Optimal Stochastic Control of Dividends and Capital Injections

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Natalie Scheer
aus Osch

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Berichterstatter: Prof. Dr. Dr. Hanspeter Schmidli, Universität zu Köln
(Gutachter)
Prof. Dr. Josef Steinebach, Universität zu Köln

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Abstract

In this thesis, we consider optimisation problems of an insurance company whose risk reserve process follows the settings of the classical risk model. The insurer has the possibility to control its surplus process by paying dividends to the shareholders. Furthermore, the shareholders are allowed to make capital injections such that the surplus process stays nonnegative. A control strategy describes the decision on the times and the amount of the dividend payments and the capital injections. To measure the risk associated with a control strategy, we consider the value of the expected discounted dividends minus the penalised expected discounted capital injections. Because of the discounting, it can only be optimal to make capital injections at times when the reserve would become negative due to a claim occurrence. Our goal is to determine the value function, which is defined as the maximal value over all proper strategies, and to find an optimal strategy which leads to this maximal value.

First, we solve the optimisation problem for the classical risk model when the capital injections are penalised by some proportional factor ϕ. We show that an optimal strategy exists and is of barrier type, i.e., all the surplus exceeding some barrier level b is paid as dividend.

The penalty factor ϕ can be interpreted as proportional costs associated with the capital injections. In the second part, we extend this model by adding fixed costs incurring any time at which capital injections are made. The optimal strategy here is not of barrier type any more, but of band type. That is a strategy where the state space of the surplus process is partitioned into three types of sets where either dividends at the premium rate, or lump sum dividends, or no dividends at all are paid.

In the third part, we allow the dynamics of the surplus process to depend on environmental conditions which are modelled by a Markov process. I.e., the frequency of the claim arrivals and the distribution of the claim amounts vary over time depending on the state of the environment process. We again maximise the difference between the expected discounted dividends and the (proportional) penalised capital injections and show that the optimal strategy for any fixed initial environment state i is of barrier type with a barrier b_i.

Zusammenfassung

In dieser Dissertation werden Optimierungsprobleme eines Versicherungsunternehmens betrachtet, dessen Überschussprozess mit einem klassischen Risikomodell beschrieben wird. Der Versicherer hat die Möglichkeit, seinen Überschussprozess durch Zahlung von Dividenden zu kontrollieren. Ausserdem dürfen die Anteilseigner Kapitalzuschüsse tätigen, damit der Überschuss nichtnegativ bleibt. Eine Kontrollstrategie enthält Entscheidungen über die Zeiten und die Höhen von Dividendenzahlungen und Kapitalzuschüssen. Um das mit einer Kontrollstrategie verbundene Risiko zu messen, betrachten wir den Wert der erwarteten diskontierten Dividenden abzüglich der erwarteten diskontierten Kapitalzuschüsse inkl. der dabei anfallenden Kosten. Wegen der Diskontierung kann es nur optimal sein, Kapitalzuschüsse zu den Zeitpunkten zu tätigen, an denen das Reservekapital aufgrund eines Schadens negativ wird. Unser Ziel ist es, die Wertefunktion zu bestimmen, die als maximaler Wert über alle geeigneten Strategien definiert ist, und eine optimale Strategie zu finden, die zu diesem maximalen Wert führt.

Zuerst lösen wir das Optimierungsproblem für das klassische Risikomodell für den Fall, dass zusätzlich zu den Kapitalzuschüssen proportionale Kosten eingerechnet werden. Wir zeigen, dass eine optimale Strategie existiert und vom Barrieretyp ist, d.h. der ganze Überschuss, der ein bestimmtes Barrierenniveau b überschreitet, wird als Dividende ausgezahlt.

Im zweiten Teil der Arbeit erweitern wir das Modell durch Hinzunahme von Fixkosten, die in den Zeiten entstehen, zu denen Kapitalzuschüsse erfolgen. Die optimale Strategie ist hier nicht mehr vom Barrieretyp, sondern vom Bandtyp. Es handelt sich also um eine Strategie, bei der der Zustandsraum des Überschussprozesses in drei Mengen unterteilt ist, in denen entweder Dividenden zur Prämienrate, eine Pauschalsumme oder keine Dividenden gezahlt werden.

Im dritten Teil lassen wir die Entwicklung des Überschussprozesses von Umweltbedingungen abhängen, die durch einen Markov-Prozess modelliert werden. Das bedeutet, dass die Schadensfrequenz und die Verteilung der Schadenshöhen in Abhängigkeit vom Zustand des Umweltprozesses in der Zeit variieren. Wir maximieren wieder die Differenz der erwarteten diskontierten Dividenden und der Kapitalzuschüsse inkl. proportionaler Kosten und zeigen, dass die optimale Strategie für jeden anfänglichen Umweltzustand i eine Barrierenstrategie mit einer Barriere b_i ist.

Acknowledgments

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Contents

Chapter 1

Introduction

Risk lies at the core of the insurance business. An insurer assumes the risk of an uncertain future loss of an insured person and promises, in exchange for payment, to compensate the policy holder if the insured event occurs. While selling safety guarantees to others, the insurance company has to insure its own economic survival. Thus, it is of primary importance for the insurer to identify and measure risks in order to control them. This is the role of risk management. Proper risk management requires good cooperation between economists and mathematicians. While the economists define risks related to a company's objectives, it is the mathematical, or actuarial, task to assign numerical values to these specified risks by building quantitative risk models and choosing appropriate risk measures. For this purpose, modern actuarial science uses not only classical risk theory but increasingly the methods and techniques of stochastic control theory.

The basic risk in insurance are the claims, since the claim sizes and occurence times are uncertain. The first step to handle this uncertaincy is to choose a probabilistic model which describes the claims adequately. Next, the insurer has to determine premia sufficient to cover the future loss. Choosing the initial capital and the premium size is one way to manage the risk. There are several other possibilities for an insurer to influence or to *control* the dynamics of the surplus process, for example by buying reinsurance, making investments, paying dividends, or a combination of these actions. A *control strategy* describes the decisions affecting the surplus: when action should be taken with regard to the surplus, what type of action, and what amounts action is taken on. By choosing a strategy, the insurer aims to maximise (minimise) some objective function connected to that strategy. In mathematical formulation, these are problems of optimal control theory (see Fleming and Soner [27] or Fleming and Rishel [26]).

The control possibilities that we will consider in this thesis are dividend payments and capital injections made by the shareholders if the surplus process becomes negative. Our goal will be to find optimal control strategies which maximise the expected discounted dividends minus the penalised expected discounted capital injections.

1.1 Risk Models and Maximisation Problems

General Model Settings

Let $X = \{X_t\}_{t\geq 0}$ be the surplus process defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Further, let $\{\mathscr{F}_t\}_{t\geq 0}$ be the natural filtration of the process X. By $U = \{U_t\}_{t\geq 0}$ we denote a control process with state space $\mathcal{U}$. We allow only controls U that are $\{\mathscr{F}_t\}$-adapted, because the decisions can only be made based on present and not on future information. Further, we may not allow all strategies from $\mathcal{U}$, but make some restrictions on them. Let $\mathscr{S}$ denote the set of *admissible* strategies, i.e., the adapted strategies that are allowed. To each initial value x and each admissible control process U we associate a value $V^U(x)$, which we call an *objective function*. The goal is now to find the *value function* which is the maximal value over all admissible controls

$$V(x) = \sup_{U\in\mathscr{S}} V^U(x)$$

and to determine - provided that it exists at all - the optimal control process U^* leading to the value function, i.e., $V(x) = V^{U^*}(x)$.

Risk Models

There are several probabilistic models for describing the development of the risk reserve process $\{X_t\}_{t\geq 0}$ of an insurance company. The most famous one goes back to Lundberg [55, 56] and Cramér [15, 16]. Over the course of time it has established itself as a "classic" and is now called the *classical risk model* (the *Cramér-Lundberg* or *compound Poisson risk model*). For the initial capital $x \geq 0$, the surplus at time t is given by

$$X_t = x + ct - \sum_{n=1}^{N_t} Y_n \,. \tag{1.1}$$

$c > 0$ is the constant premium rate. The number of the claims that have occured up to time $t \geq 0$ is described by a homogeneous Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda > 0$, i.e., $N_t \sim Poi(\lambda t)$. As a consequence, the claim inter-occurence times are exponentially distributed. The claim sizes are a sequence of positive independent and identically distributed (iid) random variables $\{Y_n\}_{n\in\mathbb{N}}$ with distribution function G and a finite mean μ. The claim sizes and the claim numbers are assumed to be independent. A sample path of the surplus process in the classical model (1.1) is illustrated in Figure 1.1.

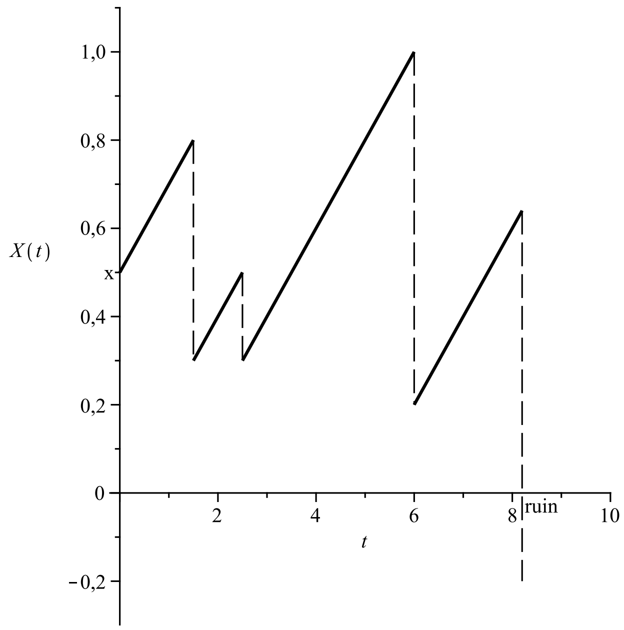

Figure 1.1: A sample path of the classical risk model

A generalised approach allows claim arrivals and the distribution of the claim amounts not to be homogeneous in time, but to depend on environmental conditions. As pointed out by Asmussen [4], in, for example, automobile insurance this could be weather conditions or traffic volume. In health insurance the outbreak of epidemics can considerably impact portfolios. Economic circumstances or political regime switchings can also be thought of as influence factors (cf. Zhu and Yang [74]). Let the environment be described by a Markov process $\{J_t\}_{t\geq 0}$ with a finite state space $\mathscr{J} = \{1, \dots, m\}$. Given the environment state $J_t = i$, premia are paid at rate c_i, the claim sizes have distribution G_i with mean μ_i, and claims occur according to a homogeneous Poisson process $\{N_t^i\}_{t\geq 0}$ with intensity λ_i. The claim counting process $\{N_t\}_{t\geq 0}$ is given by

$$N_t = \sum_{i=1}^{m} \int_0^t \mathbf{I}_{\{J_s = i\}} \, \mathrm{d}N_s^i.$$

$\{N_t\}$ is called a *Markov-modulated Poisson process* (*doubly stochastic Poisson process* or *Cox process*). The claim sizes $\{Y_n\}$ again are assumed to be independent of the claim number. The surplus at time t is then given by

$$X_t = x + \int_0^t c_{J_s} \,\mathrm{d}s - \sum_{n=1}^{N_t} Y_n. \tag{1.2}$$

The process $\{(X_t, J_t)\}_{t\geq 0}$ is called a *Markov-modulated risk model.* The idea of Cox processes goes back to Cox [14]. Reinhard [60], Asmussen [4] and Grandell [37] were the first who integrated them as models in risk theory.

There are several other alternative risk models used in the literature. In the so-called *Sparre Andersen model* proposed by Sparre Andersen [3], the claim counting process $\{N_t\}$ is modelled as a general renewal process, i.e., $\{N_t\}$ is governed by a sequence of iid inter-occurence times with some common distribution. In *diffusion models* introduced by Iglehart [45], the surplus is described by a diffusion process which can be obtained as an approximation of the classical risk models if one lets the number of claims increase and makes the claim amounts smaller (for a detailed discussion see e.g. Schmidli [66]). Gerber [29] had the idea to combine the classical risk model with a diffusion component in the *perturbed compound Poisson model.*

Dividend Strategies

We assume now that the insurance company has the possibility to control the surplus process by paying dividends to the shareholders. Let $D = \{D_t\}_{t\geq 0}$ denote a stochastic process representing the cumulated dividend payments up to time t. The controlled surplus process is defined by

$$X_t^D = X_t - D_t.$$

There are several types of dividend strategies used in the literature.

- A *band strategy* is described by the partition of the state space of the surplus process in three disjunct sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. The dividends are paid according to the set where the current surplus x is located: if $x \in \mathcal{A}$, then the incoming premium is paid as dividend; if $x \in \mathcal{B}$, then dividends of the amount $x - \max\{a : a < x, a \in \mathcal{A}\}$ are paid immediately bringing the current reserve to the next point in $\mathcal{A}$ that is smaller than x; if $x \in \mathcal{C}$, no dividends are paid. The sets $\mathcal{B}$ and $\mathcal{C}$ may consist of several disjoint intervalls dividing the state space in bands which legitimise the nomenclature. An example of a risk process controlled by a band strategy with $\mathcal{A} = \{0, b\}, \mathcal{B} = (0, a] \cup (b, \infty)$ and $\mathcal{C} = (a, b)$ is given in Figure 1.2.

- A *barrier strategy* is a special case of the band strategy where $\mathcal{A}$ consists of only one point, i.e., $\mathcal{A} = \{b\}$ for some $b \geq 0$. In this case all the surplus above the barrier b is paid as dividend. As long as the reserve process stays below b, no dividend is paid.

- A dividend strategy is called a *threshold strategy* for a fixed threshold level $b > 0$, if dividends are paid continuously at a rate a smaller than the premium rate whenever the surplus is over b, and otherwise, no dividends are paid. If there are multiple thresholds b_i with associated dividend rates a_i, then we have a multiple threshold or *multi-layer strategy.*

- A simple type of *impulse strategy* is characterised by two levels $0 \leq b_1 < b_2$. Each time the surplus process is above b_2, a dividend payment is made bringing the surplus to the level b_1. As long as the reserves are below b_2, no dividends are paid.

In general, one allows all non-decreasing càdlàg processes as dividend strategies. The optimal strategies we will be dealing with in this dissertation are of band and barrier type. Such dividend processes are examples of *feedback control strategies.* These are control processes U such that $U_t = u(X_t)$ for some measurable function u.

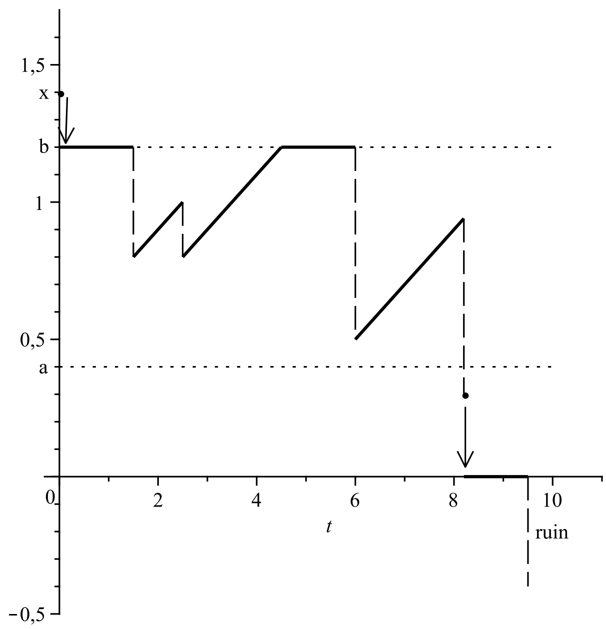

Figure 1.2: Risk process controlled by a band dividend strategy

Value Functions

The definition of the objective and accordingly the value function depends on the optimisation criterion which is used to measure the risk (or the quality) of the insurance portfolio. A traditional method to quantify the risk or the credibility of the insurance company is the probability of ruin, i.e., the probability that the surplus becomes negative in finite time. Thus, it is natural to assume that the insurer is anxious to keep the probability of ruin small. Therefore, minimising the ruin probabilities is one of the most discussed problems in risk theory. Since the first results in Lundberg [56] and Cramér [15, 16], the ruin probabilities have been extensively studied by many authors, see e.g. Rolski et al. [61], Asmussen [4], Grandell [37] for the general results, or in the context of stochastic control Schmidli [63, 64, 65, 66], Hipp and Plum [40], Hipp and Schmidli [41], Hipp and Taksar [42].

Even though the ruin probability criterion corresponds to the Value-at-Risk that is now used for financial institutions, this concept has also been criticised. Minimising the ruin probability supposes that the company should allow the reserve capital to grow to infinity, which is not realistic. This led Bruno de Finetti [17] to direct the focus not on the expected ruin but on the "expected life" (Borch [9]) and to propose measuring the risk exposure of an insurance portfolio by the expected discounted value of its future dividends which the company pays to the shareholders before ruin. This idea served as an impulse for the further developments in the field of dividend maximisation problems. Borch [9], Bühlmann [12] and Gerber [28, 30] were the first who studied de Finetti's original model in more detail. Since then there has been a lot of research activity on this optimisation criterion in various settings, for the classical risk model see Azcue and Muler [8], Albrecher and Thonhauser [1], Thonhauser and Albrecher [70] and for a diffusion approximation Shreve et al. [67], Jeanblanc-Piqué and Shiryaev [46], Radner and Shepp [59], Asmussen and Taksar [5], Højgaard and Taksar [43, 44]. For recent surveys on various dividend optimisation problems see Albrecher and Thonhauser [2], Avanzi [6] or Schmidli [66].

Unfortunately, maximising the expected dividends in de Finetti's setting results in optimal dividend strategies which lead almost surely to ruin. Therefore, Dickson and Waters [21] proposed to allow capital injections from the shareholders when the surplus falls below zero in order to cover the deficit at ruin, thereby avoiding bancruptcy. Denoting by $Z = \{Z_t\}_{t\geq 0}$ the capital injections process, the controlled surplus process is then of the form

$$X_t^{(D,Z)} = X_t - D_t + Z_t.$$

Dickson and Waters considered the value of the discounted cash flow if the dividends are paid according to a barrier strategy, see also Gerber et al. [36]

and Gerber et al. [32]. The optimal strategy in their model is a simple one because of the discounting. It is optimal to pay all capital as dividend and then let the shareholders pay the claims as they arrive keeping the surplus at zero. A different solution may be obtained if capital injections are penalised.

Having this question in mind, we will here consider the objective function

$$V^{(D,Z)}(x) = \mathbb{E}\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t | X_0 = x\Big] \tag{1.3}$$

with $\phi > 1$ and a constant discount factor $\delta > 0$. As optimisation criterion we choose the maximal value over all strategies (D, Z) such that the controlled surplus $\{X_t^{(D,Z)}\}$ remains non-negative a.s. In a diffusion setting, an analogous problem has been solved by Shreve et al. [67], see also Lokka and Zervos [54]. Avram et al. [7] showed the optimality of the barrier dividend strategies in the more general framework of spectrally negative Lévy processes. Eisenberg [24] considered the problem without dividends and minimised the expected discounted capital injections in both the classical risk model and a diffusion approximation model with the control possibility of reinsurance and investment.

The penalty factor ϕ in (1.3) can also be interpreted as proportional costs associated with capital injections. A natural extension of this model is to add fixed costs incurring any time at which capital injections are made. The next model we will investigate here is of the form

$$V^{(D,Z)}(x) = \mathbb{E}\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t - L\sum_{t\ge 0} e^{-\delta t}\mathbf{I}_{\{\Delta Z_t > 0\}} | X_0 = x\Big]. \tag{1.4}$$

In a diffusion approximation, dividend optimisation problems with transaction costs were treated by He and Liang [39] who added the possibility of proportional reinsurance and Paulsen [58] who assumed that costs incure with both dividend payments and capital injections. Loeffen [53] studied the dividend maximisation problem until ruin including transaction costs on the dividends for spectrally negative Lévy processes.

Finally, we adopt the risk measure (1.3) in the framework of a Markov-modulated risk model. For an initial environment state i and initial capital x, the objective function is of the form

$$V^{(D,Z)}(x,i) = \mathbb{E}\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t | X_0 = x, J_0 = i\Big] \tag{1.5}$$

for $\phi > 1$. Dividends problems in a Markov-modulated risk model were treated e.g. in Wei et al. [71] and Li and Lu [49, 50] who considered the value of the expected dividends until ruin. Sotomayor and Cadenillas [69] and Jiang and Pistorius [47] investigated optimal dividend strategies under a regime-switching diffusion model.

1.2 Thesis Outline

In Chapter 2, we present some known results regarding the value of a barrier dividend strategy in de Finetti's setting in the classical risk model obtained by Bühlmann [12] and Gerber [28]. Further we discuss the so-called "Gerber-Shiu penalty functions" first introduced in Gerber and Shiu [34]. These are functions of the expected discounted penalty due at ruin which depend on the severity of ruin and the surplus immediately prior to ruin. Finally, we illustrate a relationship between the expected discounted dividends until ruin and the Gerber-Shiu penalty functions referred to as the dividends-penalty identity (see Gerber et al. [32]).

In Chapter 3, we solve the optimisation problem (1.3). For this purpose, we use the two-step solution concept developed by Schmidli [66]. In the first step, we consider only strategies which are absolutely continuous processes with a density bounded by some arbitrary constant $u_0 > 0$. We derive the characterising integro-differential equation – the so-called *Hamilton–Jacobi–Bellmann equation* (HJB) – for the associated value function $V_{u_0}(x)$, show that the solution is unique, and identify the optimal dividend strategy as a barrier strategy. In the second step, we allow all càdlàg increasing dividend processes. Letting the maximal dividend rate u_0 converge to infinity, we obtain in the limit the value function for the general problem, i.e., $\lim_{u_0\to\infty} V_{u_0}(x) = V(x)$, and show that it solves the corresponding HJB equation. Unfortunately, we are not able to obtain an explicit solution to the HJB equation, nor do we have a natural initial value for the maximisation problem. Therefore, we will have to characterise the value function among other possible solutions. The optimal dividend strategy will again be of barrier type. The results obtained in this chapter have been published in the author's paper Kulenko and Schmidli [48].

In Chapter 4, we consider the model (1.4) with both fixed and proportional administration costs incurring any time at which capital injections are made. Further, at the time of an injection the company may not only inject the deficit, but also some additional capital $C \geq 0$ to prevent future capital injections. We conclude that capital injections are only carried out if the claim process falls below zero. Again, in a two-step procedure, we derive the associated HJB equation and show that the optimal strategy is not of barrier type any more, but of band type. The value function will be characterised as the smallest solution to the HJB equation, fulfilling a linear growth condition, such that

$V(C) - \phi C$ is maximal. By using Gerber–Shiu functions, we derive a method to determine the solution to the integro-differential equation and the unknown value C, if it is optimal to pay no dividends around zero. Finally, we describe an algorithm for a piecewise construction of the value function.

Chapter 5 treats the maximisation problem (1.5) in a Markov-modulated risk model. We show – first for the absolutely continuous dividend processes and then for the general case – that the value functions $V(x, i)$ for initial states $i \in \mathscr{J}$ simultaneously satisfy the system of HJB equations. The optimal strategy for every initial state i is again a barrier strategy with a barrier level b_i, i.e., if at time t the state of the environment process is $J_t = i$ and the surplus exceeds the barrier b_i, then dividends at the premium rate c_i are paid.

1.3 Conventions

- In the following, we will use a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which all stochastic quantities are defined. We supply the probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and assume it to be the smallest right-continuous filtration such that the underlying surplus process $\{X_t\}$ is adapted.

- We will work with càdlàg (right-continuous with existing limits from the left) adapted control processes because this simplifies the presentation. By the càdlàg property of the risk process $\{X_t\}$, the controlled process $\{X_t^D\}$ will be càdlàg as well. In the literature one often uses left-continuous adapted and, as a consequence, previsible controls. Then one can obtain processes with $X_{t-}^D \neq X_t^D \neq X_{t+}^D$. However, the values connected to a previsible process $\{D_t\}$ and to its right-continuous version $\{D_{t+}\}$ are the same. The latter process is càdlàg because it will be assumed to be increasing and adapted by the choice of the filtration.

  The drawback of using càdlàg dividend controls is that if D causes a jump of the controlled process, one observes not the pre-dividend but the post-dividend process (cf. Schmidli [66]). I.e., the jump size of X_t^D is composed of the claim size and the dividend. Hence, one cannot determine the size of the dividend payment without knowing the claim size. In other words, we need additionally information from the filtration about the size of the claim in order to get the size of the dividend.

- We assume that all processes we consider are already discounted. The discount factor $\delta > 0$ in the definition of the objective functions (1.3), (1.4) and (1.5) reflects the preference behaviour of the shareholders (cf. Borch [9]). They prefer to earn dividends today rather than tomorrow and, on the other side, to make capital injections tomorrow rather than today.

- Unless otherwise stated, all statements will hold almost surely. For simplified presentation, we omit the notation almost surely or with probability one.

- For notational convenience, we write

$$\mathbb{P}_x[\cdot] = \mathbb{P}[\cdot|X_0 = x] \quad \text{and} \quad \mathbb{E}_x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$$

in the classical risk model and

$$\mathbb{P}_{(x,i)}[\cdot] = \mathbb{P}[\cdot|X_0 = x, J_0 = i] \quad \text{and} \quad \mathbb{E}_{(x,i)}[\cdot] = \mathbb{E}[\cdot|X_0 = x, J_0 = i]$$

in the Markov-modulated risk model. If the connection is clear, we dismiss the letters x and (x, i).

Chapter 2

Dividends Until Ruin

In this chapter, we cite some known results about dividends until ruin (see Bühlmann [12], Gerber et al. [32]), the Gerber-Shiu functions (see Gerber and Shiu [35], Lin et al. [52]) and the dividends-penalty identity (see Gerber et al. [32]) for the classical risk model which we will use later for the calculation of the value functions.

2.1 Dividends Until Ruin for a Barrier Strategy

Suppose that the surplus process follows the classical risk model (1.1), and dividends are paid according to a barrier strategy with the barrier level b. Let X_t^b denote the controlled surplus process and

$$\begin{aligned} \tau^b &= \inf\{t \geq 0 : X_t^b < 0\}, \\ T^b &= \inf\{t \geq 0 : X_t^b = b\} \end{aligned}$$

be the corresponding time of ruin and the first time the surplus process reaches the barrier, respectively. For $x \geq 0$, let $V^b(x)$ be the expected discounted dividend payments unitl ruin, i.e.,

$$V^b(x) = \mathbb{E}_x\Big[\int_{0-}^{\tau^b-} e^{-\delta t}\,\mathrm{d}D_t\Big].$$

Bühlmann [12] showed (with the same methods which we will use in the proof of Theorem 3.2.2) that $V^b(x)$ fulfils the following integro-differential equation

$$c(V^b)'(x) + \lambda \int_0^x V^b(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V^b(x) = 0, \quad 0 \leq x \leq b,$$

with the boundary condition

$$(V^b)'(b-) = 1.$$

(If the initial value $x \geq b$, then dividends of the amount $x - b$ are paid immediately, such that $V^b(x) = x - b + V^b(b)$ holds, and therefore, $(V^b)'(b+) = 1$. Thus, the function V^b is differentiable at $x = b$, and the boundary condition is fulfilled even for the derivative at $x = b$). To solve the above equation, consider the equation

$$ch'(x) + \lambda \int_0^x h(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)h(x) = 0, \quad 0 \leq x < \infty. \tag{2.1}$$

There is a unique solution to (2.1) with initial condition

$$h(0) = 1$$

(see e.g. Schmidli [66, p. 90]). Since any multiple of $h(x)$ is another solution to (2.1), we have that

$$V^b(x) = h(x)\gamma, \quad 0 \leq x \leq b,$$

where γ does not depend on x. Now we can determine the constant γ by differentiating of $V^b(x)$ at $x = b$ getting

$$V^b(x) = \frac{h(x)}{h'(b)}, \quad 0 \leq x \leq b. \tag{2.2}$$

This formula was first obtained by Bühlmann [12, p.172], see also Schmidli [66, p. 91].

Remark 2.1.1
Equation (2.2) is well-defined because $h'(x) > 0$ for $0 \leq x < \infty$. To see this, one can use the arguments from Lin et al. [52]. Let x_0 be the largest value such that $h(x)$ is strictly increasing for $0 \leq x < x_0$. Then, $x_0 > 0$ since $h'(0) = (\lambda+\delta)/c > 0$. Suppose, x_0 is finite, then $h'(x_0) = 0$. However,

$$\begin{aligned} h'(x_0) &= -\frac{\lambda}{c}\int_0^{x_0} h(x_0-y)\,\mathrm{d}G(y) + \frac{\lambda+\delta}{c}h(x_0) \\ &> -\frac{\lambda}{c}h(x_0)G(x_0) + \frac{\lambda+\delta}{c}h(x_0) \\ &\geq -\frac{\lambda}{c}h(x_0) + \frac{\lambda+\delta}{c}h(x_0) > 0, \end{aligned}$$

which is a contradiction. Therefore, $x_0 = \infty$. ■

Define now the function

$$C^b(x) = \mathbb{E}_x[e^{-\delta T^b}\mathbf{I}_{T^b<\tau^b}] \tag{2.3}$$

as the Laplace transform of the time to reach the dividend barrier b without ruin occuring. $C^b(x)$ can also be interpreted as the expected present value of a contingent payment of 1 due at time when the surplus reaches the level b, provided that ruin has not occured yet. Since no dividends are paid until time T^b, we obtain by conditioning on $\mathscr{F}_{T^b}$,

$$V^b(x) = C^b(x)V^b(b). \tag{2.4}$$

By the same methods as above, it can be shown that $C^b(x)$ satisfies the integro-differential equation (2.1) for $0 \leq x \leq b$. Since $C^b(b) = 1$, we get

$$C^b(x) = \frac{h(x)}{h(b)} \quad \text{for } x \leq b. \tag{2.5}$$

This formula can be found by Gerber et al. [32].

2.2 Gerber-Shiu Penalty Functions

Since the first notion of severity of ruin $|X_\tau|$ by Gerber et al. [31], a number of papers investigating the distribution of the ruin time τ, the surplus just before ruin $X_{\tau-}$ or the severity of ruin $|X_\tau|$ were published, see e.g. Dickson [18], Dickson and Waters [20], Dufresne and Gerber [23], Gerber and Shiu [33]. In 1998, Gerber and Shiu [35] combined the analysis of all three quantities by studing its joint distribution via the expected discounted penalty function which depends on the time of ruin, the deficit at ruin and on the surplus immediately prior to ruin. Their seminal work provided new methods and techniques for treating the ruin problems and layed a basis for a large number of later research on the classical risk theory, see for example Lin et al. [52], Lin and Willmot [51], Schmidli [62], Willmot and Dickson [72], Cai and Dickson [13], Yuen et al. [73]. The penalty functions introduced by Gerber and Shiu became since then the name of "Gerber-Shiu penalty functions".

We present now some results from Gerber and Shiu [35]. Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function. For $x \geq 0$ and $\delta \geq 0$, the *expected discounted penalty function* $p(x)$ is defined as

$$p(x) = \mathbb{E}_x[w(X_{\tau-}, |X_\tau|)e^{-\delta\tau}\mathbf{1}_{\{\tau<\infty\}}]. \tag{2.6}$$

$p(x)$ satisfies the integro-differential equation

$$0 = cp'(x) + \lambda \int_0^x p(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)p(x) + \lambda\Pi(x) \tag{2.7}$$

with

$$\Pi(x) = \int_x^\infty w(x, y-x)\,\mathrm{d}G(y).$$

To solve Equation (2.7), consider the so-called *Lundberg's fundamental equation*

$$\lambda + \delta - cs = \lambda \hat{g}(s), \quad s \geq 0. \tag{2.8}$$

where $\hat{g}(s) = \int_0^\infty e^{-sx}\, \mathrm{d}G(x)$ denotes the Laplace(-Stieltjes) transform of the claim distribution. Let ρ be the unique positive solution to (2.8) (its existence will be shown in Remark 2.2.1 below). By multiplying (2.7) with $e^{-\rho x}$, rearranging the terms and eventually integrating from $x = 0$ to $x = z$, Gerber and Shiu obtained the following representation:

$$p(z) = \frac{\lambda}{c}\left\{\int_0^z p(x)\left[\int_{z-x} e^{\rho(z-x-y)}\, \mathrm{d}G(y)\right] \mathrm{d}x + \int_z^\infty e^{\rho(z-u)}\Pi(u)\, \mathrm{d}u\right\},$$

or, as a renewal equation,

$$p(x) = (p * \xi)(x) + \eta(x),$$

where $*$ denotes the convolution and

$$\begin{aligned}
\eta(x) &= \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)}\Pi(y)\, \mathrm{d}y = \frac{\lambda}{c}\int_x^\infty \int_0^\infty e^{-\rho(y-x)} w(y,z)\, \mathrm{d}G(y+z)\, \mathrm{d}y\,, \\
\xi(x) &= \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)}\, \mathrm{d}G(y)\,,
\end{aligned}$$

for $x \geq 0$. By the method of succesive substitution, $p(x)$ can be expressed as

$$p(x) = \eta(x) + \sum_{n=1}^{\infty} \left(\xi^{*n} * \eta\right)(x)\,.$$

It follows that the initial value $p(0) = \eta(0)$ can be calculated by

$$p(0) = \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\Pi(y)\, \mathrm{d}y = \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\int_y^\infty w(y, z-y)\, \mathrm{d}G(z)\, \mathrm{d}y. \tag{2.9}$$

Remark 2.2.1
The Laplace(-Stieltjes) transform $\hat{f}(s) = \mathbb{E}[e^{-sX}]$ of a (positive) random variable X is an analytic function for all s with $\mathrm{Re}(s) \geq 0$. It is decreasing and convex because

$$\hat{f}'(s) = \mathbb{E}[-Xe^{-sX}]) < 0$$

and

$$\hat{f}''(s) = \mathbb{E}[X^2 e^{-sX}] > 0.$$

Define $l(s) = \lambda + \delta - cs$ and consider the equation

$$l(s) = \lambda \hat{f}(s).$$

Since

$$l(0) = \lambda + \delta \geq \lambda = \lambda \hat{f}(0)$$

and $l(s)$ is a decreasing linear function, there is a unique nonnegative intersection point of $l(s)$ and $\hat{f}(s)$, say ρ, which is then the unique nonnegative root of the Lundberg's fundamental equation (2.8). ■

Alternatively, one can solve Equation (2.7) by using the Laplace transform approach (see Dickson [19]). For the Laplace(-Stieltjes) transforms

$$\hat{p}(s) = \int_0^\infty e^{-sx} p(x)\,\mathrm{d}x, \quad \hat{\Pi}(s) = \int_0^\infty e^{-sx} \Pi(x)\,\mathrm{d}x$$

we have then

$$0 = c(s\hat{p}(s) - p(0)) + \lambda \hat{p}(s)\hat{g}(s) - (\lambda + \delta)\hat{p}(s) + \lambda \hat{\Pi}(s),$$

or

$$\hat{p}(s) = \frac{\lambda \hat{\Pi}(s) - cp(0)}{\lambda + \delta - cs - \lambda \hat{g}(s)}\ .$$

The denominator is the Lundberg's fundamental equation which vanishes at $\rho > 0$. Since $\hat{p}(s)$ is an analytic function for all $s > 0$, the numerator must be zero at $s = \rho$ yielding $\lambda \hat{\Pi}(\rho) = cp(0)$, and therefore,

$$\hat{p}(s) = \frac{\lambda(\hat{\Pi}(s) - \hat{\Pi}(\rho))}{\lambda + \delta - cs - \lambda \hat{g}(s)}\ . \tag{2.10}$$

Now, for some "nice" forms of $\hat{p}(s)$ (especially, if $\hat{p}(s)$ is a rational function), $p(x)$ can be obtained by inverting its Laplace transform (2.10).

For further use, we consider three examples of the penalty function. Let $w(x, y) \equiv 1$ and define the function

$$\psi(x) = \mathbb{E}_x\Big[e^{-\delta\tau}\mathbf{I}_{\{\tau<\infty\}}\Big], \tag{2.11}$$

which is the expected discounted time of ruin, and for $w(x, y) = y$ the function

$$\sigma(x) = \mathbb{E}_x\Big[e^{-\delta\tau}|X_\tau|\mathbf{I}_{\{\tau<\infty\}}\Big] \tag{2.12}$$

as the expected discounted deficit at ruin. Finally, for the positive root ρ of the Lundberg's fundamental equation (2.8) and $w(x,y) = e^{-\rho y}$, we define

$$\chi(x) = \mathbb{E}_x\Big[e^{-\delta\tau+\rho X_\tau}\mathbf{I}_{\{\tau<\infty\}}\Big]. \tag{2.13}$$

For these penalty functions holds $\sigma(\infty) = \psi(\infty) = \chi(\infty) = 0$ and by (2.9),

$$\begin{aligned}
\psi(0) &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}(1-G(x))\,\mathrm{d}x\,, \\
\sigma(0) &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\int_x^\infty (1-G(y))\,\mathrm{d}y\,\mathrm{d}x\,, \\
\chi(0) &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}x\,\mathrm{d}G(x).
\end{aligned} \tag{2.14}$$

Gerber and Shiu [34] showed that $C^b(x) = \mathbb{E}_x[e^{-\delta T^b}\mathbf{I}_{T^b<\tau^b}]$ can be determined by

$$C^b(x) = \frac{e^{\rho x}-\chi(x)}{e^{\rho b}-\chi(b)}\,.$$

By (2.4) and the boundary condition $(V^b)'(b) = 1$ follows

$$V^b(x) = \frac{e^{\rho x}-\chi(x)}{\rho e^{\rho b}-\chi'(b)}\,. \tag{2.15}$$

In the case, when a constant dividend barrier strategy with a barrier b is applied, Lin et al. [52] investigated the corresponding penalty function

$$p^b(x) = \mathbb{E}_x[w(X^b_{\tau^b-}, |X^b_{\tau^b}|)e^{-\delta\tau^b}\mathbf{I}_{\{\tau^b<\infty\}}].$$

They showed that $p^b(x)$ satisfies Equation (2.7) with the boundary condition

$$(p^b)'(b) = 0. \tag{2.16}$$

2.3 Dividends-Penalty Identity

If dividends are paid according to a barrier strategy, then there is a connection between the Gerber-Shiu discounted penalty function with a barrier $p^b(x)$, the one without barrier $p(x)$, and the expected discounted dividends until ruin $V^b(x)$. This so-called *dividends-penalty identity* was first discovered by Lin et al. [52] for the compound Poisson model. Yuen et al. [73] extended the result to the case with interest, and Gerber et al. [32] generalised the formula for stationary Markov processes which are skip-free upwards.

We take the way of Gerber et al. [32] to motivate the dividends-penalty identity. Consider a particular sample path of the surplus process starting at $x < b$. Then the penalties at ruin (with and without the dividend barrier) can be different only, if the surplus reaches the level b before ruin. Thus, by conditioning on $\mathscr{F}_{T^b}$, we have

$$\begin{aligned} p^b(x) - p(x) &= \mathbb{E}_x[e^{-\delta T^b}\mathbf{1}_{\{T^b<\tau\}}](p^b(b) - p(b)) \\ &= C^b(x)(p^b(b) - p(b)), \quad x < b. \end{aligned}$$

By (2.16), we know that $(p^b)'(b) = 0$. Thus, replacing $C^b(x)$ by $h(x)/h(b)$ and differentiating at $x = b$ yields

$$p^b(x) - p(x) = -p'(b)\frac{h(x)}{h'(b)}.$$

Using (2.15), we finally obtain the dividends-penalty identity

$$p^b(x) = p(x) - p'(b)V^b(x), \quad x \le b. \tag{2.17}$$

Chapter 3

Optimal Control of Dividends and Capital Injections in a Classical Risk Model

3.1 Introduction

We suppose that the surplus of an insurance company is described by a classical risk process

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i$$

defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. $x \in \mathbb{R}$ is the initial capital, $c > 0$ is the premium rate. The number of claims is modelled as a Poisson process $\{N_t\}_{t\geq 0}$ with rate $\lambda > 0$, say. Claim occurrence times are denoted by $\{T_i\}_{i\geq 0}$ with $T_0 = 0$. Claim sizes $\{Y_i\}_{i\geq 1}$ are an iid sequence of positive random variables with distribution function G, independent of $\{N_t\}$. We assume that $\mathbb{E}[Y_i] = \mu < \infty$ and, for simplicity, that G is continuous. We can assume that $\{\mathscr{F}_t\}_{t\geq 0}$ is the smallest right-continuous filtration such that $\{X_t\}_{t\geq 0}$ is adapted.

We assume that the insurer controls its reserve process by paying dividends to the shareholders. Furthermore, the shareholders are supposed to inject capital if the surplus ever becomes negative. The accumulated dividends process $\{D_t\}_{t\geq 0}$ is an $\{\mathscr{F}_t\}$-adapted, non-decreasing, càdlàg process with $D_{0-} = 0$, the accumulated capital injections are denoted by $\{Z_t\}_{t\geq 0}$, which also is an adapted, non-decreasing, pure jump process with $Z_{0-} = 0$. The controlled surplus process then becomes

$$X_t^{(D,Z)} = X_t - D_t + Z_t\;, \qquad X_{0-}^{(D,Z)} = x\;.$$

The capital injections have to be chosen in such a way that $X_t^{(D,Z)} \geq 0$ a.s. for all t. Note that no positive safety loading $c > \lambda\mu$ needs to be assumed.

The value of a strategy $(D, Z) = \{(D_t, Z_t)_{t\geq 0}\}$ is defined as

$$V^{(D,Z)}(x) = \mathbb{E}_x\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t\Big]\;,$$

where $\phi > 1$ is a penalising factor and $\delta > 0$ is a discounting factor. A strategy (D, Z) is *admissible* if

$$\mathbb{P}_x[X_t^{(D,Z)} \geq 0 \text{ for all } t \geq 0] = 1\;.$$

Our goal now is to maximise $V^{(D,Z)}(x)$ and to find an optimal strategy (D^*, Z^*) with leads to the maximal value. The *value function* of our problem is

$$V(x) = \sup_{(D,Z)\in\mathscr{S}_x} V^{(D,Z)}(x)\;.$$

Note that $\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t = \delta\int_{0-}^{\infty} e^{-\delta t} D_t\,\mathrm{d}t$. Since the value of the (not admissible) strategy $D_t = x + ct$ and $Z_t = 0$ is an upper bound for the value of any admissible strategy, we get that $V^{(D,Z)}(x) < x + c/\delta < \infty$ for any admissible strategy. We denote by $\mathscr{S}_x$ the set of all admissible strategies for the initial capital x.

If we had chosen $\phi < 1$ then we could make a capital injection of size K and pay it as dividend at the same time. The value would be $K(1-\phi)$. Maximising over K shows that the value function would be infinite. The choice $\phi = 1$ would yield the trivial solution $V(x) = x + (c - \lambda\mu)/\delta$. That is the solution with $D_t = x + ct$ and $Z_t = S_t := x + ct - X_t$, i.e., any positive surplus is paid as dividend. Choosing $\delta = 0$ would lead to an infinite value function. Indeed, the shareholder would be indifferent to recieving dividend payments today or tomorrow. Then, they would prefer to wait until some high barrier b such that it is unlikely that a capital injection has to be made and the value of the dividends is larger than the value of the capital injections. In the case of positive safety loading, the barrier is reached infinitely often, and the value of the strategy would become infinite.

It is clear that, because of the discounting, it cannot be optimal to make capital injections before they really are necessary. Therefore, we need only choose the dividend process $\{D_t\}$. The corresponding capital injection process becomes

$$Z_t^D = \max(-\inf_{0\leq s\leq t}(X_s - D_s), 0)\;.$$

Therefore, we will in the following use the abbreviated notation $\{Z_t\}, \{X_t^D\}$ and $V^D(x)$ for the capital injection process $\{Z_t^D\}$, the surplus process and the value connected to a strategy $\{(D_t, Z_t^D)\}$. If the initial capital is negative, then $Z_0 = |x|$. Thus,

$$V(x) = V(0) - \phi|x| \qquad \text{for } x < 0\,. \tag{3.1}$$

A very impotant property of the value function introduced above is its concavity. We state and prove the next lemma for the model and will use it later also for restricted dividend strategies. It should be noted that the proof works for any model with independent increments and also for restricted dividend strategies as long as the allowed strategies form a convex set.

Lemma 3.1.1
The function $V(x)$ is concave.

Proof: Let $x, y > 0$ and $z = \alpha x + \beta y$ with $\alpha + \beta = 1$ and $\alpha, \beta \in (0, 1)$. Consider the strategies (D^x, Z^x) and (D^y, Z^y) for the initial capitals x and y. Define

$$D_t = \alpha D_t^x + \beta D_t^y, \quad \tilde{Z}_t = \alpha Z_t^x + \beta Z_t^y.$$

Note that $D_t = D_t^z$, but in general $\tilde{Z}_t \neq Z_t^z$. With $R_t = ct - \sum_{i=1}^{N_t} Y_i$ we have

$$\begin{aligned}
\alpha x + \beta y + R_t - D_t + \tilde{Z}_t &= \alpha x + \beta y + (\alpha + \beta) R_t - D_t + \tilde{Z}_t \\
&= \alpha \underbrace{(x + R_t - D_t^x + Z_t^x)}_{\geq 0} + \beta \underbrace{(y + R_t - D_t^y + Z_t^y)}_{\geq 0}.
\end{aligned}$$

This shows that the strategy $(D, \tilde{Z})$ is admissible and $Z_t^z \leq \tilde{Z}_t = \alpha Z_t^x + \beta Z_t^y$ (otherwise the value of the optimal strategy (D, Z^z) for the capital z were strict smaller than the value for the strategy $(D, \tilde{Z})$ for the same initial capital). It follows

$$\begin{aligned}
V(z) &\geq \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}(\,\mathrm{d}D_t - \phi\,\mathrm{d}Z_t^z)\right] \geq \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}(\,\mathrm{d}D_t - \phi\,\mathrm{d}\tilde{Z}_t)\right] \\
&\geq \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}\left((\alpha\,\mathrm{d}D_t^x + \beta\,\mathrm{d}D_t^y) - \phi(\alpha\,\mathrm{d}Z_t^x + \beta\,\mathrm{d}Z_t^y)\right)\right] \\
&= \alpha V^{D^x}(x) + \beta V^{D^y}(y).
\end{aligned}$$

Taking the supremum over all admissible strategies D we get

$$V(\alpha x + \beta y) \geq \alpha \sup_{D^x \in \mathscr{S}_x} V^{D^x}(x) + \beta \sup_{D^y \in \mathscr{S}_y} V^{D^y}(y) = \alpha V(x) + \beta V(y).$$

□

In particular, because of the concavity of V the derivatives from the left and from the right exist a.s. Moreover, $V(x)$ is absolutely continuous.

