Taut Submanifolds and Foliations

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Abstract

We give an equivalent description of taut submanifolds of complete Riemannian manifolds as exactly those submanifolds whose normal exponential map has integrable fibers. It turns out that every taut submanifold is also $\mathbb{Z}_2$-taut, so that tautness is essentially the same as $\mathbb{Z}_2$-tautness. In the case where the normal exponential map of a submanifold has integrable fibers, we explicitely construct generalized Bott-Samelson cycles for the critical points of the energy functionals on the path spaces which, generically, represent a basis for the $\mathbb{Z}_2$-cohomology. We also consider singular Riemannian foliations all of whose leaves are taut and discuss some of their main features. Using our characterization of taut submanifolds, we are able to show that tautness of a singular Riemannian foliation is actually a property of the quotient.

Kurzzusammenfassung

Wir beschreiben straffe Untermannigfaltigkeiten einer vollständigen Riemannschen Mannigfaltigkeit äquivalent durch die Eigenschaft, dass die Fasern ihrer normalen Exponentialabbildung integrierbar sind. Diese äquivalente Charakterisierung impliziert, dass eine straffe Untermannigfaltigkeit stets $\mathbb{Z}_2$-straff ist, was zeigt, dass die Begriffe von Straffheit und $\mathbb{Z}_2$-Straffheit im Wesentlichen identisch sind. In dem Fall, dass die normale Exponentialabbildung einer Untermannigfaltigkeit integrierbare Fasern hat, konstruieren wir explizit verallgemeinerte Bott-Samelson Zykel für die kritischen Punkte des Energiefunktionals auf generischen Wegeräumen, die eine Basis für die $\mathbb{Z}_2$-Kohomologie bilden. Zudem betrachten wir singuläre Riemannsche Blätterungen mit ausnahmslos straffen Blättern und untersuchen einige ihrer speziellen Merkmale. Mit Hilfe unserer Beschreibung von Straffheit können wir zeigen, dass die Eigenschaft, dass alle Blätter einer solchen Blätterung straff sind tatsächlich eine Eigenschaft des Quotienten ist.

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Contents

Introduction

The terminology of taut submanifolds was introduced by Carter and West in [CW72], where they call a submanifold L of a Euclidean space V *taut* if all the squared distance functions $d_q^2 : L \to \mathbb{R}$, $d_q^2(p) = \|p-q\|^2$, with respect to points $q \in V$ that are not focal points of L are perfect Morse functions for some field $\mathbb{F}$, i.e. if the number of critical points of index k of d_q^2 coincides with the k-th Betti number of L with respect to the field $\mathbb{F}$ for all k. If L is taut and $\mathbb{F}$ is a field as in the defintion of tautness, then L is also called $\mathbb{F}$-*taut*. Thus, geometrically, taut submanifolds are as round as possible. If one tries to generalize this defintion to submanifolds of arbitrary Riemannian manifolds, the problem arises that the squared distance function is not a priori everywhere smooth anymore. Namely, it is not differentiable in the cut locus of the submanifold.

Using different approaches, Grove and Halperin [GH91] and, independently, Terng and Thorbergsson [TT97] generalized this notion to submanifolds L of complete Riemannian manifolds M by saying that L is taut if there exists a field $\mathbb{F}$ such that every energy functional $E_q(c) = \int_{[0,1]} \|\dot{c}(t)\|^2 dt$ on the space $\mathcal{P}(M, L \times q)$ of H^1-paths $c : [0,1] \to M$ from L to a fixed point $q \in M$ is a perfect Morse function with respect to $\mathbb{F}$ if q is not a focal point of L. The critical points of E_q are exactly the geodesics parameterized proportionally to arc length that start orthogonally to L and end in q. In particular, in the case where $M = V$ is a Euclidean space, there is an obvious way to identify a submanifold L with the space of segments in $\mathcal{P}(V, L \times q)$ and under this identification the map d_q^2 corresponds to E_q. Further, it is not hard to see that in this case the path space $\mathcal{P}(V, L \times q)$ admits the subspace of segments from L to q as a strong deformation retract. So the definitions agree for submanifolds of a Euclidean space and it turns out that this is indeed the right way to generalize tautness.

It is shown in [TT97] that if $L \subset M$ is an $\mathbb{F}$-taut submanifold, then the energy functionals E_q are Morse-Bott functions for all points $q \in M$. Our first main result now states that this property actually characterize taut submanifolds.

Theorem A. *A closed submanifold L of a complete Riemannian manifold M is taut if and only if all the energy functionals $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ are Morse-Bott functions.*

In fact, if all the energy functionals are Morse-Bott functions, then the field with respect to which L is taut is $\mathbb{Z}_2$. Thus, as a direct consequence, we obtain the following result, which was, just as Theorem A, so far not even known in the case of a Euclidean space.

Theorem B. *If a submanifold is $\mathbb{F}$-taut, then it is also $\mathbb{Z}_2$-taut.*

Based on this result it is suggested to consider only $\mathbb{Z}_2$-taut submanifolds, so that we no longer distinguish between $\mathbb{Z}_2$-taut and taut.

As the definition shows, tautness is a very special property. In some sense, it is a kind of homogeneity requirement for the pair (M, L). So it is no surprise that so far not many examples of taut submanifolds are known. This makes it all the more remarkable that taut submanifolds, if at all, often occur in families which then decompose the ambient space, e.g. an orbit decomposition induced by the isotropy representation of a symmetric space. It is for this reason that we study such families as they usually appear, i.e. singular Riemannian foliations with only taut leaves. We call such families *taut foliations*. As a main result in this direction we observe that tautness of a foliation is indeed a property of the quotient of the foliation, so that it actually makes sense to talk about taut quotients as equivalence classes of quotients under isometries.

Theorem C. *Let $\mathcal{F}$ and $\mathcal{F}'$ be closed singular Riemannian foliations on complete Riemannian manifolds M and M' such that their quotients $M/\mathcal{F}$ and $M'/\mathcal{F}'$ are isometric. Then $\mathcal{F}$ is taut if and only if $\mathcal{F}'$ is taut.*

Due to this result one could think about taut foliations as foliations with *pointwise taut* quotients, where we follow [Le06] and call a manifold pointwise taut if all of its points are taut (submanifolds). Of course, in general a quotient of a singular Riemannian foliation is far from being a manifold, but as soon as it is a nice space in the sense that one could use differential geometric methods it turns out that this picture is reasonable. Viewed in this light, the largest class of spaces for which one has the appropriate tools available is the class of Riemannian orbifolds, i.e. spaces locally modelled by quotients of Riemannian manifolds modulo the action of a finite group of isometries. Now, given a taut singular Riemannian foliation $\mathcal{F}$ on M such that the quotient $M/\mathcal{F}$ is an orbifold, it follows that $M/\mathcal{F}$ is already a good Riemannian orbifold, that is to say $M/\mathcal{F}$ is isometric to N/Γ, where N is a Riemannian manifold and $\Gamma \subset I(N)$ is a discrete group of isometries. This observation together with the last theorem leads directly to our next result, which mainly motivates our picture of pointwise taut quotients.

Theorem D. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is taut and $M/\mathcal{F}$ is an orbifold if and only if $M/\mathcal{F}$ is isometric to N/Γ, where N is a pointwise taut Riemannian manifold and $\Gamma \subset I(N)$ is a discrete group of isometries of N.*

In view of applications, the more interesting direction of this result is that the existence of a pointwise taut quotient covering implies tautness of the foliation. The known examples of pointwise taut Riemannian manifolds are mainly two classes of spaces together with Riemannian products of elements of these classes. The first one is the class of symmetric spaces, which are pointwise taut by the work of Bott and Samelson [BS58] and the second class consists of manifolds without conjugate points, e.g. manifolds with non-positive curvature. In fact, if there are no conjugate points along any geodesic in a Riemannian manifold, the index of every critical point of a given energy functional is zero, hence all points in such a manifold are taut. A conjecture in [TT97] states that a compact pointwise taut Riemannian manifold that has the homotopy type of a compact symmetric space is symmetric. We want to mention as an aside that it is shown in [TT97] that in the case of a compact rank-one symmetric space this conjecture is equivalent to the Blaschke conjecture which is still not settled.

It is therefore not a surprise that in all known examples of taut foliations with orbifold quotients these quotients are isometric to a space $(N \times P)/\Gamma$, where N is a symmetric space, P is a manifold without conjugate points, and Γ is a discrete subgroup of isometries. Consider, for instance, the parallel foliation $\mathcal{F}$ of a Euclidean space V that is induced by an isoparametric submanifold L of V. Such a foliation is a singular Riemannian foliation and is also called an *isoparametric foliation*. One can show that the isoparametric foliations on V are exactly the polar ones. In this case, the quotient $V/\mathcal{F}$ is isometric to $(p + \nu_p(L))/\Gamma$, where $p \in L$ is some point and Γ is the finite Coxeter group generated by the reflections across the focal hyperplanes in $p+\nu_p(L) \subset V$. So $V/\mathcal{F}$ admits a flat orbifold covering which is a manifold, thus $\mathcal{F}$ is taut. In particular, our result implies that isoparametric submanifolds are taut, what is well known by [PT88]. More generally, we see again that the orbits of hyperpolar actions, e.g. the orbits of the action induced by the isotropy representation of a symmetric space, are taut. Since totally geodesic submanifolds of compact rank-one symmetric spaces are also compact rank-one symmetric spaces (see [Wo63] for a classification), and sections of a polar action are always totally geodesic, we also reobtain the result from [BG07] that a polar action of a compact Lie group on a compact rank-one symmetric space is taut. In fact, since the sections must also admit a Weyl group, they are always real, i.e. a sphere or a real projective space. In [GT03] Gorodski and Thorbergsson classified all taut irreducible representations of compact Lie groups as either hyperpolar and hence equivalent to the isotropy representation of a symmetric space or as one of the exceptional representations of cohomogeneity three. Let $\rho : G \to \mathbf{O}(V)$ be an exeptional representation, i.e. the induced action of G on V has cohomogeneity equal to three. Then, the restriction of this action on the unit sphere $S \subset V$ has cohomogeneity two such that S/G is isometric to a quotient S^2/Γ of the round 2-sphere with a finite Coxeter group Γ. Since it is, by linearity, not hard to see that the orbits of the G-action on V are taut if and only if the orbits of the G-action on S are taut, it follows from Theorem D again that the exeptional representations are taut.

Let us now say some words about the organization and the ideas of the present work. This work contains three chapters, all of which are devoted to one of our main results. Every chapter is subdivided into three sections, where the first two sections always provide definitions, preliminaries, and tools and the third section is intended for the respective main theorem. A detailed discussion of path spaces can be found in the appendix, where we gather some background material such as finite-dimensional approximations and formulas for the energy functional.

Practically, the only way to prove that a given submanifold $L \hookrightarrow M$ is taut is the explicit construction of so called *linking cycles* for the energy. Namely, one has to find cycles that complete the local unstable manifolds associated to some Morse chart around the critical points below the corresponding critical energy. This concept is introduced in Section 1.1. For the proof of Theorem A in Section 1.3 (see Theorem 1.3.1) we therefore first make the observation that all the energy functionals $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ are Morse-Bott functions if and only if the normal exponential map $\exp^{\perp} : \nu(L) \to M$ has integrable fibers. If so, we explicitely construct linking cycles for non-degenerated critical points, i.e. a basis for the (co-)homology of $\mathcal{P}(M, L \times q)$ if $q \in M$ is not a focal point, proving that L is taut. For this purpose, for a normal vector $v \in \nu(L)$, we define Z_v to be the set of all piecewise continuous paths $c : [0, 1] \to \nu(L)$ obtained as follows. Follow the segment tv towards the zero section up to the first focal vector $t_1 v$, then take an arbitrary normal vector v_1 in the fiber of $\exp^{\perp}$ through $t_1 v$ and follow the segment tv_1 towards the zero section up to first focal vector $t_2 v_1$, then take an arbitrary normal vector in the fiber through $t_2 v_1$ and repeat this procedure. By construction, for every $c \in Z_v$, $\exp^{\perp} \circ c$ is a broken geodesic from $\exp^{\perp}(v)$ to L and we define the space $\Delta_v \subset \mathcal{P}(M, L \times \exp^{\perp}(v))$ to consist of all broken geodesic obtained in this manner reparameterized on $[0, 1]$ after reversing the orientation. Using a powerful tool developed in Section 1.2, we then prove that Δ_v defines a linking cycle for the geodesic $\exp^{\perp}(tv) \in \mathcal{P}(M, L \times \exp^{\perp}(v))$ if the endpoint is not a focal point of L along this geodesic. Thus, if $q \in M$ is not a focal point of L, the so defined cycles indeed represent a basis for the (co-)homology of the path space $\mathcal{P}(M, L \times q)$. As mentioned above, Theorem B is then a direct consequence of this result.

In the second chapter we introduce the notion of singular Riemannian foliations and make some preliminary observations about taut foliations in the first section. As a main part of the definition, a geodesic either meets the leaves of a singular Riemannian foliation orthogonally at all or at none of its points. If a geodesic intersects one and hence all leaves perpendicularly, it is called *horizontal*. Roughly speaking, the possibilities to vary a horizontal geodesic through horizontal geodesics consist of variations of the projection of the geodesic to the quotient and of variations through horizontal geodesics all of which meeting the same leaves simultaneously. This results in an index splitting for horizontal geodesics into *horizontal* and *vertical* index that we discuss in Section 2.2, the latter one counting the intersections with the *singular leaves* (with their multiplicities). In Section 2.3 we then prove Theorem C (see Theorem 2.3.1) using Theorem A and the fact that the horizontal index is an intrinsic notion of the quotient. Using the arguments from the proof of Theorem C, we are able to construct new examples of taut submanifolds.

The third chapter deals with Theorem D, a refined version of Theorem C for a special class of singular Riemannian foliations, the so called *infinitesimally polar* foliations. These are exactly those singular Riemannian foliations whose (local) quotients are orbifolds. For this reason, we introduce the notion of orbifolds in Section 3.1. In

Section 3.2 we recall Lytchak's equivalent description of infinitesimally polar foliations as those foliations admitting a *geometric resolution* and use his construction of the canonical geometric resolution in Section 3.3 to observe that the quotient of a taut infinitesimally polar foliation is *developeable*. Instead of finishing the proof of Theorem D with Theorem C and this observation, we give a clearer and more geometric proof of Theorem D in Section 3.3 that only uses Theorem A (see Theorem 3.3.1).

Our proof of Theorem D (as well as Theorem A and Theorem C) may be viewed as a generalization of the construction of Bott and Samelson in [BS58] that proves that the orbit foliation of a variationally complete action, i.e. when the focal points of the orbits are only caused by singular orbits, is taut. Given an orbit of such an action, Bott and Samelson came up with concrete cycles associated to the critical points of the energy on the space of paths to a generic point which, generically, represent a basis for the $\mathbb{Z}_2$-(co-)homology of the corresponding path space. Let (G, K) be a symmetric pair, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Then there are three natural actions associated to the symmetric pair. Namely, the standard action of K on G/K, the action of K on $\mathfrak{p}$ induced by the adjoint representation $k \mapsto \mathrm{Ad}(k) \in \mathbf{O}(\mathfrak{p})$, and the action of $K \times K$ on G via $(k_1, k_2) \cdot g = k_1 g k_2^{-1}$. In their study of Morse theory of symmetric spaces, Bott and Samelson proved in [BS58] that all of these actions are variationally complete and therefore taut. To get a picture of their cycle construction, let us consider the action of K on $\mathfrak{p}$ given by the isotropy representation of the symmetric space $N = G/K$ and let us denote by $\mathcal{F}^K$ the induced orbit foliation. It is well known that this action of K on $\mathfrak{p}$ is hyperpolar, i.e. for all K-regular points $p \in \mathfrak{p}$, the linear subspace $\Sigma_p = p + \nu_p(K(p))$ intersects all orbits and always orthogonally, so that $\mathfrak{p}/\mathcal{F}^K = \Sigma_p/\Gamma_p$, where Γ_p is the Weyl group associated to the section Σ_p. Let $Q : \mathfrak{p} \to \mathfrak{p}/\mathcal{F}^K$ and $Q' : \Sigma_p \to \Sigma_p/\Gamma_p = \mathfrak{p}/\mathcal{F}^K$ denote the respective quotient projections. Now, given a regular orbit $K(p)$ and a point $q \in \mathfrak{p}$ that is not a focal point of $K(p)$, the work of Bott and Samelson implies that for every critical point $\gamma \in \mathcal{P}(\mathfrak{p}, K(p) \times q)$ the space Δ_γ consisting of all horizontal polygons from $K(p)$ to q that have the same projection to $\mathfrak{p}/\mathcal{F}^K$ as γ is a compact submanifold of $\mathcal{P}(\mathfrak{p}, K(p) \times q)$ of dimension equal to the index of γ (as a critical point of E_q) whose fundamental class indeed provides a linking cycle for the critical point γ.

To generalize this construction, let γ' be a lift of the orbifold geodesic $Q \circ \gamma$ to Σ_p. Then γ' is a critical point of the energy $E_{\gamma'(1)}$ on the path space $\mathcal{P}(\Sigma_p, \gamma'(0) \times \gamma'(1))$ with index equal to zero. Thus the one point set $\Delta_{\gamma'} = \{\gamma'\}$ represents a linking cycle for γ' and the linking cycle Δ_γ is nothing else than the space of all horizontal polygons c in $\mathcal{P}(\mathfrak{p}, K(p) \times q)$ with $Q \circ c = Q' \circ d$ for some $d \in \Delta_{\gamma'}$.

Therefore, this construction admits a natural generalization to the case where the quotient of a foliation $\mathcal{F}$ on a manifold M is a good orbifold N/Γ and N is pointwise taut, but not necessarily flat. Namely, for a horizontal geodesic γ starting at a regular leaf L, we consider a lift γ' of the projected orbifold geodesic $Q \circ \gamma$ to N, where again $Q : M \to M/\mathcal{F}$ denotes the quotient map. Then, the index of this lift as a critical point of the energy on $\mathcal{P}(N, \gamma'(0) \times \gamma'(1))$ equals the horizontal index of γ and one obtains a linking cycle Δ_γ for γ, intuitively having the right dimension, as follows. Let $v \in T_{\gamma'(0)}N$ denote the initial velocity and let Δ_v be the linking cycle for γ' as described in the discussion of the proof of Theorem A above. Define Δ_γ to be the space of all broken horizontal geodesics c in $\mathcal{P}(M, L \times \gamma(1))$ with $Q \circ c = Q' \circ d$ for some $d \in \Delta_v$, where $Q' : N \to N/\Gamma = M/\mathcal{F}$ is the natural projection. In this way, the space Δ_γ can be regarded, in some sense, as a singular fiber bundle over Δ_v with fibers as in the case of a flat quotient covering, having the right (co-)homological properties, so that it indeed defines a linking cycle for γ. It turns out that this construction does not depend on the fact that N is trivially foliated by points and therefore could also be used in the case where two foliations have isometric quotients proving that tautness of a foliation is in fact a property of the quotient as it is stated in Theorem C, because a dense family of taut leaves forces a foliation to be taut.

Finally, we must warn the reader that the use of the term "taut foliation" could lead to confusions. In the theory of (regular) foliations there are other definitions of tautness, such as geometrically or topologically taut foliations. But in this work, by a taut foliation, we always mean a singular Riemannian foliation all of whose leaves are taut submanifolds as defined above. Likewise, a taut action is an isometric action such that the induced orbit foliation is taut in the way we defined it.

Chapter 1

Taut Submanifolds

1.1 Linking Cycles

The terminology of tautness for submanifolds of a Euclidean space was introduced by Carter and West in [CW72]. They call a submanifold L of a Euclidean space V *taut* if there exists a field $\mathbb{F}$ such that for generic points $q \in V$, the squared distance functions $d_q^2 : L \to \mathbb{R}$, given by $d_q^2(p) = \|p - q\|^2$, are perfect with respect to the field $\mathbb{F}$. A definition similar to this can be used for submanifolds of the round sphere $S^n \subset V$.

Reminder. A Morse function on a complete Hilbert manifold P is a smooth function $f : P \to \mathbb{R}$, which is bounded below, has a discrete critical set $C(f)$ and satisfies Condition (C), i.e. if (p_n) is a sequence of points in P with $\{f(p_n)\}$ bounded and $\|df_{p_n}\| \to 0$, then (p_n) has a convergent subsequence. Thus Condition (C) can be regarded as an analogue of a compactness claim in the infinite-dimensional setting.

If p is a critical point for the Morse function f, then the index $\operatorname{ind}(p)$ is defined to be the dimension of a maximal subspace of T_pP on which the Hessian is negative definite, i.e. the number of independent directions in which f decreases. As in the finite dimensional case a Morse function gives rise to a cell complex with one cell of dimension k for each critical point with index k, which is homotopy equivalent to P. If we set $P^r = \{p \in P | f(p) \leq r\}$, then the weak Morse inequalities say that if $\nu_k(a, b)$ denotes the number of critical points of index k in $f^{-1}(a, b)$ for regular values $a < b$, then $b_k(P^b, P^a; \mathbb{F}) \leq \nu_k(a, b)$ for all k, where $b_k(P^b, P^a; \mathbb{F})$ is the k-th Betti number of (P^b, P^a) with respect to the field $\mathbb{F}$ and f is called *perfect* (*with respect to* $\mathbb{F}$) if the weak Morse inequalities are equalities for all k and all regular values $a < b$. For a detailed background we refer the reader to Part II of [PT88].

Using different approaches, Grove and Halperin [GH91] as well as Terng and Thorbergsson [TT97] defined a general notion of taut immersions into a complete Riemannian manifold. In [TT97] it has been proven that for submanifolds of a Euclidean space and the round sphere the generalized definition of tautness coincides with the one previously known. We are going to introduce this generalized notion using the exposition in [TT97].

Let M be a complete Riemannian manifold and let $H^1(I, M)$ denote the complete Hilbert manifold of H^1 paths $I = [0,1] \to M$ with its canonical differentiable structure (cf. Appendix A.1). Recall that a path is of class H^1 if and only if it is absolutely continuous with finite energy. Furnished with this differentiable structure, the map $e : H^1(I, M) \to M \times M$, given by $e(c) = (c(0), c(1))$, defines a submersion.

Now for a proper immersion $\phi : L \to M$ into a complete Riemannian manifold and a point $q \in M$, we define the path space $\mathcal{P}_{(\phi,q)}(M, L)$ to be the pullback of $H^1(I, M)$ along the map $p \mapsto (\phi(p), q)$ from L into $M \times M$, i.e. $\mathcal{P}_{(\phi,q)}(M, L)$ consists of pairs $(p, c) \in L \times H^1(I, M)$ with $\phi(p) = c(0)$ and $c(1) = q$. In particular, $\mathcal{P}_{(\phi,q)}(M, L)$ inherits a smooth structure that turns it into a complete Hilbert manifold and one can show that the induced energy functional $E_{(\phi,q)} : \mathcal{P}_{(\phi,q)}(M, L) \to \mathbb{R}$, defined by

$$E_{(\phi,q)}((p, c)) = \int_I \|\dot{c}(t)\|^2 dt,$$

is a Morse function if and only if q is not a focal point of L. The critical points of $E_{(\phi,q)}$ are exactly the pairs (p, γ), where γ is a geodesic, parameterized proportionally to arc length, which starts perpendicularly to L and ends in q. By the famous theorem of Morse, the index of a critical point (p, γ) is then given by the sum $i((p, \gamma)) = \sum_{t \in (0,1)} \mu(t)$ over the multiplicities $\mu(t)$ of the points $\gamma(t)$ as focal points of L along γ. In our setting of path spaces every energy sublevel contains a finite dimensional submanifold such that the restriction of the energy to this submanifold has the same relevant behavior, i.e. the critical points of the restriction are exactly the critical points of the energy functional and their indices and nullities coincide. In particular, the indices and nullities are finite.

For these facts and a detailed discussion on path spaces and the energy functional we refer the reader who is not familiar with these notions to the appendix (cf. A.1).

Finally, if ϕ is a closed embedding identifying L with its image $\phi(L) \subset M$, we will drop the reference to the map ϕ and simply write $\mathcal{P}(M, L \times q)$ instead of $\mathcal{P}_{(\phi,q)}(M, L)$ for the space of H^1-paths from L to q.

Terng and Thorbergsson proved that in the case where M is a Euclidean space or a round sphere a properly immersed submanifold $\phi : L \to M$ is taut if and only if the energy functional $E_{(\phi,q)} : \mathcal{P}_{(\phi,q)}(M, L) \to \mathbb{R}$ is perfect for generic $q \in M$. This led them to a natural generalization of the notion of a taut immersion into any complete Riemannian manifold M.

Definition 1.1.1. A proper immersion $\phi : L \to M$ of a manifold L into a complete Riemannian manifold (M, g) is called *taut* if there exists a field $\mathbb{F}$ such that the energy functional $E_{(\phi,q)} : \mathcal{P}_{(\phi,q)}(N, L) \to \mathbb{R}$, given by $E_{(\phi,q)}(p, c) = \int_I \|\dot{c}(t)\|^2 dt$, is a perfect Morse function with respect to the field $\mathbb{F}$ for every point $q \in M$ that is not a focal point of L. In particular, a point $p \in M$ is called a *taut point* if $\{p\}$ is a taut submanifold of M, i.e. $E_q : \mathcal{P}(M, p \times q) \to \mathbb{R}$ is perfect with respect to some field for every $q \in M$ that is not conjugated to p along some geodesic. If a submanifold L is taut and $\mathbb{F}$ is a field as in the definition of tautness, then L is also called $\mathbb{F}$-*taut*.

In [Le06] Leitschkis called a manifold with only taut points *pointwise taut* and we will continue with this notion.

Note 1.1.2. In [TT97] it is shown that a properly immersed, taut submanifold of a simply connected, complete Riemannian manifold is actually embedded. Because we will see in Section 2.1 that one can always assume that the ambient space is simply connected, we will proceed assuming that all submanifolds are embedded and closed, but all of our results will, of course, also hold in the case of a proper immersion. For this reason, if not otherwise stated, by a submanifold L of M we always mean an embedded submanifold and consider all submanifolds as subsets of M. Finally, a manifold is always assumed to be connected.

The only way that is known to prove tautness in general, i.e. that a given Morse function is perfect, is the concept of *linking cycles*, that we are going to explain now. For this reason let $f : P \to \mathbb{R}$ be again a Morse function on a complete Hilbert manifold. Then, for every $r \in \mathbb{R}$ the sublevel P^r contains only a finite number of critical points of f and we can assume that these critical points have pairwise dictinct critical values. That the latter assumption is not restrictive follows from the fact that one can lift a small neighborhood of a critical point a little without changing the relevant behavior of the function. Moreover, using the flow of the negative gradient, one sees that for small ε the sublevel sets $P^{r+\varepsilon}$ and $P^{r-\varepsilon}$ have the same homotopy type unless r is a critical value. If so, let p be the critical point of f with $f(p) = r$ and choose an ε such that $(r - \varepsilon, r + \varepsilon)$ contains no critical value except r. If we denote the index of p by i then $P^{r+\varepsilon}$ has the homotopy type of $P^{r-\varepsilon}$ with an i-cell e_i attached to $f^{-1}(r-\varepsilon)$. Consider the following part of the long exact cohomology sequence of the pair $(P^{r+\varepsilon}, P^{r-\varepsilon})$ with coefficients in a field $\mathbb{F}$:

$$\cdots \to \underbrace{H^{i-1}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{=0} \to H^{i-1}(P^{r+\varepsilon}) \to H^{i-1}(P^{r-\varepsilon}) \xrightarrow{\partial^*} \underbrace{H^{i}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{\cong\mathbb{F}}$$
$$\to H^{i}(P^{r+\varepsilon}) \to H^{i}(P^{r-\varepsilon}) \to \underbrace{H^{i+1}(P^{r+\varepsilon}, P^{r-\varepsilon})}_{=0} \to \cdots$$

Since we are using coefficients from a field, we can switch between the more common homological and our cohomological point of view by dualization, i.e. $H^*(P^r; \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H_*(P^r; \mathbb{F}), \mathbb{F})$. Anyway, we see that by passing from $P^{r-\varepsilon}$ to $P^{r+\varepsilon}$ the only possible changes in homology or cohomology occur in dimensions $i - 1$ and i. To understand this geometrically let us have a look what happens in homology. In the first case, the boundary ∂e_i of the attaching cell is an $(i-1)$-sphere in $P^{r-\varepsilon}$ which does not bound a chain in $P^{r-\varepsilon}$, i.e. e_i has as boundary the nontrivial cycle ∂e_i and so $\partial_* \neq 0$. In the second case, ∂e_i does bound a chain in $P^{r-\varepsilon}$ which we can cap with e_i to create a new nontrivial homology class in $P^{r+\varepsilon}$, that is to say $\partial_* = 0$ and $H_i(P^{r+\varepsilon}) \cong H_i(P^{r-\varepsilon}) \oplus \mathbb{F}$.

We see that the Morse inequalities are equalities if and only if

$$H^i(P^{r+\varepsilon}, P^{r-\varepsilon}) \to H^i(P^{r+\varepsilon}) \text{ is nontrivial, i.e. } \partial^* \equiv 0,$$

or, equivalently,

$$H_i(P^{r+\varepsilon}) \to H_i(P^{r+\varepsilon}, P^{r-\varepsilon}) \text{ is surjective, i.e. } \partial_* \equiv 0,$$

for all critical points p of f.

For a critical ponit $p \in P$, one can show that

$$H^*(P^{f(p)+\varepsilon}, P^{f(p)-\varepsilon}) \cong H^*(P^{f(p)}, P^{f(p)} \setminus \{p\}).$$

Thus suppose that we have a map $h_p : \Delta_p \to P^{f(p)}$ for every critical point p such that the composition

$$H^i(P^{f(p)}, P^{f(p)} \setminus \{p\}) \xrightarrow{h_p^*} H^i(\Delta_p, h_p^{-1}(P^{f(p)} \setminus \{p\})) \to H^i(\Delta_p)$$

is nontrivial. In this case, because the connecting homomorphism ∂^* is a natural transformation, we have that

$$h_p^* \circ \partial^* = \partial^* \circ h_p^*$$

and we conclude that the map $H^i(P^{f(p)}, P^{f(p)} \setminus \{p\}) \to H^i(P^{f(p)})$ cannot be zero, so that f is a perfect Morse function under this assumption. If so, we call the pair (Δ_p, h_p) a *linking cycle for* p, the critical point p *of linking type*, and we say that the function $f : P \to \mathbb{R}$ is of *linking type* if all the critical points are of linking type. Of course, if f is perfect, then the inclusions of the corresponding sublevels define linking cycles, so that f is of linking type. Thus we see that a Morse function is perfect if and only if it is of linking type.

Note 1.1.3. At the end of this section we will prove that an $\mathbb{F}$-taut submanifold is always $\mathbb{Z}_2$-taut. By this reason, and due to the fact that dealing with (co-)homology there is just a little chance to get general results with other coefficients, we restrict our attention to the case $\mathbb{F} = \mathbb{Z}_2$. From now on, saying taut we always mean $\mathbb{Z}_2$-taut and we drop the reference to the field. Because of this, we also simply write $H^*(X)$ for the singular cohomology ring $H^*(X;\mathbb{Z}_2)$ and $\check{H}^*(X)$ for the Čech cohomology groups $\check{H}^*(X;\mathbb{Z}_2)$. If we use other coefficients, e.g. in a (pre-)sheaf, we will explicitly point this out.

Remark 1.1.4. Using finite-dimensional approximations of the path space (see Appendix A.2) we see that in the setting we are interested in, singular cohomology is isomorphic to Čech cohomology (cf. Section 1.2). Because the latter groups satisfy some continuity properties and are more easy to handle, we focus on the Čhech cohomology groups in the following.

