Harmonic Maass Forms, Jacobi Forms, and Applications to Lie Superalgebras

Inaugural-Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Faultät

der Universität zu Köln

vorgelegt von

René Olivetto

aus Marostica

Berichterstatter:	Prof. Dr. Kathrin Bringmann
	Prof. Dr. Sander Zwegers

Tag der mündlichen Prüfung: 1.9.2014

To my grandmother Elsa, and the memory of my grandfather Natale

Kurzzusammenfassung

In der vorliegenden Arbeit beweisen wir eine Reihe von Aussagen über die Beschaffenheit, modularen Transformationseigenschaften und das asymptotische Verhalten von Fourier-Koeffizienten meromorpher Jacobi-Formen. Desweiteren geben wir Anwendungen in der Theorie der Lie Superalgebren. Durch Arbeiten von Kac und Wakimoto, Bringmann und Ono, Bringmann und Folsom sowie Bringmann, Folsom und Mahlburg ist bekannt, dass die Erzeugendenfunktionen von Kac-Wakimoto-Charakteren von $sl(m|n)^\wedge$ Superalgebren im Wesentlichen meromorphe Jacobi-Formen sind. Die Arbeit von Bringmann und Folsom verallgemeinernd, untersuchen wir Kac-Wakimoto-Charaktere für jede Wahl von ganzen Zahlen $m > n > 0$. Darüberhinaus beziehen wir in unsere Untersuchungen allgemeine meromorphe Jacobi-Formen von positivem Index in einer Variablen sowie Kac-Wakimoto-Charaktere in mehreren Variablen ein. Abschließend untersuchen wir das asymptotische Verhalten der Fourier-Koeffizienten von Kac-Wakimoto-Charakteren in einer Variablen, wobei wir eine Verallgemeinerung der Hardy-Ramanujan-Kreismethode verwenden.

Abstract

In this thesis, we prove several results concerning the shape, the modular properties, and the asymptotic behavior of the Fourier coefficients of meromorphic Jacobi forms, with applications to Lie superalgebras. By work of Kac and Wakimoto, Bringmann and Ono, Bringmann and Folsom, and Bringmann, Folsom, and Mahlburg it is known that the generating functions of Kac-Wakimoto characters relative to the $s\ell(m|n)^\wedge$ superalgebra are essentially meromorphic Jacobi forms. Extending previous work of Bringmann and Folsom, we investigate Kac-Wakimoto characters for any choice of integers $m > n > 0$. Subsequently, we extend the study to any single-variable meromorphic

Jacobi form of positive index, and to multivariable Kac-Wakimoto characters. Finally, we investigate the asymptotic behavior of the Fourier coefficients of single-variable Kac-Wakimoto characters using a generalization of the Hardy-Ramanujan Circle Method.

Acknowledgments

To Prof. Dr. Kathrin Bringmann I owe my deepest thanks. Without her guidance and encouragement this thesis would not have been possible. I would also like to thank Prof. Dr. Sander Zwegers for agreeing to referee this thesis and for many useful and interesting conversations. I had the opportunity to attend my Ph.D. program thanks to the Deutsche Forschungsgemeinschaft and Prof. Dr. Guido Sweers as spokesman of the DFG Graduiertenkolleg 1269.

During the last three years I had the opportunity to meet and talk with several people. I am particularly grateful to my former officemate Dr. Ben Kane. He helped me a lot especially at the beginning of my studies. I may consider him as my second advisor. For the same reason, I am truly obliged to Dr. Larry Rolen. We had very interesting conversations and he helped me a lot proofreading the final version of this thesis. I owe him deepest gratitude. I would also like to thank my former colleagues in the Graduiertenkolleg and the number theory working group. Among these people, I want to set apart Volker Genz, Dr. Matthias Waldherr, Dr. Yingkun Li, Dr. Michael Mertens, and Jose Miguel Zapata Rolon. Moreover, I thank Sabine Eisele for helping me with all the bureaucracy issues.

To conclude, I am deeply grateful to my grandmother Elsa, for growing me up and allowing me to be where I am now. I will never forget her wise suggestions and encouragements. Even if he is no longer with us, I am profoundly thankful to my grandfather Natale. His wise teaching are still inside of me. Last but not least, I deeply thank Michelle, for being constantly on my side, for supporting me every single day, for giving me the power to conclude my studies, and especially for being patient and waiting for me during these three years.

Table of Contents

List of Tables

CHAPTER 1

Introduction

1.1 Automorphic forms: Bridges for number theory

It sometimes happens that "interesting" sequences can be better understood considering their generating functions, which, in some cases, satisfy nice analytic properties. This phenomenon often appears in the context of modular and Jacobi forms. On the other hand, the rich algebraic structure of these functions gives a hint for their Fourier coefficients to encode apparently unrelated interesting and hidden informations. It is nowadays one of the main goals in number theory to understand the bridges between automorphic forms and other branches of mathematics. One of the most important and fascinating examples is given by the so-called *Monstrous Moonshine*, the hidden relation between the Fourier coefficients of the modular invariant j-function and the dimensions of irreducible representations of the Monster group $\mathbb{M}$, the largest sporadic simple group (of order $\approx 8 \cdot 10^{53}$).

The story of Moonshine started in 1978, when J. McKay made a surprising and astonishing discovery, that we may summarize as:

$$\begin{aligned}
196884 &= 196883 + 1, \\
21493760 &= 21296876 + 196883 + 1, \\
864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1.
\end{aligned}$$

In fact, these are the first three of infinitely many equations, in which the left-hand sides consists of the Fourier coefficients of the j-function

$$j(\tau) = \frac{1}{e^{2\pi i\tau}} + 744 + 196884e^{2\pi i\tau} + 21493760e^{4\pi i\tau} + 864299970e^{6\pi i\tau} + \cdots,$$

and on the right hand side, we find linear combinations of the dimensions of the smallest irreducible representations of $\mathbb{M}$. This observation suggested that an infinite-dimensional graded representation of $\mathbb{M}$ must exist. Since the graded dimension is the graded trace of the identity element of $\mathbb{M}$, J. G. Thompson [33] conjectured that the graded traces of nontrivial elements g of $\mathbb{M}$ on such a representation may be viewed in the same way. It was in 1979, when G. H. Conway and S. P. Norton [14] conjectured that certain *Hauptmoduln* (modular functions that are generators of a genus 0 modular function field) are the graded traces of infinite-dimensional representations of $\mathbb{M}$. This unexpected connection is knows as the *Monstrous Moonshine*, and it was proven by R. Borcherds [3] in 1992, using the no-ghost theorem from string theory and the theory of vertex operator algebras and generalized Kac-Moody algebras.

In the same spirit as Monstrous Moonshine, between the 70s and the 80s in a series of papers [20, 21, 23, 24], V. G. Kac and M. Wakimoto constructed a bridge between number theory (again the theory of modular forms) and Lie theory. Roughly speaking, the previously explained role of the Monster group was replaced by certain infinite dimensional Lie algebras. We describe this connection in more details in Appendix A.1.

1.2 Recent development in the theory of meromorphic Jacobi forms

The theory of holomorphic Jacobi forms was first extensively studied by M. Eichler and D. Zagier [16]. One of the main properties of a holomorphic Jacobi form φ (of positive index) is that it has a theta-decomposition, i.e., it can be written as $\varphi = \mathbf{h} \cdot \boldsymbol{\vartheta}$, where $\boldsymbol{\vartheta}$

is a vector-valued theta-function, and $\mathbf{h}$ is a vector-valued function whose components are "essentially" the Fourier coefficients of Φ (see (4.3.1)). In particular, due to the modularity of φ and $\boldsymbol{\vartheta}$, the function $\mathbf{h}$ is a vector valued modular form.

If we allow Φ to be meromorphic in the elliptic variable, i.e., Φ is a quotient of two holomorphic Jacobi forms, what we said above for holomorphic Jacobi forms is no longer true. In the third chapter of his Ph.D. thesis [36], while investigating the nature of mock theta functions, S. Zwegers showed that if φ is a meromorphic Jacobi form then its Fourier coefficients are no longer modular. More precisely, he showed that the error to the modularity can be controlled by adding a certain non-holomorphic function which depends on the poles of φ, and that if φ has only simple poles, then the Fourier coefficients are mixed mock modular forms.

Subsequently, A. Dabholkar, S. Murthy, and D. Zagier [15] reinvestigated and reformulated this phenomenon motivated by the study of the quantum theory of black holes. In their approach, they constructed a canonical decomposition of a meromorphic Jacobi form into a finite part and a polar part, which arises "naturally", and they have a concrete interpretation in the theory of black holes. Moreover, they investigated in detail the modularity of the canonical Fourier coefficients (the Fourier coefficients of the finite part) in the case that φ has poles of order at most 2. They showed that they have a mock modular behavior.

Finally, investigating the modularity properties of certain characters associated to certain Lie superalgebras (see Section A.1), K. Bringmann and A. Folsom described the structure of the Fourier coefficients of the meromorphic Jacobi form

$$\frac{\vartheta\left(z+\frac{1}{2};\tau\right)^m}{\vartheta(z;\tau)^n},$$

where ϑ is as in (2.3.1), and $m>n\geq 0$ are even integers. In this case, when the Jacobi form has poles of order $n>2$, the structure of its Fourier coefficients is more complicated and led to the definition of a new automorphic object, almost harmonic

Maass forms (see Definition 2.2.6). We also mention K. Bringmann and A. Folsom's subsequent paper in collaboration with K. Mahlburg [8], where they studied the special case of $m = n > 0$. For negative index Jacobi forms we refer the reader to [6].

1.3 The results of this thesis

We now turn to the main results of this thesis. The main goal is to extend the results explained above to more general settings. Here we state the theorems without technical details. Precise statements and discussions may be found in the relevant chapters.

Roughly speaking, the main purpose of this thesis is to determine the structure and the transformation properties of single and multivariable meromorphic Jacobi forms of positive index. As an application, we apply these results to the Kac-Wakimoto characters, in order to determine their transformation properties and the asymptotic behavior of their coefficients.

As mentioned before, in [7], K. Bringmann and A. Folsom studied the modularity and the structure of Kac-Wakimoto characters related to the Lie superalgebra $s\ell(m|n)^\wedge$, for $m \equiv n \equiv 0 \pmod 2$. We extend their results to any pair of integers (m, n), with $m > n > 0$. With this assumption, we deal with meromorphic Jacobi forms of integral or half-integral index. We shall see that studying Kac-Wakimoto characters is "equivalent" to studying the canonical Fourier coefficients h_ℓ of

$$\frac{\vartheta\left(z+\frac{1}{2};\tau\right)^m}{\vartheta\left(z;\tau\right)^n}. \tag{1.3.1}$$

Roughly speaking, these are, up to q-powers, the Fourier coefficients of Φ in a specific range on z. For a precise definition, we refer the reader to (3.1.5). We have the following result.

Theorem 1.3.1. *Let $m > n > 0$ with $n,\ m \in \mathbb{Z}$, let $\ell \in \mathbb{Z}$, and let h_ℓ be the ℓth canonical Fourier coefficient of* (1.3.1)*. Then h_ℓ is a component of the holomorphic* *part of a vector-valued almost harmonic Maass form of weight* $\frac{m-n-1}{2}$*.*

As a consequence, we have the following modularity result for Kac-Wakimoto characters.

Corollary 1.3.2. *With the assumptions as in Theorem 1.3.1, let* $tr_{L_{m,n}(\Lambda(\ell))}$ *be the* ℓ*th Kac-Wakimoto character associated to the irreducible* $s\ell(m|n)^{\wedge}$*-module of highest weight* $\Lambda(\ell)$*. Then, up to multiplication by* q *powers and a modular form,* $tr_{L_{m,n}(\Lambda(\ell))}$ *is a component of the holomorphic part of a vector-valued almost harmonic Maass form.*

For a more precise relation between the Kac-Wakimoto characters and the canonical Fourier coefficients of Theorem 1.3.1, we refer the reader to Appendix A.2.

The Jacobi form considered in Theorem 1.3.1 is not a special case. Indeed, we shall show that the same modularity properties are satisfied by the canonical Fourier coefficients of any meromorphic Jacobi form of positive index, that admits poles at torsion points.

Theorem 1.3.3. *Let* $\varphi(z;\tau)$ *be a meromorphic Jacobi form of positive index* M *and weight* k*, with poles with respect to* z *in* $\mathbb{Q}\tau+\mathbb{Q}$*. Moreover, let* $\boldsymbol{h}_{2M} := (h_\ell)_{\ell \pmod{2M}}$*, where* h_ℓ *is the* ℓ*th canonical Fourier coefficient of* φ*. Then* $\boldsymbol{h}_{2M}$ *is the holomorphic part of a vector-valued almost harmonic Maass form of weight* $k-\frac{1}{2}$*.*

Kac-Wakimoto characters as considered by Bringmann, Folsom, and Ono [7, 11, 17], and as we take into account in Corollary 1.3.2 are specializations of more general characters given in [24] as

$$\mathrm{ch}F = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}F_\ell \zeta^\ell = e^{\Lambda_0} \prod_{k\geq 1} \frac{\prod_{r=1}^{m}\left(1+\zeta\xi_r q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}\xi_r^{-1}q^{k-\frac{1}{2}}\right)}{\prod_{j=1}^{n}\left(1-\zeta\xi_{m+j}q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}\xi_{m+j}^{-1}q^{k-\frac{1}{2}}\right)}. \tag{1.3.2}$$

The function $\mathrm{ch}F$ is "essentially" the multivariable meromorphic Jacobi form

$$\Phi(z,\mathbf{u};\tau) := \frac{\prod_{r=1}^{s}\vartheta\left(z+u_r+\frac{1}{2};\tau\right)^{m_r}}{\prod_{j=1}^{t}\vartheta\left(z-w_j;\tau\right)^{n_j}}. \tag{1.3.3}$$

In joint work with K. Bringmann, we extend the notion of canonical Fourier coefficient to the multivariable setting, which we denote by $\mathbf{h}$ (see (5.3.2)). Extending the previous results to multivariable Jacobi forms, we have the following. For the notation we refer the reader to Subsection 5.1.1.

Theorem 1.3.4. *The canonical Fourier coefficient* $\boldsymbol{h}\colon \mathbb{C}^{s+t}\times\mathbb{H}\to\mathbb{C}^{m-n}$ *is the holomorphic part of a multivariable almost harmonic Maass-Jacobi form of weight* $\frac{m-n-1}{2}$.

As a special case, considering $m_r = n_j = 1$ for all r and j in (1.3.3), we show that considering the additional variables in the generating function of Kac-Wakimoto characters imposes extra structure which gives a cleaner picture for the specialized character as these are specializations of mixed H-harmonic Maass-Jacobi forms (see Definition 2.3.6).

Corollary 1.3.5. *The multivariable Kac-Wakimoto characters* chF_ℓ *are the holomorphic parts of mixed H-harmonic Maass-Jacobi form.*

Finally, in light of Theorem 1.3.1 and Theorem 1.3.3, we investigate the asymptotic behavior of the coefficients of Kac-Wakimoto characters. Let

$$\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}q^{L_0}q^{-\frac{\ell}{2}} = \sum_{t\geq 0} c_\ell(t)q^t.$$

Using a generalization of the Hardy-Ramanujan Circle Method, in Theorem 6.1.2, we determine the asymptotic behavior of $c_\ell(t)$ as $t\to+\infty$. We omit the statement of this result in this section since it is rather technical and needs heavy notation. As a consequence, we determine the main term in the asymptotic behavior $c_\ell(t)$.

Corollary 1.3.6. *As* $t\to\infty$

$$c_\ell(t) \sim \mathcal{C}t^{\frac{n}{2}-2}e^{2\pi\sqrt{t\left(\frac{n}{2}+\frac{m-n-1}{6}\right)}},$$

where
$$\mathcal{C} := \frac{\left(\frac{m-n}{2}\right)^{\frac{n}{2}-1} e^{\pi i \frac{9m-5n}{8}} \sqrt{\frac{n}{8}+\frac{m-n-1}{24}}}{2^{\frac{n+1}{2}} \pi^{\frac{n}{2}} \left(\frac{n}{2}-1\right)!}.$$

1.4 Outline of the Thesis

In Chapter 2, we give all the preliminaries necessary to prove our results. More precisely, we describe the automorphic forms we are interested in (modular form, Jacobi forms, and non-holomorphic generalizations), as well as certain differential operators acting on these forms. In Chapter 3, we prove Theorem 1.3.1 and Corollary 1.3.2. We give a unified proof considering Kac-Wakimoto characters relative to $s\ell(m|n)^\wedge$, for any possible choice of integers $m > n > 0$, without any restriction on the parity. We extend this result to any meromorphic Jacobi form of positive index in Chapter 4, proving Theorem 1.3.3. In Chapter 5, we consider multivariable Kac-Wakimoto characters, proving Theorem 1.3.4 and Corollary 1.3.5. Finally, in Chapter 6, we investigate the asymptotic behavior of single-variable Kac-Wakimoto characters. This leads to the proof of Corollary 1.3.6.

CHAPTER 2

Automorphic forms and differential operators

Here and throughout the thesis, we denote by $\mathbb{H}$ the complex upper half plane:

$$\mathbb{H} := \{\tau \in \mathbb{C} \colon \mathrm{Im}\tau > 0\}.$$

The special linear group $\mathrm{SL}_2(\mathbb{Z}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \colon a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ acts on $\mathbb{H}$ via Möbius transformations

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathbb{H} \to \mathbb{H}, \qquad \tau \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

Since the points of the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ are moduli (i.e., parameters) for the isomorphism classes of elliptic curves over $\mathbb{C}$, the group $\mathrm{SL}_2(\mathbb{Z})$ is sometimes called *modular group.* The dramatis personae of this thesis are (generalizations of) modular forms. Roughly speaking, a modular form is a function defined on $\mathbb{H}$ that transforms in a specific way under the action of $\mathrm{SL}_2(\mathbb{Z})$. However, the theory of modular forms becomes much more rich and interesting whenever one considers functions that satisfy this transformation property just for certain congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, i.e., finite index subgroups whose elements satisfies certain congruence properties. For the purposes of this thesis, we consider three kinds of congruence subgroups, namely, for a positive integer N, we define

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \colon a \equiv d \equiv 1 \pmod{N} \right\},$$

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) \colon b \equiv 0 \pmod{N} \right\}.$$

Additionally, it is also possible to define an action of the Jacobi group $\Gamma^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times \mathbb{H}$ via

$$(\gamma, (\lambda, \mu)) : \mathbb{C} \times \mathbb{H} \to \mathbb{C} \times \mathbb{H}, \qquad (z, \tau) \mapsto \left(\frac{\lambda z + \mu}{c\tau + d}, \gamma\tau \right),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We recall that the group law in Γ^J is given by $(\gamma, v)(\eta, u) := (\gamma\eta, v\eta + u)$, where the vectors have to be considered row vectors. This action gives rise to the definition of two variable relatives of modular forms, in the same spirit as before, which are called Jacobi forms. The notion of congruence subgroups extend in a natural way, and we define $\Gamma_0^J(N) := \Gamma_0(N) \ltimes \mathbb{Z}^2$ and $\Gamma_1^J(N) := \Gamma_1(N) \ltimes \mathbb{Z}^2$.

Before giving the precise definition of the objects mentioned in the previous discussion we need to introduce several differential operators, which play a key role.

2.1 Differential operators

The entire theory of non-holomorphic modular and Jacobi forms depends on the action of several differential operators on the space of $\mathcal{C}^\infty$-functions. The role of these operators is of fundamental importance for a deep understanding of the relations between number theory and representation theory. For a detailed discussion, we refer the reader to [2]. We describe separately the single-variable case and the multivariable case.

2.1.1 Single-variable differential operators

Throughout this subsection we let $f : \mathbb{H} \to \mathbb{C} \in \mathcal{C}^\infty(\mathbb{H})$. The parameter in $\mathbb{H}$ is indicated by $\tau = u + iv$, where u and v denotes respectively the real part and the imaginary part of τ. Moreover, for convenience of notation we denote

$$\partial_\tau := \frac{\partial}{\partial \tau} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v} \right), \qquad \partial_{\overline{\tau}} := \frac{\partial}{\partial \overline{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right). \tag{2.1.1}$$

As we shall see, in this thesis we consider certain classes of functions which are defined as invariant or almost invariant functions, with respect to the action of the *slash operator*, and that are eigenfunctions with respect to the action of the *hyperbolic Laplace operator*, sometimes simply called the Laplacian. For a fixed integer (resp. half integer) k and a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (resp. $\in \Gamma_0(4)$), we define the *automorphy factor of weight* k by

$$j(\gamma, \tau) := \begin{cases} \sqrt{c\tau + d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{c\tau + d} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases} \tag{2.1.2}$$

where

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker–Legendre symbol. The weight k slash operator $\big|_k$ defines an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ (resp. $\Gamma_0(4)$) on the space of $\mathcal{C}^\infty$-functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f\big|_k \gamma(\tau) := j(\gamma, \tau)^{-2k} f(\gamma\tau). \tag{2.1.3}$$

The weight k hyperbolic Laplace operator is defined by

$$\Delta_k := -4v^2 \partial_\tau \partial_{\overline{\tau}} + 2ikv\partial_{\overline{\tau}}.$$

It is a standard fact that these two operators commute.

Lemma 2.1.1. *Letting* $f \in \mathcal{C}^\infty(\mathbb{H})$ *and* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, then*

$$\Delta_k(f\big|_k \gamma) = \Delta_k(f)\big|_k \gamma.$$

Proof. To prove this lemma it is enough to apply Proposition 2.1.2 to (2.1.4). □

Fixing k (= weight), we shall consider certain $\mathbb{C}$-vector spaces of functions invariants with respect to $\big|_k$. It is possible to move between spaces of different weights thanks to the action of the *Maass raising operator* R_k *and Maass lowering operator* $\mathcal{L}_k$, which are defined by

$$R_k := 2i\partial_\tau + \frac{k}{v}, \qquad \mathcal{L}_k := -2iv^2\partial_{\overline{\tau}}.$$

The well known commutator relation

$$-\Delta_k = \mathcal{L}_{k+2}R_k + k = R_{k-2}\mathcal{L}_k \tag{2.1.4}$$

implies the following.

Proposition 2.1.2 ([28], Section 7)**.** *If* $f \in \mathcal{C}^\infty(\mathbb{H})$ *and* $\gamma \in \mathrm{SL}_2(Z)$*, then*

$$R_k(f)\big|_{k+2}\gamma = R_k(f\big|_k\gamma), \qquad \Delta_{k+2}(R_k(f)) = R_k(\Delta_k(f)) + kR_k(f),$$
$$\mathcal{L}_k(f)\big|_{k-2}\gamma = \mathcal{L}_k(f\big|_k\gamma), \qquad \Delta_{k-2}(\mathcal{L}_k(f)) = \mathcal{L}_k(\Delta_k(f)) - (k-2)\mathcal{L}_k(f).$$

We conclude this subsection describing a differential operator which plays a fundamental role in the theory of harmonic weak Maass forms, namely, the ξ-operator. For a fixed half integer k, we define ξ_k by

$$\xi_k := 2iv^k\overline{\partial_{\overline{\tau}}}. \tag{2.1.5}$$

As proven in [4], ξ_k is a map between harmonic weak Maass forms and holomorphic modular forms. We shall describe this in more details in Subsection 2.2.2. More generally, one can use this operator to classify non-holomorphic modular forms.

2.1.2 Multivariable differential operators

The theory in the multivariable case is much more rich. Not only we can extend all the single-variable operators previously described, but there exist other differential operators which play an important role and interact with each other in a marvelous way. With few exceptions, we describe the situation for 2-variable differential operators.

Here and throughout, we denote column vectors by $\mathbf{v} = (v_i)_{1\leq i\leq n} = (v_1, \cdots, v_n)$. Moreover, for an $n\times n$ matrix M and a vector $\mathbf{v}\in\mathbb{C}^n$, we denote by $\mathbf{v}^t$ the transpose row vector, and define

$$M[\mathbf{v}] := \mathbf{v}^t M\mathbf{v}.$$

Let $n\in\mathbb{N}$ and $g\colon \mathbb{C}^n\times\mathbb{H}\to\mathbb{C}\in\mathcal{C}^\infty(\mathbb{C}^n\times\mathbb{H})$. As before, the parameter in $\mathbb{H}$ is denoted by $\tau = u+iv$, while the parameter in $\mathbb{C}^n$ is denoted by $\mathbf{z} = \mathbf{x}+i\mathbf{y}$. For $n=1$ we set $z = z_1$. For a matrix $L\in\mathrm{M}_n(\mathbb{Z})$, and a half integer k, the weight k and index L slash operator is defined by

$$g\big|_{k,L}[\gamma,(\boldsymbol{\lambda},\boldsymbol{\mu})](\mathbf{z};\tau) := \frac{e^{\pi i\left(-\frac{c}{c\tau+d}L[\mathbf{z}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu}]+L[\boldsymbol{\lambda}]\tau+2\mathbf{z}^tL\boldsymbol{\lambda}\right)}}{j(\gamma,\tau)^{2k}}g\left(\frac{\mathbf{z}}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right),$$

where $j(\gamma,\tau)$ is given in (2.1.2) and $\left[\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right),(\boldsymbol{\lambda},\boldsymbol{\mu})\right]\in\Gamma^J\ltimes\mathbb{Z}^n$.

Remark. To facilitate the reader, we explicitly give the slash operator in the special case $n=1$. Let $k\in\frac{1}{2}\mathbb{Z}$ and M a positive integer. The *weight k and index M slash operator* $\big|_{k,M}$ defines an action of the Jacobi group Γ^J on $\mathcal{C}^\infty(\mathbb{C}\times\mathbb{H})$ by

$$g\big|_{k,M}[\gamma,(\lambda,\mu)](z;\tau) := \frac{e^{2\pi i\left(-\frac{Mc}{c\tau+d}(z+\lambda\tau+\mu)^2+M\lambda^2\tau+2M\lambda z\right)}}{j(\gamma,\tau)^{2k}}g\left(\frac{z}{c\tau+d};\gamma\tau\right).$$

For fixed $k\in\frac{1}{2}\mathbb{Z}$ and $M\in\mathbb{Z}$, the *Casimir operator*

$$\begin{aligned}\mathcal{C}_{k,M} := -2(\tau-\overline{\tau})^2\partial_\tau\partial_{\overline{\tau}} - (2k-1)(\tau-\overline{\tau})\partial_{\overline{\tau}} + \frac{(\tau-\overline{\tau})^2}{4\pi iM}\partial_{\overline{\tau}}\partial_z^2 + \frac{k(\tau-\overline{\tau})}{4\pi iM}\partial_z\partial_{\overline{z}}\\ + \frac{(z-\overline{z})(\tau-\overline{\tau})}{4\pi iM}\partial_{\overline{z}}\partial_z^2 - 2(w-\overline{z})(\tau-\overline{\tau})\partial_\tau\partial_{\overline{z}} + (1-k)(z-\overline{z})\partial_{\overline{z}} + \frac{(\tau-\overline{\tau})^2}{4\pi iM}\partial_\tau\partial_{\overline{z}}^2\\ + \left(\frac{(z-\overline{z})^2}{2}+\frac{k(\tau-\overline{\tau})}{4\pi iM}\right)\partial_{\overline{z}}^2 + \frac{(z-\overline{z})(\tau-\overline{\tau})}{4\pi iM}\partial_z\partial_{\overline{z}}^2\end{aligned}$$

extends the notion of Laplace operator. Moreover, we define the *Heisenberg Laplace operator* Δ_M^H and the *heat operator* H_M by

$$\begin{aligned}\Delta_M^H &:= \frac{\tau-\overline{\tau}}{2i}\partial_z\partial_{\overline{z}} + 2\pi M(z-\overline{z})\partial_{\overline{z}},\\ H_M &:= 8\pi iM\partial_\tau - \partial_z^2.\end{aligned}$$

These two operators commute with $\big|_{k,M}$, as we can see in the following proposition.

Proposition 2.1.3. *For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\varphi \in \mathcal{C}^\infty(\mathbb{H} \times \mathbb{C})$, we have*

$$\begin{aligned}\mathcal{C}_{k,M}\left(\varphi\big|_{k,M}\gamma\right) &= \left(\mathcal{C}_{k,M}\varphi\right)\big|_{k,M}\gamma,\\ \Delta_M^H\left(\varphi\big|_{k,M}\gamma\right) &= \left(\Delta_M^H\varphi\right)\big|_{k,M}\gamma.\end{aligned}$$

Proof. The proof of this proposition follows from Proposition 2.1.4 noting that both $\mathcal{C}_{k,M}$ and Δ_M^H can be written in terms of the raising and lowering operators (see (2.3) and (2.4) in [12]). $\square$

In contrast to the single variable case, it is possible to jump between different spaces of Jacobi forms whose weights differ by 1. To do so, we recall the 2-variable raising and lowering operators

$$\begin{aligned}X_+^{k,M} &:= 2i\left(\partial_\tau + \frac{z-\overline{z}}{\tau-\overline{\tau}}\partial_z + 2\pi i M\frac{(z-\overline{z})^2}{(\tau-\overline{\tau})^2} + \frac{k}{\tau-\overline{\tau}}\right),\\ Y_+^{k,M} &:= i\partial_z - 4\pi M\frac{z-\overline{z}}{\tau-\overline{\tau}},\\ X_-^{k,M} &:= -\frac{\tau-\overline{\tau}}{2i}\left((\tau-\overline{\tau})\,\partial_{\overline{\tau}} + (z-\overline{z})\,\partial_{\overline{z}}\right),\\ Y_-^{k,M} &:= -\frac{\tau-\overline{\tau}}{2}\partial_{\overline{z}},\end{aligned}$$

which were introduced by Berndt and Schmidt in [2] (Section 3.5).

Remark. As one can immediately see from the definition, the lowering operators do not depend on the weight or the index. For this reason, we will sometimes omit them from the notation.

For the purposes of this thesis we give a more general definition for X_- allowing many elliptic variables. Let $N \in \mathbb{Z}_{>0}$. For a positive definite matrix $L \in \mathrm{GL}_N(\mathbb{Z})$, define

$$X_- = X_-^{k,L} := -\frac{\tau-\overline{\tau}}{2i}\left((\tau-\overline{\tau})\,\partial_{\overline{\tau}} + (\mathbf{z}-\overline{\mathbf{z}})\cdot\partial_{\overline{\mathbf{z}}}\right),$$

where "$\cdot$" denotes the standard scalar product in $\mathbb{R}^N$, $\mathbf{z} = (z_1, \cdots, z_N)$, and $\partial_{\mathbf{z}} := (\partial_{z_1}, \cdots, \partial_{z_N})$. The following proposition summarizes their properties.

Proposition 2.1.4 ([2] Remark 3.5.2)**.** *For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\varphi \in \mathcal{C}^\infty(\mathbb{H} \times \mathbb{C})$, we have*

$$X_\pm^{k,M}\left(\varphi\big|_{k,M}\gamma\right) = \left(X_\pm^{k,M}\varphi\right)\big|_{k\pm 2,M}\gamma,$$
$$Y_\pm^{k,M}\left(\varphi\big|_{k,M}\gamma\right) = \left(Y_\pm^{k,M}\varphi\right)\big|_{k\pm 1,M}\gamma.$$

Finally, we introduce two multivariable generalizations of the ξ-operator defined in (2.1.5), namely,

$$\xi_{k,M} := \left(\frac{\tau-\overline{\tau}}{2i}\right)^{k-\frac{3}{2}}\left(-(\tau-\overline{\tau})\partial_{\overline{\tau}} - (z-\overline{z})\partial_{\overline{z}} + \frac{1}{4\pi M}\left(\frac{\tau-\overline{\tau}}{2i}\right)\partial_{\overline{z}}^2\right),$$
$$\xi_{k,M}^H := -\sqrt{\frac{\tau-\overline{\tau}}{2iM}}\, e^{-2\pi i M\frac{(z-\overline{z})^2}{\tau-\overline{\tau}}}\,\overline{\partial_{\overline{z}}}.$$

As for ξ_k, these two operators can be used to classify non-holomorphic Jacobi forms.

It will be useful for our computation to use the following compact version of the Casimir operator, in terms of the differential operators previously described. The operator $\mathcal{C}_{k,M}$ can be written in terms of the raising and the lowering operators, as described in the following proposition.

Proposition 2.1.5. *With the notation as above, we have*

$$\mathcal{C}_{k,M} = \frac{X_-^{k,M}H_M}{2\pi M} + \frac{k}{2\pi M}\Delta_M^H - \frac{1}{2\pi M}\left(X_+^{\frac{1}{2},M}\left(Y_-^{k,M}\right)^2 - \frac{4i}{(\tau-\overline{\tau})}\left(Y_-^{k,M}\right)^2\right) + \frac{2i(2k-1)}{\tau-\overline{\tau}}\xi_{k,M}.$$

Before proving this proposition, we need the following lemma describing certain commutator relations between the operators introduced before.

Lemma 2.1.6. *With the notations as above, we have*

$$H_M X_- = X_- H_M + 2\Delta_M^H + \frac{16\pi i M}{\tau-\overline{\tau}}X_-, \tag{2.1.6}$$

$$\left(\frac{\tau-\overline{\tau}}{2i}\right)^{\frac{5}{2}-k}\xi_{k,M}=X_- - \frac{1}{4\pi M}Y_-^2. \tag{2.1.7}$$

Proof. By definition

$$H_M X_- = \left(8\pi i M\partial_\tau - \partial_z^2\right)\left(-\frac{(\tau-\overline{\tau})^2}{2i}\partial_{\overline{\tau}} - \frac{(\tau-\overline{\tau})(z-\overline{z})}{2i}\partial_{\overline{z}}\right),$$

which equals

$$\begin{aligned}8\pi i M&\left(-\frac{2(\tau-\overline{\tau})}{2i}\partial_{\overline{\tau}} - \frac{(\tau-\overline{\tau})^2}{2i}\partial_{\overline{\tau}}\partial_\tau - \frac{(z-\overline{z})}{2i}\partial_{\overline{z}} - \frac{(\tau-\overline{\tau})(z-\overline{z})}{2i}\partial_{\overline{z}}\partial_\tau\right)\\ &+\left(\frac{(\tau-\overline{\tau})^2}{2i}\partial_{\overline{\tau}}\partial_z^2 + \frac{(\tau-\overline{\tau})(z-\overline{z})}{2i}\partial_{\overline{z}}\partial_z^2 + \frac{2(\tau-\overline{\tau})}{2i}\partial_{\overline{z}}\partial_z\right).\end{aligned}$$

Rearranging the terms we rewrite it as

$$\begin{aligned}&\left(-\frac{(\tau-\overline{\tau})^2}{2i}\partial_{\overline{\tau}} - \frac{(\tau-\overline{\tau})(z-\overline{z})}{2i}\partial_{\overline{z}}\right)\left(8\pi i M\partial_\tau - \partial_z^2\right) - 8\pi M(\tau-\overline{\tau})\partial_{\overline{\tau}}\\ &\qquad - 8\pi M(z-\overline{z})\partial_{\overline{z}} + 2\frac{(\tau-\overline{\tau})}{2i}\partial_{\overline{z}}\partial_z + 4\pi M(z-\overline{z})\partial_{\overline{z}},\end{aligned}$$

which gives (2.1.6). The proof of (2.1.7) is trivial. □

Proof of Proposition 2.1.5. From (2.3) in [12], we know that

$$\begin{aligned}\mathcal{C}_{k,M} =&2X_+^{k-2,M}X_-^{k,M} - \frac{1}{2\pi M}\left(X_+^{k-2,M}Y_-^{k-1,M}Y_-^{k,M} - Y_+^{k-1,M}Y_+^{k-2,M}X_-^{k,M}\right)\\ &+\frac{k-2}{2\pi M}Y_+^{k-1,M}Y_-^{k,M}\\ =&\left(2X_+^{k-2,M} + \frac{1}{2\pi M}Y_+Y_+\right)X_-^{k,M} - \frac{1}{2\pi M}X_+^{k-2,M}Y_-Y_- + \frac{k-2}{2\pi M}\Delta_M^H\\ =&\mathcal{A}X_-^{k,M} - \frac{1}{2\pi M}\mathcal{B} + \frac{k-2}{2\pi M}\Delta_M^H.\end{aligned} \tag{2.1.8}$$

We compute $\mathcal{A}$ and $\mathcal{B}$ separately.

By definition we have

$$\mathcal{A} = 4i\left(\partial_\tau + \frac{z-\overline{z}}{\tau-\overline{\tau}}\partial_z + 2\pi i M\left(\frac{z-\overline{z}}{\tau-\overline{\tau}}\right)^2 + \frac{k-2}{\tau-\overline{\tau}}\right) + \frac{1}{2\pi M}\left(i\partial_z - 4\pi M\frac{z-\overline{z}}{\tau-\overline{\tau}}\right)^2. \tag{2.1.9}$$

Expanding the square in the second summand of (2.1.9), we get

$$\mathcal{A} = 4i\left(\partial_\tau + \frac{z-\overline{z}}{\tau-\overline{\tau}}\partial_z + 2\pi i M\left(\frac{z-\overline{z}}{\tau-\overline{\tau}}\right)^2 + \frac{k-2}{\tau-\overline{\tau}}\right) \\ + \frac{1}{2\pi M}\left(-\partial_z^2 - 8\pi i M\frac{z-\overline{z}}{\tau-\overline{\tau}}\partial_z - \frac{4\pi i M}{\tau-\overline{\tau}} + (4\pi M)^2\left(\frac{z-\overline{z}}{\tau-\overline{\tau}}\right)^2\right),$$

which equals

$$4i\partial_\tau - \frac{1}{2\pi M}\partial_z^2 + \frac{2i}{\tau-\overline{\tau}}(2k-5) = \frac{1}{2\pi M}H_M + \frac{2i}{\tau-\overline{\tau}}(2k-5).$$

We now move to $\mathcal{B}$. By definition

$$X_+^{k-2,M} = X_+^{\frac{1}{2},M} + \frac{2i}{\tau-\overline{\tau}}\left(k-\frac{5}{2}\right),$$

therefore

$$\mathcal{B} = X_+^{\frac{1}{2},M}Y_-^2 + \frac{2i}{\tau-\overline{\tau}}\left(k-\frac{5}{2}\right)Y_-^2.$$

As a consequence, we rewrite (2.1.8) as

$$\left(\frac{1}{2\pi M}H_M + \frac{2i}{\tau-\overline{\tau}}(2k-5)\right)X_-^{k,M} - \frac{1}{2\pi M}\left(X_+^{\frac{1}{2},M}Y_-^2 + \frac{2i}{\tau-\overline{\tau}}\left(k-\frac{5}{2}\right)Y_-^2\right) \\ + \frac{k-2}{2\pi M}\Delta_M^H.$$

To conclude, we note that using (2.1.6), the Casimir operator equals

$$\begin{aligned}\mathcal{C}_{k,M} =& \frac{1}{2\pi M}X_-H_M + \frac{2i}{\tau-\overline{\tau}}(2k-1)X_-^{k,M} - \frac{1}{2\pi M}X_+^{\frac{1}{2},M}Y_-^2 + \frac{k}{2\pi M}\Delta_M^H \\ &+ \frac{2k-1}{2\pi i M(\tau-\overline{\tau})}Y_-^2 - \frac{4}{2\pi i M(\tau-\overline{\tau})}Y_-^2 \\ =& \frac{1}{2\pi M}X_-H_M + \frac{k}{2\pi M}\Delta_M^H - \frac{1}{2\pi M}\left(X_+^{\frac{1}{2},M}Y_-^2 - \frac{4i}{(\tau-\overline{\tau})}Y_-^2\right) + \frac{2i(2k-1)}{\tau-\overline{\tau}}\xi_{k,M},\end{aligned}$$

where in the last step we have used (2.1.7). □

We conclude this subsection by recalling a commutator relation between $\mathcal{C}_{k,M}$ and $Y_+^{k,M}$.

Proposition 2.1.7. *With the notation as above, we have*

$$\mathcal{C}_{k,M} Y_+^{k-1,M} = Y_+^{k-1,M} \mathcal{C}_{k-1,M} - (k-2) Y_+^{k-1,M}.$$

In order to prove Proposition 2.1.7, we need the following, which can be easily deduced from Proposition 3.6 in [32].

Lemma 2.1.8. *The following are true:*

1. $X_-^{k,M} Y_+^{k,M} = Y_+^{k,M} X_-^{k,M} - Y_-^{k,M}$;
2. $X_+^{k,M} Y_+^{k,M} = Y_+^{k,M} X_+^{k,M} - \frac{2i}{\tau - \overline{\tau}} Y_+^{k,M}$;
3. $Y_-^{k,M} Y_+^{k,M} = Y_+^{k,M} Y_-^{k,M} - 2\pi M$;
4. $\Delta_M^H Y_+^{k,M} = Y_+^{k,M} \Delta_M^H - 2\pi M Y_+^{k,M}$.

Proof of Proposition 2.1.7. Using the decomposition in (2.1.8) and Lemma 2.1.8, we compute each piece separately. In what follows we omit the weight and the index from the notation of each operator. We assume them to be k and M respectively everywhere.