The following lower bound, also derived in Parfumi [57], will be useful.

Lemma 3.1.2
The value of expected capital injections is bounded by $\lambda\mu/\delta$.

Proof: The worst that may happen is that one has to inject capital for all the claims. Not taking care of the dividends we find, using that the time of the k-th claim T_k is Gamma $\Gamma(\lambda, k)$ distributed,

$$\mathbb{E}\Big[\sum_{k=1}^{\infty} Y_k e^{-\delta T_k}\Big] = \mu \sum_{k=1}^{\infty}\Big(\frac{\lambda}{\lambda+\delta}\Big)^k = \mu\frac{\lambda}{\lambda+\delta}\,\frac{1}{1-\frac{\lambda}{\lambda+\delta}} = \frac{\lambda\mu}{\delta}\;.$$

□

It follows that the value of any admissible strategy is bounded from below by $-\phi\lambda\mu/\delta$.

3.2 Strategies With Restricted Densities

In this section, we only consider absolutely continuous dividend strategies which admit an adapted non-negative density process $\{U_t\}_{t\geq 0}$ such that

- $D_t = \int_0^t U_s \,\mathrm{d}s$,
- $0 \leq U_t \leq u_0 < \infty$

and denote the strategies by $\{U_t\}$. The value of such a strategy is then

$$V^U(x) = \mathbb{E}_x\Big[\int_0^{\infty} e^{-\delta t} U_t \,\mathrm{d}t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t\Big]\;.$$

Let us denote the set of the admissible strategies by $\mathscr{S}_x^r$. Then the value function is $V(x) = \sup_{U\in\mathscr{S}_x^r} V^U(x)$. Recall that Lemma 3.1.1 applies.

3.2.1 The Value Function and the HJB-Equation

Lemma 3.2.1
$V(x)$ is bounded by u_0/δ, increasing, Lipschitz continuous and therefore absolutely continuous, and it holds $\lim_{x\to\infty} V(x) = u_0/\delta$.

Proof: It is clear that $V(x)$ is increasing and that $V(x) \leq \int_0^\infty u_0 e^{-\delta t} dt = u_0/\delta$. Consider the strategy $U_t = u_0$. $\tau_x^U = \inf\{t : x + (c - u_0)t - \sum_{i=1}^{N_t} Y_i < 0\}$ converges to infinity as $x \to \infty$ and so $\mathbb{P}[\int_0^\infty e^{-\delta t}\,\mathrm{d}Z_t > \varepsilon]$ converges to zero, because $\int_0^\infty e^{-\delta t}\,\mathrm{d}Z_t$ is bounded by $e^{-\delta\tau_x^U}\lambda\mu/\delta$; see Lemma 3.1.2. So we have that

$$V(x) \geq V^U(x) \geq \mathbb{E}\left[\int_0^{\tau_x^U} u_0 e^{-\delta t}\,\mathrm{d}t - \phi\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t\right] \to \frac{u_0}{\delta}\;.$$

Let $h > 0$ be small. We choose a strategy $\tilde{U} \in \mathscr{S}^r_{x+ch}$ for initial capital $x + ch$ and define

$$\begin{aligned} U_t &= 0 \cdot \mathbb{1}_{\{T_1<h\}} + \Big(0 \cdot \mathbb{1}_{\{t\le h\}} + \tilde{U}_{t-h} \cdot \mathbb{1}_{\{t>h\}}\Big)\mathbb{1}_{\{T_1\ge h\}}, \\ Z_t &= Z^0_t \cdot \mathbb{1}_{\{T_1<h\}} + \Big(0 \cdot \mathbb{1}_{\{t\le h\}} + \tilde{Z}_{t-h} \cdot \mathbb{1}_{\{t>h\}}\Big)\mathbb{1}_{\{T_1\ge h\}}. \end{aligned}$$

where $\{Z^0_t\}$ denotes the capital injections if no dividend is paid. Recall from Lemma 3.1.2 that the value connected to Z^0 is bounded from by $\lambda\mu/\delta$.

The first claim happens with density $\lambda e^{-\lambda t}$ and T_1 is larger than h with probability $e^{-\lambda h}$. By conditioning on $\mathscr{F}_{h\wedge T_1}$, it follows

$$\begin{aligned} V(x) &\ge V^U(x) \\ &= \mathbb{E}\Big[\int_0^{h\wedge T_1} e^{-\delta t} U_t \,\mathrm{d}t - \phi \int_0^{h\wedge T_1} e^{-\delta t}\,\mathrm{d}Z_t + e^{-\delta(h\wedge T_1)} V^U(X^U_{h\wedge T_1})\Big] \\ &\ge \mathbb{P}[T_1 \ge h] e^{-\delta h} V^{\tilde{U}}(X^U_h) - \phi \mathbb{P}[T_1 < h]\mathbb{E}_x\Big[\int_0^\infty e^{-\delta s}\,\mathrm{d}Z^0_s\Big] \\ &\ge e^{-(\lambda+\delta)h} V^{\tilde{U}}(x+ch) - \frac{(1-e^{-\lambda h})\phi\lambda\mu}{\delta}\,, \end{aligned}$$

and so

$$\begin{aligned} V(x) &\ge \sup_{\tilde{U}\in\mathscr{S}^r_{x+ch}} e^{-(\lambda+\delta)h} V^{\tilde{U}}(x+ch) - \frac{(1-e^{-\lambda h})\phi\lambda\mu}{\delta} \\ &= e^{-(\lambda+\delta)h} V(x+ch) - \frac{(1-e^{-\lambda h})\phi\lambda\mu}{\delta}\,. \end{aligned}$$

The Lipschitz-continuity follows now by the boundedness of V

$$\begin{aligned} 0 &\le V(x+ch) - V(x) \le V(x+ch)(1-e^{-(\lambda+\delta)h}) + \frac{\phi\lambda\mu}{\delta}(1-e^{-\lambda h}) \\ &\le V(x+ch)(\lambda+\delta)h + \frac{\phi\lambda^2\mu}{\delta}h \le \frac{u_0}{\delta}(\lambda+\delta)h + \frac{\phi\lambda^2\mu}{\delta}h\,. \end{aligned}$$

By Doob [22, p. 164], V is also absolutely continuous. □

In the next theorem, we derive the characterising equation for the value function.

Theorem 3.2.2
The function $V(x)$ is differentiable from the left and from the right a.e. on $(0,\infty)$. The derivations from the left and from the right fulfil the Hamiltion-Jacobi-Bellman equation

$$\sup_{0\le u\le u_0}\Bigl\{(c-u)V'(x)+u-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)\,dG(y)\Bigr\}=0\;. \tag{3.2}$$

Proof: Let $h>0$ and fix $u\in[0,u_0]$. If $x=0$ we suppose $u\le c$, if $x>0$ we let h be small enough such that $x+(c-u)h\ge 0$, i.e., the reserve process does not fall below zero because of the dividend payments. Let $L>0$ be the Lipschitz-constant. Choose $\varepsilon>0$ and $n\in\mathbb{N}$ such that $L(x+(c-u)h)/n<\varepsilon/2$ and let $x_k=k(x+(c-u)h)/n$ for $0\le k\le n$. For every k there is a strategy $\{U_t^k\}$ with $V^{U^k}(x_k)>V(x_k)-\varepsilon/2$. For initial capital x' with $x_k\le x'<x_{k+1}$, we choose the strategy $\{U_t^k\}$. Then, by the Lipschitz continuity of $V(x)$, we can choose n large enough such that

$$V^{U^k}(x')\ge V^{U^k}(x_k)>V(x_k)-\varepsilon/2>V(x')-L(x'-x_k)-\varepsilon/2>V(x')-\varepsilon\;.$$

Thus, for all $x'\in[0,x+(c-u)h]$, we can find a measurable strategy $\tilde U$ such that $V^{\tilde U}(x')>V(x')-\varepsilon$.

Let $x\ge 0$. Consider now the following strategy

$$U_t=\begin{cases} u & : \; 0\le t<h\wedge T_1\\ \tilde U_{t-h\wedge T_1} & : \; t\ge h\wedge T_1\end{cases}\;,\qquad Z_t=\begin{cases} 0 & : \; 0\le t<h\wedge T_1\\ \tilde Z_{t-h\wedge T_1} & : \; t\ge h\wedge T_1\end{cases}\;.$$

By conditioning on $\mathscr{F}_{h\wedge T_1}$, it follows that

$$\begin{aligned}
V(x)&\ge V^U(x)\\
&=\mathbb{E}\Bigl[\int_0^{h\wedge T_1-}e^{-\delta t}u\,\mathrm{d}t+e^{-\delta(h\wedge T_1)}V^{\tilde U}(X^U_{h\wedge T_1})\Bigr]\\
&>\mathbb{E}\Bigl[\int_0^{h\wedge T_1-}e^{-\delta t}u\,\mathrm{d}t+e^{-\delta(h\wedge T_1)}V(X^U_{h\wedge T_1})\Bigr]-\varepsilon\\
&=e^{-\lambda h}\Bigl(\int_0^h ue^{-\delta t}\,\mathrm{d}t+e^{-\delta h}V(x+(c-u)h)\Bigr)\\
&\quad+\int_0^h\lambda e^{-\lambda t}\Bigl\{\int_0^t ue^{-\delta s}\,\mathrm{d}s+e^{-\delta t}\int_0^{x+(c-u)t}V(x+(c-u)t-y)\,\mathrm{d}G(y)\\
&\qquad\qquad+e^{-\delta t}\int_{x+(c-u)t}^\infty(V(0)-\phi(y-x-(c-u)t))\,\mathrm{d}G(y)\Bigr\}\,\mathrm{d}t-\varepsilon\\
&=e^{-\lambda h}\int_0^h ue^{-\delta t}\,\mathrm{d}t+e^{-(\delta+\lambda)h}V(x+(c-u)h)\\
&+\int_0^h\lambda e^{-\lambda t}\Bigl\{\int_0^t ue^{-\delta s}\,\mathrm{d}s+e^{-\delta t}\int_0^\infty V(x+(c-u)t-y)\,\mathrm{d}G(y)\Bigr\}\,\mathrm{d}t-\varepsilon,
\end{aligned}$$

where we used property (3.1). The constant ε is arbitrary. If we let tend it to zero, rearrange the terms and divide them by h then we get

$$\begin{aligned} 0 \;\geq\; & \frac{V(x+(c-u)h)-V(x)}{h} - \frac{1-e^{-(\lambda+\delta)h}}{h}V(x+(c-u)h) \\ & + e^{-\lambda h}\frac{1}{h}\int_0^h ue^{-\delta t}\,\mathrm{d}t + \frac{1}{h}\int_0^h \lambda e^{-\lambda t}\Big[\int_0^t ue^{-\delta s}\,\mathrm{d}s \\ & \qquad + e^{-\delta t}\int_0^\infty V(x+(c-u)t-y)\,\mathrm{d}G(y)\Big]\,\mathrm{d}t\,. \end{aligned} \tag{3.3}$$

If $c>u$, the first term converges to the derivative from the right as $h\to 0$, if $c\leq u$ to the derivative from the left (existence of the derivatives is assured by Lemma 3.1.1). Starting with initial capital $x-(c-u)h$, we get in the same way that the first term converges to the derivative from the left in the case $c>u$ and to the derivative from the right in the case $c\leq u$. We don't distinguish the notation first and get for both derivatives

$$(c-u)V'(x)+u-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}G(y)\leq 0\,,$$

as $h\to 0$, where we have

$$\begin{aligned} & \int_0^\infty V(x-y)\,\mathrm{d}G(y) \\ & = \int_0^x V(x-y)\,\mathrm{d}G(y)+\int_x^\infty (V(0)-\phi(y-x)\,\mathrm{d}G(y) \qquad (3.4)\\ & = \int_0^x V(x-y)\,\mathrm{d}G(y)+V(0)(1-G(x))-\phi\int_x^\infty (1-G(y))\,\mathrm{d}y \end{aligned}$$

because of the property (3.1). Now choose a strategy $\hat{U}=\{\hat{u}_t(h)\}$ such that $V^{\hat{U}}(x)\geq V(x)-h^2$. Denote $a(t)=\int_0^t(c-\hat{u}_s)ds$. In the same way as above we get

$$\begin{aligned} 0 \;\leq\; & h+\frac{V(x+a(h))-V(x)}{h} - \frac{1-e^{-(\lambda+\delta)h}}{h}V(x+a(h)) \\ & + e^{-\lambda h}\frac{1}{h}\int_0^h \hat{u}_t e^{-\delta t}\,\mathrm{d}t + \frac{1}{h}\int_0^h \lambda e^{-\lambda t}\Big[\int_0^t \hat{u}_s e^{-\delta s}\,\mathrm{d}s \\ & \qquad + e^{-\delta t}\int_0^\infty V(x+(c-u)t-y)\,\mathrm{d}G(y)\Big]\,\mathrm{d}t\,. \end{aligned}$$

W.l.o.g., let $\{h_n\}_{n\geq 0}$ be a sequence with $h_n\to 0$ as $n\to\infty$ such that $\lim_{h_n\to 0}a(h_n)/h_n=c-\tilde{u}$, say. Then the limit $\lim_{h_n\to 0}(V(x+a(h_n))-V(x))/h_n$ exists because of the concavity of V. Letting $h_n\to 0$ yields

$$(c-\tilde{u})V'(x)+\tilde{u}-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}G(y)\geq 0\,.$$

If $c \geq \tilde{u}$, the inequality holds for the derivative from the right, if $c \leq \tilde{u}$, for the derivative from the left, where the value of the derivative can be choosen arbitrarily if $c = \tilde{u}$. Since $u = \tilde{u}$ fulfils Equation (3.3), we conclude that equality holds. Now the supremum can be taken over all constant strategies $U = u$ with $0 \leq u \leq u_0$. So both derivatives fulfil the HJB equation.

Starting with initial capital $x - (c - u)h$, we get in the same way that Equation (3.2) holds for the derivative from the left in the case $c \geq \tilde{u}$ and for the derivative from the right in the case $c \leq \tilde{u}$, which completes the proof. □

3.2.2 The Optimal Strategy and the Characterisation of the Solution

First we show that the value function is continuously differentiable a.e. on $(0, \infty)$. Equation (3.2) is linear in u, thus, the argument $u(x)$ maximising the left-hand side of (3.2) can be determined in dependence on the derivative (from the left or from the right) as

$$u(x) = \begin{cases} 0 & : \quad V'(x) > 1 \\ \in [0, u_0] & : \quad V'(x) = 1 \\ u_0 & : \quad V'(x) < 1 \end{cases} .$$

Since V is increasing, concave, and bounded, there exists a $b := \inf\{x : V'(x) = 1\}$ with

$$u(x) = \begin{cases} 0 & : \quad x < b \quad (\Leftrightarrow V'(x) > 1) \\ u_0 & : \quad x \geq b \quad (\Leftrightarrow V'(x) \leq 1) \end{cases} . \tag{3.5}$$

By the concavity of V, we have $V'(x-) \geq V'(x+)$. If both $V'(x-)$ and $V'(x+)$ are greater or smaller than 1, then it follows by (3.2) and our assumption that $G(y)$ is continuous that $V(x)$ is continuously differentiable on $[0, b)$ and (b, ∞). Suppose now that $b > 0$. At $x = b$, considering the equation from the left and from the right, we conclude that $cV'(b-) = u_0 + (c - u_0)V'(b+)$, or

$$c(V'(b-) - V'(b+)) = u_0(1 - V'(b+)) .$$

If $u_0 < c$, we conclude that either $V'(b-) = V'(b+) = 1$ or $1 > V'(b-)$. Because the latter is impossible, we find that $V(x)$ is continuously differentiable. If $u_0 \geq c$, we have a barrier strategy. Thus the process stays at b until time T_1. Because the process does not leave the intervall $[0, b]$ and the corresponding strategy is admissible for any $u_0 \geq c$, it must be optimal for any initial value in $[0, b]$. Thus, b does not depend on the bound u_0. The expected discounted dividends until the first claim are

$$\lambda \int_0^\infty e^{-\lambda t} \int_0^t c e^{-\delta s} \, \mathrm{d}s \, \mathrm{d}t = \frac{\lambda c}{\delta} \int_0^\infty (1 - e^{-\delta t}) e^{-\lambda t} \, \mathrm{d}t = \frac{c}{\lambda + \delta} .$$

Conditioning on Y_1 yields for the time after the first claim

$$\begin{aligned}\lambda \int_0^\infty e^{-\lambda t}\Big\{\int_0^b V(b-y)\,\mathrm{d}G(y)\\ +\int_b^\infty \Big(-\phi\int_0^\infty e^{-\delta t}\,\mathrm{d}Z_t + e^{-\delta t}V(b-y+Z_t)\Big)\,\mathrm{d}G(y)\Big\}\,\mathrm{d}t\\ = \frac{\lambda}{\lambda+\delta}\int_0^\infty V(b-y)\,\mathrm{d}G(y)\;.\end{aligned}$$

Thus, we can characterise the value at b through

$$V(b) = \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^\infty V(b-y)\,\mathrm{d}G(y)\;.$$

Plugging $V(b)$ into (3.2), we find that $V'(b-) = V'(b+) = 1$, and $V(x)$ is continuously differentiable also in this case.

Remark 3.2.3
Suppose $b = \infty$. Then we would have $V'(x) > 1$ for all $x \geq 0$. From the HJB equation (3.2) would follow

$$\begin{aligned} cV'(x) &= (\lambda+\delta)V(x) - \lambda\int_0^x V(x-y)\,\mathrm{d}G(y) - \lambda V(0)(1-G(x))\\ &\qquad +\lambda\phi\int_x^\infty (1-G(y))\,\mathrm{d}y\\ &\geq \lambda V(x)(1-G(x)) + \delta V(x) - \lambda V(0)(1-G(x))\\ &= \delta V(x) + \lambda(1-G(x))(V(x)-V(0))\\ &> \delta V(x)\end{aligned}$$

and, therefore, $V(x) > e^{\delta x/c}$. Thus, V were exponentially increasing on the whole domain. However, by Lemma 3.2.1, V has a linear bound. Therefore, it must be $b < \infty$. ■

Before proving the next lemma we make the following observations.

- $\{Z_t\}$ increases only at claim times. It holds therefore in an interval (T_{i-1}, T_i) between two claims that $\mathrm{d}X_t^U = (c - U_t)\,\mathrm{d}t$.
- $X_{T_i}^U = X_{T_i-}^U - Y_i + \Delta Z_{T_i}$.
- $X_{T_i-}^U \geq X_{T_i}^U$.

- If $X^U_{T_i-} - Y_i \le 0$ then the shareholders pay so much that $X^U_{T_i} = X^U_{T_i-} - Y_i + \Delta Z_{T_i} = 0$, i.e., $Z_{T_i} = -\min(X^U_{T_i-} - Y_i, 0)$. In this case the value function fulfils

$$V(X^U_{T_i})(= V(0)) = V(X^U_{T_i-} - Y_i) + \phi \Delta Z_{T_i} \,.$$

Thus, it suffices only to consider solutions f to the HJB equation satisfying property (3.1).

Lemma 3.2.4
Let $f(x)$ be an increasing, bounded, and positive solution to the HJB equation (3.2) with property (3.1). Then for any admissible stategy U the process

$$\Big\{ f(X^U_t)e^{-\delta t} - f(x) - \phi \int_{0-}^t e^{-\delta s}\, dZ_s - \int_0^t \Big[(c - U_s) f'(X^U_s) \\ - (\lambda + \delta) f(X^U_s) + \lambda \int_0^\infty f(X^U_s - y)\, dG(y) \Big] e^{-\delta s}\, ds \Big\}$$

is an $\{\mathscr{F}_t\}-$martingale.

Proof: We have the decomposition

$$\begin{aligned} f(X^U_t)e^{-\delta t} &= f(X^U_0) + \sum_{i=1}^{N_t} \Big[f(X^U_{T_i}) e^{-\delta T_i} - f(X^U_{T_{i-1}}) e^{-\delta T_{i-1}} \Big] \\ &\quad + f(X^U_t)e^{-\delta t} - f(X^U_{T_{N_t}}) e^{-\delta T_{N_t}} \\ &= f(X^U_0) + \sum_{i=1}^{N_t} \Big[f(X^U_{T_i}) e^{-\delta T_i} - f(X^U_{T_i-}) e^{-\delta T_i -} \Big] \\ &\quad + \sum_{i=1}^{N_t} \Big[f(X^U_{T_i-}) e^{-\delta T_i -} - f(X^U_{T_{i-1}}) e^{-\delta T_{i-1}} \Big] \\ &\quad + f(X^U_t)e^{-\delta t} - f(X^U_{T_{N_t}}) e^{-\delta T_{N_t}} \\ &= f(x) + \phi Z_0 + \sum_{i=1}^{N_t} \Big[f(X^U_{T_i-} - Y_i) e^{-\delta T_i} - f(X^U_{T_i-}) e^{-\delta T_i -} \Big] \\ &\quad + \phi \sum_{i=1}^{N_t} \Delta Z_{T_i} e^{-\delta T_i} + \sum_{i=1}^{N_t} \Big[f(X^U_{T_i-}) e^{-\delta T_i -} - f(X^U_{T_{i-1}}) e^{-\delta T_{i-1}} \Big] \\ &\quad + f(X^U_t)e^{-\delta t} - f(X^U_{T_{N_t}}) e^{-\delta T_{N_t}}. \end{aligned}$$

By Theorem B.2.4 (see also Example A.2.6 in the Appendix), we know that the process

$$\Big\{\sum_{i=1}^{N_t}\Big[f(X^U_{T_i-}-Y_i)-f(X^U_{T_i-})\Big]e^{-\delta T_i} \\ -\int_0^t e^{-\delta s}\Big(\lambda\int_0^\infty f(X^U_s-y)\,\mathrm{d}G(y)-\lambda f(X^U_s)\Big)\,\mathrm{d}s\Big\}$$

is an $\{\mathscr{F}_t\}$-martingale. Noting that between the claims

$$f(X^U_{T_i-})e^{-\delta T_i-}-f(X^U_{T_{i-1}})e^{-\delta T_{i-1}}=\int_{T_{i-1}}^{T_i-}\Big((c-U_s)f'(X^U_s)-\delta f(X^U_s)\Big)e^{-\delta s}\,\mathrm{d}s$$

and using that T_i and T_{i-1} can be replaced by $T_i\wedge t$ and $T_{i-1}\wedge t$, respectively, we get that the process

$$\Big\{f(X^U_t)e^{-\delta t}-f(x)-\phi\int_{0-}^t e^{-\delta s}\,\mathrm{d}Z_s-\int_0^t\Big[(c-U_s)f'(X^U_s) \\ +\lambda\int_0^\infty f(X^U_s-y)\,\mathrm{d}G(y)-(\lambda+\delta)f(X^U_s)\Big]e^{-\delta s}\,\mathrm{d}s\Big\}$$

is an $\{\mathscr{F}_t\}-$martingale with expected value 0. $\square$

We show now that the value function is unique and the strategy (3.5) is optimal.

Theorem 3.2.5
Let $f(x)$ be an increasing, bounded, and positive solution to the HJB equation (3.2) *with property* (3.1). *Then $\lim_{x\to\infty}f(x)=u_0/\delta, f(x)=V(x)$, and an optimal strategy is given by* (3.5).

Proof: Since f is bounded, f must converge to a $f(\infty)<\infty$. Then there exists a sequence $x_n\to\infty$ such that $f'(x_n)\to 0$. Let $u_n=u(x_n)$. By the definition (3.5), we can assume $u_n=u_0$. Letting $n\to\infty$ in (3.2) yields

$$\begin{aligned} 0 &= (c-u_0)f'(x_n)+\lambda\Big[\int_0^\infty f(x_n-y)\,\mathrm{d}G(y)-f(x_n)\Big]-\delta f(x_n)+u_0 \\ &\overset{n\to\infty}{\longrightarrow} -\delta f(\infty)+u_0 \end{aligned}$$

showing $\lim_{x\to\infty}f(x)=u_0/\delta$.

Let now $U=U^*$ be the strategy given by (3.5) and the corresponding $Z^*=Z^{U^*}$. It follows from the lemma above and the HJB equation that

$$\Big\{f(X^{U^*}_t)e^{-\delta t}-f(x)+\int_0^t e^{-\delta s}U^*_s\,\mathrm{d}s-\phi\int_0^t e^{-\delta s}\,\mathrm{d}Z^*_s\Big\}$$

is a martingale with expected value 0. Then

$$f(x) = \mathbb{E}\Big[f(X_t^{U^*})e^{-\delta t} + \int_0^t e^{-\delta s} U_s^* \, \mathrm{d}s - \phi \int_0^t e^{-\delta s} \, \mathrm{d}Z_s^*\Big]$$

holds. By the boundedness of f and the bounded convergence theorem, we get $\mathbb{E}[f(X_t^{U^*})e^{-\delta t}] \to 0$ as $t \to \infty$. Since the other terms are monotone, we can interchange limit and integration and get $f(x) = V^{U^*}(x)$. For an arbitrary strategy U, (3.2) gives

$$\begin{aligned} f(x) &\geq \mathbb{E}\Big[f(X_t^U)e^{-\delta t} + \int_0^t e^{-\delta s} U_s \, \mathrm{d}s - \phi \int_0^t e^{-\delta s} \, \mathrm{d}Z_s\Big] \\ &\geq \mathbb{E}\Big[\int_0^t e^{-\delta s} U_s \, \mathrm{d}s - \phi \int_0^t e^{-\delta s} \, \mathrm{d}Z_s\Big] \,. \end{aligned}$$

Letting $t \to \infty$ shows $f(x) \geq V^U(x)$. Thus $f(x) = V(x)$. □

This shows that it is optimal to pay no dividends as the reserve process stays below a barrier b ($X_t^U < b$). As soon as the process reaches or excesses the barrier b ($X_t^U \geq b$), dividends have to be payed at the maximal rate u_0.

3.3 Unrestricted Dividends

In this section, all increasing, adapted, and càdlàg processes D are allowed. The value of a strategy D is

$$V^D(x) = \mathbb{E}\Big[\int_{0-}^{\infty} e^{-\delta t} \, \mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t} \, \mathrm{d}Z_t\Big] \,.$$

and $V(x) = \sup_{D \in \mathscr{S}_x} V^D(x)$ is the value function.

3.3.1 The Value Function and the HJB-Equation

Lemma 3.3.1
The function $V(x)$ is increasing with $V(x) - V(y) \geq x - y$ for $0 \leq y \leq x$ and Lipschitz continuous on $[0, \infty)$. For any $x \geq 0$,

$$x + \frac{c - \lambda \mu \phi}{\delta} \leq V(x) \leq x + \frac{c}{\delta} \,.$$

$V(x)$ is almost everywhere differentiable with $V'(x-) \geq 1$ and $V'(x+) \leq \phi$.

Proof: Let $y \geq 0$. Consider a strategy D with $V^D(y) \geq V(y) - \varepsilon$ for $\varepsilon > 0$. For $x \geq y$, we define a new strategy as followed: $x - y$ is paid immediately as dividend, and then the strategy D with initial capital y is followed. Then for any $\varepsilon > 0$, it holds that

$$V(x) \geq x - y + V^D(y) \geq x - y + V(y) - \varepsilon \,.$$

Because ε was arbitrary, $V(x) - V(y) \geq x - y$ follows. In particular, V is strictly increasing.

Consider the strategy D, where x is paid immediately and then the dividends are paid at rate c. We note that for any reasonable strategy, D_t is an upper bound for the accumulated dividend payments. Not taking the capital injections into account yields the upper bound:

$$V^D(x) \leq x + \mathbb{E}\Big[\int_0^\infty e^{-\delta t} c \,\mathrm{d}t\Big] = x + \frac{c}{\delta} \,.$$

The value of the capital injections is calculated in Lemma 3.1.2, which yields the lower bound. The local Lipschitz continuity follows by the local boundedness of V as in the proof of Lemma 3.2.1.

V is concave. Thus, the derivative $V'(x)$ exists almost everywhere on $[0, \infty)$. By $V(x) - V(x - \varepsilon) \geq \varepsilon$ for $\varepsilon \geq 0$, we obtain $V'(x-) \geq 1$. For the other inequality, we consider a strategy by receiving ε from the shareholders immediately and following the strategy for the initial capital $x + \varepsilon$ afterwards, so that $V(x) \geq V(x + \varepsilon) - \phi\varepsilon$. Hence, $V'(x+) \leq \phi$. This proves the (global) Lipschitz continuity. □

Note that the bounds on the derivatives were also obtained by Sethi and Taksar [68].

From the next lemma, it follows that the value function can be calculated as the limit of the value functions from the previous section. The proof is analogous to the proof in Schmidli [66, Section 2.4.2].

Lemma 3.3.2
Let $V_u(x)$ be the value function for the restricted dividend strategy in the case $u_0 = u$. Then $\lim_{u\to\infty} V_u(x) = V(x)$.

Proof: Because $V_u(x)$ is increasing in u, it is converging pointwise. The restricted strategy is admissible. Therefore $\lim_{u\to\infty} V_u(x) \leq V(x)$. We proceed in two steps. We first approximate $V(x)$ by the value of a pure jump process strategy, which on its part can be approximated by a resticted dividend strategy in the second step (see Figure 3.1).

Step 1. Let $\varepsilon > 0$. We now construct a pure jump dividend process D such that $V^D(x) \geq V(x) - 2\varepsilon$.

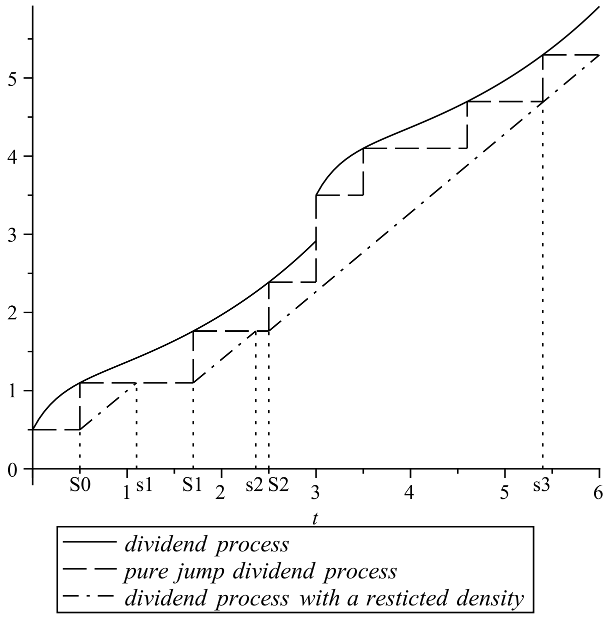

Figure 3.1: Approximation of a dividend process

Let $(\tilde{D}_t, \tilde{Z}_t)$ be a dividend process such that $V^{\tilde{D}}(x) \geq V(x) - \varepsilon$. Let $D_0 = \tilde{D}_0 \mathbf{1}_{\{\tilde{D}_0 \geq \varepsilon\}}$. On $\{\tilde{D}_t - D_{t-} < \varepsilon\}$ no dividends are paid, i.e., $\mathrm{d}D_t = 0$. On $\{\tilde{D}_t - D_{t-} \geq \varepsilon\}$ we let $D_t = \tilde{D}_t$, i.e., once the difference between the dividend processes has become larger than ε, a dividend is paid in order that the accumulated dividends become the same. Then D is a pure jump process with jumps of at least size ε.

Note that $\tilde{D}_t - D_t \leq \varepsilon$ and $\tilde{Z}_t \geq Z_t$. Using that by Fubini's theorem, $\mathbb{E}\Big[\int_{0-}^\infty e^{-\delta t}\,\mathrm{d}D_t\Big] = \mathbb{E}\Big[\int_{0-}^\infty \delta e^{-\delta t} D_{t-}\,\mathrm{d}t\Big]$, we have then

$$\begin{aligned} &V^{\tilde{D}}(x) - V^D(x) \\ &\quad = \mathbb{E}_x\Big[\int_0^\infty \delta e^{-\delta s}(\tilde{D}_s - D_s)\,\mathrm{d}s\Big] - \phi\mathbb{E}_x\Big[\int_0^\infty \delta e^{-\delta s}(\tilde{Z}_s - Z_s)\,\mathrm{d}s\Big] \\ &\quad \leq \varepsilon, \end{aligned}$$

and therefore

$$V^D(x) \geq V^{\tilde{D}}(x) - \varepsilon \geq V(x) - 2\varepsilon.$$

Step 2. We now construct a strategy $\hat{D}$ with $\hat{D}_t = \int_0^t U_s\,\mathrm{d}s$ for some bounded process U_s in order to approximate the value $V^D(x)$.

Let $s_0 = 0$ and $\hat{D}_0 = D_0$. For $n \geq 0$ and $0 \leq u_0 < \infty$, define

$$\begin{aligned} S_n &= \inf\{t \geq s_n : D_t \geq \hat{D}_{s_n} + \varepsilon\}, \\ s_{n+1} &= \inf\{t > S_n : \hat{D}_{S_n} + u_0 t = D_t\}. \end{aligned}$$

We let $U_t = 0$ on $[s_n, S_n)$ and $U_t = u_0$ on $[S_n, s_{n+1})$, i.e., as long as both processes coinside, no dividends are paid; once the process D has made a jump of at least size ε, dividends at rate u_0 are paid, until the accumulated dividends again have the same value. By construction, we have $D_t = \hat{D}_t$ for $t \in [s_n, S_n), n \in \mathbb{N}$. Further, it holds $Z_t \geq \hat{Z}_t, t \geq 0$. Thus,

$$\begin{aligned} &V^D(x) - V^{\hat{D}}(x) \\ &\quad = \mathbb{E}_x\Big[\int_0^\infty \delta e^{-\delta s}(D_s - \hat{D}_s)\,\mathrm{d}s\Big] - \phi\mathbb{E}_x\Big[\int_0^\infty \delta e^{-\delta s}(Z_s - \hat{Z}_s)\,\mathrm{d}s\Big] \\ &\quad \leq \mathbb{E}_x\Big[\sum_{n=0}^\infty \delta \int_{S_n}^{s_{n+1}} e^{-\delta s}(D_s - \hat{D}_s)\,\mathrm{d}s\Big]. \end{aligned}$$

As a function of u_0, the sum is monotonically decreasing in u_0, and as $u_0 \to \infty$ it will converge to zero. Thus, by monotone convergence we can choose u_0 such that

$$V^D(x) - V^{\hat{D}}(x) < \varepsilon.$$

This shows that

$$V(x) - V^{\hat{D}}(x) < 2\varepsilon + \varepsilon = 3\varepsilon.$$

Because ε is arbitrary, this proves the lemma. □

Now we want to determine the Hamilton-Jacobi-Bellman equation for this problem. We repeat the procedure in Schmidli [66].

Theorem 3.3.3
The function $V(x)$ is continuously differentiable on $(0, \infty)$ and fulfils the equation

$$\begin{aligned} 0 &= \\ &\max\Big\{cV'(x) + \lambda \int_0^\infty V(x-y)\,dG(y) - (\lambda+\delta)V(x), 1 - V'(x), V'(x) - \phi\Big\} \end{aligned} \tag{3.6}$$

Proof: Since we let $u \to \infty$, it is enough to consider the case $u > c$. Equation (3.2) can be written as

$$\begin{aligned} \max\Big\{&cV_u'(x) + \lambda \int_0^\infty V_u(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V_u(x), \\ &1 - V_u'(x) + \frac{\lambda \int_0^\infty V_u(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V_u(x) + c}{u-c}\Big\} = 0. \end{aligned} \tag{3.7}$$

The two parts correspond to the different cases $u = 0$ and $u = u_0$ for the restricted problem (see Eq.(3.2)), where the second equation is divided by

$u_0 - c$. We already know that $\lim_{u\to\infty} V_u(x) = V(x)$, hence, by the bounded convergence theorem,

$$\lim_{u\to\infty} (\lambda+\delta)V_u(x)-\lambda \int_0^\infty V_u(x-y)\,\mathrm{d}G(y) = (\lambda+\delta)V(x)-\lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y).$$

Assume that $V_u'(x)$ converges for every x a.e. to some function f. The limit is finite because $cV_u'(x) \le (\lambda+\delta)V_u(x) + \lambda\phi\int_x^\infty (1-G(y))\,\mathrm{d}y \le (\lambda+\delta)V(x) + \lambda\phi\int_x^\infty (1-G(y))\,\mathrm{d}y$. The function f satisfies Equation (3.7) and so

$$\max\left\{cf(x) + \lambda\int_0^\infty V(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V(x), 1-f(x)\right\} = 0.$$

It follows by the bounded convergence theorem that

$$V(x) - V(0) = \lim_{u\to\infty}\int_0^x V_u'(z)dz = \int_0^x \lim_{u\to\infty} V_u'(z)dz = \int_0^x f(z)dz.$$

I.e., f is the density of V, $V(x)$ is differentiable at all points where $f(x)$ is continuous and $f(x) = V'(x)$. By $V'(x) \le \phi$, the assertion follows.

It remains to show that $V_u'(x) \overset{u\to\infty}{\longrightarrow} f(x)$ a.e. and f is continuous.

We have seen in Section 3.2 that, for $u \ge c$ fixed, the optimal strategy is a barrier strategy, and the optimal level b does not depend on u. In particular, $V(x)$ coincides with $V_u(x)$ for $u \ge c$ and initial capital $x \le b$. Therefore, we can choose $b := \inf\{x : V_c'(x-) = 1\}$ and $V(x) = V_c(x)$ for $x < b$. Then $f(x) = V_c'(x) > 1$ and f is continuous on $[0, b)$ with $f(b-) = 1$.

Let now $x \ge b$. Then $V_u'(x) \le 1$ for all $u > c$. Let $\{u_n\}$ be a sequence tending to infinity such that $V_{u_n}'(x)$ converges to $\limsup_{u\to\infty} V_u'(x)$. From the second term on the right hand side of (3.7) we see that $\lim_{n\to\infty} V_{u_n}'(x) = 1$. Analogously, we can show that $\liminf_{u\to\infty} V_u'(x) = 1$. Thus, $f(x) = \lim_{u\to\infty} V_u'(x) = 1$. In particular, we have $f(b+) = 1$, i.e., $V(x)$ is continuously differentiable.

Condition $V'(x) - \phi \le 0$ is fulfilled by Lemma 3.3.1. □

Because the function $V(x)$ is concave, the condition $V'(x)-\phi \le 0$ is fulfilled whenever $V'(0) \le \phi$. Further, we get that $b := \inf\{x : V'(x) = 1\} < \infty$ and $V(x)$ is continuously differentiable.

3.3.2 The Optimal Strategy and the Characterisation of the Solution

We now define the following strategy:

$$\begin{aligned} D_0^* &= \max(x-b,0), \\ D_t^* &= D_0^* + \int_0^t c \cdot \mathbf{1}_{\{X_s^*=b\}}\,\mathrm{d}s \quad \text{for } t>0 \\ Z_t^* &= \max(-\inf_{0\le s\le t}(X_s - D_s^*),0) = Z_t^{D^*} \quad \text{for } t>0. \end{aligned} \tag{3.8}$$

This is a barrier strategy with the upper barrier b and the lower barrier 0. We denote by $X_t^* = X_t - D_t^* + Z_t^* \in [0,b]$ the corresponding surplus process. It is well-known that the process $\{X_t^*\}$ exists. This can be seen in the following way. Suppose $x \in [0,b]$. Let

$$\begin{aligned} D_t^0 &= \sup\{X_s - b : 0 \le s \le t\} \vee 0 \ , \\ \tau_1 &= \inf\{t : X_t - D_t^0 < 0\} \ , \\ Z_t^0 &= \sup\{D_s^0 - X_s : 0 \le s \le t\} \vee 0 \ . \end{aligned}$$

We define $X_t^1 = X_t - D_t^0 + Z_t^0$. Note that $X_t^* = X_t^1$ for $t \in [0,\tau_1]$. Suppose that we have constructed $\{X_t^n\}$. Then we let

$$\begin{aligned} D_t^n &= D_t^{n-1} + (\sup\{X_s^n - b : 0 \le s \le t\} \vee 0) \ , \\ \tau_{n+1} &= \inf\{t : X_t^n - D_t^n < 0\} \ , \\ Z_t^n &= Z_t^{n-1} + (\sup\{D_s^n - X_s^n : 0 \le s \le t\} \vee 0) \ . \end{aligned}$$

We have $X_t^* = X_t^{n+1}$ for $t \in [0,\tau_{n+1}]$. Because τ_n is a claim occurrence time and $\tau_{n+1} > \tau_n$, we have that $\tau_n \to \infty$ as $n \to \infty$. Thus, $\{X_t^n\}$ converges almost surely to $\{X_t^*\}$.

Theorem 3.3.4
The strategy (3.8) *is optimal, i.e.,* $V^*(x) = V^{D^*}(x) = V(x)$.

Proof: We can assume that $x \ge 0$, i.e. $Z_0^* = 0$. We have $V'(X_t^*) = V'(b) = 1$ on $\{X_t^* = b\}$ and $V'(X_t^*) > 1$ on $\{X_t^* < b\}$. Z_t^* is a càdlàg process with jumps at most in claim times $T_i, i > 0$. As in Lemma 3.2.4 we get that

$$\begin{aligned}
&V(X_t^*)e^{-\delta t} - V(x) \\
&\quad = \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* + \sum_{i=1}^{N_t}\Big[V(X_{T_i-}^* - Y_i) - V(X_{T_i-}^*)\Big]e^{-\delta T_i} \\
&\qquad + \sum_{i=1}^{N_t}\Big[V(X_{T_i-}^*)e^{-\delta T_i-} - V(X_{T_{i-1}}^*)e^{-\delta T_{i-1}}\Big] \\
&\qquad + V(X_t^*)e^{-\delta t} - V(X_{T_{N_t}}^*)e^{-\delta T_{N_t}} \\
&\quad = \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* + \sum_{i=1}^{N_t}\Big[V(X_{T_i-}^* - Y_i) - V(X_{T_i-}^*)\Big]e^{-\delta T_i} \\
&\qquad + \sum_{i=1}^{N_t}\Big[\int_{T_{i-1}}^{T_i-}\Big(cV'(X_s^*) - \delta V(X_s^*)\Big)\mathbf{1}_{\{X_s^*<b\}}e^{-\delta s}\,\mathrm{d}s\Big] \\
&\qquad + \int_{T_{N_t}}^t \Big(cV'(X_s^*) - \delta V(X_s^*)\Big)\mathbf{1}_{\{X_s^*<b\}}e^{-\delta s}\,\mathrm{d}s \\
&\qquad - \sum_{i=1}^{N_t}\Big[\int_{T_{i-1}}^{T_i-}\delta V(X_s^*)\mathbf{1}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\Big] - \int_{T_{N_t}}^t \delta V(X_s^*)\mathbf{1}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\,.
\end{aligned}$$

By Theorem B.2.4, the process

$$\begin{aligned}
\Big\{&\sum_{i=1}^{N_t}\Big[V(X_{T_i-}^* - Y_i) - V(X_{T_i-}^*)\Big]e^{-\delta T_i} \\
&\qquad\qquad - \lambda \int_0^t e^{-\delta s}\int_0^\infty \Big(V(X_s^* - y)\,\mathrm{d}G(y) - V(X_s^*)\Big)\,\mathrm{d}s\Big\}
\end{aligned}$$

is a martingale with expected value 0. Therefore, the same holds for the process

$$\begin{aligned}
\Big\{&V(X_t^*)e^{-\delta t} - V(x) - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* \\
&- \int_0^t \Big[cV'(X_s^*) + \lambda\int_0^\infty V(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)V(X_s^*)\Big]\mathbf{1}_{\{X_s^*<b\}}e^{-\delta s}\,\mathrm{d}s\Big\} \\
&- \int_0^t \Big[\lambda\int_0^\infty V(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)V(X_s^*)\Big]\mathbf{1}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\Big\}\,.
\end{aligned}$$

Recall that, by concavity of $V(x)$, the derivatives from the left and from the right exist, and because $V(x)$ fulfils (3.6), it is continuously differentiable by our assumption that $G(y)$ is continuous. Because $V'(X_s^*) > 1$ on $\{X_s^* < b\}$, the first term on the left hand side of (3.6) vanishes and, thus, also the integral over $\{X_s^* < b\}$. From $V'(X_s^*) = 1$ on $\{X_s^* = b\}$ and (3.6), it follows that

$$\lambda \int_0^\infty V(X_s^* - y)\,\mathrm{d}G(y) - (\lambda + \delta)V(X_s^*) = -c$$

follows. Altogether, we get that

$$\Big\{V(X_t^*)e^{-\delta t} - V(x) - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* + \int_0^t c\mathbf{I}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\Big\}$$

is a martingale with expected value 0. From the martingale property we get that

$$V(x) = \mathbb{E}\Big[V(X_t^*)e^{-\delta t} - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* + \int_0^t c\mathbf{I}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\Big]\;.$$

Since $V(X_t^*)e^{-\delta t} \le V(b)e^{-\delta t}$ converges to 0, we have that

$$\lim_{t\to\infty} \mathbb{E}[V(X_t^*)e^{-\delta t}] = 0$$

by the bounded convergence theorem. Because $\mathrm{d}D_t^* = 0$ on $\{X_t^* < b\}$ and $\mathrm{d}D_t^* = c\,\mathrm{d}t$ on $\{X_t^* = b\}$, by the monotone convergence theorem, we finally get that

$$\begin{aligned} V(x) &= \lim_{t\to\infty} \mathbb{E}\Big[\int_0^t c\mathbf{I}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^*\Big] \\ &= \mathbb{E}\Big[\int_0^\infty e^{-\delta s}\,\mathrm{d}D_s^* - \phi \int_0^\infty e^{-\delta s}\,\mathrm{d}Z_s^*\Big] \\ &= V^*(x)\;. \end{aligned}$$

□

Remark 3.3.5
The point b is characterised by the equation $V'(b) = 1$, or equivalently, $V(b) = (\lambda + \delta)^{-1}(c + \lambda \int_0^\infty V(b - y)\,\mathrm{d}G(y))$. The question is now, how b can be determined if V is not known? Candidates for b can be found in following way. Define the function

$$v(x) = \frac{1}{\lambda + \delta}\Big(c + \lambda \int_0^\infty V(x - y)\,\mathrm{d}G(y)\Big)\;.$$

If $G(y)$ is differentiable at $y = x$, then b is the smallest solution to the equation $v'(x) = 1$. This method was first proposed by Gerber [28]. ■

Because we do not have an explicit solution nor an initial value, we need to characterise the solution $V(x)$ among other possible solutions. First, we observe that it is not possible to calculate the value function from Equation (3.6) with an initial value smaller than $V(0)$. Indeed, let $f(x)$ be a solution with $f(0) < V(0)$. Define $h(x) = f(x) - V(x)$. Then $h(0) < 0$ and $h(x)$ satisfies the equation

$$ch'(x) - (\lambda+\delta)h(x) + \lambda\int_0^x h(x-y)\,\mathrm{d}G(y) + \lambda h(0)(1-G(x)) = 0.$$

We can use the same arguments as in Remark 2.1.1. It holds $h'(0) = \delta h(0)/c < 0$. Let $x_0 := \inf\{x \geq 0 : h'(x) \geq 0\}$. Then $h(x)$ is strictly decreasing on $[0, x_0)$. Suppose, x_0 is finite. Then $h'(x_0) \geq 0$. However,

$$\begin{aligned} ch'(x_0) &= (\lambda+\delta)h(x_0) - \lambda\int_0^{x_0} h(x_0-y)\,\mathrm{d}G(y) - \lambda h(0)(1-G(x_0)) \\ &\leq (\lambda+\delta)h(x_0) - \lambda h(0)G(x_0) - \lambda h(0)(1-G(x_0)) \\ &= \delta h(x_0) + \lambda(h(x_0) - h(0)) \\ &< 0 \end{aligned}$$

since $h(x)$ is negative and decreasing. This is a contradiction. Thus, $x_0 = \infty$ and $h(x)$ is strictly decreasing on $\mathbb{R}_+$. Since $V'(x) = 1$ for all $x \geq b$, we have that $0 > h'(x) = f'(x) - V'(x) = f'(x) - 1$, and therefore, $f'(x) < 1$ for all $x \geq b$. Thus, $f(x)$ cannot be the value function.

