Prehomogeneous Super Vector Spaces

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Kurzzusammenfassung

In dieser Arbeit entwickeln wir die Theorie der prähomogenen Supervektorräume. Für den Fall, dass G eine komplexe und zusammenhängende Lie Supergruppe ist, die linear auf dem Supervektorraum V wirkt und einen offen Orbit in V hat, beweisen wir, dass dieser Orbit dann, als eine offene Untersupermannigfaltigkeit betrachtet, eindeutig, zusammenhängend und dicht ist. Dies erlaubt uns prähomogene Supervektorräume zu definieren. Wir führen den Begriff der relativen Superinvarianten ein und können zeigen, dass die relativen Superinvarianten durch die Supercharaktere bestimmt sind, die unter der Isotropiesupergruppe invariant bleiben. Darüber hinaus konstruieren wir zwei Beispiele für einen prähomogenen Supervektorraum und sind in der Lage alle Supercharaktere der allgemeinen linearen Supergruppe anzugeben. Dies ermöglicht alle relativen Superinvarianten der supersymmetrischen Matrizen zu bestimmen.

Wir führen die lokalen Zeta Superfunktionen für den prähomogenen Supervektorraum der supersymmetrischen Matrizen ein und beweisen, dass diese ganze Funktionen sind. Außerdem ist die lokale Zeta Superfunktion für kompakt getragene Superfunktionen auf einer Zusammenhangskomponente V_{ij} des Orbits

$$F_{i,j}(s, \Phi_c) = \frac{1}{\gamma(s)} \int_{V_{ij}} |D(X)| \; |\mathrm{Ber}(X)|^s \; \Phi_c(X),$$

eine Regularisierung des Integral auf der rechten Seite auf den größeren Raum der Schwartz Superfunktionen. Diese sind auf dem zugehörigen cs Vektorraum definiert. Anders als die Regularisierungsmethode von Marcel Riesz und Hadamard, bei der homogene Distributionen ihre Homogenitätseigenschaft verlieren, erhält diese schon von Gelfand und Sato verwendete Regularisierungsmethode die algebraischen Eigenschaften der relativen Superinvarianten.

Es wird gezeigt, dass die Fourier Supertransformierte bis auf einen Supercharakter eine äquivariante Abbildung für die induzierte Wirkung auf dem Supervektorraum der Schwartz Superfunktionen ist. Wir beweisen auch, dass die Fourier Supertransformierte der dualen lokalen Zeta Superfunktion sich mit dem gleichen Supercharakter transformiert wie die lokale Zeta Superfunktion, geshiftet im komplexen Parameter.

Abstract

In this thesis, we develop the theory of prehomogeneous super vector spaces. In the case that G is a complex connected Lie supergroup, acting linearly on a super vector space V and if G has an open orbit in V then this orbit, as an open sub supermanifold, is unique, connected and dense. This allows us to define prehomogeneous super vector spaces. We introduce the notion of relative superinvariants and show that the relative superinvariants are already determined by the supercharacters which are invariant under the isotropy supergroup. Furthermore, we construct two examples of prehomogeneous super vector spaces. The supercharacters of the general linear supergroup will be classified and thereby all relative superinvariants for the space of supersymmetric matrices. We introduce the local zeta superfunction for the prehomogeneous super vector space of supersymmetric matrices and prove that these are entire functions. Moreover, for a compactly supported superfunction Φ_c on a connected component V_{ij} of the orbit the local zeta superfunction

$$F_{ij}(s, \Phi_c) = \frac{1}{\gamma(s)} \int_{V_{ij}} |D(X)| \, |\mathrm{Ber}(X)|^s \, \Phi_c(X),$$

is a regularization of the integral on the right-hand side to the greater space of Schwartz superfunctions. These are defined on the associated cs-vector space. Unlike the regularization methods of Marcel Riesz and Hadamard, where homogeneous distributions lose their homogeneity, this regularization method, already used by Gelfand and Sato, maintains the algebraic property of relative superinvariant.

Moreover, we demonstrate that the Fourier supertransform is, up to a supercharacter, an equivariant map for the induced action on the super vector space of Schwartz superfunctions. We also prove that the Fourier supertransform of the dual local zeta superfunctions transforms with the same supercharacter as the local zeta superfunction, shifted in the complex parameter.

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Contents

1 Introduction

Hörmander [18] considered the homogeneous function

$$x_+^s := \begin{cases} x^s & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is locally integrable when $\mathrm{Re}(s) > -1$ and discussed extensions to all $s \in \mathbb{C}$ as a distribution. There are different ways to define such a distribution: one due to Marcel Riesz and an older method due to Cauchy and Hadamard, known as Cauchy principle value and Hadamard regularization. All of these methods destroy the property of homogeneity. But the following way define a distribution for all $s \in \mathbb{C}$, which remains homogeneous. For an arbitrary Schwartz function $\Phi \in \mathscr{S}(\mathbb{R})$ the function

$$s \mapsto I_s(\Phi) = \int_0^\infty x^s \Phi(x) dx$$

is analytic when $\mathrm{Re}(s) > -1$. Using the relation

$$I_s(\Phi) = \frac{(-1)^k}{(s+1)\dots(s+k)} I_{s+k}(\Phi^{(k)}), \tag{1.1}$$

where $\Phi^{(k)} = \partial_x^k \Phi$ and which holds for $\mathrm{Re}(s) > -1$ and any integer $k > 0$, one can define the distribution I_s by analytic continuation with respect to s, if s is not a negative integer. Using the gamma function $\Gamma(s)$ one can redefine I_s as

$$s \mapsto I_s(\Phi) = \int_0^\infty \frac{x^s}{\Gamma(s+1)} \Phi(x) dx,$$

which now extends analytically using

$$I_s(\Phi) = (-1)^k I_{s+k}(\Phi^{(k)}), \tag{1.2}$$

to the whole s-plane and remains homogeneous. Note that dividing by the gamma function exactly cancels the poles which appear in Equation 1.1.

Mikio Sato generalised this idea to irreducible relative invariants of reductive prehomogeneous vector spaces. He originated the theory of prehomogeneous vector spaces around 1960 motivated by finding a testing ground for a general theory of linear differential equations, which is now called algebraic analysis.

Classically, a *prehomogeneous vector space* is a finite dimensional $\mathbb{C}$-vector space V with an $\mathbb{R}$-structure $V_\mathbb{R}$ and $\rho : G \to \mathrm{GL}(V)$ an $\mathbb{R}$-rational representation on V of a connected linear algebraic group G, such that there exists a $v \in V$ with $O_v := \rho(G)v$ an open orbit of V. This implies that O_v is unique and dense in the Zariski topology. A *relative invariant* is a homogeneous polynomial $f : V \to \mathbb{C}$ with the property $f(\rho(g)v) = \chi(g) \cdot f(v)$, where $\chi : G \to \mathrm{GL}_1$ is a rational character. For reductive prehomogeneous vector spaces, it is known that the dual triplet (G, ρ^*, V^*) and $O_v \cong O_{v^*}$, where ρ^* is the contragredient representation and $v^* \in V^*$, is also a prehomogeneous vector spaces. There is also a dual relative invariant f^* corresponding to $\chi^* = \chi^{-1}$. The sets $V_{\mathbb{R}} \cap O_v$ and $V^*_{\mathbb{R}} \cap O_{v^*}$ split into finitely many connected components denoted V_i and V^*_j respectively. In [23] the local zeta functions of the prehomogeneous vector space (G, ρ, V) and the dual prehomogeneous vector space (G, ρ^*, V^*) are defined by

$$F_i(s, \Phi) := \frac{1}{\gamma(s)} \int_{V_i} dx \; |f(x)|^s \; \Phi(x)$$

$$F^*_j(s, \Phi^*) := \frac{1}{\gamma(s)} \int_{V^*_j} dy \; |f^*(y)|^s \; \Phi^*(y).$$

The measures dx and dy are defined by the standard Lebesgue measure on $\mathbb{R}^n$ and isomorphisms $V \cong \mathbb{R}^n$ and $V \cong V^*$. Furthermore, the space of Schwartz functions on $\mathbb{R}^n$ is isomorphic to the space of Schwartz functions $\mathscr{S}(V_{\mathbb{R}})$ on $V_{\mathbb{R}}$, respectively $\mathscr{S}(V^*_{\mathbb{R}})$, with $\Phi(x) \in \mathscr{S}(V_{\mathbb{R}})$ and $\Phi^*(x) \in \mathscr{S}(V^*_{\mathbb{R}})$, (see, for more details, [23]). In the case of reductive prehomogeneous vector spaces and f a relative invariant corresponding to the character χ, we have

$$f^*(\partial_x) f(x)^{s+1} = b(s) f(x)^s.$$

The function $b(s)$ is called the Bernstein-Sato polynomial. It is due to Bernstein [5] and Sato and Shintani [32], who introduced it independently. It is a polynomial related to a differential operator. For reductive prehomogeneous vector spaces, the differential operator $f^*(\partial_x)$ is given by the dual relative invariant f^* corresponding to χ^{-1} and $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$. The Bernstein-Sato polynomial has the form

$$b(s) = b_0 \prod_{i=1}^{d} (s + \alpha_i).$$

where $\alpha_i \in \mathbb{Q}_{>0}$ for $i = 1, \ldots, d$ ([21]). The meromorphic function $\gamma(s)$ is defined as the product of gamma functions $\gamma(s) = \prod_{i=1}^{d} \Gamma(s + \alpha_i)$ and one finds that

$$f^*(\partial_x) \frac{f(x)^{s+1}}{\gamma(s+1)} = \varepsilon_i b_0 \frac{f(x)^s}{\gamma(s)},$$

where $\varepsilon_i \in \{-1, 1\}$ depends on the connected component. The purpose of the function $\gamma(s)$ is to absorb the poles that appear by analytic continuation as the gamma function did in Equation (1.2), such that the local zeta function can be extended to a holomorphic function of $s \in \mathbb{C}$ by

$$F_i(s, \Phi) = (-1)^{dm} (\varepsilon_i b_0)^{-m} \cdot F_i(s + m, f^*(D_x)^m \Phi).$$

Sato proved the following statement in 1961, which Kimura calls the Fundamental theorem of prehomogeneous vector spaces in [23, Theorem 4.17].

Theorem 1.1. *Let (G, ρ, V) be a reductive prehomogenous vector space and f a irreducible relative invariant corresponding to a character χ and f^* a relative invariant of the dual prehomogeneous vector space (G, ρ^*, V^*) corresponding to χ^{-1}. Then, the local zeta functions*

$$F_i(s, \Phi) := \frac{1}{\gamma(s)} \int_{V_i} dx\ |f(x)|^s\ \Phi(x)$$

and

$$F_j^*(s, \Phi^*) := \frac{1}{\gamma(s)} \int_{V_j^*} dy\ |f^*(y)|^s\ \Phi^*(y),$$

extend analytically to holomorphic functions on the whole s-plane. Furthermore the following holds:

$$\int_{V_j^*} dy\ |f^*(y)|^{s-\frac{n}{d}}\ \widehat{\Phi}(y) = \gamma(s - \tfrac{n}{d}) \cdot \sum_{j=1}^{l} c_{ij}(s) \int_{V_i} dx\ |f(x)|^{-s}\ \Phi(x) \quad (1.3)$$

where $c_{ij}(s)$ are entire functions which do not depend on $\Phi \in \mathscr{S}(V_{\mathbb{R}})$.

The example described at the beginning of this introduction is the simplest prehomogeneous vector space $(\mathbb{C}^*, \mathbb{C})$, with the relative invariant $f(x) = x$ and $I_s(\Phi)$ as the local zeta function for the connected component $V_1 = \mathbb{R}^*_+$.

In this thesis, we generalise the theory of prehomogeneous vector spaces to develop the theory of prehomogeneous super vector spaces. We prove that if G is a complex connected Lie supergroup, acting linearly on a super vector space V and if G has an open orbit in V then this orbit, as an open sub supermanifold, is unique, connected and dense (Theorem 3.21). Then, after translating this result into the algebraic category, this statement allows us to define a prehomogeneous super vector space. Moreover, we show that a relative superinvariant is homogeneous (Theorem 4.9) and if two relative superinvariants f and h have the same supercharacter χ, they are equal up to a constant c (Theorem 4.7). Hence, we are able to show, by using preliminary work based on algebraic geometry, (Theorem 4.10) that the set of algebraic supercharacters $X_1(G)$ that correspond to a relative superinvariant is given by the set of supercharacters, which are invariant under the isotropy supergroup G_v

$$X_1(G) = \{\chi \in X(G) \mid \chi|_{G_v} = 1\}.$$

We classify the supercharacters $\chi : \mathrm{GL}_{m|n} \to GL_1$ of the general linear supergroup $\mathrm{GL}_{m|n}$ by integer powers of the Berezinian $\mathrm{Ber}^z(.)$ and, using this, all relative superinvariants for the space of supersymmetric matrices (Theorem 4.11). Analogous to the classical theory, we define the notion of regular prehomogeneous super vector spaces, which guarantees that the dual super vector space (G, ρ^*, V^*) with the contragredient representation ρ^* is again a prehomogeneous super vector space. Further we show that $X_1(G) = X_1^*(G)$, which tells us that if we have a supercharacter χ corresponding to a relative superinvarint f there is also a dual relative superinvariant f^* on V^* corresponding to χ^{-1} (Theorem 4.14).

We derive that the flat Berezinian measure of a prehomogeneous super vector space of $\dim V = m|n$ with an irreducible relative superinvariant f, with superdegree $\deg f = d_b - d_f$ corresponding to the supercharacter χ transforms under the action of G by the factor $\chi(g)^{(m-n)/d_b - d_f}$, where $2(m-n)/d_b - d_f$ is a integer number (Lemma 4.16).

We construct two examples of prehomogeneous super vector spaces and use the prehomogeneous super vector space of supersymmetric matrices $S^2(\mathbb{C}^{p|q})$ as a toy model to examine the more general results.

Furthermore, we introduce the local zeta superfunction for the prehomogeneous super vector space of supersymmetric matrices and prove that these are entire functions of $s \in \mathbb{C}$

$$F_{i,j}(s, \Phi) := \int_{V_{ij}} |D\xi| \; F_{j,f}\left(-2(s + \frac{pq}{2}), |\det(A)|^{\frac{pq}{2}} \cdot F_{i,b}\left(s, \Phi(X_{b\text{-shift}})\right)\right)$$

for each Schwartz superfunction Φ associated to the cs-structure of the prehomogeneous super vector space. Moreover, for a compactly supported superfunction Φ_c on a connected component V_{ij} of the orbit the local zeta superfunction

$$F_{i,j}(s, \Phi_c) = \frac{1}{\gamma(s)} \int_{V_{i,j}} |D(X)| \; |\mathrm{Ber}(X)|^s \; \Phi_c(X),$$

is a regularization of the integral on the right-hand side to the greater class of Schwartz superfunctions on $\mathscr{S}(S^2(\mathbb{C}^{p|q}_{cs}))$ (Theorem 5.13). Unlike the regularization methods of Marcel Riesz or Hadamard, where homogeneous distributions lose their homogeneity, this regularization maintains the algebraic property of the Berezinian. We associate a cs-supergroup G_{cs} to an algebraic supergroup G and G^+_{cs} is the open subspace of G_{cs} such that $(G^+_{cs})_0$ contains the neutral element. Under the action of G^+_{cs}, we get

$$g.F_{i,j}(s, \Phi) = |\chi(g)|^{-(s + \frac{p-q+1}{2})} \cdot F_{i,j}(s, \Phi),$$
$$g.F^*_{i,j}(s, \Phi^*) = |\chi^*(g)|^{-(s + \frac{p-q+1}{2})} \cdot F^*_{i,j}(s, \Phi^*),$$

(Theorem 5.19).

The reason for this restriction by considering the local zeta superfunction to the example of supersymmetric matrices is that we cannot express a relative superinvariant so far in general by the classical relative invariants of the underlying prehomogeneous vector spaces. But, we believe that the prehomogeneous super vector space of supersymmetric matrices is a characteristic example for constructing general local zeta superfunctions and prove similar statements as above for the local zeta superfunction of $S^2(\mathbb{C}_{cs}^{p|q})$.

Furthermore, we show in Proposition 5.23 that the Fourier supertransform is, up to a supercharacter, an equivariant map

$$\widehat{g.\Phi}(w) = \chi(g)^{-\frac{m-n}{d_b-d_f}} \cdot g.\widehat{\Phi}(w)$$

for the induced action on the super vector space of Schwartz superfunctions. Moreover, we can show that the Fourier supertransform of the dual local zeta superfunctions, considered as tempered superdistributions, transforms with the same supercharacter

$$g.F^*_{i,j}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi}) = |\chi(g)|^{-s} \cdot F^*_{i,j}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi}),$$
$$g.F_{i,j}(-s, \Phi) = |\chi(g)|^{-s} \cdot F_{i,j}(-s, \Phi)$$

as the local zeta superfunctions, shifted in the complex parameter. Classically, this property is sufficient to show that the Fourier transform of the dual local zeta function and the local zeta function are equal, up to a constant only depending of the complex parameter s (Equation 1.3).

This thesis can be considered as a starting point for the study of prehomogeneous super vector spaces. A further question may be: Is it possible to classify all prehomogeneous super vector spaces?

In Section 2, we give the basic definitions. There the notion of super vector spaces, locally ringed superspaces, the concept of S-points, affine superschemes, complex analytic supermanifolds are defined and we prove or recall statements about open and closed sub supermanifolds that we will need throughout the thesis.

In Section 3, we define Lie supergroups and consider the quotient supermanifold of a Lie supergroup by the isotropy supergroup in order to prove Theorem 3.21. We also define the notion of algebraic supergroups and consider quotient superschemes in order to prove Theorem 4.10.

In Section 4, we develop the theory of prehomogeneous super vector spaces, and prove the results mentioned above. We define relative superinvariants, the contragredient action and the dual prehomogeneous super vector space and define the notion of a regular prehomogenous super vector space.

In Section 5, we define the local zeta superfunction for supersymmetric matrices and prove the results mentioned above about it and its Fourier supertransform.

2 Preliminaries

This section contains preparatory information. We discuss the foundation of supergeometry in order to set up the basic framework needed to develop the theory of prehomogeneous super vector spaces.

In particular, we introduce the notion of super vector spaces, tensor products, superalgebras and supermodules following the standard description in the literature. The reader may consult, for instance Ref. [10, 13, 27]. Then we define the notion of locally ringed superspaces following [2]. Moreover, we introduce generalised points, following [3, 13], in order to capture the supergeometric feature of the supergeometric objects. We will also need the following subcategories of locally ringed superspaces: complex analytic supermanifolds, cs-manifolds and affine superschemes. In introducing complex analytic supermanifolds and affine superschemes we follow [13, 28] and in the case of cs-manifolds we follow [3, 11]. Furthermore, we give, as a warm up, a proof of Theorem 2.26, which tells us that a morphism of complex analytic supermanifolds, where the Jacobian has maximal rank, admits local sections. This proof is technically similar to the proof of the inverse funtion theorem [27, Theorem 2.3.1]. Though one can often choose more general fields, we restrict our attention to $\mathbb{C}$.

2.1 Super vector spaces

Definition 2.1. A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{C}$. A *homogeneous element* $v_0 \in V_{\bar{0}}$ is said to be *even* and $v_1 \in V_{\bar{1}}$ is called *odd*. The parity function is defined on homogeneous elements by

$$\begin{gathered} |.|: (V_{\bar{0}} \cup V_{\bar{1}}) \backslash \{0\} \to \mathbb{Z}_2 \\ |v| = \bar{\imath} \quad \text{for } v \in V_i \end{gathered}$$

where $i \in \mathbb{Z}_2$. For brevity, if $\{e_j\}$ is a homogeneous basis for V we will also write $|j|$ for the parity $|e_j|$. We define the *superdimension* of the super vector space V by the pair of integers

$$\dim(V) = \dim V_{\bar{0}} \mid \dim V_{\bar{1}}.$$

A *linear map* $\phi : V \to W$ between super vector spaces V and W is called *even* if $\phi(V_{\bar{\imath}}) \subset W_{\bar{\imath}}$, and is called *odd* if $\phi(V_{\bar{\imath}}) \subset W_{\bar{\imath}+\bar{1}}$. The set of all linear maps between V and W is a super vector space with $\underline{\mathrm{Hom}}(V, W) = \mathrm{Hom}(V, W)_{\bar{0}} \oplus \mathrm{Hom}(V, W)_{\bar{1}}$, where $\mathrm{Hom}(V, W)_{\bar{0}}$ are the even maps and $\mathrm{Hom}(V, W)_{\bar{1}}$ are the odd ones. The *morphisms* from a super vector space V to a super vector space W are linear maps in $\mathrm{Hom}(V, W)_{\bar{0}}$ that preserve the grading.

The super vector spaces with their morphisms form the category **svct**. For details about categories, one may consult [33].

Definition 2.2. A real $\mathbb{Z}/2\mathbb{Z}$ graded vector space $U = U_{\bar{0}} \oplus U_{\bar{1}}$ with a fixed complex structure $U_{\bar{1}}$ will be called *cs-vector space*. Given a complex super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ has a real form, $V_{\bar{0}} = V_{\bar{0},\mathbb{R}} \oplus \mathrm{i} V_{\bar{0},\mathbb{R}}$, then the *cs*-vector space $V_{cs} := V_{\bar{0},\mathbb{R}} \oplus V_{\bar{1}}$ is called a *cs-form* of V.

We can also construct the *tensor product* of two super vector spaces

$$V \otimes W = (V \otimes W)_{\bar{0}} \oplus (V \otimes W)_{\bar{1}}$$

with

$$(V \otimes W)_{\bar{\mathrm{i}}} := \sum_{\bar{\mathrm{j}}+\bar{\mathrm{k}}=\bar{\mathrm{i}}} V_{\bar{\mathrm{j}}} \otimes W_{\bar{\mathrm{k}}} \quad \bar{\mathrm{i}}, \bar{\mathrm{j}}, \bar{\mathrm{k}} \in \mathbb{Z}_2.$$

Definition 2.3. An *associative superalgebra* A over $\mathbb{C}$ is a super vector space over $\mathbb{C}$, which is also an associative ring. Furthermore, we require that the multiplication map $A \otimes A \to A$ is even, which implies the following multiplication rule

$$a_i \cdot a_j \in A_{\bar{\mathrm{i}}+\bar{j}} \quad \text{whenever} \quad a_i \in A_{\bar{\mathrm{i}}},\ a_j \in A_{\bar{j}} \quad \bar{\mathrm{i}}, \bar{\mathrm{j}} \in \mathbb{Z}_2.$$

A superalgebra A is *supercommutative*, if the product of homogeneous elements $a, b \in A$ obeys the rule

$$ab = (-1)^{|a||b|} ba.$$

An example of a superalgebra is $M(p|q, A)$. This is the superalgebra of $(p+q) \times (p+q)$-matrices, with entries in a superalgebra A. We define

$$M(p|q, A)_{\bar{\mathrm{i}}} = \begin{pmatrix} B & \Gamma \\ \Delta & C \end{pmatrix} \quad B_{kl}, C_{mn} \in A_{\bar{\mathrm{i}}} \quad \Gamma_{kn}, \Delta_{ml} \in A_{\bar{\mathrm{i}}+\bar{1}} \quad \bar{\mathrm{i}} \in \mathbb{Z}_2$$

with

$$k, l \in \{1, ...p\} \quad m, n \in \{1, ..., q\}.$$

The superalgebra structure is given by usual matrix multiplication. Furthermore, we define the super transpose of $X = \left(\begin{smallmatrix} B & \Gamma \\ \Delta & C \end{smallmatrix}\right)$ by $X^{ST} = \left(\begin{smallmatrix} B^T & \Delta^T \\ -\Gamma^T & C^T \end{smallmatrix}\right)$.

Definition 2.4. Fix a supercommutative superalgebra A. The *standard free module* $A^{p|q}$ is the module freely generated by even elements $e_1, \ldots, e_p$ and odd elements $e_{p+1}, \ldots, e_{p+q}$. The endomorphisms of $A^{p|q}$ can be represented by elements of $M(p|q, A)_{\bar{0}}$ and the automorphisms are represented by $\mathrm{GL}(p|q, A) := \{X \in M(p|q, A)_{\bar{0}} \mid X \text{ is invertible}\}$.

2.2 Locally ringed superspaces

Here, we introduce the category of locally ringed superspaces, which includes as full sub categories affine $\mathbb{C}$-superschemes, complex analytic supermanifolds and *cs*-manifolds.

Definition 2.5. A *locally ringed superspace* $X = (X_0, \mathcal{O}_X)$ is a topological space X_0 endowed with a sheaf of superalgebras $\mathcal{O}_X$ over $\mathbb{C}$ such that the stalk at every point $x \in X_0$, denoted by $\mathcal{O}_{X,x}$, is a local superalgebra, i.e. $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$. For brevity, we will often use the term superspace instead of locally ringed superspace. A morphism $\phi : X \to T$ of superspaces is expressed by $\phi = (\phi_0, \phi^\sharp)$, where $\phi : X_0 \to T_0$ is a map of topological spaces and $\phi^\sharp : \mathcal{O}_T \to \phi_{0*}\mathcal{O}_X$ is a local sheaf morphism, so that $\phi^\sharp_x(\mathfrak{m}_{\phi_0(x)}) \subseteq \mathfrak{m}_x$. An open sub superspace U is defined by the superspace $(U_0, \mathcal{O}_X|_{U_0})$, where $U_0 \subseteq X_0$ is open in X_0. Additionally one has the *inclusion morphism* $j_U := (j_{X|_U,\bar{0}}, j^\sharp_{X|_U})$, where $j_{X|_U,\bar{0}}$ is the inclusion of U_0 in X_0 and the sheaf morphism $j^\sharp_{X|_U} : j^{-1}_{X|_U,\bar{0}}\mathcal{O}_X \to \mathcal{O}_X|_{U_0}$ is the identity. We call a morphism $\phi : T \to X$ of superspaces an *open embedding* if it factors as $\phi = j_U \circ \psi$ where $U_0 \subseteq X_0$ is an open subset and $\psi : T \to X|_U$ is an isomorphism.

An example of a superspace is $\mathbb{C}^{p|q}$, which is the model space in the category of complex analytic supermanifolds. The superspace $\mathbb{C}^{p|q} := (\mathbb{C}^p, \mathcal{O}_{\mathbb{C}^{p|q}})$ is the topological space $\mathbb{C}^p$ endowed with the following sheaf of superalgebras over $\mathbb{C}$. With any open set $U_0 \subset \mathbb{C}^p$ we associate the superalgebra

$$\mathcal{O}_{\mathbb{C}^{p|q}}(U_0) := \mathcal{H}(\mathbb{C}^p)(U_0) \otimes \bigwedge(\mathbb{C}^q)^*,$$

where $\mathcal{H}(\mathbb{C}^p)(U_0)$ consists of the algebra of complex analytic functions on U_0 and $\Lambda(\mathbb{C}^q)^*$ is the Grassmann algebra of $(\mathbb{C}^q)^*$.

2.3 The concept of S-points

In the case of supergeometric objects (or in the generality of schemes), the points of an object X do not fully capture the supergeometric (respectively scheme-theoretic) features of X. Without full knowledge of the sheaf $\mathcal{O}_X$, essential information would be omitted. In order to recapture this information, we use Grothendieck's functor of points. We consider the object X along with all morphisms $S \to X$, where S runs through all objects of the same type.

Definition 2.6. Let $\mathcal{C}$ be a category, for instance the category of locally ringed superspaces. For $S, X \in Ob(\mathcal{C})$, we denote by $X(S) := \mathrm{Hom}_{\mathcal{C}}(S, X)$ the set of morphisms from S to X and call $\alpha \in X(S)$, also written as $\alpha \in_S X$, an *S-point* of X. By definition, *the set of S-points* $X(S)$ is functorial in S, which means that a morphism $u : T \to S$ and $s \in_S X$ induces the map $s \mapsto s \circ u$ from $X(S)$ to $X(T)$. Thus, we get a contravariant functor

$$\begin{aligned} h_X : \mathcal{C} &\to \mathbf{set} \\ S &\mapsto X(S), \end{aligned}$$

where **set** is the category of sets. The functor h_X is called the *functor of points of X*.

The notion of S-points is motivated by the following: A point $x \in X_0$, where X_0 is an ordinary set, can be considered as a map from the singleton set $\{*\}$ to X_0 and $X_0(*) \cong X_0$. In this way $X_0(S)$ contains X_0 as a subset and shows that S-points serve as generalised points.

Definition 2.7. A *contravariant functor* $F : \mathcal{C} \to \mathbf{set}$ is called *representable* if there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that $F \cong h_X$ as functors. In this case, X is unique up to canonical isomorphism and is called the representative of F.

Let $\mathbf{F}_{\mathcal{C}}$ be the category of contravariant functors, from $\mathcal{C}$ to **set**, where the morphisms are natural transformations. Every category embeds in a functor category, which often has nicer properties than the original category. The use of this language is justified by the Yoneda Lemma, and can be found in any introduction to category theory, for instance [33]:

Lemma 2.8 (Yoneda Lemma). *For any $X \in Ob(\mathcal{C})$ and $F \in Ob(\mathbf{F}_{\mathcal{C}})$, we have*

$$Hom_{\mathbf{F}_{\mathcal{C}}}(h_X, F) \cong F(X).$$

In particular, the functor h is fully faithful and the full subcategory of $\mathbf{F}_{\mathcal{C}}$ consisting of the representable functors is equivalent to $\mathcal{C}$.

