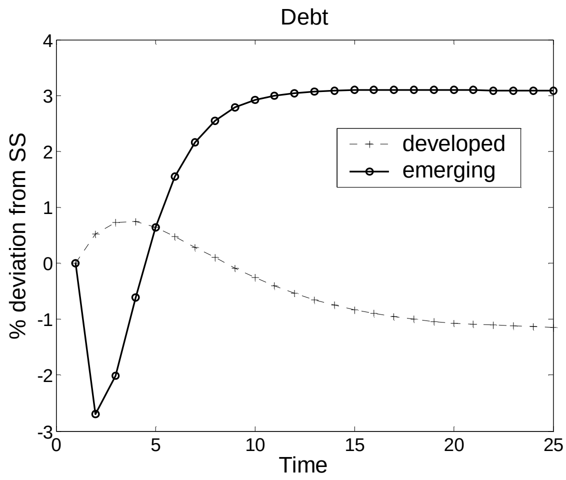
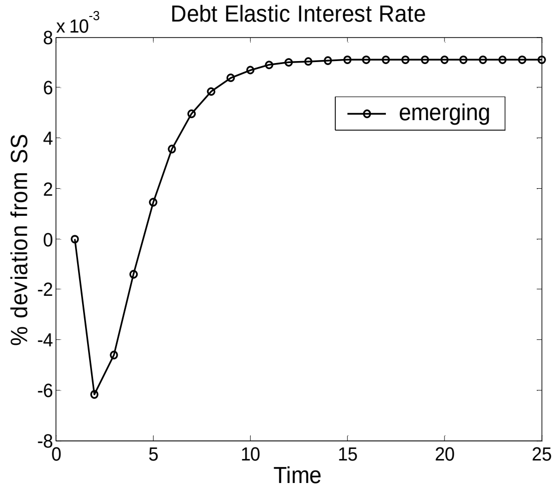


Figure 3.2 (continued):

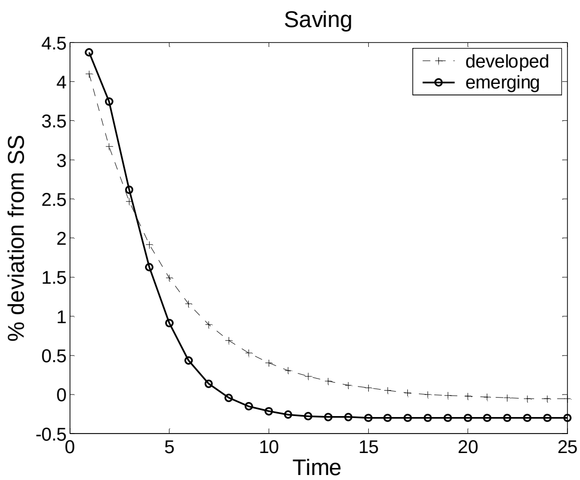
(5)



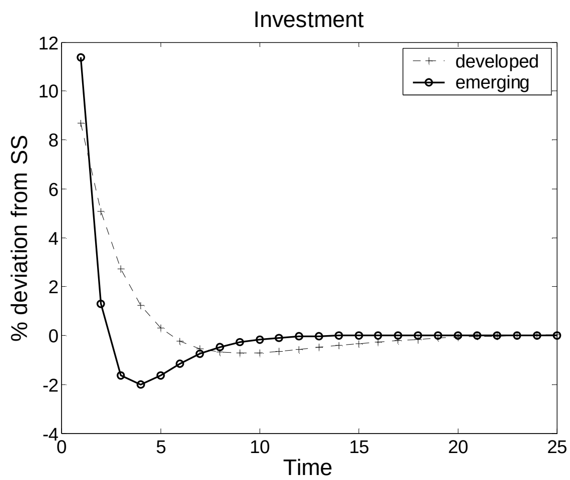
(6)



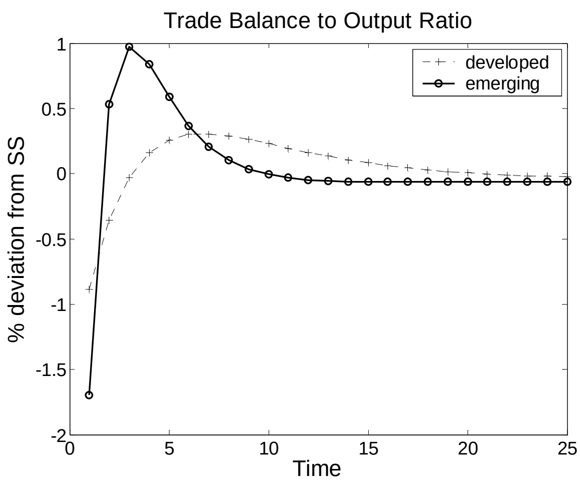
(7)



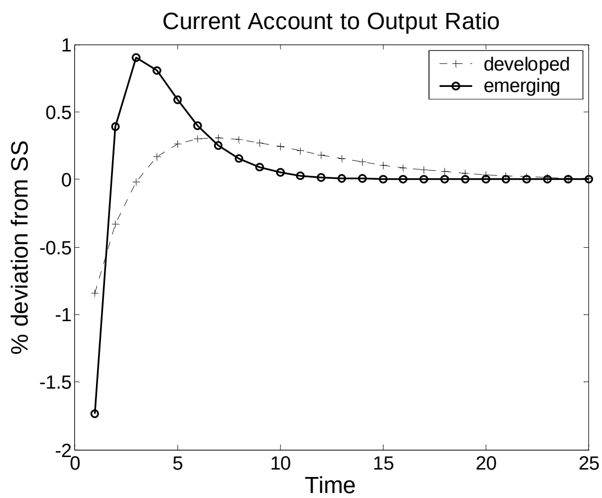
(8)



(9)



(10)



The majority of the impulse response functions show that the emerging economy's variables react stronger than the developed economy's variables, which, in turn, implies greater variable volatility in the emerging economy: The capital stock's initial response to a technological innovation in panel (4) is approximately 1 percent versus 0.4 percent in the developed economy. Interestingly, the model predicts that the positive effect on the capital stock in the emerging economy virtually disappears, while capital clearly remains above the steady state in the developed economy. The same observation holds for labor input in panel (3): The initial and intermediate dynamics are virtually the same, but in the long run, labor input in the emerging economy re-approaches its previous steady state. These two observations logically imply that output in panel (1) also follows this pattern. Consumption depicted in panel (2), however, remains above its previous steady state in the long run similar to the developed economy (by 0.1 percent and 0.2 percent respectively) The initial response of debt in panel (5) is negative, implying that the level of assets is increasing since steady state debt is given by a negative number. Therefore it must be the case that $d_{t+1}^{*} - \bar{d}^{*} < 0 \Rightarrow r_t^{*} \downarrow$ until $t \approx 4$ and $d_{t+1}^{*} - \bar{d}^{*} > 0 \Rightarrow r_t^{*} \uparrow$ thereafter. This is depicted in panel (6). As a result of output returning to its previous steady state and consumption remaining relatively higher, saving in panel (7) remains below its steady state value. Investment in panel (8) responds much stronger in the emerging economy than in the developed economy, with an initial increase of approximately 12 percent versus 9 percent in the developed economy. In the long run, investment in both types of economies re-approaches its steady state. Lastly, the trade balance and current account ratios (panel (9)–(10)) initially decrease to a greater extent in the emerging economy (1.75 percent) than in the developed one (0.7 percent) and generally display greater volatility. In the long run, both variables re-approach their former steady state in both types of economies.

3.4.2 Empirical vs. Theoretical Business Cycle Moments for Model 2

Table 3.5 presents summary business cycle statistics generated by model 2 and compares these to the relevant average statistics obtained for emerging economies in chapter 2.

Table 3.5: Business Cycle Summary Statistics: Model 2 vs. Average of Emerging Economies[(a)]

Variable *x*	Source	$\sigma(x_t^*)$	$\frac{\sigma(x_t^*)}{\sigma(y_t^*)}$	$\rho(x_t^*, y_t^*)$	$\rho(x_t^*, x_{t-1}^*)$
Output (y^*)	Model 2	1.92	1.00	1.00	0.76
	Data	1.92	1.00	1.00	0.76
Consumption (c^*)	Model 2	2.17	1.13	0.63	0.92
	Data	2.12	1.11	0.61	0.68
Labor Input (h^*)	Model 2	1.37	0.71	1.00	0.76
	Data	-	-	-	-
Capital stock (k^*)	Model 2	1.15	0.60	0.75	0.81
	Data	-	-	-	-
Saving (s^*)	Model 2	7.35	3.84	0.54	0.92
	Data	8.33	4.43	0.49	0.57
Investment (i^*)	Model 2	7.56	3.95	0.53	0.16
	Data	7.56	4.06	0.72	0.74
Trade Bal. Ratio (tby^*)	Model 2	1.97	1.03	-0.03	0.58
	Data	2.41	-	-0.43	0.73
Current Axt. Ratio (cay^*)	Model 2	1.45	0.76	-0.06	0.23
	Data	2.39	-	-0.39	0.63
Correlation (s^*, i^*)	Model 2	0.26	(a) Values reported in the data rows correspond to the values calculated in chapter 2. All standard deviations are in percent per quarter		
	Data	0.31			

Output was once again calibrated in such a way that its standard deviation (1.92) and autocorrelation (0.76) match the data. Investment also matches the empirical volatility (7.56) as a result of the calibration. The contemporaneous correlation of investment and output is somewhat understated by the former (0.53 versus 0.72 in the data) and the predicted first order autocorrelation is much lower (0.18) than in the data (0.74). Consumption again performs remarkably well (with a theoretical standard deviation of 2.17 versus an empirical one of 2.12), albeit its first order autocorrelation is too high in the model. Note that this time, however, the consumption-output correlation is not overstated as in model 1: The model predicts an autocorrelation coefficient of 0.63, the data produces a coefficient equal to 0.61.

Although no data on labor input was computed for the set of emerging economies, the volatility statistics in model 2 are extremely close to the empirical findings for model 1. For example, the volatility of labor input in the data of developed countries is given by 1.40. Model 2 actually predicts 1.37! In the sense that labor input might behave similarly across economy types, model 2 actually does a better job of forecasting volatility than model 1. The capital stock is more variable than was the case for the set of developed economies (1.15) versus (0.48) and its contemporaneous correlation with output is again overstated given the acyclical nature of the capital stock (0.75 versus 0.18 in the data on developed economies).

Even though model 2 is not able to capture the volatility of the trade balance and current account ratio completely (1.97 versus 2.41 and 1.45 versus 2.39 respectively), it performs relatively well in the sense that these variables are much more volatile in comparison to model 1. Empirically, the trade balance and current account ratio's contemporaneous correlations with output are more negative for emerging economies than for developed economies. A disappointing result is that this is by no means reflected in model 2's results. The empirical trade balance ratio-output correlation, for instance, is -0.43, model 2 predicts -0.03. Again the low first order correlations of the two international variables are correctly forecasted, although the magnitude is not altogether perfect.

The implied volatility of saving (7.35), although less than in the data (8.33), is closer to the truth than was the case in model 1, where the predicted and actual volatility deviate by almost 2 percent. The contemporaneous correlation with output is again well matched (0.54 versus 0.49 in the data). Perhaps the most welcome outcome is that the low correlation between saving and investment in emerging economies, which was overstated by model 1 for the developed economies, is properly captured by model 2: The implied statistic is 0.26, the data produces a value of 0.31.

Given the findings of the previous two sections, it can be concluded that a small open economy real business cycle model can be applied to both a developed and an emerging economy. Neither model can capture all the statistical features, but some key business cycle statistics are accurately replicated. This contradicts García-Cicco, et al.'s (2006) finding that a debt elastic interest rate model (with growth) can not forecast an emerging countries' business cycles.

3.5 Model 3: Calibration and Results for a Two-Country Model of a Developed and Emerging Economy

This section creates a two-country international real business cycle model for a developed and an emerging economy. In contrast to the previous two sections, a model with portfolio adjustment costs to debt holdings rather than an interest rate premium is used to induce stationarity. This is quantitatively and qualitatively almost identical to the debt elastic interest rate model (see Schmitt-Grohé and Uribe (2003)), except that both countries always face the same real interest rate. Obviously, this would not be the case for models in which two separate interest rate premia exist for each type of economy and in which the resultant diverging real interest rate series would not be interpretable.

The representative agents' optimization problems remain individual maximization problems, since this is again an incomplete asset market model and, as a result, the need for a social planner is eliminated. An incomplete asset market model is chosen since, empirically, output and consumption between developed and emerging markets are negligibly if not negatively correlated (see chapter 2), a fact that would not be captured by a complete asset market model (see Kollmann (1996)).

In this context, five equations of each of the previous two debt elastic interest models need to be modified and one additional equation arises: Portfolio adjustment costs to debt holdings are introduced to both budget constraints, which also affect each economy's consumption Euler and current account equation. In addition, a common steady state (world) interest rate is specified and the exogenous productivity process is now modeled as a bivariate autoregression in logarithms. Lastly, a 'size' condition needs to be included, assigning weights to the per capita debt levels of each country. Note that that for simplicity labor is assumed to be immobile.[15]

In the developed economy, the representative agent maximized (3.1) subject to (3.2) – (3.5) while taking the exogenous productivity process and starting values for state variables as given. His first order conditions were given by equations (3.7) – (3.11), and the constraints (3.2) – (3.5) as well as the flow variable and auxiliary capital stock equations (3.12) – (3.15) holding with equality. The budget constraint (3.2) for the developed economy is now replaced by a budget constraint that includes quadratic portfolio adjustment costs to debt holdings:

$$d_{t+1} = (1+r_t)d_t - y_t + c_t + i_t + \Phi(k_{t+1} - k_t) + \frac{\mu}{2}\left(d_{t+1} - \bar{d}\right)^2 \tag{3.37}$$

Note the similarity between the adjustment costs and the interest rate premia of sections 3.1 and 3.2 in the sense that in the steady state, where $d_{t+1} = \bar{d}$, adjustment costs are zero just as the interest rate premium was zero. Both the interest rate premia and the adjustment cost function therefore pin down the steady state level of debt. Due to the new budget constraint the Euler equation that now replaces (3.9) is given by:

$$\lambda_t \left[1 - \mu(d_{t+1} - \bar{d})\right] = \beta(1+r_t)E_t\lambda_{t+1} \tag{3.38}$$

Recall that the current account was defined in terms of the change in net foreign assets. As a result of the new definition of the budget constraint, it must be the case that: $-\Delta d_{t+1} = ca_t = -r_t d_t + y_t - c_t - i_t - \Phi(k_{t+1} - k_t) - \frac{\mu}{2}\left(d_{t+1} - \bar{d}\right)^2$. The current account to output ratio that replaces (3.14) is thus given by:

$$cay_t = \left(\frac{1}{y_t}\right)\left[-r_t d_t + tb_t - \frac{\mu}{2}\left(d_{t+1} - \bar{d}\right)^2\right] \tag{3.39}$$

Analogously, the representative agent in the emerging economy maximized (3.19) subject to (3.20) – (3.23) taking the exogenous productivity process and starting values for state

[15] Despite the labor mobility granted by the European Union, the assumption that labor is immobile between any two countries is standard in these types of models. A potentially new research idea would be to model labor as mobile in this context.

variables as given. His first order conditions were given by (3.25) – (3.29), and the constraints (3.20) – (3.23) and flow and auxiliary variable equations (3.30) – (3.33) holding with equality. His new budget constraint that replaces equation (3.20) is:

$$d^*_{t+1} = (1+r^*_t)d^*_t - y^*_t + c^*_t + i^*_t + \Phi(k^*_{t+1} - k^*_t) + \frac{\mu^*}{2}\left(d^*_{t+1} - \bar{d}^*\right)^2 \tag{3.40}$$

The new Euler equation that replaces (3.27) is:

$$\lambda^*_t\left[1 - \mu^*(d^*_{t+1} - \bar{d}^*)\right] = \beta(1+r^*_t)E_t\lambda^*_{t+1} \tag{3.41}$$

The new current account to output ratio that replaces (3.32) is:

$$cay^*_t = \left(\frac{1}{y^*_t}\right)\left[-r^*_t d^*_t + tb^*_t - \frac{\mu}{2}\left(d^*_{t+1} - \bar{d}^*\right)^2\right] \tag{3.42}$$

The similarity between models 1 and 2 versus model 3 arises because the log-linearized versions of equations (3.9) and (3.38) or (3.27) and (3.41) are proportionate: For the developed economy, the first Euler equation is $\hat{\lambda}_t = (1+r)^{-1}\psi\bar{d}\hat{d}_{t+1} + E_t\hat{\lambda}_{t+1}$ while the second is given by $\hat{\lambda}_t = \mu\bar{d}\hat{d}_{t+1} + E_t\hat{\lambda}_{t+1}$. For values $\mu \approx (1+r)^{-1}\psi$ the two equations will therefore produce similar results. The same holds for the emerging economy using the appropriate notation with asterisks.

Lastly, the interest rate premium equations are eliminated (equations (3.5) and (3.23)) in favour of a constant world interest rate that holds for both types of economies $\forall t$:

$$r_t = r^*_t = \bar{r} \tag{3.43}$$

One mathematical issue that arises in the context of a two-country model is that in a well defined equilibrium, the asset market must clear in all periods. This implies that one country's assets must be another country's debt. For *equal sized* countries, this usually implies an equation of the type $d_t - d^*_t = 0,\ \forall t$. In other words, the world's net per capita debt level denoted by $D_t^{tot} = d_t - d_t^* =$ 0 in each period. However, the two types of economies investigated here are not equal sized. Therefore the size constraint is given by:[16]

$$D_t^{tot} = \pi d_t + (1-\pi) d_t^* = 0 \Rightarrow d_t^* = \frac{\pi}{(\pi - 1)} d_t \text{ , } \forall t \text{ , } 0 \leq \pi \leq 1 \tag{3.44}$$

Then it must be the case that in the steady state:

$$\bar{d}^* = \frac{\pi}{(\pi-1)} \bar{d} \tag{3.45}$$

Now recall that the developed economies, on average, were characterized by a trade balance surplus, which implied an average empirical trade balance to output ratio of 0.024. The emerging economies, on the other hand, exhibited an average empirical trade balance deficit and a corresponding trade balance to output ratio of -0.021. To determine the size parameter $0 \leq \pi \leq 1$, consider the following steady state relationships:

$$\bar{d} = \frac{tby \times y}{\bar{r}} = \frac{0.024 \times y}{\bar{r}} \quad \text{and} \quad \bar{d}^* = \frac{tby^* \times y^*}{\bar{r}} = \frac{-0.021 \times y^*}{\bar{r}} \text{ .}$$

Since $\bar{d}^* = \frac{\pi}{(\pi-1)} \bar{d}$, for given values of $\bar{r}$, y and y^* it can be shown that:[17]

$$\pi = \frac{0.021 \times y^*}{\overline{rd} + 0.021 \times y^*} \approx 0.346$$

For this value of π, the steady state of each economy in this two-country model replicates the average empirical trade balance ratios of the developed and emerging economy. The respective steady state values of per capita debt are $\bar{d} \approx$ 17.570 and $\bar{d}^* \approx$ -9.285.

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[16] According to Walra's law, if asset markets clear, good markets clear as well.

[17] $\bar{d}^* = \left(\frac{\pi}{\pi-1}\right)\bar{d} = \frac{-0.021 \times y^*}{\bar{r}} \Rightarrow \left(\frac{\pi}{\pi-1}\right) = \frac{-0.021 \times y^*}{\overline{rd}} \Rightarrow \left(\frac{\pi-1}{\pi}\right) = \frac{\overline{rd}}{-0.021 \times y^*} \Rightarrow \left(-\frac{1}{\pi}\right) = \frac{\overline{rd}}{-0.021 \times y^*} - 1$

$\Rightarrow \left(\frac{1}{\pi}\right) = \frac{\overline{rd} + 0.021 \times y^*}{0.021 \times y^*} + 1 \Rightarrow \pi = \frac{0.021 \times y^*}{\overline{rd} + 0.021 \times y^*}$.

Lastly, the exogenous productivity process, which agents in both economies take as given, is now modeled as a bivariate process:

$$\begin{pmatrix} \ln A_{t+1} \\ \ln A_{t+1}^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} \ln A_t \\ \ln A_t^* \end{pmatrix} + \begin{pmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^* \end{pmatrix}, \ \forall \varepsilon_{t+1}, \varepsilon_{t+1}^* \sim NIID(0, \Sigma), \ t \geq 0 \tag{3.46}$$

The parameters ρ_{ii} ($\forall i = 1, 2$), as before, measure the intra-country persistence of the productivity shock. The parameters ρ_{ij} ($\forall i, j = 1, 2$ and $i \neq j$) capture the spillover effect that one country's productivity shock has on the other one period after the shock occurs. Note that ρ_{12} reflects the spillover effect that the emerging economy has on the developed emerging, while ρ_{21} reflects the spillover effect that the developed economy has on the emerging economy. The stochastic component of the bivariate productivity process is given by the i.i.d. shock vector ε' ($\varepsilon^{*\prime}$) with zero mean and the following variance-covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_\varepsilon^2 & \operatorname{cov}(\varepsilon') \\ \operatorname{cov}\left(\varepsilon^{*\prime}\right) & \sigma_\varepsilon^{*2} \end{pmatrix} \text{ where } \varepsilon' = \left(\varepsilon_t, \varepsilon_t^*\right)', \ \varepsilon^{*\prime} = \left(\varepsilon_t^*, \varepsilon_t\right)'$$

Note that σ_ε^2 need not but can be equal to σ_ε^{*2}, but that it must be the case that $\operatorname{cov}(\varepsilon') = \operatorname{cov}\left(\varepsilon^{*\prime}\right)$. A rational expectations equilibrium for the two-country model is therefore given by sequences $\left\{\lambda_t, \lambda_t^*, d_{t+1}, d_{t+1}^*, k_{t+1}, k_{t+1}^*, c_t, c_t^*, h_t, h_t^*\right\}_{t=0}^{\infty}$ that satisfy both countries' individual first order conditions, their individual constraints, their individual flow variable equations and the 'size' equation, given the bivariate exogenous productivity process (3.46) as well as the starting values for the state variables $A_0, A_0^*, d_0, d_0^*, k_0$ and k_0^*.[18] The calibration procedure is identical to the one of sections 3.3 and 3.4 with the following caveats and additions:

- To ensure similar dynamics in both countries' Euler equations, the portfolio adjustment cost parameter is set to $\mu \approx (1+\bar{r})^{-1}\psi$ and $\mu^* \approx (1+\bar{r})^{-1}\psi^*$.

[18] Recall that all other variables are a function of these variables and therefore need not be specified explicitly.

- The real interest rate $\bar{r}$ is now constant and set to equal the more commonly found value of 1.5 percent (versus the much higher value used in model 2 for the emerging economy where the steady state interest rate equaled 3.25 percent)
- Although the depreciation rate for the developed economy remains the same, the change in the steady state interest rate implies that the depreciation rate that now matches the empirical average for the investment to output ratio of the emerging economies decreases from 0.0895 to 0.0415.

The parameters ρ_{ii} ($\forall i = 1, 2$) are again set to match the average standard deviations of output found in chapter 2. Zimmermann's (1995) study of a three country model, consisting of a small open economy, its large neighbor and the rest of the world (RoW) concludes that smaller economies have a lower autocorrelation coefficient than their larger neighbors and the RoW. In his first experiment he uses Solow residuals to identify the parameters in equation (3.46) (i.e. autocorrelation coefficients and variances) for Switzerland versus the rest of Europe (the large neighbor) and the RoW and in his second experiment he constructs autocorrelation coefficients for Canada versus the U.S. (the large neighbor) and the RoW. Loosely translating this to the model at hand leads to the interpretation that emerging countries (the 'smaller' countries) are likely to exhibit lower autocorrelation coefficients than their developed neighbors (the 'larger' countries). Thus, $\rho_{11} > \rho_{22}$. The interpretation is simple: Perhaps developed economies are better able to retain their technology shocks than emerging economies. As it turns out in table 3.6, this inequality arises naturally within the context of model 3 by requiring that the autocorrelation coefficients for each economy type reproduce the respective standard deviations of output (given shock variances discussed below).

In terms of the spillover effects ρ_{ij} ($\forall i, j = 1, 2$ and $i \neq j$), Zimmermann states that "[a]symmetries… can reflect size. A small country may more easily take benefit of an innovation in a large country than the reverse (p.6)." Therefore it is assumed that $\rho_{ij} \neq \rho_{ji}$. Indeed he finds that the spillover effect between a small and large country (where large country refers to the large neighbor or the RoW) are higher in the direction from large to small "…hereby reflecting [the small country's] openness" (p.9). For example, he finds that the spillover effect from Europe to Switzerland is 0.188, while the spillover effect from Switzerland to Europe is 0.026. The spillover effect from the U.S. to Canada is 0.156 and 0.031 the other way around. For Switzerland versus the RoW, he even calculates very small negative spillover effects. He concludes that one can "…justify negative spillover effects in certain situations: a positive technology shock in a country gives it a competitive advantage against the others whose output, and therefore productivity, decreases while losing market shares. Such a phenomenon could be observed as Japanese manufacturers introduced the quartz movement to the detriment of the Swiss watch industry" (p. 10).

Notably model 3 is not exactly synonymous with Zimmermann's models in the sense that an emerging SMOPEC versus a developed SMOPEC 'neighbor' rather than a developed SMOPEC versus a LOPEC neighbor or the RoW is being modeled. However, the assumption that $\rho_{12} < \rho_{21}$ is not entirely counterintuitive: It simply implies that the technology shock originating in the emerging economy transfers to a lesser degree to the developed economy than the other way around.

