Field theory of disordered bosons

Feldtheorie ungeordneter Bosonen

Diplomarbeit

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1 Introduction

1.1 Motivation

Quantum field theories have in many instances be successfully applied to qualitatively and quantitatively understand fundamental questions in condensed matter physics. Effective field theories emerge in many different settings and have, in the form of non-linear σ-models, provided profound insight into the physics of disordered systems, for an example see [Efe99].

The goal of this diploma thesis is to develop a field theory, for a specific class of disordered bosonic systems, in order to study, whether disordered bosons generically have universal statistical properties, similar to fermions.[1]

For fermionic systems the universality question is well understood. There exist ten families of symmetry classes, as explained by Zirnbauer et. al. in [Zir98, AZ97, HHZ04]. These classes determine the statistics, i.e. the probability distributions of the correlation functions, of low-energy excitations of ergodic systems.

For bosons, one might doubt on physical grounds whether there is universal behaviour at low energies at all. In condensed matter, bosonic modes typically arise as Goldstone modes, where low energy usually means long wavelength. But long wavelength modes are insensitive to spatially uncorrelated weak disorder, which gets effectively averaged out on the scale of the wavelength. This argument does, however, not apply in general. Long wavelength modes might be suppressed or forbidden by boundary conditions, the disorder might be strong, in real systems the disorder will never be spatially uncorrelated and, generally speaking, the multi particle ground state might be such that low lying excitations have a comparatively short wavelength. Anyhow, as soon as low lying excitations of a wavelength comparable to the length scale of the disorder exist, it seems reasonable to assume that their behaviour will be largely determined by symmetries, rather than by microscopic details, just as in the fermionic case. In fact, hints to universal behaviour were found in [GC02] and also [GA04] for systems where the bosonic excitations are not of Goldstone type.

In order to shed some light on those questions, the general idea is to investigate a quadratic model, which should be thought of as being the lowest order approximation of a system of small oscillations around the (interacting) many particle ground state. Physical examples of such systems where the fluctuations quantise as bosons are

- vibrations of amorphous solids, see e.g. [GKK89],
- oscillations of the superfluid density of Bose-Einstein condensates,

[1] As such this thesis is part of the research project C3 'The universality question for disordered low-frequency bosons' within the SFB/TR12.

- excitations of Bose glasses, see e.g. [FWGF89],
- photons in an inhomogeneous optical medium,
- spin waves in disordered magnets, see e.g. [WW77],
- and normal modes of pinned charge density waves, see e.g. [GS88].

Now, if disordered bosons show system independent statistical features, the evident question is whether the bosonic universality classes are different from the fermionic ones. For fermions, the classification in terms of symmetric spaces is a largely algebraical one, which happens in a complexified setting before compact or non-compact real forms are specified. Hence one can rightly state that all effective bosonic non-linear σ-models are already classified within the ten fold way together with the fermionic ones. And indeed, in the system studied in [LSZ06], well known universal GUE statistics were found in the bulk and Lück finds GOE statistics in [Lü09], both for systems of disordered bosons.

But for bosons non-linear σ-models are not everything. A subtle point about non-interacting, i.e. quadratic, bosonic systems is stability. One single particle state of negative energy for a bosonic system immediately leads to an unbounded many particle Hamiltonian. For fermionic systems, a finite number of such single particle states is of no concern, due to the Pauli principle. This important difference leads to the well studied Gaussian ensembles not providing feasible distributions of bosonic Hamiltonians. If, however, more complicated probability distributions are taken into account, one has to be prepared to face a more complicated effective model than a pure non-linear σ-model in the end. In this sense, an interesting question is whether bosonic systems if they show universal behaviour lead to novel universality classes. A hint to those might be the unusual density of states near zero energy that was found in [LSZ06] or [GC02] and [GA04]. On physical grounds, one can expect the commutation relations of bosons to have important effects at low frequencies, which fits to the observations of [GC02] and [GA05].

In this thesis, we will develop and study a specific model, which is the next step after the work of Lück, Sommers and Zirnbauer, [LSZ06], towards a more realistic description of a disordered system of bosonic degrees of freedom. As in [LSZ06], our model is still purely random, i.e. there is no limit of an underlying pure system and we also stay with a model without global symmetries, such as time reversal or charge conjugation. But we turn from the homogeneous, zero-dimensional model to a spatially extended one and implement the physically important feature of locality. Furthermore, we add an additional parameter to tune the number of modes per volume. In particular we can hereby enforce a macroscopical number of zero modes. The goal is to study the density of states of the new model and therefore to develop an effective field theory. Throughout we will pay special attention to the region near zero frequency and watch out for unusual, possibly universal features.

1.2 Outline

In chapter 2 we will introduce the model to be studied in this thesis. A copy of the zero-dimensional model which was solved in [LSZ06] is placed on each vertex of a graph, which defines the spatial structure. Then we add random interactions in between neighbouring vertices. Note that, although we will specialise to a regular square lattice in chapter 3 to develop a continuum field theory, the results of chapter 2 are more general. One could also put the model on a more complicated graph to emulate, e.g., a real system consisting of a few isolated grains or dots and only allow for interactions in between specific sites. Or, a quasi one- or two-dimensional system, which is extended in one but finite in another direction, might be constructed. Anyway, no such system is considered here. As we are looking for universal features, rather than trying to model a concrete real system, we will consider the simplest spatial structure.

Furthermore, the resolvent operator is introduced in section 2.2 and super-symmetry methods are applied in order to average out the disorder, leading to the final step of superbosonisation and hence to the exact formulation of the effective model as a lattice field theory in equation (2.24).

Chapter 3 hosts the main part of this thesis. Here we will perform a continuum limit and carefully study a saddle point approximation, justified by taking the limit of a large number N of degrees of freedom per volume. Deriving the saddle point equations, considering the reachability of all possible saddle points for various parameter configurations and expanding the fluctuations to quadratic order will take the sections 3.1.2 to 3.1.4.

In section 3.2 various notions from the differential geometry of Riemannian symmetric (super-) spaces will be introduced. Finally, we will propose a general form of the effective action, describing the resolvent operator, in section 3.2.6 and determine the coefficients therein from the coordinate based calculations in 3.1.4. This concludes the derivation of the field theory as advertised in the title.

After this is finished, we turn our attention back to the zero dimensional case in chapter 4. Here we will explicitly calculate the density of states to first order in $\frac{1}{N}$ and compare to [LSZ06]. We allow a parameter which already exists in [LSZ06] to vary in a much larger range. This parameter tunes the distribution for the random coupling constants, which physically affects e.g. the number of zero modes in the model. The significance of this parameter becomes visible in the figures 4.1 and 4.2.

For a summary of the results see section 5.

1.3 Einleitung

Quantenfeldtheorien sind ein probates und erfolgreiches Instrument zur qualitativen und quantitativen Beschreibung, Untersuchung und Lösung fundamentaler Probleme im Rahmen der Physik kondensierter Materie. Effektive Feldtheorien, insbesondere in der Form nicht linearer σ-Modelle, haben in vielen Fällen zu tiefen Einsichten in die Physik ungeordneter Systeme geführt, ein Beispiel hierführ ist das Buch von Efetov [Efe99].

Das Ziel dieser Arbeit ist es eine Feldtheorie für eine bestimmte Klasse ungeordneter bosonischer Systeme zu entwickeln und zu untersuchen, ob bosonische ungeordnete Systeme universelle Eigenschaften haben, wie das bei fermionischen Systemen der Fall ist.[2]

Die universellen Eigenschaften ungeordneter fermionischer Systeme sind ausgiebig untersucht worden und es wurde insgesamt ein gutes Verständnis erreicht. Es gibt zehn Universalitätsklassen wie von Zirnbauer u.a. in [Zir98, AZ97, HHZ04] beschrieben. Die Zugehörigkeit zu einer dieser Klassen bestimmt die Statistik, d.h. die Wahrscheinlichkeitsverteilung der Korrelationsfunktionen, der niederenergetischen Anregungen ergodischer Systeme.

Es ist nicht von vornherein klar, ob auch Bosonen universelle Eigenschaften bei niedrigen Energien zeigen. Typischerweise treten bosonische Anregungen in der Festkörperphysik in Form von Goldstone-Moden, für die niedrige Energie große Wellenlänge bedeutet, auf. Solche langwelligen Moden sind unempfindlich gegenüber schwacher, räumlich unkorrelierter Unordnung, da diese auf der Skala der Wellenlänge effektiv gemittelt wird. Nun gibt es aber auch eine Vielzahl von Systemen auf die dieses Argument nicht zutrifft, zum Beispiel wenn die Anregungen elementar bosonisch sind, z.B. Photonen, wenn langwellige Anregungen durch z.B. Randbedingungen unterdrückt werden oder wenn die Unordnung hinreichend stark und oder räumlich korreliert ist, wobei zumindest letzteres in realen Systemen immer zu einem gewissen Grad der Fall sein wird. Und die thermodynamische Intuition lehrt, sobald die niedrig energetischen Anregungen hinreichend stark von Unordnung beeinflusst werden, sollten ihre Eigenschaften eher durch die Symmetrien des Systems bestimmt werden als durch mikroskopische Details. Diese grundlegende Anschauung trifft auf Bosonen ebenso zu wie auf Fermionen und in der Tat wurden in [GC02] und [GA04] Hinweise auf ein universelles Verhalten bosonischer Systeme gefunden.

Um nun die Frage nach der Universalität für Bosonen systematisch zu untersuchen betrachten wir Systeme mit Hamilton Operatoren quadratischer Ordnung. Diese sollten als erste Näherung an reale, wechselwirkende Systeme verstanden werden, d.h. als effektive Theorien kleiner Schwingungen um einen stabilen Vielteilchen-Grundzustand. Physikalische Beispiele solcher Systeme, bei denen die Anregungen als Bosonen quantisiert sind, wären

- Vibrationsmoden amorpher Festkörper, z.B. [GKK89],

[2] In diesem Sinne gliedert sich diese Arbeit in das Forschungsprojekt C3 ‘The universality question for disordered low-frequency bosons’ des SFB/TR12 ein.

- Oszillationen der Superflüssigkeitsdichte von Bose-Einstein-Kondensaten,
- Anregungen von Bose-Gläsern, z.B. [FWGF89],
- Photonen in inhomogenen optischen Medien,
- Spinwellen in ungeordneten Magneten, z.B. [WW77],
- und Normalmoden von Ladungsdichtewellen, z.B. [GS88].

Falls bosonische Systeme nun universelle Eigenschaften zeigen stellt sich die Frage, ob diese in ähnlicher Art wie die fermionischen klassifiziert werden können und ob sie sich sogar in die schon bekannten Symmetrieklassen einfügen, oder ob es zur Beschreibung bosonischer Systeme neuer Symmetrieklassen bedarf. Da die Klassifizierung fermionischer Systeme mittels symmetrischer Räume im wesentlichen algebraisch ist könnte man nun annehmen, dass der von Zirnbauer u.a. beschriebene 'ten fold way' auch die bosonischen Systeme mit einschließt. In der Tat wurde in [LSZ06] gezeigt, dass die n-Punkt Funktionen für das Modell auf dem diese Arbeit aufbaut, gerade die des gausschen unitären Ensembles sind. In [Lü09] wurden Charakteristika des orthogonalen Ensembles beobachtet.

Im allgemeinen muss man aber davon ausgehen, dass Systeme ungeordneter Bosonen nicht ohne weiteres auf reine nicht lineare σ-Modelle abgebildet werden können. Ein wichtiger Punkt hierbei ist Stabilität des Grundzustandes um den entwickelt wird. Hat der zugrundeliegende Einteilchen-Hamiltonoperator auch nur einen negativen Eigenwert, so ist das Spektrum des Vielteilchen-Operators unweigerlich nach unten hin unbeschränkt, da kein Pauli-Prinzip wie bei Fermionen den Besetzungszahloperator beschränkt. Daher eignen sich gaussche Ensemble mit unabhängig identisch verteilten Matrixeinträgen des Hamiltonians grundsätzlich nicht zur Beschreibung bosonischer Probleme. Daher ist die Frage nach neuen Universalitätsklassen nicht eine Frage nach möglichen neuen Zielräumen für nicht lineare σ-Modelle. Diese sind vollständig klassifiziert. Vielmehr lautet die Frage, ob und wie die Klasse der effektiven Feldtheorien erweitert werden muss. Hinweise auf solche neuen Klassen könnten die ungewöhnliche Zustandsdichten sein, die in [LSZ06] oder auch [GC02] und [GA04] gefunden wurden. Falls solche neuen Klassen zu neuartigen universellen Wahrscheinlichverteilungen führen, so sollten diese am ehesten bei niedrigen Energien zu beobachten sein, was zu den Beobachtungen in [GC02] und [GA05] passen würde.

In dieser Arbeit wird ein spezielles Modell ungeordneter Bosonen entwickelt und untersucht werden, dass eine Weiterentwicklung des von Lück, Sommers und Zirnbauer, [LSZ06], gelösten Problems darstellt. Das hier betrachtete Modell ist weiterhin rein zufällig, d.h. es gibt keinen Grenzfall eines deterministischen Systems. In diesem Sinne könnte man von einem Grenzwert unendlich starker Unordnung sprechen. Weiterhin werden nach wie vor keine globalen Symmetrien, wie etwa Zeitumkehrsymmetrie, betrachtet. Aber das neue Modell ist in so fern realistischer, als das nun räumliche Ausdehnung in Betracht gezogen wird und der Hamiltonoperator nur lokal wirkt. Weiterhin gibt es in unserem Modell einen zusätzlichen Parameter, d.h. wir erweitern die Familie der Wahrscheinlichkeitsverteilungen. Physikalisch bedeutet dies insbesondere, dass

Modelle mit einer makroskopischen Anzahl an Nullmoden in Betracht gezogen werden können. Das Ziel ist es, ein effektives Modell zur Beschreibung der Zustandsdichte zu entwickeln und insbesondere auf möglicherweise neuartige statistische Eigenschaften im Sinne der oben gestellten Universalitätsfrage zu untersuchen.

2 Model

In this chapter we derive a lattice field theory. All derivations should be reasonably rigorous and, apart from the last step of superbosonisation for which we refer to [LSZ07], self contained.

2.1 The setting

We extend the bosonic random matrix ensemble considered in [LSZ06] and [Lü09], chapter 3, by adding spatial structure. Therefore, we will first review the general structure of a quadratic bosonic Hamiltonian in 2.1.1 and then define what we mean by spatial structure in 2.1.2 and describe the consequences for the ensemble of feasible Hamiltonians in 2.1.3.

Independently, an additional parameter α will be introduced into the model which can be tuned to enforce a macroscopical number of zero modes in all systems of the ensemble. For the case where all neighboring modes are coupled, α can be related to the parameter[1] k, the power of a factor of $\mathrm{Det}(h)$ in the probability density. See section 2.1.3, in particular the equations (2.8) and (2.9) for further details.

Depending on the physical system under consideration, e.g. phonons in a randomly distorted lattice, it might be unclear why one should want to have a macroscopical number of zero modes. But one should keep in mind that the pure random model, which will be build in this thesis, is not supposed to be ultimately realistic. In a subsequent step in the development of a complete picture of random bosonic systems, one should add a deterministic part to the model to get closer to physically relevant systems. In such a situation the zero modes of the random model will get mixed with the deterministic modes and thus would not remain at zero. However, for the pure random model itself, the zero modes have a positive effect, in so far, as the density of states becomes more realistic. This will be discussed in detail in section 4.

2.1.1 A generic quadratic Hamiltonian

A quadratic Hamiltonian frequently arises when one linearises the equations of motion of a given system around a stable fixed point. We will now spell out what this stability means for the Hamiltonian, i.e. specify which kind of disorder is feasible for maintaining stable systems. The positivity constraint arising will be explained from the classical as well as from the quantum point of view. In any case, an isolated system can only be an idealisation of a system, which is weakly coupled to some sort of

[1] Note that our parameter k was called l in [Lü09].

environment. This idealisation has to break down for thermodynamic reasons, as soon as it is possible for the system to go to arbitrarily low energies. This is the case for a system of classical or bosonic quantum particles if there is a single particle state with negative energy. Note that, due to the Pauli principle, this is no problem for fermionic systems, where a finite number of single particle states of negative energy does not spoil thermodynamics.

As the theory developed in this thesis applies to bosonic quantum mechanical as well as classical systems, we will now introduce both viewpoints.

Classical Model

Let $\{Q_i\}$ denote the set of N canonical position variables with conjugate momenta $\{P_i\}$. I.e. Q and P form a canonical coordinate system for the $2N$-dimensional symplectic phase space such that the symplectic form reads $J = dQ_i \wedge dP_i$.

The most general quadratic Hamilton function is

$$\begin{aligned} H &= Q_i A_{ij} Q_j + Q_i B_{ij} P_j + P_i C_{ij} Q_j + P_i D_{ij} P_j \\ &= \begin{pmatrix} Q^T & P^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \end{aligned}$$

with $Q^{\mathrm{T}} := (Q_1, Q_2, \ldots, Q_N)$, similarly for P^{T}, and A, B, C, D are $N \times N$ matrices. We have $A = A^{\mathrm{T}}$, $D = D^{\mathrm{T}}$ and $B = C^{\mathrm{T}}$ and all matrix entries in the reals, because classical observables are commuting and real valued. Throughout, $()^{\mathrm{T}}$ will denote the usual matrix transpose. The important positivity constraint mentioned above is now simply a demand on

$$h := \begin{pmatrix} A & B \\ C & D \end{pmatrix} = h^T$$

to be positive semidefinite.

We stress again that this positivity is what makes the classical and bosonic random matrix problems harder to handle than the fermionic ones, because we cannot simply choose some i.i.d. Gaussian measure for the independent matrix elements of h to define our ensemble.

Another way to see why positivity is needed is that the truncation of the expansion of H at quadratic order is only sensible at a local minimum $x = (Q_0, P_0)$ of H, i.e. $dH\Big|_x = 0$ and the Hessian

$$h = \mathrm{Hess}(H)\Big|_x$$

being positive definite. Negative eigenvalues of h would correspond to unstable directions, hence the system would leave the range of applicability of the quadratic expansion and one would then have to include higher order terms, e.g. in a next step proceed to a Ginsburg Landau type model.

After considering the energy, let us now have a look at the equations of motion [2]

$$-J\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = h\begin{pmatrix} Q \\ P \end{pmatrix} \qquad \text{where} \qquad J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix} \tag{2.1}$$

We are looking for a stable fixed point of the linearised differential equation $\dot{x} = Xx$ with $X = Jh$. Physically speaking, as there is no damping, i.e. no energy dissipation mechanism, we will not find an attractive fixed point. Or, more formally, $X = -JX^TJ^{-1} \in \mathfrak{sp}(2N)$ is in the symplectic Lie algebra, which implies $\mathrm{Det}(\lambda\mathbb{1} - X) = \mathrm{Det}(\lambda\mathbb{1} + X)$, i.e. all eigenvalues come in pairs, $\pm\lambda$. Therefore demanding the fixed point to be stable, i.e. $\Re(\lambda) \leq 0$ for all eigenvalues, means that they actually all have to be purely imaginary, $\lambda = \pm i\omega$.

Having settled this reality constraint, we now come to positivity in the more formal picture. For a thermodynamic description to make sense, the action needs to be bounded from below in order for the partition function to exist. This leads to the additional constraint, which is best phrased in a geometric picture. Here positivity of the eigenfrequencies ω means that we have to restrict the domain of feasible random matrices X to a cone within the symplectic Lie algebra

$$D_X = \left\{ X = g\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} g^{-1} \mid \omega = \mathrm{diag}(\omega_1, \ldots, \omega_N), \omega_i \in \mathbb{R}^+, g \in \mathrm{Sp}(\mathbb{R}^{2N}) \right\}$$

where $\mathrm{Sp}(\mathbb{R}^{2N}) = \left\{ g \mid gJg^{\mathrm{T}} = J \right\}$ denotes the real symplectic group. Note that

$$U^{-1}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} U = i\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix} \qquad \text{for} \qquad U := \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_N & \mathbb{1}_N \\ -i\mathbb{1}_N & i\mathbb{1}_N \end{pmatrix} \in \mathrm{U}(2N)$$

i.e. this choice of D_X indeed guarantees that X is diagonalisable with all eigenvalues purely imaginary.

To see that D_X is also well chosen with respect to the positivity constraint, we write the Hamiltonian, $h = J^{-1}X$, in the new basis given by the basis change g.[3]

$$H = \begin{pmatrix} Q^T & P^T \end{pmatrix} g^T\, h\, g \begin{pmatrix} Q \\ P \end{pmatrix} = \sum_i \omega_i \left(\tilde{P}_i^2 + \tilde{Q}_i^2 \right)$$

Hence positivity of the ω_i physically means positive mass and spring constant of the harmonic oscillator modes. A negative mass term would lead to an unbounded spectrum $\{\sum_i n_i\omega_i \mid n_i \in \mathbb{N}\}$, which leads to $e^{-\beta H}$ being non-integrable and hence the partition function would be ill defined.

[2] We use the same symbol for the symplectic form and its matrix as long as there is no ambiguity about the basis.

[3] $g \in \mathrm{Sp}(\mathbb{R}^{2N}) \Leftrightarrow g^{\mathrm{T}} \in \mathrm{Sp}(\mathbb{R}^{2N})$

Bosonic model

As usual we go from the classical to the quantum system by changing the Lie algebra representation from Poisson brackets to $-i$ times the commutator, but we keep the symbols $\{Q_i\}$ for the bosonic 'position' operators with conjugate momentum operators $\{P_i\}$. I.e.

$$[Q_i, P_j] = i\delta_{i,j} \qquad [Q_i, Q_j] = [P_i, P_j] = 0$$

where $i := \sqrt{-1}$. The most general quadratic Hamiltonian now formally looks exactly the same as before.

$$H = \left(Q^T, P^T\right) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

A, B, C, D are $N \times N$ matrices and we have $A = A^T$, $D = D^T$ by the commutation relations. To ensure Hermiticity of H we also need that $B = C^\dagger$ and A, B, C, D all real, so we get the same constituents for the matrix h as in the classical setting above.

Now we can view the positivity constraint from yet another perspective. Again we diagonalise the symmetric bilinear form h by using the same symplectic[4] g as above. Now also the unitary matrix U from above will be recognised, namely as the transformation to creation and annihilation operators $a^{(\dagger)} = \frac{1}{\sqrt{2}}(\tilde{Q} \pm i\tilde{P})$ as usual. Applying both transformations leads to

$$U^{\mathrm{T}} g^T \, h \, gU = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$$

and hence to the standard quantum harmonic oscillator

$$H = \left(a, a^\dagger\right) \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \sum_i \omega_i (a_i^\dagger a_i + a_i a_i^\dagger)$$

where the ω_i are twice the standard frequencies, as in the classical case. And again we see that it is absolutely necessary to demand positivity of the ω_i. We iterate that unlike in the fermionic case, where the particle number operator is bounded (Pauli principle), a single particle eigenstate with negative energy for a bosonic operator leads immediately to an unbounded operator on Fock space.

2.1.2 Modelling the spatial structure

The underlying graph G

Throughout, we will denote the graph which defines the underlying spatial structure by $G = (V, E)$ where V is the (finite) set of vertices and $E \subset (V \times V) / \{(i, j) \sim (j, i)\}$

[4] Whilst g being a symplectic transformation was most natural in the classical phase space, in the quantum setting the reader might be more used to this group of transformations being called 'canonical' and the preservation of the commutation relations under basis change with g being emphasised.

the set of undirected edges. We use the convention $(i,i) \notin E$, such that E describes the actual spatial structure. But for the construction of the model we will mainly use $E_r := E \cup \{(i,i) \in V \times V\}$ which can be visualised by an extra edge per vertex, see Figure 2.1.

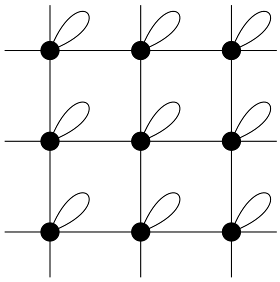

Figure 2.1: Visualisation of the two-dimensional square lattice with an extra edge per site.

Although the considerations in this work apply, to a certain extent, to any kind of graph, we will mostly think of a d-dimensional cubic lattice with periodic boundary in the following. We denote the length of the sides of the cube (counting in number of vertices) by L such that $|V| = L^d$ and $|E_r| = |E| + |V| = |V|(d+1)$.

Furthermore, for $i \in V$ and $e = (i,j) \in E_r$ we introduce the convenient 'adjacency operator' $@\,e := \{i,j\}$ and $@\,i := \{f \in E_r \mid \exists k \in V : f = (i,k)\}$. Instead of $e \in @\,i$ we will just write $e\,@\,i$.

Vector spaces on top of G

We equip the symplectic phase space with a spatial structure as follows:

At each site $i \in V$ we have a symplectic vector space S_i with symplectic form J_i to model the local degrees of freedom. For simplicity we assume all S_i to have the same dimension $2N$. The full phase space of the system is given by

$$S = \bigoplus_{i \in V} S_i$$

with the corresponding symplectic form $J = \bigoplus_{i \in V} J_i$. We denote the canonical projections by $\pi_i^S : S \twoheadrightarrow S_i$.

Additionally we introduce auxiliary Euclidean vector spaces for each edge, again all with the same dimension $\forall_{e \in E_r} O_e \simeq \mathbb{R}^M$ and a total Euclidean space

$$O = \bigoplus_{e \in E} O_e$$

with canonical projections $\pi_e^O : S \twoheadrightarrow S_e$. Those will be used to model the random interactions in between neighbouring grains by $e \in E$ and random couplings within the grains, i, by $(i,i) \in E_R$.