Each natural transformation $h_X \to h_Y$ is defined by a unique morphism $f : X \to Y$, so f induces a map f_S from $X(S)$ to $Y(S)$ functorial in S. By the Yoneda Lemma, this construction is a bijection from the set of morphisms $f : X \to Y$ to the set of maps between systems of maps $f_S : X(S) \to Y(S)$. There is a natural *topology on the set of S-points* defined as follows:

Definition 2.9. Let $U \subseteq X$ be an open sub superspace, with the inclusion morphism j_U. By composing elements of $U(S)$ with j_U, we get an embedding of $U(S)$ in $X(S)$. In this way we are able to identify $U(S)$ with the subsets of $X(S)$ as

$$U(S) = \{\psi : S \to X \mid \psi_0(S_0) \subseteq U_0\}.$$

The collection of these open sets $U(S)$, such that $U \subseteq X$ is an open sub superspace, gives a topology on $X(S)$

$$\mathrm{Top}(X(S)) := \{U(S) \mid U \subseteq X \text{ open}\}.$$

Proposition 2.10. *Let $f : X \to Y$ be a morphism of superspaces. The induced map on the S-points $f_S : X(S) \to Y(S)$ is continuous.*

Proof. Let $V \subseteq Y$ be an open sub superspace, then the continuity follows from $f_S^{-1}(V(S)) = f^{-1}(V)(S)$ with the Yoneda lemma, where $f^{-1}(V)$ is an open subspace with the underlying space $f_0^{-1}(V_0)$. This is an open subset of X_0, so $f_S^{-1}(V(S))$ is open in $X(S)$, which proves the continuity of f_S. □

Proposition 2.11. *If $W_0 \subseteq V_0$ is dense and open in V_0, then $W(S)$ is dense in $V(S)$.*

Proof. Let $u \in_S V$ and $U(S)$ be an open neighbourhood of u, then $u_0(S_0) \subseteq U_0 \subseteq V_0$. We know W_0 is dense in V_0, hence $U_0 \cap W_0$ nonempty and open. For $x \in U_0 \cap W_0$, it follows that the morphism $x_s : S \to x$ is an element of $(U \cap W)(S)$ and $x_S \in U(S) \cap W(S)$, thus $U(S) \cap W(S) \neq \emptyset$. Hence, each neighbourhood includes an element of $W(S)$. □

Now we define the main categories we will work with, the category of affine $\mathbb{C}$-superschemes, the category of complex analytic supermanifolds and the category of *cs*-manifolds.

2.4 Affine superschemes

Definition 2.12. Let **salg** be the category of supercommutative superalgebras over $\mathbb{C}$ and $A \in$ **salg**. An *affine superscheme* X (over $\mathbb{C}$) is defined as a locally ringed superspace $\underline{\mathrm{Spec}}\ A = (\mathrm{Spec}(A_{\bar{0}}), \mathcal{O}_X)$, where $\mathrm{Spec}(A_{\bar{0}})$ is the set of prime ideals of $A_{\bar{0}}$. The basic open sets of the underlying topological space $X_0 = \mathrm{Spec}(A_{\bar{0}})$ are given by

$$U_f = \{\mathfrak{p} \in \mathrm{Spec}(A_{\bar{0}}) \mid (f) \not\subset \mathfrak{p}\}$$

for $f \in A_{\bar{0}}$ and the structure sheaf $\mathcal{O}_X$ is defined on basic open sets by

$$\mathcal{O}_X(U_f) := A_f = \left\{ \frac{a}{f^n} \,\middle|\, a \in A \right\}.$$

The superalgebra of global section $\mathcal{O}_X(X_0)$ is isomorphic to A.

Remark 2.13. Here, we only deal with affine superschemes over $\mathbb{C}$, given by finitely generated superalgebras, such that the algebra $A_{\text{red}} := A/\langle A_{\bar{1}} \rangle$ has no nilpotents.

A *super vector space* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ can also be viewed *as an affine superscheme* $V = (V_{\bar{0}}, \mathcal{O}_V)$ given by the superalgebra $\mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots \xi_q]$ of polynomials in the even coordinate functions $x_1, \ldots, x_p$ on $V_{\bar{0}}$ and the odd coordinate functions $\xi_1, \ldots, \xi_q$. The underlying topological space, also called V_0, is $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots \xi_q]_{\bar{0}})$ and is built from the prime ideals of $\mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]$. Each of these prime ideals is of the form $(p, \xi_i \xi_j \mid i, j \in \{1, \ldots, q\})$, for a unique prime ideal $p \subset \mathbb{C}[x_1, \ldots, x_p]$. Furthermore, one has $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots \xi_q]_{\bar{0}}) = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_p])$ as topological spaces, so the ringed space $(V_{\bar{0}}, \mathcal{O}_V / I^{odd})$ is a usual affine variety, where the ideal I^{odd} is generated by the odd coordinates $\xi_1, \ldots, \xi_q$.

Definition 2.14. A *superfunction* h is called *regular* on the affine superscheme X if $h \in \Gamma(\mathcal{O}_X)$ or equivalently, if for all points $v \in X_0$ there is an open affine sub superscheme U with $v \in U_0$ and polynomials $p \in \mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]$ and $q \in \mathbb{C}[x_1, \ldots, x_p]$ such that q is nowhere zero on U and h can be expressed by $h = \frac{p}{q}$ on U.

Example 2.15. Let $\mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots \xi_q]$ be the superalgebra of polynomials of the super vector space V. The structure sheaf of the regular superfunctions $\mathcal{O}_V$ is defined by $\mathcal{O}_V(U_f) := \mathcal{O}_V(V_{\bar{0}})_f = \mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]_f$ for $f \in \mathbb{C}[V_{\bar{0}}]$. For every $h \in \mathcal{O}_V(U_f)$, there exists a $p \in \mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]$ and an $n \in \mathbb{N}$ such that $h = \frac{p}{f^n} \in \mathbb{C}(V_{\bar{0}})[\xi_1, \ldots, \xi_q]$, where $\mathbb{C}(V_{\bar{0}})$ is the function field of the affine scheme $V_{\bar{0}}$.

Definition 2.16. The set of *rational superfunctions* on the affine superscheme X, denoted by $\mathbb{C}(X)$, is defined as equivalence classes of pairs (U, f), where U is an open dense sub superscheme of X and f is a regular superfunction on U. Two pairs (U, f) and (V, g) are called equivalent if $f \equiv g$ on $U \cap V$.

Example 2.17. The rational superfunctions on V form the space of functions of the type $\mathbb{C}(V_{\bar{0}})[V_{\bar{1}}]$, consisting of rational functions in the even coordinates x_i and polynomials in the odd coordinates ξ_j. An element $f \in \mathbb{C}(V_{\bar{0}})[V_{\bar{1}}]$ can expressed by $f = \sum_{\mu \in \mathbb{Z}_2^q} f_\mu \xi^\mu$ with $f_\mu = \frac{p_\mu}{q_\mu}$ and p_μ, q_μ coprime polynomials in the even coordinates.

The following proposition will be important for Theorem 4.10, telling us about the relation between relative superinvariants and supercharacters.

Proposition 2.18. *Let V be a super vector space and O a Zariski open sub superscheme, then a regular superfunction on O defines a rational superfunction on V.*

Proof. Let f be a regular superfunction on O. Furthermore, let U be a Zariski open cover of $V_{\bar{0}}$, then there exist $f_i \in \mathbb{C}[V_{\bar{0}}]$ $(i \in \mathbb{N})$ with $U = \bigcup_i U_i$ and $U_i := \{x \in V_{\bar{0}} \mid f_i(x) \neq 0\}$ principal open sets. The Zariski topology is quasi compact, so there exists a $N \in \mathbb{N}$ such that $U = \bigcup_{i=1}^{N} U_i$. As the intersection of two such Zariski open and dense sets, $(O)_0 \cap U_i$ is open and dense in $V_{\bar{0}}$. Let $\mathcal{O}_V$ be the structure sheaf of regular superfunctions defined by $\mathcal{O}_V(D(g)) := \mathcal{O}_V(V_{\bar{0}})_g = \mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]_g$ for $g \in \mathbb{C}[V_{\bar{0}}]$. For every $h \in \mathcal{O}_V(D(g))$, there exists $p \in \mathbb{C}[x_1, \ldots, x_p, \xi_1, \ldots, \xi_q]$ and a $n \in \mathbb{N}$ such that $h = \frac{p}{g^n} \in \mathbb{C}(V_{\bar{0}})[\xi_1, \ldots, \xi_q]$. Let $\tilde{f} \in \mathcal{O}_V(U)$ and $\tilde{f}|_O \equiv f$, then there exists a $p_i \in \mathbb{C}[V]$ and a $q_i \in \mathbb{C}[V_{\bar{0}}]$ with $q_i \neq 0$, such that $\tilde{f}|_{U_i} = \frac{p_i}{q_i}$ and $p_i = \sum_I p_{iI}\xi^I$ for all $i \in \{1, \ldots, N\}$, where the greatest common divisor $\gcd((p_{iI})_I, q_i) = 1$. On $U_i \cap U_j$, we have that $\frac{p_i}{q_i} = \frac{p_j}{q_j} \Leftrightarrow q_j p_{iI} = q_i p_{jI}$ and q_i also divides all $q_j p_{iI}$ for all I. It follows that q_i is a divisor of q_j for all i and all j. Hence, $q_j = \alpha \cdot q_i$, where $\alpha \in \mathbb{C}[V_{\bar{0}}]^* = \mathbb{C}^*$. Therefore, $\alpha \cdot q_i p_{iI} = q_i p_{jI}$ and $q_i, q_j \neq 0$ on $U_i \cap U_j$, so that $\alpha \cdot p_{iI} = p_{jI}$ on the Zariski dense set $U_i \cap U_j$. Consequently $p_i q_j = p_j q_i$ and $\frac{p_i}{q_i} = \frac{p_j}{q_j}$ in $\mathbb{C}(V)$. Since j is arbitrary and N finite, $f = \frac{p_i}{q_i} = \tilde{f}$ on U and f is a rational superfunction on V. □

2.5 Complex analytic supermanifolds

Here, we define complex analytic supermanifolds and cs-manifolds. We show that a sufficient condition for a morphism between complex analytic supermanifolds to admit local sections is the Jacobian having maximal rank. Technically, this proof is similar to the Inverse Function Theorem [27, Theorem 2.3.1]. We will need this statement for the proof of Theorem 3.19, which ensures that if a complex connected Lie supergroup, acting linearly on a super vector space, has an open orbit, then this orbit is unique, connected and dense. Furthermore, we recapitulate when an ideal sheaf determines a closed sub supermanifold, which will be necessary in the context of orbit supermanifolds.

Definition 2.19. A *complex analytic supermanifold* of dimension $p|q$ is a locally ringed superspace $X = (X_0, \mathcal{O}_X)$ such that the topological space X_0 is Hausdorff and X admits a cover by open subspaces U_i such that every

U_i is isomorphic as a locally ringed superspace to an open subspace of $\mathbb{C}^{p|q}$.

That means, that X possesses a cover by chart domains U. By definition, a chart domain U is an open subspace that admits an isomorphism $\phi : U \to W$ to an open subspace W of $\mathbb{C}^{p|q}$. Then (ϕ, U) is called a chart. Local coordinates for U are then given by the tuple

$$y = (v_1, ..., v_p, \eta_1, ..., \eta_q) = (\varphi^\sharp(u_1), ..., \varphi^\sharp(u_p), \varphi^\sharp(\xi_1), ..., \varphi^\sharp(\xi_q)),$$

where u_j are the standard coordinates on $\mathbb{C}^p$, and $\{\xi_j\}$ form a basis of $(\mathbb{C}^q)^*$.

The structure sheaf $\mathcal{O}_X$ contains an ideal sheaf J_X, where $J_X(U) = \langle \eta_1, ..., \eta_q \rangle$ is generated by the nilpotent coordinate functions. This allows us to construct the ringed space $X_{red} = (X_0, \mathcal{O}_X/J_X)$, which is a complex analytic manifold and is often called the underlying manifold. The quotient map $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_X/J_X$, which sends a complex analytic superfunction f to a complex analytic function $\tilde{f}$, induces a morphism $\pi : X_{red} \to X$. Here π_0 is the identity map of X_0.

Definition 2.20. A *cs-manifold* X of dimension $p|q$ is a topological space X_0 with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$, such that locally $(X_0, \mathcal{O}_X)$ is isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty_p \otimes \bigwedge(\mathbb{C}^q)^*)$, where $\mathcal{C}^\infty_p$ denotes the sheaf of complex valued smooth functions on $\mathbb{R}^p$.

To each super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of complex superdimension $\dim V = p|q$ we can functorially associate a linear complex analytic supermanifold, $V = (V_{\bar{0}}, \mathcal{H}(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}})^*)$ where $V^* \subseteq \Gamma(\mathcal{O}_V)$. Analogously, one can associate to any *cs*-vector spaces V_{cs} the *cs-affine superspace*, which is also denoted by V_{cs}. The cs-affine superspace V_{cs} is given by $(V_{\bar{0},\mathbb{R}}, \mathcal{O}_{V_{cs}})$, where $\mathcal{O}_{V_{cs}} := \mathcal{C}^\infty_{V_{\bar{0},\mathbb{R}}} \otimes \bigwedge V^*_{\bar{1}}$ and $\mathcal{C}^\infty_{V_{\bar{0},\mathbb{R}}}$ is isomorphic to $\mathcal{C}^\infty_p$.

The following proposition is the obvious extension of [8, Proposition 4.6.1] to complex analytic supermanifolds or affine superschemes or *cs*-manifolds. The case of superschemes is an immediate generalisation of [14, I, 1.6.3].

Proposition 2.21. *Let be complex analytic supermanifolds, where Y is an open subspace of $\mathbb{C}^{p|q}$ or X a locally ringed superspace and Y an affine superscheme or X, Y cs-manifolds with Y an open sub superspace of $\mathbb{C}^{p|q}_{cs}$. There is a bijective correspondence*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\mathcal{O}_Y(Y_0), \mathcal{O}_X(X_0)).$$

Definition 2.22. Let X and Y be complex supermanifolds or *cs*-manifolds. Let $\phi : X \to Y$ be a morphism and charts $x = (u, \xi)$ and $y = (v, \eta)$ on the open sub superspaces $U \subseteq X$ and $V \subseteq Y$. The *Jacobian* of ϕ is then defined as

$$J^{\phi}_{x,y} = \begin{pmatrix} \frac{\partial \phi^{\sharp}(v)}{\partial u} & -\frac{\partial \phi^{\sharp}(v)}{\partial \xi} \\ \frac{\partial \phi^{\sharp}(\eta)}{\partial u} & \frac{\partial \phi^{\sharp}(\eta)}{\partial \xi} \end{pmatrix},$$

where $J^{\phi}_{x,y} \in M\left(p|q, \mathcal{O}_X(U_0 \cap \phi_0^{-1}(V_0))\right)_{\bar{0}}$. The rank of $J^{\phi}_{x,y}$ is the rank of $\pi^{\sharp}(J^{\phi}_{x,y})$, where $\pi : X_{red} \to X$ was the canonical embedding of X_{red} in X.

The *differential* $(d\phi)_p$ at $p \in X_0$ can be expressed by the matrix

$$(d\phi)_p = \pi^{\sharp}(J^{\phi}_{x,y})(p) = \begin{pmatrix} \pi^{\sharp}\left(\left. \frac{\partial \phi^{\sharp}(v)}{\partial u} \right|_p \right) & 0 \\ 0 & \pi^{\sharp}\left(\left. \frac{\partial \phi^{\sharp}(\eta)}{\partial \xi} \right|_p \right) \end{pmatrix}.$$

Definition 2.23. The supermanifold X is an immersed sub supermanifold in Y, if there exists an injective immersion $j : X \to Y$, which means that $j_0 : X_0 \to Y_0$ is injective and $(dj)_p$ is injective for all $p \in X_0$. Moreover, we say that X is an embedded sub supermanifold if $j : X \to Y$ is an injective immersion and $j_0 : X_0 \to Y_0$ is a homeomorphism onto its image.

A morphism $\phi : X \to Y$ is a submersion at $p \in X_0$ if $(d\phi)_p$ is surjective.

Definition 2.24. Let X and Y be complex analytic supermanifolds of dimension $n|s$ and $m|r$, with $n \geq m$ and $s \geq r$. Moreover, let $\phi : X \to Y$ be a morphism. For an open sub supermanifold V of Y we call a morphism $\psi : V \to X$ a local section, if $\phi_0 \circ \psi_0 = \mathrm{Id}_{V_0}$ and $\psi^{\sharp} \circ \phi^{\sharp} = \mathrm{Id}_{\mathcal{O}_Y}|_V$.

Remark 2.25. We can interpret the existence of a local section as the fact that there are neighbourhoods U and V, such that $\phi : U \to V$ is a submersion. Furthermore, $\psi : V \to U$ is an immersion (cf. [28]).

Theorem 2.26. *Let $\phi : X \to Y$ be a morphism between complex analytic supermanifolds X and Y. If the rank of the Jacobi matrix $J^{\phi}_{x,y}$ is equal to the dimension of Y for a point $p \in X_0$, then there exists a local section $\psi : V \to X$, where V is an open neighbourhood of $\phi_0(p)$ and there exists an open neighbourhood U for p, such that the map $\phi : U \to V$ is a submersion.*

Proof. The statement is purely local and therefore it is sufficient to prove the statement with $X = \mathbb{C}^{n|s}$ and $Y = \mathbb{C}^{m|r}$. Let $(x = (u, \xi), U)$ be a chart on $\mathbb{C}^{n|s}$ with $p \in U_0$, and $(y = (v, \eta), V)$ a chart on $\mathbb{C}^{m|r}$ with $\phi_0(p) \in V_0$.

The matrix $\pi^\sharp(J^\phi_{x,y})(p)$ is of the form

$$\pi^\sharp(J^\phi_{x,y})(p) = \begin{pmatrix} \pi^\sharp\left(\left.\frac{\partial\phi^\sharp(v)}{\partial u}\right|_p\right) & 0 \\ 0 & \pi^\sharp\left(\left.\frac{\partial\phi^\sharp(\eta)}{\partial\xi}\right|_p\right) \end{pmatrix}.$$

By assumption, $\mathrm{rank}(J^\phi_{x,y}(p)) = m|r$ and so, $\pi^\sharp\left(\left.\frac{\partial\phi^\sharp(v)}{\partial u}\right|_p\right) = \left.\frac{\partial\phi^\sharp_0(v_0)}{\partial u_0}\right|_p$ and $\pi^\sharp\left(\left.\frac{\partial\phi^\sharp(\eta)}{\partial\xi}\right|_p\right)$ are both of maximal rank with entries in $\mathbb{C}$.

If the tangent map $df_p : T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$ of a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^m$ has rank m, then by [25, Theorem 0.5.2] there exist open neighbourhoods U_0 and V_0 of p and $f(p)$ respectively, where $f : U_0 \to V_0$ is surjective.

After possibly shrinking our sub superspaces U and V, it follows from the last statement that ϕ_0 is a surjective map from U_0 to V_0 and V_0 is already open. Furthermore, we are using the same symbols here for the refined open sub superspaces U and V. It remains to show that a superalgebra morphism $\psi^\sharp : \mathcal{O}_X(U_0) \to \mathcal{O}_Y(V_0)$ exists such that $(\phi\circ\psi)^\sharp = \psi^\sharp\circ\phi^\sharp = Id_{\mathcal{O}_Y(V_0)}$, which immediately implies the existence of a sheaf morphism $\psi^\sharp : \mathcal{O}_X \to \mathcal{O}_Y$. Sine $\Phi_0 : U_0 \to V_0$ is surjective, we may choose cordinates such that $U_0 = V_0 \times U_0'$ and Φ_0 is the projection onto V_0. Hence, $u_i = \pi^\sharp(\phi^\sharp(v_i))$ for $i \in \{1, ..., m\}$, which implies that

$$\phi^\sharp(v_i) = u_i \bmod(J_{U_0}), \quad i \in \{1, ..., m\}.$$

Possibly shrinking U further, we may assume that the $(r\times s)$-matrix $\pi^\sharp\left(\frac{\partial\phi^\sharp(\eta)}{\partial\xi}\right)$ with entries in $\mathcal{H}(\mathbb{C}^m)(U_0)$ has rank r for all $p \in U_0$. Further, there exists a matrix $T \in \mathrm{GL}(s, \mathcal{H}(\mathbb{C}^m)(U_0))$ such that

$$A' := \left(\pi^\sharp\left(\frac{\partial\phi^\sharp(\eta)}{\partial\xi}\right)\right)T = \begin{pmatrix}\tilde{A} & 0\end{pmatrix}, \tag{2.1}$$

with $\tilde{A} \in \mathrm{GL}(r, \mathcal{H}(\mathbb{C}^m)(U_0))$. Now, we define an invertible $(s\times s)$-matrix by $A = \begin{pmatrix}\tilde{A} & 0 \\ 0 & Id\end{pmatrix} T^{-1}$ and new odd coordinates on U by

$$\xi_j' = \sum_{k=1}^{s} A_{jk}\xi_k.$$

In this new coordinate system, we have

$$\begin{aligned} \phi^\sharp(v_i) &= u_i \ (\mathrm{mod}\ J_{U_0}) \quad i \in \{1, ..., m\} \\ \phi^\sharp(\eta_j) &= \xi_j' \ (\mathrm{mod}\ J^2_{U_0}) \quad j \in \{1, ..., r\}. \end{aligned}$$

We now construct the morphism $\psi : Y \to X$, which is a right inverse to ϕ. Define a morphism $\psi_0^\sharp : \mathcal{O}_X(U_0) \to \mathcal{O}_Y(V_0)$ by

$$\begin{aligned}
\psi_0^\sharp(u_i) &= v_i \quad (i = 1, ..., m) \\
\psi_0^\sharp(u_j) &= 0 \quad (j = m+1, ..., n) \\
\psi_0^\sharp(\xi_k) &= \eta_i \quad (k = 1, ..., r) \\
\psi_0^\sharp(\xi_l) &= 0 \quad (l = r+1, ..., s),
\end{aligned}$$

and the morphisms $\psi_k^\sharp$ recursively by

$$\psi_{k+1}^\sharp(x_i) := \psi_k^\sharp(x_i) + \psi_0^\sharp(x_i - \phi^\sharp\psi_k^\sharp(x_i)),$$

where $x_i = (u, \xi)_i \in \mathcal{O}_X(U_0)$. We claim that $\psi_k^\sharp$ for $r < k$ is the required morphism. To prove this, consider the maps

$$\begin{aligned}
\Delta_k :& \mathcal{O}_Y(V_0) \to \mathcal{O}_Y(V_0) \\
\Delta_k(f) &= \psi_k^\sharp\phi^\sharp(f) - f.
\end{aligned}$$

By construction,

$$\begin{aligned}
\Delta_{k+1}(y_i) &= \psi_{k+1}^\sharp\phi^\sharp(y_i) - y_i \\
&= \psi_k^\sharp\phi^\sharp(y_i) + \psi_0^\sharp\phi^\sharp(y_i) - \psi_0^\sharp\phi^\sharp\psi_k^\sharp\phi^\sharp(y_i) - y_i \\
&= -[\psi_0^\sharp\phi^\sharp(\psi_k^\sharp\phi^\sharp(y_i) - y_i) - (\psi_k^\sharp\phi^\sharp(y_i) - y_i)] \\
&= -\Delta_0(\Delta_k(y_i)).
\end{aligned}$$

Since the homomorphism $\psi_0^\sharp\phi^\sharp : \mathcal{O}_Y(V_0) \to \mathcal{O}_Y(V_0)$ satisfies

$$\begin{aligned}
\psi_0^\sharp\phi^\sharp(v_i) &= \psi_0^\sharp(u_i \pmod{J_{U_0}}) = v_i \pmod{I_{V_0}} \quad i \in \{1, ..., m\} \\
\psi_0^\sharp\phi^\sharp(\eta_j) &= \psi^\sharp(\xi_j \pmod{J_{U_0}^2}) = \eta_j \pmod{I_{V_0}^2} \quad j \in \{1, ..., r\},
\end{aligned}$$

it follows that $\Delta_0(I_{V_0}^k) \subset I_{V_0}^{k+1}$. Hence, $\Delta_k(f) \subset I_{V_0}^k$ for all $f \in \mathcal{O}_Y(V_0)$ and $\Delta_k(f) = 0$ for $k > r$. Hence, $\psi^\sharp\phi^\sharp$ is the identity map on $\mathcal{O}_Y(V_0)$ and ψ is a right inverse to ϕ. $\square$

Definition 2.27. A $p|q \times m|n$-supermatrix $T \in \mathrm{M}(p|q, A)_{\bar{0}}$ for A a supercommutative superalgebra is said to have constant rank $(r|s)$, if there exist $G_1 \in \mathrm{GL}(m|n, A)$ and $G_2 \in \mathrm{GL}(p|q, A)$ such that

$$G_1 T G_2 = \begin{pmatrix} Id_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Id_s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.28. Notice that not all $T \in \mathrm{M}(p|q, A)_{\bar{0}}$ are of constant rank. A counter example is given by $\left(\begin{smallmatrix} 0 & 0 \\ \xi & 0 \end{smallmatrix}\right)$, because it is not possible to transform this matrix with left and right multiplication by elements of the general linear supergroup into the required form.

Now, we need two more definitions to state the next proposition.

Definition 2.29. A morphism of supermanifolds $\phi : X \to Y$ has constant rank $r|s$ in a neighbourhood U of $p \in X_0$, if the Jacobian J_ϕ has constant rank $r|s$ on U.

Definition 2.30. An ideal sheaf I determines a closed sub supermanifold N of a complex analytic supermanifold M, if for $N_0 :=$ supp $\mathcal{O}_M/I = \{x \in M_0 \mid I_x \subset \mathcal{O}_{M,x}\}$, $N := (N_0, j_0^{-1}(\mathcal{O}_{\mathcal{M}}/I))$ is a supermanifold and $j := (j_0, j^\sharp) : N \to M$ is an immersion for $j^\sharp : j_0^{-1}\mathcal{O}_M \to j_0^{-1}(\mathcal{O}_M/I)$ defined as the canonical projection.

Proposition 2.31. *Let $\phi : M \to W$ be a morphism of complex analytic supermanifolds with $q \in W_0$. For each $p \in \phi_0^{-1}(q)$ suppose there exists a neighbourhood of p where ϕ is of constant rank $r|s$. Then the ideal sheaf $I \subseteq \mathcal{O}_M$ generated by $\phi^\sharp(J_q)$, where J_q is the maximal ideal associated to the point q, determines a closed sub supermanifold N of M.*

For a proof, the reader may consult Ref. [27, Theorem 2.3.9], but we examine the idea behind the proof. One finds that $I \subseteq J_p$ for all $p \in \phi_0^{-1}(q)$. This set is a closed subset of M_0 defined by I. Since ϕ is of constant rank, there exists for each p a neighbourhood U_p and homogeneous superfunctions $f_1, ..., f_n$ in $I(U_p)$ which come via $\phi^\sharp$ from coordinates around q, such that the germs $[f_1], ...[f_n]$ generate I_x, the stalk of I in x. The $(df_1)_p, ..., (df_n)_p$ are linearly independent and therefore the f_i's can be supplemented to coordinates around p in the neighbourhood U_p. One can define locally a supermanifold structure for $U_p \cap \phi_0^{-1}(q)$, where the superfunctions are functions only in the supplemented coordinates. And by the transition function one can carry the structure over to $\phi_0^{-1}(q)$.

3 Quotient superspaces

The theory of prehomogeneous super vector spaces relies on the general theory of supergroup quotients. In this section, we give the basic definitions both in the case of real or complex supermanifolds and in the case of algebraic supergroups. Therefore, we introduce group objects in a category in order to define Lie supergroups and algebraic supergroups in an efficient way. An important step to define a prehomogeneous super vector space is to understand the orbit of a Lie supergroup through a supervector $v \in V_0$ in a super vector space V, considered as a linear supermanifold. Partially following [8, p.149ff], we show that the isotropy supergroup is a closed sub Lie supergroup and that the quotient superspace is a supermanifold.

One of the main results is Theorem 3.21, which tells us that if G is a connected Lie supergroup, acting linearly on a super vector space, and if G has an open orbit, then the orbit is a unique, connected, and dense sub supermanifold. Moreover, we are in the situation to translate this statement in the algebraic category, such that the orbit is a Zariski-open sub superscheme of V.

Furthermore, we need to handle the quotient space of an algebraic supergroup by an isotropy supergroup. The nontrival question when such a quotient of supergroups exists is answered by Masuoka and Zubkov in [29]. The authors answer this question in a very general manner, generalising ideas presented in [19] and [12], by regarding superschemes as functors from superalgebras to sets, which are sheaves on the opposite category of supercommutative superalgebras equipped with a Grothendieck topology. The existence question is answered by the representability of a functor by a superscheme. Considering this approach, we can prove that the orbit morphism is an isomorphism between the quotient scheme and the orbit (Theorem 3.44).

3.1 Group objects in a category

Now, we define a group object in a category $\mathcal{C}$ and a left action of such a group object in order to avoid repeating ourselves by defining Lie supergroups or algebraic supergroup. The reader may consult Ref.[34, p.76 ff.].