To get an impression of the concept of linking cycles we now briefly discuss the method of Bott and Samelson, which has mainly motivated our construction below. Let V be a Euclidean space and let $G \subset \mathbf{O}(V)$ be a closed, connected subgroup. Assume that $L = G(p)$ is a regular G-orbit and that $q \in V$ is a point that is not a focal point of L. Let $v \in \nu(L)$ be a normal vector such that the segment $s_v(t) = tv$ satisfies $s_v(1) = q$, i.e. s_v is a critical point of E_q, and let $0 < t_r < t_{r-1} < \cdots < t_1 < 1$ be the focal times along s_v. Then every focal point $q_i = t_i v$ comes along with an obvious contribution to its focal datum, namely, the difference $\dim(L) - \dim(G(q_i))$ of the dimensions of the orbits. To see that singular orbits indeed give rise to focal points consider the induced action of the isotropy group G_{q_i} on the normal space $T_{q_i}(G(q_i))^\perp$. This linear action is nontrivial and $\dot{s}_v(t_i)$ is a regular point of the G_{q_i}-action. Further, the dimension of the orbit $G_{q_i}(\dot{s}_v(t_i))$ equals the difference of the orbit dimensions $\dim(L) - \dim(G(q_i))$ and one obtains a smooth variation $f : G_{q_i}(\dot{s}_v(t_i))^0 \to \mathcal{P}(V, L \times q)$ of s_v through once broken geodesics, all of which have the same length as s_v, by

$$f(w)(t) = \begin{cases} -t \cdot w, & \text{if } t \in [0,1] \\ s_v(t), & \text{if } t \in [t_i, 1], \end{cases}$$

where $G_{q_i}(\dot{s}_v(t_i))^0$ denotes the connected component of $G_{q_i}(\dot{s}_v(t_i))$ that contains $\dot{s}_v(t_i)$. By construction, all of the paths c in this variation cross the same orbits simultaneously, i.e. $c(t) \in G(s_v(t))$ for all $t \in [0,1]$. Thus starting in the furthermost focal point q_1 one can glue these variations as follows. The principal isotropy group $H = G_p$ fixes the segment s_v pointwise and the r-fold product H^r acts on the product manifold $G_{q_1} \times \cdots \times G_{q_r}$ from the right by

$$(g_1, \ldots, g_r) \cdot (h_1, \ldots, h_r) = (g_1 h_1, h_1^{-1} g_2 h_2, h_2^{-1} g_3 h_3, \ldots, h_{r-1}^{-1} g_r h_r).$$

If $\Delta_v = G_{q_1} \times_H G_{q_2} \times_H \cdots \times_H G_{q_r}/H$ denotes the quotient manifold under this action, we obtain a map $h_v : \Delta_v \to L$ by $h_v(g_1, \ldots, g_r) = g_1 \cdots g_r(s_v(0))$. The space Δ_v can be viewed as the total space of an iterated fiber bundle that can be identified with the space of polygonal paths from q to $g_1 \cdots g_r(s_v(0))$ with vertices $q_1, g_1(q_2), g_1 g_2(q_3), \ldots, g_1 g_2 \ldots g_{r-1}(q_r)$ for r-tupels $(g_1, \ldots, g_r) \in G_{q_1} \times \cdots \times G_{q_r}$. Therefore, we consider h_v as a map from Δ_v into $\mathcal{P}(V, L \times q)^\kappa$, where we set $\kappa = \|v\|^2$.

For the dimension of Δ_v, we compute

$$\begin{aligned} \dim(\Delta_v) &= \sum_{i=1}^{r} (\dim(G_{q_i}) - \dim(H)) \\ &= \sum_{i=1}^{r} (\dim(L) - \dim(G(q_i))) \,. \end{aligned}$$

If all the focal points of all orbits are only caused by the singular orbits, such an action is called *variationally complete*, so that in this case the orbit differences $\dim(L) - \dim(G(q_i))$ are just the multiplicities of the points q_i as focal points of L

and the above equation yields

$$\dim(\Delta_v) = \sum_{t=1}^{r} \mu(t_i) = i(s_v).$$

Since Δ_v is a compact manifold of dimension $i(s_v)$, it has $H^{i(s_v)}(\Delta_v) \cong \mathbb{Z}_2$. Moreover, it is easy to see that $s_v \in h_v(\Delta_v) \subset E_q^{-1}(\kappa)$ and that h_v is an immersion near $(1, ..., 1) \in G_1 \times \cdots \times G_r$. Now locally in a Morse chart centered at s_v, the image $h_v(\Delta_v)$ is transversal to the ascending cell so that we can deform it into the descending cell. Therefore, if we now set $P^\kappa = \mathcal{P}(M, L \times q)$, the map

$$H^{i(s_v)}(P^\kappa, P^\kappa \setminus \{s_v\}) \xrightarrow{h_v^*} H^{i(s_v)}(\Delta_v, \Delta_v \setminus \{(1, \ldots, 1)\}) \to H^{i(s_v)}(\Delta_v)$$

is nontrivial, i.e. $h_v : \Delta_v \to \mathcal{P}(M, L \times q)$ defines a linking cycle for s_v, which is also referred to as a *Bott-Samelson cycle*.

Motivated by this example, we have introduced the concept of linking cycles, because this is exactly what we want to use to prove our main characterization of taut submanifolds, as exactly those submanifolds whose normal exponential map has an integrable kernel distribution. The problem when dealing with cycle constructions as above is the behavior of the focal data. The construction of Bott and Samelson works well, also in the general case, if the focal points along a variation corresponding to the kernel distribution of the normal exponential map do not collapse, i.e. if the cardinality of the intersections of normal geodesics with the focal set is locally constant. Unfortunately, the occurrence of focal collapses along a variation of normal geodesics cannot be avoided in general, but since these collapses depend continuously on the initial directions of the geodesics, it turns out that this indeed constitute no problem for our goal. The next section prepares Lemma 1.2.5, which provides a powerful tool in this direction. Our applications of Lemma 1.2.5 in the following demonstrate how useful this tool actually can be dealing with similar cycle constructions. Namely, as in the construction of Bott and Samelson, given a critical point of the energy functional we will construct a nice space that can be regarded as the total space of a singular fiber bundle and Lemma 1.2.5 will ensure that the occuring singularities do not pose any problems for $\mathbb{Z}_2$-cohomology, so that these spaces will indeed represent linking cycles for the energy.

1.2 The Main Tool

Given a continuous map $f : X \to B$ from a compact Hausdorff topological space X onto some nice space B with a fundamental class in $\mathbb{Z}_2$-homology, e.g. a manifold, such that all the fibers are compact manifolds of constant dimension, one would expect that the union over the base B of all the fibers defines a non-trivial $\mathbb{Z}_2$-homology class in dimension equal to the sum of the homological dimension of

B and the fiber dimension. In order to prove our main theorem later on, we are dependent on a tool like this, because we want to construct explicit (co-)cycles with specified cohomological behaviour and it will turn out how powerful this method can be, dealing with $\mathbb{Z}_2$-cohomology. The easiest way, known to the author, to prove such a statement is by means of sheaf cohomology. Therefore, we proceed with some basic definitions and facts we need for our aim, the proof of Lemma 1.2.5. We follow the notation of Bredon [B67] and Chapter 5 of [Wa83]. The reader who is familiar with this theory could directly skip ahead to Lemma 1.2.5.

A *presheaf (of Abelian groups)* on a topological space X is a contravariant functor from the category of open subsets of X, where the morphisms are just the inclusions, to the category of Abelian groups, i.e. a function that assigns to each open set U an Abelian group $A(U)$ and to each pair $U \subset V$ a homomorphism, called the *restriction*, $r_{U,V} : A(V) \to A(U)$ in such a way that $r_{U,U} = 1_{A(U)}$ and $r_{U,V} \circ r_{V,W} = r_{U,W}$ whenever $U \subset V \subset W$.

Example. For any standard cohomology theory H^* on X, the assignment given by $U \mapsto H^r(U; G)$ defines a presheaf, where the coefficients are taken to be any Abelian group G. Other well-known examples of presheaves are $U \mapsto C^0(U)$, the presheaf of (locally) continuous functions, or in the case of a manifold $U \mapsto C^r(U)$, the presheaf of (locally) r times continuous differentiable functions.

Definition 1.2.1. A *sheaf (of Abelian groups)* on X is a pair $(\mathcal{A}, \pi)$, where

1. $\mathcal{A}$ is a topological space;
2. $\pi : \mathcal{A} \to X$ is a local homeomorphism;
3. Each fiber $\mathcal{A}_x = \pi^{-1}(x)$ is an Abelian group and is called the *stalk* of $\mathcal{A}$ at x;
4. The group operations are continuous, i.e. the map

$$\begin{array}{rcl} \mathcal{A} \times_\pi \mathcal{A} & \to & \mathcal{A}, \\ (\alpha, \beta) & \mapsto & \alpha - \beta \end{array}$$

   with $\mathcal{A} \times_\pi \mathcal{A} = \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{A} | \pi(\alpha) = \pi(\beta)\}$, is continuous.

As always, if the context is clear we will drop the reference to the map π and talk about the sheaf $\mathcal{A}$. For an Abelian group G, we say that $\mathcal{A}$ is a G-sheaf on X if all the fibers $\pi^{-1}(x)$ are isomorphic to G.

If $(\mathcal{A}, \pi)$ is a sheaf on X and U is an open subset of X, then we denote by $\mathcal{A}(U)$ the space of *sections* of $\mathcal{A}$ over U, i.e. continuous maps $s : U \to \mathcal{A}$ with $\pi \circ s = \mathrm{id}_U$. By the second point in the above definition, we know that around every point in X there is a neighborhood U and a section $s \in \mathcal{A}(U)$, and the last item ensures that $s - s$ is also a section over U. Thus we see that the zero section $0 : X \to \mathcal{A}, x \mapsto 0_x$, indeed defines a global section. Again, since the group operations are continuous, for every open set U, the space $\mathcal{A}(U)$ is an Abelian group. It follows that every sheaf $\mathcal{A}$ on X induces a presheaf $U \to \mathcal{A}(U)$, where the restrictions are the usual restriction of maps.

If $(\mathcal{A}, \pi)$ is a sheaf on X, then the following properties are directly implied by the definition:

- π is an open map;
- Any section of $\mathcal{A}$ over some open set is an open map;
- Any element of $\mathcal{A}$ is in the range of some section over an open subset;
- The set of all images of sections over open sets is a basis for the topology of $\mathcal{A}$;
- For any two sections $s \in \mathcal{A}(U)$ and $t \in \mathcal{A}(V)$ the set W of points $x \in U \cap V$ with $s(x) = t(x)$ is open.

If $\mathcal{A}$ were Hausdorff, then W in the last item would also be closed. But typically $\mathcal{A}$ is not Hausdorff, what causes the main source of difficulty in dealing with sheaves.

As we have seen a sheaf $\mathcal{A}$ induces a presheaf in a natural way. Conversely, there is a canonical way to obtain a sheaf out of a given presheaf. Therefore, let us start with a presheaf A on X and consider the disjoint union

$$E = \bigsqcup_{U \subset X} U \times A(U),$$

where the union is taken over all open subsets and the sets $A(U)$ are endowed with the discrete topology. Define an equivalenve relation on E by saying that for $(x, s) \in U \times A(U)$ and $(y, t) \in V \times A(V)$

$$\begin{aligned} (x, s) \sim (y, t) \quad \Leftrightarrow \quad & x = y \text{ and there exists an open neighborhood} \\ & W \subset U \cap V \text{ of } x \text{ with } s|_W = t|_W. \end{aligned}$$

Here we use the usual notation $s|_W$ instead of $r_{W,U}(s)$. Now, let $\mathcal{A}$ be the quotient space $E/\sim$ with the quotient topology and let $\pi : \mathcal{A} \to X$ be the projection induced by the map $E \to X, (x, s) \mapsto x$. Then one can check that $(\mathcal{A}, \pi)$ actually defines a sheaf on X, which we call the *sheaf generated by the presheaf* A or the *sheafification* of A. Clearly, the stalk $\mathcal{A}_x = \pi^{-1}(x)$ is just the direct limit $\lim_{\rightarrow} A(U)$, taken over the open neighborhoods U of x, directed by inclusion. An element $s \in A(U)$ defines a section $\bar{s} : U \to \mathcal{A}$ by $\bar{s}(x) = s_x$, where we denote by s_x the equivalence class of (x, s) in $\mathcal{A}$, which we call the *germ* of s in x. In particular, the topology of $\mathcal{A}$ is generated by the sets $\{s_x | s \in A(U), x \in U\}$.

Definition 1.2.2. A *homomorphism of presheaves* $h : A \to B$ is a collection of homomorphisms $\{h_U : A(U) \to B(U)\}$ commuting with the restrictions. Two presheaves are called *isomorphic* if all the maps are isomorphisms.
A *homomorphism of sheaves* $h : \mathcal{A} \to \mathcal{B}$ is a continuous map, which commutes with the projections in such a way that the restriction to each stalk $h_x : \mathcal{A}_x \to \mathcal{B}_x$ is a homomorphism. Two sheaves $\mathcal{A}$ and $\mathcal{B}$ are called isomorphic if there is a homomorphism of sheaves that is also a homeomorphism.

If we start with a sheaf $\mathcal{A}$ on X and apply the sheafification construction, explained above, to the presheaf of sections, we obtain a sheaf, which is clearly isomorphic to $\mathcal{A}$. On the other hand, a given presheaf A on X is in general not isomorphic to presheaf of section of its sheafification. Namely, one can show that this is equivalent to the fact that A satisfies the following two properties:

1. If $U = \bigcup_i U_i$ with U_i open and $s, t \in A(U)$ are two elements with $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
2. If $U = \bigcup_i U_i$ with U_i open and $s_i \in A(U_i)$ are elements with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists an element $s \in A(U)$ with $s|_{U_i} = s_i$.

Thus we see that our definition of a sheaf is the same as to define a sheaf as a presheaf that satisfies these two properties. This equivalent definition is perhaps the more common one used in text books about this topic. It depends on the personal preference in which way one thinks about sheaves.

Example (1). Let $G \neq 0$ be an Abelian group and let A be the *constant presheaf* on X, defined by $A(U) = G$ for all open sets U and the identity as restrictions. Assume that X has at least two points which can be seperated. Then A is not the presheaf of sections of its sheafification, which is the *constant (or trivial) sheaf* $\mathcal{G} = X \times G$, because A do not satisfy the gluing property. Namely, for two disjoint open subsets U and V and two different elements s and t of G, there is obviously no element in $A(U \cup V) = G$ that restricts to s on U and t on V, respectively. Note that the presheaf of sections of the trivial sheaf $\mathcal{G} = X \times G$ is necessarily nothing else than the presheaf of locally constant functions with values in G, since G has the discrete topology.

Example (2). Let $f : Y \to X$ be a continuous map between topological spaces. Then for every $r \geq 0$ there is an associated presheaf on X given by $U \to \check{H}^r(f^{-1}(U); G)$, where we denote by $\check{H}^*$ the Čech cohomology. The sheaf $\mathcal{H}^r(f; G)$ generated by this presheaf is called the *Leray sheaf* of f on X.

The following lemma states that there are no nontrivial $\mathbb{Z}_2$-sheaves over sufficiently nice spaces.

Lemma 1.2.3. *Let X be a locally compact Hausdorff topological space. Then there are no nontrivial $\mathbb{Z}_2$-sheaves on X.*

Proof. Assume that $(\mathcal{A}, \pi)$ is a $\mathbb{Z}_2$-sheaf on X, i.e. $\pi^{-1}(x) \cong \mathbb{Z}_2$ for all $x \in X$. Then each fiber $\pi^{-1}(x)$ has exactly one element that is not zero and we denote this nontrivial element by 1_x. Because the zero section $0 : X \to \mathcal{A}$ is a global section, it remains to show that the well-defined map $1 : X \to \mathcal{A}$, given by $1(x) = 1_x$, is continuous. Because if so, the map $X \times \mathbb{Z}_2 \to \mathcal{A}$, given by $(x, \varepsilon) \mapsto \varepsilon_x$ for $\varepsilon \in \{0, 1\}$, would define an isomorphism of sheaves. Thus take a point $x \in X$ and let U be an open neighborhood of x such that there is a section $s \in \mathcal{A}(U)$ with $s(x) = 1_x$. Then $s : U \to s(U)$ is a homeomorphism onto an open neighborhood of 1_x. By assumption, there is a compact neighborhood $V \subset U$ of x, which is also closed in U since X is Hausdorff. Hence $s(V)$ is closed in $s(U)$ and contains 1_x. Now the zero section defines a topological embedding $0 : X \to \mathcal{A}$. In particular, the image $0(X)$ is open in $\mathcal{A}$. But then $s(V) \cap 0(X)$ is closed in the open set $s(U) \cap 0(X)$. Since π is an open map and the restriction $\pi : 0(X) \to X$ is a homeomorphism we conclude that $\pi(s(V) \cap 0(X))$ is closed in the open set $\pi(s(U) \cap 0(X)) \subset U$ and does not contain x. Therefore, there exists an open neighborhood $W \subset U$ around x with $W \subset U \setminus \pi(s(V) \cap 0(X))$. But this means that $s|_W \neq 0$, so $s(y) = 1_y$ for all $y \in W$ and this proves the claim. $\square$

There is a way to define cohomology theories with coefficients in a pesheaf or in a sheaf, what is the same for paracompact spaces (see, e.g. Chapter 6 of [Sp66]). But a developement of this theory would go beyond the scope of our discussion, the more so as it is not really necessary for our goal. By this reason we have to refer the reader to the literature, e.g. [B67], [Sp66] or [Wa83]. Because it is all we need, we just want to mention that in the case of a constant sheaf $X \times G$ this cohomology is exactly the same as the usual Čech cohomology with coefficients in G. Moreover, it can be shown that if X is a topological manifold of dimension n, then all the cohomology groups with coefficients in any sheaf vanish in dimensions greater than n. This is an important feature from which we make essential use of.

Example. In Example (2) above we have defined, for an arbitrary continuous map $f : Y \to X$, the Leray sheaf $\mathcal{H}^r(f; G)$ on X, as the sheaf generated by the presheaf $U \mapsto \check{H}^r(f^{-1}(U); G)$ using Čech cohomology with coefficients in G. As mentioned above, the cohomology groups $H^r(U; \mathcal{G})$ with coefficients in the constant sheaf $\mathcal{G} = X \times G$ are isomorphic with the corresponding Čech groups. Thus the generated sheaves are also isomorphic. Due to this, our definition of the Learay sheaf in this case behaves well with respect to the general definition of the Leray sheaf in the context of sheaf cohomology, which is, given a sheaf $\mathcal{A}$ on Y, generated by the presheaf $U \to H^r(f^{-1}(U); \mathcal{A})$.

Remark 1.2.4. It is shown in [Sp66] that in the cases we are interested in, the cohomology groups $H^r(X; \mathcal{G})$ with coefficients in the constant sheaf $\mathcal{G} = X \times G$ are nothing else than the Alexander-Spanier cohomology groups with coefficients in G and that the latter coincide with the Čech cohomology groups $\check{H}^r(X; G)$. In particular, it is therefore clear that we have long exact sequences, excision, and that the homotopy axiom holds. Finally, because we are dealing only with nice spaces, please recall that for compact subsets (K, L) of a manifold, the Čech cohomolgy groups $\check{H}^r(K, L)$ are isomorphic to the direct limit

$$\varinjlim \{H^r(U, V) | (K, L) \subset (U, V)\}$$

where the limit is taken over open subsets $(U, V) \supset (K, L)$.

We now come to the heart of this section, that is a very powerful tool for our use. We formulate the following lemma in an easy to handle version, adjusted accordingly to our purpose, but the reader who goes through the proof will notice that it also holds under weaker assumptions, e.g. if B as in the claim has the cohomological behavior of a manifold.

Lemma 1.2.5. *Let X be a connected, compact Hausdorff topological space and let $f : X \to B$ be a continuous map onto a manifold B of dimension k. Assume that every fiber $f^{-1}(b)$ is a connected manifold of (constant) dimension n or, more general, that $\check{H}^n(f^{-1}(b); \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\check{H}^l(f^{-1}(b); \mathbb{Z}_2) = 0$ for all $l > n$, where $\check{H}^*$ denotes the Čech cohomology. Then X has cohomological dimension $n + k$, i.e. $\check{H}^{n+k}(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\check{H}^l(X; \mathbb{Z}_2) = 0$ for $l > n + k$.*

Proof. Since X is compact and connected and f is surjective, B is compact and connected, too. If we consider cohomology with coefficients in the constant sheaf $\mathcal{Z}_2 = X \times \mathbb{Z}_2$, then, due to Theorem 6.1 in [B67], there exists a spectral sequence $\{E_r, d_r\}$ with $d_r : E_r^{m,l} \to E_r^{m+r,l-r+1}$ converging to $H^*(X; \mathcal{Z}_2)$ with E_2 page

$$E_2^{m,l} = H^m(B; \mathcal{H}^l(f; \mathcal{Z}_2)),$$

where $\mathcal{H}^l(f; \mathcal{Z}_2)$ denotes the Leray sheaf on B (cf. Example (2)), generated by the presheaf $U \mapsto H^l(f^{-1}(U); \mathcal{Z}_2)$. Further, it is proven there that in our setting the stalks $\mathcal{H}^l(f; \mathcal{Z}_2)_p$ of the Leray sheaf are isomorphic to the cohomology groups of the corresponding fibers $H^l(f^{-1}(b); \mathcal{Z}_2) \cong \check{H}^l(f^{-1}(b); \mathbb{Z}_2)$. Then, by our assumptions and Lemma 1.2.3, the n-th Leray sheaf is the constant sheaf on B, i.e. $\mathcal{H}^n(f; \mathcal{Z}_2) \cong B \times \mathbb{Z}_2$. Therefore, the entry $E_2^{k,n}$ is just given by the k-th Čech cohomology group $\check{H}^k(B; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Because B is a manifold of dimension k, all the groups $H^m(B; \mathcal{H}^l(f; \mathcal{Z}_2))$ vanish for $m > k$, by dimensional reasons mentioned above. Also all the entries $E_2^{m,l}$ with $l > n$ vanish, because $\mathcal{H}^l(f; \mathcal{Z}_2)$ is the 0-sheaf in this case. But this means that the entry $E_2^{k,n}$ survives in the spectral sequence since it is the top right entry in the nontrivial rectangle on the E_2 page. The second statement of the claim follows from the fact that if $m + l > k + n$ then $m > k$ or $l > n$. □

For every $b \in B$, the map $f : X \to B$ induces a map $X \setminus f^{-1}(b) \to B \setminus \{b\}$ by restriction. Now the inclusion $X \setminus f^{-1}(b) \hookrightarrow X$ induces a long exact sequence in cohomology and also a compatible map on each page of the respective spectral sequences. By the same arguments as in the proof of 1.2.5, there is a nontrivial class in $\check{H}^{n+k}(X, X \setminus f^{-1}(b); \mathbb{Z}_2)$. Moreover, this class is mapped onto the class in

$\check{H}^{n+k}(X;\mathbb{Z}_2)$ constructed above, because this is true for B. For this reason, we call such a cohomology class an *f-fundamental class.*

Let us state this as a corollary.

Corollary 1.2.6. *Under the assumptions in Lemma 1.2.5 the space X has an f-fundamental class.*

1.3 An Equivalent Description

Having achieved our key tool in the last section, we are now able to prove our main result after a short reminder.

Reminder. A Morse-Bott function $f : P \to \mathbb{R}$ on a complete Hilbert manifold P is a smooth function whose critical set is the union of closed submanifolds and whose Hessian is non-degenerate in the normal direction. That is to say, every critical point lies in a closed submanifold whose tangent space coincides with the kernel of the Hessian at each point. If so, the index of a critical point is defined to be the index of the restriction of the Hessian on the normal space of the corresponding critical manifold. Since the Hessian depends continuously on the points of the critical manifolds, the index is constant along the connected components of the critical set. As we already mentioned, in our case of the energy functional on the path spaces, we can replace the infinite dimensional setting by a finite dimensional one, so that of course all indices and nullities are finite (cf. A.1).

Theorem 1.3.1. *A closed submanifold $L \subset M$ of a complete Riemannian manifold M is taut if and only if all energy functionals are Morse-Bott functions.*

Proof. If L is a taut submanifold (with respect to any field $\mathbb{F}$), then all the energy functionals $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ are Morse-Bott functions by Theorem 2.8 of [TT97], which is essentially the result of Ozawa in [Oz86]. Therefore, it remains to show that if all energy functionals are Morse-Bott functions then L is taut. We prove this by constructing explicit linking cycles.

Thus let $V = \nu(L)$ be the normal bundle of L, identify L with the zero section 0 in V, and denote by $\eta : V \to M$ the restriction of the exponential map of M to the normal bundle. By the ray through $v \in V$ we mean the linear map $r_v : \mathbb{R}_0^+ \to V$, given by $r_v(t) = tv$, and by the segment to v we mean $s_v = r_v|_{[0,1]}$. Then for $\lambda \geq 1$, the point $q = \eta(v)$ is a focal point of L along the geodesic $\gamma_{\lambda v} = \eta \circ s_{\lambda v}$ if and only if $d\eta_v$ is singular and the multiplicity of q as such a point is just the dimension of the kernel of $d\eta_v$. In the case that $d\eta_v$ is not onto we will therefore also call v a focal vector and we will say that $\mu(v) = \dim(\ker(d\eta_v))$ is its multiplicity. Let us denote by C the union of all points in V where $d\eta$ is singular and call it the tangent focal locus. We call every number in $r_v^{-1}(C)$ a focal time along the ray r_v. It is a well known fact that the focal times are discrete along any ray and that they depend continuously on the rays. Namely, that every vector $v \in V$ has an open neighborhood U such that every ray that intersects U contains $\mu(v)$ focal vectors in U counted with multiplicities, i.e. if $\mathrm{im}(r_w) \cap U \neq \emptyset$, then $\sum_{t \in r_w^{-1}(U \cap C)} \mu(tw) = \mu(v)$. Finally, we call a focal vector $v \in C$ regular if there is an open neighborhood U of v such that all rays which intersect U intersect $U \cap C$ exactly once. Due to Warner [Wa65] and Hebda [Heb81], the set C_R of regular focal vectors is an open and dense subset of C that is a codimension-one submanifold of V such that $T_v V \cong T_v C_R \oplus \mathbb{R}v$ for all $v \in C_R$. Let V^R denote the set of vectors $v \in V \setminus C$ such that s_v intesects C only in C_R. Then V^R is obviously open in V and it is also dense. To see this, consider the function $n : V \setminus C \to \mathbb{Z}$, given by

$$n(v) = \# \left\{ s_v^{-1}(C) \right\},$$

which is lower semi-continuous by our above observations. Then

$$\begin{aligned} V^R &= \{ v \in V \setminus C | n \text{ is constant on a neighborhood of } v \} \\ &= \left\{ v \in V \setminus C | s_v^{-1}(C) = s_v^{-1}(C_R) \right\}. \end{aligned}$$

Since the set of regular vectors $V \setminus C$ is open and dense in V, it is enough to show that V^R is dense in $V \setminus C$. Thus assume that V^R is not dense in $V \setminus C$. Then the complement of V^R in $V \setminus C$ contains an open set U. The function n admits its maximum n_r on every intersection $U \cap B(r)$ of U with an open tube $B(r)$ of radius r around the zero section. Choose r so large that $U \cap B(r) \neq \emptyset$. Due to the semi-continuity of n, the set $n^{-1}(n_r) \cap U \cap B(r) \subset V \setminus V^R \cap V \setminus C$ defines an open subset of $V \setminus C$ on which n is constant, what clearly contradicts our definition of V^R.

Our assumption that $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is a Morse-Bott function for all q implies that the singular kernel distrinution is completely integrable, i.e. through every point $v \in V$ there is a $\mu(v)$-dimensional compact connected submanifold C_v with $T_w C_v = \ker(d\eta_w)$ for all $w \in C_v$. This follows by taking the time derivative in 0 of all paths in the corresponding critical manifolds. Moreover, we have $C_v \subset S(\|v\|)$, where $S(\|v\|)$ denotes the sphere bundle over L of normal vectors of length $\|v\|$, and the index $i(v) = \sum_{t \in (0,1)} \mu(tv)$ is constant along C_v. For a vector $v \in S(1)$ let $0 < t_1(v) \leq t_2(v) \leq \ldots$ denote the focal times along the ray r_v, counted with their multiplicities. Then these focal times (counted with multiplicities) depend continuously on $v \in S(1)$ (cf. [IT01]).

Let us now define a function $m : V \to [0, 1)$ which assigns to a vector v the number

$$m(v) = \max \{ t \in (0, 1) | \mu(tv) \neq 0 \}$$

if $i(v) > 0$, and $m(v) = 0$ if $i(v) = 0$. In particular, by our above observations, the restriction of m to each submanifold C_v is continuous.

Denote by $\mathfrak{C} = \bigcup_{v \in V} C_v$ the η-kernel decomposition of V and define $Q : V \to V/\mathfrak{C}$ to be the natural quotient map. Since the fibers of Q are compact submanifolds of V the quotient is a locally compact Hausdorff space and the restriction of the projection to $V \setminus C$ is a homeomorphism onto an open subspace of $V/\mathfrak{C}$. The fiber norms on V push down to the distance function from the image $Q(0)$ of the zero section, so that every compact subset in $V/\mathfrak{C}$ has to be of bounded distance from $Q(0)$. In particular, the map Q is proper and therefore closed.

We now define a natural cycle candidate Δ_v for every geodesic $\gamma_v = \eta \circ s_v$ with $v \in V$. Because η factorizes over $V/\mathfrak{C}$ by a map $\bar{\eta} : V/\mathfrak{C} \to M$, we will work in the quotient space and consider the space $P(V/\mathfrak{C}, Q(0) \times Q(v))$ of continuous paths $c : [0,1] \to V/\mathfrak{C}$ from $Q(0)$ to $Q(v)$ with the compact open topology. The constructed cycles will embed in an energy preserving and obvious way into $\mathcal{P}(M, L \times \eta(v))$ under the map on the path space level induced by $\bar{\eta}$, so that we renounce the reference to the latter space for the rest of the proof.

Because the η-kernel distribution on V is integrable, there is a natural cycle through $Q \circ s_v$, intuitively having the right dimension. Namly, take Z_v to be the set of all piecewise continuous maps from $[0,1]$ to V obtained (with the reversed orientation) as follows: Take a vector $w_1 \in C_v$ and follow the straight line tw_1 towards the zero section up to the first conjugate vector, then take an arbitrary conjugate vector w_2 in the corresponding leaf and follow the line tw_2 towards the zero section up to the first conjugate vector, then take an arbitrary conjugate vector w_3 in the corresponding leaf and follow the line tw_3 up to the first conjugate vector and so on. This process will end after a finite number of steps and we can push these piecewise continuous maps down via Q to continuous paths $[0,1] \to V/\mathfrak{C}$ starting in $Q(0)$ and ending in $Q(v)$ and define Δ_v to be the injective image of Z_v under this map.

To be more precise, let us say that a tuple $c = (c_1, \ldots, c_r)$ is a piecewise linear map on $[0,1]$ if there exists a partition $0 = t_r < t_{r-1} \cdots < t_1 < t_0 = 1$ of $[0,1]$ such that $c_i : [t_i, t_{i-1}] \to V$ is given by $c_i(t) = s_{w_i}(t) = tw_i$ for some vector $w_i \in V$. Then for $v \in V$, let Z_v be the set consisting of all piecewise linear maps c on $[0,1]$ recursively defined by $c_1 = s_{w_1}|_{[m(w_1),1]}$ for $w_1 \in C_v$, and for $i \geq 2$ by

$$c_i(t) = s_{\frac{\|v\|}{\|w_i\|} w_i}(t), t \in [m(w_i)\frac{\|w_i\|}{\|v\|}, m(w_{i-1})\frac{\|w_{i-1}\|}{\|v\|}] \text{ with } w_i \in C_{m(w_{i-1})w_{i-1}}.$$

Thus in the above notation $t_i = m(w_i)\frac{\|w_i\|}{\|v\|}$. Note that this is well defined because of $m(w_{i-1})\|w_{i-1}\| = \|w_i\|$ and our definition that $m(w) = 0$ if $i(w) = 0$. Moreover, we have $Z_v = Z_w$ for all $w \in C_v$. We can regard a piecewise linear map on $[0,1]$ as a piecewise continuous map $c : [0,1] \to V$, defined by $c(0) = 0$ and $c|_{(t_i,t_{i-1}]} = c_i|_{(t_i,t_{i-1}]}$, where, of course, here $c(0) = 0$ means the origin of the normal space that is uniquely defined by $c(\varepsilon)$ for some small number $\varepsilon > 0$. Anyway, there is a well-defined injective map

$$\bar{Q} : Z_v \to P(V/\mathfrak{C}, Q(0) \times Q(v)),$$

given by $\bar{Q}(c)|_{[t_i,t_{i-1}]} = Q \circ c_i$ with

$$
\begin{aligned}
E(\bar{Q}(c)) &= (1-m(w_1))\|v\|^2 + \sum_{i\geq 2} E(c_i) \\
&= (1-m(w_1))\|v\|^2 + \sum_{i\geq 2}(m(w_{i-1})\frac{\|w_{i-1}\|}{\|v\|} - m(w_i)\frac{\|w_i\|}{\|v\|})\|v\|^2 \\
&= \|v\|^2.
\end{aligned}
$$

Now we define the space Δ_v to be the image $\bar{Q}(Z_v) \subset P(V/\mathfrak{C}, Q(0)\times Q(v))$ with the relative topology, i.e. induced by the compact open topology on the path space $P(V/\mathfrak{C}, Q(0)\times Q(v))$. With this topology the space Δ_v is compact, because Q is proper, and we have that $\Delta_v = \Delta_w$ for all $w \in C_v$, so that we could also write $\Delta_{Q(v)}$ for this space. The map $\bar{Q} : Z_v \to \Delta_v$ is a bijection and we topologize Z_v by the postulation that this map is a homeomorphism. As mentioned above, we can regard Z_v as a space of piecewise continuous maps.