2.1.3 Restrictions on the Hamiltonian

The matrix h defining the characteristic frequencies as in 2.1.1 has not only to be positive, but additionally we require it to reflect the spatial structure imposed by the underlying graph.

$$h \in \mathbb{S} := \left\{ h \in \mathrm{End}(S) \mid h = h^{\mathrm{T}} \text{ positive semidefinite and } \sum_{(i,j)\in E_r} \pi_i^S \circ h \circ \pi_j^S = h \right\}$$

Note that the last condition can equivalently be written as

$$\pi_i^S \circ h \circ \pi_j^S \neq 0 \Rightarrow (i,j) \in E_r$$

and further that we only demand *semi*definiteness, i.e. we also include the boundary of our cone and pass to $\overline{D_X}$. In fact, later on we will see that only models which explicitly enforce a macroscopical number of zero modes have a finite, non vanishing density of eigenfrequencies near $\omega = 0$ for $d = 0$.

Implementing the restrictions

To get our hands on a probability measure for h, we first decompose $h = l^T l$. Thereby, instead of demanding h to be symmetric positive semidefinite, we only need to require l to be real. Secondly, to implement the spatial structure, we use the auxiliary Euclidean vector spaces on the edges, as introduced in 2.1.2. We define the following space of linear mappings

$$\mathbb{L} := \left\{ l : S \xrightarrow[\text{linear}]{} O \mid l = \sum_{\substack{i \in V \\ e\,@\,i}} \pi_e^O \circ l \circ \pi_i^S \right\} \tag{2.2}$$

Note the important summation constraint $e\,@\,i$ which means that $l(S_i) \subset \bigoplus_{e\,@\,i} O_e$ and $l^T(O_e) \subset \bigoplus_{i\,@\,e} S_i$. Therefore we get a well defined mapping

$$\begin{aligned} \phi : \mathbb{L} &\to \mathbb{S} \\ l &\mapsto h = l^T l \end{aligned}$$

The rest of this section will be spend on investigating the properties of ϕ. First, by diagonalising

$$g\, l^T l\, g^T = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2N|V|})$$

with some g in the big orthogonal group $\mathrm{O}(\mathbb{R}^{2N|V|})$, we see that $\phi(\mathbb{L})$ contains regular elements if and only if

$$\begin{aligned} & M|E_r| \geq 2N|V| \\ \Leftrightarrow\;\; & M \geq 2N\frac{|V|}{|E_r|} = \frac{2N}{d+1} \end{aligned} \tag{2.3}$$

where the last equality holds for the d-dimensional cubic lattice. This inequality motivates the introduction of the constant

$$\alpha := \frac{(d+1)M}{2N} + \mathcal{O}\left(\frac{1}{N}\right) \tag{2.4}$$

where we anticipate that we will only need α up to terms of order $\mathcal{O}\left(\frac{1}{N}\right)$ in the sequell. α gets physical meaning by rephrasing the above observation: Our ensemble contains systems without zero modes as soon as $\alpha \geq 1$. In this case the probability of having zero modes is actually zero.[5]

Note, however, that $\mathrm{O}(\mathbb{R}^{2N|V|})$ does not respect the spatial structure and ϕ will in general not be surjective onto $\mathbb{S}$. This can be seen by comparing dimensions

$$\begin{aligned} \dim(\mathbb{L}) &= 2NM \cdot (2|E| + |V|) \\ \dim(\mathbb{S}) &= \frac{2N(2N+1)}{2}|V| + (2N)^2|E| \end{aligned} \tag{2.5}$$

which means that ϕ can only be surjective for at least

$$M \geq \frac{2N+1}{2}\frac{|V|}{2|E|+|V|} + 2N\frac{|E|}{2|E|+|V|} = N + \frac{1}{4d+2}$$

i.e. $\alpha \geq \frac{d+1}{2}$.

To get a more accurate estimate, one can easily see that the differential $\mathrm{d}\phi\big|_{l_0}$ at

$$l_0 = \bigoplus_{v \in V} \left(\mathrm{diag}(1, \dots, 1) : S_v \to O_{(v,v)}\right)$$

is surjective if and only if $M \geq 2N$. This means that only for $\alpha \geq d+1$ really all modes in neighbouring grains are coupled almost surely.

Choosing a family of probability measures

Using the implementation of the restrictions from the last section, the question of what would be a natural, or at least tractable, measure on the space of Hamiltonians of interest given by $\mathbb{S}$ can now be traced back to defining a 'good' measure on $\mathbb{L}$ and then pushing this forward with ϕ.

As explained in [Lü09], chapter 3.2, it is not possible to choose a measure like $e^{-\mathrm{Tr}(Jh)^2}$, which is invariant under the full symplectic symmetry group, because Sp is non-compact. Instead, as in [Lü09], we will consider a one-parameter family of measures which is invariant under the maximal compact subgroup U of Sp.

On the level of l it is very natural to consider the Gaussian measure

$$\mathrm{d}\mu(l) \propto e^{-\frac{1}{2}\mathrm{Tr}(l^T l)}\mathrm{d}l \tag{2.6}$$

[5] Strictly speaking, for $\alpha = 1$ the $\mathcal{O}\left(\frac{1}{N}\right)$ terms matter, even in the large N limit, as can be seen in (2.3). However, for $\alpha \geq 1$ a realisation does almost surely not contain *macroscopically many* zero modes.

where dl is the flat measures on $\mathbb{L}$. This measure is in fact invariant under the product of unitary and orthogonal groups $\prod_{i\in V} \mathrm{U}(S_i) \times \prod_{e\in E_r} \mathrm{O}(O_e)$, acting by

$$l \mapsto hlg^{\mathrm{T}} \tag{2.7}$$

with $h \in \prod_{e\in E_r} \mathrm{O}(O_e)$ and $g \in \prod_{i\in V} \mathrm{U}(S_i)$ where $\mathrm{U}(S_i) \subset \mathrm{Sp}(S_i)$ is understood.

If the differential dϕ has full rank, i.e. $\alpha \geq d+1$, this measure can be pushed forward to

$$\mathrm{d}\mu_k(h) = \phi_*(\mathrm{d}\mu(l)) \propto e^{-\frac{1}{2}\operatorname{Tr} h} \operatorname{Det}(h)^k \mathrm{d}h \tag{2.8}$$

where dh is the flat measure on the subset $\phi(\mathbb{L}) \subset \mathbb{S}$. Otherwise the push forward of d$\mu(l)$ will be concentrated on the subset $\phi(\mathbb{L}) \subset \mathbb{S}$ of lower dimension. In terms of the original setting, our model is specified by d$\mu(h)$. Note that, in particular, we possibly restrict the domain of Hamiltonians defining the model further, if $\operatorname{supp}(\mathrm{d}\mu(h)) \subsetneq \mathbb{S}$. If $\alpha \geq d+1$ we can use equation (2.8) together with (2.5) to determine the relation in between the parameters M and k. By a simple scaling argument we get

$$\begin{aligned} \frac{\dim(\mathbb{L})}{2N|V|} &= 2k + \frac{2\dim(\mathbb{S})}{2N|V|} \\ \Leftrightarrow (2d+1)M &= 2k + 4Nd + 2N + 1 \\ \Leftrightarrow k &= (2d+1)\left(\frac{\alpha}{d+1} - 1\right)N - \frac{1}{2} \end{aligned} \tag{2.9}$$

where from the second line we specialise to the hyper cubic lattice.

Here we see directly that $\alpha = d + 1 + \mathcal{O}\left(\frac{1}{N}\right)$, as considered in [Lü09] and [LSZ06] for $d = 0$, is a distinguished value for α, where k is independent of N. Furthermore, we can see directly that pushing forward d$\mu(l)$ with $\alpha < d+1$ will lead to a singular measure for h.

In the sequel, the normalisation factors are chosen such that $\int_{\mathbb{S}} d\mu(h) = 1$, hence $\int_{\mathbb{L}} d\mu(l) = 1$, i.e. both are probability measures.

2.2 The resolvent operator

2.2.1 Definition

The quantity to be studied is the disorder averaged resolvent operator of the equations of motion (2.1)

$$G(z) := \langle \operatorname{Tr}(z - Jh))^{-1} \rangle := \int_{\mathbb{S}} \operatorname{Tr}(z - Jh)^{-1} d\mu(h)$$

because we are interested in the density of characteristic frequencies. We have seen in 2.1.1 that all eigenvalues of Jh are imaginary and by the so-called Dirac identity

$$\lim_{\epsilon \searrow 0} = \frac{1}{i\omega + \epsilon - i\omega_0} = \pi\delta(\omega - \omega_0) + i\,\mathrm{P}\left(\frac{1}{\omega_0 - \omega}\right)$$

where P denotes the principal part, we get

$$\rho(\omega) = \lim_{\epsilon \searrow 0} \frac{1}{\pi} \Re\left(G(i\omega + \epsilon)\right) \tag{2.10}$$

Note that G will be analytic, away from the *imaginary* axis, i.e. the conventions here might differ by a 'Wick rotation' by i from the convention for the Greens function, which the reader is used to. This is also visible in (2.10).

First we move from the original cone of operators $\mathbb{S}$ completely to $\mathbb{L}$ by using ϕ^* and rewriting the trace. Only in the following calculation we will emphasise which spaces are traced over, in the rest of the text this should be clear from the context. For large z we have

$$\begin{aligned}
\operatorname*{Tr}_{\mathbb{S}}(z - Jl^T l)^{-1} &= \frac{1}{z} \operatorname*{Tr}_{\mathbb{S}} \left(\frac{1}{1 - z^{-1} J l^T l} \right) = \frac{1}{z} \operatorname*{Tr}_{\mathbb{S}} \left(\sum_{n=0}^{\infty} (z^{-1} J l^T l)^n \right) \\
&= \frac{1}{z} \left(\operatorname*{Tr}_{\mathbb{S}}(\mathbb{1}) - \operatorname*{Tr}_{\mathbb{L}}(\mathbb{1}) + \operatorname*{Tr}_{\mathbb{L}} \left(\sum_{n=0}^{\infty} (z^{-1} l J l^T)^n \right) \right) \\
&= \frac{\dim(\mathbb{S}) - \dim(\mathbb{L})}{z} + \operatorname*{Tr}_{\mathbb{L}}(z - lJl^T)^{-1}
\end{aligned}$$

As G is analytic away from the imaginary axis, this has to hold not only for large, but for all $z \notin i\mathbb{R}$. In the following we will write

$$\Delta \dim := \dim(\mathbb{S}) - \dim(\mathbb{L}) = \left(2(2d+1) \left(1 - \frac{2\alpha}{d+1} \right) N + 1 \right) N|V|$$

for short. As mentioned already in section 2.1.3, $\alpha = \frac{d+1}{2} + \mathcal{O}\left(\frac{1}{N}\right)$ is the critical value where $\Delta \dim$ changes sign. I.e. for $\alpha < \frac{d+1}{2}$, $\Delta \dim$ is positive and $\frac{\Delta \dim}{z}$ can be interpreted as the contribution of the zero modes to the resolvent operator.

2.2.2 Gaussian integrals

The next step is to write the trace in terms of determinants.

$$\operatorname{Tr}(z - lJl^T)^{-1} = \partial_{z_1}\Big|_{z_1 = z_2 = z} \operatorname{Det}^{-1} \begin{pmatrix} z_2 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} \operatorname{Det} \begin{pmatrix} z_1 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} \tag{2.11}$$

This equality is verified by using

$$\operatorname{Det} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{Det}(D) \operatorname{Det}(A - BD^{-1}C)$$

which holds for any block matrix with invertible D, and

$$\partial_z \operatorname{Det}(z\mathbb{1} - A) = \partial_z e^{\operatorname{Tr} \ln(z - A)} = \operatorname{Det}(z - A) \operatorname{Tr}(z - A)^{-1}$$

holds for any matrix A, as long as the expressions on both sides exist, i.e. z is not in the spectrum of A. Note further, that for a differentiable function f which is non-vanishing at y

$$\partial_x\big|_{x=y}\frac{f(y)}{f(x)} = -\partial_y\big|_{x=y}\frac{f(y)}{f(x)} \tag{2.12}$$

hence we can in (2.11) as well differentiate with respect to z_2, up to a change of sign.

The determinants are then expressed in terms of Gaussian integrals over complex vectors

$$\mathrm{Det}^{-1}\begin{pmatrix} z_2 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} = \int\limits_{\mathbb{C}^{M|E_r|}} \mathrm{d}\bar{u}\mathrm{d}u \int\limits_{\mathbb{C}^{2N|V|}} \mathrm{d}\bar{v}\mathrm{d}v \; e^{-z_2 u^\dagger u + u^\dagger l v - v^\dagger l^{\mathrm{T}} u - v^\dagger J v} \tag{2.13}$$

and Grassmann variables, where we refrain from calling those complex,

$$\mathrm{Det}\begin{pmatrix} z_1 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} = \int\limits_{\mathrm{Gr}^{M|E_r|}} \mathrm{d}\rho\mathrm{d}\bar{\rho} \int\limits_{Gr^{2N|V|}} \mathrm{d}\xi\mathrm{d}\bar{\xi} \; e^{z_1\rho^\dagger\rho - \rho^\dagger l \xi + \xi^\dagger l^{\mathrm{T}}\rho + \xi^\dagger J \xi} \tag{2.14}$$

$\int_{\mathrm{Gr}^M}\mathrm{d}\rho\mathrm{d}\bar{\rho} = \partial_{\rho_M}\Big|_0 \partial_{\bar{\rho}_M}\Big|_0 \dots \partial_{\rho_1}\Big|_0 \partial_{\bar{\rho}_1}\Big|_0$ denotes Berezin integration over M Grassmann variables, which one should actually think of as differentiation. Note that $\bar{\rho}$ does not denote complex conjugation but is just another Grassmann variable independent of ρ. Note further that we have chosen to integrate first with respect to $\bar{\rho}$ and then with respect to ρ which leads to the global plus sign in the exponent of the Gaussian integral. Similarly, we denote the column vector $\rho^\dagger := (\bar{\rho}_1, \dots, \bar{\rho}_M)$ just for notational similarity with the $()^\dagger$ symbol, not implying a Hermitian product. However, when a row and a column vector meet, there is an implicit Euclidean scalar product or sum over Grassmann wedge products, as usual.

Disorder average

Now we symmetrise and rearrange the expressions from (2.13) and (2.14) involving the random matrix l

$$\begin{aligned} &\Big(u^\dagger l v - v^\dagger l^{\mathrm{T}} u - \rho^\dagger l \xi + \xi^\dagger l^{\mathrm{T}} \rho\Big) \\ &= \mathrm{Tr}(l^{\mathrm{T}} B) = \frac{1}{2}\left(\mathrm{Tr}(l^{\mathrm{T}} B) + \mathrm{Tr}(B^{\mathrm{T}} l)\right) \end{aligned}$$

with the dyadic product

$$B = \bar{u}v^{\mathrm{T}} - uv^\dagger - \bar{\rho}\xi^{\mathrm{T}} - \rho\xi^\dagger \tag{2.15}$$

and $()^{\mathrm{T}}$ still denoting the usual matrix transpose, i.e.

$$B^{\mathrm{T}} = vu^\dagger - \bar{v}u^{\mathrm{T}} + \xi\rho^\dagger + \bar{\xi}\rho^{\mathrm{T}}$$

where the minus signs are due to the interchange of Grassmann variables. Now we can easily carry out the disorder average

$$
\begin{aligned}
&\left\langle \exp\left(u^\dagger l v - v^\dagger l^{\mathrm{T}} u + \rho^\dagger l \xi - \xi^\dagger l^{\mathrm{T}} \rho\right)\right\rangle \\
&\quad = \int\limits_{\mathbb{L}} \mathrm{d}l \exp\left(-\frac{1}{2}\,\mathrm{Tr}\left(l^{\mathrm{T}} l\right) + \mathrm{Tr}\left(l^{\mathrm{T}} B\right)\right) \\
&\quad = \int\limits_{\mathbb{L}} \mathrm{d}l \exp\left(-\frac{1}{2}\,\mathrm{Tr}\left(\sum_{\substack{i\in V\\ e@i}} l^{\mathrm{T}} \pi_e^O l \pi_i^S\right) + \mathrm{Tr}\left(\sum_{\substack{i\in V\\ e@i}} l^{\mathrm{T}} \pi_e^O B \pi_i^S\right)\right) \\
&\quad = \exp\left(\frac{1}{2}\,\mathrm{Tr}\left(\sum_{\substack{i\in V\\ e@i}} B^{\mathrm{T}} \pi_e^O B \pi_i^S\right)\right)
\end{aligned}
\tag{2.16}
$$

where we used the measure (2.6) and one-dimensional Gaussian integration, involving the invariance of $\int_{\mathbb{R}} e^{-x^2} dx$ under a shift of the integration contour by a complex offset. In the second equality in (2.16) we have made the structure of $\mathbb{L}$, as in (2.2), explicit, to stress that the integration does not run over all possible matrix elements of l. This leads to the appearance of the projection operators in the last step. Another way to see $B = \sum_{\substack{i\in V\\ e@i}} B^{\mathrm{T}} \pi_e^O B \pi_i^S$ is directly from (2.15), or (2.17) below, and $u = \bigoplus_{e\in E_r} u_e$, $v = \bigoplus_{i\in V} v_i$ and similarly for the Grassmann variables.

Note that the last step looks like we have shifted l by Grassmann variables. If the reader feels uncomfortable about this, let us look at any of the one-dimensional Gaussian integrals in more detail. For b an element of the Grassmann algebra, $b = b^{\mathbb{C}} + \xi$, where $b^{\mathbb{C}}$ is the numerical part and ξ a Grassmann variable, we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}} e^{-\frac{1}{2}x^2 + bx} \mathrm{d}x &= \int\limits_{\mathbb{R}} e^{-\frac{1}{2}x^2 + b^{\mathbb{C}} x} (1 + x\xi) \mathrm{d}x \\
&= e^{-\frac{1}{2}(b^{\mathbb{C}})^2} (1 - b^{\mathbb{C}} \xi) = e^{-\frac{1}{2} b^2}
\end{aligned}
$$

Here we can see Wick's theorem at work in the second step. As $\mathrm{Tr}(l^{\mathrm{T}} B)$ contains only terms with at most one Grassmann variable this is all we need for (2.16). But Wick's Theorem of course works to all orders, so the formula generalises also to b containing an arbitrary number of Grassmann variables.

2.2.3 Superbosonisation

The general idea of superbosonisation is to exploit the symmetries of an integrand to reduce the number of integration variables. In our case we will use the symmetry (2.7) to reduce the macroscopicaly large number of integrations to 8 integrations *independent of* N, at each site. As we are going to make use of a special version of superbosonisation, which is rightly dubbed 'orthogonal' or 'real', we will now completely abandon the $()^\dagger$ symbol and explicitly use $\overline{()}^{\mathrm{T}}$ instead.

From here on we will make use of some super-mathematics, an introduction to which can be found e.g. in the book by Efetov [Efe99], chapter 2. Though we use a slightly different convention than Efetov and write the complex variables upstairs and the Grassmann variables downstairs in column vectors. We will therefore also briefly state the definitions of the super-operations used here as they appear. During this work we will always deal with four by four super-matrices

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where the four blocks are two by two each. A is called boson-boson block, D is called fermion-fermion block and both contain only even elements of the Grassmann algebra. In this section this will be products of two Grassmann variables for D and A will be numerical valued, after superbosonisation we will have just numbers in both. B and C are called fermion-boson and boson-fermion block and contain only odd elements of the Grassmann algebra. Note that this whole nomenclature does not refer to physical bosonic or fermionic particles, but to the commuting or anti-commuting nature of the variables.

To sum up what was just said, one can more concisely demand a super-matrix to represent a morphism of Z_2 graded linear spaces. These morphisms are naturally $(\mathrm{Z}_2)^2$ graded where 0 and 1 translate to 'boson-' and 'fermion-'. $\oplus : (\mathrm{Z}_2)^2 \to \mathrm{Z}_2$ where $a \oplus b = a + b \mod 2$ gives the Z_2 grading of super-morphisms, hence the diagonal blocks are even and the off diagonal ones are odd. For writing super-matrices in a concise way we introduce the elementary $(Z_2)^2$ graded matrices

$$E_{BB} := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \qquad E_{FF} := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right) \qquad \mathbb{1}_{1|1} := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

We start the derivation by decomposing B again

$$B_{e,i} := \pi^O_e B \pi^S_i = \begin{pmatrix} u & \bar{u} & \rho & \bar{\rho} \end{pmatrix}_e \begin{pmatrix} -\bar{v}^{\mathrm{T}} \\ v^{\mathrm{T}} \\ -\bar{\xi}^{\mathrm{T}} \\ -\xi^{\mathrm{T}} \end{pmatrix}_i \tag{2.17}$$

to rewrite

$$-B^{\mathrm{T}}_{e,i} B_{e,i} = \begin{pmatrix} v & -\bar{v} & \xi & -\bar{\xi} \end{pmatrix}_i P_e \begin{pmatrix} \bar{v}^{\mathrm{T}} \\ -v^{\mathrm{T}} \\ \bar{\xi}^{\mathrm{T}} \\ \xi^{\mathrm{T}} \end{pmatrix}_i$$

where we have introduced the dyadic product

$$P_e = \begin{pmatrix} \bar{u}^{\mathrm{T}} \\ u^{\mathrm{T}} \\ \bar{\rho}^{\mathrm{T}} \\ -\rho^{\mathrm{T}} \end{pmatrix}_e \begin{pmatrix} u & \bar{u} & \rho & \bar{\rho} \end{pmatrix}_e = \left(\begin{array}{cc|cc} \bar{u}^{\mathrm{T}} u & \bar{u}^{\mathrm{T}} \bar{u} & \bar{u}^{\mathrm{T}} \rho & \bar{u}^{\mathrm{T}} \bar{\rho} \\ u^{\mathrm{T}} u & u^{\mathrm{T}} \bar{u} & u^{\mathrm{T}} \rho & u^{\mathrm{T}} \bar{\rho} \\ \hline \bar{\rho}^{\mathrm{T}} u & \bar{\rho}^{\mathrm{T}} \bar{u} & \bar{\rho}^{\mathrm{T}} \rho & 0 \\ -\rho^{\mathrm{T}} u & -\rho^{\mathrm{T}} \bar{u} & 0 & \bar{\rho}^{\mathrm{T}} \rho \end{array}\right)_e$$

This P_e is the matrix of all $\mathrm{O}(M)$ invariants which we can form out of u and ρ and this is where superbosonisation will take place. Throughout we will use dividers in super-matrices to make the $(Z_2)^2$ grading visible. For further explanation see below.

One should keep in mind that we have M-dimensional objects u and ρ at each edge and $2N$-dimensional objects v and ξ at each vertex. To unclutter the notation we have pulled the indices i and e denoting the corresponding vertex or edge, outside the super-objects.

To prepare for the Gaussian integration over v and ξ we conclude rewriting

$$\frac{1}{2}\operatorname{Tr} B_{e,i}^{\mathrm{T}} B_{e,i} = -\frac{1}{2}\Phi_i^{\mathrm{T}}\left(\left(h_1 P_e^{\mathrm{ST}} h_2\right)\otimes \mathbb{1}_{2N}\right)\Phi_i \tag{2.18}$$

where

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{\mathrm{ST}} = \left(\begin{array}{c|c} A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right)$$

denotes super-transposition. Further we have composed the super-vectors

$$\Phi_i = \begin{pmatrix} v_i \\ \bar{v}_i \\ \xi_i \\ \bar{\xi}_i \end{pmatrix} \qquad \Rightarrow \qquad \Phi_i^{\mathrm{T}} = \begin{pmatrix} v_i^{\mathrm{T}} & \bar{v}_i^{\mathrm{T}} & \xi_i^{\mathrm{T}} & \bar{\xi}_i^{\mathrm{T}} \end{pmatrix}$$

The reshuffling matrices

$$h_1 = \left(\begin{array}{c|c} -i\sigma_2 & 0 \\ \hline 0 & \sigma_1 \end{array}\right) \qquad \text{and} \qquad h_2 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & \mathbb{1}_2 \end{array}\right)$$

will be of no importance later.

Next we rearrange

$$-z_2 \bar{u}_e^{\mathrm{T}} u_e + z_1 \bar{\rho}_e^{\mathrm{T}} \rho_e = -\frac{1}{2}\operatorname{STr}\left(\tilde{z} P_e\right) \tag{2.19}$$

where we introduced the four by four super-matrix

$$\tilde{z} := \left(\begin{array}{c|c} z_2\, \mathbb{1}_2 & 0_2 \\ \hline 0_2 & z_1\, \mathbb{1}_2 \end{array}\right)$$

and we are using the super-trace

$$\operatorname{STr}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \operatorname{Tr}(A) - \operatorname{Tr}(D)$$

for the first time. Note that in general this is a super-function, i.e. it takes values in the even part of the Grassmann algebra.

Finally, we also rewrite the parts of the exponent containing the symplectic J from (2.13) and (2.14).

$$\begin{aligned}
& - v_i^\dagger J_i v_i + \xi_i^\dagger J_i \xi_i \\
&= -\frac{1}{2} \operatorname*{Tr}_{\mathbb{C}^{2N}} \left(J_i \left(v\bar{v}^{\mathrm{T}} - \bar{v} v^{\mathrm{T}} + \xi\bar{\xi}^{\mathrm{T}} + \bar{\xi}\xi^{\mathrm{T}} \right)_i \right) \\
&= -\frac{1}{2} \operatorname*{Tr}_{\mathbb{C}^{2N}} \left(J_i \begin{pmatrix} v & -\bar{v} & \xi & -\bar{\xi} \end{pmatrix}_i \Sigma_3 \begin{pmatrix} \bar{v}^{\mathrm{T}} \\ -v^{\mathrm{T}} \\ \bar{\xi}^{\mathrm{T}} \\ \xi^{\mathrm{T}} \end{pmatrix}_i \right) \\
&= -\frac{1}{2} \Phi_i^{\mathrm{T}} \left((h_1 \Sigma_3^{\mathrm{ST}} h_2) \otimes J_i \right) \Phi_i
\end{aligned} \tag{2.20}$$

Here

$$\Sigma_3 = \mathbb{1}_{1|1} \otimes \sigma_3 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & \sigma_3 \end{array} \right) = \Sigma_3^{\mathrm{ST}}$$

is again a matrix in $2|2$-dimensional super-space, whilst the trace runs over the $2N$-dimensional complex space at the corresponding vertex i. All objects J, v and ξ are understood to live at this vertex.