Theorem 3.3.6
$V(x)$ is the minimal solution to (3.6). If $f(x)$ is a solution fulfilling property (3.1) and a linear growth condition $f(x) \leq \kappa_1 x + \kappa_2$ for some positive constants κ_1, κ_2 and all $x \geq 0$, and either $f'(0) > 1$ or $f(0) = (c - \lambda\phi\mu)/\delta$, then $f(x) = V(x)$.

Proof: We again suppose that $x \geq 0$. Let f be a solution to the HJB equation. Consider the process X^* under the optimal strategy. We have then, as in the proof of the previous theorem, that the process

$$\begin{aligned} &\Big\{ f(X_t^*)e^{-\delta t} - f(x) - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* \\ &\quad - \int_0^t \Big[cf'(X_s^*) + \lambda\int_0^\infty f(X_s^*-y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*)\Big]\mathbf{1}_{\{X_s^*<b\}}e^{-\delta s}\,\mathrm{d}s\Big\} \\ &\quad - \int_0^t \Big[\lambda\int_0^\infty f(X_s^*-y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*)\Big]\mathbf{1}_{\{X_s^*=b\}}e^{-\delta s}\,\mathrm{d}s\Big\} \end{aligned}$$

is a martingale with expected value 0. By (3.6)

$$cf'(X_s^*) + \lambda\int_0^\infty f(X_s^*-y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*) \leq 0$$

and

$$\lambda \int_0^\infty f(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta) f(X_s^*) \le -cf'(X_s^*) \le -c\,,$$

because $f'(x) \ge 1$. This yields

$$\begin{aligned} f(x) &\ge \mathbb{E}\Bigl[f(X_t^*)e^{-\delta t} - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* + \int_0^t c\mathbf{1}_{\{X_s^*=b\}} e^{-\delta s}\,\mathrm{d}s\Bigr] \\ &\ge \mathbb{E}\Bigl[\int_0^t e^{-\delta s}\,\mathrm{d}D_s^* - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^*\Bigr] \end{aligned}$$

and therefore $f(x) \ge V^{D^*}(x) = V(x)$.

Suppose that, additionally, f satisfies a linear growth condition. Let $\tilde{b} = \inf\{x \ge 0 : f'(x) = 1\}$. Define a barrier dividend strategy $\tilde{D}$ with barrier $\tilde{b}$ and $\tilde{Z} = Z^{\tilde{D}}$. In the same way as in the proof of Theorem 3.3.4, we find that the process

$$\Bigl\{f(X_t^{\tilde{D}})e^{-\delta t} - f(x) - \phi \int_0^t e^{-\delta s}\,\mathrm{d}\tilde{Z}_s + \int_0^t c\mathbf{1}_{\{X_s^{\tilde{D}}=\tilde{b}\}} e^{-\delta s}\,\mathrm{d}s\Bigr\}$$

is a martingale with expected value 0. Taking expectations and letting $t \to \infty$ yields the assertion, since by the linear growth condition, we have $f(X_t^{\tilde{D}})e^{-\delta t} \le f(x+ct)e^{-\delta t} \to 0$ as $t \to \infty$. Thus, $f(x) = V^{\tilde{D}}(x) \le V(x)$ and, therefore, $f(x) = V(x)$. □

3.3.3 Calculating the Value Function

Dividends at zero

We first consider the value in $x = 0$. Then

$$cV'(0) + \lambda \int_0^\infty (V(0) - \phi y)\,\mathrm{d}G(y) - (\lambda+\delta)V(0) = 0,$$

which is equivalent to

$$V(0) = \frac{cV'(0) - \phi\lambda\mu}{\delta}\,.$$

For a diffusion approximation the derivative in 0 is ϕ; see Shreve et al. [67]. This cannot hold in general for our model, since, otherwise, in the case of a positive safety loading $c > \lambda\mu$,

$$V(0) = \frac{cV'(0) - \phi\lambda\mu}{\delta} = \phi\frac{c - \lambda\mu}{\delta}$$

was increasing in ϕ, which is not possible. We guess that $V'(0) < \phi$. If the net profit condition is not fulfilled, i.e., $c \le \lambda\mu$, then

$$V(0) = \frac{cV'(0) - \phi\lambda\mu}{\delta} \le \frac{\lambda\mu(V'(0) - \phi)}{\delta} \le 0.$$

If $V'(0) = 1$, i.e., $b = 0$, then $V'(x) = 1$ for all $x > 0$ because V' is not increasing. x is immediately paid as dividend and the surplus process starts again in zero. So we have $V(x) = x + V(0) = x + (c - \lambda\mu\phi)/\delta$.

The next lemma gives a simple condition for the optimal barrier in 0. It is surprising, because the premium rate and the claim size distribution does not play any role.

Lemma 3.3.7
We have $b = 0$ if and only if $\delta \ge \lambda(\phi - 1)$.

Proof: Suppose first that $b = 0$. We already know that $V(x) = x+(c-\lambda\mu\phi)/\delta$ is equivalent to $b = 0$. V fulfils the HJB equation, so the first term on the left hand side of (3.6) is

$$c + \lambda\Big[\int_0^x \Big(x - y + \frac{c - \lambda\mu\phi}{\delta}\Big)\, \mathrm{d}G(y) + \int_x^\infty \Big(\frac{c - \lambda\mu\phi}{\delta} - \phi(y - x)\Big)\, \mathrm{d}G(y)\Big] - (\lambda + \delta)\Big(x + \frac{c - \lambda\mu\phi}{\delta}\Big) \le 0\ .$$

This is equivalent to

$$\int_0^x \Big[\lambda(\phi - 1)(1 - G(y)) - \delta\Big]\, \mathrm{d}y \le 0\ .$$

This integral is a concave function with the initial value zero. So it is negative if and only if the derivative in zero is negative. Thus, it must hold that $\lambda(\phi - 1) - \delta \le 0$.

On the other hand, if $\delta \ge \lambda(\phi-1)$ then $f(x) = x+(c-\lambda\mu\phi)/\delta$ is a solution to (3.6). Because $V(x)$ is the minimal solution, we have $V(x) \le f(x)$. But by Lemma 3.3.1, we also have $f(x) \le V(x)$. Thus, $V(x) = f(x)$, which implies that $b = 0$. □

No dividends at zero

If no dividends are paid at zero ($V'(0) > 1$) then there is a barrier $b^* = \inf\{x : V'(x) = 1\} > 0$ such that for all $x > 0$ the barrier strategy, which pays the capital exceeding b^*, is optimal. The value $V(0)$ can be found by comparing the barrier strategies with a barrier b for all b:

$$V(0) = \sup_{b \ge 0}\{f(0) : f = \text{value of a barrier strategy with barrier}\, b\}\ . \tag{3.9}$$

On $[0, b]$ we have to solve the equation

$$cf'(x) + \lambda \int_0^\infty f(x-y)\, \mathrm{d}G(y) - (\lambda+\delta) f(x) = 0 \quad \text{with} \quad f'(b) = 1\,. \tag{3.10}$$

If f is the value of a barrier strategy with a barrier in b, i.e., $f = V^{\tilde{D}}$ with $\mathrm{d}\tilde{D}_t = c \cdot \mathbf{1}_{\{X_t^{\tilde{D}} = b\}}\, \mathrm{d}t$, then we have always $f'(b) = 1$, because

$$\begin{aligned} f(b) &= \int_0^\infty \lambda e^{-\lambda t} \Big(\int_0^t c e^{-\delta s}\, \mathrm{d}s + e^{-\delta t} \int_0^\infty f(b-y)\, \mathrm{d}G(y) \Big)\, \mathrm{d}t \\ &= \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta} \int_0^\infty f(b-y)\, \mathrm{d}G(y) \end{aligned}$$

and for $h > 0$

$$f(b-ch) = e^{(-\lambda+\delta)h} f(b) + \lambda \int_0^h e^{(-\lambda+\delta)t} \int_0^\infty f(b - c(h-t) - y)\, \mathrm{d}G(y)\, \mathrm{d}t.$$

Therefore

$$c \cdot \frac{f(b) - f(b-ch)}{ch} \xrightarrow{h \to 0} (\lambda+\delta) f(b) - \lambda \int_0^\infty f(b-y)\, \mathrm{d}G(y) = c,$$

i.e., $f'(b-) = 1$. Since $f(x) = x - b + f(b)$ for $x \ge b$, we also have $f'(b+) = 1$.

Equation (3.10) can be solved for example via the Laplace transform. It can be written as

$$\begin{aligned} cf'(x) + \lambda \int_0^x f(x-y)\, \mathrm{d}G(y) + \lambda f(0)(1 - G(x)) \\ -\lambda\phi \int_x^\infty (1 - G(y))\, \mathrm{d}y - (\lambda+\delta) f(x) = 0\,. \end{aligned}$$

If we denote by $\hat{f}$ the Laplace transform of f and by $\hat{g}$ the Laplace-Stieltjes transform of the density g of G, then we obtain a linear equation for $\hat{f}$:

$$\begin{aligned} c\hat{f}(s)s - cf(0) + \lambda \hat{f}(s)\hat{g}(s) + \frac{1}{s}\lambda f(0)(1 - \hat{g}(s)) \\ -\lambda\phi \frac{1}{s^2}(s\mu - 1 + \hat{g}(s)) - (\lambda+\delta)\hat{f}(s) = 0\,. \end{aligned}$$

This yields

$$\hat{f}(s) = \frac{f(0)\Big(c - \frac{1}{s}\lambda + \frac{1}{s}\lambda\hat{g}(s)\Big) + \lambda\phi\frac{1}{s^2}(s\mu - 1 + \hat{g}(s))}{cs + \lambda\hat{g}(s) - (\lambda+\delta)}\,.$$

If $\hat{f}$ has a "nice" form, for examle if it is a rational function, then f can be determined by inversion of $\hat{f}$. f depends an $f(0)$ and so f' does also. From $f'(b) = 1$ we can determine $f(0)$ as a function of b and denote it by $f^b(0)$. By (3.9), the optimal barrier b^* is the argument which maximizes $f^b(0)$, and $V(0) = f^{b^*}(0)$ is the maximal value.

Alternatively, we could use the Gerber-Shiu penalty function. With notations from Section 2.2, we can write

$$\begin{aligned} V(x) &= \mathbb{E}_x\Big[\int_{0-}^{\tau^{b^*}} e^{-\delta t}\,\mathrm{d}D_t\Big] \\ &\quad - \phi\mathbb{E}_x\Big[e^{-\delta\tau^{b^*}}|X^D_{\tau^{b^*}}|\mathbf{I}_{\tau^{b^*}<\infty}\Big] + V(0)\mathbb{E}_x\Big[e^{-\delta\tau^{b^*}}\mathbf{I}_{\tau^{b^*}<\infty}\Big] \\ &= V^{b^*}(x) - \phi\sigma^{b^*}(x) + V(0)\psi^{b^*}(x)\ . \end{aligned}$$

Let $f^b(x)$ be the value of a barrier strategy with a barrier b. From the dividends-penalty identity (2.17), it follows

$$\begin{aligned} f^b(x) &= V^b(x) - \phi\sigma^b(x) + f^b(0)\psi^b(x) \\ &= V^b(x)(1+\phi\sigma'(b)) - \phi\sigma(x) + f^b(0)(\psi(x) - \psi'(b)V^b(x)). \end{aligned}$$

Finally, by (2.15), the initial value can be calculated by

$$f^b(0) = \frac{V^b(0)(1-\phi\sigma'(b)) - \phi\sigma(0)}{1-\psi(0)+\psi'(b)V^b(0)} \tag{3.11}$$

with

$$V^b(0) = \frac{1-\chi(0)}{\rho e^{\rho b} - \chi'(b)}\ ,$$

where ρ is the positive solution to the Lundberg's funcdamental equation (2.8). Thus, $f^b(0)$ is determined by the functions $\sigma(x), \psi(x)$ and $\chi(x)$ with inital values given in (2.14) and the barrier b. Now we can find an optimal b^* which maximises $f^b(x)$ by maximising expression (3.11). Then $V(0) = f^{b^*}(0)$.

3.3.4 Examples

Exponentially distributed claim sizes

We consider the case with exponentially distributed claim sizes, i.e., $G(y) = 1 - e^{-\alpha y}$ and $\mathbb{E}[Y] = \mu = 1/\alpha$.

Because we already know the solution in the case $\delta \geq \lambda(\phi - 1)$, we suppose that $\delta < \lambda(\phi - 1)$. There exists a $b = \inf\{x : V'(x) = 1\} > 0$. On $(0, b)$ the function $V(x)$ satisfies the equation

$$cV'(x) = (\lambda+\delta)V(x) - \lambda e^{-\alpha x}\int_0^x V(z)\alpha e^{\alpha z}\,\mathrm{d}z - \Big(\lambda V(0) - \frac{\lambda\phi}{\alpha}\Big)e^{-\alpha x}\ . \tag{3.12}$$

The right-hand side is differentiable and, therefore,

$$\begin{aligned} cV''(x) &= (\lambda+\delta)V'(x) + \alpha\lambda e^{-\alpha x}\int_0^x V(z)\alpha e^{\alpha z}\,\mathrm{d}z \\ &\qquad - \alpha\lambda V(x) + \alpha\lambda V(0)e^{-\alpha x} - \lambda\phi e^{-\alpha x} . \end{aligned}$$

Using (3.12) to remove the integral yields

$$cV''(x) - V'(x)(\lambda+\delta-\alpha c) - \alpha\delta V(x) = 0 .$$

The solution to this differential equation is

$$V(x) = C_1 \cdot e^{v_1 x} + C_2 \cdot e^{v_2 x} ,$$

where v_1 and v_2 are the solutions to

$$cv^2 - (\lambda+\delta-\alpha c)v - \alpha\delta = 0 ,$$

i.e.,

$$\begin{aligned} v_1 &= \frac{\lambda+\delta-\alpha c - \sqrt{(\lambda+\delta-\alpha c)^2 + 4\alpha\delta c}}{2c} , \\ v_2 &= \frac{\lambda+\delta-\alpha c + \sqrt{(\lambda+\delta-\alpha c)^2 + 4\alpha\delta c}}{2c} . \end{aligned}$$

Plugging $V(0)$ in (3.12) yields

$$0 = cV'(0) - \delta V(0) - \frac{\lambda\phi}{\alpha} = C_1(cv_1-\delta) + C_2(cv_2-\delta) - \frac{\lambda\phi}{\alpha} ,$$

and from $V'(b)=1$, we can conclude that

$$1 = C_1 \cdot v_1 \cdot e^{v_1 b} + C_2 \cdot v_2 \cdot e^{v_2 b} .$$

By using $v_1 \cdot v_2 = -\alpha\delta/c$, we find that

$$\begin{aligned} C_1 &= \frac{-\frac{\lambda\phi}{\alpha\delta}v_2 e^{v_2 b} - 1 + \frac{c}{\delta}v_2}{e^{v_2 b}(v_2+\alpha) - e^{v_1 b}(v_1+\alpha)} \\ C_2 &= \frac{\frac{\lambda\phi}{\alpha\delta}v_1 e^{v_1 b} - \frac{c}{\delta}v_1 + 1}{e^{v_2 b}(v_2+\alpha) - e^{v_1 b}(v_1+\alpha)} . \end{aligned} \tag{3.13}$$

Thus,

$$V(0) = C_1 + C_2 = \frac{-\frac{\lambda\phi}{\alpha\delta}v_2 e^{v_2 b} + \frac{c}{\delta}v_2 + \frac{\lambda\phi}{\alpha\delta}v_1 e^{v_1 b} - \frac{c}{\delta}v_1}{e^{v_2 b}(v_2+\alpha) - e^{v_1 b}(v_1+\alpha)}$$

with a still unknown b. By (3.9), we want to find a b which maximises the function

$$V_0(x) := \frac{-\frac{\lambda\phi}{\alpha\delta}v_2 e^{v_2 x} + \frac{c}{\delta}v_2 + \frac{\lambda\phi}{\alpha\delta}v_1 e^{v_1 x} - \frac{c}{\delta}v_1}{e^{v_2 x}(v_2+\alpha) - e^{v_1 x}(v_1+\alpha)} \ .$$

The first derivative of $V_0(x)$ is

$$V_0'(x) = \frac{e^{(v_2+v_1)x}}{\delta} \cdot \frac{\lambda\phi(v_2-v_1) - cv_2(v_2+\alpha)e^{-v_1 x} + cv_1(v_1+\alpha)e^{-v_2 x}}{(e^{v_2 x}(v_2+\alpha) - e^{v_1 x}(v_1+\alpha))^2} \ .$$

To get $V_0'(x) = 0$, the equation

$$\frac{\lambda\phi}{c}(v_2-v_1) - v_2(v_2+\alpha)e^{-v_1 x} + v_1(v_1+\alpha)e^{-v_2 x} = 0$$

must be satisfied. We define the function

$$g(x) := v_2(v_2+\alpha)e^{-v_1 x} - v_1(v_1+\alpha)e^{-v_2 x}$$

and show that there is a point of intersection with the line $\frac{\lambda\phi}{c}(v_2-v_1)$. Using $v_1 < 0 < v_2$, we find that

$$g'(x) = \underbrace{v_1 v_2}_{<0} \underbrace{\Big[(v_1+\alpha)e^{-v_2 x} - (v_2+\alpha)e^{-v_1 x}\Big]}_{<e^{-v_1 x}(v_1-v_2)<0} > 0 \ ,$$

i.e., g is increasing and

$$g(0) = v_2(v_2+\alpha) - v_1(v_1+\alpha) = (v_2-v_1)(v_2+v_1+\alpha) = (v_2-v_1)\frac{\lambda+\delta}{c} \ .$$

Thus, g and $\frac{\lambda\phi}{c}(v_2-v_1)$ intersect if and only if

$$g(0) \le \frac{\lambda\phi}{c}(v_2-v_1) \quad \Leftrightarrow \quad \lambda+\delta \le \lambda\phi \quad \Leftrightarrow \quad \delta \le \lambda(\phi-1) \ .$$

If $\delta \le \lambda(\phi-1)$ then b is the point satisfying the relation

$$v_2(v_2+\alpha)e^{-v_1 b} - v_1(v_1+\alpha)e^{-v_2 b} = \frac{\lambda\phi(v_2-v_1)}{c}, \tag{3.14}$$

where $b = 0$ if $\delta = \lambda(\phi-1)$. If $\delta > \lambda(\phi-1)$ then $g(x) > \frac{\lambda\phi}{c}(v_2-v_1)$ and $V_0'(x) < 0$, i.e., $V_0(x)$ is strictly decreasing. The maximum on $[0,\infty)$ is, therefore, reached in $x = 0 = b$. Then $V(0) = V_0(b) = V_0(0) = (c\alpha - \lambda\phi)/(\alpha\delta)$ and $V(x) = x + V(0)$ for all $x \ge 0$.

Combining the results, we get that

$$V(x) = \begin{cases} x + (c\alpha - \lambda\phi)/(\alpha\delta), & \text{if } \delta \geq \lambda(\phi - 1), \\ C_1 \cdot e^{v_1 x} + C_2 \cdot e^{v_2 x}, & \text{if } x < b \text{ and } \delta < \lambda(\phi - 1), \\ x - b + C_1 \cdot e^{v_1 b} + C_2 \cdot e^{v_2 b}, & \text{if } x \geq b \text{ and } \delta < \lambda(\phi - 1), \end{cases} \tag{3.15}$$

with C_1, C_2 calculated in (3.13) and b calculated in (3.14).

Alternatively, we can determine $V(x)$ via Laplace transform. With $\hat{g}(s) = \alpha(s+\alpha)^{-1}$, we get then the Laplace transform of V:

$$\hat{v}(s) = \frac{V(0)(cs + \alpha c - \lambda) + \frac{\lambda\phi}{\alpha}}{c(s - v_1)(s - v_2)} \,.$$

Inverting $\hat{v}(s)$ yields

$$V(x) = \frac{V(0)(cv_1 + \alpha c - \lambda) + \frac{\lambda\phi}{\alpha}}{c(v_1 - v_2)} e^{v_1 x} + \frac{V(0)(cv_2 + \alpha c - \lambda) + \frac{\lambda\phi}{\alpha}}{c(v_2 - v_1)} e^{v_2 x} \,.$$

From $V'(b) = 1$, we get that

$$V(0) = \frac{-\frac{\lambda\phi}{\alpha\delta} v_2 e^{v_2 b} + \frac{c}{\delta} v_2 + \frac{\lambda\phi}{\alpha\delta} v_1 e^{v_1 b} - \frac{c}{\delta} v_1}{e^{v_2 b}(v_2 + \alpha) - e^{v_1 b}(v_1 + \alpha)} \,.$$

To determine the maximising b, we can use the procedure above.

We now calculate the example with parameters $c = 4, \alpha = 3, \delta = 0.06, \lambda = 2$ and $\phi = 2$ in the case $\delta \leq \lambda(\phi - 1)$. Figure 3.2 shows the function $V_0(b)$ depending on the barrier b. The optimal barrier b^* is the maximising point $b^* = 1.271$. Figure 3.3 illustrates the value function.

Gamma-distributed claim sizes

We choose the $\Gamma(2, 1)$-distribution for the claim sizes, i.e., $G(x) = 1-(x+1)e^{-x}$.

Let $R_2 < R_1 < 0 < \rho$ be the solutions to Lundberg's equation

$$cs^3 + (2c - (\lambda + \delta))s^2 + (c - 2(\lambda + \delta))s - \delta = 0 \,.$$

Then we have

$$\psi(0) = \frac{\lambda(\rho + 2)}{c(\rho + 1)^2} \,, \quad \sigma(0) = \frac{\lambda(2\rho + 3)}{c(\rho + 1)^2} \,, \quad \chi(0) = \frac{2\lambda}{c(\rho + 1)^3}$$

and

$$\psi(x) = K_1 e^{R_1 x} + K_2 e^{R_2 x} \,, \sigma(x) = \tilde{K}_1 e^{R_1 x} + \tilde{K}_2 e^{R_2 x} \,, \chi(x) = \hat{K}_1 e^{R_1 x} + \hat{K}_2 e^{R_2 x}$$

with

$$\begin{aligned}
K_1 &= \frac{-c\psi(0)(1+R_1)^2 + \lambda(R_1+2)}{c(\rho - R_1)(R_1 - R_2)}\,, \\
K_2 &= \frac{c\psi(0)(1+R_2)^2 - \lambda(R_2+2)}{c(\rho - R_2)(R_1 - R_2)}\,, \\
\tilde{K}_1 &= \frac{-c\sigma(0)(1+R_1)^2 + \lambda(2R_1+3)}{c(\rho - R_1)(R_1 - R_2)}\,, \\
\tilde{K}_2 &= \frac{c\sigma(0)(1+R_2)^2 - \lambda(2R_2+3)}{c(\rho - R_2)(R_1 - R_2)}\,,
\end{aligned}$$

and

$$\hat{K}_1 = \frac{1}{c(\rho+1)^2(\rho - R_1)(R_1 - R_2)} \times \\ \Big[-c\chi(0)\Big[\rho^2(R_1+1)^2 + R_1^2(2\rho+1) + 2(\rho + 2\rho R_1 + R_1)\Big] + \lambda(\rho + R_1 + 2)\Big],$$

$$\hat{K}_2 = \frac{1}{c(\rho+1)^2(\rho - R_2)(R_1 - R_2)} \times \\ \Big[c\chi(0)\Big[\rho^2(R_2+1)^2 + R_2^2(2\rho+1) + 2(\rho + 2\rho R_2 + R_2)\Big] - \lambda(\rho + R_2 + 2)\Big].$$

By maximising (3.11), we now can determine the optimal barrier b^*.

For a numerical example, let $\lambda = 4, c = 10, \delta = 0.1$ and $\phi = 1.1$. Then $b^* = 3.2357$. Figure 3.4 and 3.5 illustrate the dependence of the initial value on the barrier and the value function, respectively.

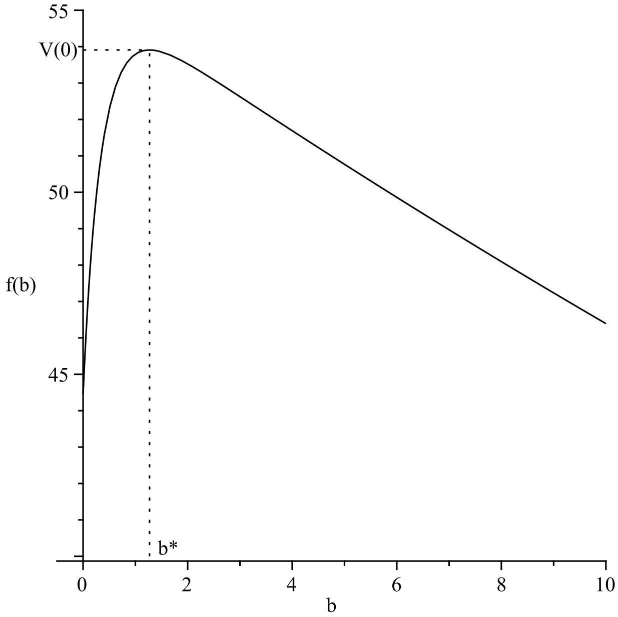

Figure 3.2: Exp(3)-claim sizes: initial value dependent on the barrier

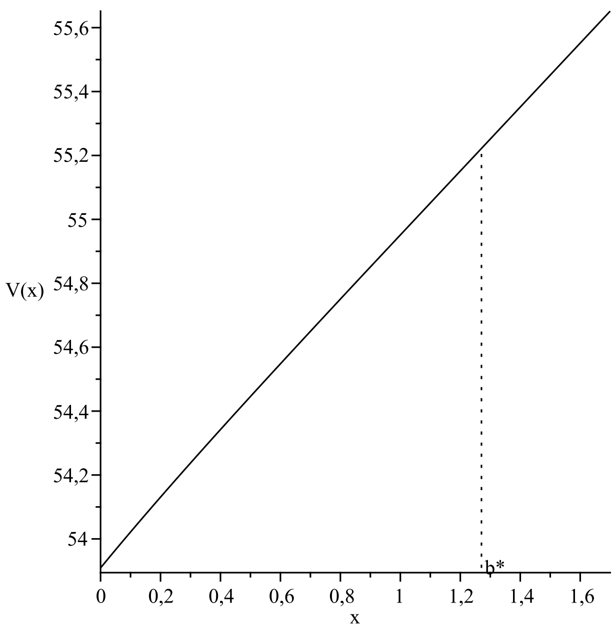

Figure 3.3: Exp(3)-claim sizes: value function

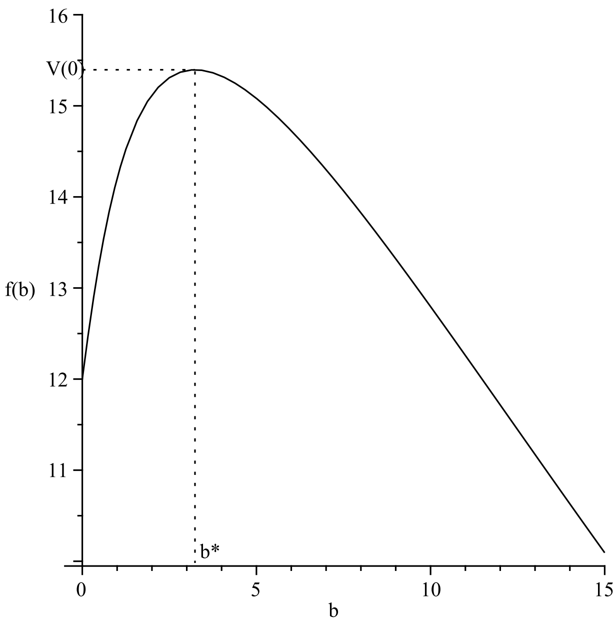

Figure 3.4: $\Gamma(2,1)$-claim sizes: initial value dependent on the barrier

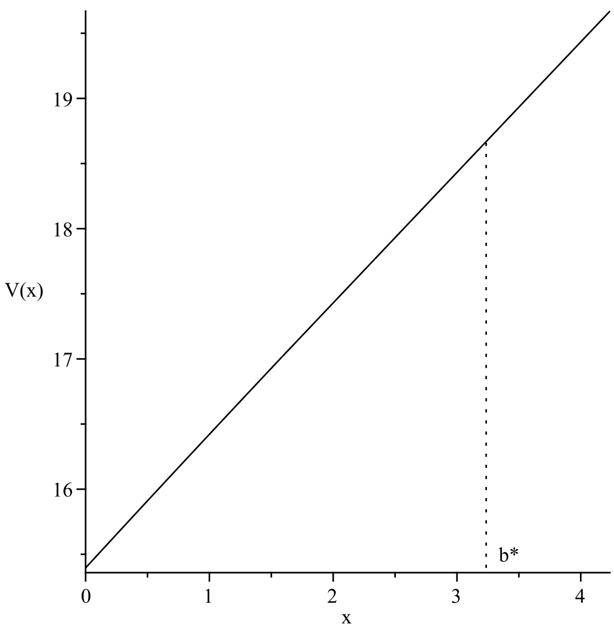

Figure 3.5: $\Gamma(2,1)$-claim sizes: value function

Chapter 4

Optimal Control of Dividends and Capital Injections with Administration Costs in a Classical Risk Model

4.1 Introduction

We assume the risk reserve process of an insurer to be a classical risk process

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i$$

on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Here $x \in \mathbb{R}$ is the initial capital, $c > 0$ is the constant premium rate, and the claim amounts are an iid sequence of strictly positive random variables $\{Y_i\}_{i\in\mathbb{N}}$ with distribution function $G(y)$. The claim number process $\{N_t\}_{t\geq 0}$ is assumed to be Poisson with intensity $\lambda > 0$ and independent of $\{Y_i\}_{i\in\mathbb{N}}$. We assume that $\mathbb{E}[Y_i] = \mu < \infty$ and, for simplicity, that $G(y)$ is continuous. We use the smallest right-continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $\{X_t\}_{t\geq 0}$ is adapted.

In the following the insurer is allowed to pay dividends. The accumulated dividends process $\{D_t\}_{t\geq 0}$ is an $\{\mathscr{F}_t\}$-adapted, non-decreasing, càdlàg process with $D_{0-} = 0$. Further, the shareholders have to inject capital in order to keep the surplus above zero. The accumulated capital injections are denoted by $\{Z_t\}_{t\geq 0}$, which also is an $\{\mathscr{F}_t\}$-adapted, non-decreasing, pure jump process with $Z_{0-} = 0$. The surplus process then becomes

$$X_t^{(D,Z)} = X_t - D_t + Z_t\;, \qquad X_{0-}^{(D,Z)} = x\;.$$

The capital injections $\{Z_t\}$ have to be chosen in such a way that $X_t^{(D,Z)} \geq 0$ for all t. Note that no positive safety loading $c > \lambda\mu$ needs to be assumed.

We assume that at any time where capital injections are made proportional costs with a factor $\phi \geq 1$ and a lump-sum penalty $L > 0$ have to be paid. As a consequence, the insurance company would possibly prefer to inject not only the minimal amount required but additional capital C for preventing future capital injections (which are costs, in fact). The value of a strategy $(D, Z) = \{(D_t, Z_t)\}_{t \geq 0}$ is defined as

$$V^{(D,Z)}(x) = \mathbb{E}_x\Bigl[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t - L\sum_{t\geq 0} e^{-\delta t}\mathbf{1}_{\{\Delta Z_t > 0\}}\Bigr] ,$$

where $\Delta Z_s = Z_s - Z_{s-}$ and $\delta > 0$ is a discount factor. A strategy (D, Z) is *admissible* if

$$\mathbb{P}_x[X_t^{(D,Z)} \geq 0 \text{ for all } t \geq 0] = 1 .$$

We denote by $\mathscr{S}_x$ the set of all admissible strategies for the initial capital x. We want to maximise $V^{(D,Z)}(x)$ over all admissible strategies and to identify the strategy (if it exists) which gives the maximal value. Thus, the *value function* of our problem is

$$V(x) = \sup_{(D,Z)\in\mathscr{S}_x} V^{(D,Z)}(x) .$$

Note that $\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t = \delta\int_{0-}^{\infty} e^{-\delta t} D_t\,\mathrm{d}t$. Since the value of the (not admissible) strategy $D_t = x + ct$ and $Z_t = 0$ is an upper bound for the value of any admissible strategy, we get that $V^{(D,Z)}(x) < x + c/\delta < \infty$ for any admissible strategy. Since $D_t = x + ct$ and $Z_t = S_t := x + ct - X_t$ is an admissible strategy, we obtain the lower bound $V(x) \geq x + (c - \lambda\mu\phi - \lambda L)/\delta$. This shows that $V(x)$ is finite.

If we had chosen $\phi < 1$ then we could make a capital injection of size K and pay it as dividend at the same time. The value would be $K(1 - \phi) - L$. This shows that the value function would be infinite. If $\delta = 0$ the value can only exceed $-\infty$ if the safety loading is positive. But then consider a barrier strategy at a high barrier b. Consider the time between the process reaches the barrier for the first time and reaches it again after leaving the barrier. If the barrier is high enough, it is unlikely that a capital injection has to be made and the value of the dividends is larger than the value of the costs. Since the barrier is reached infinitely often, the value of the strategy would become infinite. Since always a higher barrier would be better, there would not exist an optimal strategy.

We first argue that it cannot be optimal to make a capital injection unless the surplus is negative. Suppose $x \geq 0$. Let (D, Z) be a strategy allowing capital injections at any time. We assume without loss of generality that $Z_0 > 0$ and $D_0 = 0$. We construct now another strategy $(\tilde{D}, \tilde{Z})$ that pays no dividend and no capital injections until some stopping time S to be defined below. At time S the strategy $(\tilde{D}, \tilde{Z})$ is adjusted in such a way, that the surplus processes coincide from S on. I.e., $X_t^{(\tilde{D},\tilde{Z})} = X_t$ for $t < S$ and $X_t^{(\tilde{D},\tilde{Z})} = X_t^{(D,Z)}$ for $t \geq S$. Let now $\tau_0 = \inf\{t \geq 0 : X_t < 0\}$ be the first time where no control leads to a negative surplus, $\sigma_D = \inf\{t \geq 0 : D_t > Z_0\}$ be the first time where the initial injection is compensated by paying dividends, $\sigma_Z = \inf\{t \geq 0 : Z_t > Z_0\}$ the first time after zero where an injection is made, and $S = \min\{\tau_0, \sigma_D, \sigma_Z\}$; see Figure 4.1. We define

$$\tilde{D}_t = (D_t - (D_S \wedge Z_0))^+ \;, \qquad \tilde{Z}_t = (Z_t - D_S + \tilde{D}_S)\mathbf{1}_{\{t \geq S\}} \;.$$

Note that the strategy $(\tilde{D}, \tilde{Z})$ is admissible. The difference of the value between the two strategies is

$$\begin{aligned}
&L(1 - e^{-\delta S}\mathbf{1}_{\{\tilde{Z}_S > 0, Z_s = Z_0\}}) + \phi Z_0 + \phi(Z_S - Z_0 - \tilde{Z}_S)e^{-\delta S} \\
&\qquad - \int_0^S e^{-\delta t}\,\mathrm{d}D_t + e^{-\delta S}\tilde{D}_S \\
&\quad = L(1 - e^{-\delta S}\mathbf{1}_{\{\tilde{Z}_S > 0, Z_s = Z_0\}}) + \phi Z_0(1 - e^{-\delta S}) + \phi(D_S - \tilde{D}_S)e^{-\delta S} \\
&\qquad - \int_0^S e^{-\delta t}\,\mathrm{d}D_t + e^{-\delta S}\tilde{D}_S \\
&\quad \geq L(1 - e^{-\delta S}\mathbf{1}_{\{\tilde{Z}_S > 0, Z_s = Z_0\}}) + \phi Z_0(1 - e^{-\delta S}) - \int_0^S \delta e^{-\delta t} D_t\,\mathrm{d}t \\
&\quad > L(1 - e^{-\delta S}\mathbf{1}_{\{\tilde{Z}_S > 0, Z_s = Z_0\}}) + \phi Z_0(1 - e^{-\delta S}) - \int_0^S \delta e^{-\delta t} Z_0\,\mathrm{d}t \\
&\quad \geq L(1 - e^{-\delta S}\mathbf{1}_{\{\tilde{Z}_S > 0, Z_s = Z_0\}}) > 0 \;.
\end{aligned}$$

Thus, the strategy $(\tilde{D}, \tilde{Z})$ yields a larger value than (D, Z).

If the initial capital is negative, then capital injections at height $Z_0 = |x| + C$ are made; thus,

$$V(x) = V(C) - \phi(|x| + C) - L \qquad \text{for } x < 0 \;. \tag{4.1}$$

Let us now consider the decision at the time where a capital injection is necessary. Suppose the deficit is $z > 0$. The insurer decides to make a capital injection of size $z+C$. A strategy can be chosen, such that $V^{(D,Z)}(C) > V(C) - \varepsilon$. The future value of the strategy becomes then at least $V(C) - \varepsilon - \phi(z+C) - L$. The maximum that can be attained is $V(C) - \phi(z + C) - L$. Maximising over

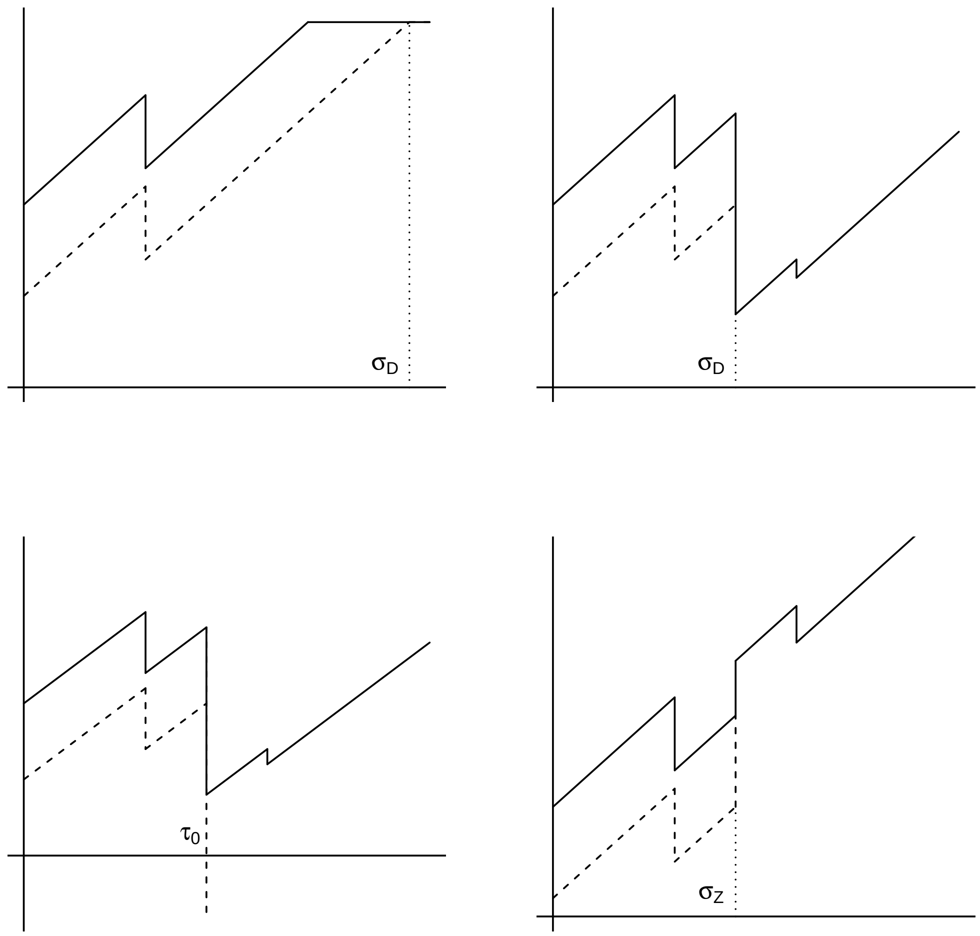

Figure 4.1: Construction of the strategy $(\tilde{D}, \tilde{Z})$ (dashed line) from the strategy (D, Z) (solid line)

C gives $V(-z) = \sup_{C \geq 0} V(C) - \phi(z + C) - L$. One therefore has to maximise $V(C) - \phi C$. Thus, the problem is independent of the deficit z.

We will see below that the function $V(x)$ is continuous and that there is a value x_1, such that $V(x) = V(x_1) + x - x_1$ for $x > x_1$. That is, capital above x_1 is paid as dividend. This implies that a choice $C > x_1$ does not make sense. Therefore, there is $C_0 \in [0, x_1]$, such that $V(C_0) - \phi C_0 = \sup_{C \geq 0} V(C) - \phi C$. It follows readily that any strategy where capital injections are made such that the process is at a level C with $V(C) - \phi C < V(C_0) - \phi C_0$ yields a lower value. In the following we will assume that we already have fixed the value $C = C_0$. We therefore only consider strategies where the surplus is at the optimal level C after a capital injection. The reader, however, should be aware that in the case where C is not unique the optimal strategy is not unique.

The capital injections process can be described in the following way. For $x \geq 0$, let $\{D_t^0\}$ be a dividend strategy for the process $\{X_t\}$ until the surplus process falls below zero for the first time at $\tau_1 := \inf\{t \geq 0 : X_t - D_t^0 < 0\}$. Then $Z_t^0 = 0$ on $[0, \tau_1)$. Now let

$$\begin{aligned}
\Delta Z_{\tau_1}^0 &= |X_{\tau_1} - D_{\tau_1}^0| + C \;, \\
Z_t^0 &= \Delta Z_{\tau_1}^0 \cdot \mathbf{I}_{t \geq \tau_1} \;, \\
X_t^1 &= X_t - D_t^0 + Z_t^0 \;.
\end{aligned}$$

Then $X_t^{(D,Z)} = X_t^1$ for $t \in [0, \tau_1]$. Suppose now, we have constructed the process $\{X_t^n\}$ on $[0, \tau_n]$. Let $\{D_t^n\}$ be a dividend strategy for $\{X_t^n\}$ until the surplus process falls below zero at $\tau_{n+1} := \inf\{t > \tau_n : X_t^n - D_t^n < 0\}$. Define

$$\begin{aligned}
\Delta Z_{\tau_{n+1}}^n &= |X_{\tau_{n+1}}^n - D_{\tau_{n+1}}^n| + C \;, \\
Z_t^n &= \sum_{k=1}^{n+1} \Delta Z_{\tau_k}^{k-1} \cdot \mathbf{I}_{t \geq \tau_k} \;, \\
X_t^{n+1} &= X_t^n - D_t^n + Z_t^n \;.
\end{aligned}$$

We have $X_t^{(D,Z)} = X_t^{n+1}$ for $t \in [0, \tau_{n+1}], n \geq 0$. By construction, the capital injections at time t depend on the dividend strategy at t and the chosen value C. Therefore, we will in the following use the short notations $\{Z_t\}, \{X_t^D\}$ and $V^D(x)$ for the capital injection process $\{Z_t^D\}$, the surplus process and the value connected to a strategy $\{(D_t, Z_t^D)\}$.

In Chapter 3, the value function with only proportional costs associated with the capital injections was concave. Including fixed costs destroys the concavity property. This makes the proofs below more complicated. The next lemma, which is analogous to Lemma 3.1.2, gives the lower bound for any admissible strategy.

Lemma 4.1.1
The value of expected capital injections is bounded by $\lambda(\phi(\mu + C) + L)/\delta$.

Proof: The worst that may happen is that one has to inject capital for all the claims. Using that the time of the k-th claim T_k is Gamma $\Gamma(\lambda, k)$ distributed, we find that

$$\begin{aligned}
&\phi\mathbb{E}\Big[\sum_{k=1}^{\infty}(Y_k + C)e^{-\delta T_k} + L\sum_{k=1}^{\infty} e^{-\delta T_k}\Big] \\
&\quad = (\phi(\mu + C) + L)\mathbb{E}\Big[\sum_{k=1}^{\infty} e^{-\delta T_k}\Big] = (\phi(\mu + C) + L)\sum_{k=1}^{\infty}\Big(\frac{\lambda}{\lambda + \delta}\Big)^k \\
&\quad = (\phi(\mu + C) + L)\frac{\lambda}{\lambda + \delta}\,\frac{1}{1 - \frac{\lambda}{\lambda+\delta}} = \frac{\lambda(\phi(\mu + C) + L)}{\delta}\;.
\end{aligned}$$

□

It follows that the value of any admissible strategy is bounded from below by $-\lambda(\phi(\mu + C) + L)/\delta$.

