Homological description of crystal structures on quiver varieties

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Zusammenfassung

Mittels Methoden der homologischen Algebra konstruieren wir einen expliziten Kristallisomorphism zwischen zwei Realisierungen der kristallinen Basen des negativen Teiles der Quantengruppe beziehungsweise der irreduziblen Höchstgewichtsdarstellungen von (fast allen) einfach verbundenen Lie Algebren. Die erste Realisierung, die wir betrachten, ist eine geometrische Konstruktion von Kashiwara und Saito mittels irreduzibler Komponenten bestimmter Köchervarietäten. Die zweite ist eine Realisierung mittels Isomorphieklassen von Köcherdarstellungen, entwickelt von Reineke unter Benutzung von Ringels Hall-Algebren-Ansatz. Den Zusammenhang der beiden Konstruktionen zeigen wir durch die Untersuchung hinreichend generischer Darstellungen der präprojektiven Algebra. Mit Hilfe der Beschreibung der kristallinen Basen durch semistandard Young Tableaux zeigen wir weiterhin, dass für Lie Algebren von Typ A der Kristallisomorphismus mit rein kombinatorischen Mitteln beschrieben werden kann.

Abstract

Using methods of homological algebra, we obtain an explicit crystal isomorphism between two realizations of crystal bases of the lower part of the quantized enveloping algebra and the irreducible highest weight representations of (almost all) simply-laced Lie algebras, respectively. The first realization we consider is a geometric construction in terms of irreducible components of certain quiver varieties established by Kashiwara and Saito. The second is a realization in terms of isomorphism classes of quiver representations obtained by Reineke using Ringel's Hall algebra approach to quantum groups. We connect the two constructions by studying certain sufficiently generic representations of the preprojective algebra. We further show that, in the type A situation, the crystal isomorphism can be described on the combinatorial level via semistandard Young tableaux.

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Contents

1 INTRODUCTION

BACKGROUND AND MAIN RESULTS

The study of configurations of linear maps between vector spaces leads naturally to the notion and study of representations of a quiver. A quiver is a finite directed graph and a representation of a quiver is given by assigning a finite dimensional vector space to each vertex of the graph and a linear map to each arrow.

In 1972, Gabriel classified all quivers of finite representation type (i.e. with finitely many isomorphism classes of indecomposable representations). He shows that the representation type of a quiver is already determined by its underlying undirected graph, called underlying diagram. In particular, a quiver Q is of finite representation type if and only if its underlying diagram is a simply-laced Dynkin diagram (i.e. of type A_n, D_n, E_6, E_7 or E_8). Furthermore, for Dynkin quivers there is a bijection between the isomorphism classes of indecomposable representations and the set of negative roots of the Lie algebra associated to the underlying Dynkin diagram.

Subsequently, the connection between representation theory of quivers and Lie algebras of simply-laced type has evolved into a rich area of research, culminating in the following result by Ringel ([21]). He shows that there is a $\mathbb{Q}(v)$-algebra isomorphism ([21]) between the (twisted, generic) Hall algebra $\mathscr{H}(\mathrm{Q})$, which is an associative algebra having as an underlying vector space a basis consisting of all isomorphism classes of representations of Q, and the quantized universal enveloping algebra $U_v(\mathfrak{n}^-)$ of the negative part $\mathfrak{n}^-$ of the semisimple Lie algebra $\mathfrak{g}$ associated to the Dynkin diagram of Q.

In [15], Lusztig geometrizes Ringel's arguments by studying the variety of modules over the preprojective algebra of a Dynkin quiver corresponding to $\mathfrak{g}$. He translates the multiplication of the Hall algebra into the language of perverse sheaves by which he obtains a unique basis $\mathcal{B}$ (known as the canonical basis) of $U_v(\mathfrak{n}^-)$ with particularly favorable properties. For instance the product of two elements of $\mathcal{B}$ is a linear combination of basis elements with coefficients in $\mathbb{Z}_{\geq 0}[v, v^{-1}]$. Furthermore, $\mathcal{B}$ has the following remarkable property. Let $V(\lambda)$ be a finite dimensional irreducible $U_v(\mathfrak{g})$-representation of highest weight λ and

$$\pi : U_v(\mathfrak{n}^-) \twoheadrightarrow V(\lambda)$$

a $U_v(\mathfrak{n}^-)$-module homomorphism sending $1 \in U_v(\mathfrak{n}^-)$ to a highest weight vector $v_\lambda \in V(\lambda)$. Lusztig shows that $\mathcal{B}(\lambda) = \{\pi(b) \mid b \in \mathcal{B} \text{ and } \pi(b) \neq 0\}$ forms a basis of $V(\lambda)$, known as the canonical basis of $V(\lambda)$.

Independently, Kashiwara ([8]) constructed a basis of $U_v(\mathfrak{n}^-)$ and $V(\lambda)$, resp., at the $v = \infty$ level by purely combinatorial means. He shows that this basis can be lifted to a global basis of $U_v(\mathfrak{n}^-)$ for a generic parameter v which coincide with the canonical basis (see [5]).

The crystal basis is the main subject of study of this thesis. Such a basis gives rise to a colored directed graph, called the crystal graph. The vertex set of this graph consists of the elements of the crystal basis while the arrows reflect the actions of the Kashiwara operators. These are modifications of the Chevalley generators, mapping a basis element of the crystal basis to another basis element or zero. The crystal graph may be seen as a combinatorial skeleton of the representation $V(\lambda)$ or of $U_v(\mathfrak{n}^-)$, respectively. For instance, the crystal graph of $V(\lambda)$ can be identified with a full subgraph of the crystal graph of $U_v(\mathfrak{n}^-)$ which reflects the projection $\pi : U_v(\mathfrak{n}^-) \twoheadrightarrow V(\lambda)$ on the combinatorial level. The crystal graph of $V(\lambda)$ gives further rise to a combinatorial way to obtain the character of the representation $V(\lambda)$. Moreover, crystal graphs behave very nicely with respect to taking tensor products and are thus very helpful for the determination of tensor product multiplicities.

Several realizations of crystal graphs have been introduced, most of them purely combinatorial (see e.g. [10] for a survey). In this thesis we focus on a geometric construction which is shown to yield the crystal basis of $U_v(\mathfrak{n}^-)$ (denoted by $B(\infty)$) in [12]. The vertices of the crystal graph are here given by the irreducible components of Lusztig's quiver varieties. This geometric realization is of particular interest because of its connection to the canonical basis which is constructed via perverse sheaves on very closely related varieties.

The main motivation of this thesis is to interpret the geometric construction of crystal graphs in a homological way in order to make it more accessible. In the geometric realization, the Kashiwara operators are given by bijections of irreducible components induced by certain geometric correspondences and it is, in general, hard to determine the images of an irreducible component under these bijections. The main result of this thesis is the construction of a crystal isomorphism between the geometric realization of crystal graphs and a realization introduced by Reineke in 1997 ([19]). In Reineke's construction, the vertices of the crystal graph of $U_v(\mathfrak{n}^-)$ are given by the isomorphism classes of representations of the Dynkin quiver associated to $\mathfrak{g}$ while the Kashiwara operators are obtained using the Hall algebra approach to quantum groups. The isomorphism between Reineke's and the geometric realization yields a homological description of the Kashiwara operators in the geometric setup in all simply-laced types except E_8. By this, we obtain an algorithm to determine the component $\tilde{f}_i X$, where X is an irreducible component of Lusztig's quiver variety.

Using ad-hoc methods, Savage ([25]) shows for $\mathfrak{g} = \mathrm{sl}_{n+1}$, that the irreducible components of Lusztig's quiver variety can be enumerated by certain combinatorial data which is in bijection to the Young tableaux appearing in the realization of the crystal graph of $U_v(\mathfrak{n}^-)$. This yields a combinatorial description of the actions of the Kashiwara operators on irreducible components of Lusztig's quiver varieties in type A. We explain how our description of the Kashiwara operators recovers his construction in this special case and are thus able to give a homological interpretation of his approach.

Moreover, for the non-exceptional simply-laced types, we consider the crystal graph of the finite dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ of highest weight λ which we denote by $B(\lambda)$. We translate the embedding of this graph as a full subgraph of the crystal graph of $U_v(\mathfrak{n}^-)$ from the geometric construction into the homological setting. We thereby get an explicit homological description of the irreducible components of Lusztig's quiver varieties which are contained in that subgraph.

We further explain how the set of components in $B(\lambda)$ can be identified with the irreducible components of varieties attached to quivers introduced by Nakajima. Using Borel-Moore homology on certain quiver varieties, Nakajima construct in [18] (a modified version of) the whole universal enveloping algebra $U(\mathfrak{g})$ such that the irreducible finite-dimensional $\mathfrak{g}$-modules are realized as the top Borel-Moore homology of Lagrangian subvarieties which we call Nakajima's quiver varieties. Using similar arguments as in [12], Saito shows in [23] that the irreducible components of Nakajima's quiver varieties give rise to the crystal graph $B(\lambda)$. This realization of the crystal graph is of particular interest since the vertices correspond to a set of naturally defined basis elements of the representation we consider. Using the identification of the vertices of the full subgraph of $B(\infty)$ with the irreducible components of Nakajima's quiver varieties, we give a criterion to determine whether the images of the Kashiwara operators on an irreducible component of Lusztig's quiver varieties are contained in the full subgraph $B(\lambda)$. Therefore we also get a homological description of the images of the Kashiwara operators on the irreducible components of Nakajima's quiver varieties.

In type A, we further recover the realization of $B(\lambda)$ in terms of semi-standard Young tableaux of shape λ introduced in [11]. Fix a dominant weight $\lambda = \sum_{i=1}^n w_i\omega_i$ where ω_i is the i-th fundamental weight of sl_{n+1}. As a byproduct, we show that the set of semi-standard Young tableaux of shape λ with entries in $\{1, 2, \ldots, n+1\}$ can be identified with functions

$$\gamma : \{(k, l) \in \mathbb{Z}^2 \mid 1 \leq k \leq l \leq n\} \to \mathbb{Z}_{\geq 0}$$

fulfilling the inequalities

$$\sum_{k=1}^{n} \gamma(i, k) - \sum_{k=i+1}^{n} \gamma(i+1, k) \leq w_i \quad \forall i \in I.$$

This yields an alternative combinatorial description of the crystal graph of $B(\lambda)$ for $\mathfrak{g} = \mathrm{sl}_{n+1}(\mathbb{C})$.

STRUCTURE AND CONTENT OF THIS THESIS

Let us now summarize the content of this thesis. In Section 2.1 we review the theory of $\mathfrak{g}$-crystals. We recall the notions of crystal lattices, crystal bases and Kashiwara's description of the crystals $B(\lambda)$ associated to the irreducible finite dimensional $\mathfrak{g}$-module of highest weight λ. In particular, we have:

$$B(\lambda) = \{\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}v_\lambda \mid i_j \in I \ \forall j\}\backslash\{0\},$$

where $\tilde{f}_i$ are the Kashiwara operators for $i \in I$ and v_λ is a highest weight vector of $V(\lambda)$. There is an arrow $b_1 \to b_2$ of color i in the crystal graph of $B(\lambda)$ if and only if $\tilde{f}_i b_1 = b_2$. Note that $B(\lambda)$ has a unique source, the element v_λ.

We further recapitulate the notion of the infinite crystal $B(\infty)$ which is associated to $U_v(\mathfrak{n}^-)$. On the crystal level, the realization of $V(\lambda)$ as a quotient of $U_v(\mathfrak{n}^-)$ is translated into the following fact: We can realize $B(\lambda)$ as a full subgraph of $B(\infty)$ which contains the unique source of $B(\infty)$. The combinatorics behind this are given in terms of certain functions ε_i^* on $B(\infty)$, namely for $\lambda = \sum_{i=1}^n w_i\omega_i$ (where ω_i are the fundamental weights of $\mathfrak{g}$), we have

$$B(\lambda) = \{b \in B(\infty) \mid \varepsilon_i^* b \leq w_i \ \forall i \in I\}.$$

In Section 2.2, we recall the notion of the Auslander-Reiten quiver of a Dynkin quiver Q which has as vertices the isomorphism classes of indecomposable representations of Q (or equivalently modules over the path algebra kQ for k the base field) while the arrows are given by irreducible morphisms between those. This quiver provides the combinatorial framework for the homological description of the crystal structure on (irreducible components of) quiver varieties. We thus review a combinatorial construction of the Auslander-Reiten quiver as well as some basic facts about its structure and representations of quivers in general that we need later on. In Section 2.3, we shortly recall the definition of the Hall algebra.

Section 3 deals with the description of the crystal $B(\infty)$ which is the main part of this thesis. In Subsection 3.1, Reineke's construction of $B(\infty)$ as the set of isomorphism classes of representations of Q is introduced. We denote this crystal structure by $B^{\mathscr{H}}(\infty)$. An element $b \in B^{\mathscr{H}}(\infty)$ is given by an isomorphism class $[M]$. Reineke shows that if there is an exact sequence of Q-representations

$$0 \to M \to X \to S(i) \to 0,$$

such that $[X]$ satisfies certain properties, we have $\tilde{f}_i[M] = [X]$. He further proves that for any Q-representation M there exists a representation X satisfying these properties by classifying the middle terms of short exact sequences of kQ-modules ending in $S(i)$. He thereby obtains an algorithm for the computation of the actions of the Kasiwara operators in terms of the given combinatorial data $(\mu_B(M))_B$, where B varies over all indecomposable Q-representations. Here $\mu_B(M)$ denotes the multiplicity of the indecomposable direct summands B of M.

We recall the geometric construction of $B(\infty)$ as irreducible components of Lusztig's quivers varieties (denoted by $B^g(\infty)$) in Section 3.2. Points in those varieties correspond naturally to representations of the preprojective algebra $\Pi(\mathrm{Q})$, which is a finite dimensional algebra associated to Q.

The Kashiwara operators can be described in the following way. Let $x \in X$ be a generic point of an irreducible component X and let $M(x)$ be the corresponding $\Pi(\mathrm{Q})$-module. The component $\tilde{f}_i X$ is given as the closure of all points y that (regarded as $\Pi(\mathrm{Q})$-modules) appear as the middle term of exact sequences

$$0 \to M(x) \to M(y) \to S(i) \to 0,$$

where $S(i)$ is the simple representation of $\Pi(\mathrm{Q})$ corresponding to the vertex i.

In Section 3.3, we give a crystal isomorphism between $B^{\mathscr{H}}(\mathrm{Q})$ and $B^g(\infty)$. We first describe how to translate the vertices. For this, a result by Lusztig is used, giving a one-to-one correspondence between the irreducible components of quiver varieties and isomorphism classes of representations of Q (see Proposition 3.30). We then work in a homological algebra setting using Ringel's description of $\Pi(\mathrm{Q})$-modules as pairs (M, ϕ) for $\phi \in \mathrm{Hom}_{k\mathrm{Q}}(\tau^{-1}M, M)$, where τ^{-1} is the inverse Auslander-Reiten translation of Q (see Section 2.2 for a definition). Let $M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ and $X_{[M]}$ be the irreducible component corresponding to the isomorphism class of M. We prove that the function ε_i on $X_{[M]}$ (which counts how many consecutive times we applied $\tilde{f}_i$ to get to the desired vertex in the crystal graph) in the geometric setting only depends on the data $(\mu_B(M))_B$. We further show that there is a dense subset of $X_{[M]}$ which is mapped to the component $X_{\tilde{f}_i[M]}$ by $\tilde{f}_i$. This is proved by constructing a certain class of points of an irreducible component which are sufficiently generic but can be handled combinatorially.

In Section 3.4 we focus on quivers of type A. We recover a result of Savage (see [25]) showing that there is a natural translation in our setting to the crystal structure of $B(\infty)$ given by semistandard Young tableaux. This is done by linking the multiplicities of direct summands of $M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ to the entries in the tableaux.

Section 4 deals with the crystal $B(\lambda)$ and the embedding

$$\rho : B(\lambda) \hookrightarrow B(\infty).$$

First, we recall Reineke's description of the functions ε_i^* on the combinatorial data $(\mu_B(M))_B$ of an isomorphism class of kQ-modules in Section 4.1. We explain in Section 4.2 how ρ is described as an embedding of irreducible components of quiver varieties. Here, the crystal graph of $B(\lambda)$ has as vertex set the irreducible components in $B^g(\infty)$ containing an open dense subset fulfilling a stability condition.

The functions ε_i^* provide a condition to check whether an irreducible component of $B^g(\infty)$ is already in $B(\lambda)$.

By dualizing the arguments of Section 3.3, we show in Section 4.3 that for an irreducible component $X_{[M]}$, the functions ε_i^* only depend on the data $(\mu_B(M))_B$. We thus get a homological description of the irreducible components in $B(\lambda)$ seen as a subgraph of $B^g(\infty)$.

In Section 4.4, we examine the connection between the functions ε_i^* on $\rho(B(\lambda)) \subset B^g(\infty)$ and the functions φ_i which counts how many consecutive times we can apply $\tilde{f}_i$ to an irreducible component $X_{[M]}$ without mapping it to zero. This allows us to decide whether the irreducible component $\tilde{f}_i X_{[M]}$ still fulfills the stability condition. We thus obtain a self-contained description of $B(\lambda)$ (not only as a subgraph of $B(\infty)$) such that the crystal operators on $B(\lambda)$ are given in explicit combinatorial terms using the data $(\mu_B(M))_B$.

Finally, Section 4.5 is again devoted to the type A situation. We show that $\rho(B(\lambda))$ naturally corresponds to the crystal structure on semistandard Young tableaux of shape λ with entries in $\{1, 2, \ldots, n+1\}$.

2 PRELIMINARIES

2.1 CRYSTAL BASES

Throughout this thesis, $\mathfrak{g}$ is a finite dimensional simple Lie algebra over the field of complex numbers $\mathbb{C}$ of simply laced type A_n, D_n, E_6 or E_7.

Let us start by recalling the notion of crystal bases briefly. Unless otherwise stated, details, proofs and precise statements can be found in [9] and [6].

A good approach to study finite dimensional $\mathfrak{g}$-modules is to find a well-behaved bases of these. A favorable property of such a bases would be to be preserved by the Chevalley generators of $\mathfrak{g}$. Unfortunately, for $\mathfrak{g} \neq \mathrm{sl}_2$ very few $\mathfrak{g}$-modules posses such a basis. This problem can be partially solved for finite dimensional highest weight representations of the quantum group $U_v(\mathfrak{g})$ (in the $v \to \infty$ limit by considering a slight renormalization of the Chevalley generators). Passing to the quantum group is possible since every finite dimension irreducible highest weight representation of $U(\mathfrak{g})$ can be lifted in an easily controlled way to an irreducible highest weight $U_v(\mathfrak{g})$-representation such that the weight space multiplicities are preserved (see [6, Theorem 3.4.6]).

We begin by fixing some notation. Let v be an indeterminate and let $A' = \mathbb{Q}(v)$ be the rational function field in v and $A = \mathbb{Z}[v, v^{-1}]$ be its subring of Laurent polynomials with integer coefficients. Let $(a_{i,j})_{1 \leq i,j \leq n}$ be the symmetric Cartan matrix of $\mathfrak{g}$ and let further Q be the free $\mathbb{Z}$-module with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ which we call the **root lattice**. We define a form on Q via $(\alpha_i, \alpha_j) = a_{i,j}$. Then

$(-,-)$ extends to an inner product on $Q_{\mathbb{R}} := Q \otimes_{\mathbb{Z}} \mathbb{R}$, the **Cartan form**. We fix a root system R of $\mathfrak{g}$ as $R = \{\alpha \in Q \mid (\alpha, \alpha) = 2\}$. Then the set of **simple roots** is precisely $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ while the set of **positive roots** is given by $R^+ = \{\alpha \in R \mid \alpha \in \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \ldots + \mathbb{Z}_{\geq 0}\alpha_n\}$ and the set of **negative roots** is given by $-R^+$. Let now $Q^*_{\mathbb{R}}$ be the dual vector space of $Q_{\mathbb{R}}$. We choose a basis $\{h_1, h_2, \ldots, h_n\}$ of $Q^*_{\mathbb{R}}$ such that $\langle h_i, \alpha_j \rangle = a_{i,j}$ where $\langle -, - \rangle$ is the dual pairing. Finally, the set of **fundamental weights** $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is the dual basis to $\{h_1, h_2, \ldots, h_n\}$. The $\mathbb{Z}$-lattice generated by the fundamental weights in Q is called the **weight lattice** P and the **dominant weights** P^+ are the elements $\lambda \in P$ such that $\langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for each $i \in I$.

Let us now recall the definition of the quantum group $U_v(\mathfrak{g})$ which is closely related to a deformation of the universal enveloping algebra $U(\mathfrak{g})$ as a Hopf algebra. Indeed we have a notion of classic limit $v \to 1$ which tends to $U(\mathfrak{g})$ (see [6, Theorem 3.4.9] for a precise statement).