Now we collect all terms from (2.18), (2.19) and (2.20) to get a formula for the ratio of determinants (2.11)

$$\begin{aligned}
&\mathrm{Det}^{-1} \begin{pmatrix} z_2 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} \mathrm{Det} \begin{pmatrix} z_1 & -l \\ l^{\mathrm{T}} & J \end{pmatrix} = \\
&\qquad \int\limits_{\mathbb{C}^{M|E_r|}} \mathrm{d}\bar{u}\mathrm{d}u \int\limits_{\mathrm{Gr}^{M|E_r|}} \mathrm{d}\rho\mathrm{d}\bar{\rho} \quad e^{-\frac{1}{2}\mathrm{STr}\left(\tilde{z}\sum_{e\in E_r} P_e\right)} \prod_{i\in V} \int\limits_{\mathbb{C}^{2N}} \mathrm{d}\bar{v}_i \mathrm{d}v_i \int\limits_{Gr^{2N}} \mathrm{d}\xi_i \mathrm{d}\bar{\xi}_i \\
&\qquad \exp\left(-\frac{1}{2} \left(\Psi_i^{\mathrm{T}} \left(\sum_{e\,@\,i} (h_1 P_e^{\mathrm{ST}} h_2) \otimes \mathbb{1}_{2N} + (h_1 \Sigma_3^{\mathrm{ST}} h_2) \otimes J_i \right) \Psi_i \right) \right)
\end{aligned} \tag{2.21}$$

Note the explicit tensor product to distinguish the 4 by 4 super-matrix space from the $2N$ by $2N$ matrix space at the vertex.

Next we can integrate out the auxiliary variables on the vertices

$$\begin{aligned}
(2.21) \propto & \int\limits_{\mathbb{C}^{M|E_r|}} \mathrm{d}\bar{u}\mathrm{d}u \int\limits_{\mathrm{Gr}^{M|E_r|}} \mathrm{d}\rho\mathrm{d}\bar{\rho} \quad e^{-\frac{1}{2}\mathrm{STr}\left(\tilde{z}\sum_{e\in E_r} P_e\right)} \\
& \prod_{i\in V} \mathrm{SDet}_{2|2} \otimes \mathrm{Det}_{2N} \left((h_1 \Sigma_3^{\mathrm{ST}} h_2) \otimes J_i + \sum_{e\,@\,i} (h_1 P_e^{\mathrm{ST}} h_2) \otimes \mathbb{1}_{2N} \right)^{-\frac{1}{2}}
\end{aligned}$$

Again we emphasise the distinct spaces over which we have to take the determinant and we use the super-determinant for the first time.

$$\mathrm{SDet}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \mathrm{Det}(A)\,\mathrm{Det}\left(D - CA^{-1}B\right)^{-1} = \mathrm{Det}\left(A - BD^{-1}C\right)\mathrm{Det}(D)^{-1} \tag{2.22}$$

Note that this is again a super-function with values in the Grassmann algebra, i.e. the ordinary determinant appearing in the definition is to be understood as a polynomial in the matrix entries, not as a real valued function. Note further that the Gaussian super-integral yield the super determinant. This is proven completely analogously to the ordinary complex or pure Grassmann Gaussian integrals.

One of the products is very simple, writing $J = \mathbb{1}_N \otimes i\sigma_2$, $\mathbb{1}_{2N} = \mathbb{1}_N \otimes \mathbb{1}_2$ and using the elementary determinant $\mathrm{Det}(x\mathbb{1}_2 + yi\sigma_2) = x^2 + y^2 = (x - iy)(x + iy)$ we get

$$(2.21) \propto \int\limits_{\mathbb{C}^{M|E_r|}} \mathrm{d}\bar{u}\mathrm{d}u \int\limits_{\mathrm{Gr}^{M|E_r|}} \mathrm{d}\rho\mathrm{d}\bar{\rho} \quad e^{-\frac{1}{2}\mathrm{STr}\left(\tilde{z}\sum_{e\in E_r} P_e\right)}$$
$$\prod_{i\in V} \mathrm{SDet}^{-\frac{N}{2}}\left(\sum_{e\,@\,i} P_e + i\Sigma_3\right)\left(\sum_{e\,@\,i} P_e - i\Sigma_3\right)$$

Now we also dropped the constant factors $\mathrm{SDet}(h_j)$ and used $\mathrm{SDet}(X) = \mathrm{SDet}(X^{\mathrm{ST}})$.

This can be simplified further by using the symmetry of P

$$P = \Gamma P^{\mathrm{ST}} \Gamma^{-1}$$

with

$$\Gamma = E_{BB} \otimes \sigma_2 + E_{FF} \otimes i\sigma_2 = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & i\sigma_2 \end{array}\right)$$

Furthermore, $\Gamma\Sigma_3^{\mathrm{ST}}\Gamma^{-1} = -\Sigma_3$, so we get $\mathrm{SDet}(P - i\Sigma_3) = \mathrm{SDet}(P + i\Sigma_3)$ and finally we can apply superbosonisation to turn the integrals over u and ρ into matrix super-integrals over P.

For a comprehensive explanation of the method see [LSZ07] and note that we are using the version for orthogonal symmetry.[6] More precisely we exploit the $\mathrm{O}(O)$ symmetry in (2.7).

$$\int\limits_{\mathbb{C}^M} \mathrm{d}\bar{u}\mathrm{d}u \int\limits_{\mathrm{Gr}^M} \mathrm{d}\rho\mathrm{d}\bar{\rho}\, F(P(u_e, \bar{u}_e, \rho_e, \bar{\rho}_e)) \propto \int\limits_{(\mathrm{Gl}_{2|2}\,/\,\mathrm{OSp}_{2|2})} \mathrm{d}\mu(P_e)\ \mathrm{SDet}^{\frac{M}{2}}(P_e)\ F(P_e)$$

holds for each of the $|E_r|$ sets of variables indexed by e. Note that the invariant measure

$$\mathrm{d}\mu(P) = \mathrm{SDet}^{\frac{q-p-1}{2}}(P)\mathrm{d}P$$

[6]For direct comparison with [LSZ07] note that our matrix P is of the form of equation (1.6), with $\beta = \mathbb{1}$ and we are going to use their result (1.13).

where p is the dimension of the boson-boson and q of the fermion-fermion block, contributes, in our case, $p = q = 2$, another $\mathrm{SDet}^{-\frac{1}{2}}(P)$ and $\mathrm{d}P$ is the flat measure on the matrix space under consideration. See (2.27) below for an explicit form of $\mathrm{d}P$ and the integration domain $\mathrm{Gl}\,/\,\mathrm{OSp}$ in coordinates in a slightly different representation.

In the end we perform the derivative in (2.11) and end up with

$$G(z) - \frac{\Delta\dim}{z}$$
$$\propto \int_{(\mathrm{Gl}_{2|2}\,/\,\mathrm{OSp}_{2|2})^{|E_r|}} (\prod_{e\in E} \mathrm{d}P_e) \sum_{e\in E} \mathrm{Tr}(P_{e,FF}) \frac{e^{-\frac{z}{2}\sum_e \mathrm{STr}\, P_e} \prod_{e\in E} \mathrm{SDet}\, P_e^{\frac{M-1}{2}}}{\prod_{i\in V} \mathrm{SDet}^N(\sum_{e@i} P_e - i\Sigma_3)}$$

where $\mathrm{Tr}(P_{e,FF}) := \mathrm{STr}\left((E_{FF}\otimes \mathbb{1}_2)P_e\right)$ denotes the trace over the fermion-fermion block only. Note that we could as well average the trace over the boson-boson block, $-\mathrm{Tr}(P_{e,BB})$, as explained in 2.2.2 above.

We perform a final change of representation to compare with an unpublished variant of [LSZ06].

$$\tilde{P} := gPg^{-1}$$

with

$$g = \frac{e^{i\phi}}{\sqrt{2}}\left(\mathbb{1}_{2|2} - i\Sigma_1\right)$$

where $\phi = -\frac{\pi}{4}$, $\Sigma_1 = \mathbb{1}_{1|1}\otimes\sigma_1$ is similar to Σ_3 and $\mathbb{1}_{2|2} = \mathbb{1}_{1|1}\otimes\mathbb{1}_2$. This redefinition leads to

$$\mathrm{SDet}^N(P - i\Sigma_3) = \mathrm{SDet}^N(\tilde{P} - i\Sigma_2)$$

with $\Sigma_2 = \mathbb{1}_{1|1}\otimes\sigma_2$ whilst SDet and STr are invariant under conjugation. As $g \propto \mathbb{1}_{1|1}$ is diagonal in super-space, also the traces over the boson-boson or fermion-fermion block are individually conserved. Or, in other words, g commutes with $\tilde{z}$, therefore we can still choose whether to average the trace over boson-boson, or fermion-fermion block. The symmetry of the new $\tilde{P}$ is given by

$$\tilde{P}^{\mathrm{ST}} = \gamma^{-1}\tilde{P}\gamma$$

with

$$\gamma = g\Gamma g = \left(\begin{array}{c|c} \mathbb{1}_2 & 0 \\ \hline 0 & \sigma_2 \end{array}\right) \tag{2.23}$$

From now on we omit ˜ and use

$$G(z) - \frac{\Delta\dim}{z}$$
$$\propto \int\cdots\int_{(\mathrm{Gl}_{2|2}\,/\,\mathrm{OSp}_{2|2})^{|E_r|}} (\prod_{e\in E} \mathrm{d}P_e) \sum_{e\in E} \mathrm{Tr}(P_{e,FF}) \frac{e^{-\frac{z}{2}\sum_e \mathrm{STr}\, P_e} \prod_{e\in E} \mathrm{SDet}\, P_e^{\frac{M-1}{2}}}{\prod_{i\in V} \mathrm{SDet}^N(\sum_{e@i} P_e - i\Sigma_2)} \tag{2.24}$$

where the new version of $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ is defined by

$$\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} = \left\{ P = \gamma P^{\mathrm{ST}} \gamma^{-1} \right\} \tag{2.25}$$

The domain of integration is given by the boson-boson block being real, symmetric and positive and the fermion-fermion block being $\mathbb{1}_2 \otimes \mathrm{U}(1)$ with arbitrary radius for $\mathrm{U}(1) \hookrightarrow \mathbb{C}$. The flat measure can now be explicitly given in coordinates $a_0, a_1, a_3 \in \mathbb{R}$ with $a_0^2 - a_1^2 - a_3^2 > 0$ and $b \in \mathrm{U}(1)$

$$P = \left(\begin{array}{c|c} a_0 \mathbb{1}_2 + a_1 \sigma_1 + a_3 \sigma_3 & F \\ \hline (F\sigma_2)^{\mathrm{T}} & b \mathbb{1}_2 \end{array} \right) \qquad \text{with} \qquad F = \begin{pmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{pmatrix} \tag{2.26}$$

where $\chi_1, \ldots \chi_4$ are independent Grassmann variables and the flat measure is simply

$$\mathrm{d}P = \mathrm{d}a_0 \mathrm{d}a_1 \mathrm{d}a_3 \mathrm{d}b \; \partial_{\chi_4} \partial_{\chi_3} \partial_{\chi_2} \partial_{\chi_1} \tag{2.27}$$

For more details about the Riemannian symmetric super-space $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ and super-integration see section 3.2, appendix A.1 and [Zir98].

In summary of this chapter, a concise form of the resolvent operator for the specified model of disordered bosons was found in (2.24), which involves only a few integrals per lattice site. This is the starting point for further development of the theory in the next chapter.

3 Field theory

In this chapter we will discuss several approximations and discuss some general aspects of differential geometry to obtain a continuum field theory in the end. The reasoning and derivations here will be more physicist style and, from a mathematical point of view, less rigorous in places. Especially for the super-mathematics in section 3.2, we are not aware of a comprehensive standard reference where those concepts are explained and proven in a mathematically rigorous fashion.

First, we will pass from the lattice field theory derived in the last chapter to a continuum field theory in section 3.1.1, i.e. we send the spacing in between the grains to 0 whilst the number of grains goes to infinity. This step involves turning from discrete integrals at each point to field integrals, involving the usual issues about existence and uniqueness of such a limit. However, those will not be discussed here.

Secondly, we will make use of the large N limit, i.e. we think of having macroscopically many degrees of freedom per grain. Thanks to superbosonisation, this limit now enables us to apply a saddle point approximation to our super-integral. In section 3.1.2 we will therefore derive the saddle point equations and investigate which saddle points are amenable. For this part of the discussion, only spatially homogeneous configurations will be considered.

In section 3.2 we review some facts about the differential geometry of symmetric spaces to finally phrase the field theory, which this thesis is aiming at, in equation (3.23) in a coordinate free fashion. On the way we dwell on the decomposition of the model at the band centre into massless and massive modes in section 3.2.4.

3.1 Approximations

3.1.1 Continuum limit

We start from the super-matrix model living on the edges E_r of our graph (2.24) and constrain our considerations to the hyper-cubic lattice. To start with a well defined model, we think of a finite lattice $\Lambda \subset \mathrm{Z}^d$ with periodic boundary conditions, but of course we will be interested in the $\Lambda \to \mathrm{Z}^d$ limit, as our system should stay of finite physical size, whilst the lattice spacing is sent to 0.

Such a continuum limit is possible if $P : \mathrm{Z}^d \to \mathrm{Gl}_{2,2} / \mathrm{OSp}_{2,2}$ is sufficiently slowly varying on the scale of the lattice spacing, which we call $2e$. I.e. we embed the hyper cubic lattice $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$, identifying a vertex with the corresponding point in real space via

$$\mathbb{Z}^d \ni v = (z_1, \ldots, z_d) \mapsto x(v) = (2e\, z_1, \ldots, 2e\, z_d) \in \mathbb{R}^d$$

Further we specify the position of the edges by interpolating to the centre point in between the bounding vertices

$$e = (v_1, v_2) \mapsto x(e) = \frac{1}{2}(x(v_1) + x(v_2)) \in \mathbb{R}^d$$

In particular the extra loops $(i, i) \in E_r$ are mapped to the position of their hosting vertices $(v, v) \mapsto x(v)$.

The crucial part of the continuum limit is lifting P to a function $P : \mathbb{R}^d \to \mathrm{Gl}_{2,2} / \mathrm{OSp}_{2,2}$. We denote both, the original function P on the lattice and its lift to a continuous function on $\mathbb{R}^d$ by the same symbol. This should not cause confusion, as we will stress which one is meant by using an index for the argument of the lattice function and brackets for the argument of the continuation. The smoothness assumption above can now be made more precise by demanding P_e to be sufficiently slow varying such that $P(x)$ exists as a differentiable function and in the limit $e \to 0$ the choice of interpolation does not matter.

Turning P into a function means turning the integrals over $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ at each edge into a field integral

$$\int\limits_{(\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2})^{|E_r|}} \cdots \int \left(\prod_{e \in E} \mathrm{d}P_e \right) \longmapsto \int\limits_{\mathbb{R}^d \to \mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}} \mathrm{D}P =: \int \mathrm{D}P$$

As usual, this step should, from a rigorous perspective, be thought of as being purely symbolic. But, as long as we carry out only Gaussian integrals in the end, we are confident that the symbolic notation provides good insight.

The next step in the continuum limit is rewriting the lattice sums as integrals. For any smooth function $f : \mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} \to \mathbb{R}$ we write

$$\begin{aligned}
\sum_{e \in E_r} f(P_e) &= \frac{1}{2} \left(\sum_{v \in V} \left(f(P_{(v,v)}) + \sum_{e \,@\, v} f(P_e) \right) \right) \\
&\to \int\limits_{\mathbb{R}^d} \frac{\mathrm{d}^d x}{(2e)^d} \left((d+1) f(P(x)) + \frac{e^2}{2} \nabla^2 (f(P(x)) + \ldots) \right) \\
&= (d+1) \int\limits_{\mathbb{R}^d} \frac{\mathrm{d}^d x}{(2e)^d} f(P(x))
\end{aligned}$$

where the ellipsis contains further derivatives of $f(P(x))$, $\nabla^2 := \sum_{i=1}^d \partial_{x^i}^2$ and in the last step we have ignored boundary terms. Further we used

$$\sum_{v \in V} \to \frac{|V|}{\mathrm{Vol}} \int\limits_{[0, 2Le]^d} \mathrm{d}^d x \to \frac{1}{(2e)^d} \int\limits_{\mathbb{R}^d} \mathrm{d}^d x$$

Note that these sums appear in the action. Therefore the prefactors are of crucial interest, while we are rather sloppy with global factors, as in the field integral

$\int \mathrm{D}P$, because those can easily be restored by demanding the resulting integrals to be normalised, see equation (2.11).

The point how to treat the term $\sum_{e\,@\,v} P_e$ which appears inside the super-determinant in the denominator is a bit more subtle. Although $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ is defined by a linear relation $P = \gamma P^{\mathrm{ST}} \gamma^{-1}$ inside $\mathrm{Gl}_{2|2}$, the real form over which we are integrating is not a vector space, but a positive cone in the boson-boson block and U(1) in the fermion-fermion block. Hence, to make sense out of a derivative of P, we have to drop these constraints for the real form. Then we can write

$$\sum_{e\,@\,v} P_e \to (2d+1)P(x(v)) + e^2 \nabla^2\big|_{x(v)} P + \mathcal{O}(e^2\nabla^4)$$

If the reader feels uncomfortable about this, the right-hand side may also be understood as a symbolical limit of the left hand side for small e. For a more precise discussion of covariant derivatives in the non linear setting see section 3.2.5.

Finally we use some more super-mathematics in rewriting

$$\mathrm{SDet}(\prod_i X_i) = e^{\sum_i \mathrm{STr}(\ln(X_i))}$$

Hence we obtain the continuum version of (2.24)

$$\partial_{z_1}\Big|_{z_1=z_2} \int \mathrm{D}P \left(e^{\frac{1}{2N}\frac{d+1}{2d+1}\mathrm{STr}(\tilde{z}P)} \frac{\mathrm{SDet}(P)^\alpha}{\mathrm{SDet}(P + \frac{e^2}{2d+1}\nabla^2 P - i\Sigma_2)} \right)^N \tag{3.1}$$

which can be written as follows

$$G(z) - \frac{\Delta \dim}{z} \propto \int \mathrm{D}P \left(\int\limits_{\mathbb{R}^d} \frac{\mathrm{d}^d x}{(2e)^d} \mathrm{Tr}(P_{FF}(x)) \right) e^{-N|V|S[P]}$$

with the action

$$S[P] = \frac{1}{\mathrm{Vol}} \int\limits_{\mathbb{R}^d} \mathrm{d}^d x \mathcal{L}(P(x)) \tag{3.2a}$$

and the Lagrangian

$$\mathcal{L}(P) = \frac{z}{2N}\frac{d+1}{2d+1}\mathrm{STr}(P) - \left(\alpha\,\mathrm{STr}\ln(P) - \mathrm{STr}\ln(P + \frac{e^2}{2d+1}\nabla^2 P - i\Sigma_2) \right) \tag{3.2b}$$

where we did not write terms of order $\mathcal{O}(e^4\nabla^4)$, scaled P by a factor of $2d+1$ and used the same parameter as in (2.4)

$$\alpha = \frac{M-1}{2N}(d+1)$$

3.1.2 Saddle point equations

In this section and the following we investigate for which parameters α and z the integral determining the resolvent operator can be treated within the saddle point method, which becomes exact in the limit $N \to \infty$.

Assuming the saddle point configuration to be spatially homogeneous, i.e. $P_S(x) \equiv P_S \in \mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$, we introduce coordinates on the skeleton

$$P = \left(\begin{array}{c|c} a_0\mathbb{1} + a_1\sigma_1 + a_3\sigma_3 & 0 \\ \hline 0 & b\mathbb{1} \end{array}\right)$$

The Grassmann part is unimportant for the saddle point discussion and will be included later.

In section 3.1.4 we will see that for slowly varying P, going from a constant to a spatially varying field increases the action, i.e. the saddle points are not degenerate in this direction. However, even if the saddle point is stable against non-constant variations, there might in general also be spatially non-constant saddle points. If those are separated from the constant ones by a region, where the action takes larger values, they will not be detected. We can think of solitons or any kind of topologically stabilised field configurations, known to be of great importance in other field theories. The possibility of such saddle points is not investigated here.

For a short review of the saddle point method see section 4.3.1. As shown above in equation (3.2), restricting to constant fields means that we consider an integrand of the form

$$F[P]\left(e^{-\hat{z}\,\mathrm{STr}(P)}\frac{\mathrm{SDet}(P)^{\alpha}}{\mathrm{SDet}(P - i\Sigma_2)}\right)^{N|V|} =: F[P]e^{-N|V|\,S[P]}$$

with $\hat{z} := \frac{(d+1)}{2(2d+1)}\frac{z}{N}$ and as before $\alpha = \frac{(d+1)M}{2N} + \mathcal{O}\left(\frac{1}{N}\right)$. Furthermore we have carried out the integration in the action, i.e. summed over V.

Remember, we have $\alpha \geq d+1$ for the model which almost surely couples all neighbouring modes and $\alpha \geq 1$ for almost surely not having (macroscopically many) zero modes. But in general we allow for any $\alpha > 0$.[1]

Bulk scaling

$\hat{z}$ will be considered to be $\mathcal{O}(1)$ in the bulk scaling limit.

$$\begin{aligned} S[P] &= 2\hat{z}(a_0 - b) + 2\alpha\ln(b) - \ln(b^2+1) \\ &\quad + \ln(a_0^2 - a_1^2 - a_3^2 + 1) - \alpha\ln(a_0^2 - a_1^2 - a_3^2) \end{aligned} \tag{3.3}$$

[1] In this section α will always be specified only up to terms of the order $\mathcal{O}\left(\frac{1}{N}\right)$, which are irrelevant for the saddle point discussion.

Now we look for saddle points, i.e. critical points of the action functional

$$\begin{aligned}
\mathrm{d}S = &\left(2\hat{z} + \frac{2a_0}{a_0^2+1-a_1^2-a_3^2} - \alpha\frac{2a_0}{a_0^2-a_1^2-a_3^2}\right)\mathrm{d}a_0\\
&+\left(-2\hat{z}+\frac{2\alpha}{b}-\frac{2b}{b^2+1}\right)\mathrm{d}b\\
&+\left(\alpha\frac{2a_1}{a_0^2-a_1^2-a_3^2}-\frac{2a_1}{a_0^2+1-a_1^2-a_3^2}\right)\mathrm{d}a_1\\
&+\left(\alpha\frac{2a_3}{a_0^2-a_1^2-a_3^2}-\frac{2a_3}{a_0^2+1-a_1^2-a_3^2}\right)\mathrm{d}a_3\\
=&\ 0\\
\Leftrightarrow &\quad (a_1=0\ \wedge\ a_3=0)\quad\vee\quad\left(\alpha\neq 1\ \wedge\ a_0^2-a_1^2-a_3^2=-\frac{\alpha}{\alpha-1}\right)\\
&\wedge\quad \hat{z}=a_0\frac{(\alpha-1)(a_0^2-a_1^2-a_3^2)+\alpha}{(a_0^2-a_1^2-a_3^2)(a_0^2+1-a_1^2-a_3^2)}=\frac{(\alpha-1)b^2+\alpha}{b(b^2+1)}\\
\Rightarrow &\quad (a_1=0\quad\wedge\quad a_3=0)\quad\vee\quad(\hat{z}=0\quad\wedge\quad\alpha\neq 1)
\end{aligned}\tag{3.4}$$

For the bulk scaling we have $\hat{z}\neq 0\Rightarrow a_1=a_3=0$.

$$\Rightarrow\quad \frac{(\alpha-1)a_0^2+\alpha}{a_0(a_0^2+1)}=\hat{z}=\frac{(\alpha-1)b^2+\alpha}{b(b^2+1)}\tag{3.5}$$

Edge scaling

In edge scaling the action (3.3) looks the same, except for $\hat{z}$ now being considered as $\mathcal{O}\left(\frac{1}{N}\right)$. Hence it can be set to 0. From (3.4) we see that now a_0, a_1 and a_3 are all equally important and the new saddle point equations are

$$b^2=-\frac{\alpha}{\alpha-1}=a_0^2-a_1^2-a_3^2\tag{3.6}$$

and we have saddle points (away from infinity) only if $\alpha\neq 1$ And, more importantly, we now have to deal with a two-dimensional manifold of saddle points.

3.1.3 Action landscape

For general energy parameter $\hat{z}$, none of the possible saddle point configurations lies on the integration contour. So we need to take a close look at the landscape of the action functional in order to decide which solutions of the saddle point equations are amenable for a detour. Further we need to make sure that the saddle points, even if they lie on the original contour, mark globally minimal values of the action along this contour. Not surprisingly, we find qualitatively different behaviour for $\alpha>1$ and $\alpha<1$.

No zero modes $\alpha \geq 1$

The saddle point equations for the bulk (3.5) are polynomials of third order and the three solutions are distinct, hence we have 9 saddle point candidates. In order to obtain the density of states we will be interested in $\hat{z} = i\hat{\omega} + \epsilon$ where ϵ is small, real and positive to ensure convergence and $\hat{\omega} = \frac{(d+1)}{2(2d+1)} \frac{\omega}{N}$ is the rescaled (real) characteristic frequency in focus. Since the spectrum is symmetric with respect to $\hat{\omega} \mapsto -\hat{\omega}$, we will restrict the discussion to $\hat{\omega} \geq 0$.