4.2 Strategies With Restricted Densities

In this section, we only consider absolutely continuous dividend strategies with an adapted non-negative density process $\{U_t\}_{t\ge0}$ such that

- $D_t = \int_0^t U_s \,\mathrm{d}s$,
- $0 \le U_t \le u_0 < \infty$,

and denote the strategies $\{(U_t, Z_t^U)\}$ by $\{U_t\}$. Then

$$X_t^U = X_t - \int_0^t U_s \,\mathrm{d}s + Z_t\;.$$

The value of such a strategy is

$$V^U(x) = \mathbb{E}_x\Big[\int_0^{\infty} e^{-\delta t}U_t \,\mathrm{d}t - \phi\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t - L\sum_{t\ge0} e^{-\delta t}\mathbf{1}_{\{\Delta Z_t>0\}}\Big]\;.$$

Let us denote the set of the admissible restricted strategies by $\mathscr{S}_x^r$. Then the value function is $V(x) = \sup_{U\in\mathscr{S}_x^r} V^U(x)$. Because of (4.1) we assume that $x \ge 0$. Then $Z_0 = 0$.

4.2.1 The Value Function and the HJB Equation

We first prove some properties of the value function.

Lemma 4.2.1
$V(x)$ is bounded by u_0/δ, increasing and Lipschitz continuous. Moreover, $\lim_{x\to\infty} V(x) = u_0/\delta$.

Proof: That $V(x)$ is increasing and that $V(x) \le \int_0^\infty u_0 e^{-\delta t}\,\mathrm{d}t = u_0/\delta$ is clear. Consider the strategy $U_t = u_0$. The first time the surplus falls below zero defined by $\tau_x^U = \inf\{t : x + (c - u_0)t - \sum_{i=1}^{N_t} Y_i < 0\}$ converges to infinity as $x \to \infty$. By bounded convergence, $\mathbb{E}[e^{-\delta\tau_x^U}]$ converges to zero. By Lemma 4.1.1, we have that

$$\begin{aligned} V(x) &\ge V^U(x) \ge \mathbb{E}\Big[\int_0^{\tau_x^U} u_0 e^{-\delta t}\,\mathrm{d}t - \phi\int_{0-}^\infty e^{-\delta t}\,\mathrm{d}Z_t - L\sum_{t\ge 0} e^{-\delta t}\mathbf{1}_{\{\Delta Z_t>0\}}\Big] \\ &\ge \mathbb{E}\Big[\int_0^{\tau_x^U} u_0 e^{-\delta t}\,\mathrm{d}t\Big] - \mathbb{E}\Big[e^{-\delta\tau_x^U}\Big](\phi(\mu+C)+L)\frac{\lambda}{\delta} \to \frac{u_0}{\delta}\,. \end{aligned}$$

Let $h > 0$ be small. We choose a strategy $\tilde U \in \mathscr{S}^r_{x+ch}$ for initial capital $x + ch$ and define the strategy

$$\begin{aligned} U_t &= 0\cdot\mathbf{1}_{\{T_1<h\}} + \Big(0\cdot\mathbf{1}_{\{t<h\}} + \tilde U_{t-h}\mathbf{1}_{\{t\ge h\}} + \Big)\mathbf{1}_{\{T_1\ge h\}}, \\ Z_t &= Z_t^0\cdot\mathbf{1}_{\{T_1<h\}} + \Big(0\cdot\mathbf{1}_{\{t<h\}} + \tilde Z_{t-h}\mathbf{1}_{\{t\ge h\}} + \Big)\mathbf{1}_{\{T_1\ge h\}}, \end{aligned}$$

where $\{Z_t^0\}$ denotes the capital injections if no dividend is paid. By Lemma 4.1.1, the value connected to Z^0 is bounded from below. The first claim happens with density $\lambda e^{-\lambda t}$ and T_1 is larger than h with probability $e^{-\lambda h}$. By conditioning on $\mathscr{F}_{h\wedge T_1}$, it follows that

$$\begin{aligned} V(x) &\ge V^U(x) \\ &\ge \mathbb{E}_x\Big[\Big(-\phi\int_0^{T_1} e^{-\delta t}\,\mathrm{d}Z_t^0 - L\sum_{0\le t\le T_1} e^{-\delta t}\mathbf{1}_{\{\Delta Z_t^0>0\}}\Big)\mathbf{1}_{\{T_1<h\}}\Big] \\ &\quad + \mathbb{E}_x\Big[\mathbf{1}_{\{T_1\ge h\}} e^{-\delta h} V^{\tilde U}(X_h^U)\Big] \\ &\ge e^{-(\lambda+\delta)h} V^{\tilde U}(x+ch) - \frac{(1-e^{-\lambda h})(\phi\mu + \phi C + L)\lambda}{\delta}\,, \end{aligned}$$

and so

$$\begin{aligned} V(x) &\ge \sup_{\tilde U\in\mathscr{S}^r_{x+ch}} e^{-(\lambda+\delta)h} V^{\tilde U}(x+ch) - \frac{(1-e^{-\lambda h})(\phi\mu+\phi C) + L)\lambda}{\delta} \\ &= e^{-(\lambda+\delta)h} V(x+ch) - \frac{(1-e^{-\lambda h})(\phi\mu+\phi C) + L)\lambda}{\delta}\,. \end{aligned}$$

The Lipschitz-continuity follows now by the boundedness of V

$$\begin{aligned} 0 &\le V(x+ch)-V(x) \\ &\le V(x+ch)(1-e^{-(\lambda+\delta)h})+\frac{(\phi(\mu+C)+L)\lambda}{\delta}(1-e^{-\lambda h}) \\ &\le V(x+ch)(\lambda+\delta)h+\frac{(\phi(\mu+C)+L)\lambda^2}{\delta}h \\ &\le \frac{u_0}{\delta}(\lambda+\delta)h+\frac{(\phi(\mu+C)+L)\lambda^2}{\delta}h\,. \end{aligned}$$

□

V is (locally) Lipschitz continuous and, therefore, absolutely continuous on $\mathbb{R}_{\ge 0}$. Furthermore, by Rademacher's Theorem (see for example [25, section 5.8.3]), V is Lebesgue a.e. differentiable with (locally) bounded derivatives. In particular, V' is the density of V and V is differentiable at all points where V' is continuous. We denote by $\mathcal{D} \subseteq \mathbb{R}_{\ge 0}$ the set of points x where $V(x)$ is differentiable. Then $\bar{\mathcal{D}} = \mathbb{R}_{\ge 0}$ and $\mathcal{D}^c = \mathbb{R}_{\ge 0}\backslash\mathcal{D}$ is a set with Lebesgue measure zero. In the next section, we show that at points of non-differentiability V has derivatives from the right and from the left.

Theorem 4.2.2
The function $V(x)$ is differentiable a.e. on $(0,\infty)$ and fulfils the Hamilton-Jacobi-Bellman equation (4.2). At points where $V(x)$ is not differentiable, the derivatives from the left and from the right exist and fulfil Equation

$$\sup_{0\le u\le u_0}\left\{(c-u)V'(x)+u-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)\,dG(y)\right\}=0\,, \quad (4.2)$$

respectively, with $V'(x-) < V'(x+)$.

Proof: Let $h>0$ and fix $u\in[0,u_0]$. If $x=0$ we suppose $u\le c$, if $x>0$ we let h be small enough such that $x+(c-u)h\ge 0$, i.e., the reserve process does not fall below zero because of the dividend payments. Let $K>0$ be the Lipschitz-constant. Choose $\varepsilon>0$ and $n\in\mathbb{N}$ such that $K(x+(c-u)h)/n<\varepsilon/2$ and let $x_k=k(x+(c-u)h)/n$ for $0\le k\le n$. For initial capital x' where $x_k\le x'<x_{k+1}$, we choose a strategy $\{U_t^k\}$ with $V^{U^k}(x_k)>V(x_k)-\varepsilon/2$. Then, by the Lipschitz continuity of $V(x)$, it holds that

$$V^{U^k}(x')\ge V^{U^k}(x_k)>V(x_k)-\varepsilon/2>V(x')-K(x'-x_k)-\varepsilon/2>V(x')-\varepsilon\,.$$

Thus, for all $x'\in[0,x+(c-u)h]$ we can find a measurable strategy $\tilde{U}$ such that $V^{\tilde{U}}(x')>V(x')-\varepsilon$.

Consider now the strategy

$$U_t = \begin{cases} u, & 0 \le t < h \wedge T_1, \\ \tilde{U}_{t-h}, & t \ge h \wedge T_1, \end{cases} \qquad Z_t = \begin{cases} 0, & 0 \le t < h \wedge T_1, \\ \tilde{Z}_{t-h}, & t \ge h \wedge T_1 \;. \end{cases}$$

Conditioning on $\mathscr{F}_{h \wedge T_1}$ yields

$$\begin{aligned}
V(x) \ge V^U(x) &= \mathbb{E}\left[\int_0^{h\wedge T_1 -} e^{-\delta t} u \,\mathrm{d}t + e^{-\delta(h\wedge T_1)} V^{\tilde{U}}(X^U_{h\wedge T_1})\right] \\
&> \mathbb{E}\Big[\Big(\int_0^h e^{-\delta t} u \,\mathrm{d}t + e^{-\delta h} V(X_h^U)\Big) \mathbf{1}_{T_1 > h} \\
&\qquad + \Big(\int_0^{T_1 -} e^{-\delta t} u \,\mathrm{d}t + e^{-\delta T_1} V(X_{T_1}^U)\Big) \mathbf{1}_{T_1 \le h}\Big] - \varepsilon \\
&= e^{-\lambda h} \int_0^h u e^{-\delta t} \,\mathrm{d}t + e^{-(\delta+\lambda)h} V(x + (c-u)h) \\
&+ \int_0^h \lambda e^{-\lambda t} \Big\{ \int_0^t u e^{-\delta s} \,\mathrm{d}s + e^{-\delta t} \int_0^{x+(c-u)t} V(x + (c-u)t - y) \,\mathrm{d}G(y) \\
&+ e^{-\delta t} \int_{x+(c-u)t}^\infty \Big(V(C) - \phi(y - (x + (c-u)t + C) - L \Big) \,\mathrm{d}G(y) \Big\} \,\mathrm{d}t - \varepsilon \\
&= e^{-\lambda h} \int_0^h u e^{-\delta t} \,\mathrm{d}t + e^{-(\delta+\lambda)h} V(x + (c-u)h) \\
&+ \int_0^h \lambda e^{-\lambda t} \Big\{ \int_0^t u e^{-\delta s} \,\mathrm{d}s + e^{-\delta t} \int_0^\infty V(x + (c-u)t - y) \,\mathrm{d}G(y) \Big\} \,\mathrm{d}t - \varepsilon \;.
\end{aligned}$$

The constant ε is arbitrary, thus, we let tend it to zero. If we rearrange the terms and divide them by h, then we get

$$\begin{aligned}
0 \;\ge\; & \frac{V(x + (c-u)h) - V(x)}{h} - \frac{1 - e^{-(\lambda+\delta)h}}{h} V(x + (c-u)h) \\
& + e^{-\lambda h} \frac{1}{h} \int_0^h u e^{-\delta t} \,\mathrm{d}t + \frac{1}{h} \int_0^h \lambda e^{-\lambda t} \Big[\int_0^t u e^{-\delta s} \,\mathrm{d}s \\
& \qquad + e^{-\delta t} \int_0^\infty V(x + (c-u)t - y) \,\mathrm{d}G(y) \Big] \,\mathrm{d}t \;. \qquad (4.3)
\end{aligned}$$

Now we choose a strategy $W(h) = \{W_t(h)\}$ with $V^{W(h)}(x) \ge V(x) - h^2$. There exists w_h such that $\mathbb{E}[\int_0^{h\wedge T_1} (W_s(h) - w_h) e^{-\delta s} \,\mathrm{d}s] = 0$. Let w_t' denote $W_t(h)$ conditioned on $T_1 > t$ and $a(t) = \int_0^t (c - w_s') \,\mathrm{d}s$. In the same way as above, we can derive

$$\begin{aligned} 0 \;\leq\; & h + \frac{V(x+a(h)) - V(x)}{h} - \frac{1-e^{-(\lambda+\delta)h}}{h} V(x+a(h)) \\ & + e^{-\lambda h} \frac{1}{h} \int_0^h w_h e^{-\delta t}\,\mathrm{d}t + \frac{1}{h} \int_0^h \lambda e^{-\lambda t} \Big[\int_0^t w_h e^{-\delta s}\,\mathrm{d}s \\ & \qquad + e^{-\delta t} \int_0^\infty V(x+a(t)-y)\,\mathrm{d}G(y) \Big]\,\mathrm{d}t\;. \end{aligned}$$

All terms with exception of the second and the forth one converge. We choose a sequence $h_n \to 0$ such that

$$\lim_{n\to\infty} \frac{V(x+a(h_n)) - V(x)}{h_n} = \limsup_{h\downarrow 0} \frac{V(x+a(h)) - V(x)}{h}\;.$$

This limit is finite by the local Lipschitz continuity. Without loss of generality, we can assume that w_{h_n} converges to some value $\tilde{u}$. Then $a(h_n)/h_n$ converges to $c - \lim w'_{h_n} = c - \lim w_{h_n} = c - \tilde{u}$ and

$$\begin{aligned} \lim_{n\to\infty} \frac{V(x+a(h_n)) - V(x)}{h_n} &= \lim_{n\to\infty} \frac{V(x+a(h_n)) - V(x)}{a(h_n)} \frac{a(h_n)}{h_n} \\ &= (c-\tilde{u})V'(x)\;. \end{aligned}$$

The sequence $\{w_{h_n}\}$ fulfils (4.3), and so equality holds for $u = \tilde{u}$.

We can repeat the above procedure for any subsequence w_{j_n} that converges to $\hat{u}$, say. Then for the limit

$$\lim_{n\to\infty} \frac{V(x+a(j_n)) - V(x)}{j_n}$$

we get

$$\limsup_{h\downarrow 0} \frac{V(x+a(h)) - V(x)}{h} + \hat{u} = (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y)$$

such that $\tilde{u} = \hat{u}$ and the limit

$$\limsup_{h\downarrow 0} \frac{V(x+a(h)) - V(x)}{h} = (c-\tilde{u}) \lim_{h\downarrow 0} \frac{V(x+(c-\tilde{u})h) - V(x)}{(c-\tilde{u})h}$$

is unique. If now $x \in \mathcal{D}$, the above limit is $(c-\tilde{u})V'(x)$. Otherwise we have shown the differentiability at x from the right if $c > \tilde{u}$ and the differentiability at x from the left if $c < \tilde{u}$. We do not distinguish the notation first and write for both derivatives the Hamilton-Jacobi-Bellman equation

$$\sup_{0\leq u\leq u_0} \Big\{ (c-u)V'(x) + u - (\lambda+\delta)V(x) + \lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y) \Big\} = 0\,,$$

where we have

$$
\begin{aligned}
&\int_0^\infty V(x-y)\,\mathrm{d}G(y) \\
&= \int_0^x V(x-y)\,\mathrm{d}G(y) + \int_x^\infty (V(C)-\phi(y-x+C)-L)\,\mathrm{d}G(y) \\
&= \int_0^x V(x-y)\,\mathrm{d}G(y) \qquad (4.4)\\
&\qquad +(V(C)-\phi C-L)(1-G(x)) - \phi\int_x^\infty (1-G(y))\,\mathrm{d}y\ ,
\end{aligned}
$$

because of the property (4.1).

Equation (4.2) is linear in u, thus, the argument $\tilde{u}=u(x)$ maximising the left-hand side of (4.2) is

$$
u(x) = \begin{cases} 0, & \text{if } V'(x)>1,\\ \in[0,u_0], & \text{if } V'(x)=1,\\ u_0, & \text{if } V'(x)<1. \end{cases}
$$

Consider now the function

$$
H(x) := (\lambda+\delta)V(x) - \lambda\int_0^\infty V(x-y)\,\mathrm{d}G(y). \tag{4.5}
$$

Since $V(x-y)\le V(x)$ for $y\ge 0$, it follows by the bounded convergence theorem and continuity of $V(x)$ that $H(x)$ is a continuous function on $\mathbb{R}$. The HJB equation (4.2) reads

$$
(c-\tilde{u})V'(x)+\tilde{u}-H(x)=0. \tag{4.6}
$$

for any $x\in\mathcal{D}$ and $\tilde{u}=u(x)$.

Let $u_0<c$. Then we have shown the differentiability from the right with

$$
\begin{array}{lclcl}
V'(x+)>1 & \Leftrightarrow & \tilde{u}=0 & \Leftrightarrow & H(x)>c,\\
V'(x+)<1 & \Leftrightarrow & \tilde{u}=u_0 & \Leftrightarrow & H(x)<c,\\
V'(x+)=1 & \Leftrightarrow & \tilde{u}\text{ arbitr.} & \Leftrightarrow & H(x)=c.
\end{array}
$$

Suppose, $H(x)>c$. Then, by continuity, there exist $\varepsilon>0$ and an interval $\mathcal{U}_\varepsilon(x)$ such that $H(z)>c$ for any $z\in\mathcal{U}_\varepsilon(x)$. Let $\{x_n\}$ be a sequence in $\mathcal{U}_\varepsilon(x)\cap\mathcal{D}$ tending to x from left. For any x_n (4.6) holds. Denote $\tilde{u}_n=u(x_n)$. Then, $H(x_n)>c$ and from (4.6) $\tilde{u}_n=0$ follows. Therefore, $u=\lim_{n\to\infty}\tilde{u}_n=0$ and we get that V is differentiable from the left with

$$
V'(x-) = \lim_{n\to\infty}\frac{H(x_n)-\tilde{u}_n}{c-\tilde{u}_n} = \frac{H(x)}{c} = V'(x+) > 1\ .
$$

If $H(x) < c$, in an analogous way we get $H(x_n) < c, \tilde{u}_n = u_0$ and therefore $u = u_0$ such that $V'(x-) = V'(x+) < 1$.

If $H(x) = c$, differentiability follows because we can choose u arbitrarily.

Thus, for $u_0 < c$, we have proved that V is continuously differentiable and fulfils (4.2). We denote this solution by $V_{u_0}(x)$.

We consider now the case $u_0 = c$. We can follow from (4.6) that

$$\begin{array}{rcl} V'(x+) > 1 \qquad \Leftrightarrow \qquad \tilde{u} = 0 & \Leftrightarrow & H(x) > c, \\ ((V'(x-) < 1 \Leftrightarrow \tilde{u} = u_0) \text{ or } (V'(x) = 1 \Leftrightarrow \tilde{u} \text{ arbitr.})) & \Leftrightarrow & H(x) = c. \end{array}$$

If $H(x) > c$, then, by similar arguments as above, we derive that V is differentiable at x with $V'(x) > 1$ and HJB equation (4.2) is fulfilled with $u = 0$.

Let now $H(x) = c$. Then, $H(z) \geq c$ for any $z \in \mathcal{U}_\varepsilon(x)$. Suppose, $V'(x-) < 1$. For a sequence $x_n \downarrow x$ with $H(x_n) > c$ we get that $\tilde{u}_n = 0$ and therefore V is differentiable from the right with $V'(x+) = 1$. In this case (4.2) is again fulfilled. If there is a sequence with $H(x_n) = c$, then we get differentiability at x.

The last case to consider is $u_0 > c$. Then

$$\begin{array}{rcl} V'(x) = 1 \qquad \Leftrightarrow \qquad \tilde{u} \text{ arbitr.} & \Leftrightarrow & H(x) = c, \\ ((V'(x-) < 1 \Leftrightarrow \tilde{u} = u_0) \text{ or } (V'(x+) > 1 \Leftrightarrow \tilde{u} = 0)) & \Leftrightarrow & H(x) > c. \end{array}$$

If $H(x) = c$, then differentiability follows similarly to above.

Let $H(x) > c$. Suppose that $V'(x+) > 1$ and $\tilde{u} = 0$. Let $\{x_n\}$ be a sequence in $\mathcal{U}_\varepsilon(x) \cap \mathcal{D}$ with $x_n \uparrow x$. If $u = \lim_{n\to\infty} \tilde{u}_n = 0$, then we get differentiability with $V'(x) > 1$. If $u = u_0$, then V is differentiable from the left with $V'(x-) < V'(x+)$ and both derivatives solve (4.2).

Suppose that $V'(x-) < 1$ and $\tilde{u} = u_0$. For a sequence $x_n \downarrow x$ we again have to choose either $u = 0$ or $u = u_0$. The choice $u = 0$ shows that V is differentiable from the right with $V'(x+) > 1$ and both derivatives solve (4.2). If $u = u_0$, then differentiability follows with $V'(x) < 1$. □

4.2.2 The Optimal Strategy and the Characterisation of the Solution

Equation (4.2) is linear in u, thus, the argument $\tilde{u} = u(x)$ maximising the left-hand side of (4.2) is

$$u(x) = \begin{cases} 0, & \text{if } V'(x) > 1, \\ \min\{c, u_0\}, & \text{if } V'(x) = 1, \\ u_0, & \text{if } V'(x) < 1, \end{cases} \tag{4.7}$$

where we let $V'(x)$ be the derivative from the left if $x \notin \mathcal{D}$. Here we used the fact that, if $u_0 < c$, any value u solves the equation (4.2). If $u_0 \geq c$, then $V'(x) = 1$ implies that $(\lambda+\delta)V(x) + \lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y) = c$ and

$$\begin{aligned} V(x) &= \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^\infty V(x-y)\,\mathrm{d}G(y) \\ &= \int_0^\infty \lambda e^{-\lambda t}\bigg\{\int_0^t ce^{-\delta s}\,\mathrm{d}s + e^{-\delta t}\int_0^x V(x-y)\,\mathrm{d}G(y) \\ &\quad + e^{-\delta t}\int_x^\infty \Big[V(C) - \phi(y-x+C) - L\Big]\,\mathrm{d}G(y)\bigg\}\,\mathrm{d}t\,, \end{aligned}$$

i.e., $V(x)$ is the value of a barrier-strategy where the incoming premium is paid as dividend until the first claim occurs. After that, the optimal strategy (if it exists) is followed. The existence of the optimal strategy has still to be shown.

We consider now the value at $x = 0$. From (4.2) and (4.4) we get

$$0 = (c-\tilde{u})V'(0) + \tilde{u} - (\lambda+\delta)V(0) + \lambda(V(C) - \phi(\mu+C) - L)$$

where $V'(0)$ is the derivative from the right. If $V'(0) < 1$, then $\tilde{u} = u_0$, i.e. the "optimal" strategy is to pay dividends at the maximal rate. In the case $u_0 > c$ this means that capital injections are needed to pay dividends from. This cannot be optimal because of the early penalty. Thus we have that $V'(0) \geq 1$ for $u_0 > c$.

Before proving the next lemma, we make the following observations.

- $\{Z_t\}$ only increases at the claim times. Therefore, it holds in an interval (T_{i-1}, T_i) between two claims that $\mathrm{d}X_t^U = (c - U_t)\,\mathrm{d}t$.
- $X_{T_i}^U = X_{T_i-}^U - Y_i + \Delta Z_{T_i}$.
- $\mathbf{1}_{\{\Delta Z_{T_i}>0\}} = \mathbf{1}_{\{Y_i > X_{T_i-}\}}$.
- If $X_{T_i-}^U - Y_i < 0$, then the shareholders pay as much that $X_{T_i}^U = X_{T_i-}^U - Y_i + \Delta Z_{T_i} = C$; i.e., $\Delta Z_{T_i} = (C + Y_i - X_{T_i-}^U)\mathbf{1}_{Y_i > X_{T_i-}}$. In this case, the value function fulfils

  $$V(X_{T_i}^U)\ (= V(C)) = V(X_{T_i-}^U - Y_i) + (\phi\Delta Z_{T_i} + L)\mathbf{1}_{\{\Delta Z_{T_i}>0\}}$$

  because of the property (4.1). Thus, it suffices to consider only solutions f to the HJB equation satisfying property (4.1).

Lemma 4.2.3
Let $f(x)$ be an increasing, bounded, and positive solution to the HJB equation (4.2) with property (4.1). Then for any admissible strategy U, the process

$$\Big\{f(X_t^U)e^{-\delta t} - f(x) - \sum_{0\le s\le t}[f(X_s^U) - f(X_s^U - \Delta Z_s)]e^{-\delta s} - \int_0^t \Big[(c-U_s)f'(X_s^U) - (\lambda+\delta)f(X_s^U) + \lambda\int_0^\infty f(X_s^U - y)\, dG(y)\Big]e^{-\delta s}\, ds\Big\}$$

is an $\{\mathscr{F}_t\}-$martingale.

Proof: We have the decomposition

$$\begin{aligned}
&f(X_t^U)e^{-\delta t} - f(X_0^U)\\
&\quad= \sum_{i=1}^{N_t}\Big[f(X_{T_i}^U - \Delta Z_{T_i})e^{-\delta T_i} - f(X_{T_{i-1}}^U)e^{-\delta T_{i-1}}\Big]\\
&\qquad+ \sum_{0\le s\le t}[f(X_s^U) - f(X_s^U - \Delta Z_s)]e^{-\delta s} + f(X_t^U)e^{-\delta t} - f(X_{T_{N_t}}^U)e^{-\delta T_{N_t}}\\
&\quad= \sum_{0\le s\le t}[f(X_s^U) - f(X_s^U - \Delta Z_s)]e^{-\delta s} + \sum_{i=1}^{N_t}[f(X_{T_i}^U - \Delta Z_{T_i}) - f(X_{T_i-}^U)]e^{-\delta T_i}\\
&\qquad+\sum_{i=1}^{N_t}[f(X_{T_i-}^U)e^{-\delta T_i-} - f(X_{T_{i-1}}^U)e^{-\delta T_{i-1}}] + f(X_t^U)e^{-\delta t} - f(X_{T_{N_t}}^U)e^{-\delta T_{N_t}}\\
&\quad= \sum_{0\le s\le t}[f(X_s^U) - f(X_s^U - \Delta Z_s)]e^{-\delta s} + \sum_{i=1}^{N_t}[f(X_{T_i-}^U - Y_i) - f(X_{T_i-}^U)]e^{-\delta T_i}\\
&\qquad+\sum_{i=1}^{N_t}[f(X_{T_i-}^U)e^{-\delta T_i-} - f(X_{T_{i-1}}^U)e^{-\delta T_{i-1}}] + f(X_t^U)e^{-\delta t} - f(X_{T_{N_t}}^U)e^{-\delta T_{N_t}}\ .
\end{aligned}$$

By Theorem B.2.4, we have that the process

$$\Big\{\sum_{i=1}^{N_t}\Big[f(X_{T_i-}^U - Y_i) - f(X_{T_i-}^U)\Big]e^{-\delta T_i} - \lambda\int_0^t e^{-\delta s}\int_0^\infty \Big(f(X_s^U - y)\,\mathrm{d}G(y) - f(X_s^U)\Big)\,\mathrm{d}s\Big\}$$

is a martingale. Now noting that

$$f(X_{T_i-}^U)e^{-\delta T_i-} - f(X_{T_{i-1}}^U)e^{-\delta T_{i-1}} = \int_{T_{i-1}}^{T_i-}\Big((c-U_s)f'(X_s^U) - \delta f(X_s^U)\Big)e^{-\delta s}\,\mathrm{d}s$$

and using that T_i- and T_{i-1} can be replaced by $T_i \wedge t$ and $T_{i-1} \wedge t$, respectively, we get that the process

$$\Big\{f(X_t^U)e^{-\delta t} - f(x) - \sum_{0 \le s \le t} [f(X_s^U) - f(X_s^U - \Delta Z_s)]e^{-\delta s} - \int_0^t \Big[(c - U_s)f'(X_s^U) + \lambda \int_0^\infty f(X_s^U - y)\,\mathrm{d}G(y) - (\lambda + \delta)f(X_s^U)\Big]e^{-\delta s}\,\mathrm{d}s\Big\}$$

is an $\{\mathscr{F}_t\}-$martingale with expected value 0. $\square$

Now we show that the value function is the unique increasing, bounded solution to (4.2) and the strategy (4.7) is optimal.

Theorem 4.2.4
Let $f(x)$ be an increasing, bounded, and positive solution to (4.2) *with property* (4.1) *and $C \ge 0$ chosen such that $f(C) - \phi C = \sup_{x \ge 0} f(x) - \phi x$. Then $\lim_{x\to\infty} f(x) = u_0/\delta$. If $u_0 \le c$ or $f'(0) \ge 1$, then $f(x) = V(x)$, and an optimal strategy is given by* (4.7).

Proof: Since f is bounded, f must converge to a $f(\infty) < \infty$. We first note that $C \le f(C) - f(0) \le f(\infty) - f(0)$. There exists a sequence $x_n \to \infty$ such that $f'(x_n) \to 0$. Let $u_n = u(x_n)$. By Definition (4.7), we can assume that $u_n = u_0$. Letting $n \to \infty$ in (4.2), yields that

$$\begin{aligned} 0 &= (c - u_0)f'(x_n) + \lambda\Big[\int_0^\infty f(x_n - y)\,\mathrm{d}G(y) - f(x_n)\Big] - \delta f(x_n) + u_0 \\ &\overset{n\to\infty}{\longrightarrow} -\delta f(\infty) + u_0\ , \end{aligned}$$

showing that $\lim_{x\to\infty} f(x) = u_0/\delta$.

Let now $U = U^*$ be the strategy given by (4.7) and the corresponding $Z^* = Z^{U^*}$. From the lemma above and the HJB equation, it follows that

$$\Big\{f(X_t^{U^*})e^{-\delta t} - f(x) + \int_0^t e^{-\delta s}U_s^*\,\mathrm{d}s - \sum_{0 \le s \le t} [f(X_s^{U^*}) - f(X_s^{U^*} - \Delta Z_s^*)]e^{-\delta s}\Big\}$$

is a martingale with expected value 0. If $\Delta Z_s^* > 0$, then $X_s^{U^*} = C$ and $f(X_s^{U^*} - \Delta Z_s^*) = f(C - \Delta Z_s^*) = f(C) - \phi(\Delta Z_s^* - C + C) - L = f(C) - \phi\Delta Z_s^* - L$. Thus

$$f(x) = \mathbb{E}\Big[f(X_t^{U^*})e^{-\delta t} + \int_0^t e^{-\delta s}U_s^*\,\mathrm{d}s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* - L\sum_{0 \le s \le t} e^{-\delta s}\mathbf{1}_{\{\Delta Z_s^* > 0\}}\Big]$$

holds. By the boundedness of f and the bounded convergence theorem, we get that $\mathbb{E}[f(X_t^{U^*})e^{-\delta t}] \to 0$ as $t \to \infty$. Since the other terms are monotone, we can interchange the limit and integration and obtain $f(x) = V^{U^*}(x)$. Here we used the condition $f'(0) \geq 1$ which is motivated by the considerations above. For an arbitrary strategy U, Equation (4.2) gives that

$$\begin{aligned} f(x) \quad &\geq \quad \mathbb{E}\Big[f(X_t^U)e^{-\delta t} + \int_0^t e^{-\delta s} U_s \,\mathrm{d}s - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s \\ &\qquad - L \sum_{0 \leq s \leq t} e^{-\delta s} \mathbf{I}_{\{\Delta Z_s > 0\}}\Big] \;, \end{aligned}$$

where we used that

$$f(X_s^U) \leq f(C) + \phi(X_s^U - C) = f(X_s^U - \Delta Z_s) + \phi \Delta Z_s + L \;.$$

Letting $t \to \infty$ shows that $f(x) \geq V^U(x)$. Thus, $f(x) = V(x)$. □

4.3 Unrestricted Dividends

In this section, all increasing, adapted and càdlàg processes $D \in \mathscr{S}_x$ are allowed. The value of a strategy D is

$$V^D(x) = \mathbb{E}_x\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t - L \sum_{t \geq 0} e^{-\delta t} \mathbf{I}_{\{\Delta Z_t > 0\}}\Big]$$

and $V(x) = \sup_{D \in \mathscr{S}_x} V^D(x)$ is the value function.

4.3.1 The Value Function and the HJB Equation

Again, we start by proving some useful properties of $V(x)$.

Lemma 4.3.1
The function $V(x)$ is increasing with

$$x - y \leq V(x) - V(y) \leq \phi(x - y) + L$$

for $0 \leq y < x$, locally Lipschitz continuous on $[0, \infty)$ and therefore absolutely continuous. For any $x \geq 0$,

$$x + \frac{c - \lambda(\phi\mu + \phi C + L)}{\delta} \leq V(x) \leq x + \frac{c}{\delta} \;.$$

Proof: Consider a strategy D with $V^D(y) \geq V(y) - \varepsilon$ for an $\varepsilon > 0$ and $y \geq 0$. For $x \geq y$, define a new strategy as follows: $x - y$ is paid immediately as dividend and then the strategy D with initial capital y is followed. Then for any $\varepsilon > 0$, it holds that

$$V(x) \geq x - y + V^D(y) \geq x - y + V(y) - \varepsilon \,.$$

Because ε was arbitrary, $V(x) - V(y) \geq x - y$ follows. In particular, V is increasing.

For the other direction, let $\varepsilon > 0$ and D an ε-optimal strategy for initial capital x. For $0 \leq y < x$, capital injections $x - y$ are made immediately and after that the strategy D is followed. Hence,

$$V(y) \geq -\phi(x-y) - L + V^D(x) \geq -\phi(x-y) - L + V(x) - \varepsilon.$$

Because this holds for all $\varepsilon > 0$, we get the inequality

$$V(x) - V(y) \leq \phi(x-y) + L \,.$$

Consider now the strategy D paying initial capital immediately and then the dividends are paid at rate c, i.e., the whole surplus exceeding 0 is paid as dividend. For such a strategy we have $C = 0$, since for $C > 0$ the sum of the dividends would be lower because of the penalty ϕ. Thus, the value of the dividends is

$$V^D(x) = x + \mathbb{E}\Big[\int_0^\infty e^{-\delta t} c \,\mathrm{d}t\Big] = x + \frac{c}{\delta} \,.$$

The value of the capital injections is calculated in Lemma 4.1.1, which yields the lower bound. We note that for any reasonable strategy, $V^D(x)$ is an upper bound for the accumulated dividend payments. Not taking the capital injections into account yields the upper bound for the value function.

The local Lipschitz continuity follows by the local boundedness of V as in the proof of Lemma 4.2.1.

By Rademacher's Theorem, the local Lipschitz continuity ensures the existence of the derivative $V'(x)$ almost everywhere on $[0,\infty)$. Then V' is a density of V. □

We note that since $(c - \lambda(\phi\mu + \phi C + L))/\delta$ is the lower bound for $V(0)$, the positivity of V can only be assured, if

$$c > \lambda(\phi\mu + \phi C + L)\,. \tag{4.8}$$

As in Chapter 3, the value function can be calculated as the limit of the value functions of the strategies with restricted densities.

Lemma 4.3.2
Let $V_u(x)$ be the value function for the restricted dividend strategy in the case $u_0 = u$. Then $\lim_{u\to\infty} V_u(x) = V(x)$.

Proof: The proof is analogous to the proof of Lemma 3.3.2. □

To prove the Hamilton–Jacobi–Bellman equation for this problem, we can repeat the procedure in Schmidli [66, Section 2.4.2].

Theorem 4.3.3
The function $V(x)$ is differentiable a.e. on $(0,\infty)$ and fulfils the Hamilton-Jacobi-Bellman equation

$$\max\Big\{cV'(x) + \lambda \int_0^\infty V(x-y)\, dG(y) - (\lambda+\delta)V(x), 1 - V'(x)\Big\} = 0\ . \quad (4.9)$$

At points where $V(x)$ is not differentiable, the derivatives from the left and from the right exist and fulfil Equation (4.9) with $V'(x-) = 1 < V'(x+)$.

Proof: Since we let $u \to \infty$, it is enough to consider the case $u > c$. Equation (4.2) can be written as

$$\max\Bigg\{cV_u'(x) + \lambda \int_0^\infty V_u(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V_u(x),$$
$$1 - V_u'(x) + \frac{\lambda \int_0^\infty V_u(x-y)\,\mathrm{d}G(y) - (\lambda+\delta)V_u(x) + c}{u-c}\Bigg\} = 0.$$

Similarly to (4.5), we let $H(x) = (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y)$ and denote the corresponding function in the case $u_0 = u$ by $H_u(x)$. Then the equation above reads

$$\max\left\{cV_u'(x) - H_u(x), 1 - V_u'(x) + \frac{c - H_u(x)}{u-c}\right\} = 0. \quad (4.10)$$

We show that $V_u'(x) \to f(x)$ a.e. for some positive function $f(x)$ and that $f(x)$ really is the density of $V(x)$.

We already know that $\lim_{u\to\infty} V_u(x) = V(x)$, hence by the bounded convergence theorem $\lim_{u\to\infty} H_u(x) = H(x)$. Assume, $V_u'(x)$ converges for a.e. x to some function f. The limit is finite because $cV_u'(x) \le (\lambda+\delta)V_u(x) \le (\lambda+\delta)V(x)$. This function satisfies the equation (4.10) and so

$$\max\{cf(x) - H(x), 1 - f(x)\} = 0.$$

It follows by the bounded convergence theorem that

$$V(x) - V(0) = \lim_{u\to\infty} \int_0^x V_u'(z)\,\mathrm{d}z = \int_0^x \lim_{u\to\infty} V_u'(z)\,\mathrm{d}z = \int_0^x f(z)\,\mathrm{d}z.$$

I.e., f is the density of V, $V(x)$ is differentiable at all points where $f(x)$ is continuous and $f(x) = V'(x)$.

It remains to show that $V_u'(x) \overset{u\to\infty}{\longrightarrow} f(x)$ a.e.

From (4.7) and (4.10) follows that $H_u(x) \geq c$, therefore the same holds for $H(x)$.

Let us first consider the case $H(x) = c$. Let $\{u_n\} \to \infty$ be a sequence such that $V_{u_n}'(x)$ converges. By (4.10) we conclude that $\lim_{n\to\infty} V_{u_n}'(x) \leq 1$ as well as $\lim_{n\to\infty} V_{u_n}'(x) \geq 1$, depending whether we take the first or the second term on the left hand side of (4.10). Since the first term is taken if $V_u'(x) \geq 1$ and the second one if $V_u'(x) \leq 1$, we find in both cases $\lim_{u\to\infty} V_u'(x) = 1$.

Suppose now $H(x) > c$. By continuity, there exist $\varepsilon > 0$ and $x_1 < x < x_2$ such that $H(z) > c + 2\varepsilon$ for all $x_1 \leq z \leq x_2$. Hence for u large enough $H_u(z) > c + \varepsilon$. In this case $V_u'(z) \neq 1$ for all $x_1 \leq z \leq x_2$.

We consider now a sequence $u_n \to \infty$ such that the optimal strategy for all $z \in [x_1, x_2]$ is to pay dividends at rate u_n. Therefore $V_{u_n}'(z) < 1$ and $\lim_{n\to\infty} V_{u_n}'(z) = 1$. By bounded convergence we have then

$$V(x_2) - V(x_1) = \lim_{n\to\infty} \int_{x_1}^{x_2} V_{u_n}'(z)\,\mathrm{d}z = x_2 - x_1.$$

Similarly we can show

$$\lim_{u\to\infty} V_u'(x) = \frac{(\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}G(y)}{c}$$

if there exists as sequence $\{u_n\}$ such that it is optimal not to pay dividends on $[x_1, x_2]$. In both cases V is differentiable at x.

The last possibility is that $V_u(z)$ not differentiable in $[x_1, x_2]$ for u large enough. Then, there is a unique point $z_u \in [x_1, x_2]$ where $V_u(z)$ is not differentiable. Moreover, $V_u'(z) \leq 1$ for $z < z_u$ and $V_u'(z) > 1$ for $z > z_u$. Choose now a sequence u_n such that z_{u_n} converges. If $y = \lim_{n\to\infty} z_{u_n} \neq x$, then the argument given above works on $[x_1, y]$ or $[y, x_2]$ if $x < y$ or $x > y$ respectively. Then $\lim_{u\to\infty} V_u'(x)$ exists. Consider therefore the case $\lim_{n\to\infty} z_{u_n} = x$. In this case the argument above shows that $f(z) = 1$ for $z < x$ and $f(z) > 1$ for $z > x$. Thus, the left and the right limits of $f(z)$ exist at x. In particular, $f(x-) = 1$. □

Remark 4.3.4
From the theorem above follows that the differentiability of V is only violated at the switching points from paying dividends to paying no dividends. ■

4.3.2 The Optimal Dividend Strategy and the Characterisation of the Solution

Motivated by the proof of Theorem 4.3.3, we consider the following three sets which will be essential for the definition of the optimal strategy. Let C be chosen optimally.

- $\mathcal{A} = \{x \in [0,\infty) : V'(x) = 1 \text{ and } H(x) = c\}$,
- $\mathcal{B} = \{x \in (0,\infty) : V'(x) = 1 \text{ and } H(x) > c\}$,
- $\mathcal{C} = (\mathcal{A} \cup \mathcal{B})^c = \{x \in [0,\infty) : V'(x) > 1 \text{ and } H(x) > c\}$.

Here we again mean the derivative from the left if the derivative does not exist. First we discuss some properties of these sets. The proof of the next lemma is based on Schmidli [66, Section 2.4.2].

Lemma 4.3.5 *1. $\mathcal{A}$ is closed.*

2. $\mathcal{B}$ is a left-open set, i.e., if $x \in \mathcal{B}$, then there exists $\varepsilon > 0$ such that $(x-\varepsilon, x] \subset \mathcal{B}$.

3. If $(x_0, x] \subset \mathcal{B}$ and $x_0 \notin \mathcal{B}$, then $x_0 \in \mathcal{A}$.

4. $\mathcal{C}$ is a right-open set, i.e., if $x \in \mathcal{C}$, then there exists $\delta > 0$ such that $[x, x+\delta) \subset \mathcal{C}$.

5. We have $(\lambda(\phi\mu + \phi C + L)/\delta, \infty) \subset \mathcal{B}$.

6. $\mathcal{A}, \mathcal{B} \neq \emptyset$.

Proof:

1. Since H is continuous, $H(x) \geq c$ for all $x \in [0,\infty)$ and $\{c\}$ is closed, the set $\{x \in [0,\infty) : H(x) = c\}$ is closed.

2. Let $x \in \mathcal{B}$. Since $\mathcal{A}$ is closed, there must be $\varepsilon > 0$ such that $(x-\varepsilon, x) \subset \mathcal{A}^c$ because, otherwise, $x \in \mathcal{A}$. Since $(x-\varepsilon, x) \subset \mathcal{C}^c$, we get $(x-\varepsilon, x) \subset \mathcal{B}$.

3. Let $\{x_n\} \subset (x_0, x]$ such that $x_n \downarrow x_0$. Then, $V'(x_n) = 1$ and $H(x_n) > c$. By continuity, $V'(x_0) = 1$ and $H(x_0) = c$, since, otherwise, $x_0 \in \mathcal{B}$.

4. If $x \in \mathcal{C}$, then, by the continuity of H, there must be a $\delta > 0$ such that $[x, x+\delta) \subset \mathcal{A}^c$. If there would be some $x_1 \in \mathcal{B}$ within this interval, we could follow the existence of an $x_0 \in \mathcal{A}$ with $x_0 < x_1$ such that $(x_0, x_1] \subset \mathcal{B}$. Since $x \notin \mathcal{B}$ this x_0 has also to be in the interval $(x, x+\delta)$ which is a contradiction. Therefore, $[x, x+\delta) \subset \mathcal{B}^c$ and $[x, x+\delta) \subset \mathcal{C}$.

5. Let C be chosen optimally. Since $V(x)$ is strictly increasing, we get $V(x) - \int_0^x V(x-y)\,\mathrm{d}G(y) \geq V(x)(1-G(x))$ and therefore

$$\begin{aligned}
&(\lambda+\delta)V(x) - \lambda \int_0^x V(x-y)\,\mathrm{d}G(y) \\
&\qquad\qquad -\lambda(V(C) - \phi C - L)(1-G(x)) + \lambda\phi \int_x^\infty (1-G(y))\,\mathrm{d}y \\
&\quad \geq \lambda V(x)(1-G(x)) + \delta V(x) - \lambda(V(C) - \phi C - L)(1-G(x)) \\
&\quad = \delta V(x) + \lambda(1-G(x))(V(x) - V(C) + \phi C + L) \\
&\quad \geq \delta V(x).
\end{aligned}$$

The last inequality holds because obviously $V(x) - V(C) \geq 0$ for $x \geq C$. For $x < C$ we have by Lemma 4.3.1 that $V(x) - V(C) \geq -\phi(C-x) - L$ and thus $V(x) - V(C) + \phi C + L \geq \phi x \geq 0$.

From Lemma 4.3.1 we can follow that for any $x > \lambda(\phi\mu + \phi C + L)/\delta$

$$\begin{aligned}
V(x) &\geq x + \frac{c - \lambda(\phi\mu + \phi C + L)}{\delta} \\
&> \frac{\lambda(\phi\mu + \phi C + L)}{\delta} + \frac{c - \lambda(\phi\mu + \phi C + L)}{\delta} = \frac{c}{\delta}
\end{aligned}$$

holds. Assume now that there is $x > \lambda(\phi\mu+\phi C+L)/\delta$ with $V'(x-) > 1$. Then $V'(z) > 1$ for all $z \geq x$. To prove this claim, we suppose that there is $z = \inf\{y > x : V'(y) = 1\} < \infty$. For this point we have

$$\begin{aligned}
1 &= V'(z) = \frac{(\lambda+\delta)V(x) - \lambda\int_0^\infty V(x-y)\,\mathrm{d}G(y)}{c} \geq \frac{\delta V(z)}{c} \\
&> \frac{\delta V(x)}{c} > 1,
\end{aligned}$$

which is a contradiction. Thus,

$$V'(z) \geq \frac{\delta}{c} V(z)$$

for all $z \geq x$, or equivalently, $\log(V(z)/V(x)) \geq (z-x)\delta/c$, i.e., $V(x)$ is exponentially increasing on $[x, \infty)$. This is a contradiction to Lemma 4.3.1. Thus, $V'(x-) = 1$.

6. The assertion follows from the forth and fifth point.

□

Now we define the following strategy D^*:

- If $X_t^{D^*} \in \mathcal{A}$, then we pay a dividend at rate c until the next claim occurs, i.e., $\mathrm{d}D_t^* = c\,\mathrm{d}t$.
- If $x := X_{T-}^{D^*} - Y \in \mathcal{B}$, then there is a $x_1 = \sup\{z < x : z \notin \mathcal{B}\}$ such that $(x_1, x] \subset \mathcal{B}$, and the sum

  $$\Delta D_t^* = x - x_1$$

  is paid as dividend. Then, $x_1 \in \mathcal{A}$.
- If $X_t^{D^*} \in \mathcal{C}$, no dividends are paid.

