Spherical representations of reductive Lie super algebras

Sphärische Darstellungen reduktiver Lie Superalgebren

Diplomarbeit

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1 Introduction

1.1 Kurzbeschreibung der Arbeit

Diese Arbeit ist Teil eines Forschungsprojektes ([AHZ08, All10, AHL11]) zur Entwicklung harmonischer Analysis auf symmetrischen Superräumen. Motiviert durch Anwendungen in der theoretischen Physik ist das übergeordnete Ziel, das Verständnis der sphärischen Funktionen, d.h. K-biinvarianter Eigenfunktionen G-invarianter Differentialoperatoren auf Quotienten, G/K, von Lie-Supergruppen. Ein solches ist unerlässlich um die Fourier-Transformation auf solche Superräume zu verallgemeinern, um dadurch etwa physikalisch relevante partielle Differentialgleichungen lösen zu können.

Wir vermuten, dass, analog zum klassischen Fall, eine Charakterisierung sphärischer Funktionen als Matrixkoeffizienten sphärischer Darstellungen möglich ist. Deshalb sind symmetrische Paare, $(\mathfrak{g}, \mathfrak{k})$, reduktiver komplexer Lie-Superalgebren mit geeigneten reellen Formen, d.h. gewöhnlichen Lie-Gruppen G_0, der Ausgangspunkt dieser Arbeit. Äquivalent zur Kategorie solcher Supergruppenpaare, $(G_0, \mathfrak{g})$, ist die Kategorie der cs-Lie-Supergruppen. Wir entwickeln im Folgenden beide Sichtweisen, da die erstere besser zum Verständnis der algebraischen Darstellungstheorie geeignet ist, die letztere hingegen sich besser für die Beschreibung von Integration über Supermannigfaltigkeiten eignet.

Das wesentliche Resultat dieser Arbeit ist unser Beweis von Theorem 3.60, dass den klassischen Satz von Cartan-Helgason, [Hel84], Kapitel V, Theorem 4.1, verallgemeinert, nämlich die Charakterisierung der sphärischen Höchstgewichtsdarstellungen V_λ: Genau dann ist V_λ sphärisch (d.h. besitzt einen K-invarianten Vektor), wenn der Höchtgewichtsvektor M-invariant ist; anders gesagt, wenn das höchste Gewicht λ auf dem toroidalen Teil $\mathfrak{h} \cap \mathfrak{k}$ einer θ-invarianten Cartanunteralgebra $\mathfrak{h}$ mit maximalem Vektorteil $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ verschwindet. Des Weiteren ist der K-invariante Vektor in diesem Fall bis auf Vielfache eindeutig. Hierbei ist $G = KAN$ die Iwasawa Zerlegung von G und $Q = MAN$ eine entsprechende minimal parabolische Untergruppe von G.

Unser Beweis verallgemeinert im Wesentlichen die Beweistechnik von [Sch84] auf den Superfall. Im Fall von $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ mit $\mathfrak{k} = \mathfrak{gl}^{q|s} \oplus \mathfrak{gl}^{0|r}$, auf den wir uns ab Abschnitt 3.3 konzentrieren, folgt in der Tat aus der K- die M-Invarianz und wir können somit eine notwendige Bedingung für sphärische endlichdimensionale Darstellungen dieses Paares angeben. Für die Umgekehrte Richtung, d.h. den Beweis, dass die M-Invarianz des Höchstgewichtsvektors auch hinreichend für die existens eines K-invarianten Vektors ist, müssen wir uns auf den Fall $s = 1$ und entweder $r > q$ oder Darstellungen von hinreichend hohem höchsten Gewicht einschränken, um ein konkretes Superintegral in Abschnitt 3.5.1 berechnen zu können. Als Korollar erhalten wir analog zum klassischen Fall eine, im eingeschränkten Fall vollständige, Charakterisierung der endlichdimensionalen sphärischen Höchstgewichtsdarstellungen über ihr höchstes Gewicht.

Ein weiteres Ergebnis ist die Entwicklung von induzierten Darstellungen und der Beweis von Frobeniusreziprozität in der Kategorie unendlichdimensionaler glatter Darstellungen von Lie-Supergruppenpaaren in Abschnitt 3.4.

Diese Arbeit soll aus sich selbst heraus verständlich sein, d.h. beginnen wir in Kapitel 2 mit einer ausführlichen Einführung der relevanten Konzepte, insbesondere der Supermathematik die im folgenden benötigt wird. Kapitel 3 beginnt mit dem Verweis auf einigen Fakten über die Darstellungstheorie von Lie-Superalgebren, meist ohne die Beweise zu reproduzieren. Unsere eigene Arbeit beginnt in Abschnitt 3.3 mit einer genauen Beschreibung von $\mathfrak{gl}^{q|r+s}$ und ihrer Wurzelraumstruktur. Es folgen die oben erwähnten Resultate und als abschließende Anwendung werden zwei Darstellungen von $\mathfrak{gl}^{1|2+1}$ diskutiert. Das letzte Kapitel 4 enthält neben einer Zusammenfassung der genauen Ergebnisse auch Hinweise auf eine Verallgemeinerung auf beliebige stark reduktive symmetrische Paare. Während sich unser Beweis für die notwendige Bedingung für sphärische Darstellungen ohne Weiteres übertragen lässt, steht und fällt die Verallgemeinerung der hinreichenden Bedingung mit der Berechnung des entsprechenden super Integrals, $\int_{K/M} \pi(k) v_\lambda \mathrm{D}k$ im verallgemeinerten Fall. Wir vermuten, dass sich dies zumindest für $\mathfrak{gl}^{q|r+s}$ mit $s > 1$ mit Hilfe einer Verallgemeinerung der Rangreduktion ähnlich dem analogen Beweis in [Sch84] bewerkstelligen lässt. Für $\mathfrak{gl}^{1|2+1}$ geben wir die Matrixkoeffizienten als Kandidaten für sphärische Funktionen an, aber die genaue Beziehung zu invarianten Differentialoperatoren bleibt das Ziel weiterführender Untersuchungen.

1.2 Introduction

The idea of describing the eigenvalues of bosonic and fermionic fields in quantum field theory by commuting and anticommuting variables led to the development of super manifolds in the 1970s. Of particular interest to condensed matter physicists studying disordered systems are symmetric super spaces, i.e. quotients G/K of Lie super groups, because those arise for example as the target spaces of non-linear σ-models, which are the effective low energy theories for disordered fermionic systems with quadratic Hamiltonians in the thermodynamic limit. See [Zir98] for an elaborate discussion of an application. An interesting development is [LSZ07] which proves that an important step in the development of these physical models is indeed mathematically rigorous. A recent example using this new technique can be found in [SZ10].

Recently, active research has been focused on the development of harmonic analysis on symmetric super spaces ([AHZ08, All10, AHL11]). One goal here is to prove a super Fourier inversion formula and hence obtain a tool to solve linear partial differential equations involving G-invariant differential operators on G/K as appear in the afore mentioned examples from physics. To this end one needs to understand spherical functions, i.e. the K-biinvariant joint eigenfunctions of such differential operators and in particular their asymptotics. We expect that, as in the classical case, those functions can be characterised as matrix coefficients of the spherical representations of G, i.e. those containing a K-invariant vector. The goal of this thesis is therefore to determine which finite dimensional irreducible G representations are spherical. For concreteness and to avoid problems stemming from non-compact real forms we restrict our attention to the Lie super algebra level and study the symmetric pair $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ with sub Lie super algebra $\mathfrak{k} = \mathfrak{gl}^{q|r} \oplus \mathfrak{gl}^{0|s}$.

Our main result, Theorem 3.60 on page 53, is a generalisation of a classical theorem due to Helgason, [Hel84], Chapter V, Theorem 4.1, which states that a representation is spherical if and only if the highest weight vector is M-invariant, where $Q = MAN$ is a minimal parabolic subgroup of G. It turns out that this is exactly the same in the super case. For the proof we use methods developed by Schlichtkrul, [Sch84], to reduce the use of integration, which in the super world holds much more pitfalls. Similar to the ordinary case we can then classify the spherical representations in terms of their highest weights in Lemma 3.61 and the subsequent corollaries. At least for $\mathfrak{gl}^{q|r+1}$ with $r > q$ this classification is complete, but see Chapter 4 for some immediate as well as conjectured generalisations.

This thesis is meant to be reasonably self contained, therefore we will start by introducing all relevant concepts in Chapter 2. In particular we will develop the notion of Lie super groups from two different perspectives as group objects in the category of super manifolds as well as in terms of super group pairs $(G_0, \mathfrak{g})$ of a classical real Lie group and a complex Lie super algebra. The category of so called cs Lie super groups is equivalent to the category of such super group pairs and hence the right one for our purpose, because we want to investigate the question about the spherical representation starting from the view point of the complex Lie super algebra $\mathfrak{g}$. The point is then to determine which $\mathfrak{g}$ representations contain a $\mathfrak{k}$-invariant vector, but, as in the classical case, we will need a global real form of $\mathfrak{g}$, namely G_0, to make use of the Iwasawa decomposition and Frobenius reciprocity and in order to formulate an integral over K/M. In fact, half of the characterisation of spherical representations, namely condition 3 in Lemma 3.61, only becomes visible on the exponentiated level.

Chapter 3 starts by reciting some facts about the representation theory of Lie super algebras and the global Iwasawa decomposition. Here we mostly omit the proofs and refer to the literature. From Section 3.3 on we narrow down the scope to $\mathfrak{gl}^{q|r+s}$ and present our own results, starting with the root space decomposition. In Section 3.4 we explain induced representations of possibly infinite dimensional representations of super group pairs and proof Frobenius reciprocity in this setting. Section 3.5 is concerned with the proof of our main theorem, classifying spherical representations. We then give two concrete examples of representations of $\mathfrak{gl}^{1|2+1}$ for which we validate the statements and conclusions from Section 3.5. Finally we conclude with some remarks about how to generalise our findings and about the future perspective for the development of harmonic analysis. In particular, the necessary condition for spherical representations, proven in Section 3.5, immediately generalises to all reductive symmetric pairs of even type for which the global Iwasawa decomposition exists, as long as only finite dimensional highest weight representations are taken into account. This includes in particular all strongly reductive pairs, but note that $\mathfrak{gl}^{q|r+s}$ is strongly reductive if and only if $q \neq r+s$. All our results apply also in the case of $q = r+s$, as long as $(q, r+s) \neq (1,1)$, as will be explicitly pointed out at the relevant points.

Although some heavy machinery using category and sheaf theory to define super Lie groups and invariant integration is developed and used, most of the results can be formulated and understood in terms of Lie super group pairs consisting of Lie super algebras and ordinary manifolds. If the reader wants to skip the more abstract parts, an accessible overview can be gained starting from some basic definitions in Sections 2.2, 2.5.1 and 2.5.2 and some representation theory in Sections 3.1.1 and 3.2.1, maybe proceeding with details about $\mathfrak{gl}^{q|r+s}$ in Section 3.3, to come to the classification results, Corollary 3.63 and 3.64 which are exemplified in Section 3.6.

2 Preliminaries

In this section we will define the objects and notions to be used in this thesis and study some of their basic properties. Most of the content can be found in [ABG$^+$10], [All10] and [All11b].

2.1 General concepts

We start by introducing the very basic notions of category and sheaf theory which will be used in the following sections.

2.1.1 Categories

In this thesis we mostly consider locally small categories which are defined as follows.

Definition 2.1. *A* category C *consists of a collection of objects* $\mathrm{Ob}(C)$ *(which does not need to be a set) and for each pair of objects* $X, Y \in \mathrm{Ob}(C)$ *there is a* set of homomorphisms $\mathrm{Hom}_C(X, Y)$*. We will write* $X \in C$ *meaning* $X \in \mathrm{Ob}(C)$ *and* $f : X \to Y$ *meaning* $f \in \mathrm{Hom}_C(X, Y)$*.*

Further we need to have an operation of concatenation

$$\begin{aligned} \circ : \mathrm{Hom}_C(Y, Z) \times \mathrm{Hom}_C(X, Y) &\to \mathrm{Hom}_C(X, Z) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

with the following properties

1. $\forall X \in C \; \exists \, \mathrm{id}_X \in \mathrm{Hom}_C(X, X)$
2. $\forall f : X \to Y : \mathrm{id}_X \circ f = f = f \circ \mathrm{id}_Y$
3. $\forall f : A \to B \; \forall g : B \to C \; \forall h : C \to D \; : (h \circ g) \circ f = h \circ (g \circ f)$

Example 2.2. *The category* Sets *has sets as objects and maps as morphisms. The subcategory* VS *consists of vector spaces with linear maps as morphisms.*

Definition 2.3. *A* functor *is a morphism of categories, i.e. for categories* C *and* D *a functor* $F : C \to D$ *is given by maps (all denoted by the same symbol)* $F : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ *and for each* $X, Y \in C$ *we have* $F : \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y))$ *such that* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ *and* $F(g \circ h) = F(g) \circ F(h)$*.*

Remark 2.4. *Note that the collection of all functors from* C *to* D *is in general not a set.*

Definition 2.5. *We will use the usual notions of endo-, iso- and automorphisms* $\forall X, Y \in C$:

$$\begin{aligned}
\mathrm{End}_C(X) &:= \mathrm{Hom}_C(X, X) \\
\mathrm{Iso}_C(X, Y) &:= \{f \in \mathrm{Hom}_C(X, Y) \big| \exists f^{-1} \in \mathrm{Hom}_C(Y, X) : \\
&\qquad\qquad f \circ f^{-1} = \mathrm{id}_Y \quad \textit{and} \quad f^{-1} \circ f = \mathrm{id}_X\} \\
\mathrm{Aut}_C(X) &:= \mathrm{Iso}_C(X, X)
\end{aligned}$$

And for $X, Y \in C$ *we write* $X \simeq Y :\Leftrightarrow \mathrm{Iso}_C(X, Y) \neq \emptyset$.

Definition 2.6. *A* subcategory D *of a category* C *is a category such that* $\forall X \in D$: $X \in C$ *and* $\forall X, Y \in D : \mathrm{Hom}_D(X, Y) \subset \mathrm{Hom}_C(X, Y)$. *It is called a* full *subcategory* $:\Leftrightarrow \forall X, Y \in D : \mathrm{Hom}_D(X, Y) = \mathrm{Hom}_C(X, Y)$.

Definition 2.7. *The* product *of* $G_1, G_2 \in C$, *if it exists, is denoted by* $G_1 \times G_2 \in C$,*comes with projections* $p_i \in \mathrm{Hom}_C(G1 \times G_2, G_i)$ *and is defined by the following property*

$$\forall H \in C \; \forall f_i \in \mathrm{Hom}_C(H, G_i) \; \exists! f \in \mathrm{Hom}_C(H, G_1 \times G_2) : p_i \circ f = f_i$$

which can be nicely expressed as a commuting diagram

$$\begin{tikzcd}
 & G_1 \\
\forall H \arrow[ru, bend left, "\forall f_1"] \arrow[r, dashed, "\exists! f"] \arrow[rd, bend right, "\forall f_2"'] & G_1 \times G_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & G_2
\end{tikzcd}$$

A terminal object, *if it exists, is defined by*

$$* \in C \; \forall G \in C \; \exists! *_G \in \mathrm{Hom}(G, *)$$

Note that by this definition all terminal objects are isomorphic. They can be thought of as the product of 0 *objects and hence by the existence of finite products we mean in particular the existence of a terminal object.*

Definition 2.8. *Let* C *be a category where all finite products exist. Then a* group object *is an object* $G \in C$ *together with a multiplication morphism,* $m \in \mathrm{Hom}(G \times G, G)$, *inversion morphism,* $i \in \mathrm{End}(G)$, *and a unit morphism,* $e \in \mathrm{Hom}(*, G)$, *such that*

1. $m \circ (\mathrm{id}_G \times m) = m \circ (m \times \mathrm{id}_G)$ *(associativity)*
2. $m \circ (\mathrm{id}_G \times i) \circ \delta_G = m \circ (i \times \mathrm{id}_G) \circ \delta_G = e \circ *_G$ *(inverse) where* $\delta_G = (\mathrm{id}_G, \mathrm{id}_G)$ *is the diagonal embedding* $G \hookrightarrow G \times G$

3. $m \circ (\mathrm{id}_G \times e) \circ (\mathrm{id}_G, *_G) = m \circ (e \times \mathrm{id}) \circ (*_G, \mathrm{id}_G) = \mathrm{id}_G$ *(neutrality)*

A morphisms of group objects $f : G \to H$ is required to intertwine multiplication, $f \circ m = m' \circ (f \times f)$, and inversion, $f \circ i = i' \circ f$, and to preserve the unit morphism $f \circ e = e'$. This defines the subcategory of group objects.

Definition 2.9. *For $X, Y \in C$ we will call the set*

$$X(Y) := \mathrm{Hom}(Y, X)$$

the Y-points of X. *This name is motivated by the $*$-points of a set being its elements and similarly the $*$-points of a topological space are its ordinary points. We will write*

$$x \in_Y X :\Leftrightarrow x \in X(Y)$$

and for $x \in_Y X$ and $f \in \mathrm{Hom}(X, Z)$ we denote

$$f(x) := (f \circ x) \in_Y Z$$

Yoneda Lemma

The purpose of this section is to reproduce the general fact that in any category a collection of maps (in between sets) $f_S : X(S) \to Y(S)$ indexed by $S \in C$ defines a morphism $f \in \mathrm{Hom}(X, Y)$ such that $\forall p \in_S X : f(p) = f_S(p)$ if and only if

$$\forall p \in_S X \; \forall g \in_T S : f_T(p(g)) = (f_S(p))(g) = f_S(p) \circ g$$

and if so, $f \in \mathrm{Hom}(X, Y)$ is unique.

Hence homomorphisms can be given on the level of generalised points, i.e. in terms of ordinary maps in the category of sets.

Definition 2.10. *For any category C there is an* opposite category C^{op} *which has the same objects but $\mathrm{Hom}_{C^{op}}(X, Y) := \mathrm{Hom}_C(Y, X)$ and $f \circ^{op} g := g \circ f$.*

Remark 2.11. *A functor $C^{op} \to D$ is called* contravariant functor.

Definition 2.12. *A* natural transformation *is a morphism of functors, i.e. for $F, G : C \to D$ a natural transformation $\theta : F \to G$ is given by morphisms (all denoted by the same symbol) $\theta \in \mathrm{Hom}_D(F(X), G(X))$ for all $X \in C$ such that*

$$\forall f \in \mathrm{Hom}_C(X, Y) : \theta \circ F(f) = G(f) \circ \theta$$

This defines the category of functors *from C to D. Note however that this category is in general not locally small, i.e. natural transformations in between two given functors might not form a set, unless C and D are small, i.e. the collections of objects are sets.*

Definition 2.13. *For any category C denote by $C^\vee$ the category of functors $C^{op} \to$* Sets. *Particular objects in this category are the* hom functors. *For $X \in C$ we define $X(.) \in C^\vee$ by $X(Y) = \mathrm{Hom}(Y, X)$ as in Definition 2.9 and for $f \in \mathrm{Hom}_C(Y, Z)$ we define $X(f) \in \mathrm{Hom}_{\mathrm{Sets}}(X(Y), X(Z))$ by $X(f)(g) := g \circ f$.*

Lemma 2.14. *The* Yoneda Lemma *states that for $X \in C$ and $F : C^{op} \to$ Sets*

$$\mathrm{Hom}_{C^\vee}(X(.), F) \simeq F(X)$$

Proof of 2.14. An isomorphism of sets is just a bijective map which we will now construct. Let $F \in C^\vee$ and

$$\begin{aligned} \Phi : \mathrm{Hom}_{C^\vee}(X(.), F) &\to F(X) \\ \theta &\mapsto \theta(\mathrm{id}_X) \end{aligned}$$

This is well defined because for a natural transformation $\theta : X(.) \to F$ we have in particular $\theta : X(X) = \mathrm{Hom}(X, X) \to F(X)$. To construct the inverse let $\Psi : F(X) \to \mathrm{Hom}_{C^\vee}(X(.), F)$ for $a \in F(X)$ be defined by

$$\begin{aligned} \Psi(a) : X(Z) &\to F(Z) \\ x &\mapsto (F(x))\,(a) \end{aligned}$$

This is a natural transformation because for $f \in \mathrm{Hom}_C(Y, Z)$ and $x \in X(Z)$ we have

$$F(f)\,(\Psi(a)(x)) = F(f)\,(F(x)(a)) = F(x \circ f)(a) = \Psi(a)(x \circ f) = \Psi(a)\,(X(f)(x))$$

and $\Phi(\Psi(a)) = \Psi(a)(\mathrm{id}_X) = F(\mathrm{id}_X)(a) = a$. Conversely

$$\Psi(\Phi(\theta))(x) = F(X)(\Phi(\theta)) = F(x)(\theta(\mathrm{id}_X)) = \theta(X(x)(\mathrm{id}_x)) = \theta(\mathrm{id}_X \circ x) = \theta(x)$$

hence $\Psi(\Phi(\theta)) = \theta$ which completes the prove. $\square$

Corollary 2.15. *In particular*

$$\mathrm{Hom}_{C^\vee}(X(.), Y(.)) \simeq Y(X) = \mathrm{Hom}_C(X, Y)$$

which is the statement at the beginning of this section.

2.1.2 Sheaves

Definition 2.16. *For a topological space X with topology $\tau(X) := \{U \subset X \text{ open}\}$ a* sheaf over X *with values in a sub category C of* Sets *is a map*

$$F : \tau(X) \to \mathrm{Ob}(C)$$

together with restriction morphisms $\forall U, V \in \tau(X)$:

$$\begin{aligned} \Big|_V^U : F(U) &\to F(V) \\ f &\mapsto f\Big|_V^U \end{aligned}$$

such that $\forall V, U \in \tau(X)$ and for any open cover $\bigcup_i U_i = U$ we have

1. $\Big|_U^U = \mathrm{id}_{F(U)}$

2. $\Big|_{U_i}^U \circ \Big|_U^V = \Big|_{U_i}^V$

3. $\forall \{f_i \in F(U_i)\} : \left(\forall i,j : f_i\Big|_{U_i\cap U_j}^{U_i} = f_j\Big|_{U_i\cap U_j}^{U_j} \Rightarrow \exists h \in F(U)\, \forall i : f_i = h\Big|_{U_i}^{U}\right)$

4. $\forall f,g \in F(U) : \left(\forall i : f\Big|_{U_i}^{U} = g\Big|_{U_i}^{U} \Rightarrow f = g\right)$

The elements of $F(U)$ are called sections over U. For the rest of this thesis we will use the notation $f\big|_V := f\Big|_V^U$ if $f \in F(U)$ is understood.

Definition 2.17. *A* homomorphism of sheaves *over X, $\theta : E \to F$, is given by homomorphisms $\theta : E(U) \to F(U)$ for all $U \in \tau(X)$ such that for all $V \in \tau(X)$*

$$\big|_V \circ \theta = \theta \circ \big|_V$$

This defines the category $\mathrm{Sh}_C(X)$ of sheaves over X with values in C.

Definition 2.18. *For F a sheaf over X and $f : X \to Y$ a continuous map of topological spaces the* direct image sheaf *f_*F is a sheaf over Y defined by $f_*F(U) := F(f^{-1}(U))$ and $\Big|_V^U := \Big|_{f^{-1}(V)}^{f^{-1}(U)}$. Note that f_* is a functor.*

Definition 2.19. *For $F \in \mathrm{Sh}_C(X)$, $U \in \tau(X)$ and $f \in F(U)$ the* germ *of f at $x \in U$ is given by the equivalence class*

$$[f]_x := \{g \big| \; \exists V, W \in \tau(X), g \in F(V), x \in W \subset V \cap U : f\big|_W = g\big|_W\}$$

The set of all germs at x, $F_x = \{[f]_x\}$, is called the stalk *of F at x. Note that $F_x \in C$ because restrictions are homomorphism.*

2.2 Super algebras

Definition 2.20. *A* super vector space *is a $\mathbb{Z}_2$ graded vector space, $V = V_0 \oplus V_1$. Morphisms of super vector spaces are degree preserving linear maps denoted by $\mathrm{Hom}_{\mathrm{SVS}}$ which defines the category of finite dimensional real super vector spaces* SVS. *Unless otherwise specified we will assume super vector spaces to be finite dimensional and the base field to be $\mathbb{R}$. The elements of $V_{0/1}$ are called even/odd respectively and both are called homogeneous. We will denote the degree by $\forall x \in V_i : |x| := i$.*

Lemma 2.21. *The product of finite dimensional super vector spaces V_1 and V_2 exists and is given by $V_1 \times V_2 = V_1 \oplus V_2$. The terminal object in* SVS *is given by $* = 0$.*

Proof of 2.21. Projections are given by

$$\begin{aligned} p_i : V_1 \times V_2 &\to V_i \\ x_1 \oplus x_2 &\mapsto x_i \end{aligned}$$

and for $W \in \mathrm{SVS}$ and $f_i : W \to V_i$ the unique f completing the diagram in 2.7 is

$$\begin{aligned} f : W &\to V_1 \times V_2 \\ y &\mapsto f_1(y) \oplus f_2(y) \end{aligned}$$

□

Definition 2.22. *Let* $\underline{\mathrm{Hom}}(V, W)$ *consist of all linear maps in between the super vector space* V *and* W*, i.e. the underline denotes forgetting the super structure and considering* V *and* W *as objects in the category of ordinary vector spaces,* VS*.* $\underline{\mathrm{Hom}}$ *inherits a super vector space structure*

$$\begin{aligned} \underline{\mathrm{Hom}}(V, W) := \mathrm{Hom}_{\mathrm{VS}}(V, W) &= (\underline{\mathrm{Hom}}(V, W))_0 \oplus (\underline{\mathrm{Hom}}(V, W))_1 \\ &= (\underline{\mathrm{Hom}}(V_0, W_0) \oplus \underline{\mathrm{Hom}}(V_1, W_1)) \oplus (\underline{\mathrm{Hom}}(V_0, W_1) \oplus \underline{\mathrm{Hom}}(V_1, W_0)) \end{aligned}$$

In the following we will indicate whether homomorphisms of super vector spaces are required to respect the grading or not by this underline and hence drop the subscript.

$$\begin{aligned} & & \mathrm{Hom}(V, W) &:= \mathrm{Hom}_{\mathrm{SVS}}(V, W) \\ \underline{\mathrm{End}}(V) &:= \underline{\mathrm{Hom}}(V, V) = \mathrm{End}_{\mathrm{VS}}(V) & \mathrm{End}(V) &:= \mathrm{End}_{\mathrm{SVS}}(V) \\ \underline{\mathrm{Iso}}(V, W) &:= \mathrm{Iso}_{\mathrm{VS}}(V, W) & \mathrm{Iso}(V, W) &:= \mathrm{Iso}_{\mathrm{SVS}}(V, W) \\ \underline{\mathrm{Aut}}(V) &:= \mathrm{Aut}(V) & \mathrm{Aut}(V) &:= \mathrm{Aut}_{\mathrm{SVS}}(V) \end{aligned}$$

Note that $\mathrm{Hom}(V) = (\underline{\mathrm{Hom}}(V, W))_0 = \mathrm{Hom}(V_0, W_0) \oplus \mathrm{Hom}(V_1, W_1)$ *and similarly for* $\mathrm{End}(V)$*,* $\mathrm{Iso}(V, W)$ *and* $\mathrm{Aut}(V)$*.*

Definition 2.23. *A* $\mathbb{Z}_2$ *graded algebra* $A = A_0 \oplus A_1$ *over* $\mathbb{C}$ *with* $A_i A_j \subset A_{i+j}$ *is called* super algebra*. It is called* unital *if there is a unit element which is then automatically even. A morphism of (unital) super algebras is a (unital) algebra morphism which preserves the degree. The category of associative unital super algebras will be denoted by* SAlg*.*

Example 2.24. $\underline{\mathrm{End}}(V)$ *as introduced above with the multiplication given by* $\circ$ *is an associative unital super algebra.*

Definition 2.25. *A super algebra is called* super commutative

$$:\Leftrightarrow \forall a, b \in A : ab = (-1)^{|a||b|} ba$$

and super anticommutative

$$:\Leftrightarrow \forall a, b \in A : ab = -(-1)^{|a||b|} ba$$

2.2.1 Lie super algebras

Definition 2.26. *A* Lie super algebra $A = A_0 \oplus A_1$ *is a graded super anticommutative (non-associative, usually non-unital) algebra with the product denoted by* $[.\,,.]$, *which additionally fulfils the super Jacobi identity*[1], *written in a mnemonic form as*

$$\forall x, y, z \in A : [x, [y, z]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] + (-1)^{|z|(|x|+|y|)}[z, [x, y]] = 0$$

This defines the category LSAlg *as a full subcategory of complex super algebras.*

Example 2.27. *The linear endomorphisms of a complex super vector space* $\underline{\mathrm{End}}(V)$ *together with the super commutator*

$$[X, Y] = X \circ Y - (-1)^{|X||Y|}\, Y \circ X$$

form a super Lie algebra which is denoted $\mathfrak{gl}(V)$.