1. For the first piece, we have

$$X_+ (X_- Y_+) = X_+ (Y_+ X_- - Y_-) = Y_+ X_+ X_- - \frac{2i}{\tau - \overline{\tau}} Y_+ X_- - X_+ Y_-.$$

2. Next, we compute

$$Y_+ Y_+ (X_- Y_+) = Y_+ Y_+ (Y_+ X_- - Y_-) = Y_+ Y_+ Y_+ X_- - Y_+ Y_+ Y_-.$$

3. Finally, we can see that

$$\begin{aligned}
X_+Y_-\left(Y_-Y_+\right) &= X_+Y_-\left(Y_+Y_- - 2\pi M\right) = X_+\left(Y_-Y_+\right)Y_- - 2\pi M X_+Y_- \\
&= X_+\left(Y_+Y_- - 2\pi M\right)Y_- - 2\pi M X_+Y_- \\
&= \left(X_+Y_+\right)Y_-Y_- - 4\pi M X_+Y_- \\
&= \left(Y_+X_+ - \frac{2i}{\tau-\overline{\tau}}Y_+\right)Y_-Y_- - 4\pi M X_+Y_- \\
&= Y_+X_+Y_-Y_- - \frac{2i}{\tau-\overline{\tau}}Y_+Y_-Y_- - 4\pi M X_+Y_-.
\end{aligned}$$

Therefore, using these three equalities and part 4 of Lemma 2.1.8, we have

$$\begin{aligned}
\mathcal{C}_{k,M}Y_+ =& 2\left(Y_+^{k,M}X_+^{k-2,M}X_-^{k,M} - \frac{2i}{\tau-\overline{\tau}}Y_+^{k-2,M}X_-^{k,M} - X_+^{k-2,M}Y_-^{k,M}\right) \\
& - \frac{1}{2\pi M}\left(Y_+^{k,M}X_+^{k-2,M}Y_-^{k-1,M}Y_-^{k,M} - \frac{2i}{\tau-\overline{\tau}}Y_+^{k-2,M}Y_-^{k-1,M}Y_-^{k,M}\right. \\
& \left. -4\pi M X_+^{k-2,M}Y_-^{k,M}\right) + \frac{k-2}{2\pi M}\left(Y_+^{k,M}\Delta_M^H - 2\pi M Y_+^{k,M}\right) \\
& + \frac{1}{2\pi M}\left(Y_+^{k,M}Y_+^{k-1,M}Y_+^{k-2,M}X_-^{k,M} - Y_+^{k,M}Y_+^{k-1,M}Y_-^{k,M}\right) \\
=& Y_+\mathcal{C}_{k,M} + i(\tau-\overline{\tau})Y_+\xi_{k,M} - (k-2)Y_+ - \frac{1}{2\pi M}Y_+\Delta_M^H.
\end{aligned}$$

To conclude the proof it is enough to check that

$$\mathcal{C}_{k,M} = \mathcal{C}_{k-1,M} - i(\tau-\overline{\tau})\xi_{k,M} + \frac{1}{2\pi M}\Delta_M^H,$$

which can be easily proven using Proposition 2.1.5.

□

2.2 Modular forms and harmonic weak Maass forms

In this section, we recall the notions of holomorphic modular form and certain non-holomorphic generalizations such as harmonic weak Maass forms. More information about these objects can be found in the path breaking papers [4, 36]. For a complete overview we refer the reader to [28, 34].

2.2.1 Weakly holomorphic Modular forms and Quasimodular forms

At the beginning of this chapter we recalled that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ via Möbius transformations. In fact, this action yields an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{C}^\infty$-functions on $\mathbb{H}$ via the slash operator, already defined in (2.1.3). Here and throughout this section, we let $k \in \frac{1}{2}\mathbb{Z}$. Moreover, here and throughout the thesis, we let $q := e^{2\pi i\tau}$.

Definition 2.2.1 (Weakly holomorphic modular forms)**.** *Let χ be a Dirichlet character modulo $N \in \mathbb{Z}_{>0}$. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a* weakly holomorphic modular form *of weight k, level N, and Nebentypus character χ if the following hold:*

1. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $f\big|_k\gamma = \chi(d)f$.*

2. *The poles of f, if any, are supported at the cusps of $\Gamma_0(N)$, i.e., f does not have poles on $\mathbb{H}$, and for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, f has a Fourier expansion of the form*

$$f\big|_k\gamma(\tau) = \sum_{n \geq n_\gamma} a(n)q^{\frac{n}{N}}, \qquad n_\gamma \in \mathbb{Z}. \tag{2.2.1}$$

As an example of modular form we recall the Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{j\geq 1} \left(1 - q^j\right). \tag{2.2.2}$$

As we shall see, this function appears as a factor in the generating function of Kac-Wakimoto characters. In the following lemma we recall the transformation properties of η (see for instance [30]).

Lemma 2.2.2. *The η-function satisfies the following modular transformation law for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:*

$$\eta\left(\gamma\tau\right) = \psi(\gamma)(c\tau + d)^{\frac{1}{2}}\eta(\tau).$$

Here, the multiplier $\psi(\gamma)$ is a 24th root of unity, which is given explicitly in [30].

The $\mathbb{C}$-vector space of weakly holomorphic modular forms of weight k and level N is denoted by $M_k^!(N)$. If $n_\gamma \geq 0$ (defined in (2.2.1)) for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then f is called *holomorphic modular form*, while if $n_\gamma > 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then f is called a *cusp form*. The $\mathbb{C}$-vector space of holomorphic modular forms (resp. cusp forms) of weight k and level N is denoted by $M_k(N)$ (resp. $S_k(N)$). Special and important examples of modular forms are given by *Eisenstein series*. In order to define them, for a positive integer k we denote the divisor function by $\sigma_{k-1}(n) := \sum_{1\leq d|n} d^{k-1}$. For even $k \geq 2$, the *weight k Eisenstein series E_k* is given by

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where B_k denotes the usual kth Bernoulli number. It is a standard fact that for each even $k \geq 4$, E_k is a holomorphic modular form of weight k and level 1. Moreover, the graded ring $M(1) := \bigoplus_k M_k(1)$ is freely generated by the Eisenstein series E_4 and E_6. The Eisenstein series E_2 fails to be modular. However, it is a standard fact that adding a simple non-holomorphic term yields a modular object. More precisely, its completion

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi v}$$

transforms as a modular form of weight 2 and level 1. Functions like $\widehat{E}_2$ play an important role in the theory, and generate the space of *almost holomorphic modular forms* (over the space of modular forms), firstly introduced by Kaneko and Zagier [25].

Definition 2.2.3 (Almost holomorphic modular forms)**.** *A function $f\colon \mathbb{H} \to \mathbb{C}$ is called* (weakly) almost holomorphic modular form *of weight k and level N if the following hold:*

1. *For each $\gamma \in \Gamma_0(N)$, $f\big|_k\gamma = f$.*

2. *The function f can be written as a polynomial in $\frac{1}{v}$ with (weakly) holomorphic*

coefficients, i.e., there exist $f_0, \cdots, f_D$ (weakly) holomorphic functions such that

$$f(\tau) = \sum_{j=0}^{D} \frac{f_j(\tau)}{v^j}.$$

The integer D is called the depth *of f, and the holomorphic function f_0 is called a* quasimodular form.

The set of quasimodular forms of a given weight k, level N, and depth D is a $\mathbb{C}$-vector space, denoted by $\mathcal{QM}_{k,D}(N)$, and it includes derivatives of holomorphic modular forms. We also denote the filtered ring of quasimodular forms by $\mathcal{QM}_k(N) := \cup_D \mathcal{QM}_{k,D}(N)$ In the following proposition we describe the basic properties of quasimodular forms. For more details see [5, 25].

Proposition 2.2.4. *1. We have*

$$\partial_\tau \left(\mathcal{QM}_{k,D}(N)\right) \subseteq \mathcal{QM}_{k+2,D+1}(N).$$

2. Every quasimodular form in $\mathcal{QM}_k(N)$ is a polynomial in E_2 with modular coefficients, namely,

$$\mathcal{QM}_{k,D}(N) = \bigoplus_{r=0}^{D} M_{k-2r}(N) \cdot E_2^r.$$

2.2.2 Harmonic weak Maass forms and almost harmonic Maass forms

In this subsection, we introduce the definition of certain non-holomorphic modular forms, introduced in 1949 by H. Maass [27], and generalized by J. Bruinier and J. Funke in [4], called harmonic weak Maass forms, as well as certain generalizations. The theory of harmonic weak Maass forms has been extensively developed in the last decade, and it has been discovered that they play a key role in the connection between number theory and other branches of mathematics and physics. For an overview we refer the reader to [15, 28, 34].

Definition 2.2.5 (Harmonic weak Maass form)**.** *A smooth function $h\colon \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight k, level N, and Nebentypus character χ if the following hold:*

1. *For each $\gamma \in \Gamma_0(N)$, $h\big|_k \gamma = \chi(d)h$, where d is the lower right entry of γ.*
2. *The function h is annihilated by the hyperbolic Laplacian, i.e.,*

$$\Delta_k h = 0.$$

3. *There exists a polynomial $P_h(q) := \sum_{n\leq 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $h(\tau) - P_h(q) = O\left(e^{-\varepsilon v}\right)$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all the cusps.*

Remark. Harmonic weak Maass forms are required to satisfy moderate growth conditions at the cusps. However, this is not the more general definition. In [4] J. Bruinier and J. Funke considered also other types of harmonic Maass forms based on varying the growth conditions at cusps. The term "weak" refer to the third condition in Definition 2.2.5.

The $\mathbb{C}$-vector space of harmonic weak Maass forms of weight k, level N, and character χ is denoted by $H_k(N,\chi)$. For convenience, we use the terminology "harmonic Maass form" instead of "harmonic weak Maass form". For $k \neq 1$ harmonic weak Maass forms have an expansion at infinity given by

$$h(\tau) = \sum_{n \gg -\infty} c^+(n)q^n + \sum_{n<0} c^-(n)\Gamma(1-k, 4\pi|n|v)q^n, \tag{2.2.3}$$

where $\Gamma(a,x) := \int_x^\infty e^{-t}t^{a-1}dt$ is the incomplete Gamma-function. The holomorphic part

$$h^+(\tau) := \sum_{n \gg -\infty} c^+(n)q^n$$

is called a *mock modular form.* We will refer to $h^- := h - h^+$ as the non-holomorphic part of h. The ξ-operator introduced in (2.1.5) defines a surjective map between harmonic Maass forms and cusp forms:

$$\xi_{2-k}\colon H_{2-k}(N,\chi) \to S_k(N,\overline{\chi}).$$

We will refer to the cusp form $\xi_{2-k}(h)$ as the *shadow* of h^+.

Remark. Since two harmonic Maass forms with the same non-holomorphic part differ by a weakly holomorphic modular form, and since the ξ-operator only sees the non-holomorphic part of h, we say that h^- and $\xi_k(h)$ are respectively the non-holomorphic part and the shadow of the harmonic Maass form h.

Remark. If the shadow is a unary theta function, we refer to the mock modular form as a mock theta function, following Zagier's definition.

We conclude this subsection by defining *almost harmonic Maass forms*, certain non-holomorphic automorphic forms recently introduced by K. Bringmann and A. Folsom [7]. These functions still need to be studied, especially the space that they generate. They extend the notion of harmonic Maass forms, almost holomorphic modular forms, and mixed harmonic Maass forms, i.e., $\mathbb{C}$-linear combinations of harmonic Maass forms multiplied by modular forms, such that the entire function satisfies the transformation property of a modular form.

Definition 2.2.6 (Almost harmonic Maass form)**.** *A smooth function $\mathcal{H}\colon \mathbb{H} \to \mathbb{C}$ is called an almost harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ and depth $r \in \mathbb{N} \cup \{0\}$ for a congruence subgroup $\widetilde{\Gamma}$ of Γ and character χ if the following hold:*

1. *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{\Gamma}$, $\mathcal{H}|_k\gamma(\tau) = \chi(d)\mathcal{H}(\tau)$.*

2. *The function $\mathcal{H}$ can be written as a finite linear combination of objects of the form $\sum\limits_{j=1}^{r} g_j R_{k+2-\nu}^{j-1}(h)$, where h is a harmonic weak Maass form of weight $k+2-\nu$,*

$\nu \in \frac{1}{2}\mathbb{Z}$ is fixed, and g_j are almost holomorphic modular forms of weight $\nu - 2j$ and character χ.

The holomorphic part of $\mathcal{H}$ is called an almost mock modular form.

Remark. Note that the first condition follows from the shape of an almost harmonic Maass form.

One can easily check that almost harmonic Maass forms generalize both harmonic weak Maass forms and almost holomorphic modular forms. Indeed, if h is trivial, then $\mathcal{H}$ is an almost holomorphic modular form. If the functions g_j are trivial and the depth $r = 1$, then $\mathcal{H} = h$ is a harmonic weak Maass form.

2.3 Holomorphic, meromorphic, and non-holomorphic Jacobi forms

The aim of this chapter is to define and give the basic properties of Jacobi forms. These are multivariable functions that are a cross between elliptic functions and modular forms. In analogy with the modular objects described in the previous section, we shall consider not only holomorphic Jacobi forms, but also certain non-holomorphic generalizations. For an extensive description of these objects we refer the reader to [16].

2.3.1 Holomorphic Jacobi forms

As in Subsection 2.1.2, throughout this section, we denote by $\mathbf{z}$ a vector of variables in $\mathbb{C}^n$, and by τ a variable in $\mathbb{H}$. Because of the transformation properties of a Jacobi form (see Definition 2.3.1) we shall refer to $\mathbf{z}$ as the elliptic variables, and to τ as the modular variable.

Definition 2.3.1 (Holomorphic Jacobi form)**.** *A holomorphic function* $\varphi\colon \mathbb{C}^n \times \mathbb{H} \to \mathbb{C}$ *is a holomorphic Jacobi form of weight* $k \in \frac{1}{2}\mathbb{Z}$*, index* $L \in \mathrm{GL}_n(\mathbb{Z})$ *(positive definite), and level* N *if it satisfies the following.*

1. *For all* $[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})] \in \Gamma_0(N)^J$*,* $\varphi\big|_{k,L}[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})](\boldsymbol{z}; \tau) = \varphi(\boldsymbol{z}, \tau)$*.*

2. *For some* $a > 0$*,* $\varphi(\boldsymbol{z}; \tau) = O(e^{aIm(\tau) + 2\pi \frac{L[Im(z)]}{Im(\tau)}})$*.*

The $\mathbb{C}$*-vector space of Jacobi forms of given weight* k*, index* L*, and level* N *is denoted by* $J_{k,L}(N)$*.*

The most famous example of Jacobi form is the so called Jacobi's theta function

$$\vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} q^{\frac{\nu^2}{2}} e^{2\pi i \nu \left(z + \frac{1}{2}\right)}. \tag{2.3.1}$$

In the following proposition we summarize the main properties of ϑ (for example, see [30] (80.31) and (80.8)).

Proposition 2.3.2. *The following are true:*

1. *For all* λ*,* $\mu \in \mathbb{Z}$ *we have*

$$\vartheta(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z} \vartheta(z;\tau).$$

2. *For all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *we have*

$$\vartheta\left(\frac{z}{c\tau + d}; \gamma\tau\right) = \psi(\gamma)^3 (c\tau + d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau + d}} \vartheta(z;\tau).$$

   Here ψ *is as in Lemma 2.2.2*

3. *Jacobi triple product identity:*

$$\vartheta(z;\tau) = -iq^{\frac{1}{8}} e^{-\pi i z} \prod_{n \geq 1} (1 - q^n)\left(1 - e^{2\pi i z} q^{n-1}\right)\left(1 - e^{-2\pi i z} q^n\right).$$

With a slight modification of ϑ it is possible to construct Jacobi forms of any positive index $M \in \frac{1}{2}\mathbb{N}$, namely, for each $\ell \pmod{2M}$,

$$\vartheta_{M,\ell}(z;\tau) := \sum_{\substack{n\in\mathbb{Z} \\ n\equiv \ell \pmod{2M}}} q^{\frac{n^2}{4M}} e^{2\pi i n z}. \tag{2.3.2}$$

We refer to these functions as index M Jacobi theta functions. The following proposition describes the transformation properties of the vector-valued[1] function $\boldsymbol{\vartheta}_M := (\vartheta_{M,\ell})_{\ell \pmod{2M}}$.

Proposition 2.3.3 (Eichler–Zagier, [16] Section 5)**.** *The function $\boldsymbol{\vartheta}_M\colon \mathbb{C}\times\mathbb{H}\to\mathbb{C}^{2M}$ is a vector-valued holomorphic Jacobi form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ with Weil representation $\varrho\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{2M}(\mathbb{C})$ defined by $\varrho_T := diag\left(e^{2\pi i\frac{\ell^2}{4M}}\right)_{0\le \ell<2M}$ and $\varrho_S := \left(e^{2\pi i\frac{-\ell r}{2M}}\right)_{0\le r,\ell<2M}$. More precisely, $\boldsymbol{\vartheta}_M$ satisfies the following transformation laws:*

$$\begin{aligned}
\boldsymbol{\vartheta}_M(z;\tau+1) &= \varrho_T \boldsymbol{\vartheta}_M(z;\tau),\\
\boldsymbol{\vartheta}_M\left(\frac{z}{\tau};-\frac{1}{\tau}\right) &= \tau^{\frac{1}{2}} e^{2\pi i\frac{Mz^2}{\tau}} \varrho_S \boldsymbol{\vartheta}_M(z;\tau).
\end{aligned}$$

Jacobi's theta functions play a fundamental role in the theory of Jacobi forms. They are not just nice examples, but they allow the so called *theta decomposition* of any holomorphic Jacobi form, as described in the following proposition. In order to state the result, we introduce the following notation, that will be often used throughout the thesis. We denote the vector of elliptic variables by $\mathbf{z} = (z_1,\cdots,z_n) =: (z,\mathbf{u})$. Moreover, we define the blocks of a matrix $L\in \mathrm{GL}_n(\mathbb{Z})$ by

$$L = \begin{pmatrix} 2M & \mathbf{b}^T \\ \mathbf{b} & \widetilde{L}\end{pmatrix}, \quad 2M\in\mathbb{N}, \quad \mathbf{b}\in\mathbb{Z}^{n-1}, \quad \widetilde{L}\in \mathrm{GL}_{n-1}(\mathbb{Z}).$$

Finally, for any $n\in\mathbb{N}$, we denote the standard scalar product between two elements $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{C}^n$ by $\mathbf{a}\cdot\mathbf{b}$. The following proposition extends Theorem 5.1 in [16], where

[1] See Subsection 2.3.12

the one-dimensional case is considered. We omit the proof since it is very similar to that of Theorem 5.1 in [16].

Proposition 2.3.4 (Theta decomposition)**.** *Let $\varphi\colon \mathbb{C}^n \times \mathbb{H} \to \mathbb{C}$ be a real-analytic function, holomorphic in the elliptic variable z. Assume that φ satisfies*

$$\varphi\big|_{k,L}\left[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),(\boldsymbol{\lambda},\boldsymbol{\mu})\right]=\varphi$$

for all $(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathbb{Z}^{2n}$, then there exists a function $\boldsymbol{h}\colon \mathbb{C}^n\times\mathbb{H}\to\mathbb{C}^{2M}$ such that

$$\varphi(\boldsymbol{z};\tau)=\varphi(z,\boldsymbol{u};\tau)=\boldsymbol{h}\left(\boldsymbol{u};\tau\right)\cdot\boldsymbol{\vartheta}_M\left(z+\frac{1}{2M}\boldsymbol{u}\cdot\boldsymbol{b};\tau\right).$$

Moreover, if $\varphi\big|_{k,L}[\gamma,(\lambda,\mu)]=\varphi$ for all γ in a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then the function $\boldsymbol{h}=\{h_\ell\}_{\ell \pmod{2M}}$ is a vector-valued real-analytic modular form of weight $k-\frac{1}{2}$ and index L^* (see (5.2.6)*), with multiplier system ϱ^{-1}, with respect to the same congruence subgroup as φ.*

We shall refer to the components h_ℓ of $\mathbf{h}$ as the *Fourier coefficients* of φ with respect to z. In fact, they differ by the standard Fourier coefficients of φ by a q-power.

2.3.2 Almost holomorphic Jacobi forms and H-Harmonic Maass Jacobi forms

As we have seen, it is possible to consider non-holomorphic functions that transform as modular forms, such as harmonic Maass forms. In the same spirit, we describe analogous non-holomorphic Jacobi forms. More precisely, we consider functions that are non-holomorphic in both the elliptic and modular variables.

The first class of non-holomorphic Jacobi forms that we consider generalizes quasi-modular forms (see Definition 2.2.3), i.e., we describe a two variable generalization of the weight 2 Eisenstein series E_2.

Definition 2.3.5 (Almost holomorphic Jacobi forms)**.** *A function $\varphi : \mathbb{C}^n \times \mathbb{H} \to \mathbb{C}$ is called an almost holomorphic Jacobi form of index $L \in GL_n(\mathbb{Z})$ and weight $k \in \mathbb{Z}$ if it is a polynomial in $\frac{z_j - \overline{z_j}}{\tau - \overline{\tau}}$ and $\frac{1}{\tau - \overline{\tau}}$, with (weakly) holomorphic coefficients in $(\boldsymbol{z}, \tau)$ that satisfies the same transformation properties of a Jacobi form. The constant term of this polynomial is called a quasi-Jacobi form.*

Remark. The definition above may be extended to congruence subgroups, vector-valued functions, multipliers, in the same way as for holomorphic Jacobi forms.

Remark. In [26], Libgober considered one-dimensional quasi-Jacobi forms of index 0.

The simplest example of almost holomorphic Jacobi form is given by the weight 1 Jacobi-Eisenstein series

$$E_1(z;\tau) := \sum_{(a,b)\in\mathbb{Z}^2}^{*} \frac{1}{(z + a\tau + b)},$$

where $\sum^*$ denotes the Eisenstein summation

$$\sum_{(a,b)\in\mathbb{Z}^2}^{*} := \lim_{A\to+\infty} \sum_{a=-A}^{A} \left(\lim_{B\to+\infty} \sum_{b=-B}^{B} \right).$$

The associated almost holomorphic Jacobi form is given by

$$\widehat{E}_1(z;\tau) := E_1(z;\tau) + \frac{z - \overline{z}}{\tau - \overline{\tau}}.$$

Another class of non-holomorphic Jacobi forms is given by H-harmonic Maass Jacobi forms. We give the definition introduced by K. Bringmann, M. Raum, and O. Richter in [12], extending previous definitions given by B. Berndt and R. Schmidt [2] and A. Pitale [29].

Definition 2.3.6 (H-harmonic Maass-Jacobi forms)**.** *A real-analytic function $\varphi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ is called a Maass-Jacobi form of weight $k \in \mathbb{Z}$ and index $M \in \frac{1}{2}\mathbb{N}$ for G^J (G^J a congruence subgroup of Γ^J) if the following conditions are satisfied:*

1. For all $[\gamma, (\lambda, \mu)] \in G^J$*, we have* $\varphi\big|_{k,2M}[\gamma, (\lambda, \mu)](z;\tau) = \varphi(z;\tau)$.

2. There exists $\lambda \in \mathbb{C}$ *such that* $\mathcal{C}_{k,M}(\varphi) = \lambda\varphi$.

3. For each fixed $z = \alpha\tau + \beta \in \mathbb{C}$*, the function* $\varphi(\alpha\tau + \beta; \tau)$ *is bounded, as* $Im(\tau) \to \infty$.

If λ *in condition (2) equals* 0*, then we say that* φ *is harmonic. If in addition* $\Delta_M^H(\varphi) = 0$*, then* φ *is called Heisenberg harmonic (H-harmonic). Finally, we call* φ *a mixed (H-)harmonic Maass-Jacobi form if it satisfies condition (1), and it can be written as a linear combination of (H-)harmonic Maass-Jacobi forms multiplied by weak Jacobi forms.*

Remark. We slightly modify the definition given in [12], relaxing the requirement on the growth condition.

In general it is not always clear how to determine the holomorphic part of a H-harmonic Maass Jacobi form. In fact, it is not always obvious that it exists. However, the functions of interest for this thesis naturally occur as holomorphic parts of (mixed) H-harmonic Maass-Jacobi forms. We thus, in analogy to mock modular forms, call them *mock Jacobi forms.* A special example of a mock Jacobi form, which plays an important role in this paper, is the Appell-Lerch sum, defined for $M \in \mathbb{N}$ and z, $w \in \mathbb{C}$ such that $w - z \notin \mathbb{Z}\tau + \mathbb{Z}$:

$$f_M(z, w; \tau) := \sum_{\alpha \in \mathbb{Z}} \frac{q^{M\alpha^2} e^{4\pi i M\alpha z}}{1 - e^{2\pi i(z-w)} q^\alpha}. \tag{2.3.3}$$

In [36], Zwegers studied and used this function to relate meromorphic Jacobi forms with Ramanujan's mock theta functions. In particular, he determined a non-holomorphic completion for f_M, in order to make it transform as a 2-variable

Jacobi form. To describe this, we need the real-analytic function $R_{M,\ell}$ defined by

$$R_{M,\ell}(w;\tau) := \sum_{\substack{\lambda\in\mathbb{Z} \\ \lambda\equiv\ell \pmod{2M}}} \left\{ \operatorname{sgn}\left(\lambda+\frac{1}{2}\right) - E\left(\left(\lambda+2M\frac{\operatorname{Im}(w)}{\operatorname{Im}(\tau)}\right)\sqrt{\frac{\operatorname{Im}(\tau)}{M}}\right)\right\} \times e^{-\pi i \frac{\lambda^2}{2M}\tau - 2\pi i \lambda w}, \quad (2.3.4)$$

where $w \in \mathbb{C}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$. The completion of f_M is the function $\widehat{f}_M$ defined by

$$\widehat{f}_M(z,w;\tau) := f_M(z,w;\tau) - \frac{1}{2}\sum_{\ell \pmod{2M}} R_{M,\ell}(w;\tau)\vartheta_{M,\ell}(z;\tau). \quad (2.3.5)$$

Zwegers proved the following.

Proposition 2.3.7 (Zwegers)**.** *For $M \in \mathbb{Z}$, the function $\widehat{f}_M$ transforms like a Jacobi form on $\mathbb{C}^2 \times \mathbb{H}$ of weight 1 and index $\left(\begin{smallmatrix} 2M & 0 \\ 0 & -2M \end{smallmatrix}\right)$ for $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.*

To explain Appell sums in the framework of harmonic Maass(-Jacobi) forms we describe the action of certain differential operators on the real-analytic functions $R_{M,\ell}(w;\tau)$.

Proposition 2.3.8. *With the notation as above, the following are true:*

1. *The function $R_{M,\ell}$ is annihilated by the Heisenberg operator Δ^H_{-M}, and the Casimir operator $\mathcal{C}_{\frac{1}{2},-M}$. Moreover, for each $(\alpha,\beta) \in \mathbb{R}^2$,*

$$H_{-M}\left[e^{-4\pi i M\alpha z} q^{-M\alpha^2} R_{M,\ell}\left(z+\alpha\tau+\beta;\tau\right)\right] = 0.$$

2. *The function $Y_+^{\frac{1}{2},-M}\left(R_{M,\ell}\right)$ is an eigenfunction with respect to $\mathcal{C}_{\frac{3}{2},-M}$ of eigenvalue $-\frac{1}{2}$.*

3. *Let α and $\beta \in \mathbb{Q}$, then $\Delta_{\frac{1}{2}}\left[q^{-M\alpha^2} R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right] = 0$.*

Proof. We start by proving part 1. We point out that $H_{-M}\left(R_{M,\ell}\right)=0$. For the special case $M=\frac{1}{2}$ the result is stated in Section 1 of [13]. A simple change of variable implies the statement for any M.

Next, we prove that $R_{M,\ell}$ is annihilated by the Heisenberg operator. In order to do so, we recall that by Lemma 1.8 of [36], we have

$$\partial_{\overline{z}}\left[R_{M,\ell+M}(z;\tau)\right]=-2i\sqrt{iM}e^{-2\pi i\left(\ell\overline{z}-\frac{\ell^2}{4M}\overline{\tau}\right)}\vartheta\left(2M\overline{z}-\frac{1}{2}+\ell\overline{\tau};-2M\overline{\tau}\right)\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}},\tag{2.3.6}$$

where $F(z;\tau):=e^{2\pi iM\frac{(z-\overline{z})^2}{\tau-\overline{\tau}}}$. The operator ∂_z acts trivially on anti-holomorphic functions in z, thus, using (2.3.6), we have that $\Delta_M^H\left(\mathcal{R}_{M,\ell+M}(z;\tau)\right)$ equals

$$-\sqrt{iM}e^{-2\pi i\ell\overline{z}-\frac{\ell^2}{4M}\overline{\tau}}\vartheta\left(2M\overline{z}-\frac{1}{2}+\ell\overline{\tau};-2M\overline{\tau}\right)\left(-\frac{\tau-\overline{\tau}}{8\pi iM}\partial_z+\frac{z-\overline{z}}{2}\right)\left(\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right).$$

A direct computation gives that

$$\left(-\frac{\tau-\overline{\tau}}{8\pi iM}\partial_z+\frac{1}{2}(z-\overline{z})\right)\left(\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right)=0.$$

Therefore $R_{M,\ell}$ is annihilated by Δ_{-M}^H.

We proceed by showing that $\mathcal{C}_{\frac{1}{2},-M}(R_{M,\ell})=0$. To do so, we make use of Proposition 2.1.5. In particular, since $\xi_{\frac{1}{2},-M}\left(R_{M,\ell}\right)=0$ (it can be easily seen with a direct computation), it remains to prove that

$$X_{+}^{\frac{1}{2},-M}\left(\partial_{\overline{z}}\partial_{\overline{z}}\left(R_{M,\ell+M}(z;\tau)\right)\right)=0.$$

For this, using (2.3.6), we compute

$$\partial_{\overline{z}}^2\left[\mathcal{R}_{M,\ell+M}(z;\tau)\right]=\partial_{\overline{z}}\left[H(\overline{z};-\overline{\tau})\right]\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}+H(\overline{z};-\overline{\tau})\partial_{\overline{z}}\left[\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right],\tag{2.3.7}$$

where

$$H(z;\tau):=-2i\sqrt{iM}e^{-2\pi i\ell z+\frac{\ell^2}{4M}\tau}\vartheta\left(2Mz-\frac{1}{2}+\ell\tau;2M\tau\right).$$

Since $H(\overline{z};-\overline{\tau})$ and $\partial_{\overline{z}}\left[H(\overline{z};-\overline{\tau})\right]$ are anti-holomorphic, applying $X_{+}^{\frac{1}{2},-M}$ to equation (2.3.7), we obtain

$$\partial_{\overline{z}}\left[H(\overline{z};-\overline{\tau})\right] X_{+}^{\frac{1}{2},-M}\left(\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right)+H(\overline{z};-\overline{\tau})X_{+}^{\frac{1}{2},-M}\left(\partial_{\overline{z}}\left[\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right]\right).$$

To conclude, a direct computation shows that

$$X_{+}^{\frac{1}{2},-M}\left(\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right)=X_{+}^{\frac{1}{2},-M}\left(\partial_{\overline{z}}\left[\frac{F(z;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right]\right)=0.$$

The proof of part 2 follows from the previous ones, using Proposition 2.1.7.

To conclude, we prove part 3. By definition of $\Delta_{\frac{1}{2}}$ the statement is equivalent of showing that

$$\partial_{\tau}\partial_{\overline{\tau}}\left[q^{-M\alpha^2}R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right]=-\frac{1}{4iv}\partial_{\overline{\tau}}\left[q^{-M\alpha^2}R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right],$$

i.e.,

$$\partial_{\tau}\partial_{\overline{\tau}}\left[R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right]=\left(2\pi i\alpha^2M-\frac{1}{4iv}\right)\partial_{\overline{\tau}}\left[R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right].$$

From Lemma 1.8 in [36] we know that

$$\begin{aligned}\partial_{\overline{\tau}}\left[R_{M,\ell}\left(\alpha\tau+\beta;\tau\right)\right]&=e^{2\pi i(\alpha\tau+\beta)(M-\ell)}q^{-\frac{(\ell-M)^2}{4M}}\frac{e^{-4\pi Mv\left(\alpha+\frac{\ell-M}{2M}\right)}}{\sqrt{4Mv}}\\&\quad\times\sum_{n\in\frac{1}{2}+\mathbb{Z}}(-1)^{n-\frac{1}{2}}\left(n+\alpha+\frac{\ell-M}{2M}\right)e^{-n^2M\overline{\tau}-n\left(2M(\alpha\overline{\tau}+\beta)+(\ell-M)\overline{\tau}-\frac{1}{2}\right)}.\end{aligned}$$

Computing the holomorphic derivative term by term we get the result. □

The function $R_{M,\ell}$ is the prototype of the non-holomorphic part of a harmonic Maass-Jacobi form. Morover, specializing the elliptic variable to torsion points, we get the non-holomorphic part of a harmonic weak Maass form. In the last part of this subsection we construct the "mock" part needed to prove our claims. To do so, we consider a slight modification of the Appell sum f_M, namely

$$\mu(z,w;\tau):=\frac{e^{\pi iz}}{\vartheta(w;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{\frac{1}{2}\left(n^2+n\right)}e^{2\pi inw}}{1-e^{2\pi iz}q^n},$$

where $z \in \mathbb{C}$ and $w \in \mathbb{C}^\times$. This function was considered and studied by Zwegers in [36], therefore we will refer to it as Zwegers' μ-function. Zwegers discovered that μ can be completed to a non-holomorphic Jacobi form $\widehat{\mu}$ by the addition of the real-analytic function

$$R(z;\tau) := \sum_{\lambda \in \frac{1}{2}+\mathbb{Z}} \left\{ \operatorname{sgn}(\lambda) - E\left(\left(\lambda + \frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)} \right) \sqrt{2\operatorname{Im}(\tau)} \right) \right\} (-1)^{\lambda - \frac{1}{2}} e^{-\pi i \lambda^2 \tau - 2\pi i \lambda z},$$

namely,

$$\widehat{\mu}(z,w;\tau) := \mu(z,w;\tau) + \frac{i}{2} R(z-w;\tau). \tag{2.3.8}$$

One can easily note that R has the same shape of $R_{M,\ell}$ (see (2.3.4)). In fact, for $\ell \in \{0,1,\cdots,2M-1\}$ we have

$$R_{M,\ell}(z;\tau) = -ie^{2\pi i z(M-\ell)} q^{-\frac{(\ell-M)^2}{4M}} R\left(2Mz - \frac{1}{2} + \tau(\ell - M); 2M\tau\right).$$

In the following proposition, we describe the transformation properties of $\widehat{\mu}$.

Proposition 2.3.9 (Zwegers, Theorem 1.11 in [36])**.** *With the notation as above, the following are true:*

1. *For all k, ℓ, m, $n \in \mathbb{Z}$, we have*

$$\widehat{\mu}\left(z + k\tau + \ell, w + m\tau + n; \tau\right) = (-1)^{k+\ell+m+n} q^{\frac{(k-m)^2}{2}} e^{2\pi i (k-m)(z-w)} \widehat{\mu}\left(z,w;\tau\right).$$

2. *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\widehat{\mu}\left(\frac{z}{c\tau+d}, \frac{w}{c\tau+d}; \gamma\tau\right) = \psi(\gamma)^{-3} (c\tau+d)^{\frac{1}{2}} e^{-\pi i \frac{c(z-w)^2}{c\tau+d}} \widehat{\mu}\left(z,w;\tau\right),$$

where ψ is the multiplier of the Dedekind η-function.

Choosing z and w such that $z - w \mapsto 2Mz - \frac{1}{2} + \tau(\ell - M)$, we define the functions

$$\mu_1^{(M,\ell)}(z;\tau) := \begin{cases} 2e^{2\pi i z(M-\ell)} q^{-\frac{(\ell-M)^2}{4M}} \mu\left(2Mz - \frac{1}{2}, (M-\ell)\tau; 2M\tau\right) & \text{if } \ell \neq M, \\ 2\mu\left(2Mz + \frac{\tau-1}{2}, \frac{\tau}{2}; 2M\tau\right) & \text{if } \ell = M, \end{cases}$$

and

$$\mu_2^{(M,\ell)}(z;\tau) := 2e^{2\pi iz(M-\ell)}q^{-\frac{(\ell-M)^2}{4M}}\mu\left(2Mz,(M-\ell)\tau+\frac{1}{2};2M\tau\right).$$

From the discussion above, we know that $\mu_1^{(M,\ell)}$ and $\mu_2^{(M,\ell)}$ can be completed to non-holomorphic Jacobi forms, and from (2.3.8) we know that they have both the same non-holomorphic part $R_{M,\ell}(z;\tau)$. Explicitly, for $j\in\{1,2\}$, we define

$$\widehat{\mu}_j^{(M,\ell)}(z;\tau) := \mu_j^{(M,\ell)}(z;\tau)+R_{M,\ell}(z;\tau).$$

Note that $\mu_1^{(M,\ell)}$ and $\mu_2^{(M,\ell)}$ are defined in different domains. The first is defined for $2Mz\notin 2M\tau\mathbb{Z}+\frac{1}{2}+\mathbb{Z}$, while the second for $2Mz\notin 2M\tau\mathbb{Z}+\mathbb{Z}$. As a consequence of Proposition 2.3.9, we have the following transformation properties for $\widehat{\mu}_1^{(M,\ell)}$ and $\widehat{\mu}_2^{(M,\ell)}$. In order to state the result, for $(\alpha,\beta)\in\mathbb{Z}^2$ we define

$$\Gamma_{\alpha,\beta} := \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z}) : \begin{array}{c}(a-1)\alpha+c\beta,\ b\alpha+(d-1)\beta\in\mathbb{Z},\\ M\left(-c\beta^2+b\alpha^2+(d-a)\alpha\beta\right)\in\mathbb{Z}\end{array}\right\}.$$

Corollary 2.3.10. *Let $M\in\frac{1}{2}\mathbb{Z}$ and $\ell\in\{0,\cdots,2M-1\}$. Then for $j\in\{1,2\}$, the following are true:*

1. *For any $[\gamma,(r,s)]\in\Gamma_1^J(4M)$,*

$$\widehat{\mu}_j^{(M,\ell)}\big|_{\frac{1}{2},-M}[\gamma,(r,s)](z;\tau) = (-1)^{2Ms}\psi(\gamma)^{-3}\widehat{\mu}_j^{(M,\ell)}(z;\tau),$$

   where ψ is the multiplier of the Dedekind η-function.

2. *Let $(\alpha,\beta)\in\mathbb{Q}^2$. For any $\gamma\in\Gamma_{\alpha,\beta}\cap\Gamma_1(M)$, the following are true:*

   (a) Let $\widehat{h}_j(\tau) := \partial_w\left[q^{-M\alpha^2}\widehat{\mu}_j^{(M,\ell)}(w+\alpha\tau+\beta;\tau)\right]_{w=0}$, then

$$\widehat{h}_j\big|_{\frac{3}{2}}\gamma(\tau) = \psi(\gamma)^{-3}\widehat{h}_j(\tau).$$

   (b) Let $\widehat{g}_j(\tau) := q^{-M\alpha^2}\widehat{\mu}_j^{(M,\ell)}(\alpha\tau+\beta;\tau)$, then

$$\widehat{g}_j\big|_{\frac{1}{2}}\gamma(\tau) = \psi(\gamma)^{-3}\widehat{g}_j(\tau).$$

In part (2) of Corollary 2.3.10 we have constructed two real-analytic modular forms, whose non-holomorphic part is respectively $q^{-M\alpha^2} R_{M,\ell}(\alpha\tau+\beta;\tau)$ and $q^{-M\alpha^2}\partial_w \left[R_{M,\ell}(w+\alpha\tau+\beta;\tau)\right]_{w=0}$. In addition, Proposition 2.3.8 implies that these functions are annihilated by the Laplacian. Combining these two results we have the following.

Corollary 2.3.11. *With the notion as above, for $j \in \{1,2\}$ the following are true:*

1. *The function $q^{-M\alpha^2}\widehat{\mu}_j^{(M,\ell)}(\alpha\tau+\beta;\tau)$ is a harmonic Maass form of weight $\frac{1}{2}$ for $\Gamma_{\alpha,\beta}$.*
2. *The function $\partial_w \left[q^{-M\alpha^2}\widehat{\mu}_j^{(M,\ell)}(w+\alpha\tau+\beta;\tau)\right]_{w=0}$ is a harmonic Maass form of weight $\frac{3}{2}$ for $\Gamma_{\alpha,\beta}$.*
3. *The function $\widehat{\mu}_j^{(M,\ell)}(z;\tau)$ is a H-harmonic Maass-Jacobi form of weight $\frac{1}{2}$ and index $-M$ for $\Gamma_1^J(4M)$.*

The definitions described so far do not always suffice in order to describe the objects we are interested in this thesis. In fact, we need to introduce a *vector-valued* notion for each of them.

Definition 2.3.12. *Given a (projective) representation $\varrho\colon \mathrm{SL}_2(\mathbb{Z}) \to GL(\mathbb{C}^d)$, where $d \in \mathbb{Z}_{>0}$, of $\mathrm{SL}_2(\mathbb{Z})$ on $GL(\mathbb{C}^d)$, we say that a function*

$$\mathbb{F}\colon \mathbb{C}^n \times \mathbb{H} \to \mathbb{C}^d$$

(where $n=0$ in the case of modular forms and harmonic weak Maass forms, while in the case of Jacobi forms $n>0$) transforms vector-valued of weight k and level N if for all $\gamma \in \Gamma_0(N)$ we have

$$\mathbb{F}\big|_k \gamma = \varrho(\gamma)\mathbb{F},$$

where on the left-hand side the slash operator is taken component-wise.

CHAPTER 3

Kac-Wakimoto characters in one variable

3.1 Introduction

Let $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}q^{L_0}$ denote the specialized character associated to the irreducible $s\ell(m|n)^\wedge$-module of highest weight $\Lambda(\ell)$[1], which we will refer to as the ℓth Kac-Wakimoto character. Here $m > n > 0$ are integers, and we fix here and throughout the chapter $M := \frac{m-n}{2}$. Letting ℓ run through the integers, we consider their generating function [24]

$$\mathrm{ch}F := \sum_{\ell\in\mathbb{Z}} \mathrm{ch}F_\ell \zeta^\ell = \prod_{k\geq 1} \frac{\left(\left(1+\zeta q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}q^{k-\frac{1}{2}}\right)\right)^m}{\left(\left(1-\zeta q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}q^{k-\frac{1}{2}}\right)\right)^n}, \tag{3.1.1}$$

with $\mathrm{ch}F_\ell$ as in (A.2.1). We recall that $\zeta = e^{2\pi i z}$. Using Jacobi's triple product identity (see part 3 of Proposition 2.3.2) one can easily rewrite $\mathrm{ch}F$ as

$$\mathrm{ch}F = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}F_\ell \zeta^\ell = (-1)^m i^{-n} \zeta^M q^{\frac{M}{3}} \eta^{-2M}(\tau) \frac{\vartheta\left(z+\frac{\tau+1}{2};\tau\right)^m}{\vartheta\left(z+\frac{\tau}{2};\tau\right)^n}, \tag{3.1.2}$$

where ϑ denotes the Jacobi theta function and η is the Dedekind η-function (2.2.2). Due to the elliptic transformation properties of ϑ, in order to study the Fourier coefficients of $\mathrm{ch}F$, which are "essentially" the Kac-Wakimoto characters $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}$, we can reduce our investigation to the meromorphic Jacobi form

$$\Phi(z;\tau) := \frac{\vartheta\left(z+\frac{1}{2};\tau\right)^m}{\vartheta\left(z;\tau\right)^n}. \tag{3.1.3}$$

[1] For more details see Chapter 1

For a more precise explanation, we refer the reader to Appendix A.2.