We follow this direction and define $e : Z_v \times [0,1] \to V$ by $e(c,0) = c(0)$ and $e(c,t) = \lim_{t' \nearrow t} c_i(t')$ if $t \in (t_i, t_{i-1}]$, so that $t \mapsto e_t(c)$ is the required map. Let us set $\bar{e} : \Delta_v \times [0,1] \to V/\mathfrak{C}$ for the continuous evaluation map, given by the prescription $\bar{e}(\bar{Q}(c), t) = \bar{Q}(c)(t)$, and consider the commutative diagram

$$
\begin{tikzcd}
Z_v \times [0,1] \arrow[r, "e"] \arrow[d, "\bar{Q}\times \mathrm{id}"] & V \arrow[d, "Q"] \\
\Delta_v \times [0,1] \arrow[r, "\bar{e}"] & V/\mathfrak{C}
\end{tikzcd}
$$

from which it follows that e is continuous in (c,t) if $\bar{Q}(c)(t) \notin Q(C)$. If we set $e_t = e(\cdot, t)$, then $e_1 : Z_v \to V$ is also continuous. To see this, we first observe that e_1 is continuous iff it is continuous considered as a map to $C_v \subset S(\|v\|)$. Then take an open subset $U \subset C_v$ and note that by definition $e_1(c) = w \in U$ iff the image of c_1 is contained in $[m(w), 1]\cdot U = \{rw' | r \in [m(w),1], w' \in U\}$. Because m is bounded away from 1 on C_v, we can find an ε such that $(1-\varepsilon, 1)\cdot C_v \subset V \setminus C$ and because Q is an embedding on $V\setminus C$ we have

$$
e_1^{-1}(U) = e_{1-\varepsilon/2}^{-1}((1-\varepsilon,1)\cdot U) = \bar{Q}^{-1}(\bar{e}_{1-\varepsilon/2}^{-1}(Q((1-\varepsilon,1)\cdot U))),
$$

which is therefore open.

The crucial point with regard to our goal is that we can write Δ_v as a twisted product over C_v, what enables us to use an inductive argument to verify the right cohomological behavior. For this reason, identify C_v with the subspace of unbroken paths in Δ_v, i.e. by the map $w \mapsto Q \circ s_w$. Set

$$
\Delta = \bigcup_{\left\{w\in V\setminus\{0\} \middle| i(w)<i(v)\right\}} \Delta_w \subset P(V/\mathfrak{C}, Q(0)\times V/\mathfrak{C}),
$$

where of course the latter space is the space, homotopy equivalent to $Q(0)$, of continuous paths $[0,1] \to V/\mathfrak{C}$ that start in $Q(0)$ with the compact open topology. Now define a map $R : \Delta_v \to \Delta$ by assigning to a path $\bar{Q}(c) \in \Delta_v$ the reparameterization of $\bar{Q}(c)|_{[0,m(c(1))]}$ on $[0,1]$, i.e. $R(\bar{Q}(c)) \in \Delta_{m(c(1))c(1)}$ is given by

$$R(\bar{Q}(c))(t) = \bar{Q}(c)(m(c(1)) \cdot t) \text{ for } t \in [0,1].$$

If $c = (c_1, c_2, \dots, c_r) \in Z_v$ is a piecewise continuous map with $c_1 = t \cdot c(1)$ on $[m(c(1)), 1]$ and $c_i : [t_i, t_{i-1}] \to V$ for $i \geq 2$ given by $t\frac{\|v\|}{\|w_i\|} \cdot w_i$ with $t_i = m(w_i)\frac{\|w_i\|}{\|v\|}$ for $w_i \in C_{m(w_{i-1})w_{i-1}}$ as above, then $R(\bar{Q}(c)) = \bar{Q}(d)$ for the piecewise linear map $d = (d_1, \dots, d_{r-1}) \in Z_{m(c(1))c(1)}$ on $[0,1]$ given by

$$\begin{aligned} d_i(t) = t\frac{\|w_2\|}{\|w_{i+1}\|}w_{i+1} &= t\frac{m(c(1))\|c(1)\|}{\|w_{i+1}\|}w_{i+1} \\ &= t \cdot m(c(1))\frac{\|v\|}{\|w_{i+1}\|}w_{i+1} \\ &= c_{i+1}(m(c(1)) \cdot t) \end{aligned}$$

for $t \in [t'_i, t'_{i-1}]$ with $t'_i = m(w_{i+1})\frac{\|w_{i+1}\|}{\|w_2\|} = m(w_{i+1})\frac{\|w_{i+1}\|}{m(c(1))\|v\|} = \frac{t_{i+1}}{m(c(1))}$.

By our previous discussion the map $\bar{m} = m \circ e_1 \circ \bar{Q}^{-1} : \Delta_v \to [0,1)$, given by $\bar{Q}(c) \mapsto m(c(1))$, is continuous and we deduce that, therefore, R is continuous, because a path $R(c) \in \Delta$ maps a compact interval J into an open set U if and only if c maps $\bar{m}(c) \cdot J$ into U. So that if we denote by $V_{J,U}$ the set of paths in $P(V/\mathfrak{C}, Q(0) \times V/\mathfrak{C})$ which map J to U and take a path $c \in R^{-1}(V_{J,U})$, then $V_{m(c) \cdot J,U}$ is an open neighborhood of c. But that $\bar{m}$ is continuous means that for some small open neighborhood W of c in Δ_v we have that $d(\bar{m}(d) \cdot J) \subset U$ for all $d \in W \cap V_{m(c) \cdot J,U}$, so that $W \cap V_{m(c) \cdot J,U}$ is an open neighborhood of c in $R^{-1}(V_{J,U})$.

Therefore, we can define a continuous map

$$\begin{aligned} T : \quad \Delta_v &\to \Delta \ {}_{\bar{e}_1}\!\times_{Q \circ \bar{m}} C_v, \\ Q(c) &\mapsto (R(Q(c)), e_1(c)), \end{aligned}$$

where the twisted product is defined by

$$\Delta \ {}_{\bar{e}_1}\!\times_{Q \circ \bar{m}} C_v = \{(d, w) \in \Delta \times C_v | \bar{e}_1(d) = d(1) = Q(m(w))\}.$$

It is now easy to see that T is bijective, so that, due to the compactness of Δ_v, it is already a homeomorphism.

Now assume that v is not a conjugate vector. Using the above identification let $pr_{m(v)v} : \Delta_{m(v)v} \to C_{m(v)v}$ be the projection onto the second factor. Then by definition, our cycle Δ_v is just $\Delta_{m(v)v} * \{Q \circ s_v|_{[m(v),1]}\}$ consisting of paths c such that $c|_{[0,m(v)]}$ is the reparameterization of a path in $\Delta_{m(v)v}$ on $[0, m(v)]$ and $c|_{[m(v),1]}$ is just $Q \circ s_v|_{[m(v),1]}$. Therefore, we can identify $\Delta_v \cong \Delta_{m(v)v}$ in this case. In particular, for $v \in V$, we have $pr_v^{-1}(v) = \{Q \circ s_v\}$ and $\Delta_v \cong C_v$ if $i(v) = 0$. If $i(v) = 1$ we have $pr_v^{-1}(w) \cong C_{m(w)w} \cong S^1$ for all $w \in C_v$ and therefore, due to Lemma 1.2.5, that $\check{H}^{1+\dim(C_v)}(\Delta_v) \cong \mathbb{Z}_2$ and $\check{H}^k(\Delta_v) = 0$ if $k > 1 + \dim(C_v) = 1 + \mu(v)$. In the general case, we first observe that for all $w \in C_v$ we have $i(w) = i(v)$ and of course $\mu(w) = \mu(v)$. Then for all $u \in C_{m(w)\cdot w}$ we have

$$i(u) = i(m(w) \cdot w) = i(w) - \mu(m(w) \cdot w) = i(v) - \dim(C_{m(w)\cdot w}) < i(v).$$

So, by induction on the index, we can therefore assume that the compact fibers $pr_v^{-1}(w) \cong \Delta_{m(w)\cdot w}$ satisfy

$$\check{H}^{i(m(w)\cdot w)+\mu(m(w)\cdot w)}(pr_v^{-1}(w)) \cong \mathbb{Z}_2$$

and

$$\check{H}^k(pr_2^{-1}(w)) = 0 \text{ if } \ k > i(m(w) \cdot w) + \mu(m(w) \cdot w).$$

Again, with Lemma 1.2.5 and $i(m(w) \cdot w) + \mu(m(w) \cdot w) = i(w) = i(v)$ as well as $\mu(v) = \dim(C_v)$, we then deduce that $H^{i(v)+\mu(v)}(\Delta_v) \cong \mathbb{Z}_2$ and $H^k(\Delta_v) = 0$ if $k > i(v) + \mu(v)$.

Coming so far, it remains to prove that the spaces Δ_v indeed represent linking cycles for $Q \circ s_v$. But this follows from the fact that, due to [Wa67] and [Heb81], $\mathfrak{C}$ defines a smooth distribution on every connected component of C_R, so that it is more or less obvious by our construction that $Q \circ s_v$ admits a manifold neighborhood in Δ_v for all $v \in V^R$. Further, this neighborhood can be deformed into the local unstable manifold in some Morse chart around $Q \circ s_v$, because of the discussion below A.2.4 and the following expression of the tangent space that is a direct consequence of our construction. Namely, $T_{Q\circ s_v}\Delta_v = \bigoplus_{k=1}^r \mathcal{J}(t_k)$, where $s_v^{-1}(C) = \{t_1, \dots, t_r\}$ and $\mathcal{J}(t_k)$ equals the vector space of continuous vector fields J along $Q \circ s_v$ such that $J|_{[0,t_k]}$ is an L-Jacobi field along $Q \circ s_v$ and $J|_{[t_k,1]} \equiv 0$. In this case, if we denote by $\mathcal{P}_{L,\eta(v)}$ the path space $\mathcal{P}(M, L \times \eta(v))$, the following commutative diagramm

$$\begin{tikzcd}
\check{H}^{i(v)}(\mathcal{P}^{\|v\|^2}_{L,\eta(v)}) \arrow[r] & \check{H}^{i(v)}(\Delta_v) \\
\check{H}^{i(v)}(\mathcal{P}^{\|v\|^2}_{L,\eta(v)}, \mathcal{P}^{\|v\|^2}_{L,\eta(v)} \setminus \{\eta \circ s_v\}) \arrow[u] \arrow[r, "\cong"] & \check{H}^{i(v)}(\Delta_v, \Delta_v \setminus \{Q \circ s_v\}) \arrow[u, "\cong"']
\end{tikzcd}$$

yields the claim if $v \in V^R$ is not a conjugate vector. Since for manifolds Čech cohomology is isomorphic to singular cohomology and V^R is dense in V, we deduce, with the same arguments as in the proof of Proposition 2.7 in [TT97], that the energy $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is $\mathbb{Z}_2$-perfect for all points q that are not focal points of L.

□

Remark 1.3.2. As we mentioned in the last section, for compact subsets (K, L) of a manifold P the Čech cohomolgy groups $\check{H}^j(K, L)$ are isomorphic to the direct limit

$$\varinjlim \left\{ H^j(U,V) | (K,L) \subset (U,V) \right\}$$

where the limit is taken over open subsets $(U,V) \supset (K,L)$. Therefore, one could also show directly that

$$H^{i(v)}(\mathcal{P}_{L,\eta(v)}^{\|v\|^2}, \mathcal{P}_{L,\eta(v)}^{\|v\|^2} \setminus \{\eta \circ s_v\}) \to \check{H}^{i(v)}(\Delta_v, \Delta_v \setminus \{\eta \circ s_v\}) \to \check{H}^{i(v)}(\Delta_v)$$

is nontrivial for all the spaces Δ_v with $v \in V \setminus C$, because using the deformation retraction of $P_{L,\eta(v)}^{\|v\|^2+\varepsilon}$ onto the Morse complex one can assume that a neighborhood base of $\eta \circ s_v$ in Δ_v is contained in some ball around the origin in $\mathbb{R}^{i(v)}$.

As a direct consequence of the proof of Theorem 1.3.1 and the above remark, we obtain the following fact, which was so far not even known in the case of a Euclidean space.

Theorem 1.3.3. *If a closed submanifold of a complete Riemannian manifold is taut with respect to some field, then it is also $\mathbb{Z}_2$-taut.*

Chapter 2

Taut Foliations

Even if there are not many examples of taut submanifolds, a remarkable observation is that they often occur, if at all, in families, which then decompose the ambient space. In this chapter we therefore focus on taut families as they usually occur, namely on singular Riemannian foliations all of whose leaves are taut. For this reason, we first recall some basic facts about singular Riemannian foliations and make some preliminary observations which we need to prove our second result in Section 2.3 that characterizes taut singular Riemannian foliations by means of their quotients.

2.1 Singular Riemannian Foliations

Definition 2.1.1. Let $\mathcal{F}$ be a partition of a manifold M^{n+k} into connected, injectively immersed submanifolds with maximal dimension n. For a point $p \in M$, let L_p denote the element of $\mathcal{F}$ which contains p. Set

$$T\mathcal{F} = \bigcup_{p \in M} T_p L_p.$$

Then the partition $\mathcal{F}$ is called a *singular foliation of M of dimension n/ codimension k* iff the $C^\infty(M)$-module $\Gamma(T\mathcal{F})$ of smooth vector fields X tangential to $\mathcal{F}$, i.e. $X_p \in T_p L_p$ for all $p \in M$, exhaust $T_p L_p$ for every $p \in M$. We call the elements of $\mathcal{F}$ *leaves*. A leaf is *regular* if it has dimension n, otherwise *singular*. A point belonging to a regular leaf is *regular*, otherwise *singular*. By M_0 we denote the set of regular points and call it the *regular stratum*. If (M, g) is a Riemannian manifold, a singular foliation is called a *singular Riemannian foliation* if every geodesic in M which intersects one leaf orthogonally intersects every leaf it meets orthogonally.

We sometimes also speak about a singular Riemannian foliation $(M, \mathcal{F})$, or $(M, g, \mathcal{F})$ if we want to abbreviate that $\mathcal{F}$ is a singular Riemannian foliation on the Riemannian manifold M, or (M, g).

Example. The set of orbits of an isometric Lie group action on a Riemannian manifold M is a singular Riemannian foliation, closed if and only if the group considered as a subgroup of the isometry group is closed.

For $d \leq n$, denote by M_d the subset of all points $p \in M$ with fixed leaf dimension $\dim(L_p) = n - d$. Since the dimension of the leaves varies lower semicontinuously, the set $\bigcup_{d' \leq d} M_{n-d'} = \{p \in M | \dim(L_p) \leq d\}$ is closed. Further, M_d is an embedded submanifold of M and the restriction of $\mathcal{F}$ to M_d is a (regular) Riemannian foliation. The main stratum M_0 is open, dense and connected if M is connected. All the other singular strata have codimension at least 2 in M.

Let p be a point in $(M, g, \mathcal{F})$ and let B be a small open ball in L_p. Then there is a number $\varepsilon > 0$ and a *distinguished tubular neighborhood* U at p so that the following holds true:

1. The foot point projection $\pi : U \to B$ is well defined;

2. U is the image of the ε-discs $\nu^\varepsilon(B)$ in the normal bundle $\nu(B)$ of B under the exponential map and the map $\exp : \nu^\varepsilon(B) \to U$ is a diffeomorphism;

3. $T_q = \pi^{-1}(q)$ is a global transversal of $(U, \mathcal{F}|_U)$ for all $q \in B$, i.e. T_q meets all the leaves of $\mathcal{F}|_U$ and always transversally;

4. For each real number $\lambda \in [-1, 1] \setminus \{0\}$ the map $h_\lambda : U \to U$, given by $h_\lambda(\exp(v)) = \exp(\lambda v)$ for all $v \in \nu^\varepsilon(B)$, preserves $\mathcal{F}$.

Indeed, the fact that the geodesics perpendicular to B remain perpendicular to the leaves implies that if $d(q, B) = \delta$, then the connected component P_q of q in the open subset $L_q \cap U$ of the leaf L_q is entirely contained in the tube S^B_δ of radius δ around B. The leaf of $\mathcal{F}_U$ through q, which is exactly P_q, is called the *plaque of $\mathcal{F}$ passing through q in the neighborhood U*. In particular, we have $B = P_p$ by construction. Moreover, we see that for all $q \in U$, the distance from q to B remains constant as q moves along the plaque P_q, thus the distance between the neighboring leaves is locally constant.

Definition 2.1.2. We say that a singular Riemannian foliation $(M, \mathcal{F})$ *has the property* P if every leaf of $\mathcal{F}$ has the property P, e.g. $\mathcal{F}$ is closed if all the leaves are closed subspaces of M.

It is well kown that the leaves of a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M admit global ε-tubes, so that the distance between two leaves is globally constant. In this case, the quotient $M/\mathcal{F}$ is a complete metric space, where the distance between two points is just the distance between the corresponding leaves, as submanifolds of M.

Lemma 2.1.3. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Then a leaf $L \in \mathcal{F}$ is embedded if it is closed.*

Proof. Let r be the dimension of L. Then M_{n-r} is an embedded submanifold of M and $\mathcal{F}|_{M_{n-r}}$ is a regular Riemannian foliation. Due to Molino (cf. p.22 of [Mol88]), the statement is true for $(M_{n-r}, \mathcal{F}|_{M_{n-r}})$, so it is true for $(M, \mathcal{F})$. □

Again, let $p \in M$ be a point and let B be a small open neighborhood in the leaf L_p through p. Then there is an $\varepsilon > 0$ and a distinguished tubular neighborhood (U, B, π) around p such that there is an embedding ϕ of U into the tangent spaces T_pM with $D_p\phi = \mathrm{Id}$ and a singular Riemannian foliation $\mathfrak{F}_p$ on T_pM, called *infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point p* that coincides with $\phi_*\mathcal{F}$ on $\phi(U)$ and such that $\mathfrak{F}_p$ is invariant under all rescalings $r_\lambda : T_pM \to T_pM, r_\lambda(v) = \lambda v$ for all $\lambda \neq 0$. In particular, $\mathfrak{F}_p$ is closed iff $\mathcal{F}$ is locally closed at p.

The diffeomorphism ϕ is constructed as follows. If $\dim(L_p) = r$, we can choose B and ε so small that there are r linear independent vector fields $X^1, \dots, X^r \in \Gamma(T\mathcal{F}|_U)$ such that $\{X^1_p, \dots, X^r_p\}$ form an orthonormal basis of T_pL_p. Let $\varphi^{X^i}_t$ denote the local flow of X^i. Denote by $B_\delta \subset \mathbb{R}^r$, resp. $D_\delta \subset \nu_p(L_p)$ the ball of radius δ around $0 \in \mathbb{R}^r$, resp. the ball of radius δ around $0 \in \nu_p(L_p)$. For δ small we get a smooth map $\psi : B_\delta \times D_\delta \to M$ by

$$\psi(t_1, \dots, t_r, v) = \varphi^{X^r}_{t_r} \circ \varphi^{X^{r-1}}_{t_{r-1}} \circ \cdots \circ \varphi^{X^1}_{t_1}(\exp_p(v)).$$

Since the differential $d\psi_{(0,0)}$ is invertible, we can find an ε such that $\psi|_{B_\varepsilon \times D_\varepsilon}$ is a diffeomorphism onto its image, which is a distinguished tubular neighborhood and after identifying $\mathbb{R}^r \cong T_pL_p, (t_1, \dots, t_r) \mapsto \sum_{i=1}^r t_i X^i_p$ we get the desired diffeomorphism by $\phi = (\psi|_{B_\varepsilon \times D_\varepsilon})^{-1}$.

One can consider $\mathfrak{F}_p$ as the blow up of $\mathcal{F}$ in the following sense. Identify U with $\phi(U)$. Set $U^\lambda = r_\lambda(U)$ for $\lambda > 0$. So $\bigcup_{\lambda>0} U^\lambda = T_pM$. Define the Riemannian metric g^λ on U^λ as $g^\lambda = \lambda^2 (r_\lambda)_* g$. Then the blow up metrics g^λ smoothly converge to the flat metric g_p (ref. [Wie08], p.43-44). By construction, the restriction of $\mathfrak{F}_p$ to U^λ is a singular Riemannian foliation with respect to g^λ. Moreover, if $\dim(L_p) = r$, then the infinitesimal singular foliation $\mathfrak{F}_p$ on $T_pM = T_pM_{n-r} \times (T_pM_{n-r})^\perp$ is a product $\mathfrak{F}^v_p \times \mathfrak{F}^h_p$, where $\mathfrak{F}^v_p$ is the trivial foliation given by parallels of T_pL_p and the main part $\mathfrak{F}^h_p$ on $(T_pM_{n-r})^\perp$ is a singular Riemannian foliation, invariant under rescalings and with the origin as the only 0-dimensional leaf. Thus $\mathfrak{F}^h_p$ is the cone over a singular Riemannian foliation of dimension $n - r$ on the unit sphere of $T_pM^\perp_{n-r}$, which is induced by the intersections of the nearby higher dimensional leaves with a slice through p.

If $\mathcal{F}$ is locally closed at p, the quotient $T_pM/\mathfrak{F}_p$ is a non-negatively curved Alexandrov space and the local quotient $U/\mathcal{F}$ is a metric space of curvature bounded below in the sense of Alexandrov. Further, the space $T_pM/\mathfrak{F}_p$ is the tangent space to this

Alexandrov space at the leaf $L \cap U \in U/\mathcal{F}$. The inclusion $U \to M$ induces a map between the quotients $U/\mathcal{F} \to M/\mathcal{F}$, which is an open finite-to-one map if $\mathcal{F}$ is closed. So, assume that $\mathcal{F}$ is closed and M is complete. Then the quotient $M/\mathcal{F}$ is a complete metric space with the metric induced by the distance of the leaves of $\mathcal{F}$ (as submanifolds). Let T be a global ε-tube around L with the same ε as in the definition of the distinguished neighborhood U, then U is saturated, i.e. it is a union of leaves, and $T/\mathcal{F}$ is a neighborhood of L in the global quotient $M/\mathcal{F}$. In this case, there is a finite group of isometries Γ acting on the local quotient $U/\mathcal{F}$ that fixes the plaque $L \cap U \in U/\mathcal{F}$ such that $U/\mathcal{F}$ is isometric to $T/\mathcal{F}$.

For a proof of the above statements and for more on singular Riemannian foliations we refer to [Mol88].

Definition 2.1.4. Let $\mathcal{F}$ be singular Riemannian foliation on a complete Riemannian manifold M. If $\mathcal{F}$ is closed, we call $\mathcal{F}$ *taut* if every leaf of $\mathcal{F}$ is taut.

Note that if $\mathcal{F}$ is closed, then by 2.1.3 all the leaves are embedded submanifolds. Further, if $\mathcal{F}$ is the trivial foliation given by the points of M and $\mathcal{F}$ is taut, then, as already defined in Section 1.1, we call M pointwise taut.

We now proceed with some properties of taut foliations. Let us start with an elementary observation on the infinitesimal foliations.

Lemma 2.1.5. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold (M, g). If $\mathcal{F}$ is taut, then for every $p \in M$, the infinitesimal foliation $\mathfrak{F}_p$ on (T_pM, g_p) is taut.*

Remark 2.1.6. Anticipating our discussion in Section 3.1, we say that a singular Riemannian foliation on a Euclidean space V is *polar* if for all regular points $v \in V$, the affine space $v + \nu_v(L_v)$ that is the image of the corresponding normal space under the endpoint map intersects all the leaves and always orthogonally. It is well known that the polar singular Riemannian foliations on Euclidean spaces are exactly the parallel foliations given by an isoparametric submanifold. Thus if the infinitesimal singular Riemannian foliation $\mathfrak{F}_p$ on (T_pM, g_p) is polar, $\mathfrak{F}_p$ is an isoparametric foliation given by the parallel foliation induced by a regular and hence isoparametric leaf. Since isoparametric foliations on a Euclidean space are taut (see, e.g. [PT88]), the infinitesimal singular foliation $\mathfrak{F}_p$ is taut in this case. For a more detailed treatment of these facts, we refer the reader to Chapter 3, where we will focus on *infinitesimally polar* foliations, namely those foliations whose infinitesimal foliations are all polar.

Proof of Lemma 2.1.5. We will use the notation from the beginning of this section. Take a point $p \in M$ with $d = \dim(L_p)$ and $k = \operatorname{codim}(M_{n-d})$. Then we have an orthogonal splitting $T_pM = T_pM_{n-d} \oplus \nu_p(M_{n-d})$ that induces a splitting of $\mathfrak{F}_p$ into a product foliation $\mathfrak{F}_p = \mathfrak{F}_p^v \times \mathfrak{F}_p^h$, where the first factor is the foliation given by parallels of T_pL_p and the main part $\mathfrak{F}_p^h$ is the cone-foliation, i.e. invariant under all homotheties $r_\lambda(v) = \lambda \cdot v$, of a singular Riemannian foliation with compact leaves of dimension at least one on the unit sphere in $\nu_p(M_{n-d})$ (if $\nu_p(M_{n-d}) \neq \{0\}$). Now for $(v, w) \in T_pM_{n-d} \times \nu_p(M_{n-d})$, let $\mathcal{L} = (v + T_pL_p) \times \mathcal{L}_w^h$ be the leaf of $\mathfrak{F}_p$ through (v, w) and let $u = (x, y)$ be not a focal point of $\mathcal{L}$ (with respect to the flat metric g_p). Then obviously a curve $c : [0, 1] \to T_pM$ is a critical point of $E_u : \mathcal{P}(T_pM, \mathcal{L} \times u) \to \mathbb{R}$ if and only if c is a straight line with $\dot{c} \in \nu_{c(0)}(\mathcal{L}) \subset W \times \nu_p(M_{n-d})$, where we again denote by W the orthogonal complement of T_pL_p in T_pM_{n-d}. So if we write $T_pM_{n-d} = W \times T_pL_p$ and $v = (v_1, v_2), x = (x_1, x_2)$ with respect to this decomposition, we have that $c(t) = (x_1 + t \cdot (v_1 - x_1), c^h(t))$ and we get a 1:1 correspondence between the critical points of E_u and the critical points of the restricted energy functional E_y on the path space $\mathcal{P}(\nu_p(M_{n-d}), \mathcal{L}_w^h \times y)$ of the compact submanifold $\mathcal{L}_w^h \subset \nu_p(M_{n-d})$. In particular, since T_pL_p is contractible, $\mathfrak{F}_p$ is taut if and only if $\mathfrak{F}_p^h$ is taut. Further, because $\mathfrak{F}_p$ as well as the set of straight lines in T_pM is invariant under homotheties it follows that $\mathfrak{F}_p$ is taut if and only if the restricted foliation $\mathfrak{F}_p^h|_D$ is taut, where D is a small ball in $\nu_p(M_{n-d})$ around the origin. Note that such a ball D is always saturated. Let U be a distinguished tubular neighborhood around p and set $V = \phi(U)$ and $h = \phi_* \, g$, where $\phi : U \to T_pM$ with $\phi_*(\mathcal{F}|_U) = \mathfrak{F}_p|_V$ is an embedding as in the definition of the infinitesimal foliation at p. Now with respect to the metric h and for a small ball D around the origin in $\nu_p(M_{n-d})$ the closed singular Riemannian foliation $\mathfrak{F}_p|_{(T_pM_{n-d} \times D) \cap \phi(U')}$ is taut, i.e. the saturation of $\mathcal{F}|_{\phi^{-1}(D)}$ in U' is taut, where $U' \subset U$ is a smaller distinguished tubular neighborhood at p that contains $\phi^{-1}(D)$. To see this, we can choose U' so small that we have to consider only critical points γ in U' with energy r such that the whole ball of radius r around $\gamma(1)$ is contained in U, so that we have

$$P(U, L_\gamma(0) \cap U \times \gamma(1))^r = P(M, L_{\gamma(0)} \times \gamma(1))^r$$

and we conclude by tautness of $\mathcal{F}$ that all the local unstable manifolds can be completed in U below the energy r. Thus since U and U' are diffeomorphic, the local unstable manifolds can also be completed in U', what implies our claim. If we now consider the blow up metrics h^λ on $V^\lambda = \{\lambda \cdot v | v \in V\}$ as defined above, i.e.

$$h^\lambda = \lambda^2 \cdot (r_\lambda)_* \, h,$$

it follows that our restricted foliation is also taut with respect to the metrics h^λ. But the constant metric g_p is just the flat limit $\lim_{\lambda \to \infty} h^\lambda$ and we deduce that $\mathfrak{F}_p$ is taut with respect to g_p, because it is not hard to see that if a sequence of perfect Morse functions converge to a Morse function, this limit has to be perfect.

□

In order to prove tautness of a foliation, one can always assume that M is simply connected. To see this, we first need the closeness property of lifts.

Lemma 2.1.7. *Let $\mathcal{F}$ be closed and let $\pi : N \to M$ be a covering map. Then the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ given by the involutive singular distribution $T\widetilde{\mathcal{F}} = \pi^*(T\mathcal{F})$ is closed.*

Remark 2.1.8. Note that the converse is false, as one can see by the dense torus foliation induced by the submersion $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto y - \lambda x$, where λ is irrational.

Proof. For every leaf L of $\mathcal{F}$, the preimage $\pi^{-1}(L) = \bigcup_i \widetilde{L}_i$ is a union of leaves of $\widetilde{\mathcal{F}}$ and the restriction $\pi|_{\widetilde{L}_i} : \widetilde{L}_i \to L$ is a covering projection for each i. Thus each leaf $\widetilde{L} \in \widetilde{\mathcal{F}}$ is a connected component of the closed saturated set $\pi^{-1}(\pi(\widetilde{L}))$ and hence closed. □

Assume that $f : N \to M$ is a Riemannian submersion between complete Riemannian manifolds and $L \subset M$ is a closed submanifold. Then, by [He60], $f : N \to M$ is a locally trivial fiber bundle and therefore for any point $\bar{q} \in f^{-1}(q)$, the spaces $\mathcal{P}(N, f^{-1}(L) \times \bar{q})$ and $\mathcal{P}(M, L \times q)$ are homotopy equivalent. Since f yields a 1:1 correspondece between the critical points and preserves their indices (cf. Lemma 6.1 in [HLO06]), we obtain

Lemma 2.1.9. *If $f : N \to M$ is a Riemannian submersion between complete Riemannian manifolds and $L \subset M$ is a closed submanifold, then L is taut if and only if $f^{-1}(L)$ is taut.*

Now, since the homology of a path connected component injects in the homology of the whole space, it is not hard to see that a union of connected, closed submanifolds is taut if and only if its components are taut. So that we deduce

Lemma 2.1.10. *Let $\pi : N \to M$ be a Riemannian covering and let M be complete. If $\mathcal{F}$ is closed, then $\mathcal{F}$ is $\mathbb{F}$-taut if and only if the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to N is $\mathbb{F}$-taut.*

Given a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M, every leaf posses a global ε-tube. For a regular leaf L with such a global tube, the restriction of the foot point projection on a nearby regular leaf induces a finite covering map onto L. We say that L has *trivial holonomy* if all these coverings are diffeomorphisms. It is well known that the set of regular points whose leaves have trivial holonomy is open and dense in M.