Definition 2.28. *The* universal enveloping algebra *of a Lie super algebra* $\mathfrak{g}$ *is defined as the associative algebra*

$$\mathfrak{U}(\mathfrak{g}) := \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}\right) / \left(x \otimes y - (-1)^{|x||y|} y \otimes x \sim [x, y]\right)$$

i.e. the tensor algebra modulo the Lie super algebra relations. $\mathfrak{U}(\mathfrak{g})$ *inherits a natural* $\mathbb{Z}_2$ *grading from that of the tensor algebra which is preserved by the equivalence relation.*

Lemma 2.29. *We have an isomorphism of vector spaces*

$$\mathfrak{U}(\mathfrak{g}_0) \otimes \mathfrak{U}(\mathfrak{g}_1) \simeq \mathfrak{U}(\mathfrak{g}_0) \otimes \bigwedge \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{U}(\mathfrak{g})$$
$$u_0 \otimes u_1 \mapsto u_0 \beta(u_1)$$

which holds by the Poincaré-Birkhoff-Witt theorem*: For any basis* $\{a_1, \ldots, a_p\}$ *of* $\mathfrak{g}_0$ *and any basis* $\{b_1, \ldots, b_q\}$ *of* $\mathfrak{g}_1$ *a basis of* $\mathfrak{U}(\mathfrak{g})$ *is given by*

$$\{a_1^{k_1} \ldots a_p^{k_p} b_{i_1} \ldots b_{i_l} \big| k_i \geq 0, 0 \leq l \leq q, i_1 < i_2 < \ldots i_l\}$$

Proof of 2.29. See Theorem 2.1 in [Ros65]. □

Definition 2.30. *We will in the following use the* antipode *of a graded algebra* $\mathfrak{U}$ *which is the even linear map* $S : \mathfrak{U} \to \mathfrak{U}$ *defined by*

$$S(1) := 1$$
$$\forall x \in \mathfrak{U}_1 : S(x) := -x$$
$$\forall x, y \in \mathfrak{U}_{n\geq 1} : S(xy) := (-1)^{|x||y|} S(y) S(x)$$

[1] Alternatively one may write the Jacobi identity as $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$ i.e. ad defined by $\mathrm{ad}(x) := [x, .]$ is a representation of $\mathfrak{g}$ on $\mathfrak{gl}(\mathfrak{g})$. Another interpretation is $\mathrm{ad}(x)([y, z]) = [\mathrm{ad}(x)(y), z] + (-1)^{|x||y|}[y, \mathrm{ad}(x)(z)]$ i.e. $\mathrm{ad}(x)$ is a super derivation on $\mathfrak{g}$.

Remark 2.31. *Note that $S([x,y]) = -[S(x), S(y)]$, in particular $S_{\mathfrak{U}(\mathfrak{g})}\big|_{\mathfrak{g}} = -\operatorname{id}_{\mathfrak{g}}$ is a Lie super algebra anti-automorphism.*

Lemma 2.32. *Let $A \in \mathrm{SAlg}$ then the Lie super algebra morphisms $\mathrm{Hom}_{\mathrm{LSAlg}}(\mathfrak{g}, A)$, where the bracket on A is given by the super commutator, are in one-to-one correspondence with morphisms of unital super algebras $\mathrm{Hom}_{\mathrm{SAlg}}(\mathfrak{U}(\mathfrak{g}), A)$.*

Remark 2.33. *This will be used later in the case of $A = \mathfrak{U}(\mathfrak{h})$.*

Proof of 2.32. The extension of a Lie super algebra homomorphism to the tensor algebra by acting on each generator simultaneously respects the equivalence relation which defines $\mathfrak{U}$. Hence this is the unique extension of a LSAlg morphism to a unital SAlg morphism. The inverse is given by restriction. □

2.3 Super ringed spaces

The first two definitions introduce the category of super (ringed) spaces, SRSp.

Definition 2.34. *A pair $X = (X_0, O_X)$ where X_0, called base, is a topological space and O_X a sheaf of super commutative super algebras over X_0 is called* super (ringed) space.

Definition 2.35. *A* morphism of super (ringed) spaces *X and Y is given by a pair of maps $f = (f_0, f^*)$ where $f_0 : X_0 \to Y_0$ is continuous and $f^* : O_Y \to (f_0)_* O_X$ is a morphism of sheaves over Y_0 (with values in super algebras).*

Definition 2.36. *For an open subset $U \subset X_0$, $X\big|_U = (U, O_x\big|_U)$ denotes the* open subspace *of X with base U. Note that the embedding $i : U \hookrightarrow X_0$ and sheave restriction $p : O_X \to O_X\big|_U$ define an embedding $(i,p) : X\big|_U \hookrightarrow X$ of super ringed spaces.*

The following lemma shows that super ringed spaces can be glued.

Lemma 2.37. *Let X_0 be a topological space with an open cover $\{U_i\}$ and sheaves F_i with values in super algebras over U_i together with isomorphisms $\phi_{i,j} : F_j\big|_{U_{i,j}} \simeq F_i\big|_{U_{i,j}}$ such that*

$$\phi_{i,i} = \operatorname{id}_{F_i} \qquad \phi_{i,j} \circ \phi_{j,i} = \operatorname{id}_{F_i|_{U_{i,j}}} \qquad \phi_{i,k} \circ \phi_{k,j} \circ \phi_{j,i} = \operatorname{id}_{F_i|_{U_{i,j,k}}}$$

where $U_{i,j} := U_i \cap U_j$. Then there exists a sheaf F and isomorphisms $\phi_i : F\big|_{U_i} \to F_i$ such that $\phi_i\big|_{U_{i,j}} = \phi_{i,j} \circ \phi_j\big|_{U_{i,j}}$ and F and the ϕ_i are unique up to isomorphism.

Proof of 2.37. See [All11b], Proposition 3.18. □

2.4 Super manifolds

In this section we define *cs manifolds*, where cs stands for complex super, as introduced in [DM99], definition 4.8.1, but, since no other type of super manifolds is considered here, we will mostly leave away the prefix cs.

2.4.1 Linear super manifolds

Definition 2.38. *With a finite dimensional real super vector space $V = V_0 \oplus V_1 \in \mathrm{SVS}$, we can associate the* linear super manifold $\underline{V} = (V_0, \mathrm{C}^\infty_{V_0} \otimes \bigwedge V_1^*)$ *where* $\mathrm{C}^\infty_{V_0}$ *denotes smooth functions from V_0 to $\mathbb{C}$ considered as an algebra over $\mathbb{C}$. I.e. although V_0 and V_1 are real, we consider a complexified structure sheaf.*

Remark 2.39. *The category* SVS *is equivalent to a category of linear super manifolds with linear morphisms, i.e. $(\phi_0, \phi^*) : \underline{V} \to \underline{W}$ consisting of a linear map $\phi_0 \in \mathrm{Hom}_{\mathrm{SVS}}(V, W)$ and ϕ^* extending $\phi^*\big|_{W^*} \in \mathrm{Hom}_{\mathrm{SVS}}(W^*, V^*)$. But as in the ordinary case the point about manifolds is the definition of smooth and not only linear maps.*

Definition 2.40. *Let $\mathbb{R}^{p|q}_{\mathbb{C}} := \underline{\mathbb{R}^p \oplus \mathbb{R}^q} = (\mathbb{R}^p, \mathrm{C}^\infty_{R^p} \otimes \bigwedge(\mathbb{R}^q)^*)$ denote the linear super manifold associated with the super vector space $\mathbb{R}^p \oplus \mathbb{R}^q$. Because we will throughout this thesis exclusively deal with manifolds with complex structure sheaf, we will drop the index and write $\mathbb{R}^{p|q} := \mathbb{R}^{p|q}_{\mathbb{C}}$ deviating from the standard notation. Note that* SVS *is equivalent to its $\mathbb{R}^{p|q}$ subcategory.*

2.4.2 General super manifolds

Definition 2.41. *Let $X = (X_0, O_X) \in \mathrm{SRSp}$. A pair (U, ϕ) with $x \in U \subset X_0$ and $\phi : X\big|_U \hookrightarrow \mathbb{R}^{p|q}$ an open embedding (i.e. an isomorphism onto an open subset) is called a chart of X around x and U a coordinate neighbourhood of x. If $\{x_i, \xi_j\}$ are the standard homogeneous coordinates of $\mathbb{R}^{p|q}$, i.e. standard coordinates of $\mathbb{R}^p$ together with the standard basis of $(\mathbb{R}^q)^*$, then $\{\phi^*(x_i), \phi^*(\xi_j)\}$ are called a* system of local coordinates *on U.*

Definition 2.42. *A collection of charts (U_i, ϕ_i) such that the U_i form an open cover of M_0 for $M = (M_0, O_M) \in \mathrm{SRSp}$ is called an* atlas *of M.*

Definition 2.43. *The category of* cs super manifolds, SMan *is a full subcategory of* SRSp *where we additionally require each super manifold to have a Hausdorf and para compact*[2] *base, M_0, and to admit an atlas. In other words, super manifolds are locally isomorphic to $\mathbb{R}^{p|q}$. We will in the following assume that super manifolds are of pure (finite)* super dimension, *i.e. $p|q =: \mathrm{SDim}(M)$ is the same for all charts.*

Remark 2.44. *For $M = (M_0, O_M) \in \mathrm{SMan}$ and $U \subset M_0$ open the sub super ringed space $M\big|_U$ is again a super manifold since we can restrict an atlas of M to one of $M\big|_U$.*

Definition 2.45. *Let $M \in \mathrm{SMan}$ of super dimension $p|q$ and (U, ϕ^*) a chart with local coordinates (x_i, χ_j). By Definition 2.41 we can uniquely express $f \in O_M(U)$ in local coordinates by*

$$f = \phi^*(\sum_{|\alpha| \geq 0} g^\alpha \xi_\alpha) = \sum_{|\alpha| \geq 0} f^\alpha \chi_\alpha$$

[2]Given that M_0 is Hausdorf, this is equivalent to M_0 being metrizable and also to each connected component of M_0 being second countable.

where $\{\xi_j\}$ *is the standard basis of* $(\mathbb{R}^q)^*$, $g^\alpha \in \mathrm{C}^\infty(U)$, $\alpha = (\alpha_1, \ldots, \alpha_{|\alpha|})$ *is an ordered multi index,* $1 \leq \alpha_i < \alpha_{i+1} \leq q$, $\chi_\alpha := \chi_{\alpha_1} \wedge \ldots \wedge \chi_{\alpha_{|\alpha|}}$ *and* $f^\alpha = \phi^*(g^\alpha) \in O_M(U)_0$. *This is called the* local expansion *of* f *with respect to these local coordinates.*

Definition 2.46. *For* $M \in \mathrm{SMan}$ *and* $x \in M_0$ *we denote by* $\mathfrak{m}_{O_M,x}$ *the maximal ideal of non-invertible elements of* $(O_M)_x$ *and by* $\mathcal{N}_{O_M,x}$ *the maximal ideal of nilpotents.*

Lemma 2.47. *We have for an open domain* $U \subset \mathbb{R}^{p|q}$ *and* $x \in U_0$ *that*

$$(O_U)_x = \mathbb{C} \oplus \mathfrak{m}_{O_U,x} \qquad\qquad \mathfrak{m}_{O_U,x} = \mathfrak{m}_{\mathrm{C}^\infty_{U_0},x} \oplus \mathcal{N}_x$$

where $\mathfrak{m}_{\mathrm{C}^\infty_{U_0},x} = \{[f]_x \in \mathrm{C}^\infty(U)_x \big| f(x) = 0\}$.

Proof of 2.47. By definition $\mathrm{C}^\infty(U)_x = \mathbb{C} \oplus \mathfrak{m}_{\mathrm{C}^\infty_{U_0},x}$ and $(O_U)_x = \mathrm{C}^\infty(U)_x \oplus \mathcal{N}_x$. □

Remark 2.48. *For* $M \in \mathrm{SMan}$, M_0 *is an ordinary smooth manifold which can be identified with the purely even super manifold* $(M_0, \mathrm{C}^\infty(M_0))$.

Definition 2.49. *For* $f \in O_M(U)$ *and* $p \in U$ *denote by* $\tilde{f}(p) \in \mathbb{C}$ *the unique number, considered as a constant section* $\tilde{f}(p) \cdot 1 \in O_M(U)$, *such that*

$$\forall V \subset U \text{ open, } p \in V : (f - \tilde{f}(p))\big|_V \text{ is not invertible}$$

which means not invertible as an element of the unital algebra $O_M(V)$.[3] *Then* $\tilde{f}(p)$ *is called the* numerical part of f *of* p.

Lemma 2.50. *With the notation of Definition 2.45 we have for* $f \in O_M(U)$ *in a chart* (U, ϕ) *around* p *that* $\tilde{f}(p) = \widetilde{f^\emptyset}(p)$.

Proof of 2.50. All χ_α for $\alpha \neq \emptyset$ are nilpotent and for any nilpotent $f_n \in O_M(U)$ we have that

$$(f + f_n)^{-1} = f^{-1} \sum_{k \geq 0} (-f_n f^{-1})^k = \sum_{k \geq 0} (-f^{-1} f_n)^k f^{-1}$$

exists if and only if f is invertible. □

Definition 2.51. *The morphism of super manifolds*

$$j_0 : M_0 \to M \qquad\qquad (j_0)_0 = \mathrm{id}_{M_0} \qquad\qquad j_0^*(f)(p) = \tilde{f}(p)$$

is called canonical embedding *of* M_0 *into* M. *Note however that this is not an open embedding in the above sense.*

Lemma 2.52. *We have that* $\mathfrak{m}_{O_M,p}$ *is finitely generated by* $\{[x_i - j_0^*(x_i)(p)]_p, [\xi_j]_p\}$ *for local coordinates* $\{x_i, \xi_j\}$ *around* p.

[3] In other words, the germ $[f - \tilde{f}(p)]_p$ is not invertible.

Proof of 2.52. The problem is local hence most has been said already in Lemma 2.47. The nilpotents $\mathcal{N}_{O_M,p}$ are generated by $\{[\xi_j]_p\}$ and if $f \in \mathrm{C}^\infty(U_0)$ for a ball $U_0 \subset \mathbb{R}^p$ vanishes at p we can write it as $f = \sum_i (x_i - x_i(p)) f_i$ with $f_i : p \mapsto \int_0^1 \partial_i f(tp)\mathrm{d}t$. □

Lemma 2.53. *For $f \in \mathrm{Hom}_{\mathrm{SMan}}(X, Y)$, $g \in O_Y(Y)$ and $p \in X_0$ we have*

$$j_0^*\big(f^*(g)\big)(p) = j_0^*\big(g\big)\big(f_0(p)\big)$$

Proof of 2.53. For $\lambda \in \mathbb{C}$ we have that $f^*(g - \lambda) = f^*(g) - \lambda$ is invertible if and only if $g - \lambda$ is invertible since f^* is a unital algebra morphism. And since f_0 is smooth, U is a neighbourhood of $p \in X$ if and only if $f_0(U)$ is a neighbourhood of $f_0(p) \in Y$. □

Corollary 2.54. *For the local expansion of $f = \sum_\alpha f^\alpha \chi_\alpha \in O_M(U)$ with $f^\alpha = \phi^*(g^\alpha)$ and $g^\alpha \in \mathrm{C}^\infty(U)$ we have*

$$j_0^*(f^\alpha) = g^\alpha$$

Corollary 2.55. *We have*

$$\mathfrak{m}_{O_M,x} = \{[f]_x \big|\, j_0^*(f)(x) = 0\}$$

by Lemma 2.47.

Corollary 2.56. *For $f \in \mathrm{Hom}_{\mathrm{SMan}}(X, Y)$ we have $f^*(\mathfrak{m}_{O_Y, f_0(x)}) \subset \mathfrak{m}_{O_X,x}$.*

Lemma 2.57. *If $f \in O_M(U)$ has the local expansion $f = \sum_\alpha f^\alpha \chi_\alpha$ for a chart (U, ϕ) with local coordinates (x_i, χ_j) and $\forall \alpha : j_0^*(f^\alpha) \equiv 0$ then $f = 0$.*

Proof of 2.57. By definition $f^\alpha = \phi^*(g^\alpha)$ for some $g^\alpha \in \mathrm{C}^\infty(\phi_0(U))$. But by Corrolary 2.54 we have $j_0^*(f^\alpha) = g^\alpha \equiv 0$ hence $f^\alpha = \phi^*(0) = 0$. □

Corollary 2.58. *If $f \in O_M(U)$ has a local expansion $f = f^\emptyset$ and $j_0^*(f) \equiv 0$ then $f = 0$.*

Corollary 2.59. *If $M \in \mathrm{SMan}$ is of super dimension $p|q$ and $f \in O_M(U)$ such that $\forall x \in U : [f]_x \in (\mathfrak{m}_{O_M,x})^{q+1}$ then $f = 0$.*

Proof of 2.59. We have

$$[f]_x \in (\mathfrak{m}_{O_M,x})^{q+1} \Rightarrow [f^\alpha]_x \in (\mathfrak{m}_{O_M,x})^{q+1-|\alpha|} \subset \mathfrak{m}_{O_M,x}$$

since $|\alpha| \leq q$ hence by Corollary 2.55 we can use Lemma 2.57. □

Lemma 2.60. *For $S \in \mathrm{SMan}$ and $V \in \mathrm{SVS}$ the S points of $\underline{V}$ are given by*

$$\phi_S : \underline{V}(S) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{SVS}}(V^*, O_S(S_0)_\mathbb{R}) = \big(O_S(S_0) \otimes V\big)_{0,\mathbb{R}}$$

where ϕ is bijective and natural, i.e. for $R \in \mathrm{SMan}$, $f \in_R S$ and $g : S \to \underline{V}$ we have

$$\phi_R(g(f)) = f^* \circ \phi_S(g)$$

Here $O_S(S_0)_{\mathbb{R}} := \{f \in O_S(S_0) \big| \ j_0^*(f)(S_0) \subset \mathbb{R}\}$ *are the super functions that take real values and*

$$\big(O_S(S_0) \otimes V\big)_{0,\mathbb{R}} := \Big\{ \sum_i f_i \otimes v_i \in O_S(S_0) \otimes V \big| \ |f_i| = |v_i|, \ j_0^*(f_i)(S_0) \subset \mathbb{R} \Big\}$$

Proof of 2.60. In the following it is important to keep in mind that V is a real super vector space.

Note that $V^* \subset O_V(V_0)$ to define

$$\phi_S(g)(\mu) := g^*(\mu)$$

for $g \in \underline{V}(S)$ and $\mu \in V^*$. Note that ϕ_S is natural by definition.

Then for $p \in S_0$ we have $\forall \mu \in V^* : j_0^*\big(\phi_S(g)(\mu)\big)(p) = j_0^*(\mu)(g_0(p)) = \mu(g_0(p))$ by Lemma 2.53, hence g_0 is uniquely determined by $\phi_S(g)$ and $\mu(g_0(p)) \in \mathbb{R}$ hence $\phi_s(g)(\mu)$ takes real values.

To show that ϕ_S is injective let $g' \in \underline{V}(S)$ with $\phi_S(g) = \phi_S(g')$ and $h = \sum_\alpha h^\alpha \xi_\alpha \in O_V(V_0) = \mathrm{C}^\infty(V_0) \otimes \bigwedge V_1^*$ where $\{\xi_i\}$ is a basis of V_1^*. Then

$$g^*\left(h\right) = \sum_\alpha g^*\left(h^\alpha\right) g^*\left(\xi_\alpha\right) = \sum_\alpha g^*\left(h^\alpha\right) g'^*\left(\xi_\alpha\right)$$

Further at each point $x \in V_0$ we can use ordinary Taylor expansion $h^\alpha = T_x^q(h^\alpha) + R_x^{q+1}(h^\alpha)$ with the Taylor polynomial $T_x^q(h^\alpha) \in \mathbb{C}[\{x_i\}]/(\{x_i\})^q$ for a basis x_i of V_0^* used as coordinates on V_0 and $R_x^{q+1}(h^\alpha) \in \mathfrak{m}^{q+1}_{\mathrm{C}^\infty(V_0),x}$ where $p|q := \mathrm{SDim}(S)$. Then $g^*(T_x^q(h^\alpha)) = g'^*(T_x^q(h^\alpha))$ and $(g^* - g'^*)(R_x^{q+1}(h^\alpha)) \in \mathfrak{m}^{q+1}_{(g_0)_*O_S,x}$. Since this can be done at each point we must have $\forall x : (g^* - g'^*)(h^\alpha) \in \mathfrak{m}^{q+1}_{(g_0)_*O_S,x}$ and hence we can use Corollary 2.59 to conclude $g = g'$.

On the other hand, $f \in \mathrm{Hom}_{\mathrm{SVS}}(V^*, O_S(S_0)_{\mathbb{R}})$ determines a map $g_0 : S_0 \to V_0$ by $\forall \mu \in V_0^* : \mu(g_0(x)) := j_0^*(f(\mu))(x) \in \mathbb{R}$ which is smooth because all $j_0^*(f(\mu))$ are.

g^* is defined using the Taylor series, i.e. for $h = \sum_\alpha h^\alpha \chi_\alpha$ and

$$h^\alpha_\beta := \partial_{x_{\beta_1}} \dots \partial_{x_{\beta_{|\beta|}}} h^\alpha$$

the ordinary $|\beta|$-fold directional derivative of $h^\alpha \in \mathrm{C}^\infty(V_0)$ we define

$$g^*(h) := \sum_\alpha \sum_{0 \le |\beta| < \infty} \frac{1}{\beta!} \left(h^\alpha_\beta \circ g_0\right) \left(f(y_\beta) - y_\beta \circ g_0\right) f(\xi_\alpha)$$

Because Taylor expansion is a unital algebra homomorphism so is g^* and by definition $\phi_S(g) = f$. So ϕ_S is also surjective.

□

Corollary 2.61. *In particular for any* $S \in \mathrm{SMan}$ *we have* $O_S(S_0) \simeq \mathrm{Hom}_{\mathrm{SMan}}(S, \mathbb{R}^{1|1})$ *which justifies calling the global sections* $O_S(S_0)$ *super functions.*

Corollary 2.62. *For $M, N \in \mathrm{SMan}$ and $h : N \to M$, then locally, i.e. on $N\big|_{h_0^{-1}(U)}$ for a chart (U, ϕ) of M with local coordinates $\{x_i, \chi_j\}$, h is in one to one correspondence with the data $\{h^*(x_i), h^*(\chi_j)\}$.*

Corollary 2.63. *More generally for $M, N \in \mathrm{SMan}$, $p|q := \mathrm{SDim}(N)$, (U, ϕ) a chart of N with local coordinates $\{x_i, \xi_j\}$, $V \subset M_0$ open, $\{z_1, \ldots, z_p\} \subset O_M(V)_0$ and $\{\zeta_1, \ldots, \zeta_q\} \subset O_M(V)_1$ there exists a unique $g \in \mathrm{Hom}_{\mathrm{SMan}}(M\big|_V, N\big|_U)$ such that $\forall i, j : z_i = g^*(x_i)$ and $\zeta_j = g^*(\chi_j)$ if and only if $\forall i : j_0^*(u_i)(V) \subset j_0^*(x_i)(U)$.*

Proof of 2.63. One direction is established by Lemma 2.53. For the other one we generalise Lemma 2.60 to open subsets of linear super manifolds and apply it to $g \circ \phi$. More details can be found in [Lei80], Theorem 2.1.7. □

Lemma 2.64. *Super manifolds are modulo isomorphism uniquely determined by their cocycles. More precisely, let X_0 be a metrizable Hausdorff topological space with an open cover $\{U_i\}$ and super manifolds M_i with $(M_i)_0 = U_i$ together with isomorphisms $\phi_{i,j} : M_j\big|_{U_{i,j}} = M_i\big|_{U_{i,j}}$ such that*

$$\phi_{i,j} = \phi_{j,i}^{-1} \qquad \phi_{i,k} \circ \phi_{k,j} \circ \phi_{j,i} = \mathrm{id}_{M_i}\big|_{U_{i,j,k}}$$

Then there exists a super manifold M with $M_0 = X_0$ and isomorphisms $\phi_i : M\big|_{U_i} \to M_i$ such that $\phi_i\big|_{U_{i,j}} = \phi_{i,j} \circ \phi_j\big|_{U_{i,j}}$ and M and the ϕ_i are unique up to isomorphism.

Proof of 2.64. This follows from the corresponding property of super ringed spaces, Lemma 2.37. □

Lemma 2.65. *Finite products of linear super manifolds exist in the category of super manifolds* SMan.

Proof of 2.65. By Lemma 2.21 the product in SVS of $V_1, V_2 \in \mathrm{SVS}$ exists. Hence we have a product of linear super manifolds $\underline{V_1}$ and $\underline{V_2}$ which is given by $\underline{V_1} \times \underline{V_2} = \underline{V_1 \oplus V_2}$ and $\underline{*} = \underline{0} = (0, \mathrm{C}^\infty_{\{0\}} \otimes \bigwedge\{0\}^*) = (0, \mathbb{C})$.

But we need to check that

$$\underline{V_1 \oplus V_2} = ((V_1)_0 \oplus (V_2)_0, \mathrm{C}^\infty_{(V_1)_0 \oplus (V_2)_0} \otimes \bigwedge((V_1)_1 \oplus (V_2)_1)^*$$

also is the direct product in SMan with projections $p_i : \underline{V_1} \times \underline{V_2} \to \underline{V_i}$ given by

$$(p_i)^* : \mathrm{C}^\infty_{(V_i)_0} \otimes \bigwedge((V_i)_1)^* \to ((p_i)_0)_* \, \mathrm{C}^\infty_{(V_1)_0 \oplus (V_2)_0} \otimes \bigwedge((V_1)_1 \oplus (V_2)_1)^*$$
$$f \otimes x \mapsto ((v, w) \mapsto f(v)) \otimes x$$

where $(p_i)_0$ is the usual projection as in Lemma 2.21 and for $U \subset (V_1)_0$ open

$$((p_1)_0)_* \, \mathrm{C}^\infty_{(V_1)_0 \oplus (V_2)_0}(U) = \mathrm{C}^\infty_{(V_1)_0 \oplus (V_2)_0}(U \times (V_2)_0)$$

So let $X \in \mathrm{SMan}$ and $f_i \in \mathrm{Hom}_{\mathrm{SMan}}(X, \underline{V_i})$. The main point is

$$\mathrm{Hom}_{\mathrm{SVS}}((V_1 \oplus V_2)^*, O_Z(Z_0)_{\mathbb{R}}) \simeq \mathrm{Hom}_{\mathrm{SVS}}(V_1^*, O_Z(Z_0)_{\mathbb{R}}) \oplus \mathrm{Hom}_{\mathrm{SVS}}(V_1^*, O_Z(Z_0)_{\mathbb{R}})$$

which is just a statement of linear algebra but by Lemma 2.60 it establishes

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SMan}}(Z, \underline{V_1}) \times \mathrm{Hom}_{\mathrm{SMan}}(Z, \underline{V_2}) &\simeq \mathrm{Hom}_{\mathrm{SVS}}((V_1 \oplus V_2)^*, O_Z(Z_0)_{\mathbb{R}}) \\ &\simeq \mathrm{Hom}_{\mathrm{SMan}}(X, \underline{V_1 \oplus V_2})\end{aligned}$$

and hence there is a unique $f_1 \oplus f_2 \simeq f \in \mathrm{Hom}_{\mathrm{SMan}}(X, \underline{V_1 \oplus V_2})$ which completes the diagram defining the product.

□

Lemma 2.66. *Finite products exist in* SMan.

Proof of 2.66. Since by definition every super manifold is locally linear, products exist locally due to Lemma 2.65. Due to Lemma 2.64 this suffice for the global existence of the product. The terminal object is the same as for linear super manifolds. □

2.4.3 Tangent and cotangent space

Definition 2.67. *A* fibre bundle over $B \in \mathrm{SMan}$ with fibre $F \in \mathrm{SMan}$ *is given by* $X \in \mathrm{SMan}$ *together with a projection* $p \in \mathrm{Hom}_{\mathrm{SMan}}(X, B)$ *such that there exists an* atlas of local trivialisations. *This is an open cover* $\bigcup_i U_i = B_0$ *together with open embeddings* $\tau_i : B\big|_{U_i} \times F \hookrightarrow X$ *such that the following diagrams commute*

$$\begin{tikzcd} B\big|_{U_i} \times F \arrow[r, hook, "\tau_i"] \arrow[d, two heads] & X \arrow[d, "p"] \\ B\big|_{U_i} \arrow[r, hook] & B \end{tikzcd}$$

where the unlabelled arrows are the canonical projection and embedding. In other words, locally we have $X\big|_{(p_x)_0^{-1}(U_i)} \simeq B\big|_{U_i} \times F$.

A morphism of fibre bundles $p_X : X \twoheadrightarrow B$ *and* $p_Y : Y \twoheadrightarrow C$ over $\phi : B \to C$ *is given by* $f \in \mathrm{Hom}_{\mathrm{SMan}}(X, Y)$ *such that* $p_Y \circ f = \phi \circ p_X$. *This defines the category of fibre bundles as a subcategory of super manifolds.*

Lemma 2.68. *We can glue fibre bundles, i.e. for an open cover* $\{U_i\}$ *of* $B \in \mathrm{SMan}$, $F \in \mathrm{SMan}$ *and* $X_i := B\big|_{U_i}$ *together with* $\phi_{i,j} \in \mathrm{Iso}_{\mathrm{SMan}}\left(X_j\big|_{U_{i,j}}, X_i\big|_{U_{i,j}}\right)$ *such that* $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}\big|_{U_{i,j,k}}$, *there exists a unique (up to isomorphism) fibre bundle* $p : X \twoheadrightarrow B$ *with an atlas* $\{(U_i, \tau_i)\}$ *such that* $\tau_i \circ \phi_{i,j} = \tau_j$.