In [7] K. Bringmann and A. Folsom described the shape and the modularity properties of the Kac-Wakimoto characters in the case of m and n positive even integers. The goal of this chapter is to give a generalization of their results for any pair of integers $m > n > 0$, regardless of the parity. With this more general setting we will encounter half-integral weight and half-integral index Jacobi forms, which also satisfy slightly different transformation properties. To be more precise, as the index of Φ is $\frac{m-n}{2}$, it has half-integral index if m and n have opposite parity. In this case, we need to construct a slightly different theta decomposition for Φ involving half-integral index theta functions. In addition, when n is odd the parity of Φ as a function of its elliptic variable changes. This modifies the group under which the Fourier coefficients transform.

3.1.1 Statement of the theorems

Let $m > n > 0$ be positive integers, and as before $M = \frac{m-n}{2} \in \frac{1}{2}\mathbb{Z}$. For a real number x we denote the fractional part of x as $\{x\}$, and for any integer ℓ we define $L := \ell + \{M\}$. In order to state the main result of this chapter, we consider the ℓth Fourier coefficient of Φ around $z_0 \in \mathbb{C}$, namely

$$h_\ell^{(z_0)}(\tau) := \int_{z_0}^{z_0+1} \Phi(z;\tau) e^{-2\pi i L z}\, \mathrm{d}z. \tag{3.1.4}$$

Remark 1. In order to make the integral in (3.1.4) well defined, we need to be more precise. To do this, we recall certain assumptions as in [7, 15]. If Φ would be holomorphic, then the integral would be independent on the path of integration and well-defined for any z_0. For meromorphic Φ, we assume the path to be the straight line, if there are no poles on it. If z_0 is a pole of Φ, then we note that the integral in (3.1.4) depends only on the height of the path, and not on the initial point of the line. Therefore, we replace the straight line $[z_0, z_0 + 1]$ with $[z_0 + \delta, z_0 + \delta + 1]$, where δ is such that $z_0 + \delta$

is not a pole of Φ. Finally, if there is a pole on the path which is not an endpoint, we define the value of the integral as the average of the integral over a path deformed to pass just above the pole and the integral over a path just below it.

Following the approach of A. Dabolkar, S. Murthy, and D. Zagier [15], and K. Bringmann and A. Folsom [7], we generalize the definition of the *ℓth canonical Fourier coefficients* of Φ to half-integral index Jacobi forms as

$$h_\ell(\tau) := h_\ell^{\left(-\frac{\ell\tau}{2M}\right)}(\tau). \tag{3.1.5}$$

The aim of this chapter is to prove the following result concerning the shape and the modularity of the functions h_ℓ.

Theorem 3.1.1. *Let $\Gamma := \Gamma_0(2)$ (resp. $\Gamma(2)$) if m and n have the same parity (resp. opposite parity). The function $\boldsymbol{h}_\ell := (h_\ell)_{\ell \pmod{2M}}$ is a vector-valued almost mock modular form of weight $M - \frac{1}{2}$ for Γ.*

As we show in Appendix A.2, for $-M \leq \ell < M$

$$\mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0} = (-1)^m i^{-n} q^{\frac{2-1M}{24}+\frac{\ell^2}{4M}} \eta^{1-2M}(\tau) h_\ell(\tau), \tag{3.1.6}$$

while for $|\ell| > M$ $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0}$ differs from $h_{\ell+M}$ by a linear combination of quasi-modular forms. In particular, we can immediately derive the transformation properties and the shape of the Kac-Wakimoto characters.

Corollary 3.1.2. *Assume the notation of Theorem 3.1.1. Up to the multiplication by $\eta(\tau)^{1-2M} q^{\frac{\ell^2}{4M}+\frac{2M-1}{24}}$, the Kac-Wakimoto characters $tr_{L_{m,n}(\Lambda(\ell))} q^{L_0}$ are the holomorphic parts of components of a vector-valued almost mock modular form of weight $M - \frac{1}{2}$ for Γ.*

3.1.2 Outline of Chapter 3

This chapter proceeds as follows. In Section 3.2, we give some preliminary results. More precisely, we describe a vector-valued half-integral index theta function that allows a theta decomposition of half-integral index Jacobi forms. Furthermore, we describe the transformation properties of certain half-integral index Appell sums. Finally, we describe the action of the heat operator and the raising operator on the non-holomorphic function $R_{M,\ell}$ (see (2.3.4)). In Section 3.3, we prove Theorem 3.1.1. To do this, we generalize the approach used in [7] to half-integral index Jacobi forms.

3.2 Preliminaries

3.2.1 Half-integral index Jacobi forms

In order to give a theta decomposition for half-integral index Jacobi forms, for each positive integer N and for $\lambda \in \{0,1\}$, we define the vector-valued Jacobi theta functions

$$\begin{aligned}
\boldsymbol{\Theta}^{\mathrm{even}}_{N,\lambda}(z;\tau) &:= \left(e^{-2\pi i\frac{\ell\lambda}{4N}}\vartheta_{N,2\ell}\left(z+\frac{\lambda}{4N};\tau\right)\right)_{0\leq\ell<N},\\
\boldsymbol{\Theta}^{\mathrm{odd}}_{N,\lambda}(z;\tau) &:= \left(e^{-2\pi i\frac{\ell\lambda}{4N}}\vartheta_{N,2\ell+1}\left(z+\frac{\lambda}{4N};\tau\right)\right)_{0\leq\ell<N},\\
\boldsymbol{\Theta}_{N,\lambda}(z;\tau) &= \left(\boldsymbol{\Theta}^{\mathrm{even}}_{N,\lambda}(z;\tau),\boldsymbol{\Theta}^{\mathrm{odd}}_{N,\lambda}(z;\tau)\right),
\end{aligned}$$

where the function $\vartheta_{N,\ell}$ was introduced in (2.3.2). The aim of this subsection is to describe their transformation properties. To do so, we define the multiplier system $\varrho_{2N}:\mathrm{SL}_2(\mathbb{Z})\to\mathrm{GL}_{2N}(\mathbb{C})$ as follows. For $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, let

$$a_{h,\ell,N}(\gamma) := \begin{cases} \frac{1}{\sqrt{2Nci}}\sum_{\substack{s \pmod{2Nc}\\ s\equiv\ell \pmod{2N}}} e^{2\pi i\frac{as^2-2hs+dh^2}{4Nc}} & \text{if } c\neq 0,\\ e^{2\pi i\frac{b\ell^2}{4N}} & \text{if } c=0.\end{cases} \tag{3.2.1}$$

We define the following matrices in $\mathrm{GL}_N(\mathbb{C})$:

$$\sigma_N^{\text{even}} = \sigma_N^{\text{even}}(\gamma) := (a_{2h,2\ell,N}(\gamma))_{0\le h,\ell<N}\,,$$
$$\sigma_N^{\text{odd}} = \sigma_N^{\text{odd}}(\gamma) := (a_{2h+1,2\ell+1,N}(\gamma))_{0\le h,\ell<N}\,,$$
$$\omega_N^{\text{odd}} = \omega_N^{\text{odd}}(\gamma) := (a_{2h,2\ell+1,N}(\gamma))_{0\le h,\ell<N}\,,$$
$$\omega_N^{\text{even}} = \omega_N^{\text{even}}(\gamma) := (a_{2h+1,2\ell,N}(\gamma))_{0\le h,\ell<N}\,.$$

Then ϱ_{2N} is defined by

$$\varrho_{2N}(\gamma) := \begin{pmatrix} \sigma_N^{\text{even}}(\gamma) & \omega_N^{\text{odd}}(\gamma) \\ \omega_N^{\text{even}}(\gamma) & \sigma_N^{\text{odd}}(\gamma) \end{pmatrix}.$$

The following lemma describes the key properties of ϱ_{2N}. Here and throughout, let $\mathbb{0}_N$ (resp. $\mathbb{1}_N$) be the $N \times N$ zero matrix (resp. identity matrix).

Lemma 3.2.1. *With the notation as above, the following are true.*

1. *If N is even, then each of the matrices σ_N^{even}, σ_N^{odd}, ω_N^{odd}, and ω_N^{even} can be written in the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$ for for certain A and $B \in M_{\frac{N}{2}}(\mathbb{C})$.*

2. *If N is even and $\gamma \in \Gamma_0(2)$, then $\omega_N^{even} = \omega_N^{odd} = \mathbb{0}_N$.*

3. *If N is odd and $\gamma \in \Gamma_0(2)$, then $\sigma_N^{even} = \sigma_N^{odd} = \mathbb{0}_N$.*

Proof. The proof follows from the following identities

$$a_{h,\ell,N}(\gamma) = e^{-2\pi i \frac{bdh^2 - 2bh\ell}{4N}} a_{0,dh+\ell,N}(\gamma),$$
$$a_{h,\ell,N}(\gamma) = a_{h,\ell+2N,N}(\gamma).$$

More precisely, these two equalities imply that

$$a_{h,\ell,N}(\gamma) = a_{h+N,\ell+N,N}(\gamma),$$
$$a_{h+N,\ell,N}(\gamma) = a_{h,\ell+N,N}(\gamma),$$

which allows us to conclude part 1. The proof of parts 2 and 3 follow by definition (3.2.1). $\square$

We now have all the ingredients to prove the transformation properties of $\mathbf{\Theta}_{N,\lambda}$. In order to state the result, in light of the previous lemma, we write

$$\sigma_N^{\text{even}} := \begin{pmatrix} \sigma_N^{\text{even},1} & \sigma_N^{\text{even},2} \\ \sigma_N^{\text{even},2} & \sigma_N^{\text{even},1} \end{pmatrix}, \qquad \sigma_N^{\text{odd}} := \begin{pmatrix} \sigma_N^{\text{odd},1} & \sigma_N^{\text{odd},2} \\ \sigma_N^{\text{odd},2} & \sigma_N^{\text{odd},1} \end{pmatrix}.$$

Proposition 3.2.2. *Let N be a positive integer and $\lambda \in \{0,1\}$. Then the following are true:*

1. *The function $\mathbf{\Theta}_{N,0}(z;\tau)$ is a vector-valued Jacobi form of weight $\frac{1}{2}$ and index N for $\mathrm{SL}_2(\mathbb{Z})$, with multiplier system ϱ_{2N}.*
2. *The function $\mathbf{\Theta}_{N,\lambda}^{even}\left(\frac{z}{2};\frac{\tau}{2}\right)$ (resp. $\mathbf{\Theta}_{N,\lambda}^{odd}\left(\frac{z}{2};\frac{\tau}{2}\right)$) is a vector-valued Jacobi form of weight $\frac{1}{2}$ and index $\frac{N}{2}$ for $\Gamma_0(2)$, with multiplier system $\sigma_{2N}^{even,1}+(-1)^\lambda\sigma_{2N}^{even,2}$ (resp. $\sigma_{2N}^{odd,1}+(-1)^\lambda\sigma_{2N}^{odd,2}$).*

Proof. The proof of part 1 is an immediate consequence of Proposition 2.3.3. We give the proof of part 2 for $\mathbf{\Theta}_{N,\lambda}^{\text{even}}$. The proof for $\mathbf{\Theta}_{N,\lambda}^{\text{odd}}$ is analogous. First of all, note that

$$\mathbf{\Theta}_{N,\lambda}^{\text{even}}\left(\frac{z}{2};\frac{\tau}{2}\right) = \mathbb{T}_\lambda\, \mathbf{\Theta}_{2N,0}^{\text{even}}\left(\frac{z}{2};\tau\right),$$

where $\mathbb{T}_\lambda := \left(\mathbb{1}_N, (-1)^\lambda \mathbb{1}_N\right)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$. Since $2N$ is even, from part 1 and from part 2 of Lemma 3.2.1, it follows that

$$\mathbf{\Theta}_{N,\lambda}^{\text{even}}\left(\frac{z}{2(c\tau+d)};\frac{\gamma\tau}{2}\right) = (c\tau+d)^{\frac{1}{2}} e^{\frac{2\pi i N c z^2}{2(c\tau+d)}}\, \mathbb{T}_\lambda \sigma_{2N}^{\text{even}}(\gamma)\, \mathbf{\Theta}_{2N,0}^{\text{even}}\left(\frac{z}{2};\tau\right). \tag{3.2.2}$$

The symmetry of $\sigma_{2N}^{\text{even}}(\gamma)$ implies

$$\mathbb{T}_\lambda \sigma_{2N}^{\text{even}}(\gamma) = \left(\sigma_{2N}^{\text{even},1} + (-1)^\lambda \sigma_{2N}^{\text{even},2}\right)\mathbb{T}_\lambda.$$

We can therefore rewrite the right-hand side of (3.2.2) as

$$(c\tau+d)^{\frac{1}{2}} e^{\frac{2\pi i N c z^2}{2(c\tau+d)}} \left(\sigma_{2N}^{\text{even},1} + (-1)^{\lambda}\sigma_{2N}^{\text{even},2}\right) \mathbb{T}_{\lambda}\, \boldsymbol{\Theta}^{\text{even}}_{2N,0}\left(\frac{z}{2};\tau\right)$$
$$= (c\tau+d)^{\frac{1}{2}} e^{\frac{2\pi i N c z^2}{2(c\tau+d)}} \left(\sigma_{2N}^{\text{even},1} + (-1)^{\lambda}\sigma_{2N}^{\text{even},2}\right) \boldsymbol{\Theta}^{\text{even}}_{N,\lambda}\left(\frac{z}{2};\frac{\tau}{2}\right),$$

which gives the desired result. □

3.2.2 Additional properties of the Appell sums

The Appell sum f_M (see (2.3.3)) plays a key role in the entire thesis. In this subsection we describe the main properties of f_M needed in this section.

The first result that we need concerns the elliptic and the modularity properties of f_M when we slightly shift the elliptic variables simultaneously.

Proposition 3.2.3. *The completion of the Appell sum $\widehat{f}_M$ satisfies the following properties:*

1. *Let $N \in \frac{1}{2}\mathbb{Z}_{>0}$. For any λ and $\mu \in \mathbb{Z}$ and $\lambda \in \{0,1\}$*
$$\widehat{f}_N\left(z + \frac{\lambda\tau+\mu}{2N}, u + \frac{\lambda\tau+\mu}{2N};\tau\right) = e^{2\pi i\lambda(u-z)}\widehat{f}_N(z,u;\tau).$$

2. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)$ and $\lambda \in \mathbb{Z}$,*
$$\widehat{f}_N\left(\frac{z}{c\tau+d} + \frac{\lambda}{4N}, \frac{u}{c\tau+d} + \frac{\lambda}{4N};\gamma\tau\right)$$
$$= (c\tau+d)e^{2\pi i \frac{cN}{c\tau+d}(z^2-u^2)}\widehat{f}_N\left(z + \frac{\lambda}{4N}, u + \frac{\lambda}{4N};\tau\right).$$

Proof. The first property comes from the fact that $\widehat{f}_N$ "almost depends" on $z-u$. To be more precise, we recall that $\widehat{f}_N$ can be written as

$$\widehat{f}_N(z,u;\tau) = e^{2\pi i N(u-z)}\widehat{A}_{2N}(z-u, 2Nz - N\tau + N;\tau),$$

where $A_{2N}(z,u;\tau) := e^{\pi i z}\sum_{n\in\mathbb{Z}} \frac{(-1)^{2Nn}q^{Nn(n+1)}e^{2\pi i n u}}{1-e^{2\pi i z}q^n}$. The elliptic transformation properties of A_M described in Theorem 2.2 in [37] immediately gives the result.

For the second property, applying Proposition 2.3.7 we obtain

$$
\begin{aligned}
&\widehat{f}_N\left(\frac{z}{c\tau+d}+\frac{\lambda}{4N},\frac{u}{c\tau+d}+\frac{\lambda}{4N};\gamma\tau\right)\\
&\quad=(c\tau+d)e^{2\pi i\frac{cN}{c\tau+d}\left(z^2-u^2+\frac{\lambda}{2N}(c\tau+d)(z-u)\right)}\widehat{f}_N\left(z+\frac{\lambda}{4N}(c\tau+d),u+\frac{\lambda}{4N}(c\tau+d);\tau\right).
\end{aligned}
$$

Part 1 of this proposition and the fact that $\gamma\in\Gamma_0(2)$ gives the desired result. $\square$

The next step consists of describing the action of certain operators introduced in Section 2.1 on the real-analytic functions $R_{M,\ell}$ defined in (2.3.4). The goal is to relate the derivatives respect to the elliptic variable of $R_{M,\ell}$ with the raising operator applied to $R_{M,\ell}$ itself. Following the proof of Theorem 3.5 in [7], it is clear that the only condition needed to make this connection possible is that the function $R_{M,\ell}$ must be annihilated by the heat operator H_M, as we show in the following proposition.

Proposition 3.2.4. *Let $g(z;\tau)$ be a smooth function in both z and τ, and assume that $H_M[g(z;\tau)]=0$. Then for any positive integer j*

$$
\partial_z^{2j}\left[e^{2\pi\frac{Mz^2}{2v}}g\left(z;\tau\right)\right]_{z=0}=(4\pi M)^j\,R_{1/2}^j\left(g(0;\tau)\right), \tag{3.2.3}
$$

$$
\partial_z^{2j+1}\left[e^{2\pi\frac{Mz^2}{2v}}g\left(z;\tau\right)\right]_{z=0}=(4\pi M)^j\,R_{3/2}^j\left(\partial_z\left[g(z;\tau)\right]_{z=0}\right). \tag{3.2.4}
$$

Remark. In the statement of Proposition 3.2.4 we consider powers of the raising operator. By that we mean

$$
R_k^j := R_{k+2(j-1)}\circ\cdots\circ R_{k+2}\circ R_k.
$$

Proof. In Theorem 3.5 in [7], Bringmann and Folsom proved (3.2.4). The proof of (3.2.3) is very similar, therefore we skip the proof here. We note that a different proof can be derived as a special case of Proposition 5.3.7, where a multivariable version of this Proposition is considered. $\square$

As a consequence of Propositions 2.3.8 and 3.2.4, we have the following key result.

Corollary 3.2.5. *For any real number $\beta \in \mathbb{R}$, the following identities hold:*

$$\partial_z^{2j}\left[e^{-\frac{2\pi M z^2}{2v}} R_{M,\ell}\left(z+\beta;\tau\right)\right]_{z=0} = \left(-4\pi M\right)^j R_{1/2}^j\left(R_{M,\ell}(\beta;\tau)\right),$$
$$\partial_z^{2j+1}\left[e^{-\frac{2\pi M z^2}{2v}} R_{M,\ell}\left(z+\beta;\tau\right)\right]_{z=0} = \left(-4\pi M\right)^j R_{3/2}^j\left(\frac{1}{2\pi i}\partial_z\left[R_{M,\ell}\left(z+\beta;\tau\right)\right]_{z=0}\right).$$

We conclude this subsection by proving an identity involving the Appell sum which will be used several times in this thesis. This result was proven by K. Bringmann and A. Folsom in [7]. First, we need the following lemma.

Lemma 3.2.6. *Let $a \in \mathbb{Q}\setminus\mathbb{Z}$ and $0 < y < 1$ such that $|x| = y^a$. Then*

$$\sum_{\ell\in\mathbb{Z}} \frac{\operatorname{sgn}(a)+\operatorname{sgn}(\ell)}{2} x^\ell = -\frac{1}{2} + \frac{1}{1-x}.$$

Proof. The left-hand side equals

$$\operatorname{sgn}(a)\left(\frac{1}{2} + \sum_{\substack{\ell\in\mathbb{Z}\\ \operatorname{sgn}(\ell)=\operatorname{sgn}(a)}} x^\ell\right) = \operatorname{sgn}(a)\left(\frac{1}{2} + \sum_{\ell>0} x^{\operatorname{sgn}(a)\ell}\right). \tag{3.2.5}$$

By assumption $|x^{\operatorname{sgn}(a)}| = y^{|a|} < 1$, thus (3.2.5) can be written as

$$\operatorname{sgn}(a)\left(-\frac{1}{2} + \frac{1}{1-x^{\operatorname{sgn}(a)}}\right) = -\frac{1}{2} + \frac{1}{1-x}.$$

□

Proposition 3.2.7. *For each $A \in \mathbb{Q}\setminus\mathbb{Z}$ positive, for $u \in \mathbb{R}$ and $Im(z) = AIm(\tau)$ the following identity holds:*

$$f_M(z,u;\tau) - \frac{1}{2}\vartheta_{M,0}(z;\tau) = \sum_{\ell\in\mathbb{Z}} e^{2\pi i \ell z} \sum_{\substack{n\in\mathbb{Z}\\ -\frac{\ell}{2M}<n<A}} q^{-Mn^2-\ell n} e^{-2\pi i u(2Mn+\ell)}.$$

Proof. By definition, the left-hand side can be written as

$$\sum_{n\in\mathbb{Z}} q^{Mn^2} e^{4\pi iMnz}\left(-\frac{1}{2}+\frac{1}{1-e^{2\pi i(z-u)}q^n}\right).$$

By assumption $|e^{2\pi i(z-u)}q^n| = e^{-2\pi\mathrm{Im}(\tau)(A+n)}$. Therefore, by Lemma 3.2.6 the term above equals

$$\sum_{n\in\mathbb{Z}} q^{Mn^2} e^{4\pi iMnz}\sum_{\ell\in\mathbb{Z}}\frac{\mathrm{sgn}(A+n)+\mathrm{sgn}(\ell)}{2}e^{2\pi i\ell(z-u)}q^{n\ell}.$$

Sending first ℓ to $\ell-2Mn$ and then n to $-n$, and interchanging the two summations, we get the desired result. $\square$

3.3 Proof of Theorem 3.1.1

3.3.1 Transformation properties of Φ

In this section we prove Theorem 3.1.1. We shall follow the ideas used in [7] and [15]. The main issue in this setting is that we have to deal with half-integral index Jacobi forms. We solve this problem by splitting Φ into an even part and an odd part. The integrality of the index dictates the vanishing of one of those parts. We can then use a similar argument to treat both of the parts, giving a unified proof for the general case.

For $r \in \mathbb{Z}_{>0}$ we define $\iota(r) := r - \left[\frac{r-1}{2}\right] \in \{1, 2\}$. The transformation properties of Φ can be deduced from the transformation properties of ϑ (see Proposition 2.3.2).

Proposition 3.3.1. *With the notation as above, the following hold:*

1. *For λ, $\mu \in \mathbb{Z}$, we have*

$$\Phi(z+\lambda\tau+\mu;\tau) = (-1)^{2M(\lambda+\mu)+m\lambda}e^{-2\pi iM(\lambda^2\tau+2\lambda z)}\Phi(z;\tau).$$

2. *For any $\gamma \in \Gamma_0(2)$, we have*

$$\Phi\left(\frac{z}{c\tau+d};\gamma\tau\right) = \chi(\gamma)(c\tau+d)^M e^{2\pi i\frac{Mcz^2}{c\tau+d}}\Phi(z;\tau),$$

where $\chi(\gamma) := \psi(\gamma)^{6M}(-1)^{\frac{mc}{4}}$, *with* $\psi(\gamma)$ *the multiplier of the* η*-function (see Lemma 2.2.2).*

3. *The function* Φ *has the same parity as* n, *namely,* $\Phi(-z;\tau) = (-1)^n\Phi(z;\tau)$.

3.3.2 Canonical Fourier coefficients and canonical decomposition

It is known by Theorem 3.9 of [36] that any meromorphic Jacobi form of positive index splits into a theta decomposition and another term which can be expressed in terms of the residues of Appell sums multiplied by the Jacobi form itself. Dabholkar, Murthy, and Zagier [15] revisited Zwegers' proof defining a canonical splitting of any meromorphic Jacobi form of positive integral index and poles of order at most 2 into two pieces: a finite part, which has the shape of a theta decomposition, and a polar part, which depends just on the poles of the Jacobi form. We follow this last approach to investigate half-integral index Jacobi forms. We will also make use of the parity of Φ to get some more information about its Fourier coefficients.

Since Φ is a meromorphic function in the elliptic variable, a global Fourier expansion does not make sense, however due to the elliptic transformation property its poles have a nice symmetry. More precisely, we know that they are exactly the points in the lattice $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$. In particular, for any $z_0 \in \mathbb{C}$ such that $\mathrm{Im}(z_0) \notin \mathbb{Z}\mathrm{Im}(\tau)$, the *$\ell$th normalized Fourier coefficient of* Φ *in* z_0

$$h_\ell^{(z_0)}(\tau) := q^{-\frac{\ell^2}{16M}} \int_{z_0}^{z_0+1} \Phi(2z;\tau)e(-\ell z)\,\mathrm{d}z$$

is well defined. If $\mathrm{Im}(z_0) \in \mathbb{Z}\mathrm{Im}(\tau)$ we modify the definition as explained in Remark 1 in Subsection 3.1.1.

Remark. We use the same notation as in [15]. However, our definition is slightly different. More precisely, it generalizes the definition in [15] since we deal with a half-integral index Jacobi form. In this setting the half-integrality of the index is equivalent to the 2-periodicity of Φ as a function of z. As we shall prove later, for 1-periodic Φ we have $h_\ell^{(z_0)}(\tau) = 0$ for ℓ odd. If we denote by $K_\ell^{(z_0)}$ the Fourier coefficient defined in [15], then a change of variables shows that $h_{2\ell}^{(z_0)}(\tau) = K_\ell^{(2z_0)}(\tau)$.

The transformation properties of Φ imply the following properties of $h_\ell^{(z_0)}$.

Lemma 3.3.2. *For each $\ell \in \mathbb{Z}$, the normalized Fourier coefficient $h_\ell^{(z_0)}$ satisfies the following periodicity properties:*

1. *For each $\lambda \in \mathbb{Z}$, $h_{\ell+4M\lambda}^{(z_0)}(\tau) = (-1)^{\lambda n} h_\ell^{\left(z_0+\frac{\lambda\tau}{2}\right)}(\tau)$.*

2. *We have $h_{-\ell}^{(z_0)}(\tau) = (-1)^n h_\ell^{(-z_0)}(\tau)$.*

Proof. First we prove part 1. By definition

$$h_{\ell+4M\lambda}^{(z_0)}(\tau) = q^{-\frac{\ell^2}{16M}} \int_{z_0}^{z_0+1} \Phi\left(2z;\tau\right) q^{-\frac{\ell\lambda}{2}-M\lambda^2} e^{-2\pi i(\ell+4M\lambda)z}\,\mathrm{d}z.$$

From Proposition 3.3.1 we know that $\Phi\left(2z+\lambda\tau;\tau\right) = (-1)^{n\lambda} q^{-M\lambda^2} e^{-8\pi i M\lambda z}\Phi\left(2z;\tau\right)$, therefore

$$\begin{aligned} h_{\ell+4M\lambda}^{(z_0)}(\tau) &= (-1)^{\lambda n} q^{-\frac{\ell^2}{16M}} \int_{z_0}^{z_0+1} \Phi\left(2z+\lambda\tau;\tau\right) q^{-\frac{\ell\lambda}{2}} e^{-2\pi i\ell z}\,\mathrm{d}z \\ &= (-1)^{\lambda n} q^{-\frac{\ell^2}{16M}} \int_{z_0+\frac{\lambda\tau}{2}}^{z_0+\frac{\lambda\tau}{2}+1} \Phi\left(2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z = (-1)^{\lambda n} h_\ell^{\left(z_0+\frac{\lambda\tau}{2}\right)}(\tau). \end{aligned}$$

We now prove part 2. By definition and the change of variable $z \mapsto -z$, we have

$$h_{-\ell}^{(z_0)}(\tau) = -q^{-\frac{\ell^2}{16M}} \int_{-z_0}^{-z_0-1} \Phi\left(-2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z. \tag{3.3.1}$$

The parity of Φ (see Proposition 3.3.1) implies that (3.3.1) equals

$$(-1)^n q^{-\frac{\ell^2}{16M}} \int_{-z_0-1}^{-z_0} \Phi\left(2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z.$$

To conclude it is enough to change the integration variable as $z \mapsto z-1$ to obtain

$$h_{-\ell}^{(z_0)}(\tau) = (-1)^n q^{-\frac{\ell^2}{16M}} \int_{-z_0}^{-z_0+1} \Phi\left(2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z = (-1)^n h_\ell^{(-z_0)}(\tau).$$

□

For a fixed z_0, Lemma 3.3.2 implies a non-periodicity for the normalized Fourier coefficients. To be more precise, a difference between two of them gives a non-trivial contribution of the residues of Φ in a certain parallelogram. However, as Dabholkar, Murthy, and Zagier showed for second order poles, it is possible to obtain periodicity making z_0 dependent on τ. This led to the definition of *canonical Fourier coefficients.* For each integer ℓ we define the ℓth canonical Fourier coefficient by

$$h_\ell(\tau) := h_\ell^{\left(-\frac{\ell\tau}{8M}\right)}(\tau).$$

Corollary 3.3.3. *For each $\ell \in \mathbb{Z}$, the canonical Fourier coefficient h_ℓ satisfies the following properties:*

1. *For each $\lambda \in \mathbb{Z}$, $h_{\ell+4M\lambda}(\tau) = (-1)^{n\lambda} h_\ell(\tau)$.*
2. *We have $h_{-\ell}(\tau) = (-1)^n h_\ell(\tau)$. In particular, if n is odd, $h_0(\tau) = 0$.*
3. *We have $h_\ell(\tau) = h_{4M-\ell}(\tau)$.*

In order to treat the integral and the half-integral index cases simultaneously, we define the even and the odd vector-valued canonical Fourier coefficients by

$$\begin{aligned}\mathbf{h}_{2M}^{\mathrm{even}}(\tau) &:= (h_0(\tau), h_2(\tau), \cdots, h_{4M-2}(\tau)) = (h_{2\ell}(\tau))_{0\le\ell<2M}\,, \\ \mathbf{h}_{2M}^{\mathrm{odd}}(\tau) &:= (h_1(\tau), h_3(\tau), \cdots, h_{4M-1}(\tau)) = (h_{2\ell+1}(\tau))_{0\le\ell<2M}\,.\end{aligned} \tag{3.3.2}$$

Lemma 3.3.4. *If $M \in \mathbb{N}$ (resp. $M \in \frac{1}{2} + \mathbb{N}$) then $\boldsymbol{h}_{2M}^{odd}(\tau) = \boldsymbol{0}$ (resp. $\boldsymbol{h}_{2M}^{even}(\tau) = \boldsymbol{0}$).*

Proof. By definition,

$$\begin{aligned}h_\ell(\tau) &= q^{-\frac{\ell^2}{16M}} \int_{-\frac{\ell\tau}{8M}}^{-\frac{\ell\tau}{8M}+1} \Phi\left(2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z \\ &= q^{-\frac{\ell^2}{16M}} \left(\int_{-\frac{\ell\tau}{8M}}^{-\frac{\ell\tau}{8M}+\frac{1}{2}} + \int_{-\frac{\ell\tau}{8M}+\frac{1}{2}}^{-\frac{\ell\tau}{8M}+1} \right) \Phi\left(2z;\tau\right) e^{-2\pi i\ell z}\,\mathrm{d}z.\end{aligned}$$

Making the substitution $z \mapsto z + \frac{1}{2}$ in the second integral, we obtain

$$h_\ell(\tau) = \left(1 + (-1)^{\ell+2M}\right) q^{-\frac{\ell^2}{16M}} \int_{-\frac{\ell\tau}{8M}}^{-\frac{\ell\tau}{8M}+\frac{1}{2}} \Phi\left(2z;\tau\right) e^{-2\pi i \ell z}\, \mathrm{d}z,$$

which equals 0 if ℓ and $2M$ have opposite parity. $\square$

We now generalize the canonical decomposition to our situation. We define the *finite part* of the meromorphic Jacobi form Φ by

$$\Phi^F(z;\tau) := \mathbf{h}_{2M}^{\mathrm{even}}(\tau) \cdot \mathbf{\Theta}_{2M,n}^{\mathrm{even}}\left(\frac{z}{2};\frac{\tau}{2}\right) + \mathbf{h}_{2M}^{\mathrm{odd}}(\tau) \cdot \mathbf{\Theta}_{2M,n}^{\mathrm{odd}}\left(\frac{z}{2};\frac{\tau}{2}\right),$$

where $\mathbf{\Theta}_{2M,n}^{\mathrm{odd}}$ and $\mathbf{\Theta}_{2M,n}^{\mathrm{odd}}$ are described in Subsection 3.2.1. As the notation suggest, the function Φ^F is no longer meromorphic. Instead, it is a holomorphic function whose Fourier coefficients with respect to z coincide (up to q-powers) with the canonical Fourier coefficients. Moreover, if Φ is holomorphic, then the integral defining $\mathbf{h}_{2M}$ no longer depends on the path of integration or on the initial point z_0. Therefore, in light of Proposition 2.3.4, in this case Φ^F coincides with Φ.

In order to define the *polar part* of Φ (see (3.3.9)), we need some more notation. As mentioned before, the function Φ has poles of order n in $\mathbb{Z}\tau + \mathbb{Z}$. We denote the Laurent expansion of $\Phi(z;\tau)$ in $z = 0$ by

$$\Phi\left(z;\tau\right) = \sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{\widetilde{D}_{\iota(n)+2j}(\tau)}{(2\pi i z)^{\iota(n)+2j}} + O(1) \qquad \text{as } z \to 0. \tag{3.3.3}$$

We recall that $\iota(n)$ was defined at the beginning of Section 3.3. As we shall see in the next lemma, the Laurent coefficients D_j are quasimodular forms (see Definition 2.2.3). Their completions turn out to be the Laurent coefficients of a similar function, this time real-analytic in τ, namely

$$e^{\frac{\pi M z^2}{v}}\Phi\left(z;\tau\right) := \sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(2\pi i z)^{\iota(n)+2j}} + O(1) \qquad \text{as } z \to 0. \tag{3.3.4}$$

In the following proposition, we describe the modular properties of the functions $\widetilde{D}_j$ and D_j. Here, $\mathcal{L}$ denotes the single-variable lowering operator.

Proposition 3.3.5. *With the notation as above, the following facts are true.*

1. *For $0 \leq j \leq \frac{n-\iota(n)}{2}$, the function $D_{\iota(n)+2j}(\tau)$ is an almost holomorphic modular form of weight $M - \iota(n) - 2j$ for $\Gamma_0(2)$, with the same multiplier system as Φ. More precisely, for each $\gamma \in \Gamma_0(2)$,*

$$D_{\iota(n)+2j}(\gamma\tau) = \chi(\gamma)(c\tau + d)^{M-\iota(n)-2j} D_{\iota(n)+2j}(\tau).$$

2. *For $0 \leq j \leq \frac{n-\iota(n)}{2}$, the function $\widetilde{D}_{\iota(n)+2j}(\tau)$ is the holomorphic part of $D_{\iota(n)+2j}(\tau)$, i.e., it is a quasimodular form.*

3. *For each $0 \leq j \leq \left[\frac{n-1}{2}\right]$*

$$D_{\iota(n)+2j}(\tau) = \left(\frac{4\pi}{M}\right)^j \mathcal{L}^j \left(D_{\iota(n)}(\tau)\right).$$

Proof. We start by proving part 1. For simplicity, we define $F(z;\tau) := e^{\frac{\pi M z^2}{v}}$ and $\widetilde{\Phi} := F\Phi$. One can easily see that F satisfies the modular transformation of a Jacobi form of index $-M$ and weight 0. In particular, $\widetilde{\Phi}$ transforms as

$$\widetilde{\Phi}\left(\frac{z}{c\tau + d}; \gamma\tau\right) = \chi(\gamma)(c\tau + d)^M \widetilde{\Phi}\left(z;\tau\right). \tag{3.3.5}$$

Writing both the right and the left-hand sides of (3.3.5) with their Laurent expansions in $z = 0$, using (3.3.4), we obtain

$$\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\gamma\tau)}{(2\pi i z)^{\iota(n)+2j}}(c\tau + d)^{\iota(n)+2j} + O(1)$$
$$= \chi(\gamma)(c\tau + d)^M \sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(2\pi i z)^{\iota(n)+2j}} + O(1) \quad \text{as } z \to 0.$$

To conclude part 1 it is enough to compare the coefficient of $z^{-\iota(n)-2j}$ for any fixed j.

To prove part 2, we first write explicitly the Laurent expansion of F in $z=0$, namely

$$F(z;\tau)=\sum_{r\geq 0}\frac{\partial_z^{2r}\left[F(z;\tau)\right]_{z=0}}{(2r)!}z^{2r}=1+\sum_{r>0}\left(\frac{2\pi M}{v}\right)^r\frac{z^{2r}}{2\cdot r!}. \tag{3.3.6}$$

Using the Laurent expansions (3.3.3), (3.3.4), and (3.3.6), we obtain

$$D_{\iota(n)+2s}(\tau)=\widetilde{D}_{\iota(n)+2s}(\tau)+\sum_{j=s+1}^{\frac{n-\iota(n)}{2}}\frac{\widetilde{D}_{\iota(n)+2j}(\tau)}{2\cdot(j-s)!}\left(-\frac{M}{2\pi v}\right)^{j-s}. \tag{3.3.7}$$

The claim is then proven in light of part 1.

We conclude the proof by showing part 3. Since the lowering operator annihilates holomorphic functions, using (3.3.7) we have

$$\mathcal{L}\left(D_{\iota(n)+2s}(\tau)\right)=\sum_{j=s}^{\frac{n-\iota(n)}{2}}\frac{\widetilde{D}_{\iota(n)+2j}(\tau)}{2\cdot(j-s)!}\left(-\frac{M}{2\pi}\right)^{j-s}\mathcal{L}\left(\frac{1}{v^{j-s}}\right). \tag{3.3.8}$$

A direct computation shows that

$$\mathcal{L}\left(\frac{1}{v^{j-s}}\right)=-\frac{j-s}{v^{j-s-1}},$$

therefore (3.3.8) becomes

$$\begin{aligned}\mathcal{L}\left(D_{\iota(n)+2s}(\tau)\right)&=\frac{M}{2\pi}\sum_{j=s+1}^{\frac{n-\iota(n)}{2}}\frac{\widetilde{D}_{\iota(n)+2j}(\tau)}{2\cdot(j-s-1)!}\left(-\frac{M}{2\pi v}\right)^{j-s-1}\\&=\frac{M}{2\pi}D_{\iota(n)+2(s+1)}(\tau).\end{aligned}$$

Iterating this computation s times, we conclude the proof. □

We now define the *polar part* of Φ by

$$\begin{aligned}\Phi^P(z;\tau):=\sum_{j=0}^{\frac{n-\iota(n)}{2}}\frac{\widetilde{D}_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1}&\left[\frac{1+(-1)^{2M}}{2}\right.\\ \times f_M\left(z+\frac{n}{4M},u+\frac{n}{4M};\tau\right)+\frac{1-(-1)^{2M}}{2}f_M&\left.\left(z+\frac{n+\tau}{4M},u+\frac{n+\tau}{4M};\tau\right)\right]_{u=0}.\end{aligned} \tag{3.3.9}$$

The functions Φ^F and Φ^P are the two pieces in the canonical decomposition of Φ, as we see in the following proposition. We note that depending on the parity of $2M$, one of the two pieces defining Φ^P vanishes. This agree with the definition of Φ^F, itself divided into an even part and an odd part.

Proposition 3.3.6. *With the notation as above we have* $\Phi = \Phi^F + \Phi^P$.

Proof. We consider the difference $\Phi - \Phi^F$, and we prove that it equals Φ^P. Let $A \in \mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}$, then for z such that $\mathrm{Im}(z) = A\mathrm{Im}(\tau)$ we have by definition

$$\Phi(z;\tau) - \Phi^F(z;\tau) = \sum_{\ell\in\mathbb{Z}} e^{\pi i\ell z}\left(\int_{A\tau}^{A\tau+1} - \int_{-\frac{\ell}{8M}}^{-\frac{\ell}{8M}+1}\right)\Phi(2u;\tau)e^{-2\pi i\ell u}\,\mathrm{d}u. \tag{3.3.10}$$

Making the change of variables $w \mapsto 2u$ and noting that

$$\Phi(u+1;\tau)e^{\pi i\ell(u+1)} = (-1)^{\ell+2M}\Phi(u;\tau)e^{\pi i\ell u},$$

Cauchy's residue theorem and the fact that $\Phi(z;\tau)$ has poles in $\mathbb{Z}\tau + \mathbb{Z}$ imply that (3.3.10) equals

$$\begin{aligned}
&2\pi i\sum_{\ell\in\mathbb{Z}} e^{\pi i\ell z}\sum_{\substack{\alpha\in\mathbb{Z}\\ -\frac{\ell}{4M}<\alpha<2A}} \operatorname*{Res}_{u=0}\left(\Phi(u+\alpha\tau;\tau)e^{-\pi i\ell(u+\alpha\tau)}\frac{1+(-1)^{\ell+2M}}{2}\right)\\
&= 2\pi i\sum_{\ell\in\mathbb{Z}}\frac{1+(-1)^{\ell+2M}}{2}e^{\pi i\ell z}\sum_{\substack{\alpha\in\mathbb{Z}\\ -\frac{\ell}{4M}<\alpha<2A}}(-1)^{n\alpha}q^{-M\alpha^2-\frac{\ell\alpha}{2}}\operatorname*{Res}_{u=0}\left(\Phi(u;\tau)e^{-\pi iu(\ell+4M\alpha)}\right),
\end{aligned} \tag{3.3.11}$$

where in the second step we used the elliptic transformation of Φ. Using the Laurent expansion of Φ in $u = 0$ (see (3.3.3)) and interchanging the summations, (3.3.11) can be written as

$$\sum_{j=0}^{J}\frac{\widetilde{D}_{n-2j}(\tau)}{(n-2j-1)!(2\pi i)^{n-2j-1}}\partial_u^{n-2j-1}\left[\sum_{\ell\in\mathbb{Z}}\frac{1+(-1)^{\ell+2M}}{2}e^{\pi i\ell z}\right.$$
$$\left.\times\sum_{\alpha\in\left(-\frac{\ell}{4M},2A\right)\cap\mathbb{Z}}(-1)^{n\alpha}q^{-M\alpha^2-\frac{\ell\alpha}{2}}e^{-\pi iu(\ell+4M\alpha)}\right]_{u=0}.$$

If we now split the sum into two pieces according to the parity of ℓ and then use (3.2.7), we can rewrite the argument of ∂_u as

$$\frac{1+(-1)^{2M}}{2} f_M\left(z+\frac{n}{4M}, u+\frac{n}{4M};\tau\right)+\frac{1-(-1)^{2M}}{2} f_M\left(z+\frac{n+\tau}{4M}, u+\frac{n+\tau}{4M};\tau\right)\\ -\frac{1}{2}\vartheta_{M,0}\left(z+\frac{n}{4M};\tau\right).$$

This conclude the proof since the extra summand given by the theta function is independent on u, and therefore annihilated by ∂_u. □

In the following proposition, we rewrite Φ^P in terms of the almost holomorphic modular forms D_j instead of their holomorphic parts $\widetilde{D}_j$. This will be useful in order to determine the completions of Φ^P and Φ^F.