To investigate the behaviour of the action functional, we fix b at one of its saddle point values and plot

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ a_0 &\mapsto \Re\left(S[a_0, b_{\text{Saddle}}]\right) \end{aligned}$$

We could also plot the action vice versa for fixed a_0 and varying b, but we observe $S[a_0, b] = -S[b, a_0]$ once we have set a_1 and a_3 to 0. Furthermore, since the action decouples nicely, inserting different b_{Saddle} will only lead to a constant shift. Hence we only need to look at one set of three saddle points to understand the full picture.

First we take a look at the model without zero modes, $\alpha > 1$. This holds in particular for $\alpha = d+1$ in $d > 0$ dimension. Here we start with at the band centre $\hat{\omega} \to 0$.[2] The action is depicted in Figure 3.1.

There is a pair of solutions which are complex conjugates of each other for real z and tend towards $\pm i\sqrt{\frac{\alpha}{\alpha-1}}$ as z goes to 0. However, the integration contour for a_0 has to start at 0 and run towards $+\infty$. Now, for small ϵ the area where the action is larger than the saddle point value is separated into disjoint regions around 0 and ∞. Hence the contour for a_0 cannot be deformed as to run through those saddle points such that it obtains its global minimum at one of those.

For b the landscape looks similar up to a global sign change of the action and the original contour now being $\mathrm{U}_1 \simeq S^1 \hookrightarrow \mathbb{C}$. Note that the only relevant singularity now is at the origin, as the action going to $+\infty$ just means that the integrand runs through 0. Only $S \to -\infty$ actually marks a pole. Hence the radius of S^1 does not matter, as it should generally be the case for the compact real form of the fermion-fermion part after superbosonisation. Again we cannot deform the contour so as to obtain a global minimum at one of the afore mentioned saddle points. Looking at the left part of Figure 3.1 we see that the contour has to run through the saddle points roughly parallel to the imaginary axis, after which one ends up in a small compact region and the contour cannot be closed without crossing the separatrix $S[b] = S_{\text{Saddle}}$ again.

The third solution is real for real $\hat{z}$ and goes to ∞ for $\hat{z} \to 0$. This solution actually lies on the contour for a_0 in the $\hat{\omega} = 0$ case and also for general $\hat{z} = i\omega + \epsilon$ we can deform the contours for both, a_0 and b, to run through this saddle point, whilst respecting the global constraints. This will in the following be examined more carefully. By a simple coordinate change, $a_0 \mapsto a_0^{-1}$, we get a better view on the important separatrix and

[2] Keep in mind that this is still the bulk picture, i.e. we are looking at $\omega \to 0$ at an $\mathcal{O}(N)$ scale. The fine scale behaviour will be investigated later.

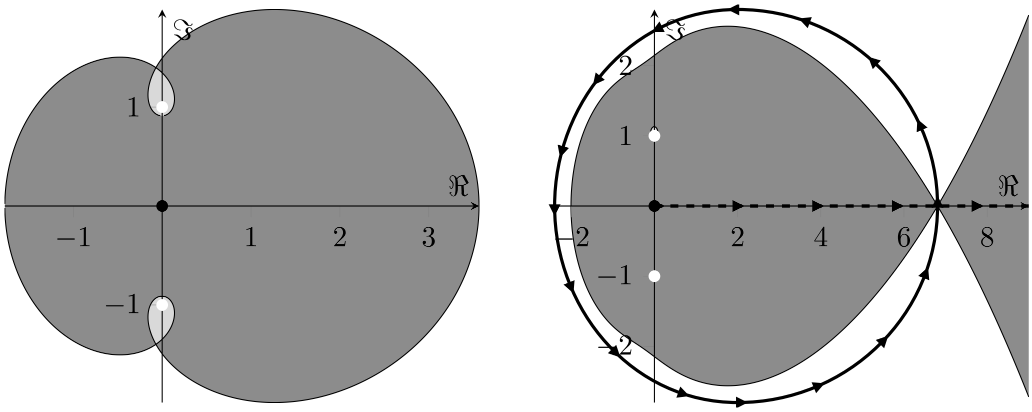

Figure 3.1: Here we plot the $a_0 \in \mathbb{C}$ plane and in there the separatrix $\{a_0 \mid \Re(S[a_0, b_s]) = \Re(S[a_s, b_s])\}$ for each of the three saddle points a_s. Regions, where the action takes larger values than at the saddle point, are shown in dark gray, i.e. integration contour must run only through these. A possible contour for the a_0 integration is depicted as the dashed path . The singularities where the action goes to $\pm\infty$ are marked with black/white dots. For $b \mapsto \Re(S[a_{0,\text{Saddle}}, b])$ we get the same picture with an overall sign change and hence the roles of the dark and light areas and dots swapped. A possible contour for b is depicted as the solid path. For this picture we chose $\alpha = 2$, $\epsilon = 0.3$ for having the saddle point in the real axis relatively close by and $\hat{\omega} = 0$, hence the other two saddle points lie on the same level.

whether it is possible to join 0 and ∞ crossing it exactly once at the saddle. Note that $\mathrm{d}a_0^{-1} = a_0^{-2}\mathrm{d}a_0$ leads only to an unimportant $\mathcal{O}\left(\frac{1}{N}\right)$ term in the action, if we change coordinates right at the level of integration. So we can as well perform the inversion of the coordinate system only at the end when plotting the action landscape.

In figure 3.3 we observe that, at some $\hat{\omega} \in \mathcal{O}(1)$, one of the two so far unimportant saddle points sinks down to the same level as the relevant saddle point. Here the separatrix opens towards $+\infty$. Hence, judging from the shape of the separatrix alone, it would become possible to run the contour for a_0 through this saddle point. But $a_0 \mapsto S[a_0, b_{\text{Saddle}}]$ goes to $-\infty$ at the singularities at $\pm i$, i.e. these are indeed singularities of the ratio of super-determinants and must not be crossed when deforming the contour. Thus the two saddle points, which were forbidden before, stay out of reach. For the integration over b, the qualitative picture stays the same as for $\hat{\omega} = 0$ and we still have a unique feasible saddle point, which stays finite for finite $\hat{\omega}$, even if $\epsilon \to 0$.

For $\alpha = 1$ and $\hat{\omega} \neq 0$ the picture still looks qualitatively the same. The only difference is that now all saddle points escape to ∞.

So for $\alpha \geq 1$ there is a unique saddle point, on which the integral localises. It is given by $a_0 = b =: s$, where the exact expression for s is given in the Appendix, (A.3).

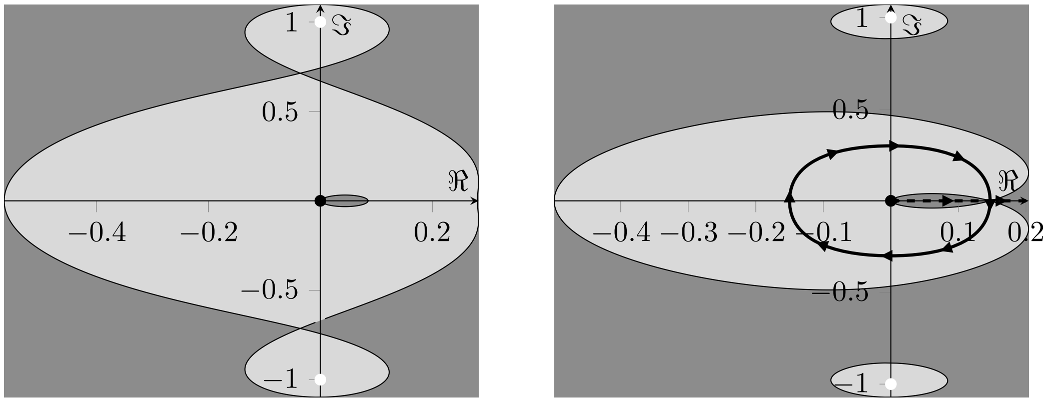

Figure 3.2: Same picture as Figure 3.1 but in an inverted coordinate system. Here the action diverges at the origin to $\pm\infty$ depending on whether it is approached from the dark or bright region. $\alpha = 2$, $\hat{\omega} = 0$, $\epsilon = 0.3$.

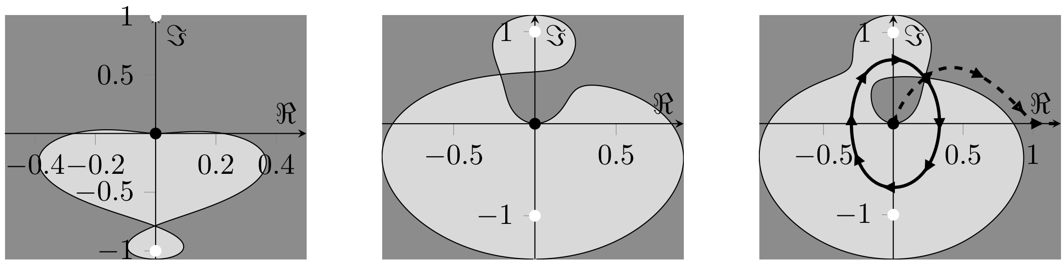

Figure 3.3: $\alpha = 2$, $\epsilon = 0.025$, $\hat{\omega} = 0.5$ in inverted coordinates.

It is the solution with the largest real part of (3.7b).

$$\boxed{P_S = s \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & \mathbb{1} \end{array} \right)} \tag{3.7a}$$

$$\boxed{\hat{z} = \frac{(\alpha-1)s^2 + \alpha}{s(s^2+1)}} \tag{3.7b}$$

The saddle point value of the action is

$$S[P_S] = 0$$

In equation (3.6) we saw that we still have two saddle points for the edge scaling case, $\hat{z} = 0$, if $\alpha \neq 1$. But as was just investigated as the $\hat{z} \to 0$ limit, we can not deform the integration contour to run through these saddle points due to global constraints. Hence the saddle point method is not applicable at the centre of the spectrum for $\alpha \geq 1$. In [LSZ06] there is a complete discussion of the $\alpha = 1$ case. This is the only value for α where also the unaccessible saddle points escape to infinity and the saddle point equations do not have any solutions for $\hat{z} = 0$. Hence the edge scaling limit with $\hat{z} = 0$ can be treated with the saddle point method only for $\alpha < 1$, the model with a macroscopic number of zero modes.

Before we continue with the $\alpha < 1$ case, we note that for large values of $\hat{\omega}$ we still have a feasible saddle point, but above a critical value it lies on the imaginary axis for $\epsilon \to 0$. In section 4 below we will see that this means that the density of states is zero and the critical value of ω, i.e. the position of the spectral edge, will be calculated. For a picture of the action landscape at large ω see figure 3.4.

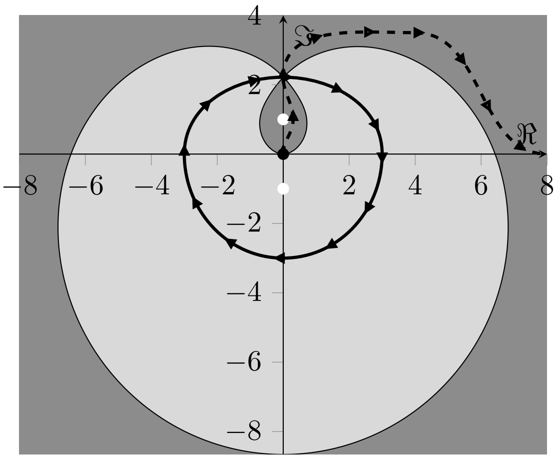

Figure 3.4: Relevant saddle point for $\alpha = 2$, $\epsilon = 0.01$, $\hat{\omega} = 5$ in inverted coordinates.

Macroscopically many zero modes $\alpha < 1$

For $\alpha < 1$ and $\hat{\omega} \gg \epsilon$ the picture does not change much compared to $\alpha > 1$, as can be seen by comparing Figure 3.5 and 3.3. For $\epsilon \to 0$ one saddle point escapes to $i\infty$ while the other two stay finite. As above there is a unique saddle point to run through and still it is the one with the largest real part.

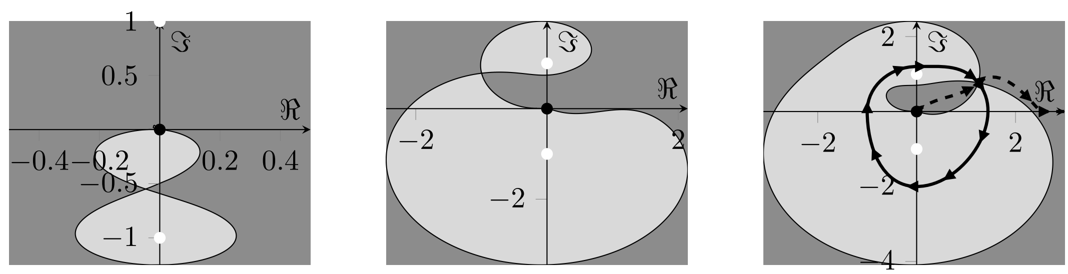

Figure 3.5: $\alpha = 0.5$, $\epsilon = 0.1$, $\hat{\omega} = 0.5$ in inverted coordinates.

For $\hat{\omega} = 0$, i.e. real $\hat{z} \neq 0$, and ϵ small all solutions become real and two of them go to $\pm\sqrt{\frac{\alpha}{1-\alpha}}$ as $\hat{z} \to 0$, whilst the third one runs to $-\infty$. See figure 3.6 for $\epsilon > 0$ and 3.7 for the $\hat{z} = 0$ case, which can now be handled.

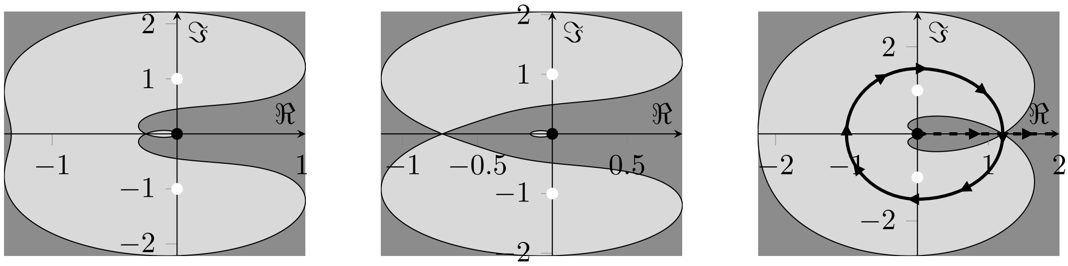

Figure 3.6: $\alpha = 0.5$, $\epsilon = 0.025$, $\hat{\omega} = 0$ in inverted coordinates.

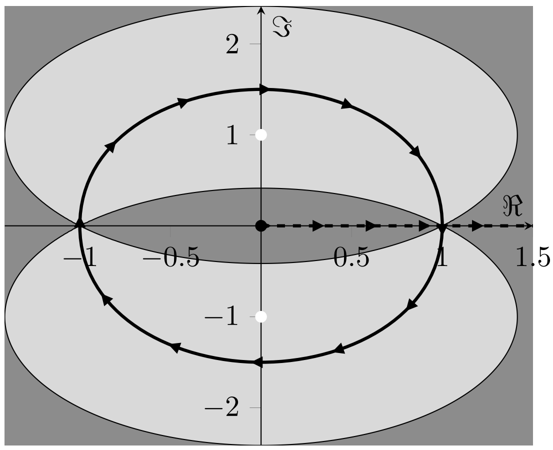

Figure 3.7: $\alpha = 0.5$, $\hat{z} = 0$ in inverted coordinates.

Again, judging from the shape of each separatrix, we could deform the integration contour to run through either of the two close to real saddle points, which survive the $\hat{z} \to 0$ limit without disappearing to ∞. But as before only the saddle point with the largest real part is relevant for the a_0 contour, due to the singularities.

For $\hat{z} = 0$ we encounter a new phenomenon. Now both saddle points, $\pm\sqrt{\frac{\alpha}{1-\alpha}}$, need to be taken into account for the b contour. This is the same for all values $0 < \alpha < 1$.

Edge scaling

By sending $\hat{z} \to 0$ and replacing a_0^2 by $a_0^2 - a_1^2 - a_3^2$ we can apply the above investigation of the action landscape also to the edge scaling case. We conclude that the manifold of solutions of the edge saddle point equations 3.1.2 is feasible for localising the integral if and only if $\alpha < 1$.

Note that the boson-boson block of P is constrained to positive matrices, which in particular means $\mathrm{Det}(P_{BB}) = a_0^2 - a_1^2 - a_3^2 > 0$, hence the integration runs through the saddle point manifold only for $\alpha < 1$. But it takes the considerations above to make sure that it is impossible to deform the integration contour such as to run through the solution manifold in the case $\alpha > 1$. This would correspond to localising at the the imaginary saddle points, which was found to be impossible in figure 3.1. So for $\alpha \geq 1$ we cannot apply saddle point analysis to the edge scaling limit. In fact we will show in section 4 below that $\omega = 0$ lies within a spectral gap for $\alpha > 1$, hence trying to look at some fine scale behaviour does not make sense here. The $\alpha = 1$ case is singular and discussed in detail in [LSZ06], as mentioned already.

For $\alpha < 1$, however, we found that

$$a_0 = +\sqrt{\frac{\alpha}{1-\alpha} + a_1^2 + a_3^2}$$

is the relevant saddle point for the boson-boson block. Again this is as expected, as the other possible saddle point manifold, $a_0 = -\sqrt{\ldots}$, would correspond to a negative definite matrix. In the discussion of the action landscape above we saw that this saddle point is prohibited due to the singularities. For the fermion-fermion block on the other hand, both $b = \pm\sqrt{\frac{\alpha}{1-\alpha}}$ need to be taken into account, see figure 3.7.

3.1.4 Fluctuations

Now that we have found the relevant saddle points, we will turn our attention to fluctuations around those. In order to complete the saddle point method, we need to expand the action to second order and integrate out the fluctuations, which leads to a functional determinant.

Of course, we cannot expect fluctuations to be spatially homogeneous so we must take the nearest neighbour interactions back into consideration. Recall the form of the Lagrangian (3.2b), which was derived within the continuum limit above. In this section we will write out $\nabla^2 = \sum_{i=1}^{d} \partial_{x^i}^2$ but the $\sum_{i=1}^{d}$ symbol will be omitted throughout and summation over indices that appear twice is understood.

Bulk scaling

We decompose the field P into its saddle point value and fluctuations

$$P(x) = P_S + \delta P(x)$$

where $P_S = s\mathbb{1}$ is given by (3.7). The situation with two saddle points, $\hat{z} \in \mathcal{O}\left(\frac{1}{N}\right)$ and $\alpha < 1$, will be treated separately below.

We start by expanding the logarithms in (3.2b) to second order in the fluctuations

$$
\begin{aligned}
&\mathrm{STr}\ln\left(P_S + \delta P(x) + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta P(x) - i\Sigma_2\right) \\
=&\,\mathrm{STr}\ln\left(P_S - i\Sigma_2\right) && (3.8a)\\
&+ \mathrm{STr}\left((P_S - i\Sigma_2)^{-1}\left(\delta P(x) + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta P(x)\right)\right) && (3.8b)\\
&- \frac{1}{2}\mathrm{STr}\left((P_S - i\Sigma_2)^{-1}\left(\delta P(x) + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta P(x)\right)\right)^2 && (3.8c)\\
&+ \mathcal{O}(\delta P)^3
\end{aligned}
$$

The part without derivatives of (3.8b) goes into the saddle point equation and the rest is a total derivative. Line (3.8a) goes into the saddle point value of the action. Using the saddle point (3.7) and writing the fluctuations as [3]

$$
\delta P = \left(\begin{array}{c|c} \delta a_0 \mathbb{1} + \delta a_1 \sigma_1 + \delta a_3 \sigma_3 & 0 \\ \hline 0 & i\delta b \mathbb{1} \end{array}\right)
$$

we obtain the quadratic part of the effective action for the fluctuations

$$
\begin{aligned}
-(s^2+1)^2 (3.8c) =& (s^2+1)\left(\left(\delta a_1 + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta a_1\right)^2 + \left(\delta a_3 + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta a_3\right)^2\right) \\
&+ (s^2-1)\left(\left(\delta b + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta b\right)^2 + \left(\delta a_0 + \frac{e^2}{2d+1}\partial_{x_i}^2 \delta a_0\right)^2\right) \\
&+ \mathcal{O}(e\partial)^4
\end{aligned}
$$

The logarithm of the numerator is more easily expanded

$$
\begin{aligned}
&\mathrm{STr}\ln\left(P_S + \delta P(x)\right) \\
=&\,\mathrm{STr}\ln\left(P_S\right) && (3.9a)\\
&+ \mathrm{STr}\left(P_S^{-1}\delta P(x)\right) && (3.9b)\\
&- \frac{1}{2}\mathrm{STr}\left(P_S^{-1}\delta P(x)\right)^2 && (3.9c)\\
&+ \mathcal{O}(\delta P)^3
\end{aligned}
$$

Again (3.9b) goes into the saddle point equation and (3.9a) into the saddle point value. The contribution of the fluctuation therefore reads

$$
(3.9c) = -\frac{a_0^2 + a_1^2 + a_3^2 + b^2}{s^2}
$$

[3] Note that we have rotated δb by a factor i, such that $\delta b \in \mathbb{R}$ is now the right direction to run through the saddle point on the real axis.

Inserting everything into the action (3.2a) we get

$$S[P] = S[P_S] + \frac{1}{\mathrm{Vol}} \int\limits_{\mathbb{R}^d} \mathrm{d}^d x \, \mathcal{L}^{(2)} + \mathcal{O}(\delta P)^3 + \mathcal{O}(e\partial)^4$$

with

$$\begin{aligned} \mathcal{L}^{(2)} = & \left| \frac{(\alpha-1)s^2+\alpha}{s^2(s^2+1)} \right| \left((\delta a_1)^2 + (\delta a_3)^2 \right) \\ & + \left| \frac{(\alpha-1)s^4 + (2\alpha+1)s^2+\alpha}{s^2(s^2+1)^2} \right| ((\delta a_0)^2 + (\delta b)^2) \\ & + \frac{1}{s^2+1} \frac{2e^2}{2d+1} e^{i\varphi_1} \Bigg((\partial_{x_i} \delta a_1)^2 + (\partial_{x_i} \delta a_3)^2 \\ & \quad + \frac{s^2-1}{s^2+1} e^{i(\varphi_2-\varphi_1)} \left((\partial_{x_i} \delta a_0)^2 + (\partial_{x_i} \delta b)^2 \right) \Bigg) \end{aligned} \tag{3.10}$$

If the reader prefers a more symmetric expression, note $\frac{(\alpha-1)s^4+(2\alpha-1)s^2+\alpha}{s^2(s^2+1)^2} = \frac{(\alpha-1)s^2+\alpha}{s^2(s^2+1)}$. The absolute value is due to the method of steepest decent. We should run the integration contour through the saddle point in such a direction that the real part of the coefficients of the second order expansion is maximal. In other words, the complex phases $e^{i\varphi_j}$ of the fluctuations a_i and b have to be chosen such that they exactly cancel the phases of the coefficients. This is most conveniently expressed by choosing the coordinates to be real valued and introducing the above absolute value.

For the derivatives, however, we need to take these phases into account writing

$$\begin{aligned} \varphi_1 &:= -\arg\left(\frac{(\alpha-1)s^2+\alpha}{s^2(s^2+1)} \right) \\ \varphi_2 &:= -\arg\left(\frac{(\alpha-1)s^4+(2\alpha+1)s^2+\alpha}{s^2(s^2+1)^2} \right) \end{aligned} \tag{3.11}$$

Those phases are discussed below in section 3.1.5.

Further we note that the saddle point value of the action is exactly zero, $S[P_S] = 0$, because the saddle point can be written as $\mathbb{1}_{1|1} \otimes \mathbb{1}_2$, where the important point is that it is proportional to $\mathbb{1}_{1|1}$ in super-space. The same holds for Σ_2, hence $\mathrm{STr}(P_S) = 0$ and $\mathrm{SDet}(P_S) = 1 = \mathrm{SDet}(P_S - i\Sigma_2)$. Hence the value of the integral is to lowest order in $\frac{1}{N}$ given by the integrand evaluated at the saddle point divided by the fluctuation determinant. This will be carried out in chapter 4.

Edge scaling

Now we have a look at the situation $\alpha < 1$, $\hat{z} \in \mathcal{O}\left(\frac{1}{N}\right)$. From the perspective of fluctuations, not much changes at the two saddle points where $a_1 = a_3 = 0$ and $b = \pm s$. Already in figure 3.7 we can see that the action landscape for $\hat{z} \to 0$ becomes symmetric upon inversion at the imaginary axis. The only thing that changes is the direction of the integration contour for the fluctuations δb, but, not surprisingly, the bilinear form $\mathcal{L}^{(2)}$ is invariant under $\delta b \mapsto -\delta b$. Of course, the value of the action at the saddle point does change, as described in section 3.1.3 above.

However, the saddle point manifold now being invariant under the action of $\widetilde{\mathrm{OSp}}_{2|2}$ is more important. For the boson-boson part, the saddle point manifold is invariant under an $\mathrm{Sl}(\mathbb{R}^2)$ action and in fact given by

$$\{g(s\mathbb{1})g^{\mathrm{T}} \mid g \in \mathrm{Sl}(\mathbb{R}^2)\} \simeq \mathrm{Sl}(\mathbb{R}^2)/\,\mathrm{SO}(2)$$

In coordinates this means $a_0^2 - a_1^2 - a_3^2 = b^2 = \frac{\alpha}{1-\alpha} =: s^2$, as calculated above. More precisely, our investigations in section 3.1.3 showed that $a_0 = +\sqrt{s^2 + a_1^2 + a_3^2}$ is the relevant part of the saddle point manifold. Hence one should change coordinates and use $\mathrm{Det}(P_{BB}) = a_0^2 - a_1^2 - a_3^2$ as the coordinate that still gets localised and treat the other two dimensions separately. This is done in chapter 4 where we will see that in fact for the edge scaling it is not sufficient to take only the contribution from the fluctuation determinant into account.