Proof of 2.68. This follows from gluing para compact Haussdorf spaces and Lemma 2.64.

□

Definition 2.69. *If* $p_X : X \twoheadrightarrow B$ *is a fibre bundle,* $Y \in \mathrm{SMan}$ *and* $p_Y \in \mathrm{Hom}_{\mathrm{SMan}}(Y, B)$ *then* $X \times_B Y \in \mathrm{SMan}$ *together with morphisms* $p_1 \in \mathrm{Hom}_{\mathrm{SMan}}(X \times_B Y, X)$ *and* $p_2 \in \mathrm{Hom}_{\mathrm{SMan}}(X \times_B Y, Y)$ *is called* fibre product of X and Y over B *if*

$$\forall Z \in \mathrm{SMan}\ \forall f_X : Z \to X\ \forall f_Y : Z \to Y\ \exists! h : Z \to X \times_B Y$$

such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, dotted, "h"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\
& X \times_B Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "p_Y"] \\
& X \arrow[r, "p_X"'] & B
\end{tikzcd}$$

Lemma 2.70. *Fibre products in* SMan *exist.*

Proof of 2.70. Since $p_X : X \twoheadrightarrow B$ is a fibre bundle we have atlas of local trivialisations $\{(U_i, \tau_i)\}$ such that $\tau_i : B\big|_{U_i} \times F \xrightarrow{\sim} X\big|_{(p_x)_0^{-1}(U_i)}$. Hence we can define $X \times_B Y$ locally to be $F \times Y$ with the canonical projection $p_2 : F \times Y \to Y$. For $S \in \mathrm{SMan}$ we define $p_1 : F \times Y \to X$ by

$$\begin{aligned} F(S) \times Y(S) &\to X(S) \\ (f, y) &\mapsto \tau_i(p_Y(y), f) \end{aligned}$$

For given $f_X \in \mathrm{Hom}_{\mathrm{SMan}}(Z, X)$ and $f_Y \in \mathrm{Hom}_{\mathrm{SMan}}(Z, Y)$ and $z \in_S Z$ such that $p_Y \circ f_Y(z) = p_X \circ f_X(z) \in_S B\big|_{U_i}$ we have the unique

$$h(z) := (p_2(\tau^{-1}(f_X(z))), f_Y(z)) \in_S F \times Y$$

(this p_2 projects onto F) completing the diagram defining the fibre product.

The global $X \times_B Y$ is then obtained by gluing.

□

Definition 2.71. *Identifying* $b \in B_0$ *with* $b : * \hookrightarrow B$ *we call* $X_b := X \times_B b$ *the* fibre *of the fibre bundle* $p : X \twoheadrightarrow B$ at b.

Corollary 2.72. *By the defining property of the fibre product we have* $\forall Z \in \mathrm{SMan}$

$$\{x \in_Z X \big| p_X(x) = b\} \simeq X_b(Z)$$

and hence each τ_i *with* $b \in U_i$ *induces an isomorphism* $X_b \simeq F$ *by Definition 2.67.*

Definition 2.73. *A fibre bundle* $p : X \twoheadrightarrow B$ *is called* vector bundle *if the fibre* F *is a linear super manifold and all induced isomorphisms* $X_b \simeq F$ *are linear. By demanding morphisms to be linear in the fibre direction we obtain the category of vector bundles as a subcategory of fibre bundles.*

Now, as in the ordinary world, one can think of the tangent space in terms of derivations or as a pointwise linearisation of the super manifold.

Definition 2.74. *Let $X \in \mathrm{SMan}$, (U, ϕ) be a chart with local coordinates $\{y_i, \chi_j\}$, and $f \in O_X(U)$ which we expand locally as*

$$f = \sum_{|\alpha| \geq 0} f^\alpha \chi_\alpha$$

Then we define even derivatives by ordinary derivatives

$$\partial_{y_i} f := \sum_{|\alpha| \geq 0} (\partial_{y_i} f^\alpha) \chi_\alpha$$

and odd derivatives by

$$\partial_{\chi_i} f := \sum_{|\alpha| \geq 0} f^\alpha \partial_{\chi_i} \chi_\alpha$$

where $[\partial_{\chi_i}, \chi_j] := \delta_{i,j}$ (and $\partial_{\chi_i}(1) = 0$). These partial derivatives *form a basis over $O_X(U)$ of the graded derivations*

$$\mathrm{Der}(O_X(U)) = \{d \in \underline{\mathrm{Hom}}(O_X\big|_U, O_X\big|_U) \big| \; d(fg) = d(f)\, g + (-1)^{|f||d|} f\, d(g)\}$$

The sheaf $\mathcal{T}X = \mathrm{Der}(O_X)$ is also called tangent sheaf *of X.*

Definition 2.75. *In the notation of Definition 2.74 above, the $O_X(U)$ dual of $\mathrm{Der}(O_X(U))$ is denoted by $\Omega^1_X(U)$ and the dual basis of $\{\partial_{y_i}, \partial_{\chi_j}\}$ by $\{\mathrm{d}y_i, \mathrm{d}\chi_j\}$, i.e.*

$$\mathrm{d}x_i(\partial x_j) = \mathrm{d}\chi_i(\partial \chi_j) = \delta_{i,j} \qquad\qquad \mathrm{d}x_i(\partial \chi_j) = \mathrm{d}\chi_i(\partial x_j) = 0$$

The corresponding sheave of $O_X(U)$ modules over X_0 is denoted by Ω^1_X and called cotangent sheaf *of X.*

Now for the linearised point of view.

Definition 2.76. *The* tangent functor T *from super manifolds to vector bundles has the following properties: $\forall X, Y, S \in \mathrm{SMan}$:*

1. *$\forall U \subset X$ open subspace, $\mathrm{T}\, U \subset \mathrm{T}\, X$ is an open subspace*
2. *$\forall V \in \mathrm{SVS} : (\mathrm{T}\, \underline{V})(S) \simeq (O_S(S_0)[\mathbb{D}] \otimes V)_{0,\mathbb{R}}$*
3. *$\forall f : X \to Y$ locally for $x \in_S U \subset_S X$ we have $\mathrm{T}\, f(x + \epsilon v) = f(x) + \epsilon(\mathrm{d}f(x))v$*

where $\mathbb{D} = \mathbb{C}[\epsilon]/\epsilon^2$ and

$$(O_S(S_0)[\mathbb{D}] \otimes V)_{0,\mathbb{R}} = \Big\{ \sum_i f_i^0 \otimes x_j^0 + \epsilon f_i^1 \otimes x_i^1 \big| \; f_i^j \in O_S(S_0), x_i^j \in V, \\ |f_i^j| = |x_i^j|, j_0^* \left(\sum_i f_i^j x_i^j \right) (S_0) \subset V_0 \Big\}$$

and $(\mathrm{d}f(x))v = \sum_i \partial_{x_i} f(x) v_i$.

Lemma 2.77. *The tangent functor* T *is uniquely determined by the properties in 2.76.*

Proof of 2.77. This follows from super manifolds being locally isomorphic to $\mathbb{R}^{p|q}$, the local expansion of super functions and the gluing Lemmas. □

2.5 Lie super groups

We start with a categorical definition, a more down to earth point of view not involving sheaves is given in Section 2.5.1. For the general linear group the picture is very concrete from both points of view, as explained in Section 2.5.2.

Definition 2.78. Lie super groups (LSG) *are group objects in* SMan. *The definition makes sense because finite products exists according to Lemma 2.66.*

Remark 2.79. *For a super Lie group* $(G_0, O_G) \in \mathrm{LSG}$*,* G_0 *is an ordinary Lie group.*

2.5.1 Super group pairs

The following two definitions assemble the category of super group pairs.

Definition 2.80. *A cs* super group pair $(G_0, \mathfrak{g})$*, is given by a classical real Lie group* G_0 *and a complex Lie super algebra* $\mathfrak{g}$ *together with a linear (even) action by Lie super algebra automorphisms, i.e. an ordinary group homomorphism* $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{\mathrm{LSAlg}}(\mathfrak{g})$*. We demand that the Lie algebra of* G_0 *is a real form of* $\mathfrak{g}_0 = \mathbb{C} \otimes \mathfrak{g}_{0,\mathbb{R}}$*, i.e.* $\mathfrak{g}_{0,\mathbb{R}} = \mathrm{Lie}(G_0)$*,* Ad *extends the adjoint action of* G_0 *on* $\mathfrak{g}_0$ *and* $\mathrm{ad} = \mathrm{d}\,\mathrm{Ad}$ *is given by the super Lie bracket, i.e. extends the action of* $\mathfrak{g}_0$ *on* $\mathfrak{g}$.

Definition 2.81. *A morphism of super group pairs is a pair*

$$(\phi_0, \mathrm{d}\phi) : (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$$

such that $\phi_0 : G_0 \to H_0$ *is a morphism of ordinary Lie groups and* $\mathrm{d}\phi$ *extends* $\mathrm{d}\phi_0 : \mathfrak{g}_0 \to \mathfrak{h}_0$ *to a Lie super algebra morphism. Further* $\mathrm{d}\phi$ *has to intertwine the* Ad *actions, i.e.*

$$\forall g \in G_0\ \forall x \in \mathfrak{g} :\ \mathrm{d}\phi(\mathrm{Ad}_G(g)x) = \mathrm{Ad}_H(\phi(g))\mathrm{d}\phi(x)$$

Definition 2.82. $(H_0, \mathfrak{h})$ *is called* subgroup pair *of* $(G_0, \mathfrak{g})$ *if* H_0 *is a Lie subgroup of* G_0*,* $\mathfrak{h}$ *is a sub Lie super algebra of* $\mathfrak{g}$ *and* $\forall h \in H_0 : \mathrm{Ad}_{G_0}(h)\big|_{\mathfrak{h}} = \mathrm{Ad}_{H_0}(h)$*. As usual, a homomorphism of super group pairs which is an isomorphism onto its image with the later being a subgroup pair is called an* embedding.

Lemma 2.83. *The categories of super group pairs and Lie super groups are equivalent. Hence we can, modulo isomorphy, identify a Lie super group with its super group pair and vice versa.*

We will not give a full prove of the equivalence of categories. But never the less we state below how to obtain the group object associated to a pair. For more details see [ABG+10], section 4.4.

Some important ideas from the proof of 2.83. How to obtain a group pair from a Lie super group amounts to computing the Lie super algebra of the group. This can be done in general as mentioned in Remark 2.87 and is explicitly carried out for the general linear super group in 2.86.

Going the other way involves a little more effort. With a super group pair $G_p = (G_0, \mathfrak{g})$ we can associate the Lie super group $G = C(G_p) = (G_0, O_G)$ where

$$\begin{aligned} O_G(U) &= \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}\left(\mathfrak{U}(\mathfrak{g}), O_{G_0}(U)\right) \\ &= \{f : \mathfrak{U}(\mathfrak{g}) \to O_{G_0}(U) \big| \forall x_0 \in \mathfrak{g}_0 \forall y \in \mathfrak{U}(\mathfrak{g}) : f(x_0 y) = \mathcal{L}_{x_0} f(y)\} \end{aligned}$$

where $\mathfrak{U}$ denotes the universal enveloping algebra. I.e. $\mathfrak{g}_0$ acts on O_{G_0} by left invariant vector fields, $\forall g \in U : f(x_0 y)(g) = \partial_t\big|_0 f(y)(g e^{t x_0})$.[4] The algebra structure is defined by

$$(f \cdot h)(y) := f(y)h(1) + f(1)h(y)$$

for $y \in \mathfrak{g}$ and $f, g \in O_G(U)$. This Lie super algebra morphism

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \\ y &\mapsto y \otimes 1 + 1 \otimes y \end{aligned}$$

is uniquely extended to a morphism of unital super algebras to define $f \cdot h$ on all of $\mathfrak{U}(\mathfrak{g})$. The sheaf structure of O_G is inherited from the sheaf of ordinary smooth functions, O_{G_0}, on the manifold G_0, i.e.

$$\left(f\big|_V\right)(g) := \left(f(g)\right)\big|_V$$

which turns O_G into a sheaf as well.

Choosing an atlas $A = \{(U_\alpha, (\phi_\alpha)_0)\}$ of G_0 we obtain one of $C(G_p)$ by completing ϕ_α to homomorphisms of SRSp, $\phi_\alpha : G\big|_U \hookrightarrow \mathbb{R}^{p|q}$. To this end we use Lemma 2.29 and consider

$$O_{\mathbb{R}^{p|q}}((\phi_\alpha)_0(U)) = \mathrm{C}^\infty_{((\phi_\alpha)_0(U)} \otimes \bigwedge \left(\mathbb{R}^{0|q}\right)^* \simeq O_{G_0}(U) \otimes \bigwedge \mathfrak{g}_1^*$$

(by definition, see 2.40) to write

$$\phi_\alpha^* : O_{G_0}(U) \otimes \bigwedge \mathfrak{g}_1^* \xrightarrow{\sim} O_G(U)$$

with

$$\begin{aligned} &\forall f \in O_{G_0}(U)\ \forall x \in \bigwedge \mathfrak{g}_1^*\ \forall u_0 \in \mathfrak{g}_0\ \forall u_1 \in \mathfrak{g}_1\ \forall g \in U : \\ &\Big(\big(\phi_\alpha^*\left(f \otimes x\right)\big)(u_0 \beta(u_1))\Big)(g) := \left(\mathcal{L}_{u_0} f\right)(g) \cdot x(u_1) \end{aligned}$$

[4] The rough idea here is that the coefficients in the local expansion are given by odd derivatives $g^\alpha = j_0^*(\partial_{\xi^\alpha} f) \in O_{G_0}$. Hence we need to know these for all directions in $\mathfrak{g}_1$ and hence for all $\xi^a \in (\mathfrak{g}_1) \subset \mathfrak{U}(\mathfrak{g})$. But since the ideal generated by $\mathfrak{g}_1$, $(\mathfrak{g}_1)$, might contain parts or even the whole of $\mathfrak{U}(\mathfrak{g}_0)$, depending on the Lie super algebra relations, we include all $y \in \mathfrak{U}(\mathfrak{g})$ in the definition of "$f(y) = \partial_y f$ " and demand that the even derivatives act as usual. This way of thinking is also useful for understanding the definition of the algebra structure which reassembles the Leibniz rule.

To see that ϕ_α^* is indeed a super algebra isomorphism it is easy to check that

$$\begin{aligned}(\phi_\alpha^*)^{-1} : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}\big(\mathfrak{U}(\mathfrak{g}), O_{G_0}(U)\big) &\to O_{G_0}(U) \otimes \bigwedge \mathfrak{g}_1^* \\ F &\mapsto \Big((g, u_1) \mapsto \big(F(\beta(u_1))\big)(g)\Big)\end{aligned}$$

is the inverse. So we have a (super) atlas and hence $C(G_p) \in \mathrm{SMan}$.

To see that $C(G_p)$ is indeed a *group* object we additionally need the structure morphisms. The multiplication morphism $m : G \times G \to G$ and inversion $i : G \to G$ are of course on the base given by group multiplication and inversion in G_0. For $f \in O_G(U); u, v, \in \mathfrak{U}(g); g, h \in G_0$ the super part $m^* : O_G \to O_G \otimes O_G$ is given by

$$((m^* f)(u \otimes v))\,(g, h) := \big(f(\mathrm{Ad}(h^{-1})(u)v)\big)\,(gh)$$

and $i^* : O_G \to O_G$ by

$$\big(i^*(f)(u)\big)(g) := \Big(f\big(\,\mathrm{Ad}(g)(S(u))\big)\Big)(g^{-1})$$

where $S : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ is the linear continuation of inversion in $\mathfrak{g}$ as introduced in 2.30.

Now to establish that these morphisms actually define a group structure, that the mappings from groups to pairs and back are actually functorial and yield an equivalence of categories, we refer to the literature, e.g. [ABG+10], Section 4.4.

□

2.5.2 The general linear super group

In this work we will be concerned with subgroups of the Lie super group of automorphisms of finite dimensional super vector spaces. Although $\underline{\mathrm{Aut}}(V)$ by definition is an ordinary group, the Lie super group structure needs some more explanation. Throughout this section let V be a real super vector space $V = V_0 \oplus V_1$ of dimension $p|q$.

Definition 2.84. *The* general linear super group pair *of $V \in \mathrm{SVS}$ is given by*

$$\mathrm{Gl}_0(V) := \mathrm{Aut}(V) = \mathrm{Gl}_0(V_0) \times \mathrm{Gl}_0(V_1)$$

where the $\mathrm{Gl}_0(V_i)$ factors are just the ordinary general linear groups, and the Lie super algebra $\mathfrak{gl}(V)$ was introduced in Example 2.27. The adjoint action is defined by considering $\mathrm{Gl}_0(V) = \underline{\mathrm{Aut}}_0(V) \subset \underline{\mathrm{End}}(V)$ and for $g \in \mathrm{Gl}_0(V)$ and $x \in \mathfrak{gl}(V)$

$$\mathrm{Ad}(g)x := g \circ x \circ g^{-1} \in \underline{\mathrm{End}}(V) = \mathfrak{gl}(V)$$

As usual this defines a Lie super algebra automorphism because

$$\mathrm{Ad}(g)(x \circ y) = \mathrm{Ad}(g)(x) \circ \mathrm{Ad}(g)(y)$$

is already an automorphism on the level of the super algebra $x, y \in \underline{\mathrm{End}}(V)$*, see Lemma 2.32. Further we have* $\mathfrak{gl}_0(V) = \mathrm{End}(V) = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1) = \mathfrak{gl}_0(V_0) \oplus \mathfrak{gl}_0(V_1) = \mathrm{Lie}(\mathrm{Gl}_0(V))$ *because* $\mathfrak{gl}_0(V_i)$ *is just the ordinary Lie algebra of* $\mathrm{Gl}_0(V_i)$*. Finally* $\mathrm{ad} = \mathrm{d}\,\mathrm{Ad}$ *extends the Lie bracket as in the ordinary case.*

$$\partial_t\big|_0 e^{tx} \circ y \circ e^{-tx} = x \circ y - y \circ x = [x, y]$$

for $x \in \mathfrak{gl}_0$*. So* $\mathrm{Gl}(V)_p := (\mathrm{Gl}_0(V), \mathfrak{gl}(V))$ *is a super group pair. Note that by construction it comes with a standard representation on* $\mathbb{C} \otimes V$*.*

Definition 2.85. *The super ringed space on which we define the* general linear super group *as a group object is an open subspace of the linear super manifold* $\underline{\mathrm{End}}(V)$*. Since* $\mathrm{Aut}(V) \subset \mathrm{End}(V) = (\underline{\mathrm{End}}(V))_0$ *is an open subset, we can define the super manifold* $\mathrm{Gl}(V) := \underline{\mathrm{End}}(V)\big|_{\mathrm{Aut}(V)}$*, i.e.*

$$\mathrm{Gl}(V) = \left(\mathrm{Gl}_0(V), \mathrm{C}^{\infty}_{\mathrm{Gl}_0(V)} \otimes \bigwedge (\underline{\mathrm{End}}_1(V))^*\right)$$

with $\mathrm{Gl}_0(V) = \mathrm{Aut}(V)$ *as above. Definition 2.89 establishes multiplication, inversion and unit by Yoneda's Lemma.*

Lemma 2.86. *The two definitions agree, i.e.* $\mathrm{Gl}(V) = C(\mathrm{Gl}_0(V), \mathfrak{gl}(V))$*.*

Proof of 2.86. Using Lemma 2.83 it suffices to calculate the Lie algebra of $\mathrm{Gl} := \mathrm{Gl}(V)$ together with the adjoint action and to show that this reproduces the pair.

Hence, in terms of Definition 2.76 and 2.71, let the Lie super algebra of $\mathrm{Gl}(V)$ be determined by $\mathfrak{g}_{\mathbb{R}} := (\mathrm{T}\,\mathrm{Gl})_{\mathbb{1}}$. By Definition 2.76 and 2.89 $\mathrm{T}\,\mathrm{Gl}$ comes with a multiplication which is on $S \in \mathrm{SMan}$ points given by

$$\begin{aligned}(\mathrm{T}\,\mathrm{Gl})(S) \times (\mathrm{T}\,\mathrm{Gl})(S) &\to (\mathrm{T}\,\mathrm{Gl})(S)\\ (g + \epsilon x, h + \epsilon y) &\mapsto gh + \epsilon(xh + gy)\end{aligned}$$

which yields the linear structure of $\mathfrak{g}_{\mathbb{R}}(S)$ by restriction

$$(1 + \epsilon x)(1 + \epsilon y) = 1 + \epsilon(x + y)$$

and further more a left and right action of Gl on T Gl. This can be composed to the adjoint action of $g \in_S \mathrm{Gl}$ on $(\mathrm{T}\,\mathrm{Gl})_{\mathbb{1}}(S)$

$$\mathrm{Ad}(g)(1 + \epsilon x) = g(1 + \epsilon x)g^{-1} = 1 + \epsilon g x g^{-1}$$

The Lie bracket is defined by $p_2 \circ \mathrm{T}\,\mathrm{Ad}$ where $p_2 : \mathrm{T}\,V \simeq V \times V \to V$ amounts to the usual identification of the tangent space of a super vector space V at a point with V itself, i.e. here $p_2 : \mathrm{T}\,\mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$. More explicitly for $(1 + \epsilon x), (1 + \delta y) \in (\mathrm{T}\,\mathrm{Gl})_{\mathbb{1}}$

$$\begin{aligned}(\mathrm{T}\,\mathrm{Ad})(1 + \epsilon x, 1 + \delta y) &= (1 + \epsilon x)(1 + \delta y)(1 + \epsilon x)^{-1}\\ &= 1 + \delta y + \delta\epsilon(xy - yx) \in (\mathrm{T}(\mathrm{T}\,\mathrm{Gl})_{\mathbb{1}})_y\end{aligned}$$

hence $[1+\epsilon x, 1+\delta y] = xy - yx$.

Now in the particular case of $S = \mathbb{R}^{1|1}$ points we have $\mathfrak{g}_{\mathbb{R}}(S) = (O_{\mathbb{R}^{1|1}}(\mathbb{R}) \otimes \mathfrak{g})_{0,\mathbb{R}} = \mathfrak{g}_{\mathbb{R}}$ due to 2.60 hence we can give the bracket more concretely. Denote the odd generator of $\mathbb{R}^{1|1}$ by ξ and let $x = 1 \otimes x_0 + \xi_x \otimes x_1$ and $y = 1 \otimes y_0 + \xi_y \otimes y_1 \in \mathfrak{g}(\mathbb{R}^{1|1})$. Then

$$\begin{aligned} xy - yx = 1 &\otimes (x_0 y_0 - y_0 x_0) + \xi_x \otimes (x_1 y_0 - y_0 x_1) + \xi_y \otimes (x_0 y_1 - y_1 x_0) \\ &+ \xi_y \xi_x \otimes (x_1 y_1 + y_1 x_1) \end{aligned}$$

So the bracket is indeed the super commutator and we get $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}$ as Lie super algebras and also the adjoint action of Gl_0 on $\mathfrak{gl}$ is the same as in 2.84.

$\square$

Remark 2.87. *An alternative definition of the Lie super algebra of any $G \in \mathrm{LSG}$ is given by*

$$\mathfrak{g} := \{x \in \underline{\mathrm{Hom}}(O_G(G_0), \mathbb{C}) \big| \; \forall v, w \in O_G(G_0) : x(vw) = x(v)e^*(w) + e^*(v)x(w)\}$$

Note that $e^*(v) = 0$ for odd v hence no sign appears.

This gisves rise to the notion of a left invariant vector field *associated to $x \in \mathfrak{g}$ via $\mathrm{id}_G \otimes x$ acting on the tensor product $O_G \otimes O_G \subset O_{G\times G}$ by*

$$\begin{aligned} \mathcal{L} : \mathfrak{g} &\to \mathrm{Der}(O_G(G_0)) \\ x &\mapsto \mathcal{L}_x := (\mathrm{id}_G^* \otimes x) \circ m^* \end{aligned}$$

Here all tensor products are meant to be graded, in particular

$$\mathrm{id}_G \otimes x(v \otimes w) = (-1)^{|v||x|} v x(w)$$

In fact one can extend $\mathcal{L}$ to all of $O_{G\times G}$ by continuous linear extension in the proper topology. Details can be found in [All10], Appendix B.

Definition 2.88. *Sub Lie super groups of* $\mathrm{Gl}(V)$ *will be called* matrix Lie super groups *because we can use Lemma 2.60 to rewrite*

$$\left(\underline{\mathrm{End}}(V)\right)(S) \simeq \left(\Gamma(O_S) \otimes \underline{\mathrm{End}}(V)\right)_{0,\mathbb{R}}$$

i.e. we can write $g \in_S \mathrm{Gl}(V) \subset_S \underline{\mathrm{End}}(V)$ as a matrix valued super function (or a matrix with coefficients in the super functions)

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

with $A \in \Gamma(O_S)_{0,\mathbb{R}} \otimes \mathrm{End}(V_0)$, $D \in \Gamma(O_S)_{0,\mathbb{R}} \otimes \mathrm{End}(V_1)$, $B \in \Gamma(O_S)_1 \otimes \underline{\mathrm{Hom}}(V_1, V_0)_1$, $C \in \Gamma(O_S)_1 \otimes \underline{\mathrm{Hom}}(V_0, V_1)_1$. Throughout we will indicate the $\mathbb{Z}_2$ grading of matrices in $\underline{\mathrm{End}}(V)$ by blocks as above.

Definition 2.89. *Similarly matrix Lie groups act on* $\underline{\mathrm{End}}(V)$ *and in particular on themselves by matrix multiplication. For* $g \in_S G \subset_S \underline{\mathrm{End}}(V)$ *and* $x \in_s \underline{\mathrm{End}}(V)$ *we have* $m(g,x) := g \circ x$ *where* $\circ$ *denotes concatenation in* $\underline{\mathrm{End}}(V)(S)$*.*

Definition 2.90. *The standard action of* $\mathrm{Gl}(V)$ *on* V*,* $\mathrm{Gl}(V) \times V \to V$*, is given for* $S \in \mathrm{SMan}$*,* $g \in_S \mathrm{Gl}(V)$ *and* $v \in_S V$ *by*

$$\pi(g)v := A(v_0) + B(v_1) \oplus C(v_0) + D(v_1) \in_S V$$

with the notation of Definition 2.88 above. Because $\underline{\mathrm{End}}(V)$ *is given by maps in between super vector spaces, this definition is natural in* S *and hence defines the action by Yoneda's lemma.*

2.6 Representations

Although the super vector spaces used to define super manifolds and groups are over the reals, we will in the following be interested on representations on complex super vector spaces.

Definition 2.91. *A* representation π of a Lie super group $G = (G_0, O_G)$ *on a finite dimensional super vector space* $V \in \mathrm{SVS}$ *is a morphism of group objects in* SMan*,* $\pi : G \to \mathrm{Gl}(V)$*. It extends linearly to a representation on* $\mathbb{C} \otimes V$*.*

For Lie super group pairs we will also want to consider infinite dimensional representations because the induced representations introduced in Section 3.4 are potentially infinite dimensional. Therefore we need to dwell a little bit on what it means for such a representation to be smooth. Note that, although we consider complex representations, we only demand smoothness, not analyticity. This follows the cs spirit of complexifying the structure sheaf without passing from smooth to analytic functions.

2.6.1 Some functional analysis

Definition 2.92. *A* locally convex topological *vector space* V *is a (possibly infinite dimensional) vector space together with a topology generated by a set of semi norms* $\{||.||_i : V \to \mathbb{R}\}_{i \in J}$ *by defining balls of radius* $\epsilon > 0$ *around* $v \in V$

$$B^i_\epsilon(v) := \{x \in V \big| ||x - v||_i < \epsilon\}$$

such that

$$\left\{ U_{I,\epsilon}(x) := \bigcap_{i \in I} B^i_\epsilon(x) \big| I \subset J, |I| < \infty, \epsilon > 0, x \in V \right\}$$

form a basis of the topology which we additionally require to be Hausdorff.

The category of locally convex vector spaces, LCVS, is the subcategory of vector spaces with objects defined above and linear continuous morphisms.