For $N, M \in \mathbb{Z}_{\geq 0}$ we introduce the following abbreviations

$$[N] = \tfrac{v^N - v^{-N}}{v - v^{-1}}, \qquad [N]! = \prod_{k=1}^{N} \frac{v^k - v^{-k}}{v - v^{-1}} \in A, \qquad \left[\begin{smallmatrix} M+N \\ N \end{smallmatrix}\right] = \tfrac{[M+N]!}{[M]![N]!} \in A.$$

Then the **quantum group** (or **quantized enveloping algebra**) $U_v(\mathfrak{g})$ is defined as the A'-algebra generated by the elements E_i, F_i K_i and K_i^{-1} for for $i \in I = \{1, 2, \ldots, n\}$ subject to the following relations

$$(i) \qquad K_i K_i^{-1} = 1; \qquad K_i^{-1} K_i = 1 \text{ and } K_i K_j = K_j K_i;$$

$$(ii) \qquad K_i E_j K_i^{-1} = v^{a_{i,j}} E_j \text{ and } K_i F_j K_i^{-1} = v^{-a_{i,j}} F_j;$$

$$(iii) \qquad E_i F_j - F_j E_i = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{v - v^{-1}};$$

$$(iv) \qquad E_i E_j = E_j E_i \text{ if } a_{i,j} = 0, \quad E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \text{ if } a_{i,j} = -1.$$

$$(v) \qquad F_i F_j = F_j F_i \text{ if } a_{i,j} = 0, \quad F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \text{ if } a_{i,j} = -1.$$

We further have a **quantized notion of the upper** (and analog of the lower) **part of the enveloping algebra** of $\mathfrak{g}$. Let therefore $U'_v(\mathfrak{n}^+)$ (resp. $U_v(\mathfrak{n}^-)$) be the A'-algebra with generators $E_i : i \in I$ (resp. $F_i : i \in I$ and relation (iv) (resp. (v)) above.

We have a triangular decomposition (as vector spaces) similar to that of $U(\mathfrak{g})$:

$$U_v(\mathfrak{g}) \cong U_v'(\mathfrak{n}^+) \otimes A'[K_i^{\pm 1} \mid i \in I] \otimes U_v'(\mathfrak{n}^-).$$

To be able to introduce a well-behaved A-subalgebra, we consider a larger set of generators. For this, we define for $N \in \mathbb{Z}_{\geq 0}$

$$E_i^{(N)} := \frac{1}{[N]!} E_i^N.$$

$$F_i^{(N)} := \frac{1}{[N]!} F_i^N.$$

and let $U_v(\mathfrak{n}^+)$ (resp. $U_v'(\mathfrak{n}^-)$) be the A-subalgebra of $U_v'(\mathfrak{n}^+)$ generated by the elements $E_i^{(M)}$ (resp. $F_i^{(M)}$) with $1 \leq i \leq n$ and $M \geq 0$.

Let us now turn our attention to representations of $U_v(\mathfrak{g})$. Let M be a finite dimensional $U_v(\mathfrak{g})$-module. Then M decomposes by analogy with $U(\mathfrak{g})$-modules as follows

$$M = \bigoplus_{\mu \in P} M_\mu, \text{ where}$$

$$M_\mu = \{m \in M \mid K_i^{\pm 1} m = v^{\langle h_i, \mu \rangle} m \ \forall i \in I\}.$$

We call M_μ the **μ-weight space** of M. If we further assume that M is irreducible, we have a unique highest weight $\lambda \in P^+$ such that $M_\lambda \neq 0$ and a unique (up to scalar multiple) highest weight vector $v_\lambda \in M_\lambda$. Hence, as in the classical case, the irreducible finite dimension $U_v(\mathfrak{g})$-modules are parametrized by their highest weights $\lambda \in P^+$. We denote the irreducible module with highest weight λ by $V(\lambda)$.

Note that, for each $i \in I$ the elements $K_i^{\pm 1}$, E_i, F_i generate a subalgebra U_i of $U_v(\mathfrak{g})$ isomorphic to $U_v(\mathrm{sl}_2)$. Let M be a irreducible finite dimensional $U_v(\mathfrak{g})$-module, then (by the representation theory of $U_v(\mathrm{sl}_2)$) for each $m \in M$ we can find an $N \in \mathbb{Z}_{\geq 0}$ unique elements $m_0, m_1, \ldots, m_N$ of M which lie in $\operatorname{Ker} E_i$ such that

$$m = m_0 + F_i^{(1)} m_1 + \ldots + F_i^{(N)} m_N.$$

We define the **Kashiwara operators** (or crystal operators) $\tilde{e}_i$ and $\tilde{f}_i$ in $\mathrm{End}_{\mathbb{Q}(v)}(M)$

$$\tilde{e}_i m := \sum_{k=1}^{N} F_i^{(k-1)} m_k \qquad \tilde{f}_i m := \sum_{k=0}^{N} F_i^{(k+1)} m_k.$$

We should now give a more precise statement of the existence of the basis of M which is preserved by the Kashiwara operators. In order to do that let $\mathcal{A}_0 = \mathbb{Q}[v^{-1}]_0 \subset \mathbb{Q}(v)$ be the localization at 0 of $\mathbb{Q}[v^{-1}]$. A free $\mathcal{A}_0$-submodule $\mathcal{L}$ of M such that $\mathcal{L} \otimes_{\mathcal{A}_0} \mathbb{Q}(v) = M$, $\mathcal{L} = \oplus_{\mu \in P} \mathcal{L}_\mu$ where $\mathcal{L}_\mu = \mathcal{L} \cap M_\mu$ and such that $\mathcal{L}$ is preserved under the Kashiwara-operators for all $i \in I$ is called a **crystal lattice** of M. A pair $(\mathcal{L}, B)$ with $\mathcal{L}$ being a crystal lattice is called a **crystal basis** if additionally

$$\begin{array}{ll} (i) & B \text{ is a } \mathbb{Q}\text{-basis of } \mathcal{L}/v^{-1}\mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{A}_0} \mathbb{Q}, \\ (ii) & B = \bigsqcup_{\mu \in P} B_\mu \text{ where } B_\mu = B \cap (\mathcal{L}_\mu / v^{-1}\mathcal{L}_\mu), \\ (iii) & \tilde{e}_i B \subset B \cup \{0\},\ \tilde{f}_i B \subset B \cup \{0\} \text{ for all } i \in I \text{ and} \\ (iv) & \text{for any } b_1, b_2 \in B \text{ and for all } i \in I \text{, we have} \\ & \tilde{e}_i b_1 = b_2 \iff \tilde{f}_i b_2 = b_1. \end{array}$$

Thus, heuristically, if $(\mathcal{L}, B)$ is a crystal bases of M, one can think of B as a basis of M in the limit $v \to \infty$.

Let $(\mathcal{L}, B)$, $(\mathcal{L}', B')$ be two crystal bases of an $U_v(\mathfrak{g})$-module M, then we say that an $\mathcal{A}_0$-linear isomorphism $\phi : \mathcal{L} \to \mathcal{L}$ is an **isomorphism of crystal bases** if ϕ commutes with the Kashiwara operators and maps B to B'.

Kashiwara proved the well-known statement that each finite dimensional irreducible highest weight $U_v(\mathfrak{g})$-module $V(\lambda)$ possesses a unique (up to isomorphism) crystal basis. Moreover, let v_λ be a highest weight vector of $V(\lambda)$, then a crystal basis $(\mathcal{L}(\lambda), B(\lambda))$ of $V(\lambda)$ is given as follows

$$\begin{aligned} \mathcal{L}(\lambda) &= \sum_{i_1, i_2, \ldots \in I} \mathcal{A}_0 \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} v_\lambda \\ B(\lambda) &= \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} v_\lambda \mid i_1, i_2, \ldots, i_l \in I\} \backslash \{0\} \end{aligned}$$

Associated to a crystal basis, we can draw the **crystal graph** which picturizes the action of the Kashiwara operators on that basis. This is the colored directed graph with vertex set B and, for each two elements b_1, b_2 of B such that $\tilde{f}_i b_1 = b_2$ for an $i \in I$, we draw an arrow colored with i:

$$b_1 \xrightarrow{i} b_2.$$

Note that (because of the uniqueness of crystal bases) each two crystal graphs of an irreducible finite dimensional highest weight $U_v(\mathfrak{g})$ module are isomorphic as colored directed graphs, i.e. the two graphs are equal up to a relabeling of vertices.

Since the Kashiwara operators act locally nilpotent on $B(\lambda)$, we can introduce two functions for $i \in I$ and $b \in B(\lambda)$:

$$\varepsilon_i(b) = \max_k \{\tilde{e}_i^k b \neq 0\},$$

$$\varphi_i(b) = \max_k \{\tilde{f}_i^k b \neq 0\}.$$

Additionally, we have a map $\mathrm{wt} : B(\lambda) \to P$ which is given on an element $b = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} v_\lambda$ as

$$\mathrm{wt}(b) = \lambda - \alpha_{i_1} - \alpha_{i_2} - \ldots - \alpha_{i_l}.$$

We further have an abstract notion of crystals which is often helpful in proofs.

Definition 2.1 A $\mathfrak{g}$**-crystal** is a set B endowed with the following maps for all $i \in I$:

$$\begin{array}{lll} \mathrm{wt}: B \to P & \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\} & \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \\ \tilde{e}_i : B \to B \sqcup \{0\} & \tilde{f}_i : B \to B \sqcup \{0\}, & \end{array}$$

such that the following axioms are satisfied

- $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for all $i \in I$,
- if $\tilde{e}_i b \neq 0$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ and $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$,
- if $\tilde{f}_i b \neq 0$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ and $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$,
- for $b_1, b_2 \in B$, we have: $\tilde{f}_i b_1 = b_2$ if and only if $\tilde{e}_i b_2 = b_1$
- if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

In the Definition above, $-\infty$ is the smallest number of $\mathbb{Z} \sqcup \{-\infty\}$ and zero is a "ghost element" not included in B (see [9, Section 7.2]).

An **isomorphism of two $\mathfrak{g}$-crystals** B_1 and B_2 is a bijection between B_1 and B_2 commuting with the actions of the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ and the maps wt, φ_i and ε_i for all $i \in I$.

Note that the crystal graph of a crystal basis gives rise to a crystal but not every abstract crystal corresponds to a crystal basis.

In this thesis we are mainly interested in a particular crystal graph, namely the one associated to the lower (resp. upper) half $U_v(\mathfrak{n}^-)$ (resp. $U_v(\mathfrak{n}^+)$) of the universal enveloping algebra itself which we denote by $B(\infty)$ (resp. $B(-\infty)$). Though this is not an integrable $U_v(\mathfrak{g})$-module, we also have the notion of a crystal basis here. This crystal is of particular importance since the crystal graph of any irreducible finite dimensional highest weight representation $B(\lambda)$ can be realized as a full subgraph of $B(\infty)$ in a concrete fashion using an involution on $B(\infty)$ (the Kashiwara-involution). This comes from the fact that we have, as in the classical case, an analogues Verma-module construction for highest weight $U_v(\mathfrak{g})$-modules, i.e. for each $\lambda \in P^+$ there is a surjective $U_v(\mathfrak{n}^-)$-linear homomorphism

$$\pi_\lambda : U_v(\mathfrak{n}^-) \to V(\lambda)$$

defined by $\pi_\lambda(u) = uv_\lambda$ for $u \in U_v(\mathfrak{n}^-)$ and v_λ a highest weight vector of $V(\lambda)$ (see also Section 8 of [9]).

To define a crystal basis for $U_v(\mathfrak{n}^-)$, we first have to slightly modify the operators E_i. We deduce from the triangular decomposition of $U_v(\mathfrak{g})$ that for any $u \in U'_v(\mathfrak{n}^-)$ there exist unique elements $R, S \in U'_v(\mathfrak{n}^-)$ such that

$$[E_i, u] = \frac{K_i S - K_i^{-1} R}{v - v^{-1}}.$$

For each $i \in I$, let e_i' be the linear endomorphism of $U_v'(\mathfrak{n}^-)$ defined by $e_i'u = R$. It is a straightforward calculation to show that for $u \in U_v(\mathfrak{n}^-)$, we have that $R \in U_v(\mathfrak{n}^-)$. Hence e_i' is a linear endomorphism of $U_v(\mathfrak{n}^-)$. For any $u \in U_v(\mathfrak{n}^-)$, we can then find $N \in \mathbb{Z}_{\geq 0}$ and unique elements $u_0, u_1, \ldots, u_N$ which lie in $\operatorname{Ker} e_i'$ such that

$$u = u_0 + F_i^{(1)} u_1 + \ldots + F_i^{(N)} u_N.$$

Thus we can define the Kashiwara operators $\tilde{e}_i,\ \tilde{f}_i \in \operatorname{End}(U_v(\mathfrak{n}^-))$ in a similar fashion as before:

$$\tilde{e}_i(u) := \textstyle\sum_{k=1}^N F_i^{(k-1)} u_k, \qquad\qquad \tilde{f}_i(u) := \textstyle\sum_{k=0}^N F_i^{(k+1)} u_k.$$

Then we find that $U_v(\mathfrak{n}^-)$ posses a unique crystal basis (up to isomorphism). It is given by the pair $(\mathcal{L}(\infty), B(\infty))$, where

$$\begin{aligned} \mathcal{L}(\infty) \quad &= \sum_{i_1, i_2, \ldots} \mathcal{A}_0 \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} 1 \\ B(\infty) \quad &= \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} 1\} \backslash \{0\} \end{aligned}$$

for $i_k \in I$ for all k.

In order to obtain a crystal structure on $B(\infty)$ we further set for $b \in B(\infty),\ b = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_l} 1$ and $i \in I$

$$\begin{aligned} \mathrm{wt}(b) &= -\alpha_{i_1} - \alpha_{i_2} - \ldots - \alpha_{i_l}; \\ \varepsilon_i(b) &= \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\}; \\ \varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle\,. \end{aligned}$$

We continue with giving a precise description of the embedding of $B(\lambda)$ in $B(\infty)$ for $\lambda \in P^*$. Therefore we introduce the aforementioned involution on $B(\infty)$.

First let us define a $\mathbb{Q}(v)$-linear algebra antiautomorhism $*$ of $U_v(\mathfrak{g})$ given by

$$\begin{aligned} E_i^* &= E_i; \\ F_i^* &= F_i; \\ K_i^* &= K_i^{-1}. \end{aligned}$$

Then $U_v(\mathfrak{n}^-)$ is clearly invariant under $*$. We further have

$$\mathcal{L}(\infty)^* = \mathcal{L}(\infty); \qquad\qquad B(\infty)^* = B(\infty).$$

We call $*$ the **Kashiwara involution**. By setting

$$\begin{aligned} \tilde{e}_i^*(b) \quad &= (\tilde{e}_i(b^*))^*; \qquad & \tilde{f}_i^*(b) \quad &= (\tilde{f}_i(b^*))^*; \\ \varepsilon_i^*(b) \quad &= \varepsilon_i(b^*); & \varphi_i^*(b) \quad &= \varphi_i(b^*) \end{aligned}$$

we obtain a new crystal structure on $B(\infty)$ which we denote by $B(\infty)^*$. Note that $\mathrm{wt}^*(b) = \mathrm{wt}(b)$.

Theorem 2.2 ([9, Proposition 8.2]) *Let $\lambda = \sum_{i \in I} w_i \omega_i \in P^+$, where $\omega_1, \omega_2, \ldots, \omega_n$ are the fundamental weights of $\mathfrak{g}$. Then the crystal graph $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ can be realized as the full subgraph of $B(\infty)$, consisting of all vertices $b \in B(\infty)$ such that $\varepsilon_i^*(b) \leq w_i$ for all $i \in I$.*

In other words: Denote by B^λ the set of all vertices lying in this subgraph. For $b \in B(\lambda)$, set

$$\begin{aligned}
\tilde{f}_i^\lambda(b) &= \tilde{f}_i(b), \\
\tilde{e}_i^\lambda(b) &= 0 \text{ for } b \notin B^\lambda \text{ and} \\
\tilde{e}_i^\lambda(b) &= \tilde{e}_i(b) \text{ for } b \in B^\lambda, \\
\mathrm{wt}^\lambda(b) &= \lambda - \mathrm{wt}(b), \\
\varepsilon_i^\lambda(b) &= \varepsilon_i(b) \text{ and} \\
\varphi^\lambda(b) &= \varphi(b) + w_i.
\end{aligned}$$

Then $(B^\lambda, \tilde{f}_i^\lambda, \tilde{e}_i^\lambda, \mathrm{wt}^\lambda, \varepsilon_i^\lambda, \varphi_i^\lambda)$ is isomorphic to $B(\lambda)$ as abstract crystals.

We end this section by noting that we have an analogues notion of crystal basis for the upper half of the quantum group $U_v(\mathfrak{n}^+)$ and finite dimensional irreducible lowest weight modules $V(-\lambda)$ for $\lambda \in P^+$. For a $\mathfrak{g}$-crystal B, let therefore $B^\vee = \{b^\vee \mid b \in B\}$ be the crystal obtained by

$$\begin{aligned}
\mathrm{wt}(b^\vee) &= -\mathrm{wt}(b); & \varepsilon_i(b^\vee) &= \varphi_i(b); & \varphi_i(b^\vee) &= \varepsilon_i(b); \\
\tilde{e}_i b^\vee &= (\tilde{f}_i b)^\vee; & \tilde{f}_i b^\vee &= (\tilde{e}_i b)^\vee.
\end{aligned}$$

We set $B(\lambda)^\vee = B(-\lambda)$ (resp. $B(\infty)^\vee = B(-\infty)$). This crystal can then be regarded as the crystal graph associated to the finite dimensional irreducible lowest weight modules $V(-\lambda)$ (resp. to the upper half of the quantum group $U_v(\mathfrak{n}^+)$).

2.2 REPRESENTATION THEORY OF QUIVERS

In Section 3, we deal with a realization of the crystal graph $B(\infty)$ via the Hall algebra approach given in [19]. An indispensable tool here is the Auslander–Reiten quiver, we therefore recall some basic facts about representations of quivers. Details and proofs can be found in [1] and [4].

A **quiver** Q is a finite directed graph and therefore given by a pair $(\mathrm{Q}_0, \mathrm{Q}_1)$ with Q_0 the set of vertices and Q_1 the set of arrows. For an arrow $a \in \mathrm{Q}_1$ we denote by $\mathrm{out}(a)$ its starting vertex and by $\mathrm{in}(a)$ its ending vertex. By fixing a labeling of the vertices we will constantly identify Q_0 with $I = \{1, 2, \ldots, n = |\mathrm{Q}_0|\}$.

For an arbitrary field k let $k\mathrm{Q}$ be the **path algebra** of Q. A basis of the underlying k-vector space of $k\mathrm{Q}$ is given by the set of all paths of the quiver along its arrows, including a trivial path of length 0 for each vertex, starting and ending at this vertex. The multiplication is given by concatenation of paths. If the starting vertex of the first path is not equal to the ending vertex of the second, their product is defined to be zero.

We define $k\mathrm{Q}-\mathrm{mod}$ to be **the category of all finite–dimensional** $k\mathrm{Q}$**–modules**, which is equivalent to the category of finite–dimensional **representations of the quiver** Q. A representation of Q is a collection $V = ((V_i)_{i\in \mathrm{Q}_0}, (x_a)_{a\in\mathrm{Q}_1})$, consisting of a vector space V_i for each vertex $i \in \mathrm{Q}_0$ and a linear map $x_a : V_{\mathrm{out}(a)} \to V_{\mathrm{in}(a)}$ for all arrows $a : \mathrm{out}(a) \to \mathrm{in}(a)$ in Q_1.

We call $M \in k\mathrm{Q}-\mathrm{mod}$ **indecomposable** if it is not isomorphic to the direct sum of two other non-trivial representations. Every $M \in k\mathrm{Q}-\mathrm{mod}$ can be uniquely decomposed (up to isomorphism) into a direct sum of indecomposable direct summands. For an indecomposable representation $N \in k\mathrm{Q}-\mathrm{mod}$, we denote by $\mu_N(M)$ the **multiplicity of** N **as a direct summand of** M.

To each representation $M = (V, x)$ of Q, we can assign the **dimension vector** $\underline{\dim}V \in \mathbb{Z}^I_{\geq 0}$ of M via $\underline{\dim}V = (\dim V_i)_{i\in I}$.

A kQ-module $M = (V, x)$ is **simple** if it has no non-trivial subrepresentations. That is a kQ-module $N = (V', x')$ such that N_i is a vector subspace of M_i for all $i \in \mathrm{Q}_0$ and for all $a \in \mathrm{Q}_1$ the restriction of $x_a : V_{\mathrm{out}(a)} \to V_{\mathrm{in}(a)}$ to $V'_{\mathrm{out}(a)}$ equals $x'_a : V'_{\mathrm{out}(a)} \to V'_{\mathrm{in}(a)}$.

A **morphism** between two representations $M = (V, x)$ and $M' = (V', x')$ of Q is a tuple $\phi = (\phi_i)_{i\in\mathrm{Q}_0}$ where $\phi_i : V_i \to V'_i$ is a homomorphism of vector spaces such that for all $a \in \mathrm{Q}_1$ the following diagram commutes:

(1)
$$\begin{tikzcd}
V_{\mathrm{out}(a)} \arrow[r, "x_a"] \arrow[d, "\phi_{\mathrm{out}(a)}"] & V_{\mathrm{in}(a)} \arrow[d, "\phi_{\mathrm{in}(a)}"] \\
V'_{\mathrm{out}(a)} \arrow[r, "x'_a"] & V'_{\mathrm{in}(a)}
\end{tikzcd}$$

A morphism ϕ between two representations of Q is called an **isomorphism** if ϕ_i is an isomorphism for all $i \in \mathrm{Q}_0$.