In particular, all regular leaves of $\mathcal{F}$ have trivial holonomy, that is to say that the quotient $M_0/\mathcal{F}$ is a Riemannian manifold, if the foliation is taut and M is simply connected. To see this, for $p, q \in M$, let $\Omega_{p,q}(M)$ denote the space of all paths from p to q. Then $\Omega_{p,q}(M) \simeq \Omega_{q,q}(M)$ and the long exact sequence of the path space fibration gives $\pi_i(M) \cong \pi_{i-1}(\Omega_{q,q}(M))$, what implies that $\Omega_{p,q}(M)$ is connected.

The fibration $\mathcal{P}(M, L \times q) \to L$ given by $c \mapsto c(0)$ gives the below part of the corresponding long exact homotopy sequence

$$\pi_0(\Omega_{q,q}(M)) \to \pi_0(\mathcal{P}(M, L \times q)) \to \pi_0(L) \to 1.$$

Thus $\mathcal{P}(M, L \times q)$ is connected. Now a leaf with nontrivial holonomy would yield at least two local minima for the energy on the path space of a neighboring generic leaf, i.e. a leaf without holonomy. By tautness, all the maps in homology are injective what clearly contradicts our connectedness observation.

We are now able to state a characterisation of taut regular foliations, that indeed also follows from our second main result, Theorem 2.3.1 below. For the notion of Riemannian orbifolds see Section 3.1.

Lemma 2.1.11. *Let $\mathcal{F}$ be a closed (regular) Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is $\mathbb{F}$-taut if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise $\mathbb{F}$-taut universal covering orbifold, i.e. $M/\mathcal{F}$ is isometric to N/Γ with a simply connected Riemannian manifold N, all of whose points are $\mathbb{F}-$taut and $\Gamma \subset I(N)$ is a discrete subgroup of isometries.*

Proof. Let $\widetilde{\mathcal{F}}$ denote the lift of $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M. Then, by Lemma 2.1.10, $\widetilde{\mathcal{F}}$ is taut if and only if $\mathcal{F}$ is taut. By the discussion above, if $\widetilde{\mathcal{F}}$ is taut, then it is simple, i.e. given by the fibers of a Riemannian submersion. So if $\mathcal{F}$ is taut, the quotient map $\widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{F}}$ is a Riemannian submersion between complete Riemannian manifolds and $\widetilde{M}/\widetilde{\mathcal{F}}$ is 1-connected by the exact sequence for fibrations. In this case, the map $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ coincide with the universal orbifold covering. On the other hand, assume that $M/\mathcal{F}$ is a good Riemannian orbifold. Then, due to Lemma 3.1.11, $\widetilde{\mathcal{F}}$ is simple, that is to say $\widetilde{M}/\widetilde{\mathcal{F}}$ is a Riemannian manifold and we can reduce the problem to Lemma 2.1.9.

□

Remark 2.1.12. In Section 3.3 we prove the corresponding statement for the class of infinitesimally polar foliations and coefficients in $\mathbb{Z}_2$.

We end this section with some preliminaries, which are useful in order to prove that a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M is taut.

Lemma 2.1.13. *Let $\mathcal{F}$ be closed. Then $L \in \mathcal{F}$ is $\mathbb{F}$-taut if and only if the energy functional $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is an $\mathbb{F}$-perfect Morse function for all non L-focal regular points $q \in M$. Further, $\mathcal{F}$ is $\mathbb{F}$-taut if and only if all regular leaves are $\mathbb{F}$-taut.*

Proof. Since the set of non-focal points of L as well as the set of regular points is open and dense in M, every neighborhood of a given point q contains a regular point that is not a focal point. Therefore, the same argument as in the proof of Proposition 2.7 in [TT97] yields the first claim.

For the second claim, assume that all regular leaves are taut. Let N be a singular leaf and let $q \in M$ be not a focal point of N. By our above observations, we can assume that the point q is regular. Let γ be a critical point of E_q^N. Then $\dot{\gamma}(0)$ is a regular vector of $\mathfrak{F}_{\gamma(0)}$, hence there exists an $\varepsilon > 0$ such that $L = L_{\gamma(\varepsilon)}$ is a regular leaf contained in a global tube of N and the point q is not a focal point of L. Denote by $\bar{\gamma}$ the restriction $\gamma|_{[\varepsilon,1]}$ after linear reparameterization on $[0,1]$. Then the horizontal geodesic $\bar{\gamma}$ is a critical point of the perfect Morse function $E_q^L : \mathcal{P}(M, L \times q) \to \mathbb{R}$. Now let $i = i(\bar{\gamma})$ be the index of the geodesic $\bar{\gamma}$ and set $\kappa = E_q^L(\bar{\gamma})$ for its energy. We also set $\mathcal{L}^c = (E_q^L)^{-1}([0,c])$, resp. $\mathcal{N}^c = (E_q^N)^{-1}([0,c])$. Denote by $\sigma \in H_i(\mathcal{L}^\kappa)$ the completion of the local unstable manifold representing a nontrivial cycle in $H_i(\mathcal{L}^\kappa, \mathcal{L}^\kappa \setminus \{\bar{\gamma}\})$ associated to $\bar{\gamma}$. We then have $i(\gamma) = i(\bar{\gamma})$ for small numbers ε, by continuity reasons.

The projection $R : L \to N, (p, v) \mapsto p$, induces a map

$$\begin{array}{rccc} \bar{R}: & \mathcal{L}^\kappa & \to & \mathcal{N}^{E_q^N(\gamma)} \\ & c & \mapsto & \tilde{c} \end{array}$$

where $\tilde{c}$ is the curve that one gets by concatenation of the unique horizontal geodesic from $R(c(0))$ to $c(0)$ with c followed by reparameterization between 0 and 1 and the map $\bar{R}$ maps level sets to level sets. Moreover, $\bar{R}$ is clearly an immersion. Therefore, $\bar{R}_*(\sigma)$ is a cycle in $\mathcal{N}^{E_q^N(\gamma)}$ that can be deformed within a morse chart around γ into a cycle z that agrees with the unstable manifold at γ above the E_q^N-level $E_q^N(\gamma) - \delta$ for small δ. It follows that the homology class of z and thus the homology class $\bar{R}_*(\sigma)$ is mapped onto a generator of $H_i(\mathcal{N}^{E_q^N(\gamma)}, \mathcal{N}^{E_q^N(\gamma)} \setminus \{\gamma\})$. Since the critical point γ was chosen arbitrary, every local unstable manifold can be completed to a cycle in $H_{i(\gamma)}(\mathcal{N}^{E_q^N(\gamma)+\delta})$, i.e. the map $H_n(\mathcal{N}^{\lambda+\delta}) \to H_n(\mathcal{N}^{\lambda+\delta}, \mathcal{N}^{\lambda-\delta})$ is surjective for all n and regular values $\lambda \pm \delta$. Hence N is taut. $\square$

Our last observation in this section is that a foliation $\mathcal{F}$ is $\mathbb{F}$-taut if and only if a dense family of regular leaves is $\mathbb{F}$-taut, where we call a family of leaves *dense* if their union is a dense set. The next lemma shows that tautness is a closed property relative to non-collapsing convergence. It is then straight forward to see that tautness of a dense family of regular leaves forces a foliation to be taut.

Lemma 2.1.14. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete manifold M and let $\{L_n\}$ be a sequence of $\mathbb{F}$-taut regular leaves converging to a regular leaf L without holonomy, i.e. for every tubular neighborhood T of L there is $n_0 \in \mathbb{N}$ such that $L_n \subset T$ and the canonical projection $\pi : T \to L$ restricted to L_n is a covering map for every $n \geq n_0$. Then L is $\mathbb{F}$-taut.*

Proof. Let T be a tubular neighborhood of L and let $\pi : T \to L$ be the canonical projection. Choose a number $n_0 \in \mathbb{N}$ so large that $L_n \subset T$ for all $n \geq n_0$. Now let $q \in M$ be not a focal point of L. Then for large n, the point q is not a focal point of L_n as well. By $f_n : \mathcal{P}(M, L \times q) \to \mathcal{P}(M, L_n \times q)$, resp. $g_n : \mathcal{P}(M, L_n \times q) \to \mathcal{P}(M, L \times q)$ we denote the induced maps between the path spaces which one gets by assigning to a curve c the curve $\gamma_{c(0)} \cdot c$ and then reparameterizing it between 0 and 1, where $\gamma_{c(0)}$ is the unique shortest geodesic between L_n and L that intersects L in $c(0)$, resp. $\gamma_{c(0)}$ is the unique shortest geodesic between L and L_n that intersects L_n in $c(0)$. Then f_n is in an obvious way a homotopy equivalence with homotopy inverse g_n.

Let γ be a critical point of E_q^L with $\kappa = E_q^L(\gamma)$. We can choose n so large, i.e. a tube T so small, that there is an $\varepsilon > 0$ such that $(\kappa - 3\varepsilon, \kappa + 3\varepsilon) \setminus \{\kappa\}$ contains only regular values and $g \circ f(P^{\kappa-2\varepsilon}) \subset P^{\kappa-\varepsilon}$ with $f = f_n, g = g_n$ and $P^r = \mathcal{P}(M, L \times q)^r$ and P_n^r defined analogous. Moreover, we can deform $g \circ f : P^{\kappa-2\varepsilon} \to P^{\kappa-\varepsilon}$ into the inclusion $j : P^{\kappa-2\varepsilon} \hookrightarrow P^{\kappa-\varepsilon}$ below the κ-level of E_q^L, i.e.

$$(g \circ f)_* = j_* : \quad H_*(P^{\kappa-2\varepsilon}) \quad \to \quad H_*(P^{\kappa-\varepsilon}).$$

Since $P^{\kappa-2\varepsilon}$ is a strong deformation retract of $P^{\kappa-\delta}$ for all $2\varepsilon > \delta > 0$, the map above is an isomorphism, i.e. the induced map in homology

$$f_* : H_*(P^{\kappa-2\varepsilon}) \to H_*(P_n^\alpha)$$

with $\alpha = \max\{E_q^{L_n}(p) : E_q^L(p) = \kappa - 2\varepsilon\}$ is injective.

Denote by $i = i(\gamma)$ the index and by $[\gamma] \in H_i(P^\kappa, P^{\kappa-2\varepsilon})$ the corresponding unstable manifold of γ and set $\tilde{\alpha} = \max\{E_q^{L_n}(p) : E_q^L(p) = \kappa\}$. Now consider the commutative diagramm which comes from the long exact sequence for pairs of spaces together with the natural behavior of the connecting homomorphism:

$$\begin{tikzcd}
H_i(P^\kappa) \arrow[r, "f_*"] \arrow[d] & H_i(P_n^{\tilde{\alpha}}) \arrow[d] \\
H_i(P^\kappa, P^{\kappa-2\varepsilon}) \arrow[r, "f_*"] \arrow[d, "\partial_*"'] & H_i(P_n^{\tilde{\alpha}}, P_n^\alpha) \arrow[d, "\tilde{\partial}_*"] \\
H_{i-1}(P^{\kappa-2\varepsilon}) \arrow[r, "f_*"] & H_{i-1}(P_n^\alpha)
\end{tikzcd}$$

Since L_n is taut, we have $\tilde{\partial}_* = 0$. So

$$f_* \circ \partial_* = \tilde{\partial}_* \circ f_* = 0.$$

As we have seen, the map $f_* : H_*(P^{\kappa-2\varepsilon}) \to H_*(P_n^\alpha)$ is injective. But this means $\partial_* = 0$, i.e. L is taut. $\square$

Now assume under the assumptions of Lemma 2.1.14 that all L_n are regular leaves without holonomy and that L is an exceptional leaf, i.e. has nontrivial holonomy. Due to Lemma 2.1.10, we can assume that M is simply connected. Then for large n, L would provide at least two local minima for L_n, because all L_n are taut and therefore have trivial holonomy, as we have seen above. Again, by tautness of L_n, the path space corresponding to L_n would be disconnected. But this is clearly a contradiction, since M is simply connected. So, L necessarily has trivial holonomy. Thus combining 2.1.10,2.1.13, and 2.1.14 together with the fact that the set of regular leaves without holonomy is open and dense in $M/\mathcal{F}$, we have

Corollary 2.1.15. *The closed singular Riemannian foliation $\mathcal{F}$ is $\mathbb{F}$-taut if and only if a dense family of leaves is $\mathbb{F}$-taut.*

2.2 Index Splitting for Horizontal Geodesics

In this section we summarize some general observations on the focal index of geodesics with respect to Lagrangian subspaces of the space of normal Jacobi fields, which we then apply to the case of a horizontal geodesic of a singular Riemannian foliation $(M, \mathcal{F})$. We will see that the focal data of the space of normal $L_{\gamma(a)}$-Jacobi fields along a regular horizontal geodesic $\gamma : [a, b] \to M$ are of two types. Namely, for $t \in (a, b)$, there is a vertical multiplicity $\dim(\mathcal{F}) - \dim(L_{\gamma(t)})$ corresponding to the focal index of t of the isotropic space of Jacobi fields that are everywhere tangent to $\mathcal{F}$, and a horizontal multiplicity that is, roughly speaking, the multiplicity of $\gamma(t)$ as a conjugate point of $\gamma(a)$ in the quotient $M/\mathcal{F}$ along the projection of γ. Our discussion is based on [L09], [LT10], and [Wil07].

If $\mathcal{F}$ is a singular Riemannian foliation on a Riemannian manifold (M, g) we call a geodesic γ *horizontal* if it meets all leaves of $\mathcal{F}$ perpendicularly. We will call such a geodesic $\gamma : [a, b] \to M$ regular, if $\gamma(a)$ and $\gamma(b)$ are regular points of $\mathcal{F}$.

A regular horizontal geodesic intersects the singular strata of $\mathcal{F}$ only in finitely many points $a < t_1 < \cdots < t_r < b$ (see Cor.4.6 in [LT10]). We set

$$c(\gamma) = \sum_{i=1}^{r} \dim(L_{\gamma(a)}) - \dim(L_{\gamma(t_i)})$$

and call this number the *crossing number* of γ.

Definition 2.2.1. Let $\gamma : [a, b] \to M$ be a horizontal geodesic. An *$\mathcal{F}$-Jacobi field* along γ is a variational field through horizontal geodesics starting on the leaf $L_{\gamma(a)}$. An *$\mathcal{F}$-vertical Jacobi field* along γ is an $\mathcal{F}$-Jacobi field J with $J(t) \in T_{\gamma(t)}L_{\gamma(t)}$ for all t.

Remark 2.2.2. In [LT07] Lytchak and Thorbergsson generalize the notion of variational completeness to the setting of singular Riemannian foliations by saying that a horizontal geodesic γ *has no horizontal conjugate points* if each $\mathcal{F}$-Jacobi field J with $J(t_0) \in T_{\gamma(t_0)}L_{\gamma(t_0)}$ for some $a < t_0 < b$ is $\mathcal{F}$-vertical. If no horizontal geodesics in M have horizontal conjugate points, $\mathcal{F}$ is called *without horizontal conjugate points*. Further, they give a description of singular Riemannian foliations without horizontal conjugate points. Namely, they show that a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M has no horizontal conjugate points if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold without conjugate points, i.e. $M/\mathcal{F} = N/\Gamma$ and N has no conjugate points. They also prove that a singular Riemannian foliation is infinitesimally polar (cf. 2.1.6 or Section 3.1) iff it is locally without horizontal conjugate points. In particular, if $\mathcal{F}$ has no horizontal conjugate points, then it is infinitesimally polar. Finally, Lytchak generalizes this result in [L10](Cor.1.4) proving that a singular Riemannian foliation on a complete Riemannian manifold M does not have horizontal conjugate points if and only if the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering $\widetilde{M}$ of M is closed and the quotient $\widetilde{M}/\widetilde{\mathcal{F}}$ is a good

Riemannian orbifold without conjugate points.

Let us recall some basic facts about the Jacobi equation, Jacobi fields, and focal points. We refer to [L09] and [Wil07] for the proofs and a more detailed discussion of the following facts. Our summary is referred to [LT10].

Let $\gamma : [a,b] \to M$ be a geodesic and let $\mathcal{N}$ be the normal bundle of γ. Let Jac denote the space of all normal Jacobi fields along γ, i.e. solutions of the equation

$$\nabla^2 J + R(J, \dot\gamma)\dot\gamma = 0,$$

where R denotes the curvator tensor. By ω we denote the canonical symplectic form on Jac, defined by $\omega(J_1, J_2) = \langle \nabla J_1, J_2\rangle + \langle J_1, \nabla J_2\rangle$. For subspaces W of Jac, we denote by $W^\perp$ the orthogonal complement with respect to ω. A subspace $W \subset \mathrm{Jac}$ is called *isotropic*, resp. *Lagrangian* if $W \subset W^\perp$, resp. $W = W^\perp$. For an isotropic subspace W and $t \in [a,b]$, we define the *W-focal index* of t to be $f^W(t) = \dim(W) - \dim(W(t))$ with $W(t) = \{J(t) \mid J \in W\}$. Note that $f^W(t) = \dim(W^t)$, where we set $W^t = \{J \in W \mid J(t) = 0\}$. One can show that the set of points with non-zero focal index is discrete and such points are called *W-focal*. The *W-index* of γ is defined by $\mathrm{ind}_W(\gamma) = \sum_{t\in[a,b]} f^W(t)$.

Example. If N is a submanifold of M through $\gamma(a)$, orthogonally to γ, then the space Λ^N of normal N-Jacobi fields is a Lagrangian. In this case, the Λ^N-focal index of a is equal to $\dim(M)-\dim(N)-1$ and a point $t \neq a$ is a Λ^N-focal point if and only if it is a focal of N along γ in the usual sense of Riemannian geometry. So we will speak, for short, of the N-focal index and N-focal points in this case. In particular, the space $\Lambda^{L_{\gamma(a)}}$ of all $\mathcal{F}$-Jacobi fields along a horizontal geodesic γ of a singular Riemannian foliation $\mathcal{F}$ is a Lagrangian. Thus the space of all vertical $\mathcal{F}$-Jacobi fields is isotropic.

We recall now the construction of Wilking [Wil07] of the transversal Jacobi equation in our situation. Again let $\gamma : [a,b] \to M$ be a geodesic and consider the normal bundel $\mathcal{N}$ of γ with the connection induced by the pull back. Let $R : \mathcal{N} \to \mathcal{N}$ denote the curvature endomorphism, defined by $R(X) = R(X, \dot\gamma)\dot\gamma$. Let Jac be as above and consider an isotropic subspace W of Jac. Set

$$\widetilde{W}(t) = W(t) \oplus \left\{\nabla J(t) \mid J \in W^t\right\}$$

and note that $\widetilde{W}(t) = W(t)$ for every non W-focal $t \in [a,b]$.

Then Wilking observed (in a more generall setting) that $\widetilde{W}$ defines a smooth subbundle of $\mathcal{N}$. If we denote by $\mathcal{H}$ the orthogonal complement of $\widetilde{W}$ and by $P : \mathcal{N} \to \mathcal{H}$ the orthogonal projection, then P defines an identification $\mathcal{H} \cong \mathcal{N}/\widetilde{W}$ and we can define a smooth endomorphism field $A : \widetilde{W} \to \mathcal{H}$, by $A(J(t)) = P(\nabla J(t))$ and $A(\nabla J(t)) = 0$ for all $J \in W^t$. Consider the field $R^{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ of symmetric endomorphisms, defined by

$$R^{\mathcal{H}}(Y) = P(R(Y)) + 3AA^*(Y)$$

and denote by $\nabla^{\mathcal{H}}$ the induced covariant derivative on $\mathcal{H}$, i.e.

$$\nabla^{\mathcal{H}}(Y) = P(\nabla Y).$$

Wilking proved in [Wil07] that for each Jacobi field $J \in W^{\perp} \subset \mathrm{Jac}$ the projection $Y = P(J)$ is an $R^{\mathcal{H}}$-Jacobi field, i.e. $(\nabla^{\mathcal{H}})^2 J + R^{\mathcal{H}}(J) = 0$. Moreover, two R-Jacobi fields $J_1, J_2 \in W^{\perp}$ have the same projection to $\mathcal{H}$ if and only if $J_1 - J_2 \in W$. Thus the induced map

$$I : W^{\perp}/W \to \mathrm{Jac}^{R^{\mathcal{H}}}$$

is injective and by dimensional reasons it is an isomorphism. Hence $R^{\mathcal{H}}$-Jacobi fields are precisely the projections of Jacobi fields in $W^{\perp}$; and Lagrangians in $\mathrm{Jac}^{R^{\mathcal{H}}}$ are projections of Lagrangians in Jac that contain W. As a consequence we obtain (cf. [L09])

Lemma 2.2.3. *For each Lagrangian $\Lambda \subset \mathrm{Jac}$ that contains W, we have the equality $\mathrm{ind}_W(\gamma) + \mathrm{ind}_{\Lambda/W}(\gamma) = \mathrm{ind}_{\Lambda}(\gamma)$.*

Example. Let $f : M \to B$ be a Riemannian submersion and let $\mathcal{F}(f)$ denote the induced foliation on M. Let γ be a horizontal geodesic in M and denote by $\bar{\gamma} = f(\gamma)$ its image in B. Consider the space W of $\mathcal{F}(f)$-vertical Jacobi fields along γ, i.e. variational fields of variations of horizontal lifts of $\bar{\gamma}$. Then W is an isotropic subspace, since it is contained in the space Λ^N of normal N-Jacobi fields, where $N = f^{-1}(f(\gamma(a)))$. In this case, for each t, the space $W(t)$ is the vertical space of f through $\gamma(t)$, $\mathcal{H}$ is canonically identified with the normal bundle of $\bar{\gamma}$ in B and the transversal operator $R^{\mathcal{H}}$ coincides with the curvatur endomorphism in the base space, i.e. the term AA^* is just the O'Neill tensor. So, the *horizontal index* $\mathrm{ind}_{\Lambda^N/W}(\gamma)$ describes the index of $\bar{\gamma}$. The *vertical index* $\mathrm{ind}_W(\gamma)$ is zero in this case, but in the much more general situation of a singular Riemannian foliation the vertical index counts the intersections of γ with singular leaves and coincides with the crossing number as we will see below.

Set $(M, \mathcal{F}, g)$ as usual and let $\gamma : [a, b] \to M$ be a horizontal geodesic. Then the space $\Lambda^{L_{\gamma(a)}}$ of all normal $\mathcal{F}$-Jacobi fields is Lagrangian but depends not only on the maximal geodesic containing γ but also on the starting point $\gamma(a)$. Let us see how we can arrange this problem. Consider the space W^{γ}, consisting of all Jacobi fields along γ with the property that these fields are variational fields through horizontal geodesics γ_s with $\gamma_s(t) \in L_{\gamma(t)}$ for all t. One can show (cf. [LT10] 4.5) that $W^{\gamma}(t) = \{J(t) \mid J \in W^{\gamma}\}$ coincides with $T_{\gamma(t)}L_{\gamma(t)}$, for all t and by definition we have $W^{\gamma} \subset \Lambda^{L_{\gamma(a)}}$. Therefore, W^{γ} is just the space of all $\mathcal{F}$-vertical Jacobi fields along γ and does not depend on the starting point in contrast to $\Lambda^{L_{\gamma(a)}}$. If $d(\gamma)$ denotes the maximal dimension of $L(\gamma(t))$, then we have $d(\gamma) = \dim(W^{\gamma})$. Moreover, the W^{γ}-focal points along γ are precisely the points t_i with $\dim(L_{\gamma(t_i)}) < d(\gamma)$ and the W^{γ}-focal index is $d(\gamma) - \dim(L_{\gamma(t_i)})$. In particular, for a regular horizontal geodesic γ its crossing number $c(\gamma)$ coincides with the vertical index $\mathrm{ind}_{W^{\gamma}}(\gamma)$. If in addition, we call a Jacobi field *horizontal* if it is the variational field of a variation of γ through horizontal geodesics one can describe the space $(W^\gamma)^\perp$ as the space consisting of normal horizontal Jacobi fields. Therefore, the singular Riemannian foliation $\mathcal{F}$ does not have horizontal conjugate points if and only if any $\mathcal{F}$-horizontal Jacobi field along any horizontal geodesic that is tangent to the leaves at two points is tangent to the leaves at all points.

Recall that two points $c < d$ in $[a, b]$ are called *conjugate* if there is a non-zero Jacobi field $J \in \mathrm{Jac}$ with $J(c) = 0 = J(d)$. Thus the statement that the point a does not have conjugate points on (a, b) for the transversal Jacobi equation on $\mathcal{H}$, where $\mathcal{H}$ is the W^γ-transversal bundle as defined above, is equivalent to the equality $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}}(\gamma_0) = \mathrm{ind}_{W^\gamma}(\gamma_0)$, where γ_0 denotes the subgeodesic $\gamma_0 : (a, b) \to M$ of γ. To see this, note that by Lemma 2.2.3 $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}}(\gamma_0) = \mathrm{ind}_{W^\gamma}(\gamma_0)$ is equivalent to the absence of focal points of $\Lambda^{L_{\gamma(a)}}/W^\gamma$ on the open interval (a, b). But $\Lambda^{L_{\gamma(a)}}/W^\gamma$ is by definition the Lagrangian in $\mathrm{Jac}(\mathcal{H})$ of all Jacobi fields Y with $Y(a) = 0$. Thus the statement that $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}/W^\gamma}(\gamma_0) = 0$ is equivalent to the fact that a does not have conjugate points with respect to the transversal Jacobi equation.

We end this section with a summarizing lemma (cf. [LT10], 5.8), which we have already proven in parts by our above discussion. For the notion of orbifolds see Section 3.1.

Lemma 2.2.4. *There are no horizontal conjugate points along γ if and only if $\mathrm{ind}_{\Lambda^{L_{\gamma(a)}}}(\gamma_0) = \mathrm{ind}_{W^\gamma}(\gamma_0)$, where $\gamma_0 : (a, b) \to M$ denotes the subgeodesic of γ. If M is complete, $\mathcal{F}$ is infinitesimally polar and closed, and $\bar{\gamma}$ denotes the projected orbifold geodesic, then there are no horizontal conjugate points along γ if and only if there are no conjugate points along $\bar{\gamma}$ in the local development of $M/\mathcal{F}$. Further, a singular Riemannian foliation is infinitesimally polar if and only if it is locally without horizontal conjugate points.*

2.3 A Property of the Quotient

Dealing with singular Riemannian foliations one focuses mainly on the horizontal geometry of the foliation, that is to say the geometry of the quotient. For this reason, one is often interested in geometric properties of the foliation that can be read off the quotient and to consider *equivalence classes* of foliations by means of isometric quotients. An example of such a quotient property is infinitesimal polarity (cf. Section 3.1), in which case the quotients are Riemannian orbifolds, i.e. every point in the quotient has a neighborhood isometric to N/Γ, where N is a Riemannian manifold and $\Gamma \subset I(N)$ is a discrete group of isometries. Our second main result now states that tautness of a foliation is actually also a property of the quotient, so that one can speak about *equivalence classes* of taut foliations by means of their leaf spaces.

We want to remind the reader that by our convention taut always means $\mathbb{Z}_2$-taut.

Theorem 2.3.1. *Let $\mathcal{F}$ and $\mathcal{F}'$ be closed singular Riemannian foliations on complete Riemannian manifolds M and M' with isometric quotients. Then $\mathcal{F}$ is taut if and only $\mathcal{F}'$ is taut.*

Since we already know that $\mathbb{F}$-tautness implies $\mathbb{Z}_2$-tautness, we directly obtain

Corollary 2.3.2. *If $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are as in Theorem 2.3.1 and one of the foliations is $\mathbb{F}$-taut, then both of them are also $\mathbb{Z}_2$-taut.*

Before we begin with the proof of the theorem let us discuss and apply this result in the context of the known examples.

If $M = S^k$ is the round sphere and $\mathcal{F}$ is the trivial foliation by points, there is a well known cycle construction for critical points of the energy functional (cf. p.95-96 of [Mi63]) which shows that S^k is pointwise taut. Terng and Thorbergsson proved in [TT97] that the standard metric on the sphere is the only one with respect to which the sphere is pointwise taut.

Now consider the more general case $M/\mathcal{F} = N/\Gamma$, where N is a symmetric space. In their study of the Morse theory of symmetric spaces, Bott and Samalson came up with concrete cycles which represent a basis in $\mathbb{Z}_2$-homology of the path space $\mathcal{P}(N, p \times q)$ and which are in fact compact connected manifolds (see [BS58]) and coincide with those we constructed in Theorem 1.3.1. In particular, symmetric spaces are pointwise taut and therefore, the foliation $\mathcal{F}$ on M has to be taut by Theorem 2.3.1.

On closer inspection, one can reduce the case of a nontrivial Γ-action to the example of the sphere as follows. If again $M/\mathcal{F} = N/\Gamma$ and N is a simply connected symmetric space, write

$$N = N_0 \times N_1 \times \cdots \times N_m$$

as a Riemannian product with N_0 of Euclidean type and irreducible symmetric spaces $N_i, i > 1$ of compact or noncompact type. Since $M/\mathcal{F}$ is a Coxeter orbifold and N is simply connected, Γ is a Coxeter group, generated by reflections on dissecting hyperplanes. Due to Corollary 3.5 of [Ko07], a totally geodesic hyperplane H in N has to be of the form $H = N_0 \times \cdots \times H_i \times \cdots \times N_m$ for some $i \geq 0$ and some totally geodesic hyperplane H_i in N_i. But this implies that also Γ splits into $\Gamma = \Gamma_0 \times \cdots \times \Gamma_m$, where Γ_i is a Coxeter group on N_i. It is well known that an irreducible symmetric space of compact or noncompact type which admits a totally geodesic hyperplane, e.g. if it allows a reflection, has constant curvature. Now for points $p = (p_0, \ldots, p_m)$ and $q = (q_0, \ldots, q_m) \in N$, we have $\pi_r(\mathcal{P}(N, p \times q)) \cong \pi_{r+1}(N)$. But the homotopy groups of a product are the products of the homotopy groups of the factors and since $\mathcal{P}(N, p \times q)$ is connected, because N is simply connected, we conclude that the map

$$\mathcal{P}(N, p \times q) \to \mathcal{P}(N_0, p_0 \times q_0) \times \cdots \times \mathcal{P}(N_m, p_m \times q_m),$$

which sends a path c to $(c_0, \dots, c_m)$ is a homotopy equivalence. The critical points of $E_{(q_0,\dots,q_m)} = \sum_{i=0}^m E_{q_i}$ are exactly the products of the geodesics in the factors and the index, as well as the nullity behaves additive. Since symmetric spaces of non-compact type have nonpositive curvature, there are no conjugate points and thus all geodesics must be local minima. Since the factors are also simply connected, there is exactly one geodesic between any two points in the factors which are not of compact type. Thus it is enough to consider the factors of compact type. But as we have seen, the factors of compact type with nontrivial Coxeter group have to be spheres and we deduce pointwise tautness again. In particular, our argumentation also works if we allow an additional constant direction of nonpositive curvature in the quotient.

We want to emphasize that the following corollary covers all known examples and that its converse in the case of a compact orbifold quotient is closely related to the Blaschke conjecture (cf. Theorem 3.3.11 below and Section 6 of [TT97]).

Corollary 2.3.3. *If $\mathcal{F}$ is a closed singular Riemannian foliation on a complete Riemannian manifold M and $M/\mathcal{F} = (N \times P)/\Gamma$ is a good Riemannian orbifold, where N is a symmetric space and P is a nonpositively curved manifold, then $\mathcal{F}$ is taut.*

Another application of Theorem 2.3.1 are foliations admitting generalized sections.

