Endogenous growth, technical change and pollution control

Insights from a Schumpeterian growth model with productivity growth and green innovation

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Dipl.-Volksw. Kerstin Burghaus

aus Waldbröl

Referent: Prof. Dr. Peter Funk

Korreferent: Prof. Dr. Alexander Ludwig

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GDP	Gross Domestic Product
IES	Intertemporal Elasticity of Substitution
IPCC	Intergovernmental Panel on Climate Change
NHIM	Normally Hyperbolic Invariant Manifold
R&D	Research and Development

Chapter 1

Technical change in environmental growth models

1.1 Introduction

This thesis studies the relation between economic growth and the environment in an endogenous growth model with a negative pollution externality of production. The aim is to find answers to some of the most pressing questions regarding the future of today's economies: "How - if at all - can economic growth be decoupled from environmental degradation?", "Is persistent economic growth socially desirable if its impact on the environment is taken into account?" and "How costly is environmental conservation in terms of consumption and economic growth?" Particular focus in this respect is given to the role of endogenous technical change.

Adverse effects of economic activity on the environment are undeniable. Certainly, the most prominent example is climate change, as measured among other indicators by the development of temperature: In its fourth assessment report, the Intergovernmental Panel on Climate Change (IPCC) observed a trend in the change of global surface temperature of 0.74 during 1906-2005, with an almost twice as large average increase in 1956-2005. According to the IPCC "(m)ost of the observed increase in global average temperatures since the mid-20th century is very likely[1] due to the observed increase in anthropogenic (greenhouse gas) concentrations(...)".[2] As the main sources of greenhouse gas emissions, the IPCC identifies energy supply (25.9%), industry (19.4%) and transport (13.1%).[3]

[1] "Very likely" describes a probability greater than 90% (IPCC (2007), 'Climate Change 2007: Synthesis Report', p. 27).

[2] IPCC (2007), 'Climate Change 2007: Synthesis Report', p. 39.

[3] IPCC (2007), 'Climate Change 2007: Synthesis Report', p. 36.

In case of a further rise in temperature by only 2-3 degrees, ecosystem loss, increased risk of species extinction and major effects on human health and wellbeing due to, for example, food and water stress are predicted by the IPCC for the end of the 21st century already.

The severe predicted impacts of human-induced environmental degradation have led scientists to account for negative side effects of economic activity on the environment in models of economic growth, to deal with the questions raised in the beginning. A lively discussion on these issues has developed since the well-known work 'The limits to growth' by Meadows et al. (1972) in particular. But only in the 1990s, scientific debate began to point to the importance of accounting for technological development in environmental-economic models to meet the challenges arising from the growth-environment relation. Technological progress can help to decouple economic growth from pollution by reducing the pollution intensity of production inputs and processes, by developing cleaner substitutes for polluting inputs or by raising input productivity, thereby allowing to produce the same amount of output with less inputs.

The IPCC and also the Stern Review on the Economics of Climate Change express their belief in technical development to decouple economic growth from environmental damages[4]. However, the question whether decoupling is possible and at what cost is subject to controversy, even when technical change is taken into account.

First, the ecological benefits of technical change are not undisputed. While in principle, increased input productivity can reduce pollution as it allows to save on polluting inputs without giving up output, in practice it may mainly raise output or even stimulate the demand for polluting inputs. This so-called rebound effect[5] of technical progress is one reason for environmental activists like Greenpeace to believe that the world economy should give up economic growth and converge towards stationary levels of consumption and production.[6]

Second, environment-friendly development is often feared to imply large costs in terms of economic growth. First, controlling the rebound effect of productiv-

[4] See Stern (2007) and IPCC (2007).

[5] A formal definition can be given along the lines of Berkhout et al. (2000): A rebound effect denotes the percentage of potential input saving that is lost due to increased input use. A rebound effect of more than 100%, which implies a net increase in input use, is sometimes referred to as backfire.

[6] Convergence to a stationary economy as demanded by environmental activists usually goes beyond merely giving up long-run growth. Environmental activists believe the world economy to have surpassed sustainable levels of economic activity so that downsizing - 'degrowth' - is unavoidable. This belief is shared by a political movement of the same name, which has its origin in France ('décroissance'), see for example Ariès (2005) and Latouche (2004).

ity growth by saving on polluting inputs obviously comes at the cost of giving up potential consumption growth. Second, the larger the fraction of research directed towards reducing the pollution intensity of inputs, the less the fraction that may be directed to raise productivity. Even if these costs are not decisive from a welfare perspective, they lower political incentives to pursue environmental policy.

To come to reliable conclusions concerning the prospects of reconciling economic growth with a clean environment, the interactions of productivity-enhancing and pollution-reducing technical change have to be studied, taking into account the possibility to control the rebound effect of productivity growth through input saving.

The model which is developed and analyzed in this thesis, to our best knowledge, is the first to explicitly and analytically do so in a framework in which perfectly clean substitutes to the polluting input are not available, but the pollution intensity of the polluting input can be reduced through innovation. In the few models which consider endogenous reductions in pollution intensity through technical change, the possibility to control the rebound effect of productivity growth is disregarded.

This omittance is not trivial: The key result of this thesis is that long-run economic growth is socially desirable for a sufficiently patient representative household but both persistent reductions in the pollution intensity of intermediates and input saving characterize long-run optimal pollution control for reasonable parameter values. Neglecting the possibility to control the rebound effect of productivity growth weakens the prospects for persistent economic growth in the long-run optimal solution considerably.

The remaining sections of this first chapter provide the theoretical background for the analysis in the subsequent chapters. We first give a short introduction to the modelling of pollution and environmental quality in models of economic growth in section 1.2, explaining the main approaches of relating the environment to the production and the consumer side of the models. In section 1.3, we survey the embodiment of technical change in theoretical environmental-economic modelling. The aim is to single out the implications of endogenous technical change for the prospect of decoupling growth from environmental deterioration, the desirability of long-run growth and the cost of environmental preservation. Section 1.4 serves to explain the mechanisms to counteract pollution growth in the model which will be presented in chapter 2 and to clarify the terminology regarding technical change and pollution control. We conclude with a short outline of the remaining chapters of this thesis in section 1.5.

1.2 Key elements in environmental-economic modelling

Whether or not technical change is taken into account, the relationship between economic growth and the environment depends crucially on the specification of household preferences and the production-environment relation. An extensive treatment of the approaches and challenges of including the environment in economic modelling has been given by Pittel (2002) for endogenous growth models and by Xepapadeas (2005) for both neoclassical and endogenous growth models. In this section, we summarize the modelling aspects relevant for the analysis of the literature in section 1.3 and the model to be presented in chapter 2.

1.2.1 Preferences

In the models to be considered in section 1.3 below, households value environmental quality positively and suffer disutility from environmental damages due to pollution. Households' preference for a clean environment is usually assumed to be attributable to non-rival services of nature from clean air and water or the recreational value of a clean environment, for example. It is assumed that households do not internalize consequences of their consumption decision on the environment. If not stated otherwise, utility in the models of the next section is increasing and concave in consumption and environmental quality or - equivalently - increasing and concave in consumption and decreasing and concave in the pollution stock. The latter implies that a decrease in environmental quality due to an increase in pollution affects household-utility the more, the lower environmental quality is already (the more polluted the environment is). Thus the instantaneous utility-function is of the general form[7]

$$\begin{aligned} u &= u(c, E) \\ u_c &> 0 \quad u_{cc} < 0 \\ u_E &> 0 \quad u_{EE} < 0 \end{aligned} \tag{1.1}$$

[7]In this and the following sections, we adapt the notation in the literature to the notation in the model which is presented in chapter 2. Further, we omit the time index t, where it does not lead to confusion.

in terms of per-capita consumption c and environmental quality E or of the form

$$\begin{aligned} u &= u(c, S) \\ u_c &> 0 \quad u_{cc} < 0 \\ u_S &< 0 \quad u_{SS} < 0 \end{aligned} \tag{1.2}$$

in terms of per-capita consumption and pollution S. u_c, u_{cc}, u_S, u_{SS} denote the first- and second-order partial derivatives with respect to c and S respectively.

We use S to refer to the stock of pollution which is the more relevant variable for household utility, for example, in the context of air pollution and climate change.[8] Several models[9] consider the flow P of pollution for simplification. We come back to the difference between flow and stock representation below.

While there is much agreement on the signs of the partial derivatives, the sign of the cross-derivative between consumption on the one hand and environmental quality or pollution on the other hand varies in the literature. If utility is additively separable ($u_{cE} = u_{cS} = 0$), the environmental externality does not affect the household's consumption decision. If, on the other hand, the marginal utility the household derives from consumption increases in environmental quality or decreases in pollution ($u_{cE} > 0$, $u_{cS} < 0$), the desired consumption growth rate is, ceteris paribus, lower than in the additively separable case when the household expects higher pollution or lower environmental quality. A third specification is the opposite case, $u_{cE} < 0$, $u_{cS} > 0$, for which the desired growth rate is higher. This specification is however seldomly used. The standard assumption is that marginal utility is non-increasing in pollution. A comparison of the different specifications for the laissez-faire equilibrium and the optimal solution in an endogenous growth model can be found in Michel and Rotillon (1995).

Bovenberg and Smulders (1995), as well as Gradus and Smulders (1996) specify requirements for the functional form of the utility function so that long-run growth at constant rates is optimal. In particular, the elasticity of marginal utility in consumption must be constant and the intertemporal elasticity of substitution between consumption and the environmental variable must be unity. A commonly

[8] Health effects, for instance, are related to concentrations of air pollutions rather than short-lived emission flows.

[9] See, e.g., Bovenberg and Smulders (1995), Elbasha and Roe (1996), Hart (2004, 2007) and Ricci (2007).

used specification which satisfies the requirements is:

$$u(c,S)=\begin{cases}\frac{\sigma_c}{\sigma_c-1}c^{\frac{\sigma_c-1}{\sigma_c}}-\psi\phi(S), & \sigma\neq 1\\ \ln(c)-\psi\phi(S), & \sigma=1\end{cases} \tag{1.3}$$

$\psi > 0$ is a parameter reflecting the strength of the preference for a clean environment and σ_c is the intertemporal elasticity of substitution in consumption. The disutility of pollution, $\psi\phi$, is assumed to be increasing as well as convex in S ($\phi_S > 0$, $\phi_{SS} > 0$). Unless stated otherwise, this specification or a utility function with similar properties is chosen in the literature in section 1.3.

1.2.2 Economy-environment representation

The main concern of this thesis regarding the economy-environment relation is the polluting impact of production, which is most commonly represented by the by-product approach: For a given level of technology, the pollution flow (sometimes also referred to as the level of emissions) P is a function of gross domestic product (GDP) Y or some input X in the production process. For computational convenience, most models assume a linear specification

$$P=\kappa_Y Y \tag{1.4}$$

where κ_Y is the emission-output coefficient or, equivalently, the pollution intensity of output.

If it is a production input rather than total GDP which causes pollution, the pollution intensity of output is determined by the pollution intensity of the input and the input's share in GDP, as the following transformation shows:

$$\begin{aligned} P &= \kappa_X X \\ &= \kappa_X \frac{X}{Y} Y \end{aligned} \tag{1.5}$$

The relation between economic growth and the environment as well as the desirability of long-run growth depend crucially on the assumptions about the emission-output coefficient. In the early growth models without technical change, which we consider in the next subsection, κ_Y and κ_X are usually exogenously fixed[10] and the

[10] Some models assume that it can be lowered by abatement investment (e.g., Keeler, Spence and Zeckhauser (1970), Van der Ploeg and Withagen (1991)), or directly chosen (Stokey (1998)). The paper by Stokey is discussed in more detail below.

input ratio X/Y is constant at least along a balanced growth path. Environmental care may then have a level effect on the pollution path through a lower X/Y, but (in the absence of ex-post abatement measures) growing GDP necessarily leads to proportionally growing pollution in the long run. In models with technical change on the contrary, there is the possibility of 'decoupling' GDP- and pollution growth: Technical change can lower κ_Y and κ_X directly, or it can help to save on polluting inputs and reduce X/Y.

As indicated before, the most severe consequences of pollution for household wellbeing do not arise from quickly dissolving pollution flows but from an accumulating stock of pollution. The development of the pollution stock over time[11] is captured in the literature by the following function:

$$\dot{S} = P^{\text{net}}(A, P) - \Delta(S) \tag{1.6}$$

Of the models we review in the next section, particularly those without endogenous technical change in subsection 1.3.1 allow for active abatement activities A to remove part of the pollution stock or flow. P^{net} denotes the pollution flow net of abatement. Further, natural regeneration also cleans up a share of S in each period, which is reflected in the function $\Delta(S)$. For computational convenience, most models assume a linear regeneration function with constant regeneration rate δ[12].

The actual stock of environmental quality can be expressed as the difference between the level of environmental quality in a non-polluted, 'virgin' state and the stock of pollution:[13,14]

$$E = E^{\max} - S \geq 0$$

It follows that

$$\dot{E} = -\dot{S}. \tag{1.7}$$

A drawback of this approach is that it necessarily requires the pollution stock to be constant in the long-run because E can neither exceed the virgin state nor fall below zero. In the model to be presented in chapter 2, we prefer the specification

$$E = S^{-1},$$

[11] Henceforth, we denote time derivatives with a dot above the variable.
[12] A more complex specification can be found in Bovenberg and Smulders (1995, 1996).
[13] See also Pittel (2002), p. 37.
[14] Aghion and Howitt (1998) define as their indicator of environmental quality the difference $E - E^{\max} \leq 0$, because along a sustainable path with declining pollution stock, it converges to a finite upper bound, zero. If pollution is allowed to rise in the long run, this specification loses its appeal, as $E - E^{\max}$ is then unbounded below.

as it does not fix the movement of the pollution stock ex ante and therefore allows to study the dependence of pollution growth on household preferences.

Besides the by-product approach to model polluting production, an approach which is found in some environmental growth models (e.g. Bovenberg and Smulders (1995)) assumes the flow of pollution P to be an input in production. This specification of the production-pollution relation suggests that more intensely polluting production techniques are more productive or, to put it differently, that avoiding pollution decreases production, ceteris paribus.[15]
Pittel (2002, p. 30) and Ricci (2007) show that the specification with pollution as a production input is equivalent to the by-product approach when the emission coefficient κ_Y is variable and has a negative effect on productivity in output production.

The environment as a factor of production is found more often in models with natural resources. The use of natural resources also imposes pressure on the environment, particularly if resource stocks are non-renewable. A separate literature has developed to consider the problem of resource scarcity in the production process.[16]
We are interested in questions arising from the external effects of pollution rather than the scarcity problem and will therefore refer to the resource literature only as far as resource use generates pollution.

1.3 The importance of endogenous technical change

This section introduces to the modelling of endogenous technical change in relevant theoretical environmental-economic literature[17]. It serves to highlight the role of endogenous technical change for the possibility of decoupling economic growth from pollution, the desirability of long-run economic growth with environmental externalities and the cost of environmental preservation.
The first subsection shortly revises results from models which do not explain tech-

[15] There may also be positive feedback from the *stock* of environmental quality E or, equivalently, negative feedback from the pollution *stock* S on productivity in the consumption-goods sector. Pollution reduction then has an additional benefit besides the positive impact on household utility. Feedback effects are not decisive for the subsequent analysis and therefore neglected here.

[16] Neoclassical models have been analyzed for example by Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974).
Models with endogenous technical change include Barbier (1999), Scholz and Ziemes (1999), Groth and Schou (2002), Grimaud and Rouge (2003), chapter 5.3.2 in Aghion and Howitt (1998), Di Maria and Valente (2008) and Hassler, Krusell and Olovsson (2012).

[17] Surveys of technical change in applied models of growth and the environment can be found, e.g., in Grubb et al. (2002) and Löschel (2002).

nical change endogenously but neglect technological progress. Among the models considered are neoclassical growth models in the line of Ramsey (1928) - Cass (1965) - Koopmans (1965) as well as endogenous growth models with AK-technology (Rebelo (1991)) or learning-by-doing (Arrow (1962)).
The results from this literature are then compared to the findings from models with endogenous technical change. In recent years, many of the standard growth models with endogenous technical change, such as disaggregated models with expanding product variety (Romer (1990), Rivera-Batiz and Romer (1991)) and vertical product differentiation (Aghion and Howitt (1992), Grossman and Helpman (1991)), have been adapted for the analysis of economic growth with environmental externalities. Lately, scientific debate has been particularly interested in the determinants of the direction of technical change, both at equilibrium and from a normative perspective. This interest is based on the awareness that technological development can occur along distinct paths which differ in their effects on the environment and economic growth. We explicitly distinguish models where the direction of technical change is exogenously given from models where it is endogenous.[18]

1.3.1 Growth and the environment without technical change

Summary: *In the models revised in this subsection, long-run growth can only be optimal if a sufficiently productive abatement technology is in place.*
Economic growth is driven by the accumulation of physical capital. Pollution is generated by either physical capital or aggregate output. Because in all models, the capital-output ratio is constant at least in the long run, pollution asymptotically grows at the same rate as consumption and GDP in the absence of pollution control. Given the standard assumptions regarding the utility function explained in the previous section, persistent economic growth with unconstrained pollution growth may be the outcome of a market equilibrium but it cannot be optimal. Without technical change however, the only way to reconcile economic growth and environmental preservation is to invest a share of output or the capital stock in ex-post abatement measures. Unless the abatement technology is sufficiently productive, this share must increase over time so that investment in physical capital is crowded out and long-run growth comes to a halt.

Due to diminishing returns to capital, neoclassical models like the Solow-Swan and the Ramsey model do not feature economic growth in the long run. Cap-

[18] We complement the aforementioned earlier surveys by Pittel (2002) and Xepapadeas (2005), as the literature on endogenous direction of technical change is briefly addressed by Pittel only.

ital, production and consumption per capita converge to a constant steady-state.[19] Naturally, neoclassical models are therefore not suited to analyze linkages between economic growth and the environment in the long run.

The central conclusion from the transition in the Ramsey-Cass-Koopmans-framework is that abatement expenditures crowd out investment in physical capital and economic growth. Capital, consumption and output grow more slowly in the transition phase and are lower in the long-run steady-state of the socially optimal solution (*Keeler, Spence and Zeckhauser (1972)*, *Forster (1973), Gruver (1975)* and *Van der Ploeg and Withagen (1991)*). The result that pollution control entails crowding out of capital investment turns out to have some relevance also for the long-run analysis of one sector endogenous growth models as is shown below.

In neoclassical models in the line of Solow (1956)-Swan (1956), there is no optimizing behavior on the household side which further limits its applicability for the analysis of environmental problems. In particular, welfare effects of pollution cannot be accounted for and a thorough analysis of environmental policy is not possible. We therefore do not consider this type of model more extensively.[20]

Simple one-sector endogenous growth models like the AK-model (Rebelo (1991)) and models with learning-by-doing also do not explain technical change endogenously. Nevertheless, they are more suitable for studying the linkages between the environment and long-run economic growth. In the absence of an environmental component, non-decreasing returns to capital allow for persistent economic growth in these models. If positive and non-decreasing returns to capital persist even if possible feedbacks from the environment on the production function are accounted for, long-run growth remains feasible. However, given the standard assumptions that instantaneous utility is increasing and concave in consumption and environmental quality (or increasing and concave in consumption and decreasing and concave in pollution) and the marginal utility of consumption non-increasing in pollution, long-run economic growth is optimal only if pollution abatement is possible and the technology productive enough. To be more precise, the abating effect of capital or output must sufficiently exceed its polluting effect (*Michel and Rotillon (1995)*, *Gradus and Smulders (1993, 1996)*).[21] Otherwise, an ever increasing share of resources is re-

[19] If the neoclassical framework is extended to allow for technical change, technology advances at an exogenously given rate. The driving force of long-run economic growth remains unexplained, so that no reliable results on the desirability of long-run economic growth, the growth-environment relation and the cost of environmental preservation can be obtained.

[20] The literature based on neoclassical growth models including the Solow-Swan framework is revised in greater detail in Xepapadeas (2005).

[21] Michel and Rotillon (1995) show that without any possibility to counteract growing pollution,

quired for pollution abatement so that capital investment is crowded out completely in the long run.

A stronger preference for a clean environment (larger ψ in equation (1.3)) leads to stronger crowding out and less growth (Gradus and Smulders (1993)), as long as the abatement technology is not so productive that pollution declines if there is balanced growth of capital and abatement.[22]

One reason for the stark dependency of model results on an exogenous abatement technology is the inability of the models to generate persistent reductions in pollution intensity (i.e., the emission-output coefficient) without decreasing the social return to capital. *Stokey (1998)* assumes in an AK-model that the emission-output coefficient can be directly chosen. At any point in time, there is a continuum of emission-output ratios available and every value of the emission-output coefficient is associated with a different existing production technology. Cleaner technologies with lower emission-output coefficient are assumed to be less productive. Stokey does however not explain the development of new, cleaner technologies. While by lowering the emission-output coefficient, the development of pollution and output can in principle be decoupled, the results with respect to the desirability of long-run economic growth are similar to the aforementioned models: Persistent reductions in the emission-output-ratio persistently decrease the marginal product of capital so that long-run growth is no longer optimal. Pollution control crowds out capital investment and long-run growth.

sustained long-run growth is optimal only if the marginal utility of consumption rises with the pollution stock. A sufficient condition is that the utility from growing consumption outweighs the disutility of growth from higher pollution.

Mohtadi (1996) finds in an AK-model that long-run growth without abatement may be optimal even though the marginal utility of consumption falls with declining environmental quality, or equivalently, rising pollution stock. This result is however driven by special assumptions on the functional form for the environment-production relation: Environmental quality is a decreasing and concave function of the capital stock. This implies that even without abatement, the emission-output-ratio falls as the capital stock grows, while the ratio is constant in the model by Michel and Rotillon.

[22] The opposite case is studied by Gradus and Smulders (1996). A stronger environmental preference may then increase growth, if households are patient and the intertemporal elasticity of substitution in consumption is large. In this case, households are not interested in smoothing utility over time and do not react to anticipated gains from declining pollution by increasing current consumption.

1.3.2 Results from models with endogenous technical change

The central result of the previous section was that without technical change, long-run growth can only be optimal if abatement is possible and the abatement technology is sufficiently productive. Otherwise abatement investment leads to a complete crowding out of capital investment and growth.

Technical change, as mentioned in the introduction, can help to decouple economic growth and pollution growth in several ways, most directly by reducing the pollution intensity of intermediates but also by developing cleaner substitutes for polluting inputs and increasing factor productivity. On the other hand, productivity growth in particular may also have adverse effects on the environment through rebound effects on the demand for polluting inputs.
A careful modelling of technical change as driven by preferences and economic conditions is therefore essential to come to reliable conclusions concerning the desirability of long-run growth and the cost of environmental preservation.

This subsection revises results from models which explain the development of technology endogenously.

Exogenous direction

Summary: *In the following, models are considered which include an endogenous representation of technical change but do not allow for an endogenous choice between different directions of technical development.*
Technical change in these models occurs in the form of productivity growth. Higher productivity allows to use polluting production inputs more efficiently without giving up output, thereby also reducing the pollution intensity of GDP (Aghion and Howitt (1998), Bovenberg and Smulders (1995, 1996)). Additionally, several authors assume that through a positive spillover, productivity growth may directly reduce the pollution intensity of production for a given amount of inputs (Elbasha and Roe (1996), Koesler (2012), Ferreira-Lopes et al. (2012)).
Overall, the conclusions concerning the possibility to reconcile growth and environmental preservation and therefore the desirability of long-run growth are more optimistic than in the previous section. Although none of the models includes abatement, long-run growth is optimal under fairly standard conditions on model parameters.
Environmental care may even have a positive effect on the long-run optimal growth rate for two reasons: First, if productivity growth at the same time reduces the pollution intensity of production and it does so at a sufficiently large rate, faster growth

leads to improved environmental quality (Elbasha and Roe (1996)). This effect is similar to the one found by Gradus and Smulders (1996) for a particularly productive abatement technology. Second, even if technical progress does not break the trade-off between growth and environmental quality, the fact that research in itself is not polluting or less polluting than production may lead to a shift of labor resources from the production to the research sector.
By the same mechanism, environmental policy can stimulate research activity and growth (Grimaud (1999)), which yields a welfare gain if research is underprovided in the laissez-faire equilibrium.
A drawback of the models to be analyzed in this subsection is the exogenous direction of technical change. Whether and how productivity-enhancing technical change influences the pollution intensity of production is determined exogenously.
While technical change clearly has the potential to facilitate environmental conservation, the environment-friendly effects of technical change may be but are by no means always a costless by-product of productivity-improvements.
Some models (Elbasha and Roe (1996), Koesler (2012)) also allow for negative instead of positive external effects of productivity growth on the pollution intensity of production. Yet just as more productive inputs are not automatically cleaner, they need not be more polluting either. Productivity growth imposes pressure on the environment only indirectly, through the rebound effect described in the introduction. This effect is, however, already accounted for by the production- and pollution accumulation function.
The assumption of spillovers from productivity on pollution intensity therefore tends to overstate either the positive or the negative environmental effects of technical change and economic growth.

Aghion and Howitt (1998, chapter 5.3.1) adapt the specification of pollution as a by-product of production with endogenous pollution intensity in Stokey (1998) to a Schumpeterian model with vertical product differentiation and creative destruction. As was pointed out in section 1.2 and will be shown below, this approach can alternatively be understood as pollution being a production input.

The production function for the consumption good is given by

$$Y = L^{1-\alpha}\kappa_Y \int_{i=0}^{1} Q_i X_i^{\alpha} di. \tag{1.8}$$

and generates a flow of pollution according to the function

$$P = \kappa_Y^{\gamma} Y, \qquad \gamma > 0. \tag{1.9}$$

As in Stokey's paper, pollution intensity κ_Y is a control variable and production technologies with a lower pollution intensity are less productive. Q_i denotes the productivity of intermediate good X_i. The intermediate is produced from capital K according to the function $X_i = K_i/Q_i$. Its productivity Q_i can be improved through costly research and development (R&D). Success is stochastic with an innovation arriving at the Poisson-arrival rate μ for the individual researcher. Average quality $Q = \int_{i=0}^{1} Q_i di$ evolves according to the function

$$\dot{Q} = \mu n q Q,$$

with n being the mass of researchers active in the R&D-sector and q the exogenously given size of innovations.
With the AK-production function in Stokey's paper, long-run growth was not optimal, because continuous reductions in the pollution intensity of the consumption good led to a persistent decrease in the social return to capital.
Aghion and Howitt show that with endogenous productivity-enhancing technical change, long-run growth with declining pollution intensity (declining κ_Y) is feasible if productivity increases sufficiently fast to avoid a decline in the social marginal product of capital. If the pollution stock is allowed to rise in the long-run optimal solution, persistent growth is optimal given an upper bound on the rate of time preference as it is standard in endogenous growth models. Because Aghion and Howitt assume a threshold for environmental quality, the pollution stock must fall over time and long-run growth is only optimal if the intertemporal elasticity of substitution is smaller than one.[23] If the intertemporal elasticity is larger than one, the marginal utility of consumption does not fall fast enough with growing consumption to make households willing to sacrifice consumption growth for pollution control.

Grimaud (1999) decentralizes the model in Aghion and Howitt (1998) to study the channels by which a stricter environmental policy (in terms of slower growth of pollution permits) influences economic growth.[24] The associated faster growth

[23] There is an additional restriction on the parameter range which implicitly defines a lower bound for the rate of natural regeneration. This restriction follows from the authors' focus on balanced growth paths. It is not needed if unbalanced paths are allowed as well.

[24] Grimaud assumes horizontal instead of vertical product differentiation. The distinction between the approaches is not essential to the analysis in this section.

in the permit price affects economic growth negatively because firms choose a less polluting (and less productive) technology and because growth in intermediate goods slows down. But there is a positive effect as well: A stricter environmental policy decreases labor demand in the consumption goods sector relative to the R&D-sector and thereby shifts the allocation of labor to research. This effect is shown to be outweighed by the negative ones but it reduces the adverse impact of a tightening of environmental policy on growth.

As suggested earlier, substitution of (1.9) into (1.8) proves the approach of Aghion and Howitt to be equivalent to a specification of the form

$$Y = L^{(1-\alpha)\frac{\gamma}{1+\gamma}} P^{\frac{1}{1+\gamma}} \left(\int_{i=0}^{1} Q_i X_i^{\alpha} di \right)^{\frac{\gamma}{1+\gamma}}, \qquad (1.10)$$

where the flow of pollution is an input in production. Productivity growth allows to reduce the use of pollution in production without giving up output. This feature of technical change is considered explicitly in a disaggregated two-sector model with pollution-augmenting knowledge by *Bovenberg and Smulders (1995, 1996)*. Further, it is a prerequisite not only for the desirability of long-run growth but for persistent growth to be feasible in the first place in models where the polluting input is a non-renewable resource which is essential for production, as, e.g., in Schou (2002).

It is important to note that the effect of productivity growth on the environment depends on how it is used. While it allows to reduce the use of polluting inputs in production and thereby the pollution intensity of output, it increases the marginal product of polluting inputs at the same time, as can be seen in (1.10).

Productivity growth does not directly affect the cleanliness of the production process. Several authors link the pollution intensity of intermediate goods or the consumption good explicitly, but exogenously, to its productivity. Even though the drawback of this approach, as pointed out in the summary, is that the relation between productivity and pollution intensity is arbitrarily fixed, it can be seen as a first attempt to account for a second, explicitly environment-friendly, component of technical development along with productivity improvements.

Among the contributions assuming a direct effect of productivity on the pollution intensity of intermediates inputs is *Elbasha and Roe (1996)*. The authors allow for a positive or negative spillover. They set up a general equilibrium model of a small open economy, where technical change takes the form of expanding variety in intermediate goods as in Romer (1990). At time t, a consumption good[25]

$$Y = Q_Y K_Y^{\alpha_1} L_Y^{\alpha_2} D^{\alpha_3} \qquad Q_Y > 0,\, 0 < \alpha_j < 1 \ \sum \alpha_j = 1$$

is produced with a constant returns to scale technology, using capital and labor along with an index D of M differentiated inputs X_i:

$$D = \left(\int_0^M X_i^\theta di \right)^{1/\theta} \qquad 0 < \theta < 1$$

Q_Y is a parameter. Intermediate goods are produced from labor and capital both assumed to be in fixed supply so that the aggregate quantity $X = \int_0^M X_i di$ of intermediates is constant. But due to the concavity of D ($\theta < 1$), D and therefore Y increase if the aggregate quantity X is divided over a larger variety M of intermediates. The number of designs of intermediates M is assumed to be proportional to a stock of knowledge. New designs of intermediate goods are produced in the R&D-sector from physical capital and labor, building on the existing stock of knowledge:

$$\dot{M} = Q_M K_M^\beta L_M^{1-\beta} M \qquad Q_M > 0,\, 0 < \beta < 1$$

Q_M is constant. Firms in the intermediate goods sector must pay for their blueprints by buying a licence from the R&D-sector. Pollution is included in the Romer-model of Elbasha and Roe as a by-product of intermediate production. Environmental quality E is a flow variable which takes the form[26]

$$E = \left(\int_0^M X_i^\gamma di \right)^{-1/\gamma} \qquad 0 < \gamma. \tag{1.11}$$

In a growing economy, as the fixed aggregate quantity of intermediates is allocated over a larger variety M, environmental quality decreases if and only if γ is smaller than one, while it stays constant whenever γ is equal to one and increases whenever γ exceeds one. For $\gamma > 1$, there are decreasing returns to scale in pollution: The

[25] Elbasha and Roe assume that there are two consumption goods - one imported and one exported good - to analyze the effects of trade on the relationship between the environment and growth. As the distinction is only relevant for the analysis of trade-effects and trade is not in the focus of this review, we explain the mechanism for a single consumption good.

[26] The authors also discuss an alternative specification where environmental quality is inversely proportional to aggregate GDP. Environmental quality then decays at a constant rate along a long-run growth path, both in the laissez-faire equilibrium and the social optimum. Long-run growth is only optimal in such a setting under special assumptions about the utility function and the production-environment relation (compare footnote 21 on page 10).

pollution flow is smaller and the flow of environmental quality, E, larger if a given quantity X of intermediates is divided over more varieties. This increase in environmental quality for $\gamma > 1$ can be understood as a reduction in pollution intensity of aggregate quantity due to expanding product variety. On the other hand, if $\gamma < 1$, there are increasing returns to intermediate quantity in pollution and pollution intensity increases in the number of varieties. Environmental quality falls with technical development. If $\gamma = 1$, environmental quality only depends on the total amount of intermediates and is independent of technology.
Elbasha and Roe find that for $\gamma > 1$, stronger environmental care increases long-run economic growth in the optimal solution if the intertemporal elasticity of substitution in utility is not smaller than one. This is due to the fact that for $\gamma > 1$, growth goes along with an increase in environmental quality, because of the technology spillover. As long as households have no desire to smooth utility, it is optimal to forego current consumption to enjoy higher consumption levels along with a cleaner environment in the future. A similar result was obtained by Gradus and Smulders (1996) in case of a strongly productive abatement technology.
With $\gamma < 1$, results are ambiguous. Depending on parameters, growth may still rise in response to a stronger environmental preference but only because of the assumption that the beneficial effect of an increase in the consumption growth rate outweighs the negative effects of the associated faster pollution growth in utility.
In fact, the assumption $\gamma < 1$ appears more intuitive than $\gamma > 1$, given that a share of total emissions may accrue independent of a firm's production level, comparable to a fixed cost. A larger number of intermediate varieties and, accordingly, firms then generates more emissions than if the same quantity is produced by fewer firms. However both in- and decreasing returns are unusual assumptions, the common specification being constant returns to scale ($\gamma = 1$).

The presence of effects, in particular of positive effects, of increasing product variety on environmental quality would be more intuitive if new varieties were cleaner than the existing. But this fact is not captured in a Romer model. It can better be accounted for in a framework with vertical product differentiation.

There exist only a few such contributions, among them two models by *Koesler (2012)* and *Ferreira-Lopes et al. (2012)*. The authors assume that the productivity level can, through a spillover effect, lower the pollution intensity of production. In Koesler (2012), the pollution flow is given by

$$P = Q^{-\gamma} \sum_{i=1}^{N} X_i \qquad -\infty < \gamma < \infty$$

so that $Q^{-\gamma}$ can be interpreted as the pollution-intensity κ_X of aggregate intermediate quantity. Whether long-run growth can be reconciled with environmental preservation naturally depends on the strength of the spillover, i.e., the size of the parameter γ. If γ is negative, the pollution intensity of intermediate goods rises with their productivity. Pollution increases faster than aggregate intermediate quantity. If γ is positive but smaller than one, productivity growth decreases the pollution intensity, but pollution still rises in a growing economy. If γ exceeds one, the spillover of productivity is strong enough to completely decouple pollution growth and economic growth so that the pollution flow declines. Calibration in Fereirra-Lopes et al. suggests that underinvestment in research and development is more likely to occur in the market equilibrium as γ increases.
Instead of a spillover of productivity to pollution intensity, *Hart (2007)* assumes that each innovation increases productivity and at the same time contributes to a second technology stock which reduces the pollution intensity of production inputs. He also finds a growth-enhancing effect of environmental policy. An extended version of Hart (2007) where the research direction is endogenously chosen is presented in more detail in the next subsection.

Endogenous direction

Summary: *In the previous subsection, it has been pointed out that it is important to model the choice between different directions of technical progress endogenously. In this subsection, models are presented which include such an endogenous representation of technical progress.*
The relation between economic growth and the environment and the cost of environmental preservation in terms of economic growth are then also influenced by the connection between different research directions: Stronger preference for the environment or stricter environmental policy may attract resources to environment-friendly research but divert them from other research directions which may be more growth-oriented. On the other hand, environment-friendly research may increase overall research activity and growth through a positive spillover.
Two approaches to model the direction of technical change exist in the literature: The first approach assumes that research can increase the productivity in two different sectors, producing two types of goods - a 'clean' and a 'dirty' one (Hung et al. (1994), Grimaud and Rouge (2008), Acemoglu et al. (2012)). In the second approach, technical change either increases productivity and/or reduces the pollution intensity of a given production input (Verdier (1995), Hart (2004), Ricci (2007)).

The models in general are still optimistic about the desirability of long-run growth from a social planner's perspective. But in particular if a clean substitute for the polluting good does not exist, they do not find a positive effect of environmental care on optimal growth as easily as the models in the previous subsection.
A drawback of the approach with a clean and a dirty production input is that it relies exclusively on input saving through factor substitution. This leads to pessimistic conclusions concerning long-run growth if substitution possibilities are limited. The second approach on the other hand neglects the possibility to save on polluting inputs. Productivity growth therefore always has a strong rebound effect, which increases polluting quantity one for one. The social benefit of long-run growth may therefore be underestimated.

Models with two distinct types of inputs - a 'clean' and a 'dirty' good have been favored in the literature to endogenize the direction of technical change. The pollution flow in these models is proportional to the dirty input, while the clean input is not polluting at all.

The production function for the final consumption good is of the form

$$Y = (\beta Y_C^{\frac{\varepsilon-1}{\varepsilon}} + (1-\beta) Y_D^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}} \qquad \beta \epsilon (0,1);\ 0 < \varepsilon < \infty,$$

where Y_C denotes the clean input, Y_D is the dirty input, ε is the elasticity of substitution between the two inputs and β is a weight parameter. Y_C and Y_D are produced in two different production sectors, using a composite of primary inputs (labor, capital and - for Y_D - possibly resources). The direction of technical change then refers to whether innovation increases productivity in the clean or the dirty production sector. The pollution intensity of the dirty input is exogenously fixed.

Models with a two-sector structure in research and production have been used primarily to study the impact of policy on the direction of technical change ('directed technical change') and the cost of environmental policy, building on Acemoglu (2002)[27].

A recent contribution is *Acemoglu, Aghion, Bursztyn and Hemous (2012)*[28]. The authors endogenize the direction of technical change in a discrete time two sector

[27] This model in turn draws on previous work by Kiley (1999) and Acemoglu and Zilibotti (2001).

[28] The approach in Acemoglu (2002) has primarily been adopted in models where the second production input Y_D is or is produced by a resource (Smulders and de Nooij (2003), Di Maria and Smulders (2004), Di Maria and Valente (2008), Grimaud and Rouge (2008) and Hassler et al. (2012)). However, with two exceptions to be cited below, these models focus exclusively on the issues arising from resource scarcity and neglect pollution and the amenity value of the environment.

model with vertical product differentiation and creative destruction. The clean and the dirty input respectively are produced with a Cobb-Douglas technology:

$$Y_s = L_{Ys}^{1-\alpha} \int_0^1 Q_{is}^{1-\alpha} X_{is}^{\alpha} di \qquad s = C,\ D;\ 0 < \alpha < 1$$

Researchers can choose between improving the productivity of intermediates in the clean or the dirty sector. There is thus a trade-off between the two research directions. Innovations for an individual researcher arrive at the exogenous rate μ_C in the clean and μ_D in the dirty sector in each period. The innovation size q is exogenously fixed and the same for both sectors. Further, research labor supply is given so that the aggregate rate of technical progress is exogenous. Sectorial productivity evolves according to the difference equation

$$Q_{st} = (1 + \mu_{st} q n_{st}) Q_{st-1},$$

where n_{st} denotes the mass of researchers in the clean and the dirty sector respectively.

Environmental quality decreases proportional to the use of the dirty input Y_D and regenerates at a positive constant rate according to (1.7) and (1.6). If the pollution stock is sufficiently large, $E_t = 0$ for all subsequent periods which is entitled an environmental disaster. In this case household utility converges to $(-\infty)$ irrespective of the consumption level.

Because the arrival rate for innovations is state-dependent, balanced growth with advances in both the dirty and the clean technology is only a knife-edge-solution. The authors assume that initially the technology in the dirty sector is sufficiently advanced relative to that in the clean sector and therefore more profitable so that innovation starts in the dirty sector.

The laissez-faire equilibrium then leads to an environmental disaster in finite time. In the optimal solution, the environmental disaster has to be avoided. Still, for a sufficiently low rate of time preference, long-run growth is optimal as long as the two inputs are substitutes ($\varepsilon > 1$). For complementary inputs, an environmental disaster can only be prevented if economic growth is given up completely.

In the case of substitutable inputs, a tax on the dirty input combined with a subsidy to clean research can reduce production in the dirty sector and make innovation in the clean sector profitable. The most notable effect of endogenous directed technical change occurs when the inputs are strong substitutes ($\varepsilon \geq 1 + \frac{\alpha}{1-\alpha}$). Policy meas-

ures may then be temporary. As the clean sector catches up and becomes the more advanced sector over time, labor is shifted to the production of the clean input. Although technical change has a rebound effect on the polluting input through an increase in the relative price, Y_D stops growing. If, on the other hand, the inputs are weak substitutes, the pollution tax has to be in place permanently. While temporary intervention is enough to redirect research activity in this case as well, the rebound effect of technology outweighs the effect of the labor-shift and encourages production of Y_D.
In the long run, the clean technology may lead to equally fast or faster growth than the dirty technology (depending on the assumption about μ_s). In a transitional period however, a switch to clean technologies lowers growth.

Similar models with a polluting production input have also been studied by Hung, Chang and Blackburn (1994) and Grimaud and Rouge (2008).

The focus in *Grimaud and Rouge (2008)* is on environmental policy, like in Acemoglu et al. (2012). The authors develop a two sector model, where Y_C is produced from a clean labor resource and Y_D from a non-renewable natural resource. Flow pollution is proportional to the use of the natural resource and decreases environmental quality.
The existence of a finite resource stock implies that resource use must ultimately decrease over time, both in the laissez-faire equilibrium and in the social optimum. Nevertheless, persistent economic growth is both feasible and optimal, because increasing productivity overcomes resource scarcity.

At the same time, as extraction flows come to a halt, so does the deterioration of environmental quality. The environmental externality vanishes asymptotically. While this implies that no environmental policy is needed in the long run, the authors show that in the short and medium term, the resource is extracted too rapidly so that pollution growth is too fast, while output growth is too slow along the equilibrium-path compared to the optimal solution. Further, research effort in labor-augmenting 'green' research is too low and technical change is too much oriented towards the non-renewable resource. This remains true even if the knowledge externality in the green research sector is internalized by an appropriate subsidy. A shift in the tax-profile on the dirty resource-input towards the early periods, which is called stricter environmental policy, can delay resource extraction, divert labor from resource-augmenting to green research and thereby increase output growth.

Hung, Chang and Blackburn (1994) compare the relation between economic growth, the environment and welfare in the laissez-faire equilibrium, the first-best

solution and a second-best solution, where all externalities but the negative external effect of pollution on utility are internalized.
Technical progress has the form of increasing variety in the intermediates used for the production of Y_C and Y_D respectively. The difference to the Romer-type models in the previous section is that there are two separate stocks of designs, M_C and M_D. The authors assume perfect substitution between Y_C and Y_D in production ($\varepsilon \to \infty$). The utility-function is specified to be additively-separable and logarithmic in both consumption and pollution so that the standard assumption of convex disutility of pollution is given up. The adverse effect of pollution growth on utility is therefore weaker than if the disutility increases in the pollution stock. Under these assumptions, the direction of technical change is determined by relative costs in research and production of the inputs and relative productivity (as measured by the ratio of knowledge stocks) of the two research sectors irrespective of households' preferences even in the optimal solution.
Hung et al. find that compared to the laissez-faire solution, long-run growth in the social optimum may be higher. Compared to the optimal solution when the environment does not affect utility however, it is lower so that environmental care depresses growth. A major weakness of the model are the strict assumptions on preferences and the production function which limit the influence of preferences on the choice of technology.

In the context of the two-sector approach as chosen by Acemoglu et al. (2012), Grimaud and Rouge (2008) and Hung et al. (1994), it is important to note that although innovations in the clean sector are often referred to as environment-friendly or 'green', they are not inherently so. As indicated in the previous section, the effect of productivity growth on the environment depends on how it is used. The entitlement of increasing productivity in the clean sector as green innovation can only be reasonable at all if higher productivity encourages substitution towards the clean input. But whether it does depends on how higher productivity in the clean sector affects the *relative* marginal product of the clean input, which in turn hinges on the elasticity of substitution between the clean and the dirty input (see also *Di Maria and Smulders (2004)*): Technical change in the clean sector raises the marginal product of the clean input relative to that of the dirty input (it is biased towards the clean input), if the inputs are gross substitutes. In this case, technical change also increases the relative cost-share of the clean input. It is then called pollution-saving. Technical change in the clean sector increases the relative cost share of the dirty input (is pollution-using) if inputs are gross complements.

Acemoglu et al. (2012) and Hung et al. (1994) base their key results on the assumption of the inputs being substitutes. In this case, technical progress in the clean sector is indeed pollution-saving and can thus be entitled environment-friendly. Grimaud and Rouge refer to labor-augmenting technical change as 'green' even though the labor- and the resource input may be complements and labor-augmenting technical change in fact increase the relative marginal product of the dirty input.

While making the models more easily analytically tractable, a shortcoming of the two-sector approach is that the pollution intensity of the dirty good is exogenously fixed. Because of this assumption, the final good can become cleaner through input substitution towards the clean input but the dirty input always remains dirty. While the approach allows for optimistic conclusions concerning the optimal relation between long-run growth and the environment if inputs are easily substitutable, the exogenously fixed pollution intensity impedes persistent long-run growth if the inputs are complements.

The endogenous choice between decreasing pollution intensity and increasing productivity has been given comparatively little attention in the literature so far, the most known exceptions being Verdier (1995), Hart (2004) and Ricci (2007).

Verdier (1995) studies environmental policy in a model with expanding product variety as in Grossman and Helpman (1991). He assumes that the quantity X_i of each variety $i = 1...M(t)$ of the differentiated consumption good generates a pollution flow $P_i = \kappa_{Xi} X_i$, similar to equation (1.5), where k_{Xi} is the emission-output ratio of variety i. While engaging in R&D-activities to develop a new variety, firms can endogenously choose κ_{Xi}. Contrary to the models by Stokey (1998) and Aghion and Howitt (1998), κ_{Xi} only changes whenever there is an innovation. Different from Koesler (2012) and Ferreira-Lopes et al. (2012), reductions in the emission-output ratio are not a by-product of productivity growth but require an additional research effort: To develop a new variety with an emission-output ratio of κ_X, an amount

$$l_R(\kappa_X, M(t)) = \frac{l_R(\kappa_X)}{M(t)}$$

of R&D-labor is required. The labor-requirement is decreasing and convex in κ_X and falls with the number $M(t)$ of products developed at time t. There is thus a positive externality from the accumulated stock of knowledge.

Under laissez-faire, the largest possible emission-output ratio is chosen as the benefits in terms of lower pollution are external to the firm. Environmental policy in form of a tax on emissions has a growth-depressing effect because it induces in-

vestment in a lower emission-output coefficient which increases R&D-labor costs. On the other hand, a tax has the countervailing positive effect already described in Grimaud (1999) as it reduces demand in the production sector and frees labor for research. If the labor requirement $l_R(\kappa_X)$ does not rise too elastically with declining pollution intensity and the tax is not too large, the latter effect outweighs the former and environmental policy increases long-run growth.
The quantity of consumption good varieties is constant in the long-run because the only factor of production, labor, is in fixed supply. The development of pollution therefore depends solely on the development of the emission-output coefficient. In Verdier's model, κ_X is also constant along the balanced growth path so that pollution does not change over time. The model cannot generate persistent reductions in pollution intensity. A more realistic representation of developing cleaner technologies would take into account a second stock of knowledge or technology and assume that environmental innovations are the outcome of an R&D-process just like productivity improvements. Among the very few contributions considering economic growth and pollution in a setting with two different technology stocks are Hart (2004) and Ricci (2007).

Hart (2004) develops a model with production vintages. At time t, there exists a number of designs for intermediate goods discovered in distinct periods i[29]. Intermediate X_i is based on a design discovered in period i, X_t is the intermediate based on the most recent design. Eventually, older designs are not used anymore because of fixed costs in intermediate good production. Designs differ not only in their productivity Q_i in production but also in their cleanliness B_i. Higher cleanliness reduces the emission-output coefficient $\kappa_{Xi} = \frac{1}{B_i}$ of the intermediate X_i. Different from Verdier (1995), B_i is a stock variable which may increase over time.
If there are v vintages in use in period t, production is

$$Y_t = \sum_{i=t-(v-1)}^{t} Q_i x_{t-i}^{\alpha} \quad \alpha < 1,$$

and it generates a pollution flow

$$P_t = \sum_{i=t-(v-1)}^{t} \frac{1}{B_i} Q_i x_{t-i}^{\alpha}.$$

[29] Contrary to Hart (2004), we distinguish vintages by the time of their discovery, not by their age relative to the most recent vintage.

Hart assumes that skilled workers can only choose between research directions, while for either direction, the impact of an innovation on the two technology stocks Q_i and B_i is exogenously given: Productivity-oriented research only increases productivity so that pollution rises at the same rate as Y_t, while environmental research increases cleanliness and productivity at the same rate so that pollution is constant. Different from Verdier, environmental research does not lead to higher costs but to a slower productivity improvement than productivity-oriented research. The change in technology following the discovery of a new design is formally given by

$$\begin{aligned} Q_{t+1} &= (1+q_Q)Q_t \qquad B_{t+1} = B_t \\ Q_{t+1} &= (1+q_B)Q_t \qquad B_{t+1} = (1+b_B)B_t \end{aligned}$$

where $q_B = b_B$ and $q_Q > q_B$. Thus the decision problem in the R&D-sector is discrete rather than continuous as there are only two alternatives. Success is stochastic with an aggregate arrival rate which is the same for both kinds of innovation.
As both in the market and from a social planner's perspective the costs and benefits of either research direction for a single researcher are independent from other researchers' decisions, the decision problem always has a corner solution where all researchers choose the same type of research. Environmental research is not chosen under laissez-faire, because it gives a lower productivity boost.

Hart derives the optimal solution and then assumes the existence of vintage-specific sales taxes on all but the newest vintage, chosen so as to implement the optimal allocation of labor across vintages. These taxes boost the market-dominance of the newest, cleanest vintage. Hart shows that through this effect, such taxes may not only divert labor from dirty to clean research but also increase total research effort. If this increase is large enough, growth accelerates despite the countervailing negative effect of a smaller innovation size ($q_B < q_Q$). From a welfare perspective, this result is of course only relevant if clean research is socially preferred. It is important to note that the positive effect of the tax on growth only arises if vintages differ in pollution intensity.[30]
A major drawback of the model in Hart (2004) is the very limited two-point technology set. *Ricci (2007)* proves that the restriction of firms' technology choice is crucial for the positive effect of environmental policy on economic growth in Hart's model.

[30] In this way, the mechanism at work differs from that in the papers by, for example, Verdier (1995) and Grimaud (1999), where the tax may increase research activity because it reduces factor-demand from the production sector. This difference is also pointed out in a similar paper by Ricci (2007), considered below and in Hart (2007).