According to Zimmermann, the variance of the technology shocks is always greater in the small countries than in the large countries, a result that makes intuitive sense and was already evident in model 1 versus model 2 (equating the emerging economy with the smaller country). Therefore model 3 is calibrated such that $\sigma_\varepsilon^{*2} > \sigma_\varepsilon^2$. For simplicity the values found in models 1 and 2 will be used as they already verify this inequality. The co-variances between technology shocks in Zimmermann's model are positive but lower than the individual variances, i.e. $0 < \operatorname{cov}(\varepsilon') = \operatorname{cov}\left(\varepsilon^{*\prime}\right) < \sigma_\varepsilon^2 < \sigma_\varepsilon^{*2}$. The best results in model 3, however, are obtained if the co-variances are actually allowed to take on very small negative values. As a matter of fact, allowing for a negative spillover effect from the emerging to the developed economy in combination with a small but negative co-variance between the technological innovations, allows for a reversal of the 'usual' cross country consumption and output correlations: In two-country international real business cycle models, it is generally the case that output is less correlated than consumption, which contrasts standard data findings where consumption is less correlated than output (i.e. more often than not the data generates output smoothing rather than consumption smoothing). Therefore allowing $\rho_{12} < 0 < \rho_{21}$ in combination with $\operatorname{cov}(\varepsilon') = \operatorname{cov}\left(\varepsilon^{*\prime}\right) \quad < 0 < \sigma_\varepsilon^2 < \sigma_\varepsilon^{*2}$ generates results where this inconsistency vis-à-vis the data is reversed.

In sum:

Table 3.6: Parameter Values for Model 3 (the Two-Country Model)[(a),(b)]

Variable	Description	Value
$\alpha = \alpha^*$	Capital share in output	0.32
$\beta = \beta^*$	rate of time preference/discount factor, set to $1/(1+\bar{r})$	0.9852
δ	Capital depreciation rate for DE, matches average i/y	0.0266
δ^*	Capital depreciation rate for EE, matches average i^*/y^*	0.0415
$\bar{d}$	SS debt for DE, matches average tb/y of SMOPECs	17.5703
$\bar{d}^*$	SS debt for EE, matches mean tb^*/y^* of EE	-9.2854
$\gamma^* = \gamma$	coefficient of relative risk aversion	2
$\phi \approx \phi^*$	capital adjustment cost parameter	0.006
$\mu \approx \mu^*$	portfolio adjustment cost parameter	0.0001
$\bar{r}$	SS world interest rate	0.015
π	weight parameter on debt size	0.346
ρ_{11}	productivity process persistence parameter in DE	0.700
ρ_{22}	productivity process persistence parameter in EE	0.635
ρ_{12}	spillover effect of productivity shock from EE to DE	-0.098
ρ_{21}	spillover effect of productivity shocks from DE to EE	0.15
σ_{ε}	standard deviation of technological innovation in DE	0.00383
σ_{ε}^*	standard deviation of technological innovation in EE	0.00633
$\text{cov}\left(\varepsilon', \varepsilon^{*\prime}\right)$	covariance of technological innovations	-1.00E-07
$\omega^* = \omega$	1 + inverse of intertemporal elasticity of substitution in labor	1.4

(a) Abbreviations used are DE=developed economy, EE=emerging economy, SS=steady state. (b)The depreciation rate for capital in the emerging economy changes relative to model 2 because the discount factor and the interest rate have changed and create different steady state values for output and capital.

3.5.1 Impulse Response Analysis for Model 3

Figures 3.3 – 3.5 displays impulse response functions for three scenarios: The first is a one percent technological innovation to productivity in the developed economy, the second is a one percent technological innovation to productivity in the emerging economy and the third is a simultaneous one percent increase in productivity in both economies. In the interest of brevity, only those variables of model 3 are graphed, where the interaction between the developed economy's and emerging economy's variables is of particular interest. The explanations for the impulse response functions are the same as for models 1 and 2, except that there now exists a positive spillover effect onto the emerging economy, if there is a productivity increase in the developed economy and a small, but negative spillover effect onto the developed economy, if there is a productivity increase in the emerging economy.

Figure 3.3 shows that the developed economy's variables respond virtually the same as in model 1 to a positive productivity shock. To understand the interaction between the linked productivity processes, consider that the log-linearized equations for the developed and emerging economies are given by $\hat{A}_{t+1} = \rho_{11}\hat{A}_t + \rho_{12}\hat{A}_t^* + \hat{\varepsilon}_{t+1}$ and $\hat{A}_{t+1}^* = \rho_{22}\hat{A}_t^* + \rho_{21}\hat{A}_t + \hat{\varepsilon}_{t+1}^*$ respectively. In period $t = 1$ it is therefore the case that:

$$\hat{A}_1 = \rho_{11} \times 0 + \rho_{12} \times 0 + 1 = 1$$

and

$$\hat{A}_1^* = \rho_{22} \times 0 + \rho_{21} \times 0 + 0 = 0$$

Since $\hat{\varepsilon}_t = 0 \ \ \forall t \geq 2$, in period $t = 2$:

$$\hat{A}_2 = \rho_{11}\hat{A}_1 + \rho_{12}\hat{A}_1^* \ = \rho_{11} \times 1 + \rho_{12} \times 0 = \rho_{11}$$

and

$$\hat{A}_2^* = \rho_{22}\hat{A}_1^* + \rho_{21}\hat{A}_1 + \hat{\varepsilon}_2^* \Rightarrow \hat{A}_2^* = \rho_{22} \times 0 + \rho_{21} \times 1 = \rho_{21}$$

In period $t = 3$:

$$\hat{A}_3 = \rho_{11}\hat{A}_2 + \rho_{12}\hat{A}_2^* = \rho_{11} \times \rho_{11} + \rho_{12} \times \rho_{21} = \rho_{11}^2 + \rho_{12} \times \rho_{21}$$

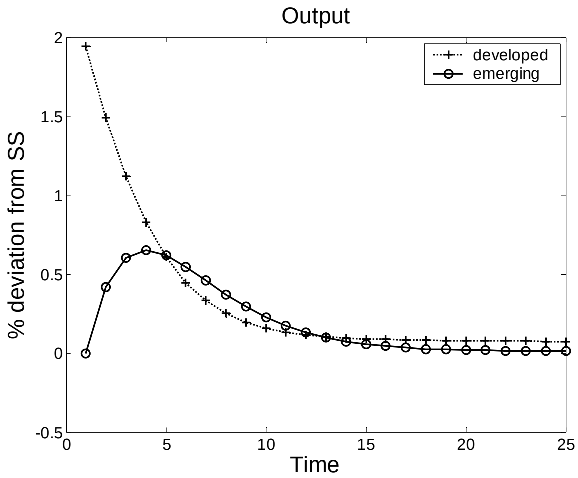
and

$$\hat{A}_3^* = \rho_{22}\hat{A}_2^* + \rho_{21}\hat{A}_2 = \rho_{22} \times \rho_{21} + \rho_{11} \times \rho_{21} = \rho_{21}\left(\rho_{11} + \rho_{22}\right)$$

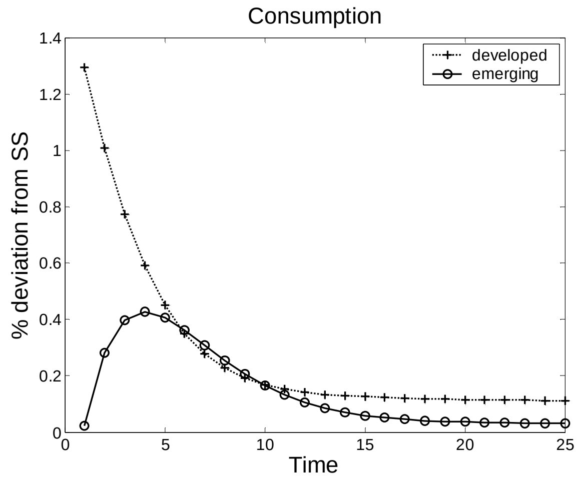
These calculations show how the productivity increase in the developed economy also translates into productivity gains in the emerging economy. With the exception of debt, the variables therefore move in the same direction for both types of economies, even though the effects on the emerging economy are not as pronounced as on the developed economy. Because the emerging economy does not experience any changes in its productivity process until period $t = 2$, all variables behave sluggishly. The only case where the variables move in opposite direction is shown for debt in panel (5). Initially debt increases above its steady state in the developed economy but in the long run adjusts to a new level below the original steady state. The explanation for this is given in section 3.3.1. Recall that the emerging economy's steady state debt level is given by a negative number, i.e. assets. Thus the initially negative movement in the emerging economy's impulse response function for 'debt', implies an accumulation of assets as the developed economy accumulates debt in the short run. In the long run, however, the equilibrium on the asset market dictates that the emerging economy has relatively less assets since the positive productivity shock in the developed economy has led to lower debt in the latter.

Figure 3.3: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of the Developed Economy Only

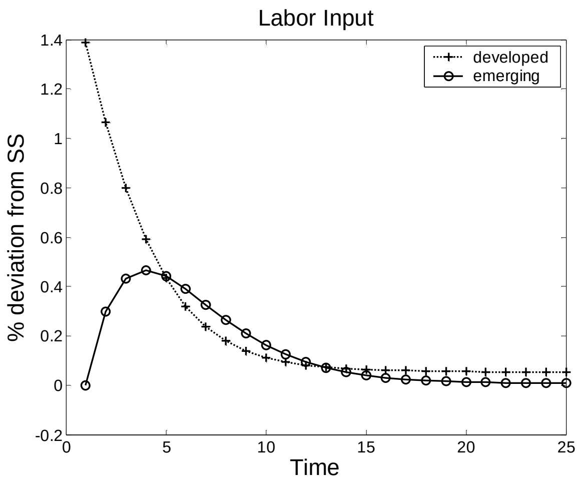
(1)



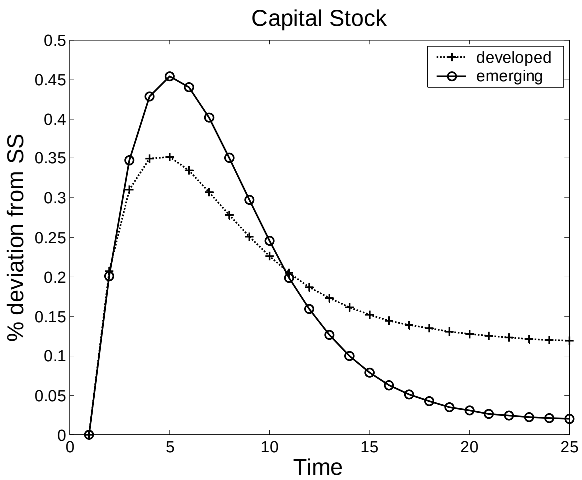
(2)



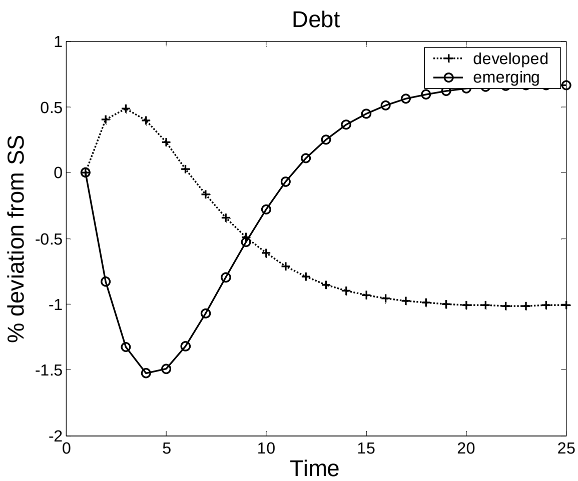
(3)



(4)



(5)



(6)

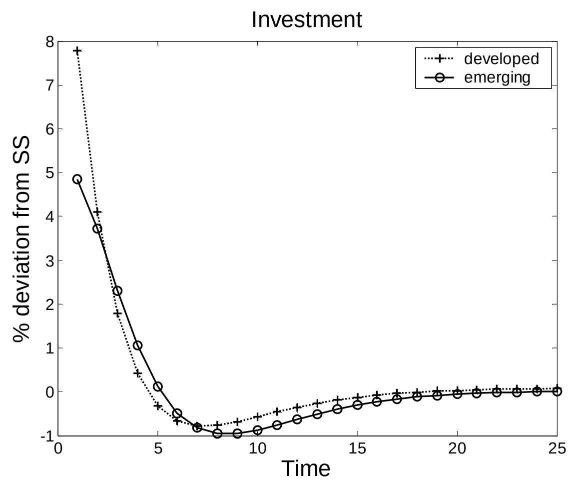


Figure 3.3 (continued)

(7)

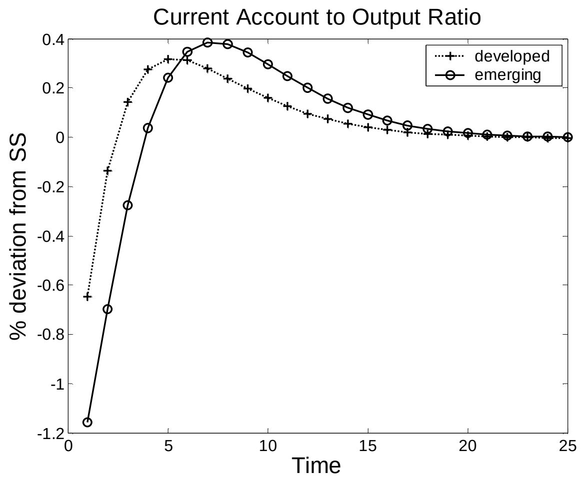


Figure 3.4 shows that the emerging economy's variables also respond in the same direction as in model 2, although the scale is slightly different. Recall that there now exists a negative spillover effect onto the developed economy that implies opposite movements in the impulse response functions (again with the exception of debt). To see this more clearly again consider the log-linearized productivity processes. In period $t=1$:

$$\hat{A}_1^* = \rho_{22}\hat{A}_0^* + \rho_{21}\hat{A}_0 + 1 = \rho_{22}\times 0 + \rho_{21}\times 0 + 1 = 1$$

and

$$\hat{A}_1 = \rho_{11}\hat{A}_0 + \rho_{12}\hat{A}_0^* + 0 = \rho_{11}\times 0 + \rho_{12}\times 0 = 0$$

Since $\hat{\varepsilon}_t^* = 0 \;\; \forall t \geq 2$, in period $t=2$:

$$\hat{A}_2^* = \rho_{22}\hat{A}_1^* + \rho_{21}\hat{A}_1 + \hat{\varepsilon}_2^* \Rightarrow \hat{A}_2^* = \rho_{22}\times 1 + \rho_{21}\times 0 = \rho_{22}$$

and

$$\hat{A}_2 = \rho_{11}\hat{A}_1 + \rho_{12}\hat{A}_1^* = \rho_{11}\times 0 + \rho_{12}\times 1 = \rho_{12} < 0$$

In period $t = 3$:

$$\hat{A}_3^* = \rho_{22}\hat{A}_2^* + \rho_{21}\hat{A}_2 = \rho_{22} \times \rho_{22} + \rho_{11} \times \rho_{12} = \rho_{22}^2 + \rho_{11} \times \rho_{12}$$

and

$$\hat{A}_3 = \rho_{11}\hat{A}_2 + \rho_{12}\hat{A}_2^* = \rho_{11} \times \rho_{12} + \rho_{12} \times \rho_{22} = \rho_{12}\left(\rho_{11} + \rho_{22}\right) < 0$$

The above shows why the increase in productivity in the emerging economy translates into productivity losses in the developed economy, potentially because the former can now better exploit its comparative advantage. Note that the emerging economy's impulse response function for debt behaves as in figure 3.3 (it initially dips below the steady state, implying an accumulation of assets, but in the long run remains above the steady state, implying that the net asset position is being run down). Conversely, the negative spillover effect from the emerging onto the developed economy is similar to a 'negative' productivity shock for the developed economy. This is the reason debt responds in an opposite fashion for the developed economy in figures 3.3 versus 3.4. In the long run, the developed economy experiences lower debt if it experiences a productivity shock itself, while it incurs slightly higher debt if the productivity shock occurs in the emerging economy. Taking the observations of figures 3.3 and 3.4 together, a positive productivity shock in a county who starts out with a positive net debt position (a borrower) can decrease its net debt holdings, while a positive productivity shock in a country who starts out with a positive net asset position (a lender) will decrease its asset holdings.

Figure 3.4: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of the Emerging Economy Only

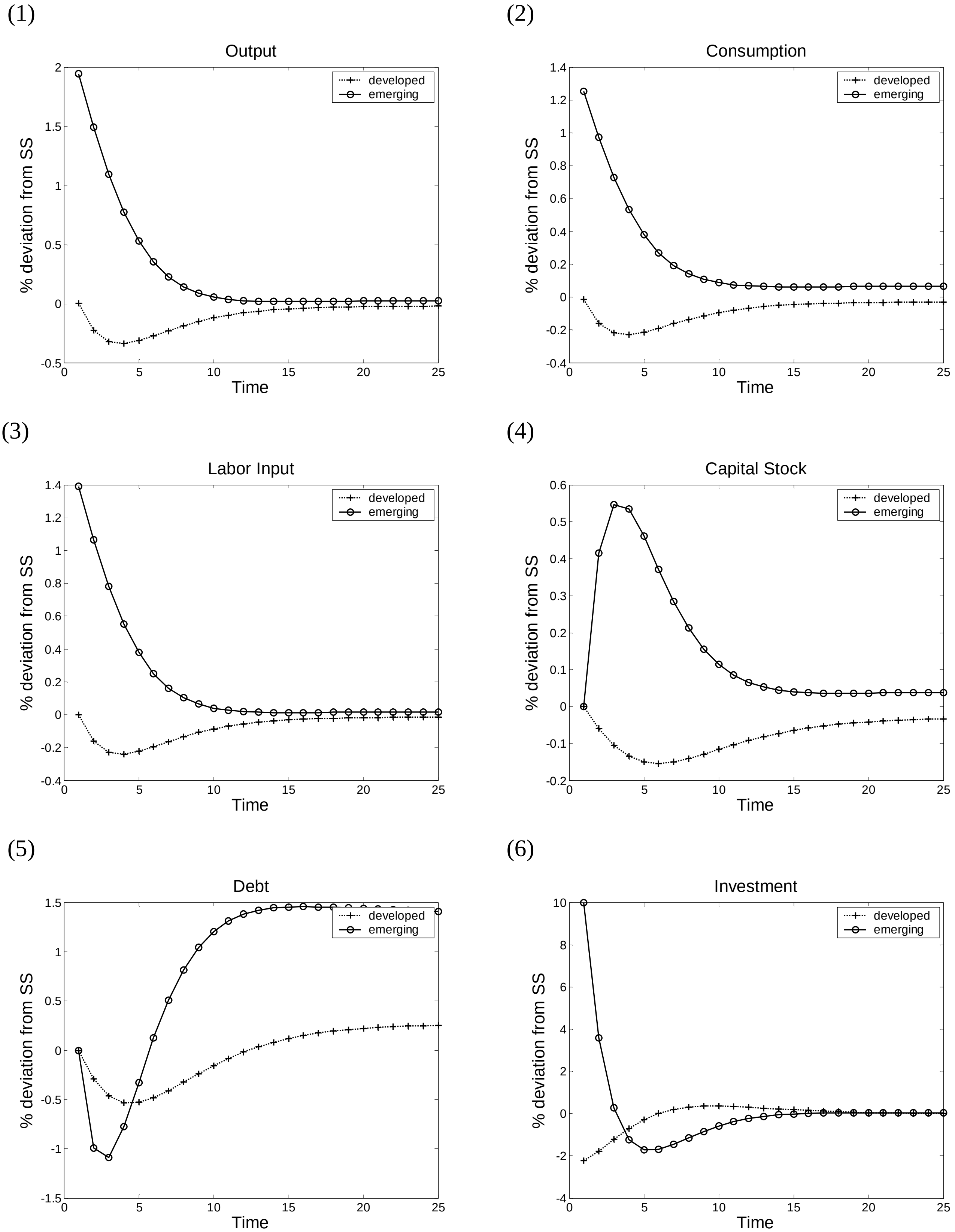
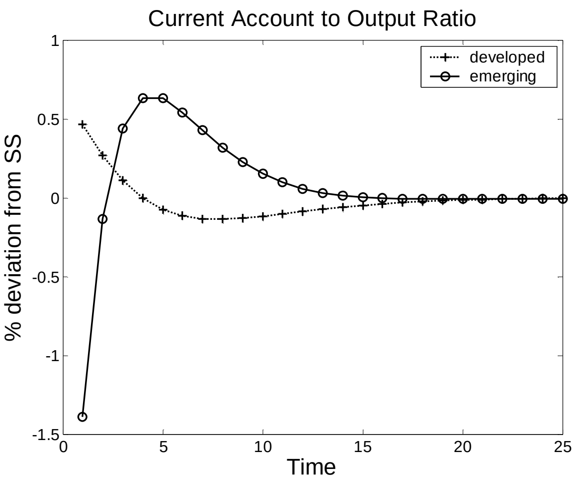


Figure 3.4 (continued)

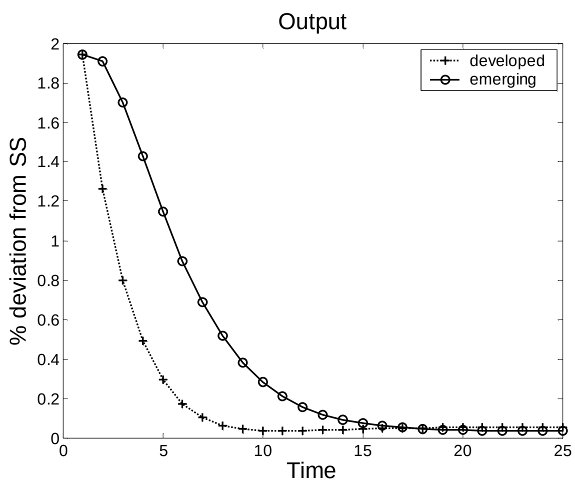
(7)



Lastly, figure 3.5 displays the impulse response functions in both economies for a simultaneous increase in productivity. Due to the parameterization, the emerging economy benefits more from this as evident, for instance, in the drawn out increases in output and consumption relative to the developed economy. The capital stock also increases significantly more for the emerging economy, while the impulse response functions for debt are similar to the one depicted in figure 3.3 (and hence the same explanation holds).

Figure 3.5: Impulse Response Functions of Model 3 after a One Percent Increase in the Technological Innovation of Both Economies

(1)



(2)

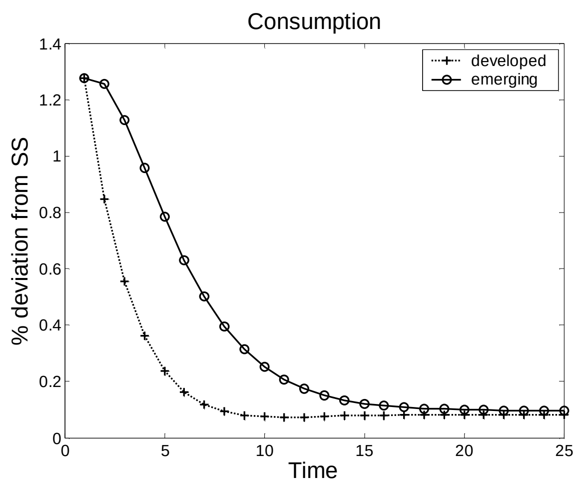
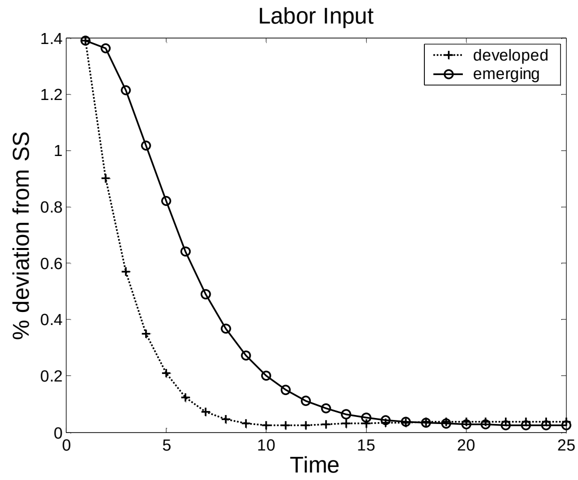
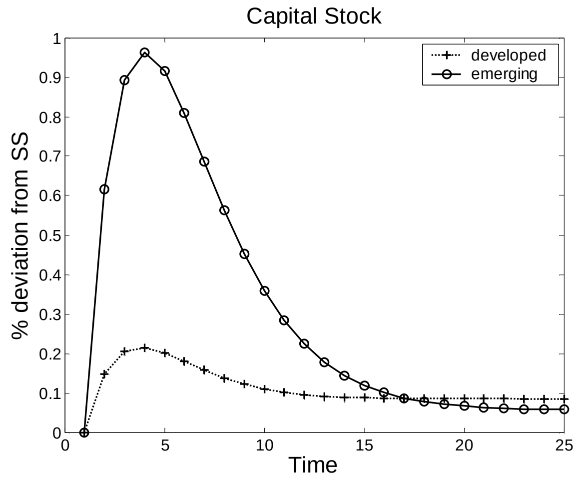


Figure 3.5 (continued)

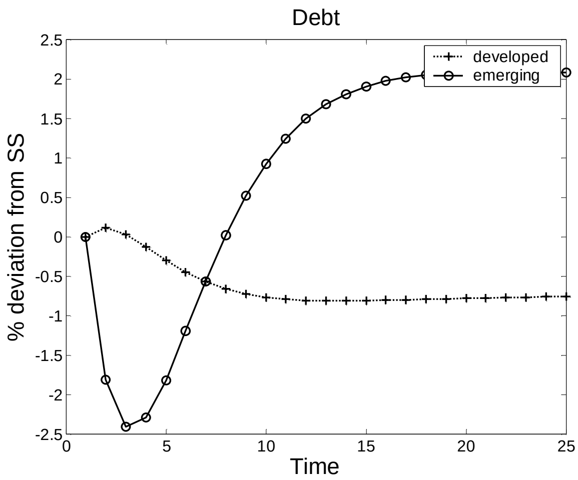
(3)



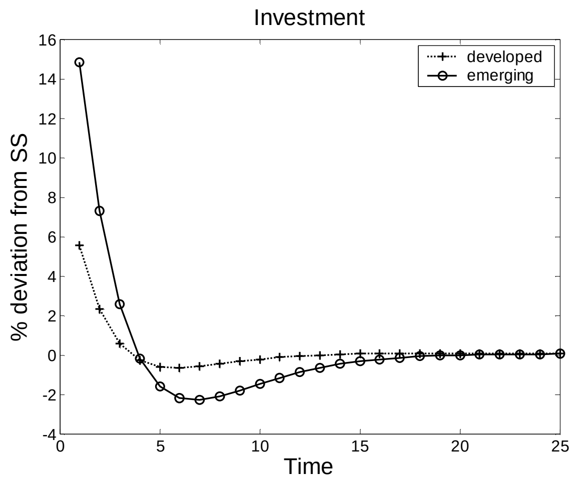
(4)



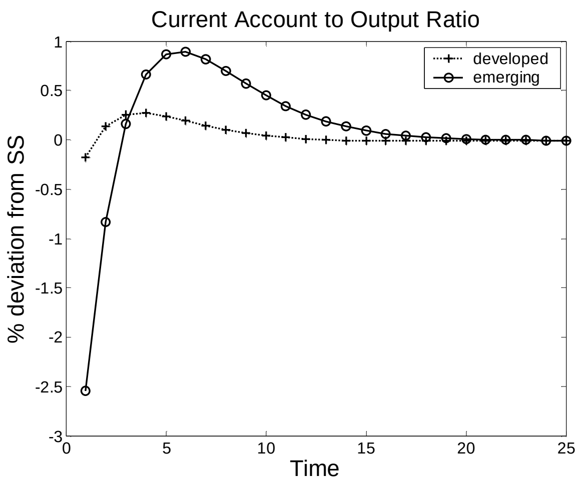
(5)



(6)



(7)



3.5.2 Empirical vs. Theoretical Business Cycle Moments for Model 3

The cross-country output correlations between the emerging and developed economies of chapter 2 generate an inauspicious wide range due to Poland's uncharacteristically (relative to other emerging economies) large positive correlation with the developed European SMOPECs (0.54). Ignoring Poland as a sample outlier, results in a more harmonized range of output correlations of $-0.07 \leq \rho(y, y^*) \leq 0.08$, where -0.07 reflects the output correlation of the Slovak Republic with the European SMOPECs (Belgium and the Netherlands) and 0.08 reflects the output correlation of both Mexico versus Canada and the Czech Republic versus the European SMOPECs . Again ignoring Poland, whose consumption correlation with the developed SMOPECs in Europe is 0.31, results in a range of consumption correlations $-0.23 \leq \rho(c, c^*) \leq -0.01$, where -0.01 reflects the Mexican-Canadian and -0.23 the Czech-EU SMOPEC consumption correlation (the Slovak Republic-EU SMOPEC consumption correlation is -0.14). The difference between the highest output and the highest consumption correlation is therefore 0.09 and the difference between the lowest output and the lowest consumption correlation is 0.16. In other words, ideally model 3 will produce a cross-country consumption correlation coefficient that is 0.09 to 0.16 points lower than the corresponding output correlation coefficient.