Remark 2.93. *This means a net $x : A \to V$ for a directed set A converges to $\bar{x}$*

$$\lim x_n = \bar{x} \Leftrightarrow \forall i \in I \,\forall \epsilon > 0 \,\exists N \in A \,\forall n \geq N : \ ||x_n - \bar{x}||_i < \epsilon$$

Lemma 2.94. *A linear mapping $f : V \to W$ in between locally convex vector spaces V and W with defining sets of semi norms $\{||.||_i^V : V \to \mathbb{R}\}_{i \in I}$ and $\{||.||_j^W : W \to \mathbb{R}\}_{j \in J}$ is* continuous *if and only if it is bounded, i.e.*

$$\forall j \in J \,\exists i_1(j), \ldots, i_{n_j}(j) \in I \,\exists c_j \in \mathbb{R} \,\forall v \in V : ||f(v)||_j^W \leq c_j \cdot \max_{k=1}^{n_j} ||v||_{i_k(j)}^V$$

Proof of 2.94. f is continuous if and only if $\lim f(x_n) = f(x)$ for any convergent net $x : A \to V$ with $\lim x_n = x$. Due to linearity of f it suffices to consider the case of $x = 0$. Then for a net converging to 0 the convergence of $f(x_n)$ is ensured by boundedness.

$$\forall j_1, \ldots, j_l \in J : \ \max_{k=1}^{l} ||f(x_n)||_{j_k}^W \leq \max_{k=1}^{l}(c_{j_k}) \cdot \max_{k=1}^{l} \max_{m=1}^{n_{j_k}} ||x_n||_{i_m(j_k)}^V$$

so using remark 2.93 we see that $||f(x_n)||$ is a zero sequence in W. On the other hand, if f is not bounded then

$$\exists j \in J \,\forall A \subset I \text{ finite } \forall c > 0 \,\exists v \in V \ : ||f(v)||_j > c \max_{a \in A} ||f(v)||_a$$

hence we can define a directed set $C = \mathbb{N} \times \{A \subset I \text{ finite}\}$ with

$$(n, A) \geq (m, B) :\Leftrightarrow n \geq m \text{ and } A \subset B$$

and a net $x : C \to V$ such that $\max_{a \in A} ||x_{(n,A)}||_a = \frac{1}{n}$ and

$$||f(x_{(n,A)})||_j > n \max_{a \in A} ||x_{(n,A)}||_a$$

Note that we can simultaneously fulfil both conditions due to linearity of f, i.e. we can rescale v without changing $||f(v)||/||v||$. Now $\lim x_n = 0$ but $||f(x_n)||_j > 1$ hence f does not converge to $f(0) = 0$, i.e. f is not continuous. □

Definition 2.95. *The* directional derivative *of $f : V \to W$ at $v \in V$ in the direction $x \in V$ is as usual defined as*

$$\partial_x\big|_v f := (\partial_x f)(v) := \partial_t f(v + tx) := \lim_{t \to 0} \frac{f(v + tx) - f(v)}{t}$$

whenever this exists. This defines the differential of f

$$\begin{aligned} \mathrm{d}f : V \times V &\to W \\ (x, v) &\mapsto \partial_x\big|_v f \end{aligned}$$

and we call f (continuously) differentiable if the map $\mathrm{d}f$ exists and is continuous.

This definition can be iterated as usual to define $f \in \mathrm{C}^k(V, W)$ if and only if $\mathrm{d}^k f$ exists and is continuous and f is called smooth if and only if $f \in \mathrm{C}^\infty(V, W)$.

Definition 2.96. *The category of* complex locally convex super vector spaces*,* LCSVS*, is a sub category of complex* LCVS *where the objects are additionally $\mathbb{Z}_2$ graded and the morphisms are additionally even.*

2.6.2 Representations of group pairs

Definition 2.97. *A smooth (linear)* representation *of a super group pair $G_p = (G_0, \mathfrak{g})$ on $V \in$ LCSVS is a pair $\pi = (\pi_0, \pi_\mathfrak{g})$ given by a representation of G_0, i.e. a morphism $\pi_0 : G_0 \to \mathrm{Gl}(V)_0$ of ordinary groups, and an even Lie super algebra morphism $\pi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{gl}(V) = \underline{\mathrm{End}}(V)$. Further we demand the following smoothness and compatibility conditions to hold.*

1. *The G_0 action, i.e.*

$$\begin{aligned} G_0 \times V &\to V \\ (g, v) &\mapsto \pi_0(g)v \end{aligned}$$

   is continuous.

2. *All vectors are smooth, i.e. $\forall v \in V$:*

$$\begin{aligned} G_0 &\to V \\ g &\mapsto \pi_0(g)v \end{aligned}$$

   is a smooth map.

3. *The $\mathfrak{g}$ action, i.e.*

$$\begin{aligned} \mathfrak{g} \times V &\to V \\ (x, v) &\mapsto \pi_\mathfrak{g}(x)v \end{aligned}$$

   is continuous.

4. *The super Lie algebra action extends the differential of the Lie group action, $\pi_\mathfrak{g}\big|_{g_{0,\mathbb{R}}} = \mathrm{d}\pi_0$, i.e.*

$$\forall x \in \mathfrak{g}_{0,\mathbb{R}} = \mathrm{Lie}(G_0) \forall v \in V : \partial_t\big|_0 \pi_0(e^{tx})v = \pi_\mathfrak{g}(x)v$$

5. *The actions are compatible with the adjoint action of the pair, i.e.*

$$\forall x \in \mathfrak{g}\ \forall g \in G_0 : \pi_\mathfrak{g}(\mathrm{Ad}(g)x) = \pi_0(g) \circ \pi_\mathfrak{g}(x) \circ \pi_0(g)^{-1}$$

Remark 2.98. *We can apply some results about ordinary representations on locally convex topological vector spaces here.*

- *By [Nee10], theorem 4.4, conditions 1 and 2 are equivalent to a smooth G_0 action.*
- *Since $(x, v) \mapsto \pi_\mathfrak{g}(x)v$ is bilinear, condition 3 is equivalent to a smooth $\mathfrak{g}$ action.*
- *By [Nee10], lemma 4.2, condition 1, 2 and 4 imply that $\pi_\mathfrak{g}\big|_{g_{0,\mathbb{R}}}$ is continuous, hence condition 3 is actually about the odd part $\mathfrak{g}_1$.*

Definition 2.99. Morphisms of G_p representations π *on* V *and* ρ *on* W *of a super group pair* G_p *are given by*

$$\begin{aligned}\mathrm{Hom}_{G_p}(V,W) := \{f \in \mathrm{Hom}_{\mathrm{LCSVS}}(V,W) \big| \forall g \in G_0 : f \circ \pi_0(g) = \rho_0(g) \circ f \text{ and} \\ \forall x \in \mathfrak{g} : f \circ \pi_{\mathfrak{g}}(x) = \rho_{\mathfrak{g}}(x) \circ f\}\end{aligned}$$

Definition 2.100. *The representation* $(\pi_0, \pi_{\mathfrak{g}})$ *of the pair* G_p *is called* irreducible *if* V *does not contain any non-trivial* $\pi_{\mathfrak{g}}(\mathfrak{g})$ *stable subspaces.*

Definition 2.101. *The* dual representation $\pi^* = (\pi_0^*, \pi_{\mathfrak{g}}^*)$ *on* V^* *to* $\pi = (\pi_0, \pi_{\mathfrak{g}})$ *on* V *for* finite dimensional V *is given for* $g \in G_0$, $x \in \mathfrak{g}$, $\mu \in V^*$ *and* $v \in V$ *by*

$$\begin{aligned}\pi_0^*(g)(\mu)v &:= \mu(\pi_0(g^{-1})v) \\ \pi_{\mathfrak{g}}^*(x)(\mu)v &:= \mu(\pi_g(S(x))v)\end{aligned}$$

Definition 2.102. *For a representation* π *of* G_p *on* $V \in \mathrm{LCSVS}$ *and* K_p *a subgroup pair we define the* K_p-invariant subspace *of* V *by*

$$V^{K_p} = \{v \in V \big| \; \pi_{\mathfrak{g}}(\mathfrak{k})v = 0 \text{ and } \pi_0(K)v = v\}$$

Lemma 2.103. *If in the definition above* V *is finite dimensional,* $G = C(G_p)$ *and* π *also denotes the corresponding representation of the Lie super group* G *on* V *then* $\forall S \in \mathrm{SMan}$

$$V^K(S) := \{v \in_S V \big| \; \forall k \in_S K : \pi(k)v = v\} = V^{K_p}(S)$$

and hence $V^K = V^{K_p}$. *We will also use the notation* $\pi(K)v = v$ *to say* $v \in V^K$.

Proof of 2.103. This follows from the equivalence of categories of Lie super groups and super group pairs. ☐

2.7 Integration

Here we introduce super integration as explained in [AHP11], Section 2, and [All10], Appendix C. Throughout we will use densities instead of volume forms to avoid questions of orientability.

2.7.1 The Berezinian sheaf

Definition 2.104. *For* $V \in \mathrm{SVS}$, $S \in \mathrm{SMan}$ *and* $g \in_S \mathrm{Gl}(V)$ *we use Definition 2.88 to define the* Berezinian *or* super determinant $\mathrm{Ber} : \mathrm{Gl}(V) \to \mathbb{R}^{1|1}$ *by*

$$\mathrm{SDet}(g) := \mathrm{Ber}(g) := \mathrm{Det}\left(A - BD^{-1}C\right)\mathrm{Det}(D^{-1}) = \mathrm{Det}(A)\,\mathrm{Det}\left(D - CA^{-1}B\right)^{-1}$$

where Det *is to be understood as a polynomial in the matrix coefficients. Further we define the* absolute Berezinian *by*

$$|\mathrm{Ber}|\,(g) := \mathrm{sign}\left(j_0^*\left(\mathrm{Det}(A)\right)\right)\mathrm{Ber}(g)$$

Definition 2.105. *For a super manifold* $X \in \mathrm{SMan}$ *of pure super dimension* $p|q$*, an atlas* $\{(U_i, \phi_i)\}$ *and the cotangent sheaf* Ω^1_X *as defined in 2.75 we define the* absolute Berezinian sheaf *of* X*,* $|\mathrm{Ber}|_X$*, as the free* O_X *module sheaf over* X_0 *of rank* $0|1$ *or* $1|0$ *if* q *is odd or even respectively by the cocycle* $\{U_{i,j}, |\mathrm{Ber}|\,(\mathrm{d}\phi_{i,j})\}$ *where*

$$\mathrm{d}\phi_{i,j} \in \mathrm{Gl}(\Omega^1_X)(U_{i,j}) := \mathrm{Gl}(\mathrm{span}(\{\mathrm{d}y_i, \mathrm{d}\chi_j\}))(X\big|_{U_{i,j}})$$

We denote the single generator with respect to the local coordinates $\{\mathrm{d}y_k, \mathrm{d}\chi_l\}$ *of* $\Omega^1_X(U)$ *by* $|\mathrm{D}(\mathrm{d}y_k, \mathrm{d}\chi_l)|$*.*

Definition 2.106. *An isomorphism* $f \in \mathrm{Iso}_{\mathrm{SMan}}(X, Y)$ *defines a sheaf morphism* $f^* : |\mathrm{Ber}|_Y \to (f_0)_* \,|\mathrm{Ber}|_X$ *in coordinates* (y_k, χ_l) *on* $U \subset Y_0$ *by*

$$f^*(h|\mathrm{D}(\mathrm{d}y_k, \mathrm{d}\chi_l)|) := f^*(h)|\mathrm{D}(\mathrm{d}f^*(y_k), \mathrm{d}f^*(\chi_l))|$$

for $h \in O_Y(U)$*.*

Hence for another coordinate system (z_i, ζ_j)

$$|\mathrm{D}(\mathrm{d}z_i, \mathrm{d}\zeta_j)| = |\mathrm{Ber}|\,(J)\,|\mathrm{D}(\mathrm{d}y_i, \mathrm{d}\chi_j)|$$

where

$$J = \left(\begin{array}{c|c} \partial_y z & \partial_y \zeta \\ \hline \partial_\chi z & \partial_\chi \zeta \end{array}\right)^{\mathrm{ST}}$$

and the operation of super transposition is given by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{\mathrm{ST}} = \left(\begin{array}{c|c} A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right)$$

Note however that $\mathrm{Ber}(J) = \mathrm{Ber}(J^{\mathrm{ST}})$*. In fact this is one of the reasons for the extra sign in the definition as compared to the ordinary case.*

2.7.2 Compact integration

Definition 2.107. *For* $X \in \mathrm{SMan}$ *of super dimension* $p|q$ *we define the integral of* $f \in O_X(U)$ *with compact support over a chart* (U, ϕ) *with local coordinates* (x_i, ξ_j) *by using the local expansion 2.45 to write down*

$$\int\limits_{X|_U} |\mathrm{D}(\mathrm{d}x_i, \mathrm{d}\xi_j)| f := (-1)^{s(p,q)} \int\limits_U j_0^*(f^{1,2,\ldots,q}) |\mathrm{d}\tilde{x}_1 \wedge \ldots \wedge \mathrm{d}\tilde{x}_p| \in \mathbb{C}$$

In other words Berezin integration means differentiation with respect to all odd coordinates followed by ordinary integration of the remaining function, up to a sign. For the sign one sometimes chooses $s(p,q) = pq$ *or, depending on the chosen ordering of the odd variables,* $s(p,q) = \frac{q(q-1)}{2}$*. Although we find* $s(p,q) = 0$ *more convenient there is no need to fix this sign for our purposes. In the following we abbreviate* $|\mathrm{D}x| := |\mathrm{d}\tilde{x}_1 \wedge \ldots \wedge \mathrm{d}\tilde{x}_p|$*.*

Remark 2.108. *The behaviour of* $|\mathrm{D}(\mathrm{d}x_i, \mathrm{d}\xi_j)|$ *under coordinate change is defined exactly in such a way as to make the integral over* U *independent of the choice of local coordinates. For the detailed proof see [Lei80], Theorem 2.4.5. Note that this is not true if we drop the assumption of* f *being compactly supported, then the value of the integral may change if* $\partial_\chi z \neq 0$*, i.e. if the even coordinates are shifted by nilpotents. See [AHP11] for the details.*

Definition 2.109. *For* $\omega \in |\mathrm{Ber}|_X (X_0)$ *compactly supported we choose an atlas* $\{(U_\alpha, \phi_\alpha)\}$ *of* X *and a subordinate partition of unity* ρ^α *in* O_X *to define*

$$\int\limits_X \omega := \sum_\alpha \int\limits_{X|_{U_\alpha}} \rho^\alpha \omega$$

the Berezin integral *of* ω.

2.7.3 Retractions and non-compact integration

Definition 2.110. *A homomorphism* $r \in \mathrm{Hom}_{\mathrm{SMan}}(M, M_0)$ *is called* retraction *if* $j_0 \circ r = \mathrm{id}_{M_0}$*. In this case* $r^*(O_{M_0}(M_0))$ *is called a* function factor.

Remark 2.111. *Note that using 2.50 and 2.54 only the* $(r^*(f))^\emptyset$ *component of the local expansion is determined by this definition, hence in general there exist many possible retractions and corresponding function factors. By [RS83], Lemma 3.2, there always exists at least one retraction on any super manifold. (Complexification of the structure sheaf as compared to the reference does not change this fact.)*

Definition 2.112. *Let* (U, ϕ_0) *be a chart of* M_0 *with local coordinates* $\{x_i\}$ *and* $r : M \to M_0$ *a retraction. Then* $\{r^*(x_i)\}$ *are called* even coordinates associated to $\{x_i\}$ and r. *Conversely if* $\{y_i, \xi_j\}$ *are local coordinates of* M *then by Corollary 2.63 there is a unique retraction* r *such that* $\forall i : y_i = r^*(j_0^*(y_i))$*. This one is called* retraction associated with the even coordinates $\{y_i\}$.

Lemma 2.113. *Let* $r : M \to M_0$ *be a retraction associated to both local coordinate systems* $\{x_i, \xi_j\}$ *and* $\{y_i, \chi_j\}$ *on* U *and let* $\omega \in |\mathrm{Ber}|_M (M_0)$ *with* $\omega|_U = |\mathrm{D}(\mathrm{d}x_i, \mathrm{d}\xi_j)|f = |\mathrm{D}(\mathrm{d}y_i, \mathrm{d}\chi_j)|g$ *then*

$$\tilde{f}^{1,2,\ldots,q}|\mathrm{D}x| = \tilde{g}^{1,2,\ldots,q}|\mathrm{D}y|$$

where each function is expanded with respect to its respective coordinate system.

Proof of 2.113. This holds because we can choose a bump function $h \in \mathrm{C}_c^\infty(U)$ and then use compact integration 2.107 to compute

$$\begin{aligned}(-1)^{s(p,q)} \int\limits_U h\tilde{f}^{1,2,\ldots,q}|\mathrm{D}x| &= \int\limits_{X|_U} |\mathrm{D}(\mathrm{d}x_i, \mathrm{d}\xi_j)| r^*(h) f = \int\limits_{X|_U} r^*(h)\omega \\ &= \int\limits_{X|_U} |\mathrm{D}(\mathrm{d}y_i, \mathrm{d}\chi_j)| r^*(h) g \\ &= (-1)^{s(p,q)} \int\limits_U h\tilde{g}^{1,2,\ldots,q}|\mathrm{D}y|\end{aligned}$$

where we used that compact integration is invariant under coordinate changes but also in the first and last step that x and y define the same retraction. The claim follows by the ordinary du Bois-Reymond Lemma (fundamental lemma of the calculus of variations). □

Remark 2.114. *Hence retractions locally define a choice of an integral of non-compactly supported densities similar to Definition 2.107 as well as a choice of even local coordinates would. The point is that retractions unlike even coordinates exist globally and hence can be used to define integration globally as we will do in the following.*

Definition 2.115. *Due to Lemma 2.113 we can use a retraction $r : M \to M_0$ to define a* fiber integration map $r_! : |\mathrm{Ber}|_M (M_0) \to |\Omega^p|_{M_0}(M_0)$*, where the latter denotes the sheaf of ordinary densities on M_0, locally on U with local even coordinates $\{x_i\}$ by*

$$(r_!(\omega))\big|_U := (-1)^{s(p,q)} \tilde{f}^{1,2,\ldots,q}|\mathrm{D}x|$$

where $\omega\big|_U = |\mathrm{D}(\mathrm{d}x_i, \mathrm{d}\xi_j)| f \in |\mathrm{Ber}|_M (U)$.

Remark 2.116. *By definition and Lemma 2.53 we have $\mathrm{supp}(r_! * \omega) \subset \mathrm{supp}(\omega)$ and $r_!(r^*(g)\omega) = g r_!(\omega)$ for $g \in \mathrm{C}^\infty(M_0)$ and $\omega \in |\mathrm{Ber}|_M (M_0)$.*

Definition 2.117. *We call $\omega \in |\mathrm{Ber}|(M)$* integrable with respect to the retraction *$r : M \to M_0$ if $r_!(\omega)$ is an integrable ordinary density on M_0 and in this case we define the* integral of ω with respect to r *by*

$$\int\limits_{M,r} \omega := \int\limits_{M_0} r_!(\omega)$$

Definition 2.118. *On a Lie super group G the* canonical retraction $r_G : G \to G_0$ *is defined by*

$$\forall x \in \mathfrak{g}_1 : \mathcal{L}_x \circ r_G^* \equiv 0$$

By the integral over G we mean the integral with respect to the canonical retraction.

Remark 2.119. *This retraction is unique because using*

$$O_G(U) = \{f : \mathfrak{U}(\mathfrak{g}) \to O_{G_0}(U)\big|\, \forall x_0 \in \mathfrak{g}_0 \forall y \in \mathfrak{U}(\mathfrak{g}) : f(x_0 y) = \mathcal{L}_{x_0} f(y)\}$$

we see that

$$\{f \in O_G(U)\big|\, \forall x \in \mathfrak{g}_1 : \mathcal{L}_x f \equiv 0\} \simeq O_{G_0}(U)$$

2.7.4 Quotients and unimodularity

Definition 2.120. *Let $G \in$ LSG and $H \subset G$ a closed subgroup (i.e. we have $j : H \to G$ such that $j_0 : H_0 \subset G_0$ a closed embedding and j^* an epimorphism) then $G/H = (G_0/H_0, O_{G/H})$ exists as explained in [AH09] and*

$$O_{G/H}(U) = \left\{ f \in O_G(\pi_0^{-1}(U) \middle|\, m^* f = p_1^* f \in O_{G\times H}(\pi_0^{-1}(U) \times H_0) \right\}$$

Definition 2.121. *G/H is called* analytically unimodular *if there exists a non-zero G bi-invariant section of $|\mathrm{Ber}|_{G/H}$. G is called analytically unimodular if $G \times G/G$ is.*

Lemma 2.122. *If G and H are analytically unimodular, so is G/H. If $\mathfrak{g}$ is strongly reductive (see Definition 3.3) or nilpotent and G_0 is connected then G is analytically unimodular.*

Proof of 2.122. See Proposition C.10 in Appendix C of [All10]. □

Lemma 2.123. $\mathrm{Gl}(\mathbb{R}^{p|q})$ *is analytically unimodular for all p, q.*

Proof of 2.123. For $p \neq q$ this follows from the previous lemma. But one can show directly that the left invariant section of $|\mathrm{Ber}|_{\mathrm{Gl}}$ is in fact bi-invariant for any p, q. To this end let $S \in$ SMan and $g \in_S \mathrm{Gl}$ and denote the left and right action of Gl on $\mathfrak{gl}$ as introduced in the proof of Lemma 2.86 by L and R. Then $\mathrm{Ad}(g) = L(g) \circ R(g)^{-1}$ and $|\mathrm{Ber}|\,(\mathrm{Ad}(g)) = |\mathrm{Ber}|\,(L(g))\,|\mathrm{Ber}|\,(R(g))^{-1}$. But on S points $L(g)$ is just matrix multiplication, i.e. $L(g)$ acts on $\mathbb{R}^{(p|q)\times(p|q)} := \left(\mathbb{R}^{p|q}\right)^{\otimes p} \otimes \left(\mathbb{R}^{0|1} \otimes \left(\mathbb{R}^{p|q}\right)^{\otimes q}\right)$. Hence $|\mathrm{Ber}|\,(L(g)) = |\mathrm{Ber}|\,(g)^{p-q} = |\mathrm{Ber}|\,(R(g))$ and $|\mathrm{Ber}|\,(\mathrm{Ad}(g)) = 1$, also for $p = q$. □

3 Representation Theory

In this section we recall some facts about the representation theory of strongly reductive Lie super algebras and about the Iwasawa decomposition of super Lie groups. Then we prove Frobenius reciprocity for induced representations of super group pairs. Section 3.5 is devoted to our main theorem, a generalisation of the Cartan-Helgason Theorem, [Hel84, Sch84]. More concretely, we prove a necessary and for $s = 1$ and $r > q$ also sufficient condition for being spherical for highest weight representations of $(\mathfrak{gl}^{q|r+s}, \mathfrak{gl}^{q|r} \oplus \mathfrak{gl}^{0|s})$. From here we establish in Corollary 3.63 and 3.64 a classification of all finite dimensional irreducible spherical highest weight representations in terms of their highest weights, similar to the classical case.

3.1 Iwasawa decomposition

3.1.1 Restricted roots

In this section, we give the necessary definitions and state some facts about the Iwasawa and restricted root space decomposition on the Lie algebra level. We will not give any proofs. Those can be found in [All10], Section 2, which we follow closely.

Definitions

Definition 3.1. *A Lie super algebra $\mathfrak{g} \in$ LSAlg together with θ an involutive automorphism which defines the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the ± 1 eigenspaces of θ is called* symmetric super pair *and denoted by $(\mathfrak{g}, \mathfrak{k})$ when θ is understood.*

Definition 3.2. *A Lie super algebra $\mathfrak{g}$ is called* reductive *if $\mathfrak{g}_0$ is reductive in $\mathfrak{g}$, the centre is even, $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}_0$, and there exists a non-degenerate, even, $\mathfrak{g}$-invariant super symmetric bilinear form b on $\mathfrak{g}$. A symmetric super pair $(\mathfrak{g}, \mathfrak{k})$ is called reductive if $\mathfrak{g}$ is reductive and b is additionally θ-invariant.*

Definition 3.3. *A reductive Lie super algebra or symmetric super pair is called* strongly reductive *if $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is the direct sum of b-non-degenerate simple graded ideals.*

Lemma 3.4. *If $\mathfrak{g}$ is strongly reductive then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$.*

Definition 3.5. *For a symmetric super pair $(\mathfrak{g}, \mathfrak{k})$ we call $\mathfrak{a} \subset \mathfrak{g}$ an* even Cartan subspace *of $\mathfrak{p}$ if $\mathfrak{a} \subset \mathfrak{p}_0$, $\mathfrak{a}$ is maximally commutative in $\mathfrak{p}$, i.e. $\mathfrak{a} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a})$, and $\mathrm{ad}(\mathfrak{a})\big|_{\mathfrak{g}_0}$ consists of semi simple endomorphisms. If such an even Cartan subspace exists we call $(\mathfrak{g}, \mathfrak{k})$* of even type.

Definition 3.6. *A* cs form *of* $(\mathfrak{g},\mathfrak{k})$ *of a reductive symmetric super pair is a* θ*-invariant real form* $\mathfrak{g}_{0,\mathbb{R}}$ *of* $\mathfrak{g}_0$ *which is* b*-non-degenerate. We will write* $X_{0,\mathbb{R}} := X \cap \mathfrak{g}_{0,\mathbb{R}}$ *for subspaces* $X \subset \mathfrak{g}$.

Definition 3.7. *An ordinary real Lie algebra* $\mathfrak{g}_0$ *is called* ρ-compact *for an ordinary representation* ρ *on a vector space* V_0 *if* $\rho(\mathfrak{g}_0)$ *is the Lie algebra of a compact analytic subgroup of* $\mathrm{Gl}(V_0)$.

Lemma 3.8. *An ordinary real Lie algebra* $\mathfrak{g}_0$ *is* $\mathrm{ad}_{\mathfrak{g}_0}$*-compact if and only if* $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ *consists of semi simple elements with imaginary spectra.*

Definition 3.9. *The real form* $\mathfrak{g}_{0,\mathbb{R}}$ *of a symmetric super pair will be called* non-compact *if* $\mathfrak{k}_{0,\mathbb{R}} \oplus i\,\mathfrak{p}_{0,\mathbb{R}}$ *is* $\mathrm{ad}_{\mathfrak{g}}$*-compact.*

Lemma 3.10. *For any strongly reductive symmetric super pair* $(\mathfrak{g},\mathfrak{k})$ *there exists a non-compact cs form. If* $(\mathfrak{g},\mathfrak{k})$ *is of even type there exists a real even Cartan subspace, i.e.* $\mathfrak{a}_{0,\mathbb{R}} \otimes \mathbb{C} = \mathfrak{a}$.

Decompositions

From now on let $(\mathfrak{g},\mathfrak{k})$ be a reductive symmetric super pair of even type with an even Cartan subspace $\mathfrak{a}$. Let $\mathfrak{m} := \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ denote the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$.

Definition 3.11. *For* $\alpha \in \mathfrak{a}^* \setminus \{0\}$ *we call*

$$\mathfrak{g}^{\alpha} := \{x \in \mathfrak{g} \big|\, \forall h \in \mathfrak{a} : [h,x] = \alpha(h)x\}$$

a restricted root space *and* α *a* restricted root *of* $\mathfrak{g}$ *with respect to* $\mathfrak{a}$ *if* $\mathfrak{g}^{\alpha} \neq 0$. *We denote the set of restricted roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{a}$ *by* $\Sigma(\mathfrak{g}:\mathfrak{a})$ *or just* Σ *if* $\mathfrak{g}$ *and* $\mathfrak{a}$ *are understood. If* $\mathfrak{g}^{\alpha} \cap \mathfrak{g}_{0/1} \neq 0$ *we will call the root* α *even/odd respectively. Note that* $\Sigma = \Sigma_0 \cup \Sigma_1$ *is the union of even and odd roots, but these are possibly not disjoint.*

Definition 3.12. *By assumption* $\mathfrak{g}$ *is a semi-simple* $\mathfrak{a}$ *module hence*

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g}:\mathfrak{a})} \mathfrak{g}^{\alpha}$$

which is called restricted root space decomposition *of* $\mathfrak{g}$ *with respect to* $\mathfrak{a}$.

Definition 3.13. *We call* $\Sigma^+ \subset \Sigma$ *a* positive system *if*

$$\Sigma = \Sigma^+ \dot{\cup} - \Sigma^+ \qquad \text{and} \qquad \left(\Sigma^+ + \Sigma^+\right) \cap \Sigma \subset \Sigma^+$$

We define by

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha} \qquad \bar{\mathfrak{n}} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^{-\alpha}$$

the nilpotent algebras with respect to Σ^+ *and* $\Sigma^- = -\Sigma^+$.