This is a strategy of *band type*. By construction, the dividend process D^* is measurable.

For the constructed strategy D^*, we denote $X_t^* = X_t^{D^*} = X_t - D_t^* + Z_t^*$ the corresponding surplus process. We again can derive the following facts.

- The process X^* only jumps at the claim times. Thus, in an interval (T_{i-1}, T_i) between two claims holds that $\mathrm{d}X_t^* = c\mathbf{1}_{\{X_t^* \in \mathcal{C}\}}\,\mathrm{d}t$.
- $X_{T_i}^* = X_{T_i-}^* - Y_i - \Delta D_{T_i}^* \mathbf{1}_{\{X_{T_i-}^* - Y_i \in \mathcal{B}\}} + \Delta Z_{T_i}^* \mathbf{1}_{\{X_{T_i-}^* - Y_i < 0\}}$.
- $\mathbf{1}_{\{\Delta Z_{T_i}^* > 0\}} = \mathbf{1}_{\{X_{T_i-}^* - Y_i < 0\}}$.
- It can not be optimal to pay dividends at claim times when the surplus falls below zero, i.e., when capital injections are needed. Thus, $\mathbf{1}_{\{\Delta D_{T_i}^* > 0\}} = 1 - \mathbf{1}_{\{\Delta Z_{T_i}^* > 0\}}$.
- If $X_{T_i-}^* - Y_i < 0$, then the shareholders pay as much that $X_{T_i}^* = C$; i.e., $\Delta Z_{T_i}^* = -(X_{T_i-}^* - Y_i) + C$. In this case, the value function fulfils

  $$V(X_{T_i}^*)(= V(C)) = V(X_{T_i-}^* - Y_i) + (\phi \Delta Z_{T_i}^* + L)\mathbf{1}_{\{\Delta Z_{T_i}^* > 0\}}$$

  because of the property (4.1). Thus, it suffices to consider only solutions f to the HJB equation with the property (4.1).

We now can show the optimality of the constructed strategy.

Theorem 4.3.6
The strategy D^* is optimal, i.e., $V^*(x) = V^{D^*}(x) = V(x)$.

Proof: We have already shown that C is chosen optimally. Similarly as in Lemma 4.2.3, we can write, using property (4.1),

$$\begin{aligned}
&V(X_t^*)e^{-\delta t}\\
&= V(X_0^*) + \sum_{i=1}^{N_t}(\phi\Delta Z_{T_i}^* + L)\mathbf{1}_{\{\Delta Z_{T_i}^*>0\}}e^{-\delta T_i}\\
&\quad + \sum_{i=1}^{N_t}\Big[V(X_{T_i-}^* - Y_i - \Delta D_{T_i}^*\mathbf{1}_{\{X_{T_i-}^*-Y_i\in\mathcal{B}\}}) - V(X_{T_i-}^*)\Big]e^{-\delta T_i}\\
&\quad + \sum_{i=1}^{N_t}\Big[V(X_{T_i-}^*)e^{-\delta T_i-} - V(X_{T_{i-1}}^*)e^{-\delta T_{i-1}}\Big]\\
&\quad + V(X_t^*)e^{-\delta t} - V(X_{T_{N_t}}^*)e^{-\delta T_{N_t}}.
\end{aligned}$$

Further,

$$\begin{aligned}
&V(X_{T_i-}^*)e^{-\delta T_i-} - V(X_{T_{i-1}}^*)e^{-\delta T_{i-1}}\\
&= \int_{T_{i-1}}^{T_i-}\Big(cV'(X_s^*) - \delta V(X_s^*)\Big)\mathbf{1}_{\{X_s^*\in\mathcal{C}\}}e^{-\delta s}\,\mathrm{d}s\\
&\quad - \int_{T_{i-1}}^{T_i-}\delta V(X_s^*)\mathbf{1}_{\{X_s^*\in\mathcal{A}\}}e^{-\delta s}\,\mathrm{d}s.
\end{aligned}$$

Not taking the jumps of D^* into the generator, we conclude from Theorem B.2.4 that the process

$$\begin{aligned}
\Big\{&\sum_{i=1}^{N_t}\Big[V(X_{T_i-}^* - Y_i - \Delta D_{T_i}^*\mathbf{1}_{\{X_{T_i-}^*-Y_i\in\mathcal{B}\}}) - V(X_{T_i-}^*)\Big]e^{-\delta T_i}\\
&\qquad - \lambda\int_0^t e^{-\delta s}\int_0^\infty\Big(V(X_s^* - y)\,\mathrm{d}G(y) - V(X_s^*)\Big)\,\mathrm{d}s\Big\}
\end{aligned}$$

is a martingale with expected value 0. Noting that on $\mathcal{B}$

$$V(X_{T_i-}^* - Y_i - \Delta D_{T_i}^*) - V(X_{T_i-}^*) = -\Delta D_{T_i}^*$$

and $V(X^*_0) = V(x) - D^*_0$ holds, it follows that the process

$$\begin{aligned}\Big\{V(X^*_t)e^{-\delta t} - V(x) + \sum_{i=0}^{N_t} \Delta D^*_{T_i} e^{-\delta T_i} - \sum_{i=1}^{N_t} (\phi \Delta Z^*_{T_i} + L)\mathbf{1}_{\{\Delta Z^*_{T_i} > 0\}} e^{-\delta T_i} \\ - \int_0^t \Big[cV'(X^*_s) \\ + \lambda \int_0^\infty V(X^*_s - y)\,\mathrm{d}G(y) - (\lambda+\delta)V(X^*_s)\Big]\mathbf{1}_{\{X^*_s \in \mathcal{C}\}} e^{-\delta s}\,\mathrm{d}s \\ - \int_0^t \Big[\lambda \int_0^\infty V(X^*_s - y)\,\mathrm{d}G(y) - (\lambda+\delta)V(X^*_s)\Big]\mathbf{1}_{\{X^*_s \in \mathcal{A}\}} e^{-\delta s}\,\mathrm{d}s\Big\}\end{aligned}$$

is a martingale. On $\mathcal{C}$ we have $V'(X^*_s) > 1$ and the first term of (4.9) vanishes. On $\mathcal{A}$ both terms of (4.9) are zero, therefore

$$\lambda \int_0^\infty V(X^*_s - y)\,\mathrm{d}G(y) - (\lambda+\delta)V(X^*_s) = -c\,.$$

Thus, we get that

$$\begin{aligned}\Big\{V(X^*_t)e^{-\delta t} - V(x) + \sum_{i=0}^{N_t} \Delta D^*_{T_i} e^{-\delta T_i} + \int_0^t c\mathbf{1}_{\{X^*_s \in \mathcal{A}\}} e^{-\delta s}\,\mathrm{d}s \\ - \sum_{i=1}^{N_t} (\phi \Delta Z^*_{T_i} + L)\mathbf{1}_{\{\Delta Z^*_{T_i} > 0\}} e^{-\delta T_i}\Big\}\end{aligned}$$

is a martingale with expected value 0. From the martingale property we get that

$$\begin{aligned}V(x) = \mathbb{E}\Big[V(X^*_t)e^{-\delta t} + \sum_{i=0}^{N_t} \Delta D^*_{T_i} e^{-\delta T_i} + \int_0^t c\mathbf{1}_{\{X^*_s \in \mathcal{A}\}} e^{-\delta s}\,\mathrm{d}s \\ - \sum_{i=1}^{N_t} (\phi \Delta Z^*_{T_i} + L)\mathbf{1}_{\{\Delta Z^*_{T_i} > 0\}} e^{-\delta T_i}\Big]\,.\end{aligned}$$

Since

$$V(X^*_t)e^{-\delta t} \le V((x+ct) \vee C^*)e^{-\delta t} \le ((x+ct) \vee C^* + c/\delta)e^{-\delta t}$$

converges to 0 as $t \to \infty$, we have that

$$\lim_{t\to\infty} \mathbb{E}[V(X^*_t)e^{-\delta t}] = 0$$

by the bounded convergence theorem. By monotone convergence, we finally get that

$$\begin{aligned} V(x) &= \lim_{t\to\infty} \mathbb{E}\Big[\int_0^t c\mathbf{I}_{\{X_s^*\in\mathcal{A}\}} e^{-\delta s}\,\mathrm{d}s + \sum_{i=0}^{N_t} \Delta D_{T_i}^* e^{-\delta T_i} \\ &\qquad\qquad -\phi\sum_{i=1}^{N_t} e^{-\delta T_i}\Delta Z_{T_i}^* - L\sum_{0\le s\le t} e^{-\delta s}\mathbf{I}_{\{\Delta Z_s^*>0\}}\Big] \\ &= \mathbb{E}\Big[\int_0^\infty e^{-\delta s}\,\mathrm{d}D_s^* - \phi\int_0^\infty e^{-\delta s}\,\mathrm{d}Z_s^* - L\sum_{0\le s<\infty} e^{-\delta s}\mathbf{I}_{\{\Delta Z_s^*>0\}}\Big] \\ &= V^*(x)\,. \end{aligned}$$

□

Remark 4.3.7
If $\phi > 1$, $C \in \mathcal{B}$ cannot be optimal. Thus $C \in \mathcal{A}$ or $C \in \mathcal{C}$. If $\phi = 1$, it follows from Lemma 4.3.1 that $V(x) - x \ge V(C) - C$ for $x \ge C$. Since C is optimal we also have $V(x) - x \le V(C) - C$, therefore equality holds, i.e. $V(x) = V(C) + x - C$ for $x \ge C$. That is, the capital injection is such that the surplus is at the maximal level in $\mathcal{A}$. ■

Because we do not have an explicit solution nor an initial value, we need to characterise the solution $V(x)$ among other possible solutions.

Theorem 4.3.8
$V(x)$ is the minimal solution to (4.9) with C chosen such that $V(C) - \phi C$ becomes maximal. If $f(x)$ is a solution with only positive jumps of its derivative fulfilling property (4.1) and a linear growth condition $f(x) \le \kappa_1 x + \kappa_2$ for some positive constants κ_1, κ_2 and all $x \ge 0$, then $f(x) = V(x)$.

Proof: Let f be a solution to the HJB equation with property (4.1) and C chosen such that $f(C) - \phi C$ becomes maximal. Then $f(x)$ is increasing. Consider the process X^* under the optimal strategy and denote the optimal surplus after a capital injection by C^*. We have then, as in the proof of

Theorem 4.3.6, that the process

$$\begin{aligned}
\Big\{f(X_t^*)e^{-\delta t} &- f(X_0^*) - \sum_{i=1}^{N_t}\Big[f(X_{T_i}^*) - f(X_{T_i}^* - \Delta Z_{T_i}^*)\Big]e^{-\delta T_i} \\
&+ \sum_{i=1}^{N_t}\Big[f(X_{T_i-}^*) - f(X_{T_i-}^* - Y_i - \Delta D_{T_i}^*)\Big]\mathbf{1}_{\{X_{T_i-}^* - Y_i \in \mathcal{B}\}}e^{-\delta T_i} \\
&- \int_0^t \Big[cf'(X_s^*) + \lambda\int_0^\infty f(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*)\Big]\mathbf{1}_{\{X_s^* \in \mathcal{C}\}}e^{-\delta s}\,\mathrm{d}s \\
&- \int_0^t \Big[\lambda\int_0^\infty f(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*)\Big]\mathbf{1}_{\{X_s^* \in \mathcal{A}\}}e^{-\delta s}\,\mathrm{d}s\Big\}
\end{aligned}$$

is a martingale with expected value 0. Since $f'(x) \geq 1$, we have $f(x) \geq f(X_0^*) + D_0^*$ and $f(X_{T_i-}^*) - f(X_{T_i-}^* - Y_i - \Delta D_{T_i}^*) \geq \Delta D_{T_i}^*$. By (4.9)

$$cf'(X_s^*) + \lambda\int_0^\infty f(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*) \leq 0$$

and

$$\lambda\int_0^\infty f(X_s^* - y)\,\mathrm{d}G(y) - (\lambda+\delta)f(X_s^*) \leq -cf'(X_s^*) \leq -c\,.$$

Noting that if $X_{T_i}^* - \Delta Z_{T_i}^* = C^* - \Delta Z_{T_i}^* = X_{T_i-}^* - Y_i < 0$, then, since $f(C) - \phi C$ is maximal, we also have

$$\begin{aligned}
f(X_{T_i}^*) - f(X_{T_i}^* - \Delta Z_{T_i}^*) &= f(C^*) - [f(C) - \phi(\Delta Z_{T_i}^* - C^* + C) - L] \\
&\leq \phi\Delta Z_{T_i}^* + L\,.
\end{aligned}$$

This yields

$$\begin{aligned}
f(x) &\geq \mathbb{E}\Big[f(X_t^*)e^{-\delta t} + \sum_{i=0}^{N_t}\Delta D_{T_i}^* e^{-\delta T_i} + \int_0^t e^{-\delta s}c\mathbf{1}_{\{X_s^* \in \mathcal{A}\}}\,\mathrm{d}s \\
&\qquad - \sum_{i=1}^{N_t}(\phi\Delta Z_{T_i}^* + L)\mathbf{1}_{\{\Delta Z_{T_i}^* > 0\}}e^{-\delta T_i}\Big] \\
&\geq \mathbb{E}\Big[\int_0^t e^{-\delta s}\,\mathrm{d}D_s^* - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* - L\sum_{0\leq s\leq t}e^{-\delta s}\mathbf{1}_{\{\Delta Z_s^* > 0\}}\Big]
\end{aligned}$$

and, therefore, by monotone convergence, $f(x) \geq V^{D^*}(x) = V(x)$.

Suppose that, additionally, f satisfies a linear growth condition. Define

$$\begin{aligned}
\tilde{H}(x) = (\lambda+\delta)f(x) - \lambda\int_0^x f(x-y)\,\mathrm{d}G(y) \\
-\lambda(f(C) - \phi C - L)(1 - G(x)) + \lambda\phi\int_x^\infty (1 - G(y))\,\mathrm{d}y
\end{aligned}$$

and the following sets:

- $\tilde{\mathcal{A}} = \{x \in [0,\infty) : f'(x) = 1 \text{ and } \tilde{H}(x) = c\}$,
- $\tilde{\mathcal{B}} = \{x \in (0,\infty) : f'(x) = 1 \text{ and } \tilde{H}(x) > c\}$,
- $\tilde{\mathcal{C}} = (\tilde{\mathcal{A}} \cup \tilde{\mathcal{B}})^c = \{x \in [0,\infty) : f'(x) > 1 \text{ and } \tilde{H}(x) > c\}$.

The results of Lemma 4.3.5 remain valid. Let $\tilde{D}$ be the strategy corresponding to $f(x)$ defined in the same way as D^*, i.e. if the current $x \in \tilde{\mathcal{A}}$, then every incoming premium is paid as dividend; if $x = X_{T-}^{\tilde{D}} - Y \in \tilde{\mathcal{B}}$, then the sum $\Delta\tilde{D}_t = x - x_1$ is paid as dividend reducing the reserve process to the next point $x_1 \in \tilde{\mathcal{A}}$ which is smaller than x; if $x \in \tilde{\mathcal{C}}$, no dividend is paid. In the same way as in Theorem 4.3.6 we find that the process

$$\Big\{f(X_t^{\tilde{D}})e^{-\delta t} - f(x) + \sum_{i=0}^{N_t} \Delta\tilde{D}_{T_i} e^{-\delta T_i} + \int_0^t c\mathbf{1}_{\{X_s^{\tilde{D}} \in \tilde{\mathcal{A}}\}} e^{-\delta s}\,\mathrm{d}s \\ - \sum_{i=1}^{N_t} (\phi \Delta Z_{T_i}^{\tilde{D}} + L)\mathbf{1}_{\{\Delta Z_{T_i}^{\tilde{D}} > 0\}} e^{-\delta T_i}\Big\}$$

is a martingale with expected value 0. Taking expectations and letting $t \to \infty$ yields the assertion since $f(X_t^{\tilde{D}})e^{-\delta t} \le f((x+ct) \vee C)e^{-\delta t}$ tends to zero as $t \to \infty$ by the linear growth condition. Thus we get $f(x) = V^{\tilde{D}} \le V(x)$ and therefore $f(x) = V(x)$. □

Remark 4.3.9
The condition that $f(C) - \phi C$ is maximal is needed in order to exclude solutions with a non-optimal choice of C. ■

4.3.3 Calculating the Value Function

Dividends at zero

We now consider the case where dividends are paid in zero. Then $0 \in \mathcal{A}$, $V'(0) = 1$ and $H(0) = c$. It follows that

$$V(0) = \frac{c + \lambda(V(C) - \phi C - L - \phi\mu)}{\lambda + \delta}.$$

By Lemma 4.3.5, there exists $x_0 \le \infty$ such that $(0, x_0) \subset \mathcal{B}$ because $\mathcal{B}$ is left-open and only elements of $\mathcal{A}$ can be lower boundaries of subsets of $\mathcal{B}$. Thus, $V(x)$ is the value of the barrier strategy with barrier at zero for $x \in [0, x_0]$, i.e., all surplus exceeding zero is paid as dividends. Of special interest is the case $C = 0$, where

$$V(0) = \frac{c - \lambda(\phi\mu + L)}{\delta} \;.$$

Since $1 - V'(x) = 0$ is fulfilled obviously, we consider the first part of the HJB equation (4.9) which reads

$$\begin{aligned} c + \lambda \int_0^x \Bigl(x - y + \frac{c - \lambda(\phi\mu + L)}{\delta}\Bigr) \,\mathrm{d}G(y) \\ + \lambda \int_x^\infty \Bigl(\frac{c - \lambda(\phi\mu + L)}{\delta} - L - \phi(y - x)\Bigr) \,\mathrm{d}G(y) \\ - (\lambda + \delta)\Bigl(x + \frac{c - \lambda(\phi\mu + L)}{\delta}\Bigr) \;\leq\; 0. \end{aligned}$$

This is equivalent to

$$\lambda L G(x) + \int_0^x (\lambda(\phi - 1)(1 - G(y)) - \delta) \,\mathrm{d}y \leq 0. \tag{4.11}$$

The condition simplifies to $\lambda L G(x) \leq \delta x$ in the case $\phi = 1$. The condition also simplifies if $G(x)$ is concave. Then the left hand side of (4.11) is concave as a function of x. The condition is fulfilled if the derivative in zero is non-positive. I.e., $\lambda L g(0) + \lambda(\phi - 1) - \delta \leq 0$, where $g(x)$ is the density of the claim size distribution. This can be written as

$$g(0) \leq \frac{\delta - \lambda(\phi - 1)}{\lambda L} \;.$$

Note that $\delta > \lambda(\phi - 1)$ is necessary. It should further be noted that condition (4.11) is necessary for a barrier in zero and $C = 0$.

No dividends at zero and $\phi > 1$

If $V'(0) > 1$, i.e., $0 \in \mathcal{C}$, then no dividends are paid in zero. We already know that $C \in \mathcal{A}$ or $C \in \mathcal{C}$. In particular, V is differentiable at C. Since C maximises $V(C) - \phi C$, we find $V'(C) = \phi$ if $C \neq 0$. It follows that $C \notin \mathcal{A}$ since $V'(x) = 1$ for all $x \in \mathcal{A}$.

If $C = 0$ and $V'(C) = V'(0) = \phi$, then, in the case of positive safety loading $c > \lambda\mu$,

$$V(0) = \frac{cV'(0) - \lambda\phi\mu - \lambda L}{\delta} = \phi\frac{c - \lambda\mu}{\delta} - \frac{\lambda L}{\delta}$$

was increasing in ϕ, which is not possible. Thus, $V'(C) \neq \phi$ for $\phi > 1$.

For small initial values, $V(x)$ is the value of a barrier strategy on $[0, b^*]$ for some (locally) optimal barrier b^*. We now denote the optimal surplus after a capital injection by C^* and assume that $C^* < b^*$. Then we could apply the method from Gerber and Shiu [34] and Gerber et al. [32] to determine the value function at least for $x \in [0, b^*]$.

Let τ be the first time the (uncontrolled) surplus process X_t falls below zero and τ^{b^*} the time of ruin of the controlled surplus process X_t^D if dividends are paid according to the barrier strategy with barrier b^*. Then we can write

$$\begin{aligned} V(x) = {} & \mathbb{E}_x\Big[\int_0^{\tau^{b^*}} e^{-\delta t}\,\mathrm{d}D_t\Big] \\ & -\phi\mathbb{E}_x\Big[e^{-\delta\tau^{b^*}}|X_{\tau^{b^*}}|\mathbf{I}_{\{\tau^{b^*}<\infty\}}\Big] + (V(C^*) - \phi C^* - L)\mathbb{E}_x\Big[e^{-\delta\tau^{b^*}}\mathbf{I}_{\{\tau^{b^*}<\infty\}}\Big]\,. \end{aligned}$$

Using notations (2.11) and (2.12) of the Gerber-Shiu penalty functions and letting $V^{b^*}(x)$ be the value of the expected discounted dividends until ruin, we can write

$$V(x) = V^{b^*}(x) - \phi\sigma^{b^*}(x) + (V(C^*) - \phi C^* - L)\psi^{b^*}(x).$$

Since we still do not know the barrier b^* nor the constant C^*, we proceed as follows. For A fixed, denote by $V^{A,b}$ the function

$$V^{A,b}(x) = V^b(x) - \phi\sigma^b(x) + (A - L)\psi^b(x)$$

for a barrier $b = b(A)$. We now have to find $A^*, b^* = b^*(A^*)$ and $C^* = C^*(A^*)$ such that $V^{A^*,b^*}(C^*) - \phi C^* = A^*$. Then $V(x) = V^{A^*,b^*}(x)$.

Note that $A^* = V(C^*) - \phi C^* \geq V(0)$. By (2.15) and the dividends-penalty identity (2.17), we have

$$\begin{aligned} V^{A,b}(x) &= V^b(x)\Big[1 + \phi\sigma'(b) - (A-L)\psi'(b)\Big] - \phi\sigma(x) + (A-L)\psi(x) \\ &= (e^{\rho x} - \chi(x))\frac{1 + \phi\sigma'(b) - (A-L)\psi'(b)}{\rho e^{\rho b} - \chi'(b)} - \phi\sigma(x) + (A-L)\psi(x)\,. \end{aligned}$$

Thus, $V^{A,b}(x)$ is determined by the functions $\sigma(x), \psi(x)$ and $\chi(x)$. Since these funtions do not depend on b, we can find an optimal $b^* = b^*(A)$ which maximises $V^{A,b}(x)$ by maximising the expression

$$\frac{1 + \phi\sigma'(b) - (A-L)\psi'(b)}{\rho e^{\rho b} - \chi'(b)}\,.$$

Then b^* is independent of x (for $x \leq b^*$). By maximising $V^{A,b^*}(x) - \phi x$, we can find an optimal $C^* = C^*(A)$. Finally, we solve the equation $V^{A,b^*}(C^*) - \phi C^* = A$ to find the correct A. For this purpose we observe the following.

Denote by A^* the correct A and τ^*, b^*, C^* the time of ruin, the optimal barrier and the optimal level for the capital injections corresponding to A^*. Let $A > A^*$ and τ^b, b, C the analogous notation for A. For A^* must hold $V^{A^*,b^*}(C^*) - \phi C^* = A^*$. Then we have

$$\begin{aligned}
&V^{A,b}(C) - \phi C - A \\
&= (V^{A,b}(C) - \phi C - A) - (V^{A^*,b^*}(C^*) - \phi C^* - A^*) \\
&\le (V^{A,b}(C) - \phi C) - (V^{A^*,b^*}(C) - \phi C) - (A - A^*) \\
&= V^{A,b}(C) - V^{A^*,b^*}(C) - (A - A^*) \\
&\le V^{A,b}(C) - V^{A^*,b}(C) - (A - A^*) \\
&= \mathbb{E}_C\Big[\int_0^{\tau^b} e^{-\delta t}\,\mathrm{d}D_t - \phi e^{-\delta\tau^b}|X_{\tau^b}|\mathbf{1}_{\{\tau^b<\infty\}} + (A-L)e^{-\delta\tau^b}\mathbf{1}_{\{\tau^b<\infty\}}\Big] \\
&\quad -\mathbb{E}_C\Big[\int_0^{\tau^b} e^{-\delta t}\,\mathrm{d}D_t - \phi e^{-\delta\tau^b}|X_{\tau^b}|\mathbf{1}_{\{\tau^b<\infty\}} + (A^*-L)e^{-\delta\tau^b}\mathbf{1}_{\{\tau^b<\infty\}}\Big] \\
&\quad -(A - A^*) \\
&= (A - A^*)\Big(\mathbb{E}_C\Big[e^{-\delta\tau^b}\mathbf{1}_{\{\tau^b<\infty\}}\Big] - 1\Big) \\
&< 0\,,
\end{aligned}$$

where the first inequality follows by the maximality property of $V^{A^*,b^*}(C^*) - \phi C^*$. The second inequality holds because the value $V^{A^*,b^*}(x)$ for the optimal barrier b^* is greater than the value $V^{A^*,b}(x)$ of the strategy with the non-optimal barrier b. If $A < A^*$, in the analogous way we get that

$$V^{A,b}(C) - \phi C - A \ge (A^* - A)\Big(1 - \mathbb{E}_{C^*}\Big[e^{-\delta\tau^*}\mathbf{1}_{\{\tau^*<\infty\}}\Big]\Big) > 0\,.$$

Thus, the function $A \longmapsto V^{A,b}(C) - \phi C - A$ is decreasing. In this way, $A^* = V(C^*) - \phi C^*$ can be found.

Remark 4.3.10
$V^{A,b}(x)$ is the unique solution to (2.1) on $[0,b]$. Thus, V^{A^*,b^*} is unique on $[0,b^*]$. ■

No dividends at zero and $\phi = 1$

In this case, we know that C is the largest value in $\mathcal{A}$. If $\mathcal{A}$ consists only of one point b, then the optimal strategy is a pure barrier strategy with a barrier b, and we have $C = b$. We can determine the value function in the same way as described above omitting calculating C.

Piecewise construction of the solution

Assume that it is optimal to pay dividends according to a barrier strategy with barrier at x_0 for some $x_0 \geq 0$ in a bounded intervall $[0, a)$ and for $x > a$ it is optimal to pay no dividends in some intervall. Additionally assume that we already know the value function v on $[0, x_0]$, i.e., $v : [0, x_0] \to [0, \infty)$ is a given continuous and increasing function. If $C \leq x_0$ (in fact, we assume that $C < \inf\{x : V'(x) = 1\}$, so C is below the lowest barrier), then for $x > x_0$, we are looking for a solution to the equation

$$\begin{aligned} 0 &= cu'(x) + \lambda \int_{x-x_0}^{x} v(x-y)\,\mathrm{d}G(y) + \lambda \int_0^{x-x_0} u(x-y)\,\mathrm{d}G(y) \\ &\quad +\lambda(v(C) - \phi C - L)(1 - G(x)) - \lambda\phi \int_x^\infty (1 - G(y))\,\mathrm{d}y \\ &\quad -(\lambda+\delta)u(x) \qquad\qquad (4.12) \\ v(x_0) &= u(x_0)\,, \end{aligned}$$

where $u \equiv v$ on $(-\infty, x_0]$.

Similarly to Albrecher and Thonhauser [1] we can show the next result.

Lemma 4.3.11
Let $x_0 \geq 0$. For any continuous and increasing function $v : [0, x_0] \to [0, \infty)$ there exists a unique, in (x_0, ∞) differentiable and strictly increasing solution $u : [x_0, \infty) \to [0, \infty)$ to (4.12) with $u(x_0) = v(x_0)$.

Proof: Let $\varepsilon = \frac{c}{2(2\lambda+\delta)}$ and denote by $CI[x_0, x_0+\varepsilon)$ the set of all continuous and increasing functions $u : [x_0, x_0+\varepsilon) \to [0, \infty)$. For a $u \in CI[x_0, x_0+\varepsilon)$, let

$$\begin{aligned} \bar{u}(x) = \frac{1}{c}\Bigg[&(\lambda+\delta)u(x) - \lambda \int_{x-x_0}^{x} v(x-y)\,\mathrm{d}G(y) - \lambda \int_0^{x-x_0} u(x-y)\,\mathrm{d}G(y) \\ &-\lambda(v(C) - \phi C - L)(1-G(x)) + \lambda\phi \int_x^\infty (1-G(y))\,\mathrm{d}y\Bigg]\,. \end{aligned}$$

By the continuity of u and v, $\bar{u}$ is continuous for $x \geq 0$. Define now for $u \in CI[x_0, x_0+\varepsilon)$

$$T_u(x) = \int_{x_0}^{x} \bar{u}(s)\,\mathrm{d}s + v(x_0)\,.$$

Because of the monotonicity of u and v and $v(x_0) = u(x_0)$ we get

$$\begin{aligned} c\bar{u}(x) &= (\lambda+\delta)u(x) - \lambda\int_{x-x_0}^{x} v(x-y)\,\mathrm{d}G(y) - \lambda\int_0^{x-x_0} u(x-y)\,\mathrm{d}G(y) \\ &\quad -\lambda(v(C)-\phi C - L)(1-G(x)) + \lambda\phi\int_x^\infty (1-G(y))\,\mathrm{d}y \\ &\geq (\lambda+\delta)u(x) - \lambda v(x_0)(G(x)-G(x-x_0)) - \lambda u(x)G(x-x_0) \\ &\quad -\lambda v(C)(1-G(x)) \\ &\geq \delta u(x) + \lambda u(x) - \lambda u(x)G(x) - \lambda v(C)(1-G(x)) \\ &= \delta u(x) + \lambda(1-G(x))(u(x)-v(C)) \\ &\geq \delta u(x) \\ &> 0\,. \end{aligned}$$

By the positivity of u and v, we get the upper bound for $\bar{u}(x)$

$$c\bar{u}(x) \leq (\lambda+\delta)u(x) + \lambda(\phi C + L + \phi\mu).$$

It follows that T_u is increasing, positive and continuous for $x \in [x_0, x_0+\varepsilon)$. For $u_1, u_2 \in CI[x_0, x_0+\varepsilon)$ holds

$$\begin{aligned} &c(\bar{u}_1(x) - \bar{u}_2(x)) \\ &\quad = (\lambda+\delta)(u_1(x)-u_2(x)) - \lambda\int_0^{x-x_0}(u_1(x-y)-u_2(x-y))\,\mathrm{d}G(y) \\ &\quad \leq (\lambda+\delta)\left\|u_1-u_2\right\| + \lambda\left\|u_1-u_2\right\| G(x-x_0) \\ &\quad \leq (2\lambda+\delta)\left\|u_1-u_2\right\|\ , \end{aligned}$$

where $\|\cdot\|$ is the supremum norm. It follows

$$T_{u_1}(x) - T_{u_2}(x) \leq \varepsilon\frac{2\lambda+\delta}{c}\left\|u_1-u_2\right\| \leq \frac{1}{2}\left\|u_1-u_2\right\|\ .$$

Interchanging u_1 and u_2 yields $\|T_{u_1} - T_{u_2}\| \leq \frac{1}{2}\|u_1-u_2\|$, i.e., T is a contraction on $CI[x_0, x_0+\varepsilon)$. This proves the existence of a $u \in CI[x_0, x_0+\varepsilon)$ such that

$$u(x) = T_u(x) = \int_{x_0}^{x} \bar{u}(s)\,\mathrm{d}s + v(x_0)\ .$$

This provides $u'(x) = \bar{u}(x)$ everywhere in $[x_0, x_0+\varepsilon)$. Thus, we get the existence of a unique solution to (4.12) with the required properties on $[x_0, x_0+\varepsilon)$.

Since ε does not depend on x_0, we have shown the existence of a unique solution on $[x_0, \infty)$. □

Now we describe an algorithm to determine the value function piecewise. The procedure is similar to Schmidli [66] (see also Albrecher and Thonhauser [1]).

Step 1. Check condition (4.11). If it is fulfilled for $x \in (0, x_0)$, then $V(x) = x + (c - \lambda(\phi\mu + L))/\delta$ for all $x \le x_0$, where x_0 is chosen maximal. If $x_0 = \infty$ we have solved the problem. If $x_0 > 0$, we expect that we have found the solution on $[0, x_0]$ and go to Step 3. If $x_0 = 0$, no dividend will be paid in zero and we proceed with Step 2.

Step 2. We expect for small x a barrier strategy. In order to fix the first barrier, we proceed as in Section 4.3.3. Let $f_0(x)$ be a solution to (4.9) with the (locally) optimal x_0 and C. Define

$$v_0(x) = \begin{cases} f_0(x) & : \quad x \le x_0 \\ x - x_0 + f_0(x_0) & : \quad x > x_0 \end{cases}$$

If now v_0 fulfils the HJB equation, then the value function is $V(x) = v_0(x)$. If not, go to Step 3.

Step 3. For $n \ge 0$, we are looking for some interval $(x_n, a) \in \mathcal{B}$. If some adjoining interval $[a, x_{n+1})$ belongs to $\mathcal{C}$, then we have to find a solution to (4.12). Suppose that we have constructed $v_n(x)$ and x_n. Let $f_{n+1}(x; y)$ be a function such that $f_{n+1}(x; y) = v_n(x)$ for $x \le y$ and $f_{n+1}(x; y)$ is a solution to (4.12) for $x > y$, i.e.,

$$\begin{aligned} 0 &= cf'_{n+1}(x; y) + \lambda \int_{x-y}^{x} v_n(x - z) \,\mathrm{d}G(z) \\ &\quad + \lambda \int_0^{x-y} f_{n+1}(x - z; y) \,\mathrm{d}G(z) + \lambda(v_n(C) - \phi C - L)(1 - G(x)) \\ &\quad - \lambda\phi \int_x^\infty (1 - G(z)) \,\mathrm{d}z - (\lambda + \delta) f_{n+1}(x; y). \end{aligned}$$

Then we have to choose the smallest $y > x_n$ such that the derivative $f'_{n+1}(\cdot; y)$ has its minimum at 1, i.e.,

$$a = \inf\{y > x_n | \inf_{z > y} f'_{n+1}(z; y) = 1\} ,$$

where the derivative is taken with respect to the first argument. If a is chosen too small then the derivative $f'_{n+1}(x; \cdot)$ will be larger than 1 and will not reach 1 again. If a is chosen too large, then the derivative will reach a value smaller than 1. The point x_{n+1} can now be determined as

$$x_{n+1} := \sup\{x \ge a | f'(x; a) = 1\} .$$

Let

$$v_{n+1}(x) = \begin{cases} f_{n+1}(x; a) & : \quad x \le x_{n+1} \\ x - x_{n+1} + f_{n+1}(x_{n+1}; a) & : \quad x > x_{n+1} \end{cases} .$$

If $v_{n+1}(x)$ solves (4.9), then it is the value function. If not, we repeat the procedure in Step 3. The algorithm terminates because of Lemma 4.3.5.

Step 4. Control, whether $V(C) - \phi C$ is maximal. If not, denote the solution obtained in Step 3 by $V_1(x)$ and let $C_1 = \arg\max\{V_1(x) - \phi x\}$. Let $V_2(x) = V_1(C_1) + \phi(x - C_1) - L$ for $x < 0$. Solve the problem with the corresponding $V_2(x)$ for $x < 0$. , i.e.,

$$\begin{aligned} 0 &= cV_2'(x) - (\lambda+\delta)V_2(x) + \lambda \int_0^x V_2(x-y)\,\mathrm{d}G(y) \\ &\quad + \lambda(V_1(C_1) - \phi C_1 - L)(1 - G(x)) - \lambda\phi \int_x^\infty (1 - G(y))\,\mathrm{d}y. \end{aligned}$$

Note that $V_2(x)$ will not be continuous in 0. The solution $V_2(x)$ has the following interpretation. After the first capital injection, one has to follow the strategy that gives $V_1(x)$. Find the optimal strategy until the first capital injection. Repeating this step give a policy improvement, that will converge to the optimal value function and therefore to the optimal strategy.

4.3.4 Examples

Exponentially distributed claim sizes

We consider the case with exponentially distributed claim sizes, i.e., $G(y) = 1 - e^{-\alpha y}$ and $\mathbb{E}[Y] = \mu = 1/\alpha$.

We start by looking for the candidate points of the set $\mathcal{A}$. Since $H(x) = c$ and $V'(x) = 1$ on $\mathcal{A}$, we have to differentiate the function

$$\begin{aligned} V(x) &= \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\left[\int_0^x V(x-y)\alpha e^{-\alpha y}\,\mathrm{d}y + (V(C) - \phi C - L)e^{-\alpha x}\right. \\ &\qquad \left. - \phi \int_x^\infty e^{-\alpha y}\,\mathrm{d}y\right] \\ &= \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}e^{-\alpha x}\left[\int_0^x V(y)\alpha e^{\alpha y}\,\mathrm{d}y + V(C) - \phi C - L - \phi\frac{1}{\alpha}\right]. \end{aligned}$$

where we used the representation (4.4). This yields

$$1 = V'(x) = \frac{c\alpha}{\lambda+\delta} - \frac{\delta\alpha}{\lambda+\delta}V(x)$$

and therefore

$$V(x) = \frac{c\alpha - (\lambda+\delta)}{\delta\alpha}. \tag{4.13}$$

Since $V(x)$ is strictly increasing, this equation can only be fulfilled for at most one point, i.e. $\mathcal{A}$ consists of at most one point, b say. Because $\mathcal{A}$ is not empty, a point b exists. By Lemma 4.3.5, b is the lower boundary of $\mathcal{B}$. Thus, a barrier strategy with a barrier b is optimal.

We now want to determine the parameters for which $b = 0$ and therefore $V(x) = x + V(0)$ for $V(0) = (c - \lambda(L + \phi/\alpha))/\delta$. Then, by (4.11), we have to check whether

$$\begin{aligned} F(x) &= \lambda(1 - e^{-\alpha x})L - \delta x + \lambda(\phi - 1)\int_0^x e^{-\alpha y}\, \mathrm{d}y \\ &= \lambda(1 - e^{-\alpha x})\Big(L + \frac{\phi - 1}{\alpha}\Big) - \delta x \le 0 \end{aligned}$$

for all $x \ge 0$. The first derivative of F,

$$F'(x) = (\lambda\alpha L + \lambda(\phi - 1))e^{-\alpha x} - \delta$$

is a decreasing function, i.e. $F(x)$ is strictly concave. Therefore, it is non-positive if and only if $F'(0) \le 0$, i.e. if

$$\delta \ge \lambda\alpha L + \lambda(\phi - 1)\ .$$

Let $\delta < \lambda\alpha L + \lambda(\phi - 1)$. Then $b > 0$. Let ρ and R be the positive and the negative solution to Lundberg's equation

$$cs^2 - (\lambda + \delta - \alpha c)s - \alpha\delta = 0\ ,$$

i.e.,

$$\begin{aligned} \rho &= \frac{\lambda + \delta - \alpha c + \sqrt{(\lambda + \delta - \alpha c)^2 + 4\alpha\delta c}}{2c}\ , \\ R &= \frac{\lambda + \delta - \alpha c - \sqrt{(\lambda + \delta - \alpha c)^2 + 4\alpha\delta c}}{2c}\ . \end{aligned}$$

By Gerber and Shiu [34], we know that $\psi(x) = \psi(0)e^{Rx}, \sigma(x) = \sigma(0)e^{Rx}$ and $\chi(x) = \chi(0)e^{Rx}$ with

$$\psi(0) = \frac{\lambda}{c(\rho + \alpha)}\ , \quad \sigma(0) = \frac{\lambda}{c\alpha(\rho + \alpha)}\ , \quad \chi(0) = \frac{\lambda\alpha}{c(\rho + \alpha)^2}\ .$$

Then, for A fixed, we can find an optimal $b(A)$ by maximising the function

$$\frac{1 + \phi\sigma(0)Re^{Rb} - (A - L)\psi(0)Re^{Rb}}{\rho e^{\rho b} - \chi(0)Re^{Rb}}\ .$$

As an illustration, we let $\lambda = 1.8, \alpha = 0.5, \delta = 0.3, c = 11, L = 2$ and $\phi = 1.2$. Then $b^* = 4.313$ is the (globally) optimal barrier height and $C^* = 0.1047$ is the optimal capital injections level which are reached for $A^* = 18.087$. Figure 4.2 shows the corresponding value function.

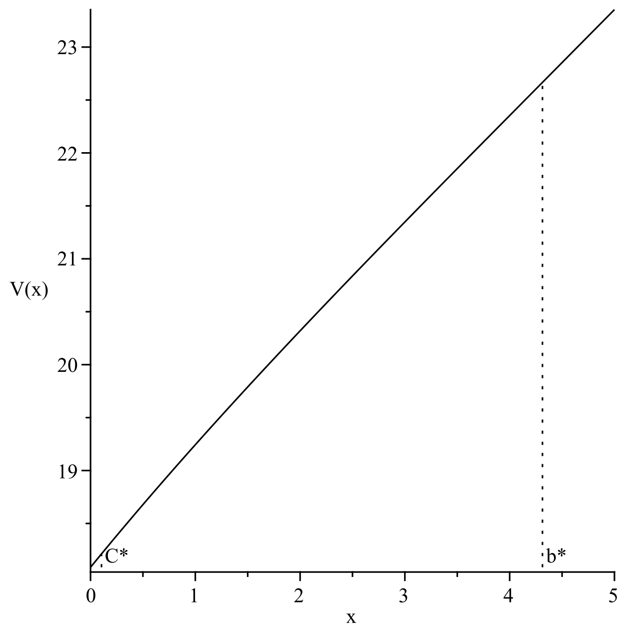

Figure 4.2: $V(x)$ for $Exp(1)$-distributed claim sizes

Gamma-distributed claim sizes

Let now the claim sizes be $\Gamma(2,1)$ distributed with $G(x) = 1-(x+1)e^{-x}$. We first check whether $f(x) = x + (c - \lambda(\phi\mu + L))/\delta$ is the value function. By (4.11), we have to verify the condition

$$\begin{aligned} F(x) &= \lambda G(x)L - \delta x + \lambda(\phi - 1)\int_0^x (y+1)e^{-y}\,\mathrm{d}y \\ &= \lambda e^{-x}\left[(1-\phi-L)x + 2(1-\phi) - L\right] \\ &\quad -\lambda(2(1-\phi)-L) - \delta x \\ &\le 0\,. \end{aligned}$$

The first derivative of $F(x)$ is $F'(x) = \lambda e^{-x}((\phi - 1 + L)x + \phi - 1) - \delta$. The second derivative is $F''(x) = \lambda e^{-x}(L - (\phi - 1 + L)x)$. We see that the function $F(x)$ is first convex and then concave. Since $F(0) = 0, F'(0) = \lambda(\phi - 1) - \delta$ and F is continuous, we can conclude that if $\delta \le \lambda(\phi - 1)$, then the derivative

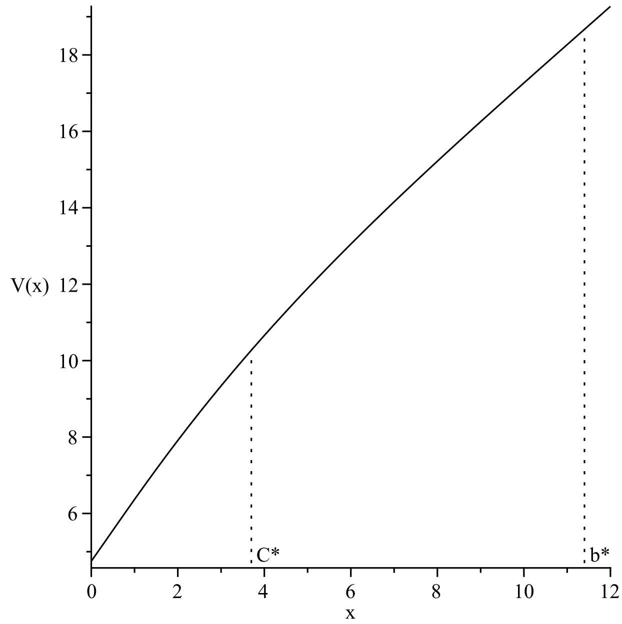

Figure 4.3: $V(x)$ for $\Gamma(2,1)$-distributed claim sizes: barrier strategy

in zero is positive and $F(x) \geq 0$ on some interval $[0, \varepsilon)$. Therefore, $f(x)$ is not the value function on $[0, \infty)$. Moreover, for the value function $V(x)$ we get $V'(0) > 1$. To find the barrier b and the capital injection level C we use the approach of Section 4.3.3.

For a numerical example, let $\lambda = 4, c = 10, \delta = 0.1, \phi = 1.3, L = 2$. Then, for $A^* = 5.463$ we get the (globally) optimal barrier level $b^* = 11.4143$ and the optimal capital level $C^* = 3.7026$. The corresponding value function is illustrated in Figure 4.3.

Chapter 5

Optimal Control of Dividends and Capital Injections in a Markov-modulated Risk Model

5.1 Introduction

In this model, we suppose that the reserve process is influenced by an external environment process $\{J_t\}_{t\geq 0}$ which is a homogeneous irreducible Markov process taking values in a finite state space $\mathscr{J} = \{1, 2, \ldots, m\}$. Denote the intensity matrix of $\{J_t\}_{t\geq 0}$ by

$$\boldsymbol{Q} = (q_{ij})_{i,j=1}^{m}\,, \quad q_{ii} = -q_i = -\sum_{i\neq j} q_{ij}\,, \quad i \in \mathscr{J}\,, \tag{5.1}$$

and its unique stationary probability distribution by $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$. Let $\{W_n\}_{n\geq 0}$ be the nondecreasing sequence of transition times of the environment process $\{J_t\}$, where $W_0 = 0$ and $W_n = \inf\{t > W_{n-1} : J_t \neq J_{t-}\}$. Denote the transition probability matrix of the embedded Markov chain $\{J_{W_n}\}_{n\geq 0}$ by

$$\boldsymbol{P} = (p_{ij})_{i,j=1}^{m}\,, \quad p_{ij} = \begin{cases} 0: & i = j, \\ \frac{q_{ij}}{q_i}: & i \neq j, \end{cases} \quad i, j \in \mathscr{J}\,. \tag{5.2}$$

It holds for all $n \geq 0, s \in \mathbb{R}_+$ and $i \in \mathscr{J}$ that

$$\mathbb{P}[W_{n+1} - W_n \leq s, J_{W_{n+1}} = j | J_{W_n} = i] = (1 - e^{-q_i s}) p_{ij}\,.$$

Assume that, given $J_t = i$, $i \in \mathscr{J}$, premia are paid at rate c_i, claims occur according to a Poisson process $\{N_t^i\}_{t\geq 0}$ with intensity λ_i and the corresponding claim size distribution is G_i with finite mean μ_i. We denote by Y_n

and T_n, respectively, the amount and the arrival time of the n-th claim, with the assumption $Y_0 = T_0 = 0$. The sequences $\{Y_n\}_{n\in\mathbb{N}}$ and $\{T_n - T_{n-1}\}_{n\in\mathbb{N}}$ are assumed to be conditionally independent given $\{J_t\}_{t\geq 0}$.