For $i \in \mathrm{Q}_0$, let $e^i \in \mathbb{Z}^I_{\geq 0}$ be such that $e^i_j = \delta_{ij}$ and let $S(i)$ be the kQ-module with dimension vector e^i. Then $S(i)$ is simple and, if we assume Q to by acyclic, every simple representation of Q is isomorphic to such an $S(i)$.

For $M \in k\mathrm{Q}-\mathrm{mod}$ we denote by $[M]$ the isomorphism class of M and recall **Gabriel's Theorem** which was the first milestone in establishing the connection between representation theory of quivers and Lie algebras.

Theorem 2.3 ([1, Theorem 5.10]) *Let* Q *be a Dynkin quiver (i.e. of type ADE) and let* α_i*,* $i \in I$*, be the set of simple roots of the Lie algebra* $\mathfrak{g}$ *associated to the* *underlying Dynkin diagram. Then the assignment*

$$[V] \to \sum_{i \in I} (\dim V_i)\alpha_i$$

is a one-to-one correspondence between the isomorphism classes of indecomposable representations of Q *and the negative roots of* $\mathfrak{g}$*. Furthermore these are the only quivers with finitely many isomorphism classes of indecomposable representations.*

Remark 2.4 For a Dynkin quiver Q, a positive root α and a field k, let us denote by $M(\alpha, k)$ a representative of the isomorphism class of indecomposable kQ-modules associated to this root via Theorem 2.3. Then we get a one-to-one correspondence between isomorphism classes of kQ-representations and functions $R^- \to \mathbb{Z}_{\geq 0}$ by mapping $[M]$ ($M \in k\mathrm{Q} - \mathrm{mod}$) to the function $\gamma_M : \alpha \mapsto \mu_{M(\alpha,k)}(M)$. Conversely, for a function $\gamma : R^- \to \mathbb{Z}_{\geq 0}$, we get a representative M of an isomorphism class of kQ-modules via $M = \oplus_{\alpha \in R^-} M(\alpha, k)^{\gamma(\alpha)}$ which we denote by $M(\gamma, k)$.

Remark 2.5 For Q an arbitrary acyclic quiver, we have the **standard duality functor** $\mathrm{D} : k\mathrm{Q} - \mathrm{mod} \to (k\mathrm{Q})^{\mathrm{op}} - \mathrm{mod}$ between the category of kQ-modules and the category of $(k\mathrm{Q})^{\mathrm{op}}$-modules where $(k\mathrm{Q})^{\mathrm{op}}$ is the opposite algebra of kQ. For $M \in k\mathrm{Q} - \mathrm{mod}$ it is given by $\mathrm{D}\, M = \mathrm{Hom}_k(M, k)$ and for $M, N \in k\mathrm{Q} - \mathrm{mod}$, $f \in \mathrm{Hom}_{k\mathrm{Q}}(M, N)$ it is given by $\mathrm{D}\, f = \mathrm{Hom}_k(f, k) : \mathrm{D}\, N \to \mathrm{D}\, M,\ \phi \mapsto \phi \circ f$ (compare with [1, Section 2.9, page 12]).

On $k\mathrm{Q} - \mathrm{mod}$ we have a non-degenerate bilinear form called the **Euler form** (also called Ringel form) given by:

$$\langle M, N \rangle_R := \dim \mathrm{Hom}_{k\mathrm{Q}}(M, N) - \dim \mathrm{Ext}^1(M, N)$$

which is known to depend only on the dimension vectors $\underline{\dim} M$ and $\underline{\dim} N$ and to be equal to

$$\sum_{j \in \mathrm{Q}_0} \dim M_j \dim N_j - \sum_{a \in \mathrm{Q}_1} \dim M_{\mathrm{out}(a)} \dim N_{\mathrm{in}(a)}.$$

For Q a Dynkin quiver, the symmetrization of the Euler form

$$(M, N)_R := \langle M, N \rangle_R + \langle N, M \rangle_R = \dim \mathrm{Hom}_{k\mathrm{Q}}(M, N) + \dim \mathrm{Hom}_{k\mathrm{Q}}(N, M)$$
$$- \dim \mathrm{Ext}^1(M, N) - \dim \mathrm{Ext}^1(N, M)$$

coincides with the Cartan form on the $\mathbb{Z}_{\geq 0}$ span of the positive roots of $\mathfrak{g}$ identifying them with kQ-representations via Remark 2.4.

Recall that a **primitive idempotent** of $k\mathrm{Q}$ is an element $e \in k\mathrm{Q}$, such that the equality $e^2 = e$ holds and e cannot be written as a sum of two other non-zero elements $e_1, e_2 \in k\mathrm{Q}$ with $e_j^2 = e_j$ for $j \in \{1,2\}$.

Two idempotents e_1, e_2 are called **orthogonal** if $e_1e_2 = e_2e_1 = 0$ holds in $k\mathrm{Q}$. A decomposition $k\mathrm{Q} = e_1k\mathrm{Q} \oplus e_2k\mathrm{Q} \oplus \ldots \oplus e_nk\mathrm{Q}$ such that $e_1, e_2, \ldots, e_n$ are primitive pairwise orthogonal idempotents with $\sum_{j=1}^{n} e_j = 1$ is called an **indecomposable decomposition** of $k\mathrm{Q}$ and such a set $\{e_1, e_2, \ldots, e_n\}$ is then called a **complete set of primitive orthogonal idempotents** of $k\mathrm{Q}$. Recall that a $k\mathrm{Q}$-module is called **projective** if it is isomorphic to a direct summand of a free $k\mathrm{Q}$-module. Dually a $k\mathrm{Q}$-module is called **injective** if it is isomorphic to a direct summand of a free $k\mathrm{Q}^{\mathrm{op}}$-module.

Note that the set of trivial paths $\{e_j\}_{i \in Q_0}$ is a complete set of primitive orthogonal idempotents of $k\mathrm{Q}$. The indecomposable projective (resp. injective) modules of $k\mathrm{Q}$ are isomorphic to $k\mathrm{Q}e_j$ (resp. $e_jk\mathrm{Q}$) for $j \in Q_0$. Note that as a vector space, the projective module $k\mathrm{Q}e_j$ has as a basis all paths starting at j. We therefore set $P(i) := k\mathrm{Q}e_i$ (resp. $I(i) = e_ik\mathrm{Q}$).

Example 1*: Let* Q *be the following quiver:*

$$1 \leftarrow 2 \leftarrow 3.$$

We have a unique indecomposable projective (resp. injective) $k\mathrm{Q}$*-module up to isomorphism for each* $i \in Q_0 = \{1,2,3\}$*. We use here the simplified notation of quiver representations instead of* $k\mathrm{Q}$*-modules:*

$$\begin{array}{lll} P(1) = k \xleftarrow{0} 0 \xleftarrow{0} 0; & P(2) = k \xleftarrow{1} k \xleftarrow{0} 0; & P(3) = k \xleftarrow{1} k \xleftarrow{1} k \\ I(1) = k \xleftarrow{1} k \xleftarrow{1} k; & I(2) = 0 \xleftarrow{0} k \xleftarrow{1} k; & I(3) = 0 \xleftarrow{0} 0 \xleftarrow{0} k. \end{array}$$

AUSLANDER-REITEN QUIVERS OF DYNKIN QUIVERS

For the rest of this section let Q be a Dynkin quiver. A lot of the structure of $k\mathrm{Q}-\mathrm{mod}$ is contained in the **Auslander–Reiten quiver** Γ_{Q}. The vertices of this quiver are given by the isomorphism classes $[V]$ of indecomposable representations of Q while there is an arrow $[V] \to [W]$ if and only if there is an **irreducible morphisms** $V \to W$ in $k\mathrm{Q} - \mathrm{mod}$. Recall that those are the non-isomorphisms in $k\mathrm{Q} - \mathrm{mod}$ that cannot be written as a composition of two non-isomorphisms.

We turn our attention to a combinatorial construction of Γ_{Q} (see [4, Section 6.5] for details). For a Dynkin quiver Q, let $\mathrm{Q}^* = (Q_0, Q_1^*)$ be the quiver with the same set of vertices and reversed arrows by setting $\mathrm{in}(a) = \mathrm{out}(a^*)$ and $\mathrm{out}(a) = \mathrm{in}(a^*)$ for each $a \in \mathrm{Q}_1$. To construct Γ_{Q}, we introduce the infinite quiver $\mathbb{Z}\mathrm{Q}^*$ which has

$\mathbb{Z} \times \mathrm{Q}_0$ as set of vertices and $\mathbb{Z} \times \{\mathrm{Q}_1 \cup \mathrm{Q}_1^*\}$ as set of arrows where for $r \in \mathbb{Z}$, $a \in \mathrm{Q}_1$

$$\begin{aligned} \operatorname{out}(r,a) &= (r, \operatorname{out}(a)); & \operatorname{out}(r,a^*) &= (r, \operatorname{in}(a)) \\ \operatorname{in}(r,a) &= (r+1, \operatorname{in}(a)); & \operatorname{in}(r,a^*) &= (r, \operatorname{out}(a)). \end{aligned}$$

Example 2: Let Q *again be the quiver:*

$$1 \leftarrow 2 \leftarrow 3.$$

Then $\mathbb{Z}\mathrm{Q}^*$ *is given as follows:*

$$\begin{tikzcd}
 & & (-1,3) \arrow[dr] & & (0,3) \arrow[dr] & & (1,3)\cdots \\
 & (-1,2) \arrow[ur] \arrow[dr] & & (0,2) \arrow[ur] \arrow[dr] & & (1,2) \arrow[ur] & \\
\cdots(-1,1) \arrow[ur] & & (0,1) \arrow[ur] & & (1,1) \arrow[ur] & &
\end{tikzcd}$$

A **slice** of $\mathbb{Z}\mathrm{Q}^*$ is a connected full subquiver which contains for each $i \in \mathrm{Q}_0$ a unique vertex of the form (r,i), $r \in \mathbb{Z}$. The isomorphism classes of indecomposable projectives of $k\mathrm{Q}$ are in bijection to the vertices of a slice of $\mathbb{Z}\mathrm{Q}^*$ starting at $(0,1)$ via $P(i) \mapsto (0,i)$. The image of this slice under the **Nakayama permutation** ν is again a slice whose vertices are in bijection with the isomorphism classes of indecomposable injectives of $k\mathrm{Q}$. Recall that ν in the various types does not depend on the orientation and is for $i \in \mathrm{Q}_0$, $r \in \mathbb{Z}$ given by:

Type A_n: $\nu(r,i) = (r+i-1, n+1-i)$.

Type D_n: n even: $\nu(r,i) = (r+n-2, i)$;
n odd: $\nu(r,i) = (r+n-2, i)$ for $i \leq n-2$ and
$\nu(r, n-1) = (r+n-2, n)$;
$\nu(r,n) = (r+n-2, n-1)$.

Type E_6: $\nu(r,i) = r+5, 6-i)$ for $i \leq 5$ and
$\nu(r,6) = (r+5,6)$.

Type E_7: $\nu(r,i) = (r+8, i)$.

Type E_8: $\nu(r,i) = (r+14, i)$.

It is well-known that the Auslander-Reiten quiver of kQ can be identified with the full subquiver of $\mathbb{Z}\mathrm{Q}^*$ formed by the vertices lying between the slice starting at $(0,1)$ and the image of this slice under ν.

Furthermore $\mathbb{Z}\mathrm{Q}^*$ has a translation structure given by the **Auslander–Reiten translation** τ which is given by translation to the left, i.e. $\tau(p,q) = (p-1,q)$. This function gives rise to a bijection between the isomorphism classes of indecomposable non–projectives and the isomorphism classes of indecomposable non–injectives when restricted to Γ_{Q}. We can describe τ as a function on the dimension vectors if we fix a labeling of Q_0 which is **adapted to** Q. That is $i > j$ if there is a path from i to j in Q_1.

For $i \in \mathrm{Q}_0$ first define a function $r_i : \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0} \to \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0}$ via

$$r_i(v) = v - \frac{2(v,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i,$$

for $v \in \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0}$. Here we denote the dimension vector of the simple kQ-module $S(i)$ by α_i. For a sequence $i_1, i_2, \ldots, i_n$ adapted to Q, we define a map $c : \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0} \to \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0}$ on $x \in \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0}$:

$$c(x) = r_{i_n} r_{i_{n-1}} \cdots r_{i_1}(x).$$

The map c is called a **coxeter element** of Q. The relation to τ is as follows: for an indecomposable non-projective kQ-module M with dimension vector v, the indecomposable kQ-module $X = \tau M$ has dimension vector v' with

$$v' = c(v).$$

Example 3: We give an example of the Auslander–Reiten quiver of the following quiver (type A_3)

$$\mathrm{Q} = 1 \leftarrow 2 \leftarrow 3.$$

Note that by Gabriel's theorem the indecomposable kQ$-$models are determined by their dimension vectors. Therefore we denote a vertex of Γ_{Q} by the dimension vector of its isomorphism class.

$$\begin{tikzcd}
 & & 111 \arrow[dr] & & \\
 & 110 \arrow[ur] \arrow[dr] & & 011 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
100 \arrow[ur] & & 010 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & 001 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

Note that the three indecomposable projective modules of kQ lie in the leftmost slice, while the three indecomposable injective modules of kQ lie in the rightmost ray of Γ_{Q}.

We further have a functorial description of τ as an equivalence between all non–injective $k\mathrm{Q}-$modules and all non–projective $k\mathrm{Q}-$modules. For this, let $k\mathrm{Q}-\mathrm{mod}_{\mathcal{P}}$ (resp. $k\mathrm{Q}-\mathrm{mod}_{\mathcal{I}}$) be the full subcategory of $k\mathrm{Q}-\mathrm{mod}$ consisting of the modules with no non-trivial projective (resp. injective) summands. Then τ can be extended to the functor

$$\tau : k\mathrm{Q}-\mathrm{mod}_{\mathcal{P}} \to k\mathrm{Q}-\mathrm{mod}_{\mathcal{I}}$$

$$\tau M = \mathrm{D}\,\mathrm{Ext}^1_{k\mathrm{Q}}(M, k\mathrm{Q})$$

with inverse

$$\tau^{-1} : k\mathrm{Q}-\mathrm{mod}_{\mathcal{I}} \to k\mathrm{Q}-\mathrm{mod}_{\mathcal{P}}$$

$$\tau^{-1} M = \mathrm{Ext}^1_{k\mathrm{Q}}(\mathrm{D}\,M, k\mathrm{Q}).$$

For $X, Y \in \mathrm{Q}-\mathrm{mod}$, the Auslander-Reiten translation τ gives rise to functorial isomorphism:

$$\mathrm{D}\,\mathrm{Hom}_{k\mathrm{Q}}(X, \tau Y) \cong \mathrm{Ext}^1_{k\mathrm{Q}}(X, Y) \cong \mathrm{D}\,\mathrm{Hom}_{k\mathrm{Q}}(\tau^{-1} X, Y).$$

This isomorphism is known as the **Auslander-Reiten formula** (or Auslander-Reiten duality).

We conclude this section with an important remark that is used extensively in the homological description of $B(\lambda)$.

Remark 2.6 From the definitions it is straightforward to see that $\mathrm{D}\,\tau M = \tau^{-1}\,\mathrm{D}\,M$. Here τ^{-1} denotes the inverse Auslander-Reiten translation in $(k\mathrm{Q})^{\mathrm{op}}-\mathrm{mod}$. We can further identify representations of $(k\mathrm{Q})^{\mathrm{op}}-\mathrm{mod}$ with representations of $k\mathrm{Q}^*-\mathrm{mod}$. Thus the Auslander-Reiten quiver of $k\mathrm{Q}^*$ can be obtained by reversing each arrow in the Auslander-Reiten quiver of $k\mathrm{Q}$ and interchanging the roles of τ and τ^{-1}.

2.3 RINGEL HALL ALGEBRAS AND THE CANONICAL BASIS

We review the notion and some facts about Hall algebras which we use in Section 3.1.

Let Q be a Dynkin quiver of type A, D, E and $\mathfrak{g}$ the associated finite dimensional simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.

Let q be a prime power, M, N, X be $k\mathrm{Q}$–modules and $\mathbb{F}_q$ be the finite field with q elements. Following Remark 2.4, we can associate to M, N and X functions γ_M, γ_N and γ_X from the negative root lattice to $\mathbb{Z}_{\geq 0}$. We define $F^X_{M,N}(q)$ as the number of submodules U of $M(\gamma_X, \mathbb{F}_q)$ over $\mathbb{F}_q$ such that $U \cong M(\gamma_N, \mathbb{F}_q)$ and

$M(\gamma_X, \mathbb{F}_q)/U \cong M(\gamma_M, \mathbb{F}_q)$ over $\mathbb{F}_q$. Recall that we denote by $M(\gamma, k)$ the kQ-module associated to the function γ.

One can show that $F^X_{M,N}(q)$ is a polynomial in q called the **Hall polynomial**. Setting $q = v^2$, we therefore obtain:

$$F^X_{M,N}(v^2) \in \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v).$$

Definition 2.7 We define the (twisted, generic) **Hall algebra** $\mathscr{H}(\mathrm{Q})$ of a quiver Q to be the $\mathbb{Z}[v, v^{-1}]$-vector space with basis elements $u_{[M]}$ indexed by the isomorphism classes $[M]$ of kQ–modules and multiplication defined by

$$u_{[M]} \cdot u_{[N]} := v^{\langle M,N \rangle_R} \sum_{[X]} F^X_{M,N}(v^2) u_{[X]}.$$

We recall the relation of Hall algebras to quantum groups and crystal basis. Therefore, we adopt the notation of Section 2.1.

We have the following fundamental theorem by Ringel (see e.g. [21]):

Theorem 2.8 *The map $\eta_{\mathrm{Q}} : U_v(\mathfrak{n}^-) \to \mathscr{H}(\mathrm{Q})$ defined by $\eta_{\mathrm{Q}}(F_i) = (u_{[S(i)]})$ induces an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras.*

By setting $f^{\mathrm{Q}}_{[M]} = v^{\dim \mathrm{End}(M) - \dim M} u_{[M]}$, we clearly get a basis of $\mathscr{H}(\mathrm{Q})$. Via η_{Q}^{-1}, this basis is sent to a PBW-basis B_{Q} of $U_v(\mathfrak{n}^-)$ corresponding to a reduced expression of the longest Weyl group element w_0 of $\mathfrak{g}$ adapted to Q ([15, 4.12]). We denote those basis elements by $F^{\mathrm{Q}}_{[M]} := \eta_{\mathrm{Q}}^{-1}(f^{\mathrm{Q}}_{[M]})$

It was shown by Lusztig that the $\mathbb{Z}[v^{-1}]$-lattice $\mathscr{L}$ spanned by B_{Q} is independent of the reduced decomposition of w_0 and thus of the orientation of Q. Furthermore the image of B_{Q} under the projection $\pi : \mathscr{L} \to \mathscr{L}/v^{-1}\mathscr{L}$ is a $\mathbb{Z}$-basis B of $\mathscr{L}/v^{-1}\mathscr{L}$ which is again independent of the orientation of Q.

Let us denote by $\overline{} : U_v(\mathfrak{n}^-) \to U_v(\mathfrak{n}^-)$ the canonical $\mathbb{Q}$-algebra involution of $U_v(\mathfrak{n}^-)$ sending F_i to F_i and v to v^{-1}. Then there is a unique $\overline{}$-invariant basis $\mathscr{B}$ of $\mathscr{L}$ whose image under π is B.

This basis is called the **canonical basis**. The elements of $\mathscr{B}$ can hence be parametrized by the isomorphism classes of kQ-modules, defining $\mathscr{F}^{\mathrm{Q}}_{[M]} \in \mathscr{B}$ by

$$\pi(\mathscr{F}^{\mathrm{Q}}_{[M]}) = \pi(F^{\mathrm{Q}}_{[M]})$$

and setting

$$\mathscr{B} = \{\mathscr{F}^{\mathrm{Q}}_{[M]} \mid M \in k\mathrm{Q} - \mathrm{mod}\}.$$

The following result is a slight reformulation of the work of Grojnowski-Lusztig.

Theorem 2.9 ([5]) *Let $\mathscr{B}$ be the canonical basis. Set*

$$\mathcal{L}' = \bigoplus_{b \in \mathscr{B}} \mathcal{A}_0 b, \qquad B' = \{b \bmod v^{-1}\mathcal{L}';\ b \in \mathscr{B}\}.$$

Then $(\mathcal{L}', B')$ is a crystal basis of $U_v(\mathfrak{n}^-)$ and hence isomorphic to $(\mathcal{L}(\infty), \mathcal{B}(\infty))$.

3 CRYSTAL GRAPHS OF ENVELOPING ALGEBRAS

3.1 THE HOMOLOGICAL CONSTRUCTION

Let Q be a Dynkin quiver associated to $\mathfrak{g}$ and k an arbitrary field. In [19], the crystal graph of $U(\mathfrak{n}^-)$ is realized as the set of isomorphism classes of kQ-modules. To state the main result of [19], we need the following definitions. We adopt the notations of Section 2.3.

Definition 3.1 • The **degree** of a non–zero Laurent polynomial $c \in \mathbb{Z}[v, v^{-1}]$ is the smallest d such that $v^{-d}c \in \mathbb{Z}[v^{-1}]$.