Table 3.7 shows that it is possible to produce lower consumption correlations than output correlations in model 3. This is a promising result, since these types of models, contrary to the data, usually generate more consumption than output smoothing. However, using a variety of different values for the spillover effects once other parameters have been assigned, never resulted in a cross-country consumption and output correlation difference larger than 0.03 points. Therefore the wedge between lower consumption correlations versus higher output correlations can not be increased beyond 0.03, no matter how negative the spillover effect from emerging to developed economy or how positive the spillover effect from developed to emerging economy.

Table 3.7: Cross Country Business Cycle Summary Statistics Implied by Model 3 vs. Data

Cross Country Correlations $\rho\left(x_t, x_t^*\right)$		
	Data Range[a]	Model 3[b]
Output	$-0.07 \leq \rho(y, y^*) \leq 0.08$	-0.05
Consumption	$-0.23 \leq \rho(c, c^*) \leq -0.01$	-0.08

(a)The data range excludes values for Poland. (b) Remaining correlations can be found in the appendix

Table 3.8: Intra-Country Business Cycle Summary Statistics: Model 3 vs. Average of Developed SMOPECs (Panel I) and vs. Average of Emerging Economies (Panel II)[a]

(I) Developed Economy					
Variable (x)	Source	$\sigma(x_t)$	$\dfrac{\sigma(x_t)}{\sigma(y_t)}$	$\rho(x_t, y_t)$	$\rho(x_t, x_{t-1})$
Output (y)	Model 3	1.28	1.00	1.00	0.80
	Data	1.28	1.00	1.00	0.84
Consumption (c)	Model 3	0.99	0.78	0.96	0.85
	Data	1.02	0.80	0.69	0.80
Investment (i)	Model 3	4.11	3.22	0.71	0.57
	Data	4.11	3.24	0.76	0.77
Current Axt. Ratio (cay)	Model 3	0.56	-	-0.07	0.62
	Data	1.21	-	-0.28	0.45
Correlation (s, i)	Model 3	0.73	(a) Remaining statistics can be found in the appendix for chapter 3.		
	Data	0.54			

Table 3.8 (continued):

(II) Emerging Economy						
Variable x	Source	$\sigma(x_t^*)$	$\frac{\sigma(x_t^*)}{\sigma(y_t^*)}$	$\rho(x_t^*, y_t^*)$	$\rho(x_t^*, x_{t-1}^*)$
Output (y^*)	Model 3	1.92	1.00	1.00	0.76
	Data	1.92	1.00	1.00	0.76
Consumption (c^*)	Model 3	1.39	0.72	0.95	0.81
	Data	2.12	1.13	0.61	0.68
Investment (i^*)	Model 3	7.56	3.94	0.63	0.420
	Data	7.56	4.06	0.72	0.74
Current Axt. Ratio (cay^*)	Model 3	1.34	-	-0.17	0.50
	Data	2.39	-	-0.39	0.63
Correlation (s^*, i^*)	Model 3	0.63	(a) Remaining statistics can be found in the appendix for chapter 3.		
	Data	0.26			

3.6 Concluding Remarks Chapter 3

The three small open economy models of the previous sections were able to replicate some key findings of chapter 2 regarding intra and cross-country stylized facts. Models 1 and 2 used a debt elastic interest rate premium to induce stationarity in a single country model of a hypothetical developed economy and a single country model of a hypothetical emerging economy respectively. Model 3 examined a two-country model with linked exogenous productivity processes and used portfolio adjustment costs to debt holdings to induce stationarity.

Model 1 correctly identified the volatility rankings of typical developed economies, such as greater output than consumption variability and greater investment than output variability. It showed that the trade balance and current account are acyclical but understated their volatility. The contemporaneous output and first order auto-correlations were all well matched with the exception that the contemporaneous correlation of capital and consumption with respect to output was overstated. Model 2, on the other hand, correctly predicted the volatility rankings of emerging economies, particularly the fact that consumption is more volatile than output and that investment as well as the trade balance and current account ratios are more volatile than in developed economies (although the volatility of the two ratios was again understated). In contrast to model 1, model 2 accurately predicted the contemporaneous correlation of consumption and output. Unfortunately, the empirical finding that the trade balance and current account are more acyclical in emerging than in developed economies could not be replicated. All in all, the model calibrated for an emerging economy performs no worse than the model calibrated for a developed economy, leading to the conclusion that these types of real business cycle models can be an appropriate tool for modeling key business cycle features of emerging economies, although room for improvement certainly exists.

The main contribution of model 3 is that it is able to reproduce the fact that countries engage in less consumption than output smoothing and the fact that there seem to exist very small and sometimes negative cross-country consumption and output correlations among developed and emerging neighboring economies or trade agreement partners. Using a combination of a negative spillover effect from the emerging onto the developed economy and a slightly negative covariance for the technological innovations reproduced the above empirical findings. Some of the successes of model 2, such as the higher consumption than output volatility, were no longer captured by model 3, although the remainder of the intra-country findings were comparable to those of models 1 and 2.

In addition, the impulse response analysis for each model led to sensible and intuitive results, of which the most prominent was that a positive productivity shock in a country who starts out as a borrower can decrease its debt holdings, while a positive productivity shock in a country who starts out as a lender will decrease its asset holdings. In addition, it showed how a productivity shock in the emerging economy combined with a negative spillover effect can imply opposite movements in business cycles.

Chapter 4

A Primer on Solving Open Economy SDGE Models

This chapter addresses the theory behind solutions for linearized real business cycle models. The discussion will be based on two versions of the debt elastic interest rate model of chapter 3. The first version is a 'reduced' variant of model 1, which contains one less state variable and allows some of the key dynamic components of the model to be examined in a two dimensional context. The second version is the 'complete' model 1 and will be used as a basis for discussing the algorithm on which computerized solutions to real business cycle models are based. The reduced model will be referred to as model 4. Section 4.1 explains the process of log-linearization. Section 4.2 develops model 4 and briefly addresses solution stability and uniqueness. Since model 4 is smaller in terms of variables and equations, solution methods such as the method of undetermined coefficients (section 4.3) and the eigenvalue decomposition (section 4.4) can be applied manually and transparently without the help of computer algorithms. The multi-dimensional model 1, containing twelve endogenous variables and equations, is too complex for a meaningful exposition of these two solution approaches. Instead the Schur-decomposition (section 4.5) presents the proper solution mechanism.

Review of Model 1

For simplicity, a quick recap of model 1 follows: The economy is populated by an infinite number of identical consumers. The representative agent solves the following optimization problem (where $A =$ exogenous productivity series, $c =$ per capita consumption, $h =$ per capita labor input, $\tilde{d} =$ aggregate per capita debt, $d =$ per capita individual debt, $k =$ per capita capital stock, $r =$ debt elastic interest rate, $y =$ per capita output, $i =$ per capita investment,

$s =$ per capita saving, $tby =$ per capita trade balance to output ratio, $cay =$ per capita current account to output ratio):

$$\max_{\{c_t, h_t, d_{t+1}, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \tag{4.1}$$

subject to:

$$d_{t+1} = (1+r_t)d_t - y_t + c_t + i_t + \Phi(k_{t+1} - k_t) \tag{4.2}$$

$$y_t = A_t F(h_t, k_t) \tag{4.3}$$

$$k_{t+1} = i_t + (1-\delta)k_t \tag{4.4}$$

$$r_t = r + \rho\left(\tilde{d}_{t+1}\right) \tag{4.5}$$

Additional equations of interest are:

$$s_t = y_t - \Phi\left(k_{t+1} - k_t\right) - c_t \tag{4.6}$$

$$tby_t = \frac{tb_t}{y_t} = 1 - \left(\frac{c_t + i_t + \Phi\left(k_{t+1} - k_t\right)}{y_t}\right) \tag{4.7}$$

$$cay_t = \frac{ca_t}{y_t} = tby_t - \frac{r_t d_t}{y_t} \tag{4.8}$$

$$k_t^a - k_{t+1} = 0 \tag{4.9}$$

The exogenous productivity process is given by:

$$\ln\left(A_{t+1}\right) = \rho \ln\left(A_t\right) + \varepsilon_{t+1};\ \forall \varepsilon_{t+1} \sim NIID(0, \sigma_\varepsilon^2),\ t \geq 0 \tag{4.10}$$

The first order conditions after maximizing (4.1) subject to (4.2) – (4.5) are:

$$U_c(c_t, h_t) = \lambda_t \tag{4.11}$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \tag{4.12}$$

$$\lambda_t = \beta(1+r_t)E_t \lambda_{t+1} \tag{4.13}$$

$$\lambda_t\left[1+\Phi'(k_{t+1}-k_t)\right]=\beta E_t\lambda_{t+1}\left[A_{t+1}F_k(h_{t+1},k_{t+1})+1-\delta+\Phi'(k_{t+2}-k_{t+1})\right] \tag{4.14}$$

$$\lim_{t\to\infty}\beta^t\lambda_t\left(k_{t+1}-d_{t+1}\right)=0 \tag{4.15}$$

and the constraints (4.2) – (4.5) and the additional equations (4.6) – (4.9) holding with equality. The rational expectations equilibrium is a sequence of processes $\left\{\lambda_t, d_t,\ \tilde{d}_t,\ k_t,\ k_{t+1}, c_t,\ h_t\right\}_{t=0}^{\infty}$ that satisfy the first order conditions (4.11) – (4.15), the constraints (4.2) – (4.5) and the additional equations of interest (4.6) – (4.9), taking equation (4.10) as well as the starting values for the state variables A_0, d_0 and k_0 as given.

Preliminary Definitions

The following concepts help to understand the remainder of this chapter and can be viewed as a supplemental guide to chapter 3, describing model 1.

Definition 1: A *choice* variable is a variable that the representative agent chooses when solving his optimization problem, implying that consumption, labor input and next period's capital stock and debt are the relevant choice variables.

Definition 2: Of the choice variables, there is a further distinction between *control* and *endogenous state* variables. For model 1, the controls are consumption and labor-input, which are non-predetermined variables each period. The endogenous state variables are the current capital stock and debt. The latter are pre-determined each period (see section 4.2.2 for intuition).

Definition 3: Endogenous and *exogenous* state variables must be differentiated. The latter are governed by an exogenously determined rule such as the *AR*(1) process defining the productivity series in equation (4.10).

Definition 4: The *flow variable* is a function of at least one of the choice variables (control or endogenous state or both), potentially at different dates. Flow variables can be eliminated from the model by substitution.

Table 4.1 summarizes the notation that will be used throughout the remainder of the chapter:

Table 4.1: Variable Classification and Notation

Notation and Size	Description	Variables
$z \quad (n_z \times 1)$	(vector of) exogenous state variables	A
$x^S \quad (n_S \times 1)$	(vector of) endogenous state variables	k, d
$x^C \quad (n_C \times 1)$	(vector of) control variables	c, h, k^a
$x^F \quad (n_F \times 1)$	(vector of) flow variables, $x^F = f\left(x^C, x^S\right)$	y, r, s, i, tby, cay

Definition 5: The *state space* is given by the set of current endogenous and exogenous state variables $\left\{\hat{x}_{jt}^S, \hat{z}_{kt}\right\}, \forall j \in \left[1,....,n_S\right], \forall k \in \left[1,....,n_Z\right]$. The vector of stochastic but known exogenous processes $\hat{z}_t$ of size $n_Z \times 1$ is subject to i.i.d. innovations $\varepsilon_t \sim NIID(0_{n_z \times 1}, \Sigma_\varepsilon)$, $t \geq 0$, where $0_{n_z \times 1}$ is the vector of zero means and Σ_ε the variance-covariance matrix of the innovations.

Definition 6: A solution for any linearized model is characterized by two types of solution functions:

(a) *Policy functions* are the optimal response of control variables to changes in the state space. They are denoted by $\hat{x}_{it}^C = g(\hat{x}_{jt}^S, \hat{z}_{kt}), \forall i \in \left[1,....,n_C\right], \forall j \in \left[1,....,n_S\right], \forall k \in \left[1,....,n_Z\right]$ for the vector of controls.

(b) *Transition functions* map the state space into itself and dictate the adjustment process of the endogenous state variables to changes in the state space over time. They are denoted by $\hat{x}_{jt+1}^S = h(\hat{x}_{jt}^S, \hat{z}_{kt}), \forall j \in \left[1,....,n_S\right], \forall k \in \left[1,....,n_Z\right]$

4.1 Log-Linearizing the Debt Elastic Interest Rate Model

In this section, the most commonly employed linearization method, log-linearization, for solving a non-linear real business cycle model is introduced. The reason models are linearized is "[to minimize]…computational costs, measured in terms of computer time and programming effort" and because there is no loss of generality since "… the linear models produce highly accurate results when the variance of the shocks hitting the system is not too large" (Oviedo, p.3).

Any generic variable x_t can be log-linearized around its steady state by defining it as $\hat{x}_t = dx_t / x$ where x denotes the steady state value.[19] A standard reference for log-linearization is given by King, et al. (2002). Additional references for obtaining a first order linear approximation to nonlinear dynamic general equilibrium models is Uhlig (1997, 2006), who shows how the equations can be obtained without explicit differentiation or Oviedo (2005). A reference for obtaining a second order approximation is provided by Schmitt-Grohé and Uribe (2004).

Before linearizing the model, define the following coefficients at the *steady state*. These coefficients can be considered the elasticities of marginal utility of consumption (the marginal disutility of labor input) with respect to consumption and labor input for the non–separable utility function introduced in section 3.1.

$$\varepsilon_{cc} = c\frac{U_{cc}}{U_c} < 0 \qquad \varepsilon_{ch} = h\frac{U_{ch}}{U_c} > 0 \qquad \varepsilon_{hh} = h\frac{U_{hh}}{U_h} > 0 \qquad \varepsilon_{hc} = c\frac{U_{hc}}{U_h} < 0$$

where $U = U(c,h)$, $U_c = \partial U / \partial c$, $U_{ch} = \partial U_c / \partial h$, etc. The calculations for each of the elasticities can be found in appendix equations (A.1) – (A.4). Note that ε_{cc} resembles the absolute value of the coefficient of relative risk aversion, but since the utility function is a composite function of consumption and labor input, it actually does not take on the same value as γ, the actual coefficient of relative risk aversion. Some other useful steady state coefficients that will be needed for the log-linearized budget constraint are given by variable to output ratios:

[19] This definition is slightly modified for the trade balance and current account to output ratio. See appendix.

$$s_c = \frac{c}{y} > 0 \qquad s_i = \frac{i}{y} > 0 \qquad s_k = \frac{k}{y} > 0 \qquad s_{tb} = \frac{tb}{y} = \frac{rd}{y} > 0$$

where $s_{tb} = tb/y = 1-(c+i)/y$ and s_{tb} is assumed to be positive (i.e. the country is a net borrower).

To obtain a linearized representation of model 1, first take natural logarithms of each side of an equation and then differentiate, evaluating each function at the steady state (again let variables without a time subscript denote steady state values). It is assumed that the necessary derivatives have been obtained (see appendix). Starting with the first order conditions (4.11) – (4.12) yields the following two equations:

$$\ln U_c(c_t, h_t) = \ln \lambda_t \;\Rightarrow \frac{1}{U_c}(U_{cc}\underbrace{\frac{c}{c}}_{\times 1}dc_t + U_{ch}\underbrace{\frac{h}{h}}_{\times 1}dh_t) = \frac{d\lambda_t}{\lambda} \;\Rightarrow \varepsilon_{cc}\hat{c}_t + \varepsilon_{ch}\hat{h}_t = \hat{\lambda}_t \tag{4.16}$$

$$\ln\left[-U_h(c_t, h_t)\right] = \ln \lambda_t + \ln A_t + \ln(1-\alpha) + \alpha \ln k_t - \alpha \ln h_t \Rightarrow -\frac{1}{U_h}(-U_{hc}\frac{c}{c}dc_t - U_{hh}\frac{h}{h}dh_t)$$

$$= \frac{d\lambda_t}{\lambda} + \frac{dA_t}{A} + \alpha\frac{dk_t}{k} - \alpha\frac{dh_t}{h} \Rightarrow \varepsilon_{hc}\hat{c}_t + \varepsilon_{hh}\hat{h}_t = \hat{\lambda}_t + \hat{A}_t + \alpha\hat{k}_t - \alpha\hat{h}_t \tag{4.17}$$

For equation (4.13), make the interest rule a function of per capita debt (because it is an equilibrium condition where aggregate per capita debt equals individual per capita debt). Also remember that in the steady state $d_{t+1} = d = \bar{d}$:

$$\ln \lambda_t = \ln \beta + \ln\left(1 + r(d_{t+1})\right) + E_t \ln \lambda_{t+1} \Rightarrow \frac{d\lambda_t}{\lambda} = E_t \frac{d\lambda_{t+1}}{\lambda} + \ln(1 + r + \psi(e^{d_{t+1}-\bar{d}} - 1)$$

$$\Rightarrow \hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \left(\frac{\psi \overbrace{e^{d-\bar{d}}}^{1} dd_{t+1}}{1 + r + \psi\underbrace{(e^{d-\bar{d}} - 1)}_{=0}}\right)\left(\frac{d}{d}\right) \Rightarrow \hat{\lambda}_t = (1+r)^{-1}\psi\bar{d}\hat{d}_{t+1} + E_t\hat{\lambda}_{t+1} \tag{4.18}$$

After both sides are logged, equation (4.14) yields:

$$\frac{d\lambda_t}{\lambda} + \frac{1}{1+\Phi'(k-k)}\Phi''\left(\Delta k_{t+1}\right)dk_{t+1}\frac{k}{k} - \frac{1}{1+\Phi'(k-k)}\Phi''\left(\Delta k_{t+1}\right)dk_t\frac{k}{k} = E_t\left\{\frac{d\lambda_{t+1}}{\lambda} + \ldots\right.$$

$$\left.\frac{dA_{t+1}F_k(k,h)+AF_{kk}(k,h)dk_{t+1}+AF_{kh}(k,h)dh_{t+1}+\Phi''\left(\Delta k_{t+2}\right)dk_{t+2}\,\frac{k}{k}-\Phi''\left(\Delta k_{t+2}\right)dk_{t+1}}{A\alpha k^{\alpha-1}h^{1-\alpha}+(1-\delta)+\Phi'(k-k)}\right\}$$

$$\Rightarrow \hat{\lambda}_t+\phi\bar{k}\left(\Delta\hat{k}_{t+1}\right)=E_t\hat{\lambda}_{t+1}+\gamma_0\left[E_t\hat{A}_{t+1}-(1-\alpha)\hat{k}_{t+1}+(1-\alpha)E_t\hat{h}_{t+1}\right]+\beta\phi\bar{k}\left(E_t\Delta\hat{k}_{t+2}\right) \tag{4.19}$$

where $\gamma_0=\beta(\beta^{-1}+\delta-1)$, $\Delta k_{t+1}=k_{t+1}-k_t$ and $\Delta k_{t+2}=k_{t+2}-k_{t+1}$.

Using the above recipe, the remaining constraints and additional equations can be expressed as follows (intermediate steps can be found in the technical appendix).[20]

$$\frac{s_{tb}}{r}\hat{d}_{t+1}=\frac{s_{tb}}{r}(1+r)\hat{d}_t+s_{tb}\hat{r}_t-\hat{y}_t+s_c\hat{c}_t+s_i\hat{i}_t \tag{4.20}$$

$$\hat{y}_t=\hat{A}_t+\alpha\hat{k}_t+(1-\alpha)\hat{h}_t \tag{4.21}$$

$$\hat{k}_{t+1}=(1-\delta)\hat{k}_t+\delta\hat{i}_t \tag{4.22}$$

$$\hat{r}_t=(r)^{-1}\psi\bar{d}\hat{d}_{t+1} \tag{4.23}$$

$$\hat{s}_t=\frac{(y\hat{y}_t-c\hat{c}_t)}{y-c} \tag{4.24}$$

$$\widehat{tby}_t=\left(s_c+s_i\right)\hat{y}_t-s_c\hat{c}_t-s_i\hat{i} \tag{4.25}$$

$$\widehat{cay}_t=\widehat{tby}_t-s_{tb}\left(\hat{r}_t+\hat{d}_t-\hat{y}_t\right) \tag{4.26}$$

$$\hat{k}_t^a-\hat{k}_{t+1}=0 \tag{4.27}$$

Lastly, the exogenous productivity process, equation (4.10), is given by:

[20] See appendix for the linearization of the trade balance and current account ratios.

$$\hat{A}_{t+1} = \rho \hat{A}_t + \varepsilon_{t+1} \tag{4.28}$$

Equations (4.16) – (4.28) are the log-linearized representation of model 1.

4.2 Derivation of Model 4

Key features of model 1 can be extracted by reducing the number of state variables from three to two. Model 4 is a variant of model 1 that simply imposes the restriction that the capital stock and investment remain constant over time. The economic justification behind this could be that a depleted capital stock is entirely replenished by investment each period, that is $i_t = \delta k_t \Rightarrow k_{t+1} = k_t = k$ and $i = \delta k, \forall t$. Put differently, both capital and investment are assumed to always take on their steady state value (Christiano, et al. (1997) also consider a model with a constant capital stock). Though this may seem awkward at first, note that King and Rebelo (2000) include a discussion on the *The (Un)Importance of Capital Formation*, in which they describe that results related to the Solow residual "…are interpreted as indicating that one should construct macroeconomic models which abstract from capital and growth, since the introduction of these features complicate[s] the analysis without helping to understand business cycle dynamics" (p. 20).[21] In terms of the linearized model of section 4.1, imposing that $i = \delta k, \forall t$ implies that neither the capital stock nor investment can ever deviate from their steady state value, i.e. $\hat{k}_t = \hat{i}_t = 0, \ \forall t$.