Denote by $N_t = \sup\{n \in \mathbb{N} : T_n \leq t\}$ the number of claims that have occurred up to time t. The counting process $\{N_t\}_{t\geq 0}$ is a Markov-modulated Poisson process (see Example B.2.2 in the Appendix) and is given by

$$N_t = \sum_{i\in\mathscr{J}} \int_0^t \mathbf{1}_{\{J_s=i\}} \,\mathrm{d}N_s^i \ .$$

Let x be the initial capital, then the corresponding surplus process is given by

$$X_t = x + C_t - \sum_{n=1}^{N_t} Y_n \quad t \geq 0 \ ,$$

where $C_t = \int_0^t c_{J_s} \,\mathrm{d}s = \sum_{i\in\mathscr{J}} \int_0^t \mathbf{1}_{\{J_s=i\}} c_i \,\mathrm{d}s$ is the cumulative premium recieved in $(0, t]$.

We suppose that all random variables and stochastic processes are defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the smallest right continuous filtration generated by $\{(X_t, J_t)\}_{t\geq 0}$. Note that the process $\{(X_t, J_t)\}_{t\geq 0}$ is a homogeneous piecewise deterministic Markov process.

We augment our model by the possibility to pay dividends to the shareholders, if the surplus is positive. Furthermore, if it is negative, the shareholders are expected to make capital injections to keep the process above zero. The accumulated dividends process is an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted, non-decreasing, càdlàg process $\{D_t\}_{t\geq 0}$ with $D_{0-} = 0$. The accumulated capital injections are denoted by $\{Z_t\}_{t\geq 0}$ which is a non-decreasing, adapted pure jump process with $Z_{0-} = 0$. The surplus process then becomes

$$X_t^{(D,Z)} = X_t - D_t + Z_t \ , \qquad X_{0-}^{(D,Z)} = x \ .$$

By the model assumptions, the capital injections have to be chosen such that $X_t^{(D,Z)} \geq 0$ for all $t \geq 0$. Note that no positive safety loading $\sum_{i\in\mathscr{J}} \pi_i(c_i - \lambda_i\mu_i) > 0$ needs to be assumed.

For initial capital x and initial environment state i, we define the value of a strategy $(D, Z) = \{(D_t, Z_t)\}_{t\geq 0}$ as the expected present value of the dividends minus the penalised capital injections, i.e.,

$$V^{(D,Z)}(x, i) = \mathbb{E}_{(x,i)}\Big[\int_{0-}^{\infty} e^{-\delta t} \,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t} \,\mathrm{d}Z_t\Big] \ , \tag{5.3}$$

where $\phi > 1$ is a penalising factor and $\delta > 0$ is a discount factor. We call a strategy (D, Z) *admissible* if

$$\mathbb{P}_{(x,i)}[X_t^{(D,Z)} \geq 0 \text{ for all } t \geq 0] = 1 \quad \text{for all } i \in \mathscr{J} \ .$$

Let $\bar{c} = \max\{c_1, \dots, c_m\}$. Because the value of the (not admissible) strategy $D_t = x + \int_0^t \bar{c}\,\mathrm{d}s$ and $Z_t = 0$ is an upper bound for the value of any admissible strategy we get that $V^{(D,Z)}(x, i) \le x + \bar{c}/\delta < \infty$ for all $i \in \mathscr{J}$ for any strategy (D, Z). We denote by $\mathscr{S}_{(x,i)}$ the set of all admissible strategies for the initial capital x and initial state i.

For all $i \in \mathscr{J}$, we now want to maximise $V^{(D,Z)}(x, i)$ and to find an optimal strategy (D^*, Z^*) – provided that it exists – which replicates the maximal value. Our value functions are

$$V(x, i) = \sup_{(D,Z)\in\mathscr{S}_{(x,i)}} V^{(D,Z)}(x, i)\ , \quad i \in \mathscr{J}\ .$$

By the same arguments as in Chapter 3, we can show that the value function would be infinite for $\phi < 1$ or $\delta = 0$. If $\phi = 1$, the optimal strategy would be a barrier strategy with barrier at zero, i.e., any positive surplus is paid as dividend. The value of such a strategy will be calculated in (5.24).

It is clear that, because of the discounting, it cannot be optimal to make capital injections before it really is necessary. We can therefore assume that $\{Z_t\}_{t\ge0}$ only increases at claim times. Then we only have to choose the dividend process $\{D_t\}_{t\ge0}$. The corresponding capital injections process then becomes

$$Z_t^D = \max(-\inf_{0\le s\le t}(X_s - D_s), 0)\ .$$

Therefore, we will in the following use the abbreviated notation $\{Z_t\}, \{X_t^D\}$ and $V^D(x)$ for the capital injection process $\{Z_t^D\}$, the controlled surplus process and the value connected to a strategy $\{(D_t, Z_t^D)\}$. If the current surplus X_t^D after a claim is negative, then $Z_t^D = |X_t^D|$ and the process starts again at zero, thus

$$V(x, i) = V(0, i) - \phi|x| \qquad \text{for } x < 0\ , i \in \mathscr{J}\ . \tag{5.4}$$

As in Chapter 2, we can show that the value functions are concave. The next lemma holds for both restricted dividend strategies and general càdlàg strategies.

Lemma 5.1.1
For all $i \in \mathscr{J}$, the function $V(x, i)$ is concave.

Proof: Let $x, y > 0$ and $\alpha \in (0, 1)$. Set $z = \alpha x + (1 - \alpha)y$. Consider the strategies (D^x, Z^x) and (D^y, Z^y) for initial capitals x and y. For initial capital z, define

$$D_t^z = \alpha D_t^x + (1 - \alpha)D_t^y, \quad \tilde{Z}_t^z = \alpha Z_t^x + (1 - \alpha)Z_t^y.$$

With $R_t = C_t - \sum_{i=1}^{N_t} Y_i$ we have

$$\begin{aligned} &z + R_t - D_t^z + \tilde{Z}_t^z \\ &\quad = \alpha x + (1-\alpha)y + R_t - \alpha D_t^x - (1-\alpha)D_t^y + \alpha Z_t^x + (1-\alpha)Z_t^y \\ &\quad = \alpha X_t^{D^x} + (1-\alpha)X_t^{D^y} \\ &\quad \geq 0 \,. \end{aligned}$$

This shows that $(D^z, \tilde{Z}^z)$ is an admissible strategy for initial capital z and initial state i. Moreover, $Z_t^{D^z} \leq \tilde{Z}^z = \alpha Z_t^x + \beta Z_t^y$ (otherwise the value of the optimal strategy (D^z, Z^{D^z}) for the capital z were strict smaller than the value of the strategy $(D^z, \tilde{Z}^z)$ for the same initial capital). It follows

$$\begin{aligned} V(z,i) &= \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}\Big(\mathrm{d}D_t^z - \phi\,\mathrm{d}Z_t^{D^z}\Big)\right] \\ &\geq \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}\Big(\mathrm{d}D_t^z - \phi\,\mathrm{d}\tilde{Z}_t^z\Big)\right] \\ &\geq \mathbb{E}\left[\int_{0-}^{\infty} e^{-\delta t}\Big((\alpha\,\mathrm{d}D_t^x + (1-\alpha)\,\mathrm{d}D_t^y) - \phi(\alpha\,\mathrm{d}Z_t^x + (1-\alpha)\,\mathrm{d}Z_t^y)\Big)\right] \\ &= \alpha V^{D^x}(x,i) + (1-\alpha)V^{D^y}(y,i). \end{aligned}$$

Taking the supremum over all admissible strategies D we get

$$\begin{aligned} V(\alpha x + (1-\alpha)y, i) &\geq \alpha \sup_{D^x \in \mathscr{S}_{(x,i)}} V^{D^x}(x,i) + (1-\alpha) \sup_{D^y \in \mathscr{S}_{(y,i)}} V^{D^y}(y,i) \\ &= \alpha V(x,i) + (1-\alpha)V(y,i)\,, \end{aligned}$$

which proves the concavity for all $i \in \mathscr{J}$. □

In particular, because of the concavity of $V(x,i)$, the derivatives from the left and from the right exist a.s. Moreover, $V(x,i)$ is absolutely continuous.

Lemma 5.1.2
The value of expected capital injections is bounded by $\bar{\lambda}\bar{\mu}/\delta$ for $\bar{\lambda} = \max_{i\in\mathscr{J}} \lambda_i$ and $\bar{\mu} = \max_{i\in\mathscr{J}} \mu_i$.

Proof: We consider the worst case that may happen, i.e., *i)* no dividends are paid, and one has to inject capital for all the claims; *ii)* the claims occur according to the maximal intensity $\bar{\lambda} = \max_{i\in\mathscr{J}} \lambda_i$; and *iii)* the claims have the maximal expected height $\bar{\mu} = \max_{i\in\mathscr{J}} \mu_i$. Denote by $\{\bar{N}_t\}$ the Poisson process with intensity $\bar{\lambda}$, and by $\{\bar{T}_n\}_{n\geq 0}$ the claim times according to the process $\{\bar{N}_t\}_{t\geq 0}$, independent of the environment state. Then the time $\bar{T}_n$ is Gamma $\Gamma(\bar{\lambda}, n)$ distributed. We now want to construct a Markov-modulated

Poisson process $\{\hat{N}_t\}_{t\geq 0}$ with the local intensity λ_{J_t}. Let $\{I_n\}_{n\geq 0}$ be a sequence of independent random variables with

$$\mathbb{P}[I_n = 1 | J_{\bar{T}_n} = i] = \frac{\lambda_i}{\bar{\lambda}} = 1 - \mathbb{P}[I_n = 0 | J_{\bar{T}_n} = i].$$

Independently for each $n = 1, 2, \ldots$, we retain $\bar{T}_n$ as a point of $\{\hat{N}_t\}$ with probability $\lambda_{J_{\bar{T}_n}}/\bar{\lambda}$ and delete it otherwise. Then, by Lemma B.1.2, the process

$$\hat{N}_t = \sum_{n=1}^{\bar{N}_t} I_n$$

is a Poisson process with intensity $\{\lambda_{J_t}\}$. Thus, for any $t \geq 0$, N_t and $\hat{N}_t$ have the same distribution. Denote by $\{\hat{T}_n\}_{n\geq 0}$ the claim times of the process $\hat{N}_t$. Then, for every n, T_n and $\hat{T}_n$ have the same distribution, and $\bar{T}_n \leq \hat{T}_n$ a.s. Thus, we have

$$\begin{aligned} \mathbb{E}_{(x,i)}\Big[\sum_{n=1}^{\infty} Y_n e^{-\delta T_n}\Big] &= \mathbb{E}_{(x,i)}\Big[\sum_{n=1}^{\infty} Y_n e^{-\delta \hat{T}_n}\Big] \leq \mathbb{E}_{(x,i)}\Big[\sum_{n=1}^{\infty} Y_n e^{-\delta \bar{T}_n}\Big] \\ &\leq \sum_{n=1}^{\infty} \bar{\mu} \mathbb{E}_{(x,i)}\Big[e^{-\delta \bar{T}_n}\Big] = \sum_{n=1}^{\infty} \bar{\mu} \left(\frac{\bar{\lambda}}{\bar{\lambda}+\delta}\right)^n \\ &= \bar{\mu} \frac{\bar{\lambda}}{\bar{\lambda}+\delta} \frac{1}{1 - \frac{\bar{\lambda}}{\bar{\lambda}+\delta}} = \frac{\bar{\lambda}\bar{\mu}}{\delta} \,. \end{aligned}$$

□

It follows, that the value of any admissible dividend strategy is bounded from below by $-\phi\bar{\lambda}\bar{\mu}/\delta$.

5.2 Strategies With Restricted Densities

In this section, we only consider absolutely continuous dividend strategies with an adapted non-negative density process $\{U_t\}_{t\geq 0}$ such that

- $D_t = \int_0^t U_s \, ds$
- $0 \leq U_t \leq u_0 < \infty$

and denote the strategies by $\{U_t\}$. The value of such a strategy for initial state i is then

$$V^U(x,i) = \mathbb{E}_{(x,i)}\Big[\int_0^\infty e^{-\delta t} U_t \,\mathrm{d}t - \phi \int_{0-}^\infty e^{-\delta t} \,\mathrm{d}Z_t\Big] \,.$$

Let us denote the set of the admissible restricted strategies by $\mathscr{S}^r_{(x,i)}$. Then we have $V(x,i) = \sup_{U\in\mathscr{S}^r_{(x,i)}} V^U(x,i)$. Recall that Lemma 5.1.1 applies.

5.2.1 The Value Function and the HJB-Equation

Lemma 5.2.1
For all $i \in \mathscr{J}$, $V(\cdot, i)$ is bounded, increasing, Lipschitz continuous and therefore absolutely continuous.

Proof: Let $J_0 = i$. It is clear that $V(x,i)$ is increasing in x and $V(x,i) \le \int_0^\infty u_0 e^{-\delta t} dt = u_0/\delta$.

Because we can express $V(x,i) = V(0,i) + \phi x$ for $x < 0$, we can assume that $x \ge 0$. Let $h > 0$ be small. We choose a strategy where no dividend is paid up to time h and then a strategy $\tilde{U} \in \mathscr{S}^r_{(x+c_i h, i)}$ with initial capital $x + c_i h$ is followed, i.e.,

$$
\begin{aligned}
U_t &= 0 \cdot \mathbf{1}_{\{T_1 \wedge W_1 < h\}} + \Big(0 \cdot \mathbf{1}_{\{t \le h\}} + \tilde{U}_{t-h} \cdot \mathbf{1}_{\{t > h\}}\Big) \mathbf{1}_{\{T_1 \wedge W_1 \ge h\}}, \\
Z_t &= Z_t^0 \cdot \mathbf{1}_{\{T_1 \wedge W_1 < h\}} + \Big(0 \cdot \mathbf{1}_{\{t \le h\}} + \tilde{Z}_{t-h} \cdot \mathbf{1}_{\{t > h\}}\Big) \mathbf{1}_{\{T_1 \wedge W_1 \ge h\}}.
\end{aligned}
$$

where $\{Z_t^0\}$ denotes the capital injections if no dividend is paid. Recall from Lemma 5.1.2 that the value connected to Z^0 is bounded by $\bar{\lambda}\bar{\mu}/\delta$. Conditioning on $\mathscr{F}_{h \wedge T_1 \wedge W_1}$, we obtain

$$
\begin{aligned}
V(x,i) &\ge V^U(x,i) \\
&= \mathbb{E}_x\Big[\int_0^{h \wedge T_1 \wedge W_1} e^{-\delta t} U_t \,\mathrm{d}t - \phi \int_0^{h \wedge T_1 \wedge W_1} e^{-\delta t} \,\mathrm{d}Z_t \\
&\qquad\qquad + e^{-\delta(h \wedge T_1 \wedge W_1)} V^{\tilde{U}}(X^U_{h \wedge T_1 \wedge W_1})\Big] \\
&\ge \mathbb{P}[T_1 \wedge W_1 \ge h] e^{-\delta h} V^{\tilde{U}}(x + c_i h, i) \\
&\qquad\qquad - \phi \mathbb{P}[T_1 \wedge W_1 \le h] \mathbb{E}_{X^U_{T_1 \wedge W_1}}\Big[\int_0^\infty e^{-\delta t} \,\mathrm{d}Z_t^0\Big] \\
&\ge e^{-(\lambda_i + q_i + \delta)h} V^{\tilde{U}}(x + c_i h, i) - (1 - e^{-(\lambda_i + q_i)h}) \frac{\phi \bar{\lambda} \bar{\mu}}{\delta}
\end{aligned}
$$

and so

$$
\begin{aligned}
V(x,i) &\ge \sup_{\tilde{U} \in \mathscr{S}^r_{(x+ch,i)}} e^{-(\lambda_i + q_i + \delta)h} V^{\tilde{U}}(x + c_i h, i) - (1 - e^{-(\lambda_i + q_i)h}) \frac{\phi \bar{\lambda} \bar{\mu}}{\delta} \\
&= e^{-(\lambda_i + q_i + \delta)h} V(x + c_i h, i) - (1 - e^{-(\lambda_i + q_i)h}) \frac{\phi \bar{\lambda} \bar{\mu}}{\delta} \,.
\end{aligned}
$$

The Lipschitz-continuity follows now by the boundedness of V

$$\begin{aligned}
0 &\le V(x+c_ih,i)-V(x,i)\\
&\le V(x+c_ih,i)(1-e^{-(\lambda_i+q_i+\delta)h})+(1-e^{-(\lambda_i+q_i)h})\frac{\phi\bar\lambda\bar\mu}{\delta}\\
&\le V(x+c_ih,i)(\lambda_i+q_i+\delta)h+(\lambda_i+q_i)h\frac{\phi\bar\lambda\bar\mu}{\delta}\\
&\le \frac{u_0}{\delta}(\lambda_i+q_i+\delta)h+(\lambda_i+q_i)\frac{\phi\bar\lambda\bar\mu}{\delta}h\;.
\end{aligned}$$

□

By monotonicity and boundedness, the limit $\lim_{x\to\infty} V(x,i) =: V(\infty,i)$ exists for all $i\in\mathscr{J}$. Then

$$V(\infty,i)=\frac{u_0}{q_i+\delta}+\frac{1}{q_i+\delta}\sum_{j\neq i} q_{ij}V(\infty,j) \tag{5.5}$$

holds. To show this, let $x\ge 0$ and consider the strategy

$$U_t=u_0\mathbf{1}_{\{t<\tau_1\wedge W_1\}}+\tilde U_{t-\tau_1\wedge W_1}\mathbf{1}_{\{t\ge\tau_1\wedge W_1\}},\quad Z_t=Z_t^U,$$

where $\tau_1=\tau_1(x,i)=\inf\{t: x+(c_i-u_0)t-\sum_{n=1}^{N_t}Y_n<0\}$ is the first time the process X_t^U falls below zero for initial environment state i and initial capital x. Then $\tau_1(x,i)$ converges to infinity as $x\to\infty$ and so, by bounded convergence, $\mathbb{E}[e^{-\tau_1}]$ converges to zero. Further, $Z_{W_1}\mathbf{1}_{W_1<\tau_1}=0$ and $Z_{\tau_1}\mathbf{1}_{\tau_1<W_1}$ is bounded by Lemma 5.1.2. Conditioning on $\mathscr{F}_{\tau_1\wedge W_1}$, we obtain

$$\begin{aligned}
&V(x,i)\ge V^U(x,i)\\
&\quad=\mathbb{E}\Big[\int_0^{\tau_1\wedge W_1}u_0e^{-\delta t}\,\mathrm{d}t-\phi e^{-\delta(\tau_1\wedge W_1)}Z_{\tau_1\wedge W_1}\\
&\qquad\qquad +e^{-\delta(\tau_1\wedge W_1)}V^{\tilde U}(X^U_{\tau_1\wedge W_1},J_{\tau_1\wedge W_1})\Big]\\
&\quad=\mathbb{E}\Big[\int_{\tau_1}^{\infty}q_ie^{-q_it}\Big(\int_0^{\tau_1}u_0e^{-\delta s}\,\mathrm{d}s-\phi e^{-\delta\tau_1}Z_{\tau_1}+e^{-\delta\tau_1}V^{\tilde U}(X^U_{\tau_1},i)\Big)\,\mathrm{d}t\\
&\qquad+\int_0^{\tau_1}q_ie^{-q_it}\Big(\int_0^t u_0e^{-\delta s}\,\mathrm{d}s+e^{-\delta t}\sum_{j\neq i}p_{ij}V^{\tilde U}(X^U_t,j)\Big)\,\mathrm{d}t\Big]\\
&\quad=\mathbb{E}\Big[e^{-q_i\tau_1}\Big(\frac{u_0}{\delta}(1-e^{-\delta\tau_1})-\phi e^{-\delta\tau_1}Z_{\tau_1}+e^{-\delta\tau_1}V^{\tilde U}(X^U_{\tau_1},i)\Big)\\
&\quad+\frac{u_0}{\delta}\Big(1-e^{-q_i\tau_1}-\frac{q_i(1-e^{-(q_i+\delta)\tau_1})}{q_i+\delta}\Big)+\int_0^{\tau_1}e^{-(q_i+\delta)t}\sum_{j\neq i}q_{ij}V^{\tilde U}(X^U_t,j)\,\mathrm{d}t\Big]\;,
\end{aligned}$$

where we used that $q_{ij} = q_i p_{ij}$. Note that $X_t^U = x + C_t - u_0 t - \sum_{n=1}^{N_t} Y_n + Z_t$ converges to infinity as $x \to \infty$. Now, taking supremum over all admissible strategies $\tilde{U}$ and letting $x \to \infty$ yields

$$\begin{aligned} V(\infty, i) &\geq \frac{u_0}{q_i + \delta} + \int_0^\infty e^{-(q_i+\delta)t} \sum_{j \neq i} q_{ij} V(\infty, j)\, \mathrm{d}t \\ &= \frac{u_0}{q_i + \delta} + \frac{1}{q_i + \delta} \sum_{j \neq i} q_{ij} V(\infty, j)\ . \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} V(x, i) &\leq \mathbb{E}_{(x,i)}\left[\int_0^{W_1} u_0 e^{-\delta t}\, \mathrm{d}t + e^{-\delta W_1} V(X_{W_1}^U, J_{W_1})\right] \\ &= \int_0^\infty q_i e^{-q_i t} \left(\int_0^t u_0 e^{-\delta s}\, \mathrm{d}s + e^{-\delta t} \sum_{j \neq i} p_{ij} V(X_t^U, j)\right) \mathrm{d}t \\ &= \frac{u_0}{q_i + \delta} + \int_0^\infty e^{-(q_i+\delta)t} \sum_{j \neq i} q_{ij} V(X_t^U, j)\, \mathrm{d}t\ , \end{aligned}$$

and, by monotonicity, also

$$V(\infty, i) \leq \frac{u_0}{q_i + \delta} + \frac{1}{q_i + \delta} \sum_{j \neq i} q_{ij} V(\infty, j)\ .$$

Thus, equality holds. Equation (5.5) can be rewritten in matrix form as

$$[\delta \boldsymbol{I} - \boldsymbol{Q}]\boldsymbol{V}(\infty) = u_0 \boldsymbol{e}\ , \tag{5.6}$$

where $\boldsymbol{V}(\infty) = (V(\infty, 1), \ldots, V(\infty, m))^T$, $\boldsymbol{I}$ is the identity matrix on $\mathbb{R}^{m \times m}$, and $\boldsymbol{e} = (1, \ldots, 1)^T$. By Lemma 5.2.2 below, the matrix $\delta \boldsymbol{I} - \boldsymbol{Q}$ is invertible, and

$$\boldsymbol{V}(\infty) = [\delta \boldsymbol{I} - \boldsymbol{Q}]^{-1} u_0 \boldsymbol{e}$$

is the unique solution to (5.6). If we denote $\tilde{\boldsymbol{Q}} = \delta \boldsymbol{I} - \boldsymbol{Q}$ and by $\tilde{\boldsymbol{Q}}^*$ the adjugate matrix (see definition C.1.1 in the Appendix), then we obtain

$$\boldsymbol{V}(\infty) = u_0 \frac{\tilde{\boldsymbol{Q}}^* \boldsymbol{e}}{\det \tilde{\boldsymbol{Q}}}\ ,$$

or

$$V(\infty, i) = \frac{u_0}{\det \tilde{\boldsymbol{Q}}} \sum_{j=1}^m \tilde{q}_{ij}^* \quad \text{for} \quad i \in \mathscr{J}\ .$$

Lemma 5.2.2
The matrix $\delta \boldsymbol{I} - \boldsymbol{Q}$ *is invertible.*

Proof: Let $\{\theta_1, \dots, \theta_n\}$ and $\{e^{\theta_1}, \dots, e^{\theta_n}\}$ be the eigenvalues of the matrices $\boldsymbol{Q}$ and $e^{\boldsymbol{Q}}$, respectively. The matrix $e^{\boldsymbol{Q}}$ is a regular stochastic matrix (see Remark A.3.7). By Perron-Frobenius theorem (see Theorem C.2.3), there is an $i_0 \in \{1, \dots, n\}$ such that $e^{\theta_{i_0}} = 1$ and $|e^{\theta_{i_0}}| < 1$ for $i \neq i_0$. Thus, we have $\theta_{i_0} = 0$ and $\mathrm{Re}(\theta_i) < 0$ for $i \neq i_0$. Since $\delta > 0$, it cannot be an eigenvalue of $\boldsymbol{Q}$. Therefore, $\det(\delta \boldsymbol{I} - \boldsymbol{Q}) \neq 0$ and the matrix is invertible. □

Theorem 5.2.3
For all $i \in \mathscr{J}$, *the function* $V(\cdot, i)$ *is differentiable a.e. on* $(0, \infty)$ *and the derivatives from the right and from the left solve the Hamilton-Jacobi-Bellman equation*

$$\sup_{0 \le u \le u_0} \Big\{ (c_i - u) V'(x,i) + u - (\lambda_i + \delta) V(x,i) \qquad (5.7)$$

$$+ \lambda_i \int_0^\infty V(x-y, i)\, dG_i(y) + \sum_{j \in \mathscr{J}} q_{ij} V(x,j) \Big\} = 0 \,.$$

Proof: Lei $J_0 = i$ be the initial state of the Markov process $\{J_t\}_{t \ge 0}$. Let $h > 0$ and fix $u \in [0, u_0]$. If $x = 0$ we suppose $u \le c_i$, if $x > 0$ we let h be small enough such that $x + (c_i - u)(h \wedge W_1) \ge 0$, i.e., the reserve process does not fall below zero because of the dividend payments. Let $L_i > 0$ be the Lipschitz-constant. Choose $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $L_i(x + (c_i - u)(h \wedge W_1))/n < \varepsilon/2$ and let $x_k = k(x + (c_i - u)(h \wedge W_1))/n$ for $0 \le k \le n$. For every k there is a strategy $\{U_t^k\}$ with $V^{U^k}(x_k, i) > V(x_k, i) - \varepsilon/2$. For initial capital x' with $x_k \le x' < x_{k+1}$, we choose the strategy $\{U_t^k\}$. Then, by the Lipschitz continuity of $V(x,i)$, it holds that

$$\begin{aligned} V^{U^k}(x', i) &\ge V^{U^k}(x_k, i) > V(x_k, i) - \varepsilon/2 > V(x', i) - L_i(x' - x_k) - \varepsilon/2 \\ &> V(x', i) - \varepsilon \,. \end{aligned}$$

Thus, for all $x' \in [0, x + (c_i - u)(h \wedge W_1)]$ we can find a measurable strategy $\tilde{U}$ such that $V^{\tilde{U}}(x', i) > V(x', i) - \varepsilon$.

Consider now a strategy such that within the interval $[0, h \wedge T_1 \wedge W_1]$ dividends at a constant rate u are paid, and thereafter, an ε-optimal strategy $\tilde{U}$ for the initial capital $x + (c_i - u)(h \wedge T_1 \wedge W_1)$ is followed, i.e.,

$$\begin{aligned} U_t &= \begin{cases} u & : \quad 0 \le t < h \wedge T_1 \wedge W_1 \\ \tilde{U}_{t - h \wedge T_1 \wedge W_1} & : \quad t \ge h \wedge T_1 \wedge W_1 \end{cases} , \\ Z_t &= \begin{cases} 0 & : \quad 0 \le t < h \wedge T_1 \wedge W_1 \\ \tilde{Z}_{t - h \wedge T_1 \wedge W_1} & : \quad t \ge h \wedge T_1 \wedge W_1 \end{cases} . \end{aligned}$$

Conditioning on $\mathscr{F}_{h\wedge T_1\wedge W_1}$, we distinguish three cases:

(i) $h < T_1 \wedge W_1$: no claim and no transition of the environment process occur in $[0,h]$,

(ii) $T_1 < h \wedge W_1$: the first claim arrives in $[0,h]$ before the first change of environment, and

(iii) $W_1 < h \wedge T_1$: the first transition of the environment occurs in $[0,h]$ before the arrival of the first claim.

It follows that

$$\begin{aligned}
&V(x,i) \geq V^U(x,i)\\
&\quad = \mathbb{E}_{(x,i)}\Big[\int_0^{h\wedge T_1\wedge W_1} e^{-\delta t}u\,\mathrm{d}t + e^{-\delta(h\wedge T_1\wedge W_1)}V^{\tilde{U}}(X^U_{h\wedge T_1\wedge W_1}, J_{h\wedge T_1\wedge W_1})\Big]\\
&\quad = A_1 + A_2 + A_3
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= \mathbb{E}_{(x,i)}\Big[\mathbf{1}_{h<T_1\wedge W_1}\Big(\int_0^h e^{-\delta t}u\,\mathrm{d}t + e^{-\delta h}V^{\tilde{U}}(X^U_h, i)\Big)\Big]\\
&= \int_h^\infty \lambda_i e^{-\lambda_i t}\int_h^\infty q_i e^{-q_i s}\Big[\int_0^h e^{-\delta v}u\,\mathrm{d}v\\
&\qquad\qquad + e^{-\delta h}V^{\tilde{U}}(x+(c_i-u)h, i)\Big]\,\mathrm{d}s\,\mathrm{d}t\\
&= e^{-(\lambda_i+q_i)h}\frac{1-e^{-\delta h}}{\delta}u + e^{-(\lambda_i+q_i+\delta)h}V^{\tilde{U}}(x+(c_i-u)h, i)\,,\\
A_2 &= \mathbb{E}_{(x,i)}\Big[\mathbf{1}_{T_1<h\wedge W_1}\Big(\int_0^{T_1} e^{-\delta t}u\,\mathrm{d}t + e^{-\delta T_1}V^{\tilde{U}}(X^U_{T_1}, i)\Big)\Big]\\
&= \int_0^h \lambda_i e^{-\lambda_i t}\int_t^\infty q_i e^{-q_i s}\Big[\int_0^t e^{-\delta v}u\,\mathrm{d}v\\
&\qquad + e^{-\delta t}\int_0^\infty V^{\tilde{U}}(x+(c_i-u)t-y, i)\,\mathrm{d}G_i(y)\Big]\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^h \lambda_i e^{-(\lambda_i+q_i)t}u\frac{1-e^{-\delta t}}{\delta}\,\mathrm{d}t\\
&\qquad + \int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)t}\int_0^\infty V^{\tilde{U}}(x+(c_i-u)t-y, i)\,\mathrm{d}G_i(y)\,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
A_3 &= \mathbb{E}_{(x,i)}\Big[\mathbf{1}_{W_1<h\wedge T_1}\Big(\int_0^{W_1} e^{-\delta t}u\,\mathrm{d}t + e^{-\delta W_1}V^{\tilde U}(X^U_{W_1}, J_{W_1})\Big)\Big] \\
&= \int_0^h q_i e^{-q_i s}\int_s^\infty \lambda_i e^{-\lambda_i t}\Big[\int_0^s e^{-\delta v}u\,\mathrm{d}v \\
&\qquad + e^{-\delta s}\sum_{j\neq i} p_{ij}V^{\tilde U}(x+(c_i-u)s, j)\Big]\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^h q_i e^{-(\lambda_i+q_i)s}u\frac{1-e^{-\delta s}}{\delta}\,\mathrm{d}s \\
&\qquad + \int_0^h q_i e^{-(\lambda_i+q_i+\delta)s}\sum_{j\neq i} p_{ij}V^{\tilde U}(x+(c_i-u)s, j)\,\mathrm{d}s\,.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
V(x,i) &\geq A_1 + A_2 + A_3 \\
&\geq e^{-(\lambda_i+q_i)h}\frac{1-e^{-\delta h}}{\delta}u + e^{-(\lambda_i+q_i+\delta)h}V(x+(c_i-u)h, i) \\
&\quad + \int_0^h (\lambda_i+q_i)e^{-(\lambda_i+q_i)t}u\frac{1-e^{-\delta t}}{\delta}\,\mathrm{d}t \\
&\quad + \int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)t}\int_0^\infty V(x+(c_i-u)t-y, i)\,\mathrm{d}G_i(y)\,\mathrm{d}t \\
&\quad + \int_0^h q_i e^{-(\lambda_i+q_i+\delta)s}\sum_{j\neq i} p_{ij}V(x+(c_i-u)s, j)\,\mathrm{d}s - \varepsilon\,.
\end{aligned}$$

The constant ε is arbitrary. If we let tend it to zero, rearrange the terms and divide them by h, then we get

$$\begin{aligned}
0 \geq\; & \frac{V(x+(c_i-u)h, i) - V(x,i)}{h} - \frac{1-e^{-(\lambda_i+q_i+\delta)h}}{h}V(x+(c_i-u)h, i) \\
& + e^{-(\lambda_i+q_i)h}\frac{1-e^{-\delta h}}{\delta h}u + \frac{1}{h}\int_0^h (\lambda_i+q_i)e^{-(\lambda_i+q_i)t}u\frac{1-e^{-\delta t}}{\delta}\,\mathrm{d}t \\
& + \frac{1}{h}\int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)t}\int_0^\infty V(x+(c_i-u)t-y, i)\,\mathrm{d}G_i(y)\,\mathrm{d}t \\
& + \frac{1}{h}\int_0^h q_i e^{-(\lambda_i+q_i+\delta)s}\sum_{j\neq i} p_{ij}V(x+(c_i-u)s, j)\,\mathrm{d}s
\end{aligned} \tag{5.8}$$

If $c_i \geq u$ the first term converges to the derivative from the right as $h \to 0$, if $c_i \leq u$ to the derivative from the left (the existence of the derivatives from the left and from the right is assured by Lemma 5.1.1). Starting with initial capital $x-(c_i-u)h$, we get in the same way that the first term converges to the derivative from the left in the case $c_i \ge u$ and to the derivative from the right in the case $c_i \le u$. We do not distinguish the notation first and, using that $q_{ij} = q_i p_{ij}$ and $-q_i = q_{ii}$, get for both derivatives

$$(c_i - u)V'(x,i) + u - (\lambda_i+\delta)V(x,i) + \lambda_i \int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) \\ + \sum_{j\in\mathscr{J}} q_{ij}V(x,j) \le 0\ ,$$

as $h \to 0$, where we have

$$\begin{aligned}
&\int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) \\
&\quad= \int_0^x V(x-y,i)\,\mathrm{d}G_i(y) + \int_x^\infty (V(0,i)-\phi(y-x)\,\mathrm{d}G_i(y) \qquad (5.9)\\
&\quad= \int_0^x V(x-y,i)\,\mathrm{d}G_i(y) + V(0,i)(1-G_i(x)) - \phi\int_x^\infty (1-G_i(y))\,\mathrm{d}y
\end{aligned}$$

because of the property (5.4).

Now choose a strategy $\hat{U}(h)$ in $\mathscr{S}_x^r$ such that $V^{\hat{U}(h)}(x,i) \ge V(x,i) - h^2$. Denote $a(t) = \int_0^t (c_i - \hat{u}_s(h))\,\mathrm{d}s$, where $\hat{u}_t(h)$ denotes $\hat{U}(h)$, if $T_1 \wedge W_1 > t$. In the same way as above we get

$$\begin{aligned}
0 \;\le\; & \frac{V(x+a(h),i)-V(x,i)}{h} - \frac{1-e^{-(\lambda_i+q_i+\delta)h}}{h}V(x+a(h),i) \\
& + e^{-(\lambda_i+q_i)h}\frac{1}{h}\int_0^h a(v)\,\mathrm{d}v + \frac{1}{h}\int_0^h (\lambda_i+q_i)e^{-(\lambda_i+q_i)t}\int_0^t a(v)e^{-\delta v}\,\mathrm{d}v\,\mathrm{d}t \\
& + \frac{1}{h}\int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)t}\int_0^\infty V(x+a(t)-y,i)\,\mathrm{d}G_i(y)\,\mathrm{d}t \\
& + \frac{1}{h}\int_0^h q_i e^{-(\lambda_i+q_i+\delta)s}\sum_{j\ne i} p_{ij}V(x+a(s),j)\,\mathrm{d}s + h\ .
\end{aligned}$$

W.l.o.g., let $\{h_n\}_{n\ge 0}$ be a sequence tending to zero such that $\lim_{n\to\infty} a(h_n)/h_n = c_i - \tilde{u}$. Then, the limit $\lim_{n\to\infty}(V(x+a(h_n),i) - V(x,i))/h_n$ exists because of the concavity of V. Letting $h_n \to 0$ yields

$$(c_i - \tilde{u})V'(x,i) + \tilde{u} - (\lambda_i+\delta)V(x,i) + \lambda_i \int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) \\ + \sum_{j\in J} q_{ij}V(x,j) \ge 0\ .$$

If $c_i \ge \tilde{u}$, the inequality holds for the derivative from the right, if $c_i \le \tilde{u}$, for the derivative from the left, where the value of the derivative can be chosen arbitrarily if $c_i = \tilde{u}$. Starting with initial capital $x - (c_i - u)h$, we get in the same way that the inequality holds for the derivative from the left in the case $c_i \ge \tilde{u}$ and for the derivative from the right in the case $c_i \le \tilde{u}$. Since $u = \tilde{u}$ fulfils (5.8), we conclude that equality holds. Now the supremum can be taken over all constant strategies $U = u$ with $0 \le u \le u_0$. Thus, both derivatives solve Equation (5.7). □

5.2.2 The Optimal Strategy and the Characterisation of the Solution

First we show that $V(\cdot, i)$ is continuously differentiable. Since $V(\cdot, i)$ is increasing, concave, and bounded, there exists a $b_i := \inf\{x : V'(x, i) \le 1\}$. Equation (5.7) is linear in u, thus, the argument maximising the left-hand side of (5.7) can be determined in dependence on the derivative (from the left or from the right) as

$$u(x,i) = \begin{cases} 0 & : \quad x < b_i \quad (\Leftrightarrow V'(x,i) > 1) \\ \in [0, u_0] & : \quad x = b_i \quad (\Leftrightarrow V'(x,i) = 1) \\ u_0 & : \quad x > b_i \quad (\Leftrightarrow V'(x,i) < 1) \end{cases} .$$

By the concavity of $V(\cdot, i)$, we have $V'(x-, i) \ge V'(x+, i)$. If both $V'(x+, i)$ and $V'(x-, i)$ are greater than 1 or both are smaller than 1, then differentiability follows from Equation (5.7). By our assumption that G_i is continuous, we have that $V'(x, i)$ is continuously differentiable on $[0, b_i)$ and (b_i, ∞). Now suppose that $b_i > 0$. At $x = b_i$, considering the equation from the left and from the right, we obtain that $c_i V'(b_i-, i) = u_0 + (c_i - u_0)V'(b_i+, i)$, or

$$c_i(V'(b_i-, i) - V'(b_i+, i)) = u_0(1 - V'(b_i+, i)) .$$

If $u_0 < c_i$, we conclude that either $V'(b_i-, i) = V'(b_i+, i) = 1$ or $1 > V'(b_i-, i)$. Because the latter is impossible, we find that $V(b_i, i)$ is continuously differentiable. If $u_0 \ge c_i$, dividends are paid according to a barrier strategy with barrier b_i keeping the surplus process at b_i until time $T_1 \wedge W_1$. Because the process does not leave the intervall $[0, b_i]$ and the corresponding strategy is admissible for any $u_0 \ge c_i$, it must be optimal for any initial value in $[0, b_i]$. Thus, b_i does not depend on the bound u_0. The expected discounted dividends until time $T_1 \wedge W_1$ are

$$\begin{aligned}&\int_0^\infty \lambda_i e^{-\lambda_i t} \int_0^t q_i e^{-q_i s} \int_0^s c_i e^{-\delta v}\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t\\ &\quad + \int_0^\infty q_i e^{-q_i t} \int_0^t \lambda_i e^{-\lambda_i t} \int_0^s c_i e^{-\delta v}\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t\\ &\quad = \frac{c_i}{\lambda_i + q_i + \delta}\,.\end{aligned}$$

For the time after $T_1 \wedge W_1$, we obtain as in the proof of Theorem 5.2.3 that

$$\begin{aligned}&\int_0^\infty q_i e^{-q_i s} \int_0^s \lambda_i e^{-\lambda_i t} e^{-\delta t} \int_0^\infty V(b_i - y, i)\,\mathrm{d}G_i(y)\,\mathrm{d}t\,\mathrm{d}s\\ &\quad + \int_0^\infty \lambda_i e^{-\lambda_i t} \int_0^t q_i e^{-q_i s} e^{-\delta s} \sum_{j\neq i} p_{ij} V(b_i, j)\,\mathrm{d}s\,\mathrm{d}t\\ &\quad = \frac{\lambda_i}{\lambda_i + q_i + \delta} \int_0^\infty V(b_i - y, i)\,\mathrm{d}G_i(y) + \frac{1}{\lambda_i + q_i + \delta} \sum_{j\neq i} q_{ij} V(b_i, j)\,.\end{aligned}$$

Thus, we can see that $V(b_i, i)$ can be characterised through

$$V(b_i, i) = \frac{c_i}{\lambda_i + q_i + \delta} + \frac{1}{\lambda_i + q_i + \delta}\Big\{\lambda_i \int_0^\infty V(b_i - y, i)\,\mathrm{d}G_i(y) + \sum_{j\neq i} q_{ij} V(b_i, j)\Big\}.$$

Plugging $V(b_i, i)$ into (5.7) shows that $V'(b_i-, i) = V'(b_i+, i) = 1$, i.e., $V(\cdot, i)$ is continuously differentiable in the case $u_0 \ge c_i$, also. With the same arguments as in Remark 3.2.3, we can show that $b_i = \inf\{x \ge 0 : V'(x, i) = 1\} < \infty$.

Let now

$$u(x, i) = \begin{cases} 0 & : \quad x < b_i \quad (\Leftrightarrow V'(x, i) > 1)\\ \min\{c_i, u_0\} & : \quad x = b_i \quad (\Leftrightarrow V'(x, i) = 1)\\ u_0 & : \quad x > b_i \quad (\Leftrightarrow V'(x, i) < 1)\end{cases}\,. \tag{5.10}$$

The surplus process controlled by the strategy $\{U_t\} = \{u(X_t^U, J_t)\}$ is then

$$\begin{aligned} X_t^U &= x + \int_0^t (c_{J_s} - U_s)\,\mathrm{d}s - \sum_{n=1}^{N_t} Y_n + Z_t\\ &= x + \int_0^t (c_{J_s} \mathbf{1}_{\{X_{s-}^U < b_{J_s}\}} + (c_{J_s} - u_0)\mathbf{1}_{\{X_{s-}^U \ge b_{J_s}\}})\,\mathrm{d}s - \sum_{n=1}^{N_t} Y_n + Z_t\,.\end{aligned}$$

Before proving the next lemma we make some observations. Between two changes of the environment the following holds:

- In an interval (T_{n-1}, T_n) between two claims the risk process X_t^U is governed by the differential equation $\mathrm{d}X_t^U = (c_{J_t} - u_{J_t})\,\mathrm{d}t$.

- $X_{T_n}^U = X_{T_n-}^U - Y_n + Z_{T_n}$.

- $X_{T_n-}^U \geq X_{T_n}^U$.

- If $X_{T_n-}^U - Y_n \leq 0$, then the shareholders pay, idependently of the state J_{T_n}, as much that $X_{T_n}^U = X_{T_n-}^U - Y_n + \Delta Z_{T_n} = 0$. In this case, because of the property (5.4), the value function fulfils

  $$V(X_{T_n}^U, i)(= V(0,i)) = V(X_{T_n-}^U - Y_n, i) + \phi \Delta Z_{T_n}\ .$$

  Thus, it suffices to only consider solutions f to the HJB equation with the property (5.4).

Lemma 5.2.4
For $i \in \mathscr{J}$, let $f(x,i)$ be an increasing, bounded, and positive solution to (5.7) *with the property* (5.4)*. Then for any admissible strategy U the process*

$$\begin{aligned}\Big\{ f(X_t^U, J_t)e^{-\delta t} - f(X_0^U, J_0) - \phi \int_0^t e^{-\delta s}\, dZ_s \\ - \int_0^t \Big[(c_{J_s} - U_s) f'(X_s^U, J_s) - (\lambda_{J_s} + \delta) f(X_s^U, J_s) \\ + \lambda_{J_s} \int_0^\infty f(X_s^U - y, J_s)\, dG_{J_s}(y) + \sum_{j \in \mathscr{J}} q_{J_s j} f(X_s^U, j) \Big] e^{-\delta s}\, ds \Big\}\end{aligned}$$

is an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale.

Proof: Let S_t be the number of the jumps of the environment process $\{J_t\}_{t\geq 0}$ in the interval $[0,t]$, i.e. $S_t = \sup\{k : W_k \leq t\}$. Note that $S_t < \infty$ and $S_0 = 0$.

Then we have the decomposition

$$
\begin{aligned}
& f(X_t^U, J_t)e^{-\delta t} - f(X_0^U, J_0) \\
&\quad = \sum_{k=1}^{S_t} \Big[f(X_{W_k}^U, J_{W_k})e^{-\delta W_k} - f(X_{W_{k-1}}^U, J_{W_{k-1}})e^{-\delta W_{k-1}} \Big] \\
&\qquad + f(X_t^U, J_t)e^{-\delta t} - f(X_{W_{S_t}}^U, J_{W_{S_t}})e^{-\delta W_{S_t}} \\
&\quad = \sum_{k=1}^{S_t} \Big[f(X_{W_k}^U, J_{W_k})e^{-\delta W_k} - f(X_{W_k-}^U, J_{W_k-})e^{-\delta W_k-} \Big] \\
&\qquad + \sum_{k=1}^{S_t} \Big[f(X_{W_k-}^U, J_{W_k-})e^{-\delta W_k-} - f(X_{W_{k-1}}^U, J_{W_{k-1}})e^{-\delta W_{k-1}} \Big] \\
&\qquad + f(X_t^U, J_t)e^{-\delta t} - f(X_{W_{S_t}}^U, J_{W_{S_t}})e^{-\delta W_{S_t}} \\
&\quad = \sum_{0<s\le t: J_s \neq J_{s-}} \Big[f(X_s^U, J_s) - f(X_{s-}^U, J_{s-}) \Big] e^{-\delta s} \\
&\qquad + \sum_{k=1}^{S_t} \Big[f(X_{W_k-}^U, J_{W_{k-1}})e^{-\delta W_k-} - f(X_{W_{k-1}}^U, J_{W_{k-1}})e^{-\delta W_{k-1}} \Big] \\
&\qquad + f(X_t^U, J_t)e^{-\delta t} - f(X_{W_{S_t}}^U, J_t)e^{-\delta W_{S_t}} \;,
\end{aligned}
\tag{5.11}
$$

where we used that $J_{W_k-} = J_{W_{k-1}}$ for $k = 1 \ldots S_t$ and $J_t = J_{W_{S_t}}$.