Example. Let M be a complete Riemannian manifold with an isometric action of a compact Lie group G. In [GOT04] the authors developed the concept of a generalized section for such an action. They call a connected, complete submanifold Σ of M a *k-section* if the following hold:

- Σ is totally geodesic;
- Σ intersects all orbits;
- for every G-regular point $p \in \Sigma$ the tangent space $T_p\Sigma$ contains the normal space $\nu_p(G(p))$ as a subspace of codimension k;
- if $p \in \Sigma$ is a G-regular point with $g(p) \in \Sigma$ for some $g \in G$ then $g(\Sigma) = \Sigma$.

Generalized sections are also called *fat sections* and the copolarity of (G, M) is defined by $\mathrm{copol}(G, M) = \min \{k \in \mathbb{N} | \text{ there is a } k\text{-section } \Sigma \subset M\}$ and measures, roughly speaking, how far the action is from being polar, i.e. admitting a 0-section. If Σ is a fat section, then it is shown in [Ma08] that there is the *fat Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ that acts on Σ with $G(p) \cap \Sigma = W(\Sigma)(p)$ if $p \in \Sigma$, inducing an isometry between the quotients $\Sigma/W(\Sigma) = M/G$. We therefore deduce that $(\Sigma, \mathcal{F}^W)$ is taut if and only if $(M, \mathcal{F}^G)$ is taut.

Before we start with the proof of Theorem 2.3.1, we now state a preparing lemma that says that focal points caused by singular leaves do not provide any difficulties when dealing with tautness. This fact was already discussed in [No08] and proves that singular Riemannian foliations without horizontal conjugate points are taut.

Lemma 2.3.4. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M and let $L \in \mathcal{F}$ be a regular leaf. For every broken horizontal geodesic $c : [0,1] \to M$ from L to a point $q \in M$ that intersects the singular stratum discretely, let $\Delta(c)$ denote the space of broken horizontal geodesics in the path space $\mathcal{P}(M, L \times q)$ which have the same projection to the quotient $M/\mathcal{F}$ as c. Then $\Delta(c)$ carries a smooth structure of a compact (possibly non-connected) manifold of dimension $\sum_{t\in[0,1]} \dim(\mathcal{F}) - \dim(L_{c(t)})$ such that the inclusion into the path space $\mathcal{P}(M, L \times q)$ becomes an embedding.*

Proof. Given a leaf $L \in \mathcal{F}$ let $\nu^\varepsilon(L)$ be a global ε-tube of L. Then the pull back of $\mathcal{F}$ by the normal exponential map is invariant under the homotheties $r_\lambda(v) = \lambda v$ for all $\lambda \in [-1, 1] \setminus \{0\}$, so that there is a unique singular foliation $\mathcal{G}(L)$ that extends the pull back to $\nu(L)$ satisfying this property. The singular foliation $\mathcal{G}(L)$ is closed if $\mathcal{F}$ is closed and it is shown in Section 4 of [LT10] that $v, w \in \nu(L)$ are in the same leaf of $\mathcal{G}(L)$ iff $\gamma_w(t) \in L_{\gamma_v(t)}$ for all t, where as usual γ_v is the unique geodesic with $\dot\gamma_v(0) = v$. Let V be a small open neighborhood of v in the leaf $\mathcal{L}_v$ of $\mathcal{G}(L)$ through v. Then the vector space of variational vector fields of variations through geodesics γ_w with $w \in V$ coincides with the space W^{γ_v} of $\mathcal{F}$-vertical Jacobi fields along γ_v, as defined in the last section. Due to [LT10], one has

$$W^{\gamma_v}(t) = \{J(t) | J \in W^{\gamma_v}\} = T_{\gamma_v(t)} L_{\gamma_v(t)},$$

so that if the dimension of L is maximal along the horizontal geodesic γ_v, i.e. $\dim(L) = \max_{t\in[0,1]}\{\dim(L_{\gamma_v(t)})\}$, we deduce that the map $\eta_t : \mathcal{L}_v \to L_{\gamma_v(t)}$, given by $\eta_t(w) = \exp(tw) = \gamma_w(t)$, is a submersion for all t, which is surjective if $\mathcal{F}$ is closed. In this case, all the preimages $\eta_t^{-1}(p)$ are compact submanifolds of $\mathcal{L}_v$ of dimension $\dim(\mathcal{L}_v) - \dim(L_{\gamma_v(t)})$. In particular, if L is a regular leaf the dimension of such a preimage equals $\dim(\mathcal{F}) - \dim(L_{\gamma_v(t)})$.

We will now describe the compact set $\Delta(c)$ as the total space of an iterated bundle. Since the general case requires no new ideas, but only some more notation, we will assume for the rest of the proof that c as in the claim is smooth. So let $L \in \mathcal{F}$ be a regular leaf and let $\gamma = \gamma_v$ be a horizontal geodesic from L to a point $q \in M$. Let $\gamma^{-1}(M \setminus M_0) = \{t_i\}_{i=1,\dots,r}$ with $0 < t_r < \cdots < t_1 \le 1$ denote the times where γ crosses the singular stratum and set $L_i = L_{\gamma}(t_i)$ and $v_i = \dim(\mathcal{F}) - \dim(L_i)$. Note that if q is a regular point, the vertical index of γ is given by $v(\gamma) = \sum_{i=1}^r v_i$. With the notation from above, let $\eta_i : \mathcal{L}_v \to L_i$ be the surjective submersion defined by $\eta_i(w) = \exp(t_i w)$. Starting with the furthermost singular leaf, we now define $V_1 = \eta_1^{-1}(\gamma(t_1)) \subset \mathcal{L}_v$ and identify this space with the subspace

$$\Delta_1 = \left\{c_w \in \Delta(\gamma) | c_w|_{[0,t_1]} = \gamma_w|_{[0,t_1]} \text{ for } w \in V_1 \text{ and } c_w|_{[t_1,1]} = \gamma|_{[t_1,1]}\right\}$$

of $\Delta(\gamma)$ of (at most) once broken geodesics in the obvious way, i.e. by $w \mapsto c_w$. With this identification Δ_1 inherits a smooth structure which turns it into an embedded submanifold of $\mathcal{P}(M, L \times q)$ of dimension v_1.

At the second step we define V_2 to be the twisted product

$$\mathcal{L}_v \times_\eta V_1 = \{(w_2, w_1) \in \mathcal{L}_v \times V_1 | \eta_2(w_2) = \exp(t_2 w_1)\},$$

which can be identified with the subspace Δ_2 of $\Delta(\gamma)$ that consists of all (at most) twice broken horizontal geodesics $c_{(w_2,w_1)}$ with $c_{(w_2,w_1)}|_{[0,t_2]} = \gamma_{w_2}|_{[0,t_2]}$ for some element $w_2 \in \mathcal{L}_v$ and $c_{(w_2,w_1)}|_{[t_2,1]} = c_{w_1}|_{[t_2,1]}$ for some $w_1 \in V_1$. With the induced smooth structure Δ_2 becomes a submanifold of $\mathcal{P}(M, L \times q)$ with

$$\dim(\Delta_2) = \dim(\mathcal{L}_v) + \dim(V_1) - \dim(L_2) = v_2 + v_1.$$

Note that all we need to ensure that Δ_2 is a submanifold is the fact that $\eta_2 : \mathcal{L}_v \to L_2$ is a submersion, so that the map $\mathcal{L}_v \times V_1 \to L_2 \times L_2$ is transversal to the diagonal in $L_2 \times L_2$.

Now assume that for some $r-1 \geq j \geq 1$ we already have defined V_j as a submanifold of dimension $\sum_{i=1}^{j} v_i$ of the j-fold product $\mathcal{L}_v^j$ together with an identification $V_j \cong \Delta_j$ given by $(w_j, \ldots, w_1) \mapsto c_{(w_j,\ldots,w_1)}$. Then we inductively define V_{j+1} and Δ_{j+1} as follows. Set $V_{j+1} = \mathcal{L}_v \times_\eta V_j$, where again the twisted product is defined by

$$\mathcal{L}_v \times_\eta V_j = \{(w_{j+1}, w_j, \ldots, w_1) \in \mathcal{L}_v \times V_j | \eta_{j+1}(w_{j+1}) = \exp(t_{j+1} w_j)\},$$

which is therefore a submanifold of $\mathcal{L}_v^{j+1}$ of dimension

$$\begin{aligned} \dim(V_{j+1}) &= \dim(\mathcal{L}_v) + \dim(V_j) - \dim(L_{j+1}) \\ &= v_{j+1} + \dim(V_j) \\ &= \sum_{i=1}^{j+1} v_i. \end{aligned}$$

Finally, define Δ_{j+1} to be the subspace of $\Delta(\gamma)$ consisting of all (at most) $(j+1)$-fold broken horizontal geodesics $c_{(w_{j+1},w_j,\ldots,w_1)}$ such that

$$\begin{aligned} c_{(w_{j+1},w_j,\ldots,w_1)}|_{[0,t_{j+1}]} &= \gamma_{w_{j+1}}|_{[0,t_{j+1}]} \text{ for some } w_{j+1} \in \mathcal{L}_v \text{ and} \\ c_{(w_{j+1},w_j,\ldots,w_1)}|_{[t_{j+1},1]} &= c_{(w_j,\ldots,w_1)}|_{[t_{j+1},1]} \text{ for some } (w_j, \ldots, w_1) \in V_j. \end{aligned}$$

By construction, it is clear that there is a 1:1 correspondence between $(j+1)$-tupels $(w_{j+1}, \ldots, w_1) \in V_{j+1}$ and paths $c_{(w_{j+1},w_j,\ldots,w_1)} \in \Delta_{j+1}$. Moreover, the identification $\Delta_{j+1} \cong V_{j+1}$ (as manifolds) via the assignment $(w_{j+1}, w_j, \ldots, w_1) \mapsto c_{(w_{j+1},w_j,\ldots,w_1)}$ turns Δ_{j+1} into a compact submanifold of $\mathcal{P}(M, L \times q)$ of dimension $\sum_{i=1}^{j+1} v_i$. In particular, this defines a smooth structure for $\Delta_r = \Delta(\gamma)$ with the desired properties. $\square$

Remark 2.3.5. The assumptions in Lemma 2.3.4 are adapted to our setting, but the conclusion also holds if the foliation is not closed, or the manifold is not complete. For this fact, because being a manifold is a local property, one only has to localize the arguments given in the proof of the lemma. Further, if $c = \gamma$ is smooth, let W^γ denotes the space of $\mathcal{F}$-vertical Jacobi fields along γ (see Section 2.2) and let $t_1 < \cdots < t_r$ be the W^γ-focal times along γ. Define W_i^γ to be the space of continuous vector fields J along γ such that $J|_{[0,t_i]} \in W^\gamma|_{[0,t_i]}$ and J vanishes on $[t_i, 1]$. Then by our description in the proof of the lemma, we conclude that the tangent space of $\Delta(\gamma)$ at γ is given by

$$T_\gamma \Delta(\gamma) = \bigoplus_{i=1}^{r} W_i^\gamma.$$

As a consequence of Lemma 2.3.4 we reprove the mentioned special case.

Corollary 2.3.6. *If $\mathcal{F}$ is a closed singular Riemannian foliation on a complete Riemannian manifold M and $\mathcal{F}$ is without horizontal conjugate points, then $\mathcal{F}$ is taut.*

Proof. By Lemma 2.3.4, there is a compact manifold $\Delta(\gamma)$ through every regular horizontal geodesic γ consisting of broken horizontal geodesics, all having the same length as γ, and $\dim(\Delta(\gamma))$ coincides with the vertical index $v(\gamma)$. But by assumption, the index of γ is just the vertical index. Moreover, the satement about $T_\gamma\Delta(\gamma)$ and the discussion below A.2.4 ensures that if we look at a finite dimensional approximation of the path space, $\Delta(\gamma)$ is transversal to the ascending cell in a Morse chart around γ, so that $\Delta(\gamma)$ can be deformed into the descending cell, hence defines a linking cycle for γ. This proves tautness of all regular leaves, so that the claim follows with Lemma 2.1.13. $\square$

Proof of Theorem 2.3.1. Let us briefly sketch the idea of the proof. For $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ as in the claim let us identify $B = M/\mathcal{F} = M'/\mathcal{F}'$ via an isometry and consider the following diagram

$$\begin{tikzcd} M \arrow[dr, "Q"'] & & M' \arrow[dl, "Q'"] \\ & B & \end{tikzcd}$$

Now assume that $\mathcal{F}$ is taut. In order to prove that $\mathcal{F}'$ is taut it suffices to prove that the normal exponential map of a generic leaf of $\mathcal{F}'$ has integrable fibers, by Theorem 1.3.1 and our genericity results from Section 2.1. Let therefore $L' \in \mathcal{F}'$ be a regular leaf without holonomy. In this case, the leaf $L = Q^{-1}(Q'(L')) \in \mathcal{F}$ is a regular leaf without holonomy, too, and its normal exponential map has integrable fibers, by assumption. For $v \in \nu(L)$, let Δ_v denote the connected component of the fiber through v that contains v and identify it with the manifold of horizontal geodesics from L to $\exp(v)$ which have initial velocity in Δ_v. Now, given a vector $v' \in \nu(L')$ with the same projection to B as v, we push Δ_v down to B and lift it to M' along

Q' to obtain a space $\Delta'_{v'}$ of horizontal geodesics that start in L' and end in $\exp(v')$. The observation that the map $\Delta'_{v'} \to \nu(L')$ which assigns to a horizontal geodesic its starting direction provides an integral manifold of the kernel distribution of the normal exponential map of L' through v' then finishes the proof.

Having sketched the proof, let us now work out the details. For this purpose, let $D(M) \subset T^1M$ be the subset of horizontal unit vectors and write $P : D(M) \to M$ for the restricted foot point projection. In [LT10] it is shown that there is a subset $O(M) \subset D_0(M) = P^{-1}(M_0)$ of horizontal unit vectors with regular foot points of full measure such that for $v \in O(M)$ the projection $Q \circ \gamma_v$ is completely contained in the open and dense orbifold part of B, where again γ_v denotes the unique horizontal geodesic with $\dot{\gamma}_v(0) = v$ (for the notion of orbifolds see Section 3.1). In this case, $Q \circ \gamma_v$ defines an orbifold geodesic. As mentioned in the proof of Lemma 2.3.4, given any leaf $L \in \mathcal{F}$ one can pull back $\mathcal{F}$ via the normal exponential map to some ϵ-neighborhood of the zero section of the normal bundel $\nu(L)$ to obtain a closed singular foliation invariant under non-zero homotheties that therefore admits a unique extension to a singular foliation $\mathcal{G}(L)$ on $\nu(L)$ with closed leaves. This induces an equivalence relation $\mathcal{R}^M$ on $D(M)$ by saying that to vectors $v, w \in D(M)$ are equivalent iff $P(w) \in L_{P(v)}$ and v and w lie in the same leaf of $\mathcal{G}(L_{P(v)})$. The set $\mathcal{D}(M)$ is stratified by the preimages of the elements of the stratification of M induced by the leaf dimension and because the main stratum $D_0(M)$ is a manifold, the restriction of $\mathcal{R}^M$ to $D_0(M)$ is given by the leaves of smooth regular foliation with closed leaves. Further, if we denote the equivalence class of $v \in D(M)$ by $\mathcal{R}^M(v)$, we have

$$\gamma_w(t) \in L_{\gamma_v(t)} \text{ for all } t \in \mathbb{R} \Longleftrightarrow w \in \mathcal{R}^M(v).$$

Now given a point $p \in M$, the infinitesimal foliation $\mathfrak{F}_p$ splits as a product foliation $\mathfrak{F}_p = T_pL_p \times \mathfrak{F}^1_p$ on the tangent space $T_pM = T_pL_p \times \nu_p(L_p)$ so that we have $T_pM/\mathfrak{F}_p = \nu_p(L_p)/\mathfrak{F}^1_p$, which is the tangent space to a local quotient $U/\mathcal{F}$ at $L \cap U$, where U is a distinguished neighborhood of p. The map $U/\mathcal{F} \to M/\mathcal{F}$, induced by the inclusion $U \to M$, is a finite-to-one open map, given by the quotient map of the action of a finite group Γ of isometries, onto a neighborhood $T^\varepsilon/\mathcal{F}$ of L_p, where T^ε is a global ε-tube around L_p with the same ε as in the definition of the distinguished neighborhood U (cf. Section 2.1). Identifying $\nu^\varepsilon(L_p) \cong T^\varepsilon$ via the normal exponential map, we see that we can identify the tangent space $T_{L_p}B$ of B at L_p with $(\nu_p(L_p)/\mathfrak{F}^1_p)/\Gamma = \nu(L_p)/\mathcal{G}(L_p) = \nu_p(L_p)/\mathcal{G}(L_p)$. We therefore define the differential $dQ_p : T_pM \to T_{Q(p)}B$ of the projection $Q : M \to B$ at the point p to be the composition of projections $T_pM \to \nu_p(L_p) \to \nu_p(L_p)/\mathcal{G}(L_p)$ and write $Q_* : TM \to TB$ for the induced map, i.e. $Q_*(v) = dQ_{P(v)}(v)$.

All the above observations are of general nature so that we use the analogous notations that we developed for $(M, \mathcal{F})$ also for $(M', \mathcal{F}')$. Moreover, the tangent spaces T_bB, as defined above, only depend on the metric structure of the space B, so that we do not distinguish between the isometric descriptions as $\nu_p(L_p)/\mathcal{G}(L_p)$ and $\nu_{p'}(L'_{p'})/\mathcal{G}(L'_{p'})$ if $Q(L_p) = Q'(L'_{p'})$ for $p' \in M'$.

Since orbifold geodesics coincide if they coincide initially, we deduce from the above that given $v \in O(M)$ and $v' \in D(M')$ with the same projection $Q_*(v) = Q'_*(v')$, we have $v' \in O(M')$ and $Q \circ \gamma_v(t) = Q' \circ \gamma_{v'}(t)$ for all $t \in \mathbb{R}$. But this implies that $Q \circ \gamma_v = Q \circ \gamma_{v'}$ for all $\mathcal{F}$-horizontal vectors $v \in TM$ and $\mathcal{F}'$-horizontal vectors $v' \in TM'$ with $Q_*(v) = Q'_*(v')$, because $O(M)$, resp. $O(M')$ is dense in $D(M)$, resp. $D(M')$ and the maps induced by Q and Q' on the corresponding path space levels are continuous.

Now take a regular leaf $L \in \mathcal{F}$ and recall that in this case $\mathcal{G}(L)$ is a regular foliation with closed leaves such that the intersection of every leaf $\mathcal{L} \in \mathcal{G}(L)$ with any normal space $\nu_p(L)$ is finite. Further, as explained in the proof of Lemma 2.3.4, for every normal vector $v \in \nu(L)$ the restriction of the normal exponential map to the leaf $\mathcal{L}_v$ induces a submersion $\eta_v : \mathcal{L}_v \to L_{\gamma_v(1)}$, so that all the fibers $\eta_v^{-1}(q)$ for $q \in L_{\gamma_v(1)}$ are (unions of) compact submanifolds of dimension $\dim(\mathcal{F}) - \dim(L_{\gamma_v(1)})$, since the normal exponential map is proper. In this case, the smooth map $\eta_v^{-1}(q) \to \nu_q(L_{\gamma_v(1)})$, given by $w \mapsto \dot{\gamma}_w(1)$, defines a smooth identification of the connected component of $\eta_v^{-1}(q)$ that contains w with the regular leaf $\mathfrak{L}_{\dot{\gamma}_w(1)}$ of $\mathfrak{F}_q^1$ through $\dot{\gamma}_w(1) \in \nu_q(L_{\gamma_v(1)})$, where $w \in \eta_v^{-1}(q)$ is any preimage of q. Moreover, due to the above observation and our discussion in Section 2.2 on the horizontal index, it is not hard to see that for $v \in \nu(L)$ and a horizontal geodesic $\gamma'_{v'} : [0,1] \to M'$ with $Q \circ \gamma_v = Q' \circ \gamma'_{v'}$ their horizontal indices coincide. That is to say, the kernel $\ker((d\exp_M^\perp)_v)$ of the differential of the normal exponential map in v contains the subspace $T_v\mathcal{L}_v \subset T_v\nu(L)$ and the dimension of $\ker((d\exp_M^\perp)_v)/T_v\mathcal{L}_v$ is independent of the foliation, or to be more precise, an intrinsic datum of the quotient.

We now finish the proof as follows. Assume that $\mathcal{F}$ is taut. Combining Lemma 2.1.15 and the proof of Theorem 1.3.1, it remains to prove that for generic leaves $L' \in \mathcal{F}'$ the normal exponential map $\exp_{M'}^\perp : \nu(L') \to M'$ has integrable fibers. Thus we can restrict our attention to a regular leaf $L' \in \mathcal{F}'$ without holonomy, i.e. $Q'(L')$ is a manifold point of B and the restriction of Q' to a tubular neighborhood of L' defines a Riemannian submersion. In particular, in this case the leaf $L = Q^{-1}(Q'(L')) \in \mathcal{F}$ is a regular leaf without holonomy, too. Let $v' \in \nu(L')$ be a horizontal vector and let us set $q' = \gamma_{v'}(1)$. Choose an $\mathcal{F}$-horizontal vector $v \in \nu(L)$ with $Q_*(v) = Q'_*(v')$ and set $q = \gamma_v(1)$. Then, by construction, $Q \circ \gamma_v = Q' \circ \gamma_{v'}$ and

$$\begin{aligned}&\dim(\ker((d\exp_{M'}^\perp)_{v'})) - (\dim(\mathcal{F}') - \dim(L_{\exp_{M'}^\perp(v')}))\\ &= \dim(\ker((d\exp_M^\perp)_v)) - (\dim(\mathcal{F}) - \dim(L_{\exp_M^\perp(v)})).\end{aligned}$$

Since $\mathcal{F}$ is taut, the connected component Δ_v of $(\exp_M^\perp)^{-1}(q)$ containing v is a compact submanifold of $\nu(L)$ that is smoothly foliated by the $(\dim(\mathcal{F}) - \dim(L_q))$-dimensional regular foliation whose leaf through a horizontal vector $w \in \Delta_v$ is given by $\mathcal{N}_w = (\exp_M^\perp)^{-1}(q) \cap \mathcal{L}_w$. Again, we can regard Δ_v as a saturated subset of the regular part of the singular Riemannian foliation $\mathfrak{F}_q^1$ on $\nu_q(L_q)$ via the map $d_v : \Delta_v \to \nu_q(L_q)$ defined by the prescription $d_v(w) = d(\exp_M^\perp)_w(w)$. Moreover, by our choice of L', the image of the composition $dQ_q \circ d_v$ is completely contained in the manifold part of $T_qM/\mathfrak{F}_q = \nu_q(L_q)/\mathfrak{F}_q^1$ (cf. Section 4 of [LT10]), so that every leaf of $\mathfrak{F}_{q'}$ through a vector $w' \in \nu_{q'}(L'_{q'})$ with $dQ'_{q'}(w') \in dQ_q(d_v(\Delta_v))$ is also regular without holonomy.

If we therefore define $K_{q'} \subset \mathfrak{F}_{q'}^1$ to be the preimage

$$K_{q'} = (dQ'_{q'})^{-1}((dQ_q \circ d_v)(\Delta_v)),$$

then $K_{q'}$ is obviously a union of regular leaves of $\mathfrak{F}_{q'}^1$ without holonomy, namely of leaves of dimension $\dim(\mathcal{F}') - \dim(L'_{q'})$, completely contained in a concentric sphere. Further, because $dQ_q(d_v(\Delta_v))$ carries a natural smooth structure that turns it into a $(\dim(\Delta_v) - (\dim(\mathcal{F}) - \dim(L_q)))$-dimensional manifold, and the restriction of $dQ'_{q'}$ to the set of points lying on regular leaves without holonomy is a submersion, the set $K_{q'}$ is a compact submanifold of $\nu_{q'}(L'_{q'})$ of dimension

$$\begin{aligned}
\dim(K_{q'}) &= \dim(\mathcal{F}') - \dim(L'_{q'}) + \dim(\ker((d\exp_M^\perp)_v)) \\
&\quad -(\dim(\mathcal{F}) - \dim(L_q)) \\
&= \dim(\mathcal{F}') - \dim(L'_{q'}) + \dim(\ker((d\exp_{M'}^\perp)_{v'})) \\
&\quad -(\dim(\mathcal{F}') - \dim(L_{q'})) \\
&= \dim(\ker((d\exp_{M'}^\perp)_{v'})).
\end{aligned}$$

Now recall that all the infinitesimal foliations are invariant under all non-zero homotheties. Thus, if we define $\Delta'_{v'} = \{-\dot\gamma_{w'}(1) \in \nu(L') | -w' \in K_{v'}\}$, it easily follows from our above discussion that $\Delta'_{v'}$ is a compact submanifold of $(\exp_{M'}^\perp)^{-1}(q')$ containing v' that satisfies $T_{w'}\Delta'_{v'} = \ker((d\exp_{M'}^\perp)_{w'})$ for all $w' \in \Delta'_{v'}$. This proves the claim. $\square$

As already mentioned before, tautness of a submanifold $L \subset M$ requires very special symmetry of the pair (M, L) around the submanifold L, what clarifies the fact that there are not many examples of taut submanifolds actually known. By this reason, it is worth mentioning that the ideas of the last proof can be used to construct lots of examples. For this purpose, consider a closed singular Riemannian foliation $\mathcal{F}$ on M such that the space of leaves $M/\mathcal{F}$ is isometric to a quotient N/Γ, where N is a Riemannian manifold and $\Gamma \subset I(N)$ is a discrete group of isometries, that is to say that $M/\mathcal{F}$ is a good Riemannian orbifold. Assume that there is a submanifold $S \subset N$ completely contained in the interior of a fundamental domain of the Γ-action which we identify with $M/\mathcal{F}$, and consider the saturated preimage $T = Q^{-1}(S)$ that is a union of regular leaves without holonomy. Now let $v \in \nu_p(T)$ be a normal vector to T. Then every $\mathcal{F}$-vertical Jacobi field along γ_v (cf. Section 2.2) is also a T-Jacobi field along γ_v, i.e. $W^{\gamma_v} \subset \Lambda^T$, and similar arguments as in the proof of Theorem 2.3.1 can be used to see that the multiplicity of $Q_*(v)$ as a focal vector of S in N is the same as the difference of the multiplicity of v as a focal vector of T in M and the number $\dim(\mathcal{F}) - \dim(L_{\gamma_v(1)})$. Thus the following lemma is obtained along the same lines as the proof of Theorem 2.3.1.

Lemma 2.3.7. *Let $\mathcal{F}$ be a closed singular Riemannian foliations on a complete Riemannian manifold M, such that the space of leaves $M/\mathcal{F}$ is isometric to a quotient N/Γ, where N is a Riemannian manifold and $\Gamma \subset I(N)$ is a discrete group of isometries. Let $N_0 \subset N$ denote a fundamental domain of the Γ-action and identify $N_0 \cong M/\mathcal{F}$. Now assume that $S \subset N$ is a taut submanifold that is completely contained in the interior of N_0. Then if $Q : M \to N_0$ denotes the projection, the submanifold $Q^{-1}(S)$ is taut, too.*

In the case where $M = \mathbb{R}^{n+k}$ is the standard Euclidean space and $\mathcal{F}$ is an n-dimensional isoparametric foliation, i.e. the parallel foliation induced by an isoparametric submanifold L of dimension n, identify a section Σ with the Euclidean space $\mathbb{R}^k$. Then take a small taut submanifold $S \subset \mathbb{R}^k$ completely contained in the interior of a Weyl chamber associated to the finite Coxeter group generated by the reflections across the L-focal hyperplanes in Σ and consider the saturated set $T = \left\{p \in \mathbb{R}^{n+k} | L_p \cap S \neq \emptyset\right\}$. Then, due to the last lemma, T is a taut submanifold of $\mathbb{R}^{n+k}$.

Chapter 3

The Infinitesimally Polar Case

This last chapter is devoted to the generalization of Lemma 2.1.11 to infinitesimally polar foliations. For the sake of completeness and for a better understanding we will give a direct prove of the corresponding statement in this special case, without the use of Theorem 2.3.1 (of course, otherwise it would be enough to observe that the quotient of a taut foliation is developeable if it is an orbifold). For this reason, we start with two introductional sections.

3.1 Riemannian Orbifolds

For our purpose, the concept of Riemannian orbifolds is closely related to a special class of singular Riemannian foliations, namely those whose infinitesimal foliations have sections. Let us recall that a singular Riemannian foliation $(M, g, \mathcal{F})$ admits *sections* if there exists a complete, immersed submanifold Σ_p through every regular point $p \in M$ that meets every leaf and always orthogonally. It is not hard to see that a section is totally geodesic in M. As an example, the set of orbits of a polar action is a singular Riemannian foliation admitting sections. Motivated by this example we also speak about *polar foliations*.

Singular Riemannian foliation with sections are well understood and were studied, for example by Alexandrino and Töben. One nice feature of this class is that one can canonically construct a blow up which has the same horizontal geometry (cf. [T06]). In [L10] it is shown that the existence of such a *geometric resolution* of a singular Riemannian foliation is equivalent to the fact that the foliation carries at the infinitesimal level the information of a singular Riemannian foliation with sections (see Section 3.2). Such foliations are called *infinitesimally polar* and were first defined and discussed by Lytchak and Thorbergsson in [LT10].

Remark 3.1.1. Please note that the polar singular Riemannian foliations on a Euclidean space are exactly the parallel foliations given by an isoparametric submanifold, which we call *isoparametric foliations* for short.

Definition 3.1.2. We call a singular Riemannian foliation $\mathcal{F}$ *infinitesimally polar* if the induced singular Riemannian foliation $\mathfrak{F}_p$ on the tangent space (T_pM, g_p) is polar for every $p \in M$, i.e. if $\mathfrak{F}_p^h$ is polar for every $p \in M$.

Example. As we have seen, the infinitesimal foliation $\mathfrak{F}_p$ on the tangent space $T_pM = T_pM_{n-r} \times \nu_p(M_{n-r})$, where again M_{n-r} denotes the stratum through p of points with fixed leaf dimension $r = \dim(L_p)$, splits as a product foliation $\mathfrak{F}_p = \mathfrak{F}_p^v \times \mathfrak{F}_p^h$. The foliation $\mathfrak{F}_p^v$ is just the trivial foliation on $T_pM_{n-r} = T_pL_p \times C_p$, where C_p denotes the orthogonal complement of T_pL_p in T_pM_{n-r}, induced by the projection onto the second factor $T_pL_p \times C_p \to C_p$. Now, if Σ is a section of $\mathfrak{F}_p^h$, then Σ must contain the origin and $\Sigma \times C_p$ is a section of $\mathfrak{F}_p$ through 0. Conversely, any section of $\mathfrak{F}_p$ through 0 is of this form and all the other sections are translates of these. Therefore, $\mathfrak{F}_p$ is polar if and only if $\mathfrak{F}_p^h$ is polar. In particular, if $\mathcal{F}$ is given by the orbits of an isometric group action, then $\mathcal{F}$ is infinitesimally polar if and only if the isotropy representation at every point is polar.

Obviously, if $\iota : \Sigma \to M$ is a section of $\mathcal{F}$ then $d\iota_p(T_p\Sigma)$ is a section of $\mathfrak{F}_{\iota(p)}$. Thus a singular Riemannian foliation with sections is infinitesimally polar. Conversely, in the general case a section Σ of $\mathfrak{F}_p$ cannot be realized as the tangent space of a local section, because this is equivalent to the fact that the horizontal distribution given by $\mathcal{H} = \bigcup_{p \in M_0} (T_pL_p)^{\perp_{g_p}}$ over the regular stratum is integrable, what, under the assumption of completeness of M, is equivalent to existence of sections. A well known example of an infinitesimally polar singular Riemannian foliation that is not polar is given by the fibers of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2(\frac{1}{2})$.

Hence, the class of infinitesimally polar foliations is in fact larger than the class of singular Riemannian foliations with sections. As we will see, this class of foliations can be described in terms of the local quotient as exactly those singular Riemannian foliations for which the local quotients are Riemannian orbifolds. We therefore proceed with a summary of basic facts about orbifolds, which can be found in [MM03] and [ALR07].