In order to condense model 1 into a system of equations containing a minimum of endogenous variables, simply apply substitution to obtain difference equations consisting of the two endogenous state variables debt and capital, the control variable consumption and the exogenous state variable productivity.[22] Note that capital and its associated equations are kept on board for now. This is because the restriction that capital and investment do not deviate

[21] They do go on to say that: "However, real business cycle analysis suggests that this conclusion [may be] unwarranted: the process of investment and capital accumulation can be very important for how the economy responds to shocks."
[22] This discussion is based on Schabert (2004).

from their steady state ($\hat{k}_t = \hat{i}_t = 0,\ \forall t$) is not imposed until the final step to ensure that the collected coefficients on the other variables are accurate. The first step is to substitute all static equations into the dynamic equations given by (4.18) – (4.20). Solve for labor input by substituting (4.16) into (4.17):

$$\hat{h}_t = \left(\frac{\varepsilon_{cc} - \varepsilon_{hc}}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha}\right)\hat{c}_t + \frac{\hat{A}_t}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} + \frac{\alpha \hat{k}_t}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha}$$

At this point, some may vaguely remember that in the steady state of section 3.2, labor input was independent of consumption because of the structure of the utility function. Why should it be any different for an equation that has been log-linearized around the steady state? It's not! Solving for ε_{cc} and ε_{hc} (see appendix), it can be shown that the two coefficients are equal and therefore cancel consumption in the current period t. Hence the last equation actually reads:

$$\hat{h}_t = \frac{\hat{A}_t}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} + \frac{\alpha \hat{k}_t}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} = \gamma_1 \hat{A}_t + \alpha \gamma_1 \hat{k}_t \tag{4.29}$$

where it is shown in the technical appendix that:

$$\gamma_1 = \frac{1}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} > 0\,.$$

In what follows, the fact that $E_t \hat{A}_{t+1} = \rho \hat{A}_t$ will repeatedly be used. The next step is to use equation (4.29) to eliminate labor input in the remaining equations. Equation (4.18) after substituting (4.16) and (4.29) can be written as:

$$\frac{\psi \bar{d}}{(1+r)}\hat{d}_{t+1} + \varepsilon_{ch}\gamma_1 \alpha \hat{k}_{t+1} + \varepsilon_{cc} E_t \hat{c}_{t+1} = \varepsilon_{ch}\gamma_1 \alpha \hat{k}_t + \varepsilon_{cc}\hat{c}_t + \varepsilon_{ch}\gamma_1 (1-\rho)\hat{A}_t \tag{4.30}$$

This represents the first difference equation in debt, capital, consumption and the productivity process. Now substitute (4.16) and (4.29) into (4.19) and collect coefficients:

$$\beta\phi k E_t \hat{k}_{t+2} + \gamma_4 \hat{k}_{t+1} + \varepsilon_{cc} E_t \hat{c}_{t+1} = [\alpha\gamma_1 - \phi k]\hat{k}_t + \varepsilon_{cc}\hat{c}_t + \gamma_5 \hat{A}_t \tag{4.31}$$

where it can be calculated that:

$$\gamma_2 = \frac{1+\varepsilon_{hh} - \varepsilon_{ch}}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} > 0$$

$$\gamma_3 = \frac{1-\alpha}{\varepsilon_{hh} - \varepsilon_{ch} + \alpha} > 0$$

$$\gamma_4 = \left[\varepsilon_{ch}\gamma_1\alpha + \gamma_0(\gamma_2\alpha - 1) - \phi k\left(\beta + 1\right)\right] > 0$$

$$\gamma_5 = [\varepsilon_{ch}\gamma_1(1-\rho) - \gamma_0\gamma_2\rho] > 0$$

This is the second difference equation in consumption, capital and the productivity process. Now substitute (4.21), (4.22), (4.28) and (4.29) into (4.20):

$$\frac{s_{tb}}{r}\hat{d}_{t+1} = \frac{s_{tb}}{r}(1 + r + \psi\bar{d})\hat{d}_t - [\alpha\gamma_2 + s_k(1-\delta)]\hat{k}_t + s_c\hat{c}_t - \gamma_2\hat{A}_t \tag{4.32}$$

This is the third difference equation in debt, capital, consumption and the productivity process. Equations (4.30) – (4.32) and the equation for the exogenous productivity process (4.28) constitute a three dimensional specification of the entire linearized model, where the productivity process is counted as a separate dimension because it is exogenous. Note that this three dimensional specification is equivalent to model 1, except that the model has been reduced in such a way that it is represented by three difference equations in three endogenous variables plus one exogenous variable. Even for a model this size, it is not an easy feat to analyze the solution using 'back of the envelope' calculations. To facilitate an intuitive discussion of solution methods, model 4 therefore imposes the restriction that capital does not deviate from the steady state ($\hat{k}_t = 0$), which is synonymous with the assumption that capital perpetually takes on its steady state value. This implies that equation (4.31), the Euler equation that results from optimization with respect to the capital stock, must also be eliminated since capital would never have been a choice variable in the first place. Eliminating capital from the remaining two equations leads to the final version of model 4:

$$\frac{s_{tb}}{r}\hat{d}_{t+1} = \frac{s_{tb}}{r}(1 + r + \psi\bar{d})\hat{d}_t + s_c\hat{c}_t - \gamma_2\hat{A}_t \qquad (4.33)$$

$$\frac{\psi\bar{d}}{(1+r)}\hat{d}_{t+1} + \varepsilon_{cc}E_t\hat{c}_{t+1} = \varepsilon_{cc}\hat{c}_t + \varepsilon_{ch}\gamma_1(1-\rho)\hat{A}_t \qquad (4.34)$$

4.2.1 Solution Stability and Uniqueness of Model 4

Now define the vector $\hat{x}_t = (\hat{d}_t \quad \hat{c}_t)'$ and note that $E_t\hat{d}_{t+1} = \hat{d}_t$ because debt is an endogenous state variable and hence predetermined in each period. Collecting coefficients creates the following matrices:

$$A_0 = \begin{pmatrix} \frac{s_{tb}}{r} & 0 \\ \frac{\psi\bar{d}}{(1+r)} & \varepsilon_{cc} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} \frac{s_{tb}}{r}(1 + r + \psi\bar{d}) & s_c \\ 0 & \varepsilon_{cc} \end{pmatrix}$$

$$C_1 = \begin{pmatrix} -\gamma_2 \\ \varepsilon_{ch}\gamma_1(1-\rho) \end{pmatrix}$$

Equations (4.33) – (4.34) can therefore be written in matrix notation as:

$$A_0\begin{pmatrix} \hat{d}_{t+1} \\ E_t\hat{c}_{t+1} \end{pmatrix} = A_1\begin{pmatrix} \hat{d}_t \\ \hat{c}_t \end{pmatrix} + C_1\hat{A}_t \Rightarrow \begin{pmatrix} \hat{d}_{t+1} \\ E_t\hat{c}_{t+1} \end{pmatrix} = A_0^{-1}A_1\begin{pmatrix} \hat{d}_t \\ \hat{c}_t \end{pmatrix} + A_0^{-1}C_1\hat{A}_t \qquad (4.35)$$

The right-hand hand side of the last expression makes use of the fact that the matrix A_0 on the left-hand side is non-singular (its determinant is non-zero) and hence invertible. This can easily be inferred from the fact that the rows of A_0 are linearly independent, which creates a non-zero determinant. Now define the square matrix $A = A_0^{-1}A_1$ and the vector $C = A_0^{-1}C_1$. In compact form, the reduced model now reads:

$$E_t\hat{x}_{t+1} = A\hat{x}_t + C\hat{A}_t \tag{4.36}$$

For equation (4.36) the inverse of A_0 such that $A_0^{-1}A_0 = I$ is needed to obtain $A = A_0^{-1}A_1$ and $C = A_0^{-1}C_1$. These matrices are listed in the technical appendix. Information about model 4's solution, its stability and its existence can be gathered by examining the characteristic polynomial of the matrix A, defined as:

$$G(\lambda) = \lambda^2 - \lambda tr(A) + \det(A)$$

where $tr(A) = a_{11} + a_{22}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, and a_{ij} denotes the coefficient located in row i, column j of the matrix A. The scalar λ takes on two eigenvalues, both of which satisfy the characteristic polynomial at zero. Blanchard and Kahn (1980) proved that, in terms of a two dimensional model, at least one eigenvalue needs to be less than one in absolute value in order for a unique and stable solution to exist. In general, they show that in any dimensional system: "…if the number of eigenvalues of A outside the unit circle is equal to the number of non–predetermined variables, then there exists a unique solution" (p.1308).[23] The characteristic polynomial of A equals:

$$G(\lambda) = \lambda^2 - \lambda\left[(2 + r + \psi\bar{d}) - \left(\frac{r}{s_{tb}}\right)\left(\frac{\psi\bar{d}s_c}{(1+r)\varepsilon_{cc}}\right)\right] + (1 + r + \psi\bar{d}) = 0$$

[23] In terms of the three dimensional model including capital, which features two predetermined variables (the capital stock and debt) and one non-predetermined variable (consumption), the Blanchard and Kahn finding implies that there must be two eigenvalues inside and one eigenvalue outside of the unit circle.

Instead of directly solving this quadratic equation, it can be 'guesstimated' what kind of roots (eigenvalues) solve the characteristic polynomial by determining its graph on the interval [0,1]:

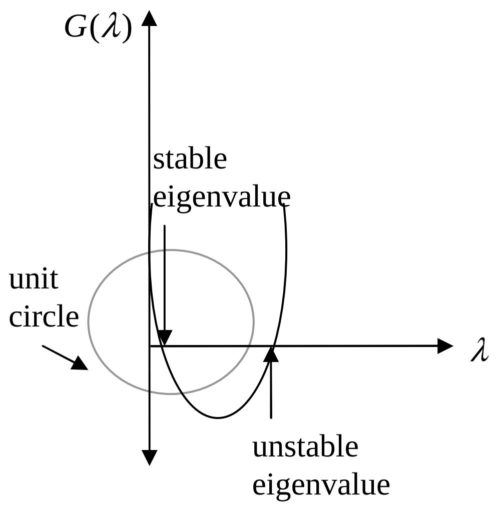
$$G(0) = 1 + r + \psi\bar{d} > 1$$

and

$$G(1) = \left(\frac{r}{s_{tb}}\right)\left(\frac{\psi\bar{d}s_c}{(1+r)\varepsilon_{cc}}\right) < 0 \;\; \text{(since } \bar{d} > 0, \varepsilon_{cc} < 0\text{)}$$

With this information it can quickly be inferred that the graph must be an upward sloping parabola that crosses the horizontal axis in $(\lambda, G(\lambda))$ space exactly once on the interval [0,1].[24] This, in turn, shows that there is one positive, stable eigenvalue that lies within the unit circle and one positive, unstable eigenvalue that lies outside of it. Since there is just one non-predetermined variable each period (consumption) and one eigenvalue outside the unit circle, the existence, stability and uniqueness of the solution for model 4 is assured.

Figure 4.1: The Characteristic Polynomial of the Matrix *A* for Model 4



[24] The above findings hold for $\bar{d} > 0$, in case of the emerging economy described by model 2, where $\bar{d} < 0$, this would simply be reversed such that $G(0) < 1$, $G(1) > 0$.

Of course, the eigenvalues of the characteristic polynomial can also be calculated directly, which is analogous to finding the roots of a quadratic equation. Given values for the coefficients contained within A, the two eigenvalues are:

$$\lambda_{1,2} = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2} \right] \tag{4.37}$$

Of these one will be less than one and the other larger than one in absolute value, as predicted by figure 4.1. The actual values can be computed using the parameters of table 3.2. As it turns out, $\lambda_1 = 0.993$ and $\lambda_2 = 1.024$.

4.2.2 Characterizing the Solution of Model 4

As mentioned in the chapter introduction, the solution is characterized by policy functions for control variables and by transition functions for next period's endogenous state variables. For flow variables, the solution function will haphazardly be termed policy function as well, even though each flow variable can be a function of just state or control variables or a combination thereof. The solution functions depend on the state space, which is simply given by the current period's endogenous and exogenous state variables. For example, model 4's state space is $\left\{\hat{d}_t, \hat{A}_t\right\}$ and model 1's state space is $\left\{\hat{d}_t, \hat{k}_t, \hat{A}_t\right\}$.

Bearing definitions 5 and 6 of this chapter's introduction in mind, the vector of $n_C \times 1$ endogenous control variables $\hat{x}_t^C$ and the vector of next period's $n_S \times 1$ endogenous state variables $\hat{x}_{t+1}^S$ is solved by policy and transition functions that depend on the state space $\left\{\hat{x}_t^S, \hat{z}_t\right\}$. For notational simplicity, assume throughout that there is only one exogenous process ($n_Z = 1$), although $\hat{z}_t$ could technically represent multiple exogenous state variables. Let the system's dimension, as in the previous section, be described by $n = n_S + n_C$, i.e. by the number of endogenous variables, while the $n_Z = 1$ exogenous process is treated separately. It is formally proven in sections 4.3 and 4.4 that both $g(\cdot)$ and $h(\cdot)$ are linear functions that can be expressed as:

$$\hat{x}_t^C = g(\hat{x}_t^S, \hat{z}_t) = V\left(\hat{x}_{jt}^S \quad \hat{z}_t\right)' = \delta_{i1}\hat{x}_{1t}^S \ldots + \delta_{ij}\hat{x}_{jt}^S \ldots + \delta_{in_S}\hat{x}_{n_S t}^S + \delta_{iz}\hat{z}_t \tag{4.38}$$

$$\hat{x}_{t+1}^S = h(\hat{x}_t^S, \hat{z}_t) = W\left(\hat{x}_{jt}^S \quad \hat{z}_t\right)' = \delta_{j1}\hat{x}_{1t}^S \ldots + \delta_{jj}\hat{x}_{jt}^S \ldots + \delta_{jn_S}\hat{x}_{n_S t}^S + \delta_{jz}\hat{z}_t \tag{4.39}$$

$\forall i \in \left[1, \ldots., n_C\right], \forall j \in \left[1, \ldots., n_S\right]$.

Some intuition on what exactly these solution functions accomplish in model 4 follows for variables in levels, although the same logic naturally translates to model 1 and for variables in percentage deviations from the steady state:[25]

Suppose that at the beginning of $t = 0$, the representative agent 'inherits' a state of the economy characterized by initial conditions for the state variables d_0 and A_0 (for model 1 this would also include an initial condition for k_0). Upon observing these values, he decides how much to consume (c_0) which determines how much he needs to borrow or can lend abroad (d_1) in the following period (note that since d_1 is decided in period $t = 0$, it is a control variable in $t = 0$ and a predetermined state variable in $t = 1$). Suppose that during period $t = 0$, he inherits positive debt ($d_0 > 0$) and a low level of productivity (A_0 low) that pushes him towards accruing even more debt to finance future consumption. Thus he decides to borrow $d_1 > d_0$ in $t = 0$. Suppose now that at the beginning of period $t = 1$, a bad shock occurs ($\varepsilon_1 < 0 \Rightarrow A_1 < A_0$). Since $d_1 > d_0$, he will again base his consumption decision (c_1) in $t = 1$ on the fact that his net debt is even higher while his productivity is even lower. The state of the economy in $t = 1$ is therefore described by the pair d_1 and A_1 while c_1 is chosen, which then implicitly determines d_2. Unless he wants to starve and no positive productivity shock occurs, he will again decide to borrow from abroad, that is $d_2 > 0$. In period $t = 2$, $d_2 > 0$ carries over since it was already pre-determined and the value of A_2 relative to A_1 depends on the persistence of the shock in period $t = 1$ (or on whether another shock occurs). The representative agent again observes these values, makes his consumption decision c_2, which

[25] This way of elucidating the policy and transition functions is inspired by Oviedo (2005) who derives the intuition for a Brock and Mirman (1972) type economy (p.13).

in turn implicitly determines his debt position d_3 and so on. At any point in time, given the state of the economy described by d_t and A_t, the agent by assumption optimally chooses consumption c_t with his *policy function* at time t, which implicitly determines d_{t+1} via the *transition function*. The policy and transition functions, in combination with the initial conditions and the exogenous rule for the productivity process thus generate sequences of d_{t+1} and c_t for the length of time the shock persists (or until another shock occurs) .

4.3 The Method of Undetermined Coefficients

This approach is based on the assumption that the solution functions take the form given by equations (4.38) – (4.39). It solves for each delta coefficient on the endogenous and exogenous state variables using the interdependent equilibrium equations of the model (equations (4.33) – (4.34)). For example, an interdependent linearized equilibrium condition for the j^{th}–state variable may be a linear function of its own past values, the i^{th}–control variable and the exogenous process (as in equation (4.34)) such that:

$$\mathrm{a}\hat{x}^S_{j,t+1} = \underbrace{\mathrm{b}\hat{x}^S_{jt}}_{\substack{\text{state}\\\text{space}}} + \mathrm{c}\hat{x}^C_{it} + \underbrace{\mathrm{d}\hat{z}_t}_{\substack{\text{state}\\\text{space}}} \tag{4.40}$$

where a – d constitute arbitrary coefficients. Note that the first and last term on the right hand side of (4.40) are already part of the state space. In order to identify the undetermined coefficients insert the proposed transition and policy functions (4.38) –(4.39) into (4.40) as follows:

$$\mathrm{a}\left(\delta_{j1}\hat{x}^S_{1t}\ldots+\delta_{jj}\hat{x}^S_{jt}\ldots+\delta_{jn_S}\hat{x}^S_{n_St}+\delta_{jz}\hat{z}_t\right)=\mathrm{b}\hat{x}^S_{jt}+\mathrm{c}\left(\delta_{i1}\hat{x}^S_{1t}\ldots+\delta_{ij}\hat{x}^S_{jt}\ldots+\delta_{in_S}\hat{x}^S_{n_St}+\delta_{iz}\hat{z}_t\right)+\mathrm{d}\hat{z}_t$$

Combining terms on this last expression and setting the right hand side equal to zero yields:

$$\left[a\delta_{j1}+c\delta_{i1}\right]\hat{x}^S_{1t}+\left[a\delta_{jj}+b+c\delta_{ij}\right]\hat{x}^S_{jt}+\left[a\delta_{jn_S}+c\delta_{in_S}\right]\hat{x}^S_{n_St}+\sum_{p=2}^{n_S-1}\left[a\delta_{jp}+c\delta_{ip}\right]\hat{x}^S_{pt}$$
$$+\left[a\delta_{jz}+c\delta_{iz}+d\right]\hat{z}_t=0,\ \forall p\neq j$$

This process needs to be continued for each interdependent equilibrium equation and can become quite cumbersome for multi-dimensional models. In the end, a new set of equations arises –each of which resembles the last expression. This new set implicitly yields restrictions on the terms in brackets by which the undetermined coefficients can be identified. Usually once such a set of new equations in coefficients and parameters has been found, a single, obvious restriction arises, namely that these interdependent equations can only simultaneously hold if all bracketed terms equal zero. Using model 4 as an example, the policy and transition function can be expressed as:

$$\hat{c}_t=g(\hat{d}_t,\hat{A}_t)=\delta_{cd}\hat{d}_t+\delta_{cA}\hat{A}_t \tag{4.41}$$
$$\hat{d}_{t+1}=h(\hat{d}_t,\hat{A}_t)=\delta_{dd}\hat{d}_t+\delta_{dA}\hat{A}_t \tag{4.42}$$

It will be shown that the method of undetermined coefficients generates a quadratic equation in δ_{dd}, which equals the characteristic polynomial of section 4. Since δ_{dd} can therefore hypothetically take on two values that satisfy the characteristic polynomial $(\lambda_1$ or $\lambda_2)$, but only the eigenvalue whose modulus is less than one in absolute value yields a bounded solution for debt, it must be the case that the stable eigenvalue λ_1 and δ_{dd} are identical (also see section 6.3.3 'on the role of the stable eigenvalue') In other words, equation (4.42) can only be a non-explosive $AR(1)$ process if the autocorrelation coefficient δ_{dd} is less than one in absolute value. If δ_{dd} were to equal the unstable eigenvalue (λ_2), equation (4.42) would not converge back to a steady state as productivity shocks dissipated over time. This would constitute a violation of the transversality condition.

The first step towards determining the policy and transition functions is to condense the two difference equations (4.33) – (4.34) further to make the algebraic manipulations more tractable. Start by defining the following coefficients:[26]

[26] These are based on the assumption that steady state debt is positive.

$$\gamma_6 = \frac{s_{tb}}{r} > 0$$

$$\gamma_7 = \frac{s_{tb}}{r}(1 + r + \psi\overline{d}) > 0$$

$$\gamma_8 = \frac{\psi\overline{d}}{(1+r)} > 0$$

$$\gamma_9 = \varepsilon_{ch}\gamma_1(1-\rho) > 0$$

Now rewrite equations (4.33) – (4.34) –in terms of these gamma coefficients:

$$\gamma_6\hat{d}_{t+1} = \gamma_7\hat{d}_t + s_c\hat{c}_t - \gamma_2\hat{A}_t \tag{4.43}$$

$$\gamma_8\hat{d}_t + \varepsilon_{cc}E_t\hat{c}_{t+1} = \varepsilon_{cc}\hat{c}_t + \gamma_9\hat{A}_t \tag{4.44}$$

Next substitute the policy/transition function (4.41) – (4.42) into (4.43) – (4.44) and rearrange to set the right hand side equal to zero:

$$\gamma_6(\delta_{dd}\hat{d}_t + \delta_{dA}\hat{A}_t) = \gamma_7\hat{d}_t + s_c(\delta_{cd}\hat{d}_t + \delta_{cA}\hat{A}_t) - \gamma_2\hat{A}_t$$

$$\Rightarrow [\gamma_6\delta_{dd} - \gamma_7 - s_c\delta_{cd}]\hat{d}_t + [\gamma_6\delta_{dA} - s_c\delta_{cA} + \gamma_2]\hat{A}_t = 0 \tag{4.45}$$

and

$$\gamma_8(\delta_{dd}\hat{d}_t + \delta_{dA}\hat{A}_t) + \varepsilon_{cc}E_t(\delta_{cd}\hat{d}_{t+1} + \delta_{cA}\hat{A}_{t+1}) = \varepsilon_{cc}(\delta_{cd}\hat{d}_t + \delta_{cA}\hat{A}_t) + \gamma_9\hat{A}_t$$

$$\Rightarrow [\gamma_8\delta_{dd} + \varepsilon_{cc}\delta_{cd}\delta_{dd} - \varepsilon_{cc}\delta_{cd}]\hat{d}_t + [\gamma_8\delta_{dA} + \varepsilon_{cc}\delta_{cd}\delta_{da} + \varepsilon_{cc}\delta_{cA}(\rho - 1) - \gamma_9]\hat{A}_t = 0 \tag{4.46}$$

After some informed staring it should become clear that (4.45) and (4.46) can only hold simultaneously $\forall t$ if the four expressions enclosed by brackets all equal zero. Thus there exist four restrictions based on which the undetermined delta coefficients can be identified:

$$0 = \gamma_6\delta_{dd} - \gamma_7 - s_c\delta_{cd} \tag{4.47}$$

$$0 = \gamma_6\delta_{dA} - s_c\delta_{cA} + \gamma_2 \tag{4.48}$$

$$0 = \gamma_8\delta_{dd} + \varepsilon_{cc}\delta_{cd}(\delta_{dd} - 1) \tag{4.49}$$

$$0 = (\gamma_8 + \varepsilon_{cc}\delta_{cd})\delta_{dA} + \varepsilon_{cc}(\rho - 1)\delta_{cA} - \gamma_9 \tag{4.50}$$

How do we make sense of these last four conditions? The simplest course of action is to invoke the one prior that has already been determined: δ_{dd} is positive, less than one and equal to the stable eigenvalue of our model. Next solve for each delta coefficient as a function of δ_{dd}. Equation (4.47) yields:

$$\delta_{cd} = \frac{(\gamma_6\delta_{dd} - \gamma_7)}{s_c} \tag{4.51}$$

For (4.48), initially only an expression of δ_{cA} as a function of δ_{dA} can be obtained:

$$\delta_{cA} = \frac{(\gamma_6\delta_{dA} + \gamma_2)}{s_c} \tag{4.52}$$

The coefficient δ_{dA} can be solved for a as a function of δ_{dd} by substituting the last two expressions into (4.50).

$$\delta_{dA} = \frac{s_c\gamma_9 - \varepsilon_{cc}(\rho - 1)\gamma_2}{s_c\gamma_8 + \varepsilon_{cc}\gamma_6\delta_{dd} - \varepsilon_{cc}\gamma_7 + \varepsilon_{cc}(\rho - 1)\gamma_6} \tag{4.53}$$

Lastly, substitute the result for δ_{dA} into (4.52):

$$\delta_{cA} = \frac{\gamma_6}{s_c}\left[\frac{s_c\gamma_9 - \varepsilon_{cc}\gamma_2(\rho - 1)}{s_c\gamma_8 + \varepsilon_{cc}\gamma_6\delta_{dd} - \varepsilon_{cc}\gamma_7 + \varepsilon_{cc}\gamma_6(\rho - 1)}\right] + \frac{\gamma_2}{s_c} \tag{4.54}$$

So far, three of the delta coefficients have been identified in terms of δ_{dd}, the stable eigenvalue. Note that substituting (4.51) into (4.49) generates a quadratic equation in δ_{dd}:

$$\delta_{dd}^2 - \left((2 + r + \psi\bar{d}) - \left(\frac{\psi\bar{d}}{1 + r}\right)\left(\frac{s_c r}{\varepsilon_{cc}s_{tb}}\right)\right)\delta_{dd} + (1 + r + \psi\bar{d}) = 0$$

The above expression can be obtained by reinserting the definitions for the gamma coefficients. It should look familiar as it is nothing but the characteristic polynomial of the matrix A discussed in section 4.2.1. As discussed above, it must be the case that $\delta_{dd} = \lambda_1$, in order for the transition function of debt to be stationary. Since values are available for all the deep parameters (table 3.2) and $\delta_{dd} = \lambda_1$, it can be shown that $\delta_{dA} < 0$, $\delta_{cA} > 0$, $\delta_{cd} < 0$ while $0 < \delta_{dd} < 1$.

4.3.1 Interpreting the Undetermined Coefficients

So far, the policy and transition function coefficients for consumption and debt have been expressed in terms of the model's deep parameters. The interpretation of these delta coefficients is twofold: On the one hand, they describe how the control variable consumption and the state variable debt respond to percentage deviations in either of the two state space variables from the steady state. This reaction of the endogenous variables will also be in terms of percentage deviations from the steady state. In the case of productivity shocks, the delta coefficients with respect to the productivity process (δ_{dA} and δ_{cA}) determine the tracing of the curve that represents the impulse response functions over time. On the other hand, the delta coefficients can also be interpreted as elasticities (see Campbell, 1992), where each delta describes the partial elasticity of the first subscript variable with respect to the second subscript variable. Thus δ_{dA} is the partial elasticity of debt with respect to the productivity process.