Proposition 3.3.7. *The function Φ^P equals*

$$\Phi^P(z;\tau) = \sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1}\left[e^{\frac{-\pi M u^2}{v}}\left(\frac{1+(-1)^{2M}}{2}\right.\right.\\ \left.\left.\times f_M\left(z+\frac{n}{4M}, u+\frac{n}{4M};\tau\right)+\frac{1-(-1)^{2M}}{2} f_M\left(z+\frac{n+\tau}{4M}, u+\frac{n+\tau}{4M};\tau\right)\right)\right]_{u=0}.$$

Proof. The proof is almost identical to the proof of Proposition 3.3.6. In (3.3.11) it is enough to replace the Laurent expansion (3.3.3) with the modified Laurent expansion (3.3.4). □

3.3.3 Modular properties of h_ℓ

In the previous subsection, more precisely in Proposition 3.3.6, we have seen how to canonically decompose the function Φ into a finite part, which is holomorphic, and a polar part, which is meromorphic and only depends on the poles of Φ. In Proposition 3.3.7, we have also seen an alternative way to write Φ^P. In that decomposition the functions D_{n-2j} transform as modular forms, as we proved in Proposition 3.3.5. Moreover, the Appell sums are well known to be closely related to Jacobi forms, as we saw in Proposition 2.3.7. Using that, we are able to complete the polar part Φ^P, i.e., to add a certain non-holomorphic function to Φ^P to get a modular object. This immediately gives a completion for the finite part Φ^F, and therefore for $\mathbf{h}_{2M}^{\text{even}}$ and $\mathbf{h}_{2M}^{\text{odd}}$.

Analyzing the shape of Φ^P, i.e., a linear combination of almost holomorphic modular forms multiplied by derivatives of the Appell function, it is natural to define its completion by substituting f_M by $\widehat{f}_M$ (see (2.3.5)), namely,

$$\widehat{\Phi}^P(z;\tau) := \sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1}\left[e^{\frac{-\pi M u^2}{v}}\left(\frac{1+(-1)^{2M}}{2}\right.\right.$$
$$\left.\left.\times \widehat{f}_M\left(z+\frac{n}{4M}, u+\frac{n}{4M};\tau\right)+\frac{1-(-1)^{2M}}{2}\widehat{f}_M\left(z+\frac{n+\tau}{4M}, u+\frac{n+\tau}{4M};\tau\right)\right)\right]_{u=0}. \tag{3.3.12}$$

Defining

$$\mathcal{R}_{2M,n}^{\text{even}}(\tau) := \frac{1+(-1)^{2M}}{4}\left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!}\right.$$
$$\left.\times\left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1}\left(e^{-\frac{\pi M u^2}{v}} R_{2M,2\ell}\left(\frac{u}{2}+\frac{n}{8M};\frac{\tau}{2}\right)\right)\right)_{0\le \ell<2M},$$

$$\mathcal{R}_{2M,n}^{\text{odd}}(\tau) := \frac{1-(-1)^{2M}}{4}\left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!}\right.$$
$$\left.\times\left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1}\left(e^{-\frac{\pi M u^2}{v}} R_{2M,2\ell+1}\left(\frac{u}{2}+\frac{n}{8M};\frac{\tau}{2}\right)\right)\right)_{0\le \ell<2M},$$

then by (2.3.5) we have that

$$\widehat{\Phi}^P(z;\tau) = \Phi^P(z;\tau) - \mathcal{R}_{2M,n}^{\text{even}}(\tau)\cdot \boldsymbol{\Theta}_{2M,n}^{\text{even}}\left(\frac{z}{2};\frac{\tau}{2}\right) - \mathcal{R}_{2M,n}^{\text{odd}}(\tau)\cdot \boldsymbol{\Theta}_{2M,n}^{\text{odd}}\left(\frac{z}{2};\frac{\tau}{2}\right).$$

Analogously, we define the completion $\widehat{\Phi}^F$ of Φ^F as

$$\widehat{\Phi}^F(z;\tau) := \Phi^F(z;\tau) + \mathcal{R}_{2M,n}^{\text{even}}(\tau)\cdot \boldsymbol{\Theta}_{2M,n}^{\text{even}}\left(\frac{z}{2};\frac{\tau}{2}\right) + \mathcal{R}_{2M,n}^{\text{odd}}(\tau)\cdot \boldsymbol{\Theta}_{2M,n}^{\text{odd}}\left(\frac{z}{2};\frac{\tau}{2}\right).$$

Since both Φ^F and the non-holomorphic piece has a theta decomposition, we define the completions of $\mathbf{h}_{2M}^{\text{even}}$ and $\mathbf{h}_{2M}^{\text{odd}}$ as the coefficients in the theta decomposition of $\widehat{\Phi}^F$. More precisely,

$$\widehat{\mathbf{h}}_{2M}^{\text{even/odd}}(\tau) := \mathbf{h}_{2M}^{\text{even/odd}} + \mathcal{R}_{2M,n}^{\text{even/odd}}(\tau),$$

and therefore we also have

$$\widehat{\Phi}^F(z;\tau) = \widehat{\mathbf{h}}_{2M}^{\text{even}}(\tau)\cdot \mathbf{\Theta}_{2M,n}^{\text{even}}\left(\frac{z}{2};\frac{\tau}{2}\right) + \widehat{\mathbf{h}}_{2M}^{\text{odd}}(\tau)\cdot \mathbf{\Theta}_{2M,n}^{\text{odd}}\left(\frac{z}{2};\frac{\tau}{2}\right). \tag{3.3.13}$$

Proposition 3.3.8. *The functions $\widehat{\Phi}^F$ and $\widehat{\Phi}^P$ satisfy the same modular transformation law as Φ. In particular, the functions $\widehat{\boldsymbol{h}}_{2M}^{even}$ and $\widehat{\boldsymbol{h}}_{2M}^{odd}$ are vector-valued non-holomorphic modular forms of weight $M-\frac{1}{2}$ for $\Gamma_0(2)$, with multiplier system described in Proposition 2.3.4.*

Proof. Looking at the definition of Φ^P in (3.3.12), we consider the modular transformation property of each summand, omitting the constants. We show that each of them satisfies the same modular transformation as Φ. Indeed, using Proposition 3.3.5 and Proposition 3.2.3, for each $\gamma \in \Gamma_0(2)$, we have

$$\begin{aligned} D_{\iota(n)+2j}\left(\gamma\tau\right)\partial_u^{\iota(n)+2j-1}&\left[e^{\frac{-\pi M u^2}{v}}\left(\frac{1+(-1)^{2M}}{2}\widehat{f}_M\left(\frac{z}{c\tau+d}+\frac{n}{4M},u+\frac{n}{4M};\gamma\tau\right)\right.\right.\\ &\left.\left.+\frac{1-(-1)^{2M}}{2}\widehat{f}_M\left(\frac{z}{c\tau+d}+\frac{n+\gamma\tau}{4M},u+\frac{n+\gamma\tau}{4M};\gamma\tau\right)\right)\right]_{u=0}\\ &=\chi(\gamma)(c\tau+d)^M D_{\iota(n)+2j}(\tau)\partial_u^{\iota(n)+2j-1}\left[e^{\frac{-\pi M u^2}{v}}\left(\frac{1+(-1)^{2M}}{2}\right.\right.\\ \times\widehat{f}_M\left(z+\frac{n}{4M},u+\frac{n}{4M};\tau\right)&\left.\left.+\frac{1-(-1)^{2M}}{2}\widehat{f}_M\left(z+\frac{n+\tau}{4M},u+\frac{n+\tau}{4M};\tau\right)\right)\right]_{u=0}. \end{aligned}$$

Since the automorphy factor is the same for each summand and equal to the automorphy factor of Φ, we conclude that $\widehat{\Phi}^P$ transforms as Φ. Since $\widehat{\Phi}^F = \Phi - \widehat{\Phi}^P$, the same statement is true for $\widehat{\Phi}^F$. The transformation properties of $\widehat{\mathbf{h}}_{2M}^{\text{even}}$ and $\widehat{\mathbf{h}}_{2M}^{\text{odd}}$ follow from (3.3.13), using Proposition 2.3.4. $\square$

3.3.4 Shape of $\widehat{\mathbf{h}}_{2M}$

To conclude the proof of Theorem 3.1.1, we need to describe the shape of $\widehat{\mathbf{h}}_{2M}^{\text{even}}$ and $\widehat{\mathbf{h}}_{2M}^{\text{odd}}$. In particular, we write their non-holomorphic pieces as described in the definition of almost harmonic Maass forms (see Definition 2.2.6). This is just a consequence of Corollary 3.2.5 and Proposition 3.3.5. We recall that each component of the vector-valued function $\mathcal{R}_{2M,n}^{\text{even/odd}}(\tau)$ is defined as

$$\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{D_{\iota(n)+2j}(\tau)}{(\iota(n)+2j-1)!} \left(\frac{\partial_u}{2\pi i}\right)^{\iota(n)+2j-1} \left[e^{-\frac{\pi M u^2}{v}} R_{2M,\ell}\left(\frac{u}{2}+\frac{n}{8M};\frac{\tau}{2}\right)\right]_{u=0},$$

for a certain integer ℓ. Using the two results cited above, we rewrite it as

$$\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{4^j}{(2\pi i)^{\iota(n)-1}} \frac{\mathcal{L}_{M-\iota(n)}^{j}\left(D_{\iota(n)}(\tau)\right)}{(\iota(n)+2j-1)!} R_{\iota(n)-\frac{1}{2}}^{j}\left(\partial_u^{\iota(n)-1}\left[R_{2M,\ell}\left(\frac{u}{2}+\frac{n}{8M};\frac{\tau}{2}\right)\right]_{u=0}\right).$$

Let $\widehat{G}_\ell(\tau)$ be a mock theta function whose non holomorphic part is $\partial_u^{\iota(n)-1}\left(R_{2M,\ell}\left(\frac{u}{2}+\frac{n}{8M};\frac{\tau}{2}\right)\right)$ (it exists by Corollary 2.3.11), and denote by G_ℓ its holomorphic part. Moreover, define the vector-valued functions

$$\mathcal{G}^{\text{even}}(\tau) := \delta_+ \left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{4^j}{(2\pi i)^{\iota(n)-1}} \frac{\mathcal{L}_{M-\iota(n)}^{j}\left(D_{\iota(n)}(\tau)\right)}{(\iota(n)+2j-1)!} R_{\iota(n)-\frac{1}{2}}^{j}\left(G_{2\ell}(\tau)\right)\right)_{0\le\ell<2M},$$

$$\mathcal{G}^{\text{odd}}(\tau) := \delta_- \left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{4^j}{(2\pi i)^{\iota(n)-1}} \frac{\mathcal{L}_{M-\iota(n)}^{j}\left(D_{\iota(n)}(\tau)\right)}{(\iota(n)+2j-1)!} R_{\iota(n)-\frac{1}{2}}^{j}\left(G_{2\ell+1}(\tau)\right)\right)_{0\le\ell<2M},$$

$$\widehat{\mathcal{G}}^{\text{even}}(\tau) := \delta_+ \left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{4^j}{(2\pi i)^{\iota(n)-1}} \frac{\mathcal{L}_{M-\iota(n)}^{j}\left(D_{\iota(n)}(\tau)\right)}{(\iota(n)+2j-1)!} R_{\iota(n)-\frac{1}{2}}^{j}\left(\widehat{G}_{2\ell}(\tau)\right)\right)_{0\le\ell<2M},$$

$$\widehat{\mathcal{G}}^{\text{odd}}(\tau) := \delta_- \left(\sum_{j=0}^{\frac{n-\iota(n)}{2}} \frac{4^j}{(2\pi i)^{\iota(n)-1}} \frac{\mathcal{L}_{M-\iota(n)}^{j}\left(D_{\iota(n)}(\tau)\right)}{(\iota(n)+2j-1)!} R_{\iota(n)-\frac{1}{2}}^{j}\left(\widehat{G}_{2\ell+1}(\tau)\right)\right)_{0\le\ell<2M},$$

where $\delta_\pm := \frac{1\pm(-1)^{2M}}{4}$ Then by construction

$$\widehat{\mathbf{h}}_{2M}^{\text{even}} = \mathbf{h}_{2M}^{\text{even}} + \mathcal{R}_{2M,n}^{\text{even}} = \mathbf{h}_{2M}^{\text{even}} - \mathcal{G}^{\text{even}} + \widehat{\mathcal{G}}^{\text{even}}.$$

To conclude it is enough to note that $\widehat{\mathcal{G}}^{\mathrm{even}}$ is a vector-valued almost harmonic Maass form, and that $\mathbf{h}_{2M}^{\mathrm{even}} - \mathcal{G}^{\mathrm{even}}$ is a vector-valued almost holomorphic modular form, and in particular a vector-valued almost harmonic Maass form.

This concludes the proof of Theorem 3.1.1. As a direct consequence, in light of Appendix A.2, one can derive the modularity of the Kac-Wakimoto characters, and the shape of their completions.

CHAPTER 4

Fourier coefficients of one variable meromorphic Jacobi forms

4.1 Introduction

In Chapter 3, we saw how to describe the modularity properties of the Fourier coefficients of the meromorphic Jacobi form Φ (see (3.1.3)). Recall that Φ has positive integral or half-integral index and has poles in the lattice $\mathbb{Z}\tau + \mathbb{Z}$, which simplifies the situation. Once again, we recall that in [15] the authors studied general positive index meromorphic Jacobi forms with poles at arbitrary torsion points, but only considered poles of order at most 2. The aim of this chapter is to generalize their result, and our result of Chapter 3, to any meromorphic Jacobi form of positive, integral index. In light of our description in Chapter 3 it is clear that everything can be easily generalized to the half-integral case and to Jacobi forms with multipliers.

4.1.1 Statement of the theorem

Throughout this chapter, we assume φ to be a meromorphic Jacobi form with positive integral index m and integral weight k for a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, whose poles lie in $\mathbb{Q}\tau + \mathbb{Q}$. Following the notation of Chapter 3, we consider the vector-valued function

$$\mathbf{h}_{2m}(\tau) := (h_\ell(\tau))_{\ell \pmod{2m}},$$

where h_ℓ is the ℓ-th canonical Fourier coefficient of φ (see (4.3.1)). We generalize the results in [7] and [15] describing the modularity and the shape of the Fourier coefficients of any positive index meromorphic Jacobi form.

Theorem 4.1.1. *With the notation as above, the function $\boldsymbol{h}_{2m}$ is a vector-valued almost mock modular form of weight $k - \frac{1}{2}$ for Γ.*

A more precise version of this theorem concerning the multiplier system occurring in the modular transformation property of $\mathbf{h}_{2m}$ is given in Proposition 4.3.3.

4.1.2 Outline of Chapter 4

In Section 4.2, we describe some properties of the poles of a meromorphic Jacobi form. Furthermore, we introduce and describe a real-analytic function which transforms as a negative index Jacobi form. This function will play a central role in the proof of the Theorem 4.1.1, which will be given in Section 4.3.

4.2 Preliminaries

The techniques used in this chapter are very similar to those of the previous one; therefore, most of the preliminaries are already described in Section 3.2. In this more general situation, we still need to describe the basic properties of the set of poles of φ, which were much simpler in the previous setting, and a non-holomorphic Jacobi form of negative index, which will play the role of the function $e^{-\frac{\pi m z^2}{v}}$ in the case of Φ.

4.2.1 The set of poles

For each fixed $\tau \in \mathbb{H}$ we denote by $S^{(\tau)}$ the set of poles of $z \mapsto \varphi(z;\tau)$. Note that this set has a nice symmetric shape. Indeed, from the elliptic transformation property of φ it follows that each pole in $S^{(\tau)}$ is equivalent to a pole in $S_0^{(\tau)} := S^{(\tau)} \cap P$ after translating by $\mathbb{Z}\tau + \mathbb{Z}$, where $P := [0,1)\tau + [0,1)$. Moreover, since P is bounded and φ is meromorphic, $S_0^{(\tau)}$ is finite. We let also

$$\mathcal{S}^{(\tau)} := \left\{(\alpha,\beta) \in \mathbb{Q}^2 : \alpha\tau + \beta \in S^{(\tau)}\right\}, \tag{4.2.1}$$

and for each $s = (\alpha,\beta) \in \mathcal{S}^{(\tau)}$ denote the associated pole by $z_s(\tau) = z_s = \alpha\tau+\beta \in S^{(\tau)}$. Finally we define $\mathcal{S}_0^{(\tau)}$ by replacing $S^{(\tau)}$ by $S_0^{(\tau)}$ in (4.2.1).

For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for each $s \in \mathcal{S}^{(\gamma\tau)}$, one has the relation

$$z_s(\gamma\tau) = \frac{z_{s\gamma}(\tau)}{c\tau + d},$$

which immediately implies

$$S^{(\tau)} = (c\tau + d)S^{(\gamma\tau)} \tag{4.2.2}$$

and

$$\mathcal{S}^{(\gamma\tau)}\gamma = \mathcal{S}^{(\tau)}. \tag{4.2.3}$$

For each Jacobi form of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$, and for each α and $\beta \in \mathbb{Q}$, Theorem 1.3 of [16] implies that the function $q^{m\alpha^2}\varphi(\alpha\tau+\beta;\tau)$ is a modular form of weight k on the finite index subgroup

$$\Gamma_{\alpha,\beta} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \begin{matrix} (a-1)\alpha + c\beta,\ b\alpha + (d-1)\beta \in \mathbb{Z}, \\ m\left(-c\beta^2 + b\alpha^2 + (d-a)\alpha\beta\right) \in \mathbb{Z} \end{matrix} \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$. Therefore, if we define the subgroup Γ_φ of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_\varphi := \bigcap_{(\alpha,\beta)\in\mathcal{S}_0^{(\tau)}} \Gamma_{\alpha,\beta}, \tag{4.2.4}$$

which can be easily seen to be of finite index, then for all $\gamma \in \Gamma_\varphi$ and for each $s \in \mathcal{S}^{(\tau)}$, $z_s(\gamma\tau) \in S^{(\gamma\tau)}$. This fact, together with (4.2.3) and the modular law of φ, implies that

$\mathcal{S}^{(\tau)}$ is Γ_φ-invariant (under right multiplication). In fact, it is straightforward to prove that for each $\gamma \in \Gamma_\varphi$, the map

$$\begin{aligned} \mathcal{S}_0^{(\tau)} \longrightarrow \mathcal{S}^{(\tau)} \longrightarrow \mathcal{S}_0^{(\tau)} \\ s \longmapsto s\gamma \longmapsto s\gamma \pmod{\mathbb{Z}^2} \end{aligned}$$

is the identity map.

4.2.2 A non-holomorphic Jacobi form of negative index

In this subsection, we introduce a non-holomorphic Jacobi form $F^{(s)}$, which plays a central role for two reasons. Firstly, it relates the Laurent coefficients of a meromorphic Jacobi form to certain almost holomorphic modular forms, whose non-holomorphic parts can be given as a linear combination of the Laurent coefficients themselves. Secondly, it allows us to relate the image of a certain class of functions under the differential operator ∂_ε to the image under the Maass raising operator. The latter property was already proven in Proposition 3.2.4.

For $s = (\alpha, \beta) \in \mathbb{Q}^2$, $u \in \mathbb{C}$, and $\tau \in \mathbb{H}$, we define

$$F^{(s)}(u;\tau) := e^{\frac{m\pi u^2}{v}} e^{2\pi i m(\alpha\beta + 2\alpha u)} q^{m\alpha^2}.$$

Note that $F^{(s)}$ is holomorphic in u and non-holomorphic in τ.

Remark. The function $e^{\frac{m\pi u^2}{v}} = F^{(0,0)}(u;\tau)$ appears in Chapter 3. In fact, s represents an element of $S_0^{(\tau)}$, and the function Φ studied in Chapter 3 has a unique pole at $0 \in S_0^{(\tau)}$.

A straightforward computation gives the following transformation properties for $F^{(s)}$.

Lemma 4.2.1. *Let $s = (\alpha, \beta) \in \mathbb{Q}^2$. Then the function $F^{(s)}$ satisfies the following transformation laws:*

1. *For all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$F^{(s)}\left(\frac{u}{c\tau+d};\gamma\tau\right) = e^{-2\pi i \frac{cm}{c\tau+d}(z_{s\gamma}+u)^2} F^{(s\gamma)}\left(u;\tau\right).$$

2. *For all* λ, $\mu \in \mathbb{Z}$

$$F^{(s+(\lambda,\mu))}\left(u;\tau\right) = e^{2\pi i m(\alpha\mu-\beta\lambda)} q^{m\lambda^2} e^{4\pi i m\lambda(z_s+u)} F^{(s)}(u;\tau).$$

Proof. We start by proving the modular property. Denoting as usual $z_s(\tau) = \alpha\tau+\beta$, we note that $F^{(s)}$ can be rewritten as

$$F^{(s)}(u;\tau) = e^{2\pi i \frac{m}{2iv}\left((z_s(\tau)+u)^2 - \overline{z_s(\tau)}(2u+z_s(\tau))\right)}.$$

Using this notation and the fact that $\frac{c\overline{\tau}+d}{c\tau+d} = 1 - \frac{2ivc}{c\tau+d}$, $\mathrm{Im}\left(\gamma\tau\right) = \frac{v}{|c\tau+d|^2}$, and $z_s(\gamma\tau) = \frac{z_{s\gamma}(\tau)}{c\tau+d}$, we get

$$\begin{aligned} F^{(s)}\left(\frac{u}{c\tau+d};\gamma\tau\right) &= e^{2\pi i \frac{m|c\tau+d|^2}{2iv}\left(\left(z_s(\gamma\tau)+\frac{u}{c\tau+d}\right)^2 - \overline{z_s(\gamma\tau)}\left(\frac{2u}{c\tau+d}+z_s(\gamma\tau)\right)\right)} \\ &= e^{2\pi i \frac{m}{2iv}\left((z_{s\gamma}(\tau)+u)^2 \frac{c\overline{\tau}+d}{c\tau+d} - \overline{z_{s\gamma}(\tau)}(2u+z_{s\gamma}(\tau))\right)} \\ &= e^{-2\pi i \frac{cm}{c\tau+d}(z_{s\gamma}+u)^2} F^{(s\gamma)}\left(u;\tau\right), \end{aligned}$$

which proves the first claim.

The second claim follows by a trivial computation, namely

$$\begin{aligned} F^{(s+(\lambda,\mu))}(u;\tau) &= e^{\frac{m\pi u^2}{v}} e^{2\pi i m((\alpha+\lambda)(\beta+\mu)+2(\alpha+\lambda)u)} q^{m(\alpha+\lambda)^2} \\ &= F^{(s)}(u;\tau) e^{2\pi i m((\alpha\mu+\beta\lambda)+2u\lambda+2\lambda\alpha\tau)} q^{m\lambda^2} \\ &= e^{2\pi i m(\alpha\mu-\beta\lambda)} q^{m\lambda^2} e^{4\pi i m\lambda(z_s+u)} F^{(s)}(u;\tau), \end{aligned}$$

which concludes the proof. □

Returning to the general problem, let φ be a meromorphic Jacobi form, and denote by $z_s = \alpha\tau+\beta$ one of its poles, where $s = (\alpha,\beta) \in \mathbb{Q}^2$. We define the Laurent coefficients $\widetilde{D}_j^{(s)}$ of φ relative to z_s by

$$\varphi(z+z_s;\tau)=\sum_{j=1}^{n_s}\frac{\widetilde{D}_j^{(s)}(\tau)}{(2\pi iz)^j}+O(1)\qquad\text{as } z\to 0 \tag{4.2.5}$$

where n_s denotes the order of the pole. Furthermore, we define the functions $D_j^{(s)}$ as the Laurent coefficients of $F^{(s)}(z;\tau)\varphi(z+z_s;\tau)$ in the elliptic variable, namely

$$F^{(s)}(z;\tau)\varphi(z+z_s;\tau)=\sum_{j=1}^{n_s}\frac{D_j^{(s)}(\tau)}{(2\pi iz)^j}+O(1)\qquad\text{as } z\to 0. \tag{4.2.6}$$

Proposition 4.2.2. *With the notation as above, the following are true:*

1. *For each $1\leq j\leq n_s$ the function $D_j^{(s)}$ is an almost holomorphic modular form of weight $k-j$ for Γ_φ (see (4.2.4)).*

2. *For each $1\leq j\leq n_s$ the holomorphic part of $D_j^{(s)}$ is given by*

$$q^{\alpha^2m}e(m\alpha\beta)\sum_{\lambda=0}^{n_s-j}\widetilde{D}_{\lambda+j}^{(s)}(\tau)(4\pi im\alpha)^\lambda.$$

   In particular, the functions $\widetilde{D}_j$ are quasimodular forms.

3. *For each $1\leq j\leq n_s$ and for $\kappa\in\{1,2\}$, we have*

$$D_{\kappa+2j}^{(s)}(\tau)=\left(\frac{4\pi}{m}\right)^j\mathcal{L}^j\left(D_\kappa^{(s)}(\tau)\right).$$

Proof. We first prove the modularity of $D_j^{(s)}$. From the definition of Jacobi forms and from Lemma 4.2.1, it follows that for each $\gamma\in\Gamma_\varphi$, we have

$$F^{(s)}\left(\frac{u}{c\tau+d};\gamma\tau\right)\varphi\left(\frac{u}{c\tau+d}+z_s(\gamma\tau);\gamma\tau\right)=(c\tau+d)^kF^{(s\gamma)}(u;\tau)\varphi(u+z_{s\gamma};\tau).$$

Using the elliptic transformation properties of both $F^{(s)}$ and φ, we shift $s\gamma$ to s, using the discussion in Subsection 4.2.1, say $s=s\gamma+(\lambda,\mu)$, for some $(\lambda,\mu)\in\mathbb{Z}^2$, obtaining

$$(c\tau+d)^kF^{(s\gamma)}(u;\tau)\varphi(u+z_{s\gamma};\tau)=e^{2\pi im(\alpha\mu-\beta\lambda)}(c\tau+d)^kF^{(s)}(u;\tau)\varphi(u+z_s;\tau).$$

Note that (λ, μ) depends on γ and s. Since $\gamma \in \Gamma_\varphi$, one can show that $e^{2\pi i m(\alpha\mu - \beta\lambda)} = 1$. In particular, writing both the right and the left hand sides in terms of the Laurent expansion, we obtain

$$D_j^{(s)}(\gamma\tau) = (c\tau + d)^{k-j} D_j^{(s)}(\tau),$$

which proves the modular property. It remains to prove that they can be written as polynomials in $\frac{1}{v}$ with weakly holomorphic coefficients. Clearly, each $D_j^{(s)}(\tau)$ can be written as a combinations of Laurent coefficients of $\varphi(u + z_s; \tau)$ and $F^{(s)}(u; \tau)$ in $u = 0$. More precisely, it is easy to see that

$$D_j^{(s)}(\tau) = \sum_{\lambda=0}^{n_s - j} \frac{1}{(2\pi i)^\lambda \lambda!} \widetilde{D}_{\lambda+j}^{(s)}(\tau) \partial_u^\lambda \left[F^{(s)}(u; \tau)\right]_{u=0}.$$

It is straightforward to show that $\partial_u^n \left[F^{(s)}(u; \tau)\right]_{u=0}$ equals $q^{\alpha^2 m}$ times a polynomial in $\frac{1}{v}$ with coefficients in $\mathbb{C}$. Furthermore, its constant term is given by $(4\pi i m\alpha)^n$. From these observations it follows that $D_j^{(s)}(\tau)$ is an almost holomorphic modular form, whose holomorphic part is given by

$$q^{\alpha^2 m} e(m\alpha\beta) \sum_{\lambda=0}^{n_s - j} \widetilde{D}_{\lambda+j}^{(s)}(\tau)(4\pi i m\alpha)^\lambda.$$

This proves parts 1 and 2. For the proof of part 3 we refer the reader to the proof of Proposition 3.3.5, which is very similar. $\square$

4.3 Proof of Theorem 4.1.1

4.3.1 Canonical Fourier coefficients and canonical decomposition

By assumption, the meromorphic Jacobi form φ has integral index and is 1-periodic in both τ and z, thus we do not need to split the Fourier coefficients into even and odd pieces, as we did in the case of the Kac-Wakimoto characters generating function Φ. We can therefore use the original definition of canonical Fourier coefficient introduced by Dabholkar, Murthy, and Zagier, namely

$$h_\ell(\tau) := q^{-\frac{\ell^2}{4m}} \int_{-\frac{\ell\tau}{2m}}^{-\frac{\ell\tau}{2m}+1} \varphi(z;\tau)\zeta^{-\ell} dz, \tag{4.3.1}$$

for each integer ℓ. We also define the vector-valued function $\mathbf{h}_{2m}$, whose components are the canonical Fourier coefficients,

$$\mathbf{h}_{2m}(\tau) := (h_\ell(\tau))_{0\leq \ell < 2m-1}.$$

Note that $\mathbf{h}_{2m}$ coincides with the vector-valued function $\mathbf{h}_{2m}^{\text{even}}$ introduced in (3.3.2). Also, if the meromorphic Jacobi form has a pole on the path of integration, which by assumption is the straight line, then we treat the integral as explained in Remark 1 of Section 3.1.1.

Remark. The definition of canonical Fourier coefficient and the canonical splitting of a meromorphic Jacobi form which we soon recall were introduced in [15] for any meromorphic Jacobi form of positive index. However, in that case the authors investigate the modularity property of the Fourier coefficients restricting to the case of poles of order at most 2.

We define the finite part φ^F of φ as

$$\varphi^F(z;\tau) := \mathbf{h}_{2m}(\tau) \cdot \boldsymbol{\vartheta}_m(z;\tau),$$

where $\boldsymbol{\vartheta}_m = (\vartheta_{m,\ell})_{\ell \pmod{2m}}$ was introduced in Subsection 2.3.1. Moreover, we give the polar part φ^P of φ as

$$\varphi^P(z;\tau) := -\sum_{z_s\in S_0^{(\tau)}} \sum_{j=1}^{n_s} \frac{\widetilde{D}_j^{(s)}(\tau)}{(j-1)!} \left(\frac{\partial_u}{2\pi i}\right)^{j-1} \Big[f_m(z, u+z_s;\tau) \\ - \mathbf{E}_m^{(s)}(u;\tau)\cdot \boldsymbol{\vartheta}_m(z;\tau)\Big]_{u=0},$$

where the functions $\widetilde{D}_j^{(s)}$ are the Laurent coefficients of φ at the pole $z_s \in S_0^{(\tau)}$, as described in (4.2.5), while the vector-valued function $\mathbf{E}_m^{(s)}(u;\tau) := \left(E_{m,\ell}^{(s)}(u;\tau)\right)_{\ell \pmod{2m}}$ is defined for $s = (\alpha, \beta)$ by

$$E_{m,\ell}^{((\alpha,\beta))}(u;\tau) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv \ell \pmod{2m}}} \frac{\operatorname{sgn}\left(r+\frac{1}{2}\right) - \operatorname{sgn}\left(r+2m\alpha\right)}{2} q^{-\frac{r^2}{4m}-r\alpha} e^{-2\pi i r(u+\beta)}.$$

Proposition 4.3.1. *With the notation as above, we have $\varphi = \varphi^F + \varphi^P$.*

Proof. The proof is very similar to that of Proposition 3.3.6 and to the original proof of Bringmann-Folsom in [7]. In this more general situation, for non zero poles s, we point out the appearance of the vector-valued function $\mathbf{E}_m$ and the importance of the "symmetries" of the poles, described in Subsection 4.2.1.

Let $\widetilde{z} := A\tau \in \mathbb{C}$ be fixed, where $A \in \mathbb{Q}$. Since both φ and φ^F are meromorphic in z, we assume without loss of generality that $\operatorname{Im}(z) = \operatorname{Im}(\widetilde{z}) = Av$. By definition

$$\varphi\left(z;\tau\right) - \varphi^F\left(z;\tau\right) = \sum_{\ell\in\mathbb{Z}} e^{2\pi i \ell z} \int_{P\left(\widetilde{z}, -\frac{\ell}{2m}\tau\right)} \varphi(u;\tau) e^{-2\pi i u \ell}, \tag{4.3.2}$$

where for x, $y \in \mathbb{C}$ $P(x, y)$ is the parallelogram of vertices $\{x, y, x+1, y+1\}$. As before, $S^{(\tau)}$ is the set of poles of $z \mapsto \varphi\left(z;\tau\right)$. Applying the Residue Theorem, we rewrite the right hand side of (4.3.2) as

$$2\pi i \sum_{\ell\in\mathbb{Z}} \sum_{z_s \in S^{(\tau)} \cap P\left(A\tau, -\frac{\ell}{2m}\tau\right)} \operatorname*{Res}_{u=0} \left(\varphi\left(u+z_s;\tau\right) e^{-2\pi i \ell(u+z_s)}\right) e^{2\pi i \ell z}. \tag{4.3.3}$$

Due to the elliptic transformation properties of φ, each pole $z_s \in S^{(\tau)} \cap P\left(\widetilde{z}, -\frac{\ell}{2m}\tau\right)$ can be written as $z_s = w_s + \lambda\tau$, for a certain $w_s \in S_0^{(\tau)}$ and $\lambda \in \mathbb{Z}$. Then, for $w_s = \alpha\tau + \beta$ we rewrite the sum over the poles as

$$\sum_{z_s \in S^{(\tau)} \cap P\left(A\tau, -\frac{\ell}{2m}\tau\right)} = \sum_{w_s \in S_0^{(\tau)}} \sum_{\lambda\in\mathbb{Z}} \frac{\operatorname{sgn}\left(\lambda - A\right) - \operatorname{sgn}\left(\lambda + \frac{\ell}{2m} + \alpha\right)}{2}.$$

Note that if $\alpha = \frac{\ell}{2m}$ for a certain integer ℓ, then this function counts the pole with multiplicity $\frac{1}{2}$, as we expect from the exceptional cases described in Remark 1. Using the Laurent expansion of φ at w_s and the elliptic transformation properties of Φ, we obtain that (4.3.3) equals

$$\sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell z} \sum_{w_s\in S_0^{(\tau)}} \sum_{\lambda\in\mathbb{Z}} \frac{\operatorname{sgn}(\lambda - A) - \operatorname{sgn}\left(\lambda + \frac{\ell}{2m} + \alpha\right)}{2}$$

$$\times \sum_{j=1}^{n_s} \frac{\widetilde{D}_j^{(s)}(\tau)}{(j-1)!} \left(\frac{\partial_u}{2\pi i}\right)^{j-1} \left[q^{-m\lambda^2-\ell\lambda} e^{-2\pi i(\ell+2m\lambda)(u+w_s)}\right]_{u=0},$$

where n_s is the order of w_s. Rearranging the order of summations and shifting the variables as in the proof of Proposition 3.3.6, this last expression can be written as

$$\varphi^P(z;\tau) = -\sum_{w_s\in S_0^{(\tau)}} \sum_{j=1}^{n_s} \frac{\widetilde{D}_j^{(s)}(\tau)}{(j-1)!} \left(\frac{\partial_u}{2\pi i}\right)^{j-1} \left[\sum_{\lambda\in\mathbb{Z}} q^{m\lambda^2} e\left(2m\lambda z\right)\right.$$

$$\left.\times \sum_{\ell\in\mathbb{Z}} \frac{\operatorname{sgn}(\lambda + A) + \operatorname{sgn}(\ell + 2m\alpha)}{2} q^{\ell\lambda} e\left(\ell\left(z - u - w_s\right)\right)\right]_{u=0},$$

To conclude the proof, it is enough to show that the term in brackets is in fact

$$f_m\left(z, u + z_s; \tau\right) - \mathbf{E}_m^{(s)}(u;\tau) \cdot \boldsymbol{\vartheta}_m(z;\tau).$$

This can be proven by splitting the summation on ℓ as

$$\sum_{\ell\in\mathbb{Z}} \frac{\operatorname{sgn}(\lambda + A) + \operatorname{sgn}\left(\ell + \frac{1}{2}\right)}{2} + \sum_{\ell\in\mathbb{Z}} \frac{\operatorname{sgn}(\ell + 2m\alpha) - \operatorname{sgn}\left(\ell + \frac{1}{2}\right)}{2}.$$

The first summation gives $f_m\left(z, u + z_s; \tau\right)$ as a consequence of Proposition 3.2.7, while a trivial computation shows that the second piece gives $\mathbf{E}_m^{(s)}(u + z_s;\tau) \cdot \boldsymbol{\vartheta}_m(z;\tau)$. □

We conclude this subsection by giving an alternative description of φ^P in terms of the almost holomorphic modular forms $D_j^{(s)}$ defined in (4.2.6). The proof is analogous to that of Proposition 3.3.7, therefore we omit it. In what follows, $F^{(s)}$ is the non-holomorphic Jacobi form described in Subsection 4.2.2.

Proposition 4.3.2. *We have*

$$\begin{aligned}&\varphi^P(z;\tau)\\&\quad = -\sum_{z_s\in S_0^{(\tau)}}\sum_{j=1}^{n_s}\frac{D_j^{(s)}(\tau)}{(j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{j-1}\left[\frac{f_m\left(z,u+z_s;\tau\right)}{F^{(s)}(u;\tau)}-\frac{\boldsymbol{E}_m^{(s)}(u;\tau)}{F^{(s)}(u;\tau)}\cdot\boldsymbol{\vartheta}_m(z;\tau)\right]_{u=0}.\end{aligned}$$

4.3.2 Modular properties of $\mathbf{h}_{2m}$

In order to understand the modular property of $\mathbf{h}_{2m}$, we use the same method as in the previous chapter. More precisely, we complete the polar and the finite parts of φ, and then we use the properties of the theta decomposition to derive the transformation laws of $\mathbf{h}_{2m}$.

We define the completion of φ^P as

$$\begin{aligned}\widehat{\varphi}^P(z;\tau) :&= -\sum_{z_s\in S_0^{(\tau)}}\sum_{j=1}^{n_s}\frac{D_j^{(s)}(\tau)}{(j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{j-1}\left[\frac{\widehat{f}_m\left(z,u+z_s;\tau\right)}{F^{(s)}(u;\tau)}\right]_{u=0}\\&= \varphi^P(z;\tau)-\mathcal{R}_m(\tau)\cdot\boldsymbol{\vartheta}_m(z;\tau),\end{aligned}\tag{4.3.4}$$

where

$$\mathcal{R}_m(\tau) := \left(\sum_{z_s\in S_0^{(\tau)}}\sum_{j=1}^{n_s}\frac{D_j^{(s)}(\tau)}{(j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{j-1}\left[\frac{E_{m,\ell}^{(s)}(u;\tau)-\frac{1}{2}R_{m,\ell}(u+z_s;\tau)}{F^{(s)}(u;\tau)}\right]\right)_{0\leq\ell<2m}.$$

Note that the second equality in (4.3.4) is a consequence of (2.3.5). Similarly, we define the completion $\widehat{\varphi}^F$ of φ^F as

$$\widehat{\varphi}^F(z;\tau) := \varphi^F(z;\tau)+\mathcal{R}_m(\tau)\cdot\boldsymbol{\vartheta}_m(z;\tau) = \widehat{\mathbf{h}}_{2m}(\tau)\cdot\boldsymbol{\vartheta}_m(z;\tau),$$

where we let

$$\widehat{\mathbf{h}}_{2m}(\tau) := \mathbf{h}_{2m}(\tau)+\mathcal{R}_m(\tau).$$

Proposition 4.3.3. *The functions φ^F and φ^P satisfy the same transformation properties as φ. In particular, the function $\widehat{\boldsymbol{h}}_{2m}$ is a vector-valued non-holomorphic modular form of weight $k-\frac{1}{2}$ for Γ with multiplier system described in Proposition 2.3.4.*

Proof. The elliptic transformation property follows from the analogous transformation for $\widehat{f}_m(z, u+z_s;\tau)$. In order to show the modular property, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we consider

$$\widehat{\varphi}^P\left(\frac{z}{c\tau+d};\gamma\tau\right) = -\sum_{z_s(\gamma\tau)\in S_0^{(\gamma\tau)}} \sum_{j=1}^{n_s} \frac{D_j^{(s)}(\gamma\tau)}{(j-1)!}\left(\frac{\partial_{\frac{u}{c\tau+d}}}{2\pi i}\right)^{j-1}\left[\frac{\widehat{f}_m\left(\frac{z}{c\tau+d}, \frac{u+z_{s\gamma}(\tau)}{c\tau+d};\gamma\tau\right)}{F^{(s)}\left(\frac{u}{c\tau+d};\gamma\tau\right)}\right]_{u=0}.$$

Using Lemma 4.2.1, Proposition 4.2.2, and the transformation properties of $\widehat{f}_m(z, u+z_s;\tau)$ in Proposition 2.3.7, we can write it as

$$-(c\tau+d)^k e\left(\frac{cmz^2}{c\tau+d}\right) \sum_{z_{s\gamma}(\tau)\in S_0^{(\tau)}} \sum_{j=1}^{n_s} \frac{D_j^{(s\gamma)}(\tau)}{(j-1)!}\left(\frac{\partial_u}{2\pi i}\right)^{j-1}\left[\frac{\widehat{f}_m(z, u+z_{s\gamma};\tau)}{F^{(s\gamma)}(u;\tau)}\right]_{u=0}.$$

Note that the sum over $z_s(\gamma\tau) \in S_0^{(\gamma\tau)}$ is the same as the sum over $z_{s\gamma}(\tau) \in S_0^{(\tau)}$ by virtue of (4.2.2).