- For a kQ-module M and $i \in I$ we define

$$a_i^{\mathrm{Q}}(M) := \max_{[Y]} \deg c_i^{\mathrm{Q}}(M, Y),$$

  where $u_{[S(i)]} \cdot f_{[M]}^{\mathrm{Q}} = \sum_{[Y]} c_i^{\mathrm{Q}}(M, Y) f_{[Y]}^{\mathrm{Q}}$.

- For $u \in U_v(\mathfrak{n}^-)$, let $\rho(u)$ be the largest integer r such that $u \in F_i^r U_v(\mathfrak{n}^-)$.

Theorem 3.2 ([19, Proposition 3.2]) *Let M be a kQ–module and $i \in I$. If X is a kQ–module such that*

$$\deg c_i^{\mathrm{Q}}(M, X) = a_i^{\mathrm{Q}}(M) \geq a_i^{\mathrm{Q}}(X) - 1,$$

then $a_i^{\mathrm{Q}}(M) = a_i^{\mathrm{Q}}(X) - 1$, $\rho(\mathscr{F}_{[M]}^{\mathrm{Q}}) = a_i^{\mathrm{Q}}(M)$ and $\mathscr{F}_{[X]}^{\mathrm{Q}} = \tilde{f}_i \mathscr{F}_{[M]}^{\mathrm{Q}} \bmod v^{-1}\mathscr{L}$.

Thus the Kashiwara operator $\tilde{f}_i$ ($i \in I$) maps the isomorphism class $[M]$ to the isomorphism class $[X]$ if the criterion on the degree of the polynomial $c_i^{\mathrm{Q}}(M, X)$ is fulfilled.

In [19] it is proved that, for certain choice of orientations for Q, we can find for any $M \in k\mathrm{Q}-\mathrm{mod}$ and any $i \in I$ such a kQ-module X fulfilling the criterion. This is done by classifying all middle terms of short exact sequences of kQ-modules of the form

$$0 \to M \to X \to S(i) \to 0,$$

which allows one to express the function a_i^{Q} in terms of multiplicities of certain indecomposable direct summands of M. Let us recall these results in more details.

Definition 3.3 A quiver Q is called **special** if $\dim \mathrm{Hom}_{k\mathrm{Q}}(X, S(i)) \leq 1$ for all $i \in I$ and all indecomposable kQ–modules X.

For the rest of this section we make the following assumption:

Q is a fixed special Dynkin quiver.

At the end of this section, we examine this condition further and state all possible orientations which yield a special Dynkin quiver. In particular, this shows that we can find at least one orientation which is special for any simply-laced type Dynkin quiver except E_8.

Fix $i \in I$, we introduce two sets of $k\mathrm{Q}$–modules which play an important role in the classification of short exact sequences ending in $S(i)$.

First we define

$$\mathscr{P}_i(\mathrm{Q}) := \{X \in k\mathrm{Q}-\mathrm{mod} \mid X \text{ is indecomposable and } \dim \mathrm{Hom}_{k\mathrm{Q}}(X, S(i)) \neq 0\}.$$

On $\mathscr{P}_i(\mathrm{Q})$ we have a relation $\preceq$ given by

$$X \preceq Y \iff \mathrm{Hom}_{k\mathrm{Q}}(X, Y) \neq 0. \tag{2}$$

It is shown in [19] that this is a partial order on $\mathscr{P}_i(\mathrm{Q})$:

Proposition 3.4 ([19, Proposition 4.3.]) *Let X, Y be in $\mathscr{P}_i(\mathrm{Q})$. If there is a path from $[X]$ to $[Y]$ in the Auslander-Reiten quiver Γ_{Q} of Q, then there exists a map $f \in \mathrm{Hom}_{k\mathrm{Q}}(X, Y)$ inducing an isomorphism $\mathrm{Hom}_{k\mathrm{Q}}(Y, S(i)) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Q}}(X, S(i))$. In particular, $\mathscr{P}_i(\mathrm{Q})$ is a poset.*

Definition 3.5 An **antichain** is a subset of a poset such that no two (distinct) elements are comparable.

We define

$$\mathscr{S}_i(\mathrm{Q}) := \{V = \bigoplus_{j=1}^{k} X_j \mid \{X_1, X_2, \ldots, X_k\} \text{ is an antichain in } \mathscr{P}_i(\mathrm{Q})\}.$$

On $\mathscr{S}_i(\mathrm{Q})$ we have a partial order $\trianglelefteq$ given by:

$V \trianglelefteq V'$ if and only if $\dim \mathrm{Hom}_{k\mathrm{Q}}(B, V') \neq 0$ for each indecomposable direct summand B of V.

Note that we always have $\mathscr{P}_i(\mathrm{Q}) \subset \mathscr{S}_i(\mathrm{Q})$ as the set of trivial antichains.

Example 4: We give examples of the sets $\mathscr{P}_i(\mathrm{Q})$ and $\mathscr{S}_i(\mathrm{Q})$. Following Proposition 3.4, the set $\mathscr{P}_i(\mathrm{Q})$ can be interpreted as a full subgraph of the Auslander-Reiten quiver.

1. *For* $\mathrm{Q} = 1 \leftarrow 2 \leftarrow 3$, *the poset* $\mathscr{P}_3(Q)$ *is the union of all framed modules:*

$$\begin{tikzcd}
 & & \boxed{\mathbf{111}} \arrow[dr] & & \\
 & 011 \arrow[ur] \arrow[dr] & & \boxed{\mathbf{110}} \arrow[ll, dashed, "\tau"] \arrow[dr] & \\
001 \arrow[ur] & & 010 \arrow[ll, dashed, "\tau"] \arrow[ur] & & \boxed{\mathbf{100}} \arrow[ll, dashed, "\tau"]
\end{tikzcd}$$

   Here $\mathscr{S}_3(\mathrm{Q}) = \mathscr{P}_3(\mathrm{Q})$, *i.e.* $\mathscr{P}_3(\mathrm{Q})$ *is a chain. The elements can be ordered as follows:*

$$111 \trianglelefteq 110 \trianglelefteq 100.$$

2. *Take* $\mathrm{Q} = 1 \leftarrow 2 \rightarrow 3$. *We find that* $\mathscr{P}_2(\mathrm{Q})$ *has the following shape:*

$$\begin{tikzcd}
100 \arrow[dr] & & \boxed{\mathbf{011}} \arrow[ll, dashed, "\tau"] \arrow[dr] & \\
 & \boxed{\mathbf{111}} \arrow[ur] \arrow[dr] & & \boxed{\mathbf{010}} \arrow[ll, dashed, "\tau"] \\
001 \arrow[ur] & & \boxed{\mathbf{110}} \arrow[ll, dashed, "\tau"] \arrow[ur] &
\end{tikzcd}$$

   Here again we put a frame around every element of $\mathscr{P}_2(\mathrm{Q})$.

   This time $\mathscr{P}_2(\mathrm{Q}) \subsetneq \mathscr{S}_2(\mathrm{Q})$. *We have a non–trivial antichain given by* $V = 011 \oplus 110$. *So we have two maximal chains in* $\mathscr{S}_2(\mathrm{Q})$:

$$111 \trianglelefteq 011 \trianglelefteq 011 \oplus 110 \trianglelefteq 010$$

$$111 \trianglelefteq 110 \trianglelefteq 011 \oplus 110 \trianglelefteq 010$$

We are now able to state the classification of middle terms of extension by $S(i)$. For that let $l(V)$ be the set of all $B \in \mathscr{P}_i(\mathrm{Q})$ which are minimal with the property that $B \ntrianglelefteq V$ minimally.

Theorem 3.6 ([19, Corallary 4.4, Proposition 4.5]) *Given a* $k\mathrm{Q}$*-module* M *and* $i \in I$, *the possible middle terms of exact sequences*

$$0 \to M \to X \to S(i) \to 0$$

are in $1:1$*-correspondence with the elements* $V \in \mathscr{S}_i(\mathrm{Q})$ *such that* τB *is a direct summand of* M *for each* $B \in l(V)$. *The bijection is given via the map*

$$V \mapsto X = N \oplus V,$$

where $M = N \oplus \bigoplus_{B \in l(V)} \tau B$.

Recall that for kQ-modules M and B, where B is indecomposable, we denote by $\mu_B(M)$ the multiplicity of B as a direct summand of M.

Definition 3.7 Fix $i \in I$. For a kQ–module M and an element $V \in \mathscr{S}_i(\mathrm{Q})$ define

$$F_i(M,V) := \sum_{B \in \mathscr{P}_i(Q);\ B \trianglelefteq V} \mu_B(M) - \mu_{\tau B}(M).$$

Proposition 3.8 ([19, Proposition 5.2]) *Let $0 \to M \to X \to S(i) \to 0$ be an exact sequence and $V \in \mathscr{S}_i(\mathrm{Q})$ obtained via the bijection of Theorem 3.6. Then* $\deg {c_i}^{\mathrm{Q}}(M,X) = F_i(M,V)$.

In this language one verifies that the criterion given in Theorem 3.2 is always fulfilled:

Proposition 3.9 ([19, Proposition 6.1]) *Fix $i \in I$. Let M be a kQ–module, $V_0 \in \mathscr{S}_i(\mathrm{Q})$ such that $F_i(M,V_0) = a_i^{\mathrm{Q}}(M)$ and V_0 is $\trianglelefteq$–maximal with this property. Then $U_0 := \oplus_{B \in l(V_0)} \tau B$ is a direct summand of M. Set $X = M' \oplus V_0$ where $M = M' \oplus U_0$. Then*

$$a_i^{\mathrm{Q}}(X) = a_i^{\mathrm{Q}}(M) + 1.$$

Remark 3.10 Note that this implies that an antichain $V_0 \in \mathscr{S}_i(\mathrm{Q})$ as in Proposition 3.9 is uniquely determined.

The rest of this section is devoted to the description of the realization of the crystal basis $B(\infty)$ as the set isomorphism classes of kQ-module. We denote the vertices of this crystal graph by $B^{\mathscr{H}}(\infty) = \{b_{[M]} \mid M \in k\mathrm{Q} - \mathrm{mod}\}$.

Corollary 3.11 *Let M be a kQ-module , then*

$$\varepsilon_i(b_{[M]}) = F_i(M,V_0) \ \textit{and}$$

$$\varphi_i(b_{[M]}) = a_i^{\mathrm{Q}} - (S(i),M)_R\,.$$

Proof. The first equation follows directly from Proposition 3.9 and the second equation from the property of crystals and the fact that the Cartan form and the symmetrized Euler form coincide on the negative root lattice. □

We recall the recipe for the computation of the action of $\tilde{f}_i$ on $B^{\mathscr{H}}(\infty)$ which is given in [19, Chapter 7].

Definition 3.12 Let M be a kQ–module.

- For all $V \in \mathscr{S}_i(\mathrm{Q})$ compute the value

$$F_i(M,V) = \sum_{B \trianglelefteq V} \mu_B(M) - \mu_{\tau B}(M).$$

- Let V_0 be the $\trianglelefteq$–maximal antichain where the maximal value of $F_i(M, V)$ is reached.
- Let U_0 be the sum of all τB such that $B \in \mathscr{P}_i(\mathrm{Q})$ and $B \ntrianglelefteq V_0$ minimally.
- Set $X = M' \oplus V_0$ where $M = M' \oplus U_0$.

We define $\mathrm{pl}_i\, M := X$.

Remark 3.13 Note that U_0 must be a direct summand of M by Proposition 3.9.

Theorem 3.14 ([19, Theorem 7.1]) *Let $b_{[M]} \in B^{\mathscr{H}}(\infty)$. Then $\tilde{f}_i b_{[M]} = b_{[\mathrm{pl}_i\, M]}$.*

Using the description of $\tilde{f}_i$, we can determine the action of $\tilde{e}_i$ on $B^{\mathscr{H}}(\infty)$.

Lemma 3.15 *Let M be a $k\mathrm{Q}$-module with the property that there exists an antichain $V \in \mathscr{S}_i(\mathrm{Q})$ such that $F_i(M, V) > 0$. Let V_0' be the $\trianglelefteq$–minimal antichain with the property that $F_i(M, V_0') = a_i^{\mathrm{Q}}(M)$. Then V_0' is a direct summand of M.*

Proof. Assume that there exists an indecomposable direct summand B of V_0', such that $\mu_B(M) = 0$. Let $V_0' = B \oplus V_0''$, we define $\tilde{V}_0$ as $\tilde{V}_0 := V_0''$ if $V_0'' \neq 0$ holds. Otherwise, we define $\tilde{V}_0$ to be any element of $\mathscr{P}_i(\mathrm{Q})$ such that $\tilde{V}_0 \triangleleft V_0'$ minimally.

Then, we clearly have $\tilde{V}_0 \in \mathscr{S}_i(\mathrm{Q})$ with $\tilde{V}_0 \trianglelefteq V_0'$ and $F_i(M, \tilde{V}_0) \geq F_i(M, V_0')$. This is a contradiction to the defining properties of V_0'. Thus V_0' must be a direct summand of M. □

Hence the following is well-defined.

Definition 3.16 Let M be a $k\mathrm{Q}$-module with the property that there exists an antichain $V \in \mathscr{S}_i(\mathrm{Q})$ such that $F_i(M, V) > 0$.

- For all $V \in \mathscr{S}_i(\mathrm{Q})$ compute the value
$$F_i(M, V) = \sum_{B \trianglelefteq V} \mu_B(M) - \mu_{\tau B}(M).$$
- Let V_0' be the $\trianglelefteq$–minimal antichain where the maximal value of $F_i(M, V)$ is reached.
- Let U_0' be the sum of all τB such that $B \in \mathscr{P}_i(\mathrm{Q})$ and $B \ntrianglelefteq V_0'$ minimally.
- Set $X = M'' \oplus U_0'$ where $M = M' \oplus V_0'$.

We define $\mathrm{m_i}\, M := X'$.

Proposition 3.17 *Let $M \in k\mathrm{Q}-\mathrm{mod}$ have the property that there exists an antichain $V \in \mathscr{S}_i(\mathrm{Q})$ such that $F_i(M,V) > 0$. Then*

$$\tilde{e}_i b_{[M]} = b_{[\mathrm{m_i}\, M]}.$$

Proof. First we show that, since we know how the Kashiwara operator $\tilde{f}_i$ acts on $B^{\mathscr{H}}(\infty)$, the action of $\tilde{e}_i$ on $B^{\mathscr{H}}(\infty)$ is already determined by the equality

$$\tilde{e}_i \tilde{f}_i b_{[M]} = b_{[M]} \tag{3}$$

for all $M \in k\mathrm{Q}-\mathrm{mod}$. For this, assume that Equation (3) holds and let N be a $k\mathrm{Q}$-module such that there exists a $V \in \mathscr{S}_i(\mathrm{Q})$ with $F_i(N,V) > 0$, i.e. $\varepsilon_i(b_{[N]}) > 0$ and $\tilde{e}_i b_{[N]} \in B^{\mathscr{H}}(\infty)$. Let N' be a $k\mathrm{Q}$-module such that

$$\tilde{f}_i \tilde{e}_i b_{[N]} = b_{[N']}.$$

Applying $\tilde{e}_i$ yields

$$\tilde{e}_i b_{[N]} = \tilde{e}_i b_{[N']}.$$

Hence $[N] = [N']$.

Let $M \in k\mathrm{Q}-\mathrm{mod}$ and let $\tilde{f}_i b_{[M]} = b_{[\mathrm{X}]}$. Then $[X] = [\mathrm{pl}_i\, M]$, i.e. $X = M' \oplus V_0$ where $M = M' \oplus U_0$ and U_0, V_0 as in Definition 3.12.

First we note that

$$F_i(X,V_0) = F_i(M,V_0) + F_i(V_0,V_0) - F_i(U_0,V_0).$$

It follows from the proof of [19, Lemma 6.3] and the considerations in [19, p. 717] (since the graph Ω defined therein has no vertices in this case) that $F_i(V_0,V_0) - F_i(U_0,V_0) = 1$, which yields

$$F_i(X,V_0) = F_i(M,V_0) + 1 = a_i^Q(M) + 1.$$

Lemma 3.9 then yields that the maximal value of $F_i(X,V)$ is reached at V_0. It remains to show, that V_0 is $\trianglelefteq$-minimal with this property.

Let $V \in \mathscr{S}_i(\mathrm{Q})$ with $V \trianglelefteq V_0$, then:

$$\begin{aligned} F_i(X,V_0) &= F_i(M,V_0) + 1 \geq F_i(M,V) + 1 \\ &= F_i(X,V) - F_i(V_0,V) + F_i(U_0,V) \\ &\geq F_i(X,V) - 1, \end{aligned}$$

where the first inequality comes from the fact that the maximal value of $F_i(M,V)$ is reached at V_0 and the second inequality follows again from [19, Lemma 6.3] and the considerations in [19, p. 717].

□

We can thus summarize

Theorem 3.18 *For $M=(V,x)$ a kQ-module, let $b_{[M]} \in B^{\mathscr{H}}(\infty)$ be the element corresponding to the isomorphism class of M. Then the following assignments define a crystal structure on $B^{\mathscr{H}}(\infty)$:*

$$\begin{aligned}
\varepsilon_i(b_{[M]}) &= a_i^Q(M),\\
\mathrm{wt}(b_{[M]}) &= -\sum_{i\in I} \dim V_i \alpha_i,\\
\varphi_i(b_{[M]}) &= \varepsilon_i(b_{[M]}) - (S(i), M)_R\,,\\
\tilde{f}_i(b_{[M]}) &= b_{[\mathrm{pl}_i\, M]},\\
\tilde{e}_i(b_{[M]}) &= \begin{cases} 0, & \text{if } a_i^Q(M)=0\\ b_{[\mathrm{m_i}\, M]}, & \text{if } a_i^Q(M)>0. \end{cases}
\end{aligned}$$

Moreover, this is isomorphic to $B(\infty)$ as abstract crystals.

Example 5*: Let* Q *be the following quiver:*

$$\mathrm{Q} = 1 \leftarrow 2 \rightarrow 3.$$

Recall that $\mathscr{S}_2(\mathrm{Q}) = \{111, 011, 110, 011\oplus 110, 010\}$ from Example 4.2. Consider the following kQ-module

$$M = 111.$$

We determine the value of $F_2(M,V)$ for all $V \in \mathscr{S}_2(\mathrm{Q})$.

$$F_2(M,111)=1;\quad F_2(M,011)=1;\qquad F_2(M,011\oplus 110)=1;\quad F_2(M,010)=0,$$

i.e. the $\trianglelefteq$-maximal element of $\mathscr{S}_2(\mathrm{Q})$ where the maximal value of $F_2(M,V)$ is reached is $V_0 = 011\oplus 110$. Hence

$$\varepsilon_2(b_{[M]}) = F_2(M,V_0) = 1$$

and

$$\tilde{f}_2 b_{[M]} = b_{[011\oplus 110]}.$$

Furthermore

$$\varphi_2(b_{[M]}) = \varepsilon_2(b_{[M]}) - (M,S_2)_R = 1 - \dim \mathrm{Hom}_{k\mathrm{Q}}(M,S_2) - \dim \mathrm{Hom}_{k\mathrm{Q}}(S_2,M) +$$
$$\dim \mathrm{Ext}^1_{k\mathrm{Q}}(M,S_2) + \dim \mathrm{Ext}^1_{k\mathrm{Q}}(S_2,M) = 1-1-0+0+1 = 1.$$

The $\trianglelefteq$-minimal element of $\mathscr{S}_2(\mathrm{Q})$ where the maximal value of $F_2(M,V)$ is reached is $V_0' = 111$, i.e.

$$\tilde{e}_2 b_{[M]} = b_{[100\oplus 001]}.$$

SPECIAL QUIVERS

We examine the property special of a Dynkin quiver Q more closely. In [20, page 14], a combinatorial description of special quivers is given. For that, let Q be a quiver. A vertex $i \in I$ is called **thick** if there exists an indecomposable kQ-representation $M = (V, x)$ such that $\dim V_i \geq 2$.

Proposition 3.19 ([20, Proposition 2.8]) *Let* Q *be a quiver. Then* Q *is special if and only if no thick vertex is a source of* Q*.*

Definition 3.20 Let $\mathfrak{g}$ be a Lie algebra of simply-laced type and $i \in I$. A fundamental weight ω_i of $\mathfrak{g}$ is call **minuscule** if

$$-\langle \alpha, \omega_i \rangle \leq 1$$

for all negative roots $\alpha \in R^-$.