Ricci studies the growth effects of environmental policy in a framework with an almost unconstrained technological menu, omitting welfare-analysis and negative external effects of pollution on household utility in particular.
The source of pollution in Ricci's model is physical capital:

$$P = \int_0^1 P_i di = \int_0^1 \kappa_i^{1/\alpha\gamma} K_i di \qquad \alpha, \gamma \in (0,1)$$

Each innovation in any of the continuum of sectors $i\epsilon[0,1]$ reduces the pollution-intensity $\underline{\kappa}_t$ of the leading-edge technology and increases its productivity $\overline{Q}_t$.

$$\begin{aligned} \dot{\overline{Q}}/\overline{Q} &= \mu n q \quad q > 0 \\ \dot{\underline{\kappa}}/\underline{\kappa} &= \zeta \mu n \quad \zeta \leq 0 \end{aligned}$$

μ is the individual arrival rate for innovations, n the economy-wide mass of research labor. The step-size q for productivity improvements is a predetermined constant, while the choice of ζ is endogenized in the last section of the paper.
Similar to Stokey (1998), Ricci assumes that a trade-off between environment-friendly innovations and output growth exists because the marginal product of intermediates in final goods production depends positively on the emission-capital-ratio. Ricci replicates Hart's result concerning a potentially positive effect of environmental policy on growth if the step-size ζ of environment-friendly innovations is exogenously fixed like in Hart's model. However, if the size of environmental innovations is endogenously determined, the positive effect of the increase in R&D-employment on total productivity growth is outweighed in numerical simulations by the negative effect of a larger step-size in environmental innovations on productivity in the consumption good sector. The divergent result found by Hart (2004) arises because the exogenous step-size of innovations sets an upper-bound to the direct negative effect of a switch in R&D-direction. This upper bound no longer applies with endogenous step-size.[31]

1.3.3 Conclusion

Technical change opens new opportunities to decouple economic growth from pollution and environmental degradation and it may reduce the cost of environmental conservation in terms of growth, output and welfare. From the models considered

[31] See Ricci (2007), p. 304.

in this section, it becomes apparent, however, that whether it indeed relaxes the trade-off between economic growth and a clean environment depends on a number of other factors. Notably, the influence of technical change on the demand for polluting inputs and whether research in environment-friendly technical change diverts resources from other - potentially more growth-stimulating - research activities is of great importance. In models where positive impacts of technical change on the environment arise as a side-effect of productivity-oriented innovation, a positive relation between growth and environmental care and a beneficial effect of technical change on the costs of environmental policy is more likely to be found than in the literature with endogenous direction where the environmental benefits of technical change cannot be obtained without costly investment.

As we have shown, most existing models with endogenous direction of technical change rely on input-substitution and input-saving technical change but neglect reductions in the pollution-intensity of inputs. Models which generate a persistently falling pollution-intensity on the other hand do not take input-savings into account. The model to be presented in chapter 2 considers reductions in pollution intensity through green innovation, while allowing for input saving to reduce the rebound effect of productivity growth. We prove that input saving and reductions in the pollution intensity of inputs simultaneously characterize long-run optimal pollution control. Neglecting either the possibility to make intermediate goods cleaner or the possibility to control the rebound effect of productivity growth leads to a slow down or even a complete halt in long-run growth in the optimal solution.

1.4 Two ways of pollution control

The preceding section highlighted the importance of technical change for the growth-environment relationship but also showed that the specification of technical change and the ways it influences growth and the environment are diverse. The aim of this section is threefold: First, we explain how technical change affects the growth-environment relation in our model and define the two ways to achieve pollution control and decoupling: green innovation and deceleration. Second, we relate and contrast our approach to the literature. Third, we clarify the terminology regarding technical change and pollution control.

1.4.1 Green Innovation

In the environmental-economic literature, the term 'green innovation' refers to different types of technological development. Our definition of green innovation is in line with Verdier (1995), Hart (2004) and Ricci (2007). We define as green innovation such innovation which is aimed at reducing the pollution-intensity of polluting inputs, κ_X (or, if total output were polluting, κ_Y): Emissions from fossil fuels can be reduced by improving filters in the combustion process. Cars become cleaner through the build-in of better catalytic converters. Green innovation in this sense directly dampens the negative effect of economic activity on the environment.

As we have indicated before, there is also a different notion of the term 'green innovation'. Increases in factor productivity can help to economize on the use of polluting inputs by increasing the relative profitability of cleaner alternatives so that the amount of pollution generated per unit of GDP falls. This kind of green innovation is represented by the productivity increase in the clean sector in the models by Hung, Chang and Blackburn (1994) and Acemoglu et al. (2012). The most prominent real-world example is probably research and development to increase the profitability of comparatively clean energy sources like solar or wind power which substitute for polluting fossil-fuels.

But, as pointed out in the previous section, this type of innovation is not inherently clean. Its effect on the environment is only indirect and depends on whether productivity growth is indeed used to reduce the use of the dirty input. Further, as indicated by Di Maria and Smulders (2004), the particular assumptions about the production function determine to what extent increasing productivity in the clean sector even encourages input-saving in the dirty sector. In general, we therefore do not speak of increases in factor productivity as green innovation, although they have the potential to be pollution-saving. In our model, factor productivity always raises the marginal product of the polluting production input and should therefore not be referred to as 'green', even according to the definition by Di Maria and Smulders.

1.4.2 Deceleration and Quantity Degrowth

Although we do not refer to productivity improvements as green, they can nevertheless help to decouple output growth and pollution growth in the optimal solution of the model which is presented in the following chapter. A prerequisite is that the rebound effect of productivity growth is restricted and higher productivity is used to save on polluting inputs rather than merely to produce more output. Reducing the ratio of intermediate inputs to GDP implies that output and consumption grow at a rate smaller than the one which could maximally be achieved given productivity growth. Potential consumption growth is thus given up. Because of its growth effect, we call this particular way of input-saving 'deceleration'. A strong form of deceleration occurs when polluting quantity decreases in absolute terms and not only per efficiency unit. We then speak of 'quantity degrowth'[32].

Quantity degrowth necessarily occurs in models where the polluting input is a non-renewable resource, if there is no substitute to the resource input, because resource use must ultimately decline as the stock gets exhausted (see Schou (2002), Grimaud and Rouge (2008)).
In the model to be presented in chapter 2, the input is an accumulative production factor and still, it may be optimal (but not an equilibrium) to let the share of intermediates in production or even their absolute quantity decline to zero. Deceleration and degrowth are not enforced by input scarcity but result from the maximization of household utility by the social planner, given household preferences and the production technology.

1.5 Outline of the remaining chapters

Chapter 2 presents the model underlying the remaining chapters of this thesis and derives the long-run laissez-faire equilibrium as well as the long-run social optimum. A Schumpeterian endogenous growth model with vertical product differentiation is extended by an environmental component by assuming that inputs are polluting and differentiated not only in their productivity but also in their pollution intensity, or cleanliness. Both productivity and cleanliness can be increased endogenously and independent of each other through costly research and development. There are no completely clean substitutes to the polluting input. Pollution growth can be controlled either by reducing the pollution intensity of a given quantity of intermediate goods by green innovation or by controlling the rebound effect of productivity growth through deceleration.

Under laissez-faire, however, neither green innovation nor deceleration is chosen at equilibrium. Productivity growth has a strong rebound-effect on polluting quantity which increases one for one in the long run so that the ratio of polluting inputs relative to GDP remains constant. Compared to this path of unconstrained pollu-

[32] If we interpret intermediate quantity as material used, quantity degrowth (and, in a weaker sense, deceleration) corresponds to what is sometimes called "**dematerialization**".

tion growth, convergence to a stationary economy would be socially preferable.
Even so, for sufficiently patient households, convergence to a stationary economy is not optimal because pollution growth can be controlled: Sustained economic growth in the optimal solution always goes along with persistent green innovation to reduce the pollution intensity of intermediates. This result is driven by the existence of fixed costs in each individual research unit. Once a research unit is opened up and the fixed costs are paid, making intermediates at least marginally cleaner while making them more productive generates almost no additional cost. If production is very elastic with respect to polluting inputs, the social planner relies exclusively on green innovation to control pollution. He keeps the share of polluting inputs in GDP constant (as under laissez-faire) to generate fast consumption growth. For reasonable parameter values, production is inelastic with respect to intermediate quantity compared to productivity. Deceleration then allows to gain from productivity growth in a relatively clean way without incurring a large loss in consumption growth. At the same time, the relative social return to green innovation is comparatively small. It is therefore optimal to choose deceleration along with reductions in the pollution intensity of intermediates.

It has been pointed out that there is controversy in the literature as to whether or not environmental care leads to slower economic growth. We show that in our model, economic growth in the long-run optimum may be faster if the representative household cares for a clean environment than when there is no such environmental externality. Similar to what is found by Hart (2004) and Ricci (2007), the positive effect of environmental care on growth in our model is driven by green innovation attracting labor to the R&D-sector, which accelerates productivity growth. The strength of the household's preference for a clean environment, given that an externality of pollution on household utility exists, does not affect long-run growth.

The restriction of the analysis in chapter 2 to a long-run perspective can only be justified if the solution of the model does indeed converge to the long-run outcome. Because our model does not include physical capital and pollution accumulation is not internalized in the market equilibrium, economic variables in the laissez-faire solution grow at their balanced growth rates without transitional dynamics even though pollution growth converges over time.
For the social optimum, stability is not as evident. Local stability properties and transitional dynamics of the long-run optimal solution are studied in **chapter 3**. The model generates a complex non-linear dynamic system in six dimensions which cannot be solved analytically. We therefore resort to numerical analysis for a large

number of parameterized examples, using the relaxation algorithm by Trimborn, Koch and Steger (2007). Results suggest that in a reasonable parameter range, there exists an optimal transition path leading towards the long-run optimal solution derived in chapter 2 for any initial state close to the long-run optimum. The focus on a long-run perspective in chapter 2 is therefore justified.

Analysis of the transitional dynamics for a benchmark parametrization confirms that green innovation and deceleration characterize the optimal solution not only in the long-run but throughout the transition as well. Further, it is apparent that the initial technology endowment of an economy is crucial for its consumption- and pollution path as well as welfare: Economies with an initially more advanced technology enjoy higher consumption levels and a cleaner environment in every period and therefore higher intertemporal welfare.

A central result of chapter 2 is that for reasonable parameter values, both green innovation to reduce the pollution intensity of polluting inputs and deceleration to dampen the rebound effect of productivity growth are chosen in the optimal solution. In **chapter 4,** we illustrate the importance of supplementing green innovation by deceleration to control pollution growth by demonstrating the consequences for long-run growth if either of the two channels for pollution control, green innovation or deceleration, is not available to the social planner.
First, we consider a constrained long-run optimum without deceleration. We impose the condition that in the long-run optimal solution, there has to be balanced growth of productivity and intermediate quantity. For parameter constellations such that the unconstrained long-run optimal solution was characterized by deceleration, there is no long-run growth in the constrained solution.
Next, we reintroduce deceleration but rule out green innovation, so that the pollution intensity of intermediates is exogenously fixed. We show that in this case, long-run consumption growth is optimal for a smaller parameter range than in the unconstrained optimum. The social planner chooses deceleration for all parameter constellations which support long-run growth. Deceleration is faster than in the unconstrained optimum while long-run growth in consumption and GDP is slower without green innovation. Further, environmental care unambiguously lowers the long-run optimal growth rates of consumption and GDP compared to the optimal solution when households do not care for the environment.

In **chapter 5**, we study a variation of the baseline model from chapter 2 in which the intermediate goods sector depends on a non-renewable resource as the only production input. As mentioned in the introduction, according to the IPCC, one of the major sources of pollution is energy supply, and the generation of energy relies heavily on non-renewable resources like fossil fuels. To reasonably interpret the polluting intermediate input in our model as energy, it is therefore important to account for the exhaustibility of energy-resources along with the pollution externality. Chapter 5 examines the robustness of the main results from the baseline model with respect to the inclusion of a non-renewable resource.

Because the resource stock is finite, resource-use and thereby the use of intermediate inputs and polluting emissions must converge to zero asymptotically, both in the long-run optimum and in the unregulated market-equilibrium: There must be quantity degrowth.

The need to reduce resource use over time leads to slower long-run consumption growth but also a declining pollution stock in the laissez-faire solution. The size of the initial resource stock, on the other hand, does not affect long-run growth rates. However, the entire paths of intermediate production, output, consumption and pollution are lower for less resource-abundant economies.

The literature on polluting non-renewable resources (Schou (2000, 2002), Grimaud and Rouge (2008)) usually assumes that the resource constraint is binding in the socially optimal solution as well. In the case of a binding natural resource constraint, we find that resource-scarcity brings about such a rapid decline in the pollution stock that green innovation is superfluous in the long run. The social planner shifts resources to the R&D-sector to spur productivity growth. The environmental externality does not influence the long-run optimal path.

Contrary to the literature, we find that, given a sufficiently large initial resource stock, the socially optimal solution with an exhaustible resource corresponds to the solution in the baseline model for reasonable parameter constellations. To be more precise, the natural resource constraint is not binding if the environmental externality requires a steep decline in the pollution stock and the factor share of intermediate goods is particularly small. In this case, the long-run optimum in the model of chapter 2 is characterized by quantity degrowth even without a constraint on resource use and therefore intermediate production. The total amount of intermediate goods used over time is then bounded so that the resource is never exhausted if the initial resource stock is sufficiently large.

Chapter 6 first summarizes the main results of this thesis. It then concludes with a short reflection on the implications and on approaches for further research.

Chapter 2

Green Innovation, Productivity Growth and GDP Deceleration[1]

In this chapter, we study economic growth and the development of environmental quality in a Schumpeterian model with polluting intermediate production and endogenous rate and direction of technical change. In particular, research effort can be directed at increasing the productivity and/or decreasing the pollution intensity of production inputs. We explicitly take into account the possibility to control the rebound effect of productivity growth through deceleration, by keeping growth in polluting inputs below productivity growth.

The first section presents the model. As is standard in models with vertical product differentiation, real GDP can either be increased by raising the quantity of intermediate inputs or the productivity of a given amount of inputs. But while producing a larger quantity of intermediates accelerates pollution growth, productivity growth affects pollution only indirectly through the effects described in the previous chapter: Higher productivity allows to reduce the share of polluting inputs in GDP but at the same time increases their marginal product, thereby setting incentives to expand intermediate production. Productivity growth has a rebound effect on polluting quantity.

We do not assume the existence of a completely clean substitute to the polluting input. There are two ways then to decouple output- and pollution growth: The first is to partially direct R&D-effort towards green innovation to reduce the pollution intensity of intermediate quantity. The second is to restrict the rebound effect of productivity growth on polluting quantity through deceleration.

The model is solved for the laissez-faire equilibrium and the social optimum.

[1] This chapter is based on Funk and Burghaus (2013).

We focus on solution paths with asymptotically constant growth rates and describe the long-run properties of these paths. It is important to note that narrowing the analysis to paths with asymptotically constant growth rates is a weaker restriction on the set of potential solutions than that of balanced growth in all periods which is usually found in the literature. We give a precise definition of both types of solutions in section 2.2 below.

In section 2.3, we determine the long-run laissez-faire equilibrium. We assume that pollution is an external effect from production on household utility. Therefore the unregulated market-equilibrium equals the standard balanced-growth outcome in endogenous growth models with creative destruction. Neither green innovation nor deceleration is chosen by agents, and pollution grows proportionally to output and consumption. We prove that a path without long-run economic growth is socially preferable to the laissez-faire equilibrium with unrestricted pollution growth.

In section 2.4, we derive the long-run social optimum. Despite the negative environmental externality of production, long-run economic growth is optimal given that the rate of time preference is not too large. The threshold is not stricter than in models of creative destruction without environmental externality. Long-run growth however has to be accompanied by persistent pollution control. For reasonable assumptions about parameters, both green innovation and deceleration are needed to restrict pollution growth.

In the introduction, we pointed out concerns in academic and political discussion that environmental care may entail substantial costs in terms of economic growth. Our analysis in this chapter suggests that even with persistent deceleration, long-run optimal consumption growth may be faster if the representative household gains utility from a clean environment than if there is no environmental externality. This result is driven by a positive effect of green innovation on overall research activity. The last section discusses the model specification.

2.1 The model

In each period, a representative household receives utility $v(c_t) = \frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}}$ from per-capita-consumption $c_t = \frac{C_t}{L}$ and utility $\psi \phi^E(E_t) = \psi \frac{\sigma_E}{\sigma_E - 1} E_t^{\frac{\sigma_E - 1}{\sigma E}}$ from environmental quality E_t. We assume, as is often done in the literature (e.g., Stokey (1998); Aghion Howitt (1998), chapter 5), that utility is additively separable. Discounted intertemporal utility is given by

$$U = \int_0^\infty e^{-\rho t} \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} + \psi \frac{\sigma_E}{\sigma_E - 1} E_t^{\frac{\sigma_E - 1}{\sigma_E}} \right) L dt, \tag{2.1}$$

where ρ is the rate of time preference, L total household labor supply and $\sigma_c, \sigma_E > 0$, $\sigma_c, \sigma_E \neq 1$ are the intertemporal substitution elasticities of consumption and environmental quality respectively. $\psi > 0$ measures the weight of environmental quality in instantaneous utility. Utility is increasing and strictly concave in both arguments.

Environmental quality is inversely related to the stock of pollution originating from the intermediate sector:

$$E_t = \frac{1}{S_t} \tag{2.2}$$

While utility is concave in E_t, the relation between environmental quality and pollution is convex. Depending on σ_E, the disutility $\psi \phi^S(S_t) = -\psi \phi^E(E_t) = \psi \frac{\sigma_E}{1-\sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}}$ of pollution can be concave or convex in S_t. We assume that it is convex, by restricting σ_E to the interval $(0, 1/2)$. As indicated in the introduction, it is reasonable to assume that the marginal disutility of pollution increases in the pollution stock. The assumption of convex disutility also rules out parameter constellations for which the utility impact of pollution asymptotically becomes negligible relative to that of consumption in a growing economy. This is not an interesting case for the analysis of questions arising out of the trade-off between economic growth and a clean environment from a long-run perspective. Not only the long-run laissez-faire but also the long-run optimal solution would be similar to those in non-environmental models of growth through creative destruction.

For the analysis in this and the following chapters, we prefer to express utility as function

$$U = \int_0^\infty e^{-\rho t} \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1 - \sigma_E} S_t^{\frac{1 - \sigma_E}{\sigma_E}} \right) L dt \tag{2.3}$$

of the pollution stock directly.

The representative household allocates an amount L_{Yt} of its labor supply L to final-good production, an amount L_{Xt} to intermediate production and an amount L_{Dt} to research:

$$L = L_{Yt} + L_{Xt} + L_{Dt} \tag{2.4}$$

Final output Y_t is produced from labor L_{Yt} and intermediate goods X_{it} of various productivity levels Q_{it}, from a continuum of sectors $i \in [0, 1]$, with the production function

$$Y_t = L_{Yt}^{1-\alpha} \int_0^1 X_{it}^{\alpha} Q_{it}^{(1-\alpha)} di, \tag{2.5}$$

where $0 < \alpha < 1$. Y_t is used for consumption only.

$$Y_t = c_t L. \tag{2.6}$$

Intermediate goods in sector i are produced with the production function

$$X_{it} = \varphi L_{Xit} Q_t, \tag{2.7}$$

where $\varphi > 0$ is a parameter and $Q_t = \int_0^1 Q_{it} di$ measures aggregate productivity.[2] X_{it} can be interpreted as any kind of non-durable production input including energy. Polluting energy inputs are however usually associated with polluting non-renewable resources which we do not consider in the baseline specification of our model. We show in chapter 5 that with only a mild restriction of the parameter range, the introduction of a non-renewable resource for intermediate production does not affect the long-run social optimum so that the main results of this model still hold.

Pollution evolves according to the equation of motion

$$\dot{S}_t = \frac{X_t}{B_t} - \delta S_t. \tag{2.8}$$

In general, we use a dot above a variable to indicate its derivative with respect to time while we mark growth rates with a cicumflex. X_t/B_t is the pollution flow generated by the quantity of intermediates. The pollution intensity of X_t decreases in the aggregate cleanliness $B_t = \int_0^1 B_{it} di$ of the inputs used. In every period, a fraction δ of the pollution stock is cleaned up by natural regeneration processes.

Because of natural regeneration, pollution growth will eventually cease if there is no growth in intermediate production. However, if X_t grows, S_t will asymptotically grow at the same rate unless the pollution intensity of intermediates is reduced over time by green innovation.

Even without green innovation, pollution growth remains below its potential if there is deceleration so that the rebound effect of productivity-growth is restricted.

[2] The dependence of X_{it} on aggregate productivity Q_t is needed to ensure that the allocation of labor supply and thereby growth rates of the aggregate variables in our model are constant in the long run. Our results would not change qualitatively if we assumed that instead of labor, a fraction of final output had to be spent on the production of intermediates (see section 2.5).

We speak of deceleration whenever growth in intermediate quantity remains below productivity growth. This is expressed by the following, more formal definition:

Definition 2.1 *There is* ***deceleration*** *whenever* $\widehat{X}_t < \widehat{Q}_t$ *so that* X_t/Q_t *declines.*

The two sources of slow pollution accumulation (besides natural regeneration) become apparent when rewriting (2.8) as $\dot{S}_t = \frac{X_t}{Q_t}\frac{Q_t}{B_t} - \delta S_t$: First, $\dot{S}_t$ is small whenever Q_t/B_t is small, which means a sufficiently large share of research must have been oriented towards green innovation in the past. Second, pollution accumulates more slowly with a smaller X_t/Q_t brought about by deceleration.

If there is no deceleration ($\widehat{X}_t = \widehat{Q}_t$), then a constant stock of pollution ($\widehat{S}_t = \frac{d\widehat{S}_t}{dt} = 0$) requires $\widehat{B}_t = \widehat{Q}_t$.[3] This suggests the definition of a natural benchmark for the direction of technical change:

Definition 2.2 *The* ***direction of technical change*** *is ecologically* ***neutral*** *if and only if* $\widehat{B}_t = \widehat{Q}_t$*,* ***productivity-oriented*** *if and only if* $\widehat{B}_t < \widehat{Q}_t$*, and* ***green*** *if and only if* $\widehat{B}_t > \widehat{Q}_t$*.*

Both productivity Q and cleanliness B change over time due to innovations from a continuum of R&D-sectors. Entry to the research sector for any intermediate X_{it} is not restricted. For research unit $j \in [0, \infty]$, increasing Q_{it} by a rate q_{ijt} and B_{it} by a rate b_{ijt} requires

$$l_{Dijt}(q_{ijt}, b_{ijt}) = q_{ijt}^2 \frac{Q_{it}}{Q_t} + b_{ijt}^2 \frac{B_{it}}{B_t} + d\frac{Q_{it}}{Q_t} \tag{2.9}$$

units of labor. We call q_{ijt} and b_{ijt} the step-size of an innovation with respect to productivity and cleanliness respectively. We denote the wage rate by w_{Dt}. Then $w_{Dt}d\frac{Q_{it}}{Q_t} > 0$ are fixed entry costs for unit j in sector i. Variable costs for each dimension of technology improvement are quadratic in the step-size. Total costs $w_{Dt}l_{Dijt}$ rise with the level of sectoral relative to aggregate productivity Q_{it}/Q_t and cleanliness B_{it}/B_t respectively. The underlying assumption is that technology improvements in a given sector are increasingly difficult the more advanced the technology in that sector is already, while there are positive spillovers from the other sectors.[4]

[3] $\widehat{S}_t = 0$ if and only if $X_t = \delta S_t B_t$ and $\frac{d\widehat{S}_t}{dt} = 0$ if, in addition, $\widehat{X}_t = \widehat{B}_t$. Since $\widehat{X}_t = \widehat{Q}_t$, this requires $\widehat{B}_t = \widehat{Q}_t$.

[4] Like the intermediate production function, labor required in R&D (equation (2.9)) must depend on the sectoral and additionally on the aggregate levels of technology to ensure asymptotically constant growth.

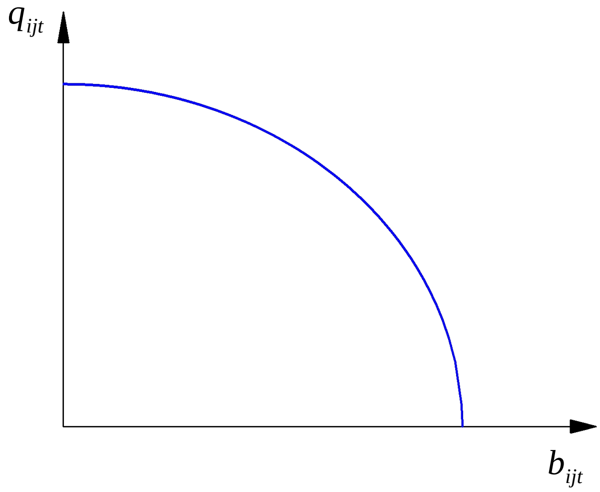

Figure 2.1: Innovation Possibilities Frontier

Given l_{Dijt}, a trade-off exists between making an intermediate more productive and making it cleaner, as is evident in figure 2.1. On the other hand, there is an indirect positive relation between research orientations as well. Once fixed costs have been paid to innovate in one direction, a comparatively small additional labor-investment is needed to increase the other technology stock as well.

If a researcher j enters into the research sector for intermediate X_i at time t, he hires labor l_{Dijt} and chooses a step-size q_{ijt} and b_{ijt} for the improvement in productivity and cleanliness respectively. The wage rate w_{Dt} is taken as given. Innovations occur at the exogenous, constant Poisson arrival-rate μ per unit of time for the individual researcher j. An innovation changes the sectoral productivity level by $q_{ijt}Q_{it}$ and the cleanliness of production by $b_{ijt}B_{it}$. The innovator obtains a patent for the production of the improved intermediate good. He then receives a profit flow from selling the intermediate which eventually ceases when a new innovation arrives and the incumbent is replaced by another firm. If n_{it} units decide to enter research sector i in t, innovations arrive at rate μn_{it} in this sector. The expected change in Q_i and B_i in period t is given by:

$$E\left[\Delta Q_{it}\right] = \int_0^{n_{it}} \mu q_{ijt} Q_{it} dj \tag{2.10}$$

$$E\left[\Delta B_{it}\right] = \int_0^{n_{it}} \mu b_{ijt} B_{it} dj \tag{2.11}$$

While the sectorial technology level faces discontinuous jumps, aggregate technology evolves continuously, because there is a continuum of sectors carrying out research.

Arguing along the lines of the law of large numbers[5], the average rates of change $\dot{Q}_t$ and $\dot{B}_t$ of Q and B approximately equal the respective expected rates of change, which are derived by aggregating over sectors in (2.10) and (2.10). Accordingly, the aggregate equations of motion are:

$$\dot{Q}_t = \int_0^1 \int_0^{n_{it}} \mu q_{ijt} Q_{it} dj di \tag{2.12}$$

$$\dot{B}_t = \int_0^1 \int_0^{n_{it}} \mu b_{ijt} B_{it} dj di \tag{2.13}$$

2.2 Definition: Balanced- and asymptotically-balanced-growth solutions

The subsequent analysis of the model in this and the following chapters extends beyond balanced growth paths to 'asymptotically-balanced growth paths'. The following definition serves to clarify the terminology, where here and in the following, z_∞ refers to the limit $\lim_{t\to\infty} z_t$ of a variable z:

Definition 2.3 *Assume that for some initial state* (Q_0, B_0, S_0)*, there exists a solution such that the sequence* $\left(\widehat{Q}_t, \widehat{B}_{t,}\widehat{S}_t\right)_{t=0}^{\infty}$ *converges towards the vector* $\left(\widehat{Q}_\infty, \widehat{B}_\infty, \widehat{S}_\infty\right)$ *for* $t \to \infty$*. We call such a solution an asymptotically-balanced growth (ABG) solution. We say that the model has an asymptotically unique ABG-solution if all ABG-solutions have the same limit vector* $\left(\widehat{Q}_\infty, \widehat{B}_\infty, \widehat{S}_\infty\right)$.
If there exist initial states (Q_0, B_0, S_0) *such that the corresponding solution paths are characterized by* $\left(\widehat{Q}_t, \widehat{B}_{t,}\widehat{S}_t\right) = \left(\widehat{Q}_\infty, \widehat{B}_\infty, \widehat{S}_\infty\right)$ *for every* t*, we call the path defined by* $\left(\widehat{Q}_t, \widehat{B}_{t,}\widehat{S}_t\right)_{t=0}^{\infty} = \left(\widehat{Q}_\infty, \widehat{B}_\infty, \widehat{S}_\infty\right)$ *the unique balanced growth (BG)-path.*

In abuse of terminology, we sometimes refer to the unique limit of all ABG-solutions for $t \to \infty$, characterized by the unique vector $\left(\widehat{Q}_\infty, \widehat{B}_\infty, \widehat{S}_\infty\right)$, as *the* ABG-solution.
Note that a BG-solution, defined by constant growth rates of Q, B and S for all t, is also an ABG-solution. The reverse is not true, because there may not exist

[5] Although it is usually cited also in this context, the law of large numbers does not in general hold for a continuum of random variables as given in this model (see Judd (1985)). Uhlig (1996), amongst others, derives conditions under which the use of the law in the case of a continuum is correct.

an initial state (Q_0, B_0, S_0) such that $\widehat{Q}_t$, $\widehat{B}_t$ and $\widehat{S}_t$ are constant for all t. We will see that while, for any set of parameters, the economy has a unique BG-equilibrium and a unique ABG-optimum, it need not have a BG-optimum. In particular, a BG-optimum does not exist in proposition 2.5 if the conditions for deceleration are satisfied, and in proposition 2.6.

The distinction between ABG-paths and BG-paths is not necessarily important for interpreting the results of the social planner's solution, as we concentrate on the long-run for both balanced- and asymptoticall-balanced-growth solutions. From a more technical point of view, an ABG-path which is not a BG-path is interesting for its own sake.

2.3 The laissez-faire equilibrium

In this section, we prove the existence of a unique balanced-growth equilibrium. The pollution stock grows without restriction at the same rate as consumption, production and productivity. There is neither green innovation nor deceleration. For any initial values of the state variables, the interest rate and the allocation of the fixed labor supply to production and research, as well as the chosen step size q_{ij} and b_{ij} are constant. All other aggregate variables except the pollution stock grow at their balanced growth rates at all times. If the relation between the state variables is unbalanced initially, the development of the pollution stock exhibits transitional dynamics.

The laissez-faire equilibrium is given by sequences of plans for per-capita consumption $\{c_t\}_0^\infty$, assets $\{A_t\}_0^\infty$, labor supply in production $\{L_{Xit}, L_{Yt}\}_0^\infty$ and research $\{L_{Dt}\}_0^\infty$, demand for intermediates $\left\{X_{it}^d\right\}_0^\infty$, demand for labor in production $\left\{L_{Xit}^d, L_{Yt}^d\right\}_0^\infty$ and research labor demand $\{l_{Dijt}\}_0^\infty$, plans for the step-size $\{q_{ijt}\}_0^\infty$ in productivity $\{b_{ijt}\}_0^\infty$ in cleanliness, as well as sequences of intermediate prices $\{p_{it}\}_0^\infty$ and wages $\{w_{Xit}, w_{Yt}, w_{Dt}\}_0^\infty$ in intermediate production, final good production and research and a path $\{r_t\}_0^\infty$ for the interest rate such that in every period t, (i) the representative household maximizes utility taking into account the budget constraint and the labor market constraint (2.4), (ii) profits from final- and intermediate goods production as well as research profits are maximized, (iii) aggregate expected profits in each research sector are zero (iv) the markets for intermediate goods, the three types of labor and assets clear (v) all variables with the possible exception of q_{ij} and b_{ij} are non-negative.

We now consider the behavior of the various agents in the model in turn before we close the model and determine the general equilibrium in section 2.3.4.

2.3.1 The representative household

The representative household earns income from labor and asset holding which he spends on consumption and the acquisition of new assets. The budget constraint is

$$c_t L + \dot{A}_t = r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt}, \tag{2.14}$$

where A_t denotes asset holdings and w_{Xit}, w_{Yt} and w_{Dt} the wage rates for labor in intermediate production, production of the consumption good and research. The interest rate is denoted by r_t.

The household maximizes (2.3) by choosing the paths for consumption, labor and asset holding while taking pollution accumulation as given[6]. He takes into account the budget-constraint (2.14) and must satisfy the no-Ponzi-condition

$$\lim_{t \to \infty} \left(e^{-\int_0^t r_v dv} A_t \right) \geq 0.$$

which rules out chain-letter finance, that is, schemes where the household borrows continuously without ever repaying his debt or interest.

As utility is additively separable, pollution accumulation does not affect the maximization problem. Solving the maximization problem yields the standard Euler-equation for per capita consumption:

$$\widehat{c}_t = \sigma_c \cdot (r_t - \rho) \tag{2.15}$$

2.3.2 Production

The production function for the consumption good is given by (2.5). Firms maximize profits over L_Y and X_i, taking the wage rate w_{Yt} and the prices p_{it} of the intermediates in sectors $i \in [0, 1]$ as given. We normalize the price of the consumption good to one. The first order condition for L_Y yields the implicit labor demand function

$$w_{Yt} = (1 - \alpha) L_{Yt}^{-\alpha} \int_0^1 X_{it}^{\alpha} Q_{it}^{1-\alpha} di. \tag{2.16}$$

[6] The maximization problem is depicted in appendix 2.A.1.

From the first-order condition for X_i, the following demand function for intermediate i is derived:

$$X_{it}^d(p_{it}) = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} Q_{it} L_{Yt} \tag{2.17}$$

Each unit of the intermediate is produced with the production function (2.7):

$$X_{it} = \varphi L_{Xit} Q_t$$

At equilibrium, wages in intermediate production must be the same in every sector i, so that marginal costs $MC_t = (1/\varphi) \cdot (w_{Xt}/Q_t)$ are the same for goods with different productivity levels. On the other hand, final good producers' demand is larger for more productive intermediates. It follows that only the owner of the patent for the intermediate design with the highest productivity will be producing in sector i, as he can always choose a price so that the firm with the next highest productivity level cannot break even. For the rest of this subsection, the firm index j is therefore omitted.

The intermediate good in sector i is sold at a price p_{it} to firms in the final good sector. The monopoly producer chooses p_{it} to maximize profits

$$\pi_{it}^X(p_{it}) = (p_{it} - MC_t) X_{it},$$

taking into account $MC_t = (1/\varphi) \cdot (w_{Xt}/Q_t)$ and the demand function (2.17). The profit maximizing monopoly price is given by a constant mark-up over marginal costs for all i[7]:

$$p_t = \frac{1}{\alpha\varphi} \cdot (w_{Xt}/Q_t)$$

The wage rate at equilibrium is obtained by substituting (2.17) in (2.16):

$$w_{Xt} = w_{Yt} = w_{Dt} = (1-\alpha)^{1-\alpha} \alpha^{2\alpha} (\varphi)^{\alpha} Q_t \tag{2.18}$$

We then derive the quantity of intermediates produced in sector i as function of the amount of labor employed in final good production, for any given sectoral level of productivity, from (2.17):

$$X_{it} = \frac{\alpha^2}{1-\alpha} \varphi L_{Yt} Q_{it} \tag{2.19}$$

[7] Monopoly pricing prevails under certain restrictions on model parameters which we derive in section 2.3.3.

Monopoly profits in sector i in period t are:

$$\pi_{it}^{X} = \frac{(1-\alpha)^{1-\alpha}}{\alpha}\alpha^{2(1+\alpha)}\varphi^{\alpha}L_{Yt}\cdot Q_{it} \tag{2.20}$$

The aggregate quantity X_t of intermediates is

$$X_t = \int_0^1 X_{it}di = \frac{\alpha^2}{(1-\alpha)}\varphi L_{Yt}Q_t, \tag{2.21}$$

where we used the definition $\int_0^1 Q_{it}di := Q_t$ of aggregate quality.

2.3.3 Research and Development

At time t, researcher j in sector i chooses l_{Dijt}, q_{ijt} and b_{ijt} to maximize expected profits from R&D. These consist of the profit flow he expects to receive as a monopolist in intermediate production less of research labor costs.

In every period and every sector, the exogenous arrival rate of innovations for the individual researcher is μ. If researcher j succeeds in innovating, he changes the productivity level in sector i from Q_{it} to $(q_{ijt}+1)\cdot Q_{it}$. After the innovation, the productivity level remains constant until the next innovation occurs and the monopoly producer is replaced by the new innovator.

The probability per unit of time of being replaced as the monopolist in sector i is exogenously given from the perspective of researcher j in every period $v > t$ and increases in the mass n_{iv} of research units active in sector i at time v. More precisely, innovations in every sector i follow a Poisson-process with arrival-rate $\mu_{iv} = \mu n_{iv}$. The probability that the incumbent monopolist is still producing in period $s > t$ is then given by $P(s) = e^{-\int_t^s \mu_{iv}dv}$.[8] His profits in period s can be deduced from (2.20), substituting the after-innovation productivity level $(q_{ijt}+1)\cdot Q_{it}$ for Q_{it}.

Expected discounted lifetime-profits are:

$$\begin{aligned} E\left[V_{ijt}(q_{ijt})\right] &= \int_t^{\infty}\pi_{ijs}^{X}(q_{ijt})\cdot P(s)e^{-\int_t^s r_v dv}ds \\ &= \frac{(1-\alpha)^{1-\alpha}}{\alpha}\alpha^{2(1+\alpha)}\varphi^{\alpha}(q_{ijt}+1)\cdot Q_{it}\int_t^{\infty}L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds \end{aligned} \tag{2.22}$$

Expected research profits are obtained by substracting research costs $w_{Dt}l_{Dijt}$:

$$E\left[\pi_{ijt}^{D}(q_{ijt},b_{ijt})\right] = \mu E\left[V_{ijt}(q_{ijt})\right] - w_{Dt}l_{Dijt}(q_{ijt},b_{ijt}), \tag{2.23}$$

[8] For a short derivation, see appendix 2.A.2.

Labor l_{Dijt} is given by (2.9) and the wage w_{Dt} by (2.18).

Researcher j maximizes (2.23) by choosing q_{ijt} and b_{ijt}.[9] Reducing the pollution intensity of intermediates by increasing B_i is costly but does not increase profits $E\left[V_{ijt}\right]$. Therefore $b_{ijt} = 0$ for all i, j, t so that the pollution intensity of intermediates is constant under laissez-faire. The first-order condition for q_{ij} can, after simplification, be written as:

$$\alpha\mu \textstyle\int_t^\infty L_{Ys} e^{-\int_t^s (r_v+\mu_{iv})dv} ds - 2q_{ijt} = 0 \tag{2.24}$$

The equation still depends on n_i through the sectoral arrival rate μ_i. To determine q_{it} and n_{it}, it must be taken into account that expected research profits in every sector i have to be zero at equilibrium. Otherwise, further research units would enter into sector i so that n_i would rise as long as the expectation value of profits was positive, while n_i would decrease if expected discounted profits were negative. The zero profit condition can be written as

$$\textstyle\int_t^\infty L_{Ys} e^{-\int_t^s (r_v+\mu_{iv})dv} ds = \frac{q_{ijt}^2 + d}{(1+q_{ijt})\alpha\mu}, \tag{2.25}$$

From (2.24) with (2.25), we determine the equilibrium value

$$q_{ijt}^{\mathrm{LF}} = q^{\mathrm{LF}} = \sqrt{1+d} - 1 \tag{2.26}$$

of q_{ijt}. q^{LF} is constant over time and across sectors. It increases in the entry cost parameter d because less entry lowers the probability of being replaced by the next innovator and therefore increases marginal profits from productivity-improvements[10].

Because q^{LF} is constant, the integral $\int_t^\infty L_{Ys} e^{-\int_t^s (r_v+\mu_{iv})dv} ds$ on the left-hand side of the free-entry condition (2.25) must be independent of t. Setting the time derivative of the integral to zero shows that the integral must be equal to $\frac{L_{Yt}}{r_t+\mu_{it}}$. This suggests that L_Y, n_i and r must be constant at equilibrium even if there is no balanced growth, which we prove in the next section. After substituting $\frac{L_{Yt}}{r_t+\mu_{it}}$ for the integral in equation (2.25), and the equilibrium value of q, (2.26), on the

[9] We summarize the essentials of the maximization problem here. See appendix 2.A.3 for a more detailed description.

[10] In the analysis, it has been assumed that the monopoly price is smaller than the limit price. This will be the case, whenever $p^{\mathrm{mon}} < \left(q^{\mathrm{LF}} + 1\right) \cdot (1/\varphi)\,(w_{Xt}/Q_t)$ which is equivalent to choosing fixed costs $d > \frac{1}{\alpha^2} - 1$.

right-hand side, we can solve the free entry condition for n_{it}:

$$n_{it} = n_t = \frac{1}{2} \frac{1}{\sqrt{1+d}-1} \cdot \alpha L_{Yt} - \frac{r_t}{\mu}. \tag{2.27}$$

2.3.4 General equilibrium

The market value of firms

Every unit of assets A in our model corresponds to a share of the market value of firms in the intermediate sector. The total stock of the representative household's assets at the beginning of period t must therefore equal the aggregate market value of firms before innovation. In each sector i, only the firm with the highest productivity level Q_{it} is active in production. The before-innovation market value of this firm can be derived from (2.22), substituting Q_{it} for the after-innovation productivity level $(q_{ijt}+1) \cdot Q_{it}$. To obtain the aggregate market value V_t of firms, we take the integral over all sectors and use (2.25) with (2.26) to replace $\int_t^\infty L_{Ys} e^{-\int_t^s (r_v + \mu_{iv}) dv} ds$:

$$\begin{aligned} V_t &= \int_0^1 E\left[V_{ijt}\right] di \\ &= 2 \frac{(1-\alpha)^{1-\alpha} \alpha^{2\alpha} \varphi^{\alpha}}{\mu} \left(\sqrt{1+d}-1\right) Q_t \end{aligned} \tag{2.28}$$

The market value is proportional to the economy-wide productivity level Q_t.

Labor market clearing

We use (2.27) along with the labor market constraint (2.4) and equation (2.21) to find the allocation of labor between final good production, intermediate production and research (L_{Yt}, L_{Xt}, L_{Dt}) and determine the mass n_t of research units in sector i for any given interest rate r_t[11]. The equilibrium n_t is:

$$n_t = \frac{\frac{1}{2} L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right) \left(\sqrt{1+d}-1\right) \frac{r_t}{\mu}}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right) \left(\sqrt{1+d}-1\right) + d} \tag{2.29}$$

The mass of research units is the same in every sector. It increases in the arrival rate μ for innovations and decreases in the interest rate r_t and the fixed labor requirement d.

[11] The derivation can be found in appendix 2.A.4.

Equilibrium growth

Taking into account that n_t and q^{LF} are the same for all research sectors and using the definition of the aggregate productivity index Q, the equation of motion (2.12) for Q simplifies to

$$\dot{Q}_t = \mu n_t q^{\mathrm{LF}} Q_t.$$

Substituting (2.26) for q^{LF} and (2.29) for n_t, we obtain the productivity growth rate in period t as a function of the interest rate r_t:

$$\widehat{Q}_t = \mu \frac{\frac{1}{2}L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\frac{r_t}{\mu}}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right) + d}\left(\sqrt{1+d}-1\right) \tag{2.30}$$

It follows from (2.21) that X_t and Q_t grow at the same rate at equilibrium because labor must be constant. From the resource constraint, it is obvious that c_t then also grows at the rate $\widehat{Q}_t$. We set (2.30) equal to (2.15) and solve for the equilibrium interest rate.

$$r^{\mathrm{LF}} = \frac{\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right) + d\right)\rho}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right) + d + \frac{1}{\sigma_c}\left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)^2} \tag{2.31}$$

With the expression for r^{LF}, equation (2.30) yields the equilibrium growth rate

$$\widehat{Q}^{\mathrm{LF}} = \frac{\frac{1}{2}\mu L - \left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\rho}{\left(\frac{1-\alpha}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right) + d + \frac{1}{\sigma_c}\left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)^2}\left(\sqrt{1+d}-1\right). \tag{2.32}$$

The growth rate decreases in the rate of time preference, ρ and increases in the intertemporal elasticity of substitution in consumption, σ_c and the arrival rate of innovations, μ. The effect of the fixed-costs-parameter d is ambiguous as q in- but entry n decreases in d. The elasticity of intermediate quantity in production, α, also has an ambiguous effect: An increase in α decreases the market-power of the monopolist in the intermediate sector. This unambiguously raises monopoly profits relative to the equilibrium wage rate and thereby increases R&D-profits, n and the equilibrium growth rate. On the other hand, more labor is used in the intermediate sector and less in research due to the increase in intermediate quantity. This second effect tends to decrease n and the equilibrium growth rate.

The growth rates of c, X, Q and B are constant for any set of initial values for the state variables. Therefore growth in c, X, Q and B is balanced without transitional dynamics.

It follows from (2.8) that the pollution stock must increase at the same rate $\widehat{Q}^{\mathrm{LF}}$ as intermediate quantity, productivity and consumption in the long run:

$$\widehat{S}^{\mathrm{LF}}_{\infty} = \widehat{Q}^{\mathrm{LF}}$$

However, contrary to the growth rates of the other variables, the growth rate of the pollution stock does not adjust to its balanced-growth level instantly if the relation between the state variables is not reconcilable with constant growth of the pollution stock initially.

Define an upper bound $\overline{\rho}^{\mathrm{LF}}$ for the rate of time preference such that $\widehat{Q}^{\mathrm{LF}} > 0$ if and only if $\rho < \overline{\rho}^{\mathrm{LF}}$. Further, define a lower bound $\rho^{\mathrm{TVC,LF}}$ such that the transversality condition for assets is satisfied if and only if $\rho > \rho^{\mathrm{TVC,LF}}$.[12] The following proposition describes the balanced-growth equilibrium for $\rho^{\mathrm{TVC,LF}} < \rho < \overline{\rho}^{\mathrm{LF}}$:

Proposition 2.1 ***BG laissez-faire equilibrium***
Assume $\rho^{TVC,LF} < \rho < \overline{\rho}^{LF}$.

There exists a unique BG laissez-faire equilibrium with positive economic growth. Productivity growth leads to equally fast expansion of polluting quantity ($\widehat{X}^{LF} = \widehat{Q}^{LF}$). There is neither deceleration nor green innovation. Pollution grows at the same rate as consumption, production and productivity.

Proof. See appendix 2.A.5. ■

At the balanced-growth equilibrium path, a one percent increase in productivity leads to an equally large expansion of polluting quantity X_t. This corresponds to the strongest possible rebound effect on polluting quantity in our model. As producers do not internalize the adverse effect of pollution on household utility and pollution does not affect the production process, there is no incentive to self-restrict in polluting intermediate production. For the same reason, no resources are invested to reduce the pollution intensity of intermediates through green innovation. In a growing economy, there is unconstrained pollution growth. We prove in appendix 2.A.6 that the strong negative utility effect from growing pollution outweighs the positive effect from consumption growth in such a way that a solution without long-run growth would be socially preferable.

[12] It follows from (2.32) that $\overline{\rho}^{\mathrm{LF}} = \frac{1}{2}\mu L \left(\left(\frac{1}{\alpha} + \frac{\alpha}{1-\alpha} \right) \left(\sqrt{1+d} - 1 \right) \right)^{-1}$.
Appendix 2.A.5 shows that the transversality condition yields the critical value $\rho^{\mathrm{TVC,\,LF}} = \frac{1}{2}\alpha(1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)(1+d)^{-1/2}\mu L$.

Proposition 2.2 ***Comparison to an economy without long-run growth***
Assume that the disutility of pollution is convex ($\sigma_E < 1/2$).

A solution without long-run growth is socially preferable to the laissez-faire equilibrium in proposition 2.1.

Proof. See appendix 2.A.6. ■

The intuition for our result follows straightforwardly from the assumption that the disutility of pollution is convex but utility is concave in consumption: The marginal utility gain from an additional unit of consumption becomes negligible relative to the marginal utility loss generated by a unit increase in the pollution stock as consumption and pollution rise at the same rate. Utility declines persistently without lower bound. If, on the contrary, long-run consumption growth is given up, the pollution stock and therefore utility converge to constant values. There is, however, a transitional welfare loss because consumption growth drops to zero instantly, but the pollution growth rate declines over time. We show in the appendix that if the switch occurs late in time (i.e. "in the long run"), the long-run effect outweighs the transitional effect.

Proposition 2.2 suggests that advocating stationary long-run levels of consumption and production as is done by environmental activist is not entirely unreasonable. If adequate regulation is not in place, giving up economic growth is indeed welfare-improving to a situation where continuous growth leads to persistent and rapid environmental degradation. We show in the next section that nevertheless, for a sufficiently patient household, a path without long-run growth is not optimal given that pollution growth can be controlled through green innovation and deceleration.

2.4 The long-run optimal solution

Having shown that unconstrained pollution growth is clearly suboptimal, we analyze the socially optimal outcome of our model in this section. In subsection 2.4.1, we combine the first-order conditions to four key equations which are central for the determination of the social planner's solution in this and the subsequent chapters. The long-run optimal solution is characterized in subsection 2.4.2. First, we prove that long-run economic growth is optimal given that the representative household is sufficiently patient. We then examine the optimal development of the pollution stock. Whether this stock declines or increases along the long-run optimal path depends on the representative household's preferences, more exactly on the intertemporal elasticities of substitution in consumption and pollution. Finally, we study long-run optimal pollution control for any set of parameters. In subsection 2.4.3, we build a numerical example which suggests that for reasonable parameter values, green innovation to reduce the pollution intensity of intermediates is optimally combined with deceleration to control the rebound effect of productivity growth. In the last subsection, we show that environmental care need not be detrimental to long-run growth. On the contrary, it may increase long-run consumption growth if green innovation induces a sufficiently strong shift of labor towards the R&D-sector.

2.4.1 Dynamic optimization problem and first-order conditions

The social planner chooses the time paths of Q_i, B_i, Q, B, S, consumption c and production x_{ij}, X_i, Y, X, as well as the allocation of labor L_{Yt}, L_{Xit}, L_{Xt}, l_{Dijt}, L_{Dt}, the mass of research units[13] n_t and the step size q_{ijt} and b_{ijt} for technology-improvements in every period t so as to maximize utility (equation (2.3)). He takes into account the labor market constraint (2.4), the production function (2.5), the aggregate resource constraint (2.6), the equation of motion for pollution (2.8), the expected change in Q_{it} (2.10) and B_{it} (2.11) and the aggregate equations of motion for Q_t (2.12) and B_t (2.13).

Before solving the dynamic optimization problem, we characterize the optimal allocation of variables across sectors and across economic agents within each sector, which can be derived through static optimization.

Because all research units j are ex ante symmetric and the labor requirement is convex in q_{ij} and b_{ij}, the social planner chooses the same q_{ijt}, b_{ijt} and therefore l_{Dijt} for every j in sector i. Further, the planner allocates intermediate production in every sector i to the latest innovator because he is the most productive and cleanest and marginal costs are the same for all j. We therefore omit the index j from now on.