Since φ and $\widehat{\varphi}^P$ satisfy the same transformation properties, the same must be true for $\widehat{\varphi}^F = \varphi - \widehat{\varphi}^P$. Finally, the modularity of $\widehat{\mathbf{h}}_{2m}$ follows since its components are the theta-coefficients of $\widehat{\varphi}^F$, as we showed in Proposition 2.3.4. □

4.3.3 Shape of $\widehat{\mathbf{h}}_m$

To conclude the proof of Theorem 4.1.1, we need to show that $\widehat{\mathbf{h}}_{2m}$ has the shape of an almost harmonic Maass form. However, this is just an immediate consequence of Proposition 4.3.2 and Corollary 3.2.5. More precisely, each component of $\mathcal{R}_m$ can be written as

$$\sum_{z_s\in S_0^{(\tau)}}\sum_{j=0}^{\left[\frac{n_s-1}{2}\right]}\frac{4^j\mathcal{L}^j\left(D_1^{(s)}(\tau)\right)}{(2j)!}R_{\frac{1}{2}}^j\left(E_{m,\ell}^{(s)}(0;\tau)-\frac{1}{2}R_{m,\ell}(z_s;\tau)\right)$$
$$+\sum_{z_s\in S_0^{(\tau)}}\sum_{j=0}^{\left[\frac{n_s}{2}\right]-1}\frac{4^j\mathcal{L}^j\left(D_2^{(s)}(\tau)\right)}{(2j+1)!}R_{\frac{3}{2}}^j\left(\frac{\partial_u}{2\pi i}\left[E_{m,\ell}^{(s)}(u;\tau)-\frac{1}{2}R_{m,\ell}(u+z_s;\tau)\right]_{u=0}\right).$$

Note that Corollary 3.2.5 deals with the function $R_{m,\ell}$. It is easy to verify that the heat operator annihilates $E_{m,\ell}$ as well, therefore, by Proposition 3.2.4, the results in Corollary 3.2.5 holds for $E_{m,\ell}-\frac{1}{2}R_{m,\ell}$. By Corollary 2.3.11, we see that for $\iota\in\{\frac{1}{2},\frac{3}{2}\}$ the function $\frac{\partial_u}{2\pi i}^{\iota-\frac{1}{2}}\left[R_{m,\ell}(u+z_s;\tau)\right]_{u=0}$ is the non-holomorphic part of a harmonic Maass form $\widehat{G}_{m,\ell}^{s,\iota}$, whose holomorphic part is denoted by $G_{m,\ell}^{s,\iota}$. Defining

$$\mathcal{G}_\iota(\tau):=\left(\sum_{z_s\in S_0^{(\tau)}}\sum_{j=0}^{n_s(\iota)}\frac{4^j\mathcal{L}^j\left(D_{\iota+\frac{1}{2}}^{(s)}(\tau)\right)}{(2j+\iota-\frac{1}{2})!}R_\iota^j\left(G_{m,\ell}^{s,\iota}(\tau)\right)\right)_{\ell\pmod{2m}},$$
$$\widehat{\mathcal{G}}_\iota(\tau):=\left(\sum_{z_s\in S_0^{(\tau)}}\sum_{j=0}^{n_s(\iota)}\frac{4^j\mathcal{L}^j\left(D_{\iota+\frac{1}{2}}^{(s)}(\tau)\right)}{(2j+\iota-\frac{1}{2})!}R_\iota^j\left(\widehat{G}_{m,\ell}^{s,\iota}(\tau)\right)\right)_{\ell\pmod{2m}},$$

where $n_s(\iota):=\left[\frac{n_s-\frac{1}{2}-\iota}{2}\right]$, we have

$$\widehat{\mathbf{h}}_{2m}=\mathbf{h}_{2m}+\mathcal{R}_m=\mathbf{h}_{2m}+\widehat{\mathcal{G}}_{\frac{1}{2}}+\widehat{\mathcal{G}}_{\frac{3}{2}}-\mathcal{G}_{\frac{1}{2}}-\mathcal{G}_{\frac{3}{2}}.$$

By construction $\widehat{\mathcal{G}}_{\frac{1}{2}}$ and $\widehat{\mathcal{G}}_{\frac{3}{2}}$ are vector-valued almost harmonic Maass forms, while $\mathbf{h}_{2m}-\mathcal{G}_{\frac{1}{2}}-\mathcal{G}_{\frac{3}{2}}$ is an almost holomorphic modular form, in particular an almost harmonic Maass form.

CHAPTER 5

Multivariable Kac-Wakimoto characters

5.1 Introduction

In the previous chapters we described the shape and the modularity of the single-variable Kac-Wakimoto characters, and more generally of the Fourier coefficients of 2-variable Jacobi forms of positive index. However, Kac-Wakimoto characters as investigated by Bringmann, Folsom, and Ono [7, 11, 17], and as we considered in Chapter 3, are specializations of more general characters given in [24] as

$$\mathrm{ch}F = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}F_\ell \zeta^\ell := e^{\Lambda_0} \prod_{k\geq 1} \frac{\prod_{r=1}^m \left(1+\zeta\xi_r q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}\xi_r^{-1}q^{k-\frac{1}{2}}\right)}{\prod_{j=1}^n \left(1-\zeta\xi_{m+j} q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}\xi_{m+j}^{-1}q^{k-\frac{1}{2}}\right)}. \tag{5.1.1}$$

We can easily see that specializing all the $\xi_r = 1$ we get (3.1.1).

In this chapter, we show that considering the additional variables in (5.1.1) imposes extra shape which gives a cleaner picture for the specialized character as these are specializations of mixed H-harmonic Maass-Jacobi forms. This construction, combined with the techniques described in Chapter 4, allows to describe the shape and the modularity of the Fourier coefficients of multivariable meromorphic Jacobi forms (Fourier coefficients with respect to a single fixed elliptic variable).

5.1.1 Statement of the results

Let $m > n > 0$ be positive even integers, and $M := \frac{m-n}{2}$. Let $(m_1, \cdots, m_s) \in \mathbb{N}^s$ and $(n_1, \cdots, n_t) \in \mathbb{N}^t$ such that $\sum_{j=1}^{s} m_j = m$ and $\sum_{j=1}^{t} n_j = n$. Moreover, we define the matrix

$$L := \begin{pmatrix} 2M & \mathbf{b}^T \\ \mathbf{b} & \widetilde{L} \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{b} = (\mathbf{b}_s, \mathbf{b}_t) &:= (m_1, \cdots, m_s, n_1, \cdots, n_t)^T, \\ \widetilde{L} &: = \operatorname{diag}(\mathbf{b}_s, -\mathbf{b}_t). \end{aligned}$$

In this chapter, we consider the function $\Phi\colon \mathbb{C}^{s+t+1} \times \mathbb{H} \to \mathbb{C}$ defined by

$$\Phi(z, \mathbf{u}; \tau) := \frac{\prod_{r=1}^{s} \vartheta\left(z + u_r + \frac{1}{2}; \tau\right)^{m_r}}{\prod_{j=1}^{t} \vartheta\left(z - w_j; \tau\right)^{n_j}}, \tag{5.1.2}$$

where ϑ is the classical Jacobi theta function (2.3.1). Here, we use the notation $\mathbf{u} := (u_1, \cdots, u_s, w_1, \cdots w_t)$ for the elliptic variables.

The function Φ is clearly a multivariable meromorphic Jacobi form, whose transformation properties will be explicitly described in Proposition 5.2.2.

The aim of this chapter is to describe the shape and the modularity properties of the canonical Fourier coefficient $\mathbf{h}$ of Φ with respect to the elliptic variable z (see (5.3.2) for the definition).

Theorem 5.1.1. *The canonical Fourier coefficient $\boldsymbol{h}$: $\mathbb{C}^{s+t} \times \mathbb{H} \to \mathbb{C}^{2M}$ is the holomorphic part of a multivariable almost harmonic Maass-Jacobi form of weight $M - \frac{1}{2}$, index L^* (see (5.2.6)) for $\Gamma_0(2)$, and with multiplier system described in Proposition 2.3.4.*

Considering the special case of Φ when $\mathbf{b} = (1, \cdots, 1)$ and using Jacobi's triple product identity (see Proposition 2.3.2), we can rewrite $\mathrm{ch}F$ as

$$\mathrm{ch}F = e^{\Lambda_0}(-1)^m i^{-n}\zeta^M \left(\prod_{r=1}^{m} e^{\pi i u_r}\right)\left(\prod_{j=1}^{n} e^{\pi i w_j}\right) q^{\frac{M}{3}}\eta(\tau)^{-2M}\Phi\left(z+\frac{\tau}{2},\mathbf{u};\tau\right).$$

Therefore, as a consequence of Theorem 5.1.1, we can deduce the shape of the multivariable Kac-Wakimoto characters.

Corollary 5.1.2. *The multivariable Kac-Wakimoto characters* $\mathrm{ch}F_\ell$ *are the holomorphic parts of mixed H-harmonic Maass-Jacobi form.*

5.1.2 Outline of Chapter 5

In Section 5.2, we generalize certain objects and tools used in the previous chapters to the multivariable setting. This will be the necessary preliminaries for the proof of Theorem 5.1.1 in Section 5.3.

5.2 Preliminaries

Most of the preliminary results needed in this chapter have already been described in the previous chapters. In this section, we generalize the function $F^{(s)}$ described in Subsection 4.2.2 to a multivariable setting.

5.2.1 An elementary non-holomorphic multivariable Jacobi form

Let $F\colon \mathbb{C}^2\times\mathbb{H}\to\mathbb{C}$ be the function defined by

$$F(z,w;\tau) := e^{2\pi i M\frac{(z+w-\overline{w})^2}{\tau-\overline{\tau}}}, \tag{5.2.1}$$

and G its normalization

$$G(z,w;\tau) := \frac{F(z,w;\tau)}{F(0,w;\tau)}. \tag{5.2.2}$$

If $z=0$, we sometimes use the notation $F(w;\tau):=F\left(0,w;\tau\right)$. This function is a non-holomorphic Jacobi form of index $-\left(\begin{smallmatrix} M & M \\ M & 0\end{smallmatrix}\right)$ and weight 0, as we see in the following lemma.

Lemma 5.2.1. *The function G satisfies the following transformation properties:*

1. *For all λ, $\mu\in\mathbb{Z}$, we have*
$$G(z,w+\lambda\tau+\mu;\tau)=e^{4\pi iM\lambda z}G(z,w;\tau).$$

2. *For all $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have*
$$G\left(\frac{z}{c\tau+d},\frac{w}{c\tau+d};\gamma\tau\right)=e^{-2\pi i\frac{Mc}{c\tau+d}\left(z^2+2zw\right)}G\left(z,w;\tau\right).$$

Proof. The proof is just a direct computation and follows immediately from the following transformation properties of F:

1. For all λ, $\mu\in\mathbb{Z}$, we have
$$F(z,w+\lambda\tau+\mu;\tau)=e^{2\pi iM\left(\lambda^2(\tau-\overline{\tau})+2\lambda(z+w-\overline{w})\right)}F(z,w;\tau).$$

2. For all $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have
$$F\left(\frac{z}{c\tau+d},\frac{w}{c\tau+d};\gamma\tau\right)=e^{-2\pi iM\left(\frac{c}{c\tau+d}(z+w)^2+\frac{c}{c\overline{\tau}+d}\overline{w}^2\right)}F\left(z,w;\tau\right).$$

To prove the first of these two claims we note that by definition
$$\begin{aligned}F(z,w+\lambda\tau+\mu;\tau)&=e^{2\pi iM\left(\frac{(z+w-\overline{w}+\lambda(\tau-\overline{\tau}))^2}{\tau-\overline{\tau}}\right)}\\&=e^{2\pi iM\left(\frac{(z+w-\overline{w})^2+\lambda^2(\tau-\overline{\tau})^2+2(z+w-\overline{w})\lambda(\tau-\overline{\tau})}{\tau-\overline{\tau}}\right)}\\&=e^{2\pi iM\left(\frac{(z+w-\overline{w})^2}{\tau-\overline{\tau}}+\lambda^2(\tau-\overline{\tau})+2(z+w-\overline{w})\lambda\right)}\\&=e^{2\pi iM\left(\lambda^2(\tau-\overline{\tau})+2(z+w-\overline{w})\lambda\right)}F(z,w;\tau).\end{aligned}$$

For the second claim, by definition we have

$$F\left(\frac{z}{c\tau+d},\frac{w}{c\tau+d};\gamma\tau\right)=e^{2\pi iM\frac{\left(\frac{z+w}{c\tau+d}-\frac{\overline{w}}{c\overline{\tau}+d}\right)^2}{\gamma\tau-\gamma\overline{\tau}}}=e^{2\pi iM\frac{((z+w)(c\overline{\tau}+d)-\overline{w}(c\tau+d))^2}{(c\tau+d)(c\overline{\tau}+d)(\tau-\overline{\tau})}}. \tag{5.2.3}$$

Using

$$\begin{aligned}\frac{c\tau+d}{c\overline{\tau}+d}&=1+(\tau-\overline{\tau})\frac{c}{c\overline{\tau}+d},\\ \frac{c\overline{\tau}+d}{c\tau+d}&=1-(\tau-\overline{\tau})\frac{c}{c\tau+d},\end{aligned}$$

the right-hand side of (5.2.3) equals

$$e^{2\pi iM\left(\frac{1}{\tau-\overline{\tau}}\left((z+w)^2\left(1-(\tau-\overline{\tau})\frac{c}{c\tau+d}\right)+\overline{w}^2\left(1+(\tau-\overline{\tau})\frac{c}{c\overline{\tau}+d}\right)-2(z+w)\overline{w}\right)\right)}$$
$$=e^{2\pi iM\left(-\frac{c(z+w)^2}{c\tau+d}+\frac{c\overline{w}^2}{c\overline{\tau}+d}\right)}F(z,w;\tau).$$

□

As in the one-variable case, the function G allows to construct the completion of the Laurent coefficients of Φ with respect to z at each pole, considering $(\mathbf{u};\tau)$ as fixed. Knowing that $\vartheta(z;\tau)$ has simple poles in z at $\mathbb{Z}\tau+\mathbb{Z}$, it follows that Φ has t poles in $z\in\{w_1,\cdots,w_t\}$. We denote the Laurent expansion of Φ as $z\to w_j$ by

$$\Phi(\varepsilon+w_j,\mathbf{u};\tau)=\sum_{\lambda=1}^{n_j}\frac{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}{(2\pi i\varepsilon)^\lambda}+O(1)\qquad\varepsilon\to0. \tag{5.2.4}$$

Analogously, we define the Laurent expansion of $G\Phi$, which clearly has the same poles, namely

$$G(\varepsilon,w;\tau)\Phi(\varepsilon+w_j,\mathbf{u};\tau)=\sum_{\lambda=1}^{n_j}\frac{D_{\lambda,j}(\mathbf{u};\tau)}{(2\pi i\varepsilon)^\lambda}+O(1)\qquad\varepsilon\to0.$$

Here, we have used a new elliptic variable w which can be expressed in terms of $\mathbf{u}$, namely,

$$w=w^{(j)}:=\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}+w_j. \tag{5.2.5}$$

Remark 2. Note that the new elliptic variable w is just a linear combination of the elliptic variables $\mathbf{u}$. In particular, if we look at F as a function of $\mathbf{u}$, applying the transformation $\mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu}$ is equivalent to applying $w+\lambda\tau+\mu$, with $\lambda := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda}+\lambda_{s+j}$ and $\mu := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\mu}+\mu_{s+j}$.

As mentioned before, the function Φ is a multivariable meromorphic Jacobi form, whose transformation properties are described in the following proposition.

Proposition 5.2.2. *The function* Φ*, defined in* (5.1.2)*, satisfies the following transformation properties:*

1. *For all* $(\lambda,\boldsymbol{\lambda})\in\mathbb{Z}^{s+t+1}$ *and* $(\mu,\boldsymbol{\mu})\in\mathbb{Z}^{s+t+1}$*, we have*

$$\Phi\left(z+\lambda\tau+\mu,\boldsymbol{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau\right)=(-1)^{\boldsymbol{b}\cdot\boldsymbol{\mu}+(\boldsymbol{0},\boldsymbol{b}_t)\cdot\boldsymbol{\lambda}}q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]}e^{-2\pi i(z,\boldsymbol{u})^T L(\lambda,\boldsymbol{\lambda})}\times\Phi\left(z,\boldsymbol{u};\tau\right).$$

2. *For all* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)$*, we have*

$$\Phi\left(\frac{z}{c\tau+d},\frac{\boldsymbol{u}}{c\tau+d};\gamma\tau\right)=\chi(\gamma)(c\tau+d)^M e^{\pi i\frac{c}{c\tau+d}L[(z,\boldsymbol{u})]}\Phi(z,\boldsymbol{u};\tau),$$

where the character $\chi(\gamma)$ *was defined in Proposition 3.3.1.*

Proof. We first prove the elliptic transformation law. If $(\lambda,\boldsymbol{\lambda})\in\mathbb{Z}^{s+t+1}$ and $(\mu,\boldsymbol{\mu})\in\mathbb{Z}^{s+t+1}$, then

$$\Phi(z+\lambda\tau+\mu,\mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau)=\frac{\prod_{r=1}^{s}\vartheta\left(z+u_r+\frac{1}{2}+(\lambda+\lambda_r)\tau+(\mu+\mu_r);\tau\right)^{m_r}}{\prod_{j=1}^{t}\vartheta\left(z-w_j+(\lambda-\lambda_{s+j})\tau+(\mu-\mu_{s+j});\tau\right)^{n_j}},$$

while, using the transformation properties of ϑ, equals

$$\left(\frac{\prod_{r=1}^{s}\left((-1)^{\lambda+\lambda_r+\mu+\mu_r}q^{-\frac{(\lambda+\lambda_r)^2}{2}}e^{-2\pi i(z+u_r+1/2)(\lambda+\lambda_r)}\right)^{m_r}}{\prod_{j=1}^{t}\left((-1)^{\lambda+\lambda_{s+j}+\mu+\mu_{s+j}}q^{-\frac{(\lambda-\lambda_{s+j})^2}{2}}e^{-2\pi i(z-w_j)(\lambda-\lambda_{s+j})}\right)^{n_j}}\right)\Phi(z,\mathbf{u};\tau).$$

Since m and n are even by assumption, the factor in parentheses equals

$$(-1)^{\sum_r m_r\mu_r+\sum_j n_j(\lambda_{s+j}+\mu_{s+j})} q^{-\frac{1}{2}\left(\sum_r m_r(\lambda+\lambda_r)^2-\sum_j n_j(\lambda-\lambda_{s+j})^2\right)} e^{2\pi i\left(-\sum_r m_r(z+u_r)+\sum_j n_j(z-w_j)\right)},$$

which, by definition of L, equals

$$(-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\mathbf{b}_t)\cdot\boldsymbol{\lambda}} q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]} e^{-2\pi i(z,\mathbf{u})^T L(\lambda,\boldsymbol{\lambda})}.$$

We now prove the modular transformation property. By the definition of Φ and using the modularity of ϑ, we have

$$\Phi\left(\frac{z}{c\tau+d},\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right)=\frac{\prod_{r=1}^{s}\vartheta\left(\frac{z+u_r}{c\tau+d}+\frac{1}{2};\gamma\tau\right)^{m_r}}{\prod_{j=1}^{t}\vartheta\left(\frac{z-w_j}{c\tau+d};\gamma\tau\right)^{n_j}}.$$

If $\gamma\in\Gamma_0(2)$, then $d-1$ is even, and we can rewrite the right-hand side as

$$\frac{\prod_{r=1}^{s}\left(\Psi(\gamma)^3(-1)^{\frac{c}{4}}(c\tau+d)^{\frac{1}{2}}e^{\frac{2\pi ic}{2(c\tau+d)}(z+u_r)^2}\vartheta\left(z+u_r+\frac{1}{2};\tau\right)\right)^{m_r}}{\prod_{j=1}^{t}\left(\Psi(\gamma)^3(c\tau+d)^{\frac{1}{2}}e^{\frac{2\pi ic}{2(c\tau+d)}(z-w_j)^2}\vartheta\left(z-w_j;\tau\right)\right)^{n_j}}$$

$$=\Psi(\gamma)^{3(m-n)}(-1)^{\frac{mc}{4}}(c\tau+d)^{\frac{m-n}{2}}e^{\frac{\pi ic}{(c\tau+d)}\left(\sum_r m_r(z+u_r)^2-\sum_j n_j(z-w_j)^2\right)}\Phi(z,\mathbf{u};\tau).$$

Similarly as before, by the definition of L we have

$$e^{\frac{\pi ic}{(c\tau+d)}\left(\sum_r m_r(z+u_r)^2-\sum_j n_j(z-w_j)^2\right)}=e^{\frac{2\pi ic}{(c\tau+d)}L[(z,\mathbf{u})]}.$$

□

The transformation properties of Φ dictate "almost" modular and elliptic transformation properties for its Laurent coefficients. The following proposition describes the transformation properties of $\widetilde{D}_{\lambda,j}$ and $D_{\lambda,j}$. In order to give the statement, we define the matrix

$$L^*:=\widetilde{L}-\frac{1}{2M}\mathbf{b}\mathbf{b}^T. \tag{5.2.6}$$

Proposition 5.2.3. *For each $j\in\{1,\cdots,t\}$ and each $\lambda\in\{1,\cdots,n_j\}$, the function $D_{\lambda,j}$ is an almost holomorphic Jacobi form of weight $M-\lambda$. More precisely it satisfies the following transformation properties:*

1. *For all $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathbb{Z}^{s+t}$ such that $\frac{1}{2M}\boldsymbol{b}\cdot\boldsymbol{\lambda}$ and $\frac{1}{2M}\boldsymbol{b}\cdot\boldsymbol{\mu} \in \mathbb{Z}$, we have*

$$D_{r,j}\left(\boldsymbol{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau\right) = (-1)^{\boldsymbol{b}\cdot\boldsymbol{\mu}+(\boldsymbol{0},\boldsymbol{b}_t)\cdot\boldsymbol{\lambda}} e^{-2\pi i M\left(\lambda^2\tau+2\lambda w\right)} q^{-\frac{1}{2}L^*[\boldsymbol{\lambda}]} e^{-2\pi i \boldsymbol{u}^T L^* \boldsymbol{\lambda}} \times D_{r,j}\left(\boldsymbol{u};\tau\right),$$

where λ and μ are defined in Remark 2.

2. *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$D_{r,j}\left(\frac{\boldsymbol{u}}{c\tau+d};\gamma\tau\right) = \chi(\gamma)(c\tau+d)^{M-r} e^{\pi i \frac{c}{c\tau+d}L^*[\boldsymbol{u}]} e^{2\pi i M\frac{c}{c\tau+d}w^2} D_{r,j}\left(\boldsymbol{u};\tau\right),$$

where χ is as in Proposition 3.3.1.

Proof. We start by proving the modular property. We recall that by definition $w = w_j + \frac{1}{2M}\mathbf{b}\cdot\mathbf{u}$ and define $z := \varepsilon + w$. With this notation, we see that the functions $D_{r,j}$ are the Laurent coefficients of

$$G(z-w,w;\tau)\Phi\left(z-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u};\tau\right), \tag{5.2.7}$$

in $z = w$. From the transformation properties of G and Φ (see Lemma 5.2.1 and Proposition 5.2.2), the function in (5.2.7) has automorphy factor

$$\chi(\gamma)(c\tau+d)^M e^{\frac{2\pi i c}{c\tau+d}\left(-Mz^2+Mw^2+\frac{1}{2}L\left[\left(z-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u}\right)\right]\right)}.$$

It is a standard fact that

$$L\left[\left(z-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u}\right)\right] = L\left[(z,\mathbf{0})\right] + L\left[\left(-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u}\right)\right].$$

Therefore, the automorphy factor can be written as

$$\chi(\gamma)(c\tau+d)^M e^{\frac{2\pi i c}{c\tau+d}\left(Mw^2+\frac{1}{2}L^*[\mathbf{u}]\right)},$$

where in the last step we have used the identity

$$L\left[\left(-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u}\right)\right] = L^*\left[\mathbf{u}\right].$$

As a consequence, we obtain

$$\sum_{\lambda=1}^{n_j} \frac{D_{\lambda,j}\left(\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right)(c\tau+d)^{\lambda}}{(2\pi i\varepsilon)^{\lambda}} + O(1)$$
$$= \chi(\gamma)(c\tau+d)^{M} e^{\frac{2\pi i c}{c\tau+d}\left(Mw^2+\frac{1}{2}L^*[\mathbf{u}]\right)} \sum_{\lambda=1}^{n_j} \frac{D_{\lambda,j}(\mathbf{u};\tau)}{(2\pi i\varepsilon)^{\lambda}} + O(1).$$

Comparing the coefficient of $\varepsilon^{-\lambda}$ on both the right- and the left-hand sides, we obtain the result.

For the elliptic property, the computation is very similar. Shifting $\mathbf{u} \mapsto \mathbf{u} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}$, (i.e., shifting w and z by $\lambda\tau+\mu$, where λ and μ are as in the hypothesis) in (5.2.7), we obtain the elliptic factor

$$(-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\mathbf{b}_t)\cdot\boldsymbol{\lambda}} q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]} e^{2\pi i\left(2M\lambda(z-w)-\left(\lambda-\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda},\boldsymbol{\lambda}\right)^T L\left(z-\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},\mathbf{u}\right)\right)}. \tag{5.2.8}$$

The second term in the exponent can be easily written as

$$(\lambda,\mathbf{0})^T L(z,\mathbf{0}) + \left(-\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda},\boldsymbol{\lambda}\right)^T L(0,\mathbf{u}) = 2M\lambda z + \boldsymbol{\lambda}^T L^*\mathbf{u},$$

where in the last equality we have used

$$\left(-\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda},\boldsymbol{\lambda}\right)^T L(0,\mathbf{u}) = \boldsymbol{\lambda}^T L^*\mathbf{u}.$$

Therefore, the entire elliptic factor in (5.2.8) turns out to be

$$(-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\mathbf{b}_t)\cdot\boldsymbol{\lambda}} q^{-Q((\lambda,\boldsymbol{\lambda}))} e^{2\pi i\left(-2M\lambda w+\boldsymbol{\lambda}^T L^*\mathbf{u}\right)}.$$

To conclude, it is enough to show that

$$q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]} = q^{-M\lambda^2-\frac{1}{2}L^*[\boldsymbol{\lambda}]}.$$

This follows easily, arguing as in the proof of the modularity property.

Finally, in order to show that the functions $D_{r,j}$ are almost holomorphic Jacobi forms, it is enough to note that

$$D_{r,j}(\mathbf{u};\tau) := \sum_{\kappa=0}^{n_j-r} \widetilde{D}_{r+\kappa,j}(\mathbf{u};\tau)\frac{1}{\kappa!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{\kappa}\left[G(\varepsilon,w;\tau)\right]_{\varepsilon=0}. \tag{5.2.9}$$

$\square$

5.3 Proof of Theorem 5.1.1

5.3.1 Canonical Fourier coefficients and canonical decomposition

In this subsection we generalize the canonical decomposition given by Dabholkar, Murthy, and Zagier [15] to the multivariable case. Here we consider the Fourier coefficients with respect to a single elliptic variable z. For a fixed $\omega \in \mathbb{C}$, we define

$$h_\ell^{(\omega)}(\mathbf{u};\tau) := q^{-\frac{\ell^2}{4M}} e^{-2\pi i \frac{\ell}{2M}\mathbf{b}\cdot\mathbf{u}} \int_\omega^{\omega+1} \Phi(z,\mathbf{u};\tau)e^{-2\pi i \ell z}dz. \tag{5.3.1}$$

The path of integration is the straight line from ω to $\omega+1$. If the there is a pole of Φ on it, we adopt the same modifications as in Remark 1. We define the ℓth *canonical Fourier coefficient of* Φ as

$$h_\ell(\mathbf{u};\tau) := h_\ell^{\left(-\frac{\ell\tau}{2M}\right)}(\mathbf{u};\tau). \tag{5.3.2}$$

With this choice, using the elliptic transformation property of Φ with respect to z, it is easy to show that

$$h_\ell(\mathbf{u};\tau) = h_{\ell+2M}(\mathbf{u};\tau).$$

In particular, we can define the vector-valued function $\mathbf{h} := (h_\ell)_{\ell \pmod{2M}}$. We define the finite part of Φ by the theta decomposition

$$\Phi^F(z,\mathbf{u};\tau) := \mathbf{h}(\mathbf{u};\tau)\cdot\boldsymbol{\vartheta}_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right).$$

Moreover, we define the polar part of Φ as

$$\Phi^P(z,\mathbf{u};\tau) := -\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j}\frac{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}{(\lambda-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{\lambda-1}\left[f_M\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M},\frac{\mathbf{b}\cdot\mathbf{u}}{2M}+w_j+\varepsilon;\tau\right)\right.$$
$$\left.-\sum_{\ell \pmod{2M}} E_{M,\ell}\left(\frac{\mathbf{b}\cdot\mathbf{u}}{2M}+w_j+\varepsilon;\tau\right)\vartheta_{M,\ell}\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M};\tau\right)\right]_{\varepsilon=0},$$

where the polynomial $E_{M,\ell}$ is given by

$$E_{M,\ell}(u;\tau) := \sum_{\substack{\lambda\in\mathbb{Z} \\ \lambda\equiv\ell \pmod{2M}}} \frac{1}{2}\left(\operatorname{sgn}\left(\lambda+\frac{1}{2}\right) - \operatorname{sgn}\left(\lambda+2M\mathrm{Im}(u)\right)\right) q^{-\frac{\lambda^2}{4M}} e^{-2\pi i\lambda u}. \tag{5.3.3}$$

The functions Φ^F and Φ^P provide the *canonical decomposition* of Φ, as shown in the following proposition.

Proposition 5.3.1. *With the notation as above, we have* $\Phi = \Phi^F + \Phi^P$.

Proof. We fix a point $W := X\tau+Y$ ($X, Y \in \mathbb{R}$), and assume $\mathrm{Im}(z) = Xv$. Furthermore, call $\mathcal{P}$ the parallelogram of vertices W, $W+1$, $-\frac{\ell\tau}{2M}+1$, and $-\frac{\ell\tau}{2M}$. We have already seen that the poles of Φ with respect to z are in w_j modulo $\mathbb{Z}\tau+\mathbb{Z}$, for all $j \in \{1, \cdots, t\}$. A pole is inside $\mathcal{P}$ if and only if

$$-\frac{\ell}{2M}v \le \mathrm{Im}\left(w_j+\mu\tau\right) \le \mathrm{Im}W = Xv,$$

for some $\mu\in\mathbb{Z}$. This is equivalent to requiring

$$-\frac{\ell}{2M} \le \frac{\mathrm{Im}(w_j)}{v} + \mu \le X.$$

As usual, the function that describes whether a pole is inside $\mathcal{P}$ or not is

$$\mathcal{S}(\mu,\ell) := \frac{1}{2}\left(\operatorname{sgn}\left(\mu+\frac{\ell}{2M}+\frac{\mathrm{Im}(w_j)}{v}\right) - \operatorname{sgn}\left(\mu+\frac{\mathrm{Im}(w_j)}{v}-X\right)\right).$$

From the Residue Theorem it follows that

$$\Phi(z,\mathbf{u};\tau) - \Phi^F(z,\mathbf{u};\tau) = 2\pi i \sum_{\ell\in\mathbb{Z}}\sum_{j=1}^{t}\sum_{\mu\in\mathbb{Z}} \mathcal{S}(\mu,\ell) \operatorname*{Res}_{w=w_j+\mu\tau}\left(\Phi(w,\mathbf{u};\tau)e^{-2\pi i w\ell}\right)\zeta^\ell$$

$$= 2\pi i \sum_{\ell\in\mathbb{Z}}\sum_{j=1}^{t}\sum_{\mu\in\mathbb{Z}} \mathcal{S}(\mu,\ell) \operatorname*{Res}_{\varepsilon=0}\left(\Phi(\varepsilon+w_j+\mu\tau,\mathbf{u};\tau)e^{-2\pi i\ell(\varepsilon+w_j+\mu\tau)}\right)\zeta^\ell.$$

Using the elliptic transformation property of Φ, one has

$$\Phi(\varepsilon+w_j+\mu\tau,\mathbf{u};\tau) = q^{-M\mu^2}e^{-2\pi i\mu\mathbf{b}\cdot\mathbf{u}}e^{-4\pi iM\mu(\varepsilon+w_j)}\Phi(\varepsilon+w_j,\mathbf{u};\tau).$$

Therefore, $\Phi - \Phi^F$ can be rewritten as

$$2\pi i \sum_{\ell\in\mathbb{Z}} \sum_{j=1}^{t} \sum_{\mu\in\mathbb{Z}} \mathcal{S}(\mu,\ell)\zeta^{\ell} q^{-M\mu^2-\mu\ell} e^{-2\pi i\mu \mathbf{b}\cdot\mathbf{u}} e^{-2\pi i(\ell+2M\mu)w_j} \\ \times \operatorname*{Res}_{\varepsilon=0}\left(\Phi(\varepsilon+w_j,\mathbf{u};\tau)e^{-2\pi i\varepsilon(\ell+2M\mu)}\right).$$

Shifting the variables as $\mu \mapsto -\mu$ and then $\ell \mapsto \ell + 2M\mu$, we obtain

$$\Phi^P(z,\mathbf{u};\tau) = -2\pi i \sum_{j=1}^{t} \sum_{\mu\in\mathbb{Z}} q^{M\mu^2}\zeta^{2M\mu} e^{2\pi i\mu\mathbf{b}\cdot\mathbf{u}} \sum_{\ell\in\mathbb{Z}} \mathcal{S}(-\mu,\ell+2M\mu) q^{\mu\ell}\zeta^{\ell} \\ \times e^{-2\pi i\ell w_j} \operatorname*{Res}_{\varepsilon=0}\left(\Phi(\varepsilon+w_j,\mathbf{u};\tau)e^{-2\pi i\varepsilon\ell}\right). \quad (5.3.4)$$

Note that the residue can be written as

$$\operatorname*{Res}_{\varepsilon=0}\left(\Phi(\varepsilon+w_j,\mathbf{u};\tau)e^{-2\pi i\varepsilon\ell}\right) = \sum_{r=1}^{n_j} \frac{\widetilde{D}_{r,j}(\mathbf{u};\tau)}{2\pi i(r-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{r-1}\left[e^{-2\pi i\ell\varepsilon}\right]_{\varepsilon=0}.$$

In particular, $\Phi - \Phi^F$ equals

$$-\sum_{j=1}^{t}\sum_{r=1}^{n_j} \frac{\widetilde{D}_{r,j}(\mathbf{u};\tau)}{(r-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{r-1}\left[\sum_{\mu\in\mathbb{Z}} q^{M\mu^2}\zeta^{2M\mu}e^{2\pi i\mu\mathbf{b}\cdot\mathbf{u}} \right. \\ \left. \times \sum_{\ell\in\mathbb{Z}} \mathcal{S}(-\mu,\ell+2M\mu)q^{\mu\ell}\zeta^{\ell}e^{-2\pi i\ell(w_j+\varepsilon)}\right]_{\varepsilon=0}.$$

A standard computation allows us to write the expression in brackets as an Appell Lerch sum and an error term, which is holomorphic. More precisely, as claimed, we can express the difference $\Phi - \Phi^F$ as

$$-\sum_{j=1}^{t}\sum_{r=1}^{n_j} \frac{\widetilde{D}_{r,j}(\mathbf{u};\tau)}{(r-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{r-1}\left[f_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}, \frac{1}{2M}\mathbf{b}\cdot\mathbf{u}+w_j+\varepsilon;\tau\right)\right. \\ \left. - \sum_{\ell \pmod{2M}} E_{M,\ell}\left(\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}+w_j+\varepsilon;\tau\right)\vartheta_{M,\ell}\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right)\right]_{\varepsilon=0}. \quad (5.3.5)$$

□

As in the single variable case, the polar part of Φ can be written in a slightly different way, where the Laurent coefficients $\widetilde{D}_{\lambda,j}$ are replaced by their completions $D_{\lambda,j}$. We describe this phenomenon in the following proposition.

Proposition 5.3.2. *With the notation as above, the polar part of* Φ *can be written as*

$$\Phi^P(z,\boldsymbol{u};\tau) = -\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j} \frac{D_{\lambda,j}(\boldsymbol{u};\tau)}{(\lambda-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{\lambda-1}\left[\frac{f_M\left(z+\frac{1}{2M}\boldsymbol{b}\cdot\boldsymbol{u}, w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)} - \sum_{\ell \pmod{2M}} \frac{E_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)\vartheta_{M,\ell}\left(z+\frac{1}{2M}\boldsymbol{b}\cdot\boldsymbol{u};\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}.$$

Here we omit the proof since it can be derive from the proof of Proposition 3.3.7.

5.3.2 The modular properties of h

As in the single variable case, the finite part and the polar part of a multivariable meromorphic Jacobi form can be completed to functions which transform as Jacobi forms, and this property will be inherited by the canonical Fourier coefficients, i.e., the components of $\mathbf{h}$. To show this, we need the non-holomorphic vector-valued function

$$\mathbf{R}\left(\mathbf{u};\tau\right) := \left(\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j} \frac{D_{\lambda,j}(\mathbf{u};\tau)}{(\lambda-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{\lambda-1}\left[\frac{\widetilde{R}_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}\right)_{\ell \pmod{2M}},$$

where $\widetilde{R}_{M,\ell} := E_{M,\ell} - \frac{1}{2}R_{M,\ell}$ and $w^{(j)}$ as in (5.2.5). We define the completion $\widehat{\Phi}^P$ of Φ^P as

$$\widehat{\Phi}^P(z,\mathbf{u};\tau) := \Phi^P(z,\mathbf{u};\tau) - \mathbf{R}(\mathbf{u};\tau)\cdot\boldsymbol{\vartheta}_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right),$$

and the completion $\widehat{\Phi}^F$ of Φ^F as

$$\widehat{\Phi}^F(z,\mathbf{u};\tau) := \Phi^F(z,\mathbf{u};\tau) + \mathbf{R}(\mathbf{u};\tau)\cdot\boldsymbol{\vartheta}_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right).$$

In the following proposition we show that these two functions are multivariable non-holomorphic Jacobi forms.

Proposition 5.3.3. *The functions $\widehat{\Phi}^F$ and $\widehat{\Phi}^P$ satisfy the same transformation properties as Φ.*

Proof. As we shall see, each summand of the double summation defining $\widehat{\Phi}^P$ gives the same automorphy factor, therefore we can reduce to showing the transformation properties of

$$\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{r-1}\left[\frac{\widehat{f}_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}, w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}$$

for fixed r and j.

We start by showing the modular transformation property. Let $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$. For a function f, let $\mathrm{Aut}(f)$ be the automorphy factor of f. Therefore, since

$$\widehat{\Phi}^P\left(\frac{z}{c\tau+d},\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right) = -\sum_{j=1}^{t}\sum_{r=1}^{n_j}\frac{D_{r,j}\left(\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right)}{(r-1)!} \\ \times\left(\frac{\partial_{\frac{\varepsilon}{c\tau+d}}}{2\pi i}\right)^{r-1}\left[\frac{\widehat{f}_M\left(\frac{z}{c\tau+d}+\frac{1}{2M}\frac{\mathbf{b}\cdot\mathbf{u}}{c\tau+d},\frac{w^{(j)}+\varepsilon}{c\tau+d};\gamma\tau\right)}{G\left(\frac{\varepsilon}{c\tau+d},\frac{w^{(j)}}{c\tau+d};\gamma\tau\right)}\right]_{\varepsilon=0}, \quad (5.3.6)$$

our goal is to show that $\frac{\mathrm{Aut}(D_{r,j})\mathrm{Aut}\left(\widehat{f}_M\right)}{\mathrm{Aut}(G)}$ equals the automorphy factor of Φ. Using Proposition 5.2.3 we have that

$$\mathrm{Aut}\left(D_{r,j}\right)=\chi(\gamma)(c\tau+d)^{M-r}e^{2\pi i\left(\frac{1}{2}\frac{c}{c\tau+d}L^*[\mathbf{u}]+\frac{Mc}{c\tau+d}w^2\right)},$$

with w as in (5.2.5). From the transformation properties of $\widehat{f}_M$, we know that

$$\mathrm{Aut}\left(\widehat{f}_M\right)=(c\tau+d)e^{2\pi i\left(\frac{Mc}{c\tau+d}\left(\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}\right)^2-(w+\varepsilon)^2\right)\right)}.$$

Finally, Lemma 5.2.1 implies that

$$\mathrm{Aut}\left(G\right)=e^{-\frac{2\pi iMc}{c\tau+d}\left(\varepsilon^2+2\varepsilon w\right)}.$$

Note that the contribution of ε in $\mathrm{Aut}\,(G)$ cancels the contribution of ε in $\mathrm{Aut}\left(\widehat{f}_M\right)$. Combining the three automorphy factors, we get

$$\frac{\mathrm{Aut}\,(D_{r,j})\,\mathrm{Aut}\left(\widehat{f}_M\right)}{\mathrm{Aut}\,(G)} = \chi(\gamma)(c\tau+d)^{M-r+1}e^{\frac{2\pi ic}{c\tau+d}\left(\frac{1}{2}L^*[\mathbf{u}]+M\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}\right)^2\right)}. \tag{5.3.7}$$

Using $L^* = \widetilde{L} - \frac{\mathbf{b}\cdot\mathbf{b}^T}{2M}$ and the block-representation of L, one can trivially see that (5.3.7) equals

$$\frac{\mathrm{Aut}\,(D_{\lambda,j})\,\mathrm{Aut}\left(\widehat{f}_M\right)}{\mathrm{Aut}\,(G)} = \chi(\gamma)(c\tau+d)^{M-r+1}e^{\frac{\pi ic}{c\tau+d}L[(z,\mathbf{u})]}.$$

As a consequence, each summand on the right-hand-side of (5.3.6) becomes

$$\chi(\gamma)(c\tau+d)^{M-r+1}e^{\frac{\pi ic}{c\tau+d}L[(z,\mathbf{u})]}\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\left(\frac{\partial_{\frac{\varepsilon}{c\tau+d}}}{2\pi i}\right)^{r-1}\left[\frac{\widehat{f}_M\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}, w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}.$$

Changing the variable $\frac{\varepsilon}{c\tau+d}\mapsto\varepsilon$, we obtain

$$\chi(\gamma)(c\tau+d)^{M}e^{\frac{\pi ic}{c\tau+d}L[(z,\mathbf{u})]}\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\left(\frac{\partial_{\varepsilon}}{2\pi i}\right)^{r-1}\left[\frac{\widehat{f}_M\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}, w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0},$$

which equals the automorphy factor of Φ.

Now we show the elliptic transformation property. Shifting the elliptic variable $\mathbf{u}$ by $\boldsymbol{\lambda}\tau+\boldsymbol{\mu}$ (where $\boldsymbol{\lambda}=(\lambda_r)_r$) is equivalent of shifting $w=w^{(j)}$ by $\lambda^*\tau+\mu^*$, where $\lambda^* := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda}+\lambda_{s+j}$ and $\mu^* := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\mu}+\mu_{s+j}$.