Definition 3.1. Let $\mathcal{C}$ be a category that admits finite products. In particular, $\mathcal{C}$ admits a terminal object, which we denote by P. For instance, if $\mathcal{C}$ is the category of locally ringed superspaces **lrss** a terminal object is $(*, \mathbb{C}_*) \in$ **lrss**. Let G be an object in $\mathcal{C}$ with morphisms $\mu : G \times G \to G$ (multiplication), $e : P \to G$ (unit) and $i : G \to G$ (inversion). If these morphisms make the following diagrams commute:

(associativity of multiplication)

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times id_G"] \arrow[d, "id_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

(two sided unit)

$$\begin{tikzcd}
P \times G \arrow[d, "e \times id_G"'] & G \arrow[l] \arrow[r] \arrow[d, "id_G"] & G \times P \arrow[d, "id_G \times e"] \\
G \times G \arrow[r, dashed, "\mu"] & G & G \times G \arrow[l, dashed, "\mu"']
\end{tikzcd}$$

(right inverse compatible to μ)

$$\begin{tikzcd}
G \times G \arrow[rr, "id_G \times i"] & & G \times G \arrow[d, "\mu"] \\
G \arrow[u, "{(id_G, id_G)}"] \arrow[r] & P \arrow[r, "e"] & G,
\end{tikzcd}$$

(left inverse compatible to μ)

$$\begin{tikzcd}
G \times G \arrow[rr, "i \times id_G"] & & G \times G \arrow[d, "\mu"] \\
G \arrow[u, "{(id_G, id_G)}"] \arrow[r] & P \arrow[r, "e"] & G,
\end{tikzcd}$$

then we call G a group object in $\mathcal{C}$.

Remark 3.2. For a group object G the functor h_G becomes group valued by the Yoneda Lemma. Making h_G group valued is the same as giving an object G a group object structure.

Definition 3.3. Let G be a group object, X an object of $\mathcal{C}$ and a morphism $a : G \times X \to X$. If the following diagrams commute:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mu \times id_X"] \arrow[d, "id_G \times a"'] & G \times X \arrow[d, "a"] \\
G \times X \arrow[r, "a"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
P \times X \arrow[rr, "e \times id_X"] \arrow[dr, "\mathrm{pr}_X"'] & & G \times X \arrow[dl, "a"] \\
& X. &
\end{tikzcd}$$

then a is called a left action of the group object G on an object X. Note that $\mathrm{pr}_X : P \times X \to X$ is the projection on to X.

The action morphism $a : G \times X \to X$ induces a natural transformation h_a, such that for all objects $T \in \mathrm{Ob}(\mathcal{C})$ the transformation satisfies the following relations:

$$1 \cdot x = x \quad \text{and} \quad (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$$

for all $g_1, g_2 \in h_G(T)$ and $x \in h_X(T)$, where 1 is the unit in $h_G(T)$. Here we set for example $h_a(T)(g_1, x) = g_1 \cdot x$ and $h_\mu(T)(g_1, g_2) = g_1 g_2$.

Definition 3.4. If $p \in_T X_0$, we define the orbit morphism a_p by the natural transformation given by

$$h_{a_p} : h_G \to h_X,\ g \mapsto g \cdot p.$$

3.2 Lie supergroups

Since we will be dealing with complex analytic supermanifolds, we will refer to them simply as "supermanifolds". Partially following [8, p.149ff], we show that the isotropy supergroup is a closed sub Lie supergroup and that the quotient superspace is a supermanifold. However, the results of this section carry over to the case of *cs*-manifolds.

Definition 3.5. A group object in the category of supermanifolds is a complex Lie supergroup.

Remark 3.6. One can associate to each complex Lie supergroup G a Lie group G_{red}. This Lie group is given by the underlying manifold of G and the canonical morphisms μ_0, e_0, i_0.

An example of a Lie supergroup is the general linear supergroup $\mathrm{GL}_{p|q}$. It illustrates the power of the approach via S-points. Using it, we can represent this Lie supergroup as a matrix group with entries in a superalgebra of global sections. This example is also important later on.

Example 3.7. Let the superspace $M_{p|q} = \mathbb{C}^{p^2+q^2|2pq}$ be the linear supermanifold corresponding to the super vector space $M(p|q, \mathbb{C})$ of $(p+q)\times(p+q)$ matrices

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{D} \end{pmatrix} + \begin{pmatrix} 0 & \tilde{B} \\ \tilde{C} & 0 \end{pmatrix}$$

with entries of the block matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ in $\mathbb{C}$. The underlying topological space $(M_{p|q})_0$ of the supermanifold $M_{p|q}$ is the direct product of $p \times p$ matrices and $q \times q$ matrices $M_p \times M_q$. The structure sheaf of $M_{p|q}$ is given by the assignment

$$V \mapsto \mathcal{O}_{M_{p|q}(\mathbb{C})}(V) = \mathcal{H}_{M_p \times M_q}(V) \otimes \bigwedge(\mathbb{C}^{2pq})^*$$

for all open subsets V in $M_p \times M_q$. The open sub superspace $GL_{p|q} := (U, \mathcal{O}_{M_{p|q}}|_U)$ is associated to the open subset U of invertible matrices in $M_p \times M_q$. It has a Lie supergroup structure which will be defined now and will be referred to as the general linear supergroup. For brevity, we write $\mathcal{O}(S)$ and $\mathcal{O}(\mathbb{C}^{p^2+q^2|2pq})$ for the superalgebras of global sections $\mathcal{O}_S(S_0)$ and

$\mathcal{O}_{\mathbb{C}^{p^2+q^2|2pq}}(\mathbb{C}^{p^2+q^2})$. By Proposition 2.21, the S-points of the supermanifold $\mathbb{C}^{p^2+q^2|2pq}$ are given by

$$\begin{aligned}\mathbb{C}^{p^2+q^2|2pq}(S) &= \mathrm{Hom}(S, \mathbb{C}^{p^2+q^2|2pq}) \\ &= \mathrm{Hom}(\mathcal{O}(\mathbb{C}^{p^2+q^2|2pq}), \mathcal{O}(S)).\end{aligned}$$

This means that a supermanifold morphism $f : S \to \mathbb{C}^{p^2+q^2|2pq}$ corresponds to a superalgebra morphism $f^\sharp : \mathcal{O}(\mathbb{C}^{p^2+q^2|2pq}) \to \mathcal{O}(S)$. By Ref. [27, Theorem 2.1.7] such a morphism is known once we know the image on the canonical chart,

$$\begin{aligned}M_{p|q}(S) &\cong \{(f^\sharp(u^1), ..., f^\sharp(u^{p^2+q^2}), f^\sharp(\xi^1), ..., f^\sharp(\xi^{2pq})) \mid \ f : S \to \mathbb{C}^{p^2+q^2|2pq}\} \\ &= M(p, q; \mathcal{O}(S))_{\bar{0}}.\end{aligned}$$

By $M(p|q, \mathcal{O}(S))_{\bar{0}}$ we mean the block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where the entries of A, D are in $\mathcal{O}(S)_{\bar{0}}$ and the entries of B, C are in $\mathcal{O}(S)_{\bar{1}}$. The group of automorphisms of

$$\mathbb{C}^{p|q}(S) = (\mathcal{O}(S) \otimes \mathbb{C}^{p|q})_{\bar{0}} = \mathcal{O}(S)^p_{\bar{0}} \oplus \mathcal{O}(S)^q_{\bar{1}}$$

is given by $GL_{p|q}(S)$, the functor of points of the general linear supergroup $GL_{p|q}$. Thereby

$$GL_{p|q}(S) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; A, D \text{ invertible} \right\} \subseteq M(p, q; \mathcal{O}(S))_{\bar{0}}$$

is an element in the category of set theoretical groups.

Now, we want to define the Lie superalgebra $\mathfrak{g}$ associated to a Lie supergroup G.

Definition 3.8. A *Lie superalgebra* is a super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with multiplication $[.,.]$ satisfying the following two axioms: For all homogeneous $X, Y, Z \in \mathfrak{g}$, we have

$$\begin{aligned}[X, Y] &= -(-1)^{|X|\cdot|Y|}[Y, X] \\ [X, [Y, Z]] &= [[X, Y], Z] + (-1)^{|X|\cdot|Y|}[Y, [X, Z]].\end{aligned}$$

The first axiom is called skew supersymmetry and the second axiom is called Jacobi superidentity.

Definition 3.9. Let M be a complex analytic supermanifold. The sheaf $\underline{\mathrm{Der}}(\mathcal{O}_M)$ of $\mathbb{C}$-linear graded derivations is defined for any open subspace U of M_0 by $\underline{\mathrm{Der}}(\mathcal{O}_M)(U) =$

$$\{D \in \underline{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{O}_M(U), \mathcal{O}_M(U)) \mid D(gf) = D(g)f + (-1)^{|D||g|} g D(f)\}.$$

A *super vector field* V on a complex analytic supermanifold M is an $\mathbb{C}$-linear gradded derivation of $\mathcal{O}_M$. The set of super vector fields of M is denoted by Vec_M.

Now, we can define the Lie superalgebra $\mathfrak{g}$ of G to be the left invariant super vector fields. A super vector field $X \in \mathrm{Vec}_G$ is said to be *left invariant* if

$$(\mathbb{1} \otimes X) \circ \mu^\sharp = \mu^\sharp \circ X.$$

Here, $\mathbb{1} \otimes X$ is the unique vector field on $G \times G$ such that

$$(\mathbb{1} \otimes X)(p_1^\sharp(f_1)p_2^\sharp(f_2)) = p_1^\sharp(f_1)p_2^\sharp(X(f_2))$$

for any local sections f_1 and f_2 of $\mathcal{O}_G$. We define $\mathfrak{g}$ to be the set of left invariant super vector fields, i.e.

$$\mathfrak{g} = \{X \in \mathrm{Vec}_G \mid (\mathbb{1} \otimes X)\mu^\sharp = \mu^\sharp X\}.$$

This is the Lie superalgebra associated to G, where for homogeneous elements $X, Y \in \mathfrak{g}$ the bracket is given by $[X, Y] = XY - (-1)^{|X|\cdot|Y|}YX$.

For example, the Lie superalgebra of $\mathrm{GL}_{p|q}$ is given by the super vector space $\mathbb{C}^{p^2+q^2|2pq}$ via the functor $A \to M(p|q, A)$ and the fact that $M(p|q, A) \cong \mathbb{C}^{p^2+q^2|2pq}(A)$.

3.3 Isotropy supergroups

An important step to define a prehomogeneous super vector space is to understand the orbit of a Lie supergroup through a supervector $v \in V_0$ in a super vector space V, considered as a linear supermanifold. To that end, we have to define the isotropy supergroup G_p of a point p, which we will do in the generality of supermanifolds. Moreover, we will see that the isotropy supergroup is a closed sub Lie supergroup.

We recall the definition of an action morphism for a Lie supergroup. Let G be a Lie supergroup, M a supermanifold and

$$a : G \times M \to M$$

a left action of G on M. Moreover, let $p \in M_0$, which can be considered as a map $p : \mathbb{C}^{0|0} \to M$. Recall that the function $\tilde{f}$ was defined by $\pi^\sharp(f)$. For $p \in M_0$ the orbit morphism is given by

$$a_p : G \to M, \quad a_p := a \circ (id_G, \tilde{p}),$$

where $\tilde{p}$ denotes the composition of the map $p : \mathbb{C}^{0|0} \to M$ given by the evaluation at the point p through $p^\sharp(f) := \tilde{f}(p)$ with the unique map $G \to \mathbb{C}^{0|0}$. The orbit morphism a_p satisfies

$$\begin{tikzcd} G \cong G \times \mathbb{C}^{0|0} \arrow[rr, "id_G \times p"] \arrow[dr, "a_p"'] & & G \times M \arrow[dl, "a"] \\ & M. & \end{tikzcd}$$

We also need the isotropy supergroup.

Definition 3.10. Let G be a Lie supergroup and $a : G \times M \to M$ an action of G on the supermanifold M. The isotropy supergroup at $p \in M_0$ is the supermanifold G_p equalizing the diagram

$$G_p \xrightarrow{j_{G_p}} G \underset{\tilde{p}}{\overset{a_p}{\rightrightarrows}} M,$$

where j_{G_p} is the canonical embbeding. The isotropy functor at p is defined by

$$G_p(S) := \{g \in_S G | g \cdot p_S = p_S\}$$

for all supermanifolds S. The set $G_p(S)$ defines a subgroup of $G(S)$. We will now show that the isotropy functor is representable by a sub supermanifold of G, which then is automatically a sub supergroup.

Theorem 3.11 ([13], Prop. 8.4.7)**.** *Let G be a Lie supergroup acting on a supermanifold M and $p \in M_0$. Then there exists a supermanifold G_p, which can be embedded as a closed sub supermanifold in G, equalizing the diagram*

$$G_p \xrightarrow{j_{G_p}} G \underset{\tilde{p}}{\overset{a_p}{\rightrightarrows}} M.$$

Further G_p is a sub Lie supergroup of G and the functor $G_p(S) = Hom(S, G_p)$ is represented by the sub supermanifold G_p of G.

Proof. We give a sketch of the proof. The orbit morphism $a_p : G \to M$ has constant rank ([13, Prop. 8.1.5]). Proposition 2.31 implies that there exists a closed embedded sub supermanifold G_p of G, such that the embedding j_{G_p} is closed and $j_{G_p}^\sharp$ is surjective. □

With the knowledge of the isotropy supergroup, we are in a position to define the quotient superspace.

3.4 Quotient supermanifolds

Definition 3.12. Let G be a Lie supergroup and H a closed sub Lie supergroup. The quotient superspace is defined by $G/H = (G_0/H_0, \mathcal{O}_{G/H})$. The quotient sheaf $\mathcal{O}_{G/H}$ is defined in the following way: For any open set $V \subset G_0/H_0$ we set $\mathcal{O}_{G/H}(V) :=$

$$= \{f \in \mathcal{O}_G(\pi_0^{-1}(V)) \mid (r_h^\sharp(f) = f) \text{ and } (Z(f) = 0) \ \forall (h \in H_0,\ Z \in \mathfrak{h})\}.$$

Here, the morphism $\pi_0 : G_0 \to G_0/H_0$ is the canonical quotient map and $r_h : G \to G$ is the right translation by h.

Theorem 3.13. *The superspace $(G_0/H_0, \mathcal{O}_{G/H})$ is a supermanifold.*

Proof. A proof can be found in [1, Theorem 5.3], where the authors use Godement's Theorem on quotients. For another proof, which uses Frobenius Theorem, the reader may consult, for instance, Ref. [8, Theorem 9.3.4]. □

Proposition 3.14. *Let $f : G \to X$ be an H-invariant morphism, i.e. $f(gh) = f(g) \quad \forall\, T,\ g \in_T G,\ h \in_T H$. Then there is a unique morphism*

$$\tilde{f} : G/H \to X,$$

so that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ G/H. \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

Now, we intend to understand the orbit through a point by the action of a Lie supergroup in more detail.

Proposition 3.15 ([13], Prop. 8.1.5)**.** *Let $a : G \times M \to M$ be the action of the Lie supergroup G on a supermanifold M through a point $p \in M_0$. The orbit morphism $a_p : G \to M$ has constant rank.*

Note that by the universal property of the quotient we get the following commuting diagram

$$\begin{tikzcd} G \arrow[r, "a_p"] \arrow[d, "\pi"'] & M \\ G/G_p \arrow[ur, dashed, "\hat{a}_p"'] & \end{tikzcd}$$

and we get a morphism $\hat{a} : G/G_p \to M$, which also has constant rank.

Proposition 3.16. *The morphism $\hat{a}_p : G/G_p \to M$ is an injective immersion.*

Proof. If $g_1, g_2 \in G_0$ are such that $g_1 \cdot p = g_2 \cdot p$, then $g_2^{-1} g_1 \in (G_p)_0$, hence $(\hat{a}_p)_0$ is injective. Moreover, the orbit map a_p is a morphism of constant rank and so $\hat{a}_p : G/G_p \to M$. The morphism $\hat{a}_p$ is an injective immersion, because $\ker T_e a_p = T_e G_p$. $\square$

Definition 3.17. We call the pair $(G/G_p, \hat{a}_p)$ the orbit of G through p. Furthermore, if G_0p is open we denote by $O_p^{an} := (G_0p, \mathcal{O}_{M|G_0p})$ the sub superspace, given by the underlying orbit $G_0p := a_0(G_0, p) \subseteq M_0$.

Remark 3.18. In general, the morphism $\hat{a}_p$ is not an embedding, as one can already see in the classical example, where one takes the irrational action of $\mathbb{R}$ on the torus.

Now, we can prove our next proposition.

Proposition 3.19. *Let G be a Lie supergroup acting on a supermanifold M. If for $p \in M_0$ the differential of the orbit map a_p has maximal rank, then $\hat{a}_p$ is an open embedding and $G/G_p \cong O_p^{an}$ is an open sub supermanifold.*

Proof. We already know by Proposition 3.16 that $\hat{a}_p : G/G_p \to M$ is an injective immersion. It remains to show that $(\hat{a}_p)_0$ is a homeomorphism onto its image. But this is the case by Theorem 2.26 and the maximal rank assumption. □

Corollary 3.20. *If for two open orbits the intersection of their underlying spaces is not the empty set, then the orbits are equal.*

Proof. Let $v, w \in M_0$ such that $(O_v^{an})_0 \cap (O_w^{an})_0 \neq \emptyset$. There are $g_1, h_1 \in G_0$ satisfying

$$\begin{aligned} g_1 v &= h_1 w \\ \Leftrightarrow (h_1^{-1} g_1) v &= w. \end{aligned}$$

Therefore $w \in (O_v^{an})_0$ and the underlying topological spaces are equal $(O_v^{an})_0 = (O_w^{an})_0$. The result follows from the fact that two sub supermanifolds of the same superdimension as M and with the same underlying topological space are equal. □

Now, we can prove the main theorem of this section, which is a crucial requirement for prehomogeneous super vector spaces.

Theorem 3.21. *Let G be a complex connected Lie supergroup, acting linearly on a super vector space V. If G has an open orbit in V, then this orbit is unique, connected, and dense.*

Proof. Let $x_1, ..., x_p, \xi_1, ..., \xi_q$ be a homogeneous basis of $\mathfrak{g}$ with $\dim \mathfrak{g} = p|q$ and $v_1^*, ..., v_m^*, \eta_1^*, ..., \eta_n^*$ a homogeneous basis of V^* with $\dim V = m|n$. The action is denoted by $a : G \times V \to V$. For each $v \in V_0$ we consider the even linear map $da_v : \mathfrak{g} \to V$ given by $X \mapsto da_v(X)$. This gives us a $\dim V \times \dim \mathfrak{g}$ matrix A_v in our selected basis. The entries of A_v are linear functions of v with values in $\mathbb{C}$ and the matrix is given by

$$A_v := \begin{pmatrix} v_i^*(da_v(x_j)) & 0 \\ 0 & \eta_k^*(da_v(\xi_l)) \end{pmatrix}.$$

Since we have assumed that G has an open orbit in V, we know for some $v = v_0$ that $da_{v_0}(\mathfrak{g}) = V$. Thus the rank of A_{v_0} is $m+n$, and so there is some non-zero $(m+n) \times (m+n)$ minor of A_{v_0}. Let P be the vector-valued function $P(v) := (p_I(v), \pi_J(v))$ with

$$p_I(v) = \det(v_i^*(da_v(x_{i_j})))_{i,j=1}^m \quad I = 1 \leq i_1 < ... < i_m \leq p,$$

$$\pi_J(v) = \det(\eta_k^*(da_v(\xi_{j_l})))_{k,l=1}^n \quad J = 1 \leq j_1 < ... < j_n \leq q.$$

on V whose value at v is the tuple of all $(m+n) \times (m+n)$ minors of A_v. This function P is a vector-valued polynomial function on V whose value at v_0 is non-zero. Let $\Omega_0 = \{v \in V_0 \mid O_v^{an} \text{ an open orbit}\}$ be the set of v for which $P(v) \neq 0$. Because P is a holomorphic polynomial function not identically zero, Ω_0 is connected, and dense. Proposition 3.15 tells us that since $da_v(\mathfrak{g}) = V$ we have $da_w(\mathfrak{g}) = V$ for $w = a(g, v)$ and $g \in G_0$. Hence, $a(g)$ maps Ω_0 onto itself. Thus Ω_0 is the union of disjoint orbits under $a(G_0)$. Since Ω_0 is connected, there is only one orbit in the open sub supermanifold $\Omega = (\Omega_0, \mathcal{O}_{V|\Omega_0})$, which is already Ω. By Theorem 2.11 we conclude that $Gv(S) \subseteq V(S)$ is dense. $\square$

Remark 3.22. For the classical case, compare [26, Theorem 10.1].

Proposition 3.23. *Let a complex connected Lie supergroup G act linearly on a super vector V. The open orbit O_v^{an} of G at $v \in V$ can be regarded as an open sub superscheme, denoted by O_v^{alg}.*

Proof. From Theorem 3.21, we know that the open orbit of a complex connected Lie supergroup in a super vector V is unique, dense and connected. Also, we saw in the proof of the Theorem 3.21, that the complement of the set $(O_v^{an})_0$ is the set of common zeros of finitely many polynomials, which were expressed by the vector-valued polynomial function P. Let $i \in \{1, \ldots, q\}$ be an index running over the components, such that P_i is a polynomial on V_0. Then the underlying open set is $(O_v^{alg})_0 := \{\mathfrak{p} \in \mathrm{Spec}(\mathbb{C}^m) \mid (P_1, \ldots, P_q) \not\subset \mathfrak{p}\}$ and the superalgebra $A = \{\frac{g}{p^n} \mid p \in (P_1, \ldots, P_q),\ g \in \mathbb{C}[x_1, \ldots, x_m; \xi_1, \ldots, \xi_n]\}$ defines an open sub superscheme of $\mathbb{C}^{m|n}$. $\square$

Remark 3.24. In the following we will refer O_v^{an} or O_v^{alg} simply as O_v.

3.5 Superschemes as functors

In this section we are preparing the proof of Theorem 4.10, which tells us about the relation between supercharacters and relative superinvariants of a prehomogeneous super vector spaces. Therefore, we need to handle the quotient space of an algebraic supergroup by an isotropy supergroup. The authors of [29] answer the question of the existence of such quotients in a very general manner by regarding superschemes as functors, which are sheaves on the opposite category of supercommutative superalgebras equipped with a Grothendieck topology, thereby following the work of Demazure-Gabriel in the ungraded case. They show that the quotient sheaf of algebraic supergroups is a Noetherian superscheme.

At first we like to motivate this approach. The functor $\underline{\mathrm{Spec}}$ is a covariant functor from the opposite category of supercommutative superalgebras $(\mathbf{salg})^{op}$ to the category of affine superschemes **sschem**. This functor has a quasi-inverse D, sending an affine superscheme X to the superalgebra of global sections $A = \mathcal{O}_X(X_0)$, which is also often called the coordinate superalgebra $\mathbb{C}[X]$. Moreover, we have as a consequence $\mathrm{Hom}_{\mathbf{sschem}}(S, X) \cong \mathrm{Hom}_{\mathbf{salg}}(A, B)$, with $S = \underline{\mathrm{Spec}}(B)$. Because of the anti-equivalence of the categories **salg** and **sschem**, and the embedding property by the Yoneda Lemma, one can consider an affine superscheme X as a functor $h^A : B \to \mathrm{Hom}_{\mathbf{salg}}(A, B)$, which is represented by the superalgebra A with $\underline{\mathrm{Spec}}(A) = X$. In this way the category of affine superschemes over $\mathbb{C}$ is embedded as a full subcategory of the category of covariant functors from **salg** to **set**, denoted by $\mathbf{F}^{\mathbf{salg}}$. We also write X or $\underline{\mathrm{Spec}}(A)$ for the functor h^A.

Let G and H be supergroup superschemes, one can easily define the quotient by the universal property of the quotient in the functor category, where the functor given by $A \mapsto G(A)/H(A)$ is the quotient. However, this functor is usually not even a sheaf in the Zariski topology, and so cannot be representable by an affine superscheme. Moreover, the inclusion of sheaves in the Zariski topology to the category of presheaves (i.e. of functors) does not have a left adjoint (i.e. a sheafification functor). Such a left adjoint functor to the inclusion functor exists, if one consider the category of functors, which are sheaves on the $fppf$-topology and will be called faisceaux below. At first we have to give some definitions.

Here, by following [19, p. 73] and [29, p.7f] we outline the sheafification of a functor in the $fppf$-topology.

Definition 3.25. We call an object X in the functor category $\mathbf{F}^{\mathbf{salg}}$ a functor. A functor $X = h^A$, which is represented by $A \in \mathbf{salg}$, is called an affine superscheme.

Remark 3.26. In general, one can work with superalgebras over a commutative ring instead of our restriction to $\mathbb{C}$-superalgebras (cf.[19, p.3]).

Definition 3.27. Let I be a superideal of $A \in \mathbf{salg}$. An *principal open subfunctor* $D(I)$ of $\underline{\mathrm{Spec}}(A)$ is defined as follows. For any $B \in \mathbf{salg}$, we set

$$\begin{aligned} D(I)(B) :=& \{x \in \underline{\mathrm{Spec}}(A)(B) \mid x(I)B = B\} \\ =& \{x \in \mathrm{Hom}_{\mathbf{salg}}(A, B) \mid x(I)B = B\}. \end{aligned}$$

Let X be a functor from **salg** to **set**. A *subfunctor* $Y \subseteq X$ is said to be *open* iff for any morphism $f : \underline{\mathrm{Spec}}(A) \to X$ in the category of functors from **salg** to **set** the preimage $f^{-1}(Y)$ is the union of principal open subfunctors of $\underline{\mathrm{Spec}}(A)$.

Example 3.28. A Zariski-open sub superscheme $\underline{\mathrm{Spec}}(A_f) = (U_f, \mathcal{O}_X|_{U_f})$ of an affine superscheme $(\mathrm{Spec}(A_{\bar{0}}), \mathcal{O}_X)$ where $U_f = \{\mathfrak{p} \in \mathrm{Spec}(A_{\bar{0}}) \mid (f) \not\subset \mathfrak{p}\}$, defined in Definition 2.12, can be considered as an example of such an open subfunctor.

A collection of open subfunctors $\{Y_i\}_{i\in I}$ of a functor X is called an *open covering* of X whenever $X(A) = \bigcup_{i\in I} Y_i(A)$ for any $A \in \mathbf{salg}$.

In order to define a sheaf on a category, it is not necessary to have a topological space in the conventional sense. The notion of a Grothendieck topology is enough, which we define in the following.

Definition 3.29. Let $\mathcal{C}$ be a category together with, for each object U of $\mathcal{C}$, a distinguished set of families of morphisms $\{U_i \to U\}_{i\in I}$ called the coverings of U with the following axioms:

1. For any U, the family $\{U \xrightarrow{\mathrm{id}} U\}$ consisting of a single morphism is a covering of U.
2. If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is any morphism, then the fiber products $\{U_i \times_U V\}_{i\in I}$ exist and the collection of projections $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.
3. If $\{U_i \to U\}_{i\in I}$ is a covering and for each index i, there is a covering $\{V_{ij} \to U_i\}_{i,j\in I}$, then the collection $\{V_{ij} \to U_i \to U\}_{i,j\in I}$ is a covering of U.

The system of coverings is then called a Grothendieck topology and a category $\mathcal{C}$ with a Grothendieck topology is called a *site*.

For example, let X be a topological space, and let $\mathcal{C}$ be the category whose objects are the open subsets of X and whose morphisms are the inclusion maps. Then the families $\{U_i \to U\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is an open covering of U. It is a Grothendieck topology on $\mathcal{C}$ and for open subsets U_1 and U_2 of V is $U_1 \times_V U_2 = U_1 \cap U_2$.

Now, we define a Grothendieck topology $\mathcal{T}_{loc}$ in $(\mathbf{salg})^{op}$ as follows: A covering in $\mathcal{T}_{loc}$ of $A \in \mathbf{salg}$ is defined to be a collection of finitely many morphisms $\{\underline{\mathrm{Spec}}(A_{f_i}) \to \underline{\mathrm{Spec}}(A)\}_{1\leq i\leq n}$, where $A \in \mathbf{salg}$ and $f_1, \dots, f_n \in A_{\bar{0}}$ such that $\sum_{1\leq i\leq n} A_{\bar{0}} f_i = A_{\bar{0}}$. Each $\underline{\mathrm{Spec}}(A_{f_i}) \to \underline{\mathrm{Spec}}(A)$ is an isomorphism onto $D(Af_i)$ and the open subfunctors $D(Af_i)$ form an open covering of $\underline{\mathrm{Spec}}(A)$. By that, the $A_{f_1}, \dots, A_{f_n}$ form an open covering of A. These coverings satisfy the conditions given in Definition 3.29, so that $\mathcal{T}_{loc}$ is indeed a Grothendieck topology.

Definition 3.30. Let $\mathcal{T}$ be a Grothendieck topology on **salg** a functor X is called a sheaf iff for any superalgebra A and any open covering $\{A_i\}_{i\in I}$ the diagram

$$X(A) \to \prod_{i\in I} X(A_i) \rightrightarrows \prod_{i,j\in I} X(A_i \otimes_A A_j)$$

is exact.

Notice, any affine superscheme is a sheaf on $\mathcal{T}_{\mathrm{loc}}$ (cf. [19], Part I, 1.8(4)).

Definition 3.31. A sheaf X on $\mathcal{T}_{\mathrm{loc}}$ is called a superscheme, if X has an open covering $\{Y_i\}_{i\in I}$ with $Y_i \simeq \underline{\mathrm{Spec}}(A_i)$ and $A_i \in \mathbf{salg}$. The full subcategory of all superschemes, defined as a sheaf on $\mathcal{T}_{\mathrm{loc}}$ with an open covering, is denoted by $\mathbf{SF}^{\mathbf{salg}}$. A superscheme X is said to be Noetherian, if it has an open covering $\{Y_i\}_{i\in I}$ with $Y_i \simeq \underline{\mathrm{Spec}}(A_i)$, such that I is finite, and each A_i is Noetherian. An affine superscheme $\underline{\mathrm{Spec}}(A)$ is Noetherian iff A is Noetherian.