Of course, the solution functions for the variables that were previously eliminated in model 4 can still be backed out using the same approach employed for consumption and debt. For instance, the policy functions for labor input, output and the interest rate can be written as:

$$\hat{h}_t = \delta_{hd}\hat{d}_t + \delta_{hA}\hat{A}_t \tag{4.55}$$

$$\hat{y}_t = \delta_{yd}\hat{d}_t + \delta_{yA}\hat{A}_t \tag{4.56}$$

$$\hat{r}_t = \delta_{rd}\hat{d}_t + \delta_{rA}\hat{A}_t \tag{4.57}$$

Next consider equations (4.29), (4.21) and (4.23) that describe each of these variables respectively. For equation (4.29), note that the constancy of the capital stock implies that $\hat{h}_t = \gamma_1\hat{A}_t$, i.e. percentage changes in labor input are entirely dependent on the percentage changes in the exogenous productivity process. For equation (4.55) it must therefore be the case that $\delta_{hd} = 0$ and the policy function for labor input is simply given by $\hat{h}_t = \delta_{hA}\hat{A}_t$. It follows that:

$$\delta_{hA} = \gamma_1 = 1/\left(\varepsilon_{hh} - \varepsilon_{ch} + \alpha\right) = 1/\left(\omega - 1 + \alpha\right) > 0 \tag{4.58}$$

In other words, labor input is positively affected by positive productivity shocks, as already discussed in chapter 3. The equation for output (4.21) without capital reads: $\hat{y}_t = \hat{A}_t + (1-\alpha)\hat{h}_t = \hat{A}_t + (1-\alpha)\delta_{hA}\hat{A}_t = \left[1 + (1-\alpha)\delta_{hA}\right]\hat{A}_t$.[27] Hence $\delta_{yd} = 0$ and:

$$\delta_{yA} = 1 + (1-\alpha)\delta_{hA} = \left(1 + (1-\alpha)\right)/\left(\omega - 1 + \alpha\right) = \omega/\left(\omega - 1 + \alpha\right) > 0 \tag{4.59}$$

As is to be expected, output profits from higher productivity. Lastly, to obtain the interest rate policy function, the transition function for debt must be substituted into equation(4.23):

$$\hat{r}_t = (r)^{-1}\psi\bar{d}\hat{d}_{t+1} \Rightarrow \hat{r}_t = (r)^{-1}\psi\bar{d}\,(\delta_{dd}\hat{d}_t + \delta_{dA}\hat{A}_t) = (r)^{-1}\psi\bar{d}\,\delta_{dd}\hat{d}_t + (r)^{-1}\psi\bar{d}\,\delta_{dA}\hat{A}_t$$

Therefore the solution coefficients for the interest rate policy function are given by:

$$\delta_{rd} = (r)^{-1}\psi\bar{d}\,\delta_{dd} > 0 \tag{4.60}$$

$$\delta_{rA} = (r)^{-1}\psi\bar{d}\,\delta_{dA} < 0 \tag{4.61}$$

In sum, the elasticities of the eliminated control variable h_t and the flow variables y_t and r_t with respect to the state space can be solved for recursively using the previously found

[27] Note that because labor input is solely dependent on the exogenous productivity process, so is output. As shown in chapter 3, this implies a one-to-one contemporaneous correlation between output and, labor input.

policy/transition function coefficients. This also extends to the remaining flow variables, which are now given a much briefer treatment. In model 4, the linearized trade balance (after eliminating investment) is given by:

$$\widetilde{tby}_t = (s_c + s_i)\,\hat{y}_t - s_c \hat{c}_t \tag{4.62}$$

To derive the policy function, again postulate a linear relationship with respect to the state space:

$$\widetilde{tby}_t = \delta_{tby,d}\hat{d}_t + \delta_{tby,A}\hat{A}_t \tag{4.63}$$

Substituting the previously found policy functions for $\hat{y}_t$ and $\hat{c}_t$ into (4.62) yields:

$$\widetilde{tby}_t = \left[(s_c + s_i)\,\delta_{yA}\hat{A}_t - s_c\left(\delta_{cd}\hat{d}_t + \delta_{cA}\hat{A}_t\right)\right] = -(s_c\delta_{cd})\,\hat{d}_t + \left[(s_c + s_i)\,\delta_{yA} - s_c\delta_{cA}\right]\hat{A}_t$$

Hence:

$$\delta_{tby,d} = -(s_c\delta_{cd}) > 0 \tag{4.64}$$

$$\delta_{tby,A} = \left[(s_c + s_i)\,\delta_{yA} - s_c\delta_{cA}\right] = \left[s_c\left(\delta_{yA} - \delta_{cA}\right) + s_i\delta_{yA}\right] > 0 \text{ (because } \delta_{yA} > \delta_{cA}\text{)} \tag{4.65}$$

Similarly, the current account to output ratio's policy function is determined by $\widetilde{cay}_t = \delta_{cay,d}\hat{d}_t + \delta_{cay,A}\hat{A}_t$. Substituting the policy functions for $\widetilde{tby}_t$ and $\hat{y}_t$ into this last expression yields:

$$\delta_{cay,d} = \delta_{tby,d} - s_{tb}\left[(r)^{-1}\psi\bar{d} + 1\right] > 0 \tag{4.66}$$

$$\delta_{cay,A} = \delta_{tby,A} + s_{tb}\delta_{yA} > 0 \tag{4.67}$$

Using the parameters values given in chapter 3, the values of each delta coefficient can be calculated. Table 4.2 summarizes these values for a one percent increase in the productivity process:

Table 4.2: Policy and Transition Function Coefficients for Productivity in Model 4 (in %)

Partial Elasticity	δ_{dA} (a)	δ_{cA}	δ_{hA}	δ_{yA}	δ_{rA}	$\delta_{tby,A}$	$\delta_{cay,A}$
Value	-0.57	1.09	1.39	1.94	-0.07	0.87	0.92

(a)Because debt is predetermined each period, this value refers to one period after the shock occurs.

4.4 The Eigenvalue Decomposition

This solution approach has been a long-time favorite for solving real business cycle models and gained prominence with the closed economy neoclassical growth model. The method is well documented by, for example, King et al (2002) or Burnside (2004). The gist of the eigenvalue decomposition is to take a system of interdependent equations, such as the one described by equations (4.33) – (4.34), and to decompose this system into a set of independent difference equations for each endogenous variable. These independent difference equations represent the policy and transition functions. As was the case for the method of undetermined coefficients, calculating the eigenvalue decomposition manually is not a quick and simple undertaking in a model that is larger than the two dimensional model 4. Instead, use of computer algorithms such as the one based on the Schur-decomposition of section 4.5 become inevitable. A brief discussion of the eigenvalue decomposition in a multi-dimensional model is deferred to the technical appendix.

4.4.1 Applying the Eigenvalue Decomposition to Model 4

Again suppose that the policy function for consumption can be described by $\hat{c}_t = g(\hat{d}_t, \hat{A}_t)$ and the transition function by $\hat{d}_{t+1} = h(\hat{d}_t, \hat{A}_t)$. Rather than postulating coefficients, the eigenvalue decomposition uses linear algebra to decouple equations (4.33) – (4.34) into independent difference equations conform with the policy and transition functions. The starting point is the matrix equation (4.36) and a definition of the *standard eigenvalue problem*:

> For any $n \times n$ matrix A, find the non–trivial solution to the equation $Ap = \lambda p \Rightarrow (A - \lambda I)p = 0$ where λ is the eigenvalue or characteristic root and p is an $n \times 1$ vector called the eigenvector. λ takes on n values each of which satisfies the characteristic equation $|A - \lambda I| = 0$. The solutions for the n eigenvectors p can be found by substituting each of the n eigenvalues into $(A - \lambda I)p = 0$. An eigenvalue is stable if $|\lambda| < 1$, it is unstable if $|\lambda| > 1$ and has a unit root if $|\lambda| = 1$.

The *eigenvalue decomposition theorem* states that the matrix A can be decomposed as $P \Lambda P^{-1} = A$ where the matrix P collects the eigenvectors (p) of A columnwise, Λ collects the eigenvalues (λ) of A on its main diagonal in ascending order (with zeroes in the off-diagonal elements) and P^{-1} simply inverts P such that $P^{-1}P = I$. Let $p_{ij} \in P(i, j)$ and $p^{ij} \in P^{-1}(i, j)$. In section 4.2.1, it was already proven that model 4 has two real and distinct eigenvalues, which implies that there are two real and distinct eigenvectors. The existence of P^{-1} is ensured by this distinctiveness of the eigenvectors. Decomposing equation (4.36) where $\hat{x}_t = (\hat{d}_t \quad \hat{c}_t)'$ yields:

$$E_t \hat{x}_{t+1} = P \Lambda P^{-1} \hat{x}_t + C \hat{A}_t \Rightarrow E_t P^{-1} \hat{x}_{t+1} = \Lambda P^{-1} \hat{x}_t + P^{-1} C \hat{A}_t$$

The next step is to define the auxiliary vector $\hat{y}_t = P^{-1} \hat{x}_t$, which implies:

$$E_t \hat{y}_{t+1} = \Lambda \hat{y}_t + Q \hat{A}_t \text{ where } Q = P^{-1} C \tag{4.68}$$

This decouples the auxiliary matrix system (4.68) into a pair of difference equations because Λ is diagonal:

$$E_t \hat{y}_{1,t+1} = \lambda_1 \hat{y}_{1,t} + q_1 \hat{A}_t \tag{4.69}$$

$$E_t \hat{y}_{2,t+1} = \lambda_2 \hat{y}_{2,t} + q_2 \hat{A}_t \tag{4.70}$$

As proven in the appendix, only the equation containing the unstable eigenvalue (due to the sorting of Λ this is equation (4.70)) needs to be solved forward in order to obtain the policy function for consumption and the transition function for debt.[28] First define the lag-operator on any generic variable x_t as $Lx_t = x_{t-1}$ while the lead-operator, the inverse of the lag-operator, is defined as $L^{-1}x_t = x_{t+1}$. In the appendix it is shown that a specific forward-looking solution of (4.70) may be written as:

$$\begin{aligned}
&\hat{y}_{2,t} = \lambda_2^{-1} E_t \hat{y}_{2,t+1} - \lambda_2^{-1} q_2 \hat{A}_t \Rightarrow \hat{y}_{2,t} - \lambda_2^{-1} E_t \hat{y}_{2,t+1} = -\lambda_2^{-1} q_2 \hat{A}_t \Rightarrow (1-\lambda_2^{-1}L^{-1}) E_t \hat{y}_{2,t} = -\lambda_2^{-1} q_2 \hat{A}_t \\
&\Rightarrow \hat{y}_{2,t} = -\lambda_2^{-1}(1-\lambda_2^{-1}L^{-1})^{-1} q_2 \hat{A}_t \Rightarrow \hat{y}_{2,t} = -\lambda_2^{-1}\left(1+\lambda_2^{-1}L^{-1}+\lambda_2^{-2}L^{-2}....\right) q_2 \hat{A}_t \\
&\Rightarrow \hat{y}_{2,t} = -\sum_{j=1}^{\infty} \lambda_2^{-j} q_2 E_t \hat{A}_{t+j-1}
\end{aligned} \tag{4.71}$$

Now expand the auxiliary matrix system as:

$$\begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \end{pmatrix} = \begin{pmatrix} p^{11} & p^{12} \\ p^{21} & p^{22} \end{pmatrix} \begin{pmatrix} \hat{x}_{1t} \\ \hat{x}_{2t} \end{pmatrix} = \begin{pmatrix} p^{11} & p^{12} \\ p^{21} & p^{22} \end{pmatrix} \begin{pmatrix} \hat{d}_t \\ \hat{c}_t \end{pmatrix}$$

The second equation yields $\hat{y}_{2t} = p^{21}\hat{d}_t + p^{22}\hat{c}_t$. Hence consumption can be expressed as:

$$\hat{c}_t = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t + \left(p^{22}\right)^{-1} \hat{y}_{2t} \tag{4.72}$$

Now simply substitute equation (4.71) into (4.72) and use successive substitution to show that the exogenous productivity process (4.28) can be solved as $E_t \hat{A}_{t+j-1} = \rho^{j-1} \hat{A}_t$:

[28] This approach is based on the closed economy model examined by Burnside (2004).

$$\hat{c}_t = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t - \left(p^{22}\right)^{-1}\left(\sum_{j=1}^{\infty}\lambda_2^{-j} q_2 E_t \hat{A}_{t+j-1}\right) = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t - \left(p^{22}\right)^{-1}\left(\sum_{j=1}^{\infty}\lambda_2^{-j} q_2 \rho^{j-1}\hat{A}_t\right)$$

$$\Rightarrow \hat{c}_t = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t - \left(p^{22}\right)^{-1}\lambda_2^{-1}\left(1+\rho\lambda_2^{-1}+\rho^2\lambda_2^{-2}\ldots\rho^{\infty}\lambda_2^{-\infty}\right)\left(q_2\hat{A}_t\right)$$

$$\Rightarrow \hat{c}_t = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t \ - \left(p^{22}\right)^{-1}\left(\frac{\lambda_2^{-1}}{1-\rho\lambda_2^{-1}}\right)\left(q_2\hat{A}_t\right) \text{ (since} \left|\rho^j\lambda_2^{-j}\right| < 1)$$

Therefore:

$$\hat{c}_t = -\left(p^{22}\right)^{-1} p^{21}\hat{d}_t - \left(\frac{\left(p^{22}\right)^{-1} q_2}{\lambda_2-\rho}\right)\hat{A}_t .$$

Or, in condensed form:

$$\hat{c}_t = \theta_{cd}\hat{d}_t + \theta_{cA}\hat{A}_t \text{ where } \theta_{cd} = -\left(p^{22}\right)^{-1} p^{21} \text{ and } \theta_{cA} = -\left(\frac{\left(p^{22}\right)^{-1} q_2}{\lambda_2-\rho}\right) \tag{4.73}$$

This is the linear policy function $\hat{c}_t = g(\hat{d}_t, \hat{A}_t) = V\begin{pmatrix}\hat{d}_t & \hat{A}_t\end{pmatrix}'$ expressing the control variable consumption as a function of the state space. The coefficient matrix on the policy function is given by $V = \begin{pmatrix}\theta_{cd} & \theta_{cA}\end{pmatrix}$. It is surprisingly simple to find the transition function for debt by returning to a generic version of equation (4.36) where the elements of $A(i, j)$ are denoted by a_{ij}.

$$\begin{pmatrix}\hat{d}_{t+1}\\ E_t\hat{c}_{t+1}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\hat{d}_t\\ \hat{c}_t\end{pmatrix} + \begin{pmatrix}c_1\\ c_2\end{pmatrix}\hat{A}_t$$

Solving for the first equation and substituting (4.73):

$$\hat{d}_{t+1} = a_{11}\hat{d}_t + a_{12}\hat{c}_t + c_1\hat{A}_t$$

$$\Rightarrow \hat{d}_{t+1} = a_{11}\hat{d}_t + a_{12}\left[-\left(p^{22}\right)^{-1} p^{21}\hat{d}_t - \left(\frac{\left(p^{22}\right)^{-1} q_2}{\lambda_2 - \rho}\right)\hat{A}_t\right] + c_1\hat{A}_t$$

$$\Rightarrow \hat{d}_{t+1} = \left[a_{11} - a_{12}\left(p^{22}\right)^{-1} p^{21}\right]\hat{d}_t + \left(\frac{(\lambda_2 - \rho)c_1 - a_{12}\left(p^{22}\right)^{-1} q_2}{\lambda_2 - \rho}\right)\hat{A}_t$$

Therefore:

$$\hat{d}_{t+1} = \theta_{dd}\hat{d}_t + \theta_{dA}\hat{A}_t \text{ where } \theta_{dd} = a_{11} - a_{12}\left(p^{22}\right)^{-1} p^{21} \text{ and } \theta_{dA} = \frac{(\lambda_2 - \rho)c_1 - a_{12}\left(p^{22}\right)^{-1} q_2}{\lambda_2 - \rho} \quad (4.74)$$

This is the linear transition function $\hat{d}_{t+1} = h(\hat{d}_t, \hat{A}_t) = W\left(\hat{d}_t \quad \hat{A}_t\right)'$ expressing the state variable debt in terms of the state space. The coefficient matrix on the transition function is given by $W = \left(\theta_{dd} \quad \theta_{dA}\right)$. In the appendix it is shown that the coefficient $\theta_{dd} = \lambda_1$, that is the coefficient on the endogenous state variable must equal the stable eigenvalue.

4.5 The Schur (QZ)-Decomposition for Higher Dimensional Models

Recall that model 4 is merely a simplified version of model 1 and that the central equation defining model 4 is given by $E_t\hat{x}_{t+1} = A\hat{x}_t + C\hat{A}_t$. This equation was derived based on the non-singularity of A_0, which in turn led to the definition of the square matrix $A = A_0^{-1}A_1$ and the vector $C = A_0^{-1}C_1$. It is important to realize that A_0 was invertible because model 4 was obtained by condensing model 1 in such a way that no deterministic equations, or rather only dynamic equations, remained. This meant that both of the difference equations defining model 4 (equations(4.33) – (4.34)) contained elements belonging to the vector $\hat{x}_{t+1}$ and therefore created linearly independent coefficients in A_0, making the latter non-singular. Had model 4 included a static equation that was not a function of $\hat{x}_{t+1}$, a row of zeroes in the matrix A_0 would have been obtained, implying a linear dependency among the rows. Both the eigenvalue decomposition and the method of undetermined coefficients were based on properties of the matrix $A = A_0^{-1}A_1$. The obvious question that arises is whether it is always necessary to reduce a system of static and dynamic equations into dynamic equations only. In other words, what are the consequences if the coefficient matrix A_0 can not be inverted to the right hand side to generate equation (4.36)? In such a situation, which is quite common for most contemporary real business cycle models, the Schur-decomposition (also known as the *QZ*-decomposition) is the correct solution approach.[29]

According to Klein, the "….generalization [allowed for by the Schur decomposition] allows static (intratemporal) equilibrium conditions to be included among the dynamic relationships,…reflecting that some equations in the original system state relationships among the variables in x_t with no reference to $[E_t x_{t+1}]$ (p.1409)." The idea behind the Schur decomposition is simply "…to try to reduce ('uncouple')… the system into a (block) triangular system of equations, and then to solve the system recursively in the sense that we first solve the second block, and then the first using the solution for the second" (Klein, p. 1410). This approach ought to ring a bell, as it is quite similar to the one taken by the eigenvalue decomposition discussed in the previous section.

Suppose now that model 1 can be described by the following matrix equation:

$$AE_t\hat{x}_{t+1} = B\hat{x}_t \qquad (4.75)$$

where A is now singular. In the terminology of section 4.2 $A = A_0$ and $B = A_1$, but A in equation (4.75) does not equal $A = A_0^{-1}A_1$ in equation (4.36)! For purposes of the following discussion, it is sensible to group the exogenous and endogenous state variables into one vector now labeled $\hat{x}_t^{ZS}$, which is of size $(n_Z + n_S)\times 1 = n_{ZS}\times 1$.[30] The vector $\hat{x}_t$ is now given

[29] This section is based on Oviedo (2005), Uribe (2005) and Klein (2000).

[30] In some applications, it may be the case that exogenous processes interact with one another. In that case the exogenous state vector $\hat{z}_t$ would be modelled with a *VAR*(*p*) rather than an *AR*(1) and we would not be able to group the exogenous and endogenous state variables together as done above. The solution for this case is presented in Klein's paper.

by $\hat{x}_t = \left(\hat{x}_t^{ZS} \quad \hat{x}_t^{C}\right)'$ where the number of variables contained within $\hat{x}_t$ is given by $m = n_{ZS} + n_C$ (to distinguish from the previous notation where $n = n_S + n_C$).

As was the case for the eigenvalue decomposition and the method of undetermined coefficients, the object of the game is to find the set of policy functions $g(\cdot)$ for the control variables such that $\hat{x}_t^{C} = V\hat{x}_t^{ZS}$ and the set of transition functions $h(\cdot)$ for the state variables such that $\hat{x}_{t+1}^{ZS} = W\hat{x}_t^{ZS}$. Note that once the law of motion for the variables contained within $\hat{x}_t$ and $\hat{x}_{t+1}$ as defined above has been found, the policy functions for the flow variables can easily be backed out (see section 4.3.1).

The focus of Klein (2000) is actually the complex generalized Schur form where A and B contain complex numbers. Since the linearized equilibrium conditions for model 1 contain no complex coefficients, it suffices for our purposes to focus on the real Schur form, which is computationally faster to obtain (see Klein, p.1411). This leads to the *principle of generalized eigenvalues*:

> For $m \times m$ matrices A and B, find the non–trivial solution to the equation $\lambda Ap = Bp$ where λ is a scalar and p is the $m \times 1$ eigenvector. λ takes on m values each of which is a *generalized* eigenvalue or characteristic root satisfying the previous equation. If A is non-singular and hence invertible, the generalized eigenvalue problem reduces to the standard eigenvalue problem described by $A^{-1}Bp = \lambda p$ (presented in section 4.4).

Similar to the eigenvalue decomposition, the Schur decomposition dissects matrices A and B by finding the square unitary (orthogonal) matrices Q and Z and the square upper triangular matrices S and T such that:[31]

$$QAZ = S \text{ and } QBZ = T \tag{4.76}$$

[31] According to Klein, it is actually sufficient for the real Schur form, if these matrices are merely block upper triangular. Block triangular matrices are square matrices divided into four even blocks, and either the upper right block contains only zeroes (lower block triangularity) or the lower left block contains only zeroes (upper block triangularity). Since this is loosening the restrictions, without loss of generality, we will assume that *S* and *T* are upper triangular as would be required by the complex generalized Schur form.

A square unitary matrix Q is one whose (conjugate) transpose Q' equals its matrix inverse Q^{-1} which guarantees the relationship $Q'Q = Q^{-1}Q = I$. A matrix S is said to be upper triangular if its entries in each row *i* and column *j* follow the rule that $s_{ij} = 0\ \forall i > j$, that is all entries below the main diagonal contain zeroes:

$$S = \begin{pmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{mm} \end{pmatrix}$$

The following rules regarding triangular matrices and the generalized eigenvalues will be useful:

- For any upper (lower) triangular matrix S, the elements on its main diagonal, denoted s_{ii}, equal the eigenvalues of the matrix.
- S^{-1} is also upper triangular and its elements are denoted by $s^{ij} \in S^{-1}(i, j)$. The elements on its main diagonal equal the inverse of the original elements on the main diagonal of S, that is $s^{ii} = s_{ii}^{-1}$ (note that this does not hold for the elements above the main diagonal, $s^{ij} \neq s_{ij}^{-1},\ \forall i \neq j$).
- The product of any two upper triangular matrices ST is also upper triangular.
- The generalized eigenvalues of A and B satisfy $\lambda_i(A, B) = t_{ii}/s_{ii}$ where $t_{ii} \in T(i,i),\ s_{ii} \in S(i,i)$ and it is assumed the ratios $|t_{ii}/s_{ii}|$ are sorted in ascending order.[32]

When A is singular, linearly dependent rows will create zero eigenvalues in the matrix S, implying that some $s_{ii} = 0 \Rightarrow \lambda_i(A, B) = \pm\infty$. In that case, consider λ_i an unstable generalized eigenvalue (even though it is technically an undefined or infinite eigenvalue). For all cases where $1 < |\lambda_i| < \infty$, consider λ_i unstable but finite, and for all cases where $|\lambda_i| < 1$ consider λ_i stable. The possibility for unit roots, i.e. $s_{ii} = t_{ii} \neq 0 \Rightarrow \lambda_i = 1$, is ignored.

Given the above information, the vector $\lambda_i(A, B) = |t_{ii}/s_{ii}|$ is sorted in such a way that the stable generalized eigenvalues come first and the unstable eigenvalues come second. Note

[32] This assumption is not natural, that is the matrices S and T do not necessarily sort themselves according to this rule. Sims' (2002) has written a MATLAB program used in Oviedo and Uhlig's algorithm that reorders the diagonal entries on the two matrices such that the ratios are presented in ascending order.

that this implies that the first block of entries on the main diagonal of S and T is such that $|s_{ii}| > |t_{ii}|,\ \forall i = [1,...,n_{zs}]$, thereby creating ratios inside the unit circle. Conversely, the second block of entries on the main diagonal of S and T is such that $|s_{ii}| < |t_{ii}|,\ \forall i = [n_{zs}+1,...,m]$; thereby creating ratios outside of the unit circle.

A quick re-labeling of notation will simplify things. Rewrite the index for the second 'control variable' block with unstable eigenvalues as $i = [n_{zs}+1,...,m] = [1,....,n_c]$, where the first entry in $i = [1,....,n_c]$ refers to the first control variable or the $(n_{zs}+1)^{th}$ variable of all m variables.

If the technical assumptions presented in Klein's paper are satisfied (these are well outside of the scope of this paper), the rule by Blanchard and Kahn mentioned in section 4.2.1 is satisfied. In other words, there are as many stable generalized eigenvalues as there are state variables and as many unstable generalized eigenvalues as there are control variables. Formally, the vector $|\lambda_i| < 1$ is of size $n_{ZS} \times 1$ and the vector $|\lambda_i| > 1$ is of size $n_C \times 1$.