Again, we compute each summand in the definition of $\widehat{\Phi}^P$ separately. We need to calculate

$$\begin{aligned}\widehat{\Phi}^P(z+\lambda\tau, \mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau) = &-\sum_{j=1}^{t}\sum_{r=1}^{n_j}\frac{D_{r,j}(\mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau)}{(r-1)!}\\ &\times\delta_\varepsilon^{r-1}\left[\frac{\widehat{f}_M\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}+\left(\lambda+\frac{\mathbf{b}\cdot\boldsymbol{\lambda}}{2M}\right)\tau+\left(\mu+\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\mu}\right), w^{(j)}+\lambda^*\tau+\mu^*+\varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)}+\lambda^*\tau+\mu^*;\tau\right)}\right]_{\varepsilon=0}.\end{aligned}$$

For a function f we denote by $\mathrm{Ell}(f)$ its elliptic factor. We start with $\mathrm{Ell}\,(D_{r,j})$, which by Proposition 5.2.3 equals

$$\mathrm{Ell}\,(D_{r,j}) = (-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\boldsymbol{b}_t)\cdot\boldsymbol{\lambda}}q^{-M(\lambda^*)^2-\frac{1}{2}L^*[\boldsymbol{\lambda}]}e^{2\pi i\left(-2M\lambda^*w-\mathbf{u}^TL^*\boldsymbol{\lambda}\right)}.$$

From the elliptic transformation properties of $\widehat{f}_M$, we have that its elliptic factor equals

$$\mathrm{Ell}\left(\widehat{f}_M\right) = q^{M\left((\lambda^*)^2 - \left(\lambda + \frac{\mathbf{b}\cdot\boldsymbol{\lambda}}{2M}\right)^2\right)} e^{2\pi i\left(2M\lambda^*(w+\varepsilon) - 2M\left(\lambda+\frac{\mathbf{b}\cdot\boldsymbol{\lambda}}{2M}\right)\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}\right)\right)}.$$

Finally, the elliptic factor of G is

$$\mathrm{Ell}\,(G) = e\left(2M\lambda^*\varepsilon\right).$$

Therefore, a direct computation gives that

$$\frac{\mathrm{Ell}\,(D_{r,j})\,\mathrm{Ell}\left(\widehat{f}_M\right)}{\mathrm{Ell}\,(G)} = (-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\boldsymbol{b}_t)\cdot\boldsymbol{\lambda}} q^{-\frac{1}{2}L^*[\boldsymbol{\lambda}] - M\left(\lambda+\frac{\mathbf{b}\cdot\boldsymbol{\lambda}}{2M}\right)^2} e^{2\pi i\left(-\mathbf{u}^T L^*\boldsymbol{\lambda} - 2M\left(\lambda+\frac{\mathbf{b}\cdot\boldsymbol{\lambda}}{2M}\right)\left(z+\frac{\mathbf{b}\cdot\mathbf{u}}{2M}\right)\right)}.$$

Using $L^* = \widetilde{L} - \frac{\mathbf{b}\mathbf{b}^T}{2M}$ and the block-definition of L, a trivial computation gives

$$\frac{\mathrm{Ell}\,(D_{r,j})\,\mathrm{Ell}\left(\widehat{f}_M\right)}{\mathrm{Ell}\,(G)} = (-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\boldsymbol{b}_t)\cdot\boldsymbol{\lambda}} q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]} e^{-2\pi i (z,\mathbf{u})^T L(\lambda,\boldsymbol{\lambda})},$$

which equals the elliptic factor of Φ. $\square$

We note that both Φ^F and $\mathbf{R}\cdot\boldsymbol{\vartheta}_M$ can be written as a theta decomposition. This implies that also $\widehat{\Phi}^F$ has this property, namely

$$\widehat{\Phi}^F(z,\mathbf{u};\tau) = \widehat{\mathbf{h}}(\mathbf{u};\tau)\cdot\boldsymbol{\vartheta}_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right),$$

where

$$\widehat{\mathbf{h}}(\mathbf{u};\tau) := \mathbf{h}(\mathbf{u};\tau) + \mathbf{R}(\mathbf{u};\tau)$$

is the completion of $\mathbf{h}$. The following result is direct consequence of Proposition 2.3.4.

Corollary 5.3.4. *The vector-valued function $\widehat{\boldsymbol{h}}$ transforms as a Jacobi form of weight $M-\frac{1}{2}$ and index L^* for $\Gamma_0(2)$, with multiplier system as in Proposition 2.3.4.*

5.3.3 Action of certain operators

A special property of the non-holomorphic functions $\widetilde{R}_{M,\ell}$ is that they are non-trivially annihilated by several differential operators which play a fundamental role in the theory of Jacobi forms. This was shown in Proposition 2.3.8, where the result was stated in terms of $R_{M,\ell}$. The fact that the same result holds for $\widetilde{R}_{M,\ell}$ is easy to check.

We proceed by describing the action of the lowering operator on the completed Laurent coefficients $D_{\lambda,j}$, and the action of the raising operator on the non-holomorphic functions $R_{M,\ell}$. This will be used to describe the shape of the vector-valued non-holomorphic Jacobi form $\widehat{\mathbf{h}}$.

Proposition 5.3.5. *For all $\lambda \in \{1, \cdots, n_j\}$, we have*

$$X_- \left(D_{\lambda,j}(\boldsymbol{u};\tau)\right) = \frac{M}{4\pi} D_{\lambda+2,j}(\boldsymbol{u};\tau).$$

In particular, for $\lambda \in \{1,2\}$ and $n \in \mathbb{N}$

$$X_-^n \left(D_{\lambda,j}(\boldsymbol{u};\tau)\right) = \left(\frac{M}{4\pi}\right)^n D_{\lambda+2n,j}(\boldsymbol{u};\tau).$$

In order to prove Proposition 5.3.5, we need the following result.

Lemma 5.3.6. *For each positive integer $r > 1$, we have*

$$X_- \left(\delta_\varepsilon^r \left[G\left(\varepsilon, w;\tau\right)\right]_{\varepsilon=0}\right) = \frac{r(r-1)M}{4\pi} \delta_\varepsilon^{r-2} \left[G\left(\varepsilon, w;\tau\right)\right]_{\varepsilon=0}.$$

Moreover, for $r \in \{0,1\}$, we have

$$X_- \left(\delta_\varepsilon^r \left[G\left(\varepsilon, w;\tau\right)\right]_{\varepsilon=0}\right) = 0. \tag{5.3.8}$$

Proof. For $r \in \{0,1\}$, the proof is straightforward. It is enough to check that 1 and $\frac{w-\overline{w}}{\tau-\overline{\tau}}$ are annihilated by X_-.

For $r=2$ we proceed by induction. We denote by $G^{(r)} := \delta_\varepsilon^r \left[G\left(\varepsilon, w; \tau\right)\right]_{\varepsilon=0}$. First, we note that for all $r>1$, one has

$$G^{(r)} = G^{(r-1)}G^{(1)} + (r-1)\frac{M}{\pi i(\tau-\overline{\tau})}G^{(r-2)}. \tag{5.3.9}$$

Let $r=2$. Then applying the lowering operator and using (5.3.8) yields

$$X_-\left(G^{(2)}\right) = X_-\left(G^{(1)}\right)G^{(1)} + \frac{M}{\pi i(\tau-\overline{\tau})}X_-\left(G^{(0)}\right) + \frac{M}{2\pi}G^{(0)}.$$

Using (5.3.8), we prove the claim.

Assume that the statement is true for all $0 \le s < r$. Applying the lowering operator and using (5.3.8) yields

$$X_-\left(G^{(r)}\right) = X_-\left(G^{(r-1)}\right)G^{(1)} + (r-1)\frac{M}{\pi i(\tau-\overline{\tau})}X_-\left(G^{(r-2)}\right) + (r-1)\frac{M}{2\pi}G^{(r-2)}. \tag{5.3.10}$$

By induction, we may rewrite the right-hand side of (5.3.10) as

$$\begin{aligned}&\frac{M(r-1)(r-2)}{4\pi}G^{(r-3)}G^{(1)} + \frac{(r-1)M^2(r-2)(r-3)}{4\pi^2 i(\tau-\overline{\tau})}G^{(r-4)} + (r-1)\frac{M}{2\pi}G^{(r-2)}\\ &= \frac{M(r-1)(r-2)}{4\pi}\left(G^{(r-3)}G^{(1)} + \frac{M(r-3)}{i\pi(\tau-\overline{\tau})}G^{(r-4)}\right) + \frac{M(r-1)}{2\pi}G^{(r-2)}.\end{aligned}$$

Using (5.3.9), this equals

$$\frac{M(r-1)(r-2)}{4\pi}G^{(r-2)} + \frac{M(r-1)}{2\pi}G^{(r-2)} = \frac{r(r-1)M}{4\pi}G^{(r-2)}.$$

This conclude the inductive step. □

Proof of Proposition 5.3.5. We only prove the first claim. The second statement follows trivially. Using (5.2.9) and the fact that $\widetilde{D}_{\lambda,j}$ are holomorphic functions, we can write

$$X_-\left(D_{\lambda,j}(\mathbf{u};\tau)\right) = \sum_{r=0}^{n_j-\lambda} \widetilde{D}_{\lambda+r,j}(\mathbf{u};\tau)\frac{1}{r!}X_-\left(\left(\frac{\partial_\varepsilon}{2\pi i}\right)^r \left[G\left(\varepsilon, w; \tau\right)\right]_{\varepsilon=0}\right).$$

By Lemma 5.3.6 this equals

$$\frac{M}{4\pi}\sum_{r=2}^{n_j-\lambda}\widetilde{D}_{\lambda+r,j}(\mathbf{u};\tau)\frac{1}{(r-2)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{r-2}\left[G\left(\varepsilon,w;\tau\right)\right]_{\varepsilon=0}.$$

Changing r into $r+2$ and applying again (5.2.9), we conclude the proof. $\square$

To conclude this subsection, we show how to use the function F defined in (5.2.1) to relate the action of the differential operator ∂_ε to the action of the raising operator $X_+^{k,M}$. This extends previous works of Bringmann and Folsom [7].

Proposition 5.3.7. *For all $\lambda\in\mathbb{N}_0$, we have*

$$F(w;\tau)\partial_w^{2\lambda}\left[\frac{\widetilde{R}_{M,\ell}\left(w;\tau\right)}{F(w;\tau)}\right]=\left(-4\pi M\right)^\lambda\left(X_+^{\frac{1}{2},-M}\right)^\lambda\left(\widetilde{R}_{M,\ell}\left(w;\tau\right)\right),$$

$$F(w;\tau)\partial_w^{2\lambda+1}\left[\frac{\widetilde{R}_{M,\ell}\left(w;\tau\right)}{F(w;\tau)}\right]=-i\left(-4\pi M\right)^\lambda\left(X_+^{\frac{3}{2},-M}\right)^\lambda\left(Y_+^{\frac{1}{2},-M}\left(\widetilde{R}_{M,\ell}\left(w;\tau\right)\right)\right).$$

In order to prove Proposition 5.3.7, we need several identities.

Lemma 5.3.8. *Let $f(w;\tau)\in\mathcal{C}^\infty(\mathbb{C}\times\mathbb{H})$. Then, the following are true:*

1. $X_+^{k+2,-M}\left(f\right)-X_+^{k,-M}(f)=\frac{4i}{\tau-\overline{\tau}}f$;
2. $X_+^{k+2,-M}\left(\partial_w\left[f\right]\right)-\partial_w\left[X_+^{k,-M}(f)\right]=\frac{2i}{\tau-\overline{\tau}}\partial_w\left[f\right]-8\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}f$;
3. $X_+^{k+2,-M}\left(\partial_w^2\left[f\right]\right)-\partial_w^2\left[X_+^{k,-M}(f)\right]=-16\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}\partial_w\left[f\right]-\frac{8\pi M}{(\tau-\overline{\tau})^2}f$.

Proof. Part 1 follows directly by definition of $X_+^{k,-M}$. We now prove part 2. By definition, we have

$$X_+^{k+2}\left(\partial_w\left[f\right]\right)=2i\left(\partial_\tau\partial_w\left[f\right]+\frac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w^2[f]+\left(-2\pi iM\left(\frac{w-\overline{w}}{\tau-\overline{\tau}}\right)^2+\frac{k+2}{\tau-\overline{\tau}}\right)\partial_w\left[f\right]\right)$$

and

$$\partial_w \left[X_+^k (f)\right] = 2i \left(\partial_\tau \partial_w [f] + \frac{w - \overline{w}}{\tau - \overline{\tau}} \partial_w^2 [f] + \frac{1}{\tau - \overline{\tau}} \partial_w [f] \right.$$
$$\left. + \left(-2\pi i M \left(\frac{w - \overline{w}}{\tau - \overline{\tau}} \right)^2 + \frac{k}{\tau - \overline{\tau}} \right) \partial_w [f] - 4\pi i M \frac{w - \overline{w}}{(\tau - \overline{\tau})^2} f \right).$$

Therefore, as claimed,

$$X_+^{k+2} (\partial_w [f]) - \partial_w \left[X_+^k (f)\right] = 2i \left(\frac{1}{\tau - \overline{\tau}} \partial_w [f] + 4\pi i M \frac{w - \overline{w}}{(\tau - \overline{\tau})^2} f \right).$$

We now prove part 3. By definition,

$$X_+^{k+2} \left(\partial_w^2 [f]\right) = 2i \left(\partial_\tau \partial_w^2 [f] + \frac{w - \overline{w}}{\tau - \overline{\tau}} \partial_w^3 [f] + \left(-2\pi i M \left(\frac{w - \overline{w}}{\tau - \overline{\tau}} \right)^2 + \frac{k+2}{\tau - \overline{\tau}} \right) \partial_w^2 [f] \right)$$

and

$$\partial_w^2 \left[X_+^k (f)\right] = 2i \left(\partial_\tau \partial_w^2 [f] + \frac{w - \overline{w}}{\tau - \overline{\tau}} \partial_w^3 [f] + \frac{2}{\tau - \overline{\tau}} \partial_w^2 [f] \right.$$
$$\left. + \left(-2\pi i M \left(\frac{w - \overline{w}}{\tau - \overline{\tau}} \right)^2 + \frac{k}{\tau - \overline{\tau}} \right) \partial_w^2 [f] - 8\pi i M \frac{w - \overline{w}}{(\tau - \overline{\tau})^2} \partial_w [f] - \frac{4\pi i M}{(\tau - \overline{\tau})^2} f \right).$$

Taking the difference

$$X_+^{k+2,-M} \left(\partial_w^2 [f]\right) - \partial_w^2 \left[X_+^{k,-M} (f)\right] = -16\pi M \frac{w - \overline{w}}{(\tau - \overline{\tau})^2} \partial_w [f] - \frac{8\pi M}{(\tau - \overline{\tau})^2} f,$$

we conclude the proof. □

Before proving Proposition 5.3.7, we need the following proposition.

Proposition 5.3.9. *For any $f(w;\tau) \in \mathcal{C}^\infty(\mathbb{C} \times \mathbb{H})$, one has*

$$F(w;\tau) \partial_w^2 \left[\frac{X_+^{k,-M} (f(w;\tau))}{F(w;\tau)} \right] = X_+^{k+2,-M} \left(F(w;\tau) \partial_w^2 \left[\frac{f(w;\tau)}{F(w;\tau)} \right] \right). \qquad (5.3.11)$$

Proof. We proceed by comparing the left and the right-hand sides of (5.3.11). In order to simplify the notation, for $\lambda \in \mathbb{N}$, we define

$$\mathcal{F}_\lambda = \mathcal{F}_\lambda(w;\tau) := F(w;\tau)\partial_w^\lambda \left[\frac{1}{F(w;\tau)}\right],$$
$$\partial_w^\lambda [f] := \partial_w^\lambda [f(w;\tau)].$$

The left-hand side of (5.3.11) explicitly becomes

$$\mathcal{F}_2 X_+^{k,-M}(f) + 2\mathcal{F}_1 \partial_w \left[X_+^{k,-M}(f)\right] + \partial_w^2 \left[X_+^{k,-M}(f)\right]. \tag{5.3.12}$$

Similarly, the right-hand side may be written as

$$X_+^{k+2,-M}\left(\mathcal{F}_2 f + 2\mathcal{F}_1 \partial_w [f] + \partial_w^2 [f]\right). \tag{5.3.13}$$

Using the general fact that for two functions g and h,

$$X_+^{k,-M}(gh) = gX_+^{k,-M}(h) + X_+^{0,0}(g)h,$$

and noting that

$$X_+^{0,0}(\mathcal{F}_1) = 0,$$

gives that (5.3.13) can be written as

$$\mathcal{F}_2 X_+^{k+2,-M}(f) + X_+^{0,0}(\mathcal{F}_2) f + 2\mathcal{F}_1 X_+^{k+2,-M}(\partial_w [f]) + X_+^{k+2,-M}\left(\partial_w^2 [f]\right). \tag{5.3.14}$$

Subtracting (5.3.12) from (5.3.13) gives that the difference between the right and the left-hand side of (5.3.11) equals

$$\begin{aligned}&\mathcal{F}_2 \left(X_+^{k+2,-M}(f) - X_+^{k,-M}(f)\right) + X_+^{0,0}(\mathcal{F}_2) f \\ &+ 2\mathcal{F}_1 \left(X_+^{k+2,-M}(\partial_w [f]) - \partial_w \left[X_+^{k,-M}(f)\right]\right) + X_+^{k+2,-M}\left(\partial_w^2 [f]\right) - \partial_w^2 \left[X_+^{k,-M}(f)\right].\end{aligned} \tag{5.3.15}$$

Using Lemma 5.3.8, we can write (5.3.15) as

$$\mathcal{F}_2\frac{4i}{\tau-\overline{\tau}}f+X_+^{0,0}\left(\mathcal{F}_2\right)f+2\mathcal{F}_1\left(\frac{2i}{\tau-\overline{\tau}}\partial_w\left[f\right]-8\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}f\right)\\ -16\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}\partial_w\left[f\right]-\frac{8\pi M}{(\tau-\overline{\tau})^2}f. \quad (5.3.16)$$

To conclude the proof, it is enough to show that (5.3.16) equals 0. This can be done with a direct computation using

$$\mathcal{F}_1=-4\pi iM\frac{w-\overline{w}}{\tau-\overline{\tau}}, \quad (5.3.17)$$

$$\mathcal{F}_2=(4\pi iM)^2\frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2}-\frac{4\pi iM}{(\tau-\overline{\tau})}, \quad (5.3.18)$$

$$X_+^{0,0}\left(\mathcal{F}_2\right)=-\frac{8\pi M}{(\tau-\overline{\tau})^2}.$$

□

We now have all the ingredients needed to prove Proposition 5.3.7.

Proof of Proposition 5.3.7. The $\lambda=0$ case is trivial for both of the statements. For $\lambda\geq 1$ we proceed by induction. To simplify the notation, throughout the proof, we omit the variables when writing the functions. We start by proving the first claim. For $\lambda=1$, the left-hand side equals

$$\delta_w^2\left[\widetilde{R}_{M,\ell}\right]+2\delta_w\left[\widetilde{R}_{M,\ell}\right]F\delta_w\left[\frac{1}{F}\right]+\widetilde{R}_{M,\ell}F\delta_w^2\left[\frac{1}{F}\right], \quad (5.3.19)$$

where we have used the notation $\delta_w:=\frac{\partial_w}{2\pi i}$. Using (5.3.17), (5.3.18), and the identity

$$\delta_\varepsilon^2\left[\widetilde{R}_{M,\ell}\left(\varepsilon+w;\tau\right)\right]_{\varepsilon=0}=-4M\delta_\tau\left[\widetilde{R}_{M,\ell}\left(w;\tau\right)\right],$$

which follows by Proposition 2.3.8, equation (5.3.19) equals

$$-\frac{2M}{\pi i}\left(\partial_\tau+\frac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w-2\pi iM\frac{\left(w-\overline{w}\right)^2}{\left(\tau-\overline{\tau}\right)^2}+\frac{1}{2(\tau-\overline{\tau})}\right)\widetilde{R}_{M,\ell}.$$

By definition of X_+, we prove the case $\lambda = 1$.

Assume now that the statement is true for $\lambda - 1$. Then

$$F\delta_w^{2\lambda}\left[\frac{1}{F}\widetilde{R}_{M,\ell}\right] = \left(\frac{M}{\pi}\right)^{\lambda-1} F\delta_w^2\left[\frac{1}{F}\left(X_+^{\frac{1}{2},-M}\right)^{\lambda-1}\left(\widetilde{R}_{M,\ell}\right)\right].$$

Applying Proposition 5.3.9 λ times yields

$$\left(\frac{M}{\pi}\right)^{\lambda-1}\left(X_+^{\frac{5}{2},-M}\right)^{\lambda-1}\left(F\delta_w^2\left[\frac{1}{F}\widetilde{R}_{M,\ell}\right]\right) = \left(\frac{M}{\pi}\right)^{\lambda}\left(X_+^{\frac{1}{2},-M}\right)^{\lambda}\left(\widetilde{R}_{M,\ell}\right),$$

as claimed.

Now we prove the second statement. For $\lambda = 1$, we have

$$\begin{aligned}F\delta_w^3\left(\frac{\widetilde{R}_{M,\ell}}{F}\right) = -\frac{1}{2\pi}F\Bigg(&\delta_w^2\left[\frac{1}{F}\right]Y_+\left(\widetilde{R}_{M,\ell}\right) + 2\delta_w\left[\frac{1}{F}\right]\delta_w\left[Y_+\left(\widetilde{R}_{M,\ell}\right)\right] \\ &+\frac{1}{F}\delta_w^2\left[Y_+\left(\widetilde{R}_{M,\ell}\right)\right]\Bigg),\end{aligned}$$

where we have used the $\lambda = 0$ case. Using (5.3.17) and (5.3.18), we write this as

$$-\frac{1}{2\pi}\left(\frac{iM}{\pi(\tau-\overline{\tau})} + (2M)^2\frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} - 4M\frac{w-\overline{w}}{\tau-\overline{\tau}}\delta_w + \delta_w^2\right)\left(Y_+\left(\widetilde{R}_{M,\ell}\right)\right). \tag{5.3.20}$$

A direct computation gives

$$\delta_w^2\left[Y_+\left(\widetilde{R}_{M,\ell}\right)\right] = -4M\delta_\tau\left[Y_+\left(\widetilde{R}_{M,\ell}\right)\right] + \frac{2Mi}{\pi(\tau-\overline{\tau})}Y_+\left(\widetilde{R}_{M,\ell}\right).$$

Thus, we rewrite (5.3.20) as

$$\frac{M}{\pi^2 i}\left(\frac{3}{2(\tau-\overline{\tau})} - 2\pi i M\frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} + \frac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w + \partial_\tau\right)\left(Y_+\left(\widetilde{R}_{M,\ell}\right)\right),$$

which by definition of $X_+^{\frac{3}{2},-M}$ concludes the proof for $\lambda = 1$. Assume that the statement is true for $\lambda - 1$, then

$$F\delta_w^{2\lambda+1}\left[\frac{1}{F}\widetilde{R}_{M,\ell}\right] = -\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda-1}F\delta_w^2\left[\frac{1}{F}\left(X_+^{\frac{3}{2},-M}\right)^{\lambda-1}\left(Y_+\left(\widetilde{R}_{M,\ell}\right)\right)\right].$$

Applying Proposition 5.3.9 λ times, we rewrite this as

$$-\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda-1}\left(X_+^{\frac{3}{2}+2,-M}\right)^{\lambda-1}\left(F\delta_w^2\left[\frac{1}{F}Y_+\left(\widetilde{R}_{M,\ell}\right)\right]\right). \tag{5.3.21}$$

By induction, firstly using the $\lambda = 0$ case, and then the $\lambda = 1$ case, we have

$$F\delta_w^2\left(\frac{1}{F}Y_+\left(\widetilde{R}_{M,\ell}\right)\right) = -2\pi F\delta_w^3\left[\frac{1}{F}\widetilde{R}_{M,\ell}\right] = \frac{M}{\pi}X_+^{\frac{3}{2},-M}\left(Y_+\left(\widetilde{R}_{M,\ell}\right)\right).$$

Thus, as claimed, (5.3.21) equals

$$-\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda}\left(\left(X_+^{\frac{3}{2},-M}\right)^{\lambda}\left(Y_+\left(\widetilde{R}_{M,\ell}\right)\right)\right).$$

$\square$

5.3.4 Shape of h

To conclude the proof of Theorem 5.1.1, we need to show that the components of $\mathbf{h}$ have the shape of an almost harmonic Maass-Jacobi form. This fact follows from the results proved in the previous subsection, more precisely from Proposition 5.3.5 and Proposition 5.3.7. Indeed, they imply that each component of the non-holomorphic function $\mathbf{R}$ can be written as

$$\sum_{j=1}^{t}\sum_{\lambda=0}^{\left[\frac{n_j-1}{2}\right]}\frac{4^\lambda}{(2\lambda)!}X_-^\lambda\left(D_{1,j}(\mathbf{u};\tau)\right)\left(X_+^{\frac{1}{2}}\right)^{\lambda}\left(\widetilde{R}_{M,\ell}(w;\tau)\right)$$
$$+\frac{1}{2\pi}\sum_{j=1}^{t}\sum_{\lambda=0}^{\left[\frac{n_j-2}{2}\right]}\frac{4^\lambda}{(2\lambda+1)!}X_-^\lambda\left(D_{2,j}(\mathbf{u};\tau)\right)\left(X_+^{\frac{3}{2}}\right)^{\lambda}\left(Y_+^{\frac{1}{2},-M}\left[\widetilde{R}_{M,\ell}(w;\tau)\right]\right).$$

In Proposition 5.1 of [12] it is shown that $\widetilde{R}_{M,\ell}$ is the non-holomorphic part of a H-harmonic Maass-Jacobi form, $\widehat{\mu}_{M,\ell}$, whose holomorphic part is denoted by $\mu_{M,\ell}$. To be more precise, for $\iota \in \{\frac{1}{2}, \frac{3}{2}\}$, we define

$$\mathcal{G}_\iota(\mathbf{u};\tau) := \left(\sum_{j=1}^{t} \sum_{\lambda=0}^{\left[\frac{n_j-\iota-1/2}{2}\right]} \frac{4^\lambda X_-^\lambda \left(D_{\iota+\frac{1}{2},j}(\mathbf{u};\tau)\right)}{(2\lambda+\iota-\frac{1}{2})!} \right.$$

$$\left. \times \left(X_+^\iota\right)^\lambda \left(\left(\frac{Y_+^{\frac{1}{2},-M}}{2\pi} \right)^{\iota-\frac{1}{2}} \mu_{M,\ell}(w;\tau) \right) \right)_{\ell \pmod{2M}},$$

$$\widehat{\mathcal{G}}_\iota(\mathbf{u};\tau) := \left(\sum_{j=1}^{t} \sum_{\lambda=0}^{\left[\frac{n_j-\iota-1/2}{2}\right]} \frac{4^\lambda X_-^\lambda \left(D_{\iota+\frac{1}{2},j}(\mathbf{u};\tau)\right)}{(2\lambda+\iota-\frac{1}{2})!} \right.$$

$$\left. \times \left(X_+^\iota\right)^\lambda \left(\left(\frac{Y_+^{\frac{1}{2},-M}}{2\pi} \right)^{\iota-\frac{1}{2}} \widehat{\mu}_{M,\ell}(w;\tau) \right) \right)_{\ell \pmod{2M}}.$$

Then we have

$$\widehat{\mathbf{h}} = \mathbf{h} - \mathcal{G}_{\frac{1}{2}} - \mathcal{G}_{\frac{3}{2}} + \widehat{\mathcal{G}}_{\frac{1}{2}} + \widehat{\mathcal{G}}_{\frac{3}{2}}.$$

By construction $\widehat{\mathcal{G}}_{\frac{1}{2}}$ and $\widehat{\mathcal{G}}_{\frac{3}{2}}$ are vector-valued almost harmonic Maass-Jacobi forms, while $\mathbf{h} - \mathcal{G}_{\frac{1}{2}} - \mathcal{G}_{\frac{3}{2}}$ is an almost holomorphic Jacobi form, therefore, in particular, an almost harmonic Maass-Jacobi form.

CHAPTER 6

Asymptotics results for Kac-Wakimoto characters

6.1 Introduction

In the previous chapters, we saw how to describe the Fourier coefficients of a meromorphic Jacobi form in terms of Appell functions and almost holomorphic modular forms. This allowed us to derive their transformation properties. In particular, in Chapter 3, we described the transformation properties of the Kac-Wakimoto characters $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}$ related to the Lie superalgebra $\mathrm{s}\ell(m|n)^\wedge$, which we showed to be described as the Fourier coefficients of a quotients of theta functions (3.1.2). Using different methods, in [9] K. Bringmann and K. Mahlburg described asymptotic formulas for the coefficients of $\mathrm{tr}_{L_{m,1}(\Lambda(\ell))}$ in the case of $\mathrm{s}\ell(m|1)^\wedge$. The aim of this chapter is to extend Bringmann-Mahlburg's result to $\mathrm{s}\ell(m|n)^\wedge$, to any $m > n > 0$. For an easier notation, we restrict to the case of $m \equiv n \equiv 0 \pmod 2$. However, this procedure applies without many differences to the other cases.

6.1.1 Statement of the Theorems

Asymptotic results for the coefficients of characters associated to affine Lie algebras were studied by V. G. Kac and D. Peterson [22]. For an affine Lie algebra $\widetilde{\mathfrak{g}}$, denote by $L(\Lambda)$ the $\widetilde{\mathfrak{g}}$-module with highest weight Λ, and consider the so-called *weight space* *decomposition*

$$L(\Lambda) = \bigoplus_{\lambda} L(\Lambda)_\lambda,$$

where λ runs through the dual roots lattice. Denoting by $\mathrm{mult}_\Lambda(\lambda)$ the multiplicity of each weight space $L(\Lambda)_\lambda$, for each λ, one defines the *character* of the module as

$$\mathrm{ch}_{L(\Lambda)} := \sum_{\lambda} \mathrm{mult}_\Lambda(\lambda) q^\lambda.$$

Using the modularity of the so called "string functions" of the character and applying Tauberian theorems, V. G. Kac and D. Peterson studied the asymptotic behaviour of the weight multiplicities for affine Lie algebras.

Theorem 6.1.1 (Kac-Peterson, [22], Section 4.7, Theorem B)**.** *If $\widetilde{\mathfrak{g}}$ is an affine Lie algebra with $\ell + 1$ simple roots, then, as $m \to \infty$,*

$$\mathrm{mult}_\Lambda(\lambda - m\delta) \sim 2^{-\frac{1}{2}} a^{\frac{\ell+1}{4}} b m^{-\frac{\ell+3}{4}} e^{4\pi\sqrt{am}},$$

where a and b are certain explicit constants that are determined by $\widetilde{\mathfrak{g}}$.

Remark. The weight expression $\lambda - m\delta$ and the constants a and b all depend on the Cartan subalgebra.

One of the most famous results in analytic number theory is the so-called Circle Method, due to Hardy and Ramanujan [18]. They developed this method in order to study the asymptotic behavior of the partition function $p(m)$, giving an asymptotic series expansion with polynomial error. Refining Hardy and Ramanujan's work, in [30], H. Rademacher obtained an exact formula for $p(m)$ in terms of Kloosterman sums and the classical modified Bessel function (for a precise statement see [30]). Similar results were then established for any weakly holomorphic modular form of negative weight by H. Rademacher and H. Zuckerman [31, 35].

In general the Kac-Wakimoto characters are not modular, thus Rademacher's method does not apply. However, in order to give asymptotic series expansion for partitions without sequences, in [10] K. Bringmann and K. Mahlburg extended this method for linear combination of modular forms multiplied by mock modular forms. Using a refinement of this "mock modular" Circle Method and the Saddle point method the same authors [9] established asymptotics for the coefficients of $\mathrm{tr}_{L_{m,1}(\Lambda(\ell))} q^{L_0}$. More precisely, writing

$$\mathrm{tr}_{L_{m,1}(\Lambda(\ell))} q^{L_0} q^{-\frac{\ell}{2}} = \sum_{t \geq 0} c_\ell(t) q^t,$$

they found an explicit asymptotic expansion for $c_\ell(t)$ as $t \to \infty$ in terms of Kloosterman sums, Bessel functions, and a certain principal part (see Theorem 1.1 in [9] for details). Surprisingly, the "continuous" principal part gives the main contribution, and the authors showed in particular that, as $t \to \infty$,

$$c_\ell(t) \sim \frac{\sqrt{m+1}}{8t\sqrt{3}} e^{2\pi\sqrt{\frac{(m+1)t}{6}}}. \tag{6.1.1}$$

Using the structure of Kac-Wakimoto characters for $n \geq 1$ (recall Chapter 3), the general structure for the Fourier coefficients of meromorphic Jacobi forms described in Chapter 4, and in view of the previous result for the case $n = 1$ [9], in this chapter we establish an asymptotic series expansion for the coefficients $c_\ell(t)$ of

$$\mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0} q^{-\frac{\ell}{2}} = \sum_{t \geq 0} c_\ell(t) q^t, \tag{6.1.2}$$

as $t \to \infty$. For notational simplicity, we only consider the case that m and n are both even. The other cases can be treated similarly. To state our theorem we need some more notation. Here and throughout, we set $M := \frac{m-n}{2}$. Moreover, we define the constants

$$\begin{aligned} \varepsilon(\ell) :=& \frac{\ell^2}{4M} + \frac{2M-1}{24} - \frac{\ell}{2}, \\ \delta_h(r) :=& -\frac{M}{4} + \frac{r^2}{4M} + \frac{2M-1}{24}, \\ \delta_g(r) :=& -\frac{n}{8} + \frac{r}{2} + \frac{r^2}{4M} + \frac{2M-1}{24}, \\ \delta_E :=& \frac{n}{8} + \frac{2M-1}{24}. \end{aligned} \tag{6.1.3}$$

For integers r and s we denote by $Q_r(s)$, $R_r(s)$, and $T_r(s)$ certain Fourier coefficients defined in Corollary 6.3.2 and Proposition 6.3.3, while $K_{k,r}$, $\widetilde{K}_{k,r}$, and $\widetilde{K}^*_{k,r}$ are certain Kloosterman-type sums defined in (6.3.2), (6.3.4), and (6.3.6). The function I_1 is the usual modified Bessel function of level 1. In general, for integer $\ell > 1$ we define

$$I_\ell(z) := \frac{1}{2\pi i} \oint e^{\frac{z}{2}\left(t+\frac{1}{t}\right)} t^{-\ell-1}\, \mathrm{d}t,$$

where the contour encloses the origin and is traversed in a counterclockwise direction. Finally, we define the principal value integral P_k by

$$P_k(A, C, J, L; W) := \int_{\mathbb{R}} \frac{(1+2i\sigma)^J e^{-2\pi\sigma A}}{\cosh(\pi\sigma)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^L e^{\frac{2\pi}{k}\left(zW+\frac{1}{z}\left(C-\frac{1}{4M}\left(\sigma-\frac{i}{2}\right)^2\right)\right)}\, \mathrm{d}\phi\, \mathrm{d}\sigma, \tag{6.1.4}$$

where J, $L \in \mathbb{N}$, A, $C \in \mathbb{R}_{>0}$, and $W \in \mathbb{R}$. For the notation $\vartheta'_{h,k}$, $\vartheta''_{h,k}$, and ϕ we refer the reader to Section 6.2.2.

Theorem 6.1.2. *Let $0 \leq \ell < 2M$. With the notation as above, as $t \to \infty$, the coefficients of $tr_{L_{m,n}(\Lambda(\ell))}$ satisfy the following asymptotic behaviour:*

$$c_\ell(\lambda) = \frac{2\pi}{\sqrt{t-\varepsilon(\ell)}} \sum_{\substack{0<k\leq\sqrt{t}\\ 2|k}} \sum_{r=0}^{2M-1} \sum_{\substack{s\in\delta_h(r)+\mathbb{Z}\\ 0<s\leq 2\delta_h(r)}} Q_r(s) K_{k,r}(-t,s) \frac{\sqrt{s}}{\sqrt{2k}} I_1\left(\frac{4\pi}{k}\sqrt{\frac{s}{2}\left(t-\varepsilon(\ell)\right)}\right)$$

$$+ \frac{2\pi}{\sqrt{t-\varepsilon(\ell)}} \sum_{\substack{0<k\leq\sqrt{t}\\ 2|k}} \sum_{r=0}^{2M-1} \sum_{\substack{s\in\delta_g(r)+\mathbb{Z}\\ 0<s\leq 2\delta_g(r)}} R_r(s) \widetilde{K}_{k,r}(-t,s) \frac{\sqrt{s}}{\sqrt{2k}} I_1\left(\frac{4\pi}{k}\sqrt{\frac{s}{2}\left(t-\varepsilon(\ell)\right)}\right)$$

$$+ \sum_{\substack{1\leq k\leq\sqrt{t}\\ 2\nmid k}} \sum_{j=1}^{\frac{n}{2}} \sum_{\substack{r \pmod{2Mk}\\ r\equiv \ell-M \pmod{2M}}} \sum_{\lambda=1}^{j} \sum_{\mu=0}^{\lambda-1} \sum_{0\leq u\leq \frac{n-2j}{2}} \sum_{\substack{s\in 2\delta_E+\mathbb{Z}\\ 0<s\leq 2\delta_E}} D^*(j,\lambda,\mu,u) \widetilde{K}^*_{k,r}(-t,s)$$

$$\times N^{2(j-\lambda+u)} k^{\mu-\frac{1}{2}} T_{2j+2u}(s) P_k\left(\frac{r}{2Mk}+\frac{1}{2}, \frac{s}{2}, 2(\lambda-\mu)-1, 2(j-\lambda)+\mu; t-\varepsilon(\ell)\right)$$

$$+ O\left(t^{\frac{1}{2}}\log t\right).$$

Using the asymptotic result for the principal value integral and the well known asymptotic behavior of the Bessel functions (see Section 6.3.4), one can see that the main contribution comes from the non-holomorphic term, as happens in the case of $n = 1$ in [9]. More precisely, we obtain the following result.

Corollary 6.1.3. *Assuming the notation as above, as* $t \to \infty$, *we have*

$$c_\ell(t) \sim \mathcal{C} t^{\frac{n}{2}-2} e^{2\pi\sqrt{t\left(\frac{n}{2}+\frac{2M-1}{6}\right)}},$$

where

$$\mathcal{C} := \frac{M^{\frac{n}{2}-1} e^{\pi i \frac{5M+2m}{4}} \sqrt{\frac{n}{8}+\frac{2M-1}{24}}}{2^{\frac{n+1}{2}} \pi^{\frac{n}{2}} \left(\frac{n}{2}-1\right)!}.$$

6.2 Preliminaries

In this section we firstly describe the transformation properties the Kac-Wakimoto characters. Secondly, we briefly recall the Circle Method and we apply it to (6.1.2).

6.2.1 Transformation properties

Let φ be a meromorphic Jacobi form of weight $k \in \mathbb{Z}$ and index $N \in \mathbb{N}$, with a unique pole (modulo $\mathbb{Z}\tau + \mathbb{Z}$) of order $n \equiv 0 \pmod 2$ in $z = 0$. Moreover, assume that φ is an even function in z. This is the setting for the generating function of the Kac-Wakimoto characters, as we shall see later in this section. We recall the main results of Chapter 4 adapted to this special setting.

We denote the Laurent expansion of φ in $z = 0$ by

$$\varphi(z;\tau) = \sum_{j=1}^{\frac{n}{2}} \frac{\widetilde{B}_{2j}(\tau)}{(2\pi i z)^{2j}} + O(1). \tag{6.2.1}$$

By Proposition 4.2.2, we know that the functions $\widetilde{B}_{2j}$ are quasimodular forms. We denote their completion by B_{2j} (see (4.2.6)). Moreover, from Proposition 4.3.1, we know that φ canonically decomposes as

$$\varphi(z;\tau) = \sum_{\ell \pmod{2N}} h_\ell(\tau)\vartheta_{N,\ell}(z;\tau) - \sum_{j=1}^{\frac{n}{2}} \frac{\widetilde{B}_{2j}(\tau)}{(2j-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[f_N(z,\varepsilon;\tau)\right]_{\varepsilon=0}, \quad (6.2.2)$$

where the functions h_ℓ are the canonical Fourier coefficients of φ (see (4.3.1)), and f_N is the level N Appell function. In Proposition 4.3.3 we saw that the functions h_ℓ can be completed to almost harmonic Maass forms $\widehat{h}_\ell$, namely,

$$\widehat{h}_\ell(\tau) := h_\ell(\tau) - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} \frac{B_{2j}(\tau)}{(2j-1)!}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[R_{N,\ell}(\varepsilon;\tau)e^{-\frac{N\pi\varepsilon^2}{v}}\right]_{\varepsilon=0}, \qquad (6.2.3)$$

where $R_{N,\ell}$ is defined in (2.3.4). Therefore, in order to understand the modularity of h_ℓ, it suffices to understand the modularity of the B_{2j}, $\widehat{h}_\ell$, and $R_{N,\ell}$. The first two objects transform as (vector-valued) modular forms, with multiplier system depending on φ. On the other hand, the functions $R_{N,\ell}$ are independent of the Jacobi form. We describe their transformation law in the following proposition. To state it, we need the level N Mordell integral

$$H_\ell^{(M)}(u;\tau) := \frac{i}{2}q^{-\frac{(\ell+M)^2}{4M}}e^{-2\pi i(\ell+M)u}H\left(2Mu+(\ell+M)\tau+\frac{1}{2};2M\tau\right),$$

where

$$H(u;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau w^2-2\pi uw}}{\cosh(\pi w)}\,\mathrm{d}w$$

is the standard Mordell integral. Moreover, let ϱ be as in Proposition 2.3.3 and let $\omega(\ell,r)$ be the entry of ϱ^{-1} in the ℓth row and rth column.