3.1.5 Consistency check

Now it is time to reconsider some assumptions that were made during the derivation of the effective action. We took care about the masses m_i^2 of a_i^2 being positive when analysing the action landscape of constant fields and we disregarded saddle points where this condition could not be fulfilled.

Now one might worry about the coefficients c_i of the kinetic terms $\sum_j (\partial_{x^j}\delta a_i)^2$. In fact, by numerical inspection, we find that for all parameters $\alpha > 0$ and $\hat{z} = \epsilon + i\hat{\omega} \neq 0$ at least one pair of coefficients have a negative real part. Now, if all modes are massive, i.e. $\hat{\omega} \neq 0$, a negative kinetic term is of no concern, as long as the fields are slowly varying. By the usual Fourier transform $\delta a_i(x) = \sum_k \delta a_{i,k} e^{ikx}$ for finite volume, or with $\sum_k \to \int_k$ for infinite volume, we can diagonalise the bilinear functional $\mathcal{L}^{(2)}$ and get

$$m_i^2(\delta a_{i,0})^2 + \sum_{k\neq 0}(m_i^2 + c_i k^2)\delta a_{i,k}\delta a_{i,-k}$$

Hence, for large positive m^2 and small k, even a negative real part of c_i will not lead to negative eigenvalues of the Hessian of the action. Consequently the saddle point is also stable against slow fluctuations. And if large momentum modes were important, we would have to reconsider the derivation of the effective action, where terms of order p^4 were ignored from the beginning, and we would even have to question whether a continuum limit makes sense in the first place.

Furthermore, we are actually free to choose an individual phase for each mode $\delta a_{i,k}$ by which we can compensate the phase of $m_i^2 + c_i k^2$.

Now, for $\hat{\omega} \to 0$ the masses $m_1 = m_3$ go to 0. In the edge scaling, where those masses are zero from the outset, the phase φ_1 of these modes is not determined by

(3.11) and can be freely chosen such as to make $c_1 = c_3$ positive. Hence the kinetic terms are of no concern here, either.

Finally we can also smoothly connect the bulk with the edge scaling by noting that the kinetic coefficients $c_1 = c_3$ in fact become positive for $\hat{\omega} \to 0$. Here $s \to \sqrt{\frac{\alpha}{1-\alpha}} \in \mathbb{R}^+$ and $\hat{z} \in \mathbb{R}^+$ and hence

$$\Re(c_1) = \Re(c_3) = \Re\left(\frac{1}{s^2+1}\frac{2e^2}{2d+1}e^{i\varphi_1}\right) \in \Re\left(\frac{s\bar{\hat{z}}}{|\hat{z}|^2(s^2+1)}\right)\mathbb{R}^+$$
$$= \frac{\epsilon}{|z|^2}\frac{s}{s^2+1}\mathbb{R}^+ \subset \mathbb{R}^+$$

Note that the seemingly divergent factor $\frac{1}{z}$ in fact comes from the angle φ_1 which becomes undetermined for $\hat{z} = 0$.

In conclusion, we found that the Hessian of the action is positive for all $\alpha < 1$ and all z, in bulk as well as in edge scaling.

3.2 Coordinate free description

To get a better understanding of the field theory, we need to take a step back and rephrase everything in a coordinate free fashion. Therefore we will first have a closer look at the relevant structures, namely the super-groups forming and acting on $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ and then describe the objects from which a super field theory involving those groups can be built.

Note, however, that we will often use results which are well known for ordinary spaces in our super context without proof of applicability. As explained in section 3.2.1 below, we have been and will be dealing only with so called split super-spaces. For such it should be possible to generalise the relevant theorems about ordinary spaces. But we will not do this here, therefore this section is from a mathematical point of view the least rigorous part of this thesis.

3.2.1 Symmetric super-spaces

In this work we understand a Riemannian super-space in the sense of [Zir98]. I.e. as a pair $(\mathcal{M}, M_r)$ where $\mathcal{M}$ is a super-manifold of complex super- (or graded) dimension $(p|q)$. The underlying ordinary complex manifold of complex dimension p(also called skeleton or support) is denoted by M_s or just M and $M_r \subset M$ is a sub manifold of real dimension p. M_r has to be chosen such that the super-geometry of $\mathcal{M}$ restricted to M_r is Riemannian. A super-manifold is locally modelled by $\mathrm{C}^\infty(\mathbb{C}^p) \otimes \bigwedge(\mathbb{C}^q)$, the algebra of analytic functions with values in the Grassmann algebra. But we will not go into details about the global structure of the most general super-manifold, because here we are in the fortunate position to consider only split super-manifolds, which is to say that the super-functions can be understood as sections of the exterior bundle of some vector bundle $\mathcal{M} = \Gamma\left(\bigwedge E\right)$ where E is a vector bundle over M of dimension q. This means that we do not have to dwell on the sheaf theory of super-functions.

We have already introduced integration over $\mathbb{C}^{p|q}$ in section 2.2.2 and over split super-spaces with respect to a given system of coordinates in section 2.2.3. There we have already seen how a super-differential operator, like $\mathrm{d}P$, of rank $(0|q)$ with values in the (real) p forms on M_r is used to define the integral of a super function f on $\mathcal{M}$ as

$$\int_{\mathcal{M}} f = \int_{M_r} \mathrm{d}P(f)$$

What was not mentioned, yet, is the transformation behaviour of $\mathrm{d}P$ under coordinate change. For two sets of coordinates, (x_i, ξ_i) and (y_i, χ_i) we do unfortunately not get a coordinate free definition of integration if we take only the Berezinian, which is the super-Jacobian, into account.

$$\begin{aligned} & dy_1 \wedge \ldots \wedge dy_p \otimes \partial_{\xi_1} \ldots \partial_{\xi_q} \\ & \qquad = \mathrm{Ber}\left(\frac{(y_i, \xi_i)}{(x_i, \chi_i)}\right) dx_1 \wedge \ldots \wedge dx_p \otimes \partial_{\chi_1} \ldots \partial_{\chi_q} + \alpha\left(\frac{(y_i, \xi_i)}{(x_i, \chi_i)}\right) \end{aligned}$$

with

$$\mathrm{Ber}\left(\frac{(y_i, \xi_i)}{(x_i, \chi_i)}\right) := \mathrm{SDet}\left(\begin{array}{c|c} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial \chi} \\ \hline \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial \chi} \end{array}\right)$$

and $\alpha = 0$ might not give the same value of the integral if the even coordinates are shifted by nilpotents, $\frac{\partial y}{\partial \chi} \neq 0$. The extra term α is called anomaly or boundary term, since it takes values in the exact p forms. For more details see [Zir98] and references therein.

3.2.2 Super-groups

In this section we will always start from the big super-group $\mathrm{Gl}_{2|2}$ of even isomorphisms of a graded $(2|2)$-dimensional complex vector space $\mathbb{C}^{2|2}$. To avoid cryptic notation we will not use a calligraphic font for super-groups, but indicate the skeleton by an index s.

One may introduce global coordinates and think of $\mathrm{Gl}_{2|2}$ as the set of invertible 4 by 4 matrices with matrix entries in the Grassmann algebra $\mathrm{Gr}^8 = \bigwedge \mathbb{C}^8$. In fact, in a very simple situation like the current one, when the super-manifold is not only split, but also the skeleton is an open subset of $\mathbb{C}^n$, it is perfectly sensible to give a super-space in terms of coordinate functions. Anyway, $\mathrm{Gl}_{2|2}$ is given by $\mathrm{C}^\infty(\mathrm{Gl}(\mathbb{C}^2) \oplus \mathrm{Gl}(\mathbb{C}^2)) \otimes \mathrm{Gr}^8$ and hence is $(8|8)$-dimensional. The boson-boson block and fermion-fermion block contain only even elements of the Grassmann algebra, while the fermion-boson and boson-fermion block contain only odd elements, which is just another way of saying that $\mathrm{Gl}_{2|2}$ consists of even morphisms.

Note that the inverse of a super-matrix is defined, like all super-functions, in terms of an expansion in the Grassmann variables, which necessarily terminates at the highest order, given by the product of all Grassmann generators. Hence, once a super-matrix $g = A + B$ is split into numerical part A and nilpotent part B, the sum

$$(A+B)^{-1} = A^{-1}\sum_{i=0}(-BA^{-1})^i \tag{3.12}$$

terminates. This is equivalent to solving the system of linear equations $gg^{-1} = \mathbb{1}$ for g^{-1}. The solution exists, i.e. g is invertible, if and only if both boson-boson *and* fermion-fermion block are invertible. From the definition of the super-determinant (2.22) or directly from (3.12) we see that in fact a super matrix is invertible if and only if the numerical part is. This confirms that the skeleton of $\mathrm{Gl}_{2|2}$ is indeed $\mathrm{Gl}(\mathbb{C}^2)\times\mathrm{Gl}(\mathbb{C}^2)$.

The geometry of $\mathrm{Gl}_{p|q}$ is defined by the invariant bilinear super-trace form $(X,Y)\mapsto \mathrm{STr}(XY)$ which is a function on the super Lie algebra $\mathfrak{gl}_{2|2}$ of $\mathrm{Gl}_{2|2}$. This trace form restricts to a multiple of the Killing form of the super-subgroups of $\mathrm{Gl}_{p|q}$ whenever the Killing form is non-degenerate. Otherwise it is preferable to use STr instead of the Killing form.

This means that $\left(\mathrm{Gl}_{2|2}\right)_r = \left(\mathrm{Gl}(\mathbb{C}^2)/\,\mathrm{U}(2)\right)\times\mathrm{U}(2)$ is a real form of $\mathrm{Gl}_{2|2}$ as $\mathrm{Tr}(X^2)\geq 0$ for $X \in \mathrm{T}_{[\mathrm{U}(2)]}(\mathrm{Gl}(\mathbb{C}^2)/\,\mathrm{U}(2)) = \{X = X^\dagger\}$ Hermitian and $-\mathrm{Tr}(Y^2)\geq 0$ for $Y\in\mathfrak{u}(2) = \{Y = -Y^\dagger\}$ anti-Hermitian. Throughout we will denote the tangent space of M at p by $\mathrm{T}_p M$. Further for H super-subgroup of $\mathrm{Gl}_{2|2}$ we will take H_r to be a Riemannian submanifold of $\left(\mathrm{Gl}_{2|2}\right)_r$.

3.2.3 Involutions and subgroups

A good way to define the Lie subgroups appearing in this section is as the fixed point sets of involutions, i.e. Lie group automorphisms that square to the identity. In the definition of the symmetric super-space $\mathrm{Gl}_{2|2}/\,\mathrm{OSp}_{2|2}$ in (2.25) we have already seen one such involution at work. $\Sigma_2 = \mathbb{1}_{1|1}\otimes\sigma_2$ maps $\gamma = E_{BB}\otimes\mathbb{1}_2 + E_{FF}\otimes\sigma_2$ from (2.23) to another important element of $\mathrm{Gl}_{2|2}$, namely

$$\tilde{\gamma} := \Sigma_2\gamma = E_{BB}\otimes\sigma_2 + E_{FF}\otimes\mathbb{1}_2$$

By conjugation with these matrices we have the following two involutions of $\mathrm{Gl}_{2|2}$:

$$\begin{aligned}\theta : \mathrm{Gl}_{2|2} &\to \mathrm{Gl}_{2|2}\\ g &\mapsto \gamma(g^{-1})^{\mathrm{ST}}\gamma^{-1}\end{aligned}$$

fixes $\mathrm{OSp}_{2|2}\hookrightarrow\mathrm{Gl}_{2|2}$ and

$$\begin{aligned}\widetilde{\theta} : \mathrm{Gl}_{2|2} &\to \mathrm{Gl}_{2|2}\\ g &\mapsto \widetilde{\gamma}(g^{-1})^{\mathrm{ST}}\widetilde{\gamma}^{-1}\end{aligned}$$

fixes $\widetilde{\mathrm{OSp}}_{2|2}\hookrightarrow\mathrm{Gl}_{2|2}$ which is in the complex picture just another embedding of $\mathrm{OSp}_{2|2}\hookrightarrow\mathrm{Gl}_{2|2}$. For more details about the similarities and differences of those two groups see appendix A.1.3.

An important point to notice is that $(g^{\mathrm{ST}})^{\mathrm{ST}} = SgS^{-1} \neq g$ with $S = (E_{BB} \otimes \mathbb{1} - E_{FF} \otimes \mathbb{1}) = S^{-1}$ being the super-parity. Therefore, a mapping of type $g \mapsto h(g^{-1})^{\mathrm{ST}}h^{-1}$ for $h \in \mathrm{Gl}$ is an involution if and only if $\forall_{g\in\mathrm{Gl}}(hg^{\mathrm{ST}}h^{-1})^{\mathrm{ST}} = h^{-1}gh$. This condition is satisfied, e.g. for $h = E_{BB} \otimes A + E_{FF} \otimes B$ with $A \in \mathrm{O}(2)$ and $B^{-1} = -B^{\mathrm{T}}$ 'skew orthogonal', or vice versa. $h \in \{\gamma, \widetilde{\gamma}, \Gamma, \ldots\}$ are all of this type. Whether A or B is skew determines which sector of the real form of the fixed group is enforced to be compact by the fixed point condition.

Now if we have a Lie super-group, like $\mathcal{G} = \mathrm{Gl}_{2|2}$ and a Lie super-subgroup, like $\mathcal{H} = \mathrm{OSp}_{2|2}$, given as the fixed point set of an involution Θ and if we furthermore have a super Riemannian structure on $\mathcal{G}$ and Θ is an isometry of this structure, then the quotient space $\mathcal{G}/\mathcal{H} \simeq \{g = \Theta(g^{-1})\}$ is a Riemannian symmetric super-space. In [Hel62], chapter IV, in particular § 3, one can find more details about ordinary Riemannian symmetric spaces. For Riemannian symmetric super-spaces see [Zir98]. The important point is that we demand the real form $(G/H)_r = G_r/H_r$ to be a Riemannian symmetric space with respect to the induced geometry. For more details about this geometry see the next section 3.2.5. Throughout we will denote elements of coset spaces $[g] \in \mathcal{G}/\mathcal{H}$ by $[g] := g\mathcal{H} := \{gh \mid h \in \mathcal{H}\}$.

Further one gets a *transitive* action of the big group G on the symmetric space G/H by twisted conjugation. In our case $\mathrm{Gl}_{2|2}$ acts on

$$\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} = \left\{P = \theta(P)^{-1}\right\} = \left\{g\mathbb{1}\theta(g)^{-1} \mid g \in \mathrm{Gl}_{2|2}\right\}$$

by twisted conjugation $P \mapsto \mathrm{AD}^{\theta}(g)P := gP\theta(g^{-1})$. The last equality is known as the Cartan embedding of $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} \hookrightarrow \mathrm{Gl}_{2|2}$. Throughout we will denote the action by twisted conjugation of the form

$$\mathrm{AD}^{\Theta}(h)g := hg\Theta(h^{-1})$$

by writing the twisting involution Θ as an upper index and AD for the action on the group or symmetric space similar to Ad for the action on the Lie algebra.

Now, why did we introduce $\widetilde{\mathrm{OSp}}$? We have seen in section 3.1 that there is an important difference in between the edge scaling and the bulk scaling limit. Or in other words, for $z \to 0$ something in the model changes qualitatively. In fact this can be seen directly from the action. There is an exact symmetry of the denominator

$$\begin{aligned}
&\mathrm{SDet}(P - i\Sigma_2) = \mathrm{SDet}(\mathrm{AD}^{\theta}(h^{-1})P - i\Sigma_2) \\
&\Leftrightarrow \mathrm{AD}^{\theta}(h)\Sigma_2 = \Sigma_2 \\
&\Leftrightarrow h\tilde{\gamma}h^{-1} = \tilde{\gamma} \\
&\Leftrightarrow h \in \widetilde{\mathrm{OSp}}
\end{aligned}$$

and the numerator $\mathrm{SDet}(P)$ is also invariant under $\mathrm{AD}^{\theta}(h)$ for $h \in \mathrm{OSp}_{2|2}$. Only the super-trace is not invariant under twisted conjugation, so the term $z\,\mathrm{STr}(P)$ breaks this symmetry. But for $z = 0$, $\widetilde{\mathrm{OSp}}_{2|2}$ is an exact symmetry of the whole action and it should shed some light on the edge scaling and the centre of the band in the bulk scaling if we could factor out this symmetry group from the symmetric space $\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$.

To do this we need to complete the picture of relations of OSp, $\widetilde{\mathrm{OSp}}$ and Gl and consider two further groups. Acting with θ on $\widetilde{\mathrm{OSp}}$ or with $\widetilde{\theta}$ on OSp we can define a new group as the fixed point sets of both involutions, namely

$$\mathrm{H}_0 = \mathrm{OSp}_{2|2} \bigcap \widetilde{\mathrm{OSp}}_{2|2} \simeq \mathrm{Gl}_{1|1}$$

Fortunately $[\gamma, \tilde{\gamma}] = 0$, therefore also the involutions θ and $\widetilde{\theta}$ commute and there is only one more important player, i.e.

$$\mathrm{H}' := \left\{ g \in \mathrm{Gl}_{2|2} \mid g = \theta \circ \widetilde{\theta}(g) \right\} \simeq \mathrm{Gl}_{1|1} \times \mathrm{Gl}_{1|1}$$

For more details about the explicit form of H_0 and H' see appendix A.1.3.

3.2.4 Bundle decomposition

Decomposition of the Lie super-algebra

The involutions $\Theta : \mathrm{Gl}_{2|2} \to \mathrm{Gl}_{2|2}$ that fix the subgroups K lead to Cartan involutions $\mathrm{d}\Theta$ of $\mathfrak{gl}_{2|2}$. Introducing two binary indices we can decompose this Lie super-algebra into

$$\begin{aligned} \mathfrak{gl}_{2|2} &= \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1} \oplus \mathfrak{g}_{1,1} \\ \text{where } \mathfrak{g}_{i,j} :&= \left\{ x \in \mathfrak{gl}_{2|2} \mid \mathrm{d}\theta(x) = (-1)^i x \text{ and } \mathrm{d}\widetilde{\theta}(x) = (-1)^j x \right\} \end{aligned}$$

with $\mathrm{d}\theta(x) = -\gamma x^{\mathrm{ST}} \gamma^{-1}$ and $\mathrm{d}\widetilde{\theta}(x) = -\widetilde{\gamma} x^{\mathrm{ST}} \widetilde{\gamma}^{-1}$ being the differentials of the Lie group involutions. This decomposition of the Lie algebra corresponds to the respective Lie groups via the following table:

Lie super-group	$\mathrm{Gl}_{2\|2}$	OSp	$\widetilde{\mathrm{OSp}}$	H'	H_0
Lie super-algebra	$\mathfrak{gl}_{2\|2}$	$\mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,1}$	$\mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,0}$	$\mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,1}$	$\mathfrak{g}_{0,0}$

Therefore we have

$$\mathrm{T}_{[\mathbb{1}]}(\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}) \simeq \mathfrak{g}_{0,1} \oplus \mathfrak{g}_{1,1} \simeq \mathrm{T}_{[\mathbb{1}]}(\widetilde{\mathrm{OSp}} / \mathrm{H}_0) \oplus \mathrm{T}_{[\mathbb{1}]}(\mathrm{H}' / \mathrm{H}_0)$$

which may be considered as the a posteriori motivation for introducing H'. For more details about these Lie super-algebras see appendix A.1.1.

Global decomposition

In the last section 3.2.4 we motivated the decomposition

$$\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} \simeq \widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathrm{H}' / \mathrm{H}_0$$

to split off the Goldstone modes from the action. On the Lie algebra level, i.e. locally, we already saw that this decomposition makes sense. To get our hands on the global situation we define the straight forward multiplication map

$$\begin{aligned}\phi :\ \widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathrm{H}' / \mathrm{H}_0 &\to \mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} \\ [\tilde{h}; h'] &\mapsto [\tilde{h}h']\end{aligned}$$

where the equivalence classes $[\tilde{h}\,; h'] = \{(\tilde{h}h_0, h_0^{-1}h'h_0') \mid h_0, h_0' \in \mathrm{H}_0\}$ and $[\tilde{h}h'] = \{\tilde{h}h'h \mid h \in \mathrm{OSp}_{2|2}\}$ denote elements of the respective coset spaces.[4] This is to say that the H_0 action is by simultaneous multiplication with h_0 from the right in the $\widetilde{\mathrm{OSp}}$ component and with h_0^{-1} from the left in the H' component which does not change the product. Now our task is to show that this mapping is a diffeomorphism. In fact, as we are only interested in this decomposition with respect to integration, it is sufficient if ϕ is a diffeomorphism only locally and almost everywhere. Which is equivalent to saying that $\mathrm{d}\phi$ should be an isomorphism almost everywhere.

By introducing normal coordinates around $[\mathbb{1}\,; \mathbb{1}]$ it is easy to see that the differential at this point $\mathrm{d}\phi\big|_{[\mathbb{1}\,;\mathbb{1}]}$ is an isomorphism. But since we are interested in the global picture, let us embed the whole situation into $\mathrm{Gl}_{2|2}$ and use the Maurer Cartan form there to get back into the Lie algebra where we have just sorted out the orthogonal decomposition into the respective tangent spaces.[5] The Maurer Cartan form is an isomorphism of vector spaces at each point, hence the differential $\mathrm{d}\phi$ at $[\tilde{h}\,; h']$ will be an isomorphism $\mathrm{T}_{[\tilde{h}\,;h']}\,\widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathrm{H}' / \mathrm{H}_0 \to \mathrm{T}_{[\tilde{h}h']}\,\mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2}$ if and only if $(\phi)^{-1}\,\mathrm{d}\phi$ is an isomorphism $\mathrm{T}_{[\tilde{h}\,;h']}\,\widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathrm{H}' / \mathrm{H}_0 \to \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}$.[6] To check whether it is, we will use orthogonal projection onto the subspaces $\mathfrak{g}_{i,j}$, denoted by $\Pi_{\mathfrak{g}_{i,j}}$.

Further we will use the matrix identity

$$e^{-X_0}\partial_t\Big|_0 e^{X_0+tX} = \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X_0))^n}{(n+1)!} X =: \frac{1-e^{-\operatorname{ad}(X_0)}}{\operatorname{ad}(X_0)} X \tag{3.13}$$

where $\operatorname{ad}(X) : Y \mapsto XY - YX$ is the matrix commutator, i.e. the differential of the adjoint action Ad. For a comprehensive derivation of this formula for principal bundles of compact Lie groups, see [BGV04], chapter 5.1.

After these introductory words we finally come to the calculation of $\mathrm{d}\phi$. We will spell out the directional derivative in the directions $\tilde{X} \in \mathfrak{g}_{1,0}$ and $X' \in \mathfrak{g}_{1,1}$ at $\tilde{h} \in \widetilde{\mathrm{OSp}}_{2|2}$ and $[h'] = e^{X_0'}\,\mathrm{H}_0 \in \mathrm{H}' / \mathrm{H}_0$ with $X_0' \in \mathfrak{g}_{1,1}$, where we used the Cartan decomposition

[4] Note that ϕ in this chapter denotes a different mapping than in section 2.1.3.

[5] The Maurer Cartan form is a differential one form with values in the Lie algebra given by the push forward of left translation in the group. $L^*_{g^{-1}} : \mathrm{T}_g\, G \xrightarrow{\sim} \mathrm{T}_e\, G \simeq \mathfrak{g}$. Note that pushing a vectorfield forward just means applying the differential, i.e. $L^*_{g^{-1}} = dL_{g^{-1}}$.

[6] To unclutter the notation we leave away L^* and write $(\phi)^{-1}\,\mathrm{d}\phi$ instead of $L^*_{\phi^{-1}} \circ \mathrm{d}\phi$, because left translation as well as its pushforward is for matrix Lie groups just given by matrix multiplication. Note however that here $(\phi)^{-1}$ always denotes Lie group, i.e. matrix inverse, not the inverse of the mapping ϕ.

for $\mathrm{H}' / \mathrm{H}_0$, which is diffeomorphic almost everywhere.

$$\begin{aligned}
&\Pi_{\mathfrak{g}_{1,0}\oplus\mathfrak{g}_{1,1}}\left(\left(\tilde{h}h'\right)^{-1}\mathrm{d}\left(\tilde{h}h'\right)\right)\\
&= \Pi_{\mathfrak{g}_{1,0}\oplus\mathfrak{g}_{1,1}}\left(\frac{1-e^{-\operatorname{ad}(X_0')}}{\operatorname{ad}(X_0')}X' + e^{-\operatorname{ad}(X_0')}\tilde{h}^{-1}\mathrm{d}\tilde{h}\right) \qquad (3.14)\\
&= \frac{\sinh(\operatorname{ad}(X_0'))}{\operatorname{ad}(X_0')}X' + \cosh(\operatorname{ad}(X_0'))\,\Pi_{\mathfrak{g}_{1,0}}\left(\tilde{h}^{-1}\mathrm{d}\tilde{h}\right) - \sinh(\operatorname{ad}(X_0'))\,\Pi_{\mathfrak{g}_{0,0}}\left(\tilde{h}^{-1}\mathrm{d}\tilde{h}\right)
\end{aligned}$$

where we used that the commutator is compatible with the Cartan involutions, i.e. $[\mathfrak{g}_{i,j},\mathfrak{g}_{k,l}] \subset \mathfrak{g}_{i+k,j+l}$ with the index sums understood in $\mathbb{Z}_2$. Now one could apply (3.13) also to $\tilde{h} = e^{\tilde{X}_0}$, using that for connected Lie groups exp is a diffeomorphism onto a dense open set, but since $\tilde{X}_0 \in \mathfrak{g}_{0,0}\oplus\mathfrak{g}_{1,0}$ has in general a component in $\mathfrak{g}_{0,0}$ this would not simplify the situation.