We get from Proposition 3.19

Corollary 3.21 *Let* Q *be a Dynkin quiver and* $\mathfrak{g}$ *the Lie algebra associated to the Dynkin diagram of* Q*. Then* Q *is special if and only if for each vertex* $i \in I$ *that is a source of* Q *the fundamental weight* ω_i *is minuscule.*

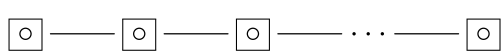
Proof. Note that if the vertex i is a source of Q, we have for α a negative root ω_i a fundamental weight of $\mathfrak{g}$

$$-\langle \alpha, \omega_i \rangle = \dim \mathrm{Hom}_{k\mathrm{Q}}(M(\alpha, k), S(i)).$$

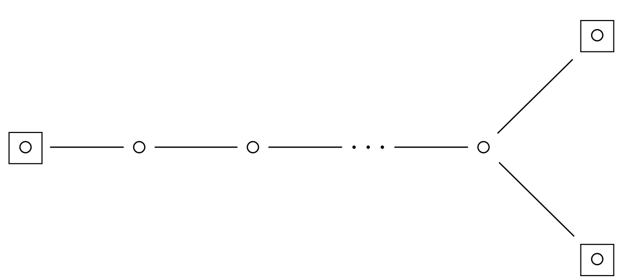
Thus no thick vertex is a source of Q if and only if letting i run over all sources, we have $-\langle \alpha, \omega_i \rangle \leq 1$ for all $\alpha \in R^-$. □

Thus there is no special quiver of type E_8. To get a special quiver Q of one of the other simply-laced types , we are only allowed to choose vertices as sources which are framed in the following diagrams (following the classification of minuscule weights given in [2, Chapter VIII, Proposition 7]):

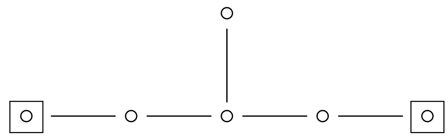
(4) A_n :



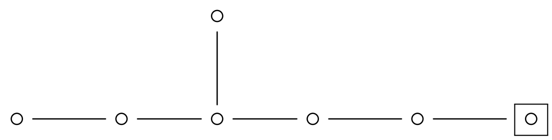
D_n :



E_6



E_7



3.2 THE GEOMETRIC CONSTRUCTION

In this section we review the realization of the crystal graph $B(\infty)$ via Lusztig's quiver varieties. Here the vertices of the crystal graph correspond to irreducible components of a variety associated to a Dynkin quiver Q of the same type as $\mathfrak{g}$, while the Kashiwara operators correspond to certain geometric operators constructed via maps with sufficiently nice fibers. This approach makes use of the fact that the elements of a basis of the weight space of $U_v(\mathfrak{n}^-)$ corresponding to the weight $\sum_{i\in I} -w_i\alpha_i$ are parametrized by the isomorphism classes of representations of Q of a given dimension vector w. On the other hand, these isomorphism classes coincide with the G_w-orbits of the variety $\mathrm{Rep}_w(\mathrm{Q})$ of representations of Q of dimension v. In this variety though, a G_w-orbit might lie in the closure of another G_w-orbit. Passing to the cotangent bundle $T^* \mathrm{Rep}_w(\mathrm{Q})$, we get again a (now symplectic) G_w-action with an associated moment map μ. The preimage of 0 under μ is then a variety (Lusztig's quiver variety) which has the property that the irreducible components of $\mu^{-1}(0)$ are parametrized by the G_w-orbits in $\mathrm{Rep}_w(\mathrm{Q})$ (see Proposition 3.30).

LUSZTIG'S QUIVER VARIETY

The points of Lusztig's quiver varieties can be identified with certain representations of the double quiver $\overline{\mathrm{Q}} = (I, H)$ associated to a Dynkin quiver $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1)$ which has the same set of vertices as Q and for each arrow of the Q, H contains two arrows with the same endpoints, one in each direction.

For an arrow $h \in H$, we denote by $\overline{h}$ the arrow with $\mathrm{out}(h) = \mathrm{in}(\overline{h})$ and $\mathrm{in}(h) = \mathrm{out}(\overline{h})$. Therefore, for a Dynkin quiver $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1)$, the associated double quiver $\overline{\mathrm{Q}} = (I, H)$ has as set of arrows $H = \mathrm{Q}_1 \sqcup \overline{\mathrm{Q}_1}$.

Example 6: We give an example of the double quiver of Dynkin type A_3:

$$\overline{\mathrm{Q}} = \; 1 \underset{h_1}{\overset{\overline{h}_1}{\rightleftarrows}} 2 \underset{h_2}{\overset{\overline{h}_2}{\rightleftarrows}} 3$$

An **orientation** is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \overline{\Omega} = H$ and $\Omega \cap \overline{\Omega} = \emptyset$. From now on we fix such an orientation and let Q be the Dynkin diagram equipped with orientation Ω. Let $\mathcal{V}$ **be the category of finite–dimensional** I**–graded vector spaces** $V = \bigoplus_{i \in I} V_i$ over $\mathbb{C}$. Fix $V \in \mathcal{V}$ (i.e. fix the vector space in a quiver representation) and let

$$E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}),$$

$$E_{V,\Omega} = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}).$$

Note that an element of E_V (resp. $E_{V,\Omega}$) together with the fixed vector space V is a representation of $\overline{\mathrm{Q}}$ (resp. Q).

We have an action of the group $G_v = \prod_i GL(V_i)$ on E_V and $E_{V,\Omega}$ by conjugation:

$$g.x = (g_i).(x_h) := g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)}.$$

The orbits of this action on E_V (resp. $E_{V,\Omega}$) are exactly the isomorphism classes of representations of the quiver $\overline{\mathrm{Q}}$ (resp. Q) with a fixed dimension vector v.

We define a symplectic form ω on E_V by

$$\omega(x, x') := \sum_{h \in H} \epsilon(h)\, \mathrm{trace}(x_h x'_{\bar{h}}) \tag{5}$$

for $x, x' \in E_V$, where

$$\epsilon(h) = \begin{cases} 1, & \text{if } h \in \Omega \\ -1, & \text{if } h \notin \Omega. \end{cases}$$

Note that $E_{V,\bar{\Omega}}$ is naturally isomorphic to the dual space of $E_{V,\Omega}$ via ω. Thus, we can identify E_V with the cotangent bundle of $E_{V,\Omega}$. Let $\mu : E_V \to gl_v = \oplus_{i \in I} \mathrm{End}(V_i)$ be the moment map associated to the G_v–action on E_V where we identify gl_v with its dual via the trace form. The i–th component of μ is for $x \in E_V$ given by

$$\mu_i(x) = \sum_{h \in H, \mathrm{in}(h) = i} \epsilon(h) x_h x_{\bar{h}} \in \mathrm{End}(V_i).$$

Definition 3.22 We define **Lusztig's quiver variety** as the set of linear maps

$$\Lambda_V := \{x \in E_V \mid \mu_i(x) = 0 \text{ for all } i \in I\}.$$

This is indeed a variety:

Proposition 3.23 ([17, Theorem 12.3 a)]) *Λ_V is a closed subvariety of E_V of pure dimension $\frac{1}{2}\dim E_V$.*

Definition 3.24 The **preprojective algebra** $\Pi(\mathrm{Q})$ of a Dynkin quiver Q is the quotient of the path algebra of the double quiver $\overline{\mathrm{Q}}$ by the ideal generated by

$$\sum_{h\in H}\epsilon(h)h\bar{h}.$$

Hence points of Λ_V can be identified with representations of the preprojective algebra with fixed vector space V.

Remark 3.25 The definition of Lusztig's quiver variety given in [12] imposes an additional nilpotency condition on the elements of Λ_V. But since we restrict ourselves to the Dynkin type this condition is automatically satisfied, so we omit it (see [17, Proposition 14.2(a)]).

Up to isomorphism, Λ_V depends only on the graded dimension $v = (\dim V_i)_{i\in I}$ of V: Let $W \in \mathcal{V}$ be another vector space with graded dimension v, then the vector space isomorphism $V \xrightarrow{\sim} W$ is unique up to right multiplication by an element of G_v which induces an isomorphism of varieties $\Lambda_V \xrightarrow{\sim} \Lambda_W$. This further yields a canonical bijection between the irreducible component of Λ_V and the irreducible components of Λ_W. Hence we also denote Λ_V by $\Lambda(v)$, regarding the graded dimension of the vector spaces attached to the vertices of the double quiver.

KASHIWARA OPERATORS

Following [12], we recall the crystal structure on the set of irreducible components of $\Lambda(v)$, which we denote by $\mathrm{Irr}\,\Lambda(v)$.

Definition 3.26 For $i \in I$ define $\varepsilon_i : \Lambda(v) \to \mathbb{Z}_{\geq 0}$ by

$$\varepsilon_i(x) := \dim \mathrm{Coker}\left(\bigoplus_{h:\mathrm{in}(h)=i} V_{\mathrm{out}(h)} \xrightarrow{x_h} V_i\right) \tag{6}$$

For $c \in \mathbb{Z}_{\geq 0}$, we further introduce the subsets

$$\Lambda(v)_{i,c} := \{x \in \Lambda(v) \mid \varepsilon_i(x) = c\}.$$

These sets form a partition of $\Lambda(v)$ into locally closed subsets. Note that the function ε_i is upper semicontinuous, thus for each $X \in \operatorname{Irr}\Lambda(v)$ there is an open dense subset of X such that ε_i is constant (namely the value of ε_i of this subset is the minimal value of ε_i on X).

To describe the actions of the Kashiwara operators, let $e^i \in \mathbb{Z}^I_{\geq 0}$ be such that $e^i_j = \delta_{ij}$ and fix $c \in \mathbb{Z}_{\geq 0}$. We denote by $\tilde{\Lambda}(v, c, i)$ the set of triples (x, x', ϕ) such that $x \in \Lambda(v)_{i,0}$, $x' \in \Lambda(v - ce^i)_{i,c}$ and $\phi : (V(v - ce^i), x') \hookrightarrow (V(v), x)$ is an injective morphism of $\Pi(\mathrm{Q})$-modules.

Consider the diagram

$$\Lambda(v - ce^i)_{i,0} \xleftarrow{p_1} \tilde{\Lambda}(v, c, i) \xrightarrow{p_2} \Lambda(v)_{i,c},$$

where $p_1(x, x', \Phi) = x'$ and $p_2(x, x', \Phi) = x$.

It is shown in [12, Lemma 5.2.3] that the map p_2 is a principal G_v-bundle and the map p_1 is smooth with a connected variety as fiber. Standard algebraic geometry arguments then yield the following Proposition.

Proposition 3.27 ([12, Proposition 5.2.4]) *Suppose $\Lambda(v)_{i,c} \neq \emptyset$. Then there is a one–to–one correspondence between the set of irreducible components of $\Lambda(v - ce^i)_{i,0}$ and the set of irreducible components of $\Lambda(v)_{i,c}$.*

Definition 3.28 For $X \in \operatorname{Irr}\Lambda(v)$, we define

$$\varepsilon_i(X) := \min_{x \in X} \varepsilon_i(x).$$

We also define for $c \in \mathbb{Z}_{\geq 0}$

$$\operatorname{Irr}\Lambda(v)_{i,c} := \{X \in \operatorname{Irr}\Lambda(v) \mid \varepsilon_i(X) = c\}.$$

We get the following bijection directly from Proposition 3.27 and [17, Theorem 12.3 b] which states that $\Lambda(v)_{i,c}$ has pure dimension $\frac{1}{2}\dim E_v$:

$$\operatorname{Irr}\Lambda(v - ce^i)_{i,0} \cong \operatorname{Irr}\Lambda(v)_{i,c}.$$

Suppose that $\bar{X} \in \operatorname{Irr}\Lambda(v - ce^i)_{i,0}$ corresponds to $X \in \operatorname{Irr}\Lambda(v)_{i,c}$ by this bijection. We define maps

$$\tilde{f}_i^c : \operatorname{Irr}\Lambda(v - ce^i)_{i,0} \to \operatorname{Irr}\Lambda(v)_{i,c},$$
$$\tilde{e}_i^{\max} : \operatorname{Irr}\Lambda(v)_{i,c} \to \operatorname{Irr}\Lambda(v)_{i,0}$$

by

$$\tilde{f}_i^c(\bar{X}) := X \text{ and } \tilde{e}_i^{\max}(X) := \bar{X}.$$

We further introduce the following notation

$$B^g(\infty) := \bigsqcup_v \operatorname{Irr}\Lambda(v)$$

and define maps

$$\tilde{e}_i, \tilde{f}_i : B^g(\infty) \to B^g(\infty) \sqcup \{0\}.$$

as follows. For $c \neq 0$, $\tilde{e}_i$ is the following composition of maps

$$\tilde{e}_i : \operatorname{Irr}\Lambda(v)_{i,c} \xrightarrow{\tilde{e}_i^{\max}} \operatorname{Irr}\Lambda(v - ce^i)_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} \operatorname{Irr}\Lambda(v - e^i)_{i,c-1}. \tag{7}$$

Moreover, we set $\tilde{e}_i(X) = 0$ for $X \in \operatorname{Irr}\Lambda(v)_{i,0}$. We define

$$\tilde{f}_i : \operatorname{Irr}\Lambda(v)_{i,c} \xrightarrow{\tilde{e}_i^{\max}} \operatorname{Irr}\Lambda(v - ce^i)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} \operatorname{Irr}\Lambda(v + e^i)_{i,c+1}.$$

We further remark that $\operatorname{Irr}\Lambda(v)_{i,c} \neq \emptyset$ implies $\operatorname{Irr}\Lambda(v - e^i)_{i,c+1} \neq \emptyset$. Hence for $X \in \operatorname{Irr}\Lambda(v)_{i,c}$, $\tilde{f}_i(X)$ is not zero. Note also that the maps $\tilde{f}_i^c$ resp. $\tilde{e}_i^{\max}$ may be considered as the c–th power of $\tilde{f}_i$ resp. the maximal power of $\tilde{e}_i$.

We also define for $X \in B^g(\infty)$

$$\begin{aligned} \operatorname{wt}(X) &:= -\sum_{i \in I} v_i \alpha_i \text{ for } X \in \operatorname{Irr}\Lambda(v), \\ \varphi_i(X) &:= \varepsilon_i(X) + \langle h_i, \operatorname{wt}(X) \rangle . \end{aligned}$$

Proposition 3.29 ([12, Theorem 5.3.2]) *$B^g(\infty)$ is a crystal isomorphic to the crystal $B(\infty)$ of $U_v(\mathfrak{n}^-)$.*

3.3 COMPARISON

In this section we give an explicit crystal isomorphism between the two crystal structures $B^{\mathscr{H}}(\infty)$ and $B^g(\infty)$. While the construction of $B^{\mathscr{H}}(\infty)$ works for isomorphism classes of kQ-modules over an arbitrary field k, we fix $k = \mathbb{C}$ in this section to relate it to the quiver representations appearing in the geometric construction.

We proceed in three steps.

<u>STEP ONE</u>: TRANSLATING THE VERTICES

We start by recalling Lusztig's description of the irreducible components of Λ_V, i.e. the vertices of $B^{\mathfrak{g}}(\infty)$.

Proposition 3.30 ([17, Proposition 14.2.(b)]) *For $\mathfrak{g}$ of type ADE, the irreducible components of Λ_V are the closures of the conormal bundles of the G_v-orbits in $E_{V,\Omega}$.*

For $M = (V, x) \in \mathbb{C}Q$, we denote conormal bundle of the orbit $G_v \cdot x$ by $\mathcal{C}_{[M]}$. Hence an irreducible component of Λ_V is given by the closure $\overline{\mathcal{C}_{[M]}}$ of $\mathcal{C}_{[M]}$.

Since the G_v-orbits in $E_{V,\Omega}$ coincide with the isomorphism classes of representations of the path algebra $\mathbb{C}Q$, we have a one-to-one correspondence between the vertices of $B^{\mathfrak{g}}(\infty)$ and the isomorphism classes of $\mathbb{C}Q$-modules. Hence the map

$$\begin{aligned} B^{\mathscr{H}}(\infty) &\to B^{\mathfrak{g}}(\infty) \\ b_{[M]} &\mapsto \overline{\mathcal{C}_{[M]}} \end{aligned} \tag{8}$$

is well-defined and bijective. In this section we prove that the map (8) is an isomorphism of crystals. We recall some results by Lusztig and Ringel by which we can describe the fibers of $\mathcal{C}_{[M]}$ as certain Hom-spaces of $\mathbb{C}Q$-modules, which is a crucial fact to draw a connection between the two crystal structures.

We use the following description of the conormal bundle $\mathcal{C}_{[M]}$.

Proposition 3.31 ([16, Lemma 9.3]) *Let $x \in E_{V,\Omega}$ and $\bar{x} \in E_{V,\overline{\Omega}}$. Then $\mu_i(x + \bar{x}) = 0$ for all $i \in I$ if and only if $\bar{x}$ is orthogonal to the tangent space to the G_v-orbit through x, regarded as a vector subspace of $E_{V,\Omega}$. Orthogonal here means with respect to the symplectic form ω given in (5).*

Hence for $M = (V, x) \in \mathbb{C}Q - \text{mod}$, a point $(x, \bar{x})$ with $\bar{x} \in E_{V,\overline{\Omega}}$ lies in the fiber ${\mathcal{C}_{[M]}}_x$ if and only if $\mu_i(x + \bar{x}) = 0$ for all $i \in I$.

In the following, we regard $E_{V,\Omega}$ as a subset of Λ_V by identifying it with the set of all elements $(x_h) = x \in \Lambda_V$ such that $x_h = 0$ whenever $h \notin \Omega$. We consider the following projection

$$\begin{aligned} \mathrm{pr} : \Lambda_V &\to E_{V,\Omega}, \\ y = (y_h)_{h \in H} &\mapsto x = (y_h)_{h \in \Omega}. \end{aligned} \tag{9}$$

Fix $M = (V, x) \in \mathrm{k}Q$. By Proposition 3.31, we have that

$$\mathcal{C}_{[M]} = \mathrm{pr}^{-1}(G_v \cdot x).$$

Recall that the points of Λ_V can be identified with representations of the preprojective algebra. As a key ingredient we use the following result by Ringel which allows us to describe the fiber ${\mathcal{C}_{[M]}}_x (= \mathrm{pr}^{-1}(x))$ in terms of $\mathbb{C}Q$-representations.

Let $\mathcal{F}, \mathcal{G} : \mathbb{C}\mathrm{Q} - \mathrm{mod} \to \mathbb{C}\mathrm{Q} - \mathrm{mod}$ be two functors and denote by $\mathbb{C}\mathrm{Q} - \mathrm{mod}(\mathcal{F}, \mathcal{G})$ the following category: its objects are pairs (M, ϕ), where $M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ and $\phi \in \mathrm{Hom}(\mathcal{F}(M), \mathcal{G}(M))$. Given two objects (M, ϕ), (M', ϕ'), a morphism $(M, \phi) \to (M', \phi')$ is a morphism $f : M \to M'$ in $\mathbb{C}\mathrm{Q} - \mathrm{mod}$ such that $\phi' \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \phi$.

Theorem 3.32 ([22, Theorem B., Theorem C., Proposition 3.]) *Let* Q *be a quiver of type A,D or E with orientation* Ω. *The categories* $\Pi(\mathrm{Q}) - \mathrm{mod}, \mathbb{C}\mathrm{Q} - \mathrm{mod}(\tau^{-1}, \mathrm{id})$ *and* $\mathbb{C}\mathrm{Q} - \mathrm{mod}(\mathrm{id}, \tau)$ *are isomorphic.*

Let furthermore $V \in \mathcal{V}$ *and* $x \in E_{V,\Omega}$. *Then* $\mathrm{pr}^{-1}(x)$ *and* $\mathrm{Hom}(\tau^{-1}M, M)$ *are isomorphic as vector spaces where* $M = (V, x) \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ *and* pr *is the projection given in (9).*

Remark 3.33 Via the Auslander-Reiten duality we furthermore have an isomorphism $\pi^{-1}(x) \cong \mathrm{D}\,\mathrm{Ext}^1(M, M)$ for $M = (V, x) \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$.

STEP TWO: COMBINATORIAL COMPARISON

From now on we assume that Q is a special Dynkin quiver. Recall that $a_i^{\mathrm{Q}}(M) = \max_{V \in \mathscr{S}_i(\mathrm{Q})} F_i(M, V)$. The main task of this step is the proof of the following proposition:

Proposition 3.34 *Let* $M = (V, x) \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$, *then* $a_i^{\mathrm{Q}}(M) = \varepsilon_i(\overline{\mathcal{C}_{[M]}})$.

Firstly, we write $\varepsilon_i(\overline{\mathcal{C}_{[M]}})$ as the dimension of certain Hom-spaces of $\Pi(\mathrm{Q})$-modules.