Remark 3.32. Any superscheme in the sense defined above is represented by a locally ringed superspace covered by affine superschemes (Ref. [29, Theorem 5.14]).

In the following we work in the finer Grothendieck topology $\mathcal{T}_{fppf}$, where $fppf$ stands for the french term with the meaning faithfully flat, finitely presented.

Let us define a Grothendieck topology $\mathcal{T}_{fppf}$ in $(\mathbf{salg})^{op}$ as follows: A covering in $\mathcal{T}_{fppf}$ of $A \in \mathbf{salg}$ is defined to be a collection of finitely many morphisms $\{\underline{\mathrm{Spec}}(A_i) \to \underline{\mathrm{Spec}}(A)\}_{1\leq i \leq n}$, where each A_i is a finitely presented A-superalgebra and $B = A_1 \times \cdots \times A_n$ is a faithfully flat A-module. An A-superalgebra is finitely presented if it is the quotient of a polynomial ring in a finite number of even and odd variables by a finitely generated ideal. Note, that for $f \in A_{\bar{0}}$ the A-superalgebra A_f is finitely presented, as $A_f \cong A[T]/(Tf - 1)$, and a flat A-module. If $f_1, \dots, f_n \in A_{\bar{0}}$ satisfy $\sum_{i=1}^{n} Af_i = A$, then $A' := \prod_{i=1}^{n} A_{f_i}$ form an $fppf$-open covering of A.

Definition 3.33. A sheaf Y on $\mathcal{T}_{fppf}$ is called a *faisceau.*

Remark 3.34. It can be seen that $\mathcal{T}_{fppf}$ is finer than $\mathcal{T}_{loc}$. Hence, a faisceau is also a sheaf on $\mathcal{T}_{\mathrm{loc}}$.

Proposition 3.35 ([29], Proposition 3.6)**.** *For any $X \in \mathbf{F}^{\mathbf{salg}}$ there is a faisceau $\tilde{X}$, called the sheafification of X, and a natural transformation $j : X \to \tilde{X}$ such that for any faisceau Y the canonical map* $\mathrm{Mor}(\tilde{X}, Y) \to \mathrm{Mor}(X, Y)$ *induced by j is a bijection.*

Definition 3.36 ([29], Definition 3.7)**.** For $A, A' \in \mathbf{salg}$ and A' an $fppf$-covering of A. A functor X is called *suitable* if it commutes with finite direct products of superalgebras and the induced map $X(A) \to X(A')$ is injective.

Remark 3.37 ([29], Remark 3.8)**.** If X is suitable, then for any $A \in \mathbf{salg}$

$$\tilde{X}(A) = \varinjlim X(B, A),$$

where $X(B, A) = \mathrm{Ker}(X(B) \rightrightarrows X(B \otimes_A B))$ and B runs over all $fppf$-coverings of A. Besides, $j : X \to \tilde{X}$ is an injection.

In order to prove Theorem 3.44, we need to prove the next proposition. Here, we are generalising ideas for affine schemes, which can be found in [19, 5.4] and [12], to the supercategory of affine superschemes.

Proposition 3.38. *Let X, Y : $\mathbf{salg} \to \mathbf{set}$ be functors, where Y is a faisceau and $f : X \to Y$ a natural transformation and X is suitable. The canonical map $j : X \to \tilde{X}$ embeds the functor X in $\tilde{X}$, the sheafification of X. It holds that the map $j(A)$ is injective for all $A \in$ $\mathbf{salg}$ and the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"'] & Y \\ \tilde{X} \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

is commutative. Moreover, there exists a unique natural transformation $\tilde{f}$: $\tilde{X} \to Y$ *and , if* $f(A)$ *is injective for all* $A \in$ **salg**, *then* $\tilde{f}(A)$ *is also injective for all* $A \in$ **salg**.

Proof. Let Y be a faisceau and B an $fppf$-covering of the superalgebra A. For any morphism $f : X \to Y$ any $f(B)x$ with $x \in X(B, A)$, where $X(B, A) := \{x \in X(B) \mid X(i_1)(x) = X(i_2)(x)\}$ with $i_1(b) = b \otimes 1$ and $i_2(b) = 1 \otimes b$, has to belong to $Y(A) \subseteq Y(B)$. Hence, one can define $\tilde{f} : \tilde{X} \to Y$ through $\tilde{f}(A)x = f(B)x \in Y(A)$. It holds that $\tilde{f}_{|X} = f$. Now, we prove the injectivity. The faisceau $\tilde{X}$ of X is defined by the direct limit $\tilde{X}(A) := \varinjlim X(B, A)$, where B runs over all $fppf$-coverings of A. Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}(A)$ with $\tilde{f}(\tilde{x}_1) = \tilde{f}(\tilde{x}_2)$, then there exists a $fppf$-covering B of A, such that $\tilde{x}_1, \tilde{x}_2 \in X(B, A) := \{x \in X(B) \mid X(i_1)(x) = X(i_2)(x)\}$ and $f(B)(\tilde{x}_1) = \tilde{f}(A)(\tilde{x}_1) = \tilde{f}(A)(\tilde{x}_2) = f(B)(\tilde{x}_2)$ and by injectivity of $f(B)$ it follows that $\tilde{x}_1 = \tilde{x}_2$. $\square$

3.6 Algebraic supergroups

We now give the definition of an algebraic supergroup.

Definition 3.39. A group object in the category of affine superschemes is called an algebraic supergroup.

From the context it should be clear whether G is considered as a functor or as a superscheme, and so we drop the notation h_G.

Recall that a morphism $f : X \to Y$ of affine superschemes is called an epimorphism if for any two morphisms $g_1, g_2 : Y \to Z$ of affine superschemes, the equality $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$. By the anti-equivalence of **salg** and **sschem**, $f : \underline{\mathrm{Spec}}(B) \to \underline{\mathrm{Spec}}(A)$ is an epimorphism if and only if $f^\sharp : A \to B$ is a monomorphism. Here, a monomorphism in **salg** algebras is an even superalgebra morphism $\varphi : A \to B$ such that for any two even superalgebra morphisms $\psi_1, \psi_2 : C \to A$, the equality $\varphi \circ \psi_1 = \varphi \circ \psi_2$ implies that $\psi_1 = \psi_2$. Certainly, any injective algebra morphism is a monomorphism. Conversely, the elements $a \in A$ are in bijection with the even superalgebra morphisms $\psi : \mathbb{C}[x, \xi] \to A$, where the correspondence is via $\psi(x) = a_{\bar{0}}$ and $\psi(\xi) = a_{\bar{1}}$. Thus, it follows easily that monomorphisms in **salg** are injective. If $\varphi : A \to B$ is an injective even superalgebra morphism and $f = (f_0, f^\sharp) : X = \underline{\mathrm{Spec}}(B) \to Y = \underline{\mathrm{Spec}}(A)$ is the corresponding epimorphism in **sschem**, then $f^\sharp$ is an injective sheaf morphism. Indeed, if $\mathfrak{p} \subseteq B_{\bar{0}}$ is a prime ideal, then $f^\sharp_{\mathfrak{p}} = \varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is injective since localization is exact. However, although one can show that f_0 has dense image, in general, f_0 is not surjective. In spite of the above remark, we say that the action a is transitive if a is an epimorphism.

The stabilizer supergroup functor at the point $v \in X_0$ with respect to the action a is defined by $G_v(A) := \{g \in G(A) \mid g \cdot v_A = v_A\}$, where v_A denoted the morphism $\{A \to v\}$.

Now, we are in a situation to state the following proposition.

Proposition 3.40. *Let G be an algebraic supergroup acting on an affine superscheme X and $v \in X_0$. Then G_v is a closed algebraic supergroup.*

Proof. A proof can be found in [13, Theorem 11.8.3]. $\square$

The general linear supergroup $\mathrm{GL}^{alg}(p|q, \mathbb{C})$ as an algebraic supergroup is given by affine superscheme

$$\underline{\mathrm{Spec}}\,(\mathbb{C}[x_{ij}, y_{kl}, t_1, t_2, \xi_{mn}, \eta_{rs}]/(t_1 \det(x_{ij}) - 1, t_2 \det(y_{kl}) - 1))$$

where $i, j, m, s \in \{1, \ldots, p\}$ and $k, l, n, r \in \{1, \ldots, q\}$.

Definition 3.41. An algebraic sub supergroup G of $\mathrm{GL}^{alg}(p|q, \mathbb{C})$, such that G is a Zariski-closed sub superscheme of $\mathrm{GL}^{alg}(p|q, \mathbb{C})$ is called a *linear algebraic supergroup.*

3.7 Quotient superschemes

We need an isomorphism from the quotient of the supergroup G by the isotropy supergroup G_v to the orbit scheme by the action of G for the proof of Theorem 4.10. The nontrival question when such quotients of supergroups exist is answered by Masuoka and Zubkov in [29]. In this context a superscheme, as is common in algebraic geometry, is considered as a functor from the category of superalgebras to the category of sets. The existence question is answered by the representability of a functor by a superscheme.

Recall that an algebraic supergroup is considered here with the structure of an affine superscheme given by a finitely generated superalgebra. Masuoka and Zubkov proved the following theorem.

Theorem 3.42 ([29],Theorem 0.1)**.** *Let G be an algebraic supergroup, and let H be a closed sub supergroup of G. Then the faisceau $\widetilde{G/H}$ is a Noetherian superscheme.*

They show that for an algebraic supergroup G and a closed sub supergroup H the functor $\widetilde{G/H}$, which is obtained by the sheafification of the naive quotient, is representable by a superscheme, also denoted by $\widetilde{G/H}$. The naive quotient functor is defined as the functor $A \mapsto G/H(A) := G(A)/H(A)$ from the category of superalgebras to category of sets. This superscheme $\widetilde{G/H}$ fulfills the universal property of a quotient.

In the following, we show that under favourable circumstance, such as are relevant in this thesis, the orbit map is an isomorphism of superschemes, which maps from the quotient supergroup to the orbit. We need the following definition, which is given in [20].

Definition 3.43. Let $f : X \to Y$ be a morphism of set-valued functors on **salg**. Then f is called *formally smooth* if for any supercommutative superalgebra A and any graded ideal $I \subseteq A$ of square $I^2 = 0$, the map

$$X(A) \to \Big\{(y, x) \in Y(A) \times X(A/I) \;\Big|\; f(A/I)(x) = y_{A/I}\Big\},$$

defined by

$$x \mapsto (f(A)(x), x_{A/I})$$

is surjective. If X and Y are superschemes and f is, in addition, locally of finite presentation, then f is called *smooth.* A $\mathbb{C}$-superscheme X is called *smooth*, if so is the structural morphism $X \to \operatorname{Spec}\mathbb{C}$.

Theorem 3.44. *Let G be a smooth algebraic supergroup acting on V, where V is a super vector space, and let G_v be the isotropy sub supergroup at some $v \in V_{\bar{0}}$ such that $\dim V = \dim G - \dim G_v$. Then the quotient $\widetilde{G/G_v}$ exists as a superscheme and $\tilde{a}_v : \widetilde{G/G_v} \to V$ induces an isomorphism of superschemes onto the open subspace corresponding to the orbit of G_0 in $V_{\bar{0}}$ at v.*

The proof of Theorem 3.44 requires some preparatory work. We will give the proof below after these preparations.

3.7.1 Effective monomorphisms and epimorphisms

We begin our preparations with some general facts on effective monomorphisms and epimorphisms.

Definition 3.45. Let **C** be a category and $f : X \to Y$ a morphism in **C**.

We say that f has a *cokernel pair* if there are an object Z and morphisms $i_1, i_2 : Y \to Z$ such that $i_1 \circ f = i_2 \circ f$, satisfying the following universal property: For any object W and any two morphisms $g_1, g_2 : Y \to W$ such that $g_1 \circ f = g_2 \circ f$, there is a unique morphism $g : Z \to W$ such that $g \circ i_1 = g_1$ and $g \circ i_2 = g_2$. When Z exists, we write $Z = Y \coprod_X Y$.

Dually, we say that f has a *kernel pair* if there are an object Z and morphisms $p_1, p_2 : Z \to X$ such that $f \circ p_1 = f \circ p_2$, satisfying the following universal property: For any object W and any two morphisms $g_1, g_2 : W \to X$ such that $f \circ g_1 = f \circ g_2$, there is a unique morphism $g : W \to Z$ such that $p_1 \circ g = g_1$ and $p_2 \circ g = g_2$. When Z exists, we write $Z = X \times_Y X$.

We say that f is an *effective monomorphism* if a cokernel pair $Y \coprod_X Y$ exists and f is moreover the equalizer of $i_1, i_2 : Y \to Y \coprod_X Y$. The latter condition amounts to the following: For every object W, any morphism $g : W \to Y$ such that $i_1 \circ g = i_2 \circ g$ factors uniquely through f.

Dually, we say that f is an *effective epimorphism* if a kernel pair $X \times_X Y$ exists and f is moreover the coequalizer of $p_1, p_2 : X \times_Y X \to X$. The latter condition amounts to the following: For every object W, any morphism $g : X \to W$ such that $g \circ p_1 = g \circ p_2$ factors uniquely though f.

Lemma 3.46. *Let* $\mathbf{C}$ *be a category and* $f : X \to Y$ *be a morphism that is either both a monomorphism and an effective epimorphism or both an effective monomorphism and an epimorphism. Then* f *is an isomorphism.*

Proof. We will show the conclusion only under the first assumption, the other one follows by categorical duality.

Firstly, $f \circ p_1 = f \circ p_2$, because f is the coequalizer of $p_1, p_2 : X \times_Y X \to X$. But then $p_1 = p_2$, since f is a monomorphism. It follows that id_Y coequalises p_1, p_2, and hence factors uniquely through f to a morphism $g : Y \to X$. That is, $g \circ f = \mathrm{id}_Y$. In particular, $f \circ g \circ f = f = \mathrm{id}_X \circ f$, but then, since f is an epimorphism, it follows that $f \circ g = \mathrm{id}_X$. Hence, f is an isomorphism. $\square$

3.7.2 Monomorphisms and epimorphisms of faisceaux

The next step is to see that in the category of faisceaux, all monomorphisms and epimorphisms are effective. Moreover, we will need a characterization of epimorphisms in terms of faithful flatness.

Proposition 3.47. *All monomorphisms and all epimorphisms in the category of faisceaux are effective. In particular, any morphism of faisceaux,* *which is at the same time a monomorphism and an epimorphism, is an isomorphism.*

Proof. The conclusion concerning isomorphisms will follow by the token of Lemma 3.46 once we have established the first two assertions.

Our proof follows [12, III, §1, no. 2, 2.1-2] closely, where the same facts are established for the case of ordinary faisceaux (without grading). Let $f : X \to Y$ be a monomorphism of faisceaux. We define

$$Z(A) := Y(A) \coprod Y(A)/\sim,$$

where $\sim$ is the equivalence relation that identifies all elements of $f(A)(X(A))$.

By construction, there are canonical morphisms $i_1, i_2 : Y \to Z$ that are equalised by f. Obviously, i_1, i_2 define a cokernel pair in the category of set-valued functors on **salg**. Since left adjoints preserve colimits, $Y \coprod_X Y := Z$ is a cokernel pairs in the category of faisceaux.

We need to show that $f : X \to Y$ is the equalizer of i_1, i_2. Sheafification preserves projective limits by [12, III, § 1, no. 1, 1.12]. Hence, it is sufficient to show that f is the equalizer of i_1, i_2 in the category of set-valued functors on **salg**.

But this is quite straightforward: Assume that $g : W \to Y$ is a morphism of faisceaux such that $i_1 \circ g = i_2 \circ g$. Let $w \in W(A)$. Then $g(w) \in f(A)(X(A))$. Since $f(A)$ is injective (f being a monomorphism), there is a unique $x \in X(A)$ such that $f(A)(x) = w$. Define $\tilde{g}(A)(w) := x$. By the uniqueness, one checks that $\tilde{g}$ is a morphism of functors such that $f \circ \tilde{g} = g$, and moreover the unique such morphism. Thus, f is indeed an effective monomorphism.

Now, assume that f is an epimorphism. Sheafification preserves projective limits by [12, III, § 1, no. 1, 1.12], so the fibre product of f with itself in the category of faisceaux exists, and is the sheafification $p_1, p_2 : X \times_Y X \to X$ of

$$Z(A) = \Big\{ (x_1, x_2) \in X(A) \times X(A) \;\Big|\; f(A)(x_1) = f(A)(x_2) \Big\},$$

together with the obvious morphisms $q_1, q_2 : Z \to X$. We need to see that f is the coequalizer of p_1, p_2. Since left adjoints preserve colimits, the coequalizer of p_1, p_2 in the category of faisceaux is the sheafification of the coequalizer q_1, q_2 in the category of set-valued functors.

The coequalizer of these morphisms is the functor defined by

$$I(A) := f(A)(X(A)),$$

together with the obvious morphism $p : X \to I$. Let $i : I \to Y$ be given by the inclusion of subsets. Since Y is a faisceau, $\tilde{i} : \tilde{I} \to Y$ is a monomorphism of faisceaux, so it is effective by the above. Moreover, $\tilde{i} \circ \tilde{p} = \tilde{f} = f$, to $\tilde{i}$ is an epimorphism by the assumption on f. Hence, by Lemma 3.46, $\tilde{i}$ is an isomorphism, and $f : X \to Y$ is indeed the coequalizer of p_1, p_2 in the category of faisceaux. Thus, f is indeed an effective epimorphism. □

Lemma 3.48. *The point-functor of any superscheme is a faisceau. The functor from superschemes to faisceaux is fully faithful.*

Proof. The first claim is a straightforward generalisation of [12, III, § 1, no. 1, 1.3 Corollaire]. The second follows from the Yoneda Lemma and the fact that faisceaux are by definition a full subcategory of set-valued functors on **salg**. □

Lemma 3.49. *Let $f : X \to Y$ be a morphism of faisceaux. Then f is an epimorphism of faisceaux if and only if for any A and $y \in Y(A)$, there is an fppf A-algebra B and an $x \in X(B)$ such that $f(B)(x) = y_B$.*

Proof. The same as [12, III, § 1, no. 2, 2.8]. □

Definition 3.50. Let $f : X \to Y$ be a morphism of superschemes.

It is called *flat* if for any $x \in X_0$, $f^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,f(x)}$-module, and *faithfully flat*, if in addition, $f_0 : X_0 \to Y_0$ is surjective.

The morphism f is called *locally of finite presentation* if for any $x \in X_0$, there are open subspaces U of X and V of Y containing x and $f(x)$, respectively, such that $f|_U : U \to Y$ factors through $j_V : V \to X$ and the superalgebra morphism $f^\sharp : \mathcal{O}_Y(V) \to O_X(U)$ exhibits $\mathcal{O}_X(U)$ as a finitely generated $\mathcal{O}_Y(V)$-module.

We call f *of finite presentation* if in addition, f is quasi-compact—i.e., for any quasi-compact subset $K \subseteq Y_0$, $f_0^{-1}(K) \subseteq X_0$ is quasi-compact—and quasi-separated—i.e., the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact. Here, a topological space is called *quasi-compact* if any open cover admits a finite subcover.

Lemma 3.51. *Let $f : X \to Y$ be a morphism of superschemes. Then f is an epimorphism of faisceaux if and only if the following is true: For any $y \in Y_0$, there are an open subspace U of Y such that $y \in U_0$, a faithfully flat morphism $g : U' \to U$ of finite presentation, and a morphism $h : U' \to X$ such that $f \circ h = j_U \circ g$.*

Proof. The same as [12, III, § 1, no. 2, 2.9]. □

3.7.3 Quotient faisceaux and orbits

Finally, we apply the general considerations given above to supergroup actions.

Let G be an affine algebraic supergroup of finite type over $\mathbb{C}$ and H a closed subsupergroup. As we have noted above, the sheafification $\widetilde{G/H}$ of the naive quotient functor $A \mapsto G(A)/H(A)$ is a Noetherian superscheme.

Lemma 3.52. *The canonical quotient morphism $\pi : G \to X$ is an epimorphism of faisceaux.*

Proof. By [29, Corollary 9.10], π is affine and faithfully flat. Then π is separated (the diagonal morphism is a closed immersion), and hence quasi-separated. By Lemma 3.51, π is an epimorphism of faisceaux. □

Lemma 3.53. *Let $a : G \times X \to X$ be an action on a scheme X, and x be $\mathbb{C}$-rational point of X. Then the morphism $\tilde{a}_x : \widetilde{G/G_x} \to X$ induced by the orbit morphism $a_x : G \to X$ is a monomorphism of faisceaux.*

Proof. Let I be the functor defined by

$$I(A) := G(A)x = a_x(A)(G(A)).$$

Then a_x factors into morphisms $p : G \to I$ and $i : I \to X$ of functors. Certainly, i is a monomorphism of set-valued functors on **salg**. Hence, $\tilde{i}$ is a monomorphism of faisceaux.

Clearly, I is isomorphic to the naive quotient functor G/G_x. Thus, $\widetilde{G/G_x}$ is isomorphic to $\tilde{I}$, and under this isomorphism, $\tilde{i}$ corresponds to $\tilde{a}_x$. Hence, $\tilde{a}_x$ is a monomorphism of faisceaux. □

Finally, we are in a position to prove Theorem 3.44.

Proof of Theorem 3.44. By Proposition 3.47 and Lemma 3.53, it is sufficient to prove that $\tilde{a}_v$ is an epimorphism of faisceaux. For this, it is sufficient that a_v be an epimorphism of faisceaux. But by Lemma 3.51, it is to that end sufficient to show that a_v is faithfully flat. By [23, Proposition 2.11], $(a_v)_0$ is surjective. Hence, it is sufficient to show that a_v is flat.

Clearly, the local rings of V are regular superrings in the sense of [35, Definition 3.2]. To see that the local rings of G are regular, we remark that [35, Lemma 3.4.4] allows us to copy in the graded case the usual proof of the standard fact that an equicharacteristic formally smooth local ring is regular [30, Chapter 10, § 28, Lemma 1].

Thus, to see that a_v is flat, it is by [35, Proposition 3.6.2] sufficient to show that for every $g \in G_0$, the morphism $\mathcal{O}_{V,gv}/J_{V,gv} \to \mathcal{O}_{G,g}/J_{G,v}$ induced by $(a_v^\sharp)_g$ is flat. Here, J_V and J_G denote the ideal sheaves of $\mathcal{O}_V$ and $\mathcal{O}_G$, respectively, that are generated by the odd part of the structure sheaf. But by [23, Proposition 2.11], this morphism is an isomorphism, and hence flat. The assertion follows. $\square$

4 Prehomogenous super vector spaces

We are now in a position to define prehomogeneous super vector spaces, which will be a natural generalisation of prehomogeneous vector spaces. A prehomogeneous vector space is a finite-dimensional vector space V together with a subgroup G of $GL(V)$ such that G has an open dense orbit in V.

In this section, we develop the theory of prehomogeneous super vector spaces. We are able to generalise the notion of relative invariants and results about them, as well as the notion of regular prehomogeneous vector spaces, to prehomogenous super vector spaces. As the previous sections indicate and as we will see in this section, these generalisations are not obvious. We define relative superinvariants and show that two relative superinvariants corresponding to the same supercharacter are equivalent up to a constant. Moreover, we will see that a relative superinvariant is, in a generalised sense, homogeneous and that the set of relative superinvariants are equal to the set of supercharacters invariant under the isotropy supergroup. For example, we determine all supercharacters of $\mathrm{GL}(m|n, \mathbb{C})$.

Furthermore, we introduce the contragredient action and answer the question: When is the dual super vector space of a prehomogeneous super vector space also a prehomogeneous super vector space? And how are the sets of relative superinvariants of V and V^* connected? We will see how one can understand how the Berezinian measure of the prehomogeneous super vector space transforms under the action of an element of a Lie supergroup. It will turn out that we can express this transformation by a suitable power of a supercharacter.

We construct two examples of prehomogeneous super vector spaces, the super vector space of supersymmetric matrices with an action of $\mathrm{GL}(p|q, \mathbb{C})$ and the super vector space $\mathrm{M}(p|2q \times m|2n, \mathbb{C})$ with an action of $\mathrm{OSp}(p|2q, \mathbb{C}) \times \mathrm{GL}(m|2n, \mathbb{C})$. The prehomogeneous super vector space of supersymmetric matrices is also used to consider the general results for prehomogeneous super vector spaces in a sufficiently complicated example, where we know all relative superinvariants.

Furthermore, we calculate explicitly the transformation of the flat Berezinian measure of the super vector space of supersymmetric matrices under the action of $\mathrm{GL}(p|q, \mathbb{C})$. We end this section by generalising the partial differential equation of the Bernstein–Sato polynomial for prehomogeneous vector spaces by introducing an operator equation for prehomogeneous super vector spaces.

4.1 Definition of prehomogeneous super vector spaces

By Theorem 3.21, we know that an open orbit is connected, unique and dense in V, which says that V is almost a homogeneous super vector space. If the orbit O_v in $v \in V_0$ is open, we call v a *generic point*. By Proposition 3.23 we know that such an open orbit, which we considered in the analytic category, can also regard as an open sub superscheme of V. Furthermore, the next proposition tells us, that one can embed an algebraic supergroup into some $\mathrm{GL}_{m|n}$, analogous to the classical case.

Proposition 4.1 ([13], Theorem 11.7.9)**.** *Let G be an algebraic supergroup. Then*

$$G \subseteq \mathrm{GL}_{m|n}$$

is a closed affine sub superscheme of the general linear supergroup $\mathrm{GL}_{m|n}$ for suitable m and n.

This motivates the following definition.

Definition 4.2. Consider a triple (G, ρ, V), where G is a connected linear algebraic supergroup, V is a complex super vector space of finite dimension $\dim V = m|n$ and $\rho : G \to \mathrm{GL}(V)$ is a rational homomorphism of supergroups. Then (G, ρ, V) is a *prehomogeneous super vector space* if there exists an open orbit of G in V. We will refer to the triple (G, ρ, V) as V when G and ρ are clear from the context.

4.2 Relative superinvariants

Here we are going to introduce the notion of relative superinvariants, for which we need to define rational supercharacters.

Definition 4.3. A *rational supercharacter* for an algebraic supergroup G is a morphism of algebraic supergroups $\chi : G \to GL_1$. The set of rational supercharacters is denoted by

$$X(G) = \{\chi : G \to GL_1 \mid \chi \text{ is a rational supercharacter}\}.$$

Definition 4.4. Let V be a prehomogeneous super vector space with an open orbit O_v. A rational superfunction f on V is called a *relative superinvariant* if f is regular on O_v and there exists a rational supercharacter χ such that we have

$$f(\rho(g) \cdot x) = \chi(g) \cdot f(x) \quad \forall x \in_T O_v,\ \forall g \in_T G \tag{4.1}$$

on T-points.

Remark 4.5. We use $\rho(g) \cdot x$ and for brevity $g \cdot x$ as notation for $a(g, x)$ on T-points.

Let $X_1(G) = \{\chi \in X(G) \mid \text{there exists a relative superinvariant } f \text{ to } \chi\}$ be the set of rational supercharacters which correspond to relative superinvariants. The set $X_1(G)$ is a subgroup of $X(G)$. A rational supercharacter $\chi : G \to GL_1$ is a *constant supercharacter* if it is constant as a superfunction on G. If $\chi \in X_1(G)$ is a constant, then the corresponding relative superinvariant is called an *absolute superinvariant.*

In the next theorems we consider how far relative superinvariants are determined by their supercharacters. First, we need the following lemma.

Lemma 4.6. *The orbit map $a_v : G \to O_v$ is an epimorphism.*

Proof. By Theorem 3.44, the induced morphism $\tilde{a}_v : G/G_v \to O_v$ is an isomorphism and in particular, an epimorphism. By Lemma 3.52, the canonical projection $\pi : G \to G/G_v$ is an epimorphism of faisceaux and in particular of superschemes. Since $a_v = \tilde{a}_v \circ \pi$, the assertion follows. $\square$

Theorem 4.7. *If V is a prehomogeneous super vector space, then all absolute superinvariants are constant superfunctions and two relative superinvariants f and h corresponding to the same supercharacter χ are equal up to a constant factor.*

Proof. Let $f \in \mathbb{C}(V_{\bar{0}})[V_{\bar{1}}]$ be an absolute superinvariant. We show that $f \equiv f(v)$.

Let $f' := f(v)$ a constant superfunction (constant on O_v). We have for all $T, g \in_T G$ the following equations

$$\begin{aligned} f(a_v(g)) =& f(g \cdot v_T) = f(v_T) \\ =& f(v)_T = f'(a_v(g)). \end{aligned}$$

Note, that $a_v(g) = a(g, v_T)$ with $v_T = T \to * \xrightarrow{v} V$. Hence, $f \circ a_v = f' \circ a_v$ and the fact that a_v is an epimorphism by Lemma 4.6, it follows $f = f'$.