The first step is to again define an auxiliary vector. This time use the unitary matrix Z to generate $\hat{y}_t = Z'\hat{x}_t = Z^{-1}\hat{x}_t$. To obtain the auxiliary system, rewrite (4.75)as follows:

$$AE_t\hat{x}_{t+1} = B\hat{x}_t \Rightarrow QAE_t\hat{x}_{t+1} = QB\hat{x}_t \Rightarrow QA\underbrace{Z^{-1}Z}_{I}E_t\hat{x}_{t+1} = QB\underbrace{Z^{-1}Z}_{I}\hat{x}_t \Rightarrow \underbrace{QAZ}_{S}E_tZ^{-1}\hat{x}_{t+1} = \underbrace{QBZ}_{T}Z^{-1}\hat{x}_t$$

$$\Rightarrow SE_t\hat{y}_{t+1} = T\hat{y}_t \qquad (4.77)$$

Next partition S, T, Z and $\hat{y}_t$ as:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

$$\hat{y}_t = \begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \end{pmatrix}$$

Note that S_{ii} and $T_{ii}, \forall i = 1,2$ are also upper triangular. Next rewrite equation (4.77) in terms of the partitioned matrices:

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} E_t \hat{y}_{1t+1} \\ E_t \hat{y}_{2t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \end{pmatrix}$$

This yields the following two equations:

$$S_{11} E_t \hat{y}_{1t+1} + S_{12} E_t \hat{y}_{2t+1} = T_{11} \hat{y}_{1t} + T_{12} \hat{y}_{2t} \tag{4.78}$$

$$S_{22} E_t \hat{y}_{2t+1} = T_{22} \hat{y}_{2t} \Rightarrow E_t \hat{y}_{2t+1} = S_{22}^{-1} T_{22} \hat{y}_{2t} \tag{4.79}$$

Consider the matrix multiplication of the second equation $S_{22}^{-1} T_{22}$.[33] According to the rules for upper triangular matrices given above, we know that S_{22}^{-1} is also upper triangular and that its elements, denoted by $\left(s^{ij}\right)_{22} \in S_{22}^{-1}(i,j)$, satisfy $\left(s^{ii}\right)_{22} = \left(s_{ii}^{-1}\right)_{22}$, $\forall i = j$ on the main diagonal. In short, entries on the main diagonal of S_{22}^{-1} are simply the inverse of the entries on the main diagonal of S_{22}. Therefore $S_{22}^{-1} T_{22}$ is given by:

$$S_{22}^{-1} T_{22} = \begin{pmatrix} s_{11}^{-1} & \cdots & s^{1n_c} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{n_c n_c}^{-1} \end{pmatrix}_{22} \begin{pmatrix} t_{11} & \cdots & t_{1n_c} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{n_c n_c} \end{pmatrix}_{22} = \begin{pmatrix} \left(t_{11} / s_{11}\right) & \cdots & \left(s^{1n_c} t_{1n_c} + \ldots + s^{1n_c} t_{n_c n_c}\right) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(t_{n_c n_c} / s_{n_c n_c}\right) \end{pmatrix}_{22}$$

Given the partitioning and sorting of the matrices S and T, all elements on the main diagonal of $S_{22}^{-1} T_{22}$ (i.e. the generalized eigenvalues) must be unstable, that is $\left(t_{ii} / s_{ii}\right)_{22} > 1$ $\forall i = [1,...,n_C]$. This implies that each equation in (4.79) is a first order difference equation where the autocorrelation coefficient is larger than one and therefore $\underbrace{\lim}_{t\to\infty} E_t \hat{y}_{2t+1}^i = \underbrace{\lim}_{t\to\infty} \left(\frac{t_{ii}}{s_{ii}}\right)^t \hat{y}_{2t}^i = \infty$. Since this is ruled out by non-explosiveness conditions for any

[33] Note that the matrix S_{22}^{-1} is guaranteed to exist, because the entries on the main diagonal of S_{22} are all nonzero by construction. Since any matrix is invertible as long as its determinant is nonzero and since the determinant of an upper triangular matrix is the product of its diagonal elements, the inverse must exist.

variables contained within our model (auxiliary or not), it must be the case that $\hat{y}_{2t}^{i}=0,\forall t,\forall i=[1,...,n_C]$. To see this more clearly, expand equation (4.79) and assume for simplicity that there are three control variables implying that the relevant index is given by $i=[1,...,n_C]=[1,2,3]$.

$$\begin{pmatrix} E_t\hat{y}_{2t+1}^{1} \\ E_t\hat{y}_{2t+1}^{2} \\ E_t\hat{y}_{2t+1}^{3} \end{pmatrix}=\begin{pmatrix} s_{11}^{-1} & s^{12} & s^{13} \\ 0 & s_{22}^{-1} & s^{23} \\ 0 & 0 & s_{33}^{-1} \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}\begin{pmatrix} \hat{y}_{2t}^{1} \\ \hat{y}_{2t}^{2} \\ \hat{y}_{2t}^{3} \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} E_t\hat{y}_{2t+1}^{1} \\ E_t\hat{y}_{2t+1}^{2} \\ E_t\hat{y}_{2t+1}^{3} \end{pmatrix}=\begin{pmatrix} \left(t_{11}/s_{11}\right) & \left(s^{12}t_{12}+s^{12}t_{22}\right) & \left(s^{13}t_{13}+s^{13}t_{23}+s^{13}t_{33}\right) \\ 0 & \left(t_{22}/s_{22}\right) & \left(s^{23}t_{13}+s^{23}t_{23}+s^{23}t_{33}\right) \\ 0 & 0 & \left(t_{33}/s_{33}\right) \end{pmatrix}\begin{pmatrix} \hat{y}_{2t}^{1} \\ \hat{y}_{2t}^{2} \\ \hat{y}_{2t}^{3} \end{pmatrix}$$

Solving for the last equation yields $E_t\hat{y}_{2t+1}^{3}=\left(t_{33}/s_{33}\right)\hat{y}_{2t}^{3}$. Since $\left(t_{33}/s_{33}\right)>1$ by construction, it must be the case that $\hat{y}_{2t}^{3}=0,\forall t$. Now solve for the second to last equation: $E_t\hat{y}_{2t+1}^{2}=\left(t_{22}/s_{22}\right)\hat{y}_{2t}^{2}+\left(s^{23}t_{13}+s^{23}t_{23}+s^{23}t_{33}\right)\hat{y}_{2t}^{3}$. Since $\hat{y}_{2t}^{3}=0\ \forall t$, $E_t\hat{y}_{2t+1}^{2}=\left(t_{22}/s_{22}\right)\hat{y}_{2t}^{2}$. Since $\left(t_{22}/s_{22}\right)>1$, it must also be the case that $\hat{y}_{2t}^{2}=0,\forall t$. Substituting these findings into the first equation yields $E_t\hat{y}_{2t+1}^{1}=\left(t_{11}/s_{11}\right)\hat{y}_{2t}^{1}$, which of course also implies that $\hat{y}_{2t}^{1}=0,\forall t$ since $\left(t_{11}/s_{11}\right)>1$.[34]

Now expand the auxiliary system of equations and substitute the above conclusion:

$$\begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \end{pmatrix}=\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}'\begin{pmatrix} \hat{x}_t^{ZS} \\ \hat{x}_t^{C} \end{pmatrix}\Rightarrow\begin{pmatrix} \hat{y}_{1t} \\ 0 \end{pmatrix}=\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}'\begin{pmatrix} \hat{x}_t^{ZS} \\ \hat{x}_t^{C} \end{pmatrix}.$$

To find the transpose of the partitioned unitary matrix Z, the entire matrix must first be transposed and then its individual sub-matrices.

[34] The reason we can not solve these unstable difference equations forward, as for the eigenvalue decomposition, is because the exogenous process is contained within the state variable vector.

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}' = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}' = \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{y}_{1t} \\ 0 \end{pmatrix} = \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} \hat{x}_t^{ZS} \\ \hat{x}_t^{C} \end{pmatrix}$$

This yields the following two equations:

$$\hat{y}_{1t} = Z'_{11}\hat{x}_t^{ZS} + Z'_{21}\hat{x}_t^{C} \tag{4.80}$$

$$0 = Z'_{12}\hat{x}_t^{ZS} + Z'_{22}\hat{x}_t^{C} \Rightarrow \hat{x}_t^{C} = -\left(Z'_{22}\right)^{-1} Z'_{12}\hat{x}_t^{ZS} \tag{4.81}$$

Clearly, equation (4.81) fits the description of a matrix policy function, as each control variable is now expressed as a function of the endogenous and exogenous state variables as governed by $V = -\left(Z'_{22}\right)^{-1} Z'_{12}$. Note that this is the exact same procedure that was used to find the policy function for $\hat{c}_t$ using the eigenvalue decomposition. To find the transition functions for the endogenous state variables substitute (4.81) into (4.80):

$$\hat{y}_{1t} = Z'_{11}\hat{x}_t^{ZS} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\hat{x}_t^{ZS}$$

$$\Rightarrow \hat{y}_{1t} = \left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]\hat{x}_t^{ZS} \tag{4.82}$$

$$\Rightarrow \hat{y}_{1t+1} = \left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]\hat{x}_{t+1}^{ZS} \tag{4.83}$$

The auxiliary system can be eliminated by returning to equation (4.78), applying the fact that $\hat{y}_{2t} = 0, \forall t$ and then inserting (4.82) – (4.83):

$$S_{11}E_t\hat{y}_{1t+1} = T_{11}\hat{y}_{1t} \Rightarrow E_t\hat{y}_{1t+1} = \left(S_{11}\right)^{-1} T_{11}\hat{y}_{1t}$$

Therefore:

$$\Rightarrow \left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]\hat{x}_{t+1}^{ZS} = \left(S_{11}\right)^{-1} T_{11}\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]\hat{x}_t^{ZS}$$

$$\Rightarrow \hat{x}_{t+1}^{ZS} = \left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]^{-1}\left(S_{11}\right)^{-1} T_{11}\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]\hat{x}_t^{ZS} \tag{4.84}$$

This expression represents the transition function for the endogenous and exogenous state variables, where $W=\left[Z'_{11}-Z'_{21}\left(Z'_{22}\right)^{-1}Z'_{12}\right]^{-1}\left(S_{11}\right)^{-1}T_{11}\left[Z'_{11}-Z'_{21}\left(Z'_{22}\right)^{-1}Z'_{12}\right]$. The coefficient matrix can be reduced to $W=Z_{11}\left(S_{11}\right)^{-1}T_{11}\left(Z_{11}\right)^{-1}$ as shown in the appendix.

Chapter 5

Conclusion

Economic interdependence of many countries participating in today's global economy has been steadily increasing during the recent decades. This often raises questions regarding the advantages and disadvantages of synchronized business cycles. For economists, the ability to model the individual business cycles of different types of economies, developed and emerging, or the interaction of their business cycles has therefore become increasingly important. The main findings of the previous analysis were that developed and emerging economies belonging to the same trade agreement and within close geographical proximity appeared to have little business cycle synchronization over as much as two and half decades of data, although the developed economies' cycles in this type of constellation were highly correlated. It was shown that it is theoretically possible to capture these cross-country findings using negative spillover effects in the productivity processes, which allow the emerging economies to exploit their comparative advantage. This in addition to individual business cycles features for both types of economies could be replicated using fairly standard international real business cycle analysis.

Chapter 2 first confirmed existing and then established some new stylized facts of business cycles for a sample of developed and emerging countries in North America and Europe (plus Australia). For the developed economies, these were given by consumption being less volatile than output, labor input being about as volatile as output, capital being about half as volatile as output and investment being is about three times more volatile than output, while the international variables were found to be acyclical. For developed economies, positive consumption and output correlations were found, although the former tended to be lower in comparison. For emerging economies the rankings of the volatility were similar to those of the developed economies except that, on the whole, variables displayed more volatility. An important finding in the literature on emerging economies was also confirmed: For both Mexico and the Eastern European economies, the relative standard deviation of consumption to output was larger than one, whereas it was lower than one for the developed economies.

Chapter 2 also showed that the developed economies in the sample display highly correlated business cycles that positively depend on geographical proximity and to a lesser extent to common membership in a trade agreement. Emerging economies belonging to the same regions or trade agreements, however, continue to display greater relative volatility and, more often than not, their business cycles were not in sync with those of the developed economies. This was not only the case for Mexico versus Canada and the U.S., but also for the set of emerging economies in Eastern Europe (the Czech Republic, Poland and the Slovak Republic) versus developed economies of the EU (Belgium, France, Germany, and the Netherlands) as well as Switzerland. The empirical implications of this were that there exist little and sometimes even negative consumption and output correlations among the set of developed and emerging countries in North America and Europe. Since positive productivity shocks should theoretically imply greater spillover effects (and hence positive output and consumption correlations) for countries with little geographical distance between them or for countries sharing membership in a regional trade agreement, it might seem surprising that there exist such little parallels between the business cycles of these countries. To account for this discrepancy between theoretical expectations and the data, the possibility that there exist positive productivity spillover effects going in the direction of developed to emerging economies but negative productivity spillover effects going in the direction of emerging to developed economies was considered. The two-country model of chapter 3 lent support to this conclusion, as the best results were obtained when parameterizing the productivity process in precisely this way.

Negative spillover effects can be interpreted as creating an opportunity for exploiting comparative advantages for the economy from which the spillover effect originates. An example of this was given by Zimmermann (1995), who explained that the development of the use of quartz in the watch industry by the Japanese could be interpreted as a negative spillover effect for the Swiss watch makers. Thus the Japanese were able to exploit a new comparative advantage at the expense of the Swiss economy, whose productivity and output in that sector declined. More generally, in both the developed EU and the developed NAFTA countries, it has been a frequent ‘phenomenon’ that production of (unskilled) labor-intensive goods is relocated to countries where labor is cheaper or direct investment more profitable. These areas constitute precisely the comparative advantages of the neighboring emerging economies. Therefore, an analogy to the Japanese-Swiss example might be warranted, which justifies the explanation of negative spillover effects.

Chapter 3 showed that the cross-country and intra-country stylized facts of the developed and emerging economies of chapter 2 could be matched using fairly standard small open economy real business cycle models. Two types of incomplete asset market models, one using an interest rate premium and the other using portfolio adjustment costs (both increasing in debt) were used to model three hypothetical economies: The interest rate premium approach was used to model a developed economy (model 1) and an emerging economy (model 2), which were then both compared to average statistics obtained in chapter 2. The portfolio adjustment cost approach (model 3) was used for a 'world economy' made up of the two previous economies, i.e. one developed and the other emerging and then compared to the appropriate intra- and cross-country statistics of chapter 2.

The one-country model for the developed economy correctly identified the volatility rankings of typical developed economies, such as greater output than consumption variability and greater investment than output variability. It showed that the trade balance and current account are acyclical but understated their volatility. The contemporaneous output and first order auto-correlations were all well matched with the exception that the contemporaneous correlation of capital and consumption with respect to output was overstated. The one-country model for the emerging economy also correctly predicted the volatility rankings of emerging economies. It was particularly successful at reproducing the fact that consumption is more volatile than output and that investment and the trade balance and current account ratios are more volatile than in developed economies (although the volatility of the international variables was again understated). In contrast to the model of the developed economy, this model accurately predicted the contemporaneous correlation of consumption and output. Unfortunately, the empirical finding that the trade balance and current account are even more acyclical in emerging than in developed economies could not be replicated. All in all, the model calibrated to an emerging country performed no worse than the model calibrated to a developed economy, leading to the conclusion that these types of real business cycle models can be an adequate tool for generating key business cycle features of emerging economies.

The main contribution of the two-country model was that it reproduced the fact that countries engage in less consumption than output smoothing and the fact that there seem to exist very small and sometimes negative cross-country consumption and output correlations among developed and emerging economies. Using a combination of a negative spillover effect from the emerging onto the developed economy and a slightly negative covariance for the technological innovations reproduced the above empirical findings. Some of the successes of the one-country emerging economy model, such as the higher consumption than output volatility, however, could no longer be captured by the two-country model.

In addition, the impulse response analysis for each of the three models led to sensible and intuitive results, of which the most prominent was that a positive productivity shock in a country who starts out as a borrower can decrease its debt holdings, while a positive productivity shock in a country who starts out as a lender will decrease its asset holdings. In addition, it showed how a productivity shock in the emerging economy combined with a negative spillover effect can imply opposite movements in business cycles which could theoretically corroborate the data findings of chapter 2.

Chapter 4 provided an in-depth treatment of solution methods for SDGE models, with an application to the small open economy real business cycle model for a developed economy. It showed how the eigenvalue decomposition and the method of undetermined coefficients could be applied to a two-dimensional version of this model. In addition, it discussed how the Schur decomposition can be used for more complex models with higher dimensions.

Chapter 6

Appendices

6.1 Data Appendix A: Chapter 2

All volatility measures (standard deviations) reported in chapter 2 are based on the longest available data span for each series within each country. All correlation measures are based on the longest common sample for each series within each country or across countries.

Sources:

The series are obtained from the OECD Economic Outlook database. Each series is described by "CC_seriesq" where "CC" stands for country code, "series" names the series and "q" stands for quarterly frequency:

Country Codes (CC):

- AUS = Australia
- BEL = Belgium
- CAN = Canada
- CZR = Czech Republic
- FRA = France
- GER = Germany
- NEL = Netherlands
- MEX = Mexico
- POL =Poland
- SLR = Slovak Republic
- SWI = Switzerland
- U.S. = United States

Name of series on a quarterly frequency:

- Output (y_t) is given by the volume of gross domestic product (CC_gdpvq)
- Household consumption (c_t) is given by the volume of private final consumption expenditure, (CC _cpvq).
- Government consumption (g_t) is given by the volume of government final consumption expenditure (CC _cgvq).
- Labor input is given by hours worked per employee in the total economy times total employment (CC _hrsq times CC _etq). When noted, it is given by hours worked per employee in the business sector times employment in the business sector (CC_hrsq times CC_etbq).
- The capital stock (k_t) is given by the volume of the total economy's capital stock (CC_ktvq). When noted, it is given by the volume of the business sector's capital stock (CC_kbvq).
- Saving (s_t) is given by $y_t - c_t - g_t$.
- Investment (i_t) is given by the volume of gross total fixed capital formation (CC_itvq).
- The trade-balance to output ratio (tby_t) is obtained by dividing the current value of net exports (CC_xgsq minus CC_mgsq) by the value of current GDP (CC_gdpq).
- The current-account to output ratio (cay_t) is the current account as a percentage of current GDP (CC_cbgdprq).
- The real interest rate series for Australia, Canada, the Netherlands and Mexico are those calculated by Neumeyer and Perri (2005).

Frequencies:

The following countries contain quarterly data for 1980:Q1-2006:Q4: Australia (AUS), Belgium (BEL), Canada (CAN), France (FRA), Mexico (MEX), Netherlands (NEL), Switzerland (SWI), and the U.S.. The Czech Republic (CZR) and the Slovak Republic (SLR) contain data from 1993:Q1–2006:Q4, Germany (GER) contains data for 1991:Q1-2006:Q4 and Poland (POL) contains data from 1990:Q1-2006:Q4. In addition, real interest rate series are available from Neumeyer and Perri (2005) for Australia (1980:Q1-2002:Q1), Canada (1980:Q1-2002:Q1), the Netherlands (1983:Q3-2002:Q2) and Mexico (1994:Q1-2002:Q2)

Estimation:

Each variable "x_t" is in constant prices (volume terms), then divided by the working age population to obtain per capita terms and finally logged. The exceptions are the trade-balance to output ratio, (tby_t), the current-account to output ratio (cay_t) and the real interest rate (r_t), which are all in percentage terms. All variables are detrended using the Hodrick-Prescott filter (setting the smoothing parameter λ=1600 for quarterly data)

6.2 Technical Appendix B: Chapter 3

6.2.1 Solving the Optimization Problem

Unless otherwise noted, all calculations refer to model 1. The representative agent's problem can be solved by partially differentiating the following Lagrangian:

$$L = \max_{\left\{\begin{matrix} c_t, h_t, \\ k_{t+1}, d_{t+1} \end{matrix}\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \left\{ \beta^t U(c_t, h_t) + \beta^t \lambda_t \left[A_t F(k_t, h_t) - c_t - k_{t+1} + (1-\delta) k_t - \Phi(k_{t+1} - k_t) - (1+r_t) d_t + d_{t+1} \right] \right\}$$

$$\frac{\partial L}{\partial c_t}: \quad U_c(c_t, h_t) = \lambda_t$$

$$\frac{\partial L}{\partial h_t}: \quad -U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t)$$

$$\frac{\partial L}{\partial d_t}: \quad \lambda_t = \beta(1+r_t) E_t \lambda_{t+1}$$

$$\frac{\partial L}{\partial k_{t+1}}: \quad \lambda_t \left[1 + \Phi'(k_{t+1} - k_t) \right] = \beta E_t \lambda_{t+1} \left[A_{t+1} F_k(h_{t+1}, k_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right]$$

The agent's first order conditions are represented by the above equations, the transversality condition and the constraints given in chapter 3. The existence of a rational expectations equilibrium for the endogenous control and state variables ensures that the equations for the flow variables can be solved, since the latter are defined as functions of endogenous control and state variables.

6.2.2 Differentiation Needed for Steady State Determination

The following shows how to calculate the derivatives of the utility function, the production function, the capital adjustment cost function and the interest rate premium function as well as the epsilon coefficients used in chapter 4.

The Utility Function

Partially differentiating $U(c,h)=\dfrac{(c-\omega^{-1}h^{\omega})^{1-\gamma}-1}{1-\gamma}$ yields:

$U_c(c,h)=(c-\omega^{-1}h^{\omega})^{-\gamma}>0$

$U_{cc}(c,h)=-\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}<0$

$U_{ch}(c,h)=\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}h^{\omega-1}>0$

$U_h(c,h)=-(c-\omega^{-1}h^{\omega})^{-\gamma}(h^{\omega-1})<0$

$U_{hh}(c,h)=-\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}(h^{\omega-1})(h^{\omega-1})-(\omega-1)(h^{\omega-2})(c-\omega^{-1}h^{\omega})^{-\gamma}<0$

$U_{hc}(c,h)=\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}h^{\omega-1}>0$

Therefore:

$$\varepsilon_{cc}=c\frac{U_{cc}}{U_c}=c\frac{[-\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}]}{(c-\omega^{-1}h^{\omega})^{-\gamma}}=-\gamma c(c-\omega^{-1}h^{\omega})^{-1}<0 \tag{A.1}$$

$$\varepsilon_{ch}=h\frac{U_{ch}}{U_c}=h\frac{[+\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}]h^{\omega-1}}{(c-\omega^{-1}h^{\omega})^{-\gamma}}=\gamma h^{\omega}(c-\omega^{-1}h^{\omega})^{-1}>0 \tag{A.2}$$

$$\varepsilon_{hh} = h\frac{U_{hh}}{U_h} = h\frac{[-\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}(h^{\omega-1})(h^{\omega-1}) - (\omega-1)(h^{\omega-2})(c-\omega^{-1}h^{\omega})^{-\gamma}]}{-(c-\omega^{-1}h^{\omega})^{-\gamma}(h^{\omega-1})} \tag{A.3}$$

$$= \gamma h^{\omega}(c-\omega^{-1}h^{\omega})^{-1} + (\omega-1) = \varepsilon_{ch} + (\omega-1) > 0$$

$$\varepsilon_{hc} = c\frac{U_{hc}}{U_h} = c\frac{[\gamma(c-\omega^{-1}h^{\omega})^{-\gamma-1}]h^{\omega-1}}{-(c-\omega^{-1}h^{\omega})^{-\gamma}h^{\omega-1}} = -\gamma c(c-\omega^{-1}h^{\omega})^{-1} = \varepsilon_{cc} < 0 \tag{A.4}$$

The production function

Partially differentiating $F(k,h) = k^{\alpha}h^{1-\alpha}$ yields:

$F_k(k,h) = \alpha k^{\alpha-1}h^{1-\alpha}$

$F_h(k,h) = (1-\alpha)k^{\alpha}h^{-\alpha}$

$F_{kk}(k,h) = \alpha\left(\alpha-1\right)k^{\alpha-2}h^{1-\alpha}$

$F_{kh}(k,h) = \alpha\left(1-\alpha\right)k^{\alpha-1}h^{-\alpha}$

Note that F_{hh} and F_{hk} are not required.

The capital adjustment cost function

Differentiating $\Phi(x) = \frac{\phi}{2}x^2$ yields:

$\Phi'(x) = \phi x$

$\Phi''(x) = \phi$

The interest rate premium function

Differentiating $\rho(\tilde{d}_t) = \psi(\mathrm{e}^{\tilde{d}_t-\bar{d}}\text{ -1})$ yields:

$\rho'(\tilde{d}_t) = \psi \mathrm{e}^{\tilde{d}_t-\bar{d}}$

6.2.3 A Theoretical Discussion of Impulse Response Functions

It might be of interest to understand the mechanism that relates policy/transition functions to impulse response functions.[35] Suppose that the linearized transition functions for the set of exogenous (*Z*) and endogenous state (*S*) variables and the linearized policy functions for the set of control (*C*) and flow (*F*) variables have been obtained for model 1. The state space, on which both types of functions must depend, is therefore given by $\left\{\hat{d}_t, \hat{k}_t, \hat{A}_t\right\}$. The functions are given by:

$$\hat{x}_{t+1}^{ZS} = h\left(\hat{x}_t^{ZS}\right) \Rightarrow \left(\hat{A}_{t+1} \quad \hat{k}_{t+1} \quad \hat{d}_{t+1}\right)' = W\left(\hat{A}_t \quad \hat{k}_t \quad \hat{d}_t\right)' \tag{A.5}$$

$$\hat{x}_t^{CF} = g\left(\hat{x}_t^{ZS}\right) \Rightarrow \left(\hat{c}_t \quad \hat{h}_t \quad \hat{y}_t \quad \hat{r}_t \quad \hat{i}_t \quad \hat{s}_t \quad \widehat{tby}_t \quad \widehat{cay}_t\right)' = V\left(\hat{A}_t \quad \hat{k}_t \quad \hat{d}_t\right)' \tag{A.6}$$

where V and W contain the policy and transition coefficients. Since productivity is solely a function of its own past values and neither depends on debt nor capital (by definition of being exogenous), it turns out that the transition function of $\hat{A}_t$ actually equals its impulse response function. Recall that in the steady state $A = 1$. If there is a one percent increase in productivity at $t = 1$ ($\hat{\varepsilon}_1 = 1$ and $\hat{\varepsilon}_t = 0 \ \forall t > 1$),[36] the law of motion for the productivity process can be described by $\hat{A}_1 = \rho\hat{A}_0 + 1 = 1 = A$ because $\hat{A}_0 = 0$. At $t = 2$, $\hat{A}_2 = \rho\hat{A}_1 = \rho A$. At $t = 3$, $\hat{A}_3 = \rho\hat{A}_2 \Rightarrow \hat{A}_3 = \rho\rho\hat{A}_1 = \rho^2 A$, etc. In general, the impulse response function of productivity to a one percent increase in the technological innovation can therefore be described as a function of the persistence parameter and the steady state $IR\left(\hat{A}_t\right) = \rho^{t-1}A, \ \forall t \geq 1$.