Lemma 3.35 *Let* $M = (V, x) \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ *and* $y \in \overline{\mathcal{C}_{[M]}}$. *Then, for* $\widetilde{M} = (V, y) \in \Pi(\mathrm{Q}) - \mathrm{mod}$, *we have*

$$\varepsilon_i(y) = \dim \mathrm{Hom}_{\Pi(\mathrm{Q})}(\widetilde{M}, S(i)).$$

Proof. Let $v_1, v_2, \dots, v_m$ be a basis of $\mathrm{Coker}\left(\bigoplus_{h \in H; \mathrm{in}(h) = i} V_{\mathrm{out}(h)} \xrightarrow{y_h} V_i\right)$. For each basis vector v_j we can find a linear map $0 \neq f_{v_j} : V_i \to \mathbb{C}$ (sending v_j to 1) such that $f_{v_j} \circ y_h = 0$ for all $h \in H$ with $\mathrm{in}(h) = i$. The maps $f_{v_1}, f_{v_2}, \dots, f_{v_m}$ can be identified with a basis of $\mathrm{Hom}_{\Pi(\mathrm{Q})}(\widetilde{M}, S(i))$ which yields the claim. □

As a consequence of Theorem 3.32, we have:

Corollary 3.36 *We have the equality*

$$\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = \min_{\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)} \dim\{f \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) \mid f \circ \phi = 0\}.$$

For $M \in \mathbb{C}\mathrm{Q}-\mathrm{mod}$ and $\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$, we define

$$\ell_i(M) := \dim \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) = \sum_{B \in \mathscr{P}_i(\mathrm{Q})} \mu_B(M).$$

$$\varepsilon_{i,\phi} := \dim\{f \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) \mid f \circ \phi = 0\} = \ell_i(\mathrm{Coker}\,\phi). \tag{10}$$

Clearly, for any $\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$, we have

$$\varepsilon_i(\overline{\mathcal{C}_{[M]}}) \leq \varepsilon_{i,\phi}.$$

Remark 3.37 For $\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ the short exact sequence

$$0 \to \mathrm{Im}\,\phi \to M \to \mathrm{Coker}\,\phi \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\mathrm{Coker}\,\phi, S(i)) \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\mathrm{Im}\,\phi, S(i)).$$

We thus obtain the inequality:

$$\ell_i(\mathrm{Coker}\,\phi) \geq \ell_i(M) - \ell_i(\mathrm{Im}\,\phi).$$

Let $V \in \mathscr{S}_i(\mathrm{Q})$. We write

$$M = M^{\trianglelefteq V} \oplus M^{\ntrianglelefteq V} \quad \text{and} \quad \tau^{-1}M = (\tau^{-1}M)^{\trianglelefteq V} \oplus (\tau^{-1}M)^{\ntrianglelefteq V},$$

where

$$M^{\trianglelefteq V} = \bigoplus_{B \in \mathscr{P}_i(\mathrm{Q}); B \trianglelefteq V} B^{\mu_B(M)}, \tag{11}$$

$$(\tau^{-1}M)^{\trianglelefteq V} = \bigoplus_{B \in \mathscr{P}_i(\mathrm{Q}); B \trianglelefteq V} B^{\mu_B(\tau^{-1}M)}. \tag{12}$$

Lemma 3.38 *For any $V \in \mathscr{S}_i(\mathrm{Q})$ and $\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ we have*

$$F_i(M, V) \leq \varepsilon_{i,\phi}.$$

Proof. Setting

$$\phi_{\trianglelefteq V} := \pi_{M^{\trianglelefteq}} \circ \phi,$$

where $\pi_{M^{\trianglelefteq}} : M \twoheadrightarrow M^{\trianglelefteq}$ denotes the canonical projection, we clearly have

$$\begin{aligned}\varepsilon_{i,\phi} &\geq \dim\{f \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) \mid f \circ \phi = 0,\ f|_{M^{\ntrianglelefteq V}} = 0\} \\ &= \dim\{f \mid\ f \circ \phi_{\trianglelefteq V} = 0,\ f|_{M^{\ntrianglelefteq V}} = 0\} = \ell_i\left(M^{\trianglelefteq V} / \operatorname{Im} \phi_{\trianglelefteq V}\right).\end{aligned}$$

Since

$$\ell_i(M^{\trianglelefteq V}) = \sum_{B \in \mathscr{P}_i(\mathrm{Q}),\ B \trianglelefteq V} \mu_B(M) \quad \text{and}$$

$$\ell_i(\operatorname{Im} \phi_{\trianglelefteq V}) \leq \ell_i\left((\tau^{-1}M)^{\trianglelefteq V}\right) = \sum_{B \in \mathscr{P}_i(\mathrm{Q});\ B \trianglelefteq V} \mu_B(\tau^{-1}(M))$$

we obtain by Remark 3.37:

$$\ell_i\left(M^{\trianglelefteq V} / \operatorname{Im} \phi_{\trianglelefteq V}\right) \geq \ell_i(M^{\trianglelefteq V}) - \ell_i(\operatorname{Im} \phi_{\trianglelefteq V}) \geq F_i(M, V)$$

$\square$

To prove Proposition 3.34, we construct a specific $\phi_0 \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ and a $V_0 \in \mathscr{S}_i(\mathrm{Q})$ such that $F_i(M, V_0) = \varepsilon_{i,\phi_0}$. We first prove a technical lemma which shows that, for a special class of morphism in $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$, we have the equality

$$\ell_i(\operatorname{Coker} \phi) = \ell_i(M) - \ell_i(\operatorname{Im} \phi).$$

For this we write

$$M = M^{\trianglelefteq S(i)} \oplus M^{\ntrianglelefteq S(i)} \quad \text{and} \quad \tau^{-1}M = (\tau^{-1}M)^{\trianglelefteq S(i)} \oplus (\tau^{-1}M)^{\ntrianglelefteq S(i)}.$$

Let $M^{\trianglelefteq S(i)} = \oplus_{j=1}^{m} B_j$ and $(\tau^{-1}M)^{\trianglelefteq S(i)} = \oplus_{k=1}^{m'} C_k$ with B_j and C_k indecomposable.

Lemma 3.39 *Assume that* $\phi : \tau^{-1}M \to M$ *is such that* $\phi|_{(\tau^{-1}M)^{\ntrianglelefteq S(i)}} = 0$ *and* $\phi|_{C_k}$ *is either zero or* $\operatorname{Im}(\phi|_{C_k}) \subset B_{j(k)}$ *for a* $j(k) \in \{1, 2, \ldots, m\}$ *such that* $\phi|_{C_k}$ *is inducing an isomorphism* $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(B_{j(k)}, S(i)) \cong \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(C_k, S(i))$. *Assume furthermore that* $j(k_1) \neq j(k_2)$ *for* $k_1 \neq k_2$. *Then*

$$\ell_i(\operatorname{Coker} \phi) = \ell_i(M) - \ell_i(\operatorname{Im} \phi) \tag{13}$$

and

$$\ell_i(\operatorname{Im} \phi) = \#\{k \in \{1, 2, \ldots, m'\} \mid \phi(C_k) \neq 0\}.$$

Proof. Since by assumption for any k the map

$$\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(B_{j(k)}, S(i)) \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\phi\left(C_k\right), S(i))$$

is surjective, we obtain that the map ψ in the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\operatorname{Coker} \phi, S(i)) \to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(M, S(i)) \xrightarrow{\psi} \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\operatorname{Im} \phi, S(i))$$

is surjective. This implies (13). Furthermore we have

$$\ell_i(\operatorname{Im}\phi) = \sum_{k:\phi(C_k)\neq 0} \ell_i(C_k) = \#\{k \in \{1,2,\dots,m'\} \mid \phi(C_k) \neq 0\}.$$

□

Fix $i \in I$ and let $\mathscr{P}$ be the Hasse diagram corresponding to the poset $\mathscr{P}_i(\mathrm{Q})$, i.e. there is a vertex v_B in $\mathscr{P}$ corresponding to each $B \in \mathscr{P}_i(\mathrm{Q})$ and an arrow $v_{B_1} \to v_{B_2}$ in $\mathscr{P}$ if and only if $B_1 \lhd B_2$ minimally. For a $\mathbb{C}\mathrm{Q}$-module M, we construct the following oriented graph $\mathscr{P}_M^\infty$: we replace each vertex v_B by a chain $v_{B^{(1)}} \to v_{B^{(2)}} \to \dots \to v_{B^{(l_B)}}$ for $l_B := \max(1, \mu_B(M), \mu_B(\tau^{-1}M))$. For each arrow $v_{B_1} \to v_{B_2}$ in $\mathscr{P}$, we add an arrow $v_{{B_1}^{(l_{B_1})}} \to v_{{B_2}^{(1)}}$ in $\mathscr{P}_M^\infty$. We further add a vertex w_∞ and for each vertex B of $\mathscr{P}$ which corresponds to a $\trianglelefteq$-maximal element of $\mathscr{P}_i(\mathrm{Q})$, we add an arrow $v_{B^{(l_B)}} \to w_\infty$ in $\mathscr{P}_M^\infty$.

Example 7: Let $\mathrm{Q} = 1 \leftarrow 2 \to 3$ *and* $i = 2$ *(compare with part 2 of Example 4). Let* $M = 111^2 \oplus 100 \oplus 011 \oplus 010$*, then* $\mathscr{P}_M^\infty$ *looks as follows.*

$$\begin{tikzcd}
 & & v_{011^{(1)}} \arrow[dr] & & & \\
v_{111^{(1)}} \arrow[r] & v_{111^{(2)}} \arrow[ur] \arrow[dr] & & v_{010^{(1)}} \arrow[r] & v_{010^{(2)}} \arrow[r] & w_\infty \\
 & & v_{110^{(1)}} \arrow[ur] & & &
\end{tikzcd}$$

Let $(\mathscr{P}_M^\infty)_0$ be the set of vertices of $\mathscr{P}_M^\infty$. We extend the ordering on $\mathscr{P}$ to an ordering on $(\mathscr{P}_M^\infty)_0$ by setting $v_1 \trianglelefteq v_2$ if and only if there is a path from v_1 to v_2 in $\mathscr{P}_M^\infty$. We define $\mathscr{A} = \mathscr{A}_{\mathscr{P}_M^\infty}$ to be the category in which the objects are subsets of $(\mathscr{P}_M^\infty)_0$. For $W_1, W_2 \in \mathscr{A}$, we define a morphism $\phi : W_1 \to W_2$ to be a map of sets $\phi : W_1 \cup w_\infty \to W_2 \cup w_\infty$, satisfying the following properties:

- for all $w \in W_1 \cup w_\infty$, we have $w \leq \phi(w)$,
- $\phi|_{W_1 \setminus \phi^{-1}(w_\infty)}$ is injective.

Let R be the subset of $(\mathscr{P}_M^\infty)_0$ corresponding to the direct summands of $\tau^{-1}M$ and W be the subset of $(\mathscr{P}_M^\infty)_0$ corresponding to the direct summands of M. For any $\phi_{\mathscr{A}} \in \operatorname{Hom}_{\mathscr{A}}(R, W)$, we define

$$\varepsilon_i(\phi_{\mathscr{A}}) := \#W \setminus \phi_{\mathscr{A}}(R).$$

Any $\phi \in \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ fulfilling the properties of Proposition 3.39 induces a morphism $\phi_{\mathscr{A}} : R \to W$ in the category $\mathscr{A}$, using the convention that a vertex

v_B of $\mathscr{P}_M^\infty$ corresponding to a direct summand of $\tau^{-1}M$ which is mapped to zero under ϕ, is mapped to the vertex w_∞ under $\phi_{\mathscr{A}}$. Note that, by Proposition 3.39, we have $\varepsilon_i(\phi_{\mathscr{A}}) = \varepsilon_{i,\phi}$.

For a subset $A = \{P_1, P_2, \cdots, P_k\}$ of $(\mathscr{P}_M^\infty)_0$, we define

$$A^\downarrow := \{P \in (\mathscr{P}_M^\infty)_0 \mid P \trianglelefteq P_j \text{ for some } 1 \le j \le k\}.$$

To $V \in \mathscr{S}_i(\mathrm{Q})$, we associate the following subset of $(\mathscr{P}_M^\infty)_0$

$$V_{\mathscr{A}} := \{v_{B^{(l_B)}} \mid B \in \mathscr{P}_i(\mathrm{Q}), \mu_B(V) \neq 0\}.$$

We define for $V \subset (\mathscr{P}_M^\infty)_0$

$$F_i(V) := \#W \cap V - \#R \cap V.$$

Note that, for $V \in \mathscr{S}_i(\mathrm{Q})$, $F_i(V_{\mathscr{A}}^\downarrow) = F_i(M, V)$ by definition.

We define the following preorder on $\mathrm{Hom}_{\mathscr{A}}(R, W)$: $\phi \preceq \psi$ if and only if there exists a $\rho \in \mathrm{Hom}_{\mathscr{A}}(W, W)$ such that we have an equality of sets

$$\psi(R) = \rho \circ \phi(R).$$

Loosely speaking, $\phi \preceq \psi$ says that we can move the vertices of $\phi(R)$ to the vertices of $\psi(R)$ along paths in $\mathscr{P}_M^\infty$. Note that this ordering is not anti-symmetric.

Example 8: We continue with Example 7. Here $W = \{v_{111^{(1)}}, v_{111^{(2)}}, v_{110^{(1)}}, v_{011^{(1)}}, v_{101^{(1)}}\}$ *and* $R = \{v_{010^{(1)}}, v_{010^{(2)}}\}$*. Let* $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathscr{A}}(R, W)$ *be given by*

$$\phi_1(v_{010^{(1)}}) = v_{010^{(1)}}, \qquad \phi_1(v_{010^{(2)}}) = w_\infty,$$
$$\phi_2(v_{010^{(1)}}) = w_\infty, \qquad \phi_2(v_{010^{(2)}}) = w_\infty.$$

For $\rho \in \mathrm{Hom}_{\mathscr{A}}(W, W)$ *given by* $\rho(v_{010^{(1)}}) = w_\infty$ *and* $\rho|_{W\setminus\{v_{010^{(1)}}\}} = \mathrm{id}_{W\setminus\{v_{010^{(1)}}\}}$

We define $\phi \in \mathrm{Hom}_{\mathscr{A}}(R, W)$ to be $\preceq$**-minimal** if for each $\psi \in \mathrm{Hom}_{\mathscr{A}}(R, W)$ such that $\psi \preceq \phi$, we also have $\phi \preceq \psi$.

Proposition 3.40 *Let* $\phi \in \mathrm{Hom}_{\mathscr{A}}(R, W)$ *be* $\preceq$*-minimal such that* $\varepsilon_i(\phi) > 0$*. Then there exists* $V^\phi \in \mathscr{S}_i(\mathrm{Q})$ *satisfying*

$$F_i((V^\phi)^\downarrow) = \varepsilon_i(\phi).$$

Furthermore, we have $W\setminus(V^\phi)^\downarrow \subset \mathrm{Im}\,\phi$.

Proof. Let $\phi \in \mathrm{Hom}_{\mathscr{A}}(R, W)$ and $V \in \mathscr{S}_i(\mathrm{Q})$. Clearly, we have

$$F_i\left(V^{\downarrow}\right) = \#W \cap V^{\downarrow} - \#R \cap V^{\downarrow} \leq \#(W \cap V^{\downarrow})\backslash\phi(R \cap V^{\downarrow}). \tag{14}$$

By the injectivity of $\phi|_{R\backslash\phi^{-1}(w_\infty)}$ we obtain that equality holds in (14) if and only if

$$\phi(R \cap V^{\downarrow}) \subset V^{\downarrow}. \tag{15}$$

Note that

$$(W \cap V^{\downarrow})\backslash\phi(R \cap V^{\downarrow}) = (W\backslash\phi(R)) \cap V^{\downarrow}.$$

Thus $F_i(V^{\downarrow}) = \varepsilon_i(\phi)$ holds if and only if $V^{\downarrow}$ satisfies Property (15) and $W\backslash\phi(R) \subset V^{\downarrow}$.

By assumption, we have $W\backslash\phi(R) \neq \varnothing$. We extend $(W\backslash\phi(R))^{\downarrow}$ to a subset of $(\mathscr{P}_M^{\infty})_0$ satisfying Property (15). For that, let $\mathcal{P}((\mathscr{P}_M^{\infty})_0)$ be the power set of $(\mathscr{P}_M^{\infty})_0$. We define the operator

$$\begin{aligned}\Phi : \mathcal{P}((\mathscr{P}_M^{\infty})_0) &\to \mathcal{P}((\mathscr{P}_M^{\infty})_0)\\ \Phi(A) &= \left(\phi\left(R \cap A^{\downarrow}\right) \cup A\right)^{\downarrow}.\end{aligned}$$

Note that, for $V_1, V_2 \subseteq (\mathscr{P}_M^{\infty})_0$ with $V_1 \subset V_2$, we have

$$V_1 \subset \Phi(V_1) \subset \Phi(V_2).$$

We therefore obtain the closure operator

$$\begin{aligned}\mathrm{H} = \mathrm{H}_\phi : \mathcal{P}((\mathscr{P}_M^{\infty})_0) &\to \mathcal{P}((\mathscr{P}_M^{\infty})_0)\\ \mathrm{H}(A) =&\Phi^{\nu}(A),\end{aligned}$$

where $\nu \in \mathbb{Z}_{\geq 0}$ is such that $\Phi^{\nu-1}(A) \neq \Phi^{\nu}(A) = \Phi^{\nu+1}(A)$. We define $V^{\phi} \subset (\mathscr{P}_M^{\infty})_0$ by

$$V^{\phi} := \mathrm{H}_\phi(W\backslash\phi(R)). \tag{16}$$

Clearly $(V^{\phi})^{\downarrow} = V^{\phi}$. Note that V^{ϕ} satisfies by construction $W\backslash\phi(R) \subseteq V^{\phi}$ as well as property (15). To obtain $F_i(V^{\phi}) = \varepsilon(\phi)$ it therefore suffices to show $w_\infty \notin V^{\phi}$. Let us now assume that ϕ is $\preceq$-minimal. If $w_\infty \notin V^{\phi}$ then the $\trianglelefteq$-maximal elements of V^{ϕ} are of the form $v_{B(\iota_B)}$ and can thus be identified with an element of $\mathscr{S}_i(\mathrm{Q})$.

If $w_\infty \in V^{\phi}$ then, by the definition of the operator H, there exists $r_1, r_2, \ldots, r_j \in R$ and a $\trianglelefteq$-maximal element w of $W\backslash\phi(R)$ such that $\phi(r_1) = w_\infty$ and $r_{k-2} \trianglelefteq \phi(r_{k-1})$ for $3 \leq k \leq j+1$ and $r_j \trianglelefteq w$.

We define $\phi' \in \operatorname{Hom}_{\mathscr{A}}(R, W)$ by

$$\begin{aligned} \phi'|_{R\setminus\{r_1,r_2,\dots r_j\}} &= \phi|_{R\setminus\{r_1,r_2,\dots r_j\}}, \\ \phi'(r_k) &= \phi(r_{k+1}) \text{ for } 1 \le k \le j-1 \\ \phi'(v_j) &= w. \end{aligned}$$

Then $\rho \circ \phi'(R) = \phi(R)$ for $\rho \in \operatorname{Hom}_{\mathscr{A}}(W, W)$ given by $\rho(w) = w_\infty$ and $\rho_{W\setminus\{w\}} = \mathrm{id}_{W\setminus\{w\}}$. By $\preceq$-minimality of ϕ, there exists $\rho' \in \operatorname{Hom}_{\mathscr{A}}(W, W)$ with $\operatorname{Im} \rho' \circ \phi = \operatorname{Im} \phi'$, yielding

$$\operatorname{Im} \phi' = \operatorname{Im} \underbrace{\rho' \circ \rho}_{=:\widetilde{\rho}} \circ \phi'.$$

Since $\# \operatorname{Im} \phi' < \infty$ this implies that $\widetilde{\rho}$ and hence $\rho|_{\operatorname{Im} \phi'}$ is injective in contradiction to $\rho(w) = \rho(w_\infty)$ with $w' \neq w_\infty = \rho'(\infty) \in \operatorname{Im} \rho' \circ \phi = \operatorname{Im} \phi'$. $\square$

Example 9: We give an example for the construction of V^ϕ for a minimal $\phi \in \operatorname{Hom}_{\mathscr{A}}(R, W)$. Assume that $\mathscr{P}_M^\infty$ is given as follows:

$$\begin{tikzcd}
 & v_{B_3^{(1)}} \arrow[r] & v_{B_3^{(2)}} \arrow[ddr, bend left] & \\
v_{B_1^{(1)}} \arrow[ur] \arrow[dr] & & & \\
 & v_{B_4^{(1)}} \arrow[r] & v_{B_4^{(2)}} \arrow[r] & w_\infty \\
v_{B_2^{(1)}} \arrow[ur] \arrow[dr] & & & \\
 & v_{B_5^{(1)}} \arrow[uurr, bend right] & &
\end{tikzcd}$$

Further assume that R and W are given as follows:

$$\begin{aligned} R &= \{v_{B_1^{(1)}}, v_{B_2^{(1)}}\} \text{ and} \\ W &= \{v_{B_3^{(1)}}, v_{B_3^{(2)}}, v_{B_4^{(1)}}, v_{B_4^{(2)}}, v_{B_5^{(1)}}\}. \end{aligned}$$

We define $\phi \in \operatorname{Hom}_{\mathscr{A}}(R, W)$ by

$$\begin{aligned} \phi(v_{B_1^{(1)}}) &= v_{B_4^{(1)}}, \\ \phi(v_{B_2^{(1)}}) &= v_{B_4^{(2)}} \end{aligned}$$

and note that ϕ is $\preceq$-minimal. We have

$$W \setminus \phi(R) = \{v_{B_3^{(1)}}, v_{B_3^{(2)}}, v_{B_5^{(1)}}\}.$$

Thus

$$\begin{aligned}\Phi(W\backslash\phi(R)) &= \big(\phi((W\backslash\phi(R))^{\downarrow}\cap R)\cup(W\backslash\phi(R))\big)^{\downarrow}\\ &= \{v_{B_1{}^{(1)}}, v_{B_2{}^{(1)}}, v_{B_3{}^{(1)}}, v_{B_3{}^{(2)}}, v_{B_4{}^{(1)}}, v_{B_4{}^{(2)}}, v_{B_5{}^{(1)}}\},\\ \Phi^2(W\backslash\phi(R)) &= \Phi(W\backslash\phi(R)).\end{aligned}$$

We conclude

$$V^{\phi} = \{v_{B_5{}^{(1)}}, v_{B_4{}^{(2)}}, v_{B_3{}^{(2)}}\}^{\downarrow}.$$

We are now able to prove Proposition 3.34:

Proof of Proposition 3.34. Let ϕ be any $\preceq$-minimal element in $\mathrm{Hom}_{\mathscr{A}}(R,W)$. We choose a corresponding element in $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M,M)$, which we also denote by ϕ by abuse of notation, the following way: for any $B,B'\in\mathscr{P}_i(\mathrm{Q})$ such that $v_B\in R$, $v_{B'}\in W$ and $\phi(v_B)=v_{B'}$, we let $\phi|_B:\tau^{-1}M\to M$ be a composition of irreducible morphisms $B\to B'$ that induces an isomorphism $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(B',S(i))\to\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(B,S(i))$. Such a homomorphism exists by Proposition 3.4.