As to the allocation of intermediate production across sectors in period t, the socially optimal amount

$$X_{it} = X_t \frac{Q_{it}}{Q_t} \tag{2.33}$$

of intermediate production in sector i is obtained from the static maximization of output Y, given aggregate intermediate quantity X[14].

[13] To allow for an analytical solution to the planner's problem, we impose the constraint $n_{it} = n_t$ for all i.

[14] See appendix 2.B.1.

The optimal q_{it} and b_{it} are the same in every sector i: The social planner chooses the step-size in every sector i so as to reach a given rate of change $\dot{Q}_t$ and $\dot{B}_t$ in the respective aggregate technology level with a minimum labor investment. From the equations of motion (2.12) and (2.13) for Q and B together with the R&D-cost function (2.9) we can conclude that the marginal gain of an increase in b_i and q_i, in terms of faster technological progress, and the additional amount of labor required increase in the sectorial technology levels Q_{it} and B_{it} in the same way. Therefore sectorial differences are irrelevant for the optimal choice of q_i and b_i.
Our reasoning yields the following lemma:

Lemma 2.1 *The optimal q_{ijt} and b_{ijt} are the same for all research units j and all sectors i: $q_{ijt} = q_t$ and $b_{ijt} = b_t$. The social planner optimally allocates a higher share of aggregate intermediate production to the sectors with relatively higher productivity according to equation (2.33).*

Proof. Proof in the text. ■

Using lemma 2.1, we can express the dynamic optimization problem in aggregate variables only: From (2.9), with $\int_0^1 Q_{it}di = Q_t$, $\int_0^1 B_{it}di = B_t$ and $n_{it} = n_t$, the total amount of labor allocated to research in period t is $L_{Dt} = n_t(q_t^2 + b_t^2 + d)$. To produce X_t units of intermediates requires $L_{Xt} = \frac{1}{\varphi}\frac{X_t}{Q_t}$ units of labor. The labor market constraint can be written as

$$L = \frac{1}{\varphi}\frac{X_t}{Q_t} + L_{Yt} + n_t(q_t^2 + b_t^2 + d). \tag{2.34}$$

Further, the equations of motion (2.12) for Q and (2.13) for B are:

$$\dot{Q}_t = \mu n q_t Q_t \tag{2.35}$$

$$\dot{B}_t = \mu n b_t B_t \tag{2.36}$$

With (2.33), output Y can be expressed as a function of aggregate variables only. The aggregate resource constraint can be written as:

$$L_{Yt}^{1-\alpha} X_t^{\alpha} Q_t^{1-\alpha} = c_t L \tag{2.37}$$

The social planner's problem can then be solved by finding the optimal paths for Q, B, S, c, X, L_Y, n, q and b subject to (2.8), (2.34), (2.35), (2.36) and the resource constraint (2.37). Further, the non-negativity constraints

$$Q_t, B_t, S_t, c_t, X_t, L_{Yt}, n_t \geq 0, \forall t$$

must hold. While we ensure that the non-negativity constraints are satisfied, we do not take them into account formally as additional constraints in the maximization problem. The current-value Hamiltonian function is:

$$\begin{aligned} H \;=\; & \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1 - \sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}} \right) L \\ & + v_{Qt} \mu n_t q_t Q_t \\ & + v_{Bt} \mu n_t b_t B_t \\ & + v_{St} \left(\frac{X_t}{B_t} - \delta S_t \right) \\ & + \lambda_{Yt} \left(X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{1-\alpha} - c_t L \right) \\ & + \lambda_{Lt} (L - \frac{1}{\varphi} \frac{X_t}{Q_t} - L_{Yt} - n_t (q_t^2 + b_t^2 + d)) \end{aligned} \tag{2.38}$$

where v_{Qt}, v_{Bt} and v_{St} are the (current-value) shadow-prices of Q_t, B_t and S_t respectively and λ_{Yt} and λ_{Lt} are the (current-value) Lagrange-multipliers for the resource constraint and the labor market constraint.

We derive the necessary first-order conditions from Pontryagin's maximum principle. These include the conditions

$$\frac{\partial H}{\partial c_t} = 0 \Leftrightarrow \lambda_{Yt} = c_t^{-1/\sigma_c} \tag{2.39}$$

$$\frac{\partial H}{\partial X_t} = 0 \Leftrightarrow \frac{v_{St}}{B_t} + \lambda_{Yt} \alpha X_t^{\alpha-1} Q_t^{1-\alpha} L_{Yt}^{1-\alpha} - \lambda_{Lt} \frac{1}{\varphi Q_t} = 0 \tag{2.40}$$

$$\frac{\partial H}{\partial q_t} = 0 \Leftrightarrow v_{Qt} \mu n_t Q_t = 2 \lambda_{Lt} n_t q_t \tag{2.41}$$

$$\frac{\partial H}{\partial b_t} = 0 \Leftrightarrow v_{Bt} \mu n_t B_t = 2 \lambda_{Lt} n_t b_t \tag{2.42}$$

$$\frac{\partial H}{\partial n_t} = 0 \Leftrightarrow v_{Qt} \mu q_t Q_t + v_{Bt} \mu b_t B_t = \lambda_{Lt} \left(q_t^2 + b_t^2 + d \right) \tag{2.43}$$

$$\frac{\partial H}{\partial L_{Yt}} = 0 \Leftrightarrow \lambda_{Yt} (1 - \alpha) X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{-\alpha} = \lambda_{Lt} \tag{2.44}$$

for the control variables, the conditions

$$
\begin{aligned}
\frac{\partial H}{\partial S_t} &= \rho v_{St} - \dot{v}_{St} \Leftrightarrow -\psi S_t^{(1-2\sigma_E)/\sigma_E} L - \delta v_{St} = \rho v_{St} - \dot{v}_{St} && (2.45) \\
\frac{\partial H}{\partial Q_t} &= \rho v_{Qt} - \dot{v}_{Qt} && (2.46) \\
&\Leftrightarrow v_{Qt}\mu n_t q_t + \lambda_{Yt}(1-\alpha) X_t^{\alpha} Q_t^{-\alpha} L_{Yt}^{1-\alpha} + \lambda_{Lt} \frac{X_t}{\varphi} \frac{1}{Q_t^2} = \rho v_{Qt} - \dot{v}_{Qt} \\
\frac{\partial H}{\partial B_t} &= \rho v_{Bt} - \dot{v}_{Bt} \Leftrightarrow v_{Bt}\mu n_t b_t - v_{St} \frac{X_t}{B_t^2} = \rho v_{Bt} - \dot{v}_{Bt} && (2.47)
\end{aligned}
$$

for the state variables, the conditions

$$
\begin{aligned}
\frac{\partial H}{\partial v_{St}} &= \dot{S}_t \Leftrightarrow \frac{X_t}{B_t} - \delta S_t = \dot{S}_t && (2.48) \\
\frac{\partial H}{\partial v_{Qt}} &= \dot{Q}_t \Leftrightarrow \mu n_t q_t Q_t = \dot{Q}_t && (2.49) \\
\frac{\partial H}{\partial v_{Bt}} &= \dot{B}_t \Leftrightarrow \mu n_t b_t B_t = \dot{B}_t && (2.50) \\
\frac{\partial H}{\partial \lambda_{Yt}} &= 0 \Leftrightarrow X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{1-\alpha} = c_t L && (2.51) \\
\frac{\partial H}{\partial \lambda_{Lt}} &= 0 \Leftrightarrow L = \frac{1}{\varphi} \frac{X_t}{Q_t} + L_{Yt} + n_t(q_t^2 + b_t^2 + d) && (2.52)
\end{aligned}
$$

for the costate variables as well as the transversality conditions[15]:

$$
\begin{aligned}
\lim_{t\to\infty} \left(e^{-\rho t} v_{Qt} Q_t\right) &= 0 \\
\lim_{t\to\infty} \left(e^{-\rho t} v_{Bt} B_t\right) &= 0 \qquad (2.53) \\
\lim_{t\to\infty} \left(e^{-\rho t} v_{St} S_t\right) &= 0
\end{aligned}
$$

Our aim is to characterize the socially optimal solution in the long run. From the set of first-order conditions, we derive four key equations which together with the equations of motion (2.48) to (2.50) for the state variables, the static constraints (2.51) and (2.52), the transversality conditions and the non-negativity constraints characterize the interior social optimum for $t \to \infty$[16].

As labor supply L is constant and we are interested in paths with at least asymp-

[15] There is some dispute about whether these transversality conditions are indeed necessary in infinite-horizon optimization problems. Counterexamples do however not exist for problems with time-discounting as in this model. The transversality conditions in (2.53) nest the weaker and unambiguously necessary transversality condition $\lim_{t\to\infty} e^{-\rho t} H_t = 0$. (Chiang (1992), p. 243-251)

[16] The derivations are shown in appendix 2.B.2. In particular, the assumption $n_\infty > 0$ is used.

totically constant growth rates, we can use that the allocation of labor to production and research as well as the number of research units n and the step-size q and b must be constant in the long run.

Asymptotically-balanced growth: The first equation is obtained from equation (2.45) with (2.40) and governs the development of the marginal utility of consumption relative to the marginal disutility of pollution so as to ensure constant growth rates in the long run:

$$\frac{\sigma_c - 1}{\sigma_c}\widehat{c}_\infty = \frac{1-\sigma_E}{\sigma_E}\widehat{S}_\infty + \left(\widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty\right) \tag{2.54}$$

The difference in parentheses is zero along a balanced growth path. Equation (2.54) is then the balanced-growth condition described in Gradus and Smulders (1996) which has become standard in environmental endogenous growth models: It requires that the ratio of instantaneous marginal utility from consumption to instantaneous marginal disutility from pollution must develop proportional to S/c so that the elasticity of substitution between c and S is unity.
If growth rates are required to be constant only asymptotically, the difference on the right hand side may be negative: If emissions X/B are decreased particularly fast, the pollution stock asymptotically falls at the rate of natural regeneration ($-\delta$), as we explain in section 2.4.2. For such a fast decline in the pollution stock to be optimal, the relative marginal utility of consumption must decline faster than the ratio c/S of consumption to pollution rises.

Consumption Euler-equation: Dividing the first-order condition (2.46) for productivity Q by its shadow-price v_Q and using (2.39), (2.41) and (2.44) to replace the variables λ_Y, λ_L, v_Q and their growth rates yields a version of the consumption Euler-equation:

$$\frac{1}{\sigma_c}\widehat{c}_\infty + \rho = \frac{\mu}{2q_\infty}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) + \mu n_\infty q_\infty + \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) \tag{2.55}$$

The Euler-equation states that the marginal social net return to higher productivity must compensate the household for shifting consumption into the future and investing in productivity-oriented research.
The first term on the right is the sum of the marginal social return (in utility units) to productivity in final good production and the marginal social benefit from lower costs in intermediate production. The second term, $\mu n_\infty q_\infty$, is the marginal contribution of an increase in the current productivity level to future productivity through

the equation of motion for Q. The third term accounts for the fact that the social value of an additional unit of Q in terms of consumption may change in response to the reallocation of resources from current consumption to productivity-oriented research. This term is equal to zero along a balanced growth path and negative whenever there is deceleration.

Research-arbitrage: An equation similar to (2.55) holds for the decision between using labor for production of the consumption good and using it for green innovation. Labor is allocated to both productivity improvements and green innovation if and only if the social net returns to both research directions are equal. This requirement is formally represented by the research-arbitrage condition

$$\frac{\mu}{2q_\infty}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) = \frac{\mu}{2b_\infty}\left(\frac{\alpha}{1-\alpha}L_{Y\infty} - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right). \tag{2.56}$$

The term on the left-hand side is the social marginal return to productivity in final good and intermediate production described above.
The term on the right-hand side denotes the social return to green innovation from slower pollution accumulation (in utility units). This return is larger if the production elasticity α of X is large and the production costs, $\frac{1}{\varphi}(X/Q)$, in terms of labor are small because the incentive to produce intermediate quantity X is stronger.
The effects of Q and B on their respective future levels and the change in the respective social value of Q and B relative to consumption cancel out.

Indifference: A fourth condition is derived from (2.41) and (2.42) with (2.43):

$$q_\infty^2 + b_\infty^2 = d \tag{2.57}$$

Equation (2.57) ensures that the social planner is indifferent between all finite, non-negative levels for the mass of research units n.

2.4.2 Characterization of the long-run optimum

Optimality of long-run economic growth

Equations (2.54) to (2.57) are relevant only for an interior solution to the optimization problem with persistent economic growth.[17] In standard endogenous growth models, long-run growth is optimal for a sufficiently patient household. A similar

[17] The four conditions were derived under the assumption $n_\infty > 0$. It will become obvious in this subsection that $n_\infty > 0$ implies persistent economic growth.

condition on the rate of time preference can be derived from the Euler-equation (2.55) in the present model with negative environmental externalities. The upper bound $\overline{\rho}$ differs depending on the parameter constellation considered and is defined in appendix 2.B.4.

Proposition 2.3 ***Positive long-run consumption growth***
Optimal growth of per capita consumption is positive in the long run, if and only if $\rho < \overline{\rho}$.

Proof. See appendix 2.B.6. ■

In proposition 2.2, we proved that compared to unconstrained pollution growth at the rate of consumption growth, a path without long-run economic growth is welfare-improving. This does, however, not imply that a solution without growth is optimal. Proposition 2.3 is not surprising given that pollution accumulation can be restricted without giving up consumption growth altogether. Yet persistent economic growth must be accompanied by continuous pollution control. Before analyzing in detail how pollution control is optimally achieved, we describe the long-run optimal relation between consumption and pollution growth in the next subsection.

The optimal relation between long-run economic growth and pollution accumulation

Although we have shown that unrestricted pollution growth cannot be optimal, it would be wrong to conclude that an optimal path for pollution in our model must exhibit constant or decreasing pollution levels. The conclusion from proposition 2.2 merely is that pollution growth must be sufficiently slower than consumption growth, so that utility from consumption is not outweighed by damages from pollution over time. It follows from the ABG-condition (2.54) that, for our assumption of convex disutility of pollution ($\sigma_E < 1/2$), whether the pollution stock de- or increases in the long-run optimum depends on the intertemporal elasticity of substitution in consumption:

Proposition 2.4 ***Development of the pollution stock***
Assume that instantaneous disutility of pollution is convex ($\sigma_E < 1/2$) and $\rho < \overline{\rho}$.

Long-run growth must be accompanied by a persistent restriction of pollution growth. The pollution stock, S_t, increases (decreases) in the long run if and only if $\sigma_c > 1$ ($\sigma_c < 1$).

Proof. The first statement follows both as a corollary from proposition 2.2 and from equation (2.54).
As to the second, note that given $\widehat{c}_\infty > 0$, the left-hand side of (2.54) is positive whenever $\sigma_c > 1$ while it is negative for $\sigma_c < 1$. Under the assumption of convex disutility of pollution, $\frac{1-\sigma_E}{\sigma_E}$ on the right-hand side is positive. Further, the difference $\widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty$ in parentheses is smaller than zero only if $\widehat{S}_\infty < 0$ and zero otherwise. Therefore the right-hand side of equation (2.54) is positive if and only if $\widehat{S}_\infty > 0$ and negative if and only if $\widehat{S}_\infty < 0$. It follows that the pollution stock must increase whenever $\sigma_c > 1$ and decrease whenever $\sigma_c < 1$.[18] ■

A similar dependency of the optimal pollution path on the intertemporal elasticity of substitution in consumption is found in Stokey (1998).

To gain a better intuition, note that if instantaneous disutility $\psi\phi^S(S_t)$ from pollution is convex, marginal instantaneous disutility converges to zero if and only if S_t decreases to zero and diverges to infinity if and only if S_t grows persistently.
For $\sigma_c > 1$, instantaneous marginal utility $v'(c_t)$ of consumption in (2.3) falls in response to an increase in consumption, but underproportionally. If the pollution stock remained constant, the ratio of marginal (dis)utilities $v'(c_t)/\left(\psi\phi'(S)\right)$ would rise relative to S_t/c_t and it would be beneficial to invest less in pollution control. If, on the other hand, the pollution stock rose at the same rate as consumption, $v'(c_t)/\left(\psi\phi'(S)\right)$ would fall relative to S_t/c_t and it would be beneficial to restrict pollution growth. It follows that the pollution stock must rise in the long run but at a rate sufficiently below the consumption growth rate for the limit of the ratio $\frac{v'(c_t)}{\psi\phi'(S)}/(\frac{S_t}{c_t})$ to be constant.
In the opposite case with $\sigma_c < 1$, instantaneous marginal utility of consumption falls overproportionally in response to an increase in the consumption level. In this case, the pollution stock must fall in the long run to satisfy the ABG-condition.

The pollution stock can at most decrease at the rate of natural regeneration $(\widehat{S}_t \geq (-\delta))$. To actually reach this rate of decrease, flow pollution would have to become zero and all economic activity would have to be given up. This path for pollution and environmental quality is clearly never optimal, given that the utility function satisfies the Inada-conditions for consumption. Still, it can be optimal to decrease the pollution flow particularly fast so that $\widehat{S}_\infty = (-\delta)$ is approached *asymp-*

[18] Note that (2.54) also suggests that under more general assumptions concerning the utility function, whether the pollution stock de- or increases depends on σ_E being smaller or larger than one as well. For $\sigma_E > 1$, pollution is allowed to rise only if $\sigma_C < 1$ while a falling pollution stock is required for $\sigma_C > 1$.

totically[19]. Such a solution becomes more likely, if the representative household is particularly patient to desire a comparatively large consumption growth rate and the rate of natural regeneration is small: It follows from (2.54) that the pollution growth rate must converge to $\widehat{S}_\infty = (-\delta)$ whenever $\widehat{c}_\infty \geq \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$. This constraint defines an upper bound ρ^{delta} for the rate of time preference so that $\widehat{S}_\infty = (-\delta)$ if and only if $\rho \leq \rho^{\text{delta}}$.[20]

Corollary 2.1 *Assume convex disutility of pollution ($\sigma_E < 1/2$).*

If and only if $\sigma_c < 1$ and the representative household is sufficiently patient ($\rho \leq \rho^{delta} < \overline{\rho}$) so that the optimal growth rate of consumption per capita is equal to or larger than $\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$, the pollution stock decreases with the rate $\widehat{S}_\infty = (-\delta)$ asymptotically.

Proof. Whenever $\sigma_c < 1$, the pollution stock decreases in the long-run optimal solution according to proposition 2.4. The pollution stock can at most decline at rate $\widehat{S}_\infty = (-\delta)$. It follows that for $\sigma_c < 1$, condition (2.54) can hold with $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$ for consumption growth rates $\widehat{c}_\infty < \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$ only. If $\rho \leq \rho^{\text{delta}}$, so that the optimal consumption growth rate is $\widehat{c}_\infty \geq \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$, pollution growth must converge towards $\widehat{S}_\infty = (-\delta)$. ■

Green innovation, deceleration and the direction of technical change

As shown in proposition 2.3, long-run growth in the optimal solution requires to persistently restrict pollution growth. In the following, we characterize the interior long-run optimal solution for any set of parameters. In proposition 2.5, we analyze in detail the long-run solution of the social planner's maximization problem for parameter constellations for which the conditions from corollary 2.1 are not satisfied so that pollution growth is given by $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$. The case with $\widehat{S}_\infty = (-\delta)$ yields similar results and is treated in proposition 2.6. Figure 2.2 gives an overview over the different cases in dependence on the rate of time preference, ρ.

We have suggested earlier that if growth rates are to be constant asymptotically, the allocation of labor and the step-size q and b must be constant in the long run.

[19] As pollution falls, the amount $(-\delta S)$ of the pollution stock which is cleaned up by natural processes also declines. As long as the pollution flow does not persistently decrease faster than $(-\delta S)$, the growth rate of the pollution stock cannot continuously fall towards its lower bound $(-\delta)$. The long-run growth rate of the pollution stock equals the growth rate of the flow in this case. For $\widehat{S}_t$ to converge towards $(-\delta)$ for $t \to \infty$, the constant long-run growth rate $\widehat{X}_\infty - \widehat{B}_\infty$ of the pollution flow X/B must not exceed the long-run growth rate of $(-\delta S)$, which is $(-\delta)$.

[20] ρ^{delta} again differs for the balanced-growth and the ABG-case. It is defined for each case in appendix 2.B.4.

In this case, equation (2.56) requires intermediate quantity in efficiency units, more precisely the ratio $(X/Q)_\infty$, to be constant in the limit as well.

Assume that there exists a balanced-growth path along which productivity and cleanliness grow at constant rates, not only asymptotically. Such a path must be characterized by a strictly positive $(X/Q)_\infty$[21] and therefore equal growth in intermediate quantity and productivity. It follows from the resource constraint that consumption c will also grow at the same rate.

Equation (2.54) then gives information about how strongly research has to be oriented towards green innovation to achieve balanced pollution growth. The orientation of research is given by

$$\widehat{B}_\infty/\widehat{Q}_\infty = 1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}. \tag{2.58}$$

Given convex disutility of pollution ($\sigma_E < 1/2$), the ratio $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ is smaller than one so that $\widehat{B}_\infty/\widehat{Q}_\infty$ is always strictly positive.

To understand (2.58), assume - besides convex disutility of pollution - that the IES in consumption, σ_c, is smaller than 1. The optimal pollution path in a growing economy must then be negatively sloped by proposition 2.4. The ratio $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ is strictly negative, so that $\widehat{B}_\infty/\widehat{Q}_\infty$ is larger than one: Along a balanced growth path, one percent growth in productivity leads to one percent growth in polluting quantity. For the pollution stock to fall, a more than one percent reduction in pollution intensity is needed. Research and technical change are oriented towards green innovation. As σ_E increases from close to zero to $1/2$, the intertemporal elasticity of substitution in pollution ($\sigma_E/(1-2\sigma_E)$) increases and the optimal pollution path becomes steeper, ceteris paribus. The ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ increases as well because relatively more green research is needed to achieve a faster pollution reduction.
If $\sigma_c > 1$, the pollution stock increases in the long run and research and technical change are oriented towards improving productivity ($\widehat{B}_\infty/\widehat{Q}_\infty < 1$). As a response to an increase in σ_E, research is now shifted more strongly towards productivity-improvements because there is less aversion towards pollution growth.

However, the research arbitrage equation (2.56) implicitly defines an upper bound for the orientation of research towards green innovation. On a balanced growth path with strictly positive $(X/Q)_\infty$, the optimal b_∞/q_∞ and therefore $\widehat{B}_\infty/\widehat{Q}_\infty$ must be smaller than $\alpha/(1-\alpha)$. If (2.58) exceeds this ratio, a balanced-growth solution of

[21] On a balanced growth path, $(X/Q)_\infty = 0$ implies $X_t/Q_t = 0$ for all t. This is only possible if $X_t = 0$ for all t which, as explained before, cannot be an optimal path for X because of the Inada-conditions for consumption.

the social planner's problem does not exist.

If we allow for growth rates to be constant only in the limit, equation (2.56) can still be satisfied by choosing the upper bound

$$b_\infty/q_\infty = \widehat{B}_\infty/\widehat{Q}_\infty = \alpha/(1-\alpha) \tag{2.59}$$

for research orientation together with persistent deceleration asymptotically. Keeping growth in intermediate quantity below productivity growth in the long run dampens the rebound effect of higher productivity and decreases the ratio X/Q towards zero.

Given the research orientation in (2.59) and using the relations $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$ and $\widehat{c}_\infty = \alpha\widehat{X}_\infty + (1-\alpha)\,\widehat{Q}_\infty$, from the resource constraint (2.52), we can derive the optimal relation between the long-run growth rates of $\widehat{X}_\infty$ and $\widehat{Q}_\infty$ from the ABG-condition (2.54) after some manipulation:

$$\widehat{X}_\infty = \frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2 - \left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} - \frac{\alpha}{1-\alpha}\right)}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\widehat{Q}_\infty \tag{2.60}$$

For $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, the fraction is smaller than one so that intermediate quantity grows indeed more slowly than productivity. The further $\frac{\alpha}{1-\alpha}$ lies below $1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, the smaller is the ratio and the faster is therefore deceleration.[22]

We substitute the expression for $\widehat{X}_\infty$ into $\widehat{c}_\infty = \alpha\widehat{X}_\infty + (1-\alpha)\,\widehat{Q}_\infty$ to find $\widehat{c}_\infty$ as function of $\widehat{Q}_\infty$:

$$\widehat{c}_\infty = \frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\widehat{Q}_\infty \tag{2.61}$$

For $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, it becomes obvious that keeping growth in intermediate quantity below productivity growth comes at a cost in terms of potential consumption growth.

We summarize our results in the following proposition, where we define the lower bound $\underline{\rho} := \left\{\begin{smallmatrix}\rho^{\mathrm{TVC}}, & \sigma_c>1\\ \rho^{\mathrm{delta}}, & \sigma_c<1\end{smallmatrix}\right.$ so that if $\rho > \underline{\rho}$, the transversality conditions in (2.53) are

[22] We say that deceleration is the faster, the smaller the growth rate of X and therefore the growth rate of c and Y is relative to the productivity growth rate. When $\widehat{X}_\infty/\widehat{Q}_\infty$ is smaller, a smaller proportion of every unit of productivity growth is used to increase polluting quantity and a smaller fraction is therefore turned into output and consumption growth.
According to this definition, faster deceleration does not imply that the ratios X/Q, c/Q and Y/Q decline faster, because if productivity growth slows sufficiently, the difference $\widehat{X}-\widehat{Q}$ does not decrease.

satisfied[23] and corollary 2.1 does not apply ($\widehat{S}_\infty > (-\delta)$).

Proposition 2.5 ***(A)BG optimum for*** $\widehat{S}_\infty > (-\delta)$
Assume $\sigma_E < 1/2$ so that the disutility of pollution is convex and $\underline{\rho} < \rho < \overline{\rho}$:

Green innovation without deceleration: *If $\alpha/(1-\alpha) > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, there exists a unique BG-path which solves the set of necessary conditions. X_t, Y_t, c_t and Q_t grow at the same constant rate. Growth in the pollution stock $\widehat{S}_\infty$ equals the growth rate of flow pollution, $\widehat{X}_\infty - \widehat{B}_\infty = \widehat{Q}_\infty - \widehat{B}_\infty$. There is green innovation ($\widehat{B}_\infty > 0$) but no deceleration ($\widehat{X}_\infty = \widehat{Q}_\infty$). The ratio of green relative to productivity-improving innovation is given by (2.58).*

Green innovation with deceleration: *For $\alpha/1-\alpha < 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, a BG-solution does not exist. There exists an asymptotically unique ABG-path which solves the necessary conditions for $t \to \infty$. Pollution growth $\widehat{S}_\infty$ equals the growth rate of flow pollution, $\widehat{X}_\infty - \widehat{B}_\infty$. $\widehat{S}_\infty$ is reduced below the potential rate $\widehat{Q}_\infty$ both by green innovation ($\widehat{B}_\infty > 0$) and through deceleration ($\widehat{X}_\infty < \widehat{Q}_\infty$, $\widehat{c}_\infty < \widehat{Q}_\infty$). The ratio of green relative to productivity-improving innovation is given by (2.59).*

Proof. See appendix 2.B.7. ■

It is intuitive that optimal pollution control always includes green innovation: Once research units are opened up, it is almost costless to make intermediate goods marginally cleaner while making them more productive. Extra costs to reduce the emissions of a new more powerful engine will be relatively low if the fixed costs (e.g. for equipment and fixed labor costs) have been paid.

Unlike green innovation, deceleration is not always optimal in a growing economy as the costs in terms of foregone consumption growth may be substantial. Reductions in pollution intensity are optimally combined with deceleration if the elasticity α of final good production $Y_t = X_t^\alpha (Q_t L_{Yt})^{1-\alpha}$ with respect to quantity is sufficiently small.

In this case, the cost of deceleration is comparatively low: A small elasticity implies that polluting quantity growth has only a small effect on output growth compared to quality growth. Quantity growth can be restricted without giving up too much consumption growth.

Further, with the relative unattractiveness of growth in polluting quantity for small α, it becomes less important to reduce the pollution intensity of intermediate goods. A small α lowers the social return to green as opposed to productivity-improving

[23] The critical value ρ^{TVC} is defined in appendix 2.B.4.

research. It becomes more likely that the ratio of green relative to productivity-oriented innovation in (2.58), needed to sustain the optimal pollution path with balanced growth, exceeds the upper bound $\alpha/(1-\alpha)$. Research then remains rather productivity-oriented but deceleration lowers the rebound effect of productivity growth and thereby helps to restrict pollution growth.

A solution without deceleration becomes less likely as σ_E increases, whenever the pollution stock falls in the long-run ($\sigma_c < 1$) and more likely if it rises ($\sigma_c > 1$). The reason is that the steeper pollution path implies a stronger orientation of R&D towards green innovation in the former and a stronger orientation towards productivity-oriented innovation in the latter case.

We now study the long-run optimal solution of our model for parameter constellations such that the condition $\rho \leq \rho^{\text{delta}}$ of corollary 2.1 is satisfied and the pollution stock must decrease with the maximum rate δ asymptotically. Recall that $\rho \leq \rho^{\text{delta}}$ requires slow natural regeneration and implies that the representative household is patient and desires a comparatively high consumption growth rate. As before, we find two types of optima - one with and one without deceleration - depending on the specification of model parameters.

Proposition 2.6 ***ABG optimum for*** $\widehat{S}_\infty = (-\delta)$
Assume $\sigma_c < 1$ *and convex disutility of pollution (*$\sigma_E < 1/2$*). Assume further, that the condition of corollary 2.1 holds (*$\rho \leq \rho^{delta}$*).*

Green innovation without deceleration: *If* $1/\sigma_c < \frac{\alpha}{1-\alpha}$ *or if* $1/\sigma_c > \frac{\alpha}{1-\alpha}$ *and* $\widehat{c}_\infty < \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c)-\frac{\alpha}{1-\alpha}}\delta$*, any ABG-path without deceleration which solves the necessary first-order conditions for* $t \to \infty$ *is characterized by a stronger orientation of research towards green innovation (larger* $\widehat{B}_\infty/\widehat{Q}_\infty$*), compared to the corresponding case in proposition 2.5.*

Green innovation with deceleration: *If* $1/\sigma_c > \frac{\alpha}{1-\alpha}$ *and* $\widehat{c}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c)-\frac{\alpha}{1-\alpha}}\delta$*, there exists an asymptotically unique ABG-path characterized by deceleration which solves the necessary first-order conditions for* $t \to \infty$*. The orientation of research is the same as in the corresponding case in proposition 2.5.*

Proof. See appendix 2.B.8. ■

As in the preceding subsection, whether there is deceleration in the long-run optimal solution depends on how elastic the production function is with respect to intermediate quantity and quality. For $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, the conditions for deceleration are still satisfied.[24] Additionally, there is deceleration in the long-run optimal solution also for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ if $1/\sigma_c > \frac{\alpha}{1-\alpha}$ and the desired consumption growth rate is sufficiently large ($\widehat{c}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c)-\frac{\alpha}{1-\alpha}}\delta$).

Because research orientation is green for $\sigma_c < 1$, the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ must exceed one. At the same time, it must remain below the upper bound $\frac{\alpha}{1-\alpha}$. It follows that the case where the long-run optimal solution is not characterized by deceleration can only occur if $\frac{\alpha}{1-\alpha} > 1$, i.e. if the production elasticity of intermediate quantity exceeds 0.5. From empirical estimates, this is not a realistic range, as we argue in section 2.4.3 below.

An overview over the different cases for the long-run optimal solution outlined in propositions 2.5 and 2.6 is given in figure 2.2.

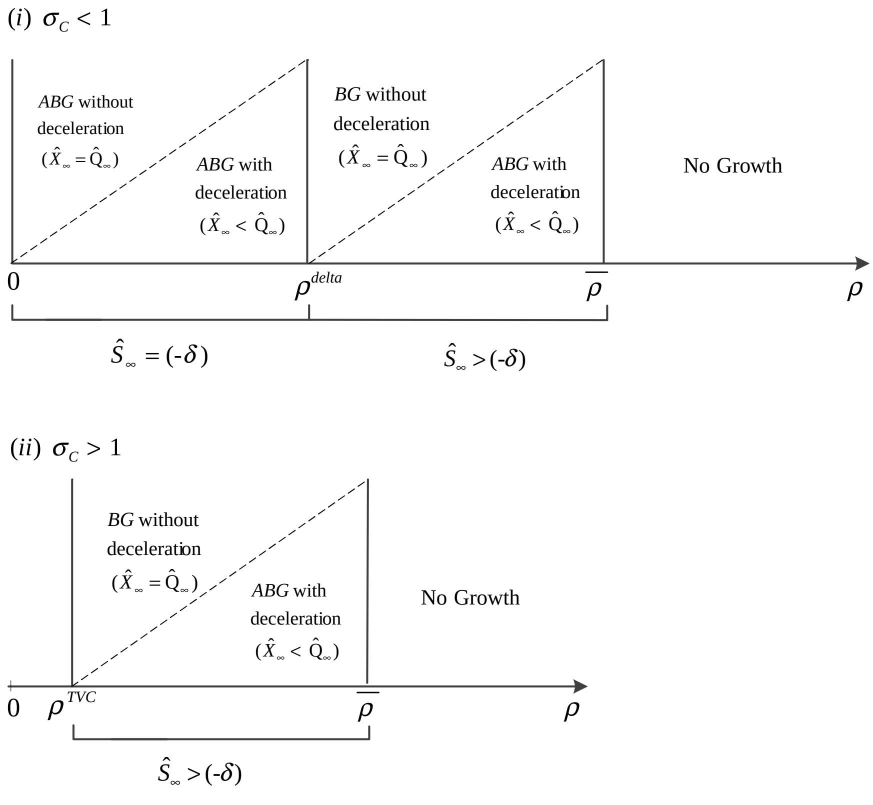

Figure 2.2: Case differentiation for the long-run optimal solution in dependence of the rate of time preference, ρ. Broken lines indicate a further partition of the parameter space in the relevant range of ρ, given by proposition 2.5 for $\widehat{S}_\infty > (-\delta)$ and by 2.6 for $\widehat{S}_\infty = (-\delta)$.

[24] First, $1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E} < 1/\sigma_C$ if $\sigma_C < 1$ and the disutility of pollution is convex ($\sigma_E < 1/2$). $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}$ and $1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E} < 1/\sigma_C$ implies that $\frac{\alpha}{1-\alpha} < 1/\sigma_C$. Given $\frac{\alpha}{1-\alpha} < 1/\sigma_C$ and $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}$, the condition for $\widehat{S}_\infty$ to converge to $(-\delta)$, which is $\widehat{c}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_C)/\sigma_C}\delta$, implies $\widehat{c}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_C)-\frac{\alpha}{1-\alpha}}\delta$.

From the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ in propositions 2.5 and 2.6, we can directly determine the direction of technical change:

Corollary 2.2 ***Direction of technical change***
Given the conditions of propositions 2.5 and 2.6, the direction of technical change is green (productivity-oriented), i.e., $\widehat{B}_\infty > \widehat{Q}_\infty$ $(\widehat{B}_\infty < \widehat{Q}_\infty)$*, if and only if* $\sigma_c < 1$ $(\sigma_c > 1)$ *when there is no deceleration. With deceleration, the direction of technical change is green (productivity oriented), i.e.,* $\widehat{B}_\infty > \widehat{Q}_\infty$ $(\widehat{B}_\infty < \widehat{Q}_\infty)$*, if and only if* $\alpha > 1/2$ $(\alpha < 1/2)$*.*

Proof. The ratio $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ in (2.58) is negative (positive) so that $\widehat{B}_\infty/\widehat{Q}_\infty > 1$ $(\widehat{B}_\infty/\widehat{Q}_\infty < 1)$ in the balanced-growth case in proposition 2.5 whenever $\sigma_c < 1$ $(\sigma_c > 1)$. Proposition 2.6 is only relevant for $\sigma_c < 1$ and the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ exceeds the value in proposition 2.5 if there is no deceleration. It follows that without deceleration, technological change is green $(\widehat{B}_\infty/\widehat{Q}_\infty > 1)$ if and only if $\sigma_c < 1$ and productivity-oriented $(\widehat{B}_\infty/\widehat{Q}_\infty < 1)$ if and only if $\sigma_c > 1$. With deceleration, $\widehat{B}_\infty/\widehat{Q}_\infty = \alpha/(1-\alpha)$ in propositions 2.5 and 2.6 so that technical change is green whenever $\alpha > 1/2$ and productivity-oriented whenever $\alpha > 1/2$. ■

Quantity degrowth A very strong form of deceleration occurs if intermediate quantity falls in absolute terms, not only per labor efficiency unit. There is then degrowth in intermediate quantity (but not in GDP). Because quantity degrowth is deceleration in its extreme, it is optimal only if the pollution stock is required to decline in the long-run optimum (for $\sigma_c < 1$) and the ratio of production elasticities is particularly small. This result follows as a corollary from propositions 2.5 and 2.6:

Corollary 2.3 ***Quantity degrowth***
Given the conditions of proposition 2.5, the long-run solution of the social planner's problem is characterized by quantity degrowth $(\widehat{X}_\infty < 0)$ *if and only if the condition* $\frac{\alpha}{1-\alpha} < (1-\alpha)\frac{(1-\sigma_c)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ *holds.*
Given the conditions of proposition 2.6, there is quantity degrowth if and only if $\frac{\alpha}{1-\alpha} < (1-\alpha)\frac{1-\sigma_c}{\sigma_c}$ *and* $\widehat{Q}_\infty$ *is sufficiently large.*

Proof. Proof follows directly from setting $\widehat{X}_\infty < 0$ in equation (2.60) and equation (2.B.48) in the appendix. ■

We will prove in chapter 5 that, given quantity degrowth, assuming intermediate goods are energy-inputs produced from a non-renewable resource leads to the same qualitative results for the social planner's solution if the initial resource stock is sufficiently large. The pollution externality induces such a strong decline in resource use over time that the stock is never exhausted and the natural resource constraint is not binding.

2.4.3 A numerical example

Which of the cases in the previous subsection is more relevant empirically? Should green innovation be complemented by deceleration or even quantity degrowth in the long-run? Although our model is too stylized to produce reliable quantitative results, we think it allows to give a plausible indicative answer at least to these general qualitative questions. The relevant parameters are α, σ_c and σ_E. While there are little reliable empirical estimates of σ_E, we believe that disutility is convex in the pollution stock ($\sigma_E < 1/2$) so that the marginal disutility of pollution is the larger, the more polluted the environment is. As for the IES of consumption σ_c, a large body of empirical literature (e.g., Hall (1988), Ogaki and Reinhart (1998)) suggests $\sigma_c \in (0,1)$. Choosing α is less straightforward. Setting α to the capital share implies $\alpha \approx 1/3$. Interpreting X_t as energy, α would be substantially smaller than the capital share. On the other hand, α is also the inverse of the mark-up in the intermediate sector. Estimates for the manufacturing sector in the U.S. (Roeger (1995)) suggest values of α of at least 0.3. We consider values of α which do not exceed 0.5 as plausible.

Corollary 2.4 *Assume $\rho < \overline{\rho}$, convex disutility of pollution ($\sigma_E < 1/2$), an intertemporal elasticity of substitution in consumption $\sigma_c \in (0,1)$ and $0 < \alpha \leq 1/2$.*

Productivity growth always has to be accompanied by both green innovation and deceleration. Research is productivity-oriented ($\widehat{Q}_\infty > \widehat{B}_\infty > 0$), but deceleration restricts the rebound effect of productivity growth.

Proof. We have shown in section 2.4.2 that for convex disutility of pollution, the condition $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ is necessary and sufficient for the long-run optimum to be characterized by deceleration in the case where $\widehat{S}_\infty > -\delta$ and a sufficient condition if $\widehat{S}_\infty$ converges towards $(-\delta)$. For $0 < \alpha \leq 1/2$, this condition holds for all $\sigma_c \in (0,1)$. ■

If we choose a smaller range for α, so that α does not exceed the capital share of 1/3, there is deceleration in the optimal solution for $\sigma_c < 2$ which covers most empirical estimates of the IES in consumption. Setting α to the energy share in real GDP[25], even extremely high values of σ_c up to 4.4 as found by Fuse (2004) for Japan do not violate the condition.

For intermediate quantity to fall in absolute terms, so that there is quantity degrowth, σ_c must be smaller than one and α should be substantially below the capital share. Quantity degrowth is most likely to be optimal if α is interpreted as the energy share in GDP: Setting $\alpha \approx 0.09$, the optimal solution is characterized by quantity degrowth for values of σ_c from almost the entire interval $(0, 1)$ if $1/3 \leq \sigma_E < 1/2$.

We conclude that for reasonable assumptions about model parameters, both green innovation and deceleration (possibly in its extreme form, i.e., quantity degrowth) contribute to optimal pollution control. In chapter 4, we illustrate the adverse effects on long-run consumption growth if only one channel of pollution control is available to the social planner.

2.4.4 Environmental care and the pace of economic growth

In our model, a stronger research orientation towards green innovation means slower productivity growth for given total research effort. Further, deceleration requires to give up potential consumption growth. Intuitively, one might therefore expect environmental care to slow down economic growth relative to the case where the negative environmental externality of intermediate goods is not taken into account. In appendix 2.B.9, we derive the long-run optimal solution of our model assuming the weight of pollution in utility is zero ($\psi = 0$), so that the representative household is not affected by pollution. Comparing the optimal solution of the baseline model to the optimum in the modified setting, we find that the above intuition is not necessarily correct. Economic growth is positive for larger rates of time preference in the baseline framework and, depending on parameters, the long-run growth rates of consumption, production and productivity may in fact be higher than in the setting without negative external effect from pollution.[26]

Moreover, the degree of the household's preference for a clean environment and therefore the strength of the negative pollution externality, as reflected in the size of ψ, does not influence long-run growth rates at all (given $\psi > 0$). The reason is that

[25] Energy expenditures as a share of GDP amounted to 8.9% in the U.S. in 2012 (EIA (2013)).

[26] A similar result can be obtained if the optimal solution for $\psi > 0$ is compared not to the optimum for $\psi = 0$ but to the laissez-faire equilibrium. It is, however, not possible in this case to attribute faster growth to the environmental externality in particular because equilibrium growth may be slower or faster than optimal as a result of several other externalities.

stronger environmental preference does not alter the social return to productivity-oriented research, which is the driver of economic growth. The long-run relation between productivity growth and growth in intermediate quantity, consumption and output is fixed independently of the environmental preference on an ABG-path.[27] We prove our claim formally for parameter constellations for which the long-run optimal solution is characterized by deceleration:

Proposition 2.7 ***Environmental care and the pace of economic growth***
Assume that the disutility of pollution is convex ($\sigma_E < 1/2$) and that the conditions for deceleration from propositions 2.5 and 2.6 are satisfied.

In the baseline setup with $\psi > 0$, compared to the optimal solution in a modified setting without negative external effect from pollution on utility ($\psi = 0$), (i) the condition for growth in per capita consumption to be positive is less strict ($\overline{\rho}^{\psi=0} < \overline{\rho}$) and (ii) optimal growth in per capita consumption is faster if and only if the rate of time preference is sufficiently large.
Given $\psi > 0$, the strength of the representative household's preference for a clean environment, as reflected in the size of ψ, has no influence on long-run optimal growth rates.

Proof. See appendix 2.B.10. ∎

The driving force behind this result is a positive link between green and productivity-oriented research. Green innovation can lead to an increase in the optimal amount of labor devoted to research which fosters also productivity- and therefore consumption growth. We show in chapter 4 that in the constrained optimum without green innovation ($b_t = 0$, $\forall t$), consumption growth is unambiguously slower than in the unconstrained optimal solution with $\psi = 0$.

2.5 Discussion of the model specification

The results of our model are affected by the parameters of the utility function and the production function for the final consumption good in particular. While the assumptions of additively separable utility and a Cobb-Douglas production function are somewhat restrictive, they are commonly used in previous literature and

[27] A similar result was found by Gradus and Smulders (1993) in a Lucas–Uzawa-model. While stronger environmental preference has no influence on long-run growth rates, it can be expected to affect the levels of the model variables along the long-run path. These effects can however not be analyzed without studying transitional dynamics.

make the model analytically convenient to handle. In this section, we consider more general specifications of the utility function and the production function for the consumption good. We also discuss alternative forms for the functions which describe production in the intermediate goods sector, pollution accumulation and environmental quality, as well as research and development. The discussion suggests that the specification of the model described above is conveniently simple, yet intuitive. The main results are robust to several changes in the model setup, notably the assumption of non-additively-separable preferences.

Our assumption of an additively-separable utility function is in line with previous important contributions (e.g., Stokey (1998), Aghion and Howitt (1998)). A more general specification, which allows for constant long-run growth rates, is proposed by Gradus and Smulders (1996):

$$U(c_t, S_t) = \begin{cases} \int_{t=0}^{\infty} e^{-\rho t} \frac{\sigma}{\sigma-1} \left(c_t S_t^{-\psi} \right)^{\frac{\sigma-1}{\sigma}} dt & \sigma \neq 1 \\ \int_{t=0}^{\infty} e^{-\rho t} \left(\ln c_t - \psi \ln S_t \right) dt & \sigma = 1 \end{cases} \tag{2.62}$$

$\sigma > 0$ is the intertemporal elasticity of substitution and $\psi > 0$ still the weight of the pollution stock in utility. For $\sigma = 1$, the utility function reduces to an additively separable form. With this specification of utility, the marginal utility of consumption is given by:

$$u_{c,t} = \left(c_t S_t^{-\psi} \right)^{\frac{\sigma-1}{\sigma}} c_t^{-1}$$

A higher pollution stock lowers (increases) the marginal utility of consumption whenever $\sigma > 1$ ($\sigma < 1$). As already explained in chapter 1, growing pollution then ceteris paribus tends to depress consumption growth relative to the additively-separable case, even under laissez-faire, whenever $\sigma > 1$ and accelerates economic growth whenever $\sigma < 1$.

Our main results concerning optimal pollution control remain unaffected by the switch to the more general specification (2.62), more precisely: Long-run growth always goes along with green innovation and, depending on parameters, requires deceleration. The non-additively-separable utility function complicates however the solution of the social planner's problem. In the case without deceleration, it is not possible to derive a complete analytical solution even for the long-run growth rates. It should be noted that the robustness of our main results does not require the standard assumptions of convex disutility of pollution and a marginal utility of consumption which is non-increasing in the pollution stock ($u_{cS} \leq 0$).

In fact, the above specification of utility does not allow for the marginal utility of consumption to decrease in the pollution stock and the disutility of pollution to be convex at the same time.[28] This shortcoming could be corrected by changing the relation between environmental quality and pollution from $E_t = S_t^{-1}$ to

$$E_t = E^{\max} - S_t.$$

With this specification, however, the pollution stock necessarily has to be constant in the long-run, as explained in chapter 1. Along a balanced-growth path without deceleration, productivity and cleanliness therefore optimally grow at the same rate. Still, conclusions concerning the desirability of long-run growth and optimal pollution control are similar to those obtained with the baseline specification.[29]

Besides the relation between environmental quality and pollution, another important equation is the accumulation function for the pollution stock. We assume that emissions are generated by polluting production inputs X. As indicated in chapter 1.2, an alternative assumption often used in the environmental-economic literature is that emissions are proportional to the entire GDP, Y, rather than to specific inputs. The pollution accumulation function is then given by

$$\dot{S}_t = \kappa_{Yt} Y_t - \delta S_t. \tag{2.63}$$

In such a setting, all inputs in GDP production contribute to pollution growth and more productive intermediates generate more emissions. Naturally, in such a world, saving on the polluting input through deceleration cannot reduce pollution growth. It can be shown that there will not be deceleration in the long-run optimum, if the pollution accumulation function is given by (2.63).

In fact, however, not the size of GDP but the extent to which its production uses polluting inputs and processes determines emissions. Higher productivity does not by itself accelerate pollution accumulation. It may only do so indirectly through rebound effects on the use of polluting inputs. However, whether it does depends on the behavior of economic agents and their willingness to save on polluting inputs despite higher productivity. We have shown that if the possibility to control the rebound effect is not taken into account, an important channel to decouple economic

[28] The convexity assumption for the disutility of pollution is satisfied if and only if $\psi(1 - 1/\sigma) < (-1)$, which implies $\sigma < 1$, while $u_{cS} < 0$ if and only if $\sigma > 1$.

[29] The laissez-faire solution with unconstrained pollution growth on the other hand will violate the non-negativity constraint $E_t \geq 0$ at some point in time if consumption grows persistently. An interesting question studied by Michel and Rotillon (1995) is whether a 'distaste effect' of pollution ($u_{cS} < 0$) will halt long-run consumption growth. The authors show that despite the distaste effect, consumption growth remains positive.

growth and environmental degradation is neglected.

Now consider the specification of the production functions. The Cobb-Douglas production function we have chosen for the consumption goods sector is a special case of the more general CES-production function

$$Y_t = \left(\alpha X_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\left(L_{Yt}Q_t\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad 0 < \varepsilon < \infty.$$

Long run growth rates and the question how pollution control is achieved in the long-run optimum now depend on the elasticity of substitution ε. If intermediate goods and the labor input are bad substitutes ($\varepsilon < 1$), deceleration cannot be optimal, as the relative marginal product of intermediates X relative to effective labor $L_Y Q$ rises overproportionally in response to a decline in the ratio X/Q. Green innovation now becomes particularly important, as pollution control can only be achieved by reducing the pollution intensity of intermediate goods. Without the possibility to develop cleaner inputs, long-run economic growth would not be optimal (see Acemoglu et al. (2012), for example). In our setup with green innovation, on the contrary, persistent growth is still socially desirable and reconcilable with environmental preservation.

For $\varepsilon > 1$, on the other hand, the polluting input is not essential for production. Intermediate quantity can be substituted by an entirely clean alternative input without incurring a loss in potential long-run output and consumption growth. Even though X/Q may fall and become zero in finite time, there is therefore no deceleration. When polluting emissions decline to zero, in finite time or asymptotically, green innovation becomes superfluous. We believe this case to be less relevant empirically, as no currently existing substitutes for polluting production inputs and processes are entirely clean.

Regarding the production function for intermediates, we have assumed that intermediate goods are produced with labor, and that overall productivity Q has a positive spillover on intermediate production. Therefore, even though labor is in fixed supply, intermediates are accumulative goods. Whether intermediate production uses (effective) labor or GDP affects long-run growth rates in the laissez-faire equilibrium but not in the optimal solution. It does not alter our main results. Given that intermediate goods are essentially inputs to their own production process when intermediate production uses GDP, the specification with labor as chosen in this model seems more appealing.

In chapter 5, we analyze how the results of our model are affected when intermediate production uses a non-renewable resource as the only input so that production is limited by the finiteness of the resource stock.