Now consider the endogenous state variables debt, which –by definition of being an endogenous state variable– can not contemporaneously adjust to the shock (the same holds for capital). After the shock at $t = 1$, it must be the case that $\hat{d}_1 = 0$ and $\hat{k}_1 = 0$. Debt can,

[35] This discussion is partially based on Uribe (2005), who gives a briefer treatment on this topic.
[36] Note that these equations are written in terms of percentage changes, i.e. “1” rather than “0.01” must be used to indicate a one percent productivity increase.

however, adjust at $t=2$. Expanding the transition function described by equation (A5) for debt yields:

$$\hat{d}_2 = \delta_{dA}\hat{A}_1 + \delta_{dk}\hat{k}_1 + \delta_{dd}\hat{d}_1 = \delta_{dA}\times 1 + \delta_{dk}\times 0 + \delta_{dd}\times 0 = \delta_{dA}$$

The scalar δ_{dA} represents the initial response of debt to the technological innovation. Similarly, the initial response of capital to the technological innovation will be given by δ_{kA} in $t=2$. After period $t=2$, the relationship between all state variables' transition functions and the impulse response functions is given by:

$$IR\left(\hat{A}_t \quad \hat{k}_t \quad \hat{d}_t\right)' = W^{t-1}\left(\bar{x}^S\right) \Rightarrow IR\left(\hat{A}_t \quad \hat{k}_t \quad \hat{d}_t\right)' = \begin{pmatrix} \rho & 0 & 0 \\ \delta_{kA} & \delta_{kk} & \delta_{kd} \\ \delta_{dA} & \delta_{dk} & \delta_{dd} \end{pmatrix}^{t-1} \left(A \quad k \quad d\right)' \ \forall t > 2,$$

with $\bar{x}^S = \left(A \quad k \quad d\right)'$ denoting the steady state values of the state variables.

Now consider the policy functions for the control and flow variables (since the impulse response function for the auxiliary capital stock is the same as the one of the capital stock, except that it lags the latter by one period, it is excluded) given by equation (A.6). Given the sorting of variables it must be the case that:

$$V = \begin{pmatrix} \delta_{cA} & \delta_{hA} & \delta_{yA} & \delta_{rA} & \delta_{iA} & \delta_{sA} & \delta_{tby,A} & \delta_{cay,A} \\ \delta_{ck} & \delta_{hk} & \delta_{yk} & \delta_{rk} & \delta_{ik} & \delta_{sk} & \delta_{tby,k} & \delta_{cay,k} \\ \delta_{cd} & \delta_{hd} & \delta_{yd} & \delta_{rd} & \delta_{id} & \delta_{sd} & \delta_{tby,d} & \delta_{cay,d} \end{pmatrix}'$$

In contrast to debt and capital, the control variables can immediately adjust when productivity shocks occur. The flow variables may or may not jump depending on what kind of variable defines them. In model 1, all flow variables can actually adjust immediately because those that are a function of endogenous state variables only (therefore making them potentially 'sluggish'), are actually defined in terms of tomorrow's endogenous state variables, e.g. $r_t = f\left(d_{t+1}\right)$ or $i_t = f\left(k_{t+1}, k_t\right)$, therefore allowing them to adjust immediately. It follows that for $t=1$:

$$\hat{x}_1^{C,F} = V\left(\hat{A}_1 \quad \hat{k}_1 \quad \hat{d}_0\right) = V\left(1 \quad 0 \quad 0\right)'$$

$$\Rightarrow \left(\hat{c}_1 \quad \hat{h}_1 \quad \hat{y}_1 \quad \hat{r}_1 \quad \hat{i}_1 \quad \hat{s}_1 \quad \widehat{tby}_1 \quad \widehat{cay}_1\right)' = \begin{pmatrix} \delta_{cA} & \delta_{hA} & \delta_{yA} & \delta_{rA} & \delta_{iA} & \delta_{sA} & \delta_{tby,A} & \delta_{cay,A} \\ \delta_{ck} & \delta_{hk} & \delta_{yk} & \delta_{rk} & \delta_{ik} & \delta_{sk} & \delta_{tby,k} & \delta_{cay,k} \\ \delta_{cd} & \delta_{hd} & \delta_{yd} & \delta_{rd} & \delta_{id} & \delta_{sd} & \delta_{tby,d} & \delta_{cay,d} \end{pmatrix}' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\hat{c}_1 \quad \hat{h}_1 \quad \hat{y}_1 \quad \hat{r}_1 \quad \widehat{tby}_1 \quad \widehat{cay}_1\right)' = \left(\delta_{cA} \quad \delta_{hA} \quad \delta_{yA} \quad \delta_{rA} \quad \delta_{iA} \quad \delta_{sA} \quad \delta_{tby,A} \quad \delta_{cay,A}\right)'$$

Using successive substitution, it can be shown that for all following periods:

$$IR\left(\hat{x}_t^{C,F}\right) = V^{t-1} W\left(\overline{x}^S\right), \forall t \geq 2 \ .$$

6.2.4 Model 3: Remaining Business Cycle Statistics

The following table displays the standard deviations, relative standard deviations, contemporaneous output correlation and first order auto-correlations as well as the contemporaneous correlations among the remaining variables not mentioned in chapter 3 for model 3.

Table 6.1: Additional Business Cycle Statistics of Model 3 for a Developed (DE) and Emerging Economy (EE)

Variable (x)	Economy Type	$\sigma(x_t)$	$\frac{\sigma(x_t)}{\sigma(y_t)}$	$\rho(x_t, y_t)$	$\rho(x_t, x_{t-1})$
Capital (k)	DE	0.69	0.54	0.60	0.99
	EE	0.90	0.70	-0.07	0.94
Labor Input (h)	DE	0.91	0.71	1.00	0.80
	EE	1.37	1.07	-0.05	0.76
Saving (s)	DE	2.54	1.99	0.93	0.76
	EE	4.44	3.47	-0.03	0.76

Table 6.1 (continued)

Contemporaneous correlations $\rho\left(x_t, x_t^*\right)$

	k (DE)	k (EE)	h (DE)	h (EE)	s (DE)	s (EE)
k (DE)	1.00	-0.04	0.60	-0.02	0.31	0.04
k (EE)	-0.04	1.00	-0.07	0.67	-0.09	0.49
h (DE)	0.60	-0.07	1.00	-0.05	0.93	-0.03
h (EE)	-0.02	0.67	-0.05	1.00	-0.06	0.93
s (DE)	0.31	-0.09	0.93	-0.06	1.00	-0.04
s (EE)	0.04	0.49	-0.03	0.93	-0.04	1.00

6.3 Technical Appendix C: Chapter 4

6.3.1 Linearizing the Trade Balance and Current Account Ratios

Since the current account is zero in the steady state, log-linearization around the steady state is not possible, because it would imply division by zero. This problem is usually circumvented by normalizing the current account by output and applying a linear rather than log-linear representation to the equations (see Uhlig (2006) on this topic). Since the trade balance is part of the current account, it is also normalized by output using the same linear representation. Now define the variable x_i's deviations from the steady state as $\hat{x}_{it} = dx_{it} = \Delta x_{it} = x_{it} - x$ in the case of the trade balance and current account but continue, as before, to define $\hat{x}_{it} = dx_{it}/x$ for all other variables. For model 1, the linear representation of the trade balance to output ratio is therefore given by

$$tby_t = 1 - \left(\frac{c_t + i_t}{y_t}\right) \Rightarrow dtby_t = \frac{(c+i)dy_t}{y^2} - \frac{dc_t}{y} - \frac{di_t}{y}$$

$$\Rightarrow \widehat{tby}_t = \left(s_c + s_i\right)\hat{y}_t - s_c\hat{c}_t - s_i\hat{i} \tag{A.7}$$

For model 4, where investment and the capital stock are held constant, the same equation is simply given by: $\widehat{tby}_t = \left(s_c + s_i\right)\hat{y}_t - s_c\hat{c}_t$. Now consider the current account to output ratio equation:

$$cay_t = tby_t - \frac{r_t d_t}{y_t} \Rightarrow dcay_t = dtby_t - dr_t\left(\frac{d}{y}\right) - dd_t\left(\frac{r}{y}\right) + dy_t\left(\frac{rd}{y^2}\right)$$

$$\Rightarrow \widehat{cay}_t = \widehat{tby}_t - s_{tb}\left(\hat{r}_t + \hat{d}_t - \hat{y}_t\right) \tag{A.8}$$

For model 4 the same equation simply substitutes the trade balance to output ratio definition of model 4 ($\widehat{tby}_t = \left(s_c + s_i\right)\hat{y}_t - s_c\hat{c}_t$) into equation (A.8).

6.3.2 The Eigenvalue Decomposition in the *n*-Variable Case

This section shows the similarity between the Schur-decomposition and the eigenvalue decomposition in higher dimensions. Recall that the only difference is that the Schur-decomposition must be applied if deterministic equations are part of $A_0 E_t \hat{x}_{t+1} = A_1 \hat{x}_t$ because they create linearly dependent rows in the matrix A_0 which make the latter non-invertible. If this is not the case, the eigenvalue decomposition can be applied.

Recall that policy functions correspond to optimal rules for a vector of endogenous control variables $\hat{x}_t^C$ while transition functions correspond to the law of motion for a vector of next period's endogenous state variables $\hat{x}_{t+1}^S$. The state space $\left\{\hat{x}_t^S, \hat{z}_t\right\}$ represents the arguments for both types of functions, where $\hat{z}_t$ is a vector of stochastic but known exogenous processes as described in the main text:

$$\hat{x}_t^C = g(\hat{x}_t^S, \hat{z}_t) \tag{A.16}$$

$$\hat{x}_{t+1}^S = h(\hat{x}_t^S, \hat{z}_t) \tag{A.17}$$

Without loss of generality, equation (4.36) of the main text can be rewritten into a standard first order auto–regressive matrix equation of the form:

$$\hat{x}_{t+1} = A\hat{x}_t + \hat{z}_{t+1} \tag{A.18}$$

where $\hat{x}_t = \left(\hat{x}_t^S \quad \hat{x}_t^C \right)'$ is an $n \times 1$ stacked vector of endogenous state variables of size $n_S \times 1$ followed by endogenous control variables of size $n_C \times 1$, where $n_S + n_C = n$. The first step in decoupling (A.18) into independent difference equations for each variable in $\hat{x}_t$ is to consider the *standard eigenvalue problem* also defined in the main text. Assuming that there are n real and distinct eigenvalues and eigenvectors,[37] the next step is to define matrices:

$$\Lambda = \underbrace{\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}}_{n \times n}$$

$$P = \left(p_1 \ldots p_n \right)' = \underbrace{\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & \cdots & p_{nn} \end{pmatrix}}_{n \times n}$$

[37] The condition for this case is simply that distinct eigenvalues lead to distinct eigenvector. It can also happen that the eigenvalues are imaginary numbers. For the eigenvalues to be real it needs to be the case that $tr(A) > 4det(A)$.

$$P^{-1} = \left(p^1 \ \dots \ p^n \right)' = \underbrace{\begin{pmatrix} p^{11} & p^{12} & \dots & p^{1n} \\ p^{21} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ p^{n1} & \dots & \dots & p^{nn} \end{pmatrix}}_{n\times n}$$

where each p_i is an $n\times 1$ column eigenvector associated with the respective eigenvalue λ_i. Hence Λ, P and P^{-1} are all $n\times n$. Also assume that the eigenvalues on the diagonal of Λ are sorted in ascending order in absolute value. In other words, the first n_S eigenvalues are strictly less than one and the remaining n_C eigenvalues are strictly larger than one in absolute value. It is common in the literature to assume that none of the eigenvalues exhibit a unit root (or simply that a unit root is considered an unstable eigenvalue). The eigenvalue decomposition theorem states that the matrix A can be decomposed as $P\Lambda P^{-1} = A$ where the existence of P^{-1} is ensured by the distinctiveness of the eigenvectors. Equation (A.18) can hence be rewritten as:

$$\hat{x}_{t+1} = P\Lambda P^{-1}\hat{x}_t + \hat{z}_{t+1} \quad \Rightarrow \quad P^{-1}\hat{x}_{t+1} = \Lambda P^{-1}\hat{x}_t + P^{-1}\hat{z}_{t+1}$$

The next step is to define the auxiliary vector $\hat{y}_t = P^{-1}\hat{x}_t$ and the auxiliary exogenous process $\hat{u}_{t+1} = P^{-1}\hat{z}_{t+1}$, which implies:

$$\hat{y}_{t+1} = \Lambda\hat{y}_t + \hat{u}_{t+1} \tag{A.19}$$

This decouples the *auxiliary matrix system* into independent difference equations because the matrix Λ is diagonal. In other words:

$$\begin{aligned} &\hat{y}_{1,t+1} = \lambda_1\hat{y}_{1,t} + \hat{u}_{1,t+1} \\ &\hat{y}_{2,t+1} = \lambda_2\hat{y}_{2,t} + \hat{u}_{2,t+1}, \\ &\dots \\ &\hat{y}_{n,t+1} = \lambda_n\hat{y}_{n,t} + \hat{u}_{n,t+1} \end{aligned} \tag{A.20}$$

Each of these transformed equations is a standard first-order difference equation. Since the first n_S eigenvalues are assumed to be strictly less than one, the first n_S equations must be solved backward (or are stable in the backward looking direction). The remaining n_C equations with eigenvalues larger than one must be solved forward (are stable in the forward looking direction). Equations with initial conditions are usually solved backward, while equations with terminal conditions are solved forward. In addition, endogenous state variables usually require initial conditions (i.e. belong to the equations with eigenvalues less than one) while control variables usually require terminal conditions. By construction, the $\hat{x}$ vector consists of state variables first and control variables second.

Forward and Backward Looking Solutions for First-Order Difference Equations[38]

Recall that the lag-operator on any generic variable is $Lx_t = x_{t-1}$ while its inverse, the lead-operator, is $L^{-1}x_t = x_{t+1}$. A *general backward-looking solution* applying to the first n_S equations $\forall i \in [1,...,n_S]$ where $|\lambda_i| < 1$ in (A.20) can be found by rewriting any of the first n_S equations as:

$$\hat{y}_{i,t+1} = \lambda_i \hat{y}_{i,t} + \hat{u}_{i,t+1} \Rightarrow \hat{y}_{i,t} - \lambda_i \hat{y}_{i,t-1} = \hat{u}_{i,t} \Rightarrow \hat{y}_{i,t} - \lambda_i L \hat{y}_{i,t} = \hat{u}_{i,t} \Rightarrow (1-\lambda_i L)\hat{y}_{i,t} = \hat{u}_{i,t}$$

Next multiply both sides of the previous expression with $\left(1+\lambda_i L + \lambda_i^2 L^2 + ... + \lambda_i^t L^t\right)$:

$$\Rightarrow \left(1+\lambda_i L + \lambda_i^2 L^2 + ... + \lambda_i^t L^t\right)\left(1-\lambda_i L\right)\hat{y}_{i,t} = \left(1+\lambda_i L + \lambda_i^2 L^2 + ... + \lambda_i^t L^t\right)\hat{u}_{i,t}$$
$$\Rightarrow \left[\left(1+\lambda_i L + \lambda_i^2 L^2 + ... + \lambda_i^t L^t\right) - \left(\lambda_i L + \lambda_i^2 L^2 + ... + \lambda_i^{t+1} L^{t+1}\right)\right]\hat{y}_{i,t} = \sum_{j=0}^{t} \lambda_i^j \hat{u}_{i,t-j}$$
$$\Rightarrow \left(1-\lambda_i^{t+1} L^{t+1}\right)\hat{y}_{i,t} = \sum_{j=0}^{t} \lambda_i^j \hat{u}_{i,t-j} \Rightarrow \hat{y}_{i,t} - \lambda_i^{t+1}\hat{y}_{i,t-(t+1)} = \sum_{j=0}^{t} \lambda_i^j \hat{u}_{i,t-j} \Rightarrow \hat{y}_{i,t} - \lambda_i^{t+1}\hat{y}_{i,-1} = \sum_{j=0}^{t} \lambda_i^j \hat{u}_{i,t-j}$$

Thus the *general backward–looking solution* is represented by:

$$\hat{y}_{i,t} = \sum_{j=0}^{t} \lambda_i^j \hat{u}_{i,t-j} + \lambda_i^{t+1}\hat{y}_{i,-1}$$

[38] A similar discussion of these solutions can found in Hamilton (1994), chapter 2.

Since the models presented in this text are infinite horizon models, the specific solution can be found by letting $\lim t \to \infty$. Since $|\lambda_i| < 1$, the term $\lambda_i^{t+1} \hat{y}_{i,-1} \to 0$ as $\lim t \to \infty$. The *specific backward–looking solution* therefore is:

$$\hat{y}_{i,t} = \sum_{j=0}^{\infty} \lambda_i^j \hat{u}_{i,t-j} \quad \forall i \in [1,...,n_S] \tag{A.21}$$

A *general forward–looking solution* applying to the last n_C equations $\forall i \in [n_S+1,...,n]$ where $|\lambda_i| > 1$ can be found by rewriting any of the last n_C equations as:

$$\begin{aligned}
&\hat{y}_{i,t+1} = \lambda_i \, \hat{y}_{i,t} + u_{i,t+1} \Rightarrow \hat{y}_{i,t} = \lambda_i^{-1} \hat{y}_{i,t+1} - \lambda_i^{-1} u_{i,t+1} \Rightarrow \hat{y}_{i,t} - \lambda_i^{-1} \hat{y}_{i,t+1} = -\lambda_i^{-1} u_{i,t+1} \\
&\Rightarrow (1 - \lambda_i^{-1} L^{-1}) \hat{y}_{i,t} = -\lambda_i^{-1} L^{-1} u_{i,t} \ \Rightarrow \hat{y}_{i,t} = -\lambda_i^{-1} L^{-1} (1 - \lambda_i^{-1} L^{-1})^{-1} u_{i,t} \\
&\Rightarrow \hat{y}_{i,t} = -\lambda_i^{-1} L^{-1} \left(1 + \lambda_i^{-1} L^{-1} + \lambda_i^{-2} L^{-2} ....\right) u_{i,t} \\
&\Rightarrow \hat{y}_{i,t} = -\sum_{j=1}^{\infty} \lambda_i^{-j} u_{i,t+j} + \kappa \lambda_i^t
\end{aligned}$$

In most economic applications requiring forward looking solutions, the constant $\kappa = 0$, because $|\lambda_i| > 1$ and therefore $\kappa \lambda_i^t$ grows unbounded as $\lim t \to \infty$. This can usually be justified by imposing an intuitively sensible terminal condition on $\hat{y}_{i,t}$ similar to the transversality condition. Thus the *specific forward–looking solution* is:

$$\hat{y}_{i,t} = -\sum_{j=1}^{\infty} \lambda_i^{-j} u_{i,t+j} \quad \forall i \in [n_S+1,...,n] \tag{A.22}$$

Using Successive Substitution to Solve for Policy and Transition Functions

So far it has been demonstrated how the decoupled equations of the auxiliary system (A.20) can be solved according to the rules for first-order difference equations. However, it remains to be shown how this translates back to the system in the original variables for $\hat{x}_t$. The recipe is simply to combine the definition of the auxiliary system and either the forward- or backward-looking solutions, depending on the type of variable under consideration. As was the case in the main body of the text, only the forward-looking specific solutions will be needed (see Burnside (2004)). By definition of $\hat{y}_t = P^{-1}\hat{x}_t$:

$$\begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \\ \cdots \\ \hat{y}_{nt} \end{pmatrix} = \begin{pmatrix} p^{11} & p^{12} & \cdots & p^{1n} \\ p^{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p^{n1} & \cdots & \cdots & p^{nn} \end{pmatrix} \begin{pmatrix} \hat{x}_{1t} \\ \hat{x}_{2t} \\ \cdots \\ \hat{x}_{nt} \end{pmatrix} \text{where } p^{ij} \in P^{-1}(i,j)$$

Any $\hat{y}_{it}$ $\forall i \in [1,...,n]$ can be written as:

$$\hat{y}_{it} = p^{i1}\hat{x}_{1t} + ... p^{ii}\hat{x}_{it} + ... + p^{in}\hat{x}_{nt}$$

Any $\hat{x}_{it}$ $\forall i \in [1,...,n]$ can therefore be transformed back to:

$$\hat{x}_{it} = -\left(p^{ii}\right)^{-1}\left(p^{i1}\hat{x}_{1t} + ... + p^{in}\hat{x}_{nt} - \hat{y}_{it}\right) \tag{A.23}$$

A solution for $\hat{y}_{it}$ in (A.23) must either be given by equation (A.21) or (A.22) depending on whether $i \in [1,...,n_S]$ or $i \in [n_S+1,...,n] = [1,....,n_C]$, i.e. whether the index belongs to the set of backward- or forward-looking equations respectively. Each backward- or forward-looking solution for $\hat{y}_{it}$, in turn, is a solution in terms of the auxiliary exogenous process $\hat{u}_{it} = P^{-1}\hat{z}_{it}$, which may be regarded as the exogenous component of the state space scaled by P^{-1}. Note, however, that each $\hat{x}_{it}$ in equation (A.23) is still a combination of remaining state *and* control variables, which does not conform to the blueprint of the policy and transition functions. More precisely, there can not be any control variables as arguments in a policy function designed for another control variable or in a transition function for an endogenous state variable.

As a result, the entire system must be solved using backward substitution (as shown in section 4.4.1 for the eigenvalue decomposition in two dimensions and for the Schur-decomposition of section 4.5). As before, the trick to solving this system of equations is to consider the control variables first, starting with the last control variable. Applying equation

(A.23) to the last control variable such that $i = n$, it must be the case that $\hat{y}_{nt}$ requires a forward looking solution because $|\lambda_n| > 1$.

$$\hat{x}_{nt}^{C} = -\left(p^{nn}\right)^{-1}\left(\underbrace{p^{n1}\hat{x}_{1t}^{S} + \dots p^{nn_S}\hat{x}_{n_S,t}^{S}}_{n_S\text{-state variables}} + \underbrace{p^{n,n_S+1}\hat{x}_{n_S+1,t}^{C} + \dots\dots + p^{n,n-2}\hat{x}_{n-2,t}^{C} + p^{n,n-1}\hat{x}_{n-1,t}^{C}}_{(n_C\text{-1) control variables excluding } \hat{x}_{n,t}^{C}} - \hat{y}_{nt}\right)$$

Substituting (A.22) and defining $\Omega_{n,t} = \sum_{j=1}^{\infty} \lambda_n^{-j}\hat{u}_{n,t+j}$ implies:

$$\hat{x}_{nt}^{C} = \left(-p^{nn}\right)^{-1}\left(p^{n1}\hat{x}_{1t}^{S} + \dots + p^{nn_S}\hat{x}_{n_S,t}^{S} + p^{n,n_S+1}\hat{x}_{n_S+1,t}^{C} + \dots + p^{n,n-2}\hat{x}_{n-2,t}^{C} + p^{n,n-1}\hat{x}_{n-1,t}^{C} + \Omega_{n,t}\right) \qquad \text{(A.24)}$$

Now consider the second to last control variable such that $i = n-1$:

$$\hat{x}_{n-1,t}^{C} = \left(-p^{n-1,n-1}\right)^{-1}\left(p^{n-1,1}\hat{x}_{1t}^{S} + \dots + p^{n-1,n_S}\hat{x}_{n_S,t}^{S} + p^{n-1,n_S+1}\hat{x}_{n_S+1,t}^{C} + \dots \underbrace{+ p^{n-1,n-2}\hat{x}_{n-2,t}^{C} + p^{n-1,n}\hat{x}_{n,t}^{C}}_{\text{without } \hat{x}_{n-1,t}^{C}} + \Omega_{n-1,t}\right)$$

Use backward substitution to insert this last expression into (A.24):

$$\hat{x}_{nt}^{C} = \left(-p^{nn}\right)^{-1}\Big\{p^{n1}\hat{x}_{1t}^{S} + \dots + p^{nn_S}\hat{x}_{n_S,t}^{S} + p^{n,n_S+1}\hat{x}_{n_S+1,t}^{C} + \dots + p^{n,n-2}\hat{x}_{n-2,t}^{C} + p^{n,n-1}\Big[\left(-p^{n-1,n-1}\right)^{-1} \dots$$
$$\left(p^{n-1,1}\hat{x}_{1t}^{S} + \dots + p^{n-1,n_S}\hat{x}_{n_S,t}^{S} + p^{n-1,n_S+1}\hat{x}_{n_S+1,t}^{C} + \dots + p^{n-1,n-2}\hat{x}_{n-2,t}^{C} + p^{n-1,n}\hat{x}_{n,t}^{C} + \Omega_{n-1,t}\right)\Big] + \Omega_{n,t}\Big\}$$

Combining terms and pulling out the expression involving $\hat{x}_{n,t}^{C}$:

$$\hat{x}_{nt}^{C} = \left(-p^{nn}\right)^{-1}\Big\{\Big[p^{n1} - \left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,1}\right)\Big]\hat{x}_{1t}^{S} + \dots + \Big[p^{n,n_S} - \left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n_S}\right)\Big]\hat{x}_{n_S,t}^{S}$$
$$+ \Big[p^{n,n_S+1} - \left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n_S+1}\right)\Big]\hat{x}_{n_S+1,t}^{C} + \dots + \Big[p^{n,n-2} - \left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n-2}\right)\Big]\hat{x}_{n-2,t}^{C}$$
$$+ \Big[\Omega_{n,t} - \left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\Omega_{n-1,t}\Big]\Big\} + \Big[\left(p^{nn}\right)^{-1}\left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n}\right)\Big]\hat{x}_{n,t}^{C}$$

Pulling the last term involving $\hat{x}^C_{n,t}$ over to the left hand side:

$$\left[1-\frac{\left(p^{n,n-1}\right)\left(p^{n-1,n}\right)}{\left(p^{nn}\right)\left(p^{n-1,n-1}\right)}\right]\hat{x}^C_{n,t}=\left(-p^{nn}\right)^{-1}\{...\}\Rightarrow\left[\frac{\left(p^{nn}\right)\left(p^{n-1,n-1}\right)-\left(p^{n,n-1}\right)\left(p^{n-1,n}\right)}{\left(p^{nn}\right)\left(p^{n-1,n-1}\right)}\right]\hat{x}^C_{n,t}=\left(-p^{nn}\right)^{-1}\{...\}$$

Now pre–multiply both sides by the inverse of the expression in brackets before $\hat{x}^C_{n,t}$:

$$\hat{x}^C_{n,t}=\left[\frac{\cancel{\left(p^{nn}\right)}\left(p^{n-1,n-1}\right)}{\left(p^{nn}\right)\left(p^{n-1,n-1}\right)-\left(p^{n,n-1}\right)\left(p^{n-1,n}\right)}\right]\left(\frac{1}{\cancel{p^{nn}}}\right)\{...\}$$

Therefore:

$$\begin{aligned}\hat{x}^C_{n,t}=&\left[\frac{\left(p^{n-1,n-1}\right)}{\left(p^{nn}\right)\left(p^{n-1,n-1}\right)+\left(p^{n,n-1}\right)\left(p^{n-1,n}\right)}\right]\left\{\left[p^{n1}-\left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,1}\right)\right]\hat{x}^S_{1t}+...\right.\\&+\left[p^{n,n_S+1}-\left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n_S+1}\right)\right]\hat{x}^C_{n_S+1,t}+...+\left[p^{n,n-2}-\left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\left(p^{n-1,n-2}\right)\right]\hat{x}^C_{n-2,t}\\&\left.+\left[\Omega_{n,t}-\left(p^{n,n-1}\right)\left(p^{n-1,n-1}\right)^{-1}\Omega_{n-1,t}\right]\right\}\end{aligned}$$

What has been accomplished by this horrendous algebraic exercise? The control variable ($\hat{x}^C_{n-1,t}$) has successfully been eliminated The next step in backward substitution would be to eliminate $\hat{x}^C_{n-2,t}$ until we have eliminated the last remaining control variable $\hat{x}^C_{n_S+1,t}$. Therefore, $\hat{x}^C_{n,t}$ will solely be a function of the endogenous state variables x^S_{it} $\forall i=[1,...,n_S]$ and the exogenous processes captured by Ω_{it}, $\forall i\in\left[n_S+1,...,n\right]$. At that point, the policy function for the control variable $\hat{x}^C_{n,t}$ has been found since $\hat{x}^C_{nt}=g(\hat{x}^S_t,\Omega(\hat{u}_t))=g(\hat{x}^S_t,\Omega(P^{-1}\hat{z}_t))$ This successive substitution procedure would then have to be repeated for all remaining control variables.