First assume $\varepsilon_{i,\phi}=0$. Let $B\in\mathscr{P}_i(\mathrm{Q})$ be a $\trianglelefteq$-minimal element. Then B is projective by the definition of the order $\trianglelefteq$, i.e. there cannot be a direct summand C of $\tau^{-1}M$ such that $C\trianglelefteq B$. Note further that B cannot be a direct summand of M: Otherwise $(W\backslash(\phi(R)))\neq\varnothing$. We set $V^{\phi}=B$ and note that

$$F_i(M,V^{\phi})=0=\varepsilon_{i,\phi},$$

which proves the claim in this special case.

Assume now that $\varepsilon_{i,\phi}>0$. By proposition 3.40 $V^{\phi}=\mathrm{H}(W\backslash(\phi(R)))$ satisfies the equality

$$\varepsilon_{i,\phi}=F_i(M,V^{\phi}).$$

Since Lemma 3.38 yields $F_i(M,V)\leq\varepsilon_{i,\psi}$ for any $V\in\mathscr{S}_i(\mathrm{Q})$ and any $\psi\in\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M,M)$, we get

$$\begin{aligned}a_i^{\mathrm{Q}}(M) &= \max_{V\in\mathscr{S}_i(\mathrm{Q})}F_i(M,V)=F_i(M,V^{\phi})=\varepsilon_{i,\phi}\\ &= \min_{\psi\in\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M,M)}\varepsilon_{i,\psi}=\varepsilon_i(\mathcal{C}_{[M]}).\end{aligned}$$

□

We say that $\phi\in\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M,M)$ is $\preceq$-minimal if ϕ satisfies the assumptions of Lemma 3.39 and induces a $\preceq$-minimal $\phi\in\mathrm{Hom}_{\mathscr{A}}(R,W)$.

Lemma 3.41 *Fix* $M\in\mathbb{C}\mathrm{Q}-\mathrm{mod}$ *such that* $a_i^{\mathrm{Q}}(M)>0$*. For any* $\preceq$*-minimal* $\phi\in\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M,M)$*,* V^{ϕ} *is a* $\trianglelefteq$*-minimal element of* $\mathscr{S}_i(\mathrm{Q})$ *such that* $F_i(M,V)$ *is maximal.*

Proof. Let V be any element in $\mathscr{S}_i(\mathrm{Q})$ such that $F_i(M,V)$ is maximal. Then $F_i(M,V) = F_i(M,V^\phi) = \varepsilon_i(\phi)$. Note that for any $V \in \mathscr{S}_i(\mathrm{Q})$, we have

$$\begin{aligned} F_i(V^\downarrow) &= \#W \cap V^\downarrow - \#R \cap V^\downarrow \\ &\leq \#W \backslash (\phi(R) \cap V^\downarrow) \\ &\leq \#W \backslash \phi(R) \\ &= \varepsilon_i(\phi) \end{aligned}$$

where the first inequality is an equality if and only if $\phi(R \cap V^\downarrow) \subset (W \cap V^\downarrow)$ and the second inequality is an equality if and only if $(W \backslash \phi(R)) \subset V^\downarrow$. By construction of the closure operator H, for any $V \in \mathscr{S}_i(\mathrm{Q})$ satisfying those properties, we have

$$(V^\phi)^\downarrow \subset \mathrm{H}(V^\downarrow) = V^\downarrow.$$

Thus $V^\phi \trianglelefteq V$. □

Consequently, the element $V^\phi \in \mathscr{S}_i(\mathrm{Q})$, defined in (16), does not depend on the choice of a $\preceq$-minimal $\phi \in \mathrm{Hom}_{\mathscr{A}}(R,W)$. We are thus able to define for any $\preceq$-minimal $\phi \in \mathrm{Hom}_{\mathscr{A}}(R,W)$

$$V^M = H_\phi(W \backslash \phi(R)).$$

We further remark that this is an alternative way to prove that there is a unique $V \in \mathscr{S}_i(\mathrm{Q})$ which is $\trianglelefteq$-minimal such that $F_i(M,V) = a_i^{\mathrm{Q}}(M)$.

We conclude this step with a Lemma that is needed in step three.

Lemma 3.42 *For each indecomposable direct summand B of V^M there exists $\phi \in \mathrm{Hom}_{\mathscr{A}}(R,W)$ with $\varepsilon_i(\phi) = a_i^{\mathrm{Q}}(M)$ and $v_B \in W \backslash \phi(R)$.*

Proof. Let B be a direct summand of V^M and $\phi \in \mathrm{Hom}_{\mathscr{A}}(R,W)$ be any $\preceq$-minimal morphism. We have that v_B is a $\trianglelefteq$-maximal element of $H(W \backslash \phi(R))$. If $v_B \in W \backslash \phi(R)$ we are done. Otherwise, by the definition of the operator H there exists $r_1, r_2, \ldots, r_j \in R$ and a $\trianglelefteq$-maximal element w of $\mathrm{H}(W \backslash \phi(R))$ such that $\phi(r_1) = v_B$, $r_{k-2} \trianglelefteq \phi(r_{k-1})$ for all $3 \leq k \leq j$ and $r_j \trianglelefteq w$. We define $\phi_1 \in \mathrm{Hom}_{\mathscr{A}}(R,W)$ by

$$\begin{aligned} \phi_1|_{R \backslash \{r_1, r_2, \ldots r_j\}} &= \phi|_{R \backslash \{r_1, r_2, \ldots r_j\}}, \\ \phi_1(r_k) &= \phi(r_{k+1}) \text{ for all } 1 \leq k \leq j-1 \\ \phi_1(v_j) &= w. \end{aligned}$$

Thus $v_B \in W \backslash \phi_1(R)$ and

$$\varepsilon_i(\phi_1) = \varepsilon_i(\phi) = a_i^{\mathrm{Q}}(M).$$

□

STEP THREE: GEOMETRICAL COMPARISON

Let $M \in \mathbb{C}\mathrm{Q}-\mathrm{mod}$ with $a_i^{\mathrm{Q}}(M) > 0$. Recall that

$$\mathrm{m_i}\, M = N \oplus \bigoplus_{B \in l(V_0)} \tau B,$$

where $M = N \oplus V_0$ with V_0 the $\trianglelefteq$-minimal element of $\mathscr{S}_i(\mathrm{Q})$ such that $F_i(M, V)$ is maximal and $l(V_0) = \{B \in \mathscr{P}_i(\mathrm{Q}) \mid B \ntrianglelefteq V_0 \text{ minimally}\}$. By Lemma 3.41, we have $V_0 \cong V^M$. We further write, as before,

$$U_0 := \bigoplus_{B \in l(V_0)} \tau B.$$

For a direct summand A of M, we denote by $\pi_A : M \twoheadrightarrow A$ the canonical projection. We introduce the following notion.

Definition 3.43 A homomorphism $\phi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ **descends via** V^M if and only if there exists a short exact sequence

$$0 \to \mathrm{m_i}\, M \xrightarrow{\iota} M \xrightarrow{f} S(i) \to 0$$

such $f \circ \phi = 0$ and $\pi_N \circ \iota|_N = \mathrm{id}_N$.

We decompose $N = N^+ \oplus N^-$, where N^- is the direct sum of all direct summands B of N for which we have $B \in \mathscr{P}_i(\mathrm{Q})$ and $B \trianglelefteq V^M$. Let

$$V^M = \bigoplus_{k \in \mathscr{V}} V_k$$

be a decomposition of V^M into indecomposable direct summands. For $j \in \mathscr{V}$, we write $V^M = V_j^\perp \oplus V_j$. We abbreviate (see (11), (12))

$$\begin{aligned} M^{\trianglelefteq} &:= M^{\trianglelefteq V^M}, \\ M^{\ntrianglelefteq} &:= M^{\ntrianglelefteq V^M}, \\ (\tau^{-1}M)^{\trianglelefteq} &:= (\tau^{-1}M)^{\trianglelefteq V^M}, \\ (\tau^{-1}M)^{\ntrianglelefteq} &:= (\tau^{-1}M)^{\ntrianglelefteq V^M}. \end{aligned}$$

Note that $M^{\trianglelefteq} = N^- \oplus V^M$. Further, we abbreviate

$$h_{S(i)}(-) := \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(-, S(i)).$$

Lemma 3.44 *Let $\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$. Assume that there exists $0 \neq f \in \mathrm{Hom}_{\mathbb{C}Q}(M, S(i))$ such that*

$$\begin{aligned} f \circ \phi &= 0, \\ f|_{N^+} &= 0 \text{ and} \\ f|_{V_k} &\neq 0 \text{ for all } k \in \mathscr{V}. \end{aligned}$$

Then ϕ descends via V^M.

Proof. By Theorem 3.6, we have a short exact sequence

$$0 \longrightarrow U_0 \xrightarrow{\ \iota_0\ } V^M \xrightarrow{\ f|_{V^M}\ } S(i) \longrightarrow 0$$

Let $N^- = \bigoplus_{k \in \mathscr{N}} B_k$ be a decomposition into indecomposable direct summands. For each $k \in \mathscr{N}$ there exists a $j \in \mathscr{V}$ such that $B_k \trianglelefteq V_j$. Thus, by Proposition 3.4, there exists $\psi_j^k \in \mathrm{Hom}_{\mathbb{C}Q}(B_k, V_j)$ and $\lambda \in \mathbb{C}$ such that

$$f|_{B_k} = \lambda f|_{V_j} \psi_j^k$$

Let $\iota : \mathrm{m_i}\, M \to M$ be the following homomorphism:

$$\begin{aligned} \iota|_{U_0} &= \iota_0, \\ \iota|_{N^+} &= \mathrm{id}_{N^+}, \\ \iota|_{B_k} &= \mathrm{id}_{B_k} - \lambda \psi_j^k \end{aligned}$$

for each $k \in \mathscr{N}$.

This yields the exact sequence

$$0 \longrightarrow \mathrm{m_i}\, M \xrightarrow{\ \iota\ } M \xrightarrow{\ f\ } S(i) \longrightarrow 0$$

with $\pi_N \circ \iota|_N = \mathrm{id}_N$. Furthermore, by assumption we have $f \circ \phi = 0$. $\square$

We obtain the following as a reformulation of Lemma 3.44

Corollary 3.45 *Let $\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ and assume that*

$$\mathrm{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}(M^{\trianglelefteq}) \setminus \bigcup_{j \in \mathscr{V}} \left(\mathrm{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}(V_j^{\perp} \oplus N^-)\right) \neq \varnothing.$$

Then ϕ descends via V^M.

Remark 3.46 Note that a $\preceq$-minimal $\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ does not necessarily descend via V^M: If $V^M \neq W \setminus (\phi(R) \cap W)^{\downarrow}$, there exists a direct summand B of V^M and a direct summand C of $\tau^{-1}M$ such that $\mathrm{Im}\, \phi|_C \subset B$ and $\phi|_C$ induces an isomorphism $\mathrm{Hom}_{\mathbb{C}Q}(B, S(i)) \cong \mathrm{Hom}_{\mathbb{C}Q}(C, S(i))$. Thus we cannot find any $f \in \mathrm{Hom}_{\mathbb{C}Q}(M, S(i))$ such that $f \circ \phi = 0$ and $f|_B \neq 0$.

For $M_1, M_2 \in \mathbb{C}Q - \text{mod}$, the functor $h_{S(i)}(-)$ yields the linear map

$$h_{S(i)} : \mathrm{Hom}_{\mathbb{C}Q}(M_1, M_2) \to \mathrm{Hom}\left(h_{S(i)}(M_2), h_{S(i)}(M_1)\right)$$
$$\phi \mapsto h_{S(i)}(\phi).$$

For $k \in \mathscr{V}$ and $\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$, we denote by

$$\pi_k : M \twoheadrightarrow V_k^{\perp} \oplus N^-$$

the canonical projection and let

$$\theta_k := \max_{\phi} \operatorname{rank} h_{S(i)}(\pi_k \circ \phi).$$

We define the following subset of $\mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$:

$$\mathscr{O} := \{\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M) \mid \operatorname{rank} h_{S(i)}(\pi_k \circ \phi) = \theta_k \ \forall k \in \mathscr{V}\}.$$

Note that $\mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ is a finite dimensional $\mathbb{C}$-vector space and therefore carries the structure of an irreducible affine variety.

Lemma 3.47 *The set $\mathscr{O}$ is a dense open subset of* $\mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$.

Proof. We have

$$\mathscr{O} = \bigcap_{k \in \mathscr{V}} \{\phi \mid \operatorname{rank} h_{S(i)}(\pi_k \circ \phi) \geq \theta_k\}.$$

Since the set

$$\mathscr{O}_k := \{\phi \mid \operatorname{rank}\left(h_{S(i)}(\pi_k \circ \phi)\right) \geq \theta_k\}$$

is non-empty for each $k \in \mathscr{V}_M$, it suffices to show that $\mathscr{O}_k \subset \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ is open. We denote by p the linear map

$$p : \mathrm{Hom}(h_{S(i)}(M), h_{S(i)}(\tau^{-1}M)) \to \mathrm{Hom}\left(h_{S(i)}\left(V_j^{\perp} \oplus N^-\right), h_{S(i)}(\tau^{-1}M)\right)$$
$$g \mapsto g|_{h_{S(i)}(V_j^{\perp} \oplus N^-)}.$$

Thus, defining

$$\mathscr{U}_k := \{\gamma \in \operatorname{Im} \operatorname{codomain} P \mid \operatorname{rank} \gamma \geq \theta_k\},$$

we obtain $\mathscr{O}_k = \left(p \circ h_{S(i)}\right)^{-1}(\mathscr{U}_k)$ as the preimage of an open set under a continuous map. $\square$

Recall that for $B \in \mathbb{C}Q - \text{mod}$, we have $\ell_i(A) = \dim \mathrm{Hom}_{\mathbb{C}Q}(B, S(i))$. We define for $k \in \mathscr{V}$

$$\nu_k := \min_{\phi} \ell_i(\operatorname{Coker}(\pi_k \circ \phi)).$$

Note that we have the equality

$$\mathscr{O} = \{\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M) \mid \ell_i(\mathrm{Coker}\,(\pi_k \circ \phi)) = \nu_k\}.$$

We further define

$$\nu^- := \min_{\phi} \ell_i(\mathrm{Coker}\,(\pi_{M^{\trianglelefteq}} \circ \phi))\,.$$

Lemma 3.48 *For $k \in \mathscr{V}_M$ we have*

$$\nu^- = a_i^{Q}(M), \text{ and}$$
$$\nu_k \leq a_i^{Q}(M) - 1.$$

Proof. Let $\phi_0 \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ be $\preceq$-minimal. By Proposition 3.40 we have $\ell_i(\mathrm{Coker}\,\pi_{M^{\trianglelefteq}} \circ \phi_0) = a_i^{Q}(M)$ implying $\nu^- \leq a_i^{Q}(M)$. Let $\phi \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M^{\trianglelefteq})$ be arbitrary. Setting $\widetilde{\phi} := \phi + \phi_0|_{(\tau^{-1}M)^{\ntrianglelefteq}}$ we obtain the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
(\tau^{-1}M)^{\ntrianglelefteq} \arrow[r, "\pi_{M^{\ntrianglelefteq}} \circ \phi_0"] \arrow[d] & M^{\ntrianglelefteq} \arrow[r] \arrow[d] & \mathrm{Coker}\,\pi_{M^{\ntrianglelefteq}} \circ \phi_0 \arrow[r] \arrow[d] & 0 \\
\tau^{-1}M \arrow[r, "\tilde{\phi}"] \arrow[d] & M \arrow[r] \arrow[d] & \mathrm{Coker}\,\tilde{\phi} \arrow[r] \arrow[d] & 0 \\
(\tau^{-1}M)^{\trianglelefteq} \arrow[r, "\phi"] \arrow[d] & M^{\trianglelefteq} \arrow[r] \arrow[d] & \mathrm{Coker}\,\phi \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

By Proposition 3.40 $\ell_i(\mathrm{Coker}\,\pi_{M^{\ntrianglelefteq}} \circ \phi_0) = 0$ which implies that the map

$$h_{S(i)}(\mathrm{Coker}\,\phi) \to h_{S(i)}\left(\mathrm{Coker}\,\widetilde{\phi}\right)$$

is an isomorphism. Hence $\ell_i(\mathrm{Coker}\,\phi) \geq a_i^{Q}(M)$.

By Lemma 3.42 there exists $\psi_0 \in \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, M)$ with

$$\ell_i(\mathrm{Coker}\,\pi_k \circ \psi_0) = a_i^{Q}(M) - 1.$$

We thus obtain $\nu_k \leq a_i^{Q}(M) - 1$. $\square$

Corollary 3.49 *For each $\phi \in \mathscr{O}$, we have*

$$\operatorname{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}(M^{\trianglelefteq}) \setminus \bigcup_{j \in \mathscr{V}} \left(\operatorname{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}(V_j^{\perp} \oplus N^{-}\right) \neq \varnothing.$$

In particular, ϕ descends via V^M.

Proof. The claim follows from Lemma 3.48, noting that

$$\operatorname{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}\left(M^{\trianglelefteq}\right) \cong h_{S(i)}(\operatorname{Coker} \pi_{M^{\trianglelefteq}} \circ \phi)$$

is an affine variety of dimension $a_i^{\mathrm{Q}}(M)$ and

$$\operatorname{Ker}\left(h_{S(i)}(\phi)\right) \cap h_{S(i)}\left(V_j^{\perp} \oplus N^{-}\right) \cong h_{S(i)}(\operatorname{Coker} \pi_k \circ \phi)$$

is an affine variety of dimension at most $a_i^{\mathrm{Q}}(M) - 1$ for all $k \in \mathscr{V}$.

By Corollary 3.45, we conclude that ϕ descends via V^M. $\square$

Let $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = c > 0$. We say that $\phi \in \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ is **compatible with** $\mathrm{m_i}^c M$ if there exists a short exact sequence

$$0 \to \mathrm{m_i}^c M \xrightarrow{\iota} M \xrightarrow{f} S(i)^c \to 0$$

and $\psi \in \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}\,\mathrm{m_i}^c M, \mathrm{m_i}^c M)$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{m_i}^c M \arrow[r, "\iota"] & M \arrow[r, "f"] & S(i)^c \arrow[r] & 0 \\
& \tau^{-1}\,\mathrm{m_i}^c M \arrow[u, "\psi"] \arrow[r, "\tau^{-1}\iota"] & \tau^{-1}M \arrow[u, "\phi"] \arrow[r] & 0. \arrow[u] &
\end{tikzcd}$$

Proposition 3.50 *There exists a dense subset $\mathscr{D}_c$ of $\operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ such that each $\phi \in \mathscr{D}_c$ is compatible with $\mathrm{m_i}^c M$.*

Proof. Let

$$\mathrm{pr}_M : \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M) \twoheadrightarrow \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}((\tau^{-1}M)^{\trianglelefteq V^M}, M^{\trianglelefteq V^M})$$

be the canonical projection.

We prove the following statement by induction on $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = c$:

There exists a dense subset $\mathscr{D}_c$ of $\operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ which is compatible with $\mathrm{m_i}\, M$ such that $\mathrm{pr}_M^{-1}(\mathrm{pr}_M(\mathscr{D}_c)) = \mathscr{D}_c$.

If $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = 1$, the set $\mathscr{O} \subset \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ has the claimed property.

Assume that $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = c+1$ and let $X := \mathrm{m_i}\, M$. By induction hypothesis there exists a dense subset $\mathscr{D}_c \subset \operatorname{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}X, X)$ which is compatible with $\mathrm{m_i}^c X$ and satisfies the equation $\mathrm{pr}_X^{-1}(\mathrm{pr}_X(\mathscr{D}_c)) = \mathscr{D}_c$.

We deduce from Theorem 3.2 that $F_i(X, V^M) = a_i^{\mathrm{Q}}(X)$ and thus, since V^X is the $\trianglelefteq$-minimal element of $\mathscr{S}_i(\mathrm{Q})$ with that property,

$$V^X \trianglelefteq V^M. \tag{17}$$

Hence

$$\tau^{-1}U_0 = \left(\bigoplus_{B \in l(V^M)} B\right) \ntrianglelefteq V^X$$

and

$$\tau^{-1}V^M \ntrianglelefteq V^X.$$

We infer that $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}((\tau^{-1}X)^{\trianglelefteq}, X^{\trianglelefteq})$ is a direct summand of $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X)$ and denote by $\mathrm{pr}_{M,X} : \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X) \twoheadrightarrow \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}((\tau^{-1}X)^{\trianglelefteq}, X^{\trianglelefteq})$ the canonical projection.