The choice of alternative specifications for the R&D-cost function is limited. Our specification is carefully chosen so as to guarantee that there exists a well-defined, interior laissez-faire equilibrium and social optimum characterized by constant growth rates asymptotically.
To ensure a unique finite choice for the step-size in technology both at the laissez-faire equilibrium and in the social optimum, research costs must be convex in the step-size q and b respectively (the innovation possibilities frontier must be concave). Fixed costs are needed to avoid that the mass of research units tends to infinity while the technology improvement within each unit converges to zero ($n \to \infty$, $q \to 0$, $b \to 0$). Finally, the specification of spillovers from sectorial and aggregate technology levels makes the setup reconcilable with (asymptotically) balanced growth.
If we assumed that given the R&D-cost function (2.9), not labor but final output was required to undertake R&D, the social planner's problem would produce the same qualitative results. There would however be no balanced growth in the laissez-faire equilibrium because, while output rises over time, research costs remain constant due to the spillovers. Changing the specification of the R&D-production function so as to make it compatible with constant growth also under laissez-faire requires a highly asymmetric function which is difficult to justify.

Overall, the discussion suggests that the model specification, while restrictive in some aspects, is intuitive and convenient to handle analytically. The main results are robust to several changes in the model setup, notably the assumption of non-additively-separable preferences.

2.6 Conclusion

In this chapter, we considered a Schumpeterian growth model with polluting intermediate production, where technical progress can increase the productivity and lower the pollution intensity of intermediate goods. Pollution accumulation can be controlled by green innovation and by deceleration. The latter means that growth in intermediate quantity is kept below productivity growth so that intermediate goods are used more efficiently over time. This goes along with a cost in terms of foregone potential growth in consumption and GDP.

We have shown that under laissez-faire, neither green innovation nor deceleration is chosen in the long run equilibrium. If utility is concave in consumption but the disutility of pollution is convex in the pollution stock, a path without long-run growth would be socially preferable.

The long-run social optimum allows for a constant positive long-run consumption growth rate whenever the representative household is sufficiently patient. However, persistent economic growth has to be accompanied by persistent pollution control. It is always optimal to allocate a part of total labor supply to green innovation. If production is sufficiently inelastic with respect to intermediate quantity, green innovation is optimally complemented by deceleration to dampen the rebound effect of productivity growth. Polluting quantity growth has only a minor effect on output growth compared to productivity growth in this case. Deceleration therefore allows to gain from productivity growth in a relatively clean way, without incurring a large loss in potential consumption growth. The relative social return to green innovation on the other hand is small, so that research is rather productivity-oriented.
We developed a numerical example showing that this case is relevant for reasonable assumptions about the parameter constellation. While deceleration in the presence of persistent productivity growth does not require that intermediate quantity falls in absolute terms, it is not unlikely that this strong form of deceleration, which we call 'quantity degrowth', is optimal. Yet even with quantity degrowth, consumption and GDP may still rise.

Because there are concerns both in scientific and public debate that environmental care may entail large costs in terms of economic growth, we were interested in singling out the effect of the environmental externality on the pace of long-run consumption growth in the long-run optimum. We therefore compared the long-run optimal consumption growth rate of the baseline model to the optimal consumption growth rate in a modified setting, when neither environmental quality nor the pollution stock enters the utility function. We proved that despite the before-mentioned concerns, consumption growth may be faster when the representative household cares for a clean environment. The underlying mechanism is that green innovation attracts labor to the R&D-sector, thereby stimulating also productivity growth. The strength of the household's preference for a clean environment, given that there exists an external effect on household utility, does not affect long-run optimal growth rates.

2.A Appendix to section 2.3

2.A.1 Consumer maximization

Taking into account the budget-constraint (2.14), and the fixed total labor supply L, we set up the current-value Hamiltonian-function

$$\begin{aligned} H &= \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1-\sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}} \right) L \\ &\quad + v_{At} \left(r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} - c_t L \right) \\ &\quad + \lambda_{Lt} \left(L - \left(\int_0^1 L_{Xit} di + L_{Yt} + L_{Dt} \right) \right) \end{aligned}$$

for the household's maximization problem. v_{At} is the current-value costate variable of assets A in t and λ_{Lt} the Lagrange-multiplier of the constraint on labor. The first-order conditions according to Pontryagin's maximum principle are:

$$\frac{\partial H}{\partial c_t} = 0 \Leftrightarrow v_{At} = c_t^{\frac{-1}{\sigma_c}} \tag{2.A.1}$$

$$\frac{\partial H}{\partial L_{Xit}} = 0 \Leftrightarrow v_{At} w_{Xit} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial L_{Yt}} = 0 \Leftrightarrow v_{At} w_{Yt} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial L_{Dt}} = 0 \Leftrightarrow v_{At} w_{Dt} = \lambda_{Lt}$$

$$\frac{\partial H}{\partial A_t} = \rho v_{At} - \dot{v}_{At} \Leftrightarrow v_{At} r_t = \rho v_{At} - \dot{v}_{At} \tag{2.A.2}$$

$$\frac{\partial H}{\partial v_{At}} = \dot{A}_t \Leftrightarrow \dot{A}_t = r_t A_t + \int_0^1 w_{Xit} L_{Xit} di + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} - c_t L \tag{2.A.3}$$

$$\frac{\partial H}{\partial \lambda_{Lt}} = 0 \Leftrightarrow L = \int_0^1 L_{Xit} di + L_{Yt} + L_{Dt}$$

The first-order conditions for the different types of labor can only be satisfied simultaneously, if firms in the different sectors of intermediate production as well as firms in final good production and research all offer the same wage. The household is then indifferent about the allocation of his labor supply.

The first-order condition for assets, A_t, can be restated as $\widehat{v}_{At} = \rho - r_t$. Log-differentiating both sides of the first-order condition for consumption yields

$$\frac{-1}{\sigma_c} \widehat{c}_t = \widehat{v}_{At}.$$

By substituting the expression for $\widehat{v}_{At}$, we obtain the Euler-equation (2.15).

Besides the first-order conditions, the transversality condition

$$\lim_{t\to\infty}\left(e^{-\rho t}v_{At}A_t\right)=0 \tag{2.A.4}$$

must hold. From (2.A.2), the path for v_{At} can be derived. It is given by $v_{At} = v_{A0}e^{-\int_0^t (r_t-\rho)dt}$. Substituting into the transversality condition shows that the transversality condition implies the no-Ponzi-condition:

$$\begin{aligned}\lim_{t\to\infty}\left(e^{-\rho t}v_{At}A_t\right) &= 0\\ \Leftrightarrow\ v_{A0}\lim_{t\to\infty}\left(e^{-\int_0^t r_v dv}A_t\right) &= 0\end{aligned}$$

As is standard in endogenous growth models, the transversality condition imposes a lower bound on the rate of time preference, ρ, to be derived in appendix 2.A.5 below.

2.A.2 Survival probability

We denoted the probability that the innovator from period t is still producing at a date $s > t$ by $P(s)$. It is the probability that $k = 0$ innovations occur in sector i in the interval $[t, s]$. As innovations in sector i follow a non-homogeneous Poisson-process with arrival rate μ_{iv}, $v \in [t, s]$, the waiting time between innovations is exponentially distributed with parameter μ_{iv}. The probability that the incumbent still holds the monopoly position in sector i at time s is then given by

$$\begin{aligned}P(s) &= \frac{m(s)^k}{k!}e^{-m(s)}\\ &= e^{-m(s)}\end{aligned} \tag{2.A.5}$$

where $m(s) = \int_t^s \mu_{iv}dv$.

2.A.3 Profit maximization and entry in the research sector

The first-order condition for q of the profit-maximization problem is:

$$\begin{aligned} \frac{\partial E\left[\pi_{ijt}^{D}\right]}{\partial q_{ijt}} &= 0 \\ \Leftrightarrow \mu\frac{(1-\alpha)^{1-\alpha}}{\alpha}\alpha^{2(1+\alpha)}\varphi^{\alpha}Q_{it}\int_{t}^{\infty} L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds &= 2w_{Dt}q_{ijt}\frac{Q_{it}}{Q_t} \end{aligned}$$

Upon substitution of w_{Dt} from (2.18) and elimination of equal terms on both sides, we obtain equation (2.24).

The zero-profit-condition

$$E\left[\pi_{ijt}^{D}\right] = \mu E\left[V_{ijt}\right] - w_{Dt}l_{Dijt} = 0$$

simplifies to

$$\alpha\mu(1+q_{ijt})\int_{t}^{\infty} L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds = \left(q_{ijt}^2 + d\right)$$

when using (2.18) for w_{Dt}, (2.9) for l_{Dijt} and (2.22), taking into account the after-innovation productivity level $(1+q_{ijt})Q_{it}$ and $b_{ijt} = 0$ for all i, j, t. We solve for the integral $\int_{t}^{\infty} L_{Ys}e^{-\int_t^s (r_v+\mu_{iv})dv}ds$ and obtain (2.25).

To derive q^{LF}, equation (2.25) is substituted into the first-order condition (2.24) for q_{ijt}. After simplification, the condition can be written as:

$$q_{ijt}^2 + 2q_{ijt} - d = 0$$

Of the two solution candidates for q_{ijt}, only the positive, i.e.

$$q^{\mathrm{LF}} = \sqrt{1+d} - 1$$

is reconcilable with the first-order condition for q_{ijt}, as μ_{it} is non-negative. Therefore $q = \sqrt{1+d} - 1$ in (2.26) is the solution of the optimization problem.

As q_{ijt} is constant over time, it follows that the integral must be constant as well, which yields the condition:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_t^{\infty} L_{Ys} e^{-\int_t^s (r_v+\mu_{iv})dv} ds\right) &= 0 \\ \Leftrightarrow \int_t^{\infty} L_{Ys} e^{-\int_t^s (r_v+\mu_{iv})dv} ds &= \frac{L_{Yt}}{r_t+\mu_{it}}\end{aligned}$$

We use the result to replace the integral on the left-hand side of (2.25), then substitute $q^{\mathrm{LF}} = \sqrt{1+d}-1$ on the right-hand side and solve for n_{it}. We obtain (2.27) in the text.

2.A.4 Allocation of labor

With the labor-market constraint (2.4) and equations (2.7) as well as (2.21), we determine entry n_t along with L_{Xt}, L_{Yt} and, using (2.9), also total research labor L_{Dt} for any given interest rate r_t.

First, it follows from (2.7) with (2.21) that

$$\begin{aligned}L_{Xt} &= \int_0^1 L_{Xit} di \\ &= \frac{\alpha^2}{1-\alpha} L_{Yt}.\end{aligned}$$

The labor-market constraint becomes:

$$\left(1+\frac{\alpha^2}{1-\alpha}\right) L_{Yt} + n_t(q^2+d) = L$$

We substitute q^{LF} and (2.27) into the labor-market constraint and solve for L_Y:

$$L_{Yt} = \frac{1}{\alpha}\frac{\left(\sqrt{1+d}-1\right)}{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d}\left(L+\frac{r_t}{\mu}\left(\left(\sqrt{1+d}-1\right)^2+d\right)\right) \tag{2.A.6}$$

In the calculations, we used $1+\frac{\alpha^2}{1-\alpha}+\frac{1}{2}\alpha\frac{\left(\sqrt{1+d}-1\right)^2+d}{\sqrt{1+d}-1} = \alpha\frac{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d}{\sqrt{1+d}-1}$.

With (2.A.6), labor use in intermediate production is:

$$\begin{aligned} L_{Xt} &= \frac{\alpha^2}{1-\alpha} L_{Yt} \qquad (2.A.7) \\ &= \frac{\alpha}{1-\alpha} \frac{\left(\sqrt{1+d}-1\right)}{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d} \left(L + \frac{r_t}{\mu}\left(\left(\sqrt{1+d}-1\right)^2+d\right)\right) \end{aligned}$$

The expression (2.29) for n given in the text follows upon substitution of (2.A.6) in (2.27), using $\frac{1}{2}\frac{\left(\sqrt{1+d}-1\right)^2+d}{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d} - 1 = \frac{\left(\frac{1}{\alpha}+\frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)}{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d}$. Labor use in R&D is given by:

$$\begin{aligned} L_{Dt} &= \int_0^1 \int_0^{n_{it}} l_{Dijt} dj di \\ &= \int_0^1 n_t \cdot \left(\left(q^{\mathrm{LF}}\right)^2 + d\right) \frac{Q_{it}}{Q_t} di \\ &= \frac{\frac{1}{2}L - \left(\frac{1}{\alpha}+\frac{\alpha}{1-\alpha}\right)\left(\sqrt{1+d}-1\right)\frac{r_t}{\mu}}{\frac{1-\alpha}{\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)\left(\sqrt{1+d}-1\right)+d} \left(\left(\sqrt{1+d}-1\right)^2+d\right) \qquad (2.A.8) \end{aligned}$$

2.A.5 Proof of proposition 2.1

The path defined by the exogenous initial values Q_0, B_0, S_0 for the state variables, the allocation of labor as given by (2.A.6) to (2.A.8) with (2.31) in the text, X_0 from (2.7), $c_0 = Y_0$ from (2.6), the growth rates $\widehat{c}^{\mathrm{LF}} = \widehat{Y}^{\mathrm{LF}} = \widehat{X}^{\mathrm{LF}} = \widehat{Q}^{\mathrm{LF}}$, $\widehat{B}^{\mathrm{LF}} = 0$ in every period t and the pollution accumulation function (2.8) satisfies all the necessary conditions for an equilibrium as defined in section 2.3. If the initial values Q_0, B_0 and S_0 for the state variables are such that with X_0 from (2.7), the pollution accumulation function (2.8) yields the balanced growth rate $\widehat{S}_{\infty}^{\mathrm{LF}}$ in $t = 0$, the path is characterized by balanced growth. It remains to be shown that the sufficient conditions for the optimization problems described in the text are satisfied and that the solution is unique.

The Hamiltonian function for the intertemporal maximization problem of the representative household is strictly concave in consumption and linear in all other variables and we have shown that the transversality condition ensures that the no-Ponzi-condition is satisfied. It follows that the household's maximization problem has a unique solution. The same is true for the static maximization problems in the R&D-sector as well as the production sectors for the consumption good and intermediates, which are concave as well. The path described in the text is therefore the unique laissez-faire equilibrium for $\rho^{\mathrm{TVC,LF}} < \rho < \overline{\rho}^{\mathrm{LF}}$ and, if the initial values Q_0, B_0 and S_0 for the state variables are reconcilable with balanced growth, the unique balanced-growth equilibrium.

The critical value $\rho^{\mathrm{TVC,LF}} := \frac{1}{2}\alpha(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\mu L\,(1+d)^{-1/2}$ is derived from the transversality condition $\lim_{t\to\infty}\left(e^{-\rho t}v_{At}A_t\right) = 0$, which has been shown to equal the condition $v_{A0}\lim_{t\to\infty}\left(e^{-\int_0^t r_v dv}A_t\right) = 0$. Substitution of $A_t = A_0 e^{\widehat{Q}^{\mathrm{LF}}t}$ with $A_0 = 2\frac{(1-\alpha)^{1-\alpha}\alpha^{2\alpha}\varphi^{\alpha}}{\mu}\left(\sqrt{1+d}-1\right)Q_0$ from (2.28) and taking into account that $r_t = r^{\mathrm{LF}}$ for all t shows that the condition can be simplified to $v_{A0}A_0\lim_{t\to\infty}e^{-(r^{\mathrm{LF}}-\widehat{Q}^{\mathrm{LF}})t} = 0$. The transversality condition is satisfied if and only if $r^{\mathrm{LF}} - \widehat{Q}^{\mathrm{LF}} > 0$. With (2.31) and (2.32), it follows that $r^{\mathrm{LF}} - \widehat{Q}^{\mathrm{LF}} > 0 \Leftrightarrow \rho > \frac{1}{2}\alpha(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\mu L\sqrt{1+d}^{-1}$.

2.A.6 Proof of proposition 2.2

For convex disutility of pollution ($\sigma_E < 1/2$), $\frac{1-\sigma_E}{\sigma_E}$ is at least one while $\frac{\sigma_c-1}{\sigma_c}$ is smaller than one. In the balanced-growth equilibrium, $\widehat{S}^{\mathrm{LF}} = \widehat{S}^{\mathrm{LF}}_{\infty} = \widehat{c}^{\mathrm{LF}}$. Instantaneous utility $u_t = \frac{\sigma_c}{\sigma_c-1}c_t^{\frac{\sigma_c-1}{\sigma_c}} - \psi\frac{\sigma_E}{1-\sigma_E}S_t^{\frac{1-\sigma_E}{\sigma_E}}$ converges to $-\psi\phi^S(S_t) = -\psi\frac{\sigma_E}{1-\sigma_E}S_t^{\frac{1-\sigma_E}{\sigma_E}}$ and declines persistently towards $(-\infty)$. The long-run growth rate is $\frac{1-\sigma_E}{\sigma_E}\widehat{S}^{\mathrm{LF}}_{\infty}$.

Now assume instead that economic growth is given up in a period $s < \infty$: Consumption growth drops to zero instantly, while pollution growth converges to zero over time. Initially, the utility loss from forgone consumption growth exceeds the gain from slower pollution growth so that there is a loss in per-period utility compared to the laissez-faire equilibrium. This loss is only transitory: In the long-run, the pollution stock is constant and so is utility. Therefore, from a certain time onwards, not growing yields a utility gain in each period which increases as $t \to \infty$.

Because of the concavity of the utility from consumption and convexity of the disutility from pollution, the transitional welfare loss is smaller, the later in time the switch occurs and converges to zero as $s \to \infty$. Giving up economic growth in the long run therefore yields an increase in intertemporal welfare.

2.B Appendix to section 2.4

2.B.1 Sectorial allocation of intermediate production

The planner solves the static maximization problem

$$\begin{aligned}\max_{X_{it}\geq 0} Y_t &= L_{Yt}^{1-\alpha}\int_{i=0}^{1} X_{it}^{\alpha}Q_{it}^{1-\alpha}di \\ \text{s.t.} \quad \int_{i=0}^{1} X_{it}di &= X_t\end{aligned}$$

We denote by λ_{Xt} the Lagrange-multiplier of the constraint. The necessary first-order condition for an interior maximum of the Lagrangian Ł is:

$$\begin{aligned}\frac{\partial Ł}{\partial X_{it}} &= 0 \\ \Leftrightarrow X_{it} &= \left(\frac{\alpha}{\lambda_{Xt}}\right)^{\frac{1}{1-\alpha}} Q_{it}\end{aligned}$$

We substitute the expression for X_{it} into $\int_{i=0}^{1} X_{it}di = X_t$ and solve for λ_{Xt}:

$$\begin{aligned}\left(\frac{\alpha}{\lambda_{Xt}}\right)^{\frac{1}{1-\alpha}}\int_{i=0}^{1} Q_{it}di &= X_t \\ \Leftrightarrow \lambda_{Xt} &= \alpha\left(\frac{Q_t}{X_t}\right)^{1-\alpha}\end{aligned}$$

Substituting λ_{Xt} back into the expression for X_{it} yields (2.33).

2.B.2 Derivation of equations (2.54) to (2.57)

Equation (2.54) - Asymptotically-balanced growth

The first-order condition (2.40) for X yields a relation between the marginal utility of consumption and the shadow price v_S of pollution:

$$\begin{aligned} v_{St} &= -B_t\left(\lambda_{Yt}\alpha X_t^{\alpha-1}L_{Yt}^{1-\alpha}Q_t^{1-\alpha} - \lambda_{Lt}\frac{1}{\varphi Q_t}\right) \\ &= -(1-\alpha)c_t^{-1/\sigma_c}B_t\left(\frac{X_t}{Q_tL_{Yt}}\right)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}L_{Yt} - \frac{1}{\varphi}\frac{X_t}{Q_t}\right)\end{aligned}$$

where we used $\lambda_{Yt} = c_t^{-1/\sigma_c}$ from (2.39) and $\lambda_{Lt} = \lambda_{Yt}(1-\alpha)X_t^{\alpha}Q_t^{1-\alpha}L_{Yt}^{-\alpha}$ from (2.44) in the second line. It follows that the relation

$$\widehat{v_{S}}_t = -\left(1/\sigma_c\right)\widehat{c}_t + \widehat{B}_t + (1-\alpha)\left(\widehat{X}_t - \widehat{Q}_t - \widehat{L_{Y}}_t\right) + \left(\widehat{\frac{\alpha}{1-\alpha}L_{Yt} - \frac{1}{\varphi}\frac{X_t}{Q_t}}\right)$$

must hold.

In the long-run, for $t \to \infty$, both L_Y and $L_X = \frac{1}{\varphi}X/Q$ are constant, so that the equation simplifies to

$$\widehat{v}_{S\infty} = -\left(1/\sigma_c\right)\widehat{c}_\infty + \widehat{B}_\infty - (1-\alpha)\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) \tag{2.B.9}$$

Taking into account that $\widehat{c}_\infty = \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ is required for the resource constraint to be satisfied in the long run, an equivalent equation is

$$\widehat{v}_{S\infty} = \left(1 - 1/\sigma_c\right)\widehat{c}_\infty + \widehat{B}_\infty - \widehat{X}_\infty.$$

Next, we divide the equation of motion for pollution, (2.45), by v_S:

$$-\psi\frac{S_t^{(1-2\sigma_E)/\sigma_E}}{v_{St}}L - \delta = \rho - \widehat{v}_{St}$$

On an ABG-path, according to the production function of intermediate quantity, X must grow at a constant rate in the long run because labor L_X is asymptotically constant and Q grows at a constant rate. It then follows from (2.B.9) that the long-run growth rate $\widehat{v}_{S\infty}$ of v_S must be constant as well. This implies that the ratio $S_t^{(1-2\sigma_E)/\sigma_E}/v_{St}$ must be asymptotically constant, so that in the long run, v_S must grow at the same rate as the (instantaneous) marginal disutility $\psi S^{(1-2\sigma_E)/\sigma_E}$

of pollution:

$$\widehat{v}_{S\infty} = ((1 - 2\sigma_E)/\sigma_E)\,\widehat{S}_\infty. \tag{2.B.10}$$

Substituting $\widehat{v}_{S\infty}$ from (2.B.9) into (2.B.10) and rearranging yields (2.54).

Equation (2.55) - Consumption Euler-equation

We derive the Euler-equation for an arbitrary period t first, and afterwards take the limit for $t \to \infty$.

We divide (2.46) by v_{Qt}, which yields the new equation

$$\mu n_t q_t + \frac{\lambda_{Yt}}{v_{Qt}}(1-\alpha)X_t^\alpha Q_t^{-\alpha} L_{Yt}^{1-\alpha} + \frac{\lambda_{Lt}}{v_{Qt}} \frac{X_t}{\varphi} \frac{1}{Q_t^2} = \rho - \widehat{v_{Q}}_t. \tag{2.B.11}$$

Next, we replace the Lagrange-multipliers λ_Y and λ_L as well as the shadow price v_Q and its growth rate:

The Lagrange-multipliers are given by (2.39) and (2.44) for all t.

In an interior solution of the optimization problem, $n_t > 0$ for all t. Given $n_t > 0$, we can derive the long-run value of the shadow-price v_Q and its long-run growth rate from (2.41): Dividing by n_t on both sides of (2.41) and solving for v_{Qt} yields

$$v_{Qt} = \frac{2\lambda_{Lt} q_t}{\mu Q_t}. \tag{2.B.12}$$

With $\lambda_{Lt} = \lambda_{Yt}(1-\alpha)X_t^\alpha Q_t^{1-\alpha} L_{Yt}^{-\alpha}$ from (2.44) and $\lambda_{Yt} = c_t^{-1/\sigma_c}$ from (2.39), the equation is equivalent to:

$$v_{Qt} = (1-\alpha)\frac{2}{\mu} c_t^{-1/\sigma_c} \left(\frac{X_t}{Q_t L_{Yt}}\right)^\alpha q_t$$

This implies that

$$\widehat{v_{Q}}_t = -(1/\sigma_c)\,\widehat{c}_t + \alpha\left(\widehat{X}_t - \widehat{Q}_t - \widehat{L_{Y}}_t\right) + \widehat{q}_t. \tag{2.B.13}$$

Substituting the expressions for v_{Qt}, λ_{Yt} and λ_{Lt} as well as $\widehat{v_Q}_t$ in 2.B.11 and simplifying yields

$$(1/\sigma_c)\,\widehat{c}_t + \rho = \frac{\mu}{2q_t}\left(L_{Yt} + \frac{1}{\varphi}\frac{X_t}{Q_t}\right) + \mu n_t q_t + \alpha\left(\widehat{X}_t - \widehat{Q}_t - \widehat{L_{Y}}_t\right) + \widehat{q}_t.$$

Taking the limit for $t \to \infty$, taking into account $\widehat{L_{Y}}_\infty = \widehat{q}_\infty = 0$ on an ABG-path proves that in the long-run, the Euler-equation is given by (2.55) in the text.

Equation (2.56) - Research arbitrage

To derive the long-run research arbitrage equation, we first manipulate (2.47) in the same way as (2.46) in the previous subsection.
Given $n_t > 0$, using (2.42) together with (2.39) and (2.44) to replace the Lagrange-multipliers, the shadow price v_{Bt} and its growth rate are given by

$$\begin{aligned} v_{Bt} &= \frac{2\lambda_{Lt} b_t}{\mu B_t} \\ &= (1-\alpha)\frac{2}{\mu} c_t^{-1/\sigma_c} \left(\frac{X_t}{Q_t L_{Yt}}\right)^{\alpha} \frac{Q_t}{B_t} b_t \end{aligned} \tag{2.B.14}$$

and

$$\widehat{v_{Bt}} = -(1/\sigma_c)\,\widehat{c}_t + \alpha\left(\widehat{X}_t - \widehat{Q}_t - \widehat{L_{Yt}}\right) + \widehat{Q}_t - \mu n_t b_t + \widehat{b}_t \tag{2.B.15}$$

respectively.

In the derivation of equation (2.54), we have shown that the relation $v_{St} = -(1-\alpha)c_t^{-1/\sigma_c} B_t \left(\frac{X_t}{Q_t L_{Yt}}\right)^{1-\alpha} \left(\frac{\alpha}{1-\alpha} L_{Yt} - \frac{1}{\varphi}\frac{X_t}{Q_t}\right)$ holds. Dividing equation (2.47) by v_{Bt}, substituting for v_{Bt}, $\widehat{v_{Bt}}$, v_{St}, λ_{Yt} and λ_{Lt} and rearranging yields:

$$(1/\sigma_c)\,\widehat{c}_t + \rho = \frac{\mu}{2b_t}\left(\frac{\alpha}{1-\alpha} L_{Yt} - \frac{1}{\varphi}\frac{X_t}{Q_t}\right) + \alpha\left(\widehat{X}_t - \widehat{L_{Yt}}\right) + (1-\alpha)\widehat{Q}_t + \widehat{b}_t$$

In the limit for $t \to \infty$, the equation becomes

$$(1/\sigma_c)\,\widehat{c}_\infty + \rho = \frac{\mu}{2b_\infty}\left(\frac{\alpha}{1-\alpha} L_{Y\infty} - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) + \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty \tag{2.B.16}$$

as b_∞ is constant.

Setting equal the right hand-sides of (2.55) and (2.B.16), we obtain (2.56).

Equation (2.57) - Indifference

Equation (2.57) is obtained directly by substituting (2.B.12) and (2.B.14) for v_{Qt} and v_{Bt} in (2.43) and taking the limit for $t \to \infty$.

2.B.3 Long-run solution to the necessary conditions for parameter constellations in proposition 2.5 ($\widehat{S}_\infty > (-\delta)$)

(1.) Balanced growth ($\widehat{X}_\infty = \widehat{Q}_\infty$)

The long-run optimal values for q and b follow from equations (2.58) (which in turn was derived from the *asymptotically-balanced growth condition* (2.54)) and the *indifference condition* (2.57): Using the accumulation functions for Q and B, the ratio $\widehat{B}_\infty/\widehat{Q}_\infty = 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ in (2.58) is equivalent to $b_\infty = \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) q_\infty$. Substituting into $q_\infty^2 + b_\infty^2 = d$ (equation (2.57)) yields the long-run solution for q_∞, which is then used to determine b_∞ from $b_\infty = \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) q_\infty$:

$$q_\infty = \left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{-1/2} d^{1/2} \tag{2.B.17}$$

$$b_\infty = \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{-1/2} d^{1/2} \tag{2.B.18}$$

n_∞ is determined from the *consumption Euler-equation* (2.55). First, we solve the labor market constraint (2.52) for $L_{Y\infty}$:

$$L_{Y\infty} = L - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty - n_\infty(q_\infty^2 + b_\infty^2 + d) \tag{2.B.19}$$

We substitute $q_\infty^2 + b_\infty^2 = d$ from (2.57) in (2.B.19) and (2.B.19) together with $\widehat{c}_\infty = \widehat{X}_\infty = \widehat{Q}_\infty = \mu n_\infty q_\infty$ in (2.55). Solving the Euler-equation for n_∞, we obtain the solution:

$$\begin{aligned} n_\infty &= \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho}{\left(d - (1-1/\sigma_c)\, q_\infty^2\right)\mu q_\infty^{-1}} \\ &= \frac{\frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} \cdot d^{-1/2}\mu L - \rho}{\left(1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)\mu d^{1/2}} \left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} \end{aligned} \tag{2.B.20}$$

In the second line, the solution for q_∞ in (2.B.17) was used.

The amount of labor devoted to R&D-activities in the long run is

$$\begin{aligned} L_{D\infty} &= n_\infty (q_\infty^2 + b_\infty^2 + d) \\ &= 2n_\infty d. \end{aligned} \tag{2.B.21}$$

The *research arbitrage equation* (2.56) yields the balanced-growth amount of labor in intermediate production $L_{X\infty} = \frac{1}{\varphi}(X/Q)_\infty$, which is proportional to the ratio $(X/Q)_\infty$: We substitute (2.B.21) in (2.B.19), (2.B.19) in (2.56) and solve the resulting equation

$$\frac{\mu}{2q_\infty}(L - 2n_\infty d) = \frac{\mu}{2b_\infty}\left(\frac{\alpha}{1-\alpha}\left(L - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty - 2n_\infty d\right) - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right)$$

for $\frac{1}{\varphi}(X/Q)_\infty$: The solution is

$$L_{X\infty} = \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty = (1-\alpha)\left(\frac{\alpha}{1-\alpha} - \frac{b_\infty}{q_\infty}\right)(L - 2n_\infty d), \tag{2.B.22}$$

with n_∞ given by (2.B.20) and $\frac{b_\infty}{q_\infty} = 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ from (2.58).

With the solutions for $L_{X\infty} = \frac{1}{\varphi}(X/Q)_\infty$ and $L_{D\infty}$ we derive $L_{Y\infty}$ from (2.B.19).

$$\begin{aligned} L_{Y\infty} &= (1-\alpha)\left(1 + \frac{b_\infty}{q_\infty}\right)(L - 2n_\infty d) \qquad (2.B.23) \\ &= \left(2 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) \cdot \frac{2\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{1/2}\rho - (1 - 1/\sigma_c)\mu L}{\frac{1}{1-\alpha}\left(1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)\mu} \end{aligned}$$

Next, we determine the long-run growth rates for c, Y, X, Q, B and S. We first derive the common growth rate $\widehat{Q}_\infty$ of Q, c, Y and X. With the solutions for n_∞ and q_∞ we find that the growth rate of productivity is:

$$\begin{aligned} \widehat{Q}_\infty &= \mu n_\infty q_\infty \qquad (2.B.24) \\ &= \frac{1}{1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2}\left(\frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right) \end{aligned}$$

The growth rate of intermediate cleanliness can be deduced directly from (2.58). The long-run pollution growth rate follows from (2.54), taking into account that $\widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty = 0$ holds under the assumption that corollary 2.1 does not apply so that $\widehat{S}_\infty > (-\delta)$. The long-run growth rate $\widehat{S}_\infty$ is given by:

$$\begin{aligned}\widehat{S}_\infty &= \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\widehat{c}_\infty \qquad (2.B.25)\\ &= \frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2} \cdot \left(\frac{1}{2}\sqrt{1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2} d^{-1/2}\mu L - \rho\right)\end{aligned}$$

It remains to be proven that $\widehat{S}_\infty$ is indeed larger than $(-\delta)$ for $\sigma_c < 1$, and that the non-negativity constraints as well as the transversality conditions are satisfied. The condition on model parameters to ensure $\widehat{S}_\infty > (-\delta)$ for $\sigma_c < 1$ is:

$$\begin{aligned}\rho > \rho_{\mathrm{BG}}^{\mathrm{delta}} &= \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L \qquad (2.B.26)\\ &- \left(\frac{1}{\sigma_c} + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta\end{aligned}$$

Equation (2.B.26) also ensures that $(X/(BS))_\infty > 0$. Whenever $\sigma_c > 1$ so that $\widehat{S}_\infty > 0$ in a growing economy, $\widehat{S}_\infty > (-\delta)$ and $(X/(BS))_\infty > 0$ is satisfied for any ρ which allows for positive long-run consumption growth.

In the derivations of the above equations, an interior solution with $n_\infty > 0$ was presumed. Equation (2.B.20) shows that $n_\infty > 0$ if and only if

$$\rho < \overline{\rho}_{\mathrm{BG}} = \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L. \qquad (2.B.27)$$

For the remaining constraints, it is sufficient to prove that the transversality conditions are satisfied: As is standard in endogenous growth models, the transversality conditions guarantee $L_{Y\infty} > 0$. Because $(X/Q)_\infty$ in (2.B.22) is proportional to $L_{Y\infty}$, it is strictly positive as well.

To prove that the transversality conditions are satisfied, we use (2.B.13) to express the long-run growth rate of the shadow price v_Q on a balanced growth path as function $\widehat{v_Q}_\infty = -(1/\sigma_c)\widehat{Q}_\infty$ of the state variable. $\lim_{t\to\infty}(e^{-\rho t}v_{Qt}Q_t) = 0$ holds whenever the rate of time preference exceeds the long-run growth rate of $v_{Qt}Q_t$. Using $\widehat{v_Q}_\infty = -(1/\sigma_c)\widehat{Q}_\infty$, the formal condition is $\rho > (1 - 1/\sigma_c)\widehat{Q}_\infty$. Upon substitution of (2.B.24) and rearranging, this is equivalent to

$$\rho > \rho_{\mathrm{BG}}^{\mathrm{TVC}} = \frac{1}{2}\frac{1-1/\sigma_c}{1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2}\left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L \tag{2.B.28}$$

Note that for $\sigma_c < 1$, the right hand side is negative so that the condition is satisfied for any positive ρ.

Using (2.B.9) and (2.B.15) to derive $\widehat{v_{S\infty}}$ and $\widehat{v_{B\infty}}$, the same condition is obtained from the transversality conditions for B and S.

(2.) Asymptotically-balanced growth with deceleration ($\widehat{X}_\infty < \widehat{Q}_\infty$)

The long-run values q_∞ and b_∞ are derived from (2.59) in the text, which follows from the *research arbitrage equation* (2.56), and the *indifference condition* (2.57):

$$q_\infty = \left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2} \tag{2.B.29}$$

$$b_\infty = \frac{\alpha}{1-\alpha}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2} \tag{2.B.30}$$

n_∞ is determined from the *consumption Euler-equation* (2.55) as in the previous subsection. First, the labor market constraint (2.52) is used again to express $L_{Y\infty}$ as function of n_∞. With $\lim_{t\to\infty}(X/Q) = 0$, labor use in intermediate production converges to zero asymptotically, i.e.

$$L_{X\infty} = \frac{1}{\varphi}\left(X/Q\right)_\infty = 0,$$

while the total amount of labor in the R&D-sector is still given by $L_{D\infty} = 2n_\infty d$ from (2.B.21). The labor market constraint becomes

$$L_{Y\infty} = L - 2n_\infty d. \tag{2.B.31}$$

We substitute (2.60), (2.61) and (2.B.31) in the Euler-equation (2.55) and solve for

n_∞. The solution is:

$$\begin{aligned} n_\infty &= \frac{\frac{1}{2}\mu L - \rho q_\infty}{\left(d - \frac{(1-1/\sigma_c)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} q_\infty^2\right)\mu} \qquad (2.B.32) \\ &= \frac{\frac{1}{2}\mu L - \rho\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2}}{\left(1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right) d\mu} \left(1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right) \end{aligned}$$

In the second line, q_∞ from (2.B.29) was used.

$L_{D\infty}$ and $L_{Y\infty}$ are obtained by substituting n_∞ in (2.B.21) and (2.B.31).

$$\begin{aligned} L_{Y\infty} &= L - 2n_\infty d \qquad (2.B.33) \\ &= \frac{2\left(1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1/2} d^{1/2}\rho - (1-1/\sigma_c)\mu L}{\left(1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\mu} \end{aligned}$$

With (2.B.32) and (2.B.29), we derive the long-run productivity growth rate:

$$\begin{aligned} \widehat{Q}_\infty &= \mu n_\infty q_\infty \qquad (2.B.34) \\ &= \frac{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}{\left(\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right) \end{aligned}$$

The long-run growth rates of cleanliness B, intermediate quantity and consumption (and GDP) can be found by substituting $\widehat{Q}_\infty$ in (2.59), (2.60) and (2.61) in the text respectively:

$$\widehat{B}_\infty = \frac{\frac{\alpha}{1-\alpha}\left(1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)}{\left(\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right) \qquad (2.B.35)$$

$$\widehat{X}_\infty = \frac{1+\frac{\alpha}{1-\alpha}^2 - \left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} - \frac{\alpha}{1-\alpha}\right)}{\left(\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right) \qquad (2.B.36)$$

$$\widehat{c}_\infty = \frac{1}{\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \rho\right) \qquad (2.B.37)$$

The growth rate of the pollution stock is $\widehat{S}_\infty = \widehat{X}_\infty - \widehat{B}_\infty$, which is equivalent to

$$\widehat{S}_\infty = \frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \left(\frac{1}{2}\sqrt{1+\left(\frac{\alpha}{1-\alpha}\right)^2} d^{-1/2}\mu L - \rho\right). \quad (2.B.38)$$

The condition on the rate of time preference needed to ensure $\widehat{S}_\infty > (-\delta)$ for $\sigma_c < 1$ is given by:

$$\begin{aligned}\rho \;>\; \rho_{\text{Dec}}^{\text{delta}} &= \frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L \\ &- \left(\frac{1}{\sigma_c} + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\right)\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta \qquad (2.B.39)\end{aligned}$$

Similar to the previous paragraph, $\widehat{S}_\infty > (-\delta)$ is satisfied for any ρ reconcilable with positive long-run consumption growth if $\sigma_c > 1$.

The upper bound on ρ to guarantee positive long-run growth is:

$$\overline{\rho}_{\text{Dec}} = \frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L \qquad (2.B.40)$$

Similar to the previous subsection, it remains to be shown that the transversality conditions are satisfied, because $L_{Y\infty} > 0$ then follows from (2.B.33). Taking the long-run growth rates of v_S, v_Q and v_B from (2.B.9), (2.B.13) and (2.B.15) into account, we derive the condition $\rho > (1-1/\sigma_c)\widehat{c}_\infty$ which, with (2.B.37), is equivalent to

$$\rho > \rho_{\text{Dec}}^{\text{TVC}} = \frac{1}{2}\frac{1-1/\sigma_c}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L. \qquad (2.B.41)$$

Again, for $\sigma_c < 1$, the condition is satisfied for any positive ρ.

2.B.4 Definition of boundary values for the rate of time preference

From (2.B.26) to (2.B.27) and (2.B.39) to (2.B.41), we define the following boundary values for the rate of time preference:

$$\overline{\rho} := \begin{cases} \frac{1}{2}\left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} > 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} < 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases} \tag{2.B.42}$$

$$\rho^{\text{delta}} := \begin{cases} \frac{1}{2}\left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_1 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} > 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}, \\ & \sigma_c < 1 \\ \frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_2 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} < 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}, \\ & \sigma_c < 1 \end{cases} \tag{2.B.43}$$

$$\rho^{\text{TVC}} := \begin{cases} \frac{1}{2}\frac{1-1/\sigma_c}{1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2}\left(1+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} > 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\frac{1-1/\sigma_c}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} < 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases}$$

$\kappa_1 = \frac{1}{\sigma_c}+\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2$ and $\kappa_2 = \frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)$ are positive constants.

2.B.5 Long-run solution to the necessary conditions for parameter constellations in proposition 2.6 ($\widehat{S}_\infty = (-\delta)$)

(1.) Asymptotically-balanced growth without deceleration

With $\widehat{X}_\infty = \widehat{Q}_\infty$, it still follows from the resource constraint that

$$\widehat{c}_\infty = \widehat{Y}_\infty = \widehat{X}_\infty = \widehat{Q}_\infty.$$

Substituting into the *asymptotically-balanced-growth condition* (2.54) with $\widehat{S}_\infty = (-\delta)$ and solving for $\widehat{B}_\infty$, we obtain

$$\widehat{B}_\infty = (1/\sigma_c)\,\widehat{Q}_\infty - ((1-2\sigma_E)/\sigma_E)\,\delta. \tag{2.B.44}$$

As $\sigma_c < 1$, the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ in (2.B.44) is larger than $\widehat{B}_\infty/\widehat{Q}_\infty = 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ in the case with $\widehat{S}_\infty > (-\delta)$ for every constellation of parameters that satisfies the condition $\widehat{Q}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$ for $\widehat{S}_\infty = (-\delta)$.

The ratio $\widehat{B}_\infty/\widehat{Q}_\infty$ implied by (2.B.44) is only reconcilable with the *research-arbitrage equation* (2.56) if it does not exceed $\frac{\alpha}{1-\alpha}$. If $(1/\sigma_c) < \frac{\alpha}{1-\alpha}$, this is true for any $\widehat{c}_\infty > 0$. If $(1/\sigma_c) > \frac{\alpha}{1-\alpha}$, $\widehat{B}_\infty/\widehat{Q}_\infty < \frac{\alpha}{1-\alpha}$ is guaranteed by the condition

$$\widehat{Q}_\infty < \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c) - \frac{\alpha}{1-\alpha}}\delta, \tag{2.B.45}$$

which implies a lower bound on the rate of time preference ρ.

With $\widehat{B}_\infty = \mu n_\infty b_\infty$ and $\widehat{Q}_\infty = \mu n_\infty q_\infty$ and using the *indifference condition* (2.57), $q_\infty^2 + b_\infty^2 = d$, to express b_∞ as function of q_∞, equation (2.B.44) can also be written as

$$\mu n_\infty \sqrt{d - q_\infty^2} = (1/\sigma_c)\,\mu n_\infty q_\infty - ((1-2\sigma_E)/\sigma_E)\,\delta \tag{2.B.46}$$

The *consumption Euler-equation* (2.55) does not depend on $\widehat{S}_\infty$ or the ratio $\widehat{B}_\infty/\widehat{Q}_\infty$. It yields the same relation

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1}L - \rho}{(d - (1-1/\sigma_c)\,q_\infty^2)\,\mu} q_\infty \tag{2.B.47}$$

between n_∞ and q_∞ as in the balanced-growth case.

Equations (2.B.46) and (2.B.47) form a system of two equations in the two unknowns q_∞ and n_∞. n_∞ as as function of q_∞ in equation (2.B.47) is unchanged from the balanced-growth case. However after substituting for n_∞ in (2.B.46), it is not possible to solve (2.B.46) for q_∞ analytically due to the mixture of exponents.

Depending on parameters, there may be a unique solution, two solutions or no solution. To prove this claim, consider equation (2.B.46), where $n_\infty = n_\infty(q_\infty)$ is given by (2.B.47). We divide both sides of equation (2.B.46) by $n_\infty(q_\infty)$:

$$\mu\sqrt{d - q_\infty^2} = (1/\sigma_c)\,\mu q_\infty - n_\infty(q_\infty)^{-1}\,((1-2\sigma_E)/\sigma_E)\,\delta$$

The left hand side of the modified equation is non-negative as well as decreasing and concave in q_∞. The right-hand side is positive whenever the condition for $\widehat{S}_\infty = (-\delta)$ is satisfied, because $\mu n_\infty q_\infty = \widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}\delta$ is a weaker condition than $\widehat{Q}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$. For $\sigma_c < 1$, in the relevant range with $\widehat{Q}_\infty > \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$, the right-hand side is concave and first increasing, then decreasing in q_∞ because the first term is linear and $n_\infty (q_\infty)^{-1}$ is decreasing and convex in q_∞ whenever $\sigma_c < 1$.

A unique solution exists if and only if the value $q_\infty = \sqrt{d}$, which sets the left-hand side of the equation to zero, lies between the two zeros of the right-hand side. An equivalent condition is that at $q_\infty = \sqrt{d}$, the right-hand side is positive. This is true if and only if

$$\rho < \frac{1}{2}\mu d^{-1/2} L - \left(\left(1 - 2\sigma_E\right)/\sigma_E\right)\delta.$$

(2.) Asymptotically-balanced growth with deceleration

As in the case where $\widehat{S}_\infty > (-\delta)$, the *research arbitrage equation* (2.56) with $(X/Q)_\infty = 0$ yields

$$\widehat{B}_\infty / \widehat{Q}_\infty = \frac{\alpha}{1-\alpha}.$$

Using the *indifference condition* (2.57), the long-run solutions for q_∞ and b_∞ are given by (2.B.29) and (2.B.30).

With $\widehat{S}_\infty = (-\delta)$, the relation between $\widehat{X}_\infty$ and $\widehat{Q}_\infty$ differs from the one in appendix 2.B.3: Substituting $\widehat{c}_\infty = \alpha \widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$, $\widehat{B}_\infty = \frac{\alpha}{1-\alpha}\widehat{Q}_\infty$ as well as $\widehat{S}_\infty = (-\delta)$ in the *balanced growth condition* (2.54) we obtain the following expression for $\widehat{X}_\infty$ as function of $\widehat{Q}_\infty$:

$$\widehat{X}_\infty = \frac{1}{1-\alpha}\frac{\left(1-2\sigma_E\right)/\sigma_E}{\frac{\alpha}{1-\alpha}\left(1/\sigma_c\right)+1}\delta + \frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2 - \left(1/\sigma_c - \frac{\alpha}{1-\alpha}\right)}{\frac{\alpha}{1-\alpha}\left(1/\sigma_c\right)+1}\widehat{Q}_\infty \tag{2.B.48}$$

Whenever the two conditions $1/\sigma_c > \frac{\alpha}{1-\alpha}$ and $\widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c)-\frac{\alpha}{1-\alpha}}\delta$, which together are necessary and sufficient for deceleration, are satisfied, the growth rate $\widehat{X}_\infty$ is smaller than $\widehat{Q}_\infty$.

With $\widehat{X}_\infty$ from (2.B.48), $\widehat{c}_\infty = \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ yields

$$\widehat{c}_\infty = \frac{\alpha}{1-\alpha}\frac{\left(1-2\sigma_E\right)/\sigma_E}{\frac{\alpha}{1-\alpha}\left(1/\sigma_c\right)+1}\delta + \frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2}{\frac{\alpha}{1-\alpha}\left(1/\sigma_c\right)+1}\widehat{Q}_\infty. \tag{2.B.49}$$

In the same way as in appendix 2.B.3, we derive n_∞ from the *consumption Euler-equation* (2.55):

$$n_\infty = \frac{\frac{1}{2}\mu q_\infty^{-1} L - \rho + \left(1 - 1/\sigma_c\right)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta}{\left(d - \frac{(1-1/\sigma_c)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}q_\infty^2\right)\mu q_\infty^{-1}}$$

$L_{D\infty}$ and $L_{Y\infty}$ are given by (2.B.21) and (2.B.31) respectively.

The growth rate of productivity is:

$$\begin{aligned}
\widehat{Q}_\infty &= \mu n_\infty q_\infty \\
&= (1-\alpha)\left(\frac{\alpha}{1-\alpha}+\sigma_c\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{-1} \qquad (2.B.50)\\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L-\rho+(1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)
\end{aligned}$$

The growth rates of intermediate quantity and consumption (and GDP) follow from (2.B.48) and (2.B.49).

$$\begin{aligned}
\widehat{X}_\infty &= \frac{1}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta+(1-\alpha)\sigma_c\frac{1+\left(\frac{\alpha}{1-\alpha}\right)^2-\left(1/\sigma_c-\frac{\alpha}{1-\alpha}\right)}{1+\left(\frac{\alpha}{1-\alpha}\right)^2} \qquad (2.B.51)\\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L-\rho+(1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)
\end{aligned}$$

$$\begin{aligned}
\widehat{c}_\infty &= \frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta+(1-\alpha)\sigma_c \qquad (2.B.52)\\
&\cdot\left(\frac{1}{2}\left(1+\left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L-\rho+(1-1/\sigma_c)\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)
\end{aligned}$$

As $\widehat{S}_\infty = (-\delta)$ only occurs for $\sigma_c < 1$ and sufficiently large values of the long-run consumption growth rate, it can readily be verified that the transversality conditions and the non-negativity constraints for $L_{Y\infty}$ and n_∞ are satisfied for any $\rho \leq \rho^{\text{delta}}$.

2.B.6 Proof of proposition 2.3

Given that $q_\infty > 0$ and $\widehat{c}_\infty$ is proportional to $\widehat{Q}_\infty$ in both cases of appendix 2.B.3, $\widehat{c}_\infty > 0 \Leftrightarrow n_\infty > 0$. It follows that for parameter constellations as assumed in proposition 2.5, long-run consumption growth is positive if and only if $\rho < \overline{\rho}$.

The cases in proposition 2.6 only occur for sufficiently fast consumption growth so that whenever proposition 2.6 applies, $\widehat{c}_\infty > 0$ is necessarily satisfied and no additional restriction of the parameter range is needed.

2.B.7 Proof of proposition 2.5

The allocation of labor and the long-run growth rates derived in appendix 2.B.3 for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ and $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ respectively satisfy all the necessary first-order conditions as well as the non-negativity constraints and the transversality conditions, given the parameter restriction $\underline{\rho} < \rho < \overline{\rho}$ in proposition 2.5. They therefore describe a solution of the set of necessary conditions. It remains to be shown that the solution is unique.

The only other solution candidate which has so far been excluded by the assumption of an interior solution is a solution with $n_\infty = 0$. To prove that $n_\infty = 0$ cannot be an optimal choice for n under the parameter restriction $\rho < \overline{\rho}$, we show that, given $n_\infty = 0$ and $\rho < \overline{\rho}$, the partial derivative of the Hamiltonian-function with respect to n is positive in the limit, i.e. $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} \mid_{n_\infty=0} > 0$. This condition is satisfied, if and only if

$$v_{Q\infty}\mu q_\infty Q_\infty + v_{B\infty}\mu b_\infty B_\infty > \lambda_{L\infty}\left(q_\infty^2 + b_\infty^2 + d\right). \tag{2.B.53}$$

Given $n_\infty = 0$, the first-order conditions (2.41) and (2.42) for q and b are always satisfied and the social planner is indifferent between any levels of q_∞ and b_∞. Because every choice of q_∞ and b_∞ must yield the same level of intertemporal welfare, any particular pair can be selected as solution. We define the limits $\lim\limits_{n_\infty\to 0} q(n_\infty)$ and $\lim\limits_{n_\infty\to 0} b(n_\infty)$, obtained from the first-order conditions given $n_\infty > 0$ as the solutions in this case. The limit for q can be derived by solving the Euler-equation (2.55) for q instead of n. The limit for b follows from (2.58) or (2.59) respectively. It differs between the balanced-growth case and the case with deceleration.