First we consider the behavior of the risk process between two changes of the environment in $[W_{k-1}, W_k), k = 1, \ldots, S_t$. In the same way as in the proof of Lemma 3.2.4 we can show that the processes

$$
\begin{aligned}
M_t^k := {} & f(X_{W_k-\wedge t}^U, J_{W_k-\wedge t})e^{-\delta(W_k-\wedge t)} - f(X_{W_{k-1}\wedge t}^U, J_{W_{k-1}\wedge t})e^{-\delta(W_{k-1}\wedge t)} \\
& - \phi \sum_{(W_{k-1}\wedge t)<s\le(W_k-\wedge t)} e^{-\delta s} \mathbf{1}_{\{J_s = J_{W_{k-1}\wedge t}\}} \Delta Z_s \\
& - \int_{W_{k-1}\wedge t}^{W_k-\wedge t} e^{-\delta s} \Big[(c_{J_s} - U_s) f'(X_s^U, J_s) + \lambda_{J_s} \int_0^\infty f(X_s^U - y, J_s)\, \mathrm{d}G_{J_s}(y) \\
& - (\lambda_{J_s} + \delta) f(X_s^U, J_s) \Big]\, \mathrm{d}s
\end{aligned}
\tag{5.12}
$$

and

$$
\begin{aligned}
M'_t := {} & f(X_t^U, J_t) e^{-\delta t} - f(X_{W_{S_t}}^U, J_{W_{S_t}}) e^{-\delta W_{S_t}} \\
& - \phi \sum_{W_{S_t} < s \le t} e^{-\delta s} \mathbf{1}_{J_s = J_{W_{S_t} \wedge t}} \Delta Z_s \\
& - \int_{W_{S_t}}^t e^{-\delta s} \Big[(c_{J_s} - U_s) f'(X_s^U, J_s) + \lambda_{J_s} \int_0^\infty f(X_s^U - y, J_s) \, \mathrm{d}G_{J_s}(y) \\
& - (\lambda_{J_s} + \delta) f(X_s^U, J_s) \Big] \, \mathrm{d}s \, .
\end{aligned} \tag{5.13}
$$

are $\{\mathscr{F}_t\}_{t \ge 0}$-martingales.

Next we consider the changes of the environment. Since the dividend strategies are absolutely continuous, there are no jumps of the risk process because of the environmental change. Further, $f(X_s^U, J_s) = f(X_{s-}^U, J_s)$. We follow from Theorem B.2.4 that the process

$$
\begin{aligned}
M''_t := {} & \sum_{0 < s \le t : J_s \ne J_{s-}} \Big[f(X_{s-}^U, J_s) - f(X_{s-}^U, J_{s-}) \Big] e^{-\delta s} \\
& - \int_0^t \sum_{j \in \mathscr{J} : j \ne J_s} \Big[f(X_s^U, j) - f(X_s^U, J_s) \Big] e^{-\delta s} q_{J_s j} \, \mathrm{d}s \\
= {} & \sum_{0 < s \le t : J_s \ne J_{s-}} \Big[f(X_{s-}^U, J_s) - f(X_{s-}^U, J_{s-}) \Big] e^{-\delta s} \\
& - \int_0^t \sum_{j \in \mathscr{J}} f(X_s^U, j) e^{-\delta s} q_{J_s j} \, \mathrm{d}s
\end{aligned} \tag{5.14}
$$

is a martingale, where we used that $\sum_{j \in \mathscr{J} : j \ne i} q_{ij} = -q_{ii}$ for all $i \in \mathscr{J}$.

Now combining (5.11), (5.12), (5.13) and (5.14) yields that

$$
\begin{aligned}
& \sum_{k=1}^{S_t} M_t^k + M'_t + M''_t \\
& \quad = f(X_t^U, J_t) e^{-\delta t} - f(X_0^U, J_0) - \phi \int_0^t e^{-\delta s} \, \mathrm{d}Z_s \\
& \quad - \int_0^t \Big[(c_{J_s} - U_s) f'(X_s^U, J_s) - (\lambda_{J_s} + \delta) f(X_s^U, J_s) \\
& \qquad + \lambda_{J_s} \int_0^\infty f(X_s^U - y, J_s) \, \mathrm{d}G_{J_s}(y) + \sum_{j \in \mathscr{J}} q_{J_s j} f(X_s^U, j) \Big] e^{-\delta s} \, \mathrm{d}s
\end{aligned}
$$

is an $\{\mathscr{F}_t\}_{t \ge 0}$-martingale with expected value 0. $\square$

We show now that the value function is unique and the strategy (5.10) is optimal.

Theorem 5.2.5
For $i \in \mathscr{J}$, let $f(x,i)$ be an increasing, positive, and bounded solution to (5.7) *with the property* (5.4)*. Then $f(x,i) = V(x,i)$, and an optimal strategy is given by* (5.10)*.*

Proof: Since $f(x,i)$ is bounded, f must converge to a $f(\infty,i) < \infty$. Then there exists a sequence $x_n \to \infty$ such that $f'(x_n,i) \to 0$. Let $u_n = u(x_n,i)$. By the definition (5.10) we can assume $u_n = u_0$. Letting $n \to \infty$ in (5.7) yields

$$\begin{aligned} 0 &= (c_i - u_0)f'(x_n,i) + \lambda\Big[\int_0^\infty f(x_n-y,i)\,\mathrm{d}G_i(y) - f(x_n,i)\Big] \\ &\quad + \sum_{j\in\mathscr{J}:j\neq i} q_{ij}f(x_n,j) + (q_{ii}-\delta)f(x_n,i) + u_0 \\ &\overset{n\to\infty}{\longrightarrow} \sum_{j\in\mathscr{J}:j\neq i} q_{ij}f(\infty,j) + (q_{ii}-\delta)f(\infty,i) + u_0\,. \end{aligned}$$

Denote $\boldsymbol{f}(\infty) = (f(\infty,1),\dots,f(\infty,m))^T$. Then we have that $\boldsymbol{f}(\infty)$ is a solution to

$$(\delta\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}(\infty) = u_0\boldsymbol{e},$$

(compare with (5.6)).

Denote by U^* the strategy given by (5.10) and the corresponding $Z^* = Z^{U^*}$. It follows from Lemma 5.2.4 and the HJB equation (5.7) that

$$\Big\{f(X_t^{U^*},J_t)e^{-\delta t} - f(x,i) + \int_0^t e^{-\delta s}U_s^*\,\mathrm{d}s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^*\Big\}$$

is a martingale with expected value 0. Then

$$f(x,i) = \mathbb{E}_{(x,i)}\Big[f(X_t^{U^*},J_t)e^{-\delta t} + \int_0^t e^{-\delta s}U_s^*\,\mathrm{d}s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s^*\Big]$$

holds. By the boundedness of f and the bounded convergence theorem, we get $\mathbb{E}_{(x,i)}[f(X_t^{U^*},J_t)e^{-\delta t}] \to 0$ as $t \to \infty$. Since the other terms are monotone, we can interchange limit and integration and get $f(x,i) = V^{U^*}(x,i)$. For an arbitrary strategy U, Equation (5.7) gives

$$\begin{aligned} f(x,i) &\geq \mathbb{E}_{(x,i)}\Big[f(X_t^U,J_t)e^{-\delta t} + \int_0^t e^{-\delta s}U_s\,\mathrm{d}s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s\Big] \\ &\geq \mathbb{E}_{(x,i)}\Big[\int_0^t e^{-\delta s}U_s\,\mathrm{d}s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z_s\Big]\,. \end{aligned}$$

Letting $t \to \infty$ shows $f(x,i) \geq V^U(x,i)$. Thus $f(x,i) = V(x,i)$. □

This shows that, if the environment process at time t is at state i, then it is optimal to pay no dividends while the reserve process stays below barrier b_i ($X_t^U < b_i$). As soon as the process reaches or excesses the barrier b_i ($X_t^U \geq b_i$), dividends have to be payed at the maximal rate u_0.

5.3 Unrestricted Dividends

In this section all increasing, adapted, and càdlàg processes D are allowed. The value of a strategy D is

$$V^D(x,i) = \mathbb{E}_{(x,i)}\Big[\int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}D_t - \phi \int_{0-}^{\infty} e^{-\delta t}\,\mathrm{d}Z_t\Big] .$$

and $V(x,i) = \sup_{D\in\mathscr{S}_{(x,i)}} V^D(x,i)$ is the value function for the initial state $J_0 = i$.

5.3.1 The Value Function and the HJB Equation

Lemma 5.3.1
For all $i \in \mathscr{J}$, the function $V(\cdot,i)$ is increasing with $V(x,i) - V(y,i) \geq x - y$ for $0 \leq y \leq x$ and Lipschitz continuous on $[0,\infty)$. For any $x \geq 0$,

$$x + \frac{c_i - \phi\lambda_i\mu_i}{\delta + q_i} \leq V(x,i) \leq x + \frac{\overline{c}}{\delta}$$

with $\overline{c} = \max_{i\in\mathscr{J}} c_i$. The function $V(\cdot,i)$ is almost everywhere differentiable with $V'(x-,i) \geq 1$ and $V'(x+,i) \leq \phi$.

Proof: For initial capital y consider a strategy D with $V^D(y,i) \geq V(y,i) - \varepsilon$ for a $\varepsilon > 0$. For $x \geq y$ define a strategy where $x - y$ is paid immediately as dividend, and then the strategy D is followed. Then it holds for any $\varepsilon > 0$

$$V(x,i) \geq x - y + V^D(y,i) \geq x - y + V(y,i) - \varepsilon .$$

Because ε was arbitrary, $V(x,i) - V(y,i) \geq x - y$ follows. In particular, $V(\cdot,i)$ is strictly increasing.

Consider a strategy D such that initial capital is paid immediately and after that the dividends at rate c_i are paid until the first transition of the environment state. Conditioning on $\mathscr{F}_{W_1}$, we have

$$\begin{aligned} V(x,i) &\geq V^D(x,i) \\ &\geq x + \mathbb{E}\Big[\int_0^{W_1} c_i e^{-\delta t}\,\mathrm{d}t - \phi \int_0^{W_1} e^{-\delta t}\,\mathrm{d}Z_t + e^{-\delta W_1} V^D(X^D_{W_1}, J_{W_1})\Big] \\ &\geq x + \mathbb{E}\Big[\int_0^{W_1} c_i e^{-\delta t}\,\mathrm{d}t - \phi \int_0^{W_1} e^{-\delta t}\,\mathrm{d}Z_t\Big]\,. \end{aligned}$$

The value of the dividends is

$$\mathbb{E}\Big[\int_0^{W_1} c_i e^{-\delta t}\,\mathrm{d}t\Big] = \int_0^\infty q_i e^{-q_i s} \int_0^s e^{-\delta t} c_i\,\mathrm{d}t\,\mathrm{d}s = \frac{c_i}{\delta + q_i}\,.$$

By Fubini's theorem, we get for the value of the capital injections

$$\begin{aligned} \mathbb{E}\Big[\int_0^{W_1} e^{-\delta t}\,\mathrm{d}Z_t\Big] &= \mathbb{E}\Big[\int_0^\infty q_i e^{-q_i s} \int_0^s e^{-\delta t}\,\mathrm{d}Z_t\,\mathrm{d}s\Big] \\ &= \mathbb{E}\Big[\int_0^\infty \int_t^\infty q_i e^{-q_i s}\,\mathrm{d}s\; e^{-\delta t}\,\mathrm{d}Z_t\Big] = \mathbb{E}\Big[\int_0^\infty e^{-(\delta+q_i)t}\,\mathrm{d}Z_t\Big] \\ &\leq \mathbb{E}\Big[\sum_{n=1}^\infty e^{-(\delta+q_i)T_n} Y_{T_n}\Big] = \sum_{n=1}^\infty \mu_i \Big(\frac{\lambda_i}{\lambda_i + \delta + q_i}\Big)^n = \frac{\lambda_i \mu_i}{\delta + q_i}\,, \end{aligned}$$

where we applied the same arguments as in the proof of Lemma 3.1.2. Thus, we obtain the lower bound

$$V(x,i) \geq V^D(x,i) \geq x + \frac{c_i - \phi\lambda_i\mu_i}{\delta + q_i}\,.$$

Consider now the strategy D where x is paid immediately and then the dividends are paid at rate $\overline{c}$. We note that for any reasonable strategy, D_t is an upper bound for the accumulated dividend payments. Not taking the capital injections into account yields the upper bound

$$V^D(x,i) \leq x + \mathbb{E}\Big[\int_0^\infty e^{-\delta t}\overline{c}\,\mathrm{d}t\Big] = x + \frac{\overline{c}}{\delta}\,.$$

The local Lipschitz-continuity follows by the local boundedness of $V(\cdot,i)$ as in the proof of Lemma 5.2.1.

From Lemma 5.1.1 we know that $V(\cdot,i)$ is concave, and therefore, $V(\cdot,i)$ is differentiable almost everywhere on $[0,\infty)$. At the points where $V(\cdot,i)$ is not differentiable, the derivatives from the left and from the right exist. By $V(x,i) - V(x-\varepsilon,i) \geq \varepsilon$ for $\varepsilon \geq 0$ we obtain $V'(x-,i) \geq 1$. For the other inequality we consider a strategy by receiving ε from the shareholders immediately and following the strategy for the initial capital $x+\varepsilon$ afterwards, so that $V(x,i) \geq V(x+\varepsilon,i) - \phi\varepsilon$. Hence $V'(x+,i) \leq \phi$. This proves the global Lipschitz-continuity. Moreover, it follows that $V(\cdot,i)$ is absolutely continuous. □

Remark 5.3.2
It is possible to calculate a finer upper bound for the value functions. Consider a strategy such that x is paid immediately and then dividends at premium rate c_{J_t} are paid, i.e., $D_t = x + \int_0^t c_{J_s}\,\mathrm{d}s$. Then the value of the dividends is

$$\begin{aligned} x + \mathbb{E}\Big[\int_0^\infty e^{-\delta t} c_{J_t}\,\mathrm{d}t\Big] &= x + \int_0^\infty e^{-\delta t} \sum_{j=1}^m c_j \mathbb{P}[J_t = j | J_0 = i]\,\mathrm{d}t \\ &= x + \int_0^\infty e^{-\delta t} (e^{\boldsymbol{Q}t})_i \boldsymbol{c}\,\mathrm{d}t = x + \Big(\int_0^\infty e^{(-\delta \boldsymbol{I} + \boldsymbol{Q})t}\,\mathrm{d}t\Big)_i \boldsymbol{c}, \end{aligned}$$

where $(e^{\boldsymbol{Q}t})_i$ denotes the i-th row of the matix $e^{\boldsymbol{Q}t}$ and $\boldsymbol{c} = (c_1, \cdots, c_m)^T$. From Lemma 5.2.2, we know that the matrix $-\delta\boldsymbol{I} + \boldsymbol{Q}$ is invertible and for all eigenvalues holds $\mathrm{Re}(-\delta + \theta_i) < 0$. By Lemma C.3.3, it follows

$$V(x,i) \leq x + [\delta\boldsymbol{I} - \boldsymbol{Q}]_i^{-1}\boldsymbol{c}.$$

■

As in the case $m = 1$, the value function can be calculated as the limit of the value functions from the previous section. The proof is analogous to the proof of Lemma 3.3.2.

Lemma 5.3.3
For $i \in \mathscr{J}$, let $V_u(\cdot,i)$ be the value function for the restricted dividend strategy in the case $u_0 = u$. Then $\lim_{u\to\infty} V_u(x,i) = V(x,i)$ for all $x \geq 0$.

Now we want to determine the Hamilton-Jacobi-Bellman equation for this problem. We repeat the procedure in Schmidli [66].

Theorem 5.3.4
For all $i \in \mathscr{J}$, the function $V(\cdot,i)$ is continuously differentiable on $(0,\infty)$ and fulfils the equation

$$\begin{aligned} \max\Big\{c_i V'(x,i) + \lambda_i \int_0^\infty V(x-y,i)\,dG_i(y) - (\lambda_i+\delta)V(x,i) \\ + \sum_{j\in\mathscr{J}} q_{ij} V(x,j), 1 - V'(x,i), V'(x,i) - \phi\Big\} = 0. \end{aligned} \tag{5.15}$$

Proof: Since we let $u \to \infty$, it is enough to consider the case $u > c_i$. Equation (5.7) can be written as

$$\begin{aligned} 0 = \max\Big\{ & c_i V_u'(x,i) - (\lambda_i + \delta) V_u(x,i) \\ & + \lambda_i \int_0^\infty V_u(x-y,i)\,\mathrm{d}G_i(y) + \sum_{j\in\mathscr{J}} q_{ij} V_u(x,j)\;, 1 - V_u'(x,i) \\ & + \frac{\sum_{j\in\mathscr{J}} q_{ij} V_u(x,j) + \lambda_i \int_0^\infty V_u(x-y,i)\,\mathrm{d}G_i(y) - (\lambda_i+\delta)V_u(x,i) + c_i}{u - c_i} \Big\}\,. \end{aligned} \tag{5.16}$$

The two parts correspond to the different cases $u = 0$ and $u = u_0$ for the restricted problem (see Equation (5.7)), where the second equation is divided by $u - c_i$. We already know that $\lim_{u\to\infty} V_u(x,i) = V(x,i)$. Hence, by the bounded convergence theorem,

$$\begin{aligned} \lim_{u\to\infty} & (\lambda_i+\delta)V_u(x,i) - \lambda_i \int_0^\infty V_u(x-y,i)\,\mathrm{d}G_i(y) - \sum_{j\in\mathscr{J}} q_{ij} V_u(x,j) \\ &= (\lambda_i+\delta)V(x,i) - \lambda_i \int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) - \sum_{j\in\mathscr{J}} q_{ij} V(x,j). \end{aligned}$$

Assume that $V_u'(\cdot,i)$ converges pointwise to some function $f(\cdot,i)$ a.e. The limit is finite because $c_i V_u'(x,i) \le (\lambda_i+\delta+q_i)V_u(x,i) + \lambda_i\phi\int_x^\infty(1-G_i(y))\,\mathrm{d}y \le (\lambda_i+\delta+q_i)V(x,i) + \lambda_i\phi\int_x^\infty(1-G_i(y))\,\mathrm{d}y$. From Equation (5.16) follows that the function f satisfies

$$\begin{aligned} \max\Big\{ c_i f(x,i) - (\lambda_i+\delta)V(x,i) + \lambda_i \int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) & \\ + \sum_{j\in\mathscr{J}} q_{ij} V(x,j), 1 - f(x,i)\Big\} &= 0. \end{aligned}$$

It follows by the bounded convergence theorem that

$$V(x,i) - V(0,i) = \lim_{u\to\infty} \int_0^x V_u'(z,i)\,\mathrm{d}z = \int_0^x \lim_{u\to\infty} V_u'(z,i)\,\mathrm{d}z = \int_0^x f(z,i)\,\mathrm{d}z.$$

I.e., f is the density of V, V is differentiable at all points where f is continuous, and $f = V'$. By $V'(x,i) \le \phi$, Equation (5.15) follows.

It remains to show that $V_u'(x,i) \overset{u\to\infty}{\longrightarrow} f(x,i)$ a.e. and f is continuous.

We have seen in Section 5.2 that, for u fixed, the optimal strategy is a barrier strategy, and the optimal level b_i does not depend on u. In particular, $V(x,i)$ coincides with $V_u(x,i)$ for $u \ge c_i$ and initial capital $x \le b_i$. Therefore, we can choose $b_i := \inf\{x : V'_{c_i}(x-, i) = 1\}$ and $V(x) = V_{c_i}(x)$ for $x < b_i$. From (5.16) we get that

$$\begin{aligned} V'(x,i) &= V'_{c_i}(x,i) \\ &= \frac{(\lambda_i+\delta)V(x,i) - \lambda_i \int_0^\infty V(x-y,i)\,\mathrm{d}G_i(y) - \sum_{j\in\mathscr{J}} q_{ij}V(x,j)}{c_i} \\ &> 1\,, \end{aligned}$$

i.e., (5.15) is valid. By the continuity of $V(\cdot,i)$ and G_i, $V'(\cdot,i)$ is continuous on $[0,b_i)$.

Let now $x \ge b_i$. Then $V'_u(x,i) \le 1$ for all $u > c_i$. Let $\{u_n\}$ be a sequence tending to infinity such that $V'_{u_n}(x,i)$ converges to $\limsup_{u\to\infty} V'_u(x,i)$. From the second term on the right hand side of (5.16) we see that $\lim_{n\to\infty} V'_{u_n}(x,i) = 1$. Analogously, we can show that $\liminf_{u\to\infty} V'_u(x,i) = 1$. Thus, $f(x,i) = \lim_{u\to\infty} V'_u(x,i) = 1$. In particular, we have $V'(b_i-,i) = V'(b_i+,i) = 1$. □

5.3.2 The Optimal Strategy and the Characterisation of the Solution

Because the function $V(x,i)$ is concave, the condition $V'(x,i) - \phi \le 0$ is fulfilled whenever $V'(0,i) \le \phi$. Since $V(x,i)$ is continuously differentiable, we have $b_i := \inf\{x : V'(x,i) = 1\}$.

We now define the following strategy:

$$\begin{aligned} D_0^* &= \max(x - b_{J_0}, 0)\,, \\ D_t^* &= D_0^* + \int_0^t c_{J_s} \mathbf{1}_{\{X_s^{D^*} = b_{J_s}\}}\,\mathrm{d}s + \sum_{0<s\le t: J_{s-}\ne J_s} \Delta D_s^{*J_{s-}J_s}, \\ Z_t^* &= \max(-\inf_{0\le s\le t}(X_s - D_s^*), 0) = Z_t^{D^*} \quad \text{for } t > 0\,, \end{aligned} \tag{5.17}$$

where $\Delta D_s^{*J_{s-}J_s}$ is a lump sum payment due to the change in the environment from state J_{s-} to state J_s, i.e.,

$$\Delta D_s^{*J_{s-}J_s} = \max\{(X_{s-}^{D^*} - b_{J_s})\mathbf{1}_{\{J_{s-}\ne J_s\}}, 0\}.$$

We denote $X_t^* = X_t^{D^*} = X_t - D_t^* + Z_t^*$.

Theorem 5.3.5
The strategy (5.17) *is optimal, i.e.,* $V^{D^*}(x,i) = V(x,i)$ *for all* $i \in \mathscr{J}$.

Proof: We can assume that $x \ge 0$, i.e. $Z_0^* = 0$. The process D^* only jumps if the state of the environment changes. The process $\{(X_t^*, J_t)e^{-\delta t}\}$ is then a piecewise deterministic Markov process. As in Lemma 5.2.4, we decompose the value $V(X_t^*, J_t)e^{-\delta t}$ at the times W_k of the transition of the environment state and obtain

$$\begin{aligned}
&V(X_t^*, J_t)e^{-\delta t} - V(X_0^*, J_0) \\
&\quad = \sum_{0<s\le t: J_s \ne J_{s-}} \Big[V(X_{s-}^* - \Delta D_s^{*J_{s-}J_s}, J_s) - V(X_{s-}^*, J_s)\Big] e^{-\delta s} \\
&\qquad + \sum_{0<s\le t: J_s \ne J_{s-}} \Big[V(X_{s-}^*, J_s) - V(X_{s-}^*, J_{s-})\Big] e^{-\delta s} \\
&\qquad + \sum_{k=1}^{S_t} \Big[V(X_{W_k-}^*, J_{W_k-})e^{-\delta W_k-} - V(X_{W_{k-1}}^*, J_{W_{k-1}})e^{-\delta W_{k-1}}\Big] \\
&\qquad + V(X_t^*, J_t)e^{-\delta t} - V(X_{W_{S_t}}^*, J_t)e^{-\delta W_{S_t}} \ .
\end{aligned} \tag{5.18}$$

Since $V'(X_{s-}^* - \Delta D_s^{*J_{s-}J_s}, J_s) = V'(b_{J_s}, J_s) = 1$, we find first that

$$\begin{aligned}
&\sum_{0<s\le t: J_s \ne J_{s-}} \Big[V(X_{s-}^* - \Delta D_s^{*J_{s-}J_s}, J_s) - V(X_{s-}^*, J_s)\Big] e^{-\delta s} \\
&\quad = - \sum_{0<s\le t: J_s \ne J_{s-}} V'(X_{s-}^* - \Delta D_s^{*J_{s-}J_s}, J_s)\Delta D_s^{*J_{s-}J_s} e^{-\delta s} \\
&\quad = - \sum_{0<s\le t: J_s \ne J_{s-}} \Delta D_s^{*J_{s-}J_s} e^{-\delta s} \ .
\end{aligned} \tag{5.19}$$

By Theorem B.2.4, we follow for the second sum in (5.18) that the process

$$\begin{aligned}
&\sum_{0<s\le t: J_s \ne J_{s-}} \Big[V(X_{s-}^*, J_s) - V(X_{s-}^*, J_{s-})\Big] e^{-\delta s} \\
&\qquad\qquad - \int_0^t \sum_{j\in\mathscr{J}} q_{J_s j} V(X_s^*, j) e^{-\delta s}\, \mathrm{d}s
\end{aligned} \tag{5.20}$$

is an $\{\mathscr{F}_t\}_{t\ge 0}$-martingale with expected value 0.

Again, because $V(x,i)$ is absolutely continuous, we have between two changes of the environment that the process defined as

$$\begin{aligned}&\sum_{k=1}^{S_t}\Big[V(X^*_{W_k-},J_{W_k-})e^{-\delta W_k-}-V(X^*_{W_{k-1}},J_{W_{k-1}})e^{-\delta W_{k-1}}\Big] \qquad (5.21)\\ &\quad +V(X^*_t,J_t)e^{-\delta t}-V(X^*_{W_{S_t}},J_t)e^{-\delta W_{S_t}}-\phi\sum_{0\le s\le t}e^{-\delta s}\Delta Z^*_s\\ &\quad -\int_0^t\Big[c_{J_s}V'(X^*_s,J_s)+\lambda_{J_s}\int_0^\infty V(X^*_s-y,J_s)\,\mathrm{d}G_{J_s}(y)\\ &\qquad\qquad -(\lambda_{J_s}+\delta)V(X^*_s,J_s)\Big]\mathbf{I}_{\{X^*_s<b_{J_s}\}}e^{-\delta s}\,\mathrm{d}s\\ &\quad -\int_0^t\Big[\lambda_{J_s}\int_0^\infty V(X^*_s-y,J_s)\,\mathrm{d}G_{J_s}(y)\\ &\qquad\qquad -(\lambda_{J_s}+\delta)V(X^*_s,J_s)\Big]\mathbf{I}_{\{X^*_s=b_{J_s}\}}e^{-\delta s}\,\mathrm{d}s\end{aligned}$$

is a martingale with expected value 0. Recall that for all $i\in\mathscr{J}$, $V(x,i)$ is continuously differentiable by our assumption that $G_i(y)$ is continuous. Because $V'(X^*_s,J_s)>1$ on $\{X^*_s<b_{J_s}\}$, the first term on the left hand side of the HJB equation (5.15) vanishes and, thus, the integral over $\{X^*_s<b\}$ collapses to

$$-\int_0^t\sum_{j\in\mathscr{J}}q_{J_sj}V(X^*_s,j)\mathbf{I}_{\{X^*_s<b_{J_s}\}}e^{-\delta s}\,\mathrm{d}s\;.$$

From $V'(X^*_s,J_s)=1$ on $\{X^*_s=b_{J_s}\}$ and (5.15), it follows that

$$\begin{aligned}&\lambda_{J_s}\int_0^\infty V(X^*_s-y,J_s)\,\mathrm{d}G_{J_s}(y)-(\lambda_{J_s}+\delta)V(X^*_s,J_s)\\ &\qquad\qquad=-c_{J_s}-\sum_{j\in\mathscr{J}}q_{J_sj}V(X^*_s,j)\;.\end{aligned}$$

Finally, putting (5.18), (5.19), (5.20), and (5.21) together shows that the process

$$\begin{aligned}&V(X^*_t,J_t)e^{-\delta t}-V(X^*_0,J_0)\\ &\quad-\phi\int_0^t e^{-\delta s}\,\mathrm{d}Z^*_s+\int_0^t c_{J_s}\mathbf{I}_{\{X^*_s=b_{J_s}\}}e^{-\delta s}\,\mathrm{d}s+\sum_{0<s\le t:J_s\neq J_{s-}}\Delta D^{*J_{s-}J_s}_s e^{-\delta s}\end{aligned}$$

is an $\{\mathscr{F}_t\}_{t\ge0}$-martingale with expected value 0. For $J_0=i$, using that $X^*_0=x-D^*_0$, we get from the martingale property

$$\begin{aligned}V(x,i)&=\mathbb{E}_{(x,i)}\Big[V(X^*_t,J_t)e^{-\delta t}-\phi\int_0^t e^{-\delta s}\,\mathrm{d}Z^*_s+D^*_0\\ &\qquad+\int_0^t c_{J_s}\mathbf{I}_{\{X^*_s=b_{J_s}\}}e^{-\delta s}\,\mathrm{d}s+\sum_{0<s\le t:J_s\neq J_{s-}}\Delta D^{*J_{s-}J_s}_s e^{-\delta s}\Big]\;.\end{aligned}$$

Since $V(X_t^*, J_t)e^{-\delta t} \le V(b_{J_t}, J_t)e^{-\delta t}$ converges to 0, we have that

$$\lim_{t\to\infty} \mathbb{E}_{(x,i)}[V(X_t^*, J_t)e^{-\delta t}] = 0$$

by the bounded convergence theorem. Finally, by monotone convergence, we get that

$$\begin{aligned} V(x,i) &= \lim_{t\to\infty} \mathbb{E}_{(x,i)}\Big[D_0^* + \int_0^t c_{J_s} \mathbf{1}_{\{X_s^*=b_{J_s}\}} e^{-\delta s}\,\mathrm{d}s \\ &\qquad + \sum_{0<s\le t: J_s \ne J_{s-}} \Delta D_s^{*J_{s-}J_s} e^{-\delta s} - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^*\Big] \\ &= \mathbb{E}_{(x,i)}\Big[\int_{0-}^\infty e^{-\delta s}\,\mathrm{d}D_s^* - \phi \int_0^\infty e^{-\delta s}\,\mathrm{d}Z_s^*\Big] \\ &= V^{D^*}(x,i)\ . \end{aligned}$$

□

If the environmental process at time t is in the state $J_t = i$ for $i \in \mathscr{J}$, then the optimal strategy is a barrier dividend strategy with reflecting barriers b_i and 0. Then the corresponding surplus process $X_t^{D^*}$ does not leave the interval $[0, \bar{b}]$ with $\hat{b} := \max\{b_i : i = 1 \dots m\}$. The existence of such a process $\{X_t^*\}$ can be seen in the following way. Let

$$\begin{aligned} D_t^0 &= \sup\{X_s - b_{J_s} : 0 \le s \le t\} \vee 0\ , \\ \tau_1 &= \inf\{t : X_t - D_t^0 < 0\}\ , \\ Z_t^0 &= \sup\{D_s^0 - X_s : 0 \le s \le t\} \vee 0\ . \end{aligned}$$

We define $X_t^1 = X_t - D_t^0 + Z_t^0$. Note that $X_t^* = X_t^1$ for $t \in [0, \tau_1]$. Suppose we have constructed $\{X_t^n\}$. Then we let

$$\begin{aligned} D_t^n &= D_t^{n-1} + (\sup\{X_s^n - b_{J_s} : 0 \le s \le t\} \vee 0)\ , \\ \tau_{n+1} &= \inf\{t : X_t^n - D_t^n < 0\}\ , \\ Z_t^n &= Z_t^{n-1} + (\sup\{D_s^n - X_s^n : 0 \le s \le t\} \vee 0)\ . \end{aligned}$$

We have $X_t^* = X_t^{n+1}$ for $t \in [0, \tau_{n+1}]$. Because τ_n is a claim occurence time and $\tau_{n+1} > \tau_n$ we have that $\tau_n \to \infty$ as $n \to \infty$. Thus $\{X_t^n\}$ converges almost surely to $\{X_t^*\}$.

Because we do not have an explicit solution nor an initial value, we need to characterise the solution $V(x,i)$ among other possible solutions.

Theorem 5.3.6
For all $i \in \mathscr{J}$, $V(x,i)$ is the minimal solution to the HJB equation (5.15)*. If $f(x,i)$ is an increasing, positive solution to* (5.15) *such that $f(x,i) \le \kappa_1^i x + \kappa_2^i$ for some $\kappa_1^i, \kappa_2^i \ge 0$ and all $x \ge 0$, then $f(x,i) = V(x,i)$.*

Proof: We again suppose that $x \ge 0$. Let $f(x,i)$ be a solution to the HJB equation. Then $f(x,i)$ is increasing. Consider the process X^* under the optimal strategy. We have then, as in the proof of the previous theorem, that the process

$$\begin{aligned}
f(X_t^*, J_t)e^{-\delta t} &- f(X_0^*, J_0) - \phi \int_0^t e^{-\delta s}\,\mathrm{d}Z_s^* \\
&- \sum_{0<s\le t: J_s \ne J_{s-}} \Big[f(X_{s-}^* - \Delta D_s^{*J_{s-}J_s}, J_s) - f(X_{s-}^*, J_s) \Big] e^{-\delta s} \\
&- \int_0^t \Big[c_{J_s} f'(X_s^*, J_s) + \lambda_{J_s} \int_0^\infty f(X_s^* - y, J_s)\,\mathrm{d}G_{J_s}(y) \\
&\qquad -(\lambda_{J_s} + \delta) f(X_s^*, J_s) + \sum_{j\in\mathscr{J}} q_{J_s j} f(X_s^*, j) \Big] \mathbf{1}_{\{X_s^* < b_{J_s}\}} e^{-\delta s}\,\mathrm{d}s \\
&- \int_0^t \Big[\lambda_{J_s} \int_0^\infty f(X_s^* - y, J_s)\,\mathrm{d}G_{J_s}(y) - (\lambda_{J_s} + \delta) f(X_s^*, J_s) \\
&\qquad + \sum_{j\in\mathscr{J}} q_{J_s j} f(X_s^*, j) \Big] \mathbf{1}_{\{X_s^* = b_{J_s}\}} e^{-\delta s}\,\mathrm{d}s
\end{aligned}$$

is a martingale with expected value 0. By (5.15), we have

$$\begin{aligned}
c_{J_s} f'(X_s^*, J_s) &+ \lambda_{J_s} \int_0^\infty f(X_s^* - y, J_s)\,\mathrm{d}G_{J_s}(y) \\
&-(\lambda_{J_s} + \delta) f(X_s^*, J_s) + \sum_{j\in\mathscr{J}} q_{J_s j} f(X_s^*, j) \le 0
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{J_s} \int_0^\infty & f(X_s^* - y, J_s)\,\mathrm{d}G_{J_s}(y) - (\lambda_{J_s} + \delta) f(X_s^*, J_s) \\
&+ \sum_{j\in\mathscr{J}} q_{J_s j} f(X_s^*, j) \le -c_{J_s} f'(X_s^*, J_s) \le -c\,,
\end{aligned}$$

because $f'(x,i) \geq 1$. For the same reason, we have

$$\begin{aligned}
\sum_{0<s\leq t: J_s \neq J_{s-}} & \Big[f(X^*_{s-} - \Delta D^{*J_{s-}J_s}_s, J_s) - f(X^*_{s-}, J_s)\Big] e^{-\delta s} \\
&= - \sum_{0<s\leq t: J_s \neq J_{s-}} f'(X^*_{s-} - \Delta D^{*J_{s-}J_s}_s, J_s) \Delta D^{*J_{s-}J_s}_s e^{-\delta s} \\
&\leq - \sum_{0<s\leq t: J_s \neq J_{s-}} \Delta D^{*J_{s-}J_s}_s e^{-\delta s}
\end{aligned}$$

and $f(x,J_0) \geq f(x - D^*_0, J_0) + D^*_0 = f(X^*_0, J_0) + D^*_0$. Taking expectations yields

$$\begin{aligned}
f(x,i) &\geq \mathbb{E}_{(x,i)}\Big[f(X^*_t, J_t)e^{-\delta t} - \phi\int_0^t e^{-\delta s}\, \mathrm{d}Z^*_s + \int_0^t c_{J_s}\mathbf{1}_{\{X^*_s = b_{J_s}\}} e^{-\delta s}\,\mathrm{d}s \\
&\qquad + D^*_0 + \sum_{0<s\leq t: J_s \neq J_{s-}} \Delta D^{*J_{s-}J_s}_s e^{-\delta s}\Big] \\
&\geq \mathbb{E}_{(x,i)}\Big[\int_{0-}^t e^{-\delta s}\,\mathrm{d}D^*_s - \phi\int_0^t e^{-\delta s}\,\mathrm{d}Z^*_s\Big]
\end{aligned}$$

and therefore, by monotone convergence, $f(x,i) \geq V^{D^*}(x,i)$ for all $i \in \mathscr{J}$.

Let now $\tilde{b}_i := \inf\{x : f(x,i)' = 1\}$ and define a strategy $\tilde{D}$ analogously to the strategy D^* as

$$\begin{aligned}
\tilde{D}_0 &= \max\{x - \tilde{b}_{J_0}, 0\}, \\
\tilde{D}_t &= \tilde{D}_0 + \int_0^t c_{J_s}\mathbf{1}_{\{X^{\tilde{D}}_s = \tilde{b}_{J_s}\}}\,\mathrm{d}s + \sum_{0<s\leq t: J_{s-} \neq J_s} \Delta\tilde{D}^{J_{s-}J_s}_s \\
\tilde{Z}_t &= \max(-\inf_{0\leq s\leq t}(X_s - \tilde{D}_s), 0) = \tilde{Z}^{\tilde{D}}_t \quad \text{for } t > 0\ ,
\end{aligned}$$

with $\Delta\tilde{D}^{J_{s-}J_s}_s = \max\{(X^{\tilde{D}}_{s-} - \tilde{b}_{J_s})\mathbf{1}_{J_{s-}\neq J_s}, 0\}$. With the same arguments as in Theorem 5.3.5 we can show that the process

$$\begin{aligned}
& f(X^{\tilde{D}}_t, J_t)e^{-\delta t} - f(X^{\tilde{D}}_0, J_0) \\
& \quad -\phi\int_0^t e^{-\delta s}\,\mathrm{d}\tilde{Z}_s + \int_0^t c_{J_s}\mathbf{1}_{\{X^{\tilde{D}}_s = \tilde{b}_{J_s}\}} e^{-\delta s}\,\mathrm{d}s + \sum_{0<s\leq t: J_s \neq J_{s-}} \Delta\tilde{D}^{J_{s-}J_s}_s e^{-\delta s}
\end{aligned}$$

is an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale with expected value 0. Taking expectations and letting $t \to \infty$, we obtain $f(x,i) = V^{\tilde{D}}(x,i)$, since $f(X^{\tilde{D}}_t, J_t)e^{-\delta t} \leq f(\tilde{b}_{J_t}, J_t)e^{-\delta t}$ tends to zero as $t \to \infty$ by the linear growth condition. Thus, $f(x,i) \leq V(x,i)$ and, therefore, equality holds. $\square$

5.3.3 Calculating the Value Function

General case

Suppose that $0 \le b_1 \le \cdots \le b_m$. We have to calculate the value functions on each intervall $[0, b_1], \ldots, [b_{m-1}, b_m]$ separately. For $0 \le x \le b_1$, the value functions $V(x,i)$ are solutions to the sytem of the integro-differential equations

$$c_i f'(x,i)+\lambda_i \int_0^\infty f(x-y,i)\,\mathrm{d}G_i(y)-(\lambda_i+\delta)f(x,i)+\sum_{j\in\mathscr{J}} q_{ij} f(x,j) = 0,\ i \in \mathscr{J}.$$

Let $k \in \{2, \ldots, m\}$. Suppose that we have constructed solutions $V(x,i), i \in \mathscr{J}$, for $x \in [0, b_k]$. For any $j \le k$ and $x \ge b_j$, the value function is linear, i.e., $V(x,j) = x - b_j + V(b_j, j)$. Thus, for $b_k \le x \le b_{k+1}$ and $i = k+1, \ldots, m$, we have to solve the reduced system of the integro-differential equations

$$\begin{aligned} c_i f'(x,i) &+ \lambda_i \int_{x-b_k}^x V(x-y,i)\,\mathrm{d}G_i(y) + \lambda_i \int_0^{x-b_k} f(x-y,i)\,\mathrm{d}G_i(y) \\ &-(\lambda_i+\delta) f(x,i) + \lambda_i f(0,i)(1-G_i(x)) - \lambda_i \phi \int_x^\infty (1-G_i(y))\,\mathrm{d}y \\ &+ \sum_{j=1}^k q_{ij}(x - b_j + V(b_j,j)) + \sum_{j=k+1}^m q_{ij} f(x,j) = 0. \end{aligned}$$

This procedure is complicate and requires intricate numerical calculations. For simplicity reasons, we consider two special cases when we have only one barrier.