Proposition 6.2.1. *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $c>0$ the function $R_{M,\ell}$ satisfies the following transformation property:*

$$R_{M,\ell}(u;\tau) = \sum_{r \pmod{2M}} \frac{\omega(\ell,r)}{(c\tau+d)^{\frac{1}{2}}}e^{2\pi i\frac{Mcu^2}{c\tau+d}}R_{M,r}\left(\frac{u}{c\tau+d};\gamma\tau\right)$$
$$+2\sum_{\substack{r \pmod{2Mc} \\ r\equiv\ell \pmod{2M}}} e^{2\pi i\frac{dr^2}{4Mc}}H_r^{(Mc)}(u;c\tau+d).$$

Proof. The following relations are either well known or straightforwardly proven:

1. For all $c \in \mathbb{N}$, we have

$$f_M(z,u;\tau) = \sum_{r \pmod{c}} q^{Mr^2} e^{4\pi i Mrz} f_{Mc}\left(z + r\tau, u; c\tau\right).$$

2. The function f_M is $\mathbb{Z}$-invariant in τ.

3. We have the following:

$$f_M\left(\frac{z}{\tau}, \frac{u}{\tau}; -\frac{1}{\tau}\right) = \tau e^{\frac{2\pi i M\left(z^2-u^2\right)}{\tau}} \left(f_M(z,u;\tau) - \sum_{r \pmod{2M}} H_r^{(M)}(u;\tau)\vartheta_{M,r}(z;\tau)\right).$$

Note that part 1 is related to Proposition 6 of [1], and that part 3 is proven in [36] Proposition 3.3. Using 1. and 2., we have

$$\begin{aligned}
f_M\left(\frac{z}{c\tau+d}, \frac{u}{c\tau+d}; \gamma\tau\right) &= \sum_{r \pmod{c}} e^{2\pi i\left(Mr^2\gamma\tau + \frac{2Mrz}{c\tau+d}\right)} f_{Mc}\left(\frac{z + r(a\tau+b)}{c\tau+d}, \frac{u}{c\tau+d}; c\gamma\tau\right)\\
&= \sum_{r \pmod{c}} e^{2\pi i\left(Mr^2\gamma\tau + \frac{2Mrz}{c\tau+d}\right)} f_{Mc}\left(\frac{z + r(a\tau+b)}{c\tau+d}, \frac{u}{c\tau+d}; -\frac{1}{c\tau+d}\right)
\end{aligned}$$

Then, using 3., we obtain

$$\begin{aligned}
\sum_{r \pmod{c}} e^{2\pi i\left(Mr^2\gamma\tau + \frac{2Mrz}{c\tau+d}\right)} e^{2\pi i \frac{Mc}{c\tau+d}\left((z+r(a\tau+b))^2-u^2\right)}(c\tau+d)\Big(f_{Mc}\left(z + r(a\tau+b), u; c\tau+d\right)\\
- \sum_{s \pmod{2Mc}} H_s^{(Mc)}(u;c\tau+d)\vartheta_{Mc,s}(z+r(a\tau+b);c\tau+d)\Big)
\end{aligned}$$

Since a and c are coprime, using again the first fact in the opposite direction, we obtain

$$\begin{aligned}
\sum_{r \pmod{c}} e^{2\pi i\left(Mr^2\gamma\tau + \frac{2Mrz}{c\tau+d} + \frac{Mc}{c\tau+d}\left((z+r(a\tau+b))^2-u^2\right)\right)} f_{Mc}\left(z + r(a\tau+b), u; c\tau+d\right)\\
= e^{\frac{Mc}{c\tau+d}\left(z^2-u^2\right)} f_M(z,u;\tau).
\end{aligned}$$

Therefore, we can write

$$
\begin{aligned}
&f_M\left(\frac{z}{c\tau+d}, \frac{u}{c\tau+d}; \gamma\tau\right)\\
&= (c\tau+d)e^{2\pi i\frac{Mc}{c\tau+d}\left(z^2-u^2\right)} f_M(z,u;\tau) - (c\tau+d)\sum_{r \pmod{c}} e^{2\pi i\left(Mr^2\tau+2Mrz+\frac{Mc}{c\tau+d}\left(z^2-u^2\right)\right)}\\
&\qquad\qquad\times \sum_{s \pmod{2Mc}} e^{2\pi i\frac{ds^2}{4Mc}} H_s^{(Mc)}(u;c\tau+d)\vartheta_{Mc,s}(z+r\tau;c\tau),
\end{aligned}
$$

which equals

$$
\begin{aligned}
f_M\left(\frac{z}{c\tau+d}, \frac{u}{c\tau+d}; \gamma\tau\right) &= (c\tau+d)e^{2\pi i\frac{Mc}{c\tau+d}\left(z^2-u^2\right)} f_M(z,u;\tau)\\
&\quad - (c\tau+d)e^{2\pi i\frac{Mc}{c\tau+d}\left(z^2-u^2\right)} \sum_{s \pmod{2Mc}} e^{2\pi i\frac{ds^2}{4Mc}} H_s^{(Mc)}(u;c\tau+d)\vartheta_{M,s}(z;\tau).
\end{aligned}
$$

On the other hand, we know that

$$
\begin{aligned}
&f_M\left(\frac{z}{c\tau+d}, \frac{u}{c\tau+d}; \gamma\tau\right)\\
&= \widehat{f}_M\left(\frac{z}{c\tau+d}, \frac{u}{c\tau+d}; \gamma\tau\right) + \frac{1}{2}\sum_{r \pmod{2M}} R_{M,r}\left(\frac{u}{c\tau+d}; \gamma\tau\right)\vartheta_{M,r}\left(\frac{z}{c\tau+d}; \gamma\tau\right)\\
&\qquad = (c\tau+d)e^{2\pi i\frac{Mc}{c\tau+d}\left(z^2-u^2\right)} \widehat{f}_M\left(z,u;\tau\right)\\
&\qquad\quad + \frac{1}{2}\sum_{r \pmod{2M}} R_{M,r}\left(\frac{u}{c\tau+d}; \gamma\tau\right)\vartheta_{M,r}\left(\frac{z}{c\tau+d}; \gamma\tau\right),
\end{aligned}
$$

where in the last step we have used the transformation properties of $\widehat{f}_M$. Comparing the two equalities above one can write

$$
\begin{aligned}
&\frac{1}{2}\sum_{r \pmod{2M}} R_{M,r}\left(u;\tau\right)\vartheta_{M,r}\left(z;\tau\right) - \sum_{s \pmod{2Mc}} e^{2\pi i\frac{ds^2}{4Mc}} H_s^{(Mc)}(u;c\tau+d)\vartheta_{M,s}(z;\tau)\\
&\quad = (c\tau+d)^{-1}e^{2\pi i\frac{Mc\left(u^2-z^2\right)}{c\tau+d}}\frac{1}{2}\sum_{r \pmod{2M}} R_{M,r}\left(\frac{u}{c\tau+d}; \gamma\tau\right)\vartheta_{M,r}\left(\frac{z}{c\tau+d}; \gamma\tau\right),
\end{aligned}
$$

which can be written as

$$\sum_{r \pmod{2M}} \left(\frac{1}{2} R_{M,r}\left(u;\tau\right) - \sum_{\substack{\ell \pmod{2Mc} \\ \ell \equiv r \pmod{2M}}} e^{2\pi i \frac{d\ell^2}{4Mc}} H_\ell^{(Mc)}(u; c\tau+d) \right) \vartheta_{M,r}(z;\tau)$$
$$= (c\tau+d)^{-\frac{1}{2}} e^{2\pi i \frac{Mcu^2}{c\tau+d}} \frac{1}{2} \sum_{r \pmod{2M}} R_{M,r}\left(\frac{u}{c\tau+d}; \gamma\tau\right) \sum_{\ell \pmod{2M}} \omega(\ell, r) \vartheta_{M,\ell}\left(z;\tau\right).$$

From the linear independence of the Jacobi theta functions $\vartheta_{M,\ell}$ one concludes the proof. □

From now on we consider the following meromorphic Jacobi forms:

$$\Phi(z;\tau) := \frac{\vartheta\left(z+\frac{1}{2};\tau\right)^m}{\vartheta\left(z;\tau\right)^n}, \tag{6.2.4}$$

$$\Psi(z;\tau) := \frac{\left(q^{\frac{1}{8}}\zeta^{\frac{1}{2}}\vartheta\left(z+\frac{\tau}{2};\tau\right)\right)^m}{\vartheta(z;\tau)^n}. \tag{6.2.5}$$

Note that Φ is the same function studied in Chapter 3. Moreover, the function (Φ, Ψ) is a vector valued Jacobi form for $\mathrm{SL}_2(\mathbb{Z})$. Due to the transformation properties of ϑ (see Proposition (2.3.2)), we deduce the transformation properties of Φ and Ψ, as described in the following proposition. Here, χ^*, $\widetilde{\chi}$ and $\widetilde{\chi}^*$ are certain characters that can be easily made explicit, although it is not necessary for our purposes.

Proposition 6.2.2. *The functions Φ and Ψ satisfy the following modular transformation laws:*

1. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, we have $\Phi\big|_{M,M}\gamma(z;\tau) = \chi^*(\gamma)\Phi(z;\tau)$.*
2. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(2)$, we have $\Psi\big|_{M,M}\gamma(z;\tau) = \widetilde{\chi}(\gamma)\Psi(z;\tau)$.*
3. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $a \equiv 0 \pmod 2$, we have $\Psi\big|_{M,M}\gamma(z;\tau) = \widetilde{\chi}^*(\gamma)\Phi(z;\tau)$.*

It is straightforward to show that Φ and Ψ satisfy also the elliptic transformation law of a Jacobi form, although it is not in the interest of this chapter. Since ϑ has a simple pole in $z = 0$, Φ and Ψ are meromorphic Jacobi forms with a unique pole of order n in $z = 0$. In particular, their Laurent coefficients and their canonical Fourier coefficients satisfies the same properties as the general function φ described before. In the following table we fix the notation for the canonical Fourier coefficients and Laurent coefficients of Φ and Ψ, as well as their completions.

	Φ	Ψ
canonical Fourier coeff.	h_ℓ	g_ℓ
completion of canonical Fourier coeff.	$\widehat{h}_\ell$	$\widehat{g}_\ell$
Laurent coeff.	$\widetilde{D}_{2j}$	$\widetilde{E}_{2j}$
completion of Laurent coeff.	D_{2j}	E_{2j}

Table 6.1: Fourier and Laurent coefficients

In light of Proposition 4.2.2, Proposition 4.3.3, and Proposition 6.2.2 we have the following.

Proposition 6.2.3. *With the notation as above the functions $\widehat{h}_\ell$ and D_{2j} satisfy the following transformation properties:*

1. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, we have*

$$\widehat{h}_\ell(\tau) = \frac{1}{\chi^*(\gamma)}(c\tau + d)^{\frac{1}{2}-M} \sum_{r \pmod{2M}} \omega(\ell, r)\widehat{h}_r(\gamma\tau).$$

2. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $a \equiv 0 \pmod 2$, we have*

$$\widehat{h}_\ell(\tau) = \frac{1}{\widetilde{\chi}^*(\gamma)}(c\tau + d)^{\frac{1}{2}-M} \sum_{r \pmod{2M}} \omega(\ell, r)\widehat{g}_r(\gamma\tau).$$

3. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, we have

$$D_{2j}(\tau) = \frac{1}{\chi^*(\gamma)}(c\tau+d)^{2j-M} D_{2j}(\gamma\tau).$$

4. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $a \equiv 0 \pmod 2$, we have

$$D_{2j}(\tau) = \frac{1}{\widetilde{\chi}^*(\gamma)}(c\tau+d)^{2j-M} E_{2j}(\gamma\tau).$$

Here the $\omega(\ell, r)$s are the same as in Proposition 6.2.1.

As a consequence, we can now derive explicitly the modularity of the Fourier coefficients h_ℓ under the action of the full modular group.

Proposition 6.2.4. *The functions h_ℓ satisfy the following modular transformation properties:*

1. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$

$$\begin{aligned} h_\ell(\tau) = {} & \frac{1}{\chi^*(\gamma)}(c\tau+d)^{\frac{1}{2}-M} \sum_{r \pmod{2M}} \omega(\ell, r) h_r(\gamma\tau) + \frac{1}{\chi^*(\gamma)} \sum_{j=1}^{\frac{n}{2}} \frac{(c\tau+d)^{2j-M}}{(2j-1)!} \\ & \times D_{2j}(\gamma\tau) \sum_{\substack{r \pmod{2Mc} \\ r \equiv \ell \pmod{2M}}} e^{2\pi i \frac{dr^2}{4Mc}} \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[e^{-\frac{M\pi\varepsilon^2}{v}} H_r^{(Mc)}(\varepsilon; c\tau+d) \right]_{\varepsilon=0}. \end{aligned}$$

2. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with a even

$$\begin{aligned} h_\ell(\tau) = {} & \frac{1}{\widetilde{\chi}^*(\gamma)}(c\tau+d)^{\frac{1}{2}-M} \sum_{r \pmod{2M}} \omega(\ell, r) g_r(\gamma\tau) + \frac{1}{\widetilde{\chi}^*(\gamma)} \sum_{j=1}^{\frac{n}{2}} \frac{(c\tau+d)^{2j-M}}{(2j-1)!} \\ & \times E_{2j}(\gamma\tau) \sum_{\substack{r \pmod{2Mc} \\ r \equiv \ell \pmod{2M}}} e^{2\pi i \frac{dr^2}{4Mc}} \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[e^{-\frac{M\pi\varepsilon^2}{v}} H_r^{(Mc)}(\varepsilon; c\tau+d) \right]_{\varepsilon=0}. \end{aligned}$$

Proof. Here we prove part 1. The proof of part 2 is very similar. Writing h_ℓ according to (6.2.3) and using Proposition 6.2.3, we have that

$$h_\ell(\tau) = \frac{(c\tau+d)^{\frac{1}{2}-M}}{\chi^*(\gamma)} \sum_{r \pmod{2M}} \omega(\ell,r)\widehat{h}_r(\gamma\tau) \\ - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} \frac{(c\tau+d)^{2j-M}}{\chi^*(\gamma)} \frac{D_{2j}(\gamma\tau)}{(2j-1)!} \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[R_{M,\ell}(\varepsilon;\tau)e^{-\frac{M\pi\varepsilon^2}{v}}\right]_{\varepsilon=0}.$$

Using again (6.2.3), the expression above equals

$$\frac{(c\tau+d)^{\frac{1}{2}-M}}{\chi^*(\gamma)} \sum_{r \pmod{2M}} \omega(\ell,r)\Bigg(h_\ell(\gamma\tau) \\ + \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} \frac{D_{2j}(\gamma\tau)}{(2j-1)!} \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[R_{N,\ell}(\varepsilon;\gamma\tau)e^{-\frac{N\pi\varepsilon^2}{\mathrm{Im}(\gamma\tau)}}\right]_{\varepsilon=0}\Bigg) \\ - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} \frac{(c\tau+d)^{2j-M}}{\chi^*(\gamma)} \frac{D_{2j}(\gamma\tau)}{(2j-1)!} \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[R_{M,\ell}(\varepsilon;\tau)e^{-\frac{M\pi\varepsilon^2}{v}}\right]_{\varepsilon=0}.$$

To conclude, we rewrite $R_{M,\ell}$ in the last summand according to Proposition 6.2.1, and we note that the contribution of $R_{M,\ell}$ from the first and the second summands cancel. This can be also argued considering the non-holomorphicity of $R_{M,\ell}$, being $h_\ell(\tau)$ holomorphic. $\square$

We now have all the ingredients needed to describe the transformation property of the Kac-Wakimoto characters. As alluded to in Section 3.1, the generating function for Kac-Wakimoto characters (6.1.2) is essentially the meromorphic Jacobi form $\Phi\left(z+\frac{\tau}{2};\tau\right)$. More precisely, one can see that for each $\ell \in \mathbb{Z}$ the ℓ-th Kac-Wakimoto character is given by

$$\mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0} = h_{\ell-M}(\tau) q^{\frac{\ell^2}{4M}} \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{2M-1}. \tag{6.2.6}$$

The transformation of Kac-Wakimoto character arise directly from Proposition 6.2.4 and Lemma 2.2.2.

Proposition 6.2.5. *The Kac-Wakimoto character $tr_{L_{m,n}(\Lambda(\ell))} q^{L_0}$ satisfies the following modular transformation laws:*

1. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$*

$$tr_{L_{m,n}(\Lambda(\ell))} q^{L_0} = q^{\frac{\ell^2}{4M} + \frac{2M-1}{24}} \frac{\psi(\gamma)^{2M-1}}{\widetilde{\chi}^*(\gamma)} \sum_{r \pmod{2M}} \omega(\ell - M, r) \frac{h_r(\gamma\tau)}{\eta(\gamma\tau)^{2M-1}}$$
$$+ q^{\frac{\ell^2}{4M} + \frac{2M-1}{24}} \frac{\psi(\gamma)^{2M-1}}{\chi^*(\gamma)} \sum_{j=1}^{\frac{n}{2}} \frac{(c\tau + d)^{2j-\frac{1}{2}}}{(2j-1)!} \frac{D_{2j}(\gamma\tau)}{\eta(\gamma\tau)^{2M-1}}$$
$$\times \sum_{\substack{r \pmod{2Mc} \\ r \equiv \ell - M \pmod{2M}}} e^{2\pi i \frac{dr^2}{4Mc}} \left(\frac{\partial_\varepsilon}{2\pi i} \right)^{2j-1} \left[e^{-\frac{M\pi\varepsilon^2}{v}} H_r^{(Mc)}(\varepsilon; c\tau + d) \right]_{\varepsilon=0}.$$

2. *For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with a even*

$$tr_{L_{m,n}(\Lambda(\ell))} q^{L_0} = q^{\frac{\ell^2}{4M} + \frac{2M-1}{24}} \frac{\psi(\gamma)^{2M-1}}{\widetilde{\chi}^*(\gamma)} \sum_{r \pmod{2M}} \omega(\ell - M, r) \frac{g_r(\gamma\tau)}{\eta(\gamma\tau)^{2M-1}}$$
$$+ q^{\frac{\ell^2}{4M} + \frac{2M-1}{24}} \frac{\psi(\gamma)^{2M-1}}{\widetilde{\chi}^*(\gamma)} \sum_{j=1}^{\frac{n}{2}} \frac{(c\tau + d)^{2j-\frac{1}{2}}}{(2j-1)!} \frac{E_{2j}(\gamma\tau)}{\eta(\gamma\tau)^{2M-1}}$$
$$\times \sum_{\substack{r \pmod{2Mc} \\ r \equiv \ell - M \pmod{2M}}} e^{2\pi i \frac{dr^2}{4Mc}} \left(\frac{\partial_\varepsilon}{2\pi i} \right)^{2j-1} \left[e^{-\frac{M\pi\varepsilon^2}{v}} H_r^{(Mc)}(\varepsilon; c\tau + d) \right]_{\varepsilon=0}.$$

6.2.2 The Circle Method

In this section we briefly recall the Circle Method, we fix the notation, and we apply it to Kac-Wakimoto characters. This will give us four main terms, which we analyze separately in the following sections.

Given a holomorphic modular form f, the Circle Method allows to understand the asymptotic behavior of its Fourier coefficients. Roughly speaking, we can say that it is a "nice" parametrization of the Cauchy integral of f, that allows an optimal approximation of the Fourier coefficients of f due to its modular properties. For a classical description of this method we refer the reader to the primary work of G. Hardy and S. Ramanujan [18], and to the revisitation of H. Iwaniec and E. Kowalski in Chapter 20 of [19].

6.2.3 Notation

To explain the Circle Method, we need some definitions and notation. All the material in this section can be found in Chapter 20 of [19]. For a positive integer N consider the *Farey series of order* N defined by

$$F_N := \left\{ \frac{h}{k} : \gcd(h,k) = 1,\ 0 \leq h < k \leq N \right\}.$$

It is a standard fact that the Farey series can be used to split the unit interval into disjoint subintervals as follows: Given three consecutive elements in F_N

$$\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2},$$

one defines $\vartheta'_{h,k} := \frac{1}{k(k+k_1)}$ and $\vartheta''_{h,k} := \frac{1}{k(k+k_2)}$. It is straightforward to check that the intervals $\left[\frac{h}{k} - \vartheta'_{h,k},\ \frac{h}{k} + \vartheta''_{h,k}\right)$ are all disjoint as $\frac{h}{k}$ runs through the elements of the Farey series. Furthermore, the union of these intervals gives the whole interval $(0,1)$. Furthermore, it is well known that for $j \in \{1,2\}$ and $N \in \mathbb{N}$, the following facts hold:

1. $N - k < k_j \leq N$;

2. $hk_j \equiv (-1)^{j+1} \pmod{k}$;

3. $\frac{1}{2Nk} < \vartheta'_{h,k}, \vartheta''_{h,k} < \frac{1}{Nk}$.

For a fixed $\frac{h}{k} \in F_N$, we define the variable

$$\tau_{h,k} := \frac{h}{k} + \frac{iz}{k},$$

where $z \in \mathbb{C}$ is a complex variable that satisfies $\mathrm{Re}(z) > 0$. This clearly implies $\tau_{h,k} \in \mathbb{H}$. Let $[a]_b$ denotes the inverse of a modulo b. Then for each element $\frac{h}{k} \in F_N$ we choose $[-h]_k$ to have parity opposite to that of k. We point out that this is always possible. Indeed, from the properties above we can choose $[-h]_k = k_2$ if k is even, and either $[-h]_k = k_2$ or $[-h]_k = k_2 + k$ if k is odd. With these choices, the matrix

$$\gamma_{h,k} := \begin{pmatrix} [-h]_k & -\frac{[-h]_k h+1}{k} \\ k & -h \end{pmatrix}$$

lies in $\mathrm{SL}_2(\mathbb{Z})$. We define the image of $\tau_{h,k}$ under $\gamma_{h,k}$ by

$$\widetilde{\tau}_{h,k} := \gamma_{h,k}\tau_{h,k} = \frac{[-h]_k}{k} + \frac{i}{zk}.$$

Note that the automorphy factor $k\tau_{h,k} - h$ is equal to iz. Finally, we define $q_{h,k} := e^{2\pi i\tau_{h,k}}$ and $\widetilde{q}_{h,k} := e^{2\pi i\widetilde{\tau}_{h,k}}$.

6.2.4 The Circle Method

Applying Cauchy's Theorem to (6.1.2), we obtain

$$c_\ell(t) = \frac{1}{2\pi i}\int_{|q|=r} \mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0} q^{-\frac{\ell}{2}} q^{-(t+1)}\,\mathrm{d}q, \tag{6.2.7}$$

where r is any fixed real number in $(0,1)$. For our purposes the best choice for r turns out to be

$$r := e^{-\frac{2\pi}{N^2}},$$

where $N := \lfloor t^{1/2} \rfloor$. Parametrizing the path of integration as

$$u \mapsto e^{-\frac{2\pi}{N^2}+2\pi i u},$$

with $u \in [0,1]$, and using the partition of the unit interval as described in the previous subsection, one can rewrite (6.2.7) as

$$c_\ell(t) = \sum_{\frac{h}{k}\in F_N} \int_{\frac{h}{k}-\vartheta'_{h,k}}^{\frac{h}{k}+\vartheta''_{h,k}} \mathrm{tr}_{L_{m,n}(\Lambda(\ell))} q^{L_0} q^{-\left(t+\frac{\ell}{2}\right)}\,\mathrm{d}u.$$

In the integral above, note that τ depends on u. Finally, we rewrite each integral as

$$c_\ell(t) = \sum_{\frac{h}{k}\in F_N} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \operatorname{tr}_{L_{m,n}(\Lambda(\ell))} q_{h,k}^{L_0} q_{h,k}^{-\left(t+\frac{\ell}{2}\right)} \,\mathrm{d}\phi, \tag{6.2.8}$$

where we have defined the variables z and ϕ as

$$\begin{aligned}
\tau = \tau_{h,k} = u + \frac{i}{N^2} &=: \frac{h}{k} + \frac{iz}{k}, \\
\phi = u - \frac{h}{k} &= \frac{iz}{k} - \frac{i}{N^2}.
\end{aligned}$$

Using Proposition 6.2.5, we split $c_\ell(t)$ into four pieces, splitting the Mordell integral from the q-series and splitting the Farey series based on the parity of k. Before giving the splitting, we define the following characters:

$$\begin{aligned}
\chi_r(\gamma) &:= \frac{\psi(\gamma)^{2M-1}}{\chi^*(\gamma)}\omega(\ell-M,r) \\
\widetilde{\chi}_r(\gamma) &:= \frac{\psi(\gamma)^{2M-1}}{\widetilde{\chi}^*(\gamma)}\omega(\ell-M,r) \\
\chi_r^*(\gamma) &:= \frac{\psi(\gamma)^{2M-1}}{\chi^*(\gamma)}e^{2\pi i\frac{dr^2}{4Mc}} \\
\widetilde{\chi}_r^*(\gamma) &:= \frac{\psi(\gamma)^{2M-1}}{\widetilde{\chi}^*(\gamma)}e^{2\pi i\frac{dr^2}{4Mc}}
\end{aligned}$$

We obtain

$$c_\ell(t) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,$$

where

$$\Sigma_1 := \sum_{\substack{\frac{h}{k}\in F_N \\ 2|k}} \sum_{r \pmod{2M}} \chi_r(\gamma_{h,k}) \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{h_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}} q_{h,k}^{\varepsilon(\ell)-t}\, \mathrm{d}\phi, \tag{6.2.9}$$

$$\Sigma_2 := \sum_{\substack{\frac{h}{k}\in F_N \\ 2\nmid k}} \sum_{r \pmod{2M}} \widetilde{\chi}_r(\gamma_{h,k}) \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{g_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}} q_{h,k}^{\varepsilon(\ell)-t}\, \mathrm{d}\phi,$$

$$\Sigma_3 := \sum_{\substack{\frac{h}{k}\in F_N \\ 2|k}} \sum_{j=1}^{\frac{n}{2}} \sum_{\substack{r \pmod{2Mk} \\ r\equiv \ell-M \pmod{2M}}} \frac{\chi_r^*(\gamma_{h,k})}{(2j-1)!} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} (iz)^{2j-\frac{1}{2}} q_{h,k}^{\varepsilon(\ell)-t} \frac{D_{2j}(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}}$$
$$\times \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[e^{-M\pi\varepsilon^2N^2} H_r^{(Mk)}(\varepsilon; iz)\right]_{\varepsilon=0} \mathrm{d}\phi,$$

$$\Sigma_4 := \sum_{\substack{\frac{h}{k}\in F_N \\ 2\nmid k}} \sum_{j=1}^{\frac{n}{2}} \sum_{\substack{r \pmod{2Mk} \\ r\equiv \ell-M \pmod{2M}}} \frac{\widetilde{\chi}_r^*(\gamma_{h,k})}{(2j-1)!} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} (iz)^{2j-\frac{1}{2}} q_{h,k}^{\varepsilon(\ell)-t} \frac{E_{2j}(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}}$$
$$\times \left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[e^{-M\pi\varepsilon^2N^2} H_r^{(Mk)}(\varepsilon; iz)\right]_{\varepsilon=0} \mathrm{d}\phi.$$

We recall that $\varepsilon(\ell)$ was defined in (6.1.3).

Remark. Note that the formula representing the contribution arising from the case of k even is the same as the one associated to the case of k odd, up to the term in $\widetilde{\tau}_{h,k}$. In particular, these terms are $\widetilde{q}_{h,k}$-series. Since

$$|\widetilde{q}_{h,k}| \leq e^{-\pi},$$

the main contribution for our final bound will come from the term that contains the smaller $\widetilde{q}_{h,k}$-power. We shall see that this occurs in the case of k odd, so we are mainly interested in Σ_2 and Σ_4. In fact, from [9], we may expect that the main contribution will comes from Σ_4.

6.3 Proof of the main results

6.3.1 The principal parts

In this section, we explicitly compute the principal parts of the canonical Fourier coefficients $\frac{h_r(\tau)}{\eta(\tau)^{2M-1}}$ and $\frac{g_r(\tau)}{\eta(\tau)^{2M-1}}$, and those of the Laurent coefficients $\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}}$ and $\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}}$.

We start by computing the q-expansion of $\frac{h_r(\tau)}{\eta(\tau)^{2M-1}}$ and $\frac{g_r(\tau)}{\eta(\tau)^{2M-1}}$. In the following proposition we calculate the q-expansion of the numerator.

Proposition 6.3.1. *Let the h_r's and the g_r's be as in Table 6.1. For each $r \in \mathbb{Z}$, let $\widehat{r} \in (-M, M]$ and $\widetilde{r} \in (-2M, 0]$ be congruent to r modulo $2M$. Then,*

$$h_r(\tau) = q^{\frac{M}{4} - \frac{\widehat{r}^2}{4M}} \sum_{s \geq 0} Q_r^*(s) q^s$$

and

$$g_r(\tau) = q^{\frac{m}{8} - \frac{(\widetilde{r}+M)^2}{4M}} \sum_{s \geq 0} R_r^*(s) q^{\frac{s}{2}},$$

for certain complex numbers $Q_r^*(s)$ and $R_r^*(s)$.

Proof. By definition

$$h_r(\tau) = q^{-\frac{r^2}{4M}} \int_{-\frac{r\tau}{2M}}^{-\frac{r\tau}{2M}+1} \frac{\vartheta\left(z + \frac{1}{2}; \tau\right)^m}{\vartheta(z;\tau)^n} \zeta^{-r} \, \mathrm{d}z.$$

The integrand is a power series in ζ whose coefficients are functions of q. In particular, the only term of this power series that gives a non-zero contribution to the integral is the coefficient of ζ^0. Since our goal is to understand the minimal power of q appearing in the expansion of h_r, it is enough to find the minimal power of q in the coefficient of ζ^0 in the ζ-expansion of the integrand. In order to do so, we rewrite it using the Jacobi's triple product identity (see Proposition 2.3.2) obtaining

$$\zeta^{-M-r} q^{\frac{M}{4}} \frac{\prod_{\lambda \geq 1} \left(1 + \zeta q^{\lambda - 1}\right)^m \left(1 + \zeta^{-1} q^{\lambda}\right)^m}{\prod_{\lambda \geq 1} \left(1 - \zeta q^{\lambda-1}\right)^n \left(1 - \zeta^{-1} q^{\lambda}\right)^n}.$$

To write the denominator as a power series, using the geometric expansion, we need both $|\zeta| < 1$ and $|\zeta^{-1}q| < 1$. Since inside the integral $\mathrm{Im}(z) = -\frac{r}{2M}\mathrm{Im}(\tau)$, this is equivalent of requiring $-2M < r < 0$. We are allowed to make this choice since h_r depends just on r modulo $2M$. By symmetry, to obtain the minimal power of q, we can assume $\lambda = 1$. In particular, we look for the minimal power of q in the coefficient of ζ^0 in

$$q^{\frac{M}{4}}\zeta^{-(M+r)}\left(1+\zeta\right)^m\left(1+\zeta^{-1}q\right)^m\left(\sum_{\lambda\geq 0}\zeta^{\lambda}\right)^n\left(\sum_{\lambda\geq 0}\left(\zeta^{-1}q\right)^{\lambda}\right)^n.$$

It is easy to check that if $r + M \geq 0$ (i.e., $r = \widehat{r}$) the minimal power of q is $q^{\frac{M}{4}}$, since the term ζ^0 appears in the expansion of $\zeta^{-(r+M)}\left(1+\zeta\right)^m$. If $r + M < 0$, then to cancel $\zeta^{-(r+M)}$ one requires the expansion of $\left(1+\zeta^{-1}q\right)^m$, giving the extra term $q^{-(r+M)}$. Thus, the minimal power of q in h_r is

$$q^{-\frac{r^2}{4M}+\frac{M}{4}-(r+M)} = q^{\frac{M}{4}-\frac{\widehat{r}^2}{4M}}.$$

The computation for g_r is similar. For the seek of completeness we give the entire proof also in this case. Using again the Jacobi's triple product identity, we obtain

$$g_r(\tau) = q^{-\frac{r^2}{4M}}\int_w^{w+1} q^{-\frac{n}{8}}\zeta^{\frac{n}{2}-r}\frac{\prod_{\lambda\geq 1}\left(1-\zeta q^{\lambda-\frac{1}{2}}\right)^m\left(1-\zeta^{-1}q^{\lambda-\frac{1}{2}}\right)^m}{\prod_{\lambda\geq 1}\left(1-\zeta q^{\lambda}\right)^n\left(1-\zeta^{-1}q^{\lambda}\right)^n}dz.$$

In this case, we may also assume $r \in (-2M, 0]$ in order to write the denominator in power series. Arguing as above and noting that $\frac{n}{2} - r > 0$, it turns out that the minimal power of q in the coefficient of ζ^0 is the minimal power of q in the coefficient of $\zeta^{r-\frac{n}{2}}$ of

$$q^{-\frac{r^2}{4M}-\frac{n}{8}}\left(1-\zeta^{-1}q^{\frac{1}{2}}\right)^m\left(\sum_{\lambda\geq 0}\left(\zeta^{-1}q\right)^{\lambda}\right)^n.$$

Since $m > \frac{n}{2} - r$, the minimal power of q is

$$q^{-\frac{r^2}{4M}-\frac{n}{8}}q^{\frac{n}{4}-\frac{r}{2}} = q^{\frac{m}{8}-\frac{(r+M)^2}{4M}}.$$

□

We have computed the q-expansion of the numerator of $\frac{h_r(\tau)}{\eta(\tau)^{2M-1}}$ and $\frac{g_r(\tau)}{\eta(\tau)^{2M-1}}$. From the definition of η, we can easily compute the principal part of the entire functions. For convenience of notation, we define the numbers

$$\delta_h(r) := -\frac{M}{4} + \frac{\widehat{r}^2}{4M} + \frac{2M-1}{24},$$
$$\delta_g(r) := -\frac{m}{8} + \frac{\left(\widetilde{r}+M\right)^2}{4M} + \frac{2M-1}{24},$$

where $\widehat{r}$ and $\widetilde{r}$ are as in Proposition 6.3.1.

Corollary 6.3.2. *The functions* $\frac{h_r(\tau)}{\eta(\tau)^{2M-1}}$ *and* $\frac{g_r(\tau)}{\eta(\tau)^{2M-1}}$ *have the following q–expansion:*

$$\frac{h_r(\tau)}{\eta(\tau)^{2M-1}} = q^{-\delta_h(r)} \sum_{s\geq 0} Q_r(s) q^{\frac{s}{2}}$$

and

$$\frac{g_r(\tau)}{\eta(\tau)^{2M-1}} = q^{-\delta_g(r)} \sum_{s\geq 0} R_r(s) q^{\frac{s}{2}},$$

for some complex numbers $Q_r(s)$ and $R_r(s)$.

In the remainder of the section, we compute the principal part of the Laurent coefficients of Φ and Ψ. For convenience of notation, we define the constants

$$\delta_D := -\frac{M}{6} - \frac{1}{24}$$
$$\delta_E := \frac{n}{8} + \frac{2M-1}{24}.$$

Proposition 6.3.3. *The functions* $\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}}$ *and* $\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}}$ *have the following q-expansion:*

$$\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}} = q^{-\delta_D} \sum_{0\leq r\leq \frac{n-2j}{2}} \frac{(-1)^r}{r!} \left(\frac{M}{4\pi v}\right)^r \sum_{s\geq 0} V_{2j+2r}(s) q^{\frac{s}{2}},$$

$$\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}} = q^{-\delta_E} \sum_{0\leq r\leq \frac{n-2j}{2}} \frac{(-1)^{r+M}}{r!} \left(\frac{M}{4\pi v}\right)^r \sum_{s\geq 0} T_{2j+2r}(s) q^{\frac{s}{2}},$$

for some complex numbers $V_{2j+2r}(s)$ and $T_{2j+2r}(s)$.

Proof. We only prove the result for $\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}}$, since the other case is analogous. If we define the function

$$\vartheta^*(z;\tau) := \frac{\vartheta(z;\tau)}{z},$$

and we denote by $\vartheta^{(2\nu)}$ the derivative of ϑ with respect to z (respectively for ϑ^*), then we can write

$$\begin{aligned}\Phi(z;\tau) =& \frac{\left(\vartheta\left(\frac{1}{2};\tau\right) + \vartheta^{(2)}\left(\frac{1}{2};\tau\right)\frac{z^2}{2} + \cdots + O(z^n)\right)^m}{z^n\left(\vartheta^*\left(0;\tau\right) + \vartheta^{*(2)}\left(0;\tau\right)\frac{z^2}{2} + \cdots + O(z^n)\right)^n}\\ =& \frac{\vartheta\left(\frac{1}{2};\tau\right)^m}{z^n\vartheta^*\left(0;\tau\right)^n}\frac{\left(1 + \frac{\vartheta^{(2)}\left(\frac{1}{2};\tau\right)}{\vartheta\left(\frac{1}{2};\tau\right)}\frac{z^2}{2} + \cdots + O(z^n)\right)^m}{\left(1 + \frac{\vartheta^{*(2)}(0;\tau)}{\vartheta^*(0;\tau)}\frac{z^2}{2} + \cdots + O(z^n)\right)^n}\\ =& \frac{\vartheta\left(\frac{1}{2};\tau\right)^m}{z^n\vartheta^*\left(0;\tau\right)^n}\left(1 + \sum_{j\geq 1} d_{2j}(\tau)z^{2j}\right),\end{aligned}$$

for some coefficients $d_{2j}(\tau)$. It is a standard fact that

$$\vartheta\left(\frac{1}{2};\tau\right) = -2\frac{\eta(2\tau)^2}{\eta(\tau)}$$

and

$$\vartheta^*\left(0;\tau\right) = \vartheta^{(1)}(0;\tau) = -2\pi\eta(\tau)^3.$$

In particular, the Laurent coefficients $\widetilde{D}_{2j}$ of Φ in $z=0$ can be written as

$$\widetilde{D}_n(\tau) = 2^m i^n\frac{\eta(2\tau)^{2m}}{\eta(\tau)^{m+3n}}$$

and for $1 \leq j \leq \frac{n-2}{2}$

$$\widetilde{D}_{n-2j}(\tau) = 2^{m-2j}\pi^{-2j}i^{n-2j}\frac{\eta(2\tau)^{2m}}{\eta(\tau)^{m+3n}}d_{2j}(\tau).$$

By construction the functions d_{2j} are combination (sum and multiplication) of $\frac{\vartheta^{(2\nu)}\left(\frac{1}{2};\tau\right)}{\vartheta\left(\frac{1}{2};\tau\right)}$ and $\frac{\vartheta^{*(2\nu)}(0;\tau)}{\vartheta^*(0;\tau)}$, for arbitrary $\nu \in \mathbb{Z}$. Therefore, they are q-series without principal part. It follows that

$$\widetilde{D}_{n-2j}(\tau) = q^{\frac{M}{4}}\sum_{s\geq 0} U_{n-2j}(s)q^{\frac{s}{2}},$$

for certain constants $U_{n-2j}(s)$. To conclude, it is enough to note that

$$\begin{aligned} D_{2j}(\tau) = &\sum_{0\le r\le \frac{n-2j}{2}} \frac{(-1)^r}{r!}\left(\frac{M}{4\pi v}\right)^r \widetilde{D}_{2r+2j}(\tau) \\ =& q^{\frac{M}{4}} \sum_{0\le r\le \frac{n-2j}{2}} \frac{(-1)^r}{r!}\left(\frac{M}{4\pi v}\right)^r \sum_{s\ge 0} U_{2r+2j}(s) q^{\frac{s}{2}}, \end{aligned}$$

and so

$$\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}} = q^{\frac{M}{6}+\frac{1}{24}} \sum_{0\le r\le \frac{n-2j}{2}} \frac{(-1)^r}{r!}\left(\frac{M}{4\pi v}\right)^r \sum_{s\ge 0} V_{2r+2j}(s) q^{\frac{s}{2}},$$

for certain constants $V_j(s)$. □

6.3.2 The holomorphic part

In this section, we give an asymptotic estimate of the terms Σ_1 and Σ_2 defined in (6.2.9), using the Circle Method.

6.3.2.1 The even case

We start by studying Σ_1, which we recall to be defined by

$$\Sigma_1 = \sum_{\substack{\frac{h}{k}\in F_N \\ 2|k}} \sum_{r=0}^{2M-1} \chi_r(\gamma_{h,k}) \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{h_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}} q_{h,k}^{\varepsilon(\ell)-t}\, \mathrm{d}\phi.$$

Remark 3. We split the $\widetilde{q}_{h,k}$-series $\frac{h_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}}$ into two pieces, accordingly to the sign of the exponent of $\widetilde{q}_{h,k}$. The non-principal part is uniformly bounded over the outer sum, since $|\widetilde{q}_{h,k}| \le e^{-\pi}$. As a consequence, its contribution is smaller than

$$\sum_{\substack{\frac{h}{k}\in F_N \\ 2|k}} \sum_{r=0}^{2M-1} \chi_r(\gamma_{h,k}) \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}(t-\varepsilon(\ell))} d\phi. \tag{6.3.1}$$

Since $\mathrm{Re}(z) = \frac{k}{t}$, the integrand is bounded, and since $\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \mathrm{d}\phi \le \frac{2}{\sqrt{t}k}$ (6.3.1) can be bounded again by

$$\sum_{\substack{0\le k<\sqrt{t}\\ 2|k}}\ \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \frac{1}{\sqrt{t}k} \le 1,$$

where we have used known inequalities involving Euler's totient function. In particular, the non-principal part gives a contribution bounded by $O(1)$.