Balanced growth

Substituting the labor market constraint (2.B.19) into the Euler-equation (2.55) and taking the limit for $n_\infty \to 0$ on both sides yields $\lim\limits_{n_\infty\to 0} q(n_\infty) = \frac{\mu}{2} L/\rho$. Accordingly, the limit for b is $\lim\limits_{n_\infty\to 0} b(n_\infty) = \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) \frac{\mu}{2} L/\rho$ from (2.58). The research-arbitrage condition (2.56) requires that $\lim\limits_{n_\infty\to 0} \frac{X}{Q}(n_\infty) = (1-\alpha)\,\varphi\left(\frac{\alpha}{1-\alpha} - \frac{b_\infty}{q_\infty}\right) L$ in the limit.

We then determine the values of the shadow prices $v_{Q\infty}$ and $v_{B\infty}$ for $n_\infty = 0$ from (2.46) and (2.47) with (2.40) and (2.52), taking into account that X, c, Q, B and S are constant in the long run. We obtain the expressions $v_{Q\infty} = \lambda_{L\infty} Q_\infty^{-1} \left(L_{Y\infty} + \frac{1}{\varphi} \left(\frac{X}{Q}\right)_\infty \right) \frac{1}{\rho}$ and $v_{B\infty} = \lambda_{L\infty} B_\infty^{-1} \left(\frac{\alpha}{1-\alpha} L_{Y\infty} - \frac{1}{\varphi} \left(\frac{X}{Q}\right)_\infty \right) \frac{1}{\rho}$.

Substituting $v_{Q\infty}$, $v_{B\infty}$, $\lim\limits_{n_\infty \to 0} q(n_\infty)$, $\lim\limits_{n_\infty \to 0} b(n_\infty)$ and $\lim\limits_{n_\infty \to 0} \frac{X}{Q}(n_\infty)$ as well as $L_{Y\infty} = L - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty$ in (2.B.53) and simplifying yields

$$
\begin{aligned}
\lim_{t\to\infty} \frac{\partial H}{\partial n} &> 0 \\
\Leftrightarrow \rho &< \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L.
\end{aligned}
$$

Because $\frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L = \overline{\rho}_{\mathrm{BG}}$, which is the upper limit for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, we have shown that given $\rho < \overline{\rho}$ and $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, no solution to the set of necessary first-order conditions with $n_\infty = 0$ exists.

Asymptotically-balanced growth with $\widehat{X}_\infty < \widehat{Q}_\infty$

In the asymptotically-balanced-growth case, it can readily be verified from (2.55) that $\lim\limits_{n_\infty \to 0} q(n_\infty) = \frac{\mu}{2} L/\rho$ as before. The limit for b changes to $\lim\limits_{n_\infty \to 0} b(n_\infty) = \frac{\alpha}{1-\alpha}\frac{\mu}{2} L/\rho$ and $\lim\limits_{n_\infty \to 0} \frac{X}{Q}(n_\infty) = 0$.

Proceeding as in the balanced-growth case, we find that $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} > 0 \Leftrightarrow \rho < \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L$. The right-hand side corresponds to the upper bound $\overline{\rho}$ for $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$. Again, $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} > 0$ proves that $n_\infty = 0$ cannot be an optimal solution in the given parameter range.

2.B.8 Proof of proposition 2.6

In the case without deceleration, depending on parameters, one, two or no interior solution to the set of first-order conditions with the properties described in proposition 2.6 may exist as shown in appendix 2.B.5.

The solution derived in appendix 2.B.5 for the case with deceleration ($1/\sigma_c > \frac{\alpha}{1-\alpha}$ and $\widehat{Q}_\infty > \frac{(1-2\sigma_E)/\sigma_E}{(1/\sigma_c) - \frac{\alpha}{1-\alpha}}\delta$) satisfies all the necessary first-order conditions given $\rho \leq \rho^{\mathrm{delta}}$. Uniqueness follows from appendix 2.B.7. No further proof has to be provided for the case with deceleration.

2.B.9 Long-run growth in the model without pollution externality ($\psi = 0$)

Without pollution externality, utility depends on consumption only:

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} L dt$$

The first-order condition for S becomes

$$\frac{\partial H}{\partial S_t} = \rho v_{St} - \dot{v}_{St} \Leftrightarrow -\delta v_{St} = \rho v_{St} - \dot{v}_{St} \tag{2.B.54}$$

This condition can only be satisfied if $v_{St} = 0$ and $\dot{v}_{St} = 0$ for all t. The second solution $v_{St} > 0$ with $\widehat{v}_{St} = \rho + \delta$ violates the transversality condition for S for all feasible long-run growth rates $\widehat{S}_\infty \geq (-\delta)$.
The first-order condition (2.40) for X then directly yields aggregate intermediate production

$$X_t = \frac{\alpha}{1-\alpha} \varphi Q_t L_{Yt} \tag{2.B.55}$$

for any given labor supply L_{Yt} and productivity level Q_t.

With $v_{St} = 0$, it follows from the first-order condition

$$v_{Bt} \mu n_t b_t = \rho v_{Bt} - \dot{v}_{Bt}$$

for B_t that $v_{Bt} = 0$ and $\dot{v}_{Bt} = 0$ for all t[30].
If $v_{Bt} = 0$, it is optimal to set $b_t = 0$ for all t as can be seen from (2.42). Then the optimal long-run level of q is

$$q_\infty^{\psi=0} = \sqrt{d} \tag{2.B.56}$$

from (2.57).

As $L_{Y\infty}$ is constant, we conclude from (2.B.55) and the resource constraint that $\widehat{X}_\infty = \widehat{c}_\infty = \widehat{Q}_\infty$. We can still determine n_∞ from the Euler-equation (2.55). Using $\widehat{X}_\infty = \widehat{c}_\infty = \widehat{Q}_\infty$, $q = \sqrt{d}$, the labor market constraint (2.52) and the indifference condition (2.57), the solution is

$$n_\infty^{\psi=0} = \frac{1}{(1/\sigma_c)\sqrt{d}\mu} \left(\frac{1}{2} d^{-1/2} \mu L - \rho \right).$$

[30] Again, there is a second solution $\widehat{v}_{Bt} = \rho - \mu n_t b_t$ but like the non-zero solution for v_{St}, it does not satisfy the transversality condition for the associated state-variable (B).

The growth rate of production, consumption and productivity is

$$
\begin{aligned}
\widehat{Q}_{\infty}^{\psi=0} &= \mu n_{\infty} q_{\infty} \\
&= \frac{1}{1/\sigma_c}\left(\frac{1}{2}d^{-1/2}\mu L - \rho\right)
\end{aligned}
\tag{2.B.57}
$$

which is positive whenever $\rho < \overline{\rho}^{\psi=0} = \frac{1}{2}\mu d^{-1/2}L$.

2.B.10 Proof of proposition 2.7

(1.) Comparison of consumption growth in the baseline model to the model with $\psi = 0$

(i) **Parameter region for positive consumption growth:** Comparison of $\overline{\rho} = \frac{1}{2}\mu\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}L$ and $\overline{\rho}^{\psi=0}$ shows that positive consumption growth is optimal for larger values of the rate of time preference with $\psi > 0$ than with $\psi = 0$ because $\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} > 1$.

(ii) **Consumption growth rate:** If proposition 2.5 applies ($\widehat{S}_{\infty} > (-\delta)$) and there is deceleration in the long-run optimal solution, the long-run consumption growth rate in the baseline model is given by (2.B.37). If proposition 2.6 is relevant ($\widehat{S}_{\infty} = (-\delta)$), equation (2.B.52) displays the consumption growth rate. Comparison of (2.B.37) and (2.B.52) respectively to the growth rate in (2.B.57) proves the claim in the proposition.

(2.) Influence of the size of ψ

Given $\widehat{S}_{\infty} > (-\delta)$, it follows from equations (2.B.34) to (2.B.38) that long-run growth rates are not affected by the parameter ψ. If $\widehat{S}_{\infty} = (-\delta)$, the relevant equations are (2.B.50) to (2.B.52) and $\widehat{B}_{\infty} = \frac{\alpha}{1-\alpha}\widehat{Q}_{\infty}$.

Chapter 3

Local stability analysis and transitional dynamics

Our propositions from the previous chapter are only relevant if at least for an initial state of the model-economy close to its long-run path, there exists a transition path which leads towards the long-run solution. We therefore study local stability properties and transitional behavior of our model in this chapter.

At the laissez-faire equilibrium, all variables except the pollution stock grow at their balanced growth rates at all times and the pollution growth rate converges for any initial conditions. It follows that the balanced-growth laissez-faire equilibrium is globally asymptotically stable.

For the long-run social optimum derived in the previous chapter, the stability properties are not as apparent. Unless there exists a balanced-growth optimum, and initial conditions are such that the economy starts on the balanced-growth path in the social planner's solution, all variables exhibit transitional behavior. Due to the complexity of the non-linear model, the transition paths cannot be found analytically. For the same reason, global stability analysis of the long-run optimum is beyond the scope of this thesis.

We examine local stability of the social planner's solution close to the long-run optimum numerically for a set of 88 different parameter constellations. For ease of exposition, the analysis is limited to parameter constellations for which the conditions of corollary 2.1 in chapter 2 are not satisfied ($\widehat{S}_\infty > (-\delta)$). In a reasonable parameter range, we find that for any initial state of the economy, there exists an optimal path which converges towards the long-run solution derived in the previous chapter.

Transitional dynamics are studied for an exemplary parametrization with deceleration. The analysis of the trajectories suggests that the optimal solution is characterized by green innovation and deceleration not only in the long run but also throughout the transition path. Further, it becomes apparent that the initial technology endowments of an economy are crucial for the optimal consumption- and pollution profiles. Economies with an initially more advanced technology enjoy persistently lower pollution levels, higher consumption levels and larger intertemporal welfare.

In preparation of the numerical analysis, we reduce the set of necessary first-order conditions for the social planner's problem in chapter 2 to a dynamic system of six non-linear first-order differential equations in six variables. To study local stability properties, variables which are not constant in the long run have to be rescaled so that instead of asymptotically balanced growth, the dynamic system displays a stationary solution in the long run.

For the numerical analysis, we choose the relaxation algorithm by Trimborn, Koch and Steger (2008). The algorithm requires as input the dynamic system, initial values for the state variables and an initial guess for the long-run steady-state and the transition path. Based on this information, it iteratively searches for the true transition path of the scale-adjusted system. Moreover, it computes the eigenvalues of the Jacobian-matrix at steady-state, which reveal the local stability properties of the long-run solution.

A major advantage of the relaxation algorithm over more established numerical procedures such as backward integration (Brunner and Strulik (2002)), projection methods (Judd (1992); Judd (1998), chapter 11) or time elimination is that it is well-suited to deal with two characteristics of the present dynamic system which complicate numerical analysis: center manifolds and multi-dimensional stable manifolds.[1]

A center manifold is tangent to the eigenspace spanned by the eigenvalues of the Jacobian-matrix with real part zero[2]. In the dynamic system considered, the center manifold represents the continuum of steady-states into which scale-adjustment converts the asymptotically unique ABG-path of the social planner's solution. In the balanced-growth case, the center manifold is one-dimensional, while it is two-dimensional if the long-run social optimum is characterized by deceleration. Accordingly, the Jacobian-matrix evaluated at the center has one or two eigenvalues with zero real part respectively. Both with balanced growth and with deceleration,

[1]For a more detailed comparison to alternative procedures, see Trimborn et al. (2008).
[2]See, for example, chapter 9.4 in Tu (1994).

there exists a three-dimensional stable manifold in which trajectories converge to the center.

This chapter is organized as follows: The scale-adjusted dynamic system and the center manifold are described in sections 3.1 and 3.2. In sections 3.3 and 3.4, we explain the relaxation algorithm and its implementation. Section 3.5 presents the results.

3.1 The scale-adjusted dynamic system

In general, there are different ways to transfer the original dynamic system, which is characterized by balanced growth asymptotically, into a system with a long-run steady-state[3]: One approach, to which we resort here, has amongst others been used in the analysis of the Lucas (1988)–Uzawa (1965)-model. All variables with long-run growth rates different from zero are rescaled, so that their motion is slowed according to the respective long-run growth rates. The second option is to build ratios of variables which are constant in the long run, as is usually done in the Ramsey-Cass-Koopmans-model, for example, and define theses ratios as new variables.
We choose scale-adjustment, because it has the advantage, also pointed out by Trimborn (2007), that the paths of the unscaled variables can be recovered from the development of the scaled variables.

As explained in chapter 2, the allocation of labor to research and production of final output and intermediates is stationary in the long-run optimum ($L_{Y\infty}$, $L_{X\infty}$, n_∞, q_∞, b_∞ are constant). All other aggregate variables z grow at constant growth rates $\widehat{z}_\infty$ different from zero. We rescale these variables so that $\widetilde{z}_t = z_t \cdot e^{-\widehat{z}_\infty \cdot t}$ denotes the new variable, which converges to a constant value as t tends to infinity.[4]

The set of scale-adjusted first-order conditions resembles the original conditions except for an exponential expression $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ in the scale-adjusted first-order conditions for X and Q and in the labor market constraint:

$$\begin{aligned}
\frac{\widetilde{v_{St}}}{\widetilde{B}_t} + \widetilde{\lambda}_{Yt}\alpha\widetilde{X}_t^{\alpha-1}L_{Yt}^{1-\alpha}\widetilde{Q}_t^{1-\alpha} - \frac{1}{\varphi}\frac{\widetilde{\lambda}_{Lt}}{\widetilde{Q}_t}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} &= 0 \\
-\mu n_t q_t - \frac{\widetilde{\lambda}_{Yt}}{\widetilde{v_{Q}}_t}(1-\alpha)\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{-\alpha}L_{Yt}^{1-\alpha} - \frac{\widetilde{\lambda}_{Lt}}{\widetilde{v_{Q}}_t}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t^2}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} & \\
+\frac{\widehat{c}_\infty}{\sigma_c} - \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) + \rho &= \widehat{\widetilde{v_{Q}}}_t
\end{aligned}$$

[3]See also Trimborn (2007), pp. 85/86.
[4]Appendix 3.A.1 shows the details of the scale-adjustment.

$$L_{Yt} + \frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} + n_t(q_t^2 + b_t^2 + d) = L$$

Along the long-run balanced-growth path, $\widehat{X}_\infty = \widehat{Q}_\infty$ and $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} = 1$ for all t. Otherwise $\widehat{X}_\infty < \widehat{Q}_\infty$, so that $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ converges to zero as $t \to \infty$.

In appendix 3.A.2, we combine the first-order conditions so as to eliminate the Lagrange-multipliers and costate variables as well as the control variables n and c. The remaining equations form a system of six non-linear first-order differential equations in the three control variables b, L_Y and $\widetilde{X}$ as well as the three state variables $\widetilde{Q}$, $\widetilde{B}$ and $\widetilde{S}$.

$$\frac{\dot{b}_t}{b_t} = \frac{1}{2}\mu \frac{d - b_t^2}{d}\frac{1}{b_t} L_{Yt}\left[\left(\frac{b_t}{\sqrt{d - b_t^2}} - \frac{\alpha}{1-\alpha}\right) + \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\left(\frac{b_t}{\sqrt{d - b_t^2}} + 1\right)\right] \quad (3.1)$$

$$\begin{aligned}\frac{\dot{\widetilde{X}}_t}{\widetilde{X}_t} = {} & \frac{1}{\alpha\left(1 - \frac{1}{\sigma_c}\right)}\rho - \widehat{X}_\infty - \frac{1-\alpha}{\alpha}\frac{\mu}{2d}\sqrt{d - b_t^2}L \\ & + \frac{1}{\alpha\left(1 - \frac{1}{\sigma_c}\right)}\frac{\mu}{2d}\sqrt{d - b_t^2}\left(\begin{array}{c}\frac{b_t}{\sqrt{d - b_t^2}} \\ -(1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{1-\alpha}\right)\end{array}\right) L_{Yt}\left(1 + \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right) \\ & - \frac{1}{\alpha(1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)}\frac{\mu}{2d} b_t L_{Yt} + \frac{1-\alpha}{\alpha}\frac{1}{\left(1 - \frac{1}{\sigma_c}\right)}\left(\frac{1}{\sigma_c} + \frac{\alpha}{1-\alpha}\right)\widehat{L_{Yt}} \quad (3.2)\end{aligned}$$

$$\begin{aligned}\frac{\dot{L}_{Yt}}{L_{Yt}} = {} & -\sigma_c\rho + \frac{(\sigma_c - 1)\psi \widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}} L^{1-\frac{1}{\sigma_c}}}{\widetilde{X}_t^{\alpha\left(1-\frac{1}{\sigma_c}\right)-1}\left(L_{Yt}\widetilde{Q}_t\right)^{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)}\widetilde{B}_t} \\ & - (1-\alpha)(\sigma_c - 1)\left(\frac{\alpha}{1-\alpha} - \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\delta \\ & + \frac{\mu\left((1-\alpha)(\sigma_c - 1)L + L_{Yt}\right)}{2d}\sqrt{d - b_t^2}\left(\begin{array}{c}\left(\frac{\alpha}{1-\alpha} - \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\frac{b_t}{\sqrt{d - b_t^2}} \\ + \left(1 + \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\end{array}\right) \quad (3.3)\end{aligned}$$

$$\frac{\dot{\widetilde{S}}_t}{\widetilde{S}_t} = \frac{\widetilde{X}_t}{\widetilde{B}_t \widetilde{S}_t} - \delta - \widehat{S}_\infty \quad (3.4)$$

$$\frac{\dot{\widetilde{Q}}_t}{\widetilde{Q}_t} = \frac{1}{2}\frac{\mu}{d}\sqrt{d - b_t^2}\left(L - L_{Yt}\left(1 + \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\right) - \widehat{Q}_\infty \quad (3.5)$$

$$\frac{\dot{\widetilde{B}}_t}{\widetilde{B}_t} = \frac{1}{2}\frac{\mu}{d} b_t\left(L - L_{Yt}\left(1 + \frac{1}{\varphi}\widetilde{X}_t \widetilde{Q}_t^{-1} L_{Yt}^{-1} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\right) - \widehat{B}_\infty \quad (3.6)$$

The expression $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ occurs in all equations except for equation (3.4). If the parameter space is such that the long-run optimal solution is characterized by deceleration ($\widehat{X}_\infty < \widehat{Q}_\infty$), the scale-adjusted dynamic system is therefore non-autonomous, i.e., explicitly dependent on time. The relaxation algorithm by Trimborn et al. allows to analyze also non-autonomous dynamic systems. While in principle, time dependence may complicate local stability analysis, it does not impair the analysis of the present system of differential equations because the system converges towards an autonomous system asymptotically.[5] Li and Löfgren (2000) show, building on Benaim and Hirsch (1996), that if the time-dependent term declines exponentially, as it does in the above equations, any solution to the non-autonomous system converges to the solution of the autonomous limit system asymptotically. The two systems share the same local stability properties near the long-run steady-state.

3.2 Steady-state: The center manifold

The scale-adjustment described in the previous section turns the asymptotically balanced growth path of the social planner's solution derived in chapter 2 into a continuum of steady-states. The manifold of steady-states is a center manifold.[6]
We now implicitly characterize the center manifold by solving for the steady-states of the scale-adjusted dynamic system, both when the original system is characterized by balanced growth in the long run and in the case with deceleration. The steady-state levels of the unscaled variables b and L_Y are known from chapter 2. The levels for the scale-adjusted variables have yet to be determined.

From the steady-state solutions, it becomes obvious that in the balanced-growth case, the center manifold is one-dimensional, while it is of dimension two in the case with deceleration.

3.2.1 Balanced growth

When there is balanced growth in the unscaled system, $\widehat{Q}_\infty = \widehat{X}_\infty$. It follows that $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} = 1$ for all t. The steady-state solutions for b_∞ and $L_{Y\infty}$ are given by (2.B.18) and (2.B.23) in chapter 2. The long-run level of the ratio $\widetilde{X}_\infty/\widetilde{Q}_\infty$, can, for $\widehat{Q}_\infty = \widehat{X}_\infty$, be inferred from equation (2.B.22) in chapter 2. It can be verified when setting the right-hand sides of (3.1), (3.5) and (3.6) to zero, that b_∞, $L_{Y\infty}$ and $\widetilde{X}_\infty/\widetilde{Q}_\infty$ as determined in chapter 2 are the unique solutions to this homogenous

[5] The concept of asymptotically autonomous equations goes back to Markus (1956).
[6] A formal proof can be found in corollary 10 in Trimborn (2007).

equation system.

The ratio $\widetilde{X}_\infty/\widetilde{Q}_\infty$ defines $\widetilde{X}_\infty$ as a function of $\widetilde{Q}_\infty$. Presuming $\dot{L}_{Y\infty}/L_{Y\infty} = 0$ must hold at the long-run steady-state, it becomes obvious that the right-hand side of (3.2) is a linear combination of the right-hand sides of (3.1), (3.3), (3.5) and (3.6): $\dot{\widetilde{X}}_\infty/\widetilde{X}_\infty = 0$ is satisfied for any $\widetilde{Q}_\infty$, given $\dot{b}_\infty/b_\infty = \dot{L}_{Y\infty}/L_{Y\infty} = \dot{\widetilde{Q}}_\infty/\widetilde{Q}_\infty = \dot{\widetilde{B}}_\infty/\widetilde{B}_\infty = 0$ and $\widehat{Q}_\infty$ from (2.B.24).

From the two equations $\dot{L}_{Y\infty}/L_{Y\infty} = 0$ and $\dot{\widetilde{S}}_\infty/\widetilde{S}_\infty = 0$, the three remaining variables $\widetilde{Q}_\infty$, $\widetilde{B}_\infty$ and $\widetilde{S}_\infty$ cannot be uniquely determined. There is a continuum of solutions. We solve $\dot{L}_{Y\infty}/L_{Y\infty} = 0$ and $\dot{\widetilde{S}}_\infty/\widetilde{S}_\infty = 0$ for $\widetilde{B}_\infty$ and $\widetilde{S}_\infty$ as functions of $\widetilde{Q}_\infty$. The set of long-run steady-state solutions is:

$$b_\infty = \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)\left(1 + \left(1 - \frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{-1/2} \cdot d^{1/2} \tag{3.7}$$

$$L_{Y\infty} = (1-\alpha)\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}} + 1\right)\left(L - 2\frac{d}{\mu}\sqrt{d-b_\infty^2}^{-1}\widehat{Q}_\infty\right) \tag{3.8}$$

$$\widetilde{X}_\infty = \varphi\frac{\frac{\alpha}{1-\alpha} - \frac{b_\infty}{\sqrt{d-b_\infty^2}}}{\frac{b_\infty}{\sqrt{d-b_\infty^2}} + 1} L_{Y\infty}\widetilde{Q}_\infty \tag{3.9}$$

$$\begin{aligned}\widetilde{B}_\infty = {} & \left(\frac{\frac{b_\infty}{\sqrt{d-b_\infty^2}}\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1\right)^{-1}\delta + \frac{\sigma_c}{\sigma_c - 1}\rho}{-\frac{\mu}{2d}\sqrt{d-b_\infty^2}\left(\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}\right)^2+1\right)\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1\right)^{-1}\left(L + \frac{1}{1-\alpha}\frac{L_{Y\infty}}{\sigma_c - 1}\right)}\right)^{-\frac{\sigma_E}{1-\sigma_E}} \\ & \cdot\left(\frac{1}{\delta+\widehat{S}_\infty}\right)^{\frac{1-2\sigma_E}{1-\sigma_E}}\left(\psi L^{1-\frac{1}{\sigma_c}}\right)^{\frac{\sigma_E}{1-\sigma_E}}\left(\varphi\frac{\frac{\alpha}{1-\alpha} - \frac{b_\infty}{\sqrt{d-b_\infty^2}}}{\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1}\right)^{1-\alpha\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}} \\ & \cdot\left(L_{Y\infty}\widetilde{Q}_\infty\right)^{1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\widetilde{S}_\infty = {} & \left(\frac{\frac{b_\infty}{\sqrt{d-b_\infty^2}}\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1\right)^{-1} + \frac{\sigma_c}{\sigma_c - 1}\rho}{-\frac{\mu}{2d}\sqrt{d-b_\infty^2}\left(\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}\right)^2+1\right)\left(\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1\right)^{-1}\left(L + \frac{1}{1-\alpha}\frac{L_{Y\infty}}{\sigma_c - 1}\right)}\right)^{\frac{\sigma_E}{1-\sigma_E}} \\ & \cdot\left(\frac{1}{\delta+\widehat{S}_\infty}\right)^{\frac{\sigma_E}{1-\sigma_E}}\left(\psi L^{1-\frac{1}{\sigma_c}}\right)^{\frac{-\sigma_E}{1-\sigma_E}}\left(\varphi\frac{\frac{\alpha}{1-\alpha} - \frac{b_\infty}{\sqrt{d-b_\infty^2}}}{\frac{b_\infty}{\sqrt{d-b_\infty^2}}+1}\right)^{\alpha\frac{\sigma_c-1}{\sigma_c}/\frac{1-\sigma_E}{\sigma_E}} \\ & \cdot\left(L_{Y\infty}\widetilde{Q}_\infty\right)^{\frac{\sigma_c-1}{\sigma_c}/\frac{1-\sigma_E}{\sigma_E}}\end{aligned} \tag{3.11}$$

Note that the steady-state values of the unscaled variables b_∞ and $L_{Y\infty}$, given by (3.7) and (3.8), do not depend on $\widetilde{Q}_\infty$ and are therefore the same in each steady-state of the center manifold.

3.2.2 Deceleration

If there is deceleration, the long-run growth rate of intermediate quantity lies below the productivity growth rate ($\widehat{X}_\infty < \widehat{Q}_\infty$) so that $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ converges to zero for $t \to \infty$. In this case, the steady-state values (2.B.30) for b_∞ and (2.B.33) for $L_{Y\infty}$ from chapter 2 solve the three equations $\dot{b}_\infty / b_\infty = 0$, $\dot{\widetilde{Q}}_\infty / \widetilde{Q}_\infty = 0$ and $\dot{\widetilde{B}}_\infty / \widetilde{B}_\infty = 0$.

Given $\dot{L}_{Y\infty} / L_{Y\infty} = 0$ holds, taking into account $\lim_{t\to\infty} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} = 0$ and substituting the expressions for b_∞ and $L_{Y\infty}$, as well as (2.B.34) and (2.B.36) from chapter 2 for the long-run growth rates $\widehat{Q}_\infty$ and $\widehat{X}_\infty$, the equation $\widehat{\widetilde{X}}_\infty = 0$ is satisfied as well.

The equation system is underdetermined by two equations. We solve $\dot{L}_{Y\infty} / L_{Y\infty} = 0$ and $\dot{\widetilde{S}}_\infty / \widetilde{S}_\infty = 0$ for $\widetilde{X}_\infty$ and $\widetilde{S}_\infty$ as functions of $\widetilde{Q}_\infty$ and $\widetilde{B}_\infty$. The continuum of steady-states is described by:

$$b_\infty = \frac{\alpha}{1-\alpha} \left(1 + \left(\frac{\alpha}{1-\alpha} \right)^2 \right)^{-1/2} d^{1/2} \tag{3.12}$$

$$L_{Y\infty} = L - 2\frac{d}{\mu} \sqrt{d - b_\infty^2}^{-1} \widehat{Q}_\infty \tag{3.13}$$

$$\begin{aligned} \widetilde{X}_\infty = & \left(\frac{\alpha\delta + \frac{\sigma_c}{\sigma_c - 1}\rho}{-\frac{\mu}{2d}(1-\alpha)\sqrt{d-b_\infty^2}\left(\frac{\alpha}{1-\alpha}\frac{b_\infty}{\sqrt{d-b_\infty^2}} + 1 \right)\left(L + \frac{L_{Y\infty}}{(1-\alpha)(\sigma_c - 1)} \right)} \right)^{\frac{\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \\ & \cdot \left(\delta + \widehat{S}_\infty \right)^{\frac{1-2\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \left(\psi L^{1-\frac{1}{\sigma_c}} \right)^{\frac{-\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \\ & \cdot \left(L_{Y\infty} \widetilde{Q}_\infty \right)^{\frac{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \widetilde{B}_\infty^{\frac{1-\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \end{aligned} \tag{3.14}$$

$$\begin{aligned} \widetilde{S}_\infty = & \left(\frac{\alpha\delta + \frac{\sigma_c}{\sigma_c - 1}\rho}{-\frac{\mu}{2d}(1-\alpha)\sqrt{d-b_\infty^2}\left(\frac{\alpha}{1-\alpha}\frac{b_\infty}{\sqrt{d-b_\infty^2}} + 1 \right)\left(L + \frac{L_{Y\infty}}{(1-\alpha)(\sigma_c - 1)} \right)} \right)^{\frac{\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \\ & \cdot \left(\delta + \widehat{S}_\infty \right)^{\frac{\left(\alpha\left(1-\frac{1}{\sigma_c}\right)-1\right)\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \left(\psi L^{1-\frac{1}{\sigma_c}} \right)^{\frac{-\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \\ & \cdot \left(L_{Y\infty} \widetilde{Q}_\infty \right)^{\frac{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \widetilde{B}_\infty^{\frac{\alpha\left(1-\frac{1}{\sigma_c}\right)\sigma_E}{1-\left(1+\alpha\left(1-\frac{1}{\sigma_c}\right)\right)\sigma_E}} \end{aligned} \tag{3.15}$$

The next subsection summarizes the essentials of the relaxation procedure.[7]

3.3 Description of the solution algorithm

The relaxation algorithm by Trimborn et al. (2008) requires as inputs the system of differential equations, initial conditions for the state variables, a set of final boundary conditions and an initial guess for the transition path towards the final steady-state. If no explicit guess for the path is provided, the code by default sets all variables constant at their final steady-state values. The N_1 initial and N_2 boundary conditions must sum up to the number N of differential equations.

To make the continuous-time dynamic system with infinite time horizon suitable for numerical processing, the Matlab-code first rescales the time range $\mathbb{R}_+$ to map the interval $[0, 1]$ before choosing a grid $\{\tau_1, ..., \tau_M\}$ from the transformed time interval. With the vector of the dependent variables $z_k = (z_{1k}, ..z_{Nk})$ at each time τ_k, $k = 1...M$, a set of M meshpoints (τ_k, z_k) is obtained.

Afterwards, the differential equations are discretized using the midpoint of each interval (τ_k, τ_{k+1}). Along the true solution path, the difference in the values of the dependent variables between two neighboring meshpoints k and $k+1$ approximately equals the slope $f(\overline{\tau}_k, \overline{z}_k) = \dot{z}(\overline{\tau}_k, \overline{z}_k)$ at the midpoint $(\overline{\tau}_k,\overline{z}_k)$ as derived from the differential equation system times the length of the time interval. Formally this relation is given by the vector equation

$$z_{k+1} - z_k = (\tau_{k+1} - \tau_k)f(\overline{\tau}_k, \overline{z}_k)$$

with $\overline{\tau}_k = \frac{\tau_k+\tau_{k+1}}{2}, \overline{z}_k = \frac{z_k+z_{k+1}}{2}$. The discretization error is small, given a sufficiently large number of meshpoints. The difference

$$H_k = z_{k+1} - z_k - (\tau_{k+1} - \tau_k)f(\overline{\tau}_k, \overline{z}_k)$$

therefore reflects the error which occurs when the trial solution is not the true solution path. This error can be computed between any two meshpoints $k = 1, .., M-1$, yielding $M-1$ vector equations. With the initial conditions $I : \mathbb{R}^N \to \mathbb{R}^{N_1}$ and the

[7] A more detailed description can be found in Trimborn (2007) and Trimborn et al. (2008). We adjust the notation to make it compatible with the notation in our model.

final boundary conditions $F : \mathbb{R}^N \rightarrow \mathbb{R}^{N_2}$, an $MxN-$dimensional system

$$E(z) = \begin{pmatrix} I(z_1) \\ . \\ . \\ . \\ H_k(z_k, z_{k+1}) \\ . \\ . \\ . \\ F(z_M) \end{pmatrix}$$

of $M + 1$ non-linear vector-equations is obtained.

The relaxation procedure then uses the Newton-algorithm to iteratively find the vector of variables for which $E(z) = 0$ so that the error between the trial solution path and the path predicted by the differential equation system is zero. In every iteration, the Newton algorithm computes a vector of change $\Delta z = (\Delta z_1, .., \Delta z_M)$ by solving the linear equation

$$D_z E(z) \cdot \Delta z = -E(z).$$

$D_z E(z)$ is an MxN-matrix which contains the Jacobi-matrices of each of the $M+1$ vector-equations of $E(z)$. The trial solution is then adjusted by Δz or a fraction of it. The algorithm continues until the error E is sufficiently small or the maximum number of iterations is reached.

3.4 Implementation

We run the Matlab-code for 88 parameter constellations.
As suggested in our numerical example of chapter 2, we constrain σ_E to the open interval $(0, 1/2)$. We extend the range of values for α and σ_c, so as to cover parameter constellations with balanced growth besides such with deceleration. We consider values of α from the interval $(0, 1)$ and $\sigma_c \, \epsilon (0, 2)$. For the rate of time preference, ρ, we pick values between 0.001 as suggested by Stern et al. (2007) and 0.015 as assumed by Nordhaus (see, for example, Nordhaus (2007)).
It is more difficult to decide over reasonable values for the rate of natural regeneration, δ: The regenerative capacity of nature typically depends on the type of

pollutant considered and the existing pollution level. Moreover, regeneration may vary over time. While our aggregated specification of the pollution accumulation function cannot account for such differences, the numerical analysis shows that the size of the regeneration rate is not crucial for local stability. We choose δ from the interval $(0.001, 0.2)$. The fixed-cost parameter d in the R&D-cost function is set to $d = 1/\alpha^2$.[8]

As to the population size L and the individual arrival rate μ for innovations, we arbitrarily fix $\mu = 0.03$ and adjust the value for L so as to obtain a value for the long-run GDP growth rate between 1% and 4% and to satisfy the transversality condition. The parameter φ in the production function for intermediates and the weight ψ of the pollution stock in the utility function only affect steady-state levels but not long-run growth rates. Neither of the parameters is relevant for the distinction between balanced growth and deceleration. We set $\varphi = 0.5$ and consider values between 0.5 and 2 for the weight ψ of the pollution stock.

All variables of the dynamic system with the exception of b have to be non-negative at any point in time. While the long-run steady-state of the scale-adjusted system satisfies the non-negativity constraints, the relaxation procedure does not account for the constraints during iterations. We therefore run the algorithm using logarithmized variables to ensure that the original variables are non-negative.

A guess for the final steady-state for the parameter constellations without deceleration is constructed from equations (3.7) to (3.11), with an arbitrary choice of $\widetilde{Q}_\infty$. Almost any value for $\widetilde{Q}_\infty$ can be chosen[9], because the algorithm only requires a final guess which lies on the center manifold but the particular steady-state to which the system converges does not need to be known in advance. In the cases with deceleration, we use equations (3.12) to (3.15), with $\widetilde{Q}_\infty$ and $\widetilde{B}_\infty$ set to an arbitrary value.

We do not provide a precise initial guess for the solution path, so that the algorithm assumes all variables to be constant at their final steady-state values at all mesh-points as the trial solution.

As final boundary conditions, we require the three control variables to correspond to their respective steady-state levels.

[8] This value guarantees that the condition $d > 1/\alpha^2 - 1$ for monopoly pricing in the laissez-faire equilibrium is satisfied, even though it is not relevant for the social planner's solution.

[9] The sole exception is $\widetilde{Q}_\infty = 1$. Setting $\widetilde{Q}_\infty = 1$ is not viable, because we use logarithmized variables and initial values of the state variables are given as multiple of the long-run steady-state values. With $\widetilde{Q}_\infty = 1$, the starting value for the logarithmized $\widetilde{Q}$ would be zero.

3.5 Results

This section displays the results of the numerical analysis. First, local stability close to the center manifold of steady-states is examined. In a reasonable parameter range, the dynamic system has a three-dimensional stable manifold in the examples considered. This implies that for any set of initial conditions for the state variables, there exists an optimal transition path leading towards the long-run optimal growth path. In section 3.5.2, the transition path is analyzed in an example with deceleration and trajectories with different initial technology endowments and pollution stocks are compared. It is shown that green innovation and deceleration characterize the optimal solution not only in the long-run but along the entire path. Differences in the initial pollution stock lead to only small divergence in the long-run optimal levels of technology, consumption and pollution. Comparing optimal paths for economies with different initial technology endowments, on the other hand, suggests that the optimal path for the economy with the initially more advanced technology exhibits higher consumption as well as lower pollution levels and therefore larger intertemporal welfare.

3.5.1 Local stability

While in general, the stability properties of the non-linear model cannot be derived from studying the eigenvalues of the Jacobian-matrix at steady-state when there exists a center manifold[10], a center manifold of steady states is a so-called normally hyperbolic invariant manifold (NHIM) which can be treated analogously to a hyperbolic fixed point.[11]

The fundamental theorem of normally hyperbolic invariant manifolds (Hirsch et al. (1977), see also Li et al. (2003)) is a generalization of the Hartman-Grobman theorem for hyperbolic fixed points. The most important conclusions from the theorem of NHIM for our purpose are twofold: First, close to the center, the non-linear system and its linearization are topologically equivalent. Local stability properties

[10] The reason is that the linearized system does not give enough information about the development of the system on the center manifold (see also Trimborn (2007)).

[11] Normally hyperbolic invariant manifolds (NHIM) extend the concept of hyperbolic fixed points, which are in fact zero-dimensional manifolds, to dimensions ≥ 1: The main characteristic of a NHIM is that along the manifold, movement of the dynamic system is slower than offside. Locally, there exist stable and unstable manifolds to the NHIM just as to a fixed point.
Along a center-manifold of steady-states, there is no movement at all, so that the main requirement for a NHIM is naturally satisfied (see also Trimborn (2007), p. 16). Trimborn (2007) proves in corollary 12 that a center manifold of stationary points is a NHIM.

near the center manifold can therefore be examined by studying the eigenvalues of the Jacobian-matrix evaluated at the center. Second, a stable manifold to the center exists, the dimension of which is determined by the number of eigenvalues with negative *and zero* real part.[12]

The stable manifold in the present dynamic system is three-dimensional, both with balanced growth and with deceleration, in a reasonable parameter range[13]: In the balanced-growth case, the Jacobian-matrix of the dynamic system has one eigenvalue with zero real part, corresponding to the one-dimensional center manifold. Further, it has two eigenvalues with negative real part. With deceleration, two eigenvalues have zero real part and there is one eigenvalue with negative real part. The real parts of the three remaining eigenvalues are positive. We conclude that locally, there exists an optimal transition path converging to the center manifold for any set of initial conditions (Q_0, B_0, S_0). The path is uniquely identified by the three initial conditions for the state variables.

While each trajectory in the stable manifold converges to some steady-state on the center manifold, the stable manifold does not characterize the convergence region for a particular point on the center. According to the fundamental theorem of NHIM, information on convergence to a particular steady-state is given by stable submanifolds, so-called fibers.[14] The dimension of the fibers is determined by the number of eigenvalues with negative real parts. The stable fibers of the present dynamic system are therefore two-dimensional in the balanced-growth case and one-dimensional in the case with deceleration.

3.5.2 Transitional dynamics

The center manifold for parameter constellations with deceleration is derived by defining a mesh for the variables Q and B and computing the values for S from equation (3.15) above, using Matlab. Figure 3.1 displays the center manifold along with several trajectories in (Q, B, S)-space in an example with deceleration, where the values $\alpha = 0.3$, $\sigma_c = 0.5$, $\sigma_E = 0.3$, $\delta = 0.1$, $\rho = 0.01$ and $\psi = 0.5$ are chosen for the variable parameters. In the example, all eigenvalues are real so that there is

[12] The stable manifold to the center is tangent to the subspace spanned by the eigenvectors associated with the eigenvalues with negative real part and the center subspace.

[13] The algorithm fails for some parameter constellations where the value of the intertemporal elasticity of substitution in consumption, σ_c, is either smaller than or equal to 0.2, or at least 1.3. Excluding this part of the range for σ_c leaves an interval which still covers most of the empirical estimates for the IES in consumption.

[14] See also Trimborn (2007), pp. 20-21.

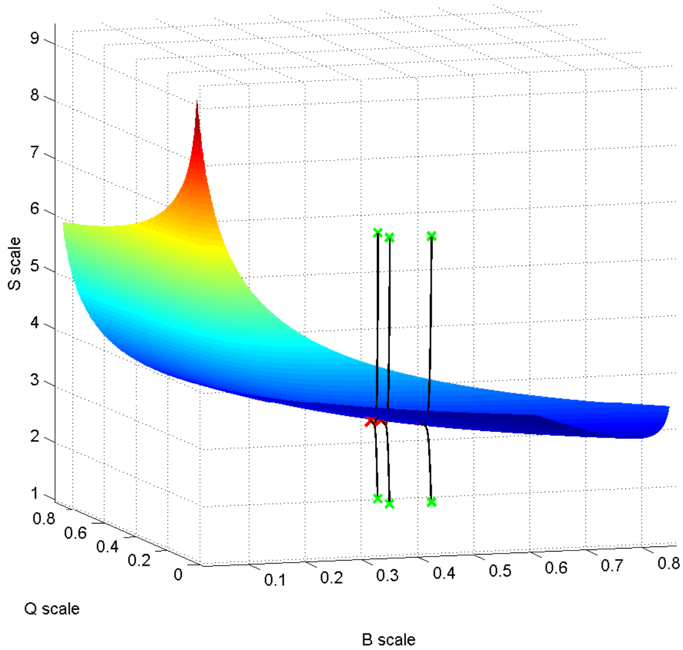

Figure 3.1: Center manifold and transition paths with deceleration for $\alpha = 0.3$, $\sigma_C = 0.5$, $\sigma_E = 0.3$, $\delta = 0.1$, $\rho = 0.01$, $\psi = 0.5$

no oscillation. A green cross in figure 3.1 marks the starting point of a trajectory at time $t = 0$, a red cross the final state (for $t = 500$).

It becomes obvious that the initial levels of technology and pollution matter for the steady-state levels of the scale-adjusted variables: Each of the trajectories takes the economy to a different steady-state on the center manifold, which implies that trajectories represent distinct fibers.

However, for the pairs of starting points which differ only in the initial value of the pollution stock, fibers are very close, so that the final steady-states appear to coincide in the figure. This suggests that initial technology levels matter significantly more for the long-run levels of pollution and productivity than initial pollution levels. The transition is studied in more detail below. It is shown that not only the optimal paths for pollution and productivity but also the optimal consumption paths for economies with different initial pollution stock almost coincide in the medium- and long term if economies share the same technology endowment initially.

It can be concluded from the almost vertical progression of the trajectories in figure 3.1 that, similar to the laissez-faire equilibrium, the growth rate of unscaled pollution adjusts more strongly over time than the growth rates of the unscaled technology stocks. The growth rates of productivity and cleanliness are close to

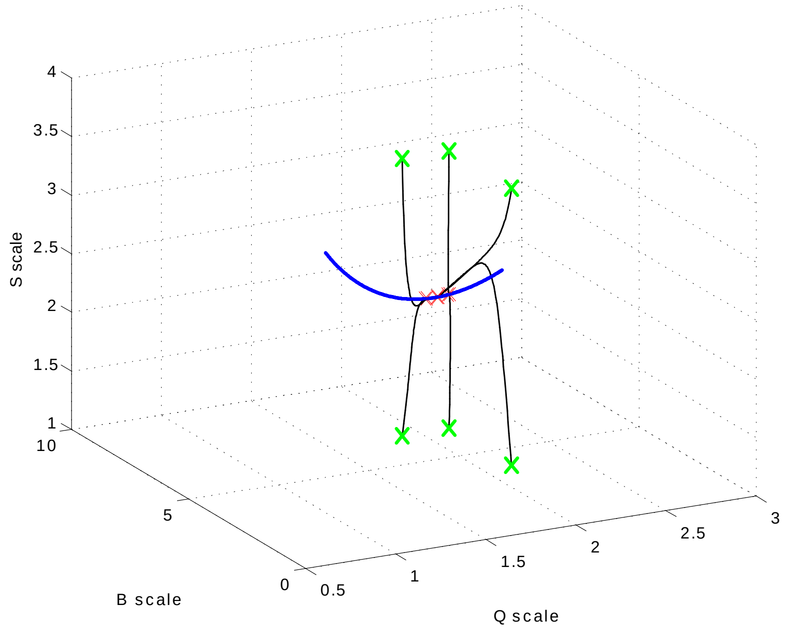

Figure 3.2: Center manifold and transition paths with balanced growth for $\alpha = 0.9$, $\sigma_c = 0.5$, $\sigma_E = 0.3$, $\delta = 0.1$, $\rho = 0.01$, $\psi = 0.5$

their respective long-run values from the beginning.[15]

The convergence properties of the optimal solution in the case with deceleration extend to the balanced-growth case. Figure 3.2 shows the transition of the state variables towards the center manifold in this case, for parameters $\alpha = 0.9$, $\sigma_c = 0.5$, $\sigma_E = 0.3$, $\delta = 0.1$, $\rho = 0.01$ and $\psi = 0.5$.[16]

We now study transitional dynamics in the example with deceleration from figure 3.1 in more detail. Figure 3.3 shows the time paths of the six variables $\widetilde{Q}$, $\widetilde{B}$, $\widetilde{S}$, b, L_Y and $\widetilde{X}$ in the dynamic system as well as the paths for scale-adjusted per-capita consumption $\widetilde{c}$ and the ratio of unscaled intermediate quantity to unscaled productivity, X/Q, along two different trajectories with high and low initial

[15] Note that the development of a scale-adjusted variable mirrors the relation of the unscaled growth rate to its long-run value, not the behavior of the unscaled variable directly. If the scale-adjusted variable remains constant, the unscaled variable grows at its long-run rate. If the scale-adjusted variable increases (decreases), the growth rate of the original variable lies above (below) its long-run limit.

[16] The figure displays two strongly bent trajectories with overshooting in the scaled pollution stock. These trajectories show a behavior often encountered in systems with multi-dimensional stable manifolds, when the stable eigenvalues are sufficiently unequal in size (Trimborn (2007)). The solution is then first attracted by the submanifold associated with the eigenvalue which is smaller in absolute terms. The reason is that the component of the solution with this eigenvalue decays more slowly. Overshooting may also occur for the deceleration-case, as will become obvious in figure 3.3 below.

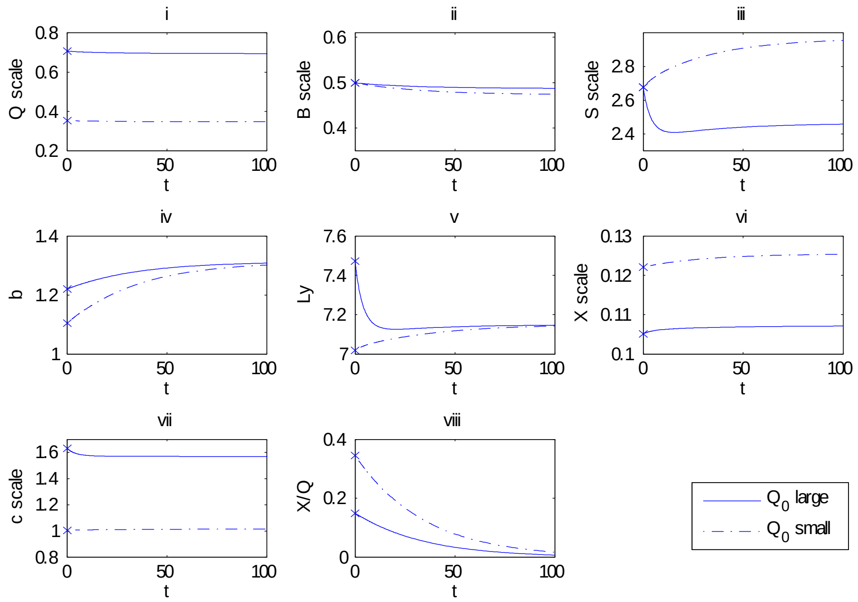

Figure 3.3: Transitional dynamics for 'high' and 'low' initial productivity level

productivity level respectively.

Considering the transition process along either of the trajectories confirms that the optimal solution is characterized by green innovation and deceleration not only in the long-run but along the entire transition path: b in subplot (iv) is positive throughout and subplot (viii) shows that X/Q declines persistently.

Subplot (iv) also suggests that for high and low initial productivity, research is more strongly oriented towards productivity-enhancing technical change in the beginning and becomes greener over time. However, it can be concluded from the fact that $\widetilde{Q}$ in (i) is almost constant over time and $\widetilde{B}$ in (ii) shows only a mild decrease, that the growth rates of unscaled productivity and cleanliness remain close to their respective long-run levels despite the shift in research orientation. This is in line with the results from figure 3.1 above.

Comparing the two trajectories, it becomes evident from subplots (iii) and (vii), that the social optimum for the economy with the higher initial productivity level is characterized by larger consumption and lower pollution levels over the whole time path. It follows that utility in every period t and intertemporal welfare are higher for the economy which starts out more productive.

The higher consumption levels are a direct consequence of the larger initial productivity level, as the initial difference in productivity persists over time (see (i)).

The lower pollution levels in the initially more productive economy are explained by lower emissions and, consequently, a smaller pollution growth rate during transition.[17] Emissions are lower both because the use of polluting intermediate inputs is restricted more strongly in every period and because intermediates are cleaner: Greater productivity, as can be concluded from (vi), is used optimally not only to consume more but also to use less polluting intermediate quantity in every period. At the same time, subplot (ii) shows that, while both economies start with the same $\widetilde{B}$, cleanliness grows slightly faster during transition along the trajectory with the larger initial productivity level[18]. Research is more strongly oriented towards green innovation in this economy in the short and medium term (see (iv)).[19]

It is interesting that in the economy with high initial productivity level, the scale-adjusted pollution stock $\widetilde{S}$ in (iii) first slightly undershoots its long-run value before it rises towards its steady-state level. The behavior of $\widetilde{S}$ suggests that the pollution growth rate is below its long-run value initially, then rises for some time and overshoots its steady-state value before it declines again.
It appears that in the economy which starts out with a comparatively low pollution growth rate, it is optimal to let emissions X/B rise for some time - although not in such a way that the pollution growth rate becomes positive. The comparatively fast growth in polluting intermediate quantity relative to the reduction in its pollution intensity dampens the negative effect from the decline in production labor (see (v)) on consumption. In the long run, as innovation becomes greener, emissions fall and the pollution growth rate declines towards its steady-state level.