Definition 3.14. *A positive root $\alpha \in \Sigma^+$ is called* simple *if it can not be decomposed into a sum of positive roots. We denote the set of simple roots by*

$$\Pi = \{\alpha_1, \dots, \alpha_r\} := \Sigma^+ \setminus \left(\Sigma^+ + \Sigma^+\right)$$

Lemma 3.15. *We have*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

which we call Iwasawa decomposition *of $\mathfrak{g}$ with respect to θ, $\mathfrak{a}$ and Σ^+.*

3.1.2 Global Iwasawa decomposition

Definition 3.16. *We call a super group pair $(G_0, \mathfrak{g})$ with connected G_0 together with an involutive automorphism ϑ of G_0 a* global cs form *of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ if $\mathfrak{g}_{0,\mathbb{R}} = \mathrm{Lie}(G_0)$ the ordinary real Lie algebra of G_0 is a real form of $(\mathfrak{g}, \mathfrak{k})$ and $\theta\big|_{\mathfrak{g}_{0,\mathbb{R}}} = \mathrm{d}\vartheta$ such that*

$$\forall g \in G_0 : \mathrm{Ad}(\vartheta(g)) \circ \theta = \theta \circ \mathrm{Ad}(g)$$

A global cs form is called non-compact *if $\mathfrak{g}_{0,\mathbb{R}}$ is non-compact and $\mathrm{Ad}(K_0) \subset \mathrm{Aut}_{LSAlg}(\mathfrak{g})$ is compact for the analytic subgroup $K_0 \subset G_0$ with Lie algebra $\mathfrak{k}_{0,\mathbb{R}}$.*

Lemma 3.17. *For a strongly reductive symmetric super pair $(\mathfrak{g}, \mathfrak{k})$ of even type with real even Cartan subspace $\mathfrak{a}_{0,\mathbb{R}}$ and Σ^+ a positive system, there exists a so called* standard global cs form $(G_0, \mathfrak{g})$. *Using the analytic subgroups K_0, A_0 and N_0 of G_0 with Lie algebras $\mathfrak{k}_{0,\mathbb{R}}$, $\mathfrak{a}_{0,\mathbb{R}}$ and $\mathfrak{n}_{0,\mathbb{R}}$, respectively, we define the Lie super subgroups $K := C(K_0, \mathfrak{k})$, $A = (A_0, \mathrm{C}^\infty(A_0))$ and $N = C(N_0, \mathfrak{n})$ of $G = C(G_0, \mathfrak{g})$. Then the multiplication morphism of G yields an isomorphism of super manifolds*

$$m \circ (m \times \mathrm{id}_G) : K \times A \times N \xrightarrow{\sim} G$$

Lemma 3.18. *More generally we have the Iwasawa Isomorphism $G \simeq K \times A \times N$ for any global cs form G of a reductive symmetric super pair $(\mathfrak{g}, \mathfrak{k})$ of even type whenever the ordinary Iwasawa decomposition of the base $G_0 \simeq K_0 \times A_0 \times N_0$ exists.*

Remark 3.19. *Lemma 3.17 ensures the existence of a global Iwasawa decomposition of $\mathfrak{gl}^{p+q|r+s}$ for $p+q \neq r+s$ which is strongly reductive. But also for $p+q = r+s > 1$ we have a global decomposition because each factor of $G_0 = U(p,q) \times U(r,s)$ has an ordinary Iwasawa decomposition (see [Hel84]). Note that in the particular case of $\mathfrak{gl}^{q|r+s}$ the first factor is contained in K and only the second one needs to be decomposed.*

3.2 Irreducible representations of reductive Lie super algebras

In this section we recapitulate some basics of the representation theory of ordinary Lie algebras and facts from [Kac78] about Lie super algebras. Although Kac considers only basic classical Lie super algebras, we can apply his results also to strongly reductive ones due to Lemma 3.4. More concretely, we are interested in $\mathfrak{gl}^{q|r+s}$ with the bilinear form b given by the super trace form $b(x, y) = \mathrm{STr}(xy)$ and since for $q \neq r+s$ we have $\mathfrak{gl}^{q|r+s} = \mathrm{span}\{\mathbb{1}\} \oplus \mathfrak{sl}^{q|r+s}$, where $\mathfrak{sl}$ is the subspace of $\mathrm{STr} \equiv 0$ matrices, we essentially need what is known about the so called A series from [Kac78]. But all statements below also hold for $q = r + s \neq 1$, although here $\mathfrak{z}_{\mathfrak{g}} = \mathrm{span}(\mathbb{1}) \subset \mathfrak{g}' = \mathfrak{sl}^{q|r+s}$ and hence b becomes degenerate upon restriction to $\mathfrak{g}'$, which is why one better stays with $\mathfrak{gl}$ or, as in [Kac78], proceed to $\mathfrak{sl}/\mathfrak{z}_{\mathfrak{g}}$. We prefer the former. Hence let $\mathfrak{g}$ in the following be a strongly reductive Lie super algebra or $\mathfrak{g} = \mathfrak{gl}^{q|q}$ with $q \neq 1$ which is reductive but not strongly reductive (see Lemma 3.46).

3.2.1 Root space decomposition

Definition 3.20. *In Section 3.1.1 we introduced the notion of the restricted roots of a symmetric pair. If we perform this for the particular pair $(\mathfrak{g}, 0)$, we drop the attribute 'restricted' and denote the Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h} \subset \mathfrak{g}_0$ and the set of* roots of $\mathfrak{g}$ *by $\Delta = \Sigma(\mathfrak{g} : \mathfrak{h})$. By definition we now have $\mathfrak{m} = 0$ and we call the decomposition from Definition 3.12 in this case* root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$$

Note that $\mathfrak{z}_{\mathfrak{g}} \subset \mathfrak{h}$.

Lemma 3.21. *For a root space decomposition of $\mathfrak{g}$ we have $\forall \alpha, \beta \in \Delta$,*

- $\dim \mathfrak{g}^{\alpha} = 1$
- $\Delta_0 \cap \Delta_1 = \emptyset$
- $[g_{\alpha}, g_{\beta}] \subset g_{\alpha+\beta}$
- $\alpha \notin \{\pm\beta\} \Rightarrow b(\mathfrak{g}^{\alpha}, g^{\beta}) = 0$
- $\mathbb{C}\alpha \cap \Delta \subset \{\pm 2\alpha, \pm\alpha\}$
- $\pm 2\alpha \in \Delta \Leftrightarrow \alpha \in \Delta_1$ *and* $b(\alpha, \alpha) \neq 0$

Note that $2\Delta_1 \cap \Delta \subset \Delta_0$ hence for $\alpha \in \Delta_1$ we have $b(\alpha, \alpha) = 0$ if and only if $2\alpha \notin \Delta_0$.

Definition 3.22. *A positive root system is called* distinguished *if there is exactly one simple odd root $\Pi \cap \Delta_1 = \{a_o\}$. We can always choose such a distinguished positive root system.*

Lemma 3.23. *We can for each simple root α_i choose*

$$e_i \in \mathfrak{g}^{\alpha_i} \qquad f_i \in \mathfrak{g}^{-\alpha_i} \qquad h_i \in \mathfrak{h} \setminus \{0\}$$

such that

$$[e_i, f_j] = \delta_{i,j} h_i \ , \quad [h_i, h_j] = 0 \ , \quad [h_i, e_j] = a_{i,j} e_j \ , \quad [h_i, f_j] = -a_{i,j} f_j$$

with $a_{i,j} \in \mathbb{Z}$, $a_{i,i} \in \{0, 2\}$ *and* $a_{i,i} = 0 \Rightarrow a_{i,i+k} = 1$ *for* $k = \min\{k | a_{i,i+k} \neq 0\}$. *Further* $\{e_i, f_i, h_i\}$ *generate* $\mathfrak{g}/\mathfrak{z}_\mathfrak{g}$ *and* $\{h_i\}$ *is a basis of* $\mathfrak{h}/\mathfrak{z}_\mathfrak{g}$ *upon projection. The correspondence between* α_i *and the* coroot h_i *is given by* $\forall h \in \mathfrak{h} : \alpha_i(h) = b(h_i, h)$, *in this sense* $h_i = \alpha_i^*$. *Note however that for isotropic roots* $\alpha_i(\alpha_i^*) = 0$.

Definition 3.24. *The matrix* $(a_{i,j})$ *is called* Cartan matrix *of* $\mathfrak{g}$ *with respect to* Δ^+. *It can be translated into a* Dynkin diagram *by drawing a white node for every even simple root* α_i *and a crossed or black node for each odd simple root that squares to* 0 *or* 2 *and then connecting each pair of nodes by* $|a_{i,j} a_{j,i}|$ *many lines. Note that* $a_{i,j} a_{j,i} = 0 \Rightarrow a_{i,j} = a_{j,i} = 0$ *and* $a_{i,i} = 2 \Rightarrow a_{i,k \neq i} \leq 0$.

Restricting roots

As we introduced root space decomposition as a special case of restricted root space decomposition one might wonder how the adjective is justified. The following Lemma explains how one can indeed obtain a restricted root space decomposition by restricting the roots of the full decomposition, provided the Cartan subalgebra was chosen accordingly.

Lemma 3.25. *For a symmetric super pair* $(\mathfrak{g}, \mathfrak{k})$ *of even type with even Cartan subspace* $\mathfrak{a} \subset \mathfrak{p}_0$ *one can always choose an even Cartan subspace of* $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}_0$, *such that* $\mathfrak{a} \subset \mathfrak{h}$ *and* $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{a}$ *where* $\mathfrak{h} \cap \mathfrak{k} \subset \mathfrak{m}$. *Then*

$$\Sigma(\mathfrak{g} : \mathfrak{a}) = \{\alpha\big|_\mathfrak{a} \ \Big| \ \alpha \in \Sigma(\mathfrak{g} : \mathfrak{h}) = \Delta\} \setminus \{0\}$$

and for $\beta \in \Sigma(\mathfrak{g} : \mathfrak{a})$

$$g^\beta = \bigoplus_{\alpha|_\mathfrak{a} = \beta} g^\alpha$$

Similarly $\mathfrak{m}$ *contains those root spaces whose roots restrict to* 0

$$\mathfrak{m} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \sum_{\alpha|_\mathfrak{a} = 0} g^\alpha$$

in particular $\mathfrak{z}_\mathfrak{g} \subset (\mathfrak{h} \cap \mathfrak{k}) \subset \mathfrak{m}$.

Definition 3.26. *Given a symmetric super pair we call a system of positive roots* Δ^+ *of* $\mathfrak{g}$ *for a root space decomposition as above* compatible with the pair *if* $\Delta^+\big|_\mathfrak{a} = \Sigma^+$ *is a positive system of restricted roots. In the following it will be important to choose such a compatible system.*

3.2.2 Highest weight representations

Let in this section $\mathfrak{g}$ be a strongly reductive Lie super algebra or $\mathfrak{gl}^{q|q}$ with $q \neq 1$, $\mathfrak{h} \subset \mathfrak{g}_0$ a Cartan subalgebra and Δ^+ a distinguished system of positive roots. We assume in the following that all appearing representations of $\mathfrak{g}$ are semi simple as representations of $\mathfrak{h}$. For $\mathfrak{g}$ strongly reductive this is equivalent to the action of the centre being semi simple.

Definition 3.27. *For a subalgebra $\mathfrak{q} \subset \mathfrak{g}$ and a representation $\pi_\mathfrak{q}$ of $\mathfrak{q}$ on V we define the* induced Lie super algebra representation

$$\mathrm{Ind}_\mathfrak{q}^\mathfrak{g}(V) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} V = (\mathfrak{U}(\mathfrak{g}) \otimes V) / \sim$$

where $gh \otimes v \sim g \otimes \pi_\mathfrak{q}(h)v$ for $g \in \mathfrak{g}$, $h \in \mathfrak{q}$, $v \in V$. $p \in \mathfrak{g}$ acts on $\mathrm{Ind}_\mathfrak{q}^\mathfrak{g}(V)$ by

$$\rho(p)\,(g \otimes v) := (pg) \otimes v$$

Lemma 3.28. *If $\rho_\mathfrak{g}$ is an irreducible representation of $\mathfrak{g}$ on V then V is a factor module of $\mathrm{Ind}_\mathfrak{q}^\mathfrak{g}(V)$.*

Lemma 3.29. *If $\mathfrak{q}$ contains $\mathfrak{g}_0$ and we have $\{g_1, \ldots, g_t\} \subset \mathfrak{g}_1$ such that their projections into $\mathfrak{g}/\mathfrak{q}$ form a basis then the sum*

$$\mathrm{Ind}_\mathfrak{q}^\mathfrak{g}(V) = \bigoplus_{1 \leq i_1 < \ldots < i_s \leq t} g_{i_1} \cdots g_{i_s} V$$

is direct as a sum of vector spaces, in particular $\dim \mathrm{Ind}_\mathfrak{q}^\mathfrak{g}(V) = 2^t \dim V$.

Definition 3.30. *For $\lambda \in \mathfrak{h}^*$ let $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ act on the even super vector space $V^\lambda := \mathrm{span}\{v_\lambda\}$ by $h(v_\lambda) = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$ and $\mathfrak{n}(v_\lambda) = 0$. Then the $\mathfrak{g}$ representation $\mathrm{Ind}_\mathfrak{b}^\mathfrak{g} V^\lambda$ contains a unique maximal ideal I and*

$$V_\lambda := \mathrm{Ind}_\mathfrak{b}^\mathfrak{g} V^\lambda / I$$

is called the irreducible representation of highest weight λ.

Definition 3.31. *For $\mu \in \mathfrak{h}^*$ we call*

$$V^\mu = \{v \in V \big| \, \forall h \in \mathfrak{h} : \pi(h)(v) = \mu(v)v\}$$

weight space of weight μ *and μ a* weight of V *if $V^\mu \neq 0$.*

We denote the set of all weights of V by $P(V)$. Further we denote by $P(V) = P^+ \dot\cup P^- = P_+ \dot\cup P_- = P_0 \dot\cup P_1$ various subsets of the weights, namely $P^-(\mathfrak{a}) = 0$, $P^\pm = P(V) \cap (\mathfrak{h}^*)^\pm$ where $(\mathfrak{h}^*)^\pm$ is spanned by the positive / negative roots and $P_{0/1}$ are the even / odd weights, respectively. Intersections will be denoted by multiple indices, for example $P^+_{-,0}$ are the even positive weights that vanish on $\mathfrak{a}$. Note that the roots are the weights of the adjoint representation $\Delta(\mathfrak{g}) = P(\mathfrak{g})$.

Lemma 3.32. *For the highest weight representation V_λ we have*

- $\{v \in V_\lambda \big| \, \mathfrak{n}v = 0\} = V_\lambda^\lambda$ *and* $\dim(V_\lambda^\lambda) = 1$
- $V_\lambda \simeq V_{\lambda'} \Leftrightarrow \lambda = \lambda'$
- $V_\lambda = \bigoplus_{\mu \in P(V_\lambda)} V^\mu$ *and all* V^μ *are finite dimensional*

Further more, all finite dimensional representations of $\mathfrak{g}$ *on which the* $\mathfrak{h}$ *action is semi simple are highest weight representations. This is in particular the case for strongly reductive Lie super algebras if the action of the center is semi simple.*

Lemma 3.33. *For* $\mathfrak{g} = \mathfrak{gl}^{p|q}$ *we have that* V_λ *is finite dimensional if and only if for a distinguished positive system*

$$\forall i \neq o : \lambda(h_i) \in \mathbb{N}$$

with $\{h_i\}$ *as in Lemma 3.23. Note that the weight of the odd simple root,* $\lambda(h_o) \in \mathbb{C}$, *is arbitrary.*

Corollary 3.34. *If* V_λ *is finite dimensional then* $\lambda(\Delta) \subset \mathbb{Z} + \lambda(h_o)\mathbb{Z}$. *If* $\lambda(\Delta) \subset \mathbb{Z}$ *then* V_λ *is finite dimensional. Note however that it depends on the chosen positive system which weight of a given representation will be the highest.*

Lemma 3.35. *If* V_λ *is finite dimensional and* V_λ^0 *is the ordinary highest weight module with highest weight* λ *of* $\mathfrak{g}_0$ *considered as a* $\mathfrak{g}_0 \oplus \mathfrak{n}_1$ *module via* $\mathfrak{n}_1 V_\lambda^0 = 0$ *then*

$$\bar{V}_\lambda := \operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{n}_1}^{\mathfrak{g}} V_\lambda^0$$

is finite dimensional and has a unique maximal submodule I *and*

$$V_\lambda = \bar{V}_\lambda / I$$

Definition 3.36. *A finite dimensional highest weight module* V_λ *is called* typical *if* $V_\lambda = \bar{V}_\lambda$.

3.2.3 Weyl group and odd reflections

The following facts about the (even) Weyl group are well known from classical representation theory. The statements about odd reflections are in general for basic Lie super algebras proven in [CW] but in the simple case of $\mathfrak{gl}^{q|r+s}$ we can check them directly, also for $q = r + s$, with the details about the root system given in Section 3.3.

Definition 3.37. *For each even root* $\alpha \in \Delta_0 \subset \mathfrak{h}^*$ *the reflection at the hyperplane which is b-orthogonal to* α *is denoted by* $r_\alpha \in \mathrm{O}(\mathfrak{h}^*)$. *The finite group generated by these reflections is called the* Weyl group of $\mathfrak{g}_0$ *and denoted by* $W_0(\mathfrak{g})$ *or simply* W_0.

Lemma 3.38. *The Weyl group is generated by simple reflections, i.e. by* $\{r_\alpha \big| \, \alpha \in \Pi_0\}$.

Lemma 3.39. *The Weyl group stabilizes the even and odd weights of any finite dimensional representation* V *of* $\mathfrak{g}$, *i.e.* $W_0(P_0) = P_0$ *and* $W_0(P_1) = P_1$.

Definition 3.40. *For an odd isotropic positive root $\alpha \in \Delta_1^+$, $|\alpha|^2 := b(\alpha,\alpha) = 0$, we denote by r_α the change of positive root system from Δ^+ to $r_\alpha(\Delta^+) := \{-\alpha\} \cup \Delta^+ \setminus \{\alpha\}$. This is called* odd reflection *with respect to α.*

Lemma 3.41. *If Π is the set of simple roots for Δ^+ then*

$$r_\alpha(\Pi) := \{\beta \in \Pi \setminus \{\alpha\} \big| \ b(\beta,\alpha) = 0\} \cup \{\beta + \alpha \big| \ \beta \in \Pi \text{ and } b(\beta,\alpha) \neq 0\} \cup \{-\alpha\}$$

is the one for $r_\alpha(\Delta^+)$.

Remark 3.42. *Although r_α is called odd reflection and $r_{-\alpha} \circ r_\alpha = \mathrm{id}$, it can* not *be extended to a linear map in $\mathrm{O}(\mathfrak{h}^*)$ which sends Π to $r_\alpha(\Pi)$ and Δ^+ to $r_\alpha(\Delta^+)$.*

Remark 3.43. *For a distinguished positive root system $\Delta^+(\mathfrak{gl}^{p|q})$ the single odd simple root α_o is isotropic. Further all $\Delta'^+ \in W_0(\Delta^+)$ are distinguished but $r_\alpha(\Delta^+)$ is not for $(p,q) \neq (1,1)$.*

Lemma 3.44. *If π on V_λ is a highest weight representation with highest weight vector v_λ of weight λ and $\alpha \in \Pi$ then*

$$\lambda(h_\alpha) = 0 \Rightarrow \pi(f_\alpha)v_\lambda = 0$$

Proof of 3.44. For even roots this is a classical result due to the invariance of the roots under Weyl group reflections. If α is an isotropic odd simple root the statement follows from [CW], Lemma 1.36.

But one can in general also just compute directly

$$\forall \beta \in \Pi : \pi(e_\beta)\pi(f_\alpha)v_\lambda = \pi([e_\beta, f_\alpha])v_\lambda = \delta_{\alpha,\beta}\pi(h_\alpha)v_\lambda = \delta_{\alpha,\beta}\lambda(h_\alpha)v_\lambda = 0$$

and $f_\alpha V^\lambda \subset V^{\lambda-\alpha}$ hence by Lemma 3.32 we have $\pi(f_\alpha)v_\lambda \in V^\lambda \cap V^{\lambda-\alpha} = 0$. □

3.3 Concretion: $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$

As announced earlier, we will primarily focus on the case $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ in this thesis. This is sloppy notation for the symmetric super pair $(\mathfrak{g}, \mathfrak{k} = \mathfrak{gl}^{q+r} \oplus \mathfrak{gl}^{0|s})$, the concrete details of which will be spelled out in this section. We will use the notation introduced in Section 3.1.1 and 3.2.1.

Let throughout this section $q, r, s \in \mathbb{N}_0$, w.l.o.g. $r \geq s$, $V := \mathbb{C}^{q|r+s} = \mathbb{C}^q \oplus \mathbb{C}^r \oplus \mathbb{C}^s$ which is $\mathbb{Z}_2$ graded by considering $\mathbb{C}^q$ as even and $\mathbb{C}^r \oplus \mathbb{C}^s$ as odd. In order to give explicit matrix expressions for the constructions below, we choose the standard homogeneous basis for $\mathbb{C}^{q|r+s}$, denoted by $\{e_1, \ldots, e_{q+r+s}\}$, which yields the dual basis $\{e_1^*, \ldots, e_{q+r+s}^*\}$ and the corresponding basis $\{E_{i,j} = e_i \otimes e_j^*\}$ of $V \otimes V^* \simeq \underline{\mathrm{End}}(V)$. Note that $E_{i,j}^\dagger = E_{j,i}$ with respect to the standard hermitian scalar product which will be used to define a real form.

Definition 3.45. *The involution defining $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is*

$$\theta : \begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{g} \\ X & \mapsto & \sigma X \sigma^{-1} \end{array} \qquad \text{with } \sigma = \left(\begin{array}{c|cc} \mathbb{1}_q & 0 & 0 \\ \hline 0 & \mathbb{1}_r & 0 \\ 0 & 0 & -\mathbb{1}_s \end{array}\right)$$

Hence we have the following block decomposition

$$\mathfrak{g} = \left\{ \left(\begin{array}{c|cc} \mathfrak{k}_0 & \mathfrak{k}_1 & \mathfrak{p}_1 \\ \hline \mathfrak{k}_1 & \mathfrak{k}_0 & \mathfrak{p}_0 \\ \mathfrak{p}_1 & \mathfrak{p}_0 & \mathfrak{k}_0 \end{array}\right) \right\}$$

The $\underline{\mathrm{End}}(\mathbb{C}^q)$ *part in the depicted direct sum decomposition which is even in all respects, will be called* boson-boson block *and the* $\underline{\mathrm{End}}(\mathbb{C}^{r+s})$ *part, which is even with respect to the $\mathbb{Z}_2$ grading but contains even and odd parts with respect to θ, will be called* fermion-fermion block.

Lemma 3.46. *The super trace form, $b(X,Y) := \mathrm{STr}(XY)$, is a non-degenerate, even, super symmetric, $\mathfrak{g}$ and θ-invariant bilinear form on $\mathfrak{g}$. Further $(\mathfrak{g}, \mathfrak{k})$ together with b is a strongly reductive symmetric super pair if and only if $q \neq r+s$. If $q = r+s \neq 1$ then $(\mathfrak{g}, \mathfrak{k})$ is never the less reductive.*

Proof of 3.46. The facts about the super trace form follow from its definition, it is always non degenerate on $\mathfrak{gl}^{q|r+s}$ also for $q = r+s$.

We have $\mathfrak{g}' = \mathfrak{sl}^{q|r+s}$ which is b-non-degenerate and basic classical for $q \neq r+s$ and $\mathfrak{z}_{\mathfrak{g}} = \mathrm{span}(\mathbb{1}_{q+r+s}) \subset \mathfrak{g}_0$. For $q = r+s$ however $\mathfrak{z}_{\mathfrak{g}} \subset \mathfrak{g}'$ because

$$\left[\left(\begin{array}{c|c} 0 & \mathbb{1}_q \\ \hline \mathbb{1}_q & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & \mathbb{1}_q \\ \hline \mathbb{1}_q & 0 \end{array}\right) \right] = \mathbb{1}_{2q}$$

hence $\mathfrak{gl}^{q|q}$ is not strongly reductive by Lemma 3.4.

□

Lemma 3.47. *A non-compact cs form of $(\mathfrak{g}, \mathfrak{k})$ is given by*

$$\mathfrak{g}_{0,\mathbb{R}} = \left\{ \left(\begin{array}{c|cc} C & 0 & 0 \\ \hline 0 & B & A \\ 0 & D & E \end{array}\right) = \left(\begin{array}{c|cc} -C^\dagger & 0 & 0 \\ \hline 0 & -B^\dagger & D^\dagger \\ 0 & A^\dagger & -E^\dagger \end{array}\right) \right\}$$

Proof of 3.47. With this choice, θ-invariance comes by definition. The spectrum of $\mathrm{ad}(\mathfrak{g}_{0,\mathbb{R}})$ can be red of from the root space decomposition in the next section, see Corollary 3.54. □

Remark 3.48. *Note that $b\big|_{\mathfrak{gl}^{q|0}_{0,\mathbb{R}}}$ and $b\big|_{\mathfrak{p}_{0,\mathbb{R}}}$ are negative definite and $b\big|_{\mathfrak{gl}^{0|r}_{0,\mathbb{R}} \oplus \mathfrak{gl}^{0|s}_{0,\mathbb{R}}}$ is positive definite. Hence $\mathfrak{k}_{0,\mathbb{R}} = \mathfrak{gl}^{q|0}_{0,\mathbb{R}} \oplus \mathfrak{gl}^{0|r}_{0,\mathbb{R}} \oplus \mathfrak{gl}^{0|s}_{0,\mathbb{R}}$ is non-degenerate as an orthogonal sum of non degenerate subalgebras.*

Lemma 3.49. *We can choose a real even Cartan subspace subspace $\mathfrak{a}_{0,R} \subset \mathfrak{p}_{0,\mathbb{R}}$ using the following notation*

$$A: \begin{array}{ccl} \mathbb{C}^s & \to & \underline{\mathrm{Hom}}(\mathbb{C}^s, \mathbb{C}^r) \hookrightarrow \mathfrak{g} \\ a & \mapsto & A(a) = \sum_{i=1}^s a_i \, E_{i+q,i+r+q} \end{array} \qquad i.e.\ A(a) = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}(a) \\ 0 & 0 & 0 \end{array}\right)$$

to set $\mathfrak{a} = \{(A + A^\dagger)(i\mathbb{R}^s)\}$.[1]

Proof of 3.49. The computation of the root space decomposition below shows that $\mathfrak{a}$ is maximally commutative and $\mathrm{ad}(\mathfrak{a})$ is semisimple. $\square$

Lemma 3.50. *The following choice of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{h}$. We use the notation*

$$B : C^s \times C^{r-s} \to \underline{\mathrm{End}}(\mathbb{C}^{r+s}) \hookrightarrow \mathfrak{g}$$
$$(b^{(1)}, b^{(2)}) \mapsto \sum_{i=1}^{s} b_i^{(1)} (E_{i+q,i+q} + E_{i+q+r,i+q+r}) + \sum_{i=1}^{r-s} b_i^{(2)} E_{i+q+s,i+q+s}$$

and

$$C : C^q \to \underline{\mathrm{End}}(\mathbb{C}^q) \hookrightarrow \mathfrak{g}$$
$$c \mapsto \sum_{i=1}^{q} c_i E_{i,i}$$

to write down

$$\mathfrak{h}_{0,\mathbb{R}} = \Big\{ h(a,b,c) := A(a) + A^\dagger(a) + B(b) + C(c) \Big|\ a \in i\mathbb{R}^s, b \in i\mathbb{R}^r, c \in i\mathbb{R}^q \Big\}$$

which in matrix form looks like

$$h(a,b,c) = \left(\begin{array}{c|ccc} \mathrm{diag}(c) & 0 & 0 & 0 \\ \hline 0 & \mathrm{diag}(b^{(1)}) & 0 & \mathrm{diag}(a) \\ 0 & 0 & \mathrm{diag}(b^{(2)}) & 0 \\ 0 & -\mathrm{diag}(a) & 0 & \mathrm{diag}(b^{(1)}) \end{array}\right)$$

Remark 3.51. *Using the elementary super commutator*

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - (-1)^{|E_{i,j}||E_{k,l}|} \delta_{l,i} E_{k,j}$$

it is now a straight forward task to diagonalise $\mathrm{ad}\,\mathfrak{h}$. *Since $\mathfrak{h}$ is even we actually always use the ordinary commutator and hence obtain the known ordinary results adjusted to our choice of $\mathfrak{h}$.*

In the notation introduced in Definition 3.31 we will distinguish between roots which vanish on restriction to $\mathfrak{a}$, Δ_-, and those which have non vanishing overlap with $\mathfrak{a}^*$, denoted by Δ_+.

[1] Note that any $\mathfrak{a}' = \{(A + A^\dagger)(e^{i\phi}\mathbb{R}^s)\}$ would also be a possible choice, but $[\mathfrak{a}, \mathfrak{a}'] \neq 0$ unless $\mathfrak{a}' = \mathfrak{a}$ as the Pauli matrices σ_x and σ_y do not commute.

Even roots

Lemma 3.52. *The even roots containing an $\mathfrak{a}$ component, $\Delta_{+,0}$, and the corresponding root spaces are given by*

root $\alpha(h)$	$\mathfrak{g}^\alpha$ spanned by	for
$b^{(2)}_{i-q-s} - b^{(1)}_{j-q} \pm i a_{j-q}$	$E_{i,j} \pm i E_{i,j+r}$	$q+s < i \leq q+r$ $q < j \leq q+s$
$b^{(1)}_{i-q} - b^{(2)}_{j-q-s} \pm i a_{i-q}$	$E_{i,j} \pm i E_{i+r,j}$	$q < i \leq q+s$ $q+s < j \leq q+r$
$b^{(1)}_{i-q} - b^{(1)}_{j-q} \pm i\,(a_{i-q} - a_{j-q})$	$E_{i,j} \pm i\,(E_{i+r,j} - E_{i,j+r}) + E_{i+r,j+r}$	$q < i \neq j \leq q+s$
$b^{(1)}_{i-q} - b^{(1)}_{j-q} \pm i\,(a_{i-q} + a_{j-q})$	$E_{i,j} \pm i\,(E_{i+r,j} + E_{i,j+r}) - E_{i+r,j+r}$	$q < i, j \leq q+s$

Note that in particular $\pm 2 i a_i$ is a root for $s > 0$ (otherwise we have the standard root space decomposition of the ordinary $\mathfrak{gl}$ with respect to the standard $\mathfrak{h}$ of diagonal matrices).