Let $f,\ h$ are relative superinvariants with the same supercharacter χ, then we have

$$\begin{aligned}f(a_v(g)) =&f(\rho(g)v_T) = (f(v_T)/h(v_T))h(\rho(g)v_T)\\ =&(f(v_T)/h(v_T))h(a_v(g)).\end{aligned}$$

Since $h(v_T)$ is invertible and a_v is an epimorphism, we get that f is equal to h on the orbit and by the first statement $f = c \cdot h$. $\square$

Definition 4.8. We define a map $\deg : \mathbb{C}(V_0)[V_1] \to \mathbb{Z}$, by

$$\deg(f_\mu \xi^\mu) = \deg_{cl}(p_\mu) - \deg_{cl}(q_\mu) + |\mu|,$$

where $\deg_{cl}$ stands for the standard degree of a polynomial and $f_\mu = \frac{p_\mu}{q_\mu}$. Moreover, we set $\deg(f) = \max_\mu(\deg(f_\mu \xi^\mu))$. We call a rational superfunction $\sum_\mu f_\mu \xi^\mu$ *homogeneous* if $\deg(f) = \deg(f_\mu \xi^\mu)$ and p_μ, q_μ are homogeneous polynomials for all $\mu \in \mathbb{Z}_2^n$.

For the next theorem, we introduce the Lie supergroup $\tilde{G} = G \times \mathrm{GL}_1$ with the extended homomorphism $\tilde{\rho} : G \times \mathrm{GL}_1 \to \mathrm{GL}(V)$ which is given by $\tilde{\rho}(\tilde{g}) = \rho(g) \cdot (t \cdot \mathrm{Id}_V)$ for $\tilde{g} = (g,t) \in_T G \times \mathrm{GL}_1$.

Theorem 4.9. *A relative superinvariant f is homogeneous of degree* $\deg f = \deg f_0$.

Proof. Let f be a relative superinvariant with supercharacter χ. Fix $\lambda \in \mathbb{C}^*$. Then the rational superfunction, defined by $h(x) := f(\lambda_T \cdot x)$ for all T and $x \in_T O_v$ and $\lambda_T := T \to * \xrightarrow{\lambda} \mathbb{C}^*$, has the following relation for all $g, g' \in_T \tilde{G}$

$$\begin{aligned}h(g \cdot a_v(g')) =&h(g \cdot g' \cdot v_T) = f(\lambda_T \cdot g \cdot g' \cdot v_T)\\ =&\chi(g) f(\lambda_T \cdot g \cdot g' \cdot v_T) = \chi(g) h(a_v(g')).\end{aligned}$$

By the fact that a_v is an epimorphism, it follows that h is a relative superinvariant corresponding to χ with

$$h(g \cdot x) = \chi(g) \cdot h(x).$$

Hence by Theorem 4.7, there exists a constant $c(\lambda)$ such that

$$c(\lambda) f = h. \tag{4.2}$$

In particular, $(f_0(\lambda \cdot v))_T = f(\lambda_T \cdot v_T) = h(v_T) = c(\lambda) f_0(v_T) = (c(\lambda) f_0(v))_T$, so that

$$f_0(\lambda v) = c(\lambda) f_0(v).$$

Since the latter holds for any λ, we find by Equation 4.2 that f_0 has to be a homogenous relative invariant as described in Ref. [23, Corollary 2.7]. This implies that $c = \lambda^{\deg f_0}$. $\square$

We are in the situation to reap the benefits of our labour by using Theorem 3.44 to prove the following theorem. It plays a crucial role in telling us when the dual super vector space V^* is also a prehomogeneous super vector space and how the relative superinvariants are related.

The following theorem tells us that the set of supercharacters corresponding to a relative superinvariant is equal to the set of supergroup homomorphisms from G to $\mathbb{C}^*$ invariant under the isotropy supergroup G_v.

Theorem 4.10. *Let (G, ρ, V) be a prehomogeneous super vector space and $v \in V_{\bar{0}}$ a generic point. Then*

$$X_1(G) = \{\chi \in X(G) \mid \chi|_{G_v} = 1\}.$$

Proof. Let $\chi \in X_1(G)$, then there exists a relative superinvariant $f \neq 0$ corresponding to χ. There exists $y \in (\tilde{O}_v)_0$ with $f(y) \neq 0$ and for all $x \in (\tilde{O}_v)_0$ there exist a $g \in G_0$ with $x = gy$, such that

$$f(x) = \chi(g)f(y) \neq 0.$$

Hence $f(x) \neq 0$ for all $x \in (\tilde{O}_v)_0$. In particular $0 \neq f(v_T) = f(gv_T) = \chi(g)f(v_T)$ for all $g \in_T G_v$ and it follows that $\chi(g) = 1$ for all $g \in_T G_v$.

On the other hand, let $\chi \in X_1(G)$ with $\chi|_{G_v} = 1$. Then by the universal property of $\widetilde{G/G_v}$, there exists an $f' \in \mathbb{C}[\widetilde{G/G_v}]$ with $\pi^\sharp(f') = \chi$. Moreover, $\widetilde{a_v} : \widetilde{G/G_v} \xrightarrow{\simeq} \tilde{O}_v$ is an isomorphism in **ssch** by Theorem 3.44, so there exists $f \in \mathbb{C}[\tilde{O}_v]$ with $\widetilde{a_v}^\sharp(f) = f'$. Then for all $g \in_T G$ we have $f(gv) = f(a_v(g)) = f'(\pi(g)) = \chi(g) = \chi(g)f(v)$. Since, by Proposition 2.18, regular superfunctions on Zariski open sub superschemes are rational superfunctions on V, the statement follows. □

We now consider the example $G = \mathrm{GL}_{m|n}$. For a super commutative superalgebra A recall from Definition 2.3, the invertible elements in $\mathrm{Mat}(m|n, A)_{\bar{0}}$ are denoted by $\mathrm{GL}(m|n, A)$. The Berezinian is then a morphism

$$\mathrm{Ber} : \mathrm{GL}(m|n, A) \to A^*_{\bar{0}}$$

defined by

$$\mathrm{Ber}\begin{pmatrix} R & S \\ T & V \end{pmatrix} := \det(R - SV^{-1}T)\det V^{-1}.$$

The multiplicative property of the Berezinian [27, Theorem 1.7.4], implies that it is supercharacter of the general linear supergroup $\mathrm{GL}_{m|n}$.

In the classical case, a character $\chi : GL_m \to GL_1$ of the general linear group must have the form

$$\chi(g) = (\det g)^z \quad z \in \mathbb{Z},$$

see for example Ref. [23, p.44]. Now we state a similiar theorem for the general linear supergroup.

Theorem 4.11. *The rational supercharacters $\chi : GL_{m|n} \to GL_1$ are exactly integer powers of the Berezinian.*

Proof. Let χ be a supercharacter of $GL_{m|n}$ and Q be an arbitrary affine superscheme. For brevity, we also write $\chi : GL_{m|n}(Q) \to GL_1(Q)$. Let $X \in GL_{m|n}(Q) = GL(m|n|\mathcal{O}(Q))$. Since each supermatrix $X = \left(\begin{smallmatrix} R & S \\ T & V \end{smallmatrix}\right)$, can be decomposed as

$$X = \begin{pmatrix} 1 & SV^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R - SV^{-1}T & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ V^{-1}T & 1 \end{pmatrix},$$

we have

$$\chi(X) = \chi \begin{pmatrix} 1 & SV^{-1} \\ 0 & 1 \end{pmatrix} \cdot \chi \begin{pmatrix} R - SV^{-1}T & 0 \\ 0 & V \end{pmatrix} \cdot \chi \begin{pmatrix} 1 & 0 \\ V^{-1}T & 1 \end{pmatrix}.$$

The supercharacter χ is a rational function in the matrix entries and equals 1 for the identity matrix. We can express

$$\chi \begin{pmatrix} 1 & SV^{-1} \\ 0 & 1 \end{pmatrix} = 1 + f(SV^{-1}),$$

where f is a rational superfunction of degree d independent of Q. The relation

$$\chi \begin{pmatrix} 1 & SV^{-1} \\ 0 & 1 \end{pmatrix}^n = \chi \begin{pmatrix} 1 & n \cdot SV^{-1} \\ 0 & 1 \end{pmatrix}$$

gives

$$(1 + f(SV^{-1}))^n = 1 + f(n \cdot SV^{-1}).$$

The function f has degree d, so the expression on the left-hand side has degree nd. The only solution to this equation is $f \equiv 0$. Analogously,

$$\chi \begin{pmatrix} 1 & 0 \\ V^{-1}T & 1 \end{pmatrix} = 1.$$

Since, in the classical case the character is an integer power of the determinant, the most general form of the supercharacter, which is still a character of the underlying classical group, is

$$\chi(X) = \det(R - SV^{-1}T)^{z_1} \cdot \det(V)^{-z_2} \quad z_1, z_2 \in \mathbb{Z}$$

which is the same as

$$\chi(X) = \mathrm{Ber}\begin{pmatrix} R & S \\ T & V \end{pmatrix}^{z_1} \cdot \det(V)^{(z_1 - z_2)}. \tag{4.3}$$

Furthermore, a supercharacter χ has to fulfill for two supermatrix X_1, X_2 the relation

$$\chi(X_1 \cdot X_2) = \chi(X_1) \cdot \chi(X_2)$$

with

$$X_1 \cdot X_2 = \begin{pmatrix} R_1R_2 + S_1T_2 & R_1S_2 + S_1V_2 \\ T_1R_2 + V_1T_2 & T_1S_2 + V_1V_2 \end{pmatrix}.$$

The Berezinian fulfills this relation, but

$$\det(V_1V_2 + T_1S_2)^{z_1 - z_2} = \det(V_1)^{(z_1 - z_2)} \det(V_2)^{(z_1 - z_2)}$$

only if $z_1 = z_2$. It follows by Equation 4.3, that the most general form of a supercharacter is an integer power of the Berezinian. □

4.3 Contragredient actions and dual prehomogeneous super vector spaces

In this subsection, we define the notion of the contragredient action and of a regular prehomogeneous super vector space. As main results, we prove Theorem 4.14, which tells us that each regular prehomogeneous super vector space has a dual prehomogeneous super vector space and the set of relative superinvariants is equal to the set of dual relative superinvariants. Lemma 4.16 is crucial to understand how the Berezinian measure on V transforms under the action of the supergroup G induced by the representation ρ.

Let V be a prehomogeneous super vector space with the data (G, ρ, V) and $V^* = \{v^* : V \to \mathbb{C} \mid v^* \text{ is an even linear mapping}\}$ the dual super vector space of V. The contragredient representation of the Lie supergroup G on V^*, written as $\rho^* : G \to \mathrm{GL}(V^*)$, is defined by the relation

$$\langle \rho^*(g)v^*, \rho(g)w \rangle = \langle v^*, w \rangle, \tag{4.4}$$

with $g \in_T G$, $w \in_T V$ and $v^* \in_T V^*$. For a homogeneous basis $v_1, \ldots, v_{p+q}$ of $V(T)$ and the dual homogeneous basis $v^1, \ldots, v^{p+q}$ of $V^*(T)$ the canonical pairing is defined by $\langle v^i, v_j \rangle := \delta^i_j(T)$, where δ^i_j is the Kronecker delta. Here we use Einstein's summation convention and get for $v^* \in V^*$ and $w \in V$

$$\langle v^*, w \rangle = \langle y_i v^i, v_j x^j \rangle = y_i x^i.$$

$$\begin{aligned}\langle \rho^*(g)v^*, \rho(g)w\rangle &= \langle \rho^*(g)(y_i v^i), \rho(g)(v_j x^j)\rangle = y_i\langle \rho^*(g)(v^i), \rho(g)(v_j)\rangle x^j \\ &= y_i \langle v^k \rho^*(g)_{ki}, v_l \rho(g)_{lj}\rangle x^j \\ &= y_i(-1)^{|k|(|k|+|i|)}\rho^*(g)_{ki}\langle v^k, v_l\rangle \rho(g)_{lj} x^j \\ &= y_i(-1)^{|k|(|k|+|i|)}\rho^*(g)_{ki}\rho(g)_{kj} x^j\end{aligned}$$

The relation in Equation 4.4 is fulfilled, if

$$(-1)^{|k|(|k|+|i|)}\rho^*(g)_{ki}\rho(g)_{kj} = \delta_{ij}(T),$$

or equivalently $(\rho^*(g))^{ST^3}\rho(g) = I$, which gives the expression $\rho^*(g) = (\rho(g)^{-1})^{ST}$ for the contragredient representation of G on V^*.

4.4 Regular prehomogeneouses super vector spaces

Here, we answer the question when the dual vector space with the contragredient action is a prehomogeneous super vector space. Let $x \in_T O_v$, we define $\phi_f(x) \in_T V^*$

$$\phi_f(x) = v^i \frac{1}{f(x)} \sum_{i=1}^{p+q} \frac{\partial f}{\partial x_i}(x). \tag{4.5}$$

This defines a morphism $\phi_f : O_v \to V^*$. This definition is independent of the choice of a basis on V. Indeed, let $v'_1, \ldots, v'_{p+q}$ be another homogeneous basis of V and $v'^1, \ldots, v'^{p+q}$ the dual basis. Then there exist unique matrices $A \in \mathrm{GL}(V)$ and $B \in \mathrm{GL}(V^*)$ such that $v'^i = v^k a^i_k$ and $v'_j = v_l b^l_j$ with $A = (a^i_j)$ and $B = (b^i_j)$, so we have

$$\begin{aligned}\delta_{ij} =&\langle v'^i, v'_j\rangle = \langle v^k a^i_k, v_l b^j_l\rangle \\ =&(-1)^{|k|(|i|+|k|)} a^i_k \langle v^k, v_l\rangle b^j_l \\ =&(A^{ST^3}B)_{ij}\end{aligned}$$

Let $v = v_i x_i = v'_j x'_j$, then we have $v'_j x'_j = v_i b_{ij} x'_j = v_i x_i$ and $x_i = b_{ij} x'_j$. Moreover, let $\frac{\partial}{\partial x_i} = c_{ik}\frac{\partial}{\partial x'_k}$, then

$$\delta_{ij} = \frac{\partial}{\partial x_i}(x_j) = c_{ik}\frac{\partial}{\partial x'_k}(b_{jt}x'_t) = c_{ik}\frac{\partial}{\partial x'_k}(x'_t)(-1)^{|t|(|j|+|t|)} b_{jt} = (CB^{ST})_{ij}.$$

Hence, one gets for $v \in_T V$:

$$\begin{aligned}f(v)\phi_f(v) =&v^i \frac{\partial f}{\partial x_i}(v) \\ =&v'^k B^{ST}_{ki}(B^{ST})^{-1}_{il}\frac{\partial f}{\partial x'_l}(v) \\ =&v'^k \frac{\partial f}{\partial x'_k}(v),\end{aligned}$$

which shows that ϕ_f is independent of the choice of basis.

Let $x = \mathrm{Id}_{O_v} \in O_v(O_v)$ and $g \in_T G$. Note, that $x^\sharp(x_i) = x_i$ and $g \cdot x = a(g,x) \in O_v(T \times O_v)$. In order to express the next calculation in a streamlined fashion, we can identify $\frac{\partial f}{\partial x_i}(x) = x^\sharp\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial f}{\partial x_i} \in \Gamma(\mathcal{O}_{O_v})$ and note that

$$f(\rho(g)x) = (\rho(g)x)^\sharp(f) \in \mathbb{C}^*(T \times O_v) = \Gamma(\mathcal{O}_{T\times O_v})^*_{\bar{0}}.$$

Now we get the following proposition.

Proposition 4.12. *Let $g \in_T G$ and $x = \mathrm{Id}_{O_v} \in O_v(O_v)$, then*

$$\phi_f(\rho(g)x) = \rho^*(g)\phi_f(x).$$

Proof. For simplicity we may assume $G(T) \subseteq \mathrm{GL}(\mathcal{O}^{p|q}(T))$, so by differentiating $f(gx) = \chi(g)f(x)$ we obtain

$$\begin{aligned}
\chi(g)\frac{\partial f}{\partial x_i}(x) =&\partial_{x_i} f(gx) = \frac{\partial (gx)_k}{\partial x_i}\frac{\partial f}{\partial x_k}(gx)\\
=&(\partial_{x_i} g_{kj}x_j)\frac{\partial f}{\partial x_k}(gx)\\
=&(-1)^{|i|(|k|+|j|)}g_{kj}\delta_{ij}\frac{\partial f}{\partial x_k}(gx)\\
=&(g^{ST})_{ik}\frac{\partial f}{\partial x_k}(gx).
\end{aligned}$$

Here, we used the chain rule in the second step and expanded the linear action in coordinates. With the previous calculation we obtain

$$\begin{aligned}
\frac{1}{f(x)}\frac{\partial f}{\partial x_i}(x) =&\frac{1}{f(x)}\frac{\chi(g)}{\chi(g)}\frac{\partial f}{\partial x_i}(x)\\
=&\frac{1}{f(gx)}(g^{ST})_{ik}\frac{\partial f}{\partial x_k}(gx)\\
=&(g^{ST})_{ik}\left(\frac{1}{f}\frac{\partial f}{\partial x_k}\right)(gx).
\end{aligned}$$

It follows $\phi_f(\rho(g)x) = (g^{-1})^{ST}\phi_f(x) = \rho^*(g)\phi_f(x)$. □

Now we define the matrix $H = (H_{ij})_{i,j=1,\dots,p+q}$ by

$$H_{ij}(x) := \frac{\partial}{\partial x_i}\left(\frac{1}{f}\frac{\partial f}{\partial x_j}\right)(x) \quad \text{for } i,j = 1,\dots,p+q$$

to state the following proposition.

Proposition 4.13. $H(gx) = (g^{ST})^{-1}H(x)(g^{ST^2})^{-1}$

Proof. Using the same ideas as the last proposition, we calculate

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\frac{1}{f}\frac{\partial f}{\partial x_j}\right)(x) =& \frac{\partial}{\partial x_i}(g^{ST})_{jk}\left(\frac{1}{f}\frac{\partial f}{\partial x_k}\right)(gx)\\ =& \frac{\partial}{\partial x_i}\left(\frac{1}{f}\frac{\partial f}{\partial x_k}\right)(gx)(-1)^{|k|(|j|+|k|)}(g^{ST})_{jk}\\ =&(g^{ST})_{it}\frac{\partial}{\partial x_t}\left(\frac{1}{f}\frac{\partial f}{\partial x_k}\right)(gx)(-1)^{(|j|+|k|)}(g^{ST^2})_{kj}\end{aligned}$$

and it follows that $H(gx) = (g^{ST})^{-1}H(x)(g^{ST^2})^{-1}$. □

Now we see that the sheafification of the functors $\phi_f(O_v(T)),\ \phi_f(\rho(G(T))v)$, and $\rho^*(G(T))\phi_f(v)$ are equal and give an orbit of V^*. If this orbit is dense, we call the relative superinvariant f *nondegenerate* and (G, ρ, V) a *regular prehomogeneous super vector space.* Moreover, (G, ρ^*, V^*) becomes a prehomogeneous super vector space.

Theorem 4.14. *If (G, ρ, V) is a regular prehomogeneous super vector space, then the dual triplet (G, ρ^*, V^*) is also a regular prehomogenous super vector space. In this case, if $X_1^*(G)$ denotes the group of supercharacters corresponding to relative superinvariants of (G, ρ^*, V^*), we have $X_1(G) = X_1^*(G)$. Moreover, $O_v \cong O^*_{v^*}$.*

Proof. We are going to show that for all $A \in \mathfrak{g}$ and $x \in_S V,\ y \in_S V^*$ the following holds

$$\langle d\rho^*(A)y, x\rangle + \langle y, d\rho(A)x\rangle = 0.$$

We define $F \in \Gamma(\mathcal{O}_{G\times V^*\times V})$ through $F(g) := \langle \rho^*(g)y, \rho(g)x\rangle = \langle y, x\rangle$ with $g = \mathrm{id}_G \in_G G,\ y = \mathrm{id}_{V^*} \in_{V^*} V^*,\ x = \mathrm{id}_V \in_V V$.

Recall, that the tangent super vector space T_pX at $p \in X_0$ of X is defined by $T_pX := \mathrm{Der}(\mathcal{O}_{X,p}, \mathbb{C})$ and we can identify $\mathfrak{g} \cong T_1G$. Furthermore, one can express

$$T_pX(S) = \{\phi \in_{S[t|\tau]} X \mid \phi|_{t=\tau=0} = p_S\},$$

with $S[t|\tau] = (S_0, \mathcal{O}_S \otimes \mathbb{C}[t, \tau]/(t^2, t\tau))$.

For all $A \in_S T_1G$ there exists a unique $g \in_{S[t|\tau]} G$, such that $g|_{t=\tau=0} = 1$ and $\frac{\partial}{\partial t}g^\sharp(f)|_{t=\tau=0} = A_{\bar{0}}(f)$ and $\frac{\partial}{\partial \tau}g^\sharp(f)|_{t=\tau=0} = A_{\bar{1}}(f)$ with $f \in \mathcal{O}_G$.

We know $F(g) = \langle y, x\rangle$ is independent of g, so $dF(A) = 0$, such that we get

$$\begin{aligned}0 = A(F) =&(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})\langle \rho^*(g)y, \rho(g)x\rangle_{t=\tau=0}\\ =&(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})\langle \rho^*(g)y, \rho(1)x\rangle_{t=\tau=0} + (\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})\langle \rho^*(1)y, \rho(g)x\rangle_{t=\tau=0}\\ =&\langle d\rho^*(A)y, x\rangle + \langle y, d\rho(A)x\rangle,\end{aligned}$$

where $d\rho(A)f := A(\rho^\sharp(f))$ for $f \in \mathcal{O}_{\mathrm{GL(V)}}$.

If (G, ρ, V) is a prehomogeneous super vector space and f a relative super-invariant with the supercharacter χ, then we have

$$\langle \phi_f(x), d\rho(A)x\rangle = d\chi(A)$$

for $x \in_T V$.

We define $H \in \Gamma(\mathcal{O}_{G\times V})$ by $H(g) = f(\rho(g)x) = \chi(g)f(x)$ for $x = \mathrm{id}_V \in_V V$ and $g = \mathrm{id}_G \in_G G$ and set $dH(A) := A(H)$ and $d\chi(A)f(x) := A(H)$. Since

$$\begin{aligned}A(H) =&(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial \tau})f(\rho(g)x)\\ =&(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial \tau})(\rho(g)x)^\sharp f\\ =&((\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial \tau})\rho(g)x)_i \cdot \tfrac{\partial f}{\partial x_i}(x)\\ =&\langle d\rho(A)x, \mathrm{grad} f(x)\rangle,\end{aligned}$$

with $\mathrm{grad} f(x) := \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_{p+q}}(x)\right)^T$, it follows that

$$\begin{aligned}d\chi(A) = A(\chi) =&(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial \tau})\chi(g)f(x)\tfrac{1}{f(x)}\\ =&\langle d\rho(A)x, \phi_f(x)\rangle\\ =&\langle \phi_f(x), d\rho(A)x\rangle.\end{aligned}$$

Let f be a nondegenerate relative superinvariant. We now show that the map $(\phi_f)_0 : (O_v)_0 \to (V^*)_{\bar{0}}$ is injective. If $x, x' \in (O_v)_0$ and $\phi_f(x) = \phi_f(x')$, then we have

$$\begin{aligned}0 = d\chi(A) - d\chi(A) &= \langle \phi_f(x), d\rho(A)x\rangle - \langle \phi_f(x'), d\rho(A)x'\rangle\\ &= \langle \phi_f(x), d\rho(A)(x - x')\rangle = \langle d\rho^*(A)\phi_f(x), x - x'\rangle\end{aligned}$$

Since f is nondegenerate, $\phi_f(x)$ is a generic point of (G, ρ^*, V^*) and in the proof of Theorem 3.21, we saw that for a generic point

$$\{d\rho^*(A)\phi_f(x) \mid A \in \mathrm{Lie}(G)\} = V^*,$$

which implies $x = x'$ and $(\phi_f(x))_0$ is injective.

Since $\phi_f(\rho(g)v) = \rho^*(g)\phi_f(v)$, it is clear that $G_v(T) \subseteq G_{\phi_f(v)}(T)$. On the other hand, if $g \in_T G_{\phi_f(v)}(T)$ then $\phi_f(\rho(g)v) = \rho^*(g)\phi_f(v) = \phi_f(v)$ and by the injectivity of $(\phi_f)_0$ we get $(G_{\phi_f(v)})(T) \subseteq (G_v)(T)$. Hence $G_v = G_{\phi_f(v)}$ and by Theorem 4.10, $X_1(G) = X_1^*(G)$. Let χ be the supercharacter corresponding to the relative superinvariant f, then $\chi^{-1} \in X_1(G) = X_1^*(G)$. Hence, there exists a relative superinvariant f^* of (G, ρ^*, V^*) corresponding to the supercharacter χ^{-1}. Let $\phi^*_{f^*} : O^*_{v^*}(T) \to V(T)$. We have $\langle d\rho^*(A)y, \phi^*_{f^*}(y)\rangle = -d\chi(A)$ for $A \in \mathrm{Lie}(G)$ and $y = \phi_f(x) \in_T O^*_{v^*}(T)$. Using $\langle -x, d\rho^*(A)\phi_f(x)\rangle = d\chi(A)$ we get

$$\langle \phi^*_{f^*}(\phi_f(x)) - x, d\rho^*(A)\phi_f(x)\rangle = 0,$$

and since $\{d\rho^*(A)\phi_f(x) \mid A \in \mathrm{Lie}(G)\} = V^*(T)$, we have $\phi^*_{f^*}(\phi_f(x)) = x \in_T O_v(T)$. In particular, (G, ρ^*, V^*) is a regular prehomogeneous super vector space. □

Corollary 4.15. *Let (G, ρ, V) be a regular prehomogeneous super vector space. Then there exists a relative superinvariant corresponding to the supercharacter $Ber(\rho)^2$, such that $Ber(\rho)^2 \in X_1(G)$.*

Proof. Let f be a nondegenate relative superinvariant, then $\phi_f : O_v(T) \to O_{v^*}(T)$ is an isomorphism and $(d\phi_f)$ is a bijective linear mapping given by the matrix

$$H(x) = \left(\frac{\partial}{\partial x_i} \left(\frac{1}{f} \frac{\partial f}{\partial x_j} \right)(x) \right)_{i,j=1\ldots p+q.}$$

Hence, $\mathrm{Ber}(H(x)) \neq 0$ for $x = \mathrm{Id}_{O_v} \in_{O_v} O_v$. By Proposition 4.13, $\mathrm{Ber}(H(x))$ is not identically zero and a relative superinvariant corresponding to the supercharacter $\mathrm{Ber}(\rho(x))^{-2}$. □

Lemma 4.16. *Let (G, ρ, V) be a regular prehomogeneous super vector space, f a relative superinvariant and χ the corresponding supercharacter. Assume that any relative superinvariant is a constant multiple of f^m, for some $m \in \mathbb{Z}$ and that $\mathbb{C}^* \subseteq \rho(G_0)$. Let $r|s = \dim V$ and $d := \deg f := \deg f_0$. Then*

$$\mathrm{Ber}(\rho^2) = \chi^{2(r-s)/d}$$

where $2(r-s)/d$ is an integer.

Proof. By Corollary 4.15, there is a relative superinvariant corresponding to the supercharacter $\mathrm{Ber}(\rho)^2$, which is of the form $c \cdot f^m$. Hence, there exists an integer m such that $\mathrm{Ber}(\rho)^2 = \chi^m$. We may assume that there exists an element $g \in G_0$ such that $\rho(g) = tI_V$ with $t \in \mathbb{C}^*$. Hence, we get $\mathrm{Ber}(\rho(g)) = t^{r-s}$. Further $\chi(g)f(x) = f(\rho(g)x) = f(tx) = t^d f(x)$ for any x, such that $\chi(g) = t^d$. Hence $t^{2(r-s)} = t^{dm}$, and it follows that $m = 2(r-s)/d \in \mathbb{Z}$ and $\mathrm{Ber}(\rho)^2 = \chi^{2(r-s)/d}$. □

4.5 Examples of prehomogeneous super vector spaces

We present the construction of two examples of prehomogeneous super vector spaces. The first example is the super vector space of supersymmetric matrices with an action of $\mathrm{GL}(p|q, \mathbb{C})$. We will use this example in order to make some of our statements explicit. The second example is the super vector space $\mathrm{M}(p|2q \times m|2n, \mathbb{C})$ with an action of $\mathrm{OSp}(p|2q, \mathbb{C}) \times \mathrm{GL}(m|2n, \mathbb{C})$.

Example 4.17. Let $V = \mathbb{C}^{p|q}$, with q an even number and $S^2(V)$ be the super vector space of supersymmetric matrices. We define an action of $G = \mathrm{GL(V)}$ on $S^2(V)$ by first defining one on $\bigotimes^2 V$. The action on the latter is simply the natural extension of the action on V. In terms of the canonical basis (e_i) of V, it is explicitly given, for $g \in_T G$ and $R = e_i R_{ij} \otimes e_j \in_T \bigotimes^2 V$, by

$$\begin{aligned} gR &= g(e_i)R_{ij} \otimes g(e_j) \\ &= e_k g_{ki} R_{ij} \otimes e_l g_{lj} = e_k (g_{ki} R_{ij} (-1)^{(|l|+|j|)|l|} g_{lj}) \otimes e_l \\ &= e_k (g_{ki} R_{ij} (g^{(ST)^3})_{jl}) \otimes e_l. \end{aligned}$$

The super symmetric matrices $S^2(V)$ are given as the fixed point set on T-valued points of the right linear map ϑ, defined by

$$\begin{aligned} \vartheta : V \otimes V &\to V \otimes V \\ \vartheta(e_i R_{ij} \otimes e_j) :=& (-1)^{|i||j|+(|R|+1)(|i|+|j|)} e_j R_{ji} \otimes e_i. \end{aligned}$$

So the coefficents R_{ij} have to fulfill the relation

$$R_{ij} = R_{ji}(-1)^{|i||j|+(|R|+1)(|i|+|j|)}.$$

Written as a block matrix $R = \left(\begin{smallmatrix} S & B \\ C & A \end{smallmatrix}\right)$ this means that $\left(\begin{smallmatrix} S & B \\ C & A \end{smallmatrix}\right) = \left(\begin{smallmatrix} S^T & -C^T \\ -B^T & -A^T \end{smallmatrix}\right)$, i.e. $S = S^T$; $C^T = -B$; $A = -A^T$.