The law of motion for each state variable $\hat{x}^S_{i,t+1}=g(\hat{x}^S_{it},\hat{z}_t)$ $\forall i\in\left[1,...,n_S\right]$ remains to be found. This turns out to be simpler than the previous exercise, assuming that the policy functions for all control variables have been found. The first equation of (A.18) can be obtained by expanding the matrix and vectors.

$$\begin{pmatrix} \hat{x}^S_{t+1} \\ \hat{x}^C_{t+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \hat{x}^S_t \\ \hat{x}^C_t \end{pmatrix} + \begin{pmatrix} \hat{z}_{1,t+1} \\ \hat{z}_{2,t+1} \end{pmatrix} \Rightarrow \hat{x}^S_{t+1} = A_{11}\hat{x}^S_t + A_{12}\hat{x}^C_t + \hat{z}_{1,t+1} \qquad \text{(A.25)}$$

where A_{ij}, z_i and z_j are partitioned matrices and vectors conform with the size of the control or state variable vector. Since all policy functions for the controls have been found, rewrite (A.25) as:

$$\hat{x}^S_{t+1} = A_{11}\hat{x}^S_t + A_{12}g(\hat{x}^S_t, \hat{z}_t) + \hat{z}_{1,t+1}$$

which almost corresponds to the blueprint for the transition function $\hat{x}^S_{t+1} = h(\hat{x}^S_t, \hat{z}_t)$ except for the fact that there still exists a period $t+1$ exogenous state variable vector. To circumvent this problem, take expectations of both sides and use (1) the fact that endogenous state variables are by definition predetermined, that is $E_t\hat{x}^S_{t+1} = \hat{x}^S_{t+1}$ and (2) the fact that the exogenous vector $\hat{z}_{t+1}$ follows a known process, i.e. $E_t\hat{z}_{t+1}$ can be expressed in terms of some forcing process $f(\hat{z}_t)$.

$$\hat{x}^S_{t+1} = a_{11}\hat{x}^S_t + a_{12}g(\hat{x}^S_t, \hat{z}_t) + f(\hat{z}_t) = h(\hat{x}^S_t, \hat{z}_t) \qquad \text{(A.26)}$$

If it has not become apparent already, it should now be emphasized that the eigenvalue decomposition can be a lot of strenuous work for large(r) state spaces and multiple control variables, such as is the case of model 1. One may be ill advised to try and solve these models manually (say, for $n > 3$).

6.3.3 The Eigenvalue Decomposition: On the Role of the Stable Eigenvalue in Transition Functions in Model 4

There is one minor detour worth taking regarding the endogenous state variable's coefficient $\theta_{dd} = a_{11} - a_{12}\left(p^{22}\right)^{-1} p^{21}$ in the transition function given by (4.74). Of course, since $\theta_{dd} = \delta_{dd}$, the same applies to the coefficient found using the method of undetermined coefficients.

Proposition I

The coefficient $\theta_{dd} = a_{11} - a_{12}\left(p^{22}\right)^{-1} p^{21}$ equals the stable eigenvalue λ_1.

Proof I

Using the definition of the eigenvalue decomposition in two dimensions:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} p^{11} & p^{12} \\ p^{21} & p^{22} \end{pmatrix} \text{ where } p^{ij} \in P^{-1}(i, j)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 p_{11} p^{11} + \lambda_2 p_{12} p^{21} & \lambda_1 p_{11} p^{12} + \lambda_2 p_{12} p^{22} \\ \lambda_1 p_{21} p^{11} + \lambda_2 p_{22} p^{21} & \lambda_1 p_{21} p^{12} + \lambda_2 p_{22} p^{22} \end{pmatrix}$$

First find an expression that reflects the above proposition, i.e. first find a_{11} and then $a_{12}\left(p^{22}\right)^{-1} p^{21}$ and then the difference between the two. The determinant needed for finding the inverse matrix P^{-1} is given by $|P| = p_{11}p_{22} - p_{12}p_{21}$, such that:

$$P^{-1} = \begin{pmatrix} p^{11} & p^{12} \\ p^{21} & p^{22} \end{pmatrix} = \begin{pmatrix} p_{22}/|P| & -p_{12}/|P| \\ -p_{21}/|P| & p_{11}/|P| \end{pmatrix}$$

Then:

$$a_{11} = \lambda_1 p_{11} p^{11} + \lambda_2 p_{12} p^{21} \Rightarrow a_{11} = \lambda_1 \frac{p_{11}p_{22}}{|P|} - \lambda_2 \frac{p_{12}p_{21}}{|P|}$$

Second:

$$a_{12}\left(\frac{p^{21}}{p^{22}}\right)=\lambda_1 p_{11}p^{12}\left(\frac{p^{21}}{p^{22}}\right)+\lambda_2 p_{12}p^{22}\left(\frac{p^{21}}{p^{22}}\right)=\lambda_1 p_{11}p^{12}\left(\frac{p^{21}}{p^{22}}\right)+\lambda_2 p_{12}p^{21}$$

$$\Rightarrow a_{12}\left(\frac{p^{21}}{p^{22}}\right)=\lambda_1 p_{11}\left(-\frac{p_{12}}{|P|}\right)\left(\frac{-p_{21}/|P|}{p_{11}/|P|}\right)+\lambda_2 p_{12}\left(-\frac{p_{21}}{|P|}\right)$$

$$\Rightarrow a_{12}\left(\frac{p^{21}}{p^{22}}\right)=\lambda_1\left(\frac{p_{12}p_{21}}{|P|}\right)-\lambda_2\left(\frac{p_{12}p_{21}}{|P|}\right)=\left(\frac{p_{12}p_{21}}{|P|}\right)(\lambda_1-\lambda_2)$$

Hence:

$$a_{11}-a_{12}\left(p^{22}\right)^{-1}p^{21}=$$

$$\lambda_1\frac{p_{11}p_{22}}{|P|}-\lambda_2\frac{p_{12}p_{21}}{|P|}-\left(\frac{p_{12}p_{21}}{|P|}\right)(\lambda_1-\lambda_2)=\lambda_1\left(\frac{p_{11}p_{22}}{|P|}-\frac{p_{12}p_{21}}{|P|}\right)-\left(\frac{p_{12}p_{21}}{|P|}\right)(\lambda_2-\lambda_2)=\lambda_1\frac{|P|}{|P|}=\lambda_1$$

Therefore it has been shown that the coefficient on the endogenous state variable in the transition function equals the stable eigenvalue.

6.3.4 The Schur-Decomposition: Reducing the Coefficient Matrix *W*

In section 4.5, it was shown that the transition functions for the endogenous and exogenous state variables are described by the matrix: [39]

$$W=\left[Z'_{11}-Z'_{21}\left(Z'_{22}\right)^{-1}Z'_{12}\right]^{-1}\left(S_{11}\right)^{-1}T_{11}\left[Z'_{11}-Z'_{21}\left(Z'_{22}\right)^{-1}Z'_{12}\right].$$

[39] This is based on Uribe (2005 classnotes).

The bracketed term $\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]$ can be reduced by considering the entries of:

$$I = Z'Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}' \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} Z'_{11}Z_{11} + Z'_{21}Z_{21} & Z'_{11}Z_{12} + Z'_{21}Z_{22} \\ Z'_{12}Z_{11} + Z'_{22}Z_{21} & Z'_{12}Z_{12} + Z'_{22}Z_{22} \end{pmatrix}$$

First solve for the entry $Z'Z(2,1) = I_{21}$ and then pre- and post-multiply by $\left(Z'_{22}\right)^{-1}$ and $\left(Z_{11}\right)^{-1}$ respectively:

$$I_{21} = Z'_{12}Z_{11} + Z'_{22}Z_{21} \Rightarrow Z'_{12}Z_{11} = -Z'_{22}Z_{21} \text{ (because } I_{21} = 0\text{)}$$
$$\Rightarrow \left(Z'_{22}\right)^{-1} Z'_{12} \underbrace{Z_{11}\left(Z_{11}\right)^{-1}}_{I} = -\underbrace{\left(Z'_{22}\right)^{-1} Z'_{22}}_{I} Z_{21}\left(Z_{11}\right)^{-1} \Rightarrow \left(Z'_{22}\right)^{-1} Z'_{12} = -Z_{21}\left(Z_{11}\right)^{-1}$$

This expression can be inserted in the bracketed term to give:

$$\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right] \Rightarrow \left[Z'_{11} + Z'_{21}Z_{21}\left(Z_{11}\right)^{-1}\right]$$

Now use $Z'Z(1,1)$ and insert this into this last expression:

$$I_{11} - Z'_{11}Z_{11} = Z'_{21}Z_{21}$$
$$\Rightarrow \left[Z'_{11} + Z'_{21}Z_{21}\left(Z_{11}\right)^{-1}\right] = \left[Z'_{11} + \left(I_{11} - Z'_{11}Z_{11}\right)\left(Z_{11}\right)^{-1}\right] = \left[Z'_{11} + \left(Z_{11}\right)^{-1} - Z'_{11}\underbrace{Z_{11}\left(Z_{11}\right)^{-1}}_{I}\right]$$
$$\Rightarrow \left[Z'_{11} + \left(Z_{11}\right)^{-1} - Z'_{11}\right] = \left(Z_{11}\right)^{-1}$$

Therefore $\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right] = \left(Z_{11}\right)^{-1}$ and $\left[Z'_{11} - Z'_{21}\left(Z'_{22}\right)^{-1} Z'_{12}\right]^{-1} = Z_{11}$ and the coefficient matrix is given by $W = Z_{11}\left(S_{11}\right)^{-1} T_{11}\left(Z_{11}\right)^{-1}$.

6.4 Computer Algorithms

This section conveys the relationship between the theoretical solution approaches of chapter 4 and applied computer algorithms for real business cycle models. On the most rudimentary level, the algorithms find the policy and transition functions $g(\cdot)$ and $h(\cdot)$ defined by the coefficient matrices V and W. Based on these solution functions, the algorithms are able to generate impulse response functions and business cycle summary statistics that would virtually be impossible to obtain manually for a complex model such as model 1.

When applying computer algorithms, the necessity to reduce the model's size –by substituting static conditions into dynamic conditions and backing out policy and transition functions for eliminated variables later on– hardly ever arises. Recall that this was precisely the approach taken for model 4. Uhlig (1997) observes that: "…[O]ne often sees researchers exploiting…equilibrium conditions to 'get rid off' some variables, and have only a few variables remaining…[T]here is no reason to go through the hassle of 'eliminating' variables by hand…since this is all just simple linear algebra applied to a system of equations, it is far easier…[to] leave it to the formulas to sort it all out (pp.33 - 34)."

Since the computer algorithms introduced below are both based on the solution method provided by the Schur decomposition of section 4.5, let model 1 be described by:

$$0 = AE_t\hat{x}_{t+1} - B\hat{x}_t \tag{A.27}$$

where for now $\hat{x}_t = \left(\hat{z}_t, \hat{x}_t^S, \hat{x}_t^{C,F}\right)$ and A and B collect the coefficients from the *linearized* equations (4.16) -(4.28). Analogously, the *non-linear* interdependent system of equations that defines model 1 in section 3.1 can be written as

$$0 = E_t f\left(x_{t+1}, x_t\right) \tag{A.28}$$

Two available 'toolboxes' for solving more elaborate models such as model 1 are due to Uhlig (1997) and Oviedo (2005). Both algorithms essentially perform the same tasks – computing policy and transition functions, graphing impulse response functions and calculating business cycle summary statistics for the simulated data.

The main difference between the two algorithms is that the interdependent *linear* system's coefficient matrices A and B need to be specified by the user in order to apply Uhlig's algorithm, i.e. the log-linearization needs to be carried out beforehand. Oviedo's algorithm, on the other hand, log-linearizes the equilibrium conditions of the non-linear system of equations $0 = E_t f\left(x_{t+1}, x_t\right)$ and therefore creates the two coefficient matrices needed for $0 = AE_t \hat{x}_{t+1} - B\hat{x}_t$ on its own. A second, subtler difference between the two toolboxes is that each algorithms defines the vector $\hat{x}_t$ and hence partitions A and B differently. This is due to notational differences involving: (1) the treatment of flow variables and (2) the treatment of exogenous processes. Let's briefly examine each of these points, as they have important implications for how the system is partitioned:

(1) Flow variables: Uhlig's algorithm does not include an explicit provision for flow variables and instead groups them either with the endogenous control or state variables. Recalling the definition of flow variables, there are some that are solely a function of endogenous state variables, such as the auxiliary capital stock at the end of period t, the debt-elastic interest rate and investment. These are intuitively categorized with the state variables and will be referred to as 'type 1' flow variables. [40] Those flow variables that are a function of both control and state variables will be referred to as 'type 2' flow variables and may be grouped with the control variables. The latter include output, saving, the trade balance ratio, etc. Thus the vector $\hat{x}_t^F$ is entirely usurped. Oviedo's algorithm, on the other hand, does differentiate between control, state and flow variables and creates a separate matrix equation for $\hat{x}_t^F$.

(2) Exogenous processes: Uhlig's algorithm uses a separate provision for the exogenous state vector $\hat{z}_t$, while Oviedo's algorithm includes $\hat{z}_t$ in the vector $\hat{x}_t$.

In sum:

- Uhlig's algorithm requires the user to specify A and B of the linearized system $AE_t \hat{x}_{t+1} = B\hat{x}_t$. It then partitions the latter to accommodate the vector $\hat{x}_t = \left(\hat{x}_t^S \quad \hat{x}_t^C\right)'$,

[40] Note that if the auxiliary capital stock is grouped with the state variables, then investment can also be written solely in terms of state variables and hence becomes a type 1 flow variable.

where $\hat{x}_t^{F1} \in \hat{x}_t^S$ and $\hat{x}_t^{F2} \in \hat{x}_t^C$ (F1= 'type 1' flow variable; F2 = 'type 2' flow variable). In addition it creates a separate matrix equation for the exogenous state vector $\hat{z}_t$.

- Oviedo's algorithm creates the system $AE_t\hat{x}_{t+1} = B\hat{x}_t$ by log-linearizing each of the non-linear conditions $0 = E_t f\left(x_{t+1}, x_t\right)$. It then partitions the linear system to accommodate the vector $\hat{x}_t = \left(\hat{z}_t \quad \hat{x}_t^S \quad \hat{x}_t^C\right)'$ and creates a separate matrix equation for the flow variable vector $\hat{x}_t^F$.
- Both algorithms apply the Schur-decomposition of section 4.5 as the solution mechanism to find the policy and transition functions.

6.4.1 Uhlig's Toolbox

This algorithm requires us to decompose equation (A.27) into three types of matrix equations: (1) Those that group the deterministic and backward-looking equations together, (2) those that group the forward-looking equations together and (3) those that group the equations that describe the potentially multiple exogenous processes together. The three matrix equations are given by:

$$0 = AA\hat{x}_t^S + BB\hat{x}_{t-1}^S + CC\hat{x}_t^C + DD\hat{z}_t \tag{A.29}$$

$$0 = E_t\left[FF\hat{x}_{t+1}^S + GG\hat{x}_t^S + HH\hat{x}_{t-1}^S + JJ\hat{x}_{t+1}^C + KK\hat{x}_t^C + LL\hat{z}_{t+1} + MM\hat{z}_t\right] \tag{A.30}$$

$$0 = E_t\left[\hat{z}_{t+1} - NN\hat{z}_t\right] \tag{A.31}$$

Matrices AA and BB must both be of size $\left(b \times n_S\right)$, CC must be of size $\left(b \times n_C\right)$ and DD must be of size $\left(b \times n_Z\right)$, where b equals the number of backward-looking or static equations. Matrices FF, GG and HH must all be of size $\left(f \times n_S\right)$ where f equals the number of forward-looking equations (potentially) involving the expectations operator.

Similarly, matrices JJ and KK must both be of size $\left(f \times n_C\right)$ and LL and MM of size $\left(f \times n_Z\right)$. Lastly, NN is a diagonal square matrix of size $\left(n_Z \times n_Z\right)$ containing the autocorrelation coefficients of the stochastic processes, which are assumed to be less than one in absolute value. Note that equation (A.31) is synonymous with $\hat{z}_{t+1} = NN\hat{z}_t + \hat{\varepsilon}_{t+1}$; where $E_t(\varepsilon_{i,t+1}) = 0$ and $\mathrm{Var}(\varepsilon_{it}) = \sigma^2_{i\varepsilon}$, $\forall t$, $\forall i \in (1,...,n_Z)$.

One requirement on the matrix CC is that $b \geq n_C$, that is the number of deterministic equations b must be at least as large as the number of control variables. If this is not the case, Uhlig suggests redefining some of the flow variables as state variables, which will increase n_S and lower n_C until $b = n_C$.[41] A second requirement on CC is that it has rank n_C, that is it must have at least n_C linearly independent columns or rows. A requirement on the matrix FF is that it must of size $\left(\left(n_S + n_C - b\right) \times n_S\right) = f \times n_S \Rightarrow f + b = n_S + n_C$. This simply says that there must be as many endogenous variables as there are equations in the model.

Programming Uhlig's algorithm requires modification of any one of his example files contained within the toolbox available online. First the user must declare the parameters and the steady state of model 1. Second he must specify the variables by creating a stacked vector of endogenous states, followed by control variables and then by exogenous state variables. Lastly he must define the coefficients in the matrices $AA - NN$ of equations (A.29) – (A.31). It is prudent to check whether the requirements on the matrices CC and FF listed above are being met before adapting the algorithm to one's own model. This is a simple counting exercise of equations and variables.

[41] As a matter of fact, this is precisely the reason we had to categorize the auxiliary capital stock and hence investment as state variables. Loosely speaking, Uhlig's algorithm favours state variables, and it makes sense to have as many of them as possible. Had we considered the auxiliary capital stock a control variable, investment would have also been labelled a control variable. Even if we still considered the interest rate a state variable, we would not have had enough state variables to satisfy the requirements on *CC*.

Table 6.2: Decomposing Uhlig's Toolbox

Variable Type		
Endo. State ($\hat{x}_t^S$)	Control ($\hat{x}_t^C$)	Exo. State ($\hat{z}_t$)
$\hat{\lambda}_t$ (a), $\hat{k}_t$, $\hat{k}_t^a$ (b), $\hat{i}_t$, $\hat{d}_t$, $\hat{r}_t$	$\hat{c}_t$, $\hat{h}_t$, $\hat{y}_t$, $\hat{s}_t$, $\widehat{tby}_t$, $\widehat{cay}_t$	$\hat{A}_t$
($\Rightarrow n_S = 6$)	($\Rightarrow n_C = 6$)	($\Rightarrow n_Z = 1$)
Equation Type		
Deterministic	Forward-Looking	Exo. Process
(4.16), (4.17), (4.21), (4.22), (4.24), (4.25), (4.26)	(4.18), (4.19), (4.20), (4.23), (4.27)	(4.28)
($\Rightarrow b = 7$)	($\Rightarrow f = 5$)	

(a) Optimal control theory tells us that the Lagrangian multiplier (also known as the shadow price of consumption) is considered a co–state variable. Therefore it is included in the list of state variables although it is not an empirically observable entity.
(b) In the main text of this paper, the auxiliary capital stock has been considered a control variable. Due to the wide range of dates included in Uhlig's algorithm, it is possible to relabel it and therefore investment as endogenous state variables.

Since $b = 7 > n_C = 6$, the requirement on CC is satisfied. Since there are twelve equations in twelve endogenous unknowns plus an exogenous equation, the requirement on FF is also satisfied.

6.4.2 Oviedo's Toolbox

This algorithm creates the system $0 = AE_t \hat{x}_{t+1} - B\hat{x}_t$ by log-linearizing the original, nonlinear system $0 = E_t f\left(x_{t+1}, x_t\right)$ given by equations (4.2) – (4.14). The following discussion presupposes that the appropriate nonlinear equations have been specified for the algorithm and that the model can therefore already be summarized by $0 = AE_t \hat{x}_{t+1} - B\hat{x}_t$. The linearized system, in this case, must be partitioned into two matrix systems: The first contains the equations defining the state (endogenous and exogenous) and control variables and the second contains equations defining the flow variables. These matrix systems of the linearized model are described by:

$$0 = PPE_t\hat{x}_{t+1} - QQ\hat{x}_t \tag{A.32}$$

$$0 = RR\hat{x}_{t+1} + SS\hat{x}_t - \hat{x}_t^F \tag{A.33}$$

Here, $\hat{x}_t$ must be defined as $\hat{x}_t = \begin{pmatrix}\hat{z}_t & \hat{x}_t^S & \hat{x}_t^C\end{pmatrix}'$. The only requirement on the matrices PP and QQ is that they must be square, i.e. contain the same number of equations as there are variables in $\hat{x}_t$. The size of $\hat{x}_t$ is $(n_Z + n_S + n_C)\times 1 = m\times 1$. Therefore PP and QQ must both be of size $m\times m$. Since $\hat{x}_t^F$ contains $n_F\times 1$ flow variables, matrices RR and SS must be of order $n_F\times m$.

To stick with Oviedo's paper, the Lagrangian multiplier (the co-state variable) is eliminated. In terms of the non-linearized system this implies replacing all equations involving λ_t with the marginal utility of consumption $(c_t - \omega^{-1}h_t^{\omega})^{-\gamma}$. In terms of the linearized model that has hypothetically already been created, we would simply replace all equations involving $\hat{\lambda}_t$ with $\varepsilon_{cc}\hat{c}_t + \varepsilon_{ch}\hat{h}_t$ (see equation (4.16)).

Just as before, each variable needs to be classified according to type. This is analogous to the variable classification undertaken for Uhlig's algorithm in table 6.2, except that the auxiliary capital stock is now a control variable and that all flow variables are separated from the vector $\hat{x}_t$. Whereas for Uhlig's algorithm, exogenous processes, deterministic equations and forward-looking equations needed to be differentiated, this time a distinction between control and state variable equations ('not flow') on the one hand and flow-variable equations on the other is required. The roster of model 1 participants is summarized in the following table.

Table 6.3: Decomposing Oviedo's Toolbox

Variable Type			
Exo. State ($\hat{z}_t$)	Endo. State ($\hat{x}_t^S$)	Endo. Control ($\hat{x}_t^C$)	Endo. Flow ($\hat{x}_t^F$)
$\hat{A}_t$	$\hat{k}_t$, $\hat{d}_t$	$\hat{c}_t$, $\hat{h}_t$, $\hat{k}_t^a$	$\hat{y}_t$, $\hat{r}_t$, $\hat{i}_t$, $\hat{s}_t$, $\widehat{tby}_t$, $\widehat{cay}_t$
$(\Rightarrow n_Z = 1)$	$(\Rightarrow n_S = 2)$	$(\Rightarrow n_C = 3)$	$(\Rightarrow n_F = 6)$

Equation Type	
Not-Flow $\left(n_Z + n_S + n_C = 6\right)$	Flow $\left(n_F = 6\right)$
Non-linearized System: (4.2), (4.10), (4.11), (4.12), (4.13), (4.14)	Non-linearized System: (4.3),(4.4)[(a)], (4.5), (4.6), (4.7), (4.8)

(a) Solving this for investment

In order to program Oviedo's algorithm, the parameters and the steady state need to be declared first. Next, model 1 is separated into two sets of (potentially) non-linear equations—those that define flow variable equations and those that do not. The algorithm then calculates the log-linearization of each of these equations, transforming model 1 from $0 = E_t f(x_{t+1}, x_t)$ to $PPE_t \hat{x}_{t+1} = QQ\hat{x}_t$ for non-flow variables and to $\hat{x}_t^F = RR\hat{x}_{t+1} + SS\hat{x}_t$ for flow variables. This again leads to results for the policy and transition functions, based on which the impulse response functions can be generated and the business cycle summary statistic calculated.

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