The even roots not containing an $\mathfrak{a}$ component, $\Delta_{-,0}$, are given by

root $\alpha(h)$	$\mathfrak{g}^\alpha$ spanned by	for
$c_i - c_j$	$E_{i,j}$	$1 \leq i \neq j \leq q$
$b^{(2)}_{i-q-s} - b^{(2)}_{j-q-s}$	$E_{i,j}$	$q+s < i \neq j \leq q+r$

Odd roots

Lemma 3.53. *The odd roots containing an $\mathfrak{a}$ component, $\Delta_{+,1}$, are given by*

root $\alpha(h)$	$\mathfrak{g}^\alpha$ spanned by	for
$c_i - b^{(1)}_{j-q} \pm i a_{j-q}$	$E_{i,j} \pm i E_{i,j+r}$	$i \leq q$, $q < j \leq q+s$
$b^{(1)}_{i-q} - c_j \pm i a_{i-q}$	$E_{i,j} \pm i E_{i+r,j}$	$q < i \leq q+s$, $j \leq q$

The odd roots not containing an $\mathfrak{a}$ component, $\Delta_{-,1}$, are given by

root $\alpha(h)$	$\mathfrak{g}^\alpha$ spanned by	for
$c_i - b^{(2)}_{j-q-s}$	$E_{i,j}$	$i \leq q$, $q+s < j \leq q+r$
$b^{(2)}_{i-q-s} - c_j$	$E_{i,j}$	$q+s < i \leq q+r$, $j \leq q$

Corollary 3.54. *By inspection we see* $\mathrm{Spec}(\mathrm{ad}(\mathfrak{a}_{0,\mathbb{R}})) \subset \mathbb{R}$ *and* $\mathrm{Spec}(\mathrm{ad}(\mathfrak{h}_{0,\mathbb{R}} \cap \mathfrak{k})) \subset i\mathbb{R}$. *This proves Lemma 3.47 using Lemma 3.8.*

Corollary 3.55. *By Lemma 3.25 we obtain the restricted roots $\Sigma(\mathfrak{g} : \mathfrak{a})$ from the roots calculated above.*

$$\Sigma(\mathfrak{g} : \mathfrak{a}) = \{\pm i a_i, i(a_i - a_j), \pm i(a_i + a_j)\}$$

with the multiplicities $m_\alpha := \mathrm{SDim}(\mathfrak{g}^\alpha)$ given by

$$m_{\pm i a_i} = (2(r-s)|2q) \qquad m_{i(a_i - a_j)} = (2|0) \qquad m_{\pm i(a_i + a_j)} = (1|0)$$

and $\mathrm{SDim}(\mathfrak{m}) = (q(q-1)+(r-s)(r-s-1)|2q(r-s)) + \mathrm{SDim}(\mathfrak{h}\cap\mathfrak{m})$ *with* $\mathrm{SDim}(\mathfrak{h}\cap\mathfrak{m}) = (q+r|0)$ *where the latter includes the centre.*

Remark 3.56. *For $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ we have $\mathfrak{g}_0 = \mathfrak{g}_{FF} \oplus \mathfrak{g}_{BB}$, where we denote the 'fermion-fermion block' by $\mathfrak{g}_{FF} = \mathfrak{gl}^{q|0}$ and the 'boson-boson block' by $\mathfrak{g}_{BB} = \mathfrak{gl}^{0|r+s}$. Since the sum is a direct sum of ordinary Lie algebras and $\mathfrak{g}_0$ preserves the $\mathbb{Z}_2$ grading we immediately get for an finite dimensional $\mathfrak{g}$ module V*

$$V = \bigoplus_{s\in\mathbb{Z}_2} \bigoplus_{(\lambda,\mu)\in\mathfrak{t}_0^*} \mathbb{C}^{m_{(\lambda,\mu)}} \otimes V^s_\lambda \otimes V^s_\mu$$

where the sums are direct as sums of $\mathfrak{g}_0$ modules and $V^{0/1}_\lambda$ is the even/odd isotopic component with respect to $\mathfrak{g}_{BB}$ of highest weight λ and V^s_μ likewise for $\mathfrak{g}_{FF}$ and highest weight μ.

This is a nice starting point for investigating the structure of $\mathfrak{g}$ representations. For typical highest weight representations V_λ, Kac even has a formula for the multiplicities $m_{(\lambda,\mu)}$, see [Kac78] Proposition 2.11. Unfortunately it turns out that spherical representations of $\mathfrak{gl}^{q|r+s}$ are always atypical.

Note further that taking together Lemma 3.35 and 3.29 (Proposition 2.4 and 2.1 of [Kac78]) we also get a decomposition of $\bar{V}_\lambda$ as a vector space into homogeneous subspaces but this is not a decomposition into modules and in particular not the nice decomposition into representations of the even part above, even if $V_\lambda = \bar{V}_\lambda$ is typical.

3.4 Induced representations

Although we will later on start of from a finite dimensional representation space V, the induced representation will in general be infinite dimensional, hence in this section we will be general and consider representations in LCSVS as introduced in 2.97 right away.

Definition 3.57. *Let $H_p = (H_0, \mathfrak{h})$ be a subgroup pair of $G_p = (G_0, \mathfrak{g})$ and $\pi = (\pi_0, \pi_\mathfrak{g})$ a representation oh H_p on $W \in$* LCSVS. *From this data we can construct a G_p representation which is called* induced representation. *The underlying super vector* *space is given by*

$$\begin{aligned}
&\mathrm{Ind}_{H_p}^{G_p}(W) := \underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_0), H_p} \\
&= \{f : \mathfrak{U}(\mathfrak{g}) \to \mathrm{C}^\infty(G_0, W) \big| \forall x \in \mathfrak{U}(\mathfrak{g})\ \forall x_0 \in \mathfrak{g}_0\ \forall g \in G_0\ \forall y \in \mathfrak{h}\ \forall h \in H_0 : \\
&\quad (f(x_0 x))(g) = \mathcal{L}_{x_0}(f(x))(g) = \partial_t\big|_0 f(x)(g e^{t x_0}) \qquad (3.1a) \\
&\quad \wedge\ f(x)(g) = \pi_0(h)\Big((f(\mathrm{Ad}(h^{-1})x))(gh)\Big) \qquad (3.1b) \\
&\quad \wedge\ (f(xy))(g) = (-1)^{|y|(|x|+|f|)} \pi_{\mathfrak{h}}(S(y))\big(f(x)(g)\big)\} \qquad (3.1c)
\end{aligned}$$

The action of G_p is given by $\rho = (\rho_0, \rho_{\mathfrak{g}})$ where $\forall g, p \in G_0\ \forall u \in \mathfrak{U}(\mathfrak{g})\ \forall x \in \mathfrak{g}$:

$$\begin{aligned}
\big(\rho_0(g) f\big)(u)(p) &:= f(u)(g^{-1} p) \\
\big(\rho_{\mathfrak{g}}(x) f\big)(u)(p) &:= (-1)^{|f||x|} f\left(\mathrm{Ad}\left(p^{-1}\right)\big(S(x)\big)u\right)(p)
\end{aligned}$$

Lemma 3.58. *The induced representation is a representation of G_p.*

Proof of 3.58. Firts we show that the above action defines a representation of the group pair G_p on $f \in V := \underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_0)}$ without the H_p-invariance constraints. The homomorphism properties can easily be checked.

$$\begin{aligned}
&\forall g, h, p \in G_0\ \forall x, y, z \in \mathfrak{U}(\mathfrak{g})\ \forall \lambda \in \mathbb{C} : \\
&\big(\rho_0(gh) f\big)(z)(p) = f(z)((gh)^{-1} p) = \big(\rho_0(g)(\rho_0(h) f)\big)(z)(p) \\
&\big(\rho_0(\mathbb{1}) f\big)(z)(p) = f(z)(p) \\
&\big(\rho_{\mathfrak{g}}(xy) f\big)(z)(p) = (-1)^{|f|(|x|+|y|)} f\Big(\mathrm{Ad}(p^{-1})\big(S(xy)\big) z\Big)(p) \\
&\quad = (-1)^{|f|(|x|+|y|)} f\Big((-1)^{|x||y|} \mathrm{Ad}(p^{-1})\big(S(y)\big) \mathrm{Ad}(p^{-1})\big(S(x)\big) z\Big)(p) \\
&\quad = (-1)^{|f|(|x|+|y|)+|x||y|-(|f|+|y|)|x|} \big(\rho_{\mathfrak{g}}(x) f\big)(\mathrm{Ad}(p^{-1})\big(S(y)\big) z)(p) \\
&\quad = \big(\rho_{\mathfrak{g}}(x) \rho_{\mathfrak{g}}(y) f\big)(z)(p)
\end{aligned}$$

where we made use of 2.32 to extend $\rho_{\mathfrak{g}}$ to $\mathfrak{U}(\mathfrak{g})$.

Then we need to check that the $\mathfrak{U}(\mathfrak{g}_0)$-invariance as in (3.1a) is preserved. Therefore consider $f \in V$ and $x \in \mathfrak{U}(\mathfrak{g})$, $x_0 \in \mathfrak{g}_0$, $g \in G_0$, and $v \in \mathfrak{g}$ and compute

$$\begin{aligned}
&(-1)^{|f||v|} (\rho_{\mathfrak{g}}(v) f)(x_0 x))(g) \\
&\quad = f(\mathrm{Ad}(g^{-1})(S(v)) x_0 x)(g) \\
&\quad = -f([\mathrm{Ad}(g^{-1})(S(v)), x_0] x)(g) + f(x_0 \mathrm{Ad}(g^{-1})(S(v)) x)(g) \\
&\quad = \partial_t\big|_0 \left(f(\mathrm{Ad}(e^{-t x_0} g^{-1})(S(v)) x)(g) + f(\mathrm{Ad}(g^{-1})(S(v)) x)(g e^{t x_0})\right) \\
&\quad = (-1)^{|f||v|} \mathcal{L}_{x_0}((\rho_{\mathfrak{g}}(v) f)(x))(g)
\end{aligned}$$

hence $\mathfrak{U}(\mathfrak{g}_0)$-invariance is indeed preserved by the $\rho_{\mathfrak{g}}$ action and trivially also by the ρ_0 action since for $p \in G_0$

$$(\rho_0(g) f)(x_0 x))(p) = \partial_t\big|_0 f(x)(g^{-1} p e^{t x_0}) = \mathcal{L}_{x_0}(\rho_0(g) f(x))(p)$$

Next we need to check smoothness and compatibility, therefore we define the locally convex topology on V by the set of semi norms indexed by $(j, u, K) \in J \times \mathfrak{U}(\mathfrak{g}) \times \{K \subset G_0 \big|$ compact$\}$, where J is the index set of a family of semi norms defining the topology on W.

$$||f||_{j,u,K} := \sup_{p \in K} ||f(u)(p)||_j^W$$

1. Let $(g, f) : A \to G_0 \times V$ be a net converging to $(\bar{g}, \bar{f})$. Now since the action is linear
$$\rho_0(g_n)f_n - \rho_0(\bar{g})\bar{f} = \rho_0(g_n)\left(f_n - \bar{f}\right) + (\rho_0(g_n) - \rho_0(\bar{g}))(\bar{f})$$
and we can show convergence in two steps. First we need to show that $\rho_0(g_n)f_n$ converges to 0 for $\bar{f} = 0$. Now for some $j \in J$, $u \in \mathfrak{U}(\mathfrak{g})$ and $p \in G_0$
$$||\rho_0(g_n)f_n(u)(p)||_j^W = ||f_n(u)(g_n^{-1}p)||_j^W$$
and all $f_n(u)$ are *smooth* functions on G_0. To exploit this we introduce normal coordinates around $(\bar{g})^{-1}p$, denote $x_n := \log(g_n^{-1}p)$ from the $N \in A$ on where it exists and note that x_n is a net converging to 0 in $\mathrm{T}_{\bar{g}^{-1}p}G_0 \simeq \mathfrak{g}_{0,\mathbb{R}}$. Now we have
$$||f_n(u)(g_n^{-1}p)||_j^W \leq ||f_n(u)(\bar{g})||_j^W + \sup_{t \in [0,1]} ||D\big|_{e^{-tx}p}\left(f_n(u)\right)(x_n)||_j^W$$
where we identify $\mathrm{T}_{e^{-tx}p}W \simeq W$ as usual for vectorspaces. Finally $D(f_n(u))$ is smooth, hence bounded, x_n converges to zero and so does $||f_n(u)(\bar{g})||_j^W$ because f_n is a net converging to 0, hence $\lim\left(\rho_0(g_n)f_n(u)(p)\right) = 0$.

   The second part to consider is $(\rho_0(g_n) - \rho_0(\bar{g}))(\bar{f})$ where we can assume $\bar{g} = \mathbb{1}$ and hence need to show that $\lim \rho_0(g_n)(\bar{f}) = \bar{f}$. Like above we can write $g_n = e^{x_n}$ in exponential coordinates around $\mathbb{1}$ for $n \geq N$ for some $N \in A$. Then
$$\begin{aligned}
&||\left(\rho_0(g_n) - 1\right)\bar{f}(u,p)||_j^W = ||\bar{f}(u)(e^{-x_n}p) - \bar{f}(u)(p)||_j^W \\
&= ||\int_0^1 \partial_s\big|_0 \bar{f}(u)(e^{-(t+s)x_n}p)\mathrm{d}t||_j^W = ||-\int_0^1 \bar{f}(\mathrm{Ad}(p)(x_n)u)(e^{-tx_n}p)\mathrm{d}t||_j^W \\
&\leq \sup_{t\in[0,1]} ||\bar{f}(\mathrm{Ad}(p)(x_n)u)(e^{-tx_n}p)||_j^W \\
&\leq ||\,\mathrm{Ad}(p)||_{\mathrm{Op},\mathfrak{g}}||x_n||_{\mathfrak{g}} \sum_{i=1}^{\dim \mathfrak{g}_{0,\mathbb{R}}} \sup_{t\in[0,1]} |\bar{f}(e_i u)(e^{-tx_n}p)|
\end{aligned} \tag{3.2}$$
for some basis e_i of $\mathfrak{g}_{0,\mathbb{R}}$ and $||\sum_i y_i e_i||_{\mathfrak{g}} := \max_i y_i$. Now since we need to consider the supremum over $p \in K$ for some compactum K, $\mathrm{Ad} : G_0 \to \mathrm{Aut}(\mathfrak{g})$ is smooth and all $f(e_i u)$ are smooth, all terms in (3.2) are bounded and $||x_n||_{\mathfrak{g}}$ converges to 0. Hence $\lim \rho_0(g_n)(\bar{f}) = \bar{f}$ and all in all the action $G_0 \times V \to V$ is continuous.

2. In part 1 we already used the fact that inversion and the left action on G_0 are smooth and we start from smooth functions hence everything is smooth in G_0. More explicitly we can Taylor expand

$$f(u)(e^{-tx}p) = e^{t\partial_t}\big|_0 \left(f(u)(e^{-tx}p)\right)$$

and

$$\partial_t\big|_0 f(u)(e^{-tx}p) = -D\big|_p \left(f(u)\right)(x)$$

is smooth in p and likewise are all higher derivatives due to smoothness of f, so every term in the Taylor expansion is a member of V. Hence all vectors f are smooth with respect to the ρ_0 action.

3. Take a convergent net $\lim(x_n, f_n) = (\bar{x}, \bar{f})$ in $\mathfrak{g} \times V$. Like in 1 we can decompose

$$\rho_{\mathfrak{g}}(x_n)f_n - \rho_{\mathfrak{g}}(\bar{x})\bar{f} = \rho_{\mathfrak{g}}(x_n)(f_n - \bar{f}) + (\rho_{\mathfrak{g}}(x_n) - \rho_{\mathfrak{g}}(\bar{x}))\bar{f}$$

and for the first step assume $\bar{f} = 0$ due to linearity. Then for $u \in \mathfrak{U}(\mathfrak{g})$, $p \in K \subset G_0$ and $j \in J$ we have

$$\begin{aligned} ||(\rho_{\mathfrak{g}}(x_n)f_n)(u)(p)||_j^W &= ||f_n(\mathrm{Ad}(p^{-1})(S(x_n))u)(p)||_j^W \\ &\leq ||\,\mathrm{Ad}(p^{-1})||_{\mathrm{Op},\mathfrak{g}} ||x_n||_{\mathfrak{g}} \sum_{i=1}^{\dim \mathfrak{g}_{0,\mathbb{R}}} ||f_n||_{j,e_i u,K} \end{aligned} \tag{3.3}$$

where the last step is similar to (3.2) and f_n converging to 0 means that all $||f_n||_{j,u,K}$ and in particular $||f_n||_{j,e_i u,K}$ converge to 0. And since x_n is converging so is $\lim ||x_n||_{\mathfrak{g}} = ||\bar{x}||_{\mathfrak{g}}$ hence the whole (3.3) converges to 0.

For the second step we may assume $\bar{x} = 0$. Then we get

$$||(\rho_{\mathfrak{g}}(x_n)\bar{f})(u)(p)||_j^W \leq ||\,\mathrm{Ad}(p^{-1})||_{\mathrm{Op},\mathfrak{g}} ||x_n||_{\mathfrak{g}} \sum_{i=1}^{\dim \mathfrak{g}_{0,\mathbb{R}}} ||\bar{f}||_{j,e_i u,K}$$

just like above but now $||\bar{f}||_{j,e_i u,K}$ is fixed and $||x_n||_{\mathfrak{g}}$ is converging to zero which again yields $\lim \rho_{\mathfrak{g}}(x_n)\bar{f} = 0$. Taking both steps together the action $\mathfrak{g} \times V \to V$ is continuous.

4. Let $x \in \mathfrak{g}_{0,\mathbb{R}}$, $f \in V$, $u \in \mathfrak{U}(\mathfrak{g})$, $p \in G_0$ then

$$\begin{aligned} \partial_t\big|_0 f(u)(e^{-tx}p) &= \partial_t\big|_0 f(u)(pe^{-t\,\mathrm{Ad}(p^{-1})x}) \\ &= \mathcal{L}_{\mathrm{Ad}(p^{-1})S(x)} f(u)(p) \\ &= f(xu)(p) = \left(\rho_{\mathfrak{g}}(x)f\right)(u)(p) \end{aligned}$$

hence $\rho_{\mathfrak{g}}$ extends the derivative of ρ_0.

5. Let $x \in \mathfrak{g}$, $f \in V$, $u \in \mathfrak{U}(\mathfrak{g})$, $p, g \in G_0$ then

$$\begin{aligned}\left(\rho_0(g)\circ\rho_{\mathfrak{g}}(x)\circ\rho_0(g^{-1})f\right)(u)(p) &= \left(\rho_0(g)\circ\rho_{\mathfrak{g}}(x)f\right)(u)(gp)\\ &= (-1)^{|f||x|}\left(\rho_0(g)f\right)(\mathrm{Ad}((gp)^{-1})(S(x))u)(gp)\\ &= (-1)^{|f||x|}f(\mathrm{Ad}(p^{-1})\circ\mathrm{Ad}(g)(S(x))u)(p)\\ &= (-1)^{|f||x|}f(\mathrm{Ad}(p^{-1})(S(\mathrm{Ad}(g)((x))))u)(p)\\ &= \left(\rho_{\mathfrak{g}}(\mathrm{Ad}(g)x)f\right)(u)(p)\end{aligned}$$

using that S commutes with the adjoint action. Hence the action intertwine adjunction as demanded.

So all in all V carries indeed a representation of the group pair G_p.

Secondly, to see that $\underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_\circ),H_p}$ is a subrepresentation we need to show that the action of the group preserves the H_p-invariance. To see that the H_0-invariance as in (3.1b) is preserved take $h \in H_0$, $f \in \underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_\circ),H_p}$ and $x \in \mathfrak{U}(\mathfrak{g})$, $x_0 \in \mathfrak{g}_0$, $g \in G_0$, $y \in \mathfrak{h}$ and $v \in \mathfrak{g}$ and compute

$$\begin{aligned}(\rho_{\mathfrak{g}}(v)f)(x))(g) &= (-1)^{|f||v|}f(\mathrm{Ad}(g^{-1})(S(v))x)(g)\\ &= (-1)^{|f||v|}\pi_0(h)\Big(f(\mathrm{Ad}(h^{-1})(\mathrm{Ad}(g^{-1})(S(v))x))(gh)\Big)\\ &= (-1)^{|f||v|}\pi_0(h)\Big(f(\mathrm{Ad}((gh)^{-1})(S(v))\,\mathrm{Ad}(h^{-1})(x))(gh)\Big)\\ &= \pi_0(h)\Big((\rho_{\mathfrak{g}}(v)f)(\mathrm{Ad}(h^{-1}x)(gh)\Big)\end{aligned}$$

The ρ_0 action does also not interfere with the H_0-invariance because it leaves the first argument of f alone and in the second argument acts from the left whilst the H_0-invariance acts on the right hand side.

Similarly, since $\rho_{\mathfrak{g}}$ multiplies the argument of f from the left hand side and $\mathfrak{h}$-invariance is defined in (3.1c) by acting on the right hand side, the two do not interfere. Finally $\mathfrak{h}$-invariance is not affected by ρ_0 since the two act on different arguments.

Hence we see that indeed the G_p action preserves $\underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_\circ),H_p}$ inside $\underline{\mathrm{Hom}}_{\mathrm{SVS}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty\left(G_0, W\right)\right)^{\mathfrak{U}(\mathfrak{g}_\circ)}$ and hence the former is a subrepresentation of the latter.

□

3.4.1 Frobenius reciprocity

Theorem 3.59. *For the induced representation Frobenius reciprocity occurs, i.e. for a super group pair G_p with subgroup pair H_p we have*

$$\mathrm{Hom}_{G_p}(V, \mathrm{Ind}_{H_p}^{G_p}(W)) \simeq \mathrm{Hom}_{H_p}(V, W)$$

Proof of 3.59. On the right hand side V is naturally also considered as a representation of H_p via $\pi^V\big|_{H_p}$ and we will omit the explicit restriction symbol in the following.

Consider the linear map

$$\begin{aligned}\Phi : \mathrm{Hom}_{G_p}(V, \mathrm{Ind}_{H_p}^{G_p}(W)) &\to \mathrm{Hom}_{H_p}(V, W)\\ T &\mapsto \Big(v \mapsto ((T(v))(1))(\mathbb{1})\Big)\end{aligned}$$

where $1 \in \mathfrak{U}(\mathfrak{g})$ and $\mathbb{1} \in G_0$. To see that this map is well defined take $T \in \mathrm{Hom}_{G_p}(V, \mathrm{Ind}_{H_p}^{G_p}(W))$, $v \in V$, $h \in H_0$ and $y \in \mathfrak{h}$ and compute

$$\begin{aligned}\Phi(T)(\pi_0^V(h)v) &= T(\pi_0^V(h)v)(1)(\mathbb{1})\\ &= (\rho_0(h)T(v))(1)(\mathbb{1}) \qquad (3.4a)\\ &= T(v)(1)(h^{-1}) = T(v)(\mathrm{Ad}(h)(1))(\mathbb{1}h^{-1})\\ &= \pi_0^W(h)\big(T(v)(1)(\mathbb{1})\big) \qquad (3.4b)\\ &= \pi_0^W(h)\Phi(T)(v)\end{aligned}$$

where we used that T intertwines G_p representations in (3.4a) and (3.4b) is a defining property of the induced representation. Similarly

$$\begin{aligned}\Phi(T)(\pi_{\mathfrak{g}}^V(y)v) &= (\rho_{\mathfrak{g}}(y)T(v))(1)(\mathbb{1})\\ &= (-1)^{|T(v)||y|}\, T(v)(\mathrm{Ad}(\mathbb{1})S(y))(\mathbb{1})\\ &= (-1)^{|T(v)||y|}(-1)^{(|T(v)|+|1|)|y|}\,\pi_{\mathfrak{h}}^W(y)(T(v)(1)(\mathbb{1})) = \pi_{\mathfrak{h}}^W(y)(T(v)(1)(\mathbb{1}))\end{aligned}$$

So we see that indeed $\Phi(T) \in \mathrm{Hom}_{H_p}(V, W)$.

Further for $u \in \mathfrak{U}(\mathfrak{g})$ and $g \in G_0$ note that

$$\begin{aligned}T(v)(u)(g) &= (-1)^{|v||u|}\left(\rho_{\mathfrak{g}}(S(u)) \circ \rho_0(g^{-1})T(v)\right)(1)(\mathbb{1})\\ &= (-1)^{|v||u|}T\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(g^{-1})v\right)(1)(\mathbb{1})\\ &= (-1)^{|v||u|}\Phi(T)\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(g^{-1})v\right)\end{aligned}$$

which means $\Phi^{-1} \circ \Phi(T) = T$ for

$$\begin{aligned}\Phi^{-1} : \mathrm{Hom}_{H_p}(V, W) &\to \mathrm{Hom}_{G_p}(V, \mathrm{Ind}_{H_p}^{G_p}(W))\\ R &\mapsto \Big(v \mapsto \Big(u \mapsto \Big(g \mapsto (-1)^{|v||u|}R\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(g^{-1})v\right)\Big)\Big)\Big)\end{aligned}$$

This definition of Φ^{-1} also yields $\Phi \circ \Phi^{-1}(R) = R$. But it remains to be shown that Φ^{-1} is well defined in the first place. Therefore take $p \in G_0$ and $u \in \mathfrak{U}(\mathfrak{g})$ and compute

$$\begin{aligned}\Phi^{-1}(R)(\pi_0^V(g)v)(u)(p) &= (-1)^{|v||u|}R\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(p^{-1})\pi_0^V(g)v\right)\\ &= (-1)^{|v||u|}R\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V((g^{-1}p)^{-1})v\right)\\ &= \Phi^{-1}(R)(v)(u)(g^{-1}p) = \left(\rho_0(g)\Phi^{-1}(R)(v)\right)(u)(p)\end{aligned}$$

hence $\Phi^{-1}(R)(\pi_0^V(g)v) = \rho_0(g)\Phi^{-1}(R)(v)$ and with $x \in \mathfrak{g}$

$$\begin{aligned}
\Phi^{-1}(R)(\pi_{\mathfrak{g}}^V(x)v)(u)(p) &= (-1)^{(|v|+|x|)|u|} R\left(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(p^{-1})\pi_{\mathfrak{g}}^V(x)v\right) \\
&= (-1)^{(|v|+|x|)|u|} R\left(\pi_{\mathfrak{g}}^V(S(u)\operatorname{Ad}(p^{-1})(x))\pi_0^V(p^{-1})v\right) \\
&= (-1)^{(|v|+|x|)|u|+|x||u|} R\left(\pi_{\mathfrak{g}}^V(S(S(\operatorname{Ad}(p^{-1})(x))u))\pi_0^V(p^{-1})v\right) \\
&= \Phi^{-1}(R)(v)(\operatorname{Ad}(p^{-1})(S(x))u)(p) \\
&= \left(\rho_{\mathfrak{g}}(x)\Phi^{-1}(R)(v)\right)(u)(p)
\end{aligned}$$

so we also have $\Phi^{-1}(R)(\pi_{\mathfrak{g}}^V(x)v) = \rho_{\mathfrak{g}}(x)\Phi^{-1}(R)(v)$ and $\Phi^{-1}(R)$ is indeed equivariant.

The last thing to be shown is that $\Phi^{-1}(R)(v) \in \operatorname{Ind}_{H_p}^{G_p}(W)$. Linearity and smoothness, i.e. $\Phi^{-1}(R)(v) \in \operatorname{Hom}_{\mathrm{SVS}}(\mathfrak{U}(\mathfrak{g}), \mathrm{C}^\infty(G_0, W)$, are due to the linearity and smoothness of R, $\pi_{\mathfrak{g}}^V$ and smoothness of π_0^V. To check (3.1a) let $x_0 \in \mathfrak{g}_0$, $u \in \mathfrak{U}(\mathfrak{g})$ and $g \in G_0$ and compute

$$\begin{aligned}
\Phi^{-1}(R)(v)(x_0u)(g) &= (-1)^{|v||u|} R(\pi_{\mathfrak{g}}^V(S(x_0u))\pi_0^V(g^{-1})v) \\
&= (-1)^{|v||u|} R(\pi_{\mathfrak{g}}^V(S(u))\pi_{\mathfrak{g}}^V(-x_0)\pi_0^V(g^{-1})v) \\
&= \partial_t\big|_0 (-1)^{|v||u|} R(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(e^{-tx_0}g^{-1})v) \\
&= \partial_t\big|_0 \Phi^{-1}(R)(v)(u)(ge^{tx_0})
\end{aligned}$$

For (3.1b) let $h \in H_0$ then

$$\begin{aligned}
\pi_0^W(h)\Phi^{-1}(R)(v)(\operatorname{Ad}(h^{-1}u)(gh) &= (-1)^{|v||u|}\pi_0^W(h) \\
&\quad R\left(\pi_{\mathfrak{g}}^V(S(\operatorname{Ad}(h^{-1}u))\pi_0^V(h^{-1}g^{-1})v\right) \\
&= (-1)^{|v||u|} R(\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(g^{-1})v) \\
&= \Phi^{-1}(R)(v)(u)(g)
\end{aligned}$$

where we used that R is a morphism of H_p representations. Finally to check (3.1c) let $y \in \mathfrak{h}$ then

$$\begin{aligned}
\Phi^{-1}(R)(v)(uy)(g) &= (-1)^{|v||u|} R(\pi_{\mathfrak{g}}^V(S(uy))\pi_0^V(g^{-1})v) \\
&= (-1)^{|v||u|+|u||y|} R(\pi_{\mathfrak{g}}^V(S(y))\pi_{\mathfrak{g}}^V(S(u))\pi_0^V(g^{-1})v) \\
&= (-1)^{|u||y|}\pi_{\mathfrak{g}}^V(S(y))\Phi^{-1}(R)(v)(uy)(g)
\end{aligned}$$

where we used the intertwining property of R again.