Special case: one constant barrier $\boldsymbol{b = 0}$

We assume that $b_1 = \cdots = b_m = 0$. Then $V'(0,i) = 1$ and

$$V(x,i) = x + V(0,i) \quad \text{for all} \quad i \in \mathscr{J}. \tag{5.22}$$

Consider the value in $x = 0$. Then Equation (5.15) reads

$$c_i - \phi\lambda_i\mu_i - \delta V(0,i) + \sum_{j\in\mathscr{J}} q_{ij} V(0,j) = 0\ , \tag{5.23}$$

or in matrix form,

$$\boldsymbol{C}\boldsymbol{e} - \phi\boldsymbol{\Lambda}\boldsymbol{\mu} = [\delta\boldsymbol{I} - \boldsymbol{Q}]\boldsymbol{V}(0)\ ,$$

where

$$\begin{array}{rcllrcl} \boldsymbol{e} &=& (1,\ldots,1)^T\ , & & \boldsymbol{V}(0) &=& (V(0,1),\ldots,V(0,m))^T\ , \\ \boldsymbol{C} &=& \mathrm{diag}(c_1,\ldots,c_m)\ , & & \boldsymbol{\Lambda} &=& \mathrm{diag}(\lambda_1,\ldots,\lambda_m)\ , \\ \boldsymbol{\mu} &=& (\mu_1,\ldots,\mu_m)^T\ . & & & & \end{array}$$

By Lemma 5.2.2, the matrix $\delta\boldsymbol{I} - \boldsymbol{Q}$ is invertible. The value at zero is given by

$$\boldsymbol{V}(0) = [\delta\boldsymbol{I} - \boldsymbol{Q}]^{-1}[\boldsymbol{C}\boldsymbol{e} - \phi\boldsymbol{\Lambda}\boldsymbol{\mu}].$$

Denoting $\tilde{\boldsymbol{Q}} = \delta\boldsymbol{I} - \boldsymbol{Q}$ and by $\tilde{\boldsymbol{Q}}^*$ the adjugate matrix, we obtain

$$\boldsymbol{V}(0) = \frac{\tilde{\boldsymbol{Q}}^*[\boldsymbol{C}\boldsymbol{e} - \phi\boldsymbol{\Lambda}\boldsymbol{\mu}]}{\det \tilde{\boldsymbol{Q}}} \ ,$$

or,

$$V(0,i) = \frac{1}{\det \tilde{\boldsymbol{Q}}} \sum_{j=1}^{m} \tilde{q}^*_{ij}(c_j - \phi\lambda_j\mu_j) \quad \text{for} \quad i \in \mathscr{J}. \tag{5.24}$$

We can now derive a necessary condition for (5.24). Plugging (5.22) in the HJB equation (5.15) yields

$$\begin{aligned}
c_i &+ \lambda_i \int_0^x (x - y + V(0,i))\,\mathrm{d}G_i(y) + \lambda_i V(0,i)(1 - G_i(x)) \\
&\quad -\phi\lambda_i \int_x^\infty (1 - G_i(y))\,\mathrm{d}y - (\lambda_i + \delta)(x + V(0,i)) + \sum_{j\in\mathscr{J}} q_{ij}(x + V(0,j)) \\
&= c_i - \phi\lambda_i\mu_i - \delta V(0,i) + \sum_{j\in\mathscr{J}} q_{ij} V(0,j) \\
&\quad +\lambda_i \int_0^x (x-y)\,\mathrm{d}G_i(y) + \phi\lambda_i \int_0^x (1 - G_i(y))\,\mathrm{d}y - (\lambda_i + \delta)x + \sum_{j\in\mathscr{J}} q_{ij} x \\
&= \int_0^x \Big[\lambda_i(\phi - 1)(1 - G_i(y)) - \delta\Big]\,\mathrm{d}y \ ,
\end{aligned}$$

where we used (5.23) and $\sum_{j\in\mathscr{J}} q_{ij} = 0$. The latter integral is a convex function in x. Thus, it is non-positive if and only if the derivative at zero is non-positive, i.e., if

$$\lambda_i(\phi - 1) - \delta \le 0 \quad \text{for all} \quad i \in \mathscr{J}.$$

Thus, a necessary condition for all barriers to be at zero is

$$\max_{i\in\mathscr{J}} \lambda_i \le \frac{\delta}{\phi - 1} \ . \tag{5.25}$$

Remark 5.3.7
Since in the general case $V'(0,i) \ge 1$ and $q_{ij} > 0$ for $i \ne j$, we obtain from (5.23) that

$$V(0,i) \ge \frac{c_i - \phi\lambda_i\mu_i}{\delta - q_{ii}} = \frac{c_i - \phi\lambda_i\mu_i}{\delta + q_i}$$

and therefore

$$V(x,i) \geq x + V(0,i) \geq x + \frac{c_i - \phi\lambda_i\mu_i}{\delta + q_i} .$$

This is the lower bound in Lemma 5.3.1. ■

Special case: one constant barrier $b > 0$

Suppose that $b_1 = \cdots = b_m = b$ for some $b > 0$. Then we have to solve the system of integro-differential equations

$$\begin{aligned} c_i f'(x,i) + \lambda_i \int_0^x f(x-y,i)\,\mathrm{d}G_i(y) + \lambda_i f(0,i)(1-G_i(x)) \qquad (5.26) \\ -\lambda_i\phi \int_x^\infty (1-G_i(y))\,\mathrm{d}y - (\lambda_i+\delta)f(x,i) + \sum_{j\in\mathscr{J}} q_{ij} f(x,j) = 0,\ i \in \mathscr{J}, \end{aligned}$$

for $0 \leq x \leq b$. We apply the method of Laplace transforms. Denote for $\mathrm{Re}(s) \geq 0$ the Laplace(-Stieltjes) transforms ($i = 1,\ldots,m$)

$$\begin{aligned} \hat{f}_i(s) &:= \int_0^\infty e^{-sx} f(x,i)\,\mathrm{d}x, \quad \hat{g}_i(s) := \int_0^\infty e^{-sx}\,\mathrm{d}G_i(x) \\ \hat{r}_i(s) &:= \int_0^\infty e^{-sx} \int_x^\infty (1-G_i(y))\,\mathrm{d}y\,\mathrm{d}x . \end{aligned}$$

Then we obtain for $i = 1,\ldots,m$,

$$\begin{aligned} &\Big(c_i s + \lambda_i \hat{g}_i(s) - \lambda_i - \delta\Big)\hat{f}_i(s) + \sum_{j\in\mathscr{J}} q_{ij}\hat{f}_j(s) \\ &\quad = \Big(c_i - \frac{\lambda_i}{s} + \frac{\lambda_i \hat{g}_i(s)}{s}\Big) f(0,i) + \phi\lambda_i \hat{r}_i(s), \end{aligned}$$

or in matrix form,

$$\Big[s\boldsymbol{C} + \boldsymbol{\Lambda}\hat{\boldsymbol{G}}(s) - \boldsymbol{\Lambda} - \delta\boldsymbol{I} + \boldsymbol{Q}\Big]\hat{\boldsymbol{f}}(s) = \Big[\boldsymbol{C} + \frac{1}{s}\boldsymbol{\Lambda}\hat{\boldsymbol{G}}(s) - \frac{1}{s}\boldsymbol{\Lambda}\Big]\boldsymbol{f}(0) + \phi\boldsymbol{\Lambda}\hat{\boldsymbol{r}}(s),$$

where $\boldsymbol{I}$ is the identity matrix and

$$\begin{aligned} \hat{\boldsymbol{f}}(s) &= (\hat{f}_1(s),\ldots,\hat{f}_m(s))^T, & \boldsymbol{f}(0) &= (f(0,1),\ldots,f(0,m))^T, \\ \boldsymbol{C} &= \mathrm{diag}(c_1,\ldots,c_m), & \boldsymbol{\Lambda} &= \mathrm{diag}(\lambda_1,\ldots,\lambda_m), \\ \hat{\boldsymbol{G}}(s) &= \mathrm{diag}(\hat{g}_1(s),\ldots,\hat{g}_m(s)), & \hat{\boldsymbol{r}}(s) &= (\hat{r}_1(s),\ldots,\hat{r}_m(s))^T . \end{aligned}$$

If we denote

$$\begin{aligned} \boldsymbol{A}(s) &= s\boldsymbol{C} + \boldsymbol{\Lambda}\hat{\boldsymbol{G}}(s) - \boldsymbol{\Lambda} - \delta\boldsymbol{I} + \boldsymbol{Q}, \\ \boldsymbol{B}(s) &= \boldsymbol{C} + \frac{1}{s}\boldsymbol{\Lambda}\hat{\boldsymbol{G}}(s) - \frac{1}{s}\boldsymbol{\Lambda}, \\ \boldsymbol{r}(s) &= \boldsymbol{\Lambda}\hat{\boldsymbol{r}}(s), \end{aligned}$$

then we have to solve

$$\boldsymbol{A}(s)\hat{\boldsymbol{f}}(s) = \boldsymbol{B}(s)\boldsymbol{f}(0) + \phi\boldsymbol{r}(s)\ .$$

We have

$$\begin{aligned}\hat{\boldsymbol{f}}(s) &= [\boldsymbol{A}(s)]^{-1}\Big[\boldsymbol{B}(s)\boldsymbol{f}(0) + \phi\boldsymbol{r}(s)\Big] \\ &= \frac{\boldsymbol{A}^*(s)\Big[\boldsymbol{B}(s)\boldsymbol{f}(0) + \phi\boldsymbol{r}(s)\Big]}{\det \boldsymbol{A}(s)}\ , \end{aligned} \tag{5.27}$$

where $\boldsymbol{A}^*(s)$ is the adjugate of the matrix $\boldsymbol{A}(s)$. By inverting the Laplace transform, we obtain the function $\boldsymbol{f}$ which depends on $\boldsymbol{f}(0)$ and so $\boldsymbol{f}'$ does also. From $\boldsymbol{f}'(b) = 1$ we can determine $\boldsymbol{f}(0)$ as a function of b and denote it by $\boldsymbol{f}^b(0)$. For the stationary initial distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ consider the function

$$\sum_{i\in\mathscr{J}} \pi_i V(x,i) = \boldsymbol{\pi}\boldsymbol{V}(x).$$

Then we can determine the initial value $\boldsymbol{V}(0)$ by

$$\boldsymbol{\pi}\boldsymbol{V}(0) = \sup_{b\geq 0}\{\boldsymbol{\pi}\boldsymbol{f}^b(0)\}.$$

The optimal barrier b^* is the argument which maximises $\boldsymbol{\pi}\boldsymbol{f}^b(0)$, and $\boldsymbol{V}(0) = \boldsymbol{f}^{b^*}(0)$ is the maximal value.

5.3.4 Illustrations for a Two-State Model

We consider the case of one constant barrier and a two-state Markov process, i.e., $m = 2$. The two states of the environment process correspond to the different environmental effects due to, probably, "high risk" versus "low risk" conditions. We have

$$\boldsymbol{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

Let $\boldsymbol{\pi}^T = (\pi_1, \pi_2)^T$ be the unique stationary distribution of $\{J_t\}$. Then

$$\boldsymbol{\pi}^T = \left(\frac{q_2}{q_1+q_2}\ ,\quad \frac{q_1}{q_1+q_2}\right)^T\ ;$$

(see Example A.3.8).

Exponentially distributed claim sizes

Assume now that the claim sizes are exponentially distributed with parameters $\alpha_1 > 0$ and $\alpha_2 > 0$, i.e., $g_1(x) = \alpha_1 e^{-\alpha_1 x}$ and $g_2(x) = \alpha_2 e^{-\alpha_2 x}$. Their Laplace transforms are of the form

$$\hat{g}_1(s) = \frac{\alpha_1}{s+\alpha_1}\,, \quad \hat{g}_2(s) = \frac{\alpha_2}{s+\alpha_2}\,.$$

Then we have

$$\begin{aligned}
\boldsymbol{A}(s) &= \begin{pmatrix} sc_1 - \frac{\lambda_1 s}{s+\alpha_1} - \delta - q_1 & q_1 \\ q_2 & sc_2 - \frac{\lambda_2 s}{s+\alpha_2} - \delta - q_2 \end{pmatrix}, \\
\boldsymbol{B}(s) &= \begin{pmatrix} c_1 - \frac{\lambda_1}{s+\alpha_1} & 0 \\ 0 & c_2 - \frac{\lambda_2}{s+\alpha_2} \end{pmatrix}, \\
\boldsymbol{r}(s) &= \begin{pmatrix} \frac{\lambda_1}{\alpha_1(s+\alpha_1)} \\ \frac{\lambda_2}{\alpha_2(s+\alpha_2)} \end{pmatrix}.
\end{aligned}$$

By Corrolar C.1.3, the adjugate matrix $\boldsymbol{A}^*(s)$ is

$$\boldsymbol{A}^*(s) = [\boldsymbol{A}(s)]^{-1} \det \boldsymbol{A}(s) = \begin{pmatrix} sc_2 - \frac{\lambda_2 s}{s+\alpha_2} - \delta - q_2 & -q_1 \\ -q_2 & sc_1 - \frac{\lambda_1 s}{s+\alpha_1} - \delta - q_1 \end{pmatrix}.$$

Let $R_2 < R_1 < 0 < \rho_1 < \rho_2$ be the solutions to the Lundberg's equation

$$\det \boldsymbol{A}(s) = (sc_2 - \frac{\lambda_2 s}{s+\alpha_2} - \delta - q_2)(sc_1 - \frac{\lambda_1 s}{s+\alpha_1} - \delta - q_1) - q_1 q_2 = 0.$$

Then the functions $f(x,1)$ and $f(x,2)$ obtained by inverting of (5.27) are of the form

$$f(x,i) = K_1^i e^{R_2 x} + K_2^i e^{R_1 x} + K_3^i e^{\rho_1 x} + K_4^i e^{\rho_2 x}, \quad i = 1,2.$$

The constants $K_n^i, n = 1,\ldots,4$ depend on the unknown initial values $f(0,1)$ and $f(0,2)$ which can now be determined by solving the system

$$f'(b,i) = 1 = K_1^i R_2 e^{R_2 b} + K_2^i R_1 e^{R_1 b} + K_3^i \rho_1 e^{\rho_1 b} + K_4^i \rho_2 e^{\rho_2 b}, \quad i = 1,2.$$

For a numerical example, let $c_1 = 11, c_2 = 10, \lambda_1 = 1, \lambda_2 = 4, \delta = 0.1, q_1 = 1/4, q_2 = 1/2, \alpha_1 = 1, \alpha_2 = 0.5, \phi = 1.5$. Then $\pi_1 = 2/3, \pi_2 = 1/3$ and we get the optimal barrier level is $b^* = 4.1668$. Figure 5.1 shows the function $\boldsymbol{\pi f}^b(0)$ dependent on the barrier. The value functions $V(x,1)$ and $V(x,2)$ can be seen in Figure 5.2.

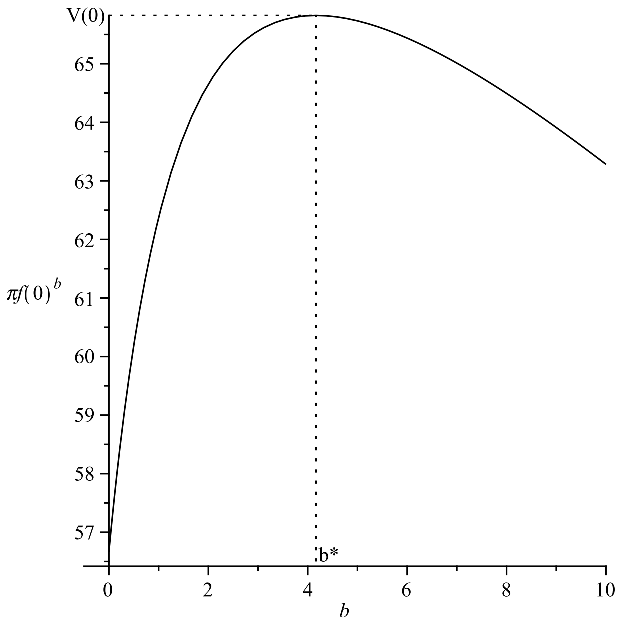

Figure 5.1: The function $\boldsymbol{\pi f}^b(0)$

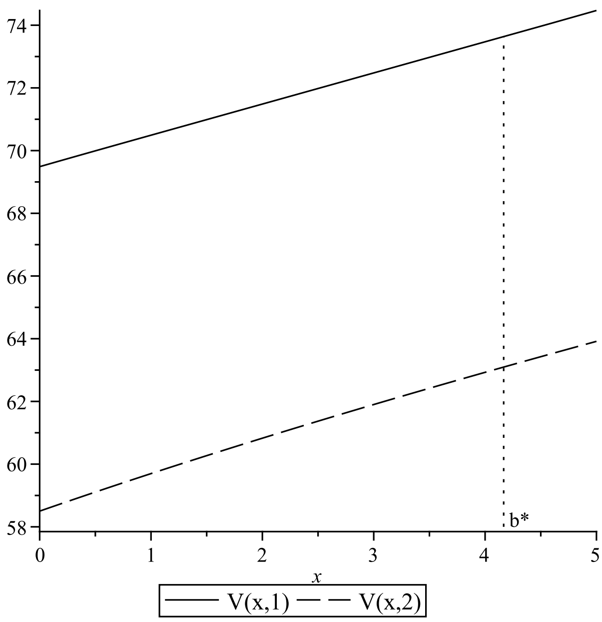

Figure 5.2: The value functions

Gamma-distributed claim sizes

We assume that the claim sizes are $\Gamma(2,1)-$ and $\Gamma(2,2)-$distributed with the density functions $g_1(x) = xe^{-x}$ and $g_2(x) = 4xe^{-2x}$. The Laplace transforms are

$$\hat{g}_1(s) = \frac{1}{(s+1)^2}\ , \quad \hat{g}_2(s) = \frac{4}{(s+2)^2}\ .$$

Then we have

$$\begin{aligned}
\boldsymbol{A}(s) &= \begin{pmatrix} sc_1 + \frac{\lambda_1}{(s+1)^2} - \lambda_1 - \delta - q_1 & q_1 \\ q_2 & sc_2 + \frac{4\lambda_2}{(s+2)^2} - \lambda_2 - \delta - q_2 \end{pmatrix}, \\
\boldsymbol{B}(s) &= \begin{pmatrix} c_1 - \frac{\lambda_1(2+s)}{(s+1)^2} & 0 \\ 0 & c_2 - \frac{\lambda_2(s+4)}{(s+2)^2} \end{pmatrix}, \\
\boldsymbol{r}(s) &= \begin{pmatrix} \frac{\lambda_1(3+2s)}{(s+1)^2} \\ \frac{\lambda_2(s+3)}{(s+2)^2} \end{pmatrix}.
\end{aligned}$$

The Lundberg's equation

$$\det \boldsymbol{A}(s) = (sc_1 + \frac{\lambda_1}{(s+1)^2} - \lambda_1 - \delta - q_1)(sc_2 + \frac{4\lambda_2}{(s+2)^2} - \lambda_2 - \delta - q_2) - q_1 q_2 = 0$$

has six solutions $R_4 < R_3 < R_2 < R_1 < 0 < \rho_1 < \rho_2$ such that the functions $f(x,1)$ and $f(x,2)$ obtained by inverting (5.27) are of the form

$$f(x,i) = K_1^i e^{R_4 x} + K_2^i e^{R_3 x} + K_3^i e^{R_2 x} + K_4^i e^{R_1 x} + K_5^i e^{\rho_1 x} + K_6^i e^{\rho_2 x}, \quad i = 1,2.$$

For a numerical example, let $c_1 = 11, c_2 = 10, \lambda_1 = 1, \lambda_2 = 4, \delta = 0.1, q_1 = 1/4, q_2 = 1/2, \phi = 1.5$. Then $\pi_1 = 2/3, \pi_2 = 1/3$ and we get the optimal barrier level is $b^* = 3.2806$. Figures 5.3 and 5.4 show the functions $\boldsymbol{\pi f}^b(0)$ dependent on the barrier and the value functions $V(x,1)$ and $V(x,2)$, respectively.

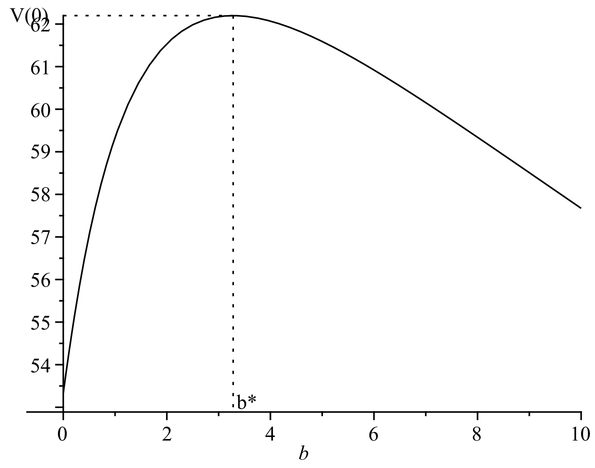

Figure 5.3: The function $\boldsymbol{\pi} \boldsymbol{f}^b(0)$

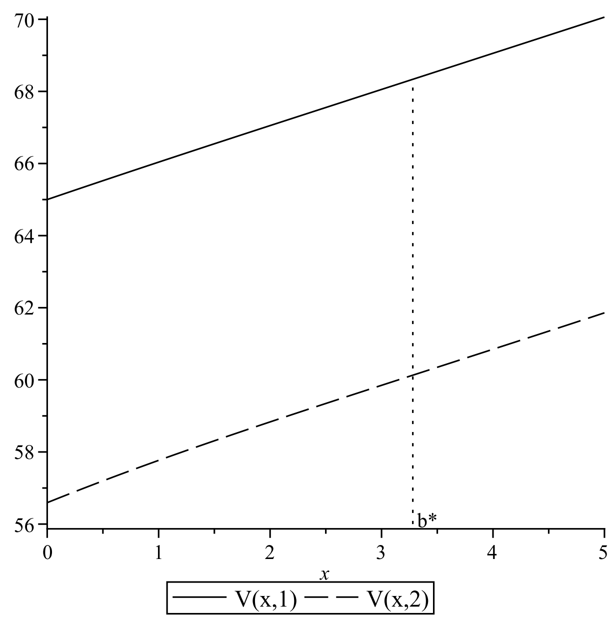

Figure 5.4: The value functions

Appendix A

Markov Processes

Let $E \subseteq \mathbb{R}$. Let $\mathcal{B}(E)$ denote the σ-algebra of Borel sets in E, $M(E)$ the family of all real-valued measurable functions on E, and $M_b(E) \subset M(E)$ the subfamily of all bounded functions on E endowed with the supremum norm $\|g\| = \sup_{x \in E} |g(x)|$ for $g \in M_b(E)$.

A.1 Definition of Markov Processes

Let $\{X_t\}_{t \geq 0}$ be a (càdlàg) process with values in E.

Definition A.1.1
A real-valued stochastic process $\{X_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an $\{\mathscr{F}_t\}_{t \geq 0}$-Markov process if it is adapted to $\{\mathscr{F}_t\}_{t \geq 0}$ and the Markov property holds, i.e.,

$$\mathbb{P}[X_{s+t} \in B | \mathscr{F}_t] = \mathbb{P}[X_{s+t} \in B | X_t], \quad \mathbb{P}\text{-}a.s.,$$

*for all $s, t \geq 0$ and $B \in \mathcal{B}(E)$. The **transition function** is defined by*

$$P_t(s, x, B) = \mathbb{P}[X_{t+s} \in B | X_t = x].$$

*If the transition function does not depend on t, we call the Markov process **homogeneous**.*

We assume in the following that the Markov process is homogeneous and omit the index t. The transition function has the following properties:

- $P(s, x, \cdot)$ is a probability measure on $\mathcal{B}(E)$,
- $P(0, x, \{x\}) = 1$,
- $P(\cdot, \cdot, B) \in M(\mathbb{R}_+ \times E)$ for fixed $B \in \mathcal{B}(E)$,

- $P(s_1 + s_2, x, B) = \int_E P(s_2, y, B)P(s_1, x, \mathrm{d}y)$.

A probability measure ν on $(E, \mathcal{B}(E))$ is called **initial distribution** of $\{X_t\}$ if $\mathbb{P}[X_0 \in B] = \nu(B)$ for all $B \in \mathcal{B}(E)$. We have then

$$\begin{aligned}&\mathbb{P}[X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n]\\&\quad = \int_{B_0}\int_{B_1}\cdots\int_{B_n} P(t_n - t_{n-1}, x_{n-1}, \mathrm{d}x_n)\cdots P(t_1, x_0, \mathrm{d}x_1)\nu(\mathrm{d}x_0)\end{aligned}$$

for all $n \geq 0, B_0, B_1, \ldots, B_n \in \mathcal{B}(E)$,$t_0 = 0 \leq t_1 \leq \ldots \leq t_n$.

We call $\{X_t\}$ a **strong Markov process** with respect to its history $\{\mathscr{F}_t^X\}$, if with probability 1

$$\mathbb{P}[X_{\tau+h} \in B | \mathscr{F}_\tau^X] = P(h, X_\tau, B)$$

on $\{\tau < \infty\}$, for all $h \geq 0, B \in \mathcal{B}(E)$ and each $\{\mathscr{F}_t^X\}$-stopping time τ.

Lemma A.1.2
Every stochastic process $\{X_t\}_{t\geq 0}$ with stationary and independent increments is a Markov process.

A.2 Generators

Definition A.2.1
Let $\{T(h)\}_{h\geq 0}$ be a family of bounded mapping from $M_b(E)$ to $M_b(E)$. $\{T(h)\}$ is called a ***contraction semigroup*** *on $M_b(E)$ if*

- *$T(0) = I$, where I denotes the identity mapping on $M_b(E)$,*
- *$T(h_1 + h_2) = T(h_1)T(h_2)$,*
- *$\|T(h)f\| \leq \|f\|$*

for all $h, h_1, h_2 \geq 0$ and $f \in M_b(E)$.

Lemma A.2.2
Let $\{X_t\}$ be an E-valued Markov process with transition functions $P(h, x, B)$. Let

$$T(h)f(x) = \int_E f(y)P(h, x, dy) = \mathbb{E}[f(X_h)|X_0 = x] \tag{A.1}$$

for any $f \in M_b(E)$. Then $\{T(h)\}$ is a contraction semigroup on $M_b(E)$.

Proof: Refer Rolski et al. [61, p. 440]. □

Definition A.2.3 (Generator)
*The **infinitesimal generator** of a contraction semigroup $\{T(h)\}$ is a linear operator defined by*

$$\mathcal{A}f = \lim_{h \downarrow 0} \frac{T(h)f - f}{h}$$

*for each function $f \in M_b(E)$ for which the limit exists in the supremum norm and belongs to $M_b(E)$. We call the class $\mathcal{D}(\mathcal{A}) \subset M_b(E)$ of functions with these properties the **domain** of $\mathcal{A}$.*

For the semigroup given in (A.1) we have

$$\mathcal{A}f(x) = \lim_{h \downarrow 0} \frac{\mathbb{E}[f(X_h) - f(x)|X_0 = x]}{h}$$

for all functions $f \in \mathcal{D}(\mathcal{A})$.

Theorem A.2.4 (Dynkin's Formula)
Assume that $\{X_t\}$ is an E-valued Markov process with transition functions $P(h, x, B)$. Let $\{T(h)\}$ denote the semigroup defined in (A.1) *and let $\mathcal{A}$ be its generator. Then, for each $f \in \mathcal{D}(\mathcal{A})$ the stochastic process $\{M_t\}_{t \geq 0}$ is an $\{\mathscr{F}_t^X\}$-martingale, where*

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds.$$

Proof: Refer Rolski et al. [61, p. 442]. □

We now extend the definition of the generator and allow it acting on unbounded functions.

Definition A.2.5 (Full Generator)
*A multilinear operator $\mathcal{A} \subset \{(f, g) \in M(E) \times M(E)\}$ is called the **full generator** of the Markov process $\{X_t\}$ if*

$$\Big\{ f(X_t) - f(X_0) - \int_0^t g(X_s)\, ds \Big\}_{t \geq 0}$$

*is an $\{\mathscr{F}_t^X\}$-martingale. The set $\mathcal{D}(\mathcal{A}) = \{f \in M(E) : (f, g) \in \mathcal{A}$ for some $g \in M(E)\}$ is called the **domain** of the operator $\mathcal{A}$.*

Theorem A.2.4 implies that the domain of the infinitesimal generator of a Markov process is always contained in the domain of its full generator. We write $\mathcal{A}f$ if we mean a function g such that $(f, g) \in \mathcal{A}$. Note that g is not uniquely defined.

Example A.2.6
Let $X_t = ct - \sum_{i=1}^{N_t} Y_i$ be a compound Poisson process with drift and f be a (bounded or unbounded) continuous function. We want to analyse the process $\{f(X_t)e^{-\delta t}\}$. For this purpose, we consider the process between the jumps and compensate for the jumps:

$$\begin{aligned}
f(X_t)e^{-\delta t} &= f(X_0) + \sum_{i=1}^{N_t} \Big[f(X_{T_i})e^{-\delta T_i} - f(X_{T_{i-1}})e^{-\delta T_{i-1}} \Big] \\
&\quad + f(X_t)e^{-\delta t} - f(X_{T_{N_t}})e^{-\delta T_{N_t}} \\
&= f(X_0) + \sum_{i=1}^{N_t} \Big[f(X_{T_i})e^{-\delta T_i} - f(X_{T_i-})e^{-\delta T_i-} \Big] \\
&\quad + \sum_{i=1}^{N_t} \int_{T_{i-1}}^{T_i-} \Big[cf'(X_s) - \delta f(X_s) \Big] e^{-\delta s} \, \mathrm{d}s + \int_{T_{N_t}}^{t} \Big[cf'(X_s) - \delta f(X_s) \Big] e^{-\delta s} \, \mathrm{d}s.
\end{aligned}$$

Consider now the jump part. We want to find a measurable function g such that the process

$$\Big\{ \sum_{i=1}^{N_t} \Big[f(X_{T_i-} - Y_i) - f(X_{T_i-}) \Big] e^{-\delta T_i} - \int_0^t g(X_s) \, \mathrm{d}s \Big\}$$

becomes a martingale with the zero expected value. Since the above expression can be written as

$$\sum_{i=1}^{N_t} \Big[\Big(f(X_{T_i-} - Y_i) - f(X_{T_i-}) \Big) e^{-\delta T_i} - \int_{T_{i-1}}^{T_i-} g(X_s) \, \mathrm{d}s \Big] - \int_{T_{N_t}}^{t} g(X_s) \, \mathrm{d}s$$

it is enough to replace t by $T_1 \wedge t$, i.e.

$$\Big(f(X_{T_1-} - Y_1) - f(X_{T_1-}) \Big) e^{-\delta T_1} \mathbb{1}_{T_1 \le t} - \int_0^{t \wedge T_1} g(X_s) \, \mathrm{d}s.$$

T_1 is exponentially distributed. By the lack of memory property of the exponential distribution we do not take the conditioned expectation with regard to $\mathscr{F}_s$ but consider the expected value. We are looking for a function g with

$$\mathbb{E}\Big[\Big(f(X_{T_1-} - Y_1) - f(X_{T_1-}) \Big) e^{-\delta T_1} \mathbb{1}_{T_1 \le t} - \int_0^{t \wedge T_1} g(X_s) \, \mathrm{d}s \Big] = 0.$$

The expected value of the first term is

$$\int_0^t \lambda e^{-\lambda s} e^{-\delta s} \int_0^\infty \Big\{ f(x + cs - y) - f(x + cs) \Big\} \, \mathrm{d}G(y) \, \mathrm{d}s$$

and of the second one

$$\int_0^t \lambda e^{-\lambda s} \int_0^s g(x+cv)\,\mathrm{d}v\,\mathrm{d}s + e^{-\lambda t}\int_0^t g(x+cs)\,\mathrm{d}s = \int_0^t e^{-\lambda s} g(x+cs)\,\mathrm{d}s,$$

where in the last step we used integration by parts. Thus we can choose

$$\begin{aligned} g(X_t) &= \lambda e^{-\delta t}\int_0^\infty \Big(f(X_t-y)-f(X_t)\Big)\,\mathrm{d}G(y) \\ &= \lambda e^{-\delta t}\int_0^\infty f(X_t-y)\,\mathrm{d}G(y) - \lambda e^{-\delta t} f(X_t)\,. \end{aligned} \tag{A.2}$$

Another posibility to determine g is to use Theorem B.2.4. Since

$$\sum_{i=1}^{N_t}\Big(f(X_{T_i-}-Y_i)-f(X_{T_i-})\Big)e^{-\delta T_i} = \int_0^t \Big(f(X_{s-}-Y)-f(X_{s-})\Big)e^{-\delta s}\,\mathrm{d}N_s\,,$$

we have that

$$\begin{aligned} \mathbb{E}\Big[\int_0^t g(X_s)\,\mathrm{d}s\Big] &= \mathbb{E}\Big[\int_0^t \Big(f(X_{s-}-Y)-f(X_{s-})\Big)e^{-\delta s}\,\mathrm{d}N_s\Big] \\ &= \mathbb{E}\Big[\int_0^t \lambda e^{-\delta s}\int_0^\infty \Big(f(X_s-y)-f(X_s)\Big)\,\mathrm{d}G(y)\,\mathrm{d}s\Big] \end{aligned}$$

and therefore (A.2) holds a.s. ■

A.3 Continuous-Time Markov Chains with Finite State Space

Let E be a finite countable set and $\{X_t\}_{t\geq 0}$ be an E-valued stochastic process.

Definition A.3.1
*$\{X_t\}_{t\geq0}$ is called a **continuous-time Markov chain** if for all $i,j,i_1,\dots,i_k \in E$, all $t,s\geq 0$, and all $s_1,\dots,s_k\geq 0$ with $s_l \leq s$ for all $l\in[1,k]$, the Markov-property*

$$\mathbb{P}[X_{t+s}=j|X_s=i,X_{s_1}=i_1,\dots,X_{s_k}=i_k] = \mathbb{P}[X_{t+s}=j|X_s=i] \tag{A.3}$$

*holds, whenever both sides are well-defined. The Markov chain is called **homogeneous** if the right-hand side of* (A.3) *is independent of s.*

Let $\boldsymbol{P}(t) := (p_{ij}(t))_{i,j\in E}$ with

$$p_{ij}(t) := \mathbb{P}[X_{t+s}=j|X_s=i].$$

The family $\{\boldsymbol{P}(t)\}_{t\geq 0}$ is called a **matrix transition function**. It fulfils the Chapman-Kolmogorov equation

$$\boldsymbol{P}(t+s)=\boldsymbol{P}(t)\boldsymbol{P}(s)$$

and is continuous at zero, that is,

$$\lim_{t\downarrow 0}\boldsymbol{P}(t)=\boldsymbol{P}(0)=\boldsymbol{I}.$$

The distribution at time t of X_t is given by the vector $\nu(t)=(\nu_i(t))_{i\in E}$ with $\nu_i(t)=\mathbb{P}[X_t=i]$. It is obtained from the initial distribution by

$$\nu(t)^T=\nu(0)^T\boldsymbol{P}(t),$$

and it holds

$$\mathbb{P}[X_{t_1}=i_1,\ldots,X_{t_n}=i_n]=\sum_{i_0\in E}\nu(0)_{i_0}p_{i_0i_1}(t_1)p_{i_1i_2}(t_2-t_1)\cdots p_{i_{n-1}i_n}(t_n-t_{n-1}),$$

for all $n=0,1,,\ldots,i_0,i_1,\ldots,i_n\in E, 0\leq t_1\leq\ldots\leq t_n$.

Theorem A.3.2
If $\{\boldsymbol{P}(t)\}_{t\geq 0}$ is a matrix transition function, then the following limits exist and are finite:

$$q_{ij}=\lim_{t\downarrow 0}\frac{p_{ij}(t)-\delta_{ij}}{t}.$$

Proof: Refer Rolski et al. [61, p. 311]. □

Corollary A.3.3
For each $i\neq j, q_{ij}\geq 0$ and $q_{ii}\leq 0$. Furthemore, for each $i\in E$,

$$\sum_{j\in E}q_{ij}=0.$$

The matrix $\boldsymbol{Q}=(q_{ij})_{i,j\in E}$ is called the **intensity matrix**. It is convenient to set $q_i=-q_{ii}=\sum_{j\neq i}q_{ij}$.

Remark A.3.4
The family $\{\boldsymbol{P}(t)\}_{t\geq 0}$ of the transition matrices is a contraction semigroup of the Markov chain, and the intensity matrix $\boldsymbol{Q}$ is the infinitesimal generator, since by Theorem A.3.2,

$$\boldsymbol{Q}=\lim_{h\downarrow 0}\frac{\boldsymbol{P}(h)-\boldsymbol{P}(0)}{h}.$$

■

Theorem A.3.5
For all $i, j \in E$ and $t \geq 0$, the transition functions $p_{ij}(t)$ are differentiable and satisfy the following system of differential equations:

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{P}(t)$$

with the initial condition $\boldsymbol{P}(0) = \boldsymbol{I}$.

Proof: Refer Rolski et al. [61, p. 313]. □

Theorem A.3.6
The matrix transition function $\{\boldsymbol{P}(t)\}_{t\geq 0}$ can be represented by its intensity matrix $\boldsymbol{Q}$ via

$$\boldsymbol{P}(t) = e^{t\boldsymbol{Q}}.$$

Proof: Refer Rolski et al. [61, p. 316]. □

The Markov process $\{X_t\}_{t\geq 0}$ is called **irreducible** if for all $i \neq j$, $p_{ij}(t) > 0$ for all $t > 0$, or, equivalently, if for each pair $i \neq j$ there exists a sequence $i_1, \ldots, i_n \in E$ such that $q_{ii_1}q_{i_1i_2}\cdots q_{i_{n-1}j} > 0$.

Remark A.3.7
If the Markov process $\{X_t\}_{t\geq 0}$ is irreducible, then the transition matrix $\boldsymbol{P}(t)$ is regular for each $t > 0$. ■

A probability function $\boldsymbol{\pi}$ on E is called a **stationary distribution** if

$$\boldsymbol{\pi}^T\boldsymbol{P}(t) = \boldsymbol{\pi}^T$$

for all $t \geq 0$. On the finite state space E the condition

$$\boldsymbol{\pi}^T\boldsymbol{Q} = \boldsymbol{0}$$

is necessary and sufficient for the probability distribution $\boldsymbol{\pi}$ to be a stationary distribution (refer Brémaud [11, p. 343]).

Example A.3.8
Let $E = \{1, 2\}$ and $\{X_t\}_{t\geq 0}$ an irreducible Markov chain with the intensity matrix $\boldsymbol{Q}$ given by

$$\boldsymbol{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

Then the stationary distribution is given by

$$\boldsymbol{\pi}^T = (\pi_1, \pi_2)^T = \left(\frac{q_2}{q_1 + q_2}\ ,\quad \frac{q_1}{q_1 + q_2}\right)^T.$$

■

Theorem A.3.9
If the Markov process $\{X_t\}_{t\geq 0}$ is irreducible, then for each $i \in E$,

$$\lim_{t\to\infty} \mathbb{P}[X_t = i] = \pi_i,$$

where $\boldsymbol{\pi}$ is the stationary initial distribution of $\{X_t\}_{t\geq 0}$.

Proof: Refer Rolski et al. [61, p. 323]. □

Example A.3.10 (**Embedded Markov chain**)
Let $\{\tau_n\}_{n\geq 0}$ be the nondecreasing sequence of transition times of the Markov chain $\{X_t\}_{t\geq 0}$ with $\tau_0 = 0$ and $\tau_n = \inf\{t > \tau_{n-1} : X_t \neq X_{t-}\}$. We set $\tau_n = \infty$ if there are strictly fewer than n transitions in $(0, \infty)$.

Let Δ be an arbitrary element not in E with the convention $X_\infty = \Delta$. The process $\{X_{\tau_n}\}_{n\geq 0}$ is called the **embedded process**.

Theorem A.3.11
Let $\{X_t\}_{t\geq 0}$ be a homogeneous Markov chain with intensity matrix $\boldsymbol{Q}$ and transition times $\{\tau_n\}_{n\geq 0}$. Then

1. *The embedded process $\{X_{\tau_n}\}_{n\geq 0}$ is a discrete-time homogeneous Markov chain with state space $E_\Delta = E \cup \{\Delta\}$ and transition matrix given by*

$$p_{ij} = \frac{q_{ij}}{q_i}$$

if $q_i > 0$ and $j \neq i$; by $p_{\Delta\Delta} = 1, p_{i\Delta} = 1$ if $i \in E$ and $q_i = 0$; and by $p_{i\Delta} = 0$ if $i \in E$ and $q_i > 0$.

2. *Given $\{X_{\tau_n}\}_{n\geq 0}$, the sequence $\{\tau_{n+1} - \tau_n\}_{n\geq 0}$ is independent, and for all $n \geq 0$ and all $a \in \mathbb{R}_+$,*

$$\mathbb{P}[\tau_{n+1} - \tau_n \leq a | \{X_{\tau_k}\}_{k\geq 0}] = 1 - e^{-q_{X_{\tau_n}} a}.$$

Proof: Refer Brémaud [11, p. 348] ■

Appendix B

Poisson Processes

Let$\{T_k\}_{k\geq 0}$ be a sequence of nonnegative random variables on the positive half-line such that, almost surely,

(i) $T_0 = 0$,

(ii) $0 < T_1 < T_2 < \ldots,$

(iii) $\lim_{k\to\infty} T_k = \infty$.

We call $\{T_k\}$ **arrival** (or occurrence) **times** and consider the **random counting measure** $\{N(B) : B \in \mathcal{B}(\mathbb{R})\}$ with

$$N(B) = \sum_{k=1}^{\infty} \mathbb{I}_{\{T_k \in B\}}.$$

We define a **counting process** $\{N_t\}_{t\geq 0}$ by

$$N_t = N((0,t]) = \sum_{k=1}^{\infty} \mathbb{I}_{\{T_k \leq t\}}.$$

B.1 Homogeneous Poisson Processes

Theorem B.1.1
Let $\{N_t\}_{t\geq 0}$ be a counting process with occurrence times $0 = T_0 < T_1 < \cdots$ and $\lambda > 0$. The following conditions are equivalent:

1. *$\{N_t\}$ has stationary and independent increments such that*

$$\mathbb{P}[N_h = 0] = 1 - \lambda h + o(h), \quad \mathbb{P}[N_h = 1] = \lambda h + o(h) \quad \text{as} \quad h \downarrow 0.$$

2. *$\{N_t\}$ has independent increments and N_t is Poisson-distributed with parameter λt for each $t > 0$.*

3. *The interarrival times $\{T_k - T_{k-1}\}_{k\geq 1}$ are independent and exponentially distributed with parameter λ.*

Proof: Refer Rolski et al. [61, pp. 157-160]. □

$\{N_t\}$ is called a **homogeneous Poisson process** with intensity λ.

Let $\{Y_i\}_{i\geq 1}$ be a sequence of iid random variables independent of $\{N_t\}$. The process $\Big\{\sum_{i=1}^{N_t} Y_i\Big\}$ is called a **compound Poisson process** and has independent and stationary increments. By Lemma A.1.2, the compound Poisson process (and, as a special case with $Y_i = 1$, the Poisson process) is a Markov process.

Lemma B.1.2 (Thinning)
Let $\{N_t\}$ be a Poisson process with parameter λ. Let $\{I_k\}_{k\in\mathbb{N}}$ be a sequence of iid random variables independent of $\{N_t\}$ with $\mathbb{P}[I_k = 1] = 1 - \mathbb{P}[I_k = 0] = p$ for some $p \in (0,1)$. Then $\Big\{\sum_{k=1}^{N_t} I_k\Big\}$ and $\Big\{\sum_{k=1}^{N_t}(1-I_k)\Big\}$ are independent Poisson processes with parameters λp and $\lambda(1-p)$, respectively.

Proof: Refer Grandell [37]. □

B.2 Cox Processes

Definition B.2.1
Let $\{\lambda(t)\}_{t\geq 0}$ be a nonnegative measurable stochastic process such that

$$\int_0^t \lambda(s)\,ds < \infty, \quad \mathbb{P}\text{-}a.e., \quad t \geq 0.$$

*We call $\{\lambda(t)\}$ an **intensity process**, and $\{\Lambda(B) : B \in \mathcal{B}(\mathbb{R})\}$ with $\Lambda(B) = \int_B \lambda(v)\,dv$ a **cumulative intensity measure**. A counting measure $N(B)$ is called a **Cox process** or a **doubly stochastic Poisson process** if for all $n = 1, 2, \ldots,$ for $k_1, \ldots, k_n \in \mathbb{N}$, and $a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n$*

$$\mathbb{P}\Big[\bigcap_{i=1}^{n}\{N((a_i, b_i]) = k_i\}\Big] = \mathbb{E}\Big[\prod_{i=1}^{n} \frac{(\int_{a_i}^{b_i} \lambda(v)\,dv)^{k_i}}{k_i!} \exp\Big(-\int_{a_i}^{b_i} \lambda(v)\,dv\Big)\Big].$$

Thus, conditioning on $\{\lambda(t)\}$, $\{N_t\}$ is a Poisson process with intensity function $\{\lambda(t)\}$. If $\{\lambda(t)\}$ is deterministic, then $\{N_t\}$ is a (non-homogeneous) Poisson process. If $\lambda(t) \equiv \Lambda$ for some nonnegative random variable Λ, then $\{N_t\}$ is called a mixed Poisson process.

Example B.2.2 (Markov-modulated Poisson process)
Let $\{J_t\}_{t\geq 0}$ be a Markov process with state space $E = \{1, \ldots, m\}$ and intensity matrix $\boldsymbol{Q} = (q_{ij})_{i,j\in E}$. By a **Markov-modulated Poisson process** we mean a Cox process whose intensity process $\{\lambda(t)\}$ is given by

$$\lambda(t) = \lambda_{J_t}.$$

■

Lemma B.2.3
A Markov-modulated Poisson process has stationary increments if $\{J_t\}$ has a stationary initial distribution.

Proof: Refer Rolski et al. [61]. □

Theorem B.2.4
Let $\{N_t\}$ be an $\{\mathscr{F}_t\}$-adapted (doubly stochastic) Poisson process with intensity function $\{\lambda(t)\}$. Then

1. *$M_t = N_t - \int_0^t \lambda(s)\, ds$ is an $\{\mathscr{F}_t\}$-local martingale;*
2. *if $\{X_t\}$ is an $\{\mathscr{F}_t\}$-predictable process such that*

$$\mathbb{E}[\int_0^t |X_s|\lambda(s)\, ds] < \infty, \quad t \geq 0,$$

then $\int_0^t X_s\, dM_s$ is an $\{\mathscr{F}_t\}$-martingale.

Proof: Refer Brémaud [10, p. 27]. □

Appendix C

Linear Algebra

C.1 Determinants

Let $\boldsymbol{A} = (a_{ij})_{i,j=1}^{n}$ be a $n \times n$-matrix. We denote by $\boldsymbol{A}_{ij}$ the $(n-1) \times (n-1)$-matrix resulting from A by removing the $i-$th row and the $j-$th column.

Definition C.1.1
*The (i,j)-**minor** M_{ij} of the matrix $\boldsymbol{A}$ is defined by*

$$M_{ij} = \det \boldsymbol{A}_{ij} \ .$$

*The **adjugate** $\boldsymbol{A}^*$ of the matrix $\boldsymbol{A}$ is defined by*

$$a^*_{ij} = (-1)^{i+j} M_{ji} \ .$$

For the special case $n = 2$, the adjugate can be easily calculated as

$$\boldsymbol{A}^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \ .$$

Theorem C.1.2 (Laplace's formula)
For all $j = 1, \ldots, n$ holds

$$\det \boldsymbol{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \ .$$

Corollary C.1.3
For any invertible matrix $\boldsymbol{A}$ holds

$$\boldsymbol{A}^{-1} = \frac{\boldsymbol{A}^*}{\det \boldsymbol{A}} \ .$$

For $n = 2$ we have

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \ .$$

C.2 Eigenvalues

Let $\boldsymbol{A}$ be a $n \times n$-matrix, $\boldsymbol{\phi}, \boldsymbol{\psi}$ be n-dimensional non-zero vectors and $\theta \in \mathbb{R}$ or $\theta \in \mathbb{C}$.

Definition C.2.1
If

$$\boldsymbol{A}\boldsymbol{\phi} = \theta \boldsymbol{\phi},$$

*then θ is said to be an **eigenvalue** of $\boldsymbol{A}$ and $\boldsymbol{\phi}$ a **right eigenvector** corresponding to θ. If*

$$\boldsymbol{\psi}^T \boldsymbol{A} = \theta \boldsymbol{\psi}^T,$$

*then $\boldsymbol{\psi}$ is said to be a **left eigenvecto**r corresponding to θ.*

The eigenvalues are exactly the solutions to the **characteristic equation**

$$\det(\boldsymbol{A} - \theta \boldsymbol{I}) = 0.$$

This is an algebraic equation of order n, i.e. there are n eigenvalues $\theta_1, \dots, \theta_n$, which can be complex and some of them can coincide. We assume that the eigenvalues are numbered such that

$$|\theta_1| \geq |\theta_2| \geq \cdots |\theta_n|$$

and denote the set of eigenvalues of $\boldsymbol{A}$ by $\mathrm{sp}(\boldsymbol{A})$.

Definition C.2.2
*A nonnegative matrix $\boldsymbol{A}$ is called **stochastic**, if $\sum_j a_{ij} = 1$ for all $i = 1, \dots, n$. A nonnegative matrix $\boldsymbol{A}$ is called **regular**, if there exists some $n_0 \geq 1$ such that all entries of $\boldsymbol{A}^{n_0}$ are strictly positive.*

Theorem C.2.3 (Perron-Frobenius theorem)
If $\boldsymbol{A}$ is a regular stochastic matrix, then $\theta_1 = 1$ and $|\theta_i| < 1$ for $i = 2, \dots, n$.

Proof: See Rolski et al. [61, p.284].

C.3 Matrix-exponentials

The **exponential** $e^{\boldsymbol{A}}$ of a $n \times n$-matrix $\boldsymbol{A}$ is defined by the series expansion

$$e^{\boldsymbol{A}} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!} \ .$$

The next Lemma shows that the series is a well-defined matrix.

Lemma C.3.1
For $h \in \mathbb{R}$, *the series* $\sum_{n=0}^{\infty} \frac{(h\boldsymbol{A})^n}{n!}$ *converges uniformly with respect to* $h \in [-h_0, h_0]$, *for each* $h_0 > 0$.

Proof: See Rolski et al. [61, p. 314]. $\square$

Lemma C.3.2
On the whole real line holds

$$\frac{de^{h\boldsymbol{A}}}{dh} = \boldsymbol{A}e^{h\boldsymbol{A}} = e^{h\boldsymbol{A}}\boldsymbol{A}.$$

Proof: See Rolski et al. [61, p. 315]. $\square$

Lemma C.3.3
If all eigenvalues of an invertible matrix $\boldsymbol{A}$ *have negative real parts, then*

$$\int_0^{\infty} e^{t\boldsymbol{A}}\, dt = -\boldsymbol{A}^{-1}.$$

Proof: See Rolski et al. [61, p. 328]. $\square$

For the eigenvalues of the exponential $e^{\boldsymbol{A}}$ holds

$$\mathrm{sp}(e^{\boldsymbol{A}}) = \{e^{\theta} : \theta \in \mathrm{sp}(\boldsymbol{A})\}.$$

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