To understand (3.14) better we should rather diagonalise the adjoint action $\operatorname{ad}(X_0')$. So we choose a maximal (in the even part of $\mathfrak{g}_{1,1}$) commutative subalgebra (of $\mathfrak{g}_{0,0}\oplus\mathfrak{g}_{1,1}$), call it $\mathfrak{a}_{1,1}\subset\mathfrak{g}_{1,1}$, and decompose $X_0' = \operatorname{Ad}(h_0)a$ with $h_0 \in \mathrm{H}_0/Z_{\mathfrak{a}_{1,1}}$ and $a\in\mathfrak{a}_{1,1}$. On the global scale this corresponds to $\mathrm{H}'/\mathrm{H}_0 \simeq e^{\mathfrak{g}_{1,1}} \simeq (\mathrm{H}_0/Z_{\mathfrak{a}_{1,1}})\times e^{\mathfrak{a}_{1,1}}$, where $Z_{\mathfrak{a}_{1,1}}$ is the centraliser of $\mathfrak{a}_{1,1}$ in H_0 with respect to the Ad action. We use

$$f(\operatorname{ad}(X_0')) = \operatorname{Ad}(h_0)\circ f(\operatorname{ad}(a))\circ\operatorname{Ad}(h_0^{-1})$$

for any analytic function f. Further we pass from the directional derivative to the differential itself

$$\mathrm{d}X_0' = \operatorname{Ad}(h_0)\left(\mathrm{d}a - \operatorname{ad}(a)(h_0^{-1}\mathrm{d}h_0)\right)$$

Note that $\operatorname{Ad}(h_0)$ commutes with all projectors $\Pi_{\mathfrak{g}_{i,j}}$ since $h_0\in\mathrm{H}_0$ so we get

$$\begin{aligned}
(3.14) = \operatorname{Ad}(h_0)\Bigg(&\mathrm{d}a - \sinh(\operatorname{ad}(a))(h_0^{-1}\mathrm{d}h_0)\\
&+\cosh(\operatorname{ad}(a))\,\Pi_{\mathfrak{g}_{1,0}}\left((\tilde{h}h_0)^{-1}\mathrm{d}(\tilde{h}h_0) - h_0^{-1}\mathrm{d}h_0\right)\\
&-\sinh(\operatorname{ad}(a))\,\Pi_{\mathfrak{g}_{0,0}}\left((\tilde{h}h_0)^{-1}\mathrm{d}(\tilde{h}h_0) - h_0^{-1}\mathrm{d}h_0\right)\Bigg) \qquad (3.15)\\
= \operatorname{Ad}(h_0)\Bigg(&\mathrm{d}a + \cosh(\operatorname{ad}(a))\,\Pi_{\mathfrak{g}_{1,0}}\left(\tilde{h}^{-1}\mathrm{d}(\tilde{h})\right) - \sinh(\operatorname{ad}(a))\,\Pi_{\mathfrak{g}_{0,0}}\left(\tilde{h}^{-1}\mathrm{d}(\tilde{h})\right)\Bigg)
\end{aligned}$$

where in the last equation we shifted $\tilde{h}\mapsto\tilde{h}h_0^{-1}$. One should keep in mind that we are working in $\widetilde{\mathrm{OSp}}_{2|2}\times_{\mathrm{H}_0}\mathrm{H}'/\mathrm{H}_0$, hence shifting $h'\mapsto h_0e^ah_0^{-1}$ and simultaneously $\tilde{h}\mapsto\tilde{h}h_0^{-1}$ does not change $[\tilde{h};h']$.

Now one might worry about the vanishing of $\mathrm{d}h_0$. But a closer look at the above computation shows that we in fact decomposed

$$\widetilde{\mathrm{OSp}}\times_{\mathrm{H}_0}\mathrm{H}'/\mathrm{H}_0 \simeq \widetilde{\mathrm{OSp}}/Z_{\mathfrak{a}_{1,1}}\times e^{\mathfrak{a}} \qquad (3.16)$$

even further than originally intended. The $\mathrm{d}h_0$ component is now included in $\tilde{h}^{-1}\mathrm{d}\tilde{h}$ of $\widetilde{\mathrm{OSp}}$, which did not have this component on the left hand side of (3.16) due to the $\times_{\mathrm{H}_0}$. This is equivalent to our original statement $\tilde{X} \in \mathfrak{g}_{1,0}$.

So from (3.15) one can see that $\mathrm{d}\phi$ will indeed be an isomorphism as long as $\cosh(\mathrm{ad}(a))$ and $\sinh(\mathrm{ad}(a))$ are, because the adjoint group action $\mathrm{Ad}(\mathrm{H}_0)$ is always an isomorphism which does not mix the $\mathfrak{g}_{i,j}$ components, so there is no cancellation of different terms, and the Maurer Cartan form $\tilde{h}^{-1}\mathrm{d}\tilde{h}$ of $\widetilde{\mathrm{OSp}}_{2|2}$ is at each point an isomorphism onto $\mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,0}$, so nothing gets lost in the projection.

An additional benefit from the above calculation is that we can now immediately read off the Berezinian for changing coordinates and decomposing the integral.[7]

$$\begin{aligned}
&\int\limits_{\mathrm{Gl}_{2|2}\,/\,\mathrm{OSp}_{2|2}} f(g\,\mathrm{OSp}_{2|2})\mathrm{d}\mu(g) \\
&\quad \propto \int\limits_{\mathfrak{a}_{1,1}} \mathrm{d}a \int\limits_{\widetilde{\mathrm{OSp}}_{1|1}/Z_{\mathfrak{a}_{1,1}}} \mathrm{d}\mu(\tilde{h}) f(\tilde{h}e^{a}\,\mathrm{OSp}_{2|2})\,\mathrm{SDet}\left(\sinh(\mathrm{ad}(a))\right)\mathrm{SDet}\left(\cosh(\mathrm{ad}(a))\right)
\end{aligned} \tag{3.17}$$

The determinants can be computed by diagonalising the adjoint action, and we get

$$\mathrm{SDet}_{\mathfrak{g}_{1,0}}\left(\cosh(\mathrm{ad}(a))\right) = \frac{1}{\cosh^2(x-y)}$$

with $x - y = \frac{1}{2}\,\mathrm{STr}(a)$, see equation (A.1), and

$$\mathrm{SDet}\left(\sinh(\mathrm{ad}(a))\right) = -\frac{1}{\sin^2(x-y)}$$

see equation (A.2). Here the determinant is understood with respect to the volume forms induced by STr, i.e. $\mathrm{d}\,\mathrm{Vol}_{\mathfrak{g}_{0,0}/Z_{\mathfrak{a}_{1,1}}} =: \mathrm{Det}(f)f^*\left(\mathrm{d}\,\mathrm{Vol}_{\mathfrak{g}_{1,1}/\mathfrak{a}_{1,1}}\right)$ for $f : \mathfrak{g}_{0,0}/Z_{\mathfrak{a}_{1,1}} \to \mathfrak{g}_{1,1}/\mathfrak{a}_{1,1}$.

Finally, specifying the real form of $e^{\mathfrak{a}_{1,1}}$ by the STr being positive, we get

$$(3.17) \propto -\int\limits_{\mathbb{R}} \mathrm{d}x \int\limits_{[0;2\pi]} \mathrm{d}y \int\limits_{\widetilde{\mathrm{OSp}}_{1|1}/Z_{\mathfrak{a}_{1,1}}} \mathrm{d}\mu(\tilde{h}) \frac{f(\tilde{h}e^{xE_{BB}+yE_{FF}}\,\mathrm{OSp}_{2|2})}{\sin^2(x-y)\cosh^2(x-y)}$$

Unfortunately, this Berezinian is not the full truth. As mentioned in section 3.2.1, we get additional anomalous terms, if the integrand does not vanish at the boundaries, $x \to \pm\infty$. This leads to the decomposition not being applicable to the field theory, as we would get anomalous terms at every point in space.

[7] Note that a similar computation can be found in appendix A of [ZH95].

3.2.5 Differential super-geometry

In this section we introduce some general concepts about Riemannian symmetric (super-) spaces, mainly taken from [Zir98], which we need to finally write down the action in the next section.

Vector fields and metric

In general, for $\mathcal{G}$ a super Lie group ($\mathrm{Gl}_{2|2}$) with invariant bilinear form B (STr) and Riemannian real form G_r and with $\mathcal{H}$ a Lie super-subgroup of $\mathcal{G}$ (OSp) given as the fixed point set of an isometric involution (θ) we have the following:

We already used that the super Lie algebra of $\mathcal{G}$ is split into an orthogonal sum with respect to the super-trace form, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h} = \{X \in \mathfrak{g} \mid \mathrm{d}\Theta(X) = X\}$ is the super Lie algebra of $\mathcal{H}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \mathrm{d}\Theta(X) = -X\}$ is isomorphic to the tangent super-space $\mathrm{T}_{[\mathcal{H}]}\,\mathcal{G}/\mathcal{H}$. The sum is orthogonal because for $X \in \mathfrak{h}$ and $Y \in \mathfrak{p}$ we have $\mathrm{STr}(XY) = \mathrm{STr}(\mathrm{d}\Theta(X)\mathrm{d}\Theta(Y)) = -\,\mathrm{STr}(XY)$ by Θ being an isometry.

We have already seen that the Maurer Cartan form maps tangent vectors, or left invariant vector fields, into Lie algebra elements. Now, of course, we can also go the other way and associate Lie algebra elements with left invariant vector fields. To do so we again use that $\mathcal{G}$ acts on $\mathcal{G}/\mathcal{H}$ by left translation. In combination with the exponential map $\exp : \mathfrak{g} \to \mathcal{G}$ we can use this to define a mapping

$$\begin{aligned} \mathfrak{g} &\to \Gamma(\mathcal{G}/\mathcal{H}) \\ X &\mapsto \hat{X} \\ \hat{X}\Big|_{[g]} f &:= \partial_s\Big|_0 f(\exp(sX)g\mathcal{H}) \end{aligned}$$

The super-geometry on $\mathcal{G}/\mathcal{H}$ is then given by the unique (up to a multiplicative constant) invariant bilinear form which is induced by the one on $\mathcal{G}$ via

$$\langle \hat{X}, \hat{Y} \rangle\Big|_g = B\left(\Pi_{\mathfrak{p}}\left(\mathrm{Ad}(g^{-1})X\right), \Pi_{\mathfrak{p}}\left(\mathrm{Ad}(g^{-1})Y\right)\right) \tag{3.18}$$

The adjoint action is due to the coset structure as described in (3.19) and particularly (3.20) below. Additionally one needs to fix a maximal Riemannian (in the geometry that was just described) submanifold of $G_r/H_r \subset G_s/H_s$ to complete the data for a Riemannian symmetric super-space. In our setting this means that we have a metric on $\widetilde{\mathrm{OSp}}_{2|2}/\,\mathrm{H}_0$ and $\mathrm{H}'\,/\,\mathrm{H}_0$ which are both induced by the same super-geometry of $\mathrm{Gl}_{2|2}$ which restricts to super-geometry of the subgroups. On the Lie algebra level this is more simply said, we are just using the same invariant bilinear form STr on all components $\mathfrak{g}_{i,j}$ of $\mathfrak{gl}$.

Associated vector bundles

In section 3.2.3 we saw that it might be a good idea to split a $\widetilde{\mathrm{OSp}}_{2|2}$ component off from our super-field P, which becomes massless as z goes to zero, and in 3.2.4

the remaining $\mathrm{H}' / \mathrm{H}_0$ component was identified. As the latter will stay massive, it should be sensible to expand the action locally around the saddle point value, hence we pass from $\mathrm{H}' / \mathrm{H}_0$ to $\mathrm{T}_{P_s} \mathrm{H}' / \mathrm{H}_0 \simeq \mathfrak{g}_{1,1}$. And instead of the full non-linear field $P : \mathbb{R}^d \to \mathrm{Gl}_{2|2} / \mathrm{OSp}_{2|2} \simeq \widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathrm{H}' / \mathrm{H}_0$ we will consider $\Psi : \mathbb{R}^d \to \widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1}$, where the associated vector bundle $\widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1}$ is defined via the adjoint action Ad of H_0 on $\mathfrak{g}_{1,1}$. Therefore we will recall some basic facts about sections of associated vector bundles in the following two subsections. First note

$$\begin{aligned} \mathrm{T}(G/H) &\simeq G \times_H \mathfrak{p} \\ \partial_t \Big|_0 (g e^{tx}) &\hookleftarrow [g; x] \\ y &\mapsto [g; (g^{-1} dg)_{\mathfrak{p}}(y)] \end{aligned} \tag{3.19}$$

where in the last line $g \in \pi(y)$ and '$(g^{-1}dg)_{\mathfrak{p}}$' denotes the $\mathfrak{p}$ component of the Maurer Cartan form at g. The action of H in $G \times_H \mathfrak{p}$ is given by

$$[g \;\; y] = [gh\,; \mathrm{Ad}(h^{-1})y] \tag{3.20}$$

from which one immediately sees that both maps are well defined and evidently inverse to each other, which establishes the diffeomorphism. This was already used when formulating the metric above and will again be useful in section 3.2.6.

For sections of an associated vector bundle we further have

$$\begin{aligned} \Gamma(G \times_H V) &\simeq C^\infty(G, V)^H \\ \text{by } \Theta : ([g] \mapsto [g; \varphi(g)]) &\mapsto \varphi \end{aligned} \tag{3.21}$$

where V is some left H module and $C^\infty(G, V)^H$ denotes smooth functions from G to V that are H equivariant, i.e. $\varphi(gh) = h^{-1}.\varphi(g)$. Here by $H.V$ we denote the action (representation) of H on V. (3.21) is a Diffeomorphism because for $s = ([g] \mapsto [g, \varphi(g)]) \in \Gamma(G \times_H V)$ a section and $h \in H$ we have

$$\begin{aligned} s([g]) = [g; \varphi(g)] &= [gh; h^{-1}.\varphi(g)] && \text{by definition of } G \times_H V \\ = s([gh]) &= [gh; \varphi(gh)] && \text{by definition of } G/H \end{aligned}$$

therefore φ indeed has to be an equivariant function and conversely every equivariant function defines a section of $\Gamma(G \times_H V)$.

Covariant derivative on associated vector bundles

Now we define a covariant derivative on vector fields via the Lie derivative L on equivariant functions by completing the following diagram

$$\begin{tikzcd} \Gamma(G \times_H V) \arrow[r, "\Theta"] \arrow[d, dotted, "\nabla"'] \arrow[dr, phantom, "\circlearrowright"] & C^\infty(G, V)^H \arrow[d, "\mathrm{L}"] \\ \Gamma(G \times_H V) & C^\infty(G, V)^H \arrow[l, "\Theta^{-1}"] \end{tikzcd}$$

For a vector $[g,x] \in \mathrm{T}_{[g]}(G/H)$ and a section $s : [g] \mapsto [g,\psi(g)]$ we take the Lie derivative with respect to the left invariant vector field associated to $x \in \mathfrak{g}_{1|1}$.

$$\nabla_{[g,x]}s := [g; \partial_t\Big|_0 \psi(ge^{tx})] \tag{3.22}$$

But note that this notation is deceptive in that it does not look well defined when another representative of $s \equiv s_h = ([g] \mapsto [gh(g), h(g)^{-1}.\psi(g)])$ for any smooth $h : G \to H$ is taken into account. However, by the above diagram, i.e. writing $\nabla = \Theta^{-1} \circ \mathrm{L} \circ \Theta$ it becomes clear that

$$\begin{aligned}\nabla_{[g,x]}s_h &= [g, \partial_t\Big|_0 h(t).\psi(gh(t)e^{t\,\mathrm{Ad}(h(t)^{-1})x})] = \nabla_{[g,x]}s \\ &\neq [gh(t), \partial_t\Big|_0 \psi(gh(t)e^{t\,\mathrm{Ad}(h(t)^{-1})x})]\end{aligned}$$

In our setting the Ψ is not just a pull back of a section of $\widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1}$ but more generally

$$\begin{aligned}\Psi : \mathbb{R}^d &\to \widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1} \\ x &\mapsto [\tilde{h}(x), \psi(x)]\end{aligned}$$

Fortunately we can still adapt the notion of covariant derivative (3.22) on $\widetilde{\mathrm{OSp}} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1}$ to the pullback bundle by

$$\nabla_y \Psi(x) := [\tilde{h}(x), \partial_t\Big|_0 e^{t\tilde{h}_* y}\psi(x)]$$

3.2.6 The action

So now finally we have collected all the structures to write down a general form of the action $S = \int \mathcal{L}$ where $\mathcal{L}$ is a d-form which is by the structure of the theory fixed up to numerical constants c_i, m and $\tilde{m}$ and terms of higher order.

$$\boxed{\mathcal{L} = c_1\,||\mathrm{d}\tilde{h}||^2 + c_2\,||\nabla\Psi||^2 + \tilde{m}\,||\varphi||^2 + m'\,||\psi||^2} \tag{3.23}$$

Here we have introduced a d form valued product $\bullet$ of

$$\mathrm{T}\left[\widetilde{\mathrm{OSp}}_{2|2} \times_{\mathrm{H}_0} \mathfrak{g}_{1,1}\right] \simeq \widetilde{\mathrm{OSp}}_{2|2} \times_{\mathrm{H}_0} (\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1})$$

valued forms on $\mathbb{R}^d$ and wrote $X \bullet X =: ||X||^2$. By (3.18) we get the metric on this bundle via the STr form on $\mathfrak{gl}$.

$$(X \bullet Y)\Big|_{\tilde{h}} := \mathrm{STr}\left(\mathrm{Ad}\left(\tilde{h}^{-1}\right) X \wedge * \mathrm{Ad}\left(\tilde{h}^{-1}\right) Y\right)$$

where $*$ is the Hodge star operator on $\mathbb{R}^d$. Note that $\mathrm{d}\tilde{h} \bullet \nabla\Psi = 0$ therefore the above terms are all that one can possibly write down up to higher orders in the fields.

The individual terms have the following physical meaning: $\mathrm{d}\tilde{h}$ is a one form with values in $\mathrm{T}\,\widetilde{\mathrm{OSp}}_{2|2}/\,\mathrm{H}_0$. $\mathrm{d}\tilde{h} \bullet \mathrm{d}\tilde{h}$ is the usual kinetic term of a non-linear σ-model given by $\tilde{h} : \mathbb{R}^d \to \widetilde{\mathrm{OSp}}_{2|2}/\,\mathrm{H}_0$ alone. For $\hat{z} \neq 0$, i.e. in the bulk scaling limit, also these $\widetilde{\mathrm{OSp}}$ modes will oscillate with a finite mass around their fixed point value $\tilde{h}_s$. In this case $\tilde{h} = e^{\varphi}\tilde{h}_s$ is a sensible expansion and $\tilde{m}$ the mass of these modes. Here φ is a zero form, hence the wedge product is just multiplication and $*1$ yields the volume form. For $\hat{z} \to 0$ the mass $\tilde{m}$ has to go to zero, so it does not matter that φ will not be a well defined field any more.

In addition to this non-linear σ-model with emerging mass term for $z \neq 0$, we have the additional massive field ψ, again a 0 form. The mass is given by m' and should be positive everywhere for the expansion to make sense.

$\nabla\Psi \bullet \nabla\Psi$ finally is the kinetic term of this new field which also couples $\tilde{h}$ and ψ in a gauge field like fashion. The covariant derivative $\nabla\Psi$, as explained above, is here understood as a one form on $\mathbb{R}^d$.

Coefficients

To fix the coefficients in the general form of the action we now go back to the quadratic expansion (3.10) and change coordinates using the explicit forms of the Lie algebras as in A.1.1 to write

$$P_S + \delta P = \mathrm{Ad}^\theta(e^\varphi)\,\mathrm{Ad}^\theta(e^\psi)P_S = e^\varphi e^\psi P_S \gamma e^{\psi^{\mathrm{ST}}} e^{\varphi^{\mathrm{ST}}}\gamma$$

with

$$\varphi = \left(\begin{array}{c|c} \varphi_1\sigma_1 + \varphi_2 i\sigma_2 + \varphi_3\sigma_3 & 0 \\ \hline 0 & \varphi_{f,2}\sigma_2 \end{array}\right) \in \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,0}$$

and

$$\psi = \left(\begin{array}{c|c} \psi\mathbb{1} + \psi_2 i\sigma_2 & 0 \\ \hline 0 & \psi_f\mathbb{1} + \psi_{f,2}\sigma_2 \end{array}\right) \in \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{1,1}$$

We get

$$\begin{aligned} a_0 &= 2s\psi_0 + \mathcal{O}(\psi,\varphi)^2 \\ a_1 &= 2s\varphi_1 + \mathcal{O}(\psi,\varphi)^2 \\ a_3 &= 2s\varphi_3 + \mathcal{O}(\psi,\varphi)^2 \\ b &= 2s\psi_f + \mathcal{O}(\psi,\varphi)^2 \end{aligned}$$

And inserting this into the second order Lagrangian (3.10) we can read off the coefficients

$$
\begin{aligned}
\mathcal{L}^{(2)} = &\left|4\frac{(\alpha-1)s^2+\alpha}{(s^2+1)}\right|\left(\varphi_1^2+\varphi_3^2\right)\\
&+\left|4\frac{(\alpha-1)s^4+(2\alpha+1)s^2+\alpha}{(s^2+1)^2}\right|\left(\psi_0^2+\psi_f^2\right)\\
&+\frac{4s^2}{s^2+1}\frac{2e^2}{2d+1}e^{i\phi_1}\Bigg(\left(\partial_{x_i}\varphi_1\right)^2+\left(\partial_{x_i}\varphi_3\right)^2\\
&\quad+\frac{s^2-1}{s^2+1}e^{i(\phi_2-\phi_1)}\left(\left(\partial_{x_i}\psi_0\right)^2+\left(\partial_{x_i}\psi_f\right)^2\right)\Bigg)
\end{aligned}
$$

Hence the mass of the $\widetilde{\mathrm{OSp}}$ modes is

$$
\tilde{m} = \left|4\frac{(\alpha-1)s^2+\alpha}{(s^2+1)}\right|
$$

and we remember from the saddle point equation for s, (3.7b), that $(\alpha-1)s^2+\alpha$ indeed goes to 0 for $\hat{z}\to 0$.

The mass of the H' modes is given by

$$
m' = \left|4\frac{(\alpha-1)s^4+(2\alpha+1)s^2+\alpha}{(s^2+1)^2}\right|
$$

and does indeed not vanish. Further we can read off the coefficients of the kinetic terms

$$
c_1 = \frac{8s^2}{s^2+1}\frac{e^2}{2d+1}e^{i\phi_1} \qquad \text{and} \qquad c_2 = \frac{8s^2(s^2-1)}{(s^2+1)^2}\frac{e^2}{2d+1}e^{i\phi_2}
$$

This completes the derivation of the effective action.

4 Density of states in dimension zero

In this chapter we turn back to the zero-dimensional model, where we can calculate the density of states explicitly. Here we will investigate the effect of the zero modes controlled by the parameter α to compare with [LSZ06] and look for universal behaviour.

The original definition of super-integration tells us that one should first integrate out the Grassmann variables. This is done by differentiation, i.e. one expands the integrand super-function in Grassmann variables and then picks the coefficient of the product of those Grassmann variables, which we are integrating out. In equation (2.24) only the super-determinants are super-functions, everything else is numerical valued as long as boson-boson and fermion-fermion block contain only numbers. So to integrate out the Grassmann variables one needs to expand

$$\frac{\mathrm{SDet}(P)^{\alpha N}}{\mathrm{SDet}(P - i\Sigma_2)^N}$$

and pick the coefficients of the highest order nilpotent element, $\chi_4\chi_3\chi_2\chi_1$ where the χ_i are the Grassmann variables in (2.26). In this chapter we will write N instead of $N|V|$ as in dimension zero $|V| = 1$ anyway. For finite $|V|$, scaling $N|V| \mapsto N$ also looks harmless but in the spatially extended case one should think more carefully about the two limits $N \to \infty$ and $|V| \to \infty$. However this will not be discussed here.

The expansion in Grassmann variables is carried out explicitly in the appendix, (A.4). The result is

$$\begin{aligned}\partial_{\chi_4}\partial_{\chi_3}\partial_{\chi_2}\partial_{\chi_1}&\left(\frac{\mathrm{SDet}(P)^{\alpha}}{\mathrm{SDet}(P - i\Sigma_2)}\right)^N \\ &=: Gr(\mathrm{Det}(A), \mathrm{Det}(B), N)\left(\frac{\mathrm{Det}(A)}{\mathrm{Det}(B)}\right)^{\alpha N}\left(\frac{\mathrm{Det}(A)+1}{\mathrm{Det}(B)+1}\right)^{-N} \qquad (4.1)\end{aligned}$$

where Gr is a rational function, in particular it does not contain any powers of N. The exact form of Gr can be read off from equation (A.4), but more importantly the action has, to order $\mathcal{O}(N)$, the same form as in 3.1.2 where we started by just setting the Grassmann variables to 0. Hence we can use the saddle point equations and the global facts about the action landscape that we found in 3.1.2 and 3.1.3.

4.1 Bulk scaling

For $N \to \infty$ we can evaluate the remaining ordinary integrals within the saddle point approximation. For a short review of this method see section 4.3.1 below and note that in bulk scaling the first order term does not vanish, so using this standard version of the saddle point method is sufficient. The result is

$$G(z) - \frac{\Delta\dim}{z} \propto s\ Gr(s^2, s^2, N)\frac{e^{NS[P_S]}}{N^2\sqrt{D}}$$

where D is the fluctuation determinant, which can be read off from (3.10).

$$\sqrt{D} = \left| \frac{(\alpha-1)s^2+\alpha}{s^2(s^2+1)}\ \frac{(\alpha-1)s^4+2(\alpha+1)s^2+\alpha}{s^2(s^2+1)^2} \right|$$

Now its time to restore the proportionality factors which have been dropped every now and then. By looking at (2.11) and (2.21), we see that our integral should be $= 1$ if not for the derivative, which leads to the factor $\frac{d+1}{4d+2}\operatorname{Tr}(P_{S,FF}) = \frac{d+1}{2d+1}s$. This means, in $d = 0$ and for $N \to \infty$ the proportionality factor C must be such that

$$Gr(s^2, s^2, N)\frac{e^{NS[P_S]}}{N^2\sqrt{D}} = \frac{1}{C}$$

and a quick check yields $C = \frac{1}{4}$, i.e. all s and α dependent terms cancel, as expected.