Definition 3.1.3. Let B be a topological space. An *orbifold chart of dimension k on B* is a triple (U, G, ϕ), where U is a connected open subset of $\mathbb{R}^k$, G is a finite subgroup of $\mathrm{Diff}(U)$ and $\phi : U \to B$ is an open map which induces a homeomorphism $U/G \to \phi(U)$. If (V, H, ψ) is another orbifold chart on B, an *embedding* $\lambda : (V, H, \psi) \to (U, G, \phi)$ between orbifold charts is an embedding $\lambda : V \to U$ such that $\phi \circ \lambda = \psi$.

Lemma 3.1.4. *Let M be a manifold and G a finite subgroup of* $\mathrm{Diff}(M)$. *For any smooth map $f : V \to M$ defined on a non-empty open subset V of M, satisfying $f(p) \in Gp$ for each $p \in V$, there exist a unique $g \in G$ such that $f = g|_V$.*

Let $\lambda : (V, H, \psi) \to (U, G, \phi)$ be an embedding between orbifold charts. Then the image $\lambda(V)$ is a G-stable open subset of U, so that for every $h \in H$ there is a unique element $\bar{\lambda}(h) \in G_{\lambda(V)} = \{g \in G \mid g(\lambda(V)) = \lambda(V)\}$ which extends the embedding $\lambda \circ h \circ \lambda^{-1}$ to U, by Lemma 3.1.4. Moreover, the so defined homomorphism $\bar{\lambda} : H \to G_{\lambda(V)}$ satisfying $\lambda(hx) = \bar{\lambda}(h)\lambda(x)$ is unique.

Lemma 3.1.5. *The following statements hold true.*

1. *The composition of two embeddings between orbifold charts is an embedding between orbifold charts.*

2. *For any orbifold chart (U, G, ϕ), any diffeomorphism $g \in G$ defines an embedding of (U, G, ϕ) into itself with $\bar{g}(g') = gg'g^{-1}$.*

3. *If $\lambda, \mu : (V, H, \psi) \to (U, G, \phi)$ are two embeddings between the same orbifold charts, there exist a unique $g \in G$ with $\lambda = g \circ \mu$.*

Remark 3.1.6. Because of the identity $G_{g(p)} = gG_pg^{-1}$ for the isotropy groups, there is, up to an isomorphism, a well defined isotropy group for each point $b \in \phi(U)$. Note that for an embedding $\lambda : (U, G, \phi) \to (V, H, \psi)$ betwenn orbifold charts with $\lambda(p) = q$ we have $\bar{\lambda}(G_p) = H_q$. Points with nontrivial isotropy group are called *singular*, otherwise they are called *regular*.

Definition 3.1.7. We say that two orbifold charts $(U, G, \phi), (V, H, \psi)$ of dimension k on B are *compatible* if for any $z \in \phi(U) \cap \psi(V)$ there exists an orbifold chart (W, K, ρ) on B with $z \in \rho(W)$ together with two embeddings $\lambda : (W, K, \rho) \to (U, G, \phi)$ and $\mu : (W, K, \rho) \to (V, H, \psi)$.

Definition 3.1.8. An *orbifold atlas of dimension* k of a topological space B is a collection of pairwise compatible orbifold charts $\mathfrak{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ of dimension k on B such that $B = \bigcup_{i \in I} \phi_i(U_i)$. Two orbifold atlases of B are equivalent if their union is an orbifold atlas. An *orbifold of dimension* k is a pair $(B, \mathfrak{U})$, where B is a second countable Hausdorff topological space and $\mathfrak{U}$ is a maximal orbifold atlas of dimension k of B. If in addition B is a metric space and there is a Riemannian metric on the U_i such that $G \subset I(U_i)$ and the homeomorphisms $U_i/G_i \to \phi_i(U_i)$ are isometric, then we call B a *Riemannian orbifold*.

If the context is clear we often just speak about the orbifold B instead of $(B, \mathfrak{U})$, keeping the additional structure in mind. But as the example of the teardrop, i.e. the round 2-sphere with one cone point with cone angle π/n, shows, one should be careful in doing so not to confuse the properties of the underlying space with those of the orbifold.

Example. Let N be a Riemannian manifold and let Γ be a discrete subgroup of isometries of N. Then local linearization of the action gives us a natural structure of N/Γ as a Riemannian orbifold. We always think about global quotients as orbifolds furnished with this structure.

Definition 3.1.9. A *covering orbifold* of an orbifold $(B, \mathfrak{U})$ is an orbifold $(\widetilde{B}, \widetilde{\mathfrak{U}})$, with a map $P : \widetilde{B} \to B$ between the underlying spaces such that each point $b \in B$ has a neighborhood $V = U/G$ for which each component $\widetilde{V}_i$ of $P^{-1}(V)$ is isomorphic to U/G_i for some subgroup $G_i \subset G$ and such that the isomorphisms commute with P.

It is well known that any orbifold admits a *universal orbifold covering*, i.e. for a regular base point $b_0 \in B$, there exists a pointed connected covering orbifold $P : \widetilde{B} \to B$ with base point $\tilde{b}_0$ projecting to b_0 such that for any other covering orbifold $P' : B' \to B$ with base point b_0' and $P'(b_0') = b_0$, there is a lift $Q : \widetilde{B} \to B'$ of P along P' to an orbifold covering. In particular, a universal orbifold covering is regular in the sense that its group of deck transformations acts simply transitive on a generic fiber. This group is denoted by π_1^{orb} and is called the *orbifold fundamental group.*

Definition 3.1.10. An orbifold is called *good* if it is a global quotient or, equivalently, if the universal covering orbifold is a manifold, i.e. there are no singular points.

In the case of a Riemannian orbifold all the definitions are to be modified in the obvious manner, so that we can speak about *Riemannian orbifold coverings* and *good Riemannian orbifolds.* Of course, the statement about the universal orbifold covering also holds in the Riemannian case.

Example. Again let N be a Riemannian manifold and let Γ be a discrete group of isometries of N. If $\widetilde{N}$ denotes the universal covering of N, then $\widetilde{N}$ is the universal Riemannian covering orbifold of N/Γ. This can be seen, for instance, by the observation that every covering orbifold of N/Γ has to be of the form $\widetilde{N}/\widetilde{\Gamma}'$, where $\widetilde{\Gamma}'$ is a subgroup of the group $\widetilde{\Gamma}$ of deck transformations of $\widetilde{N}$ over N/Γ. Hence the two definitions of a good orbifold are indeed equivalent.

Because we will use it frequently in the following, we will formulate the next observation as a lemma.

Lemma 3.1.11. *Let $\mathcal{F}$ be a closed (regular) Riemannian foliation on a complete Riemannian manifold M and let $\widetilde{\mathcal{F}}$ denote its lift to the universal Riemannian covering $\widetilde{M}$ of M. Then the quotient $\widetilde{M}/\widetilde{\mathcal{F}}$ is a complete Riemannian manifold, i.e. $\widetilde{\mathcal{F}}$ is simple, if $M/\mathcal{F}$ is a good Riemannian orbifold. In particular, the orbifold covering $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ coincides with the universal Riemannian orbifold covering.*

Proof. Since $\mathcal{F}$ is closed its lift $\widetilde{\mathcal{F}}$ is closed too (cf. Lemma 2.1.7), and the leaves of $\widetilde{\mathcal{F}}$ admit global ε-tubes, because $\widetilde{M}$ is complete. Due to [Hae88] or [Sal88], there is a surjective homomorphism $\pi_1(\widetilde{M}) \to \pi_1^{orb}(\widetilde{M}/\widetilde{\mathcal{F}})$, where the latter group is the group of deck transformations of the universal orbifold covering of $\widetilde{M}/\widetilde{\mathcal{F}}$. Now, if $M/\mathcal{F}$ is a good Riemannian orbifold, its branched cover $\widetilde{M}/\widetilde{\mathcal{F}}$ is a good Riemannian orbifold, too. But then $\pi_1^{orb}(\widetilde{M}/\widetilde{\mathcal{F}}) = 1$ implies that $\widetilde{M}/\widetilde{\mathcal{F}}$ already coincide with its universal covering orbifold and is therefore a manifold. $\square$

Let B be a Riemannian orbifold. Then B is locally isometric to a finite quotient of a smooth Riemannian manifold. Since tangent spaces and geodesics are invariant under isometries, one gets corresponding notions on B. Namely, the tangent bundle TB being a disjoint union $TB = \bigcup_{b \in B} T_b B$ of spaces of directions, locally being a finite quotient of the tangent bundle of a covering Riemannian manifold. This tangent bundle comes along with a foot point projection $\pi : TB \to B$, a locally compact (quotient) topology and a local geodesic flow ϕ. For $v \in TB$, we set $\eta_v(t) = \pi(\phi_t(v))$, i.e. the curve η_v is locally the image of a geodesic in a Riemannian manifold under the quotient map. We call η_v the *orbifold-geodesic* in direction v.

For each orbifold-geodesic η_v, the curvature endomorphism along η_v is well defined. Therefore, the notions of Jacobi fields and conjugate points are also well-defined. Let us now assume that B is complete as a metric space. Then each orbifold-geodesic is defined on $\mathbb{R}$ and the local geodesic flow is a global flow. Denote by B_0 the regular stratum and note that B is stratified by Riemannian manifolds, where the unique maximal stratum B_0 is open and dense in B. Take a regular point $b \in B_0$ and consider the orbifold exponential map $\exp : T_b B \to B$, given by $\exp(tv) = \eta_v(t)$. This map (since defined in metric terms) factors over local branched covers of B, i.e. for each $w \in T_b B$ there is a finite quotient $N/\Gamma_w = O \subset B$ with $\exp(w) \in O$, such that exp lifts on a neighborhood of w to a smooth map to N. The vector $w = tv$ is a conjugate vector along the geodesic η_v if and only if this lift has a non-injective differential at w. For a detailed discussion about orbifolds see [ALR07].

In [LT10]Lytchak and Thorbergsson proved the following

Theorem 3.1.12. *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. Let $p \in M$ be a point and let $\mathfrak{F}_p$ be the infinitesimal singular Riemannian foliation induced by $\mathcal{F}$ on the tangent space T_pM. Then the following are equivalent:*

1. *The infinitesimal singular Riemannian foliation $\mathfrak{F}_p$ is polar;*
2. *$\mathcal{F}$ is locally closed at p and a local quotient $U/\mathcal{F}$ of a neighborhood U of p is a Riemannian orbifold.*

In fact, in [LT10] it is shown that the statements above are equivalent to the non-explosion of the curvature in the local quotients as one approaches a boundary point p of M_0.

Now assume $(M, g, \mathcal{F})$ is a closed singular Riemannian foliation of dimension n on a complete, simply connected manifold M^{n+k} and that $\mathcal{F}$ is infinitesimally polar.

Let p be a point in M and let $\mathfrak{F}_p$ be the infinitesimal foliation. Then $\mathfrak{F}_p$ is an isoparametric foliation on (T_pM, g_p). Let $\Sigma \subset \nu_p(L_p)$ be a section of $\mathfrak{F}_p$ through the origin. Then Σ is a totally geodesic submanifold of (T_pM, g_p), hence a linear subspace which we can identify with $\mathbb{R}^k$. Moreover, there is a finite Coxeter group W^p, generated by reflections on the focal hypersurfaces of a regular leaf of $\mathfrak{F}_p$ such that $\Sigma/W^p = T_pM/\mathfrak{F}_p$. In particular, Σ/W^p is a Weyl chamber. Due to [LT10], there is a metric h^p on a ball U_p around 0 in Σ such that each element of W^p acts as an isometry on (U_p, h^p) and $(U_p, h^p)/W^p$ is isometric to a neighborhood of the point $\bar{p}$ in the local quotient. Now assume that all regular leaves have trivial holonomy. Then the local quotient can be identified with a neighborhood of $\bar{p} \in B = M/\mathcal{F}$ in the global quotient (see proof of Theorem 1.6 in [L10]) and this construction yields a collection $\mathfrak{U} = \{(U_p, h^p, W^p, \phi_p)\}$ of orbifold charts. In this case, the quotient $B = M/\mathcal{F}$ is a Riemannian orbifold with the property that the local groups W^p are finite Coxeter groups. Such an orbifold is called *Coxeter orbifold*. Thus the absence of holonomy on the regular part is a sufficient condition for B to be a Coxeter orbifold. That this is also necessary is one of the statements of the next theorem.

We refer to [L10] where Lytchak shows the following

Theorem 3.1.13. *Let M be a complete, simply connected Riemannian manifold and let $\mathcal{F}$ be a closed infinitesimally polar singular Riemannian foliation on M with quotient $B = M/\mathcal{F}$. Then the following are equivalent.*

1. *There are no exceptional leaves;*
2. *The regular part $B_0 = M_0/\mathcal{F}$ is a good orbifold;*
3. *The quotient B is a Coxeter orbifold;*
4. *All non-manifold points of the orbifold B are contained in the closure ∂B of the stratum of codimension 1 points.*

Example. Let (M^m, g) be a complete, simply connected Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$ of codimension 2 and $M/\mathcal{F} = S^2/\Gamma$. Then $\mathcal{F}$ is infinitesimally polar and Γ is a finite Coxeter group. Further, $\mathcal{F}$ has no exceptional leaves by 3.1.13. Let $L \in M/\mathcal{F}$ be a point of codimension 2, i.e. a corner. Take a point $p \in L^{m-k}$ and consider the infinitesimal singular Riemannian foliation $\mathfrak{F}_p^h$ on $(\nu_p(L), g_p)$. Since L has codimension 2 in $M/\mathcal{F}$, the singular Riemannian foliation $\mathfrak{F}_p^h$ is the cone foliation over a singular Riemannian foliation of codimension 1 on the unit sphere S^{k-1} in $\nu_p(L)$. By a result of Münzner (see [Mü80], [Mü81]), one has therefore $S^{k-1}/\mathfrak{F}_p^h = I_d$ for an interval I_d of length $|I_d| = \pi/d$ with $d \in \{1, 2, 3, 4, 6\}$, i.e. $\nu_p(L)/\mathfrak{F}_p^h$ is an open cone over I_d with angle π/d. Note that to obtain a local isometry between $\nu_p(L)/\mathfrak{F}_p^h$ and $U/\mathcal{F}$ for a neighborhood U around p, indeed one has to change the metric on $\nu_p(L)$, but this has no influence on the possible values of the angle, because the metrics coincide in 0. Now the finite to one mapping $U/\mathcal{F} \to M/\mathcal{F}$ between the local and global quotient is given by the quotient $(U/\mathcal{F})/W$, where W is a group acting on U by isometries. But the absence of exceptional leaves implies that W acts trivially, so that a neighborhood of L in $M/\mathcal{F}$ is isometric to $U/\mathcal{F}$. It follows by the known classification of S^2/Γ that the quotient $M/\mathcal{F}$ is either the whole sphere S^2, the hemisphere $S^2/\mathbb{Z}_2$, a sickle S^2/D_i with $i \in \{2, 3, 4, 6\}$, or it is a spherical triangle with angles $(\pi/n_1, \pi/n_2, \pi/n_3)$ and $(n_1, n_2, n_3) \in \{(2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 2, 6), (2, 3, 3), (2, 3, 4)\}$

3.2 Geometric Resolution

In the last section we have seen how infinitesimally polar singular Riemannian foliations can be described by means of their local quotients. In some sense, the quotient $M/\mathcal{F}$ carries the whole information about the *horizontal geometry* of a singular Riemannian foliation $(M, g, \mathcal{F})$. So, if one is intrested in the horizontal geometry of such a foliation, one is naturally lead to the question whether there is a regular Riemannian foliation $(\widehat{M}, \hat{g}, \widehat{\mathcal{F}})$ having the same horizontal geometry, because the singular leaves are the main source of difficulties if one tries to understand the geometric and topological properties of a singular foliation.

Example. In the standard picture of an isometric action of a Lie group G on a Riemannian manifold M, one could ask if there exists another Riemannian manifold $(\widehat{M}, \hat{g})$, canonically related to M, on which G acts by isometries in such a way that all orbits of this action are regular. There are ways, known before, to resolve actions preserving some information, but none preserving the transverse geometry.

Keeping the above example in mind, the following definition generalizes this setting.

Definition 3.2.1. Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold (M, g). A *geometric resolution* of $(M, g, \mathcal{F})$ is a smooth surjective map $F : \widehat{M} \to M$ from a Riemannian manifold $(\widehat{M}, \hat{g})$ with a regular foliation $\widehat{\mathcal{F}}$ such that the following holds true: For all smooth curves c in $\widehat{M}$ the transverse lengths of c with respect to $\widehat{\mathcal{F}}$ and of $F(c)$ with respect to $\mathcal{F}$ coincide.

Remark 3.2.2. The transverse length of a smooth curve $c : [a, b] \to M$ is defined as the length of the projection to local quotients

$$L_T(c) = \int_a^b \|P_{c(t)}(\dot{c}(t))\| dt,$$

where $P_q : T_qM \to (T_qL_q)^\perp =: H_q$ denotes the orthogonal projection. In particular, F, as in the definition of a geometric resolution, sends leaves of $\widehat{\mathcal{F}}$ to leaves of $\mathcal{F}$, such a map is called *foliated*, and induces a length preserving map between the quotients.

As the main result in [L10] Lytchak proves

Theorem 3.2.3. *Let (M, g) be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Then $(M, \mathcal{F})$ has a geometric resolution if and only if $\mathcal{F}$ is infintesimally polar. If $\mathcal{F}$ is infinitesimally polar, then there is a canonical resolution $F : \widehat{M} \to M$ with the following properties*

1. $\dim(\widehat{M}) = \dim(M)$*;*
2. *F induces a bijection between the spaces of leaves;*
3. *$F|_{F^{-1}(M_0)} : F^{-1}(M_0) \to M_0$ is a diffeomorphism;*
4. *F is proper and 1-Lipschitz.*

In particular, the resolution $\widehat{M}$ is compact or complete if M has the corresponding property. The isometry group Γ of $(M, \mathcal{F})$ acts by isometries on $(\widehat{M}, \widehat{\mathcal{F}})$ and the map $F : \widehat{M} \to M$ is Γ-equivariant. If $\mathcal{F}$ is given by the orbits of a group G of isometries of M, then G acts by isometries on $\widehat{M}$, and $\widehat{\mathcal{F}}$ is given by the orbits of G. If M is complete, then the singular Riemannian foliation $\mathcal{F}$ has no horizontal conjugate points if and only if $\widehat{\mathcal{F}}$ has no horizontal conjugate points and f has sections if and only if $\widehat{\mathcal{F}}$ has sections.

For the notion of horizontal conjugate points see Section 2.2.

To get an idea what this canonical resolution is about let us consider some constructions (cf. [L10]). For the rest of this section $\mathcal{F}$ always denotes an infinitesimally polar singular Riemannian foliation of codimension k on a Riemannian manifold (M, g).

Recall that the *Grassmannian bundle* $\mathfrak{G}_k(M)$ of a given manifold M^{n+k} consists fiberwise of the Grassmaniann manifolds

$$G_k(T_pM) = \{\sigma \subset T_pM | \sigma \text{ is a } k\text{-plane}\}$$

of the k-dimensional linear subspaces of the tangent space T_pM, that is to say $\mathfrak{G}_k(M) = \bigcup_{p \in M} G_k(T_pM)$. For a detailed discussion of the Grassmanian bundle with its natural metric we refer the reader to [Wie08].

For a singular Riemannian foliation $\mathcal{F}$ of codimension k on M which has sections, Boulam defined in [B93] the set

$$\widehat{M}' = \{T_p\Sigma | \Sigma \text{ is a section through } p\}$$

of the Grassmannian bundle. Let $P : \widehat{M}' \to M$ denote the restriction of the canonical map $\mathfrak{G}_k(M) \to M$. Boualem constructed a differentiable structure on

$\widehat{M'}$ and showed that there is some Riemannian metric on N such that the partition $\widehat{\mathcal{F}'} = \{P^{-1}(L)|L \in \mathcal{F}\}$ becomes a regular Riemannian foliation on $\widehat{M'}$. The foliation $\widehat{\mathcal{F}'}$ is called the *blow up of* $\mathcal{F}$.

In [T06] Töben proved this result again with another technique and gives the following amplification: If we denote by h the natural Riemannian metric on $\mathfrak{G}_k(M)$ and by $\hat{g}' = \iota^* h$ the pull back on $\widehat{M'}$, then the pair $(\widehat{\mathcal{F}'}, \widehat{\mathcal{F}'}^{\perp})$ is a bi-foliation on $\widehat{M'}$ with a Riemannian foliation $\widehat{\mathcal{F}'}$ and totally geodesic foliation $\widehat{\mathcal{F}'}^{\perp}$.

Therefore, in some sense, the sections of $\mathcal{F}$ play the role of a global benchmark and give rise to a resolution of the singularities. In [L10] Lytchak generalized this construction replacing $\widehat{M'}$ by

$$\widehat{M} = \{\Sigma \subset T_pM | \Sigma \text{ is a section of } \mathfrak{F}_p \text{ through } 0\}$$

to infinitesimally polar singular Riemannian foliation. Thus what is really needed is just the infinitesimal geometric information of a singular Riemannian foliation with sections, but not the actual existence of sections.

Example. Consider the standard S^1-action on $M = \mathbb{R}^2$ by rotation with its orbit foliation $\mathcal{F}$. In this case, the canonical resolution is just the subset

$$\widehat{M} = \{0\} \times \mathbb{R}P^1 \cup \bigcup_{x \in \mathbb{R}^2} (x, \mathbb{R} \cdot x)$$

of the Riemannian product $\mathfrak{G}_k(\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{R}P^1$ with the induced metric and the foliation is given by $\widehat{\mathcal{F}} = F^*\mathcal{F}$, where $F : \widehat{M} \to \mathbb{R}^2$ is the projection onto the first factor.

By reasons of a detailed understanding, we follow [L10] and recapitulate the main steps of Lytchak's construction of $\widehat{M}$.

Let V be a finite-dimensional real vector space with scalar products g and g^+ and let $I_{g,g^+} : V \to V$ be the linear map, defined by

$$g^+(I_{g,g^+}(v), w) = g(v, w) \text{ for all } v, w \in T.$$

A quick calculation easily shows that for a linear subspace H of V we have the identity $H^{\perp_g} = (I_{g,g^+}(H))^{\perp_{g^+}}$ and that the equality $I_{g,g^+} \circ I_{g^+,g} = \mathrm{id}_V$ holds.

If M is a manifold with Riemannian metrics g and g^+ we get a canonically defined smooth bundle automorphism

$$I_{g,g^+} : TM \to TM$$

and also a smooth bundle automorphism

$$I_{g,g^+} : \mathfrak{G}_k(M) \to \mathfrak{G}_k(M)$$

of the Grassmannian bundle, which we will denote by the same symbol.

Now, if $\mathcal{F}$ is a singular Riemannian foliation on M with respect to g and g^+, the map I_{g,g^+} satisfies $I_{g,g^+}(H_p(g)) = H_p(g^+)$ by construction, with $H_p(h) = (T_pL_p)^{\perp_h}$ for $h \in \{g, g^+\}$. In fact, since the tangent spaces of the leaves do not depent on the adapted Riemannian metrics, we have

$$T_pL_p = (H_p(g))^{\perp_g} = (I_{g,g^+}(H_p(g)))^{\perp_{g^+}} \iff \\ (T_pL_p)^{\perp_{g^+}} = H_p(g^+) = I_{g,g^+}(H_p(g)).$$

Definition 3.2.4. We define the subset $\widehat{M} \subset \mathfrak{G}_k(M)$ of the Grassmannian bundle of k-planes to be

$$\widehat{M} = \{\Sigma_p | \Sigma_p \text{ is a section of } \mathfrak{F}_p \text{ through } 0\}.$$

Remark 3.2.5. As we have seen, the infinitesimal foliation $\mathfrak{F}_p$ on T_pM is a singular Riemannian foliation with respect to the flat metric g_p of the same codimension as $\mathcal{F}$. Since any section is totally geodesic and the totally geodesic submanifolds of Euclidean spaces are just the affine linear subspaces, every section of $\mathfrak{F}_p$ through the origin is a k-dimensional linear subspace of H_p and these sections are in 1:1-correspondence with sections of $\mathfrak{F}_p^h$.

Assume that $\mathcal{F}$ is an infinitesimally polar singular Riemannian foliation with respect to another Riemannian metric g^+ and let $\widehat{M}^+$ be defined as above with respect to g^+. Then it is not hard to see that the map I_{g,g^+} sends $\widehat{M}$ to $\widehat{M}^+$.

For every point $p \in M$, we can pull back the flat metric g_p via the foliated diffeomorphism $\phi : (U, \mathcal{F}|_U) \to (\phi(U), \mathfrak{F}_p|_{\phi(U)}) \subset (T_pM, \mathfrak{F}_p)$ from Section 2.3. Since the metric $g^+ = \phi^* g_p$ is adapted to $\mathcal{F}$, we see that the foliation $\mathcal{F}$ is locally equivalent to an isoparametric foliation on an open set of a Euclidean space. Therefore, after we identify locally $\widehat{M}^+ = I_{g,g^+}(\widehat{M})$, using the results from [T06] we get a differentiable structure for $\widehat{M}$ such that $\widehat{M}$ is an immersed submanifold of $\mathfrak{G}_k(M)$, satisfying the desired properties. Moreover, it is clear that $\widehat{\mathcal{F}} = P^*(\mathcal{F})$ is a regular foliation.

The main observation by Lytchak in order to define the distribution $\widehat{\mathcal{H}}$ normal to $T\widehat{\mathcal{F}}$ is that the assignment of a regular horizontal unit vector to the corresponding section is a smooth submersion. Let us say some explaining words about that. Let $D^0 \subset T^1M$ be the space of all regular horizontal unit vectors. Then D^0 is a smooth, injectively immersed submanifold of the unit tangent bundle T^1M that is invariant under the geodesic flow. Recall that a horizontal vector $v \in H_p$ is called regular iff the geodesic starting in the direction of v contains at least one regular point. If so, the set of singular points on this geodesic is discrete. Equivalently, one can say that a vector $v \in H_p$ is regular iff v is a regular point of the infinitesimal singular Riemannian foliation $\mathfrak{F}_p$. If $\mathcal{F}$ is infinitesimally polar, then a horizontal vector v is regular if and only if it is contained in exactly one section Σ_v of the isoparametric foliation $\mathfrak{F}_p$. If we let $m : D^0 \to \widehat{M}, v \mapsto \Sigma_v$, denote this map, the assertion is that m is a smooth submersion. In particular, m has local sections and this is the observation Lytchak uses. Since the restriction to the preimage of the regular part $P|_{P^{-1}(M_0)} : P^{-1}(M_0) \to M_0$ is a diffeomorphism there is a smooth distribution $\widehat{\mathcal{H}}_0$ that is sent to the horizontal distribution $\mathcal{H}_0$ of $\mathcal{F}|_{M_0}$ by P_*. Thus it remains to show that this distribution can uniquely be extended. For the reason of understanding we give the proof of [L10].

Lemma 3.2.6. *There is a unique smooth k-dimensional distribution $\widehat{\mathcal{H}}$ on $\widehat{M}$ that extends $\widehat{\mathcal{H}}_0$.*

Proof. Since $P^{-1}(M_0)$ is dense in $\widehat{M}$, it is clear that if there exists such an extension, then it is unique. Therefore, it remains to show that for each point $\Sigma \in \widehat{M}$ there are k linearly independent smooth vector fields V_i defined on an open neighborhood O of Σ such that the restriction of each V_i to $O \cap P^{-1}(M_0)$ is a section of $\widehat{\mathcal{H}}_0$.
Thus let $\Sigma \in \widehat{M}$ be given and let $p = P(\Sigma) \in M$ be the foot point of Σ. Let $v \in T_pM$ be a regular horizontal unit vector contained in Σ. Since the map $m : D^0 \to \widehat{M}$ is a smooth submersion, we find an open neighborhood O of Σ in $\widehat{M}$ and a smooth section $X : O \to D^0$ with $m \circ X = \mathrm{id}_O$ and $X(\Sigma) = v$.
There is a small interval I around 0 such that the map $\bar{\xi} : O \times I \to D^0$, given by $\bar{\xi}(\Sigma', t) = \varphi_t(X(\Sigma'))$, where φ_t denotes the restriction of the geodesic flow to D^0, is defined. Since $\bar{\xi}$ is smooth, the composition $\xi : O \times I \to O$, given by $\xi = m \circ \bar{\xi}$, is smooth, too. By construction, ξ satisfies $\xi(\Sigma', 0) = \Sigma'$ for all $\Sigma' \in O$ and the map $V(\Sigma') = \frac{d}{dt}\big|_{t=0}\xi(\Sigma', t)$ defines a smooth vector field on the open neighborhood O.
We have $P(m(v)) = P(v)$ for all $v \in D^0$, i.e. the map m commutes with the foot point projections. Thus the projection of any ξ-trajectory to M is the projection of the corresponding $\bar{\xi}$-trajectory to M. Because of the fact that the $\bar{\xi}$-trajectories are flow lines of the geodesic flow, the ξ-trajectory of $\Sigma' \in O$ is sent by P to the regular horizontal geodesic that starts at $P(\Sigma')$ in the direction of $X(\Sigma')$. In particular, we deduce that the restriction of V to $O \cap P^{-1}(M_0)$ is a section of $\widehat{\mathcal{H}}_0$ with $P_*(V(\Sigma)) = v$.
Applying this construction to a basis $v_1, \ldots, v_k$ of Σ that consists of regular unit vectors, we get k linearly independent smooth vector fields V_i as in the claim. $\square$

Finally, let h denote the canonical Riemannian metric on $\mathfrak{G}_k(M)$ and also its restriction to $\widehat{M}$. Then we define the Riemannian metric $\hat{g}$ on $\widehat{M}$ uniquely by the following three properties:

1. On $T\widehat{\mathcal{F}}$ the metric $\hat{g}$ coincides with the canonical metric h;
2. The distributions $T\widehat{\mathcal{F}}$ and $\widehat{\mathcal{H}}$ are orthogonal with respect to $\hat{g}$;
3. On $\widehat{\mathcal{H}}$ the restriction of the differential P_* to $\widehat{\mathcal{H}}$ induces an isometry between $\widehat{H}_\Sigma$ and $\Sigma \subset H_{P(\Sigma)}$.

By construction, $\hat{g}$ is a smooth Riemannian metric on $\widehat{M}$. Moreover, if we identify $P^{-1}(M_0)$ and M_0 the metric $\hat{g}$ arises from the metric g by changing g only on $T\mathcal{F}$.

Therefore, $\widehat{\mathcal{F}}$ is a Riemannian foliation on $P^{-1}(M_0)$. Since $P^{-1}(M_0)$ is dense in $\widehat{M}$, the regular foliation $\widehat{\mathcal{F}}$ is Riemannian on the whole manifold $(\widehat{M}, \hat{g})$.

Having the constructions of $(\widehat{M}, \hat{g})$, the foliation $\widehat{\mathcal{F}}$, and the distribution $\widehat{\mathcal{H}}$ in mind, the properties stated in Theorem 3.2.3 are an immediate consequence.

3.3 Pointwise Taut Quotients

Coming so far, we now prove the announced result in the infinitesimally polar case. Altough the analogous statement of Lemma 2.1.11 for arbitrary coefficient fields might be false in the general case of a singular Riemannian foliation, we give an affirmative answer in the $\mathbb{Z}_2$-case. We want to remind the reader once again that, by our convention, taut always means $\mathbb{Z}_2$-taut.

Theorem 3.3.1. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. Then $\mathcal{F}$ is infinitesimally polar and taut if and only if the quotient $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold.*

Unfortunately, there are examples of taut singular Riemannian foliations that are not infinitesimally polar, already in the homogenous case. So there is no chance to drop the condition of infinitesimal polarity in the theorem. However, since, due to Lemma 2.1.5, the infinitesimal foliations are taut if the foliation is taut, it might be possible to decide if a foliation is infinitesimally polar in particular cases.