So $\Phi^{-1}(R)(v) \in \operatorname{Ind}_{H_p}^{G_p}(W)$ and Φ^{-1} is indeed the well defined inverse of Φ which is thereby an isomorphism.

□

3.5 Spherical representations

In this section we prove our main Theorem 3.60 which is a generalisation to certain super cases of the classical Theorem by Helgason, [Hel84], Chapter V, Theorem 4.1, using methods introduced by Schlichtkrull, [Sch84].

We want to make a statement about the spherical representations of the symmetric super pair $(\mathfrak{g}, \mathfrak{k})$ of complex Lie super algebras $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ and $\mathfrak{k} = \mathfrak{gl}^{q|r} \oplus \mathfrak{gl}^{s}$, i.e. we seek to characterise which finite dimensional irreducible $\mathfrak{g}$ representations contain a non trivial $\mathfrak{k}$-invariant subspace. As in the classical case we will make use of a global real (in our case cs) form with base $G_0 = \mathrm{U}(q)\times\mathrm{U}(r,s)$ and $K_0 = \mathrm{U}(q)\times\mathrm{U}(r)\times\mathrm{U}(s)$, in the proof. More precisely we will employ the Iwasawa decomposition $G := C(G_0, \mathfrak{g}) = KAN$ with $K = C(K_0, \mathfrak{k})$ as explained generally in Section 3.1.

A sufficient condition for a representation to be spherical can be given under rather general conditions using algebraic arguments. To show that this condition is also necessary we unfortunately need to employ super integration which currently restricts the range of applicability to $s = 1$, although a generalisation to higher rank cases along the line of in [Sch84] should be possible.

We will use the same notation as in Section 3.3. The minimal parabolic subgroup will be denoted by $Q = MAN$ and the super Lie algebras of the Lie super groups are denoted by the corresponding German letters. By an irreducible representation of $\mathfrak{gl}^{q|r+s}$ we mean $\mathfrak{sl}^{q|r+s}$-irreducible or in the case of $q = r+s$, $\mathfrak{sl}^{q|q}/\mathfrak{z}$-irreducible. This ensures that irreducible finite dimensional representations are highest weight representations. Then the only case to be excluded in the following is $(q, r+s) = (1,1)$.

Theorem 3.60. *Let $\pi_{\mathfrak{g}}$ be a finite dimensional irreducible representation of $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ on V. Then $\pi_{\mathfrak{g}}$ is a highest weight representation. Let G be a global cs form such that the global Iwasawa decomposition $G = KAN$ exists. Let π be a representation on V which extends $\pi_{\mathfrak{g}}$ and denote the highest weight by $\lambda \in \mathfrak{h}^*$ and a highest weight vector by $v_\lambda \in V^N \setminus \{0\}$.*

Consider the following conditions:

1. $v_\lambda \in V^M$
2. $V^K \neq \{0\}$

Then for any $q, r, s \in \mathbb{N}$ we have that 2 implies 1. Conversely if $s = 1$ and $b(\lambda, \alpha_{\mathfrak{a}}) > 2(q-r)$ for $\{\alpha_{\mathfrak{a}}\} := \Sigma^+ \setminus 2\Sigma^+$ then 1 implies 2.

In any case $\dim(V^K) \leq 1$.

Proof of 3.60. The correspondence of irreducible and highest weight representations is a basic fact, see Lemma 3.32 and [Kac78], Proposition 2.2.

We start by proving the second implication for which we will use the notion of generalised points for $S \in \mathrm{SMan}$ as introduced in Definition 2.9 and integration on Lie super groups as explained in Section 2.7. So let $v_\lambda \in V^M$. For $v \in_s V$ denote by

$$\begin{aligned} \pi v &: G \to V \\ G \ni_S g &\mapsto \pi(g)(v) \in_S V \end{aligned}$$

and define

$$v_0 := \int_{K/M} \pi v_\lambda$$

Now we need to show that this integral exists and is non zero. Then we get $\pi(K)v_0 = v_0$ by invariance of the measure. To this end, let $w_\mu^* \in_S V^*$ be the *lowest* weight vector of π^* and consider the following calculation which we will validate step by step.

$$\begin{aligned} v_0 &= \int_{\bar{N}} \pi\left(k(\bar{n})\right) v_\lambda \, e^{-2\rho(H(\bar{n}))} \mathrm{D}\bar{n} && \text{(3.5a)} \\ &= \int_{\bar{N}} \pi(\bar{n}) v_\lambda e^{-(\lambda+2\rho)(H(\bar{n}))} \mathrm{D}\bar{n} && \text{(3.5b)} \\ \Rightarrow w_\mu^*(v_0) &= \int_{\bar{N}} w_\mu^*(\pi(\bar{n})v_\lambda) e^{-(\lambda+2\rho)(H(\bar{n}))} \mathrm{D}\bar{n} \\ &= w_\mu^*(v_\lambda) \int_{\bar{N}} e^{-(\lambda+2\rho)(H(\bar{n}))} \mathrm{D}\bar{n} \neq 0 && \text{(3.5c)} \end{aligned}$$

To see that (3.5a) holds we need a generalisation of Helgason's proof ([Hel84], Thm. 5.20, p. 198) to the super case. This is given in [All11a], section 4, and uses the following maps. First the commutative diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, hook] & K \arrow[r, two heads] \arrow[d, hook] & K/M \arrow[d, dashed, "\varphi"] \\ Q \arrow[r, hook] & G \arrow[r, two heads] & G/Q \end{tikzcd}$$

defines φ which is in fact an isomorphism due to the Iwasawa decomposition being one. Then, due to $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{q}$, there is a local isomorphism $\phi : \bar{N} \to G/Q$. Composing the two we get a local isomorphism $\vartheta^{-1} \circ \phi : \bar{N} \to G/Q$. The next step is the super version of [Hel84] Prop. 5.1, p. 181, namely [All10] Proposition 3.4, Equation (3.4),

$$\int_G f(g) \mathrm{D}g = \int_{K\times A\times N} f(kan) e^{2\rho(\log a)} \mathrm{D}k \mathrm{D}a \mathrm{D}n$$

where

$$e^{2\rho(x)} := \operatorname{SDet} \operatorname{Ad} e^x \Rightarrow 2\rho = \sum_{\alpha\in\Sigma^+(\mathfrak{g}:\mathfrak{a})} \alpha \, m_\alpha$$

Here $m_\alpha = \operatorname{SDim} g^\alpha$ denotes the multiplicity of the root α, which might be negative for odd roots and we used

$$\operatorname{SDet} \operatorname{Ad} e^x = e^{\operatorname{STr} \operatorname{ad} x}$$

Finally [All11a] Proposition 4.4 explains how to derive (3.5a) from these facts.

For (3.5b) we use the Iwasawa decomposition of $\bar{n} \in_S \bar{N} \subset_S G$

$$\bar{n} = k(\bar{n}) e^{H(\bar{n})} n(\bar{n}) \Rightarrow k(\bar{n}) = \bar{n} \left(n(\bar{n})\right)^{-1} e^{-H(\bar{n})}$$

hence

$$\pi(k(\bar{n}))v_\lambda = \pi(\bar{n})\pi\left((n(\bar{n}))^{-1}\right)\pi\left(e^{-H(\bar{n})}\right)v_\lambda = \pi(\bar{n})e^{-\lambda(H(\bar{n}))}v_\lambda$$

where we used that v_λ is a highest weight vector in the last step.

For (3.5c) suppose $w_\mu^*(v_\lambda) = 0$.

$$\begin{aligned} &\forall n \in \mathfrak{n} : \pi_{\mathfrak{g}}(n)v_\lambda = 0 \\ \Rightarrow &\forall u \in \mathfrak{U}(\mathfrak{n}) : 0 = w_\mu^*(v_\lambda) = w_\mu^*(\pi_{\mathfrak{g}}(u)v_\lambda) = (\pi_{\mathfrak{g}}^*(S(u))w_\mu^*)(v_\lambda) \end{aligned}$$

But, since w_μ^* is the lowest weight vector and the dual of an irreducible representation is irreducible, $\pi_{\mathfrak{g}}^*(\mathfrak{U}(\mathfrak{n}))w_\mu^* = V^*$ and hence $(v_\lambda^*)^* = 0$ which is incompatible with $v_\lambda \neq 0$. We can even be a little more concrete here, by Corollary 3.70 we have in fact that $w_\mu^*(v_\lambda) = 1$ for a properly chosen dual w_μ^*.

Now finally we need to restrict to $s = 1$ and $\lambda(h_{\mathfrak{a}}) > 2(q - r)$ to use the computation in Section 3.5.1 to shows that

$$\int_{\bar{N}} e^{-(\lambda+2\rho)H(\bar{n})}\mathrm{D}\bar{n} \neq 0 \tag{3.6}$$

$\boxed{1 \Rightarrow 2}$

To show that 2⇒1 no restrictions on $q, r, s \in \mathbb{N}$ are needed.

The outline of the argument is as follows. First we need to embed $V := V_\lambda$ by

$$\alpha : V \hookrightarrow \mathrm{Ind}_{MAN}^G(V^N) \tag{3.7a}$$

which turns out to be rather lengthy and is therefore postponed to Section 3.5.2 below. Then the multiplicity of any irreducible K representation W is given by the dimension of

$$\begin{aligned} \mathrm{Hom}_K\left(W, \mathrm{Ind}_{MAN}^G(V^N)\right) &= \mathrm{Hom}_M\left(W, V^N\right) && (3.7\mathrm{b}) \\ \Rightarrow \dim\left(V^K\right) &\in \{0, 1\} && (3.7\mathrm{c}) \end{aligned}$$

where (3.7b) holds by Frobenius reciprocity, Theorem 3.59.

Now for (3.7c) note that V^N is M irreducible. Hence the multiplicity of the one dimensional trivial K representation, which is of course also the irreducible trivial $M \subset K$ representation, in $\mathrm{Ind}_{MAN}^G(V^N)$ can be only 0 or 1. Using the inclusion $V \hookrightarrow \mathrm{Ind}_{MAN}^G(V^N)$ this means that also the multiplicity in V, which is $\dim V^K$, is at most 1 as claimed.

Now, assuming that there is an invariant vector v_0, we have $V^K = \mathrm{span}\{v_0\}$. Hence with $W = V^K$ the left hand side of (3.7b) has dimension (at least) one. This means that the right hand side contains a non zero element $f \in \mathrm{Hom}_M\left(V^K, V^N\right)$. Observe $M \subset K$ so we have

$$\forall w \in_S V^K, m \in_S M : \ \pi(m)f(w) = f(\pi(m)w) = f(w)$$

and since f is in particular surjective, any vector in V^N is M-invariant.

□

Lemma 3.61. *Let $(\mathfrak{g}, \mathfrak{k})$ be a reductive pair of even type with a non-compact global cs form $G = KAN$ and corresponding minimal parabolic $Q = MAN$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and Δ^+ a positive root system compatible with the pair. Let $\lambda \in \mathfrak{h}^*$ such that π on V_λ is a finite dimensional highest weight module with highest weight vector v_λ. Then $v_\lambda \in V^M$ if and only if*

1. $\lambda\big|_{\mathfrak{h}\cap\mathfrak{k}} \equiv 0$
2. $\forall \alpha \in \Sigma^+ : b(\lambda, \alpha) \in 2\mathbb{N}$
3. $\pi_{\mathfrak{g}}\left(\sum_{\beta\in\Delta^+_{-,1}} g^{-\beta}\right) v_\lambda = 0$

where $\Delta^+_{-,1}$ denotes the positive odd roots which restrict to zero on $\mathfrak{a}$.

Proof of 3.61. We have by definition $v_\lambda \in V^M$ if and only if $\pi_{\mathfrak{g}}(\mathfrak{m})v_\lambda = 0$ and $\pi_0(M_0)v_\lambda = v_\lambda$.

Since
$$\mathfrak{m} = \mathfrak{k} \cap \mathfrak{h} \oplus \sum_{\beta\in\Delta_-} g^\beta$$
(see 3.25) and $\forall h \in \mathfrak{h} : \pi_{\mathfrak{g}}(h)v_\lambda = \lambda(h)v_\lambda$ we see immediately that $\pi_{\mathfrak{g}}(\mathfrak{m})v_\lambda = 0$ implies 1 and 3.

For the converse note that for $\beta \in \Delta^+$, $\pi_{\mathfrak{g}}(\mathfrak{g}^\beta)v_\lambda \subset V_{\lambda+\beta} = 0$ because λ is a highest weight. If $\beta \in \Delta^+_{-,0}$ is even, hence a weight of V_λ considered as a $\mathfrak{g}_0$ representation, then there is an associated Weyl group reflection s_β in the ordinary Weyl group W_0 of $\mathfrak{g}_0$ such that $\lambda - \beta = s_\beta(\lambda + \beta)$ since $\lambda\big|_{\mathfrak{h}\cap\mathfrak{k}} = 0$. But since the weights are W_0 stable this means that $\pi_{\mathfrak{g}}(g^{-\beta})v_\lambda \subset V_{\lambda-\beta} = 0$. So the additional condition 3 ensures $\pi_{\mathfrak{g}}\left(\sum_{\beta\in\Delta_-} g^\beta\right)(v_\lambda) = 0$. Hence 1 and 3 imply $\pi_{\mathfrak{g}}(\mathfrak{m})v_\lambda = 0$.

For the M_0 part we can use the result from the ordinary case (see Halgason's prove of Theorem 4.1 in Chapter 5 of [Hel84])
$$\{x \in i\mathfrak{a}_{0,\mathbb{R}} : e^x \in M_0\} = \mathrm{span}_{\mathbb{Z}}\{i\pi\alpha^* \big| \alpha \in \Sigma^+\}$$
Note that we have to look at the coroots of the restricted roots instead of the coroots $\{h_i\}$ from Lemma 3.23. Hence given $\pi_{\mathfrak{g}}(\mathfrak{m})v_\lambda = 0$ we have $\pi_0(M_0)v_\lambda = v_\lambda$ if and only if $\forall \alpha \in \Sigma^+ : \lambda(\alpha^*) \in 2\mathbb{N}$. □

Remark 3.62. *By Lemma 3.44 we have that 1 implies 3 since for $\alpha \in \Delta_-$ we have $\alpha^* \in \mathfrak{h}\cap\mathfrak{k}$ and for Δ^+ compatible with the pair we have that Δ^+_- is generated by $\Delta^+_-\cap\Pi$.*

Corollary 3.63. *Let $\mathfrak{g} = \mathfrak{gl}^{q|r+1}$, $\lambda \in \mathfrak{h}^*$ with V_λ finite dimensional and $\lambda\big|_{\mathfrak{h}\cap\mathfrak{k}} \equiv 0$, $\lambda(\alpha^*) \in 2\mathbb{N}$ and $\lambda(\alpha^*) > 2(q - r)$ for the single simple restricted root $\alpha \in \Sigma^+ \setminus 2\Sigma^+$. Then V_λ is a* spherical *representation.*

Corollary 3.64. *For $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ and V_λ a finite dimensional irreducible* spherical *representation we have $\lambda\big|_{\mathfrak{h}\cap\mathfrak{k}} \equiv 0$ and $\forall \alpha \in \Sigma^+ : \lambda(\alpha^*) \in 2\mathbb{N}$.*

Corollary 3.65. *All spherical representations of* $\mathfrak{g}$ *are atypical if* $\Delta^+_{-,1} \neq \emptyset$. *This is the case for* $\mathfrak{g} = \mathfrak{gl}^{q|r+s}$ *if* $\mathfrak{g}_1 \neq 0$.

Proof of 3.65. By Lemma 3.35 and 3.29 we have

$$\bar{V}_\lambda = \bigoplus_{s=0}^{|\Delta_1^+|} \bigoplus_{\beta_i \in \Delta_1^+} f_{\beta_1} \dots f_{\beta_s} V_\lambda^0$$

where $f_\beta \in \mathfrak{g}^{-\beta} \setminus \{0\}$. In particular $f_\beta v_\lambda \in f_\beta V_\lambda^0 \subset \bar{V}_\lambda$ for $\beta \in \Delta^+_{-,1} \subset \Delta_1^+$ but by Lemma 3.61 $f_\beta v_\lambda \notin V_\lambda \setminus \{0\}$ hence $V_\lambda \neq \bar{V}_\lambda$. $\square$

3.5.1 Computation of the super integral

Most of computation in this section is due to Wolfgang Palzer and came to our knowledge in private communication. It is to appear in one of his future publications.

To compute the integral (3.6) we first need to determine $H(\bar{n})$, i.e. the $\mathfrak{a}$ component of the global Iwasawa decomposition. More concretely $K \times \mathfrak{a} \times N \xrightarrow{\sim} G$ can be inverted to define morphisms of super manifolds $(K, H, N) : G \to K \times \mathfrak{a} \times N$ such that for $S \in \mathrm{SMan}$ and $g \in_S G$ we have $g = k(g)e^{H(g)}N(g)$. In particular this can be applied to decompose $\bar{n} \in_S \bar{N} \subset_S G$. To compute the morphism H explicitly we get rid of the K component using the defining involutive automorphism $\theta : G \to G$ that fixes K.

$$\theta(\bar{n})^{-1}\bar{n} = \theta(N(\bar{n}))^{-1} e^{2H(\bar{n})} N(\bar{n}) \tag{3.8}$$

Using Section 3.3, in particular Corollary 3.55, we choose a system of positive restricted roots $\Sigma^+ = \{ia_i, i(a_i + a_j), i(a_i - a_j)\}$ where for the third term $i > j$. Then

$$\mathfrak{n} = \left(\begin{array}{c|ccc} 0 & C & 0 & iC \\ \hline D & G+E+F & B & i(G+F-E) \\ 0 & A & 0 & iA \\ iD & i(E+F-G) & iB & E-F+G \end{array}\right)$$

Here $A \in \mathbb{C}^{(r-s)\times s}$, $B \in \mathbb{C}^{s\times(r-s)}$, $C \in \mathbb{C}^{q\times s}$ and $D \in \mathbb{C}^{s\times q}$ parametrise $\bigoplus_i \mathfrak{g}^{ia_i}$, $E, G \in \mathbb{C}^{s\times s}$ and E strictly upper and G strictly lower triangular parametrise $\bigoplus_{i<j} \mathfrak{g}^{i(a_i-a_j)}$ and $F \in \mathbb{C}^{s\times s}$ parametrises $\bigoplus_{i,j} \mathfrak{g}^{i(a_i+a_j)}$.

So $\mathfrak{n}$ is nilpotent of degree $s+1$ and hence for $s = 1$ things become quite computable. Let $x \in \bar{\mathfrak{n}}$ be parametrised by

$$x = \left(\begin{array}{c|ccc} 0 & C & 0 & -iC \\ \hline D & F & B & -iF \\ 0 & A & 0 & -iA \\ -iD & -iF & -iB & -F \end{array}\right)$$

then solving (3.8) yields

$$H(e^x) = \frac{1}{2}\log\left((1-(BA+DC))^2 - 4F^2\right)h$$

with the single generator of $\mathfrak{a}$ being

$$h_{\mathfrak{a}} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array}\right)$$

Using the non compact cs form of Lemma 3.47 for the even part means $B = -A^\dagger$ and $F \in i\mathbb{R}$.

Hence we can express the relevant integrand in coordinates. For $S \in \mathrm{SMan}$ and $\bar{n} \in_S \bar{N}$ we have

$$e^{\lambda(H(\bar{n}))} = \left((1 + \sum_{i=1}^{r-1} z_i z_i^* + \sum_{j=1}^{q} \xi_{2j-1}\xi_{2j})^2 + 4x^2\right)^{\frac{\lambda(h_{\mathfrak{a}})}{2}}$$

From here Palzer computes

$$\begin{aligned}\int_{\bar{N}} e^{-(\lambda+2\rho)H(\bar{n})}\mathrm{D}\bar{n} &\propto \int_{\mathbb{R}^{2r-1|2q}} \left((1 + \sum_{i=1}^{2r-2} x_i^2 + \sum_{j=0}^{q} \chi_j\chi_j^*)^2 + y^2\right)^{-(\frac{1}{2}\lambda+\rho)(h_{\mathfrak{a}})} |\mathrm{D}(x,y,\chi,\chi^*)| \\ &= \frac{\pi^r 2^{2-(\lambda+2\rho)(h_{\mathfrak{a}})}\Gamma((\lambda+2\rho)(h_{\mathfrak{a}}) - r + q)}{\Gamma^2((\frac{1}{2}\lambda+\rho)(h_{\mathfrak{a}}))} \\ &\propto \frac{\Gamma(\lambda(h_{\mathfrak{a}}) + r - q)}{\Gamma^2((\frac{1}{2}\lambda(h_{\mathfrak{a}}) + r - q)}\end{aligned}$$

where the proportionality factors are non zero, $\rho(h_{\mathfrak{a}}) = \frac{1}{2}(2 + (2(r-s) - 2q)) = r - q$ and $\lambda(h_{\mathfrak{a}}) \geq 0$. By Corollary 3.64 we have $\lambda(h_{\mathfrak{a}})$ even, hence the integral is non zero if and only if $\frac{1}{2}\lambda(h_{\mathfrak{a}}) > q - r$.

3.5.2 Embedding

In this section let $G = KAN$ be the Iwasawa decomposition of G (see Section 3.1), $Q = MAN$ the corresponding minimal parabolic subgroup and $\pi = (\pi_0, \pi_{\mathfrak{g}})$ an irreducible finite dimensional highest weight representation of G_p on V. In this section we will drop the subscript and write $G = G_p$ and similarly for the other super group pairs and work only with those.

Lemma 3.66. *There is an isomorphism of G representations*

$$\mathrm{Ind}_Q^G\left(V^N\right) \simeq \mathrm{Ind}_Q^G\left(\left((V^*)^{\bar{N}}\right)^*\right)$$

Proof of 3.66. We will show that

$$\begin{aligned}\Phi : V^N &\to \left((V^*)^{\bar N}\right)^* \\ v &\mapsto (\mu \mapsto \mu(v))\end{aligned}$$

is an isomorphism of Q representations which implies the above. Therefore note that Φ intertwines the representation π of Q on V with π^{**} on V^{**} by Definition 2.101 of the dual representation. Further observe that Φ is injective because

$$\begin{aligned}&\Phi(v) = 0 \\ &\Rightarrow \forall \mu \in V^{*\bar N} : \Phi(v)(\mu) = \mu(v) = 0 \\ &\Rightarrow \forall x \in \mathfrak{n}\ \forall \mu \in V^{*\bar N} : \mu(\pi_{\mathfrak{g}}(S(x)v)) = (\pi_{\mathfrak{g}}^*(x)\mu)(v) = 0 \\ &\Rightarrow \mathfrak{U}(\pi_{\mathfrak{g}}^*(\mathfrak{n}))\left(V^{*\bar N}\right)(v) = V^*(v) = 0\ \Rightarrow v = 0\end{aligned}$$

where we used that $v \in V^N$ and the lowest weight vector of V^* is contained in $V^{*\bar N}$. Hence $\dim V^{N^*} \leq \dim V^{*\bar N}$, but interchanging the roles of N and $\bar N$ we can also construct an injective map in the other direction, hence the dimensions are equal and Φ is an isomorphism.

□

Lemma 3.67. *There is an embedding of G representations*

$$\begin{aligned}\alpha : V &\hookrightarrow \mathrm{Ind}_Q^G\left(\left((V^*)^{\bar N}\right)^*\right) \\ \alpha(v)(u)(g)(\mu) &:= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\Big(\pi_0(g^{-1})\pi_{\mathfrak{g}}(\mathrm{Ad}(g)S(u))\ v\Big)\end{aligned}$$

Proof of 3.67. The map α is well defined, because for $g \in G_0$, $v \in V$, $u \in \mathfrak{U}(\mathfrak{g})$, $\mu \in V^{*\bar N}$ and $p \in Q_0$

$$\begin{aligned}\left(\pi_0^{**}(p)\alpha(v)(\mathrm{Ad}(p^{-1})u)(gp)\right)(\mu) &= (-1)^{|u||v|+|\mu|(|u|+|v|)} \\ &\quad \mu\Big(\pi_0(p)\,\pi_0\left((gp)^{-1}\right)\pi_{\mathfrak{g}}\left(\mathrm{Ad}(gp)S(\mathrm{Ad}(p^{-1})u)\right)v\Big) \\ &= \alpha(v)(u)(g)(\mu)\end{aligned}$$

and similarly for $y \in \mathfrak{q}$

$$\begin{aligned}\alpha(v)(uy)(g)(\mu) &= (-1)^{(|u|+|y|)|v|+|\mu|(|u|+|y|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(uy))\pi_0(g^{-1})v\right) \\ &= \Big((-1)^{(|u|+|v|)|y|}\pi_{\mathfrak{g}}^{**}(S(y))\alpha(v)(u)(g)\Big)(\mu)\end{aligned}$$

and also for $x_0 \in \mathfrak{g}$

$$\begin{aligned}\alpha(v)(x_0u)(g)(\mu) &= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(x_0u))\pi_0(g^{-1})v\right) \\ &= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(u))\pi_{\mathfrak{g}}(-x_0)\pi_0(g^{-1})v\right) \\ &= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(u))\partial_t\big|_0\pi_0(e^{-tx_0}g^{-1})v\right) \\ &= \partial_t\big|_0\alpha(v)(u)(ge^{tx_0})(\mu)\end{aligned}$$

Further more α is a map of G_p representations because for $h \in G_0$ we have

$$\begin{aligned}\alpha(\pi_0(h)v)(u)(g)(\mu) &= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(u))\pi_0(g^{-1})\pi_0(h)v\right)\\ &= (-1)^{|u||v|+|\mu|(|u|+|v|)}\mu\left(\pi_{\mathfrak{g}}(S(u))\pi_0((h^{-1}g)^{-1})v\right)\\ &= (\rho_0(h)\alpha(v))(u)(g)(\mu)\end{aligned}$$

and for $x \in \mathfrak{g}$

$$\begin{aligned}\alpha(\pi_{\mathfrak{g}}(x)v)(u)(g)(\mu) &= (-1)^{|u|(|v|+|x|)+|\mu|(|u|+|x|+|v|)}\\ &\quad \mu\left(\pi_0(g^{-1})\pi_{\mathfrak{g}}(\mathrm{Ad}(g)S(u))\pi_{\mathfrak{g}}(x)v\right)\\ &= (-1)^{|u||v|+|\mu|(|u|+|x|+|v|)}\\ &\quad \mu\left(\pi_0(g^{-1})\pi_{\mathfrak{g}}(\mathrm{Ad}(g)S\left(\mathrm{Ad}(g^{-1})(S(x))u\right))v\right)\\ &= (-1)^{|x||v|}\alpha(v)(\mathrm{Ad}(g^{-1})(S(x))u)(g)(\mu)\\ &= (\rho_{\mathfrak{g}}(x)\alpha(v))(u)(g)(\mu)\end{aligned}$$

Finally $\alpha \neq 0$ because

$$\alpha(v_\lambda)(1)(\mathbb{1})(\mu_{-\lambda}) = \mu_{-\lambda}(v_\lambda) \neq 0$$

for the highest weight vector v_λ of V and the lowest weight vector $\mu_{-\lambda}$ of V^* and V is G irreducible, so $V \simeq \alpha(V)$ as G representations.

□

Lemma 3.68. *For a G_p representation π on $V \in$ LCSVS there is an isomorphism of K_p representations*

$$\mathrm{Ind}_P^G(V) \simeq \mathrm{Ind}_M^K(V)$$

Proof of 3.68. The restriction to $\mathfrak{U}(\mathfrak{k}) \subset \mathfrak{U}(\mathfrak{g})$ and $K_0 \subset G_0$ denoted by $f \mapsto \Psi(f)$ is an injective map of K_p representations because $G_0 = K_0A_0N_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus n$ hence $\mathfrak{U}(\mathfrak{g}) \simeq \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a} \oplus \mathfrak{n})$ as vector spaces and

$$f(xy)(kan) = (-1)^{(|f|+|x|)(|y|)}\pi_{\mathfrak{g}}(S(y))\pi_0(an)^{-1}f(x)(k)$$

for $x \in \mathfrak{U}(\mathfrak{k})$, $y \in \mathfrak{U}(\mathfrak{a} \oplus \mathfrak{n})$,$k \in K_0$, $a \in A_0$ and $n \in N_0$, so $f \in \mathrm{Ind}_P^G(V)$ is uniquely determined by $\Psi(f)(x)(k) = f(x)(k)$.