So we obtain

$$G(z) = \frac{\Delta\dim}{z} + \frac{d+1}{2d+1}s$$

which means that the density of eigenfrequencies for $\omega \neq 0$ is already given by the saddle point s alone. And, conforming our observation in 2.2.2, $2s = \operatorname{Tr}(P_{BB}) = -\operatorname{Tr}(P_{FF})$, i.e. we could perform the derivative with respect to the bosonic as well as the fermionic energy parameter. Now (2.10) yields

$$\rho(\omega) = \frac{2}{\pi}\Re\left(s(i\omega+\epsilon)\right)$$

For $\omega = 0$, however, we get an additional divergent contribution from the $\frac{\Delta\dim}{z}$ term. This can be interpreted as the contribution of the zero modes if $\Delta\dim > 0$, i.e. $\alpha < \frac{d+1}{2}$. For now we will drop this term from the discussion.

We have plotted the density of eigenfrequencies, normalised to $\int_{\mathbb{R}^+}\rho(\omega) = 1$, for some values of $\alpha < 1$ in Figure 4.1 and for $\alpha > 1$ in Figure 4.2. Note that the spectrum is symmetric with respect to reflection at 0, therefore we plot only the positive half. The relevant solution of the saddle point equation $s(\hat{z})$ is explicitly given in (A.3).

Obviously the most striking difference is that for $\alpha > 1$ the spectrum consists of two bands, which melt at $\alpha = 1$ in the singular fashion that was investigated in [LSZ06]. For $\alpha < 1$ we have a continuous single band spectrum without singularity at 0. Note however that the $\frac{\Delta\dim}{z}$ term was ignored. This splitting of the spectrum explains why we were unable to find a feasible saddle point at $\hat{\omega} = 0$ for $\alpha \geq 1$ in section 3.1.3.

We also note that the maximal density of states for all cases lies at low frequencies. For $\alpha \leq 1$ the density is maximal at $\hat{\omega} = 0$, for $\alpha > 1$ there is even an expressed peak at moderately low frequencies. This might be a hint at the so-called boson peak

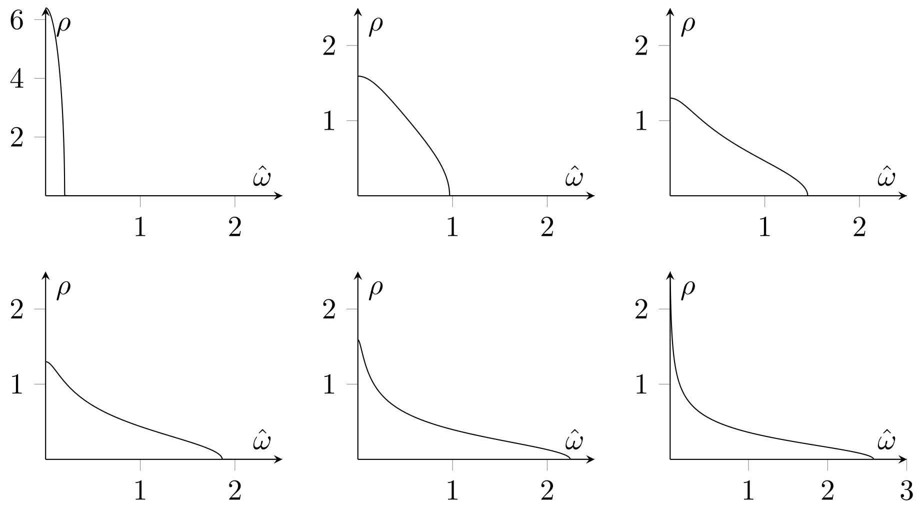

Figure 4.1: Density of eigenfrequencies for $\alpha = 0.01$, $\alpha = 0.2$, $\alpha = 0.4$ and $\alpha = 0.6$, $\alpha = 0.8$, $\alpha = 0.99$

as found in [GA05]. We did not rescale our plots by $\hat{\omega}^2$, as [GA05] did, because our dispersion is obviously not quadratic. In fact our model does not include a limit of a pure system with any specific dispersion what so ever, but describes the pure random part. So finding a peak in the spectrum itself seems reasonable.

4.2 Spectral edges

We can underline the above observations by computing the positions of the band edges. By the implicit function theorem, the saddle point equation (3.7b) can (locally) be inverted to produce the saddle point as a function $s(\hat{z})$ as long as $\hat{z}$ is a regular value. The singular points of (3.7b) are given by

$$s^2 = \frac{1 + 2\alpha \pm \sqrt{8\alpha + 1}}{2(1 - \alpha)} \tag{4.2}$$

This means that for $\alpha > 1$ we have $s^2 < 0$, i.e. 4 singular points on the imaginary axis. If s is purely imaginary, so is $\hat{z}(s)$, hence we have four singular eigenfrequencies. These are the edges of the spectrum. For α close to 1 two of the singular points s diverge, i.e. the corresponding inner spectral edges $\hat{z}$ go to 0. The other two go to $s = \mp i \left(\frac{1}{\sqrt{3}} + \frac{2(\alpha-1)}{9\sqrt{3}} + \mathcal{O}(\alpha - 1)^2 \right)$. Hence $\hat{z}$ goes to $\pm i \left(\frac{3\sqrt{3}}{2} + \sqrt{3}(\alpha - 1) + \mathcal{O}(\alpha - 1)^2 \right)$. Note that $\hat{z} = \frac{d+1}{2(2d+1)} \frac{z}{N} = \frac{z}{2N}$ for $d = 0$, so for $\alpha = 1$ the band edges lie at $\omega = 3\sqrt{3}N$, in agreement with [Lü09], chapter 3.5.

For $\alpha < 1$ two of the singular points (4.2) and also the corresponding singular values $\hat{z}$ lie on the real axis. Hence they do not interfere with the spectrum, which is therefore

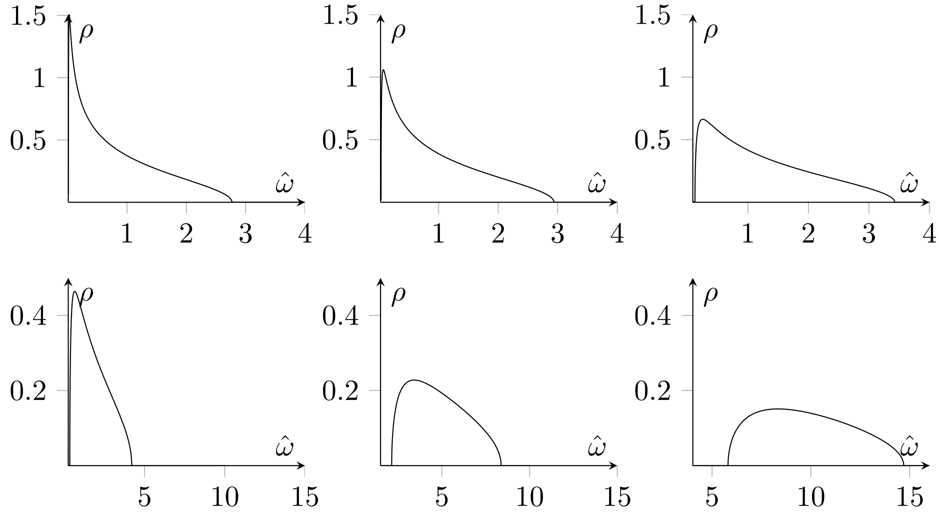

Figure 4.2: Density of eigenfrequencies for $\alpha = 1.1$, $\alpha = 1.2$, $\alpha = 1.5$, $\alpha = 2$, $\alpha = 5$ and $\alpha = 10$

supported on a single interval, as observed above. It remains unclear whether there is a physical interpretation of those real singular values.

4.3 Edge scaling

We have seen in 3.1.3 that if we want to look at the centre of the spectrum at $\hat{\omega} = 0$ at a higher resolution, i.e. not scale z with N, we have to deal with a saddle point manifold in the boson-boson sector, given by (3.6). In the fermion-fermion sector we have to sum contributions from two separate saddle points, as visualised in figure 3.7.

4.3.1 Saddle point method reviewed

A much more severe point is that the saddle point method itself is not applicable as straight forwardly as for the bulk scaling. To see this we need to review the local part of the prove of why the saddle point method yields the right result for $N \to \infty$ and to which order.

So let us consider the general situation of smooth functions $f, S \in C^\infty(\mathbb{R})$ and we want to calculate the integral

$$\int\limits_{\mathbb{R}} f(x, N) e^{-NS(x)} \mathrm{d}x \tag{4.3}$$

to a given order in the large parameter N. Note that in our case f also depends on N which is why we need to worry about higher orders. But still f should be of finite order in N, i.e. $f \in \mathcal{O}(N^k)$ for some k, which in the case of interest will be $k = 2$.

There are global constraints for the applicability of the saddle point method, as usual. In essence one has to estimate the contributions of the integral far away from the saddle point x_s and show that they are small, which e.g. leads to the condition of the saddle point being a global minimum of S. This part of the proof will not be reviewed here, but this in particular rules out such cases where (4.3) is not convergent. What will be done now is calculating the contribution to the integral which comes from a region close to the saddle point. We start by expanding

$$(4.3) = e^{-NS_s} \int\limits_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{f_s^{(n)}}{n!} y^n \right) e^{-N\left(S_s + \sum_{n=3}^{\infty} \frac{S_s^{(n)}}{n!} y^n\right)} e^{-N\frac{1}{2}S_s^{(2)}y^2} \mathrm{d}y \tag{4.4}$$

where we have shifted the integral to $y = x - x_s$, a subscript s stands for evaluation at the saddle point and a superscript (n) for the nth derivative, i.e. $f_s^{(n)} := \partial_x\big|_{x_s} f(x)$.

Now we use Wick's theorem

$$\int\limits_{\mathbb{R}} x^n e^{-\frac{N}{2}\lambda x^2} \mathrm{d}x = \begin{cases} 0 & n \text{ odd} \\ \left(-\frac{2}{N}\partial_\lambda\right)^{\frac{n}{2}} \sqrt{\frac{2\pi}{N\lambda}} = \frac{(n-1)!!}{(N\lambda)^{n/2}} \sqrt{\frac{2\pi}{N\lambda}} & n \text{ even} \end{cases}$$

where $(-1)!! := 1 =: 0!$ to get

$$\begin{aligned}(4.4) = e^{-NS_s} \sqrt{\frac{2\pi}{NS_s^{(2)}}} \Bigg(& \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2n!} \frac{f_s^{(2n)}}{\left(NS_s^{(2)}\right)^n} \\ & - N \sum_{\substack{n\geq 1, m\geq 3 \\ n+m \text{ even}}} \frac{(n+m-1)!!}{n!m!} \frac{f_s^{(n)} S_s^{(m)}}{(NS_s^{(2)})^{\frac{n+m}{2}}} + \dots \Bigg)\end{aligned}$$

Here the ellipsis stands for further terms similar to the last one and starting with $+N^2 \sum_{\substack{n\geq 0, m\geq 3, p\geq 3 \\ n+m+p \text{ even}}}$. So we finally get

$$(4.3) = e^{-NS_s} \sqrt{\frac{2\pi}{NS_s^{(2)}}} \left(f_s + \frac{1}{2} \frac{f_s^{(2)}}{NS_s^{(2)}} - \frac{1}{2} \frac{f_s^{(1)} S_s^{(3)}}{N\left(S_s^{(2)}\right)} + \mathcal{O}\left(\frac{f}{N^2}\right) \right) \tag{4.5}$$

which includes all terms scaling with a positive power of N if $f \in \mathcal{O}\left(N^2\right)$.

If the reader is interested in more details about this expansion, note that, after we calculated the term above, we found a paper by Kirwin, [Kir08], which gives a complete discussion of higher order terms and even extends to non-smooth functions. In their appendix one can find the terms which we have just calculated, although they did not pull out the factor e^{-NS_s} in their definition of the coefficients which leads to additional terms. [1]

[1] Below we have $S_s = 0$ anyway.

4.3.2 Density of frequencies

So how does this apply to our integral for the resolvent operator? Starting from (2.24) with bosonic coordinates as in (2.26) and integrating out Grassmann variables as in (4.1) we get

$$\int\limits_{A=A^{\mathrm{T}}>0} \mathrm{d}a_0\mathrm{d}a_1\mathrm{d}a_3 \int\limits_{\mathrm{U}(1)} \mathrm{d}b\; e^{-z_0a_0+z_1b}\, Gr(\mathrm{Det}(A),b^2,N)\; e^{-NS(\mathrm{Det}(A),b^2)} \tag{4.6}$$

To separate massive from massless modes we will use $\mathrm{Det}(A) = a_0^2 - a_1^2 - a_3^2$ as a coordinate instead of $a_0 = \sqrt{\mathrm{Det}(A) + a_1^2 + a_3^2}$. This means the integral is of the form (4.3) with

$$\begin{aligned} f &= \frac{1}{2a_0} Gr(\mathrm{Det}(A),b^2,N)e^{-z_0a_0+z_1b} \\ S &= \alpha\left(\ln(b^2) - \ln(\mathrm{Det}(A))\right) - \ln(b^2+1) + \ln(\mathrm{Det}(A)+1) \end{aligned}$$

Now the crucial point is that f_s is not of order $\mathcal{O}(N^2)$ at the saddle point $s^2 = \frac{\alpha}{1-\alpha}$, because this contributions cancel, but only of order $\mathcal{O}(N)$. This means that all terms in (4.5) are of the same order and hence need to be taken into account.

Applying the saddle point approximation for both b and $\mathrm{Det}(A)$ we get

$$\begin{aligned} (4.6) = \int\limits_{\mathbb{R}^2} \mathrm{d}a_1\mathrm{d}a_3 \sum_{b=\pm s} \frac{2\pi}{N} \frac{1}{\sqrt{S_s^{(2,0)} S_s^{(0,2)}}} \\ \left(f_s + \frac{1}{2N}\left(\frac{f_s^{(2,0)}}{S_s^{(2,0)}} + \frac{f_s^{(0,2)}}{S_s^{(0,2)}}\right) - \frac{1}{2N}\left(\frac{f_s^{(1,0)} S_s^{(3,0)}}{\left(S_s^{(2,0)}\right)^2} + \frac{f_s^{(0,1)} S_s^{(0,3)}}{\left(S_s^{(0,2)}\right)^2}\right)\right) \end{aligned} \tag{4.7}$$

where $f_s^{(m,n)} := \partial^m_{\mathrm{Det}(A)}\partial^n_b\big|_s f$. Now the computation of all terms is an algorithmic task and one gets

$$(4.7) \propto \int\limits_{\mathbb{R}^2} \frac{\mathrm{d}a_1\mathrm{d}a_3}{a_0} \sum_{b=\pm s} \frac{e^{-z_0a_0+z_1b}}{a_0^2} \left(s(1+a_0z_0) + \frac{a_0^2 z_1 b}{s}\right) \tag{4.8}$$

where only some factors of 2 and π have been dropped, i.e. the terms containing N and α nicely cancel. Now we can perform the integration over the saddle point manifold. As this is the orbit of a symmetry group, integrating out fluctuations has to produce the invariant measure with respect to this group action. And in fact, we compute the Sl_2 invariant measure in appendix A.5 and find that it is

$$\mathrm{d}\mu = \frac{\mathrm{d}a_1\mathrm{d}a_3}{a_0}$$

which appears above. The integrals (4.8) finally yield

$$s\cosh(z_1 s)\int\limits_{\mathbb{R}^2} \mathrm{d}a_1\mathrm{d}a_3 \frac{1+a_0 z_0}{a_0^3} e^{-z_0 a_0} + z_1 \sinh(z_1 s)\int\limits_{\mathbb{R}^2} \mathrm{d}a_1\mathrm{d}a_3 \frac{1}{a_0} e^{-z_0 a_0}$$
$$= \frac{e^{-z_0 s}}{z_0 s}\left(z_0 s\cosh(z_1 s) + z_1 s\sinh(z_1 s)\right) =: G(z_0, z_1)$$

Note that $G(z,z) = 1$, which means that this is the right normalisation as can be seen from (2.11). Also the stronger condition $\partial_{z_1}\big|_z G(z, z_1) = -\partial_{z_0}\big|_z G(z_0, z)$ is fulfilled as demanded from the start, see (2.12).

So we finally obtain the density of states at the centre of the spectrum

$$\rho(\omega) = \Re\left(\partial_{z_1}\Big|_z G(z,z_1)\Big|_{z=i\omega}\right) = \Re\left(s\left(1+\frac{1-e^{-2sz}}{2sz}\right)\Big|_{z=i\omega}\right) = s\left(1+\frac{\sin(2\omega s)}{2\omega s}\right)$$

See figure 4.3 for a plot and note that for $z \to \infty$, G approaches $s = \sqrt{\frac{\alpha}{1-\alpha}}$ and hence joins the $\tilde{z} \to 0$ limit in bulk scaling, as it should. Further note that this is the universal density of states for symmetry class D.

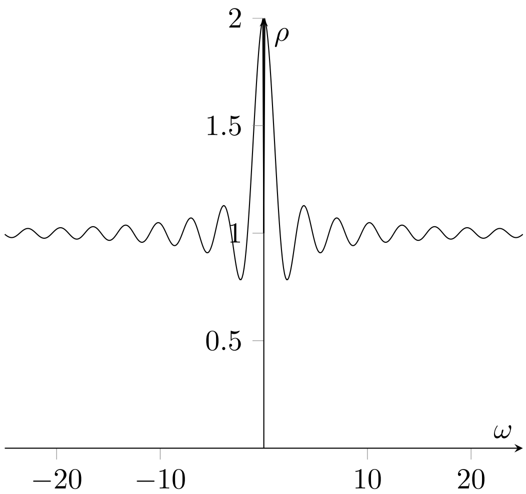

Figure 4.3: Density of frequencies on the edge scale for $\alpha = \frac{1}{2}$, i.e. $s = 1$.

5 Discussion

In summary, we accomplished the derivation of a lattice field theory for a certain one parameter family of disordered bosonic systems which are a generalised and improved version of the model considered in [LSZ06]. We developed this discrete model to a continuum field theory and studied the mean field solution as well as spatially homogeneous fluctuations. Although our field theory describes only the density of states, the generalisation to n-point functions should be straightforward. We confirm the observation of Lück, Sommers and Zirnbauer, [LSZ06], that our bosonic systems do show universal behaviour. We found well known universal statistics of the density of states at low frequencies which fits to the observed universality of correlation functions in [LSZ06] and in fact refines their classification.

In the first part, chapter 2, we started from a reasonable, discrete model for a physical system of very strongly disordered bosons and rigorously derived a lattice super-field theory for the resolvent operator. Due to superbosonisation the target space of the fields is of fixed dimension, independent of N, the number of degrees of freedom per lattice site. The price to be paid for this low dimensionality are odd directions and a non-trivial geometry, i.e. the target space was found to be the Riemannian symmetric super-space $\mathrm{Gl}_{2|2} \,/\, \mathrm{OSp}_{2|2}$. The super-dimension, $(4|4)$, is rather small, because we considered only the resolvent operator, i.e. the one-point function. For higher n-point functions one would have to deal with larger super-spaces and one might want to use more elaborate techniques than integrating out the Grassmann variables 'by hand' as done in chapter 4. Therefore it might be fruitful if one could get the boundary terms, which were not calculated in section 3.2.4, under control.

Apart from writing a symmetric positive semidefinite matrix as $h = l^{\mathrm{T}} l$, which was already done in [LSZ06], the main idea in chapter 2 was to think of the auxiliary matrices l as mappings from the physical phase space at the vertices to auxiliary spaces, located at the edges of a graph. Another point was the introduction of an extra edge per vertex, which nicely embeds the zero-dimensional model as a special case into the spatially extended one. The technical challenge which was overcome in section 2.2 was to apply the super-symmetry techniques, in particular to derive a form of the resolvent operator amenable to superbosonisation.

In chapter 3 we applied physically motivated approximations, namely a continuum limit along with a gradient expansion in the lattice spacing and saddle point expansion in $\frac{1}{N}$ to lowest order. We carefully investigated global constraints for the saddle point method and found the relevant saddle point for all energies and throughout the range of the model parameter in section 3.1.3.

In section 3.2 we developed the actual continuum super-field theory from general principles together with the structure of the target space. It turns out to consist of a non-linear σ-model with an additional mass term, for non-zero bulk scale frequencies. As in [LSZ06], we looked at different scaling limits, namely scaling the eigenfrequencies as $\hat{\omega} = \frac{\omega}{N} \in \mathcal{O}(1)$, the so-called bulk scaling, in which the spectrum is compactly supported, or $\omega \in \mathcal{O}(1)$, the so called edge scaling. At the edge scale the σ-model modes become massless, which is not surprising, since these are the Goldstone-Modes of an $\widetilde{\mathrm{OSp}}_{2|2}$ invariance of the action, hence the target space for this field is isomorphic to the saddle point manifold, $\widetilde{\mathrm{OSp}}_{2|2}/\,\mathrm{H}_0$. But this σ-model is coupled in a gauge field like fashion to a second type of modes, which stay massive throughout the spectrum and hence can be linearised. All in all we see that, as speculated in the beginning, the effective field theory for our class of disordered bosonic systems is not only given by a non-linear σ-model living in one of the well known symmetric spaces, but there are additional massive modes in the picture to which the massless fields are coupled. From our previous coordinate based calculations we could also read off the coupling constants of the novel field theory in terms of the original microscopic parameters.

Then in chapter 4 we turned back to the zero dimensional or spatially homogeneous case. Here we were able to calculate the density of states exactly to lowest order in $\frac{1}{N}$ at the bulk as well as at the edge scale and for the whole model parameter range, except for the point which was studied in [LSZ06] and which is singular in some respects. To be more precise, already in [LSZ06] a one parameter-family of measures was considered. But in our work we allow the same parameter to run through a larger range, in particular it is allowed to be negative and of order N. In equation (2.9) we pass from Lück's notation in which our parameter is $k \in \mathcal{O}(N)$, to the more convenient $\alpha \in \mathcal{O}(1)$. And, in fact, we largely ignored terms of order $\frac{1}{N}$ in α, hence our approach is in this respect complementary to [LSZ06]. The significance of this parameter is that it controls how many Gaussian random variables are combined into one entry of the random matrix that defines the Hamiltonian. The straight forward generalisation of the previous approach would read $\alpha = d + 1$ in terms of the new parameter, i.e. the model investigated in [LSZ06] corresponds to $\alpha = 1$. We found that only at this point the density of states near $\hat{z} = 0$ diverges on the bulk scale and our saddle point method is not applicable on the edge scale for $\alpha \geq 1$. The overall picture is that there are two qualitatively different regimes for $\alpha < 1$ and $\alpha > 1$, separated by this singular $\alpha = 1$ case, which is characterised by a divergence of the density of eigenfrequencies near zero. For $\alpha \neq 1$ the density of eigenfrequencies on the bulk scale stays smooth and finite throughout the spectrum. Note, however, that a $\frac{\Delta \dim}{z}$ term was removed from the model to get this smooth picture.

The effect of $\alpha > 1$ is clearly visible in figure 4.1. A gap around zero energy opens and the spectrum is split into two bands. In this case the single auxiliary space that provides the Gaussian random variables was enlarged beyond the minimal number of variables needed to potentially couple all modes. The effect of $\alpha > d + 1$ on the probability distribution can be seen directly from equations (2.8) and (2.9). Configurations with large eigenfrequencies become more probable by a macroscopically large power and at the same time the probability to have zero modes is suppressed by a factor of frequency to the same power. From a technical point of view, this explains the gap in the density of eigenfrequencies around $\omega = 0$.

For $\alpha < 1$ the spectrum consists of a single band and the density of eigenfrequencies is non zero and finite at $\hat{\omega} = 0$. It can be seen already on the level of model building in section 2.1.3 that $\alpha < 1$ leads to a macroscopic number of zero modes in any sample system of the ensemble. One should, however, keep in mind the divergent $\frac{\Delta \dim}{z}$ contribution that was removed from the model, which explains the finite density observed in figure 4.1. For this case it makes sense to look at the model within the edge scaling limit and investigate the behaviour of the density of states near $\omega = 0$ more closely.

In this scaling limit it turned out that we not only have to deal with a saddle point manifold, but we also need to take lower order terms in $\frac{1}{N}$ into account, since the highest order vanishes. This makes the saddle point method more laborious than in the bulk scaling case, but aided by computer algebra we were able to compute the density of states also in edge scaling and interestingly found the universal behaviour of class D. This nicely fits to the observations in [LSZ06] where GUE statistics (class A) was found for all correlation functions in bulk scaling, since at the bulk scale class A and D cannot be distinguished.

We have further calculated the positions of the spectral edges for all α in section 4.2 and found those, as well as the whole density of eigenfrequencies in the bulk scaling for $\alpha = 1$, to agree with the results of [LSZ06]. We did, however, not check the edge scaling results at $\alpha = 1$, which in [LSZ06] were found to depend on terms of order $\mathcal{O}\left(\frac{1}{N}\right)$ in α which we excluded from our discussion throughout.

Interpretation

All in all we found that the field theory for our spatially extended system of disordered bosons contains more than just a non-linear σ-model. On the one hand we found for the zero dimensional case, in agreement with [LSZ06], universal statistics that are well known from the fermionic symmetry classes. On the other hand the non-linear modes couple to the massive linear ones only via derivatives, hence it is not surprising that they decouple in zero dimensions and the low energy range is dominated by the Goldstone modes. This explains why the behaviour of our model at small energies fits into one of the well known universality classes for $0 < \alpha < 1$ in $d = 0$.