Comparing trajectories with 'high' and 'low' initial value of $\widetilde{B}$ or, equivalently, unscaled B yields similar results as the comparison for differences in initial productivity concerning consumption- and pollution levels. Greater cleanliness allows the economy to consume more and enjoy a less polluted environment at the same time. However, the difference in consumption levels is apparently driven by different levels of intermediate production, not by productivity differentials (see subplot (vi) of figure 3.4): The economy which is initially cleaner can 'afford' to produce a larger quantity of intermediate goods in every period without incurring higher pollution

[17] Because $\sigma_C < 1$, the pollution growth rate is negative so that 'smaller' refers to a faster decline in the unscaled pollution stock.

[18] Even though the deviation in both $\widetilde{B}$ and $\widetilde{X}$ is small in value, it causes a divergence in pollution paths which is clearly visible.

[19] During transition, a higher step-size b in green innovation necessarily goes along with a lower step-size q in productivity-oriented research because the indifference condition, $q_t^2 + b_t^2 = d$, must hold in every period t, not only for $t \to \infty$.
In the long run, as was explained in section 3.2, b, the orientation of research, as well as the allocation of labor to research production are the same in every steady-state on the center manifold and therefore independent of the initial state.

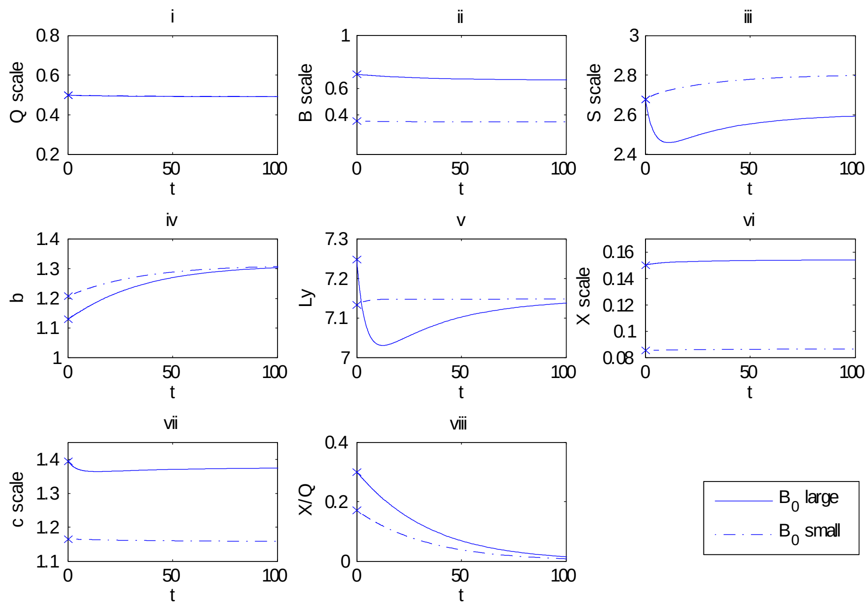

Figure 3.4: Transitional dynamics for 'high' and 'low' initial pollution intensity

levels. The paths for scale-adjusted productivity, on the other hand, almost coincide, as is obvious from (i).

If trajectories differ in initial pollution levels but not in technology endowments, it can be seen from figure 3.5 (subplots (iii) and (vii)) that the time paths of (scaled) consumption and pollution converge, as was suggested in figure 3.1. In the medium and long term, consumption is only slightly larger and pollution marginally lower in the initially less polluted economy.

For the less polluted economy, it is optimal to choose larger levels of intermediate production initially (see (vi)) but at the same time more labor in research and less in the production of the consumption good (v). This spurs innovation only slightly, as can be seen in (i) and (ii). Because of the smaller labor force in the consumption-good sector, scaled consumption levels in the less polluted economy are below those in the more polluted economy in the beginning. After only a few periods, the gap almost closes as in the initially cleaner economy, labor from the intermediate sector is shifted towards the consumption good sector and in the initially more polluted economy, production labor is shifted to the research sector.

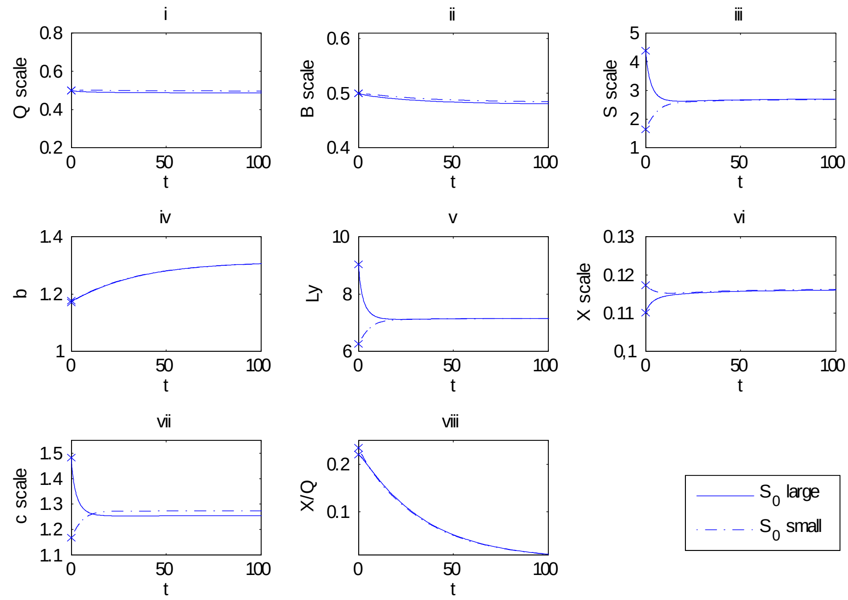

Figure 3.5: Transitional dynamics for 'high' and 'low' initial pollution stock

3.6 Conclusion

In this chapter, we have studied numerically the local stability properties of the long-run optimal solution for a large set of parameter constellations, using the relaxation algorithm by Trimborn, Koch and Steger (2007). Further, we examined transitional dynamics in an exemplary parametrization with deceleration.

The results of the stability-analysis suggest that the socially optimal solution converges to the long-run (asymptotically) balanced growth path of the unscaled system. This justifies the focus on ABG-solutions in chapter 2.

Analysis of the transition towards the long-run solution confirms that the social optimum is characterized by green innovation and deceleration, not only in the long run but also throughout the transition path. Transitional dynamics were studied for different technology endowments and initial pollution stocks. Differences in the initial pollution stock lead to only small divergence in the optimal paths. The initial technology endowment, on the other hand, affects crucially optimal pollution and consumption levels during transition and in the long run. Economies with initially more advanced technology enjoy lower pollution levels, higher consumption levels and larger intertemporal welfare.

3.A Appendix to chapter 3

3.A.1 Scale adjustment

We adjust the scale of all variables but the control variables q_t, b_t, n_t, and L_{Yt} which are constant in the long-run optimum without scale-adjustment. The scale-adjusted variables are defined as follows:
Control variables:

$$\begin{aligned}\widetilde{c}_t &:= c_t e^{-\widehat{c}_\infty \cdot t}\\ \widetilde{X}_t &:= X_t e^{-\widehat{X}_\infty \cdot t}\end{aligned}$$

State variables:

$$\begin{aligned}\widetilde{Q}_t &:= Q_t e^{-\widehat{Q}_\infty \cdot t}\\ \widetilde{B}_t &:= B_t e^{-\widehat{B}_\infty \cdot t}\\ \widetilde{S}_t &:= S_t e^{-\widehat{S}_\infty \cdot t}\end{aligned}$$

Lagrange-multipliers:

$$\begin{aligned}\widetilde{\lambda}_{Yt} &:= \lambda_{Yt} e^{-\widehat{\lambda_Y}_\infty \cdot t}\\ \widetilde{\lambda}_{Lt} &:= \lambda_{Lt} e^{-\widehat{\lambda_L}_\infty \cdot t}\end{aligned}$$

Costate variables:

$$\begin{aligned}\widetilde{\upsilon_Q}_t &:= \upsilon_{Qt} e^{-\widehat{\upsilon_Q}_\infty \cdot t}\\ \widetilde{\upsilon_B}_t &:= \upsilon_{Bt} e^{-\widehat{\upsilon_B}_\infty \cdot t}\\ \widetilde{\upsilon}_{St} &:= \upsilon_{St} e^{-\widehat{\upsilon_S}_\infty \cdot t}\end{aligned}$$

We then use the above definitions to express the first-order conditions (FOC) in the scale-adjusted variables only:

FOC for the control variables

With the definitions of $\widetilde{c}_t$ and $\widetilde{\lambda}_{Yt}$ and taking into account that the first-order condition for consumption implies the long-run relation $\widehat{\lambda_Y}_\infty = -\frac{1}{\sigma_C}\widehat{c}_\infty$ between consumption growth and growth in the Lagrange-multiplier of the resource constraint, the first-order condition for consumption becomes:

$$\widetilde{c_t}^{-\frac{1}{\sigma_C}} = \widetilde{\lambda_{Yt}}$$

Using the definitions of the relevant scale-adjusted variables in the first-order condition for X (equation (2.40) in chapter 2) yields

$$\frac{\widetilde{v_{St}}}{\widetilde{B}_t} e^{\left(\widehat{v_{S\infty}} - \widehat{B}_\infty\right)t} + \widetilde{\lambda}_{Yt} \alpha \widetilde{X}_t^{\alpha-1} L_{Yt}^{1-\alpha} \widetilde{Q}_t^{1-\alpha} e^{\left(\widehat{\lambda_{Y\infty}} - (1-\alpha)(\widehat{X}_\infty - \widehat{Q}_\infty)\right)t} - \frac{1}{\varphi} \frac{\widetilde{\lambda}_{Lt}}{\widetilde{Q}_t} e^{\left(\widehat{\lambda_{L\infty}} - \widehat{Q}_\infty\right)t} = 0.$$

We substitute $\widehat{\lambda_{L\infty}} = \widehat{\lambda_{Y\infty}} + \alpha \widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ obtained by taking growth rates in the first-order condition for L_Y (equation (2.44) in chapter 2) and divide by $\widehat{\lambda_{Y\infty}} - (1-\alpha)(\widehat{X}_\infty - \widehat{Q}_\infty)$. As from the first-order condition for X, it follows that $\widehat{v_{S\infty}} - \widehat{B}_\infty - \widehat{\lambda_{Y\infty}} + (1-\alpha)(\widehat{X}_\infty - \widehat{Q}_\infty) = 0$ both with and without deceleration (in the latter case, $\widehat{X}_\infty - \widehat{Q}_\infty = 0$), the exponent of the first term becomes zero:

$$\frac{\widetilde{v_{St}}}{\widetilde{B}_t} + \widetilde{\lambda}_{Yt} \alpha \widetilde{X}_t^{\alpha-1} L_{Yt}^{1-\alpha} \widetilde{Q}_t^{1-\alpha} - \frac{1}{\varphi} \frac{\widetilde{\lambda}_{Lt}}{\widetilde{Q}_t} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} = 0$$

The exponent $\widehat{X}_\infty - \widehat{Q}_\infty$, on the contrary, is only zero in the cases without deceleration and is negative otherwise. Therefore $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ is either one for all t or converges to zero.

Next, we use the definitions of $\widetilde{v_{Q_t}}$, $\widetilde{Q}_t$ and $\widetilde{\lambda}_{Lt}$ in the first-order condition (2.41) for q:

$$\mu \widetilde{v_{Q_t}} \widetilde{Q}_t e^{\left(\widehat{v_{Q\infty}} + \widehat{Q}_\infty\right)t} = 2\widetilde{\lambda_{Lt}} q_t e^{\widehat{\lambda_{L\infty}} t}$$

We then replace $\widehat{v_{Q_\infty}}$ by $\widehat{\lambda_{L\infty}} - \widehat{Q}_\infty$ from (2.41). The exponential expressions cancel out:

$$\mu \widetilde{v_{Q_t}} \widetilde{Q}_t = 2\widetilde{\lambda}_{Lt} q_t$$

The same is true for the first-order condition (2.42) for b, where we use the definitions of $\widetilde{v_{Bt}}$, $\widetilde{B}_t$, $\widetilde{\lambda_{Lt}}$ and the relation $\widehat{v_{B\infty}} = \widehat{\lambda_{L\infty}} - \widehat{B}_\infty$:

$$\mu \widetilde{v_{Bt}} \widetilde{B}_t = 2\widetilde{\lambda}_{Lt} b_t$$

In the first-order condition (2.43) for n, the variables v_{Qt}, v_{Bt}, Q_t, B_t and λ_{Lt} have to be scale-adjusted to their respective steady-state values. Using $\widehat{\lambda_{L\infty}} = \widehat{v_{Q_\infty}} + \widehat{Q}_\infty = \widehat{v_{B\infty}} + \widehat{B}_\infty$, it becomes obvious that all exponential terms cancel out and the scale-adjusted first-order condition is:

$$\widetilde{v_Q}_t \mu q_t \widetilde{Q}_t + \widetilde{v_{Bt}} \mu b_t \widetilde{B}_t = \widetilde{\lambda}_{Lt} \left(q_t^2 + b_t^2 + d \right)$$

In the scale-adjusted first-order condition for L_Y, the exponential terms also vanish if we take the relation $\widehat{\lambda_{L\infty}} = \widehat{\lambda_{Y\infty}} + \alpha \widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty$ into account. The scale-adjusted equation is:

$$\widetilde{\lambda}_{Yt}(1-\alpha)\widetilde{X}_t^\alpha \widetilde{Q}_t^{1-\alpha} L_{Yt}^{-\alpha} = \widetilde{\lambda}_{Lt}$$

FOC for the state variables

When rescaling the first-order conditions for the state variables, the growth rates of the costate variables have to be adjusted as well. Because $\widetilde{v_{St}} = v_{St} e^{-\widehat{v_S}_\infty t}$, the growth rate $\widehat{\widetilde{v_{St}}}$ is the difference between the growth rate $\widehat{v_{St}}$ of the unscaled variable at time t and the long-run growth rate $\widehat{v_{S\infty}}$. Rearranging yields $\widehat{v_{St}} = \widehat{\widetilde{v_{St}}} + \widehat{v_{S\infty}}$. We divide the first-order condition for S by v_{St} and use the definition of $\widetilde{v_{St}}$ and $\widehat{v_{St}} = \widehat{\widetilde{v_{St}}} + \widehat{v_{S\infty}}$:

$$-\psi \frac{\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}}{\widetilde{v_{St}}} e^{\left(\frac{1}{\sigma_E}\widehat{S}_\infty - \widehat{v_{S\infty}}\right)t} L - \delta = \rho - \widehat{\widetilde{v_{St}}} - \widehat{v_{S\infty}}$$

We know that in the long-run social optimum, $\frac{1}{\sigma_E}\widehat{S}_\infty = \widehat{v_{S\infty}}$, so that the exponential expression is one:

$$-\psi \frac{\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}}{\widetilde{v_{St}}} L - \delta = \rho - \widehat{\widetilde{v_{St}}} - \widehat{v_{S\infty}}$$

We proceed similarly with the first-order conditions (2.46) and (2.47) for Q and B. From the first-order conditions (2.41) for q and (2.44) for L_Y, we can see that $\widehat{\lambda_{Y\infty}} - \widehat{v_{Q_\infty}} + \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) = \widehat{\lambda_{L\infty}} - \widehat{v_{Q_\infty}} - \widehat{Q}_\infty = 0$. The scale-adjusted first-order condition for Q is:

$$\mu n_t q_t + \frac{\widetilde{\lambda}_{Yt}}{\widetilde{v_{Q_t}}}(1-\alpha)\widetilde{X}_t^\alpha \widetilde{Q}_t^{-\alpha} L_{Yt}^{1-\alpha} + \frac{1}{\widetilde{v_{Q_t}}} \widetilde{\lambda}_{Lt} \frac{1}{\varphi} \frac{\widetilde{X}_t}{\widetilde{Q}_t^2} e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} = \rho - \widehat{\widetilde{v_{Q_t}}} - \widehat{v_{Q_\infty}}$$

Again, the last exponent is zero if and only if there is no deceleration and strictly negative otherwise.

When adjusting the scale of the first-order condition for B, we use $\widehat{v_{B\infty}} = \widehat{\lambda_{L\infty}} - \widehat{B}_\infty = \widehat{\lambda_{Y\infty}} + \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty - \widehat{B}_\infty$ and the relation $\widehat{v_{S\infty}} = \widehat{\lambda_{Y\infty}} + \widehat{B}_\infty - (1-\alpha)\,(\widehat{X}_\infty - \widehat{Q}_\infty)$:

$$-\frac{\widetilde{v_{St}}}{\widetilde{v_{Bt}}}\frac{\widetilde{X}_t}{\widetilde{B}_t^2} + \mu n_t b_t = \rho - \widehat{\widetilde{v_{Bt}}} - \widehat{v_{B\infty}}$$

FOC for the costate variables and Lagrange-multipliers

In the first-order conditions for the costate variables, we must take into account that the growth rate of the state variable is the sum of its long-run growth rate and the growth rate of the scale-adjusted state variable. This leads to the following scale-adjusted first-order condition for v_S:

$$\frac{\widetilde{X}_t}{\widetilde{B}_t\widetilde{S}_t}e^{(\widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty)t} - \delta = \widehat{\widetilde{S}}_t + \widehat{S}_\infty$$

In the first-order conditions (2.49) and (2.50) for v_Q and v_B, we only need to adjust the growth rates:

$$\mu n_t q_t = \widehat{\widetilde{Q}} + \widehat{Q}_\infty$$
$$\mu n_t b_t = \widehat{\widetilde{B}} + \widehat{B}_\infty$$

Finally, we derive the scale-adjusted first-order conditions for the Lagrange-multipliers λ_Y and λ_L. For λ_Y, we obtain

$$\widetilde{c}_t L = \widetilde{X}_t^\alpha \widetilde{Q}_t^{1-\alpha} L_{Yt}^{1-\alpha},$$

using $\alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty = \widehat{c}_\infty$. In the scale-adjusted first-order condition for λ_L, the exponential expression $e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}$ occurs once more:

$$L = L_{Yt} + \frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} + n_t(q_t^2 + b_t^2 + d)$$

The set of scale-adjusted first-order conditions is:

$$
\begin{aligned}
\widetilde{c}_t^{-\frac{1}{\sigma_C}} &= \widetilde{\lambda}_{Yt} && (3.A.1)\\
\frac{\widetilde{v_{St}}}{\widetilde{B}_t} + \widetilde{\lambda}_{Yt}\alpha\widetilde{X}_t^{\alpha-1}L_{Yt}^{1-\alpha}\widetilde{Q}_t^{1-\alpha} - \frac{1}{\varphi}\frac{\widetilde{\lambda}_{Lt}}{\widetilde{Q}_t}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} &= 0 && (3.A.2)\\
\mu\widetilde{v_{Q}}_t\widetilde{Q}_t &= 2\widetilde{\lambda}_{Lt}q_t && (3.A.3)\\
\mu\widetilde{v_{Bt}}\widetilde{B}_t &= 2\widetilde{\lambda}_{Lt}b_t && (3.A.4)\\
\widetilde{v_{Q}}_t\mu q_t\widetilde{Q}_t + \widetilde{v_{Bt}}\mu b_t\widetilde{B}_t &= \widetilde{\lambda}_{Lt}\left(q_t^2+b_t^2+d\right) && (3.A.5)\\
\widetilde{\lambda}_{Yt}(1-\alpha)\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{1-\alpha}L_{Yt}^{-\alpha} &= \widetilde{\lambda}_{Lt} && (3.A.6)\\
-\psi\frac{\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}}{\widetilde{v_{St}}}L - \delta &= \rho - \widehat{\widetilde{v_{St}}} - \widehat{v_{S\infty}} && (3.A.7)\\
\mu n_t q_t + \frac{\widetilde{\lambda}_{Yt}}{\widetilde{v_{Q}}_t}(1-\alpha)\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{-\alpha}L_{Yt}^{1-\alpha} + \frac{1}{\widetilde{v_{Q}}_t}\widetilde{\lambda}_{Lt}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t^2}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} &= \rho - \widehat{\widetilde{v_{Q}}_t} - \widehat{v_{Q\infty}} && (3.A.8)\\
-\frac{\widetilde{v_{St}}}{\widetilde{v_{Bt}}}\frac{\widetilde{X}_t}{\widetilde{B}_t^2} + \mu n_t b_t &= \rho - \widehat{\widetilde{v_{Bt}}} - \widehat{v_{B\infty}} && (3.A.9)\\
\frac{\widetilde{X}_t}{\widetilde{B}_t\widetilde{S}_t}e^{(\widehat{X}_\infty-\widehat{B}_\infty-\widehat{S}_\infty)t} - \delta &= \widehat{\widetilde{S}}_t + \widehat{S}_\infty && (3.A.10)\\
\mu n_t q_t &= \widehat{\widetilde{Q}} + \widehat{Q}_\infty && (3.A.11)\\
\mu n_t b_t &= \widehat{\widetilde{B}} + \widehat{B}_\infty && (3.A.12)\\
\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{1-\alpha}L_{Yt}^{1-\alpha} &= \widetilde{c}_t L && (3.A.13)\\
L_{Yt} + \frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} + n_t(q_t^2+b_t^2+d) &= L && (3.A.14)
\end{aligned}
$$

3.A.2 Derivation of the scale-adjusted dynamic system

Elimination of the Lagrange-multipliers $\widetilde{\lambda_Y}$ and $\widetilde{\lambda_L}$

$\widetilde{\lambda}_{Yt}$ and $\widetilde{\lambda}_{Lt}$ are directly given by (3.A.1) and (3.A.6). Their growth rates are:

$$
\begin{aligned}
\widehat{\widetilde{\lambda}_{Yt}} &= -\frac{1}{\sigma_c}\widehat{\widetilde{c}}_t\\
\widehat{\widetilde{\lambda}_{Lt}} &= \widehat{\widetilde{\lambda}_{Yt}} + \alpha\widehat{\widetilde{X}}_t + (1-\alpha)\widehat{\widetilde{Q}}_t - \alpha\widehat{L_{Yt}}\\
&= -\frac{1}{\sigma_c}\widehat{\widetilde{c}}_t + \alpha\widehat{\widetilde{X}}_t + (1-\alpha)\widehat{\widetilde{Q}}_t - \alpha\widehat{L_{Yt}}.
\end{aligned}
$$

At the center manifold of the scale-adjusted dynamic system, these growth rates are zero, while the unscaled variables grow at the rates:

$$\begin{aligned}\widehat{\lambda_{Y\infty}} &= -\frac{1}{\sigma_c}\widehat{c}_\infty \\ \widehat{\lambda_{L\infty}} &= -\frac{1}{\sigma_c}\widehat{c}_\infty + \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty\end{aligned}$$

Elimination of the costate variables

From (3.A.2), it follows with (3.A.1) for $\widetilde{\lambda_{Yt}}$ and (3.A.6) for $\widetilde{\lambda_{Lt}}$ that the scale-adjusted costate variable for pollution is given by

$$\widetilde{v_{St}} = (1-\alpha)\left(\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} - \frac{\alpha}{(1-\alpha)}\right)\widetilde{c}_t^{-\frac{1}{\sigma_c}}\widetilde{X}_t^{\alpha-1}L_{Yt}^{1-\alpha}\widetilde{Q}_t^{1-\alpha}\widetilde{B}_t.$$

The growth rate is

$$\begin{aligned}\widehat{\widetilde{v}_{St}} &= -\frac{1}{\sigma_c}\widehat{\widetilde{c}}_t + \widehat{\widetilde{B}}_t + \alpha\frac{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+1}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\left(\widehat{\widetilde{X}}_t - \widehat{\widetilde{Q}}_t - \widehat{L_{Yt}}\right) \\ &\quad + \frac{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\left(\widehat{X}_\infty - \widehat{Q}_\infty\right).\end{aligned}$$

The unscaled variable v_S grows at the rate

$$\widehat{v_{S\infty}} = -\frac{1}{\sigma_c}\widehat{c}_\infty + \widehat{B}_\infty - (1-\alpha)\left(\widehat{X}_\infty - \widehat{Q}_\infty\right)$$

at the steady-state of the scale-adjusted system.

The costate-variables $\widetilde{v_{Q_t}}$ and $\widetilde{v_{Bt}}$ and their growth rates are determined from (3.A.3) and (3.A.4) respectively, using $\widetilde{\lambda_{Lt}}$ from (3.A.6). The levels are:

$$\begin{aligned}\widetilde{v_{Q_t}} &= \frac{2}{\mu}(1-\alpha)\widetilde{c}_t^{-\frac{1}{\sigma_c}}\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{-\alpha}L_{Yt}^{-\alpha}q_t \\ \widetilde{v_{Bt}} &= \frac{2}{\mu}(1-\alpha)\widetilde{c}_t^{-\frac{1}{\sigma_c}}\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{1-\alpha}L_{Yt}^{-\alpha}\frac{b_t}{\widetilde{B}_t}\end{aligned}$$

Computing the growth rates yields:

$$\begin{aligned}\widehat{\widehat{v_{Q_t}}} &= -\frac{1}{\sigma_c}\widehat{\widehat{c}}_t - \alpha\left(\widehat{\widehat{Q}}_t - \widehat{\widehat{X}}_t + \widehat{L_{Yt}}\right) + \widehat{q}_t \\ \widehat{\widehat{v_{Bt}}} &= -\frac{1}{\sigma_c}\widehat{\widehat{c}}_t - \alpha\left(\widehat{\widehat{Q}}_t - \widehat{\widehat{X}}_t + \widehat{L_{Yt}}\right) + \widehat{\widehat{Q}}_t - \widehat{\widehat{B}}_t + \widehat{b}_t\end{aligned}$$

From the unscaled first-order conditions, we obtain:

$$\begin{aligned}\widehat{v_{Q\infty}} &= -\frac{1}{\sigma_c}\widehat{c}_\infty - \alpha\left(\widehat{Q}_\infty - \widehat{X}_\infty\right) \\ \widehat{v_{B\infty}} &= -\frac{1}{\sigma_c}\widehat{c}_\infty + \alpha\widehat{X}_\infty + (1-\alpha)\widehat{Q}_\infty - \widehat{B}_\infty\end{aligned}$$

Substituting $\widetilde{v_{Q_t}}$, $\widetilde{v_{Bt}}$ and $\widetilde{\lambda_{Lt}}$ into equation (3.A.5) and collecting terms yields:

$$q_t^2 + b_t^2 = d$$

We rearrange equation (3.A.7), $-\psi\frac{\widetilde{S}_t^{(1-2\sigma_E)/\sigma_E}}{\widetilde{v_{St}}}L - \delta = \rho - \widehat{\widetilde{v}}_{St} - \widehat{v}_{S\infty}$, and substitute $\widetilde{v}_{St}$, $\widehat{\widetilde{v}}_{St}$ and $\widehat{v}_{S\infty}$:

$$\begin{aligned}&\frac{\psi\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}}{(1-\alpha)\left(\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} - \frac{\alpha}{(1-\alpha)}\right)\widetilde{c}_t^{-\frac{1}{\sigma_c}}\widetilde{X}_t^{\alpha-1}L_{Yt}^{1-\alpha}\widetilde{Q}_t^{1-\alpha}\widetilde{B}_t}L + \frac{1}{\sigma_C}\widehat{c}_\infty - \widehat{B}_\infty \\ &-\alpha\frac{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+1}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} - \frac{\alpha}{(1-\alpha)}}(\widehat{X}_\infty - \widehat{Q}_\infty) + \delta + \rho \\ = \;& -\frac{1}{\sigma_c}\widehat{\widehat{c}}_t + \widehat{\widehat{B}}_t + \alpha\frac{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+1}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} - \frac{\alpha}{(1-\alpha)}}\left(\widehat{\widehat{X}}_t - \widehat{\widehat{Q}}_t - \widehat{L_{Yt}}\right)\end{aligned} \quad (3.A.15)$$

We then proceed similarly with equation (3.A.8), using the expressions for $\widetilde{\lambda}_{Yt}$, $\widetilde{\lambda_{Lt}}$, $\widetilde{v_{Q_t}}$, $\widehat{\widetilde{v_{Q_t}}}$ and $\widehat{v_{Q_\infty}}$. The equation becomes

$$\begin{aligned}&-\mu n_t q_t - \frac{\mu}{2}\frac{1}{q_t}L_{Yt} - \frac{\mu}{2}\frac{1}{q_t}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t} + \frac{1}{\sigma_C}\widehat{c}_\infty - \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) + \rho \\ &\qquad = -\frac{1}{\sigma_c}\widehat{\widehat{c}}_t + \alpha\left(\widehat{\widehat{X}}_t - \widehat{\widehat{Q}}_t - \widehat{L_{Yt}}\right) + \widehat{q}_t\end{aligned} \quad (3.A.16)$$

Finally, we replace $\widetilde{v_S}_t$, $\widetilde{v_B}_t$, $\widehat{\widetilde{v_B}}_t$ and $\widehat{v_B}_\infty$ in (3.A.9):

$$\begin{aligned}
&\frac{\mu}{2}\frac{1}{b_t}\left(\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}L_{Yt}\right)-\mu n_t b_t+\rho-\frac{1}{\sigma_c}\widehat{\widetilde{c}}_t \qquad (3.A.17)\\
= \;& \alpha\left(\widehat{\widetilde{X}}_t-\widehat{\widetilde{Q}}_t-\widehat{L_{Y}}_t\right)+\widehat{\widetilde{Q}}_t-\widehat{\widetilde{B}}_t+\widehat{b}_t+\frac{1}{\sigma_C}\widehat{c}_\infty-(\alpha\widehat{X}_\infty+(1-\alpha)\widehat{Q}_\infty)+\widehat{B}_\infty
\end{aligned}$$

Elimination of $\widetilde{c}$

We use the scale-adjusted resource constraint to obtain $\widetilde{c}_t$ and the growth rate $\widehat{\widetilde{c}}_t$. The long-run growth rate $\widehat{c}_\infty$ of the unscaled variable c was derived earlier. We substitute

$$\begin{aligned}
\widetilde{c}_t &= \frac{\widetilde{X}_t^{\alpha}\widetilde{Q}_t^{1-\alpha}L_{Yt}^{1-\alpha}}{L}\\
\widehat{\widetilde{c}}_t &= \alpha\widehat{\widetilde{X}}_t+(1-\alpha)\left(\widehat{\widetilde{Q}}_t+\widehat{L_{Y}}_t\right)\\
\widehat{c}_\infty &= \alpha\widehat{X}_\infty+(1-\alpha)\widehat{Q}_\infty
\end{aligned}$$

in equations (3.A.15) to (3.A.17). Equation (3.A.15) becomes:

$$\begin{aligned}
&\frac{\psi\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}L^{1-\frac{1}{\sigma_c}}}{(1-\alpha)\left(\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}\right)\widetilde{X}_t^{\alpha\left(1-\frac{1}{\sigma_c}\right)-1}\left(L_{Yt}\widetilde{Q}_t\right)^{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)}\widetilde{B}_t}\\
&+\alpha\left(\frac{\left(\frac{1}{\sigma_C}-1\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\left(\frac{1}{\sigma_C}\frac{\alpha}{(1-\alpha)}+1\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{X}_\infty \qquad (3.A.18)\\
&+(1-\alpha)\left(\frac{\left(\frac{1}{\sigma_C}+\frac{\alpha}{(1-\alpha)}\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+\frac{\alpha}{(1-\alpha)}\left(1-\frac{1}{\sigma_C}\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{Q}_\infty-\widehat{B}_\infty+\delta+\rho\\
= \;& -\alpha\left(\frac{\left(\frac{1}{\sigma_C}-1\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\left(\frac{1}{\sigma_C}\frac{\alpha}{(1-\alpha)}+1\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{\widetilde{X}}_t\\
&-(1-\alpha)\left(\frac{\left(\frac{1}{\sigma_C}+\frac{\alpha}{(1-\alpha)}\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+\frac{\alpha}{(1-\alpha)}\left(1-\frac{1}{\sigma_C}\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\left(\widehat{\widetilde{Q}}_t+\widehat{L_{Y}}_t\right)+\widehat{\widetilde{B}}_t
\end{aligned}$$

Equation (3.A.16) can be written as:

$$\begin{aligned}
& -\mu n_t q_t - \frac{\mu}{2}\frac{1}{q_t}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} - \frac{\mu}{2}\frac{1}{q_t}L_{Yt} + (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\widehat{Q}_\infty \\
= & -\alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{X}_\infty - (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\left(\widehat{\widetilde{Q}}_t + \widehat{L_{Yt}}\right) \\
& +\alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widetilde{X}}_t + \widehat{q}_t - \rho
\end{aligned} \tag{3.A.19}$$

And the modified equation (3.A.17) is:

$$\begin{aligned}
& \frac{\mu}{2}\frac{1}{b_t}\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} - \frac{\mu}{2}\frac{1}{b_t}\frac{\alpha}{(1-\alpha)}L_{Yt} - \mu n_t b_t - (1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)\widehat{Q}_\infty \\
= & -\alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{X}_\infty + \widehat{B}_\infty + (1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widetilde{Q}}_t - (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\widehat{L_{Yt}} \\
& +\alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widetilde{X}}_t - \widehat{\widetilde{B}}_t + \widehat{b}_t - \rho
\end{aligned} \tag{3.A.20}$$

Elimination of q

To eliminate q and its growth rate from the equations, we use $q_t^2 + b_t^2 = d$ to express q_t and $\widehat{q}_t$ as functions of b_t and $\widehat{b_t}$

$$\begin{aligned}
q_t &= \sqrt{d - b_t^2} \\
\widehat{q}_t &= -\frac{b_t^2}{d - b_t^2}\widehat{b}_t
\end{aligned}$$

Equations (3.A.18) and (3.A.20) remain unchanged. Equation (3.A.19) becomes:

$$\begin{aligned}
& -\mu n_t\sqrt{d - b_t^2} - \frac{\mu}{2}\frac{1}{\sqrt{d - b_t^2}}L_{Yt} - \frac{\mu}{2}\frac{1}{\sqrt{d - b_t^2}}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} - \alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{X}_\infty \\
= & (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\widehat{Q}_\infty - (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\left(\widehat{\widetilde{Q}}_t + \widehat{L_{Yt}}\right) \\
& +\alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widetilde{X}}_t - \rho - \frac{b_t^2}{d - b_t^2}\widehat{b}_t
\end{aligned}$$

Further, we replace q_t and $\widehat{q}_t$ in the scale-adjusted first-order conditions (3.A.11) and (3.A.12):

$$\mu n_t \sqrt{d-b_t^2} - \widehat{Q}_\infty = \widehat{\widehat{Q}}_t$$

$$\mu n_t b_t - \widehat{B}_\infty = \widehat{\widehat{B}}_t$$

Elimination of n

With the labor market constraint $L_{Yt} + \frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} + n_t(q_t^2 + b_t^2 + d) = L$ and $q_t^2 + b_t^2 = d$, we substitute n_t out of the system. Equation (3.A.18) does not depend on n_t directly. The modified equations (3.A.19) and (3.A.20) are:

$$\begin{aligned}
&-\frac{\mu}{2d}\left(L - L_{Yt} - \frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\sqrt{d-b_t^2} - \frac{\mu}{2}\frac{1}{\sqrt{d-b_t^2}}L_{Yt} \qquad (3.A.21)\\
&-\frac{\mu}{2}\frac{1}{\sqrt{d-b_t^2}}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} - \alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{X}_\infty + (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\widehat{Q}_\infty\\
= \;& -(1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\left(\widehat{\widehat{Q}}_t + \widehat{L_{Yt}}\right) + \alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widehat{X}}_t - \frac{b_t^2}{d-b_t^2}\widehat{b}_t - \rho
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{\mu}{2d}\left(L - L_{Yt} - \frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)b_t - \frac{\mu}{2}\frac{1}{b_t}\frac{\alpha}{(1-\alpha)}L_{Yt}\\
&+\frac{\mu}{2}\frac{1}{b_t}\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t} - \alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{X}_\infty - (1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)\widehat{Q}_\infty + \widehat{B}_\infty \qquad (3.A.22)\\
= \;& (1-\alpha)\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widehat{Q}}_t - (1-\alpha)\left(\frac{1}{\sigma_c} + \frac{\alpha}{(1-\alpha)}\right)\widehat{L_{Yt}} + \alpha\left(1 - \frac{1}{\sigma_c}\right)\widehat{\widehat{X}}_t - \widehat{\widehat{B}}_t + \widehat{b}_t - \rho
\end{aligned}$$

And equations (3.A.11) and (3.A.12) become:

$$\frac{\mu}{2d}\left(L - L_{Yt} - \frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)\sqrt{d-b_t^2} - \widehat{Q}_\infty = \widehat{\widehat{Q}}_t \qquad (3.A.23)$$

$$\frac{\mu}{2d}\left(L - L_{Yt} - \frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty - \widehat{Q}_\infty)t}\right)b_t - \widehat{B}_\infty = \widehat{\widehat{B}}_t \qquad (3.A.24)$$

Construction of the dynamical system

First, we replace $\widehat{\widehat{Q}}$ by $\frac{\mu}{2d}\left(L-L_{Yt}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\sqrt{d-b_t^2}-\widehat{Q}_\infty$ and $\widehat{\widehat{B}}$ by $\frac{\mu}{2d}\left(L-L_{Yt}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)b_t-\widehat{B}_\infty$ in equations (3.A.18), (3.A.21) and (3.A.22). The resulting equations are:

$$\begin{aligned}
&\frac{\psi\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}L^{1-\frac{1}{\sigma_c}}}{(1-\alpha)\left(\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}\right)\widetilde{X}_t^{\alpha\left(1-\frac{1}{\sigma_c}\right)-1}\left(L_{Yt}\widetilde{Q}_t\right)^{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)}\widetilde{B}_t} \\
&+\alpha\left(\frac{\left(\frac{1}{\sigma_c}-1\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\left(\frac{1}{\sigma_c}\frac{\alpha}{(1-\alpha)}+1\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{X}_\infty+\delta+\rho \\
= \; &-\alpha\left(\frac{\left(\frac{1}{\sigma_c}-1\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\left(\frac{1}{\sigma_c}\frac{\alpha}{(1-\alpha)}+1\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{\widetilde{X}}_t \\
&-(1-\alpha)\left(\frac{\left(\frac{1}{\sigma_c}+\frac{\alpha}{(1-\alpha)}\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+\frac{\alpha}{(1-\alpha)}\left(1-\frac{1}{\sigma_c}\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\right)\widehat{L_{Yt}} \\
&+\left\{\begin{array}{c}\left(b_t-(1-\alpha)\frac{\left(\frac{1}{\sigma_c}+\frac{\alpha}{(1-\alpha)}\right)\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}+\frac{\alpha}{(1-\alpha)}\left(1-\frac{1}{\sigma_c}\right)}{\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\frac{\alpha}{(1-\alpha)}}\sqrt{d-b_t^2}\right) \\ \cdot\frac{\mu}{2d}\left(L-L_{Yt}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\end{array}\right\}
\end{aligned} \tag{3.A.25}$$

and

$$\begin{aligned}
&-(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\frac{\mu}{2d}\left(L-L_{Yt}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\sqrt{d-b_t^2} \\
&-\frac{\mu}{2}\frac{1}{\sqrt{d-b_t^2}}L_{Yt}-\frac{\mu}{2}\frac{1}{\sqrt{d-b_t^2}}\frac{1}{\varphi}\frac{\widetilde{X}_t}{\widetilde{Q}_t}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\alpha\left(1-\frac{1}{\sigma_c}\right)\widehat{X}_\infty \\
= \; &-(1-\alpha)\left(\frac{1}{\sigma_c}+\frac{\alpha}{(1-\alpha)}\right)\widehat{L_{Yt}}+\alpha\left(1-\frac{1}{\sigma_c}\right)\widehat{\widetilde{X}}_t-\frac{b_t^2}{d-b_t^2}\widehat{b}_t-\rho
\end{aligned} \tag{3.A.26}$$

and

$$\begin{aligned}
&-(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)\frac{\mu}{2d}\left(L-L_{Yt}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\sqrt{d-b_t^2} \\
&-\frac{\mu}{2}\frac{1}{b_t}\frac{\alpha}{(1-\alpha)}L_{Yt}+\frac{\mu}{2}\frac{1}{b_t}\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}-\alpha\left(1-\frac{1}{\sigma_c}\right)\widehat{X}_\infty \\
= \; &-(1-\alpha)\left(\frac{1}{\sigma_c}+\frac{\alpha}{(1-\alpha)}\right)\widehat{L_{Yt}}+\alpha\left(1-\frac{1}{\sigma_c}\right)\widehat{\widetilde{X}}_t+\widehat{b}_t-\rho
\end{aligned} \tag{3.A.27}$$

Next, we collect the equal terms in (3.A.26) and (3.A.27) and equate.

Solving for $\widehat{b}_t$ yields:

$$\widehat{b}_t = \frac{1}{2}\mu\frac{d-b_t^2}{d}\frac{1}{b_t}L_{Yt}\left[\left(\frac{b_t}{\sqrt{d-b_t^2}} - \frac{\alpha}{1-\alpha}\right) + \frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\left(\frac{b_t}{\sqrt{d-b_t^2}}+1\right)\right] \tag{3.A.28}$$

We then substitute the expression for $\widehat{b}_t$ back in (3.A.27) and solve for $\widehat{\widetilde{X}}_t$:

$$\begin{aligned}
\widehat{\widetilde{X}}_t &= \frac{1}{\alpha\left(1-\frac{1}{\sigma_c}\right)}\rho - \widehat{X}_\infty - \frac{1-\alpha}{\alpha}\frac{\mu}{2d}\sqrt{d-b_t^2}L \\
&\quad + \frac{1}{\alpha\left(1-\frac{1}{\sigma_c}\right)}\frac{\mu}{2d}\sqrt{d-b_t^2}\begin{pmatrix} \frac{b_t}{\sqrt{d-b_t^2}} \\ -(1-\alpha)\left(\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\right)\end{pmatrix}L_{Yt}\left(1+\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right) \\
&\quad - \frac{1}{\alpha(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)}\frac{\mu}{2d}b_tL_{Yt} + \frac{1-\alpha}{\alpha}\frac{1}{\left(1-\frac{1}{\sigma_c}\right)}\left(\frac{1}{\sigma_c}+\frac{\alpha}{1-\alpha}\right)\widehat{L_{Yt}}
\end{aligned} \tag{3.A.29}$$

We substitute $\widehat{\widetilde{X}}_t$ in (3.A.25), simplify and solve for $\widehat{L_{Yt}}$:

$$\begin{aligned}
\widehat{L_{Yt}} &= -\sigma_c\rho + \frac{(\sigma_c-1)\psi\widetilde{S}_t^{\frac{1-2\sigma_E}{\sigma_E}}L^{1-\frac{1}{\sigma_c}}}{\widetilde{X}_t^{\alpha\left(1-\frac{1}{\sigma_c}\right)-1}\left(L_{Yt}\widetilde{Q}_t\right)^{(1-\alpha)\left(1-\frac{1}{\sigma_c}\right)}\widetilde{B}_t} \qquad (3.A.30)\\
&\quad - (1-\alpha)(\sigma_c-1)\left(\frac{\alpha}{1-\alpha}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\delta \\
&\quad + \frac{\mu}{2d}\left((1-\alpha)(\sigma_c-1)L+L_{Yt}\right)\sqrt{d-b_t^2}\begin{pmatrix}\left(\frac{\alpha}{1-\alpha}-\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\frac{b_t}{\sqrt{d-b_t^2}} \\ +\left(1+\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\end{pmatrix}
\end{aligned}$$

Taking into account the definition of the growth rate of a variable z, $\widehat{z}_t = \frac{\dot{z}_t}{z_t}$, the differential equation system is given by equations (3.A.28) to (3.A.30) and equations (3.A.10), (3.A.23) and (3.A.24), namely

$$\begin{aligned}
\widehat{\widetilde{S}}_t &= \frac{\widetilde{X}_t}{\widetilde{B}_t\widetilde{S}_t}e^{(\widehat{X}_\infty-\widehat{B}_\infty-\widehat{S}_\infty)t} - \delta - \widehat{S}_\infty && ((3.A.10)) \\
\widehat{\widetilde{Q}} &= \frac{1}{2}\frac{\mu}{d}\sqrt{d-b_t^2}\left(L-L_{Yt}\left(1+\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\right) - \widehat{Q}_\infty && ((3.A.23)) \\
\widehat{\widetilde{B}} &= \frac{1}{2}\frac{\mu}{d}b_t\left(L-L_{Yt}\left(1+\frac{1}{\varphi}\widetilde{X}_t\widetilde{Q}_t^{-1}L_{Yt}^{-1}e^{(\widehat{X}_\infty-\widehat{Q}_\infty)t}\right)\right) - \widehat{B}_\infty && ((3.A.24))
\end{aligned}$$

As we exclude parameter constellations such that corollary 2.1 in chapter 2 is satisfied, we set $e^{(\widehat{X}_\infty-\widehat{B}_\infty-\widehat{S}_\infty)t} = 1$ in (3.A.10).

Chapter 4

Constraining pollution control

In chapter 2, it was shown that for reasonable parameter values, the long-run optimal solution is characterized by both persistent green innovation and deceleration to dampen the rebound effect of productivity growth. In this chapter, we illustrate the importance of using both channels simultaneously to control pollution growth.[1]

For this purpose, we consider in turn two constrained optimization problems, where either green innovation or deceleration is not available to the social planner. While we cannot conduct a proper welfare analysis without studying transitional dynamics, we can analyze the consequences of either constraint for economic growth in the long run.

First, we rule out the possibility to dampen the rebound effect of productivity growth through deceleration in section 4.1. More precisely, we assume that the social planner must accomplish balanced growth of intermediate quantity and productivity in the long run. For parameter constellations such that the unconstrained long-run optimum requires deceleration, persistent economic growth is no longer optimal in the constrained solution.

Next, we make the alternative assumption that green innovation is not available to the social planner in section 4.2. The pollution intensity of intermediate goods is exogenously fixed so that pollution growth can only be restricted through deceleration. The constraint on green innovation is binding for any parameter constellation with positive long-run growth in the unconstrained model. Without green innovation, long-run growth is still positive for sufficiently low rates of time preference. However, the social planner chooses faster deceleration in this case and consumption growth slows down relative to the unconstrained solution. Further, contrary to what was found in chapter 2, consumption growth is unambiguously slower than in

[1]Excerpts from this and the following chapter are contained in Funk and Burghaus (2013).

the model without environmental externality if technical progress cannot be used to reduce the pollution intensity of intermediate goods.

4.1 The model without deceleration

An interior solution to the social planner's problem requires deceleration in the long run for reasonable assumptions about model parameters, as suggested in chapter 2.4.3. Still, models with a similar structure of the R&D-sector do not take the possibility to control the rebound effect of productivity growth into account (see Hart (2004), Ricci (2007)). In this section, we analyze the implications of confining the analysis to solutions without deceleration.

We therefore assume that the social planner is forced to choose $\widehat{X}_\infty = \widehat{Q}_\infty$. The constraint implies that in the long run, every increase in productivity leads to a one for one percentage increase in polluting quantity.

Consider again the research arbitrage equation (2.56) from chapter 2:

$$\frac{\mu}{2q_\infty}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) = \frac{\mu}{2b_\infty}\left(\frac{\alpha}{1-\alpha}L_{Y\infty} - \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right)$$

With the constraint $\widehat{X}_\infty = \widehat{Q}_\infty$, the ratio $(X/Q)_\infty$ is strictly positive and the ratio $b_\infty/q_\infty = \widehat{B}_\infty/\widehat{Q}_\infty$ must be sufficiently smaller than $\frac{\alpha}{1-\alpha}$. At the same time, condition (2.54) for balanced growth

$$\frac{\sigma_c - 1}{\sigma_c}\widehat{c}_\infty = \frac{1-\sigma_E}{\sigma_E}\widehat{S}_\infty + \widehat{X}_\infty - \widehat{B}_\infty - \widehat{S}_\infty$$

must hold.

Naturally, if deceleration is not chosen in the unconstrained long-run optimal solution, the constraint $\widehat{X}_\infty = \widehat{Q}_\infty$ is not binding and the constrained solution equals the unconstrained optimum.

If the conditions for deceleration in propositions 2.5 and 2.6 are satisfied, we know that the ratio $\widehat{B}_\infty/Q_\infty$ obtained from (2.54) under the assumption $\widehat{X}_\infty = \widehat{Q}_\infty$ is not compatible with $(X/Q)_\infty > 0$ in (2.56). The only solution to (2.56) and (2.54) with $\widehat{X}_\infty = \widehat{Q}_\infty$ occurs when q_∞, b_∞ and all long-run growth rates are zero.

Proposition 4.1 ***Constrained BG-optimum without deceleration***
Assume that the social planner must choose $\widehat{X}_\infty = \widehat{Q}_\infty$, so that pollution growth can only be restricted by green innovation in the long run.

If the conditions for deceleration in propositions 2.5 and 2.6 are not satisfied, the constraint $\widehat{X}_\infty = \widehat{Q}_\infty$ is not binding and the constrained solution equals the unconstrained optimum. Whenever the parameter constellation satisfies the conditions for deceleration in the unconstrained solution, the constraint is binding. There exists a constrained solution where the long-run growth rates of all variables are zero.

Proof. See appendix 4.A.1. ■

With the constraint $\widehat{X}_\infty = \widehat{Q}_\infty$, bringing about the balanced-growth relation (2.54) between growth in consumption and pollution requires research to be rather strongly oriented towards reductions in pollution intensity. But the social planner chooses deceleration in the unconstrained solution precisely when the relative return to green research is too small for such a strong orientation towards green innovation to be optimal. With less environmentally-oriented research on the other hand, pollution growth is suboptimally fast. The only solution is to give up long-run growth.

The implications of the result in proposition 4.1 are twofold: First, neglecting the possibility to control the rebound effect of productivity growth leads to the conclusion that long-run economic growth is not optimal for a larger parameter region than is actually the case. Second, the result reinforces the statement of proposition 2.2: We have shown in this proposition, that stationary consumption levels as demanded by certain green movements are socially preferable to the unregulated market solution. Proposition 4.1 indicates that stationary consumption levels may dominate long-run growth in terms of welfare even if there is government intervention and the externalities in green innovation in particular are internalized, if adequate policy-measures to control the rebound effect of productivity-growth are not in place.

4.2 The model without green innovation

In this section, we assume that b_t is exogenously fixed to zero for all t. We first derive and characterize the constrained long-run optimal solution in subsections 4.2.1 and 4.2.2. In subsection 4.2.3, the constrained optimum is compared to the unconstrained long-run optimal solution from chapter 2, both in the baseline model and in the modified setting where the representative household does not suffer a negative external effect from pollution.