To define the inverse we choose a homogeneous basis $\{e_i\}$ of $\mathfrak{U}(\mathfrak{a} \oplus \mathfrak{n})$ and denote

$$\begin{aligned}\mathfrak{U}(\mathfrak{g}) &\xrightarrow{\sim} \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a} \oplus \mathfrak{n})\\ x &\mapsto \sum_i [x]_i \otimes e_i\end{aligned} \tag{3.9}$$

where the sum is finite for all x and we will drop the sum sign in the following, summation over the index i always understood. Then

$$\begin{aligned}&\Phi : \mathrm{Ind}_M^K(V) \to \mathrm{Ind}_P^G(V)\\ &\Phi(f)(x)(kan) := (-1)^{(|f|+|[\mathrm{Ad}(an)(x)]_i|)|e_i|}\pi_0(an)^{-1}\pi_{\mathfrak{g}}(S(e_i))f([\mathrm{Ad}(an)(x)]_i)(k)\end{aligned}$$

Note that we can as well use any decomposition of $x = \sum_{i,j} \mu_{i,j}\, x_i \otimes y_j$ with $x_i \in \mathfrak{U}(\mathfrak{k})$ and $y_j \in \mathfrak{U}(\mathfrak{a}\oplus\mathfrak{n})$ but using a basis saves us a bit of writing. Clearly $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Ind}_M^K(V)}$. But to see that Φ is well defined we need to check the equivariance properties (3.1a) to (3.1c) which we will do for mostly the rest of this proof. Therefore let $g = kan$ and $p = m'a'n' \in P_0 = M_0A_0N_0$, then

$$kanm'a'n' = \big(km'\big)\,\big(aa'\big)\,\big((m'a')^{-1}n(m'a')n'\big) \in M_0A_0N_0$$

where we used that M_0 and A_0 commute and we note $\mathrm{Ad}(MA)(N) \subset N$. As a shorthand we will use for $x \in \mathfrak{U}(\mathfrak{g})$

$$z := \mathrm{Ad}\,\big(\big(aa'\big)\,\big((m'a')^{-1}n(m'a')n'\big)\big)\,\big(\mathrm{Ad}(m'a'n')^{-1}(x)\big) = \mathrm{Ad}\,\big(a(m')^{-1}n\big)\,(x)$$

in the following computation. With $f \in \mathrm{Ind}_M^K(V)$ we have

$$\begin{aligned}
&\pi_0(m'a'n')\,\big(\Phi(f)(\mathrm{Ad}(m'a'n')^{-1}(x))(kanm'a'n')\big)\\
&= \pi_0(m'a'n')\,\Big((-1)^{(|f|+|[z]_i|)(|e_i|)}\pi_0(a(m')^{-1}nm'a'n')^{-1}\pi_{\mathfrak{g}}(S(e_i))f([z]_i)(km')\Big)\\
&= (-1)^{(|f|+|[z]_i|)(|e_i|)}\pi_0(a(m')^{-1}n)^{-1}\pi_{\mathfrak{g}}(S(e_i))\pi_0(m')^{-1}f(\mathrm{Ad}(m')([z]_i))(k) \qquad (3.10a)\\
&= (-1)^{(|f|+|[\mathrm{Ad}(an)(x)]_i|)(|e_i|)}\pi_0(an)^{-1}\pi_{\mathfrak{g}}(S(e_i))f([\mathrm{Ad}(an)(x)]_i))(k)\\
&= \Phi(f)(x)(kan)
\end{aligned}$$

where we used M-equivariance of f in (3.10a) and further $am' = m'a$ and $\mathrm{Ad}(M_0)$ preserves the decomposition (3.9) and hence commutes with the projections. So we have verified (3.1b). To check equivariance also on the algebra level, i.e. (3.1c), let $y \in \mathfrak{U}(\mathfrak{g})$ and denote $K_i := [\mathrm{Ad}(an)x]_i$ and $M_j := [\mathrm{Ad}(an)y]_j$ and note $K_ie_iM_je_j = K_i[e_i, M_j]e_j + (-1)^{|e_i||M_j|}K_iM_je_ie_j$ with $[e_i, M_j], e_ie_j \in \mathfrak{U}(\mathfrak{a}\oplus\mathfrak{n})$ and $K_i, K_iM_j \in \mathfrak{U}(\mathfrak{k})$. For the following computation summation over i and j is understood.

$$\begin{aligned}
\Phi(f)(xy)(kan) &= (-1)^{(|f|+|K_i|)(|e_i|+|M_j|+|e_j|)}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S([e_i, M_j]e_j)\right)f(K_i)(k)\\
&\quad + (-1)^{(|f|+|K_i|+|M_j|)(|e_i|+|e_j|)+|e_i||M_j|}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S(e_ie_j)\right)f(K_iM_j)(k)\\
&= (-1)^{(|f|+|K_i|)(|e_i|+|M_j|+|e_j|)}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S(e_iM_je_j)\right)f(K_i)(k)\\
&\quad - (-1)^{(|f|+|K_i|)(|e_i|+|M_j|+|e_j|)+|e_i||M_j|}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S(M_je_ie_j)\right)f(K_i)(k)\\
&\quad + (-1)^{(|f|+|K_i|+|M_j|)|e_j|+(|f|+|K_i|)|e_i|}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S(e_ie_j)\right)\\
&\qquad (-1)^{|M_j|(|K_i|+|f|)}\pi_{\mathfrak{g}}(S(M_j))f(K_i)(k)\\
&= (-1)^{(|f|+|K_i|)|y|+|e_i||y|}\pi_{\mathfrak{g}}(S(y))\\
&\qquad (-1)^{(|f|+|K_i|)|e_i|}\pi_0(an)^{-1}\pi_{\mathfrak{g}}\left(S(e_i)\right)f(K_i)(k)\\
&= (-1)^{(|f|+|x|)|y|}\pi_{\mathfrak{g}}(S(y))\Phi(f)(x)(kan)
\end{aligned}$$

To check (3.1c) let now $x \in \mathfrak{g}_0$ and denote K_0 and AN_0 part of the Iwasawa decomposition of $ke^{t\,\mathrm{Ad}(an)x}an$ for small $t \in \mathbb{R}$ by $k(t)(an)(t)$. This defines curves in K_0 and AN_0 for which we denote the derivative (or velocity) at $t = 0$ by $\dot{k}_0$ and $\dot{an}_0$. Further denote $M_j(t)e_j := \mathrm{Ad}(an(t))y$ where $M_j(t)$ are curves in a finite dimensional subspace of $\mathfrak{U}(\mathfrak{k})$ for which we denote the initial velocities by $\dot{M}_j$. Then

$$\begin{aligned}
\partial_t\big|_0\Phi(f)(y)(kane^{tx}) &= \partial_t\big|_0(-1)^{(|f|+|M_j(t)|)|e_j|}\pi_0(an(t))^{-1}\pi_{\mathfrak{g}}(S(e_j))f(M_j(t))(k(t)) \\
&= (-1)^{(|f|+|M_j|)|e_j|}\pi_0(an)^{-1}\Big(-\pi_{\mathfrak{g}}(\dot{an}_0)\pi_{\mathfrak{g}}(S(e_j))f(M_j)(k) \\
&\qquad + \pi_{\mathfrak{g}}(S(e_j))f(\dot{M}_j)(k) \\
&\qquad + \pi_{\mathfrak{g}}(S(e_j))\mathcal{L}_{\dot{k}_0}f(M_j)(k)\Big) \\
&= (-1)^{(|f|+|M_j|)|e_j|}\pi_0(an)^{-1}\Big(\pi_{\mathfrak{g}}(S(e_j\ \dot{an}_0))f(M_j)(k) \\
&\qquad + \pi_{\mathfrak{g}}(S(e_j))f(\dot{M}_j)(k) \\
&\qquad + \pi_{\mathfrak{g}}(S(e_j))f(\dot{k}_0M_j)(k)\Big) \\
&= \Phi(f)\big(\mathrm{Ad}(an)^{-1}\big(M_je_j\ \dot{an}_0 + \dot{M}_je_j + \dot{k}_0M_je_j\big)\big)(kan) \\
&= \Phi(f)\big(y\,\mathrm{Ad}(an)^{-1}(\dot{an}_0) + [\mathrm{Ad}(an)^{-1}\dot{an}_0, y] \\
&\qquad\quad + \mathrm{Ad}(an)^{-1}(\dot{k}_0)y\big)(kan) \\
&= \Phi(f)\big(\mathrm{Ad}(an)^{-1}(\dot{k}_0 + \dot{an}_0)y\big)(kan) \\
&= \Phi(f)\big(xy\big)(kan)
\end{aligned}$$

So all in all Φ is well defined.

Finally we check

$$\begin{aligned}
(\Phi\circ\Psi(f))\,(x)(kan) &= \pi_0(an)^{-1}f([\mathrm{Ad}(an)x]_{\mathfrak{k}}[\mathrm{Ad}(an)x]_{\mathfrak{an}})(k) \\
&= \pi_0(an)^{-1}f(\mathrm{Ad}(an)x)(k) \\
&= f(x)(kan)
\end{aligned}$$

hence $\Phi\circ\Psi = \mathrm{id}_{\mathrm{Ind}_{G_p}^{P_p}(V)}$.

We did not mention smoothness of the action, but, due to our definition of representations of Lie super group pairs, restriction of smooth representations is not an issue and in the definition of ϕ only smooth functions appear, as π is smooth and the induced representations are defined using smooth functions $\mathrm{C}^\infty(G_0)$. I.e. by using that the induced representations are smooth in the first place, see Lemma 3.58, we indeed have $\mathrm{Ind}_P^G(V) \simeq \mathrm{Ind}_M^K(V)$ also for infinite dimensional V.

□

Corollary 3.69. *Taking together all three lemmas in this section we obtain an embedding of K representations*

$$V \hookrightarrow \mathrm{Ind}_{MAN}^G(V^N)$$

as demanded in (3.7a) *for the proof of Theorem 3.60.*

3.6 Example: $\mathfrak{gl}^{1|2+1}$

The simplest example which is not purely even and where $r > s \geq 1$ is $\mathfrak{gl}^{1|2+1}$. So we now explicitly write down the root space decomposition 3.3 in this low dimensional case. Then we will explicitly compute one non-trivial example each of a spherical and non-spherical representations.

So let in this section $\mathfrak{g} = \mathfrak{gl}^{1|3}$ with involution $\theta(x) = \sigma x \sigma^{-1}$ and $\sigma = \mathrm{diag}(1, 1, 1, -1)$. We use $b(A, B) := \mathrm{STr}(AB)$ as the invariant bilinear form and the global cs form

$$G_0 = U(1) \times U(2, 1)$$

hence the base of the symmetric space is

$$\begin{aligned} G_0/K_0 &= (\mathrm{U}(1)/\,\mathrm{U}(1)) \times (\mathrm{U}(2,1)/\,(\mathrm{U}(2) \times \mathrm{U}(1))) \\ &\simeq \mathrm{SU}(2,1)/\,\mathrm{S}\,(\mathrm{U}(2) \times \mathrm{U}(1)) \simeq \mathbb{CH}^2 \end{aligned} \tag{3.12}$$

where $\mathbb{CH}^2$ denotes the complex two dimensional hyperbolic space.

3.6.1 Roots and root spaces

In accordance with Section 3.3 we choose

$$\mathfrak{a}_{0,\mathbb{R}} = \mathrm{span}\left(\left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ & i & 0 & 0 \end{array}\right)\right)$$

$$\mathfrak{h}_{0,\mathbb{R}} \cap \mathfrak{k} = \mathrm{span}\left(\left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & 0 & i \end{array}\right), \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|c} i & 0 \\ \hline & \\ 0 & 0 \\ & \end{array}\right)\right)$$

and we will use the coordinates

$$\mathfrak{h} = \left\{\left(\begin{array}{c|ccc} c & 0 & 0 & 0 \\ \hline 0 & b_1 & 0 & a \\ 0 & 0 & b_2 & 0 \\ 0 & -a & 0 & b_1 \end{array}\right) \,\middle|\, a, b_i, c \in \mathbb{C}\right\}$$

Applying the results from Section 3.3 we have the even roots

$$\Delta_0 = \{b_2 - b_1 \pm ia, b_1 - b_2 \pm ia, \pm 2ia\}$$

and odd roots

$$\Delta_1 = \{c - b_1 \pm ia, b_1 - c \pm ia, \pm(c - b_2)\}$$

from which we choose a system of positive roots

$$\begin{aligned}\Delta^+ = \big\{&\alpha_1 = b_1 - c + ia,\\ &\alpha_2 = c - b_2,\\ &\alpha_3 = b_2 - b_1 + ia,\\ &\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\big\}\end{aligned} \tag{3.13}$$

where the α_i are the simple roots.

Note that this positive system is not distinguished, but it features $\Delta^+\big|_{\mathfrak{a}} \setminus \{0\} = \Sigma^+$ which is essential for the application of Corollary 3.63 and 3.64. In fact there is no distinguished positive system which is compatible with the Iwasawa decomposition in this sense.

The corresponding raising and lowering operators, f_i and e_i, can be read of from the root spaces. The relations

$$[e_i, f_j] = \delta_{i,j} h_i\ , \quad [h_i, h_j] = 0\ , \quad [h_i, e_j] = a_{i,j} e_j\ , \quad [h_i, f_j] = -a_{i,j} f_j \tag{3.14}$$

for the three sub algebras then define h_i which is dual[2] to α_i and we get

$$e_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{c|ccc} 0 & \multicolumn{3}{c}{0} \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right) \qquad h_1 = \frac{1}{2}\left(\begin{array}{c|ccc} 2 & \multicolumn{3}{c}{0} \\ \hline & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ & i & 0 & 1 \end{array}\right) \tag{3.15a}$$

$$e_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline & & & \\ 0 & & 0 & \\ & & & \end{array}\right) \qquad h_2 = \left(\begin{array}{c|ccc} 1 & \multicolumn{3}{c}{0} \\ \hline & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{array}\right) \tag{3.15b}$$

$$e_3 = \frac{1}{\sqrt{2}}\left(\begin{array}{c|ccc} 0 & \multicolumn{3}{c}{0} \\ \hline & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ & 0 & 0 & 0 \end{array}\right) \qquad h_3 = \frac{1}{2}\left(\begin{array}{c|ccc} 0 & \multicolumn{3}{c}{0} \\ \hline & -1 & 0 & -i \\ 0 & 0 & 2 & 0 \\ & i & 0 & -1 \end{array}\right) \tag{3.15c}$$

and $f_i = e_i^\dagger$. Note that the anti commutator had to be used here for the first time to determine h_1 and h_2.

The normalisation has been chosen such that the $a_{i,j}$ from (3.14) form the Cartan matrix

$$(a_{i,j}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Hence we have the following Dynkin diagram (see Definition 3.24)

[2]In the STr sense.

$$\otimes\!\!-\!\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\!-\!\!\bigcirc$$

Note that the white, i.e. even, subdiagram is *not* that of the ordinary $\mathfrak{sl}^3$, because we did not choose a distinguished positive system. Bu it is well suited for the Iwasawa decomposition since it is the one of $\mathfrak{sl}^2 \subset \mathfrak{k}_0 = \mathfrak{gl}^1 \oplus \mathfrak{gl}^2 \oplus \mathfrak{gl}^1$. Note further that the roots do not span $\mathfrak{h}^*$ but $(\mathfrak{h}/\mathfrak{z})^*$ as mentioned in Lemma 3.23. On the exponential level there is a trivial U(1) subgroup corresponding to the centre which will drop out on passing to the quotient. This can be seen already in Equation (3.12).

3.6.2 Invariant vectors

For our example we will determine directly whether there is an $\mathfrak{k}$-invariant vector to check the validity of the general results in this simple setting. But to compare with the general results we need the positive restricted roots $\Sigma^+ = \{ia, 2ia\}$ and in particular the coroot of the simple restricted root $h_{ia} = h_3 + h_1 - 2h_2$ as well as $\mathfrak{h} \cap \mathfrak{k} = \operatorname{span}\{h_2, h_1 - h_3\}$. Hence by Corollary 3.63 and 3.64 we expect a highest weight representation of weight λ to be spherical if and only if $\lambda(h_2) = 0$ and $\lambda(h_1) = \lambda(h_3) \in \mathbb{N}$ because this already implies $\lambda(h_{ia}) \in 2\mathbb{N}$.

Further note that $\mathfrak{k}$-invariant vectors need to be invariant under h_2 and $h_1 - h_3$, which means that they can only appear in

$$V^K \subset \bigoplus_{\mu(h_2)=\mu(h_1-h_3)=0} V^\mu \tag{3.16}$$

and invariance under all of

$$\mathfrak{k} = \mathfrak{m} \oplus \operatorname{span}\{(e_{2,3} + f_1), (e_1 - f_{2,3}), (e_{1,2} + f_3), (e_3 + f_{1,2}), (e_{1,2,3} - f_{1,2,3})\}$$

with

$$\mathfrak{m} = \operatorname{span}\{\mathbb{1}, h_2, h_1 - h_3, e_2, f_2\}$$

will be checked directly. Here we use the notation

$$e_{i_1,\dots,i_k} := [e_{i_1}, e_{i_2,\dots,i_k}] \tag{3.17}$$

and similarly for $f_{i_1,\dots,i_k}$. For completeness note that the explicit matrices are given by

$$e_{1,2} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & i & 0 \end{array}\right) \quad e_{2,3} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|ccc} 0 & 1 & 0 & i \\ \hline & & & \\ 0 & & 0 & \\ & & & \end{array}\right) \quad e_{1,2,3} = \frac{1}{2}\left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ & i & 0 & -1 \end{array}\right)$$

and $e_{1,3} = 0 = f_{1,3}$. Noting that $f_i = e_i^\dagger$ gives $f_{1,2} = e_{1,2}^\dagger$, $f_{2,3} = -e_{2,3}^\dagger$ and $f_{1,2,3} = -e_{1,2,3}^\dagger$. Altogether this specifies the basis of $\mathfrak{gl}^{1|3}$ which we are going to use.

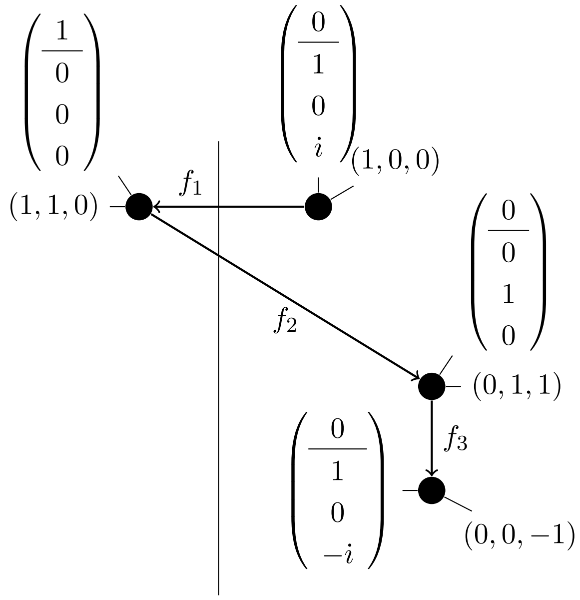

Figure 3.1: Standard representation of $\mathfrak{gl}^{1|3}$

3.6.3 Standard representation

The defining representation of $\mathfrak{gl}^{1|3}$ is the irreducible module $\mathbb{C}^{1|3}$. All weight spaces here are one-dimensional. Generators and weights are depicted in the weight diagram in Figure 3.1 as column and row vectors, respectively. We do not try to adequately depict the three dimensional geometry but place the dots rather arbitrarily, except for the bar dividing even from odd representations and the even $\mathfrak{sl}^2$ sub representations being aligned vertically. The action of the simple lowering operators is depicted by the arrows and all non vanishing actions of those are depicted.

The highest weight is $\lambda(h_1, h_2, h_3) = (1, 0, 0)$ as the corresponding vector is annihilated by all raising operators. The representation is atypical since typical representations need to have at least dimension $2^3 = 8$.

From (3.16) we see immediately that the standard representation does not contain any $\mathfrak{k}$-invariant vectors. Since $\lambda(h_1) \neq \lambda(h_3)$ for the highest weight λ this is in accordance with the general theory, Corollary 3.64. As explained in Section 3.6.2 the standard representation must indeed not be spherical.

3.6.4 Adjoint representation

Using the notation introduced in (3.17) for non-simple raising and lowering operators we produce the diagram indicating the weights for the adjoint representation in Figure 3.2. The arrows again depict the action of of the lowering operators. Horizontal arrows stand for f_1, vertical ones for f_3 and the diagonal or bended ones for f_2. Again all non zero actions are depicted. The representation is decomposable even as a $\mathfrak{g}$

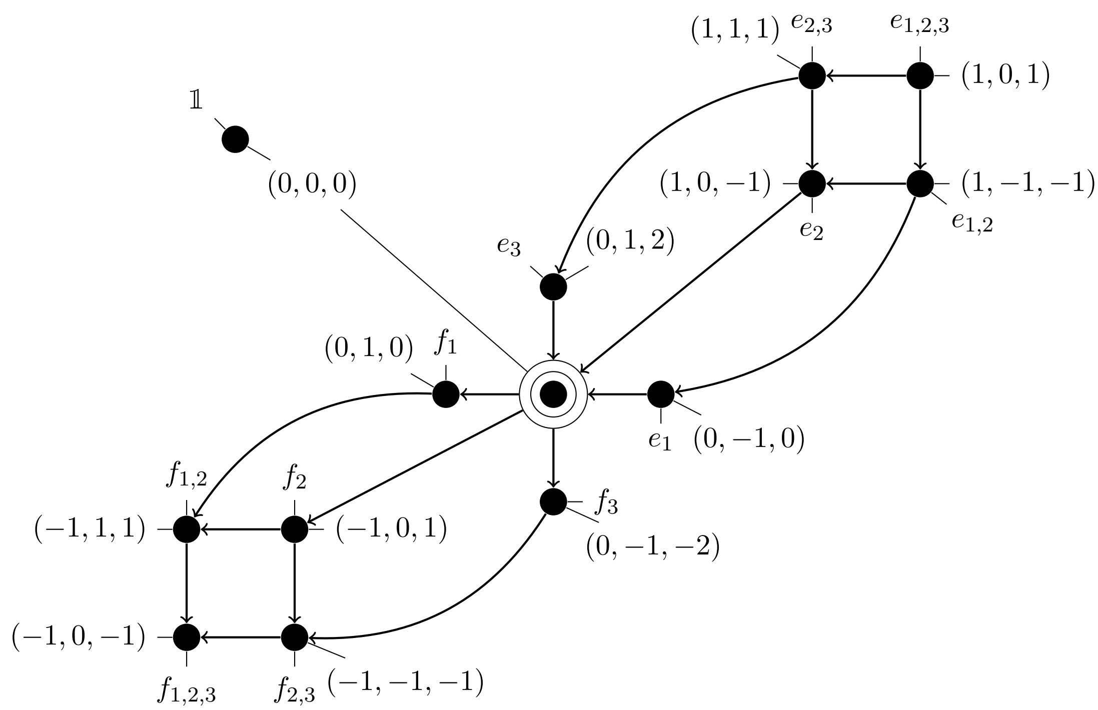

Figure 3.2: Adjoint representation of $\mathfrak{gl}^{1|3}$

representation into the trivial representation ($\mathbb{1}$) and the $\mathfrak{g}$ irreducible complement $\mathfrak{sl}^{1|3}$. All weight spaces are one dimensional, except for $(0,0,0)$, which is spanned by $\{\mathbb{1}, h_1, h_2, h_3\}$. Since $[e_i, e_{1,2,3}] = 0$ the highest weight is $\lambda(h_1, h_2, h_3) = (1,0,1)$. Also the adjoint representation is atypical since all simple odd lowering operators annihilate the highest weight.

Invariant vectors

Using (3.16) again, we identify $(0,0,0)$ and $\pm(1,0,1)$ as the only weights for which the corresponding weight space might contribute to the $\mathfrak{k}$-invariant sector. Now $\mathbb{1}$ is trivially invariant, so interesting candidates are $v_0 \in \mathrm{span}\{h_1, h_2, h_3, e_{1,2,3}, f_{1,2,3}\}$. But demanding $[e_2, v_0] = [e_{1,2} + f_3, v_0] = 0$ yields

$$v_0 = h_2 + e_{1,2,3} + f_{1,2,3} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

And indeed v_0 spans the centre of $\mathfrak{k}/\mathfrak{z}_{\mathfrak{gl}^{1|3}} = (\mathfrak{gl}^{1|2} \oplus \mathfrak{gl}^{0|1}) \cap \mathfrak{sl}^{1|3}$ and hence is $\mathfrak{k}$-invariant.

As explained in Section 3.6.2 the existence of a spherical vector is enforced by the highest weight fulfilling $\lambda(h_1) = \lambda(h_3)$ and $\lambda(h_2) = 0$. Evidently, the two irreducible spherical components of the adjoint representation are the spherical representations with the lowest possible highest weights, i.e. with the smallest possible dimensions.

3.6.5 Dual representations

Corollary 3.70. *The dual representation of the highest weight representation V_λ of highest weight λ is the lowest weight representation of lowest weight $-\lambda$.*

Proof of 3.70. This holds in general for highest weight representations by Definition 2.101 of the dual representation. The argument goes as follows: Choose a basis B of V_λ consisting of weight vectors. Then $v_\lambda \in B$ and we denote by $\mu_{-\lambda}$ the dual of v_λ with respect to this basis. Hence

$$\forall v \in V_\lambda : \pi^*(f_i)(\mu_{-\lambda})(v) = -\mu_{-\lambda}(\pi(f_i)v) = 0$$

because $\pi(f_i)v \in V^\lambda \Rightarrow v \in V_{\lambda+\alpha_i} = 0$, so $\mu_{-\lambda}$ is a lowest weight vector. We have $\pi^*(h_i)(\mu_{-\lambda})(v_\lambda) = -\lambda(h_i)$ hence $\mu_{-\lambda} \in (V_\lambda^*)^{-\lambda}$. $\square$

Corollary 3.71. *All finite dimensional spherical representations of $\mathfrak{gl}^{1|2+1}$ are self dual.*

Proof of 3.71. With the positive system of (3.13) we have $-\Delta^+ = r_{\alpha_2} \circ r_{\alpha_1+\alpha_2+\alpha_3}\Delta^+$ where r_{α_2} is the odd reflection at $\alpha_2 = r_{\alpha_1+\alpha_2+\alpha_3}(\alpha_2)$ and $r_{\alpha_1+\alpha_2+\alpha_3} \in W_0$ is an ordinary Weyl group element. This means the lowest weight of V_λ is $\nu = r_{\alpha_2} \circ r_{\alpha_1+\alpha_2+\alpha_3}(\lambda)$. More concretely $\nu(h_1, h_2, h_3) = \lambda(-h_1, h_2, -h_3)$. But since $\lambda(h_2) = 0$ for spherical representations by Corollary 3.64 this means $\nu = -\lambda$. By the previous Corollary 3.70 this establishes self duality. $\square$

4 Summary and outlook

As stressed throughout, the main point of this thesis is the generalisation of Helgason's theorem, $v_\lambda \in V_\lambda^M \Leftrightarrow V_\lambda^K \neq 0$, to the super case. For $(\mathfrak{g} = \mathfrak{gl}^{q|r+1}, \mathfrak{k} = \mathfrak{gl}^{q|r} \oplus \mathfrak{gl}^{0|1})$ with $r > q$ this has been accomplished and for high enough highest weight λ we can also drop the restriction on r and q. The weak point in the proof causing the limitation of the range of dimensions is the computation of the integral in Section 3.5.1. The next step for improving this part would be to work out a reduction of rank argument from $s > 1$ to the rank 1 case in a similar way as done in [Sch84]. We conjecture that hereby our result should be generalisable to higher rank cases, i.e. to $(\mathfrak{gl}^{q|r+s}, \mathfrak{gl}^{q|r} \oplus \mathfrak{gl}^{s})$ also for $s \geq 1$ without problems.

We have made an effort to stress that the general theory developed up until Section 3.3 is valid for any strongly reductive symmetric super pair of even type, in particular also for $(\mathfrak{gl}^{p+q|r+s})$ and $\mathfrak{osp}^{m|n}$ with a suitable involution. In fact our proof of $\dim(V_\lambda^K) \leq 1$ and $V_\lambda^K \neq 0 \Rightarrow v_\lambda \in V_\lambda^M$ only relies on the existence of a global Iwasawa decomposition of the super group pair and hence applies as it stands to any strongly reductive symmetric super pair of even type. Hence by Lemma 3.61 we can in fact conclude $\lambda\big|_{\mathfrak{h}\cap\mathfrak{k}} \equiv 0$ and $\forall \alpha \in \Sigma^+ : b(\lambda, \alpha) \in 2\mathbb{N}$ with respect to a compatible positive system for any finite dimensional spherical highest weight representation of any such symmetric super pair. For the converse, however, the crucial point for further generalisation of the method of proof by Schlichtkrul is again to show that $\int_{K/M} \pi v_\lambda \neq 0$. Here some further work on either the concrete computation of super integrals of the form of (3.6) for more general symmetric pairs or a more general argument is needed.

An interesting side development of this thesis was presented in Sections 3.4, namely the generalisation of induced representations to the realm of infinite dimensional modules of Lie super groups pairs. In Lemma 3.68 we showed that also in this setting induction from a minimal parabolic subgroup to the full group is on the level of K representations the same thing as induction from M to K. Further we could prove Frobenius reciprocity in this rather general case.

The next step on the route to harmonic analysis of symmetric super spaces is to establish the relation to spherical functions and hence to invariant differential operators. For $\mathfrak{gl}^{1|2+1}$ we have shown that all spherical representations are self dual, hence we can form the matrix coefficient, i.e. for $g \in_S \mathrm{Gl}^{1|2+1}$ we can define $\phi(g) := v_0^* (\pi(g)(v_0))$ where v_0 is the K-invariant vector. By definition ϕ is K-biinvariant and it is easily seen to be an eigenfunction of the super Laplacian, since V_λ is irreducible. It remains to be shown that all spherical functions are matrix coefficients and to investigate their asymptotics.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig verfasst habe. Ich habe alle Zitate und Quellen nach bestem Wissen und Gewissen kenntlich gemacht. Die Abbildungen sind von mir selbst mit Hilfe von 'TikZ' und 'PGF' erstellt worden. Es wurden keine weiteren Hilfsmittel benutzt.

Diese Arbeit ist in gleicher oder ähnlicher Form noch bei keiner anderen Prüfungsbehörde eingereicht worden.

Köln, den 04.10.2011,

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