Hence we conclude that

1. bosonic disordered systems indeed show universal features and
2. there is room for novel bosonic universality classes.

Our general form of the field theory in terms of a σ-model which in higher dimension acts as a gauge field on a second massive field might be a hint of how to generalise the picture of universality classes to include bosonic models.

A Appendix

A.1 Explicit forms of the super-spaces

In this section we briefly state explicit forms of the groups introduced in section 3.2.2 to compare with the coordinate based computations. Therefore we will consider the real forms embedded into the real form $(\mathrm{Gl}_{2|2})_r$ as introduced in 3.2.2. For getting the real forms right it is useful to first look at the tangent spaces.

A.1.1 Lie super-algebras

In the following we will spell out the decomposition of $\mathfrak{gl}_{2|2}$ as introduced in 3.2.4 explicitly. Additionally we determine the possible maximal Abelian subalgebras of the Grassmann even parts $\mathfrak{a}_{ij} \subset (\mathfrak{g}_{ij})_0$. Two ad actions of those will be diagonalised below. In addition to the usual notation $\mathrm{span}_{\mathbb{C}}\{e_i\}$ for the complex vector space spanned by the vectors $\{e_i\}$ we use $\mathrm{span}_{\mathrm{Gr}}\{e_i\}$ for the Grassmann algebra generated by $\{\xi_i \otimes e_i\}$ with the number of independent Grassmann generators ξ_i given by $|\{e_i\}|$.

$$\mathfrak{g}_{0,0} = \mathrm{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_2 \end{array}\right) \right\} \oplus \mathrm{span}_{\mathrm{Gr}} \left\{ \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \sigma_2 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \sigma_2 \\ \hline -\mathbb{1} & 0 \end{array}\right) \right\}$$

The even part (w.r.t. the 'super' grading) of $\mathfrak{g}_{0,0}$ is already Abelian.

$$\mathfrak{a}_{0,0} = \mathrm{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_2 \end{array}\right) \right\}$$

$$\mathfrak{g}_{0,1} = \mathrm{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_1 \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_3 \end{array}\right) \right\} \oplus \mathrm{span}_{\mathrm{Gr}} \left\{ \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline -i\sigma_3 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \sigma_3 \\ \hline i\sigma_1 & 0 \end{array}\right) \right\}$$

Maximal Abelian are exactly the one-dimensional subspace of the even part.

$$\mathfrak{g}_{1,0} = \mathrm{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right), \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right) \right\} \oplus \mathrm{span}_{\mathrm{Gr}} \left\{ \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline i\sigma_3 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \sigma_3 \\ \hline -i\sigma_1 & 0 \end{array}\right) \right\}$$

One may choose an arbitrary one-dimensional subspace as the maximally Abelian one.

$$\mathfrak{g}_{1,1} = \operatorname{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1} \end{array} \right) \right\} \oplus \operatorname{span}_{\mathrm{Gr}} \left\{ \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\sigma_2 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & \sigma_2 \\ \hline \mathbb{1} & 0 \end{array} \right) \right\}$$

As for $\mathfrak{g}_{0,0}$ the even part is already Abelian

$$\mathfrak{a}_{1,1} = \operatorname{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1} \end{array} \right) \right\}$$

A.1.2 Generalised 'root space' or 'Iwasawa' decomposition

In this section we will diagonalise the ad action of $\mathfrak{a}_{1,1}$ on $\mathfrak{g}_{k,l}$. This leads to the decomposition

$$\mathfrak{g}_{k,l} = \bigoplus_{\alpha \in \Sigma(\mathfrak{g}_{k,l}:\mathfrak{a}_{1,1})} \mathfrak{g}_{k,l}^{\alpha}$$

where $\Sigma(\mathfrak{g}_{k,l} : \mathfrak{a}_{i,j}) \subset \mathfrak{a}_{i,j}^*$ is the set of (generalised) roots. For $i = j = 0$ a root $\alpha \in \mathfrak{a}_{i,j}^*$ is defined by

$$\mathfrak{g}_{k,l}^{\alpha} := \{X \in \mathfrak{g}_{k,l} \mid \forall_{a \in \mathfrak{a}_{i,j}} \operatorname{ad}(a)X = \alpha(a)X\} \neq \{0\}$$

and $\mathfrak{g}_{k,l}^{\alpha}$ is called root space. Otherwise we have $\operatorname{ad}(a) : \mathfrak{g}_{k,l} \to \mathfrak{g}_{k+i,l+j} \neq \mathfrak{g}_{k,l}$ and we have to look at *generalised* roots $\alpha \in \mathfrak{a}_{i,j}^*$ given by

$$\mathfrak{g}_{k,l}^{\alpha^2} := \{X \in \mathfrak{g}_{k,l} \mid \forall_{a \in \mathfrak{a}_{i,j}} \operatorname{ad}^2(a)X = \alpha^2(a)X\} \neq \{0\}$$

where $\mathfrak{g}_{k,l}^{\alpha^2}$ is called generalised root space. Drawing the square root is not an issue, as roots come in pairs $\pm\alpha$ anyway. But note that $\mathfrak{g}^{\alpha^2} = \bigoplus \mathfrak{g}^{\pm\alpha}$ if the latter exist. Hence, generalised rootspaces have twice the dimension of rootspaces (which are at least in a semi simple setting one dimensional).

In 3.2.4 we will in particular need $\Sigma(\mathfrak{a}_{1,1} : \mathfrak{g}_{1,0})$ which is easily computed. There are two generalised root spaces given by the Grassmann odd and even part of $\mathfrak{g}_{1,0}$. The corresponding roots are

$$\alpha_1^2 = 0 \qquad \text{with} \qquad \mathfrak{g}_{1,0}^{0} = \operatorname{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array} \right) \right\}$$

which means that α_1 is an even root of multiplicity $+2$ and

$$\alpha_2^2\,(xE_{BB} + yE_{FF}) = (x-y)^2$$

$$\text{with} \qquad \mathfrak{g}_{1,0}^{\alpha_2^2} = \operatorname{span}_{\mathrm{Gr}} \left\{ \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline i\sigma_3 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & \sigma_3 \\ \hline -i\sigma_1 & 0 \end{array} \right) \right\}$$

i.e. α_2 is an odd root of multiplicity -2. So the upshot of this diagonalisation is

$$\operatorname{SDet}_{\mathfrak{g}_{1,0}} f(\operatorname{ad}(a)) = (f(\alpha_2(a)))^{-2} \tag{A.1}$$

for any even analytic function f.[1]

The action of $\mathfrak{a}_{1,1}$ on $\mathfrak{g}_{0,0}$ will be needed even more explicitly. Note that an orthonormal basis (in a graded sense) of $\mathfrak{g}_{0,0}$ is given by

$$\left\{ \frac{1}{2}\left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & 0 \end{array}\right), \frac{1}{2}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_2 \end{array}\right), \right.$$
$$\left. \frac{i}{2\sqrt{2}}\left(\begin{array}{c|c} 0 & \xi_1\mathbb{1} + \xi_2\sigma_2 \\ \hline \xi_1\sigma_2 - \xi_2\mathbb{1} & 0 \end{array}\right), \frac{1}{2\sqrt{2}}\left(\begin{array}{c|c} 0 & \xi_1\mathbb{1} - \xi_2\sigma_2 \\ \hline \xi_1\sigma_2 + \xi_2\mathbb{1} & 0 \end{array}\right) \right\}$$

where orthogonality is understood as usual, $\mathrm{STr}(XY) = 0$, but by 'normalised' odd elements we mean such for which $\mathrm{STr}(X^2) = \xi_1\xi_2$.

Again all of the even part of $\mathfrak{g}_{0,0}$ is the centraliser, i.e. gets annihilated by $\mathrm{ad}\,\mathfrak{a}_{1,1}$. In the odd part we observe

$$\mathrm{ad}\left(\begin{array}{c|c} x\mathbb{1} & 0 \\ \hline 0 & y\mathbb{1} \end{array}\right)\left(\begin{array}{c|c} 0 & \xi_1\mathbb{1} \pm \xi_2\sigma_2 \\ \hline \xi_1\sigma_2 \mp \xi_2\mathbb{1} & 0 \end{array}\right)$$
$$= (x-y)\left(\begin{array}{c|c} 0 & \xi_1\mathbb{1} \pm \xi_2\sigma_2 \\ \hline -(\xi_1\sigma_2 \mp \xi_2\mathbb{1}) & 0 \end{array}\right)$$

Of course $\mathrm{ad}(\mathfrak{a}_{1,1})\mathfrak{g}_{0,0} \subset \mathfrak{g}_{1,1}$, but

$$\mathrm{ad}(\mathfrak{a}_{1,1}) : \mathfrak{g}_{0,0}/Z_{\mathfrak{a}_{1,1}} \to \mathfrak{g}_{1,1}/\mathfrak{a}_{1,1}$$

is an isomorphism and maps an orthonormal basis to and orthogonal one, normalised to $-\xi_1\xi_2$. This sign yields an extra factor i, hence choosing the volume form on each quotient to be $\xi_1\xi_2$, we get

$$\mathrm{SDet}(f(\mathrm{ad}(a))) = \left(f\left(\frac{i}{2}\mathrm{STr}(a)\right)\right)^{-2} \tag{A.2}$$

for an odd function f.

A.1.3 Lie super-groups

Here we give explicit forms of the super-groups considered in section 3.2.2. We start by considering the underlying complex manifold of

$$\mathrm{OSp}_{2|2} = \left\{\tilde{h} = \gamma\tilde{h}^{\mathrm{ST}}\gamma^{-1}\right\}$$

which is given by

$$\left(\mathrm{OSp}_{2|2}\right)_s = \left(\begin{array}{c|c} \mathrm{SO}(\mathbb{C}^2) & \\ \hline & \mathrm{Sp}(\mathbb{C}^2) \end{array}\right)$$

[1] Note that the determinant here is understood for $\mathfrak{g}_{1,0} \to \mathfrak{g}_{1,0}$ mappings, i.e. f has to be even.

By demanding the super-trace form to be positive definite on the Lie algebra and exponentiating we get the real form

$$\left(\mathrm{OSp}_{2|2}\right)_r = \left(\begin{array}{c|c} \mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2) & \\ \hline & \mathrm{U\,Sp}(2) \end{array}\right)$$

where $\mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2)$ is a non-compact form of $\mathrm{SO}(2)$ given by

$$\mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2) = \exp\{X = -X^{\mathrm{T}} = X^\dagger\} = \{\cosh(\lambda)\mathbb{1}_2 + \sinh(\lambda)\sigma_2 \mid \lambda \in \mathbb{R}\}$$

and $\mathrm{U\,Sp}(2)$ is the unitary symplectic group, a compact form of $\mathrm{Sp}(\mathbb{R}^2)$.[2] Now, since we are in just 2 dimensions, we notice $\mathrm{Sp}(\mathbb{C}^2) = \mathrm{Sl}(\mathbb{C}^2)$, which leads to $\mathrm{U\,Sp}(2) = \mathrm{SU}(2)$.

Next we have a look at

$$\widetilde{\mathrm{OSp}}_{2|2} = \left\{\tilde{h} = \tilde{\gamma}\tilde{h}^{\mathrm{ST}}\tilde{\gamma}^{-1}\right\}$$

which has skeleton

$$\left(\widetilde{\mathrm{OSp}}_{2|2}\right)_s = \left(\begin{array}{c|c} \mathrm{Sp}(\mathbb{C}^2) & \\ \hline & \mathrm{SO}(\mathbb{C}^2) \end{array}\right)$$

The real form is given by

$$\left(\widetilde{\mathrm{OSp}}_{2|2}\right)_r = \left(\begin{array}{c|c} \mathrm{Sp}(\mathbb{C}^2)/\,\mathrm{U\,Sp}(2) & \\ \hline & \mathrm{SO}(\mathbb{R}^2) \end{array}\right)$$

where we encounter yet another version of the symplectic group $\mathrm{Sp}(\mathbb{C}^2)/\,\mathrm{U\,Sp}_2 = \mathrm{Sl}(\mathbb{C}^2)/\,\mathrm{SU}_2$ which is non compact.

The skeleton of

$$\mathrm{H}' = \left\{h' = \Sigma_2 h' \Sigma_2^{-1}\right\} \simeq \mathrm{Gl}_{1|1} \times \mathrm{Gl}_{1|1}$$

is

$$\mathrm{H}'_s = \left(\begin{array}{c|c} \mathrm{Gl}(\mathbb{C}) \times \mathrm{Gl}(\mathbb{C}) & \\ \hline & \mathrm{Gl}(\mathbb{C}) \times \mathrm{Gl}(\mathbb{C}) \end{array}\right)$$

with the embedding

$$\mathrm{Gl}(\mathbb{C}) \times \mathrm{Gl}(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0\right\} = \mathbb{C}^* \times \mathrm{SO}(\mathbb{C}^2)$$

The real form is given by

$$\mathrm{H}'_r = \left(\begin{array}{c|c} R^* \times \mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2) & \\ \hline & \mathrm{U}(1) \times \mathrm{SO}(\mathbb{R}^2) \end{array}\right)$$

[2] $\mathrm{U\,Sp}(2)$ is sometimes also denoted by $\mathrm{Sp}(1)$ and considered as the quaternionic unitary group.

with

$$\mathrm{U}(1)\times\mathrm{SO}(\mathbb{R}^2)=\left\{e^{i\phi}\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix} \mid \phi,\theta\in[0,2\pi]\right\}$$

and similarly

$$\mathbb{R}^*\times\mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2)=\{\mu\,(\cosh(\lambda)\mathbb{1}_2+\sinh(\lambda)\sigma_2)\mid \mu\in\mathbb{R}^*,\lambda\in\mathbb{R}\}$$

Finally $\mathrm{H}_0=\mathrm{OSp}_{2|2}\cap\widetilde{\mathrm{OSp}}_{2|2}\simeq\mathrm{Gl}_{1|1}$ is very simple. Exponentiating $\mathfrak{g}_{0,0}$ we immediately get

$$(\mathrm{H}_0)_s=\left(\begin{array}{c|c}\mathrm{SO}(\mathbb{C}^2) & \\ \hline & \mathrm{SO}(\mathbb{C}^2)\end{array}\right)$$

with real form

$$(\mathrm{H}_0)_r=\left(\begin{array}{c|c}\mathrm{SO}(\mathbb{C}^2)/\,\mathrm{SO}(\mathbb{R}^2) & \\ \hline & \mathrm{SU}(2)\end{array}\right)$$

A.2 Saddle point

The relevant saddle point as computed using Mathematica is the following solution s of the Saddlepoint equation (3.5):

$$\begin{aligned}12\hat{z}\,s=\;&4(\alpha-1)\\&+\frac{2^{\frac{7}{3}}\left(3\hat{z}^2-(\alpha-1)^2\right)}{\left(-2(\alpha-1)^3-9\hat{z}^2(1+2\alpha)+3\sqrt{3}\sqrt{\hat{z}^2\left(4\hat{z}^4+4(\alpha-1)^3\alpha+\hat{z}^2(4\alpha(5+2\alpha)-1)\right)}\right)^{\frac{1}{3}}}\\&+2^{\frac{2}{3}}(1+i\sqrt{3})\Bigg(-2(\alpha-1)^3-9\hat{z}^2(1+2\alpha)\\&+3\sqrt{3}\sqrt{\hat{z}^2\left(4\hat{z}^4+4(\alpha-1)^3\alpha+\hat{z}^2(4\alpha(5+2\alpha)-1)\right)}\Bigg)^{\frac{1}{3}}\end{aligned}\tag{A.3}$$

A.3 Grassmann integral of the ratio of super-determinants

In this section we expand the ratio of super-determinants $\left(\frac{\mathrm{SDet}(P)^\alpha}{\mathrm{SDet}(P-i\Sigma_2)}\right)^N$ in Grassmann variables. This was first done by hand and then checked with the mathematica package to handle Grassmann variables, which was written by Tobias Lück, [Lü09], and slightly extended by the author. Here we use the following coordinates

$$P=\left(\begin{array}{c|c}A & F\\ \hline -\sigma_2F^T & B\end{array}\right)$$

$$A=a_0\mathbb{1}+a_1\sigma_1+a_3\sigma_3\qquad B=b\mathbb{1}\qquad F=\xi_0\mathbb{1}+\xi_1\sigma_2+\xi_2 i\sigma_2+\xi_3\sigma_3$$

The computation is best organised by first computing for arbitrary $\beta \in \mathbb{R}$ and $n \in \mathbb{Z}$

$$\begin{aligned}
\mathrm{SDet}\,(P - i\beta\Sigma_2)^n = \left(\frac{\mathrm{Det}(A)+\beta^2}{\mathrm{Det}(B)+\beta^2}\right)^n \Bigg(\\
1 + \frac{4in}{(\mathrm{Det}(A)+\beta^2)(\mathrm{Det}(B)+\beta^2)} \Big(a_0 b\,(\xi_1\xi_3 - \xi_0\xi_2) + a_1 b\,(\xi_1\xi_2 - \xi_0\xi_3) \\
+ a_3 b\,(\xi_0\xi_1 - \xi_2\xi_3) + \beta^2\,(\xi_1\xi_3 + \xi_0\xi_2) \Big) \\
+ \frac{8n}{((\mathrm{Det}(A)+\beta^2)(\mathrm{Det}(B)+\beta^2))^2} \Big((2N-1)\beta^4 + (\mathrm{Det}(A) - 3\,\mathrm{Det}(B))\beta^2 \\
- (2N+1)\,\mathrm{Det}(A)\,\mathrm{Det}(B) \Big) \xi_0\xi_1\xi_2\xi_3 \Bigg)
\end{aligned}$$

For $\beta = 1$, $n = -N$ and $\beta = 0$, $n = \alpha N$, respectively, this yield the two terms to be multiplied. Then we pick the term which contains the product of all four Grassmann variables.

$$\begin{aligned}
\partial_{\chi_4}\partial_{\chi_3}\partial_{\chi_2}\partial_{\chi_1} \left(\frac{\mathrm{SDet}(P)^\alpha}{\mathrm{SDet}(P - i\Sigma_2)} \right)^N = \\
8N \left(\frac{\mathrm{Det}(A)}{\mathrm{Det}(B)} \right)^{\alpha N} \left(\frac{\mathrm{Det}(B)+1}{\mathrm{Det}(A)+1} \right)^N \\
\Bigg(\frac{2N + 1 - \mathrm{Det}(A) + 3\,\mathrm{Det}(B) - \mathrm{Det}(A)\,\mathrm{Det}(B)(2N-1)}{(\mathrm{Det}(A)+1)^2(\mathrm{Det}(B)+1)^2} \\
+ \frac{4N\,\mathrm{Det}(A)\,\mathrm{Det}(B) - (\mathrm{Det}(A)+1)(\mathrm{Det}(B)+1)}{\mathrm{Det}(A)(\mathrm{Det}(A)+1)\,\mathrm{Det}(B)(\mathrm{Det}(B)+1)}\alpha - \frac{2N\alpha^2}{\mathrm{Det}(A)\,\mathrm{Det}(B)} \Bigg)
\end{aligned} \tag{A.4}$$

Inserting the saddle point $\mathrm{Det}(A) = s^2 = \mathrm{Det}(B)$ we get

$$(\text{A.4}) = 8N \left(\frac{2N + 1 - s^2(-2 + s^2(2N-1))}{(s^2+1)^4} - \frac{(s^2+1)^2 - 4Ns^4}{s^4(s^2+1)^2}\alpha - \frac{2N\alpha^2}{s^4} \right)$$

A.4 Residue

The U(1) integral in the fermion-fermion sector in equation (3.1) can be performed exactly in $d = 0$ dimension using the residue theorem, once the Grassmann variables have been integrated out. We consider

$$\oint_{\mathrm{U}(1)} \mathrm{d}b e^{-z_0 a_0 + z_1 b} \left(\frac{\mathrm{Det}(A)^\alpha (b^2+1)}{b^{2\alpha}(\mathrm{Det}(A)+1)} \right)^N \frac{p(\mathrm{Det}(A), b^2)}{b^2(b^2+1)^2\,\mathrm{Det}(A)(\mathrm{Det}(A)+1)} \tag{A.5}$$

where p is a polynomial which can be read off from equation (A.4). It is of second order in b^2, i.e. $p(A, b^2) = p_0(A) + p_1(A)b^2 + p_2(A)b^4$. So up to factors which depend on A we have

$$(A.5) \propto \oint_{\mathrm{U}(1)} \mathrm{d}b e^{z_1 b} \left(\frac{b^2+1}{b^{2\alpha}}\right)^N \frac{p(\mathrm{Det}(A), b^2)}{b^2(b^2+1)^2}$$

$$= \frac{2\pi i}{(2\alpha N+1)!} \partial_b^{2\alpha N+1}\Big|_0 \left(e^{z_1 b}(b^2+1)^{N-2} p(\mathrm{Det}(A), b^2)\right) \quad (A.6)$$

Now we can spell out all series and get

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{N-2} \sum_{i=0}^{2} \frac{z_2^n}{n!} \binom{N-2}{k} p_i(A) \delta_{(n+2(k+i)),(2\alpha N+1)}$$

$$= 2\pi i \sum_{i=0}^{2} p_i(A) \sum_{k=0}^{\alpha N - i} \frac{z_2^{2\alpha N+1-2(i+k)}}{(2\alpha N+1-2(i+k))!} \binom{N-2}{k}$$

Where one should probably consider $2\alpha N$ as odd, due to equation (2.24) which means that the integral is an even function of z_2. Unfortunately we cannot say much more about the resulting series.

A.5 Invariant measure

In this section we explicitly compute the invariant measure on $\mathrm{Sl}(\mathbb{R}^2)/\,\mathrm{SO}(\mathbb{R}^2)$ which is the real form of the boson-boson part of $\widetilde{\mathrm{OSp}}_{2|2}/\,\mathrm{H}_0$. Since the fermion-fermion part is zero-dimensional this actually yields the invariant measure on the whole real form.

We now stop emphasising that the symplectic and orthogonal groups in this section are always understood over the reals and start by introducing coordinates for $g \in \mathrm{Sl}_2$.

$$g = \sqrt{1 + a_1^2 - a_2^2 + a_3^2}\ \mathbb{1} + a_1\sigma_1 + a_2 i\sigma_2 + a_3\sigma_3$$

and compute the invariant Maurer-Catrtan form $g^{-1}dg$ in these coordinates. Since this form is Lie-algebra, $\mathfrak{sl}_2$, valued, it has 3 components. The wedgeproduct of those yields the volume form on Sl_2 which is invariant under left translation.

$$\mathrm{d}\mu(g) \propto \frac{\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \mathrm{d}a_3}{\sqrt{1 + a_1^2 - a_2^2 + a_3^2}} \propto \frac{\mathrm{d}[g]}{\mathrm{Tr}(g)}$$

where we denote the measure given by the wedge product of the differentials of independent matrix entries by $\mathrm{d}[g]$.

Next we introduce similar coordinates for $p \in \mathrm{Sl}_2\,/\,\mathrm{SO}_2$.

$$p = \sqrt{1 + p_1^2 + p_3^2}\ \mathbb{1} + p_1\sigma_1 + p_3\sigma_3$$

And via the Cartan embedding

$$\begin{aligned} \mathrm{Sl}_2 / \mathrm{SO}_2 &\hookrightarrow \mathrm{Sl}_2 \\ p = gg^{\mathrm{T}} &\leftarrow\!\shortmid g \end{aligned}$$

we obtain the change of coordinates

$$p_1 = 2a_1\sqrt{1+a_1^2+a_3^2} \qquad p_3 = 2a_3\sqrt{1+a_1^2+a_3^2}$$

Note that the Sl_2 action on $\mathbb{1}$ by twisted conjugation with respect to the involution $g \mapsto (g^{-1})^{\mathrm{T}}$ that generates this orbit is the descendant of the AD^{θ} action, as introduced in section 3.2.3.

Further we note

$$\frac{\mathrm{d}p_1 \wedge \mathrm{d}p_3}{\sqrt{1+p_1^2+p_3^2}} = 4\ \mathrm{d}a_1 \wedge \mathrm{d}a_3$$

and integrating $\mathrm{d}a_2$ out of the invariant measure $\mathrm{d}\mu(g)$ yields a constant,

$$\int\limits_{-\sqrt{1+a_1^2+a_3^2}}^{\sqrt{1+a_1^2+a_3^2}} \frac{\mathrm{d}a_2}{\sqrt{1+a_1^2+a_3^2-a_2^2}} = \pi$$

hence the measure on $\mathrm{Sl}_2 / \mathrm{SO}_2$ in the above representation and coordinates which is invariant under twisted conjugation is given by

$$\mathrm{d}\mu(p) \propto \frac{\mathrm{d}p_1 \wedge \mathrm{d}p_3}{\sqrt{1+p_1^2+p_3^2}} \propto \frac{\mathrm{d}[p]}{\mathrm{Tr}(p)}$$

In fact, one can check by direct computation that

$$\forall_{g \in \mathrm{Sl}_2} \mathrm{d}\mu(gpg^t) = \mathrm{d}\mu(p)$$

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig angefertigt habe. Ich habe alle Zitate und Quellen nach bestem Wissen und Gewissen kenntlich gemacht. Die Zeichnungen und Abbildungen sind von mir selbst mit Hilfe von 'PGFPlots' erstellt worden. Grafen zugrundeliegende Daten wurden mit 'Wolfram Mathematica' berechnet. Es wurden keine weiteren Hilfsmittel benutzt.

Diese Arbeit ist in gleicher oder ähnlicher Form noch bei keiner anderen Prüfungsbehörde eingereicht worden.

Köln, den 28.10.2010,

Sebastian E. Schmittner

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