4.2.1 Optimization problem and first-order conditions

Without green innovation, there is no possibility to reduce the pollution intensity of polluting inputs. Cleanliness B is no longer a state-variable but a constant so that the equation of motion for B is irrelevant.
The labor requirement in the R&D-sector depends on the step-size for productivity-oriented innovation and fixed costs only. The labor market constraint becomes:

$$L = L_{Yt} + \frac{1}{\varphi}\frac{X_t}{Q_t} + n_t(q_t^2 + d) \tag{4.1}$$

The current-value Hamiltonian-function reduces to:

$$\begin{aligned} H \;=\; & \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c-1}{\sigma_c}} - \psi \frac{\sigma_E}{1-\sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}}\right) L \\ & + v_{St}\left(\frac{X_t}{B} - \delta S_t\right) \\ & + v_{Qt}\mu n_t q_t Q_t \\ & + \lambda_{Yt}\left(X_t^{\alpha} Q_t^{1-\alpha} L_{Yt}^{1-\alpha} - c_t L\right) \\ & + \lambda_{Lt}\left(L - L_{Yt} - \frac{1}{\varphi}\frac{X_t}{Q_t} - n_t(q_t^2 + d)\right) \end{aligned}$$

Three changes arise in the necessary first-order conditions compared to the unconstrained model: First, the first-order conditions for B and its shadow price v_B are dropped from the system. Second, the first-order condition for λ_{Lt}, which restates the labor market constraint, is given by (4.1) instead of (2.34). Finally, the first-order condition for n reduces to

$$\frac{\partial H}{\partial n_t} = 0 \Leftrightarrow v_{Qt}\mu q_t Q_t = \lambda_{Lt}\left(q_t^2 + d\right), \tag{4.2}$$

because both the marginal social benefit and the marginal social costs of an increase in the mass of research units do no longer depend on how much an innovation reduces pollution intensity.

Of the four key equations (2.54) to (2.57) for the long-run solution from chapter 2, the balanced-growth condition (2.54) and the consumption Euler-equation (2.55),

$$(1/\sigma_c)\,\widehat{c}_\infty + \rho = \frac{\mu}{2q_\infty}\left(L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_\infty\right) + \widehat{Q}_\infty + \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right),$$

remain unchanged. The research arbitrage equation (2.56) on the other hand is no longer relevant. The indifference condition (2.57), $q_\infty^2 + b_\infty^2 = d$, reduces to

$$q_\infty^2 = d. \tag{4.3}$$

4.2.2 Characterization of the constrained long-run optimum

In the baseline model, given that the rate of time preference is low enough to make positive economic growth desirable, it is optimal to allocate some labor to green innovation in the long run and, as shown in chapter 3, also in the short and medium term. The constraint $b_t = 0$, $\forall t$ is therefore binding for any constellation of parameters which supports positive growth rates in the unconstrained solution.

If the social planner cannot reduce the pollution intensity of intermediates through green innovation, any rise in intermediate quantity leads to a proportional increase in emissions. The long-run growth rate of the pollution stock is given by

$$\widehat{S}_\infty = \max\left[\widehat{X}_\infty, -\delta\right] \tag{4.4}$$

From the balanced-growth equation (2.54) with $\widehat{B}_\infty = 0$ and (4.4), we obtain the relation between the long-run growth rates of intermediate quantity and productivity, which is

$$\widehat{X}_\infty^{b=0} = \frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1 + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\widehat{Q}_\infty^{b=0} \tag{4.5}$$

if $\widehat{S}_\infty > (-\delta)$.[2] As the numerator $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ of the ratio in (4.5) is smaller than one, while the denominator exceeds one, it is obvious that the constrained long-run optimum is always characterized by deceleration. For $\sigma_c < 1$, the ratio $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ is negative. There is then quantity degrowth.

Using the resource constraint, consumption growth as function of productivity

[2] For ease of exposition, we display long-run growth rates only for parameter constellations such that $\widehat{S}_\infty > (-\delta)$. The growth rates for the opposite case, $\widehat{S}_\infty = (-\delta)$, are shown in appendix 4.B.1. The results of proposition 4.2 below pertain also to the case with $\widehat{S}_\infty = (-\delta)$.

growth is given by

$$\widehat{c}_{\infty}^{b=0} = \frac{1}{1 + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \widehat{Q}_{\infty}^{b=0}. \tag{4.6}$$

From the consumption Euler-equation (2.55) with (4.3), we derive the long-run productivity growth rate

$$\widehat{Q}_{\infty}^{b=0} = \frac{1 + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}{(1/\sigma_c) + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} \left(\frac{1}{2}\mu d^{-1/2} L - \rho\right) \tag{4.7}$$

It is evident from (4.7) that productivity growth is positive if and only if the rate of time preference is smaller than $\overline{\rho}^{b=0} = \frac{1}{2}\mu\frac{1}{\sqrt{d}}L$.

Similar to chapter 2, a lower bound $\rho^{\mathrm{TVC},b=0}$ is implicitly defined by the transversality conditions for the state variables[3]. The following proposition then describes the constrained long-run optimum:

Proposition 4.2 ***Constrained ABG-optimum without green innovation*** *Assume $\sigma_E < 1/2$, $\rho^{TVC,b=0} < \rho < \overline{\rho}^{b=0}$ and that $b_t = 0$ $\forall t$ is exogenously fixed.*

There exists an asymptotically unique ABG-solution which satisfies the necessary conditions of the constrained optimization problem for $t \to \infty$. The ABG-solution is characterized as follows: (i) Long-run growth in productivity, output and per capita consumption is positive. (ii) Pollution growth is restricted by deceleration, i.e. $\widehat{X}_{\infty} < \widehat{Q}_{\infty}$. (iii) Whenever $\sigma_c < 1$, there is quantity degrowth ($\widehat{X}_{\infty} < 0$).

Proof. Most of the proof for the case $\widehat{S}_{\infty} > (-\delta)$ is given in the text. Appendix 4.B.2 proves the remaining results. ■

A long-run growth path with unconstrained pollution growth, along which consumption and the pollution stock grow at the same rate, cannot be optimal according to proposition 2.2. Therefore, when there is no green innovation, polluting quantity growth must remain below consumption and output growth. This means that the ratio of polluting intermediate quantity in GDP must decrease over time in a growing economy to decouple consumption- and pollution growth. The constrained long-run optimal solution must be characterized by deceleration if there is positive economic growth. From equation (4.5), it can be seen that deceleration is

[3] To satisfy the transversality conditions, it is still required that $\rho > (1 - 1/\sigma_C)\,\widehat{Q}_{\infty}$ which, upon substitution of $\widehat{Q}_{\infty}^{b=0}$ yields $\rho^{\mathrm{TVC},b=0} = \frac{(1-1/\sigma_C)\left(1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}\right)\right)}{1+\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}\right)+(1-1/\sigma_C)\frac{\alpha}{1-\alpha}\left(1-\frac{(\sigma_C-1)/\sigma_C}{(1-\sigma_E)/\sigma_E}\right)}\frac{1}{2}\mu\frac{1}{\sqrt{d}}L$. As before, the condition is satisfied for any $\rho > 0$ if $\sigma_C < 1$.

the stronger ($\widehat{X}_\infty^{b=0}/\widehat{Q}_\infty^{b=0}$ is the smaller), the smaller $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, or equivalently, the greener innovation (the larger $\widehat{B}_\infty^{b=0}/\widehat{Q}_\infty^{b=0}$) in the unconstrained model.

Contrary to the findings in the previous section, long-run growth remains optimal for sufficiently low values of the rate of time preference despite the constraint on pollution control. The reason is that the optimal relation between consumption and pollution growth given by (2.54) can be reached by choosing sufficiently fast deceleration without giving up consumption growth completely.

Because equation (2.54) still holds, proposition 2.4 in chapter 2 implies that the pollution stock must decline in the long run for $\sigma_c < 1$. It follows that in this case, the quantity of intermediates must decrease when there is no green innovation.

4.2.3 Comparison to the unconstrained solution

Comparison to the solution in the baseline model

To assess the implications of the constraint on innovation for output- and consumption growth, we compare the constrained solution to the unconstrained long-run optimum from chapter 2.[4] Proposition 4.3 summarizes the results:

Proposition 4.3 ***Comparison to the solution in the baseline model***
Compare the constrained solution as described in proposition 4.2 to the unconstrained solution from proposition 2.5.

In the constrained solution with $b_t = 0$, $\forall t$, (i) the upper bound on the rate of time preference to guarantee positive long-run growth in productivity, output and per capita consumption is smaller, (ii) deceleration is faster and (iii) the long-run optimal growth rates of intermediate quantity, output and consumption are smaller than in the unconstrained solution.

Proof. See appendix 4.B.3. ■

It is intuitive that without the option to decrease the pollution intensity of intermediate goods, quantity growth is restricted relative to the unconstrained case and that deceleration is faster. Of every unit of productivity growth, less is used to increase polluting quantity so that less of it results in consumption growth. $\widehat{X}_\infty^{b=0}/\widehat{Q}_\infty^{b=0}$ and $\widehat{c}_\infty^{b=0}/\widehat{Q}_\infty^{b=0}$ are therefore smaller than in the long-run optimum of chapter 2.

[4] We focus our comparison on the parameter region with $\rho > \rho^{\text{delta}}$ where ρ^{delta} is given by (2.B.43). Under this assumption, $\widehat{S}_\infty > (-\delta)$ not only in the unconstrained optimum of chapter 2, but also in the constrained solution of this chapter. We thereby avoid parameter constellations for which the long-run social optimum cannot be derived analytically. The case with $\rho \leq \rho^{\text{delta}}$ is treated in the appendix.

Comparing productivity growth yields ambiguous results. On the one hand, labor resources in the R&D-sector can entirely be used for productivity improvements when there is no green innovation, so that the step-size q is larger. On the other hand, the optimal mass of research units, n, decreases in q for most parameter constellations. An additional effect of the constraint $b_t = 0, \forall t$ on long-run productivity growth arises from the fact that deceleration is faster without green innovation. If $\sigma_c > 1$, optimal productivity growth tends to increase so as to dampen the negative effect of faster deceleration on consumption growth. The optimal productivity growth rate decreases if $\sigma_c < 1$ so that consumption smoothing is desired. Because of these countervailing effects, $\widehat{Q}_\infty^{b=0}$ may exceed or fall below productivity growth in the unconstrained case. But even if productivity growth is faster without green innovation, the larger productivity growth rate cannot outweigh the negative effect of stronger deceleration on consumption growth.

Comparison to the solution in the unconstrained model without environmental preference ($\psi = 0$)

If the representative household does not suffer utility losses from higher pollution ($\psi = 0$), naturally, the social planner does not choose any pollution control in the unconstrained model. The optimal solution for $\psi = 0$ is therefore unaffected by the restriction $b_t = 0$, $\forall t$. In chapter 2.4.4, we have shown, by comparing growth rates in the baseline model and the model without pollution externality, that environmental care need not be detrimental to long-run economic growth but may in fact accelerate it. For the constrained solution without innovation, this result is no longer true:

Proposition 4.4 ***Comparison to the model with*** $\psi = 0$
Compare the constrained solution from proposition 4.2 to the solution of the model without environmental externality ($\psi = 0$) as described in appendix 2.B.9.

In the constrained solution with $b_t = 0$, $\forall t$, and $\psi > 0$, (i) the upper bound on the rate of time preference to guarantee positive long-run growth in productivity, output and per capita consumption is the same, but (ii) the long-run growth rates of intermediate quantity, output and consumption per capita are smaller and (iii) the productivity growth rate is smaller (larger) if and only if $\sigma_c > 1$ ($\sigma_c < 1$), compared to the setting where the representative household does not suffer from pollution ($\psi = 0$).

Proof. See appendix 4.B.5. ■

The positive effect of environmental care in the unconstrained solution occurs when green innovation strongly increases the overall amount of labor in R&D. In

the constrained solution with $b_t = 0$, this positive effect vanishes. Further, deceleration in the constrained solution without green innovation is faster than in the unconstrained solution. Output and consumption growth are therefore always lower if the representative household cares for a clean environment.
Even without the positive spillover effect from green innovation, productivity growth can be faster with environmental externality. More precisely, this case occurs when there is quantity degrowth, i.e., for $\sigma_c < 1$, so that faster productivity growth is used to accelerate the decline in polluting quantity. If $\sigma_c > 1$, although deceleration ensures that polluting quantity does not grow proportionally to productivity, productivity growth still enhances quantity growth and thereby pollution growth. Consequently, the social planner chooses slower productivity growth if $\sigma_c > 1$.

4.3 Conclusion

In this chapter, it has been shown that neglecting either the possibility to reduce the pollution intensity of polluting production inputs through green innovation, or the possibility to dampen the rebound effect of productivity growth through deceleration considerably weakens long-run economic growth in the optimal solution.

If the rebound effect cannot be controlled, long-run economic growth is no longer optimal for any constellation of parameters which requires deceleration in the baseline model. The relative social return to green innovation is then too small to achieve the optimal restriction in pollution growth.

If, on the other hand, the pollution intensity of intermediate goods cannot be reduced by green innovation, long-run growth remains optimal if the representative household is sufficiently patient. However, there is stronger deceleration so that the growth rates of consumption and output fall short of those in the unconstrained solution. Long-run economic growth is also unambiguously lower compared to the model without environmental externality described in chapter 2, both because the positive effect of green innovation on overall research activity is no longer present and because of faster deceleration.

4.A Appendix to section 4.1

4.A.1 Proof of proposition 4.1

To prove the proposition, it has to be shown that the path without long-run growth satisfies all the necessary first-order conditions for $t \to \infty$, as well as the transversality conditions.

The balanced-growth condition (2.54) and the research arbitrage equation (2.56) have already been taken into account in the text. Condition (2.57), $q_\infty^2 + b_\infty^2 = d$, is no longer relevant. With $q_\infty = b_\infty = 0$, the social marginal return to n is zero while marginal costs are strictly positive due to the existence of fixed costs, so that $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n_t} < 0$. It follows that $n_\infty = 0$ is reconcilable with the Kuhn-Tucker-condition for n. The first-order conditions for q and b (equations (2.41) and (2.42) in chapter 2) hold for any q_∞ and b_∞ whenever $n_\infty = 0$. The consumption Euler-equation (2.55) and the first-order conditions for B and S determine the long-run levels of the shadow prices v_Q, v_B and v_S.

As all variables and shadow prices are constant in the long run, time discounting ($\rho > 0$) guarantees that the transversality conditions for Q, B and S are satisfied, so that the path without long-run growth indeed solves the necessary conditions of the constrained maximization problem.

4.B Appendix to section 4.2

4.B.1 Solution to the necessary conditions for $\widehat{S}_\infty = (-\delta)$

We first substitute $\widehat{S}_\infty = (-\delta)$ into (2.54) and solve for $\widehat{X}_\infty$:

$$\widehat{X}_\infty^{b=0} = \frac{1}{1-\alpha} \frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + \frac{1-1/\sigma_c}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\widehat{Q}_\infty^{b=0} \tag{4.B.1}$$

The resulting relation between consumption growth and productivity growth is

$$\begin{aligned} \widehat{c}_\infty^{b=0} &= \alpha \widehat{X}_\infty^{b=0} + (1-\alpha)\widehat{Q}_\infty^{b=0} \\ &= \frac{\alpha}{1-\alpha} \frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta + \frac{1}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\widehat{Q}_\infty^{b=0}. \end{aligned} \tag{4.B.2}$$

Using the same steps as in chapter 2, we obtain the long-run optimal value $n_\infty^{b=0}$ from (2.55). $n_\infty^{b=0}$ is given by:

$$n_\infty^{b=0} = \frac{(1-\alpha)\sigma_c}{\sqrt{d}\mu}\left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)\left(\frac{1}{2}\mu\frac{1}{\sqrt{d}}L-\rho-\frac{1-\sigma_c}{\sigma_c}\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)$$

The productivity growth rate is:

$$\begin{aligned}\widehat{Q}_\infty^{b=0} &= \mu n_\infty^{b=0} q_\infty^{b=0} \\ &= (1-\alpha)\sigma_c\left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)\left(\frac{1}{2}\mu\frac{1}{\sqrt{d}}L-\rho-\frac{1-\sigma_c}{\sigma_c}\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)\end{aligned} \tag{4.B.3}$$

We substitute (4.B.3) in (4.B.1) and (4.B.2) to derive the long-run growth rates of c and X:

$$\begin{aligned}\widehat{X}_\infty^{b=0} &= \frac{1}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta \\ &\quad +(1-\alpha)(\sigma_c-1)\left(\frac{1}{2}\mu\frac{1}{\sqrt{d}}L-\rho-\frac{1-\sigma_c}{\sigma_c}\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right)\end{aligned} \tag{4.B.4}$$

$$\widehat{c}_\infty^{b=0} = \frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta+(1-\alpha)\sigma_c\left(\frac{1}{2}\mu\frac{1}{\sqrt{d}}L-\rho-\frac{1-\sigma_c}{\sigma_c}\frac{\alpha}{1-\alpha}\frac{(1-2\sigma_E)/\sigma_E}{\frac{\alpha}{1-\alpha}(1/\sigma_c)+1}\delta\right) \tag{4.B.5}$$

Because $\sigma_c < 1$, the transversality conditions are satisfied.

4.B.2 Proof of proposition 4.2

1. **Uniqueness:** The path characterized in the text for $\widehat{S}_\infty > (-\delta)$ and in appendix 4.B.1 for $\widehat{S}_\infty = (-\delta)$ satisfies all the necessary conditions for an interior optimum ($n_\infty > 0$) for $\rho^{\text{TVC},b=0} < \rho < \overline{\rho}^{b=0}$. It is still to prove that the interior solution is unique and the only other solution candidate, $n_\infty = 0$, solves the necessary conditions only if $\rho \geq \overline{\rho}^{b=0}$. To do so, it has to be shown that $\lim_{t\to\infty} \frac{\partial H}{\partial n} \mid_{n_\infty=0} \leq 0$ if and only if $\rho \geq \overline{\rho}^{b=0}$. $\lim_{t\to\infty} \frac{\partial H}{\partial n} \mid_{n_\infty=0} \leq 0$ if and only if

$$v_{Q\infty}\mu q_\infty Q_\infty \leq \lambda_{L\infty}\left(q_\infty^2+d\right). \tag{4.B.6}$$

As q_∞ is not uniquely determined for $n_\infty = 0$, we define the limit $\lim_{n_\infty\to 0} q(n_\infty) = \frac{\mu}{2}L/\rho$ as the long-run solution for q (see appendix 2.B.7). The long-run solution for the shadow-price of Q follows from the first-order condition for Q (equation

(2.46)). With $L_{Y\infty} + \frac{1}{\varphi}\left(\frac{X}{Q}\right)_{\infty} = L$, $v_{Q\infty}$ is given by $v_{Q\infty} = \lambda_{L\infty} Q_{\infty}^{-1} \frac{L}{\rho}$. After substituting $\lim\limits_{n_\infty \to 0} q(n_\infty) = \frac{\mu}{2} L/\rho$ and $v_{Q\infty} = \lambda_{L\infty} Q_{\infty}^{-1} \frac{L}{\rho}$ in (4.B.6) and solving for ρ, it is obvious that $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} |_{n_\infty=0} \leq 0$ if and only if $\rho \geq \overline{\rho}^{b=0}$.

2. **Properties for $\widehat{S}_\infty = (-\delta)$:**

   (ii) **Deceleration:** There is deceleration in the long-run optimal solution if and only if $\widehat{X}_\infty < \widehat{Q}_\infty$. From (4.B.1), it follows that in the case where $\widehat{S}_\infty = (-\delta)$, $\widehat{X}_\infty^{b=0} < \widehat{Q}_\infty^{b=0}$ if and only if

   $$\widehat{Q}_\infty^{b=0} > \frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}\delta$$

   This condition is satisfied because $\widehat{Q}_\infty \geq \widehat{c}_\infty$ and the threshold $\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$ which $\widehat{c}_\infty$ must exceed for $\widehat{S}_\infty$ to converge to $(-\delta)$ is larger than $\frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}\delta$: Given $\sigma_c < 1$ and $\sigma_E < 1/2$, the numerators and denominators of both ratios are positive. The numerator of $\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}$ is larger than the numerator of $\frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}$ and the denominator of $\frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}$ is smaller than the denominator of $\frac{(1-2\sigma_E)/\sigma_E}{1/\sigma_c}$.

   (iii) **Quantity degrowth:** There is quantity degrowth in the long-run optimal solution if and only if $\widehat{X}_\infty < 0$. If (4.B.1) is the relevant equation for $\widehat{X}_\infty$, quantity degrowth requires:

   $$\widehat{Q}_\infty^{b=0} > \frac{(1-2\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$$

   By a similar reasoning as in (ii), it can be concluded that as $\frac{(1-2\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta < \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$ for $\sigma_c < 1$ and $\sigma_E < 1/2$, the condition is satisfied whenever $\widehat{S}_\infty = (-\delta)$.

4.B.3 Proof of proposition 4.3

1. **Parameter restriction for positive long-run growth:** The upper bound on the rate of time preference which guarantees positive growth in the unconstrained model is given by (2.B.42):

$$\overline{\rho} = \begin{cases} \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L, & \frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases}$$

In the constrained solution, it is $\overline{\rho}^{b=0} = 1/2\mu d^{-1/2}L$. As the radiant in $\overline{\rho}$ from the unconstrained solution is larger than one, it follows that $\overline{\rho} > \overline{\rho}^{b=0}$.

2. **Comparison of $\widehat{X}_\infty/\widehat{Q}_\infty$:** For $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, there is no deceleration in the unconstrained solution. It follows straightforwardly that $\widehat{X}_\infty^{b=0}/\widehat{Q}_\infty^{b=0} < \widehat{X}_\infty/\widehat{Q}_\infty = 1$.
For $\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, comparison of (4.5) with (2.60) in chapter 2 shows that $\widehat{X}_\infty^{b=0}/\widehat{Q}_\infty^{b=0} < \widehat{X}_\infty/\widehat{Q}_\infty$ if and only if $\frac{\alpha}{(1-\alpha)^2} > 0$, which is true.

3. **Comparison of $\widehat{X}_\infty$:** Substituting (4.7) in (4.5) yields the long-run growth rate of X for $b = 0$:

$$\widehat{X}_\infty^{b=0} = \frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\left(\frac{1}{2}\mu d^{-1/2}L - \rho\right) \tag{4.B.7}$$

From equation (4.B.7) and equation (2.B.24) in chapter 2, we find that for $\frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$, a sufficient condition for $\widehat{X}_\infty^{b=0} < \widehat{X}_\infty$ is

$$\frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} < \frac{1}{1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2}$$

This condition is satisfied as the numerator of the fraction on the left-hand side is smaller than one and the denominator exceeds one, given $\frac{\alpha}{1-\alpha} > 1-\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$. With deceleration ($\frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}$), the relevant equation from chapter 2 is (2.60) and a sufficient condition for $\widehat{X}_\infty^{b=0} < \widehat{X}_\infty$ is:

$$\begin{aligned}
\frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E} &< \frac{1 + \left(\frac{\alpha}{1-\alpha}\right)^2 - \left(\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) - \frac{\alpha}{1-\alpha}\right)}{1 + \left(\frac{\alpha}{1-\alpha}\right)^2} \\
\Leftrightarrow \alpha\frac{(\sigma_c - 1)/\sigma_c}{(1-\sigma_E)/\sigma_E} &< 1
\end{aligned}$$

As the right-hand side is negative and the left-hand side is positive, the condition is satisfied.

4. **Comparison of $\widehat{c}_\infty \left(= \widehat{Y}_\infty\right)$:** Substituting (4.7) in (4.6) yields the long-run growth rate of c for $b = 0$:

$$\widehat{c}_\infty^{b=0} = \frac{1}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)}\left(\frac{1}{2}\mu d^{-1/2}L - \rho\right) \tag{4.B.8}$$

If $\frac{\alpha}{1-\alpha} > \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)$, comparing (4.B.8) to (2.B.24), we find that a sufficient condition for consumption growth in the constrained solution to be slower is

$$\frac{1}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} < \frac{1}{1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2},$$

which is satisfied given the above restriction of the parameter range.
For $\frac{\alpha}{1-\alpha} < \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)$, it follows from equations (4.B.8) and (2.61) that long-run consumption growth is slower in the constrained solution, if and only if

$$\frac{1}{2}\mu d^{-1/2}L < \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L,$$

which is true because the radiant on the right-hand side is larger than one.

4.B.4 Comparison to the unconstrained solution of proposition 2.6 ($\rho \leq \rho^{\text{delta}}$)

The critical value ρ^{delta}, applying in the unconstrained model if $\sigma_c < 1$, is

$$\rho^{\text{delta}} = \begin{cases} \frac{1}{2}\left(1 + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_1 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} > 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \\ \frac{1}{2}\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} d^{-1/2}\mu L - \kappa_2 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta, & \frac{\alpha}{1-\alpha} < 1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} \end{cases},$$

with $\kappa_1 = 1/\sigma_c + \left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)^2$ and $\kappa_2 = 1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)$ (see equation (2.B.43)).

Using (4.B.7) and $\widehat{S}_\infty^{b=0} = \widehat{X}_\infty^{b=0}$, it follows that the critical value for $\sigma_c < 1$ in the constrained model is:

$$\rho^{\text{delta},b=0} = \frac{1}{2}d^{-1/2}\mu L - \kappa_1 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta \tag{4.B.9}$$

Comparison shows that $\rho^{\text{delta, } b=0} < \rho^{\text{delta}}$. It follows that for $\rho \leq \rho^{\text{delta}}$, the following cases can arise:

(1.) $\rho^{\text{delta, } b=0} < \rho \leq \rho^{\text{delta}}$: In this case, $\widehat{S}_\infty = (-\delta)$ and $\widehat{S}_\infty^{b=0} > (-\delta)$.

(2.) $\rho \leq \rho^{\text{delta, } b=0} < \rho^{\text{delta}}$: In this case, $\widehat{S}_\infty = \widehat{S}_\infty^{b=0} = (-\delta)$.

A formal comparison of the unconstrained and the constrained solution is only possible if the parameter range is such that the conditions for deceleration in proposition 2.6 are satisfied, because the unconstrained social planner's problem cannot be solved analytically otherwise.

It is straightforward that growth in output and consumption is slower without green innovation in case (1.): $\rho^{\text{delta, } b=0} < \rho < \rho^{\text{delta}}$ implies that $\widehat{c}_\infty^{b=0} < \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta \leq \widehat{c}_\infty$. In the second case, proof that $\widehat{c}_\infty^{b=0} < \widehat{c}_\infty$ is obtained straightforwardly from (4.B.8) and (2.B.52), as $\left(1 + \left(\frac{\alpha}{1-\alpha}\right)^2\right)^{1/2} > 1$.

Comparison of (4.B.1) and (2.B.48) from chapter 2 shows that in case (2.), $\widehat{X}_\infty/\widehat{Q}_\infty$ is smaller in the constrained solution, so that deceleration is unambiguously faster without green innovation if and only if $\frac{\alpha}{(1-\alpha)^2} > 0$, which is true.

The comparison of the unconstrained and the constrained solution otherwise yields ambiguous results.

4.B.5 Proof of proposition 4.4

1. **Parameter restriction for positive long-run growth**: The upper bound for positive long-run growth in the optimal solution without preference for the environment is $\rho^{\psi=0} = \frac{1}{2}\mu d^{-1/2}L$ which is equal to $\rho^{b=0}$.

2. **Comparison of $\widehat{X}_\infty$:** From equation (4.B.7) and equation (2.B.57) in chapter 2, we find that for $\widehat{S}_\infty^{b=0} > (-\delta)$, $\widehat{X}_\infty^{b=0} < \widehat{X}_\infty^{\psi=0}$ if and only if

$$\frac{\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} < \frac{1}{1/\sigma_c},$$

   which is true, as $\frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E} < 1$ and $1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) > 1/\sigma_c$ given $\sigma_E < 1/2$.
   If $\widehat{S}_\infty^{b=0}$ converges to $(-\delta)$, $\widehat{X}_\infty^{b=0} < 0$ while $\widehat{X}_\infty^{\psi=0} > 0$ so that obviously $\widehat{X}_\infty^{b=0} < \widehat{X}_\infty^{\psi=0}$.

3. **Comparison of $\widehat{c}_\infty$:** Comparing (4.B.8) to (2.B.57) in chapter 2, we find that consumption growth in the constrained solution is slower for $\widehat{S}_\infty^{b=0} > (-\delta)$ if

and only if

$$\frac{1}{1/\sigma_c + \frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right)} < \frac{1}{1/\sigma_c},$$

which is true.
If $\widehat{S}_\infty^{b=0} = (-\delta)$, comparing (4.B.5) to (2.B.57) and collecting terms proves that $\widehat{c}_\infty^{b=0} < \widehat{c}_\infty^{\psi=0}$ if and only if

$$\frac{1}{2}\mu d^{-1/2}L - \rho - \frac{1-2\sigma_E}{\sigma_E}\delta > 0$$

This condition is satisfied whenever $\widehat{S}_\infty^{b=0} = (-\delta)$:
If $\widehat{S}_\infty^{b=0} = (-\delta)$, then $\rho < \rho^{\mathrm{delta},b=0} = \frac{1}{2}\mu d^{-1/2}L - \kappa_1 \frac{(1-\sigma_E)/\sigma_E}{(1-\sigma_c)/\sigma_c}\delta$. As $(1-2\sigma_E)/\sigma_E < (1-\sigma_E)/\sigma_E$, it suffices to show that

$$\kappa_1 \frac{1}{(1-\sigma_c)/\sigma_c} > 1.$$

Using the definition of κ_1 and rearranging yields

$$\frac{\alpha}{1-\alpha}\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) > -1,$$

which is true.

4. **Comparison of $\widehat{Q}_\infty$:** From (4.7) and (2.B.57), it can be concluded that given $\widehat{S}_\infty^{b=0} > (-\delta)$, $\widehat{Q}_\infty^{b=0} < \widehat{Q}_\infty^{\psi=0}$ if and only if

$$\frac{\alpha}{1-\alpha}(1-\sigma_c)\left(1 - \frac{(\sigma_c-1)/\sigma_c}{(1-\sigma_E)/\sigma_E}\right) < 0.$$

This inequality is satisfied if and only if $\sigma_c > 1$. For $\sigma_c < 1$, $\widehat{Q}_\infty^{b=0} > \widehat{Q}_\infty^{\psi=0}$. The case where $\widehat{S}_\infty^{b=0} = (-\delta)$ occurs only if $\sigma_c < 1$. It still has to be proven that $\widehat{Q}_\infty^{b=0} > \widehat{Q}_\infty^{\psi=0}$ in this case. We compare (2.B.57) to (4.B.3). It follows after division by σ_c and collecting terms that $\widehat{Q}_\infty^{b=0} < \widehat{Q}_\infty^{\psi=0}$ in case $\widehat{S}_\infty^{b=0} = (-\delta)$ if and only if

$$\left(\frac{1}{\sigma_c} - 1\right)\left(\frac{1}{2}\mu\frac{1}{\sqrt{d}}L - \rho - \frac{1-2\sigma_E}{\sigma_E}\delta\right) < 0.$$

As $\sigma_c < 1$, the inequality is equivalent to:

$$\frac{1}{2}\mu d^{-1/2}L - \rho - \frac{1-2\sigma_E}{\sigma_E}\delta < 0$$

We have shown before that $\frac{1}{2}\mu d^{-1/2}L - \rho - \frac{1-2\sigma_E}{\sigma_E}\delta > 0$ whenever $\widehat{S}_{\infty}^{b=0} = (-\delta)$. This proves that $\widehat{Q}_{\infty}^{b=0} > \widehat{Q}_{\infty}^{\psi=0}$ also for $\widehat{S}_{\infty}^{b=0} = (-\delta)$.

Chapter 5

Pollution and resource scarcity

A large share of worldwide emissions results from the use of polluting non-renewable resources like fossil fuels in energy generation.[1] Because our baseline model does not consider non-renewable resources, intermediate goods could not explicitly be interpreted as energy inputs. This chapter extends the baseline model of chapter 2 by a non-renewable resource to examine the robustness of our main results with respect to the consideration of resource scarcity. In particular, it is assumed that the resource is the only input to intermediate production and that its use as production input generates pollution.

Exhaustibility of the resource stock demands that resource use must ultimately decline to zero. This implies on the one hand, that the adverse effect of production on the environment automatically vanishes with the resource stock as polluting emissions converge to zero asymptotically. The total amount of emissions remains finite and the pollution stock decreases over time. On the other hand, intermediate quantity must also decline with resource use. Quantity degrowth is unavoidable at least in the long run, not only in the optimal solution but also in the laissez-faire equilibrium.

We find that because the finiteness of the resource stock requires intermediate quantity to fall along the equilibrium path, it slows growth in consumption and output in the laissez-faire equilibrium. Growth rates are not affected by the particular size of the initial resource stock. However, less resource-abundant economies face lower consumption, output and pollution levels on the entire equilibrium path compared to economies with a large natural resource endowment.

While for the laissez-faire equilibrium, the natural resource constraint is always

[1] The energy share in total anthropogenic greenhouse gas emissions in 2004 in terms of CO_2-equivalent was 25.9 % according to the IPCC Synthesis Report (IPCC (2007)).

binding, we have shown in chapter 2 that the social planner voluntarily chooses quantity degrowth if preferences are such that a fast decline in the pollution stock is desired and production is sufficiently inelastic with respect to intermediates. We prove that in this case, contrary to what is assumed in the literature[2], resource scarcity is no constraint to the long-run social optimum and the results from the baseline model still hold if the initial resource stock is large enough.
We also characterize the optimal solution for the case where the natural resource constraint is binding. The need to save on the exhaustible resource then leads to such a fast decline in resource use and therefore pollution that green innovation is no longer optimal in the long run.

Section 5.1 describes the modifications to the model setup compared to the baseline model. In section 5.2, we derive the laissez-faire equilibrium while the long-run social optimum is analyzed in section 5.3.

5.1 Setup

We denote the resource stock in period t by F_t. Starting from a finite positive initial level F_0, the resource stock is depleted proportionally to resource use:

$$\dot{F}_t = -R_t \tag{5.1}$$

For simplification, we assume that the resource can be extracted at zero cost. This assumption is in line with previous literature (see for example Barbier (1999), Schou (2002) and Groth and Schou (2002)).

The resource stock F_t must be non-negative for any t. Therefore total extraction must not exceed the initial stock F_0, a requirement which is formally represented by the condition

$$\int_0^\infty R_t dt \leq F_0. \tag{5.2}$$

Suppose that one unit of intermediate goods is produced by one unit of the non-renewable resource so that

$$X_{it} = R_{it} \tag{5.3}$$

is resource input in sector i and $X_t = \int_{i=0}^1 X_{it} di = R_t$ is aggregate resource use. As suggested in the introduction, equations (5.2) and (5.3) imply that intermediate

[2] See, for example, Grimaud and Rouge (2008) and Schou (2000, 2002) for models with a polluting non-renewable resource.

quantity must decline in the long run as the stock of the natural resource gets exhausted, both in the laissez-faire equilibrium and the social optimum:

Lemma 5.1 *If intermediate goods are produced with a non-renewable resource according to equation (5.3), the growth rate $\widehat{X}$ of intermediate quantity is negative in the long-run. Any solution path is characterized by quantity degrowth for $t \to \infty$.*

Proof. It follows from (5.3), that aggregate resource use is $R_t = X_t$. Substitution into equation (5.2) yields $\int_0^\infty X_t dt \leq F_0$. To satisfy the condition, the integral must converge, which requires $\lim_{t\to\infty} \widehat{X}_t = \widehat{X}_\infty < 0$ as a necessary condition. ■

We now study in detail the laissez-faire equilibrium and the long-run social optimum.[3]

5.2 The laissez-faire equilibrium

The laissez-faire equilibrium is defined similar to chapter 2, with the exception that there is an additional market for resources which must clear at equilibrium.

We prove in this section that because of the exhaustibility of the resource stock, resource use and intermediate production fall along the entire equilibrium path. Quantity degrowth may induce consumption to decrease as well. On the other hand, emissions decline with resource use and the long-run growth rate of the pollution stock is negative. The total amount of pollution generated over time is limited by the scarcity of the polluting resource. Subsections 5.2.1 to 5.2.3 describe the changes in the individual optimization problems. Subsection 5.2.4 characterizes the equilibrium.

In subsection 5.2.5, the effects of resource scarcity on the growth rates and levels of production, consumption and pollution are examined more closely. It is shown that the growth rate of the resource price, which reflects the progressing scarcity of the resource over time given an initial stock, depresses growth in GDP and consumption for any constellation of parameters. The size of the initial resource stock does not affect growth rates. The less resource-abundant an economy is, however, the higher is the level of the resource price in every period and the lower are the levels of production, consumption and pollution along the equilibrium path.

[3] As in the previous chapters, we focus on balanced and asymptotically-balanced growth solutions.

5.2.1 The representative household

We assume that the resource stock is owned by the household and that there are well-defined property rights. Firms pay a price p_{Rt} per unit of the resource. The modified budget constraint of the representative household is given by

$$C_t + \dot{A}_t = r_t A_t + w_{Yt} L_{Yt} + w_{Dt} L_{Dt} + p_{Rt} R_t. \tag{5.4}$$

As in chapter 2, the no-Ponzi condition must hold to rule out chain-ladder financing.

Besides the budget constraint, the representative household takes (5.2) into account when maximizing utility. We denote the current-value of the Lagrange-multiplier for the new constraint by $\lambda_{Rt} = \widetilde{\lambda}_{Rt} e^{\rho t}$. Because the constraint is the same in every period t, the present-value $\widetilde{\lambda}_R$ is constant so that $\widehat{\lambda}_R = \rho$.

The set of first-order conditions in the baseline model from chapter 2 is extended by the first-order condition for resource extraction R_t:

$$\frac{\partial H}{\partial R_t} = 0 \Leftrightarrow v_{At} p_{Rt} = \lambda_{Rt} \tag{5.5}$$

and the Kuhn-Tucker condition

$$\frac{\partial H}{\partial \lambda_{Rt}} \leq 0 \Leftrightarrow \int_0^\infty R_t dt \leq F_0 \quad \lambda_{Rt} \geq 0 \quad \lambda_{Rt} \left(F_0 - \int_0^\infty R_t dt \right) = 0. \tag{5.6}$$

v_{At} is the shadow price of assets A in t.

Condition (5.5) states that the representative household is indifferent about the amount of resource extraction in period t when the marginal utility gain ($v_{At} p_{Rt}$) from selling the resource and using the proceeds to accumulate more assets equals the marginal utility loss (λ_{Rt}) the household incurs because an additional unit extracted in t reduces the amount which can be extracted in future periods.

Because the household gains from selling the resource while he does not take into account the pollution caused by its use in production, the resource stock is fully exhausted asymptotically. This follows formally from the Kuhn-Tucker condition (5.6) together with (5.5): If $\int_0^\infty R_t dt < F_0$, λ_{Rt} must be zero for all t[4] by the complementary-slackness condition $\lambda_{Rt} \left(F_0 - \int_0^\infty R_t dt \right) = 0$. From (5.5), it follows that the resource price p_{Rt} would have to be zero for all t[5]. This cannot be an equi-

[4] Because $e^{\rho t}$ cannot become zero, $\lambda_{Rt} = 0$ can only be satisfied if $\widetilde{\lambda}_R = 0$. But in this case, $\lambda_{Rt} = 0$ for all t.

[5] v_{At} cannot be zero for $t < \infty$ as it must equal the marginal utility of consumption by the first-order condition for c (equation (2.A.1) in appendix 2.A.1).

librium as resource demand would become infinitely large and it would be beneficial to slightly raise the price and extract more of the resource in every period.

The Euler-equation

$$\widehat{c}_t = \sigma_c (r_t - \rho) \tag{5.7}$$

for per capita consumption from chapter 2 still holds.

Taking growth rates on both sides of (5.5) and using $\widehat{\lambda}_R = \rho$ as well as $\dot{v}_{At} = (\rho - r_t) v_{At}$ from the first-order condition for assets[6], we show that the consumption Euler-equation is supplemented by the Hotelling-rule

$$\widehat{p}_{Rt} = r_t. \tag{5.8}$$

Equation (5.8) states that growth in the resource price must compensate the consumer for not selling the resource today and investing in assets at the interest rate r_t.

5.2.2 Production

The demand function for intermediate goods remains unchanged vis-à-vis the baseline model:

$$X_{it}^d(p_{it}, L_{Yt}, Q_{it}) = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} Q_{it} L_{Yt} \tag{5.9}$$

In the profit function

$$\pi_{it}^X = (p_{it} - MC_t) X_{it},$$

it has to be taken into account that marginal production costs correspond to the price p_{Rt} for the resource instead of marginal labor costs, as one unit of intermediate goods is now produced from one unit of the non-renewable resource. Marginal costs are still the same for every firm j so that, as in chapter 2, only the firm with the latest patent will be active in production. The profit-maximizing monopoly price in period t, given by the constant mark-up $\frac{1}{\alpha}$ over marginal costs, is

$$p_{it} = p_t = \frac{1}{\alpha} \cdot p_{Rt} \tag{5.10}$$

in every sector. With (5.9), (5.10) and $MC_t = p_{Rt}$, profits in sector i are given by

$$\pi_{it}^X = \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} p_{Rt}^{-\frac{\alpha}{1-\alpha}} L_{Yt} \cdot Q_{it}. \tag{5.11}$$

[6] See equation (2.A.2) in appendix 2.A.1.

Profits depend negatively on the resource price. The effect is stronger, the more price-elastic intermediate demand from the consumption goods sector is (i.e., the larger α).

5.2.3 Research and Development

After-innovation profits in a period $s > t$ are $\pi^X_{ijs} = \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}} p_{Rs}^{-\frac{\alpha}{1-\alpha}} L_{Ys} \cdot (1+q_{ijt})Q_{it}$. As in chapter 2, $P(s) = e^{-\int_t^s \mu_{iv} dv}$ is the probability that the incumbent monopolist is still producing in period s. It follows that expected discounted profits from intermediate production in sector i are given by:

$$\begin{aligned} E\left[V_{ijt}(q_{ijt})\right] &= \int_t^\infty \pi^X_{ijs}(q_{ijt}) P(s) e^{-\int_t^s r_v dv} ds \qquad (5.12) \\ &= \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}} p_{Rt}^{-\frac{\alpha}{1-\alpha}} (1+q_{ijt}) Q_{it} \int_t^\infty L_{Ys} e^{-\int_t^s \left(\frac{1}{1-\alpha} r_v + \mu_{iv}\right) dv} ds \end{aligned}$$

In the second line, it was used that at time $s > t$, $p_{Rs} = p_{Rt} e^{\int_t^s r_v dv}$ according to the Hotelling-rule (equation (5.8)).

Expected discounted profits are affected negatively both by the level of the resource price in t and by its growth rate, which raises the effective discount rate: The term $\frac{1}{1-\alpha} r_v$ is the sum of the standard discount effect of the interest rate, r_v, and the term $\frac{\alpha}{1-\alpha} r_v$.. The influence of the growth rate results from the fact that, contrary to the baseline model, unit production costs in the intermediate sector rise over time because resource scarcity drives up the price of the resource input.

At any time t, as in the baseline model, a researcher j in sector i maximizes expected returns $\mu E\left[V_{it}\right]$ from an innovation in t less of research costs $w_{Dt} l_{Dijt}$. The labor requirement l_{Dijt} is given by (2.9) in chapter 2.

resource scarcity does not affect the step-size of innovations or the orientation of research in the laissez-faire equilibrium. As in chapter 2, there is no incentive to invest in green research because pollution is not internalized so that profit maximization yields $b^{\mathrm{LF}}_{ijt} = b^{\mathrm{LF}} = 0$. The profit-maximizing choice of q_{ijt} in the baseline model in chapter 2 is only dependent on d and in particular not affected by the effective discount rate. But upon substitution of the wage rate[7] in the first-order conditions

[7] As in the baseline model, it is used that w_{Dt} must equal the wage in the production sector for the consumption good at equilibrium. w_{Yt} is equal to the marginal product of labor L_{Yt} which is $(1-\alpha)\,\alpha^{\frac{2\alpha}{1-\alpha}} p_{Rt}^{-\frac{\alpha}{1-\alpha}} Q_t$.

and the zero profit condition, it becomes evident that both conditions differ from those in chapter 2 precisely by the effective discount rate. The equilibrium value $q_{ijt}^{\mathrm{LF}} = q^{\mathrm{LF}} = \sqrt{1+d} - 1$ from chapter 2 therefore pertains to the model of this chapter as well.

The discount factor on the other hand does affect n_{it}. Applying the same steps as in the baseline model, the zero-profit condition can be simplified to

$$n_{it} = n_t = \alpha L_{Yt} \frac{1}{\sqrt{1+d} - 1} - \frac{1}{1-\alpha} \frac{r_t}{\mu}. \tag{5.13}$$

The increasing resource price, ceteris paribus, crowds out overall research investment through the aforementioned stronger discounting of profits in the intermediate sector: Entry is smaller for any given interest rate r_t and labor used in final good production L_{Yt}.

5.2.4 General equilibrium

The market value of firms

As in chapter 2, the value of firms must equal the stock of the representative household's assets. Proceeding similar to the baseline model, we obtain the expression

$$\begin{aligned} V_t &= \int_0^1 E\left[V_{ijt}\right] di \\ &= 2 \frac{(1-\alpha)\, \alpha^{\frac{2\alpha}{1-\alpha}}}{\mu} \left(\sqrt{1+d} - 1\right) p_{Rt}^{-\frac{\alpha}{1-\alpha}} Q_t \end{aligned}$$

for the market value. Different from chapter 2, the market value of firms grows slower than productivity because of the increasing resource price.

Resource market clearing

Because the resource stock is fully exhausted, total resource demand $\int_0^\infty R_t dt = \int_0^\infty X_t^d dt$ must equal total supply F_0. Integrating (5.9) over all sectors i and using the Hotelling-rule to describe the development of the resource price, the condition can be written as

$$\left(\frac{\alpha^2}{p_{R0}}\right)^{\frac{1}{1-\alpha}} \int_0^\infty e^{-\frac{1}{1-\alpha}\int_0^t r_v dv} L_{Yt} Q_t dt = F_0. \tag{5.14}$$

Given the paths for productivity Q, labor L_Y and the interest rate, condition (5.14) fixes the resource price at $t = 0$ for any given initial resource stock F_0. It thereby determines the level of the path $\{p_{Rt}\}_0^\infty$. The more resource-abundant the economy is, the smaller is the resource price in every period t.

Labor market clearing

Using the labor market constraint, $L = L_{Yt} + n_t\,(q_t^2 + d)$, to replace L_{Yt} in (5.13), we express n_t as function of the interest rate only:

$$n_t = \frac{\frac{1}{2}L - \frac{1}{\alpha}\frac{1}{1-\alpha}\left(\sqrt{1+d}-1\right)\frac{r_t}{\mu}}{\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d} \tag{5.15}$$

Equilibrium growth

Substituting (5.15) and the solution for $q^{\mathrm{LF,R}}$ in the equation of motion for Q[8] and dividing by Q_t yields

$$\widehat{Q}_t = \frac{\frac{1}{2}\left(\sqrt{1+d}-1\right)\mu L - \frac{1}{\alpha}\frac{1}{1-\alpha}\left(\sqrt{1+d}-1\right)^2 r_t}{\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d}.$$

On a balanced growth path, r must be constant for all t, so that n does not change over time and Q grows at a constant rate. Integrating (5.9) over i, computing the growth rate and using the Hotelling-rule (5.8) then yields the relation between productivity growth and growth in intermediate quantity along an asymptotically-balanced growth path:

$$\widehat{X}^{\mathrm{LF,R}} = \widehat{Q}^{\mathrm{LF,R}} - \frac{1}{1-\alpha}r \tag{5.16}$$

Equation (5.16) shows that the growth rate of the resource price drives a wedge between productivity growth and growth in intermediate quantity. From lemma 5.1, we know that intermediate quantity does not only grow more slowly than productivity but must ultimately decline along a long-run equilibrium path. This is guaranteed by the transversality condition for Q as will be shown in the proof of proposition 5.1 below.

Because intermediate quantity decreases over time, output and consumption

[8] See equation (2.12) in chapter 2.

growth must fall short of productivity growth as well:

$$\begin{aligned}\widehat{c}^{\mathrm{LF,R}} = \widehat{Y}^{\mathrm{LF,R}} &= (1-\alpha)\widehat{Q}^{\mathrm{LF,R}} + \alpha\widehat{X}^{\mathrm{LF,R}} \\ &= \widehat{Q}^{\mathrm{LF,R}} - \frac{\alpha}{1-\alpha}r \end{aligned} \tag{5.17}$$

Setting equal (5.17) and (5.24), the interest rate and the equilibrium growth rates of Q, c and X can be determined as in chapter 2. The equilibrium interest rate is given by:

$$r^{\mathrm{LF,\ R}} = \frac{\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)\rho}{\frac{1}{\alpha(1-\alpha)}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)\left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)} \tag{5.18}$$

The equilibrium productivity growth rate is:

$$\widehat{Q}^{\mathrm{LF,R}} = \frac{\left(\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)\mu L - \frac{1}{\alpha(1-\alpha)}\left(\sqrt{1+d}-1\right)\rho\right)\left(\sqrt{1+d}-1\right)}{\frac{1}{\alpha(1-\alpha)}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)\left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)} \tag{5.19}$$

The growth rates of the economic and technology variables and the equilibrium interest rate are constant for any initial state. Therefore, the path described by $\widehat{Q}_t = \widehat{Q}^{\mathrm{LF,R}}$, $\widehat{B}_t = 0$, $\widehat{X}_t = \widehat{X}^{\mathrm{LF,R}}$, $\widehat{S}_t = \widehat{X}^{\mathrm{LF,R}}$ and $\widehat{Y}_t = \widehat{c}_t = \widehat{c}^{\mathrm{LF,R}}$ for all t is a balanced growth path and all variables except the pollution stock grow at their balanced growth rates without transitional dynamics. For an initial pollution stock not reconcilable with balanced growth, the pollution growth rate adjusts to $\widehat{S}^{\mathrm{LF,R}}_\infty = \widehat{X}^{\mathrm{LF,R}}$ over time.

Contrary to our baseline model, it is possible that $\widehat{X}^{\mathrm{LF,R}} < (-\delta)$ so that $\widehat{S}^{\mathrm{LF,R}}_\infty = (-\delta) > \widehat{X}^{\mathrm{LF,R}}$. In this case the equilibrium path described above is only asymptotically balanced because their exist no initial conditions such that $\widehat{S}^{\mathrm{LF,R}} = (-\delta)$ for all t. The ABG-equilibrium is otherwise equal to the BG-equilibrium, as the development of the pollution stock does not affect the decision of economic agents under laissez-faire.

Both along a BG- and along an ABG-path, the pollution stock declines with intermediate quantity in the long run. Further, the total amount of pollution emitted is necessarily finite and equal to the initial resource stock F_0.