For $g \in_T G$, we compute

$$\begin{aligned} \vartheta(g(e_i) \otimes g(e_j)) &= \vartheta(e_k g_{ki} \otimes e_l g_{lj}) = \vartheta(e_k \otimes e_l)(-1)^{|g_{ki}||l|} g_{ki} g_{lj} \\ &= (e_l \otimes e_k)(-1)^{|k||l|+|g_{ki}||l|} g_{ki} g_{lj} \\ &= (e_l \otimes e_k) g_{lj} g_{ki} (-1)^{|g_{lj}||g_{ki}|+|k||l|+|g_{ki}||l|} \\ &= (e_l g_{lj} \otimes e_k g_{ki})(-1)^{|k||g_{lj}|+|g_{lj}||g_{ki}|+|k||l|+|g_{ki}||l|} \\ &= g(e_j) \otimes g(e_i)(-1)^{|i||j|} = g\vartheta(e_i \otimes e_j). \end{aligned}$$

Hence, G leaves $S^2(V)$ invariant. This action is compatible with matrix multiplication. For an even super matrix $R = \left(\begin{smallmatrix} S & B \\ C & A \end{smallmatrix}\right) \in_T S^2(V)$ the action $g.R = gRg^{ST^3}$ can be calculated for concrete matrices $g \in_T G$ as

$$g.R = \begin{pmatrix} G_0 & G_1 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} S & B \\ -B^T & A \end{pmatrix} \begin{pmatrix} G_0^T & -G_3^T \\ G_1^T & G_4^T \end{pmatrix}.$$

The isotropy supergroup of $\mathcal{J} = \left(\begin{smallmatrix} I_p & 0 \\ 0 & J_q \end{smallmatrix}\right)$ under this action, where I_p is the $p \times p$ identity matrix and $J_q = \left(\begin{smallmatrix} 0 & I_{q/2} \\ -I_{q/2} & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, is isomorphic to $\mathrm{OSp}(V)$. Its T-valued points are given by

$$\begin{pmatrix} I_p & 0 \\ 0 & J_q \end{pmatrix} = \begin{pmatrix} G_0G_0^T + G_1JG_1^T & -G_0G_3^T + G_1JG_4^T \\ G_3G_0^T + G_4JG_1^T & G_4JG_4^T - G_3G_3^T \end{pmatrix}.$$

The Lie superalgebra is defined by the relation $X\mathcal{J} + \mathcal{J}X^{(ST)^3} = 0$, which is isomorphic to $\mathfrak{osp}(V)$ via the Chevalley automorphism

$$\tau : X \mapsto \tau(X) := -X^{ST}.$$

A dimension calculation gives

$$\begin{aligned} &\dim G - \dim \mathrm{OSp}(V) \\ &\quad = (p^2 + q^2 \mid 2pq) - \left(\frac{p(p+1)}{2} + \frac{q(q-1)}{2} \;\middle|\; pq \right) \\ &\quad = \frac{p(p-1)}{2} + \frac{q(q+1)}{2} \;\middle|\; pq \\ &\quad = \dim S^2(V). \end{aligned}$$

By this calculation and Theorem 3.21, the next proposition follows directly.

Proposition 4.18. *The action of* $\mathrm{GL}(V)$ *on* $S^2(V)$ *admits an open orbit, whose* T*-valued points are given by the invertible elements* $R \in S^2(V)(T)$*.*

The irreducible relative superinvariants for the space of supersymmetric matrices is given by the Berezinian and all integer powers of it give a relative superinvariant. One calculates

$$\begin{aligned} \mathrm{Ber}(G.R) &= \mathrm{Ber}(G) \cdot \mathrm{Ber}(R) \cdot \mathrm{Ber}(G^{ST^3}) \\ &= \mathrm{Ber}(G)^2 \cdot \mathrm{Ber}(R). \end{aligned}$$

to confirm this statement.

Now we consider the second example of a prehomogeneous super vector space.

Example 4.19. Let $p \geq m \geq 2$ and $q \geq n \geq 2$. We define an action of $\mathrm{OSp}(p|2q,\mathbb{C}) \times \mathrm{GL}(m|2n,\mathbb{C})$ on $\mathrm{M}(p|2q \times m|2n,\mathbb{C})$ by $\rho(A,B)(X) = AXB^{ST^3}$, where $A \in \mathrm{OSp}(p|2q,\mathbb{C})$ and $B \in \mathrm{GL}(m|2n,\mathbb{C})$ and $X \in \mathrm{M}(p|2q\times m|2n,\mathbb{C})$. The infinitesimal representation $d\rho$ is given by $d\rho(Y,Z)X = YX + XZ^{ST^3}$ with $(Y,Z) \in \mathfrak{osp}(p|2q,\mathbb{C}) \oplus \mathfrak{gl}(m|2n,\mathbb{C})$. Now, we calculate the isotropy algebra at

$$I = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{pmatrix}.$$

The isotropy superalgebra is given by the elements for which the equation $d\rho(Y,Z)I = 0$ holds. An element $Y \in \mathfrak{osp}(p|2q,\mathbb{C})$ is given by the equation $Y\mathcal{J} + \mathcal{J}Y^{ST^3} = 0$, compare with Example 4.17, and has the general form

$$Y = \begin{pmatrix} A_1 & A_2 & C_1^T & C_3^T & -B_1^T & -B_3^T \\ A_3 & A_4 & C_2^T & C_4^T & -B_2^T & -B_4^T \\ B_1 & B_2 & -F_1^T & -F_3^T & D_1 & D_2 \\ B_3 & B_4 & -F_2^T & -F_4^T & D_3 & D_4 \\ C_1 & C_2 & E_1 & E_2 & F_1 & F_2 \\ C_3 & C_4 & E_3 & E_4 & F_3 & F_4 \end{pmatrix},$$

where the square submatrix, denoted by $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ is skew-symmetric and the submatrices $\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$, $\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$ are symmetric. We write a general element $Z \in \mathfrak{gl}(m|2n,\mathbb{C})$ in the form

$$Z = \begin{pmatrix} G_1 & H_1 & K_1 \\ G_2 & H_2 & K_2 \\ G_3 & H_3 & K_3 \end{pmatrix}.$$

Now we get that the elements of the isotropy superalgebra satisfy:

$$YI + IZ^{ST^3} = 0$$

or equivalently

$$\begin{pmatrix} A_1 & C_1^T & -B_1^T \\ A_3 & C_2^T & -B_2^T \\ B_1 & -F_1^T & D_1 \\ B_3 & -F_2^T & D_3 \\ C_1 & E_1 & F_1 \\ C_3 & E_3 & F_3 \end{pmatrix} + \begin{pmatrix} G_1^T & G_2^T & G_3^T \\ 0 & 0 & 0 \\ -H_1^T & H_2^T & H_3^T \\ 0 & 0 & 0 \\ -K_1^T & K_2^T & K_3^T \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that the general form of an element from $\mathfrak{g}_I$ is

$$\left(\begin{pmatrix} A_1 & 0 & C_1^T & 0 & -B_1^T & 0 \\ 0 & A_4 & 0 & C_4^T & 0 & -B_4^T \\ B_1 & 0 & -F_1^T & 0 & D_1 & 0 \\ 0 & B_4 & 0 & -F_4^T & 0 & D_4 \\ C_1 & 0 & E_1 & 0 & F_1 & 0 \\ 0 & C_4 & 0 & E_4 & 0 & F_4 \end{pmatrix}, \begin{pmatrix} A_1 & B_1^T & C_1^T \\ -C_1 & F_1 & -E_1^T \\ B_1 & -D_1^T & -F_1^T \end{pmatrix} \right)$$

and we see that $\mathfrak{g}_I \cong \mathfrak{osp}(p-m|2q-2n,\mathbb{C}) \oplus \mathfrak{osp}(m|2n,\mathbb{C})$. A dimension calculation shows that

$$\begin{aligned}
&\dim(\mathrm{OSp}(p|2q,\mathbb{C}) \times \mathrm{GL}(m|2n,\mathbb{C})) - \dim(\mathrm{M}(p|2q \times m|2n,\mathbb{C})) - \dim(\mathfrak{g}_I) \\
&\quad = p(p+1)/2 + q(2q-1) + 2pq + (m+2n)^2 - (p+2q)(m+2n) \\
&\qquad - ((p-m)(p-m+1)/2 + (q-n)(2(q-n)-1) \\
&\qquad + (p-m)2(q-n) + m(m+1)/2 + n(2n-1) + 2mn) \\
&\quad = 0,
\end{aligned}$$

and this defines a prehomogeneous super vector space.

In order to construct a relative superinvariant, we define a super bilinear form by $b(X,Y) := X^{ST^3}\mathcal{J}^{-1}Y$. Then $b(\rho(A,B)X, \rho(A,B)Y)$ is

$$\begin{aligned}
(AXB^{ST^3})^{ST^3}\mathcal{J}^{-1}(AXB^{ST^3}) &= B^{ST^2}X^{ST^3}(A^{ST^3}\mathcal{J}^{-1}A)XB^{ST^3} \\
&= B^{ST^2}X^{ST^3}(A^{ST^3}\mathcal{J}^{-1}A)XB^{ST^3} \\
&= B^{ST^2}(X^{ST^3}(A^{-1}\mathcal{J}(A^{-1})^{ST^3})^{-1}XB^{ST^3} \\
&= B^{ST^2}(X^{ST^3}\mathcal{J}^{-1}X)B^{ST^3},
\end{aligned}$$

where we used the relation

$$A\mathcal{J}A^{ST^3} = \mathcal{J} \Leftrightarrow (A^{-1})^{ST^3}\mathcal{J}^{-1}(A^{-1}) = \mathcal{J}^{-1},$$

since $A \in \mathrm{OSp}(p|2q,\mathbb{C})$. Hence, we get by

$$f(X) := \mathrm{Ber}(X^{ST^3}\mathcal{J}^{-1}X)$$

a relative superinvariant with $f(\rho(A,B)(X)) = \mathrm{Ber}(B)^2 f(X)$.

Now we go back to the first example and calculate the contragredient action of $\mathrm{GL}(p|q,\mathbb{C})$ on $S^2(V)^*$ from Example 4.17.

Let $e_1,\ldots,e_{p+q}$ be a homogeneous basis of $\mathbb{C}^{p|q}$ and $e^1,\ldots,e^{p+q}$ be the dual homogeneous basis.

$$\begin{aligned}
X =& v \otimes \tilde{v} = e_i v_i \otimes \tilde{v}_j e_j \\
Y =& w \otimes \tilde{w} = e_i w_i \otimes \tilde{w}_j e_j.
\end{aligned}$$

Let us define

$$\begin{aligned}
\langle X, Y\rangle &= \langle v\otimes\tilde{v}, w\otimes\tilde{w}\rangle = \langle v, w\rangle \cdot \langle \tilde{v}, \tilde{w}\rangle \\
&= (e^i v_i e_k w^k)(\tilde{v}_j e^j \tilde{w}^k e_k) \\
&= (-1)^{|i|+|j|} v_i w^i \tilde{v}_j \tilde{w}^j = (-1)^{|i|+|j|+|i||j|} v_i \tilde{v}_j w^i \tilde{w}^j \\
&= (-1)^{|j|}(-1)^{(|i|+|j|)|i|} v_i \tilde{v}_j w^i \tilde{w}^j = \mathrm{str}(X^{ST^3}Y),
\end{aligned}$$

where $X^{ST^3}_{ji} = (-1)^{(|i|+|j|)|i|}X_{ij}$.

Proposition 4.20. *The contragredient action for the pairing*

$$\langle X, Y\rangle = \mathrm{str}(X^{ST^3} Y)$$

on $S^2(\mathbb{C}^{p|q})^* \otimes S^2(\mathbb{C}^{p|q}) \to \mathbb{C}$ *has the form*

$$\rho^*(g).Y = (g^{-1})^{ST^3} Y (g^{-1})^{ST^2}.$$

Proof. The action of $GL(p|q; \mathcal{O}(T))$ on $S^2(\mathcal{O}^{p|q}(T))^*$, which fulfills the relation

$$\langle \rho(g).X, \rho^*(g).Y\rangle = \langle X, Y\rangle$$

is given by

$$\rho^*(g).Y = (g^{-1})^{ST^3} Y (g^{-1})^{ST^2}.$$

□

We also like to calculate the transformation of the flat Berezinian measure on $S^2(V)$ under $\mathrm{GL}(p|q, \mathbb{C})$.

Let us define the flat Berezinian measure on $S^2(\mathbb{C})$. A globally defined super coordinate system is given by (s_{ij}, a_{kl}, b_{il}), where the indices are $i, j \in \{1, ..., p\}$ and $k, l \in \{1, ..., q\}$ with $i \leq j$ and $k < l$. The flat Berezinian measure on $S^2(\mathbb{C})$ is defined by

$$d\mu(X) = D(s, a, b) = \prod_{i \leq j} ds_{ij} \prod_{k<l} da_{kl} \prod_{i,l} \frac{\partial}{\partial b_{il}},$$

where we took the wedge product for the even coordinate differentials.

The following definition is needed.

Definition 4.21. Let $(s_{i,j})$, $(a_{i,j})$ and $(x_{i,j})$ the standard coordinate systems of the vector spaces of symmetric matrices, skew-symmetric and arbitrary $n \times n$-matrices. We define the Lebesgue measures on these vector spaces $\mathrm{Sym}(n, \mathbb{C})$, $\mathrm{Skew}(n, \mathbb{C})$ and $\mathrm{Mat}(n, \mathbb{C})$ by

$$d\mu(S) := \prod_{i \leq j} ds_{ij}, \;\; d\mu(A) := \prod_{k<l} da_{kl} \;\; \text{and} \;\; d\mu(B) := \prod_{k,l} db_{kl},$$

with $i, j, k, l \in \{1, ..., n\}$, where the wedge product is used as necessary.

The next proposition follows directly by Lemma 4.16.

Proposition 4.22. *Let $X \in_T S^2(\mathbb{C}^{p|q})$ and $g \in_T \mathrm{GL}(p|q, \mathbb{C})$, then the flat Berezinian measure transforms under the action given by $\rho(g).X = gXg^{ST^3}$ as*

$$d\mu(\rho(g).X) = \mathrm{Ber}(\rho(g))d\mu(X) = \mathrm{Ber}(g)^{p-q+1}d\mu(X).$$

Proof. In order to apply Lemma 4.16, we have to calculate two numbers, $r - s$ where $\dim S^2(\mathbb{C}^{p|q}) = r|s$ and $d = \deg \mathrm{Ber}$. For the first number we get

$$r - s = \frac{p^2 + p}{2} + \frac{q^2 - q}{2} - pq$$

and $d = p - q$, so that

$$\begin{aligned} \mathrm{Ber}(\rho(g)) &= \chi(g)^{(p^2-2pq+q^2+p-q)/2(p-q)} \\ &= \chi(g)^{(p-q+1)/2} = \mathrm{Ber}(g)^{(p-q+1)}. \end{aligned}$$

□

We also give a more elementary proof for Proposition 4.22, which shows how powerful Lemma 4.16 is and which gives some insight to Proposition 4.22 from another point of view.

Now we give the proof.

Proof. First, we have to calculate the Jacobian of the coordinate transformation. Second, we have to take the Berezinian from this Jacobian and express it as the Berezinian of $g \in_S \mathrm{GL}(p|q, \mathbb{C})$ to some power. Let T be the transformation map $X' = T(X) = gXg^{ST^3}$. With $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ we get $X' = T(X) =$

$$\begin{pmatrix} \alpha S\alpha^T - \beta B^T\alpha^T + \alpha B\beta^T + \beta A\beta^T & -\alpha S\gamma^T + \beta B^T\gamma^T + \alpha B\delta^T + \beta A\delta^T \\ \gamma S\alpha^T - \delta B^T\alpha^T + \gamma B\beta^T + \delta A\beta^T & -\gamma S\gamma^T + \delta B^T\gamma^T + \gamma B\delta^T + \delta A\delta^T \end{pmatrix}.$$

Now, we introduce the following notation $c_{x,y}(z) := xzy^T$, where x, y, z are matrices. Because we are interested in the coordinate transformation we write it as

$$\begin{pmatrix} c_{\alpha,\alpha} & c_{\beta,\beta} & c_{\alpha,\beta} - c_{\beta,\alpha} \circ (.)^T \\ c_{\gamma,\gamma} & c_{\delta,\delta} & c_{\gamma,\delta} + c_{\delta,\gamma} \circ (.)^T \\ c_{\alpha,\gamma} & c_{\beta,\delta} & c_{\alpha,\delta} + c_{\beta,\gamma} \circ (.)^T \end{pmatrix} \cdot \begin{pmatrix} S \\ A \\ B \end{pmatrix},$$

which also gives us the Jacobian by linearity in X. Notice that one has a decomposition for an invertible super matrix $g \in_T \mathrm{GL}(p|q, \mathbb{C})$. Such a matrix can be written as a product of an upper triangular matrix, a diagonal matrix and a lower triangular matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta^{-1}\gamma & 1 \end{pmatrix}, \tag{4.6}$$

with $\alpha \in \mathrm{GL}(p, \mathcal{O}_{T,\bar{0}})$, $\delta \in \mathrm{GL}(q, \mathcal{O}_{T,\bar{0}})$, $\beta \in \mathcal{O}_{T,\bar{1}}^{p\times q}$ and $\gamma \in \mathcal{O}_{T,\bar{1}}^{q\times p}$. We have to check the statement for each matrix in the decomposition case by case.

But first we need the following classical lemma.

Lemma 4.23. *Let the invertible $n \times n$-matrices $\mathrm{GL}(n, \mathbb{C})$ act on these vector spaces $\mathrm{Sym}(n, \mathbb{C})$, $\mathrm{Skew}(n, \mathbb{C})$ and $\mathrm{Mat}(n, \mathbb{C})$ by $g.X := gXg^T$ for $g \in \mathrm{GL}(n, \mathbb{C})$. The corresponding Lebesgue measures transform as follows: In the symmetric and the skew-symmetric case, we have*

$$d\mu(gSg^T) = det(g)^{n+1} d\mu(S) \quad \textit{and} \quad d\mu(gAg^T) = det(g)^{n-1} d\mu(A)$$

and in the case of arbitrary $n \times n$-matrices by

$$d\mu(gBg^T) = det(g)^{2n} d\mu(B)$$

for $S \in \mathrm{Sym}(n)$, $A \in \mathrm{Skew}(n)$ and $B \in \mathrm{Mat}(n, \mathbb{C})$.

Proof. Every invertible matrix is the product of finitely many elementary matrices. In order to prove the lemma, it is enough to show it for elementry matrices of type $S_{ii}(\lambda) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $R_{ij}(\lambda) = \left(\begin{smallmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, where $\lambda \in \mathbb{C}^*$ is positioned in the row of the first index and in the column of the second. Acting by g gives a linear coordinate transformation for all of these three vector spaces.

We have to calculate the Jacobian of these transformations. We give the proof for symmetric matrices, the other cases are similiar. First let $g = S_{ii}(\lambda)$. Multiplying an arbitrary matrix B from the right-hand side with $S_{ii}(\lambda)$ multiplies the i^{th} column of B by λ and multiplying B from the left-hand side with $S_{ii}(\lambda)$ multiplies the i^{th} row of B by λ, so that

$$gBg^T = \begin{pmatrix} b_{1,1} & \dots & \lambda b_{1,i} & \dots & b_{1,n} \\ \vdots & & \vdots & & \vdots \\ \lambda b_{i,1} & \dots & \lambda^2 b_{i,i} & \dots & \lambda b_{i,n} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \dots & \lambda b_{n,i} & \dots & b_{n,n} \end{pmatrix}.$$

Hence, the coordinate transformation is a diagonal matrix in this case. Here, a λ appears when the entry comes from a coordinate from the i^{th} column or the i^{th} row of the matrix gBg^T. The entry associated with the coordinate function with the indices ii gives a λ^2 and all the other diagonal entries are one.

Now we regard B as a symmetric matrix S. Hence, we get

$$d\mu(gSg^T) = \lambda^{(n+1)} d\mu(S),$$

where λ is the determinant of the elementary matrix $S_{ii}(\lambda)$.

We also have to prove the statement for $g = R_{ij}(\lambda)$. Again let $B \in \mathrm{Mat}(n, \mathbb{C})$. Multiplying B from the left-hand side with $R_{ij}(\lambda)$ adds λ times the j^{th} row of B to the i^{th} row of B. Multiplying B from the right-hand side with $R_{ij}(\lambda)^T$ adds λ times the i^{th} column of B to the j^{th} column of B. The Lebesgue density of $\mathrm{Sym}(n, \mathbb{C})$ is an alternating form in the coordinate functions, so it vanishes if a coordinate appears twice and we get $d\mu(gSg^T) = d\mu(S)$, which proves the lemma. □

Now, we can proceed to prove the proposition. Consider the case where

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \delta' \end{pmatrix} := \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix}$$

The transformation is given by

$$T = \begin{pmatrix} c_{\alpha',\alpha'} & 0 & 0 \\ 0 & c_{\delta',\delta'} & 0 \\ 0 & 0 & c_{\alpha',\delta'} \end{pmatrix},$$

which gives

$$\begin{aligned} \mathrm{Ber}(T) &= (\det \alpha')^{p+1} \cdot (\det \delta')^{q-1} \cdot \big((\det \alpha')^q \cdot (\det \delta')^p\big)^{-1} \\ &= (\det \alpha')^{p+1-q} \cdot (\det \delta')^{-(p-q+1)} \end{aligned}$$

and proves the first case. In the second case we have

$$\begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix}$$

and we get

$$T = \begin{pmatrix} 1 & c_{\beta',\beta'} & c_{1,\beta'} - c_{\beta',1}(.)^T \\ 0 & 1 & 0 \\ 0 & c_{\beta',1} & 1 \end{pmatrix}$$

so that

$$\begin{aligned}\operatorname{Ber}(T) &= \det\left(\begin{pmatrix}1 & c_{\beta',\beta'}\\ 0 & 1\end{pmatrix} - \begin{pmatrix}c_{1,\beta'} - c_{\beta',1}(.)^T\\ 0\end{pmatrix}\cdot\begin{pmatrix}0 & c_{\beta',1}\end{pmatrix}\right)\cdot\det(1)^{-1}\\ &= \det\left(\begin{pmatrix}1 & c_{\beta',\beta'}\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}\right) = 1 = \operatorname{Ber}\begin{pmatrix}1 & \beta'\\ 0 & 1\end{pmatrix}^{p-q+1}\end{aligned}$$

for the second case. The proof for the third case is the same. □

Now, we consider the relative superinvariant under such a supermatrix decomposition. An invertible supersymmetric matrix $X \in_T O_v$ can be decomposed by Corrollary 4.18 as $X = g\mathcal{J}g^{ST^3}$ with $g \in_T \mathrm{GL}_{p|q}$ and such a g can also be decomposed as in Equation 4.6. Let $g = \left(\begin{smallmatrix}1 & BA^{-1}\\ 0 & 1\end{smallmatrix}\right)$, then we can express X as

$$X = \begin{pmatrix}S & B\\ -B^T & A\end{pmatrix} = g\begin{pmatrix}S+BA^{-1}B^T & 0\\ 0 & A\end{pmatrix}g^{ST^3} = \rho(g)\begin{pmatrix}S+BA^{-1}B^T & 0\\ 0 & A\end{pmatrix}.$$

For a supercharacter χ in general, we obtain:

Lemma 4.24. *For $y \in_T \mathfrak{g}_{\bar{1}}$, it holds that $\chi(\exp(Y)) = 1$.*

Proof. We know χ is an even morphism of Lie supergroups, and that the differential $d\chi : \mathfrak{g} \to \mathbb{C}$ satisfies

$$d\chi|_{\mathfrak{g}_{\bar{1}}} = 0$$

and hence $d\chi(Y) = 0$ for all $Y \in_T \mathfrak{g}_{\bar{1}}$. Furthermore, one has for Lie supergroups the commutative diagram

$$\begin{tikzcd}\mathfrak{g} \arrow[r, "d\chi"] \arrow[d, "\exp_G"'] & \mathbb{C} \arrow[d, "\exp_{\mathbb{C}}"] \\ G \arrow[r, "\chi"] & \mathbb{C}^*,\end{tikzcd}$$

so that $\chi(\exp_G(Y)) = \exp_{\mathbb{C}}(d\chi(Y)) = 1$ for $Y \in_T \mathfrak{g}_{\bar{1}}$. □

We know that the Berezinian to any integer power is an relative superinvariant of the prehomogeneous super vector space of supersymmetric matrices. On the underlying even space, we have a product of two prehomogeneous vector spaces of symmetric matrices Sym(p) and skew-symmetric matrices Skew(q). Their irreducible relative invariants are the determinant (det) and the pfaffian (pf) respectively, (of course $\mathrm{pf}^2 = \det$ is also a relative invariant of $\mathrm{Skew}(q, \mathbb{C})$). By Lemma 4.24, we have $\chi(g) = 1$ for $g = \exp(x)$ with $x \in_T \mathfrak{g}_{\bar{1}}$, and from Equation 4.6 we see that

$$\mathrm{Ber}\left(\begin{smallmatrix} S & B \\ -B^T & A \end{smallmatrix}\right) = \det(S + BA^{-1}B^T) \cdot \det(A)^{-1}.$$

Note, that we express the relative superinvariant Ber as a nilpotent (in the boson–boson sector) of the relative invariants of the underlying prehomogeneous vector spaces. Of course, we also have

$$\mathrm{Ber}\left(\begin{smallmatrix} S & B \\ -B^T & A \end{smallmatrix}\right) = \det(S) \cdot \det(A + B^T S^{-1} B)^{-1}$$

with a shift in the fermion–fermion sector. In order to extend the local zeta function to an entire function, one needs to consider the Bernstein–Sato polynomial of a relative invariant. The notion of the local zeta function of a prehomogeneous vector spaces is explained in the next section.

The function $b(s)$ is called the Bernstein–Sato polynomial. It is attributed to Bernstein [5] and Sato and Shintani [32], who introduced it independently. It is a polynomial related to a differential operator. Here for reductive prehomogeneous vector spaces, the differential operator $f^*(\partial_x)$ is given by the dual relative invariant f^* corresponding to the character χ^{-1} and $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$. The Bernstein–Sato polynomial is of the form

$$b(s) = b_0 \prod_{i=1}^{d} (s + \alpha_i).$$

By [21] it is known that $\alpha_i > 0$ and $\alpha_i \in \mathbb{Q}$.

For regular prehomogeneous vector spaces we have the following proposition.

Proposition 4.25 ([23], Proposition 2.22). *Let f be the relative invariant of a regular prehomogeneous vector space, which we denote by f_b for* $\mathrm{Sym}(\mathrm{p}, \mathbb{C})$ *or by f_f for* $\mathrm{Skew}(q, \mathbb{C})$*, corresponding to the character χ and f^* the dual relative invariant corresponding to χ^{-1}, then*

$$f^*(\partial_x) f(x)^{s+1} = b(s) f(x)^s,$$

where $x = (x_{ij})$ is a vector of the coordinate functions and $\partial_x = (\frac{\partial}{\partial x_{ij}})$ is the corresponding vector of partial derivatives.

In order to derive a similar relation for the relative superinvariant $f(X) = \mathrm{Ber}(X)$, we introduce suitable coordinates. Let $X = \left(\begin{smallmatrix} S & B \\ -B^T & A \end{smallmatrix}\right)$ be the standard coordinates on $S^2(\mathbb{C}^{p|q})$. Let $x = S + BA^{-1}B^T$, $y = A$, $z = B$.