In [GT03] Gorodski and Thorbergsson classified the taut irreducible representations of compact Lie groups as either variationally complete and hence hyperpolar or as one of the exeptional cases of cohomogeneity equal to three that are not polar. With this result and Lemma 2.1.5 one easily obtain

Corollary 3.3.2. *Let M be a complete Riemannian manifold with an action of a closed subgroup $G \subset I(M)$. Assume that for all $p \in M$ the isotropy representation of G_p on $\nu_p(G(p))$ contains no irreducible factor equivalent to one of the following:*

- (standard) $\otimes_{\mathbb{R}}$ (spin) : $\mathbf{SO}(2) \times \mathbf{Spin}(9) \to \mathbf{O}(32)$*;*
- (standard) $\otimes_{\mathbb{C}}$ (standard) : $\mathbf{U}(2) \times \mathbf{Sp}(n) \to \mathbf{O}(8n)$*;*
- (standard)^3 $\otimes_{\mathbb{H}}$ (standard) : $\mathbf{SU}(2) \times \mathbf{Sp}(n) \to \mathbf{O}(8n)$.

Then the partition $\mathcal{F}^G$ into the orbits is infinitesimally polar. If, in addition, the G-action on M is taut, the quotient M/G is developable with a pointwise $\mathbb{Z}_2$-taut universal covering orbifold.

With respect to Theorem 3.3.1 let us now formulate the following statement:

(S) $\mathcal{F}$ is taut and infinitesimally polar $\Leftrightarrow$
$M/\mathcal{F}$ is a good Riemannian orbifold N/Γ and N is pointwise taut.

In order to discuss this, let us fix a setting. Throughout this section, M is always assumed to be a complete, connected $(n+k)$-dimensional Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$ of dimension n. If $\mathcal{F}$ is infinitesimally polar, we will denote by $F : (\widehat{M}, \widehat{\mathcal{F}}) \to (M, \mathcal{F})$ the canonical geometric resolution, discussed in Section 3.2. Recall that $\mathcal{F}$ is infinitesimally polar iff it is locally without horizontal conjugate points (see Section 2.2), e.g. if $M/\mathcal{F}$ is a Riemannian orbifold. Since we already know that $\mathcal{F}$ is taut and infinitesimally polar iff its lift to the universal covering of M is and the right hand side of **(S)** is, of course, equivalent to the corresponding statement for this lift, we see that **(S)** is invariant under the transition to the universal covering. Therefore, we can and will additionally assume from now on that M is simply connected.

Assume that $\mathcal{F}$ is infinitesimally polar and let $\gamma : I = [0,1] \to M$ be a horizontal geodesic starting from a regular leaf $L \in \mathcal{F}$ to a point $q \in M$. Then $\dot\gamma(t)$ is a regular horizontal vector for all $t \in I$. Thus γ admits a lift $\hat\gamma$ to $\widehat{M}$ given by $\hat\gamma(t) = \Sigma_{\dot\gamma(t)}$, where $\Sigma_{\dot\gamma(t)}$ is the unique section of $\mathfrak{F}_{\gamma(t)}$ which contains $\dot\gamma(t)$ and by construction of $(\widehat{M}, \widehat{\mathcal{F}})$ it follows that $\hat\gamma : I \to \widehat{M}$ is a horizontal geodesic. On the other hand, if $\gamma : I \to \widehat{M}$ is a horizontal geodesic, then $F \circ \gamma : I \to M$ is a horizontal geodesic, too. Accordingly, if L is a regular leaf of $\mathcal{F}$ and q is a point in M and we set $\widehat{L} = F^{-1}(L)$ and choose a point $\hat q \in F^{-1}(q)$, we see that the induced map, which we still denote by F,

$$F : \mathcal{P}(\widehat{M}, \widehat{L} \times \hat q) \to \mathcal{P}(M, L \times q)$$

provides a bijective correspondence between the critical points of the respective energy functionals. Further, if we let $i(\gamma)$ be the index of γ as a critical point of E_q, then ,as we have seen in Lemma 2.2.3 and in A.2.4, we have

$$\begin{aligned} i(\gamma) &= \mathrm{ind}_{\Lambda^{L_{\gamma(0)}}}(\gamma_0) \\ &= \mathrm{ind}_{W^\gamma}(\gamma_0) + \mathrm{ind}_{\Lambda^{L_{\gamma(a)}}/W^\gamma}(\gamma_0), \end{aligned}$$

where as usual $\gamma_0 : (0,1) \to M$ denotes the subgeodesic.

Definition 3.3.3. We set $v(\gamma) = \mathrm{ind}_{W^\gamma}(\gamma_0)$, resp. $h(\gamma) = \mathrm{ind}_{\Lambda^{L_{\gamma(a)}}/W^\gamma}(\gamma_0)$ and call this number the *vertical index*, resp. the *horizontal index* of γ.

In the case of a regular horizontal geodesic, we have seen in Section 2.2 that, geometrically, the vertical index coincides with the crossing number, i.e. counts the intersection with the singular leaves (counted with their multiplicity) and that the horizontal index can be described in terms of conjugate points in the local developement of the quotient. Since $\widehat{\mathcal{F}}$ is a regular foliation and the quotients $M/\mathcal{F}$ and $\widehat{M}/\widehat{\mathcal{F}}$ are isometric, we conclude that with the above notation $i(\hat{\gamma}) = h(\gamma)$.

Consider the following diagramm **(D)**:

$$\begin{tikzcd}
(\widetilde{M},\widetilde{\mathcal{F}}) \arrow[rr, "\pi"] \arrow[d, "\widetilde{Q}"'] & & (\widehat{M},\widehat{\mathcal{F}}) \arrow[d, "F"] \arrow[ddl, "\widehat{Q}"'] \\
\widetilde{M}/\widetilde{\mathcal{F}} \arrow[dr, "P"] & & (M,\mathcal{F}) \arrow[dl, "Q"] \\
 & M/\mathcal{F} = \widehat{M}/\widehat{\mathcal{F}} &
\end{tikzcd}$$

where $\pi : \widetilde{M} \to \widehat{M}$ is the universal covering with $\widetilde{\mathcal{F}} = \pi^*(\widehat{\mathcal{F}})$ and $Q, \widehat{Q}, \widetilde{Q}$ denote the respective quotient maps and $P : \widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ is the induced orbifold covering.

If $M/\mathcal{F}$ is a good Riemannian orbifold, it is a Coxeter orbifold, because M is assumed to be simply connected. Then $\mathcal{F}$ is infinitesimally polar and $\widetilde{M}/\widetilde{\mathcal{F}}$, which is an orbifold covering of $M/\mathcal{F}$, is a good Coxeter orbifold, too. Thus, by Lemma 3.1.11, the map $P : \widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ coincides with the universal orbifold covering and the quotient $\widetilde{M}/\widetilde{\mathcal{F}}$ is a manifold. In particular, the projection $\widetilde{Q} : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{F}}$ is a Riemannian submersion between complete Riemannian manifolds. Moreover, if $M/\mathcal{F} = N/\Gamma$ is a good Riemannian manifold with N pointwise taut, then $\widetilde{M}/\widetilde{\mathcal{F}}$ is pointwise taut. But in this case, by the discussion at the beginning of the previous section, we conclude that $\widetilde{\mathcal{F}}$ and thus, by Lemma 2.1.7, $\widehat{\mathcal{F}}$ is taut. Conversely, assume that $\widehat{\mathcal{F}}$ exists and is taut. Then $\widetilde{\mathcal{F}}$ is taut, too. But $\widetilde{\mathcal{F}}$ is a regular foliation on a simply connected manifold and again, as in the proof of 2.1.11, tautness implies that all leaves have trivial holonomy, i.e. $\widetilde{M}/\widetilde{\mathcal{F}}$ is a complete Riemannian manifold and $\widetilde{Q}$ a Riemannian submersion. Hence, all points of $\widetilde{M}/\widetilde{\mathcal{F}}$ have to be taut. We conclude that if $\mathcal{F}$ is infinitesimally polar, the statement **(S)** is equivalent to the following statement:

(S') $\quad \mathcal{F}$ is taut $\Longleftrightarrow$ $\widehat{\mathcal{F}}$ is taut.

Remark 3.3.4. Uncoupled from our geometric setting one could translate the statements **(S)** and **(S')** into an abstract one as follows. Let $f : P \to \mathbb{R}$ be a Morse function on a closed, finite-dimensional manifold and let $K \subset P$ be a submanifold which contains all the critical points of f and such that the gradient ∇f is tangent to K along K, i.e. $\nabla f|_K \in \Gamma(TK)$. Under which assumptions can one deduce $\mathbb{F}$-perfectness of f from $\mathbb{F}$-perfectness of $f|_K$ or the other way round?

The fact that **(S)** holds for $\mathbb{Z}_2$-coefficients, but that without further assumptions there are counterexamples for the abstract problem, even in the $\mathbb{Z}_2$-case, and that it is not at all clear how to give special conditions, at least for one implication, shows the special geometry of our setting.

With the notation used in the diagram let $M/\mathcal{F}$ be a good Riemannian orbifold with universal orbifold covering N and recall that, by our convention, M is always assumed to be simply connected. Let $p, q \in N$ be two regular points for the Γ-action and let $c : [0,1] \to N$ be a broken geodesic between p and q which intersects the singular Γ-strata of N in finitely many times. Then the projection $\bar{c} = P \circ c$ of c to $M/\mathcal{F}$ is a broken orbifold geodesic which intersects the singular orbifold strata in exactly the same times. We can associate to such a broken geodesic a generalized crossing number $v(c) = \sum_{t \in [0,1]} \dim(\mathcal{F}) - \dim(\bar{c}(t))$ that coincides with the generalized version of the vertical index of a horizontal lift of $\bar{c}$ to M and which we therefore also denote by v. Since the dimension of the leaves is locally constant along the strata in $M/\mathcal{F}$ we can associate to a stratum S of $M/\mathcal{F}$ a multiplicity v_S, which is just $\dim(\mathcal{F}) - \dim(L)$ for any $L \in S$. This function behaves additve, i.e. if S_i are strata with $S_k = \bigcap_i \bar{S}_i$ then $v_{S_k} = \sum_i v_{S_i}$, because this is true for isoparametric foliations. The next lemma shows that we can extend this function to the set of hypersurfaces $\mathcal{H}$ associated with the Γ-action, i.e. the supports of the walls of a fixed chamber and their Γ-translates.

Lemma 3.3.5. *If $P : N \to M/\mathcal{F}$ denotes the orbifold covering map, the function $N \to \mathbb{Z}, p \mapsto \dim(P(p))$, is constant on the set $H \setminus \bigcup_{H' \in \mathcal{H} : H' \neq H} H'$.*

Proof. Given a point $p \in H$, we fix a point $\tilde{p} \in P(p)$ in M. Then we can identify a small ball U around p with a small ball V in a section Σ of the isoparametric foliation $\mathfrak{F}_{\tilde{p}}$. Identify $\Sigma \cong T_x\mathfrak{L}_x^{\perp}$ for a regular point $x \in \Sigma$, where $\mathfrak{L}_x \in \mathfrak{F}_{\tilde{p}}$ is the leaf through x. Under this identifications, $H \cap U$ corresponds to the intersection of V with a focal hyperplane in Σ. Since $\mathfrak{F}_{\tilde{p}}$ is the parallel foliation induced by the regular leaf $\mathfrak{L}_x$, the claim follows from the slice theorem for isoparametric submanifolds (cf. Section 6.5 of [PT88]). $\square$

The lemma ensures that we can associate to each element $H \in \mathcal{H}$ a multiplicity v_H which is just $\dim(\mathcal{F}) - \dim(P(p))$ for a generic point $p \in H$, and the crossing number $v(c)$ as above counts the intersections of c with the singular Γ-strata with respect to their multiplicities.

Definition 3.3.6. In the above setting, we call a path $c \in H^1([0,1], N)$ *transversal to $\mathcal{H}$* if it intersects the Γ-singular strata in N discretely and if $\lim_{t' \searrow t} \dot{c}(t')$ and $\lim_{t' \nearrow t} \dot{c}(t')$ lie in the same chamber of the $\Gamma_{c(t)}$-action on $T_{c(t)}N$ for all $t \in [0,1]$, i.e. if for small ε, the points $c(t-\varepsilon)$ and $c(t+\varepsilon)$ lie on opposite sides of H for all $H \in \mathcal{H}$ containing $c(t)$.

Lemma 3.3.7. *Let N be a complete, simply connected and pointwise taut Riemannian manifold on which a reflection group Γ acts. Denote by $\mathcal{H}$ the set of reflection* *hyperplanes related to the Γ-action and associate to each element $H \in \mathcal{H}$ a number $v_H \in \mathbb{N}$. Extend this assignment to N by $v(p) = \sum_{H \in \mathcal{H}: p \in H} v_H$, what can be done because the set of hyperplanes is locally finite. Then for every Γ-regular point $p \in N \setminus \bigcup_{H \in \mathcal{H}} H$, all the linking cycles Δ_w with $w \in T_pN$, as constructed in Theorem 1.3.1, consist of transversal broken geodesics and the assignment $v : \Delta_w \to \mathbb{N}$, given by $v(c) = \sum_{t \in [0,1]} v(c(t))$, is constant on every cycle Δ_w.*

Proof. Since the Γ-hyperplanes are totally geodesic and each cycles Δ_w consists of broken geodesics, either an element $c \in \Delta_w$ intersects $\mathcal{H}$ discretely or there exists a hyperplane $H \in \mathcal{H}$ and two breaking points $t_0 < t_1$ such that $c([t_0, t_1]) \subset H$. In the latter case, it follows that $c(0) \in H$ because by construction $c|_{[t_2,t_1]}(t) = \exp_{c(0)}(tu)$ for some $t_0 \leq t_2 < t_1$ and $u \in T_{c(0)}N$, so that $c(0) = \exp_{c(0)}(0) \in H$. Thus, if $p \in N$ is a Γ-regular point, then every path in Δ_w intersects $\mathcal{H}$ discretely for every $w \in T_pN$ and we have to check the transversality condition only at the breaking points of such paths. But if $C_u \subset T_pN$ is a kernel leaf with $q = \exp_p(C_u) \in H$, consider the smooth map $\partial_1 : C_u \to T_qN, \partial_1(x) = (d\exp_p)_x(x)$. Then, due to our construction of the cycles, the transversality condition is verified by the fact that, with the same argument as before, $\partial_1(x) \notin T_qH$ and therefore the image $\partial_1(C_u)$ is completely contained in one half of $T_qN \setminus T_qH$. Further, we have

$$v(c) = \sum_{t \in [0,1]} v(c(t)) = \sum_{H \in \mathcal{H}} \#\{c^{-1}(H)\} \cdot v_H,$$

so that v is constant if and only if $c \mapsto \#\{c^{-1}(H)\}$ is constant on Δ_w for all $H \in \mathcal{H}$. Thus, the second claim follows by generalized transversality arguments. Namely, for the continuous evaluation map $e : \Delta_w \times [0,1] \to N$ and the projection onto the second factor, $pr : \Delta_w \times [0,1] \to \Delta_w$, our transversality condition implies that the image $pr(e^{-1}(H))$ is open in Δ_w for every reflection hyperplane H and that the function $\Delta_w \to \mathbb{Z}$, given by $c \mapsto \#\{c^{-1}(H)\}$, is locally constant. Since, by compactness of Δ_w, the image $pr(e^{-1}(H))$ is always closed, we have $pr(e^{-1}(H)_j) = \Delta_w$ for every connected component $e^{-1}(H)_j$ of $e^{-1}(H)$, so that the map $e^{-1}(H) \to \Delta_w$ is a local homeomorphism and the restriction of this map to each connected component is a covering projection. In particular, the number

$$v(c) = \sum_{H \in \mathcal{H}} \#\{c^{-1}(H)\} \cdot v_H = \sum_{H \in \mathcal{H}} \sum_j \#\{(pr|_{e^{-1}(H)_j})^{-1}(c)\} \cdot v_H$$

is constant on Δ_w.

$\square$

Corollary 3.3.8. *If N is the universal orbifold covering of $M/\mathcal{F}$ and additionally pointwise taut, then for generic points $p \in N$ the generalized crossing number is constant on each linking cycle Δ_v with $v \in T_pN$.*

After this observations, which are needed in equal parts for our understanding of the next proof and for further constructions, we are now able to state the first half of Theorem 3.3.1 from the beginning of this section.

Theorem 3.3.9. *If $\mathcal{F}$ is a closed infinitesimally polar singular Riemannian foliation on a complete Riemannian manifold M and $\mathcal{F}$ is $\mathbb{F}$-taut, then $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold. In particular, $\hat{\mathcal{F}}$ is taut in this case.*

Proof. As always we will assume that M is simply connected. Consider the above diagram **(D)** with the notation used there. We first show that $M/\mathcal{F}$ is good by showing that the Riemannian foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ has no exceptional leaves. Let us therefore assume that $\widetilde{L} \in \widetilde{\mathcal{F}}$ is a leaf with nontrivial holonomy. Then for a nearby regular leaf $\widetilde{L}_1$ and a point $\tilde{q} \in \widetilde{L}$, the point $\tilde{q}$ is not a focal point of $\widetilde{L}_1$ and there are at least two minimal horizontal geodesics γ_1 and γ_2 from $\widetilde{L}_1$ to $\tilde{q}$. These two minima yield two 0-cells for the cell decomposition of $\mathcal{P}(\widetilde{M}, \widetilde{L}_1 \times \tilde{q})$ induced by the energy functional. But the latter space is connected since $\widetilde{M}$ is simply connected, so that there must be a critical point γ_3 of index 1 in such a way that the corresponding 1 cell e_1 satisfies $\partial e_1 \neq 0$. On the other hand, if we push down γ_3 via $F \circ \pi$, we obtain a horizontal geodesic from the leaf $L = F \circ \pi(\widetilde{L}_1)$ to the point $q = F \circ \pi(\widetilde{q})$ and after moving $\widetilde{L}_1$ slightly, we can assume that if $\gamma_3(t_0)$ denotes the focal point on γ_3, the point $F \circ \pi(\gamma_3(t_0)$ does not lie on a singular leaf. Due to Theorem 2.8 of [TT97], the energy functional $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is a Morse Bott function for every point $q \in M$, so that if we denote by γ the restriction $F \circ \pi \circ \gamma_3|_{[0,t_0]}$, there exists a closed, connected 1-dimensional submanifold S of horizontal geodesics parameterized on $[0, t_0]$ through γ, which start in L and end in $\gamma(t_0)$, all having the same length as γ. If we concatenate the elements of S with the segment $F \circ \pi \circ \gamma_3|_{[t_0,1]}$, we get a one dimensional variation of $F \circ \pi \circ \gamma_3$, which we can lift along $F \circ \pi$ to a one dimensional variation $\widetilde{S}$ of γ_3 with constant energy. By the classification of one dimensional manifolds, $\widetilde{S}$ is the 1-sphere and therefore represents a linking cycle for γ_3 with respect to any field $\mathbb{F}$, which is clearly a contradiction, because this implies $\partial e_1 = 0$. It follows that $\widetilde{\mathcal{F}}$ has trivial holonomy and therefore, $\widetilde{M}/\widetilde{\mathcal{F}}$ is a complete Riemannian manifold, which is also simply connected by the exact homotopy sequence. In particular, $\widetilde{M}/\widetilde{\mathcal{F}} \to M/\mathcal{F}$ is the universal orbifold covering.

In order to prove that $N = \widetilde{M}/\widetilde{\mathcal{F}}$ is pointwise taut we will deduce that for every Γ-regular point $\widetilde{p} \in N$, the energy functional $E_{\widetilde{q}} : \mathcal{P}(N, \widetilde{p} \times \widetilde{q}) \to \mathbb{R}$ is a Morse-Bott function for all points $\widetilde{q} \in N$. This being the case, we then finish the proof with Theorem 1.3.1 and Lemma 2.1.13.

So, let $\widetilde{p}, \widetilde{q} \in N$ be two points as above and consider a regular path space sublevel $\widetilde{P}^b = \mathcal{P}(N, \widetilde{p} \times \widetilde{q})^b = E_{\widetilde{q}}^{-1}((-\infty, b])$. Let L denote the leaf $P(\widetilde{p}) \in \mathcal{F}$, fix a point $q \in P(\widetilde{q})$, and set $P^b = \mathcal{P}(M, L \times q)^b$. Using a common sufficiently fine subdivision $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = 1$ of the interval $[0, 1]$ as in Section A.2, we can regard $\widetilde{P}^b(t_1, \ldots, t_r)$ as a submanifold of $P^b(t_1, \ldots, t_r)$ as follows. Recall that $F : \widehat{M} \to M$ restricted to the preimage of the regular part of M is a diffeomorphism and set $\widehat{L} = F^{-1}(L)$ and choose a point $\hat{q} \in F^{-1}(q)$. Let $\widetilde{L}$ be the union of leaves $\pi^{-1}(\widehat{L})$ and fix a point $q' \in \pi^{-1}(\hat{q})$. Then we can uniquely lift an absolutely continuous path that starts in $\widetilde{p}$ and ends in $\widetilde{q}$ to a horizontal absolutely continuous path in $\widetilde{M}$ of the same energy that starts in $\widetilde{L}$ and ends in q'. Since lifting paths is a continuous operation, this defines a continuous map between the corresponding path spaces, which we can compose with the map on the path space levels induced by $F \circ \pi$ to push down the paths to M. In this manner we obtain a continuous map $j : \widetilde{P}^b \to P^b$ that restricts to an energy preserving smooth injective immersion on the finite-dimensional approximations of the path spaces and therefore defines an embedding $j : \widetilde{P}^b(t_1, \ldots, t_r) \to P^b(t_1, \ldots, t_r)$, because $\widetilde{P}^b(t_1, \ldots, t_r)$ is compact. For notational reasons, set $X = \widetilde{P}^b(t_1, \ldots, t_r)$ and $Y = P^b(t_1, \ldots, t_r)$. We will identify X with its image $j(X)$ which consists exactly of the F-liftable broken horizontal geodesics in Y, i.e. those broken horizontal geodesics $c \in Y$ such that $\lim_{t \nearrow t_i} \dot{c}(t)$ and $\lim_{t \searrow t_i} \dot{c}(t)$ lie in the same section of the isoparametric foliation $\mathfrak{F}_{c(t_i)}$ on $T_{c(t_i)}M$ for all i. With this identification, we have that $E_{\widetilde{q}} = E_q|_X$ and from A.2 it follows that the gradient of E_q is tangent to X along X. Thus, the critical points of $E_{\widetilde{q}}$ are exactly those of E_q lying in X and, as we have seen above, their indices as critical points of $E_{\widetilde{q}}$ are just their horizontal indices considered as critical points of E_q.

It is shown in [TT97] that under the assumption that $\mathcal{F}$ is taut, E_q is always a Morse-Bott function. Now let $\gamma \in X$ be a horizontal geodesic with index $i(\gamma) = h(\gamma) + v(\gamma)$ as a critical point of E_q, where as usual $h(\gamma)$, resp. $v(\gamma)$ denotes the horizontal index, resp. vertical index of γ. Let $n(\gamma) = h(1) + v(1)$ with $v(1) = \dim(\mathcal{F}) - \dim(L_q)$ denote its nullity. Then $h(1)$ is just the multiplicity of $\widetilde{q}$ as a conjugate point of $\widetilde{p}$ along γ. Since E_q is a Morse-Bott function, the connected component of the critical points containing γ is an $n(\gamma)$-dimensional closed manifold Z in $E^{-1}(E(\gamma))$ through γ consisting of horizontal geodesics all of which having the same index as γ. Moreover, for $c \in Z$, the tangent space T_cZ consists of all L-Jacobi fields along c which vanish in q and contains the $v(1)$-dimensional subspace of $\mathcal{F}$-Jacobi fields along c (cf. Section 2.2). Consider the smooth injective evaluation map $\widetilde{e} : Z \to \nu_q(L_q), \widetilde{e}(c) = \dot{c}(1)$. Since L is a regular leaf, every tangent vector $\dot{c}(1)$ lies in some regular leaf of the retsricted ioparametric foliation $\mathfrak{F}_q^1$ on $\nu_q(L_q)$. So let $l = \dim(L_q)$ denote the dimension of the leaf through q and set M_{n-l} for the connected component of the stratum of all l-dimensional leaves that contains the point q. Then this foliation is a product foliation $\mathfrak{F}_q^1 = \mathfrak{F}_q^2 \times \mathfrak{F}_q^h$ on the orthogonal product $\nu_q(L_q) = W \times (T_qM_{n-l})^\perp$, where W is the orthogonal complement of T_qL_q in T_qM_{n-l} (cf. Section 2.1) and $\mathfrak{F}_q^2$ is the trivial foliation of W by points. In particular, every section Σ of $\mathfrak{F}_q$ is of the form $\Sigma = W \times \Sigma^h$ for a section Σ^h of the isoparametric main part $\mathfrak{F}_q^h$, the leaves of which through a point x we will denote by $\mathcal{L}_x$. Let $pr_2 : \nu_q(L_q) \to (T_qM_{n-l})^\perp$ be the orthogonal projection onto the second factor and set $e = pr_2 \circ \widetilde{e} : Z \to (T_qM_{n-l})^\perp$. Then, due to our discussion in the proof of Lemma 2.3.4, it is clear that for all $c \in Z$ we have $T_{pr_2(\dot{c}(1))}\mathcal{L}_{pr_2(\dot{c}(1))} \subset (de)_c(T_cZ)$, what implies that (Z, e) is transversal to $\Sigma^h_{\dot{\gamma}(1)}$, where as always $\Sigma_{\dot{\gamma}(1)} = W \times \Sigma^h_{\dot{\gamma}(1)}$ denotes the unique section of $\mathfrak{F}_q$ that contains $\dot{\gamma}(1)$. By our description of the image of X in Y, we conclude that $Z \cap X = e^{-1}(\Sigma^h_{\dot{\gamma}(1)})$ and that it is therefore a submanifold of Z of the same codimension as $\Sigma^h_{\dot\gamma(1)}$ in $(T_qM_{n-l})^\perp$. Thus

$$\begin{aligned}
\dim(\mathfrak{F}^h_q) + h(1) - \dim(Z\cap X) &= \dim(\mathcal{F}) - \dim(L_q) + h(1) - \dim(Z\cap X)\\
&= v(1) + h(1) - \dim(Z\cap X)\\
&= \dim(Z) - \dim(Z\cap X)\\
&= \operatorname{codim}(M_{n-l}) - \dim(\Sigma^h_{\dot\gamma(1)})\\
&= \dim(\mathfrak{F}^h_q) + \dim(\Sigma^h_{\dot\gamma(1)}) - \dim(\Sigma^h_{\dot\gamma(1)})\\
&= \dim(\mathfrak{F}^h_q),
\end{aligned}$$

so that as we expected

$$\dim(Z\cap X) = h(1)$$

is just the multiplicity of $\widetilde{q}$ as a conjugate point of $\widetilde{p}$ along γ. This proves that $E_{\widetilde{q}}$ is also a Morse-Bott function in that case. The proof of the theorem now follows with Theorem 1.3.1 and Lemma 2.1.13. □

Remark 3.3.10. Of course, constancy of the vertical index on Z could also be deduced directly from the above setting from which one could obtain the Morse-Bott property by regular value arguments using the splitting of the index along $Z\cap X$ and the fact that the gradient of E_q is tangent to X along X.

For the rest of this section, we will now apply ourself to the remaining direction of the statement **(S)**, which is perhaps the more interesting one in view of applications. Namely, we will show that a closed singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M is $\mathbb{Z}_2$-taut if the quotient $M/\mathcal{F}$ is good with a pointwise taut orbifold covering.

Theorem 3.3.11. *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M such that $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold. Then $\mathcal{F}$ is taut.*

As already mentioned above, for the round sphere there is a well known cycle construction for critical points of the energy functional (cf. p.95-96 of [Mi63]), what shows that the sphere is pointwise taut. Now let G be one of the groups listed in 3.3.2 acting on a Euclidean space V via the indicated representation. Then the action induces an action of G on the unit sphere $S(V)$ and, by linearity, one easily sees that $\mathcal{F}^G$ is taut if and only if $\mathcal{F}^G|_{S(V)}$ is taut. In all of these cases, the quotient $S(V)/\mathcal{F}^G|_{S(V)}$ is isometric to a spherical triangle S^2/Γ with angles $(\pi/2,\pi/2,\pi/n)$ for $n\in\{2,3\}$ and multiplicities for the edges given by

$$(v_1,v_2,v_3)\in\{(1,6,7),(1,2,\dim(V)/2-5),(\dim(V)/2-5,1,1)\}\,.$$

In particular, as a special case of our result, we prove again that the orbit foliations of the exceptional representations listed in Corollary 3.3.2 are taut. For more applications, we refer to the beginning of Section 2.1, where we have already discussed, as a special case of Theorem 2.3.1, the most common application examples of Theorem 3.3.11.

Let us now say some words about the proof of Theorem 3.3.11. Given a regular horizontal geodesic γ, we will again construct a nice space Δ_γ which is, in some sense, the total space of a stratified fiber bundle over some Δ_v, as constructed in the last section, and consists of broken horizontal geodesics. The idea is the following. Assume that $M/\mathcal{F}$ is a good Riemannian orbifold with a pointwise taut universal covering orbifold N. Then a developement $\widetilde{\gamma}$ of $Q \circ \gamma$ in this space is a geodesic whose index coincides with the horizontal index of γ. Now, if v denotes the tangent vector of $\widetilde{\gamma}$ at 0, the space Δ_v represents a linking cycle for $\widetilde{\gamma}$, which we can push down to $M/\mathcal{F}$ and define Δ_γ to be the connected component that contains γ of the set of all broken horizontal geodesics which start in $L_{\gamma(0)}$ and end in $\gamma(1)$ and whose projection down to $M/\mathcal{F}$ lie in the image of Δ_v. Since we know from Corollary 3.3.8 that the vertical index is constant on Δ_v, this, together with Lemma 1.2.5, would identify Δ_γ as a linking cycle for γ if we could show that each fiber of Δ_γ, i.e. the subspace of broken horizontal geodesics c in Δ_γ which are mapped down to the same orbifold geodesic, is a compact manifold of dimension equal to the generalized vertical index $v(c)$. So this should be our first goal.

As in the first part of this section, we will assume that M is simply connected and we continue with the notation used there, i.e. from diagramm **(D)**. We also denote maps on the occuring path spaces, which are induced by maps in the diagram by the same symbols. Now again, by Lemma 3.1.11, the orbifold covering $P : \widetilde{M/\mathcal{F}} \to M/\mathcal{F}$ coincides with the universal orbifold covering, so that we identify $\widetilde{M/\mathcal{F}} \cong N$. Let $L \in \mathcal{F}$ be a regular leaf and let $q \in M$ be a regular point which is not a focal point of L. Set $\widehat{L} = F^{-1}(L)$, $\widehat{q} = F^{-1}(q)$, $\widetilde{L} = \pi^{-1}(\widehat{L})$, and fix a point $\widetilde{q} \in \pi^{-1}(\widehat{q})$. If $\gamma \in \mathcal{P}(M, L \times q)$ is a critical point for the energy and $\bar{\gamma}$ denotes the developement of the orbifold geodesic $\widetilde{Q \circ \gamma}$ in N, then by our convention $\bar{\gamma}$ coincides with $\widetilde{Q} \circ \widetilde{\gamma}$, where $\widetilde{\gamma}$ is the unique lift to $\widetilde{M}$ which ends in $\widetilde{q}$ of the horizontal geodesic $\widehat{\gamma} : [0,1] \to \widehat{M}$, given by $\widehat{\gamma}(t) = \Sigma_{\dot{\gamma}(t)}$, itself being the lift of γ to $\widehat{M}$. Then $\bar{\gamma} : [0,1] \to N$ is a geodesic with $\bar{\gamma}(1) = \widetilde{Q}(\widetilde{q})$ and $\bar{\gamma}(0) \in \widetilde{Q}(\widetilde{L})$. Note that $\widetilde{L}$ is a union of leaves. If v denotes the initial tangent vector of $\bar{\gamma}$, then, by assumption, the space Δ_v, as constructed in Theorem 1.3.1, represents a linking cycle for $\bar{\gamma}$ as a critical point of index $h(\gamma)$ for the energy on $\mathcal{P}(N, \bar{\gamma}(0) \times \bar{\gamma}(1))$. We define $\Delta_\gamma \subset \mathcal{P}(M, L \times q)$ to be the connected component of the space of broken horizontal geodesics c with $Q \circ c = P \circ d$ for some $d \in \Delta_v$ that contains γ. Then Δ_γ is compact and we define for $c \in \Delta_\gamma$ the compact subset $\Delta_\gamma(c)$ to be

$$\Delta_\gamma(c) = Q^{-1}(Q \circ c) \cap \Delta_\gamma = \{d \in \Delta_\gamma | Q \circ d = Q \circ c\}$$

and we call it the *fiber through* c. Now we can lift Δ_v along $\widetilde{Q}$ to horizontal broken geodesics in $\mathcal{P}(\widetilde{M}, \widetilde{L} \times \widetilde{q})$ and then push it down to $\mathcal{P}(M, L \times q)$ via $F \circ \pi$ to obtain a homeomorphic image of Δ_v contained in Δ_γ. To be more precise, the image of Δ_v under this map consists of all broken horizontal geodesics $c \in \Delta_\gamma$ such that $\lim_{t' \nearrow t} \dot{c}(t')$ and $\lim_{t' \searrow t} \dot{c}(t')$ lie in the same section of the infinitesimally polar foliation $\mathfrak{F}_{c(t)}$. For this reason, we will identify Δ_v with its image in Δ_γ.