From (5.17) and (5.19), define an upper bound $\overline{\rho}^{\mathrm{LF,R}}_{\mathrm{Q}}$ so that $\widehat{Q}^{\mathrm{LF,R}}_\infty > 0$ if and only if $\rho < \overline{\rho}^{\mathrm{LF,R}}_{\mathrm{Q}}$ and a second critical value $\overline{\rho}^{\mathrm{LF,R}}_{\mathrm{c}}$ so that $\widehat{c}^{\mathrm{LF,R}}_\infty > 0$ if and only if $\rho < \overline{\rho}^{\mathrm{LF,R}}_{\mathrm{c}}$. Further, define $\rho^{\mathrm{TVC,R}}$ such that the transversality condition is satisfied if and only if $\rho > \rho^{\mathrm{TVC,R}}$.[9]

The following proposition describes the laissez-faire equilibrium:

Proposition 5.1 ***(A)BG-laissez-faire equilibrium***
Assume $\sigma_E < 1/2$ and $\rho > \rho^{TVC,R}$, so that the disutility from pollution is convex and the transversality conditions are satisfied. Assume further that intermediate goods are produced with a non-renewable resource according to equation (5.3).

There exists either a unique BG- or a unique ABG-equilibrium with the following characteristics: There is no green innovation but persistent quantity degrowth ($\widehat{X}^{LF,R} < 0$). Pollution declines at the rate of intermediate quantity in the long run. For parameter constellations such that $\rho \geq \overline{\rho}_Q^{LF,R}$, there is no productivity growth in the (A)BG-laissez-faire equilibrium. Growth in per capita consumption and output is negative. For $\overline{\rho}_Q^{LF,R} > \rho \geq \overline{\rho}_c^{LF,R}$, productivity growth is positive, while growth in output and per capita consumption is negative. For $\rho < \overline{\rho}_c^{LF,R}$, growth in productivity, output and per capita consumption is positive.

Proof. See appendix 5.A.1. ∎

Contrary to the long-run equilibrium in the baseline model, it is possible that the growth rates of consumption and output are negative (see also Schou (2002) for a similar result). Degrowth does then not only occur in polluting quantity but in consumption and output. The reason is that even if the representative household is patient enough to lend to firms for R&D-investment, productivity growth is too slow to compensate for the decline in intermediate production.

5.2.5 The effects of resource scarcity

While it is obvious from the preceding analysis that resource scarcity not only slows growth in the intermediate production sector but even leads to persistent quantity degrowth, it is interesting to study the impact of resource scarcity on the (A)BG-laissez-faire equilibrium more extensively.

For any given initial resource stock F_0, the fact that the resource becomes scarcer over time is reflected in the positive growth rate of the resource price. The rise in

[9] From (5.17) and (5.19), it follows that $\overline{\rho}_{\mathrm{Q}}^{\mathrm{LF,R}} = \frac{\left(1+\frac{\alpha}{1-\alpha}\frac{1}{\sigma_C}\right)\frac{1}{2}\mu L}{\frac{1}{\alpha(1-\alpha)}\left(\sqrt{1+d}-1\right)}$ and that the upper bound for consumption growth is $\overline{\rho}_{\mathrm{c}}^{\mathrm{LF,R}} = \frac{\frac{1}{2}\mu L}{\frac{1}{\alpha(1-\alpha)}\left(\sqrt{1+d}-1\right)+\frac{1}{\alpha^2}+d/\left(\sqrt{1+d}-1\right)}$.
As the first-order condition (2.A.2) for assets is unchanged compared to chapter 2, the transversality condition still requires $r^{\mathrm{LF,R}} - \widehat{Q}^{\mathrm{LF,R}} > 0$. With (5.18) and (5.19), the critical value $\rho^{\mathrm{TVC,R}} = \frac{1}{2}\frac{\frac{\alpha}{1-\alpha}\frac{1}{\sigma_C}+1-\frac{1}{\sigma_C}}{\left(\frac{1}{\alpha(1-\alpha)}+1\right)\sqrt{1+d}-\frac{1}{1-\alpha}}\mu L$ is obtained.

the price induces the decline in intermediate quantity along the equilibrium path. Further, resource scarcity affects economic variables through the level of the resource price, which depends negatively on the size of the initial stock F_0.

While the size of F_0 is exogenous, the growth rate of the resource price is endogenous. Still, it is possible to single out the effects not only of F_0 but also the impact of the growing resource price on equilibrium growth and the levels of technology, production, consumption and pollution.

The following proposition summarizes the results:

Proposition 5.2 ***Effects of resource scarcity***
Consider the equilibrium in proposition 5.1.

Growth effects: *For any given initial resource stock F_0, the exhaustibility of the resource is reflected in a positive growth rate of the resource price. Resource scarcity thereby (i) unambiguously decreases the growth rates of X, Y and c as well as the long-run growth rate of S in the (A)BG-equilibrium, (ii) lowers (increases) the productivity growth rate whenever $\sigma_c > 1$ ($\sigma_c < 1$), given $\rho < \overline{\rho}_Q^{LF,R}$, and (iii) restricts the parameter range for which there is positive growth in per capita consumption in the (A)BG-equilibrium.*
Level effects: *The size of the initial resource stock F_0 (i) does not affect the allocation of labor to production and research, growth rates of the economic variables and the long-run pollution growth rate, nor the paths of the technology stocks Q and B. However, the smaller F_0, (ii) the larger is the level of the resource price along the entire equilibrium path and (iii) the smaller are, accordingly, the levels of intermediate production, output, per capita consumption and pollution in every period t.*

Proof. See appendix 5.A.2. ■

The growth rate of the resource price has two opposing effects on the size of the productivity growth rate, $\widehat{Q}^{\mathrm{LF,R}}$: On the one hand, the negative effect on entry for a given interest rate, apparent in (5.13), tends to slow growth. On the other hand, the decrease in entry causes a countervailing general-equilibrium effect: The equilibrium interest rate is smaller, which slows the price increase, as can be seen from (5.8), and stimulates entry and productivity growth.
If $\sigma_c < 1$, the representative household desires to smooth consumption over time and reacts inelastically to changes in the interest rate. The decline in the interest rate is therefore more pronounced than for $\sigma_c > 1$, so that the positive effect on productivity growth predominates in the former, and the negative effect predominates in the latter case.

The growing resource price depresses consumption growth along the equilibrium path because it induces quantity degrowth. The increase in productivity growth for $\sigma_c < 1$ dampens the decline in intermediate quantity and stimulates output and consumption growth also directly. Nevertheless, the overall effect of resource scarcity on consumption and output growth is unambiguously negative.

On the other hand, resource scarcity has a beneficial effect on household utility through the pollution externality: The growing resource price ensures that the total amount of emissions at equilibrium is bounded and the pollution stock declines along the (asymptotically) balanced growth path.

The initial resource stock affects the laissez-faire equilibrium only through the level of the resource price, according to equation (5.14). The price level does not influence research profits because profits from intermediate production and research costs decline in the price level in the same way. It follows that growth rates are unaffected by the size of the initial resource stock. The implication is that two economies with different initial resource endowments share the same long-run growth rates and the same technology paths.
On the other hand, the price level is relevant for the determination of intermediate production levels in each period (see (5.9)). The higher the resource price, the higher is the price firms in the consumption goods sector pay for intermediate goods and the lower are intermediate demand and the equilibrium quantity of intermediates. Because productivity and labor in the consumption goods sector are independent of the initial resource stock, it follows that the paths for output and consumption in an economy with small initial resource stock are below those of a more resource-abundant economy. At the same time, there is less pollution in every period as less of the polluting input is produced.

5.3 The long-run social optimum

We now derive the long-run optimal solution under the constraint resource scarcity imposes on intermediate production. Subsection 5.3.1 describes the changes in the key equations. In subsection 5.3.2, we first characterize the social optimum in case of a binding natural resource constraint, which is the case most commonly studied in the literature. As in the previous section, we focus on (asymptotically) balanced growth. We then suggest that for a sufficiently large initial resource stock and parameters such that the baseline model features quantity degrowth, the natural resource constraint is not binding so that the results from the baseline model still apply.

5.3.1 Optimization problem and first-order conditions

The current-value Lagrange-multiplier for the natural resource constraint, (5.2), is again denoted by λ_{Rt}. The equations of motion for S, Q and B as well as the allocation of intermediate production across sectors and the economic resource constraint remain unchanged compared to chapter 2. The modified production structure in the intermediate goods sector does, however, alter the labor market constraint because labor is not used in intermediate production anymore. The modified labor market constraint is

$$L = L_{Yt} + n_t(q_t^2 + b_t^2 + d). \tag{5.20}$$

We use the production function (5.3) for intermediate goods to eliminate the variable R from the optimization problem: Aggregate resource-use in period t equals the aggregate quantity of intermediates produced, i.e., $R_t = X_t$. The natural resource constraint (5.2) can therefore be rewritten as

$$\int_0^\infty X_t dt \leq F_0.$$

The new current-value Hamiltonian function is:

$$\begin{aligned} H \quad = \quad & \left(\frac{\sigma_c}{\sigma_c - 1} c_t^{\frac{\sigma_c - 1}{\sigma_c}} - \psi \frac{\sigma_E}{1 - \sigma_E} S_t^{\frac{1-\sigma_E}{\sigma_E}}\right) L \\ & + v_{St}\left(\frac{X_t}{B_t} - \delta S_t\right) \\ & + v_{Qt}\mu n_t q_t Q_t \\ & + v_{Bt}\mu n_t b_t B_t \\ & + \lambda_{Yt}\left(X_t^\alpha Q_t^{1-\alpha} L_{Yt}^{1-\alpha} - c_t L\right) \\ & + \lambda_{Rt}\left(F_0 - \int_0^\infty X_t dt\right) \\ & + \lambda_{Lt}(L - L_{Yt} - n_t(q_t^2 + b_t^2 + d)) \end{aligned} \tag{5.21}$$

The Lagrange-multiplier for the natural resource constraint affects only the first-order condition for X, reflecting the social cost of intermediate production arising from the depletion of the non-renewable resource:

$$\frac{\partial H}{\partial X_t} = 0 \Leftrightarrow \frac{v_{St}}{B_t} + \lambda_{Yt}\alpha X_t^{\alpha-1} L_{Yt}^{1-\alpha} Q_t^{1-\alpha} - \lambda_{Rt} = 0 \tag{5.22}$$

As under laissez-faire, the set of first-order conditions is supplemented by the Kuhn-Tucker-condition

$$\frac{\partial H}{\partial \lambda_{Rt}} \leq 0 \Leftrightarrow F_0 - \int_0^\infty X_t dt \geq 0 \quad \lambda_{Rt} \geq 0 \quad \lambda_{Rt}\left(F_0 - \int_0^\infty X_t dt\right) = 0 \tag{5.23}$$

for the Lagrange-multiplier λ_{Rt}, which expresses that either the resource stock is fully depleted in infinite time or λ_{Rt} is zero for all t. Contrary to the laissez-faire equilibrium, it is not clear a priory that it is optimal to extract the resource completely. The social planner may choose not to extract the whole resource stock in order to restrict the amount of polluting emissions.

Of the four key equations which have been derived in chapter 2 to describe the (A)BG-path, the Euler-equation (2.55) and the indifference condition (2.57) are not affected by the resource constraint. Equation (2.57) remains unchanged. The Euler-equation differs from its equivalent in chapter 2 only because productivity does no longer affect the costs of intermediate production, so that the term $1/\varphi(X/Q)_\infty$ is dropped from the equation:

$$(1/\sigma_c)\,\widehat{c}_\infty + \rho = \frac{\mu}{2q_\infty} L_{Y\infty} + \widehat{Q}_\infty + \alpha\left(\widehat{X}_\infty - \widehat{Q}_\infty\right) \tag{5.24}$$

The resource constraint does affect the research arbitrage equation and the balanced-growth equation through its effect on the first-order condition (5.22) for intermediate goods. The new research arbitrage equation is:

$$\frac{\mu}{2q_\infty} L_{Y\infty} = \frac{\mu}{2b_\infty} L_{Y\infty}\left(\frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha}\left(\frac{\lambda_R}{\lambda_Y}\right)_\infty \left(\frac{X}{Q}\right)_\infty^{1-\alpha} L_{Y\infty}^{\alpha-1}\right) \tag{5.25}$$

As we show in the next subsection, the condition for asymptotically-balanced growth differs depending on whether or not the natural resource constraint is binding.

5.3.2 Characterization of the long-run optimum

Binding natural resource constraint

The Lagrange-multiplier λ_{Rt} reflects the social costs of producing one unit of intermediates - it is the social price of the non-renewable resource. λ_{Rt} increases over time according to the modified Hotelling-rule

$$\widehat{\lambda}_{Rt} = \rho. \tag{5.26}$$

While the social price λ_{Rt} of the non-renewable resource increases with progressing resource scarcity, the shadow price v_{St} of pollution moves along with the marginal disutility of pollution on an asymptotically balanced growth path[10]. The shadow price therefore falls towards $v_{S\infty} = 0$ as the stock of the polluting resource gets exhausted and the pollution stock declines.

The balanced-growth condition from chapter 2 is replaced by the requirement that asymptotically, the social marginal product of intermediates in output production, $\lambda_{Yt}\alpha X_t^{\alpha-1}L_{Yt}^{1-\alpha}Q_t^{1-\alpha}$, must grow at the same rate as the social price λ_{Rt} of the resource to satisfy the first-order condition (5.22) for X. This relation implies that the long-run optimal growth rate of X as function of $\widehat{Q}_\infty$ is given by

$$\widehat{X}_\infty^{\mathrm{R}} = \frac{1}{\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c} + 1}\left(\left(1 - \frac{1}{\sigma_c}\right)\widehat{Q}_\infty^{\mathrm{R}} - \frac{1}{1-\alpha}\rho\right). \tag{5.27}$$

The increasing social resource price decreases growth in intermediate quantity below productivity growth. The price effect is reflected in the term $\frac{1}{1-\alpha}\rho$, similar to the term $\frac{1}{1-\alpha}r$ in the laissez-faire growth rate (5.16).

For $\sigma_c < 1$, it is obvious from (5.27) that intermediate quantity declines whenever $\widehat{Q}_\infty \geq 0$. If $\sigma_c > 1$, the transversality condition for Q is sufficient to guarantee quantity degrowth as will be shown in appendix 5.B.1.

When using the relation $\lambda_{Y\infty}\alpha X_\infty^{\alpha-1}L_{Y\infty}^{1-\alpha}Q_\infty^{1-\alpha} = \lambda_{R\infty}$ between the social marginal product of intermediates and the social price of the resource in the research arbitrage equation, (5.25), it becomes obvious that green innovation is no longer optimal in the long-run. The difference in square brackets in (5.25) becomes zero and the equation can only be satisfied if[11]

$$b_\infty^{\mathrm{R}} = \widehat{B}_\infty^{\mathrm{R}} = 0.$$

Asymptotically, all labor resources in the research sector is shifted towards productivity improvements. From equation (2.57), $q_\infty^2 + b_\infty^2 = d$, it follows that the optimal step-size is

$$q_\infty^{\mathrm{R}} = \sqrt{d}. \tag{5.28}$$

From the Euler-equation, we derive the mass of research units n_∞^{R}, which with (5.28)

[10] See equation (2.B.10) in the appendix of chapter 2.

[11] $L_{Y\infty} = 0$ and $q_\infty = \infty$ are no viable outcomes of the social planner's problem.

yields the long-run productivity growth rate

$$\widehat{Q}^{\mathrm{R}}_{\infty} = \frac{\frac{1}{2}(1-\alpha)\left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1\right)q_{\infty}^{-1}\mu L - \rho}{1/\sigma_c}. \tag{5.29}$$

Productivity growth dampens the adverse effects from quantity degrowth on output and consumption growth. The relation between long-run growth in output and consumption and productivity growth becomes evident after substituting (5.27) into the production function:

$$\widehat{c}^{\mathrm{R}}_{\infty} = \widehat{Y}^{\mathrm{R}}_{\infty} = \frac{1}{\frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}+1}\left(\widehat{Q}^{\mathrm{R}}_{\infty} - \frac{\alpha}{1-\alpha}\rho\right) \tag{5.30}$$

Long-run output and consumption growth in the optimal solution is positive if and only if productivity rises sufficiently fast to outweigh the decline in intermediate quantity.

Pollution growth is given by

$$\widehat{S}^{\mathrm{R}}_{\infty} = \max[\widehat{X}^{\mathrm{R}}_{\infty}, -\delta]$$

Contrary to the baseline model, the long-run optimal growth rates of the economic variables have the same functional form for $\widehat{S}_{\infty} > (-\delta)$ and $\widehat{S}_{\infty} = (-\delta)$. Pollution growth has feedbacks on economic variables only through the shadow price v_S of the pollution stock in the first-order condition for X. But we have shown that with a binding resource-constraint, the shadow price does not affect the first-order condition asymptotically.

For the subsequent proposition, define the upper bounds $\overline{\rho}^{\mathrm{R}}_{\mathrm{Q}}$ and $\overline{\rho}^{\mathrm{R}}_{\mathrm{c}}$ on the rate of time preference for positive long-run productivity growth and consumption growth respectively. Further, as in the previous chapters, define $\rho^{\mathrm{TVC,R}}$ so that the transversality conditions are satisfied if and only if $\rho > \rho^{\mathrm{TVC,R}}$.[12]

[12] From (5.29), it follows that $\overline{\rho}^{\mathrm{R}}_{\mathrm{Q}} = \frac{1}{2}(1-\alpha)\left(1+\frac{\alpha}{1-\alpha}\frac{1}{\sigma_C}\right)d^{-1/2}\mu L$ and from (5.30), the upper bound $\overline{\rho}^{\mathrm{R}}_{\mathrm{c}} = \frac{1}{2}\frac{(1-\alpha)\left(1+\frac{\alpha}{1-\alpha}\frac{1}{\sigma_C}\right)}{1+\frac{1}{1-\alpha}\frac{1}{\sigma_C}}d^{-1/2}\mu L$ is obtained.

The critical value $\rho^{\mathrm{TVC,R}}$ is given by $\rho^{\mathrm{TVC,R}} = \frac{1}{2}(1-\alpha)\left(\frac{\alpha}{1-\alpha}\frac{1}{\sigma_C}+1\right)(1-1/\sigma_C)d^{-1/2}\mu L$. Whenever $\sigma_C < 1$, the expression on the right-hand side is negative and the transversality conditions are satisfied for every $\rho > 0$.

The results from the above analysis can be summarized as follows:

Proposition 5.3 ***ABG-optimum with binding resource constraint***
Assume $\sigma_E < 1/2$ and $\rho > \rho^{TVC,R}$. Assume further that intermediates are produced with a non-renewable resource according to equation (5.3) and that the resource constraint is binding.

There exists an asymptotically unique ABG-solution which solves the necessary conditions for $t \to \infty$. While the long-run optimum is always characterized by quantity degrowth, green innovation is not optimal in the long-run ($\widehat{B}^R_\infty = 0$). For parameter constellations such that $\rho \geq \overline{\rho}^R_Q$, there is no productivity growth in the long-run social optimum. Long-run growth in per capita consumption and output is negative. For $\overline{\rho}^R_Q > \rho \geq \overline{\rho}^R_c$, long-run growth in output and per capita consumption is negative, while long-run productivity growth is positive. For $\rho < \overline{\rho}^R_c$, long-run growth in productivity, output and per capita consumption is positive.

Proof. See appendix 5.B.1. ■

In the social optimum, as in the laissez-faire equilibrium, resource scarcity may lead to degrowth in consumption and output in the long run. Intuitively, the binding resource constraints demands a comparatively fast reduction in intermediate production. Positive consumption growth is then only possible with a sufficient research investment because productivity must increase fast enough to outweigh the decline in intermediate quantity. But this requires a reduction in production labor and thus current consumption which is not optimal if the rate of time preference is large.

When the resource constraint is binding, the social planner is forced to save on polluting inputs to such an extent that investing in green innovation to bring about an even faster decline in the pollution stock is not optimal in the long run. The environmental externality does not influence the social planner's decision asymptotically: Neither the intertemporal elasticity of substitution in pollution, σ_E, nor the weight of pollution in utility, ψ, affects long-run growth rates. On the other hand, resource scarcity becomes an increasing threat to economic growth over time. Therefore, asymptotically, all labor resources in the research sector are shifted towards productivity improvements, which help to use the resource more efficiently.

Non-binding natural resource constraint

A necessary condition for the natural resource constraint to be non-binding is that resource use and therefore intermediate production decrease without the constraint, at least in the long run. Such a necessary condition is given in corollary 2.3 in the baseline model which characterizes the parameter range for the long-run social optimum to be characterized by quantity degrowth. We prove in appendix 5.B.2 that given quantity degrowth, the total amount $\int_0^\infty X_t dt$ extracted of the resource remains finite so that for a sufficiently large initial stock F_0, the resource is not fully exhausted.[13]

If the resource stock is not binding, it follows from (5.23), that the shadow price λ_R equals zero for all t. In this case, taking into account $(X/Q)_\infty = 0$, the first-order condition (5.22) for intermediate quantity X reduces to the corresponding equation in chapter 2. With the requirement that the shadow price v_{St} of pollution must move along with the marginal disutility of pollution, the same condition (2.54) for (asymptotically) balanced growth as in the social optimum of the baseline model is obtained. Given $\lambda_{R\infty} = \lambda_R = 0$ and $(X/Q)_\infty = 0$, the consumption Euler-equation (5.24) and the research arbitrage equation (5.25) also equal the respective conditions in the baseline model.

As all four key equations (2.54), (2.57), (5.24), and (5.25) correspond to those from the baseline model, it follows that for a non-binding resource constraint, the long-run optimal solution of the resource model is the same as in chapter 2.[14] In particular, some labor is always allocated to green innovation in a growing economy so that

$$\widehat{B}_\infty = \frac{\alpha}{1-\alpha}\widehat{Q}_\infty \tag{5.31}$$

and long-run output- and consumption growth is non-negative.

Proposition 5.4 ***ABG-optimum with non-binding resource constraint***
Besides $\sigma_E < 1/2$ and $\rho > \rho^{TVC}$, assume that intermediates are produced with a non-renewable resource according to equation (5.3) and that the conditions for quantity degrowth in corollary 2.3 are satisfied. Assume further that the path $\{X_t\}_0^\infty$ for intermediate quantity is continuous.

Given a sufficiently large (but finite) initial resource stock F_0, the natural resource constraint is not binding for the social planner's problem. There exists an asymptotically unique ABG-solution solving the necessary conditions for $t \to \infty$ which is identical to the ABG-solution described in section 2.4.2 of chapter 2. More precisely,

[13] It is not possible to express the necessary and sufficient condition $\int_0^\infty X_t dt < F_0$ for the resource constraint to be non-binding in terms of the model parameters as this requires knowledge of the entire path of X, which cannot be derived analytically.

[14] In the short run, the optimal solution will differ from the one in chapter 2, because labor is no longer used in the intermediate production sector. This effect vanishes in the long run, as labor in intermediate production converges to zero for $X/Q \to 0$.

there is quantity degrowth so that intermediate quantity X and resource use R fall over time. Growth in output and consumption is positive, given $\rho < \overline{\rho}$, and entirely driven by productivity growth. The pollution stock S declines both due to quantity degrowth and because the pollution intensity of intermediate goods is reduced by green innovation. The orientation of research and technical change is given by (5.31).

Proof. The proof requires to show convergence of $\int_0^\infty X_t dt$, see appendix 5.B.2. The remaining claims of the proposition follow from the text and the analysis in chapter 2. ■

The condition $\frac{\alpha}{1-\alpha} < (1-\alpha)\frac{(1-\sigma_c)/\sigma_c}{(1-\sigma_E)/\sigma_E}$ in corollary 2.3 is sufficient for the long-run optimal solution to be characterized by quantity degrowth both for $\widehat{S}_\infty = (-\delta)$ and $\widehat{S}_\infty > (-\delta)$. The condition is satisfied if the factor elasticity α of intermediate quantity is small, the value of the intertemporal elasticity of substitution (IES) in consumption, σ_c, is below one and the IES in pollution, σ_E, is large. In this case, a decline in intermediate quantity has no strong effect on growth in output and consumption, the relative social return to green research is small and a steep decline in the pollution stock is desired. Because of the negative environmental externality of production, optimal resource use is restricted to such an extent that the stock is never exhausted.

In our numerical example in section 2.4.3 in chapter 2, we have suggested that the parameter constellations for which there is quantity degrowth in the long-run optimal solution are well in line with empirical evidence. In particular, quantity degrowth was shown to be a likely outcome of the social planner's optimization problem if the intermediate good is interpreted as energy input and its production elasticity α as the energy share in GDP.

We conclude that without too strong restrictions on the parameter range, the long-run results from the socially optimal solution of the baseline model extend to a model with a non-renewable resource.

5.4 Conclusion

If intermediate goods are produced from a polluting non-renewable resource as the only input, resource scarcity forces intermediate production and therefore emissions and the pollution stock to decrease asymptotically, even in the laissez-faire equilibrium. We have shown that the decline in intermediate production depresses equilibrium growth in output and consumption. Growth rates in the (A)BG-equilibrium do not depend on the particular size of the initial resource stock. However, less resource-abundant economies face lower consumption, output and pollution levels on the entire growth path compared to economies with a large natural resource endowment.

While including a non-renewable resource drastically affects the laissez-faire equilibrium, we have pointed out that for reasonable parameter constellations, the resource constraint is not binding in the social planner's solution and the results from the baseline model without resources continue to hold.

It is straightforward and widely acknowledged in the literature (Grimaud and Rouge (2008) and Schou (2000, 2002)) that if the source of pollution is a non-renewable resource, the finiteness of the resource stock alleviates the pollution problem in the long run. The analysis in this chapter suggests that the causality may work in the opposite direction as well: The preference for a clean environment may make it optimal to restrict resource use in a way that the resource stock is never exhausted and the resource constraint is not binding.

5.A Appendix to section 5.2

5.A.1 Proof of proposition 5.1

1. **Existence and Uniqueness:** Given $\rho^{\mathrm{TVC}} < \rho < \overline{\rho}_{\mathrm{Q}}^{\mathrm{LF,R}}$, existence and uniqueness follow along the lines of the proof of proposition 2.1 in chapter 2. For $\rho^{\mathrm{TVC,R}} < \overline{\rho}_{\mathrm{Q}}^{\mathrm{LF,R}} \leq \rho$, it needs to be shown that $n_t = n_{it} = 0$ for all t is an equilibrium. $n = 0$ implies $\widehat{Q}^{\mathrm{LF,R}} = 0$ which leads to $\widehat{c}^{\mathrm{LF,R}} = -\frac{\alpha}{1-\alpha} r$ by (5.17). Substituting this growth rate in the Euler-equation and solving for r yields $r = \rho / \left(1 + \frac{\alpha}{1-\alpha} \frac{1}{\sigma_c}\right)$. $n = 0$ is an equilibrium if and only if $w_{Yt} > w_{Dt}$ for this value of r.
   With $r = \rho / \left(1 + \frac{\alpha}{1-\alpha} \frac{1}{\sigma_c}\right)$ and $n_{it} = 0$, the integral in the expression for expected discounted intermediate profits is given by $\int_t^\infty L_{Ys} e^{-\int_t^s \left(\frac{1}{1-\alpha} r_v + \mu_{iv}\right) dv} ds$ $= \frac{1-\alpha}{\rho} \left(1 + \frac{\alpha}{1-\alpha} \frac{1}{\sigma_c}\right) L$. From the first-order condition with respect to q_{ijt} for the maximization of expected research profits and the zero-profit condition $\mu E[V_{ijt}] = w_{Dt} l_{Dijt}$, using $b_{ijt} = b^{\mathrm{LF,R}} = 0$, we derive $q_{ijt} = q^{\mathrm{LF,R}} = \sqrt{1+d} - 1$ and $w_{Dt} = \mu \frac{(1-\alpha)^2}{\alpha\rho} \alpha^{\frac{2}{1-\alpha}} \left(1 + \frac{\alpha}{1-\alpha} \frac{1}{\sigma_c}\right) \frac{1}{\sqrt{1+d}-1} p_{Rt}^{-\frac{\alpha}{1-\alpha}} QL$. The wage in the consumption good sector is $w_{Yt} = (1-\alpha) \alpha^{\frac{2\alpha}{1-\alpha}} p_{Rt}^{-\frac{\alpha}{1-\alpha}} Q$. Comparison proves that $w_{Yt} > w_{Dt}$ if and only if $\rho > \overline{\rho}_{\mathrm{Q}}^{\mathrm{LF,R}}$.

2. **Quantity degrowth:** As $\frac{1}{1-\alpha} > 1$, the transversality condition is sufficient for $\widehat{X}^{\mathrm{LF,R}}$ in (5.16) to be negative.

3. **Consumption and productivity growth:** Because $\left(1 + \frac{\alpha}{1-\alpha} \frac{1}{\sigma_c}\right) \frac{1}{2} \mu L >$ $\frac{1}{2} \mu L$ and $\frac{1}{\alpha(1-\alpha)} \left(\sqrt{1+d} - 1\right) < \frac{1}{\alpha(1-\alpha)} \left(\sqrt{1+d} - 1\right) + \frac{1}{\alpha^2} + d / \left(\sqrt{1+d} - 1\right)$, it is evident that $\overline{\rho}_{\mathrm{c}}^{\mathrm{LF,R}} < \overline{\rho}_{\mathrm{Q}}^{\mathrm{LF,R}}$. The results in proposition 5.1 then follow straightforwardly.

5.A.2 Proof of proposition 5.2

1. Growth effects

(i) As to the effect of the growth rate $\widehat{p_R}$ of the resource price on the equilibrium growth rates of X and S, it is sufficient to note that $\widehat{X}^{\mathrm{LF,R}} < 0$ and $\widehat{S}_\infty^{\mathrm{LF,R}} = \max[\widehat{X}^{\mathrm{LF,R}}, -\delta] < 0$ only because the price grows over time.

From the consumption Euler-equation

$$\widehat{c} = \sigma_c \cdot (r - \rho),$$

taking into account $\widehat{Y} = \widehat{c}$, it follows that $\widehat{p_R}$ lowers output and consumption growth for any σ_c if and only if it decreases the equilibrium interest rate. The interest rate in (5.18) can be written as

$$r^{\text{LF, R}} = \frac{\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)\rho}{\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d+\kappa_{R,1}}$$

where $\kappa_{R,1} := \frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}\left(\frac{1}{\alpha}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)$. The term $\kappa_{R,1}$ in the denominator reflects the influence of $\widehat{p_R}$. As $\kappa_{R,1} > 0$, the fact that the price rises over time decreases the equilibrium interest rate and therefore growth in output and consumption.

(ii) To prove the result for productivity growth, note that the productivity growth rate in (5.19) can be rewritten as

$$\widehat{Q}^{\text{LF,R}} = \frac{\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho + k_{R,2}}{\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d+\kappa_{R,1}}\left(\sqrt{1+d}-1\right),$$

with $\kappa_{R,2} = \frac{\alpha}{1-\alpha}\left(\frac{1}{\sigma_c}\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho\right)$. Both $\kappa_{R,1}$ and $\kappa_{R,2}$ are attributable to the growth rate of the resource price. Setting $\kappa_{R,1}$ and $\kappa_{R,2}$ to zero and comparing the resulting expression to $\widehat{Q}^{\text{LF,R}}$ proves that the growing resource price decreases the productivity growth rate if and only if

$$\begin{aligned} k_{R,2}&\left(\frac{1}{\alpha}\frac{1}{\sigma_c}\left(\sqrt{1+d}-1\right)^2 + \frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right) \\ &< \left(\frac{1}{2}\mu L - \frac{1}{\alpha}\left(\sqrt{1+d}-1\right)\rho\right)\kappa_{R,1} \quad . \end{aligned}$$

Substituting the expressions for $\kappa_{R,1}$ and $k_{R,2}$, the condition becomes

$$\frac{1}{\alpha}\frac{1-\sigma_c}{\sigma_c}\left(\sqrt{1+d}-1\right)\left(\frac{1}{2}\frac{1}{\sigma_c}\mu L\left(\sqrt{1+d}-1\right) + \left(\frac{1-\alpha}{\alpha}\left(\sqrt{1+d}-1\right)+d\right)\rho\right) < 0$$

which is equivalent to $\sigma_c > 1$. For $\sigma_c < 1$, the growth rate of the resource price increases the productivity growth rate.

(iii) The upper bound on ρ which guarantees positive consumption growth can be rewritten as $\overline{\rho}_c^{\mathrm{LF,R}} = \frac{1}{2}\mu L \frac{1}{\frac{1}{\alpha}\left(\sqrt{1+d}-1\right)+\kappa_{R,3}}$. The expression $\kappa_{R,3}$, defined as $\kappa_{R,3} := \frac{1}{1-\alpha}\left(\sqrt{1+d}-1\right)+\frac{1}{\alpha^2}+d/\left(\sqrt{1+d}-1\right)$, results from the growth rate of the resource price. As $\kappa_{R,3} > 0$, the increasing resource price lowers the upper bound on ρ.

2. Level effects

(i) It is obvious from equations (5.16) to (5.19), as well as $\widehat{B}^{\mathrm{LF,R}} = 0$ and $\widehat{S}_\infty^{\mathrm{LF,R}} = \max[\widehat{X}^{\mathrm{LF,R}}, -\delta]$ that the initial resource stock F_0 does not influence the growth rates of c, Y, X, Q, B and S along the (A)BG-equilibrium path. Because the initial values for Q and B are given and the growth rates of Q and B jump to their respective ABG-levels directly, it follows that the entire paths of Q and B do not depend on F_0.

(ii) Taking into account that $r = r^{\mathrm{LF,R}}$ and L_Y are constant along the equilibrium path and Q grows at the constant rate $\widehat{Q}^{\mathrm{LF,R}}$ for all t, equation (5.14) can be written as $-\left(\frac{\alpha^2}{p_{R0}}\right)^{\frac{1}{1-\alpha}} Q_0 L_Y \left(\widehat{Q}^{\mathrm{LF,R}} - \frac{1}{1-\alpha} r^{\mathrm{LF,R}}\right)^{-1} = F_0$. It follows that the resource price in $t = 0$ is

$$p_{R0} = \alpha^2 \left(\frac{Q_0 L_Y}{F_0}\right)^{1-\alpha} \left(\frac{1}{1-\alpha} r^{\mathrm{LF,R}} - \widehat{Q}^{\mathrm{LF,R}}\right)^{1-\alpha},$$

with $r^{\mathrm{LF,R}}$ and $\widehat{Q}^{\mathrm{LF,R}}$ given by (5.18) and (5.19). Using the Hotelling-rule (5.8), the resource price can be determined at any point in time. A decline in F_0 increases the price for all t.

(iii) It has been shown in (i) that the path for Q is unaffected by a variation in F_0. The same is true for the constant $L_Y = L - n^{\mathrm{LF,R}}\left(\left(q^{\mathrm{LF,R}}\right)^2 + d\right)$ because neither $q^{\mathrm{LF,R}}$ nor $n^{\mathrm{LF,R}}$ (as given by (5.15) with (5.18)) depends on F_0. Intermediate demand in every period t decreases in the resource price according to equation (5.9). It follows that by increasing the resource price for all t, a decline in F_0 shifts the path for intermediate quantity downwards.

Because $\{L_Y\}_0^\infty$ and $\{Q_t\}_0^\infty$ are independent of F_0, the path for output and consumption shifts downwards with the path for X.

Further, because $\{B_t\}_0^\infty$ is not affected by F_0, emissions X_t/B_t are lower for all t. The path for the pollution stock S_t is given by the solution to the differential equation (2.8), $\dot{S}_t = X_t/B_t - \delta S_t$. From the general solution, it can be concluded that due to the decline in emissions, the pollution stock S is lower in every period.

5.B Appendix to section 5.3

5.B.1 Proof of proposition 5.3

1. **Uniqueness:** The interior ABG-solution described in the text satisfies all the necessary first-order conditions for $\rho < \overline{\rho}_{\mathrm{Q}}^{\mathrm{R}}$. Similar to chapter 2, it remains to be shown that it is the unique solution of the first-order conditions for $\rho < \overline{\rho}_{\mathrm{Q}}^{\mathrm{R}}$ (the only other solution candidate being $n_\infty = 0$), and that $n_\infty = 0$ is a solution of the first-order conditions for $\rho \geq \overline{\rho}_{\mathrm{Q}}^{\mathrm{R}}$.
   It therefore has to be shown that $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} |_{n_\infty=0} \leq 0$ if and only if $\rho \geq \overline{\rho}_{\mathrm{Q}}^{\mathrm{R}}$. The condition $\lim\limits_{t\to\infty} \frac{\partial H}{\partial n} \leq 0$ is equivalent to

   $$v_{Q\infty}\mu q_\infty Q_\infty + v_{B\infty}\mu b_\infty B_\infty \leq \lambda_{L\infty}(q_\infty^2 + b_\infty^2 + d) \tag{5.B.1}$$

   From the first-order condition for B (equation (2.47) in chapter 2), it follows with $n_\infty = 0$ and $v_{S\infty} = 0$ that $v_{B\infty} = 0$[15].

   The first-order condition for Q yields $\widehat{v_{Q}}_t = \rho - (1-\alpha)\lambda_{Yt}X_t^\alpha Q_t^{-\alpha} L_{Yt}^{1-\alpha}/v_{Qt}$. On an asymptotically-balanced growth path, $(1-\alpha)\lambda_{Y\infty}X_\infty^\alpha Q_\infty^{-\alpha} L_{Y\infty}^{1-\alpha}/v_{Q\infty}$ must be a positive constant. If $n_\infty = 0$, Q_∞ is constant and $L_{Y\infty} = L$, while X and λ_Y (which equals the marginal product of consumption by the first-order condition for c) still decline. Therefore $(1-\alpha)\lambda_{Y\infty}X_\infty^\alpha Q^{-\alpha} L_Y^{1-\alpha}/v_{Q\infty}$ is constant if and only if $\widehat{v_{Q}}_\infty = \widehat{\lambda_{Y}}_\infty + \alpha\widehat{X}_\infty$. With (5.27) and (5.30), which must hold also if $n_\infty = 0$, we obtain $\widehat{v_{Q}}_\infty = \frac{\alpha}{1-\alpha}(1/\sigma_c - 1)\rho/\left(\frac{\alpha}{1-\alpha}1/\sigma_c + 1\right)$. Substituting into $\widehat{v_{Q}}_t = \rho - (1-\alpha)\lambda_{Yt}X_t^\alpha Q_t^{-\alpha} L_{Yt}^{1-\alpha}/v_{Qt}$ for $t = \infty$ and rearranging yields $v_{Q\infty} = (1-\alpha)^2\lambda_{Y\infty}X_\infty^\alpha Q_\infty^{-\alpha} L^{1-\alpha}\left(\frac{\alpha}{1-\alpha}1/\sigma_c + 1\right)/\rho$.

   From the first-order condition for L_Y which is unchanged from chapter 2, it follows that $\lambda_{L\infty} = (1-\alpha)\lambda_{Y\infty}X_\infty^\alpha Q_\infty^{1-\alpha} L^{-\alpha}$ and $\widehat{\lambda_{L}}_\infty = \widehat{\lambda_{Y}}_\infty + \alpha\widehat{X}_\infty$.

   As in chapter 2, q_∞ and b_∞ are not uniquely determined for $n_\infty = 0$ and we define the limits $\lim\limits_{n_\infty\to 0} q(n_\infty) = (1-\alpha)\left(\frac{\alpha}{1-\alpha}1/\sigma_c + 1\right)(1/2)\mu L/\rho$ and $\lim\limits_{n_\infty\to 0} b(n_\infty) = 0$ obtained from the Euler-equation and the research-arbitrage equation respectively as the solutions in this case.

[15] A second potential solution, $\widehat{v}_B = \rho$, violates the transversality condition for B.

Substituting $v_{B\infty} = 0$ along with the expressions for $v_{Q\infty}$, $\lambda_{L\infty}$, $\lim_{n_\infty \to 0} q(n_\infty)$ and $\lim_{n_\infty \to 0} b(n_\infty) = 0$ into (5.B.1) yields after simplification and rearrangement the condition $\rho \geq \frac{1}{2}(1-\alpha)\left(1 + \frac{\alpha}{1-\alpha}\frac{1}{\sigma_c}\right)\mu L d^{-1/2} = \overline{\rho}^{\mathrm{Q,R}}$. We have thus shown that $\lim_{t\to\infty} \frac{\partial H}{\partial n_t} \leq 0$ and $n_\infty = 0$ is a solution of the necessary first-order conditions if and only if $\rho \geq \overline{\rho}^{\mathrm{Q,R}}$.

2. **Quantity degrowth:** For $\sigma_c < 1$, $\widehat{X}^{\mathrm{R}}_\infty < 0$ follows straightforwardly from (5.27). For $\sigma_c > 1$, the transversality condition $\lim_{t\to\infty} v_{Qt} Q_t e^{-\rho t} = 0$ is sufficient for $\widehat{X}^{\mathrm{R}}_\infty < 0$: For the transversality condition to be satisfied, ρ must exceed $(1 - 1/\sigma_c)\widehat{Q}^{\mathrm{R}}_\infty$. As $\frac{1}{1-\alpha} > 1$, $\rho > (1 - 1/\sigma_c)\widehat{Q}^{\mathrm{R}}_\infty$ implies that $\widehat{X}^{\mathrm{R}}_\infty$ in (5.16) is negative.

3. **Productivity and consumption growth:** Because $1 + \frac{1}{1-\alpha}\frac{1}{\sigma_c} > 1$, $\overline{\rho}^{\mathrm{R}}_{\mathrm{Q}} > \overline{\rho}^{\mathrm{R}}_{\mathrm{c}}$. The results concerning the signs of $\widehat{Q}_\infty$ and $\widehat{c}_\infty$ follow straightforwardly.

5.B.2 Proof of proposition 5.4

The integral $\int_0^\infty X_t dt$ can be written as the sum of the two integrals $\int_0^T X_t dt$ and $\int_T^\infty X_t dt$. It converges if and only if both integrals in the sum converge.

Because X_t is finite for every t, the definite integral $\int_0^T X_t dt$ is equal to a finite value.

Consider the second integral: In any solution to the social planner's problem for which growth rates converge to the growth rates of the asymptotically-balanced growth solution with quantity degrowth, the sequence $\left\{\widehat{X}_t\right\}_0^\infty$ converges to the constant $\widehat{X}_\infty < 0$. Assuming continuity, convergence implies that there exists a time T such that $\widehat{X}_t < \overline{\widehat{X}} < 0$ for all $t > T$. Therefore if the integral $\int_T^\infty X_T e^{\overline{\widehat{X}}\cdot t} dt$ converges, then so does the integral $\int_T^\infty X_t dt$. The limit of the integral $\int_T^\infty X_T e^{\overline{\widehat{X}}\cdot t} dt$ is $X_T[1/\overline{\widehat{X}} \cdot e^{\overline{\widehat{X}}\cdot t}]_T^\infty = -X_T/\overline{\widehat{X}} \cdot e^{\overline{\widehat{X}}\cdot T} > 0$ as $\overline{\widehat{X}} < 0$. Because $X_T < \infty$, the limit is finite. It follows that the integral $\int_T^\infty X_t dt$ converges.

We have thus proven that $\int_0^\infty X_t dt = \int_0^T X_t dt + \int_T^\infty X_t dt$ converges.

Chapter 6

Concluding remarks

6.1 Summary of results

The present thesis studied the equilibrium and in particular the optimal relation between economic growth and the environment. The central aim was to find answers to the questions "How - if at all - can economic growth be decoupled from environmental degradation?", "Is persistent economic growth socially desirable if its impact on the environment is taken into account?" and "How costly is environmental conservation in terms of consumption and economic growth?" Particular focus in this respect was given to the role of endogenous technical change.

Chapter 1 summarized the main conclusions from existing theoretical environmental-economic literature. Technical development was found to be crucial for long-run growth to remain a desirable social aim in the presence of environmental externalities. It was suggested that technical progress may help to reconcile economic growth and environmental preservation in several ways: e.g., by reducing the pollution intensity of inputs or processes, by developing cleaner substitutes to polluting inputs, or by raising productivity, which allows to reduce the amount of polluting inputs in production without giving up output.
However, the analysis in chapter 1 also pointed out that whether technical progress indeed relaxes the growth-environment trade-off depends on whether and how strong there are rebound effects of productivity growth on GDP and the demand for polluting inputs in particular, and on how environmentally beneficial technical change affects other, more growth-enhancing, research alternatives.

The main lesson from the first chapter has been that, in order to come to reliable conclusions concerning the prospects of reconciling economic growth with a clean environment and the desirability of long-run economic growth, the interaction of pollution-reducing, 'green' innovation and productivity-enhancing technical change has to be modelled endogenously. Further, it must be taken into account that the rebound effect of productivity growth on the demand for polluting inputs is endogenous as well and can be restricted by using higher productivity to save on polluting inputs.

In chapter 2, we presented a model which meets these requirements. Economic growth is driven by productivity improvements. Decoupling of economic growth and pollution growth is possible by reducing the pollution intensity of polluting production inputs through green innovation and by restricting the rebound effect of productivity growth through deceleration.

In the laissez-faire equilibrium of our model, neither green innovation nor deceleration is chosen. Polluting quantity increases one for one with productivity in the long run so that the ratio of the polluting input relative to GDP remains constant. Productivity growth has a strong rebound effect on polluting quantity. It was shown that compared to this path of unconstrained pollution growth, convergence to a stationary economy would be socially preferable.
Even so, we have shown that for sufficiently patient households, convergence to a stationary economy is not optimal. Persistent growth must, however, be accompanied by both green innovation and deceleration, or even quantity degrowth, to restrict pollution growth for reasonable parameter values.

It has been pointed out earlier, that in the environmental-economic literature, there is controversy as to whether or not environmental care entails a large cost in terms of economic growth. In our model, economic growth in the long-run optimum may be faster if households care for a clean environment than when there is no such environmental externality. The positive effect of environmental care on growth in our model is driven by green innovation attracting labor to the R&D-sector, which accelerates productivity growth.

In chapter 3, we examined local stability properties of the long-run solution derived in chapter 2 and studied the transitional behavior of the economy.
In the laissez-faire equilibrium, the economy jumps to its balanced growth path for any initial conditions, even though the pollution stock may take time to adjust. The numerical analysis of the social optimum suggests that for any set of initial values for the state variables, there exists an optimal transition path leading to the long-run optimal solution. The focus on a long-run perspective in chapter 2 is therefore justified.
Analysis of the transitional dynamics of the social planner's solution for an exem-

plary parametrization confirmed that green innovation and deceleration characterize the optimal solution not only in the long run but throughout the transition as well. Moreover, it became apparent that the initial technology endowment of an economy is crucial for its further development: Economies with an initially more advanced technology enjoy higher consumption levels and a less polluted environment in every period and therefore higher intertemporal welfare.

Chapter 4 underlined the importance of using both green innovation and deceleration to control pollution growth. It illustrated the consequences for long-run growth if either of the two channels for pollution control, green innovation or deceleration, is not available to the social planner.

Depriving the social planner of the possibility to control the rebound effect of productivity growth through deceleration is particularly detrimental to long-run growth: Persistent economic growth is no longer desirable for any parameter constellation which requires deceleration in the baseline model of chapter 2.

If, on the other hand, the pollution intensity of intermediates cannot be reduced by green innovation, long-run consumption growth may still be optimal, but for a smaller parameter range than in the unconstrained optimum. Further, the fixed pollution intensity must be compensated by faster deceleration which leads to lower long-run optimal growth rates of consumption and GDP.

In chapter 5, we examined the robustness of the main results with respect to the consideration of resource scarcity. While the dependency of production on a scarce resource alters the laissez-faire equilibrium, we have shown that the results from the socially optimal solution in the baseline model extend to the long-run social optimum with an exhaustible resource for reasonable parameter constellations. The negative pollution externality of intermediate production then reduces optimal resource use in a way that the resource is never exhausted if the initial stock is large enough.

6.2 Implications and extensions

The model presented in this thesis contributes to the ongoing debate on whether technical progress can resolve the growth-environment trade-off. The results above suggest that there is reason to believe technical development alone will not solve the pollution problem. For parameter constellations which are well in line with empirical evidence, fostering productivity growth while investing in green innovation to decrease the pollution intensity of production is not sufficient to achieve the optimal balance between consumption growth and pollution growth. Green innovation has to be supplemented by persistent deceleration in order to ensure that productivity growth is used to decrease the share of polluting inputs in GDP and does not merely lead to a faster expansion of production. The rebound effect of productivity growth must be restricted.

Nevertheless, technical change is crucial for decoupling economic growth from environmental degradation. Productivity growth accompanied by green innovation and deceleration allows for consumption growth to occur in a relatively clean way. It is therefore due to technical change that despite the environmental externality and under parameter restrictions no stricter than in standard endogenous growth models, economic growth is a desirable long-run aim in the model presented in this thesis.

Technical change should not be expected to free environmental preservation of all costs: First, controlling the rebound effect of productivity growth by saving on polluting inputs comes at the cost of giving up potential consumption growth. Second, the larger the fraction of R&D-resources directed towards environment-friendly research, the less the fraction that may be directed to raise productivity. This slows consumption growth, ceteris paribus.

Yet even though technical change does not turn environmental preservation into a free-lunch, it may lower the cost of pollution control considerably: Although for a given amount of labor used in R&D, a stronger orientation towards green innovation implies slower productivity growth, green innovation may at the same time induce a reallocation of labor resources from production to the research sector, thereby stimulating growth in productivity and consumption. Without green innovation, environmental care unambiguously lowers long-run optimal consumption growth.

While pollution control is optimal, it is not chosen in an unregulated market equilibrium. A straightforward extension of our work is to analyze how the optimal path for the economy can be implemented in the market.

A first-best policy has to stimulate green innovation and - if productivity-enhancing research is underprovided - also productivity growth while at the same time controlling its rebound effect on the polluting production inputs. All three aims could be achieved by supplementing the two policy instruments which are standard in models of growth through creative destruction, namely a subsidy to productivity-enhancing research and a subsidy to intermediate production to correct the distortion from monopolistic competition, with a research subsidy to green innovation and a tax on emissions. The subsidy on green research must internalize the intertemporal spillovers in the generation of green knowledge. The tax must correct for the distortion between equilibrium and optimal solution caused by the pollution externality. For this purpose, the tax must internalize the marginal social loss in utility units from the increase in the pollution stock caused by an additional unit of emissions. On a balanced-growth path, the tax is constant. If the optimal solution requires deceleration, the tax rate must increase along the long-run growth path to induce a persistent decline in polluting intermediate quantity.[1]

It would be interesting to examine the effects of such an emission tax on economic growth, pollution and welfare if a full set of research subsidies is not available. Further, the impacts of technology standards instead of subsidies in productivity-oriented and green research could be analyzed.

For the analysis, however, the entire time path of the economy has to be known. In this respect, a second starting-point for future work is the extension of the numerical examples from chapter 3 into a proper calibration of the model to allow for a meaningful policy analysis.
Calibration would also allow to quantify the welfare losses which occur, when, as in chapter 4, either the rebound effect of productivity growth cannot be dampened through deceleration, or the pollution intensity of production inputs cannot be reduced by green innovation.

[1] Instead of by imposing an emission tax, the pollution externality could probably be internalized by a tax on intermediate production which decreases in the cleanliness of intermediate goods.

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