In these coordinates we have $f(X) = \det(x)\det(y)^{-1}$. The corresponding coordinate derivatives are $\partial_x = (\frac{\partial}{\partial x_{ij}})_{1\le i\le j\le p}$, $\partial_y = (\frac{\partial}{\partial y_{ij}})_{1\le i\le j\le q}$ and $\partial_z = (\frac{\partial}{\partial z_{ij}})_{\substack{1\le j\le q\\ 1\le i\le p}}$, which are uniquely determined by $\frac{\partial}{\partial x_{ij}}(x_{kl}) = \delta_{ik}\delta_{jl}$ and $\frac{\partial}{\partial x_{ij}}(y_{kl}) = \frac{\partial}{\partial x_{ij}}(z_{kl}) = 0$ and similarly for y and z, see [27, 3.3.13]

We have by Ref. [23, p. 261]

$$\det(\partial_y)\,(\det(y))^{-s} = (-s)\cdots(-s-1+q)\,(\det(y))^{-(s+1)}\,,$$

so that we get the relation

$$\det(\partial_x)\left(\frac{\det(x)}{\det(y)}\right)^{s+1} = \frac{(s+1)\cdots(s+p)}{(-s)\cdots(-s-1+q)}\ \det(\partial_y)\left(\frac{\det(x)}{\det(y)}\right)^{s}.$$

The Bernstein–Sato relation for Sym(p) is given by

$$\begin{aligned}\det(\partial_x)\,(\det(x))^{s+1} &= (s+1)\left(s+\frac{3}{2}\right)\cdots\left(s+\frac{p+1}{2}\right)(\det(x))^{s}\\ &= \prod_{k=1}^{p}\left(s+\frac{k+1}{2}\right)(\det(x))^{s}\end{aligned}$$

and for Skew(q), it is given by

$$\begin{aligned}\mathrm{pf}(\partial_y)\,(\mathrm{pf}(y))^{s+1} &= (s+1)(s+3)\cdots(s+q-1)\,(\mathrm{pf}(y))^{s}\\ &= \prod_{k=1}^{q}(s+2k-1)\,(\mathrm{pf}(y))^{s}\,.\end{aligned}$$

The reader may, for instance, consult Ref. [23, p.262]. It follows that

$$\begin{aligned}\det(\partial_y)\,(\det(y))^{s} &= \mathrm{pf}(\partial_y)\mathrm{pf}(\partial_y)\mathrm{pf}(y)^{2s}\\ &= \prod_{j=1}^{q}(2s+2j-2)\mathrm{pf}(\partial_y)\mathrm{pf}(y)^{2s-1}\\ &= \prod_{j=1}^{q}(2s+2j-2)\prod_{k=1}^{q}(2s+2k-3)\mathrm{pf}(y)^{2s-2}\\ &= \prod_{j=1}^{q}(2s+2j-2)\prod_{k=1}^{q}(2s+2k-3)\det(y)^{s-1},\end{aligned}$$

and thus

$$\det(\partial_x)\left(\frac{\det(x)}{\det(y)}\right)^{s+1} = \frac{b_b(s)}{b_f(-2s-1)b_f(-2s-2)}\det(\partial_y)\left(\frac{\det(x)}{\det(y)}\right)^{s}$$

$$= \frac{\prod_{k=1}^{p}(s+\frac{k+1}{2})}{\prod_{j=1}^{q}(2j-2s-2)\cdot\prod_{k=1}^{q}(2k-2s-3)}\det(\partial_y)\left(\frac{\det(x)}{\det(y)}\right)^{s},$$

where $b_b(s) = \prod_{k=1}^{p}(s+\frac{k+1}{2})$ and $b_f(s) = \prod_{k=1}^{q}(s+2k-1)$. One can rewrite this relation using the gamma function. Let us define the bosonic gamma function by

$$\gamma_b(s) := \prod_{k=1}^{p} \Gamma(s + \tfrac{k+1}{2}) \tag{4.7}$$

and the fermionic gamma function by

$$\gamma_f(s) := \prod_{k=1}^{q} \Gamma(s + 2k - 1), \tag{4.8}$$

then we can rewrite the relation to get the following theorem.

Theorem 4.26. *For the prehomogeneous super vector space of supersymmetric matrices we have in the super coordinate system (x, y, z) the relation*

$$\frac{\det(\partial_x)}{\gamma_b(s+1)\gamma_f(-2(s+1))}\left(\frac{\det(x)}{\det(y)}\right)^{s+1} = \frac{\det(\partial_y)}{\gamma_b(s)\gamma_f(-2s)}\left(\frac{\det(x)}{\det(y)}\right)^{s}.$$

5 Local zeta superfunctions and Fourier supertransform

In this section, we consider the prehomogeneous super vector spaces of supersymmetric matrices, although our methods are in principle not restricted to this example. We introduce the local zeta superfunctions as regularizations of the integral

$$F_{i,j}(s, \Phi_c) = \frac{1}{\gamma(s)} \int_{V_{ij}} |D(X)| \; |\mathrm{Ber}(X)|^s \; \Phi_c(X),$$

which is a priori defined only for superfunctions with compact support contained in a connected component V_{ij} of the intersection of O_v with a *cs*-form V_{cs} of V. As we show, the regularizations form families of tempered superdistributions, analytic on the whole complex plane. Moreover, the regularizations preserve the relative superinvariance of the Berezinian.

Furthermore, we show in Proposition 5.23 for an abitrary regular prehomogeneous super vector space that the Fourier supertransform is, up to a supercharacter, an equivariant map:

$$\widehat{g.\Phi}(w) = \; |\chi(g)|^{-\frac{m-n}{d_b-d_f}} \cdot g.\widehat{\Phi}(w).$$

We are able to show that the Fourier supertransform of the dual local zeta superfunctions $\widehat{F^*_{k,l}}(s - \frac{m-n}{d_b-d_f}, .)$, considered as tempered superdistributions, transform by the same supercharacter as the local zeta superfunction $F_{i,j}(-s, .)$ and that

$$\begin{aligned} g.F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi}) &= |\chi(g)|^{-s} \cdot F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi}), \\ g.F_{i,j}(-s, \Phi) &= |\chi(g)|^{-s} \cdot F_{i,j}(-s, \Phi). \end{aligned}$$

Classically, this property is sufficient to show that the Fourier transform of the dual local zeta function and the local zeta function are equal, up to a constant depending only on the complex parameter. Besides the fact that the local zeta functions are entire functions, this is the main content of the Fundamental Theorem of Prehomogeneous Vector Spaces proved 1961 by Sato [23, Theorem 4.17].

5.1 Preliminaries

We recall the definition of a *cs-form* and a *cs-affine superspace* of V. Given a complex super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ has a real form,

$V_{\bar{0}} = V_{\bar{0},\mathbb{R}} \oplus \mathrm{i}V_{\bar{0},\mathbb{R}}$, then the cs-vector space $V_{cs} := V_{\bar{0},\mathbb{R}} \oplus V_{\bar{1}}$ is called a cs-form of V. The cs-affine superspace, associated to a cs-vector space and also denoted by V_{cs}, is given by $V_{cs} = (V_{\bar{0},\mathbb{R}}, \mathcal{O}_{V_{cs}})$, where $\mathcal{O}_{V_{cs}} := \mathcal{C}^{\infty}_{V_{\bar{0},\mathbb{R}}} \otimes \bigwedge V_{\bar{1}}^*$ and $\mathcal{C}^{\infty}$ denotes the sheaf of complex-valued smooth functions.

From now on, we assume that (G, ρ, V) is a regular prehomogeneous super vector space (recall that $(G_0, \rho_0, V_{\bar{0}})$ is a regular prehomogeneous vector space) and analogous to the classical case

- a real form $V_{\bar{0},\mathbb{R}}$ of $V_{\bar{0}}$,
- a real form $G_{0,\mathbb{R}} := G_0 \cap (\mathrm{GL}(\mathrm{m}, \mathbb{R}) \times \mathrm{GL}(\mathrm{n}, \mathbb{R}))$ of G_0, and
- a linear action $a_0 : G_{0,\mathbb{R}} \times V_{\bar{0},\mathbb{R}} \to V_{\bar{0},\mathbb{R}}$

such that the following diagram commutes

$$\begin{array}{ccc} G_0 \times V_{\bar{0}} & \xrightarrow{\;a_0\;} & V_{\bar{0}} \\ \uparrow & & \uparrow \\ G_{0,\mathbb{R}} \times V_{\bar{0},\mathbb{R}} & \xrightarrow{\;a_0\;} & V_{\bar{0},\mathbb{R}}. \end{array}$$

By definition, the cs-supergroup $G_{cs} = (G_{0,\mathbb{R}}, \mathcal{O}_{G_{cs}})$ with $\mathcal{O}_{G_{cs}} := \mathcal{O}_G|_{G_{0,\mathbb{R}}}$ associated to G (where $G \subseteq \mathrm{GL}(m|n, \mathbb{C})$) leaves $V_{cs} := V_{\bar{0},\mathbb{R}} \oplus V_{\bar{1}}$ invariant. Let G_{cs}^+ be the open subsupergroup whose underlying group is the connected component of the identity in $G_{0,\mathbb{R}}$. The underlying vector space $V_{\bar{0}}$ is then a classical regular prehomogeneous vector space with $\mathbb{R}$-structure $V_{\bar{0},\mathbb{R}}$. The $\mathbb{R}$-structure of the open orbit $(O_v)_0$ in $V_{\bar{0}}$ is $O_{v,\mathbb{R}}$ and can be decomposed into $O_{v,\mathbb{R}} = \bigcup_{i=1}^{l} V_{\bar{0},i}$ (compare with [23, Proposition 4.5]). This decomposition induces a decomposition of the cs-orbit

$$(O_v)_{cs} = \bigcup_{i=1}^{l} V_i$$

with $V_i = (V_{\bar{0},i}, \mathcal{O}_{V_i})$ and $\mathcal{O}_{V_i} := \mathcal{O}_{V_{cs}}|_{V_{\bar{0},i}}$. Moreover,

$$a_0(G_{0,\mathbb{R}}^+, v_i) = \rho_0(G_{0,\mathbb{R}}^+)v_i = V_{\bar{0},i}$$

with $v_i \in V_{\bar{0},i}$, which induces for the corresponding cs-supergroup G_{cs}^+ that

$$\rho(G_{cs}^+(T))v_i = V_i(T)$$

for $v_i \in_T V_i$.

Let V_{cs} be a cs-form of V, considered as a cs-affine superspace. Furthermore, let $x = (u, \xi)$ and $y = (v, \eta)$ with $x_i, y_j \in \Gamma(\mathcal{O}_{V_{cs}})$ be a global supercoordinate system of V_{cs}, where the underlying coordinates are equally oriented.

Now, we need the notion of a retraction.

Definition 5.1. A morphism $\gamma : X \to X_0$ is called a *retraction* if it is a right inverse of the canonical embedding j_X, i.e.

$$\gamma \circ j_X = \mathrm{id}_X.$$

Let $u_1, \ldots, u_p$ be a coordinate system of X_0 and $u_1, \ldots, u_p, \xi_1, \ldots, \xi_q$ a coordinate system of X, the associated retraction γ is called the standard retraction.

A compactly supported superfunction f can be written as $f = \sum_{I \in \mathbb{Z}_2^n} \gamma^*(f_I)\eta^I$ with $f_I \in \Gamma_c(\mathcal{C}^\infty_{V_{\bar{0},\mathbb{R}}})$ and γ the standard retraction. Here, the letter c stands for compactly supported. The flat Berezinian measure is given by

$$D(v, \eta) = dv_1 \wedge \cdots \wedge dv_m \otimes \frac{\partial}{\partial \eta_n} \cdots \frac{\partial}{\partial \eta_1}$$

and the Berezinian integral is defined by

$$\begin{aligned} \int_{V_{cs}} D(v, \eta)\ f(v, \eta) &:= \int_{V_{cs}} dv_1 \cdots dv_m\ \frac{\partial}{\partial \eta_n} \cdots \frac{\partial}{\partial \eta_1}\ f(v_1, \ldots, v_m, \eta_1, \ldots, \eta_n) \\ &= \int_{V_{0,\mathbb{R}}} dv_1 \cdots dv_m\ f_{1,\ldots,1}(v_1, \ldots, v_m). \end{aligned}$$

The Berezinian of a coordinate transformation, which is a map $\alpha : V_{cs} \to V_{cs}$ with $\alpha^*(y_i)$ a superfunction in the coordinates x_i, is defined by

$$\frac{D(v, \eta)}{D(u, \xi)} := \mathrm{Ber}(J^\alpha_{x,y}) = \mathrm{Ber}\begin{pmatrix} \frac{\partial \alpha^*(v)}{\partial u} & -\frac{\partial \alpha^*(v)}{\partial \xi} \\ \frac{\partial \alpha^*(\eta)}{\partial u} & \frac{\partial \alpha^*(\eta)}{\partial \xi} \end{pmatrix}.$$

The following proposition tells us how the Berezinian integral for compactly supported superfunctions transforms under such a coordinate transformation.

Proposition 5.2 ([27], Theorem 2.4.5). *Let $x = (u, \xi)$ and $y = (v, \eta)$ be two coordinate systems on V_{cs}, with the same orientation, and f compactly supported, then*

$$\int_{V_{cs}} D(v, \eta)\ f = \int_{V_{cs}} D(u, \xi)\ \frac{D(v, \eta)}{D(u, \xi)}\ \alpha^*(f).$$

Remark 5.3. Without the assumption that f is compactly supported, one has to regard a boundary term, which would appear by transforming the measure. Applied to the superfunction f this boundary term gives after integrating out the odd variables an exact m-form.

Now we introduce the space of Schwartz superfunctions and the space of tempered superdistributions for a cs-vector space V_{cs}. We are following Appendix C in [3]. Let $S(V)$ be the symmetric superalgebra and $j_{V_{\bar{0}}}$ the embedding of $V_{\bar{0},\mathbb{R}}$ in the cs-affine superspace V_{cs}.

Definition 5.4. A superfunction $\Phi \in \Gamma(\mathcal{O}_{V_{cs}})$ will be called tempered if for every $u \in S(V_{cs})$, there exist $N > 0$ such that

$$\sup_{x \in V_{\bar{0}}} ||x||^N |j^*_{V_{\bar{0}}}(\partial_u \Phi)(x)| < \infty. \tag{5.1}$$

Definition 5.5. Similarly, a superfunction $\Phi \in \Gamma(\mathcal{O}_{V_{cs}})$ is a Schwartz superfunction if for any $u \in S(V_{cs})$ and any $N > 0$

$$p_{N,u}(\Phi) := \sup_{x \in V_{\bar{0}}} ||x||^N |j^*_{V_{\bar{0}}}(\partial_u \Phi)(x)| < \infty. \tag{5.2}$$

The totality of all tempered superfunctions is denoted by $\Gamma_{\text{temp}}(\mathcal{O}_{V_{cs}})$ and the space of Schwartz superfunctions are denoted by $\mathscr{S}(V_{cs})$. The space $\mathscr{S}(V_{cs})$ is endowed with the locally convex topology defined by the semi-norms $p_{N,u}$. By $\mathscr{S}'(V_{cs})$ we denote the topological dual space of $\mathscr{S}(V_{cs})$, with the strong topology. The elements of $\mathscr{S}'(V_{cs})$ are called tempered superdistributions.

The space of *Schwartz superfunctions on* V_{cs} is also given by

$$\mathscr{S}(V_{cs}) := \mathscr{S}(V_{\bar{0},\mathbb{R}}) \otimes \bigwedge V^*_{\bar{1}},$$

where $\mathscr{S}(V_{\bar{0},\mathbb{R}})$ is the classical space of Schwartz function on the vector space $V_{\bar{0},\mathbb{R}}$.

Let $V_{\bar{0},\mathbb{R}} = U_{\bar{0},\mathbb{R}} \oplus W_{\bar{0},\mathbb{R}}$ be the even part of the cs-vector space, then by [3, Corollary C.10], there is an isomorphism between the locally convex super vector spaces

$$\mathscr{S}(V_{cs}) \cong \left(\mathscr{S}(U_{\bar{0},\mathbb{R}}) \widehat{\otimes} \mathscr{S}(W_{\bar{0},\mathbb{R}})\right) \otimes \bigwedge V^*_{\bar{1}}, \tag{5.3}$$

where $\widehat{\otimes}$ denotes the completed projective tensor product topology, for instance the reader may consult [37, Theorem 51.6].

Let us define an action of G^+_{cs} on the space of superfunctions $\Gamma(\mathcal{O}_{V_{cs}})$.

Definition 5.6. For $g \in_T G_{cs}^+$, $\Phi \in \Gamma(\mathcal{O}_{V_{cs}})$ and $\Phi^* \in \Gamma(\mathcal{O}_{V_{cs}^*})$, we define $\rho(g).\Phi \in \Gamma(\mathcal{O}_{T\times V_{cs}})$ by

$$(\rho(g).\Phi)(p,v) := \Phi(\rho(g_{T'})v),$$

and $\rho^*(g).\Phi^* \in \Gamma(\mathcal{O}_{T\times V_{cs}^*})$ by

$$(\rho^*(g).\Phi^*)(p,v) := \Phi(\rho^*(g_{T'})v),$$

$p : T' \to T$, $v \in_{T'} V_{cs}$. Here, $g_{T'} := g \circ p$.

This definition automatically give actions of G_{cs} on the spaces of Schwartz superfunctions $\mathscr{S}(V_{cs})$ and $\mathscr{S}(V_{cs}^*)$ respectivley. If $\Phi \in \mathscr{S}(V_{cs})$, then $\rho(g).\Phi \in \Gamma(\mathcal{O}_T)\widehat{\otimes}\mathscr{S}(V_{cs})$ and if $\Phi^* \in \mathscr{S}(V_{cs}^*)$, then $\rho^*(g).\Phi^* \in \Gamma(\mathcal{O}_T)\widehat{\otimes}\mathscr{S}(V_{cs}^*)$ respectively by [3, Appendix C.2].

Induced by the standard pairing we also get actions of G_{cs} on the spaces of tempered superdistributions $\mathscr{S}'(V_{cs})$ and $\mathscr{S}'(V_{cs}^*)$ respectivley.

Definition 5.7. For $F \in \mathscr{S}'(V_{cs})$ we define for all $\Phi \in \mathscr{S}(V_{cs})$ the action on the tempered superdistributions $\rho(g).F \in \Gamma(\mathcal{O}_T)\widehat{\otimes}\mathscr{S}'(V_{cs})$ with $g \in_T G_{cs}^+$ by

$$\langle \rho(g).F, \Phi\rangle := \langle F, \rho(g).\Phi\rangle \in \Gamma(\mathcal{O}_T),$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing $\mathscr{S}'(V_{cs}) \otimes \mathscr{S}(V_{cs}) \to \mathbb{C}$.

Analogously for $F^* \in \mathscr{S}'(V_{cs}^*)$ and $g \in_T G_{cs}^+$, we define for all $\Phi^* \in \mathscr{S}(V_{cs}^*)$ the action on the tempered superdistributions $\rho^*(g).F^* \in \Gamma(\mathcal{O}_T)\widehat{\otimes}\mathscr{S}'(V_{cs}^*)$ by

$$\langle \rho^*(g).F^*, \Phi^*\rangle := \langle F^*, \rho^*(g).\Phi^*\rangle \in \Gamma(\mathcal{O}_T),$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing $\mathscr{S}'(V_{cs}^*) \otimes \mathscr{S}(V_{cs}^*) \to \mathbb{C}$.

Now we extend Proposition 5.2 to Schwartz superfunctions. By [3, Appendix C.9] is the space of compactly supported superfunctions dense in the space of Schwartz superfunctions. We assume that the transformation $\alpha : V_{cs} \to V_{cs}$ is polynomial. Hence, $\alpha^*(\Phi)$ is obviously a Schwartz superfunction. Under this assumption and the fact that the Berezinian integral is a continuous map between the space of Schwartz superfunctions to $\mathbb{C}$, we can extend Proposition 5.2 to Schwartz superfunctions.

Definition 5.8. The morphism $\alpha : V_{cs} \to V_{cs}$ is called a polynomial transformation, if α is an isomorphism and $\alpha^\sharp(x_i) \in \mathbb{C}[x_1, \ldots, x_{p+q}]$ for all $i \in \{1, \ldots, p+q\}$ with x a supercoordinate system of V_{cs}.

Corollary 5.9. *For all $\Phi \in \mathscr{S}(V_{cs})$ and a polynomial transformation $\alpha : V_{cs} \to V_{cs}$ we have*

$$\int_{V_{cs}} D(v,\eta)\ \Phi = \int_{V_{cs}} D(u,\xi)\ \frac{D(v,\eta)}{D(u,\xi)}\ \alpha^\sharp(\Phi).$$

Before we define the local zeta superfunctions, let us define $|f|$. Note, that one has $\Gamma(\mathcal{O}_{V_{cs}}) \cong \mathrm{Mor}(V_{cs}, \mathbb{C}^{1|1})$ and even superfunctions can be regarded as $\mathrm{Mor}(V_{cs}, \mathbb{C})$.

Definition 5.10. Let $f \in \Gamma(\mathcal{O}_{\bar{0},V_{cs}})^*$ an invertible superfunction on V_{cs} and $|.| : \mathbb{C} \to \mathbb{R}^{\geq 0}$ the usual norm of the complex numbers, then we define

$$|f| := f^\sharp(|.|) \in \mathrm{Mor}(V_{cs}, \mathbb{R}^+).$$

5.2 The local zeta superfunctions

In this section, we define the local zeta superfunctions for the prehomogeneous super vector space $S^2(\mathbb{C}^{p|q})$ of supersymmetric matrices, where q is even. Let (S, A, ξ) be a global supercoordinate system, where $S = (s_{1,1}, \ldots, s_{i,j}, \ldots, s_{p,p})$, $A = (a_{1,2}, \ldots, a_{k,l}, \ldots, a_{q-1,q})$ and $\xi = (\xi_{1,1}, \ldots, \xi_{p,q})$ with $i \leq j$ and $k < l$. We denote by $\mathrm{Sym}(p)$ the complex vector space of $p \times p$ symmetric matrices, which we also call the boson–boson sector; and we denote by $\mathrm{Skew}(q)$ the complex vector space of skew-symmetric matrices, also called the fermion–fermion sector. Together $\mathrm{Sym}(p) \times \mathrm{Skew}(q)$ form the even part of $S^2(\mathbb{C}^{p|q})$. A supermatrix $X \in_T S^2(\mathbb{C}^{p|q})$ can be represented by a $p \times p$ symmetric matrix S, a $q \times q$ skew-symmetric matrix A, both with entries in $\mathcal{O}_T(T)_{\bar{0}}$, and a $p \times q$ matrix ξ with entries in $\mathcal{O}_T(T)_{\bar{1}}$. It is of the form

$$X = \begin{pmatrix} S & \xi \\ -\xi^T & A \end{pmatrix}.$$

(The subscript b is related to the boson–boson sector and the subscript f to the fermion–fermion sector of the supermatrix X.) The aim of this section is to find a superdistribution, that regularizes the expression

$$\int_{V_{ij}} |D(X)|\ |\mathrm{Ber}(X)|^s\ \Phi_c(X), \tag{5.4}$$

where $|D(X)|$ is the flat Berezinian measure, $\mathrm{Ber}(X)$ a relative superinvariant and $\Phi_c(X)$ a compactly supported superfunction on the connected component V_{ij} of the cs-orbit.

The vector spaces of $\mathrm{Sym}(p)$ and $\mathrm{Skew}(q)$ are prehomogeneous vector spaces and we recall some facts about them.

The local zeta functions for Sym(p) are

$$F_{i,b}(s,\Phi) := \frac{1}{\gamma_b(s)} \int_{V_{i,b}} |dS| \, |\det(S)|^s \; \Phi(S)$$

$$F^*_{i,b}(s,\Phi^*) := \frac{1}{\gamma_b(s)} \int_{V^*_{i,b}} |dS'| \, |\det(S')|^s \; \Phi^*(S'),$$

where $\gamma_b(s) := \prod_{k=1}^{p} \Gamma(s + \frac{k+1}{2})$, as in Equation 4.7. For this prehomogeneous vector space we have the determinant as the irreducible relative invariant, the real symmetric matrices $\mathrm{Sym}(p,\mathbb{R})$ as the real form, $\Phi \in \mathscr{S}(\mathrm{Sym}(p,\mathbb{R}))$ and $\Phi^* \in \mathscr{S}(\mathrm{Sym}(p,\mathbb{R})^*)$, and the intersection of the open orbit $O_{\mathrm{Sym}(p)}$ with $\mathrm{Sym}(p,\mathbb{R})$ are the invertible real symmetric matrices $O_{\mathrm{Sym}(p,\mathbb{R})}$. They decompose into $p+1$ connected components $V_{1,b},\dots,V_{p+1,b}$.

The local zeta functions of Skew(q), where q is even, are

$$F_{j,f}(s,\Phi) := \frac{1}{\gamma_f(s)} \int_{V_{j,f}} |dA| \; |\mathrm{pf}(A)|^s \Phi(A)$$

$$F^*_{j,f}(s,\Phi^*) := \frac{1}{\gamma_f(s)} \int_{V^*_{j,f}} |dA'| \; |\mathrm{pf}(A')|^s \; \Phi^*(A'),$$

where $\gamma_f(s) := \prod_{l=1}^{\frac{q}{2}} \Gamma(s + 2l - 1)$ as in Equation 4.8. For this prehomogeneous vector space we have the Pfaffian pf as irreducible relative invariant, the real skew-symmetric matrices $\mathrm{Skew}(q,\mathbb{R})$ as the real form, $\Phi \in \mathscr{S}(\mathrm{Skew}(q,\mathbb{R}))$ and $\Phi^* \in \mathscr{S}(\mathrm{Skew}(q,\mathbb{R})^*)$, and the intersection of the open orbit $O_{\mathrm{Skew}(q)}$ with $\mathrm{Skew}(q,\mathbb{R})$ are the invertible real skew-symmetric matrices. They decompose in two connected components $V_{1,f}, V_{2,f}$.

Sato proved the following statement in 1961, which Kimura calls the Fundamental theorem of prehomogeneous vector spaces in [23, Theorem 4.17].

Theorem 5.11. *Let (G,ρ,V) be a reductive prehomogenous vector space and f a irreducible relative invariant corresponding to a character χ and f^* a relative invariant of the dual prehomogeneous vector space (G,ρ^*,V^*) corresponding to χ^{-1}. Then, the local zeta functions*

$$F_i(s,\Phi) := \frac{1}{\gamma(s)} \int_{V_i} dx \; |f(x)|^s \; \Phi(x)$$

and

$$F^*_j(s,\Phi^*) := \frac{1}{\gamma(s)} \int_{V^*_j} dy \; |f^*(y)|^s \; \Phi^*(y),$$

extend analytically to holomorphic functions on the whole s-plane. Furthermore the following holds:

$$\int_{V^*_j} dy \; |f^*(y)|^{s-\frac{n}{d}} \; \widehat{\Phi}(y) = \gamma(s - \tfrac{n}{d}) \cdot \sum_{j=1}^{l} c_{ij}(s) \int_{V_i} dx \; |f(x)|^{-s} \; \Phi(x) \qquad (5.5)$$

where $c_{ij}(s)$ are entire functions which do not depend on $\Phi \in \mathscr{S}(V_{\mathbb{R}})$.

By the last theorem we can consider $F_{i,b}$ and $F_{j,f}$ as entire functions. Let $V_{ij,\bar{0}} := V_{i,b} \times V_{j,f}$ and V_{ij} be the corresponding open sub superspace of V_{cs}.

Moreover, let $X \in_T S^2(\mathbb{C}^{p|q})$ be the supermatrix

$$X = \begin{pmatrix} S & \xi \\ -\xi^T & A \end{pmatrix}.$$

We define, when A is invertible, the bosonically-shifted supermatrix $X_{b\text{-shift}}$ of X by

$$X_{b\text{-shift}} := \begin{pmatrix} S - \xi A^{-1} \xi^T & \xi \\ -\xi^T & A \end{pmatrix}$$

and when S is invertible, the fermionically-shifted supermatrix $X_{f\text{-shift}}$ of X by

$$X_{f\text{-shift}} := \begin{pmatrix} S & \xi \\ -\xi^T & A - \xi^T S^{-1} \xi \end{pmatrix}.$$

Below we will also use for the flat Berezinian measure $|D(X)|$ the notation $|D\xi||dS||dA|$. The norm of an invertible superfunction $|f|$ is defined in Definition 5.10. Now, we can define the local zeta superfunctions for the prehomogeneous super vector space $S^2(\mathbb{C}^{p|q})$ and its dual.

Definition 5.12. The local zeta superfunctions for the prehomogeneous super vector space $S^2(\mathbb{C}^{p|q})$ are defined by

$$F_{i,j}(s, \Phi) := \int_{V_{\bar{1}}} |D\xi| \; F_{j,f}\left(-2(s + \tfrac{pq}{2}), |\det(A)|^{\frac{pq}{2}} \cdot F_{i,b}\left(s, \Phi(X_{b\text{-shift}})\right)\right),$$

$$\widetilde{F}_{i,j}(s, \Phi) := \int_{V_{\bar{1}}} |D\xi| \; F_{i,b}\left(s - \tfrac{pq}{2}, |\det(S)|^{\frac{pq}{2}} \cdot F_{j,f}\left(-2s, \Phi(X_{f\text{-shift}})\right)\right).$$

For the dual case, we define the local zeta superfunction by

$$F^*_{i,j}(s, \Phi^*) := \int_{V^*_{\bar{1}}} |D\xi'| \, F^*_{j,f}\left(-2(s + \frac{pq}{2}), |\det(A')|^{\frac{pq}{2}} \cdot F^*_{i,b}\left(s, \Phi^*(X'_{b\text{-shift}})\right)\right),$$

$$\widetilde{F}^*_{i,j}(s, \Phi^*) := \int_{V^*_{\bar{1}}} |D\xi'| \; F^*_{i,b}\left(s - \frac{pq}{2}, |\det(S')|^{\frac{pq}{2}} \cdot F^*_{j,f}\left(-2s, \Phi^*(X'_{f\text{-shift}})\right)\right).$$

Theorem 5.13. *The functions $F_{i,j}(s, \Phi)$ and $F^*_{i,j}(s, \Phi^*)$ are entire functions on the whole s-plane for each $\Phi \in \mathscr{S}(S^2(\mathbb{C}^{p|q}_{cs}))$. Moreover, for any compactly supported superfunction Φ_c on V_{ij}, we have*

$$F_{i,j}(s, \Phi_c) = \frac{1}{\gamma(s)} \int_{V_{ij}} |D(X)| \; |\mathrm{Ber}(X)|^s \; \Phi_c(X).$$

Therefore, $F_{i,j}(s,\Phi)$ is a regularization of the right-hand side to the greater class of Schwartz superfunctions on $\mathscr{S}(S^2(\mathbb{C}_{cs}^{p|q}))$.

Remark 5.14. Later we show that this regularization still respects the algebraic properties of the Berezinian.

Before we prove the statement of the theorem, we need the following lemma.