Now each fiber intersects Δ_v in a finite set and those intersections are encoded in the Coxeter group Γ as some different word representations of a given element or, equivalently, given some path in Δ_v, as the finite number of possibilities to obtain different broken geodesics in Δ_v by reflections of its segments. Indeed each fiber intersects Δ_v in exactly one point. To see this, let us denote by

$$(M/\mathcal{F}, L_1, L_2)^{([0,1],\{0\},\{1\})}$$

the space of continuous paths $[0,1] \to M/\mathcal{F}$ from L_1 to L_2 with the compact open topology. Then we get a commutative diagram

$$\begin{tikzcd} \Delta_v \arrow[r, hook] \arrow[dr, "P"'] & \Delta_\gamma \arrow[d, "Q"] \\ & (M/\mathcal{F}, L, L_q)^{([0,1],\{0\},\{1\})} \end{tikzcd}$$

that induces a homeomorphism

$$\Delta_v/\sim_P \cong \Delta_\gamma/\sim_Q,$$

where of course $\sim_R$ denotes the equivalence relation $c \sim_R d \Leftrightarrow R \circ c = R \circ d$ for $R \in \{P, Q\}$. But, as we observed in Lemma 3.3.7, all the paths in Δ_v are transversal to the family $\mathcal{H}$ of Γ-hypersurfaces, so that we deduce that the map $P : \Delta_v \to (M/\mathcal{F}, L, L_q)^{([0,1],\{0\},\{1\})}$ is injective. In particular, each fiber $\Delta_\gamma(c)$ intersects Δ_v exactly once. Thus, Δ_γ is the disjoint union

$$\Delta_\gamma = \bigcup_{c \in \Delta_v} \Delta_\gamma(c).$$

Because one can show that in the infinitesimally polar case all the fibers are connected, Δ_γ is connected, too. We further obtain a retraction $R : \Delta_\gamma \to \Delta_v$ via $R(c) = \Delta_\gamma(c) \cap \Delta_v$.

Remark 3.3.12. Given a real number $\kappa > 0$, for generic points q, all the critical points in $\mathcal{P}(M, L \times q)^\kappa$ intersect the singular stratum only in points with quotient codimension one, i.e. the projected orbifold geodesics cross the singular orbifold strata only in the codimension one strata. In this case, the proofs of Theorem 1.3.1 and Lemma 2.3.4 show that such a critical point γ has a manifold neighborhood in Δ_γ of dimension $i(\gamma)$ and that the tangent space $T_\gamma\Delta_\gamma$ splits as

$$T_\gamma\Delta_\gamma = T_\gamma\Delta_v \oplus T_\gamma\Delta_\gamma(\gamma),$$

the space $T_\gamma\Delta_v$ being the direct sum

$$T_\gamma\Delta_v = \bigoplus_{j=1}^{s} (\Lambda^L/W^\gamma)_j,$$

where $0 < t_1 < \cdots < t_s < 1$ are the conjugate times along $\bar{\gamma}$, that is to say the Λ^L/W^γ-focal times along γ (cf. Section 2.2) and $(\Lambda^L/W^\gamma)_j$ is the vector space consisting of continuous vector fields J along γ such that $J|_{[0,t_j]} \in (\Lambda^L/W^\gamma)|_{[0,t_j]}$ and J vanishes on $[t_j, 1]$. In particular, due to our observations in Section A.2, using finite dimensional approximations a neighborhood of γ in Δ_γ can be deformed to coincide locally with the descending cell in some Morse chart around γ.

Together with our genericity results from Section 2.1, the next lemma finishes the proof of Theorem 3.3.11. Finally, combining Theorem 3.3.9 and Theorem 3.3.11, we obtain Theorem 3.3.1.

Lemma 3.3.13. *Let L be a regular leaf and let $q \in M$ be a regular point. Then, if $\gamma \in \mathcal{P}(M, L \times q)$ is a critical point for the energy such that it crosses the singular stratum only in quotient codimension one points, the space Δ_γ, as defined above, defines a linking cycle with respect to $\mathbb{Z}_2$-coefficients if q is not a focal point of L along γ.*

Proof. As in the proof of Theorem 1.3.1, by induction on the horizontal index, we can assume that the 0-sheaf on Δ_v has trivial cohomology. Then, as in Lemma 1.2.5, the map $R : \Delta_\gamma \to \Delta_v$ gives rise to an explicit generator of the Čech chomology group $\check{H}^{i(\gamma)}(\Delta_\gamma) \cong \mathbb{Z}_2$, which is in the image of the map

$$\check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \Delta_\gamma(\gamma)) \to \check{H}^{i(\gamma)}(\Delta_\gamma).$$

Now, due to 3.3.12, we can choose a small manifold neighborhood U of γ in Δ_v such that

$$R^{-1}(U) = \bigcup_{c \in U} \Delta_\gamma(c)$$

is a manifold neighborhood of $\Delta_\gamma(\gamma)$ of dimension $h(\gamma) + v(\gamma) = i(\gamma)$. In this case, we deduce by excision that the inclusion

$$(\Delta_\gamma, \Delta_\gamma \setminus \Delta_\gamma(\gamma)) \hookrightarrow (\Delta_\gamma, \Delta_\gamma \setminus \{\gamma\})$$

induces an isomorphism

$$\check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \{\gamma\}) \to \check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \Delta_\gamma(\gamma)).$$

If we set $\mathcal{P} = \mathcal{P}(M, L \times q)$ and denote by $j : \Delta_\gamma \hookrightarrow \mathcal{P}$ the inclusion, then the map j induces an isomorphism

$$\check{H}^{i(\gamma)}(\mathcal{P}^{E(\gamma)}, \mathcal{P}^{E(\gamma)} \setminus \{\gamma\}) \to \check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \{\gamma\}),$$

by 3.3.12, so that the commutative diagram

$$\begin{tikzcd}
\check{H}^{i(\gamma)}(\mathcal{P}^{E(\gamma)}) \arrow[rr, "j^*"] & & \check{H}^{i(\gamma)}(\Delta_\gamma) \\
& & \check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \Delta_\gamma(\gamma)) \arrow[u, "\cong"'] \\
\check{H}^{i(\gamma)}(\mathcal{P}^{E(\gamma)}, \mathcal{P}^{E(\gamma)} \setminus \{\gamma\}) \arrow[uu] \arrow[rr, "\cong"] & & \check{H}^{i(\gamma)}(\Delta_\gamma, \Delta_\gamma \setminus \{\gamma\}) \arrow[u, "\cong"']
\end{tikzcd}$$

shows that Δ_γ indeed defines a linking cycle for the critical point γ.

$\square$

Appendix A

Path Spaces

A.1 The Energy Functional

We start this section with a brief discussion of the Hilbert manifold of H^1-paths of a complete Riemannian manifold (M^n, g). We refer the reader who is not familiar with this setting to [Kl82],[Pa63].

Let $c : I \to M$ be a path in M, where $I = [0, 1]$ is the unit interval. If we assume that c is almost everywhere differentiable and that the time derivative $\dot{c}$ is square integrable, then the *energy integral*

$$E(c) = \int_I \|\dot{c}(t)\|^2 dt$$

of c is well defined. The largest class for which the energy integral make sense (for all paths in this class) is the class of H^1-paths, i.e. absolutely continuous paths $c : I \to M$ with $E(c) < \infty$. To be more precise, we recall that a path $c : [a, b] \to \mathbb{R}^n$ is called *absolutely continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $a \leq t_0 < \cdots < t_{2k+1} \leq b$ and $\sum_{i=0}^k |t_{2i+1} - t_{2i}| < \delta$ imply $\sum_{i=0}^k \|c(t_{2i+1}) - c(t_{2i})\| < \varepsilon$. So, if c is absolutely continuous, then it is continuous and the time derivative is defined almost everywhere. Moreover, $H^1(I, \mathbb{R}^n)$ is exactly the completion of the piecewise differentiable curves $C'^\infty(I, \mathbb{R}^n)$ with respect to the norm obtained by the scalar product

$$\langle c_1, c_2 \rangle = \int_I \langle c_1(t), c_2(t) \rangle \, dt + \int_I \langle \dot{c}_1(t), \dot{c}_2(t) \rangle \, dt.$$

A path $c : I \to M$ is then called absolutely continuous if for every chart (U, ϕ) of M with $c^{-1}(U) \neq \emptyset$ and every closed intervall $I' \subset c^{-1}(U)$ the path $\phi \circ c|_{I'} : I' \to \mathbb{R}^n$ is absolutely continuous. If we denote the set of all paths in M of class H^1 by $H^1(I, M)$, we have the following canonical inclusions:

$$C'^\infty(I, M) \hookrightarrow H^1(I, M) \hookrightarrow C^0(I, M),$$

where $C'^{\infty}(I,M)$ denotes the set of all piecewise differentiable curves in M and $C^0(I,M)$ is the space of continuous curves $c : I \to M$ endowed with the metric $d_{\infty}(c,\tilde{c}) = \sup_{t\in I} d(c(t),\tilde{c}(t))$. Moreover, it is not hard to see that $C'^{\infty}(I,M)$ is dense in $C^0(I,M)$.

Now let $c \in H^1(I,M)$ be piecewise differentiable and consider the induced bundle $c^*(TM)$ over I. Let $C'^{\infty}(c^*TM)$ denote the vector space of piecewise differentiable sections of c^*TM and let $\langle\cdot,\cdot\rangle_1$ be the scalar product on $C'^{\infty}(c^*TM)$ defined by

$$\langle X,Y\rangle_1 = \int_I g(X(t),Y(t))dt + \int_I g(\nabla X(t),\nabla Y(t))dt.$$

By $H^1(c^*TM)$ we denote the completion of $C'^{\infty}(c^*TM)$ with respect to the induced norm. As a standard result, there is a small open neighborhood $\mathcal{O}$ around the zero section of the tangent bundle $TM \to M$ such that for every point $p \in M$ the exponential map of M restricted to $\mathcal{O}_p = T_pM \cap \mathcal{O}$ is a diffeomorphism onto its image. Let $\mathcal{O}_c$ denote the preimage $(\bar{c})^{-1}(\mathcal{O})$, where $\bar{c}$ is the canonical bundle map over c. If we denote by $H^1(\mathcal{O}_c)$ the set of all sections $X \in H^1(c^*TM)$ satisfying $X(t) \in \mathcal{O}_{c(t)}$ for all $t \in I$, then $H^1(\mathcal{O}_c)$ is an open subset of $H^1(c^*TM)$ and we get an injective map $\widetilde{\exp}_c : H^1(\mathcal{O}_c) \to H^1(I,M)$, with $\widetilde{\exp}_c(X)(t) = \exp_{c(t)} \circ \bar{c}(X(t))$. Since I is contractible, the bundle $c^*(TM)$ is trival. So $H^1(c^*TM)$ does not depend on c and one easily check the differentiability of $\widetilde{\exp}_c^{-1} \circ \widetilde{\exp}_d$, for $c, d \in C'^{\infty}(I,M)$ as a map between open subsets of the Hilbert spaces $H^1(d^*TM)$ and $H^1(c^*TM)$. The requirement that those charts are diffeomorphisms yields a differentiable structure on $H^1(I,M)$ which turns $H^1(I,M)$ into a Hilbert manifold, locally modeled on $H^1(I,\mathbb{R}^n)$. In [Kl82] it is shown that the scalar product $\langle\cdot,\cdot\rangle_1$ on the tangent space $T_cH^1(I,M) \cong H^1(c^*TM)$ for $c \in C'^{\infty}(I,M)$, as defined above, extends to a complete Riemannian metric on the Hilbert manifold $H^1(I,M)$ which we still denote by $\langle\cdot,\cdot\rangle_1$.

Next consider the differentiable map $e : H^1(I,M) \to M \times M$ which is defined by $e(c) = (c(0),c(1))$, and let $f : N \to M \times M$ be a smooth map from a complete Riemannian manifold N. Since e is a submersion the pull back of $H^1(I,M)$ to N along f, which we denote by

$$\mathcal{P}_f(M,N) = f^*(H^1(I,M)) = \big\{(p,c) \in N \times H^1(I,M) | f(p) = e(c)\big\}$$

is a complete submanifold of $N \times H^1(I,M)$. In particular, if $f_i : L_i \to M$ are immersed submanifolds of M and we set $N = L_0 \times L_1$ and $f = (f_0,f_1)$, the space $\mathcal{P}_f(M, L_0 \times L_1)$ is formed by tuples $((p_0,p_1),c) \in N \times H^1(I,M)$ with $f_i(p_i) = c(i)$ and the tangent space $T_{((p_0,p_1),c)}\mathcal{P}_f(M, L_0 \times L_1)$ consists of H^1-vector fields $X(t)$ along $c(t)$ with $X(i) \in (df_i)_{p_i}(T_{p_i}L_i)$ for $i = 0,1$.

Now consider the pull back diagram

$$\begin{tikzcd}
N \times H^1(I,M) \arrow[drr, "\mathrm{pr}_2"] \arrow[ddr, "\mathrm{pr}_1"'] & & & \\
& \mathcal{P}_f(M,N) \arrow[ul, hook] \arrow[r, "\bar{f}"'] \arrow[d, "f^*e"] & H^1(I,M) \arrow[r, "E"'] \arrow[d, "e"] & \mathbb{R} \\
& N \arrow[r, "f"'] & M \times M &
\end{tikzcd}$$

We get a smooth function on $\mathcal{P}_f(M,N)$ by the induced energy integral $E_f = E \circ \bar{f}$ whose critical points are characterized in [G73] as exactly those $(p,\gamma) \in \mathcal{P}_f(M,N)$ such that the path $\gamma : [0,1] \to M$ is a geodesic with $(\dot\gamma(0), \dot\gamma(1)) \perp \mathrm{im}(df_p)$.

As a main result in [GH91], Grove and Halperin prove the following theorem.

Theorem A.1.1. *The energy integral $E_f : \mathcal{P}_f(M,N) \to \mathbb{R}$ satisfies Condition (C) (cf. Section 1.1) if and only if* $\mathrm{dist} \circ f : N \to \mathbb{R}$ *is proper.*

Since we want to apply critical point theory in the special case induced by an immersion $i : L \to M$, i.e. $N = L \times \{q\}$ and $f = (i,q)$ for a point $q \in M$, the theorem says that we have to focus our attention on proper immersions, because in this case $\mathrm{dist} \circ f$ is proper if and only if i is proper. Moreover, it is well known that E_f is a Morse function if and only if q is not a focal point of L. If in addition i is injective, i.e. $i : L \to M$ is a closed embedding, we will identify L with $i(L) \subset M$, drop the reference to the map i and replace it by the reference to the point q, i.e. we just write $\mathcal{P}(M, L \times q)$ instead of $\mathcal{P}_{(i,q)}(M,L)$.

For the rest of this section we will assume that $L \subset M$ is a closed embedded submanifold and refer to [Kl82] or [Sak96] for of the next propositions.

Proposition A.1.2. *The energy functional $E_q : \mathcal{P}(M, L \times q) \to \mathbb{R}$ is a differentiable function, bounded below and satisfies Condition (C) with differential*

$$\begin{aligned}
\frac{1}{2}(dE_q)_c(X) = &- \int_0^1 g(\nabla \dot c(t), X(t))dt + \sum_{i=1}^{k-1} g(\dot c(t_i^-) - \dot c(t_i^+), X(t_i)) \\
&- g(\dot c(0), X(0)),
\end{aligned}$$

for every piecewise differentiable curve c such that the restriction $c|_{[t_{i-1},t_i]}$ is smooth for $i = 1, \dots, k$. The critical points of E_q are exactly the geodesics $\gamma \in \mathcal{P}(M, L \times q)$ parameterized proportional to arc length with $\dot\gamma(0) \perp T_{\gamma(0)}L$.

The expression for dE_q is also called the *first variation formula.* The *second variation formula* yields an analogous expression for the Hessian at a critcal point γ.

Proposition A.1.3. *The Hessian of E_q at a critical point γ is given by*

$$\begin{aligned}\frac{1}{2}H(E_q)(\gamma)(X,Y) = \int_0^1 &(g(\nabla X(t), \nabla Y(t)) - g(R(X(t),\dot\gamma(t))\dot\gamma(t), Y(t)))dt \\ &- g(A_{\dot\gamma(0)}X(0), Y(0)),\end{aligned}$$

where A denotes the shape operator of L, R denotes the curvature tensor, and $X, Y \in T_\gamma\mathcal{P}(M, L \times q)$ are piecewise differentiable vector fields along γ.

Remark A.1.4. If $0 = t_0 < \cdots < t_k = 1$ is a subdivision of $[0,1]$ such that $X|_{[t_i,t_{i+1}]}$ is smooth, integration by parts together with the identity

$$\frac{d}{dt}g(\nabla X, Y) = g(\nabla^2 X, Y) + g(\nabla X, \nabla Y)$$

yields

$$\begin{aligned}\frac{1}{2}H(E_q)(\gamma)(X,Y) = &-\int_0^1 g(\nabla^2 X(t) - R(X(t),\dot\gamma(t))\dot\gamma(t), Y(t))dt \\ &+ \sum_{i=1}^{k-1} g(\nabla X(t_i^-) - \nabla X(t_i^+), Y(t_i)) \\ &- g(A_{\dot\gamma(0)}X(0) + \nabla X(0), Y(0)).\end{aligned}$$

These identities are formulated just for $c \in C'^\infty(I, M)$, respectively, for vector fields $X, Y \in C'^\infty(\gamma^*TM)$. But $C'^\infty(I, M)$ is dense in $H^1(I, M)$ and $C'^\infty(\gamma^*TM)$ is dense in $H^1(\gamma^*TM)$. Further, in [Kl82] it is shown that it is possible to extend the above notions to $\mathcal{P}(M, L\times q)$, resp. $T\mathcal{P}(M, L\times q)$. Therefore, by an abuse of notation the above equations hold for $c \in H^1(I, M)$, resp. tangent vectors $X, Y \in T_\gamma\mathcal{P}(M, L\times q)$. As a direct consequence we get

Proposition A.1.5. *A tangent vector $X \in T_\gamma\mathcal{P}(M, L\times q)$ belongs to the null space of $H(E_q)(\gamma)$ if and only if X is a Jacobi field along γ which satisfies the boundary condition $X(0) \in T_{\gamma(0)}L$ and $\nabla X(0) + A_{\dot\gamma(0)}X(0) \in T_{\gamma(0)}L^\perp$, i.e. X is an L-Jacobi field along γ. In particular, E_q is a Morse function if and only if q is not a focal point of L.*

Note A.1.6. It is well known that the path space $\mathcal{P}_f(M, N)$ as defined above is homotopy equivalent to the space $C_f(M, N)$, which is the pull back via f of the space $C^0(I, M)$ of continuous paths $c : I \to M$ endowed with the compact open topology. Since we are only interested in the path space structure up to homotopy, we will not distinguish these spaces in our notation during the discussion. If we use topological methods, such as fibrations, we will implicitly deal with $C_f(M, N)$ and if we use differential geometric methods we really mean $\mathcal{P}_f(M, N)$.

A.2 Finite-Dimensional Approximation

As we have seen in the last section, the path space $\mathcal{P}(M, L\times q)$ carries the structure of an infinite-dimensional Riemannian Hilbert manifold locally modelled on $H^1(I,\mathbb{R}^n)$. Therefore, we can use differential geometric methods for computations, e.g. the energy functional is bounded below and satisfies Condition C, so we can use infinite-dimensional critical point theory to get a picture of the topology of the CW-complex $\mathcal{P}(M, L\times q)$ for a generic point $q \in M$. But in general it is more convenient to work with another description of $\mathcal{P}(M, L\times q)$ namely a finite-dimensional approximation of the sub levels

$$\mathcal{P}(M, L\times q)^d = \{c \in \mathcal{P}(M, L\times q) \mid E_q(c) \leq d\} = E_q^{-1}((-\infty, d])$$

and $\mathcal{P}(M, L\times q)^{d-} = E_q^{-1}((-\infty, d))$. This description is the aim of this section, namely for every $d > 0$, we will construct a finite-dimensional manifold that is homotopically equivalent to $\mathcal{P}(M, L\times q)^{d-}$ and consists of broken geodesics between q and L of length less than $\sqrt{d}$.

Let (M, g) be a complete Riemannian manifold of dimension n and let $\iota : L \to M$ be a properly immersed submanifold. For simplicity, we will assume that ι is an embedding and identify $L \cong \iota(L) \subset M$. We proceed on this assumption just to make the notation easier. The following arguments hold true in the more general setting of a proper immersion. For a point $q \in M$ and a real number $d \in \mathbb{R}$ we set $P^d = \mathcal{P}(M, L\times q)^{d^2}$, respectively $P^{d-} = \mathcal{P}(M, L\times q)^{d^2}_-$ for short.

As a basic result in differential geometry, there exists for any point $p \in M$ an $\varepsilon > 0$ and a small open neighborhood W of p such that for every point $p' \in W$ the restricted exponential map $\exp_{p'}|_{B_\varepsilon(0)} : T_{p'}M \supset B_\varepsilon(0) \to M$ is a diffeomorphism onto its image with $W \subset \exp_{p'}(B_\varepsilon(0))$. In particular, every point p in M posses a neighborhood W which is geodesically convex, in the sense that any two points in W are joint by a unique minimal geodesic which has image totally contained in W. Further, recall that by the Hopf-Rinow theorem completeness of (M, g) is equivalent to the compactness of the closure of all distance balls $\overline{B_r(p)}$. Therefore, given an $d > d(L, q)$ we can choose an $r > 0$ such that the following statements hold:

1. For any point p of the compact set $\overline{B_d(q)}$ the open distance ball $B_r(p)$ is geodesically convex;
2. For the compact subset $K = L \cap \overline{B_d(q)}$ of L, the normal exponential map $\exp^\perp$ restricted to an open neighborhood of the normal r-tube defined by $B_r(0_K) = \left\{v \in T_pL^\perp | p \in K, \|v\| \leq r\right\}$ is a diffeomorphism onto its image.

After we have choosen such an $r > 0$ we fix a subdivision of the unit interval $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ such that $t_{i+1} - t_i < r^2/d^2$ for $i = 0, \ldots, k-1$. Now we define the subset $P^d(t_1, \ldots, t_k)$ of P^d as

$$\begin{aligned} P^d(t_1, \ldots, t_k) = \{c \in P^d | c|_{[t_i,t_{i+1}]} &\text{are minimal geodesics for } 1 \leq i \leq k, \\ c|_{[t_0,t_1]} &\text{is a minimal geodesic joining } L \text{ to } c(t_1)\}. \end{aligned}$$

The subset $P^{d-}(t_1,\dots,t_k)$ of P^{d-} is defined analogously. By definition, for $c \in P^d$, we have $d(c(t),q) \le L(c) \le d$, i.e. $c(t) \in \overline{B_d(q)}$ for all $t \in I$ and

$$d(c(t_i), c(t_{i+1})) \le L(c|_{[t_i,t_{i+1}]}) \le \sqrt{(t_{i+1}-t_i)E_q(c)} < r \text{ for } i = 1,\dots,k-1,$$

as well as $d(L, c(t_1)) < r$. Thus, there exist unique minimal geodesics joining $c(t_i)$ to $c(t_{i+1})$ and L to $c(t_1)$, which together define a point in $P^d(t_1,\dots,t_k)$.

Proposition A.2.1. *The subset $P^d(t_1,\dots,t_k)$ is a strong deformation retract of P^d. The deformation retraction $H : P^d \times [0,1] \to P^d$ is given by*

$$H(c,s)(t) = \begin{cases} \sigma_s^i(t), & t_i \le t \le t_i + s(t_{i+1}-t_i), \\ c(t), & t_i + s(t_{i+1}-t_i) \le t \le t_{i+1}, \end{cases}$$

where σ_s^i denotes the unique minimal geodesic from $c(t_i)$ to $c(t_i + s(t_{i+1}-t_i))$, from L to $c(t_0 + s(t_1 - t_0))$ when $i = 0$, respectively.

Proof. By the previous consideration H is well defined, i.e. $H_s(P^d) \subset P^d$ for all $s \in [0,1]$. Since H satisfies $H_0(c) = H(c,0) = c$, $H_1(P^d) \subset P^d(t_1,\dots,t_k)$ and fixes $P^d(t_1,\dots,t_k)$ pointwise, the only thing we need to show is continuity and this is straight forward to check. □

Having achieved this result as a first step, we will show that $P^{d-}(t_1,\dots,t_k)$ carries the structure of a smooth manifold of dimension $k\cdot n$, where n denotes the dimension of M. For this reason, consider the product M^k of k copies of M and the function $\bar{E}_q : M^k \to \mathbb{R}$, given by

$$\bar{E}_q(p_1,\dots,p_k) = \frac{d^2(p_1,L)}{t_1-t_0} + \sum_{i=1}^{k} \frac{d^2(p_i,p_{i+1})}{t_{i+1}-t_i},$$

where we set $q = p_{k+1}$. We define

$$\mathcal{M}_k^d = \left\{(p_1,\dots,p_k) \in M^k | \bar{E}_q(p_1,\dots,p_k) \le d^2\right\}$$

and

$$\mathcal{M}_k^{d-} = \mathcal{M}_k^d \setminus \bar{E}_q^{-1}(d^2).$$

The set $\mathcal{M}_k^{d-}$ is open in M^k and, by construction, $\bar{E}_q$ is a smooth function on $\mathcal{M}_k^{d-}$. Thus for almost all d, i.e. for all non-critical values, $\mathcal{M}_k^d$ is a submanifold of M^k with boundary $\bar{E}_q^{-1}(d^2)$.

Now we define a bijective map $\Phi : \mathcal{M}_k^d \to P^d(t_1,\dots,t_k)$ by assigning a tuple $(p_1,\dots,p_k) \in \mathcal{M}_k^d$ to the curve $\Phi(p_1,\dots,p_k)$ such that $\Phi(p_1,\dots,p_k)|_{[t_0,t_1]}$ is the minimal geodesic from L to p_1 and $\Phi(p_1,\dots,p_k)|_{[t_i,t_{i+1}]}$ is the unique minimal geodesic joining p_i to p_{i+1}.

Proposition A.2.2. *The space $P^{d-}(t_1, \ldots, t_k)$ carries a unique smooth structure such that the map Φ defined as above becomes a diffeomorphism and E_q is a proper C^∞ function on it.*

As stated in the introduction to this section we are intrested in studying the topology of the path space by means of Morse theory. The next lemma shows that it suffices to consider the corresponding space of polygons. For a proof we refer to [Sak96].

Lemma A.2.3. *With the notation introduced above we have*

1. *The tangent space of $P^{d-}(t_1, \ldots, t_k)$ at $c \in P^{d-}(t_1, \ldots, t_k)$ is given by*

$$T_c P^{d-}(t_1, \ldots, t_k) = \{Y \in T_c\mathcal{P}(M, L \times q) \big| Y|_{[t_0,t_1]} \textit{ is an } L\textit{-Jacobi field}, \; Y|_{[t_i,t_{i+1}]} \textit{ are Jacobi fields along } c|_{[t_i,t_{i+1}]}\}.$$

2. *$\gamma \in P^{d-}(t_1, \ldots, t_k)$ is a critical point of E_q if and only if γ is an $L \times q$-geodesic, i.e. γ starts from L perpendicularly and ends at q.*

3. *If γ is a critical point of E_q and $X, Y \in T_\gamma P^{d-}(t_1, \ldots, t_k)$, then*

$$\frac{1}{2} H(E)(\gamma)(X, Y) = \sum_{i=1}^{k} g(\nabla X(t_i^-) - \nabla X(t_i^+), Y(t_i)).$$

4. *At a critical point γ of E_q, the null space of $H(E_q)(\gamma)|_{(T_\gamma P^{d-}(t_1,\ldots,t_k))^2}$ coincides with the null space of $H(E_q)(\gamma)$ that is given by*

$$\{Y \in T_\gamma \mathcal{P}(M, L \times q) | Y \textit{ is an } L\textit{-Jacobi field along } \gamma \textit{ with } Y(1) = 0\}.$$

   Moreover, the index $\operatorname{ind}(\gamma)$ *of the critical point γ is equal to the index of $H(E_q)(\gamma)|_{(T_\gamma P^{d-}(t_1,\ldots,t_k))^2}$. In particular,* $\operatorname{ind}(\gamma)$ *is finite.*

Finally, we give a geometric interpretation of the index of the energy functional, the famous Morse index theorem, which can also be seen as a way to compute the index.

Theorem A.2.4. *Let M be a Riemannian manifold and let L be a submanifold of M. Let $\gamma : [0, 1] \to M$ be an $L \times q$-geodesic, i.e. a geodesic from L to q emanating perpendicularly from L. Denote by $\gamma(t_1), \ldots, \gamma(t_k)$ the (isolated) focal points of L along $\gamma|_{(0,1)}$, ordered by $0 < t_1 < \cdots < t_k < 1$. Let $\mu(t_i)$ be the multiplicity of $\gamma(t_i)$ as a focal point of L along γ. Then*

$$\operatorname{ind}(\gamma) = \sum_{i=1}^{k} \mu(t_i).$$

A proof of this theorem can be found in almost any text book about Riemannian geometry. The idea of the proof is the following. If $\gamma(t_i)$ is a L-focal point of multiplicity $\mu(t_i)$, then there exist $\mu(t_i)$ linearly independent L-Jacobi fields $Y_1, \ldots, Y_{\mu(t_i)}$

along $\gamma|_{[0,t_i]}$ with $Y_j(t_i) = 0$, which are the variational fields of variations through L-geodesics with length less than or equal to the length of $\gamma|_{[0,t_i]}$. If we extend the Y_j to broken Jacobi fields along γ by $Y_j|_{[t_i,1]} \equiv 0$, we get vector spaces $W(t_i)$, which we put together to obtain a vector space $W = \bigoplus_i W(t_i)$ of dimension $\sum_{i=1}^k \mu(t_i)$. After choosing d large and fixing a subdivision of $[0,1]$ that includes the t_i, we get an injective linear map $G : W \to T_\gamma P^{d-}(t_1, \ldots, t_k)$.

Let V_-, resp. V_+ be the direct sum of eigenspaces corresponding to negative, resp. non-negative eigenvalues of $H(E_q)(\gamma)|_{(T_\gamma P^{d-}(t_1,\ldots,t_k))^2}$. Denote by

$$p_- : T_\gamma P^{d-}(t_1, \ldots, t_k) = V_- \oplus V_+ \to V_-$$

the orthogonal projection. The observation that $H(E_q)(\gamma)(G(W(t_i)), G(W(t_j))) = 0$ together with the last lemma ensures that $p_- \circ G$ is injective. Because if not, we can write $G(Y) = X = X_- + X_+ \neq 0$ with $p_-(X) = 0$, i.e. $X = X_+$. Then we have $0 = H(E_q)(\gamma)(X, X) = H(E_q)(\gamma)(X_+, X_+) \geq 0$. Since $H(E_q)(\gamma)$ is positive semidefinite on V_+, it follows that $X = X_+$ belongs to the null space of $H(E_q)(\gamma)$ and is therefore a smooth L-Jacobi field along γ with $X(1) = 0$. But because of $X(1) = \nabla X(1) = 0$, we then have $X = 0$, what is a contradiction. The proof of the fact that $p_- \circ G$ is also surjective is somewhat more difficult but similar.

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