Next we compute the contribution of the principal part. Defining the Kloosterman-type sum $K_{k,r}(\alpha,\beta)$ by

$$K_{k,r}(\alpha,\beta) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \chi_r(\gamma_{h,k}) e^{2\pi i \frac{h\varepsilon(\ell)}{k}} e^{\frac{2\pi i}{k}(\alpha h+\beta[-h]_k)}, \tag{6.3.2}$$

and writing the principal part of $\frac{h_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}}$ as

$$\sum_{\substack{s\in\delta_h(r)+\mathbb{Z}\\ 0<s\le\delta_h(r)}} Q_r(\delta_h(r)-s)\widetilde{q}_{h,k}^{\,-s},$$

we can rewrite Σ_1 as

$$\Sigma_1 = \sum_{\substack{0<k\le\sqrt{t}\\ 2|k}} \sum_{r=0}^{2M-1} \sum_{\substack{s\in\delta_h(r)+\mathbb{Z}\\ 0<s\le\delta_h(r)}} Q_r(\delta_h(r)-s)K_{k,r}(-t,s)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k}\left(z(t-\varepsilon(\ell))+\frac{s}{2z}\right)} d\phi + O(1).$$

We finally recall the well known integral evaluation as $N\to+\infty$ [31]:

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k}\left(za+\frac{b}{z}\right)}\,\mathrm{d}\phi = \frac{2\pi}{k}\sqrt{\frac{b}{a}}I_1\left(\frac{4\pi}{k}\sqrt{ab}\right) + O\left(\frac{1}{Nk}\right),$$

where $a,\ b>0$, $\frac{h}{k}\in F_N$, and where I_1 is the level 1 modified Bessel function. The error term coming from the integral evaluation gives a contribution $O(1)$ to Σ_1 (it is a similar computation as in Remark 3). As a consequence, we conclude that

$$\Sigma_1 = \frac{2\pi}{\sqrt{t-\varepsilon(\ell)}} \sum_{\substack{0<k\le\sqrt{t}\\ 2|k}} \sum_{r=0}^{2M-1}$$
$$\sum_{\substack{s\in\delta_h(r)+\mathbb{Z}\\ 0<s\le\delta_h(r)}} Q_r(\delta_h(r)-s)K_{k,r}(-t,s)\frac{\sqrt{s}}{\sqrt{2}k} I_1\left(\frac{4\pi}{k}\sqrt{\frac{s}{2}(t-\varepsilon(\ell))}\right) + O(1). \tag{6.3.3}$$

6.3.2.2 The odd case

The analysis in the odd case is exactly the same as in the even case. Defining

$$\widetilde{K}_{k,r}(\alpha,\beta) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \widetilde{\chi}_r(\gamma_{h,k}) e^{2\pi i \frac{h\varepsilon(\ell)}{k}} e^{\frac{2\pi i}{k}\left(\alpha h + \frac{\beta}{2}[-h]_k\right)}, \tag{6.3.4}$$

replacing $\delta_h(r)$ by $\delta_g(r)$, and denoting the principal part of $\frac{g_r(\widetilde{\tau}_{h,k})}{\eta(\widetilde{\tau}_{h,k})^{2M-1}}$ as

$$\sum_{\substack{s\in 2\delta_g(r)+\mathbb{Z}\\ 0<s\le 2\delta_g(r)}} R_r(\delta_g(r)-s)\widetilde{q}_{h,k}^{\,-\frac{s}{2}},$$

one obtains

$$\Sigma_2 = \frac{2\pi}{\sqrt{t-\varepsilon(\ell)}} \sum_{\substack{0<k\le\sqrt{t}\\ 2\nmid k}} \sum_{r=0}^{2M-1}$$
$$\sum_{\substack{s\in 2\delta_g(r)+\mathbb{Z}\\ 0<s\le 2\delta_g(r)}} R_r(\delta_g(r)-s)\widetilde{K}_{k,r}(-t,s)\frac{\sqrt{s}}{\sqrt{2}k} I_1\left(\frac{4\pi}{k}\sqrt{\frac{s}{2}\left(t-\varepsilon(\ell)\right)}\right) + O(1). \tag{6.3.5}$$

6.3.3 The non-holomorphic part

In this section, we determine an asymptotic estimate for Σ_3 and Σ_4 (6.2.9).

6.3.3.1 The odd case

We begin by rewriting explicitly the derivatives of the Mordell integral.

Lemma 6.3.4. *With the notation as above, the following equality holds:*

$$\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1} \left[e^{-\pi M\varepsilon^2 N^2} H_r^{(Mk)}(\varepsilon; iz)\right]_{\varepsilon=0} = \sum_{\nu=1}^{j}\sum_{\mu=0}^{\nu-1} D(j,\nu,\mu)\frac{N^{2(j-\nu)}k^{\mu-\frac{1}{2}}}{z^{2\nu-\mu-\frac{1}{2}}} e^{2\pi i\frac{r}{4Mk}} e^{\frac{\pi}{8Mkz}}$$
$$\times \int_{\mathbb{R}} \frac{(1+2i\sigma)^{2(\nu-\mu)-1} e^{-\frac{\pi\sigma^2}{2Mkz}-2\pi\sigma\left(\frac{r}{2Mk}+\frac{1}{2}+\frac{1}{4Mkiz}\right)}}{\cosh(\pi\sigma)}\, d\sigma,$$

where
$$D(j,\nu,\mu) := \frac{(2j-1)!}{(2(\nu-\mu)-1)!(j-\nu)!\mu!}\frac{M^{j-\nu+\mu-\frac{1}{2}}i^{3-2\nu}}{\pi^{j-\nu+\mu}2^{2j-\mu+\frac{1}{2}}}.$$

Proof. By definition
$$H_{\ell-M}^{(M)}(z;\tau) = \frac{i}{2}q^{-\frac{\ell^2}{4M}}e^{-2\pi iz\ell}H\left(2Mz+\tau\ell+\frac{1}{2};2M\tau\right),$$
where H is the Mordell integral defined in (5.3.9).

Using the transformation (see Proposition 1.2 of [36])
$$H\left(z;\tau\right) = \frac{1}{\sqrt{-i\tau}}e^{\pi i\frac{z^2}{\tau}}H\left(\frac{z}{\tau};-\frac{1}{\tau}\right),$$
one can write
$$H_{\ell-M}^{(M)}(z;\tau) = \frac{i}{2\sqrt{-2iM\tau}}e^{2\pi i\frac{\left(2Mz+\frac{1}{2}\right)^2}{4M\tau}}e^{2\pi i\frac{\ell}{4M}}H\left(\frac{z}{\tau}+\frac{\ell}{2M}+\frac{1}{4M\tau};-\frac{1}{2M\tau}\right).$$
Replacing τ by iz, z by ε and M by Mk, we obtain
$$H_{\ell-Mk}^{(Mk)}\left(\varepsilon;iz\right) = \frac{i}{2\sqrt{2Mkz}}e^{2\pi i\frac{\ell}{4Mk}+\frac{\pi}{2Mkz}\left(2Mk\varepsilon+\frac{1}{2}\right)^2}H\left(\frac{\varepsilon}{iz}+\frac{\ell}{2Mk}+\frac{1}{4Mkiz};-\frac{1}{2Mkiz}\right).$$
On the other hand, a direct computation gives
$$\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2(j-\nu)}\left[e^{-\pi Mt\varepsilon^2}\right]_{\varepsilon=0} = \left(\frac{Mt}{4\pi}\right)^{j-\nu}\frac{(2(j-\nu))!}{(j-\nu)!},$$
and
$$\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2\nu-1}\left[e^{\frac{2Mk\pi}{z}\varepsilon^2+\frac{\pi\varepsilon}{z}(1+2ix)}\right]_{\varepsilon=0} = \sum_{\mu=0}^{\nu-1}\binom{2\nu-1}{2\mu}\frac{(2\mu)!}{\mu!}\left(-\frac{Mk}{2\pi z}\right)^{\mu}\left(\frac{1+2ix}{2iz}\right)^{2\nu-2\mu-1}.$$
This gives
$$\begin{aligned}
&\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2j-1}\left[e^{-\pi Mt\varepsilon^2}H_{\ell-Mk}^{(Mk)}\left(\varepsilon;iz\right)\right]_{\varepsilon=0}\\
&\quad= \sum_{\nu=1}^{j}\binom{2j-1}{2\nu-1}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2(j-\nu)}\left[e^{-\pi M\varepsilon^2t}\right]_{\varepsilon=0}\left(\frac{\partial_\varepsilon}{2\pi i}\right)^{2\nu-1}\left[H_{\ell-Mk}^{(Mk)}\left(\varepsilon;iz\right)\right]_{\varepsilon=0}\\
&= \sum_{\nu=1}^{j}\binom{2j-1}{2\nu-1}\left(\frac{Mt}{4\pi}\right)^{j-\nu}\frac{(2j-2\nu)!}{(j-\nu)!}\frac{ie^{2\pi i\frac{\ell}{4Mk}}}{2\sqrt{2Mkz}}e^{\frac{\pi}{8Mkz}}\int_{\mathbb{R}}\frac{e^{-\frac{\pi\sigma^2}{2Mkz}}e^{-2\pi\sigma\left(\frac{\ell}{2Mk}+\frac{1}{4Mkiz}\right)}}{\cosh\left(\pi\sigma\right)}\\
&\qquad\times\sum_{\mu=0}^{\nu-1}\binom{2\nu-1}{2\mu}\frac{(2\mu)!}{\mu!}\left(-\frac{Mk}{2\pi z}\right)^{\mu}\left(\frac{1+2i\sigma}{2iz}\right)^{2\nu-2\mu-1}\mathrm{d}\sigma.
\end{aligned}$$

Reordering the terms and replacing ℓ by $r+Mk$, we conclude the proof. $\square$

We define the Kloosterman-type sum

$$\widetilde{K}^*_{k,r}(\alpha,\beta) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \widetilde{\chi}^*_r(\gamma_{h,k}) e^{2\pi i \frac{r}{4Mk}} e^{2\pi i \frac{h\varepsilon(\ell)}{k}} e^{\frac{2\pi i}{k}\left(\alpha h + \frac{\beta}{2}[-h]_k\right)}, \tag{6.3.6}$$

and the constant term

$$D^*(j,\nu,\mu,u) := D(j,\nu,\mu)\frac{i^{2j-\frac{1}{2}}}{(2j-1)!}\left(\frac{M}{4\pi}\right)^u \frac{(-1)^{u+M}}{u!}.$$

Furthermore, according to Proposition 6.3.3, we write the principal part in the expansion of $\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}}$ as

$$\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}} = \sum_{0\le u\le \frac{n-2j}{2}} \frac{(-1)^{u+M}}{u!}\left(\frac{M}{4\pi v}\right)^u \sum_{\substack{s\in 2\delta_E+\mathbb{Z}\\ 0<s\le 2\delta_E}} T_{2j+2u}(s)\widetilde{q}_{h,k}^{\,-\frac{s}{2}}.$$

Thus, we can rewrite Σ_4 as

$$\begin{aligned}\Sigma_4 = &\sum_{\substack{1\le k\le\sqrt{t}\\ 2\nmid k}} \sum_{j=1}^{\frac{n}{2}} \sum_{\substack{r \pmod{2Mk}\\ r\equiv \ell-M \pmod{2M}}} \sum_{\lambda=1}^{j}\sum_{\mu=0}^{\lambda-1} \sum_{0\le u\le\frac{n-2j}{2}} \sum_{\substack{s\in 2\delta_E+\mathbb{Z}\\ 0<s\le 2\delta_E}} D^*(j,\lambda,\mu,u)\widetilde{K}^*_{k,r}(-t,s)\\ &\times N^{2(j-\lambda+u)}k^{\mu-\frac{1}{2}}T_{2j+2u}(s)P_k\left(\frac{r}{2Mk}+\frac{1}{2},\frac{s}{2},2\lambda-2\mu-1,2j-2\lambda+\mu;t-\varepsilon_\ell(M)\right)\\ &+O\left(t^{\frac{1}{2}}\log t\right),\end{aligned} \tag{6.3.7}$$

where P_k is the principal value integral defined in (6.1.4).

Remark 4. The error term $O\left(t^{\frac{1}{2}}\log t\right)$ comes from the non-principal part of $\frac{E_{2j}(\tau)}{\eta(\tau)^{2M-1}}$. More precisely, for this summand, one has the asymptotic expansion

$$\begin{aligned}&\sum_{\substack{1\le k\le\sqrt{t}\\ 2\nmid k}} \sum_{j=1}^{\frac{n}{2}} \sum_{\substack{r \pmod{2Mk}\\ r\equiv \ell-M \pmod{2M}}} \sum_{\lambda=1}^{j}\sum_{\mu=0}^{\lambda-1} \sum_{0\le u\le\frac{n-2j}{2}} D^*(j,\lambda,\mu,u)\widetilde{K}^*_{k,r}(-t,s)N^{2(j-\lambda+u)}\\ &\qquad\times k^{\mu-\frac{1}{2}}P_k\left(\frac{r}{2Mk}+\frac{1}{2},0,2\lambda-2\mu-1,2j-2\lambda+\mu;t-\varepsilon_\ell(M)\right),\end{aligned}$$

which is $O\left(t^{\frac{1}{2}}\log t\right)$ in light of Lemma 6.3.5. Note that for the same reason, the same error term will come from the contribution of Σ_3, as we shall see in Subsubsection 6.3.3.2

6.3.3.2 The even case

For Σ_3 we can use the same analysis as for Σ_4. We point out that in this case the function $\frac{D_{2j}(\tau)}{\eta(\tau)^{2M-1}}$ has no principal part, which makes this term of lower exponential decay. More precisely, one has

$$\Sigma_3 = \sum_{\substack{1\leq k\leq\sqrt{t}\\ 2|k}}\sum_{j=1}^{\frac{n}{2}}\sum_{\substack{r \pmod{2Mk}\\ r\equiv \ell-M \pmod{2M}}}\sum_{\lambda=1}^{j}\sum_{\mu=0}^{\lambda-1}\sum_{0\leq u\leq\frac{n-2j}{2}} D^*(j,\lambda,\mu,u)K^*_{r,k}(-t,0)N^{2(j-\lambda+u)}$$
$$\times k^{\mu-\frac{1}{2}}P_k\left(\frac{r}{2Mk}+\frac{1}{2},0,2\lambda-2\mu-1,2j-2\lambda+\mu;t-\varepsilon(\ell)\right),$$

where the Kloosterman sum K^* is defined by

$$K^*_{k,r}(\alpha,\beta) := \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} \chi^*_r(\gamma_{h,k})e^{2\pi i\frac{r}{4Mk}}e^{2\pi i\frac{h\varepsilon(\ell)}{k}}e^{\frac{2\pi i}{k}\left(\alpha h+\frac{\beta}{2}[-h]_k\right)}.$$

6.3.4 The principal value integral

In this subsection, we provide the main term in the asymptotic expansion of the principal value integral P_k. To do that, we split the integral into two parts, one in terms of the finite integral

$$\mathcal{P}_L^{(J)}(\alpha,\beta;\mu) := \int_{-1}^{1}\frac{w^J e^{-2\pi w\beta}}{\sinh(\pi w\alpha)}\left(1-w^2\right)^{\frac{L}{2}} I_L\left(\mu\sqrt{1-w^2}\right)\mathrm{d}w, \tag{6.3.8}$$

and the second one which gives an error term, as we show in the following lemma. In (6.3.8), we assume J, $L\in\mathbb{N}$, and α, β, and $\mu\in\mathbb{Q}$.

Lemma 6.3.5. *Let $W \sim N^2$. Then, as $N \to +\infty$, we have*

$$P_k(A, C, J, L; W) = -(2i)^J e^{-\pi i A} \frac{2\pi i C^{\frac{L+1}{2}} (4MC)^{\frac{J+1}{2}}}{kW^{\frac{L+1}{2}}} \\ \times \mathcal{P}_{L+1}^{(J)}\left(\sqrt{4MC}, A\sqrt{4MC}; \frac{4\pi}{k}\sqrt{CW}\right) + O\left(\frac{e^{\frac{2\pi W}{N^2}}}{kN^{L+1}}\right).$$

Moreover, if $C = 0$, then, as $N \to +\infty$, we have

$$P_k(A, 0, J, L; W) = O\left(\frac{e^{\frac{2\pi W}{N^2}}}{kN^{L+1}}\right).$$

To prove this result, we need the following well-known asymptotic behavior for the Bessel function (see e.g. Proposition 3.3 in [9]).

Lemma 6.3.6. *Let I_ℓ be the classical modified I-Bessel function. For positive real A, B and for half-integral ℓ, as $N \to +\infty$, we have*

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\ell-1} e^{\frac{2\pi}{k}\left(Az+\frac{B}{z}\right)}\, \mathrm{d}\phi = \frac{2\pi}{k}\left(\frac{B}{A}\right)^{\frac{\ell}{2}} I_\ell\left(\frac{4\pi}{k}\sqrt{AB}\right) + e^{8\pi B + \frac{2\pi A}{N^2}} O\left(\frac{1}{kN^\ell}\right).$$

Furthermore, for positive A and negative B, as $N \to +\infty$, we have

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\ell-1} e^{\frac{2\pi}{k}\left(Az+\frac{B}{z}\right)}\, \mathrm{d}\phi = e^{2\pi B + \frac{2\pi A}{N^2}} O\left(\frac{1}{kN^\ell}\right).$$

Proof of Lemma 6.3.5. Making the change of variables $\sigma \mapsto \sigma + \frac{i}{2}$, P_k equals

$$-i(2i)^J e^{-\pi i A} \int_{\mathbb{R}} \frac{\sigma^J e^{-2\pi A\sigma}}{\sinh(\pi\sigma)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^L e^{\frac{2\pi}{k}\left(zW + \frac{1}{z}\left(C - \frac{\sigma^2}{4M}\right)\right)}\, \mathrm{d}\phi\, \mathrm{d}\sigma. \tag{6.3.9}$$

We point out that the path of integration remains $\mathbb{R}$ since the integrand is a holomorphic function. Furthermore, it is straightforward to check that the vertical contribution goes to 0. Splitting the integral in $\mathrm{d}\sigma$ as $\int_{\mathbb{R}} = \int_{|\sigma|<\sqrt{4MC}} + \int_{|\sigma|>\sqrt{4MC}}$ and using Lemma 6.3.6, we rewrite (6.3.9) as

$$P_k(A,C,J,L;W) = \int_{|\sigma|<\sqrt{4MC}} \frac{\sigma^J e^{-2\pi A\sigma}}{\sinh(\pi\sigma)} \frac{2\pi}{k} \left(\frac{C-\frac{\sigma^2}{4M}}{W}\right)^{\frac{L+1}{2}}$$
$$\times I_{L+1}\left(\frac{4\pi}{k}\sqrt{W\left(C-\frac{\sigma^2}{4M}\right)}\right) \mathrm{d}\sigma + O\left(\frac{e^{\frac{2\pi W}{N^2}}}{kN^{L+1}}\right) \int_{\mathbb{R}} \frac{\sigma^J e^{-2\pi A\sigma} e^{8\pi\left(C-\frac{\sigma^2}{4M}\right)}}{\sinh(\pi\sigma)} \mathrm{d}\sigma. \tag{6.3.10}$$

Considering the second integral, note that the function $\frac{\sigma^J}{\sinh(\pi\sigma)}$ is bounded on all of $\mathbb{R}$. Therefore the integral turn out to be a Gaussian integral, which is also bounded. Rescaling the integration variable of the first integral as $w = \frac{\sigma}{\sqrt{4MC}}$ and using (6.3.8), we conclude the proof. $\square$

In the reminder of the section, we determine the main asymptotic term for $\mathcal{P}_{L+1}$. To do this, we make use of the Laplace method.

Proposition 6.3.7. *With the notation as above, as $T \to \infty$, we have*

$$P_\ell^{(1)}(\alpha,\beta;T) = \frac{e^T}{\pi\alpha T} + O\left(\frac{e^T}{T^2}\right).$$

Furthermore, for $J > 1$, as $T \to +\infty$, we have

$$P_\ell^{(J)}(\alpha,\beta;T) = O\left(\frac{e^T}{T^2}\right).$$

Proof. We split the integral into two pieces as

$$P_\ell^{(J)}(\alpha,\beta;T) = \int_{|w|<\frac{1}{2}} \frac{w^J e^{-2\pi w\beta}}{\sinh(\pi w\alpha)} \left(1-w^2\right)^{\frac{\ell}{2}} I_\ell\left(T\sqrt{1-w^2}\right) \mathrm{d}w$$
$$+ \int_{\frac{1}{2}<|w|<1} \frac{w^J e^{-2\pi w\beta}}{\sinh(\pi w\alpha)} \left(1-w^2\right)^{\frac{\ell}{2}} I_\ell\left(T\sqrt{1-w^2}\right) \mathrm{d}w.$$

It is a standard fact that, as $x \to +\infty$, we have

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{6.3.11}$$

Therefore, the main contribution in the first summand is given by

$$\int_{|w|<\frac{1}{2}} \frac{w^J e^{-2\pi w\beta}}{\sinh(\pi w\alpha)} \left(1-w^2\right)^{\frac{\ell}{2}-\frac{1}{4}} \frac{e^{T\sqrt{1-w^2}}}{\sqrt{2\pi T}} \left(1+O\left(\frac{1}{T}\right)\right) \mathrm{d}w. \tag{6.3.12}$$

Furthermore, since I_ℓ is monotonically increasing in $[0,\infty)$, (6.3.11) implies that $\int_{\frac{1}{2}<|y|<1}$ is of exponentially lower order than (6.3.12). To finish the proof, it is enough to apply Laplace method to (6.3.12). More precisely, if

$$f_J(w) := \frac{w^J e^{-2\pi w\beta}}{\sinh(\pi w\alpha)} \left(1-w^2\right)^{\frac{\ell}{2}-\frac{1}{4}}, \tag{6.3.13}$$

$$g(w) := \sqrt{1-w^2}, \tag{6.3.14}$$

then (6.3.12) equals

$$\frac{1}{\sqrt{2\pi T}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_J(w) \left(1+O\left(\frac{1}{T}\right)\right) e^{Tg(w)} \,\mathrm{d}w. \tag{6.3.15}$$

Since g is an even function with a local maximum in 0, expanding f and g in Taylor expansion, we rewrite (6.3.15) as

$$\frac{1}{\sqrt{2\pi T}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(f_J(0) + f_J''(0)\frac{w^2}{2}\right) e^{Tg(0)+Tg''(0)\frac{w^2}{2}} \,\mathrm{d}w + O\left(\frac{e^{Tg(0)}}{T^2}\right). \tag{6.3.16}$$

Remark. We consider the Taylor expansion of g until the third term since the other terms gives an exponentially lower order. Furthermore, the other terms in the Taylor expansion of f_J gives a higher power of N at the denominator, which fall into the error.

Remark. The odd Taylor coefficients of f_J give no contribution because g is even, so the integral becomes 0.

The integral in (6.3.16) is a Gaussian integral, thus it can be written as

$$\frac{e^{Tg(0)}}{\sqrt{2\pi T}} f_J(0) \frac{1}{\sqrt{T}} \sqrt{-\frac{2\pi}{g''(0)}} + O\left(\frac{e^{Tg(0)}}{T^2}\right)$$

Note that $g(0)=1$, $g''(0)=-1$. Moreover, for $J=1$ we have

$$f_1(0) = \lim_{w\to 0} \frac{w}{\sinh(\pi\alpha w)} = \frac{1}{\pi\alpha},$$

while for $J > 1$

$$f_J(0) = 0.$$

This gives the desired result. $\square$

6.3.5 Proof of Theorem 6.1.2 and Corollary 6.1.3

In the previous sections we have provided all the ingredients necessary to prove Theorem 6.1.2 and Corollary 6.1.3.

Proof of Theorem 6.1.2. In Section 6.2.4 we splitted $c_\ell(t)$ into four pieces, denoted by Σ_j, for $j \in \{1, 2, 3, 4\}$. From Remark 4 it follows that the contribution of Σ_3 falls in the error term. To conclude the proof, it is enough to recall (6.3.3), (6.3.5), and (6.3.7). $\square$

We now proceed with the proof of Corollary 6.1.3.

Proof of Corollary 6.1.3. From Lemma 6.3.5 and Proposition 6.3.7 it follows that, as $N \to +\infty$,

$$P_k\left(A, C, 1, L; W\right) \sim \frac{2e^{-\pi i A} M^{\frac{1}{2}} C^{\frac{L+1}{2}}}{W^{\frac{L+2}{2}} \pi} e^{\frac{4\pi}{k}\sqrt{CW}}. \tag{6.3.17}$$

Remark. Note that on the left-hand side of (6.3.17) the parameter N is implicit in the definition of P_k. More precisely, it is related to the Farey series.

From the asymptotic behavior of the Bessel function I_ℓ and (6.3.17), we see that each of the terms Σ_1, Σ_2, and Σ_4 has a main asymptotic term which grows exponentially, which is maximized for $k = 1$. More precisely, the exponential contribution is respectively

$$e^{4\pi\sqrt{\delta_h(r)(t-\varepsilon(\ell))}}, \quad e^{4\pi\sqrt{\delta_g(r)(t-\varepsilon(\ell))}}, \quad e^{4\pi\sqrt{\delta_E(t-\varepsilon(\ell))}}.$$

Since we are free to choose r in any range of length $2M$, for the first summand we choose $r \in [-M, M]$, while in the second summand we choose $r \in [-2M, 0]$. With this choice it is straightforward to check that $\delta_E > \max_r\{\delta_h(r), \delta_g(r)\}$, and this implies that the major contribution comes from Σ_4. It remains to compute the asymptotic of this term. One can easily see that the major contribution is obtained for $j = 1$, $\lambda = 1$, $\mu = 0$, and $u = \frac{n}{2} - 1$. In particular, the major contribution comes from the term

$$D^*\left(1, 1, 0, \frac{n}{2} - 1\right) \widetilde{K}^*_{1,\ell-M}(-t, 2\delta_E) t^{\frac{n}{2}-1} T_n(2\delta_E) P_1\left(\frac{\ell}{2M}, \delta_E, 1, 0, t\right),$$

where

$$D^*\left(1, 1, 0, \frac{n}{2} - 1\right) = \frac{M^{\frac{n-3}{2}} i^{\frac{1}{2}} (-1)^{\frac{m}{2}}}{2^{\frac{5}{2}} (4\pi)^{\frac{n}{2}-1} \left(\frac{n}{2} - 1\right)!}$$

$$\widetilde{K}^*_{1,\ell-M}(-t, 2\delta_E) = e^{2\pi i \frac{\ell-M}{4M}} i^{\frac{M+1}{2}}$$

$$T_n(2\delta_E) = (-1)^M$$

$$P_1\left(\frac{\ell}{2M}, \delta_E, 1, 0, t\right) \sim \frac{2M^{\frac{1}{2}} e^{-2\pi i \frac{\ell}{4M}}}{\pi t} \sqrt{\frac{n}{8} + \frac{2M-1}{24}} e^{4\pi \sqrt{t\left(\frac{n}{8} + \frac{2M-1}{24}\right)}}.$$

Gluing together these data, we get the desired result. $\square$

APPENDIX A

Kac-Wakimoto characters

A.1 Kac-Wakimoto characters and Jacobi forms: A brief overview

Given a simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ endowed with a suitably normalized invariant symmetric bilinear form $(\cdot|\cdot)$, one associates an *affine Lie algebra* $\widehat{\mathfrak{g}}$, i.e., the infinite-dimensional Lie algebra over $\mathbb{C}$ defined by

$$\widehat{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

satisfying certain commutator relations. Here K denotes the central element of $\widehat{\mathfrak{g}}$, while d is an outer derivation of $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$. By identifying $\mathfrak{g}$ with the subalgebra $1 \otimes \mathfrak{g}$, one extends the bilinear form to $\widehat{\mathfrak{g}}$, which is still non-degenerate, symmetric, and invariant. Furthermore, given a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$ (here $\mathfrak{n}_+$ is a maximal nilpotent subalgebra of $\mathfrak{g}$), it is possible to define a Cartan subalgebra and a Borel subalgebra of $\widehat{\mathfrak{g}}$ by

$$\begin{aligned} \widehat{\mathfrak{h}} &= \mathbb{C}d \oplus \mathfrak{h} \oplus \mathbb{C}K, \\ \widehat{\mathfrak{b}} &= \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \left(\oplus_{n>0} \mathfrak{g} t^n\right). \end{aligned}$$

Extending a linear function $\Lambda \in \widehat{\mathfrak{h}}^*$ (the dual space) to $\widehat{\mathfrak{b}}$ by zero on all other summands, let $L(\Lambda)$ be the highest weight integrable module over $\widehat{\mathfrak{g}}$, i.e., the irreducible module that admits an eigenvector of $\widehat{\mathfrak{b}}$ with weight Λ. Since K is the central element of $\widehat{\mathfrak{g}}$, it is represented by a scalar $\Lambda(K)$, called the level of $L(\Lambda)$. Using the following coordinates on $\widehat{\mathfrak{h}}$

$$\widehat{\mathfrak{h}} \ni h = 2\pi i(-\tau d + z + tK), \text{ where } \tau,\, t \in \mathbb{C}, \text{ and } z \in \mathfrak{h}\ ,$$

one can define define the *character* of $L(\Lambda)$ corresponding to the weight space decomposition with respect to $\widehat{\mathfrak{h}}$ as

$$\mathrm{ch}_{L(\Lambda)}(\tau, z, t) := \mathrm{tr}_{L(\Lambda)} e^{2\pi i(-\tau d + z + tK)},$$

which converges on the domain

$$X = \{h \in \widehat{\mathfrak{h}} \colon \mathrm{Re}(h|K) > 0\} = \{(\tau, z, t) \colon \mathrm{Im}(\tau) > 0\}.$$

We denote by P_K the finite set of highest weights Λ of level $\Lambda(K)$ modulo $\mathbb{C}K$. Defining the normalized character

$$\mathrm{ch}_\Lambda(\tau, z, t) := e^{2\pi i m_\Lambda \tau} \mathrm{ch}_{L(\Lambda)}(\tau, z, t),$$

for a suitable rational m_Λ, the *Weyl-Kac character formula* implies that the finite set $\{\mathrm{ch}_\Lambda \colon \Lambda \in P_K\}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant under the following action:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{c(t|t)}{c\tau + d}\right).$$

This property is called *modular invariance*, and using a number-theoretical lexicon, it is equivalent to saying that the vector-valued function $\{\mathrm{ch}_\Lambda \colon \Lambda \in P_K\}$ is a Jacobi form (we address the reader to Subsection 2.3 for details).

A natural question might be whether these arguments extend to the case of finite-dimensional simple *Lie superalgebras*. A Lie superalgebra is a (non-associative) $\mathbb{Z}_2$-graded algebra, or superalgebra, over a commutative ring (typically $\mathbb{R}$ or $\mathbb{C}$) whose product, called the Lie superbracket or supercommutator, satisfies two conditions (analogs of the usual Lie algebra axioms, with grading), called the super skew-symmetry and the super Jacobi identity. In this case, the situation is slightly different. Indeed, the previous explained modular invariance property becomes now a “mock-modular invariance” property, i.e., the vector-valued Kac-Wakimoto characters transforms as mock modular (or Jacobi) forms. For a precise definition of mock modular transformation property, we refer to Section 2.2. Roughly speaking, these functions are not invariant under the action of a certain congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, but we are able to control the modularity by adding a certain “nice” non-holomorphic piece. In fact, there exist another way to describe this phenomenon, which is the heart of this thesis. Denoting by $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}$ the ℓth Kac-Wakimoto character relative to the Lie superalgebra $\mathrm{s}\ell(m|n)^\wedge$ for each integer ℓ, we considering their generating function

$$\mathrm{tr}_{L_{m,n}}(z;\tau) := \sum_{\ell\in\mathbb{Z}} \mathrm{tr}_{L_{m,n}(\Lambda(\ell))}(\tau)e^{2\pi i\ell z}.$$

The mock modularity of the Kac-Wakimoto characters is equivalent to say that $\mathrm{tr}_{L_{m,n}}$ is a meromorphic Jacobi form.

A.2 Kac-Wakimoto characters as canonical Fourier coefficients

Let F be the Lie superalgebra considered in [24], Section 3, and for each $\ell \in \mathbb{Z}$ denote by F_ℓ the ℓth $\mathrm{gl}(m|n)^\wedge$-module arising in the charge decomposition of F (see (3.4) in [24]), with $m > n > 0$. Also, fix $M := \frac{m-n}{2}$. Let $\mathrm{ch}F_\ell$ be the associated character. Its generating function $\mathrm{ch}F$ is given in (5.1.1). Moreover, let $L(\Lambda(\ell))$ be the irreducible $\mathrm{s}\ell(m|n)^\wedge$-module of highest weight $\Lambda(\ell)$, and denote by $\mathrm{ch}L(\Lambda(\ell))$ the associated character. Assuming the basic specialization as in (4.7) of [24], we denote the specialized character by $\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}q^{L_0}$. As showed in Section 4 of [24], we have

$$\mathrm{tr}_{L_{m,n}(\Lambda(\ell))}q^{L_0} = \mathrm{ch}F_\ell \cdot \prod_{k\geq 1}\left(1-q^k\right). \tag{A.2.1}$$

The generating function $\mathrm{ch}F(z;\tau)$ of the Kac-Wakimoto characters $\mathrm{ch}F_\ell(\tau)$ is a meromorphic function in z, therefore its Fourier coefficients are not globally well defined.

However, we can define the functions $\mathrm{ch}F_\ell(\tau)$ locally, as the Fourier coefficients of $\mathrm{ch}F(z;\tau)$ for $-\frac{\mathrm{Im}(\tau)}{2} < z < \frac{\mathrm{Im}(\tau)}{2}$, namely

$$\mathrm{ch}F_\ell(\tau) := \int_0^1 \mathrm{ch}F(z;\tau)e^{-2\pi i\ell z}\,\mathrm{d}z. \tag{A.2.2}$$

Since $\mathrm{ch}F(z;\tau)$ is meromorphic Jacobi form, we shall see that changing the range correspond to choose a different Fourier coefficient, say $\mathrm{ch}F_{\ell+2Mk}$ instead of $\mathrm{ch}F_\ell$, for some $k \in \mathbb{Z}$. Moreover, the difference between $\mathrm{ch}F_\ell$ and $\mathrm{ch}F_{\ell+2Mk}$ is a linear combination of quasimodular forms.

Let Φ as in (3.1.3). In the following Lemma, we show the relation between $\mathrm{ch}F_\ell$ and the canonical Fourier coefficients of Φ.

Lemma A.2.1. *Let $-M \leq \ell < M$ and consider the ℓth canonical Fourier coefficient h_ℓ of Φ. Then, we have*

$$chF_\ell(\tau) = (-1)^m i^{-n}\eta(\tau)^{n-m}q^{\frac{\ell^2}{2(m-n)}+\frac{m-n}{12}}h_{\ell+M}(\tau).$$

Proof. From (3.1.2), we know that

$$\mathrm{ch}F(z;\tau) = T(\tau)e^{2\pi iMz}\Phi\left(z+\frac{\tau}{2};\tau\right), \tag{A.2.3}$$

where $T(\tau) := (-1)^m i^{-n}q^{\frac{m-n}{6}}\eta(\tau)^{n-m}$. Plugging (A.2.3) into (A.2.2) and changing the variable of integration as $z \mapsto z - \frac{\tau}{2}$, we get

$$\mathrm{ch}F_\ell(\tau) = T(\tau)q^{\frac{\ell^2}{2(m-n)}-\frac{m-n}{8}}h_{\ell-M}^{\left(\frac{\tau}{2}\right)}(\tau) = T(\tau)q^{\frac{\ell^2}{2(m-n)}-\frac{m-n}{8}}h_{\ell+M}^{\left(-\frac{\tau}{2}\right)}(\tau), \tag{A.2.4}$$

where $h_\ell^{(z_0)}(\tau)$ is defined in (3.3.1). In the second equality we have used Lemma (3.3.2). To conclude the proof it is enough to note that for $-M \leq \ell < M$, we have $h_{\ell+M}^{\left(-\frac{\tau}{2}\right)} = h_{\ell+M}$. □

For $\ell \notin \{-M, -M+1, \cdots, M-1\}$, the Kac-Wakimoto characters $\mathrm{ch}F_\ell$ can not be written in terms of the canonical Fourier coefficients of Φ. However, their difference can be expressed as a linear combination of quasimodular forms. More precisely, these quasimodular forms are the Laurent coefficients of Φ near $z = 0$, as we show in the following lemma. We describe the result for $m \equiv n \equiv 0 \pmod 2$. Analogous results can be obtained for the other cases.

Lemma A.2.2. *Let $-M \leq \ell < M$ and $k \in \mathbb{N}$. Then, we have*

$$chF_{\ell+2Mk}(\tau) = (-1)^m i^{-n} \eta(\tau)^{-2M} q^{\frac{(\ell+2Mk)^2}{4M}+\frac{M}{12}} \Bigg(h_{\ell+M}(\tau) \\ + \sum_{\alpha=0}^{k-1} \sum_{j=1}^{n/2} \frac{q^{-M\alpha^2-\alpha(\ell+M)} \left(-(\ell+M(2\alpha+1))\right)^{2j-1}}{(2j-1)!} \widetilde{D}_{2j}(\tau) \Bigg).$$

Proof. With the same notation as in Lemma A.2.1, from (A.2.4) we know that

$$\mathrm{ch}F_{\ell+2Mk}(\tau) = T(\tau) h_{\ell+M+2Mk}^{\left(-\frac{\tau}{2}\right)}(\tau) = T(\tau) h_{\ell+M}^{\left(-\frac{\tau}{2}+k\tau\right)}(\tau),$$

where in the second equality we use part 1 of Lemma 3.3.2. Using the Residue Theorem, we have

$$h_{\ell+M}^{\left(-\frac{\tau}{2}+k\tau\right)}(\tau) = h_{\ell+M}(\tau) + 2\pi i \sum_{\alpha=0}^{k-1} \operatorname*{Res}_{z=0} \left(\Phi(z+\alpha\tau;\tau) e^{-2\pi i(\ell+M)z} q^{\alpha(\ell+M)}\right).$$

Using the elliptic transformation properties and then Laurent expansion of Φ in $z = 0$ (see (3.3.3)), we conclude the proof. $\square$

References

[1] C. Alfes and T. Creutzig, *The mock modular data of a family of superalgebras*, Proc. Amer. Math. Soc., accepted for publication.

[2] R. Berndt and R. Schmidt, *Elements of the representation theory of the Jacobi group*, Progr. Math. **163**. Birkhäuser, Basel, (1998).

[3] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992) 405–44.

[4] J. Bruinier and J. Funke, *On Two Geometric Theta Lifts*, Duke Math. Journal **125** (2004), 45–90.

[5] J. Bruinier, G. van der Geer, G. Harder, and D. Zagier *The 1-2-3 of Modular Forms*, Universitext, Springer-Verlag, Berlin-Heidelberg-New York (2008).

[6] K. Bringmann, T. Creutzig, and L. Rolen, *Negative index Jacobi forms and quantum modular forms*, preprint.

[7] K. Bringmann and A. Folsom, *Almost harmonic Maass forms and Kac-Wakimoto characters*, J. Reine Angew. Math., doi:10.1515/crelle-2012-0102.

[8] K. Bringmann, A. Folsom, and K. Mahlburg, *Quasimodular forms and* $s\ell(m|m)^\wedge$ *characters*, Ramanujan Journal, special volume in honor of Basil Gordon.

[9] K. Bringmann and K. Mahlburg, *Asymptotic formulas for coefficients of Kac-Wakimoto characters*, Mathematical Proceedings of the Cambridge Philosophical Society **155** (2013), 51–72.

[10] K. Bringmann and K. Mahlburg, *An extension of the Hardy-Ramanujan Circle Method and applications to partitions without sequences*, American Journal of Mathematics **133** (2011), pages 1151–1178.

[11] K. Bringmann and K. Ono, *Some characters of Kac and Wakimoto and nonholomorphic modular functions*, Mathematische Annalen **345** (2009), 547–558.

[12] K. Bringmann, M. Raum, and O. Richter, *Harmonic Maass-Jacobi forms with singularities and a theta-like decomposition*, Transactions of the AMS, accepted for publication.

[13] K. Bringmann and S. Zwegers, *Rank-crank type PDE's and non-holomorphic Jacobi forms*, Mathematical Research Letters **17** (2010), 589–600.

[14] G. H. Conway and S. P. Norton, textitMonstrous moonshine, Proc. Amer. Math. Soc. **132** (2004) 2233–40.

[15] A. Dabholkar, S. Murthy, and D. Zagier, *Quantum black holes, wall crossing, and mock modular forms*, to appear in Cambridge Monographs in Mathematical Physics.

[16] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math. **55**, Birkhäuser-Verlag, Basel-Boston (1985).

[17] A. Folsom, *Kac-Wakimoto characters and universal mock theta functions*, Transactions of the American Mathematical Society **363** no. 1 (2011), 439–455.

[18] G. Hardy and S. Ramanujan, *Asymptotic Formulae in Combinatory Analysis*, Proc. London Math. Soc. **17** (1918), 75-115.

[19] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society, Colloquium Publications **53**, 2004.

[20] V. G. Kac, *Infinite-dimensional Lie algebras and Dedekinds eta function*, Funct. Anal. Appl. **8** (1974), 68–70.

[21] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[22] V. G. Kac and D. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), 125-264.

[23] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Progress in Math. **123**, Birkhäuser, Boston, 1994, 415–456.

[24] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appells function*, Comm. Math. Phys. **215** (2001), 631–682.

[25] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, in The Moduli Spaces of Curves (R. Dijkgraaf, C. Faber, G. v.d. Geer, eds.), Prog. in Math. **129**, Birkhäuser, Boston (1995) 165–172.

[26] A. Libgober, *Elliptic genera, real algebraic varieties and quasi-Jacobi forms*, In Topology of stratified spaces, volume 58 of Math. Sci. Res. Inst. Publ., pages 95–120. Cambridge Univ. Press, Cambridge, 2011.

[27] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[28] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference, International Press, Somerville, MA, (2009), 347–454.

[29] A. Pitale, *Jacobi Maass Forms*, Abh. Math. Sem. Univ. Hamburg **79** (2009), 87–111.

[30] H. Rademacher, *Topics in analytic number theory*, Die Grundlehren der math. Wiss., Band **169**, Springer-Verlag, Berlin, (1973).

[31] H. Rademacher and H. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. **39** (1938), 433-462.

[32] M. Raum, *Dual weights in the theory of harmonic Siegel modular forms*, Ph.D. Thesis, University of Bonn (2012).

[33] J. G. Thompson, *Some numerology between the FischerGriess Monster and the elliptic modular function*, Bull. Lond. Math. Soc. **11** (1979) 352–353.

[34] D. Zagier, *Ramanujan's mock theta functions and their applications [d'aprs Zwegers and Bringmann-Ono]*, Séminaire Bourbaki, 60ème année, 2007–2008, no 986, Astérisque **326** (2009), Soc. Math. de France, 143–164

[35] H. Zuckerman, *Certain functions with singularities on the unit circle*, Duke Math. J. **10** (1943), 381–395.

[36] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, (2002).

[37] S. Zwegers, *Multivariable Appell functions*, preprint, (2010).

Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlicht vor Abschluss des Promotionsverfahrens nicht vornehmen werde.

Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Kathrin Bringmann betreut worden.

Es liegt die folgende Teilpublikation vor: On the Fourier coefficients of meromorphic Jacobi forms, International Journal of Number Theory.

Köln, den 10. Juli 2014

Lebenslauf

Persönliche Daten

Name	René Olivetto
Geburtsdatum	26. November 1987
Geburtsort	Marostica (Italien)
Nationalität	italienisch

Ausbildung

7/2006	Abitur am Liceo Gimnasio statale G.B. Brocchi, Bassano del Grappa (Italien)
10/2006–9/2009	Bachelor-Studium in Mathematik an der Università degli studi di Padova (Italien)
10/2009–7/2011	Master-Studium in Mathematik an der Università degli studi di Padova (Italien) und Universitè Bordeaux 1 (Frankreich)
11/2011–9/2014	Promotionsstudium im Graduiertenkolleg "Globale Strukturen in Geometrie und Analysis" an der Universität zu Köln