Lemma 5.15. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree d, $\Phi \in \mathscr{S}(\mathbb{R}^n)$ a Schwartz function, and $s \in \mathbb{C}$. Then the function $|h|^s\Phi$ is integrable over $\mathbb{R}^n$ for $\mathrm{Re}(s) > -\frac{n}{d}$.*

Proof. In the first step, we transform the integral to spherical coordinates and obtain

$$\left|\int_{\mathbb{R}^n} |h(x)|^s \Phi(X)dx\right| = \left|\int_{\mathbb{R}^+}\int_{S^n} |h(rw)|^s\Phi(rw)r^{n-1}dw\ dr\right|.$$

Because the integrand is a continuous function and attains its maximum on a compact set, we can define $\widetilde{\Phi}(r) := \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \cdot \max_{w\in S^n} |h(w)|^s|\Phi(rw)|$. Here, the factor $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the surface area of S^n. Hence,

$$\left|\int_{\mathbb{R}^+}\left(\int_{S^n} |h(w)|^s\Phi(rw)dw\ \right) r^{sd+n-1}dr\right| \ \leq\ \int_{\mathbb{R}^+} \widetilde{\Phi}(r)r^{\mathrm{Re}(s)d+n-1}dr.$$

In the following expression the second integral is absolutely convergent for all $s \in \mathbb{C}$

$$\int_0^1 \widetilde{\Phi}(r)r^{\mathrm{Re}(s)d+n-1}dr + \int_1^\infty \widetilde{\Phi}(r)r^{\mathrm{Re}(s)d+n-1}dr$$

and the first integral is absolutely convergent for $\mathrm{Re}(s) > -\frac{n}{d}$, since

$$\begin{aligned}\int_0^1 \widetilde{\Phi}(r)r^{\mathrm{Re}(s)d+n-1}dr \ &\leq C\int_0^1 r^{\mathrm{Re}(s)d+n-1}dr \\ &=\ C\left[\frac{r^{\mathrm{Re}(s)d+n}}{\mathrm{Re}(s)d+n}\right]_0^1\end{aligned}$$

where $C := \max_{r\in[0,1]} \widetilde{\Phi}(r)$. $\square$

Remark 5.16. The statement of Lemma 5.15 holds also for Φ with values in some Banach space.

As a corollary, we obtain the following:

Corollary 5.17. *Let $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ be homogeneous polynomials of degree d_h and d_g, then the function $|h|^s|g|^{-s}\Phi$ with $\Phi \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m)$ is integrable over $\mathbb{R}^n \times \mathbb{R}^m$ for $-\frac{n}{d_h} < Re(s) < \frac{m}{d_g}$.*

Proof of Theorem 5.13. We set $\gamma(s) := \gamma_b(s)\gamma_f(-2(s + \frac{pq}{2}))$ and define for $\det(A) \neq 0$, the parameter-dependent integral

$$F_{j,f}(A, s, \Phi) = \\ = \frac{1}{\gamma(s)} |\det(A)|^{-s} \int_{V_{b,i} \times V_{\bar{1}}} |D(S, \xi)| \ \left|\det(S + \xi A^{-1}\xi^T)\right|^s \Phi(S, A, \xi).$$

This integral is absolutely convergent for $-\frac{p+1}{2} < \mathrm{Re}(s)$ by Lemma 5.15. Furthermore, the measure $|D(S, \xi)|$ is invariant under the translation $S' = S + \xi A^{-1}\xi^T,\ \xi' = \xi$, which Jacobian has the Berezinian equal to one. Since the transformation is polynomial

$$F_{j,f}(A, s, \Phi) = \\ \frac{1}{\gamma(s)} |\det(A)|^{-s} \int_{V_{b,i} \times V_{\bar{1}}} |D(S, \xi)| \ |\det(S)|^s \Phi(S - \xi A^{-1}\xi^T, A, \xi),$$

by Corollary 5.9. We express $\Phi(S - \xi A^{-1}\xi^T, A, \xi)$ as a Taylor expansion, for which we will use the following multi-index notation.

Let $\beta = (\beta_{11}, \ldots, \beta_{ij}, \ldots, \beta_{pp}) \in \mathbb{N}^{p\times p}$ be a multi-index with

$$|\beta| := |\beta_{11}| + \cdots + \beta_{ij} + \cdots + |\beta_{pp}| \ \text{ and } \ \beta! := \beta_{11}! \cdots \beta_{ij}! \cdots \beta_{pp}!.$$

Moreover, we abbreviate $\partial_S^\beta := \partial_{S_{11}}^{\beta_{11}} \cdots \partial_{S_{ij}}^{\beta_{ij}} \cdots \partial_{S_{pp}}^{\beta_{pp}}$ and

$$(\xi A^{-1}\xi^T)^\beta := (\xi A^{-1}\xi^T)_{11}^{\beta_{11}} \cdots (\xi A^{-1}\xi^T)_{ij}^{\beta_{ij}} \cdots (\xi A^{-1}\xi^T)_{pp}^{\beta_{pp}}.$$

The Taylor expansion is then

$$\Phi(S - \xi A^{-1}\xi^T, A, \xi) = \sum_{\beta=0} \frac{1}{\beta!}(\partial_S^\beta \Phi)(S, A, \xi) \cdot (-\xi \cdot A^{-1} \cdot \xi^T)^\beta.$$

The inverse matrix of A can expressed in terms of the adjugate $\mathrm{adj}(A)$ of A, ie. $A^{-1} = \det(A)^{-1} \cdot \mathrm{adj}(A)$. Hence, we get

$$\sum_{\beta=0} \frac{(-1)^{|\beta|}}{\beta!} \det(A)^{-|\beta|}(\partial_S^\beta \Phi)(S, A, \xi) \cdot \left(\xi \cdot \mathrm{adj}(A) \cdot \xi^T\right)^\beta.$$

The sum is finite because for $|\beta| > \frac{pq}{2}$, $(\xi A^{-1}\xi^T)^\beta$ is zero. The expression is then equal to

$$\det(A)^{-\frac{pq}{2}} \sum_{0\leq|\beta|\leq\frac{pq}{2}} \frac{(-1)^{|\beta|}}{\beta!} \det(A)^{\frac{pq}{2}-|\beta|} (\partial_S^\beta \Phi)(S,A,\xi) \cdot \left(\xi \cdot \operatorname{adj}(A) \cdot \xi^T\right)^\beta .$$

For brevity, we define

$$\widetilde{\Phi}(S,A,\xi) := \sum_{0\leq|\beta|\leq\frac{pq}{2}} \frac{(-1)^{|\beta|}}{\beta!} \det(A)^{\frac{pq}{2}-|\beta|} (\partial_S^\beta \Phi)(S,A,\xi) \cdot \left(\xi \cdot \operatorname{adj}(A) \cdot \xi^T\right)^\beta .$$

As the sum of products of superpolynomials with Schwartz superfunctions, it is a Schwartz superfunction. By this calculation, we obtain

$$\begin{aligned} F_{j,f}(A,s,\Phi) =& \frac{1}{\gamma_f(-2(s+\frac{pq}{2}))} \operatorname{sign}(\det(A))^{\frac{pq}{2}} |\det(A)|^{-(s+\frac{pq}{2})} \\ & \frac{1}{\gamma_b(s)} \int_{V_{b,i}\times V_{\bar{1}}} |D(S,\xi)| \, |\det(S)|^s \, \widetilde{\Phi}(S,A,\xi). \end{aligned}$$

From Equation 5.3 we have the isomorphism

$$\mathscr{S}(S^2(\mathbb{C}^{p|q})_{cs}) \cong \left(\mathscr{S}(\operatorname{Sym}(p,\mathbb{R})) \widehat{\otimes} \mathscr{S}(\operatorname{Skew}(q,\mathbb{R}))\right) \otimes \bigwedge (V_{\bar{1}})^*.$$

This and Theorem 5.11 imply that the function

$$\psi(s,A,\xi) = \frac{1}{\gamma_b(s)} \int_{V_{b,i}} |dS| \, |\det(S)|^s \, \widetilde{\Phi}(S,A,\xi)$$

with values in the space of Schwartz superfunction $\mathscr{S}(\operatorname{Skew}(q,\mathbb{R})) \otimes \bigwedge(V_{\bar{1}})^*$ extends as an entire function in s to the whole s-plane. Integrating out the odd variables, the function

$$\psi(s,A) := \frac{1}{\gamma_b(s)} \int_{V_{b,i}\times V_{\bar{1}}} |D(S,\xi)| \, |\det(S)|^s \, \psi(s,A,\xi)$$

is a Schwartz function in A. Now, we consider the integral

$$\frac{1}{\gamma_f(-2(s+\frac{pq}{2}))} \int_{V_{f,j}} |dA| \; |\operatorname{pf}(A)|^{-2(s+\frac{pq}{2})} \; \psi(s,A),$$

where $\operatorname{pf}(A)$ is the Pfaffian of A. This integral is absolutly convergent for $s < -\frac{pq}{2}$ and again extends analytically by Theorem 5.11. It follows that

$$F_{i,j}(s,\Phi) := \int_{V_{\bar{1}}} |D\xi| \; F_{j,f}\left(-2(s+\tfrac{pq}{2}), |\det(A)|^{\frac{pq}{2}} \cdot F_{i,b}\left(s, \Phi(X_{b\text{-shift}})\right)\right),$$

extends as an entire function to the whole s-plane. As we have seen, it regularizes the integral

$$\int_{V_{ij}} |D(X)| \, |\operatorname{Ber}(X)|^s \; \Phi_c(X).$$

The other local zeta superfunctions are analogously entire functions. □

The next expression is a representation of the function $F_{i,j}(s,\Phi)$, which is integrable for $-\frac{p+1}{2} < \mathrm{Re}(s) < \frac{q-1}{2}$. It is given by

$$F_{i,j}(s,\Phi) = \tfrac{(-1)^{d_f \cdot pq}}{\gamma_b(s)\gamma_f(-2s)} \int_{V_{ij}} |D(S,A,\xi)| \left|\frac{\det(S)}{\det(A)}\right|^s (\mathrm{pf}(\partial_A))^{pq}\, \widetilde{\Phi}(S,A,\xi),$$

with $d_f = \deg(\mathrm{pf})$ and $\partial_A = (\partial_{a_{1,2}}, \dots, \partial_{a_{q-1,q}})$. Here, we have used

$$\begin{aligned} F_{j,f}(-2s-pq,\Phi) &= \tfrac{1}{\gamma_f(-2s-pq)} \int_{V_{f,j}} |dA|\; |\mathrm{pf}(A)|^{-2s-pq}\; \Phi(A) \\ &= \tfrac{(-1)^{d_f \cdot pq}}{\gamma_f(-2s)} \int_{V_{f,j}} |dA|\; |\mathrm{pf}(A)|^{-2s} \mathrm{pf}(\partial_A)^{pq}\; \Phi(A), \end{aligned}$$

which follows from [23, Proposition 4.7] for $pq \in \mathbb{N}$, and Corollary 5.17.

Furthermore, one can express, using the same idea, the local zeta superfunction $F_{i,j}(s,\Phi)$ for $m > s + pq$ by

$$F_{i,j}(s,\Phi) = \tfrac{(-1)^{d_f \cdot m}}{\gamma_b(s)\gamma_f(2m-2s-pq)} \int_{V_{ij}} |D(S,A,\xi)| \cdot$$

$$\cdot |\det(S)|^s |\det(A)|^{m-s-\frac{pq}{2}} \mathrm{pf}(\partial_A)^{2m}\; |\det(A)|^{\frac{pq}{2}} \Phi(X_{b\text{-shift}}). \tag{5.6}$$

It is absolutly convergent for $s > 0$ and $m > s + pq$. The following lemma will be needed for the proof of the next theorem.

Lemma 5.18 ([23], Proposition 2.21)**.** *The differential operator* $\mathrm{pf}(\partial_A)$ *transforms under the* $\mathrm{GL}(q,\mathbb{C})$ *action* $\rho_f(g)A := dAd^T$ *on the underlying fermionic prehomogeneous vector space by*

$$\mathrm{pf}(\partial_{\rho_f(g)A}) = \mathrm{pf}(\partial_{dAd^T}) = \det(d)^{-1}\mathrm{pf}(\partial_A) = \chi_f(g)^{-1}\mathrm{pf}(\partial_A).$$

Now we state the relative superinvariance of the local zeta superfunctions.

Theorem 5.19. *The local zeta superfunctions, considered as superdistributions, are relatively superinvariant under the action of* G^+_{cs}*, with*

$$\begin{aligned} g.F_{i,j}(s,\Phi) &= |\chi(g)|^{-(s+\frac{p-q+1}{2})} \cdot F_{i,j}(s,\Phi), \\ g.F^*_{i,j}(s,\Phi^*) &= |\chi^*(g)|^{-(s+\frac{p-q+1}{2})} \cdot F^*_{i,j}(s,\Phi^*). \end{aligned}$$

Proof. By analyticity, it is sufficient to consider $s > 0$ and use Equation 5.6 for some $m > s + pq$. The transformations we consider are polynomial isomorphisms, which means that we can use Corollary 5.9. In order to prove the statement we have to calculate $g.F_{i,j}(s,\Phi)$. First consider

$(g.\Phi)(X_{b\text{-shift}}) = \Phi(\rho(g).X_{b\text{-shift}})$. An arbitrary element $g \in_T G_{cs}^+$ can be expressed as

$$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), g^{ST^3} = \left(\begin{smallmatrix} a^T & -c^T \\ b^T & d^T \end{smallmatrix}\right) \text{ and } X_{b\text{-shift}} = \left(\begin{smallmatrix} S-\xi A^{-1}\xi^T & \xi \\ -\xi^T & A \end{smallmatrix}\right).$$

Explicitly, $\rho(g).X = gXg^{ST^3} =$

$$\left(\begin{smallmatrix} a(S-\xi A^{-1}\xi^T)a^T+\ a\xi b^T+\ bAb^T-\ b\xi^T a^T & a\xi d^T-\ a(S-\xi A^{-1}\xi^T)c^T+\ b\xi^T c^T+bAd^T \\ c(S-\xi A^{-1}\xi^T)a^T+\ c\xi b^T+\ dAb^T-\ d\xi^T a^T & c\xi d^T-\ c(S-\xi A^{-1}\xi^T)c^T+\ d\xi^T c^T+dAd^T \end{smallmatrix}\right).$$

An arbitrary element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be decomposed into matrices of the form $g_1 = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, and $g_3 = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. In the first case, we get

$$g_1 X g_1^{ST^3} = \left(\begin{smallmatrix} a(S-\xi A^{-1}\xi^T)a^T & a\xi d^T \\ -d\xi^T a^T & dAd^T \end{smallmatrix}\right).$$

In the next calculations, we drop the factor $\frac{(-1)^{d_f \cdot m}}{\gamma_b(s)\gamma_f(2m-2s-pq)}$.

$$\begin{aligned} g_1.F_{i,j}(s,\Phi) = \int_{V_{ij}} & |D(S,A,\xi)|\ |\det(S)|^s |\det(A)|^{m-s-\frac{pq}{2}} \mathrm{pf}(\partial_A)^{2m} \\ & \cdot |\det(A)|^{\frac{pq}{2}} \Phi(a(S-\xi A^{-1}\xi^T)a^T, dAd^T, a\xi d^T). \end{aligned}$$

Under the polynomial transformation $\{S \mapsto S' := aSa^T, A \mapsto A' := dAd^T, \xi \mapsto \xi' := a\xi d^T\}$ Lemma 4.22 implies that the measure transforms as $|D(S,A,\xi)| = |\chi(g_1)|^{-\frac{p-q+1}{2}} |D(S',A',\xi')|$. Using Lemma 5.18 we obtain

$$\begin{aligned} g_1.F_{i,j}(s,\Phi) &= |\chi(g_1)|^{-\left(s+\frac{p-q+1}{2}\right)} \int_{V_{ij}} |D(S',A',\xi')|\ |\det(S')|^s |\det(A')|^{m-s-\frac{pq}{2}} \\ &\qquad \cdot \mathrm{pf}(\partial_{A'})^{2m} |\det(A')|^{\frac{pq}{2}} \Phi(S'-\xi'(A')^{-1}(\xi')^T, A', \xi') \\ &= |\chi(g_1)|^{-\left(s+\frac{p-q+1}{2}\right)} F_{i,j}(s,\Phi). \end{aligned}$$

In the second case,

$$g_2 X g_2^{ST^3} = \left(\begin{smallmatrix} (S-\xi A^{-1}\xi^T)+\xi b^T-b\xi^T+bAb^T & \xi+bA \\ -\xi^T+Ab^T & A \end{smallmatrix}\right)$$

and we have to consider the following integral

$$\begin{aligned} \int_{V_{ij}} & |D(S,A,\xi)||\det(S)|^s |\det(A)|^{m-s-\frac{pq}{2}} \mathrm{pf}(\partial_A)^{2m} |\det(A)|^{\frac{pq}{2}} \\ & \cdot \Phi((S-\xi A^{-1}\xi^T)+\xi b^T - b\xi^T + bAb^T,\ A\ ,\ \xi+bA). \end{aligned}$$

Making the polynomial transformation $\xi \mapsto \xi + bA$, which has Berezinian equal to one, we get

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)|^s|\det(A)|^{m-s-\frac{pq}{2}}\mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}$$
$$\cdot\ \Phi(S-(\xi-bA)A^{-1}(\xi^T+Ab^T)+\xi b^T-b\xi^T-bAb^T,\ A\ ,\ \xi).$$

which is

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)|^s|\det(A)|^{m-s-\frac{pq}{2}}\mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}$$
$$\cdot\ \Phi(S-\xi A^{-1}\xi^T,\ A,\ \xi).$$

Now we come to the third and hardest case

$$g_3Xg_3^{ST^3}=\begin{pmatrix} (S-\xi A^{-1}\xi^T) & \xi-(S-\xi A^{-1}\xi^T)c^T \\ c(S-\xi A^{-1}\xi^T)-\xi^T & A+c\xi+\xi^Tc^T-c(S-\xi A^{-1}\xi^T)c^T \end{pmatrix}.$$

For $g_3'=\begin{pmatrix} 1 & 0 \\ c' & 1\end{pmatrix}$ we have $g_3\cdot g_3'=\begin{pmatrix} 1 & 0 \\ c+c' & 1\end{pmatrix}$, and we can assume, without loss of generality, that the entries of the matrix c are zero up to one entry $\eta\in(\bigwedge(\mathbb{C}^{pq})^*)_1$. For such a matrix c, we know that $c(S-\xi A^{-1}\xi^T)c^T=0$, which means $\eta^2=0$ and setting $D:=(A^{-1}c\xi+A^{-1}\xi^Tc^T)$, we get the following equations

$$\det(1+D)=1+\mathrm{tr}(D)\ \text{ and }\ (1+D)^{-1}=(1-D).$$

Therefore, the integral expression is

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)|^s|\det(A)|^{m-s-\frac{pq}{2}}\mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}$$
$$\cdot\ \Phi(S-\xi A^{-1}\xi^T,A(1+D),\xi-(S-\xi A^{-1}\xi^T)c^T).$$

Making the transformation $A\mapsto A(1+D)$, polynomial in A, which has Berezinian equal to one, we get, by replacing A by $A(1-D)$,

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)|^s|\det(1-D)|^{-s}|\det(A)|^{m-s-\frac{pq}{2}}\mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}$$
$$\cdot\ \Phi(S-\xi(1+D)A^{-1}\xi^T,A,\xi-(S-\xi A^{-1}\xi^T)c^T).$$

The next transformation, polynomial in ξ, is $\xi\mapsto\xi-(S-\xi A^{-1}\xi^T)c^T$, which again has Berezinian equal to one. Setting

$$\begin{aligned} S'(S,A,\xi) &:= S-(\xi+(S-\xi A^{-1}\xi^T)c^T)(1+D)A^{-1}(\xi^T+c(S-\xi A^{-1}\xi^T)) \\ &= S(\mathrm{Id}-S^{-1}\xi A^{-1}cS-c^TA^{-1}\xi^T)-\xi A^{-1}\xi^T \end{aligned}$$

this expression is

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)|^s|\det(1-D)|^{-s}|\det(A)|^{m-s-\frac{pq}{2}}\mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}$$
$$\cdot\ \Phi(S'(S,A,\xi),A,\xi).$$

With the transformation $S \mapsto S(\mathrm{Id} - S^{-1}\xi A^{-1}cS - c^T A^{-1}\xi^T)$ polynomial in S, which has again Berezinian equal to one, and setting $D' := (S^{-1}\xi A^{-1}cS + c^T A^{-1}\xi^T)$, we get, after replacing S by $S(\mathrm{Id} + D')$,

$$\int_{V_{ij}} |D(S,A,\xi)||\det(S)\det(\mathrm{Id}+D')|^s|\det(1-D)|^{-s}|\det(A)|^{m-s-\frac{pq}{2}}$$
$$\cdot\, \mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}\Phi(S - \xi A^{-1}\xi^T, A, \xi).$$

We calculate

$$\begin{aligned}\frac{\det(S)\det(\mathrm{Id}+D')}{\det(A)\det(\mathrm{Id}-D)} &= \frac{\det(S)(\mathrm{Id} + \mathrm{tr}(S^{-1}\xi A^{-1}cS) + \mathrm{tr}(c^T A^{-1}\xi^T))}{\det(A)(\mathrm{Id} - \mathrm{tr}(A^{-1}c\xi) - \mathrm{tr}(A^{-1}\xi^T c^T))} \\ &= \frac{\det(S)(\mathrm{Id} + \mathrm{tr}(\xi A^{-1}c) + \mathrm{tr}(c^T A^{-1}\xi^T))}{\det(A)(\mathrm{Id} - \mathrm{tr}(A^{-1}c\xi) - \mathrm{tr}(A^{-1}\xi^T c^T))} = \frac{\det(S)}{\det(A)},\end{aligned}$$

where in the last step we used that the trace is invariant under cyclic permutations. Furthermore, the trace is invariant under similarity transformations, so that we finally get

$$\int_{V_{i,j}} |D(S,A,\xi)||\det(S)|^s|\det(A)|^{m-s-\frac{pq}{2}}$$
$$\cdot\, \mathrm{pf}(\partial_A)^{2m}|\det(A)|^{\frac{pq}{2}}\Phi(S - \xi A^{-1}\xi^T, A, \xi).$$

For the other local zeta superfunctions, the proof is entirely analogous. □

5.3 Fourier supertransform on prehomogeneous super vector spaces

In this subsection, we recall the basics about the Fourier supertransform on cs-vector spaces and consider the Fourier supertransform of the local zeta superfunction. Presenting the Fourier supertransform, we follow [3].

Let V_{cs} be a cs-vector space of $\dim V_{cs} = m|n$, endowed with a homogeneous basis (v_a, η_b), where it is assumed $v_a \in V_{\bar{0},\mathbb{R}}$. Let (v^a, η^b) be the dual basis. The Lebesgue measure $|dv|$ is the unique translation invariant measure on $V_{\bar{0},\mathbb{R}}$ such that the unit cube spanned by v^a has volume 1. Moreover, there is a unique Berezinian measure $|Dv|$ on the cs-vector space V_{cs}, considered as a cs-manifold such that

$$\int_{V_{cs}} |Dv| f = \int_{V_{\bar{0},\mathbb{R}}} |dv| \cdot \frac{\partial}{\partial \eta^n} \cdots \frac{\partial}{\partial \eta^1} \Phi \quad \text{for all } \Phi \in \Gamma(\mathcal{O}_{V_{cs}}). \tag{5.7}$$

Let V_{cs}^* be the dual cs-vector space, with measures $|dv^*|$ and $|Dv^*|$ associated with the dual basis (v^a, η^b). For $\Phi \in \mathscr{S}(V_{cs})$, the Fourier supertransform $\mathcal{F}(\Phi) = \widehat{\Phi} \in \mathscr{S}(V_{cs}^*)$ is defined by

$$\mathcal{F}(\Phi) := \frac{1}{\pi^{p/2}} \int_{V_{cs}} |Dv| e^{-i\langle \cdot, v\rangle} \Phi(v), \tag{5.8}$$

where $\langle \cdot, \cdot \rangle : V_{cs}^* \times V_{cs} \to \mathbb{C}$ denotes the canonical pairing.

Proposition 5.20 ([3], Proposition C.17). *The Fourier supertransform $\mathcal{F} : \mathscr{S}(V_{cs}) \to \mathscr{S}(V_{cs}^*)$ is an isomorphism of locally convex super vector spaces.*

Definition 5.21. For any homogeneous $F \in \mathscr{S}'(V_{cs})$ and $\Phi \in \mathscr{S}(V_{cs})$, the distributional Fourier supertransform $\widehat{F} \in \mathscr{S}'(V_{cs}^*)$ is defined by

$$\langle \widehat{F}, \Phi \rangle := F(\widehat{\Phi}). \tag{5.9}$$

Proposition 5.22 ([3]). *The Fourier supertransform $\mathcal{F} : \mathscr{S}'(V) \to \mathscr{S}'(V^*)$ is an isomorphism of locally convex vector spaces.*

For the prehomogeneous super vector space of $S^2(\mathbb{C}^{p|q})$, for instance, the Fourier supertransform is

$$\widehat{\Phi}(X') = \int_{V_{cs}} |DX| \; e^{-\mathrm{i}\langle X', X\rangle} \; \Phi(X),$$

where $\langle X', X \rangle = \mathrm{str}(X^{ST^3} X') = \mathrm{tr}(SS') - 2\mathrm{tr}(\xi\xi'^T) + \mathrm{tr}(AA')$.

The next proposition shows that the Fourier supertransform is, up to a supercharacter, an equivivariant map.

Proposition 5.23. *The Fourier supertransform acts on the space of Schwartz superfunctions as follows:*

$$\widehat{\rho(g).\Phi}(w) = \ |\chi(g)|^{-\frac{m-n}{d_b-d_f}} \cdot \rho^*(g).\widehat{\Phi}(w).$$

Proof. For $\Phi \in \Gamma_c(V_{cs})$

$$\widehat{\rho(g).\Phi}(w) = \int_{V_{cs}} |Dv| \exp(-i\langle w, v\rangle))\Phi(\rho(g)v).$$

By denoting $\rho(g)v = v'$, we obtain $|Dv| = |\chi(g)|^{-\frac{m-n}{d_b-d_f}}(g)|Dv'|$ by Lemma 4.16 and $\langle w, v\rangle = \langle \rho^*(g)w, v'\rangle$, so that

$$\begin{aligned}
\widehat{\rho(g).\Phi}(w) &= \int_{V_{cs}} |Dv| \exp(-i\langle w, v\rangle)\Phi(\rho(g)v) \\
&= |\chi(g)|^{-\frac{m-n}{d_b-d_f}} \cdot \int_{V_{cs}} |Dv'| \exp(-i\langle \rho^*(g)w, v'\rangle)\Phi(v') \\
&= |\chi(g)|^{-\frac{m-n}{d_b-d_f}} \cdot \widehat{\Phi}(\rho^*(g)w) \\
&= |\chi(g)|^{-\frac{m-n}{d_b-d_f}} \cdot \rho^*(g).\widehat{\Phi}(w).
\end{aligned}$$

□

Remark 5.24. The exponent $\frac{m-n}{d_b-d_f}$ is not singular, because $m-n$ is divisible by $d_b - d_f$, and moreover a natural number by Lemma 4.16. For the prehomogeneous super vector space $S^2(\mathbb{C}^{p|q})_{cs}$ we have $\frac{m-n}{d_b-d_f} = \frac{p-q+1}{2}$.

Proposition 5.25. *Let $F^*_{k,l}(s, .)$ be the local zeta superdistributions on the prehomogeneous super vector space $\mathscr{S}(S^2(\mathbb{C}^{p|q})^*_{cs})$ and Φ^* a Schwartz superfunction. Let $\chi^*(g) = \chi^{-1}(g)$. The Fourier supertransforms of $F^*_{k,l}$ are relatively superinvariant superdistributions on $\mathscr{S}(V_{cs})$, with*

$$g.F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi}) = |\chi(g)|^{s-\frac{m-n}{d_b-d_f}} \cdot F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \widehat{\Phi})$$

and transform like the local zeta superdistributions

$$g.F_{i,j}(-s, \Phi) = |\chi(g)|^{s-\frac{m-n}{d_b-d_f}} \cdot F_{i,j}(-s, \Phi).$$

Proof. By Theorem 5.19 the dual local zeta superdistribution has the following property

$$g.F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \Phi^*) = |\chi^*(g)|^{-s} \cdot F^*_{k,l}(s - \tfrac{m-n}{d_b-d_f}, \Phi^*),$$

such that with Proposition 5.23 we get

$$\begin{aligned}
\langle g.\widehat{F^*_{k,l}}(s-\tfrac{m-n}{d_b-d_f},.),\Phi\rangle =&\widehat{F^*_{k,l}}(s-\tfrac{m-n}{d_b-d_f},g.\Phi)\\
=&F^*_{k,l}(s-\tfrac{m-n}{d_b-d_f},\widehat{g.\Phi})\\
=&F^*_{k,l}(s-\tfrac{m-n}{d_b-d_f},|\chi(g)|^{-\frac{m-n}{d_b-d_f}}\cdot g.\widehat{\Phi})\\
=&|\chi(g)|^{s-\frac{m-n}{d_b-d_f}}\cdot F^*_{k,l}(s-\tfrac{m-n}{d_b-d_f},\widehat{\Phi})\\
=&|\chi(g)|^{s-\frac{m-n}{d_b-d_f}}\cdot\langle\widehat{F^*_{k,l}}(s-\tfrac{m-n}{d_b-d_f},.),\Phi\rangle.
\end{aligned}$$

□

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Köln, den 12. November 2013

(Mike Mücke)