Let $\mathscr{I}$ be the set of all injective morphisms $\iota : X \hookrightarrow M$ such that $\pi_N \circ \iota|_N = \mathrm{id}_N$ and let $\Lambda(M, X) = \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X) \times \mathscr{I}$ be the variety of pairs (ξ, ι) where $\xi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X)$ and $\iota \in \mathscr{I}$. We define

$$\begin{aligned} p_1 : \Lambda(M, X) &\to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X) \\ (\xi, \iota) &\mapsto \xi \end{aligned}$$

and

$$\begin{aligned} p_2 : \Lambda(M, X) &\to \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M) \\ (\xi, \iota) &\mapsto \iota \circ \xi. \end{aligned}$$

This yields the following diagram

$$\begin{tikzcd} & & & \Lambda(M,X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}X, X) \arrow[dr, "\mathrm{pr}_X"] & & \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, X) \arrow[dl, "\mathrm{pr}_{M,X}"'] & & \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M). \\ & \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}((\tau^{-1}X)^{\trianglelefteq}, X^{\trianglelefteq}) & & & \end{tikzcd}$$

We define

$$\mathscr{D}_{c+1} := p_2(p_1^{-1}(\mathrm{pr}_{M,X}^{-1}(\mathrm{pr}_X(\mathscr{D}_c)))).$$

Note that pr_X, $pr_{M,X}$ and p_1 are projections and therefore continuous and open. Further $\mathscr{O} \subset \mathrm{Im}\, p_2$ by Corollary 3.49, thus p_2 and pr_X are continuous with dense image. Hence $\mathscr{D}_{c+1}$ is a dense subset of $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}X, X)$.

We show that $\pi_M^{-1}\pi_M(\mathscr{D}_{c+1}) = \mathscr{D}_{c+1}$. Let $\widetilde{\phi} \in \pi_M^{-1}\pi_M(\mathscr{D}_{c+1})$. By construction, there exists $\phi \in \mathscr{D}_{c+1}$ and $\lambda \in \mathrm{Ker}\, \pi_M$ such that $\widetilde{\phi} = \phi + \lambda$. Since by (17) $p_1(p_2^{-1}(\mathrm{Ker}\, \pi_M)) \subseteq \mathrm{Ker}\, \pi_{M,X}$, it follows that $\widetilde{\phi} \in \mathscr{D}_{c+1}$.

It remains to show that any $\phi \in \mathscr{D}_{c+1}$ is compatible with $\mathrm{m_i}^{c+1}$. Let therefore $\bar{\phi} \in \mathrm{pr}_{M,X}^{-1} \mathrm{pr}_X(\mathscr{D}_c)$ and $\iota \in \mathscr{I}$ such that

$$\phi = \iota \circ \bar{\phi}.$$

We show that $\bar{\phi} \circ \tau^{-1}\iota \in \mathscr{D}_c$ which is equivalent to

$$\mathrm{pr}_X(\bar{\phi} \circ \tau^{-1}\iota) = \mathrm{pr}_{M,X}(\bar{\phi}) \tag{18}$$

by induction hypothesis. Equation (18) holds since by (17)

$$\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}V^X, N^{\trianglelefteq V^X}) = 0 = \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}V^M, N^{\trianglelefteq V^X})$$

and $\pi_{\tau^{-1}N} \circ \tau^{-1}\iota|_{\tau^{-1}N} = \mathrm{id}_{\tau^{-1}N}$. $\square$

Proposition 3.51 *For $M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$, with $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = c > 0$, we have*

$$\tilde{e}_i^c\overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[\mathrm{m_i}^c M]}}.$$

Proof. We have shown in Proposition 3.50 that there exists a dense subset $\mathscr{D}_c \subset \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ such that for every $\phi \in \mathscr{D}$, the (up to isomorphism) unique $\Pi(\mathrm{Q})$-submodule of (M, ϕ) with quotient isomorphic to $S(i)^c$ is of the form $(\mathrm{m_i}^c M, \psi)$ for a $\psi \in \mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}\mathrm{m_i}^c M, \mathrm{m_i}^c M)$. Recall from Theorem 3.32 that $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(\tau^{-1}M, M)$ can be identified with the fiber of x (for $M = (V, x)$) of the conormal bundle $\mathcal{C}_{[M]}$. Thus $\mathscr{D}_c$ can be identified with a dense subset of that fiber. The set $G_v\mathscr{D}_c$ is therefore a dense subset of $\mathcal{C}_{[M]}$ and hence of $\overline{\mathcal{C}_{[M]}}$. On the other hand, $g \in G_v$ maps (M, ϕ) to an isomorphic $\Pi(\mathrm{Q})$-module. We conclude $\tilde{e}_i^{\max}(G_v \cdot \mathscr{D}_c) \subset \mathcal{C}_{[\mathrm{m_i}^c M]}$. $\square$

Proposition 3.52 *Let Q be a special Dynkin quiver. Take $(V, x) = M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ and let $\overline{\mathscr{C}_{[M]}} \in B^g(\infty)$ be the element of the crystal corresponding to the closure of the G_v–orbit through M. Then*

$$\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = a_i^{\mathrm{Q}}(M), \tag{19}$$

$$\mathrm{wt}(\overline{\mathcal{C}_{[M]}}) = -\sum_{i \in I} \dim(V_i)\alpha_i, \tag{20}$$

$$\varphi_i(\overline{\mathcal{C}_{[M]}}) = a_i^{\mathrm{Q}}(M) + \left\langle h_i, \mathrm{wt}(\overline{\mathcal{C}_{[M]}})\right\rangle, \tag{21}$$

$$\tilde{e}_i(\overline{\mathcal{C}_{[M]}}) = \begin{cases} \overline{\mathcal{C}_{[\mathrm{m_i}\, M]}} & \textit{if } a_i^{\mathrm{Q}}(M) > 0, \\ 0 & \textit{if } a_i^{\mathrm{Q}}(M) = 0, \end{cases} \tag{22}$$

$$\tilde{f}_i(\overline{\mathcal{C}_{[M]}}) = \overline{\mathcal{C}_{[\mathrm{pl}_i\, M]}}, \tag{23}$$

with $\mathrm{pl}_i\, M$ as in Definition 3.12 and $\mathrm{m_i}\, M$ as in Definition 3.16.

Proof. We have already shown Equality (19) in Proposition 3.34. Furthermore (20) follows from the construction and we get (21) from the properties of crystals.

Let $a_i^{\mathrm{Q}}(M) = c > 0$. We have shown in Proposition 3.51 that $\tilde{e}_i^c \overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[\mathrm{m_i}^c\, M]}}$.

Note from the construction that $\overline{\mathcal{C}_{[\mathrm{pl}_i\, M]}}$ is never zero. For $\overline{\mathcal{C}_{[M]}}$ with $\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = 0$, we have

$$\overline{\mathcal{C}_{[\mathrm{m_i}^c\, \mathrm{pl}_i^c\, M]}} = \overline{\mathcal{C}_{[M]}}.$$

Which yields

$$\tilde{f}_i^c \overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[\mathrm{pl}_i^c\, M]}}.$$

This proves Equations (22) and (23).

□

We have thus proved:

Theorem 3.53 *For $M \in k\mathrm{Q}-\mathrm{mod}$ the map $B^{\mathscr{H}}(\infty) \to B^g(\infty)$ given by $b_{[M]} \mapsto \overline{\mathcal{C}_{[M]}}$ is an isomorphism of crystals.*

3.4 TYPE A

For this section, we fix the following quiver

$$\mathrm{Q} = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n.$$

In [24], Savage gives for $\mathfrak{g} = \mathrm{sl}_{n+1}$ a crystal isomorphism between $B^g(\infty)$ and the combinatorial realization of $B(\infty)$ via semistandard Young Tableaux introduced in [14]. In this section, we recover this result in the homological setup. Let therefore $V(k,l)$ $(1 \leq k \leq l \leq n)$ be the indecomposable $\mathbb{C}\mathrm{Q}$-module corresponding to the negative root $-\alpha_{k,l} = -\alpha_k - \alpha_{k+1} - \ldots - \alpha_l$. Let $M(\gamma(T), \mathbb{C})$ be the $\mathbb{C}\mathrm{Q}$-module obtained via the rule

$$\mu_{V(k,l)}(M) = \#\{(l+1)\text{-entries in the } k\text{- row of } T\}.$$

We show in Theorem 3.65 that map sending a (large, see Definition 3.60) semistandard Young tableau Y to the element $b_{[M(\gamma(T),\mathbb{C})]}$ is an isomorphism of sl_{n+1}-crystals.

HOMOLOGICAL CONSTRUCTION

Since the Auslander-Reiten quiver has a particularly symmetric shape in this special case, we note that $\mathscr{P}_i(\mathrm{Q})$ is a chain, i.e. $\mathscr{P}_i(\mathrm{Q}) = \mathscr{S}_i(\mathrm{Q})$, for $1 \leq i \leq n$, and the elements of $\mathscr{P}_i(\mathrm{Q})$ are of the following form:

$$\mathscr{P}_i(\mathrm{Q}) = \{V(k,i) \mid 1 \leq k \leq i\}.$$

Furthermore, we have for $2 \leq k \leq l \leq n$ the following description of the Auslander-Reiten translation on an indecomposable $\mathbb{C}Q$-module:

$$\tau V(k,l) = V(k-1, l-1).$$

Recall from Remark 2.4 that we have a one-to-one correspondence between the isomorphism classes of kQ-modules and function $\gamma : R^- \to \mathbb{Z}_{\geq 0}$. For $\mathfrak{g} = \mathrm{sl}_{n+1}$, the negative roots R^- are of the form $-\alpha_{k,l}$ for $1 \leq k \leq l \leq n$, we can thus identify R^- with the set of pairs $\{(k,l) \in \mathbb{Z}^2_{\geq 0} \mid 1 \leq k \leq l \leq n\}$. We write $\gamma(k,l)(M)$ (or short $\gamma(k,l)$ if M is fixed) for the multiplicity of $V(k,l)$ as a direct summand of M.

Definition 3.54 Let M be a kQ-module, $M \cong \bigoplus_{1 \leq k \leq l \leq n} V(k,l)^{\gamma(k,l)}$. For $j \leq i$, we define sums

$$F_{j,i}(M) := \sum_{k=1}^{j} \gamma(k,i) - \sum_{k=1}^{j-1} \gamma(k, i-1).$$

Note that for $k \leq k'$, we have $V(k,i) \trianglelefteq V(k',i)$. Thus $F_{k,i}(M) = F_i(M, V(k,i))$ with $F_i(M, V(k,i))$ given in Definition 3.7. For $1 \leq j \leq i \leq n$, we also write $F_{j,i}$ for short if the module $M \in k\mathrm{Q} - \mathrm{mod}$ is fixed.

Theorem 3.18 simplifies in the following way.

Corollary 3.55 *Let $M \in k\mathrm{Q} - \mathrm{mod}$, $M \cong \bigoplus_{1 \leq k \leq l \leq n} V(k,l)^{\gamma(k,l)}$. Fix $1 \leq i \leq n$ and let $j_0 \leq i$ be maximal such that $F_{j,i}(M)$ is maximal and let $j_0' \leq i$ be minimal such that $F_{j,i}(M)$ is maximal. Then*

$$\begin{aligned}
\varepsilon_i(b_{[M]}) &= F_{j_0,i}(M); \\
\mathrm{wt}(b_{[M]}) &= - \sum_{1 \leq k \leq i \leq l \leq n} \gamma(k,l)\alpha_i; \\
\varphi_i(b_{[M]}) &= \varepsilon_i(b_{[M]}) - (M, S(i))_R; \\
\tilde{f}_i b_{[M]} &= b_{[M']}; \\
\tilde{e}_i b_{[M]} &= \begin{cases} 0 & \text{if } \varepsilon_i(b_{[M]}) = 0 \\ b_{[M'']} & \text{else} \end{cases}
\end{aligned}$$

with $M' = \bigoplus_{1 \leq k \leq l \leq n} V(k,l)^{\gamma'(k,l)}$, where

$$\gamma'(k,l) = \begin{cases} \gamma(k,l) + 1, & \text{if } k = j_0, l = i \\ \gamma(k,l) - 1, & \text{if } k = j_0, l = i-1 \\ \gamma(k,l) & \text{else;} \end{cases}$$

and $M'' = \bigoplus_{1\le k\le l\le n} V(k,l)^{\gamma''(k,l)}$, *where*

$$\gamma''(k,l) = \begin{cases} \gamma(k,l)-1, & \text{if } k=j_0', l=i \\ \gamma(k,l)+1, & \text{if } k=j_0', l=i-1 \\ \gamma(k,l) & \text{else.} \end{cases}$$

Example 10*: We give an example of type* A_3*. As before*

$$\mathrm{Q} = 1 \leftarrow 2 \leftarrow 3.$$

We recall the Auslander-Reiten quiver:

$$\begin{tikzcd}[column sep=small] & & 111 \arrow[dr] & & \\ & 110 \arrow[ur] \arrow[dr] & & 011 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\ 100 \arrow[ur] & & 010 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & 001 \arrow[ll, dashed, "\tau"'] \end{tikzcd}$$

For $i \in \{1,2,3\}$ *and* $M \cong \bigoplus_{1\le k\le l\le 3} V(k,l)^{\gamma(k,l)} \in \mathbb{C}\mathrm{Q}-\mathrm{mod}$*, the value of* $F_{k,i}$ *is determined as follows:*

$$\begin{array}{lll} F_{1,1} = \gamma(1,1); & F_{1,2} = \gamma(1,2); & F_{2,2} = F_{1,2} + \gamma(2,2) - \gamma(1,1); \\ F_{1,3} = \gamma(1,3); & F_{2,3} = F_{1,3} + \gamma(2,3) - \gamma(1,2); & F_{3,3} = F_{2,3} + \gamma(3,3) - \gamma(2,2). \end{array}$$

Let $M = V(2,2) \oplus V(2,3)$*. We have, for* $k = i = 3$*:*

$$F_{1,3}(M) = 0;\ \ F_{2,3}(M) = 1;\ \ F_{3,3}(M) = -1.$$

Thus $\varepsilon_3(b_{[M]}) = 1$, $\tilde{f}_3 b_{[M]} = b_{[V(2,3)\oplus V(2,3)]}$ *and* $\tilde{e}_3 b_{[M]} = b_{[V(2,2)\oplus V(2,2)]}$.

COMBINATORICS FOR QUIVER VARIETIES

Following [24] we define an ordering on pairs $(k,l), 1 \le k \le l \le n$ given by:

$$(k,l) < (k',l') \text{ if } k < k' \text{ or if } k = k' \text{ and } l > l'.$$

For $i \in I$, Savage uses this order to define the i-signature by which he can determine the value of ε_i and the actions of the crystal operators in a combinatorial way.

Definition 3.56 Let M be a $\mathbb{C}\mathrm{Q}$-module, $M \cong \bigoplus_{1\le k\le l\le n} V(k,l)^{\gamma(k,l)}$ and $i \in I$.

1. Write all pairs of the form (k, i) or $(k, i-1)$ ordered from left to right with respect to $<$.
2. Under pairs of the form (k, i) write $\gamma(k, i)$ "-"'s and under pairs of the form $(k, i-1)$ write $\gamma(k, i-1)$ "+"'s. This is called the (+,-)-**sequence of** M.
3. Cancel all $(+, -)$ pairs.

The remaining (possibly empty) sequence of "-" followed be the remaining (possibly empty) sequence of "+" is then called the i**-signature**.

Example 11: *Fix $n = 2$, we have three possible pairs (k, l) with $1 \leq k \leq l \leq n$. They are ordered as follows:*

$$(1, 2) < (1, 1) < (2, 2).$$

Take $M = V(1, 1) \oplus V(2, 2) \oplus V(2, 2)$. There is one $(+, -)$-cancellation in the formation of the 2-signature of M:

$$\begin{array}{ccc} (1,2) & (1,1) & (2,2) \\ & (+ & -)- \end{array}$$

Thus the 2-signature of M just consists of one "-" corresponding to a direct summand of M isomorphic to $V(2, 2)$.

Let $\gamma : \{(k, l) \in \mathbb{Z}_{\geq 0} \mid 1 \leq k \leq l \leq n\} \to \mathbb{Z}_{\geq 0}$. Following Remark 2.4, we denote by $M(\gamma, \mathbb{C})$ the following $\mathbb{C}Q$-module:

$$M(\gamma, \mathbb{C}) = \bigoplus_{1 \leq k \leq l \leq n} V(k, l)^{\gamma(k,l)}.$$

Proposition 3.57 ([24, Theorem 4.1, Lemma 4.2]) *Let X_γ be the element of $B^g(\infty)$ corresponding to the irreducible component $\overline{\mathcal{C}_{M(\gamma,\mathbb{C})}}$ of $B^g(\infty)$. If there is a "+" (resp. "-") in the i-signature of $M(\gamma, \mathbb{C})$, let $(k_0, i-1)$ be the pair corresponding to the leftmost "+" (resp. let (k_0', i) be the pair corresponding to the rightmost "-"). Then*

$$\begin{aligned} \varepsilon_i(X_\gamma) &= \#\{\text{"-" in the } i\text{-signature of } M(\gamma, \mathbb{C})\}; \\ \mathrm{wt}(X_\gamma) &= \sum_{1 \leq k \leq i \leq l \leq n} -\gamma(k, l)\alpha_i; \\ \varphi_i(X_\gamma) &= \varepsilon_i(X_\gamma) + \langle h_i, \mathrm{wt}(X_\gamma) \rangle; \\ \tilde{f}_i X_\gamma &= X_{\gamma'}; \\ \tilde{e}_i X_\gamma &= \begin{cases} 0 & \text{if } \varepsilon_i(X_\gamma) = 0 \\ X_{\gamma''} & \text{else;} \end{cases} \end{aligned}$$

with

$$\gamma'(k,l) = \begin{cases} \gamma(k,l)+1, & \text{if } k=k_0, l=i \\ \gamma(k,l)-1, & \text{if } k=k_0, l=i-1 \\ \gamma(k,l) & \text{else} \end{cases} \qquad (*)$$

If there is no "+" in the i-signature, we ignore equation $(*)$ and set $k_0 = i$.

Further

$$\gamma''(k,l) = \begin{cases} \gamma(k,l)-1, & \text{if } k=k'_0, l=i \\ \gamma(k,l)+1, & \text{if } k=k'_0, l=i-1 \\ \gamma(k,l) & \text{else} \end{cases} \qquad (**)$$

If there is no "-" in the i-signature, we ignore equation $(**)$ and set $k'_0 = i$.

COMPARISON

Let us compare the both constructions. We prove the following theorem.

Theorem 3.58 *Let M be a $k\mathrm{Q}$-module. Then the map*

$$\vartheta : b_{[M]} \mapsto X_{\gamma_M}$$

is an isomorphism of sl_{n+1}-crystals.

Fix $M \in \mathbb{C}\mathrm{Q}-\mathrm{mod}$, $M \cong \bigoplus_{1\leq k\leq l\leq n} V(k,l)^{\gamma(k,l)}$ and fix $1 \leq i \leq n$. Before we are able to prove Theorem 3.58, we prove the following lemma.

Lemma 3.59 *Let j_0 be maximal such that $F_{j,i}$ is maximal. Then the value of $F_{j_0,i}$ is equal to the number of "-" in the i-signature of M. If there is at least one "-" in the i-signature of M then $V(j_0,i)$ corresponds to the rightmost "-" in the i-signature of M. Otherwise, $j_0 = 1$.*

Proof. In the case that there is no "-" in the $(+,-)$-sequence of M, there is no indecomposable direct summand of M isomorphic to a $V(k,i)$ for $k \leq i$. Thus $F_i(M,V) \leq 0$ for each $V \in \mathscr{P}_i(\mathrm{Q})$. Since $F_{1,i} = 0$, the statement follows for this case.

Assume for the rest of the proof that there is at least one "-" in the $(+,-)$-sequence of M. Each "-" appearing in the $(+,-)$-sequence of M belongs to a direct summand of M of the form $V(k,i)$ for some $k \leq i$ and each "+" appearing belongs to a summand of the form $V(k,i-1)$ for some $k \leq i-1$. We thus have $\sum_{k=1}^{i-1} \gamma(k,i-1)$ many "+" and $\sum_{k=1}^{i} \gamma(k,i)$ many "-" in the $(+,-)$-sequence of M.

Let $V(j'_0, i)$ be the direct summand of M that belongs to the rightmost "-" i-signature of M. If there is no "-" in the i-signature of M, set $j'_0 = 1$. Consider the following sum:

$$F = \sum_{k=1}^{j'_0} \underbrace{1+1+\ldots+1}_{\gamma(k,i)} - \left(\sum_{k=1}^{j'_0-1} \underbrace{1+1+\ldots+1}_{\gamma(k,i-1)} \right), \tag{24}$$

Note that a cancellation of a summand on the left hand side with a summand on the right hand side belongs to a $(+,-)$-cancellation in the formation of the $i-$signature. Furthermore, we have the equality $F{=}F_{j'_0,i}$. and the value of F is the number of "-" appearing in the $i-$signature.

It remains to show that $j'_0 = j_0$. In the case $j'_0 = 1$, we clearly have

$$F_{j,i} \leq 0 \text{ for all } j \text{ and } F_{j'_0,i} = 0.$$

The claim follows for this case.

Assume that there is at least one "-" in the i-signature of M. The fact that $V(j'_0, i)$ belongs to the rightmost "-" of the i-signature of M implies for all $j \geq j'_0$

$$F_{i,j} - F_{i,j'_0}, \leq 0,$$

Assume that there is a $j' \leq j'_0$, such that $F_{i,j_0} \leq F_{i,j'}$. This implies

$$V(j', i) \trianglelefteq V(j'_0, i)$$

and

$$F_{i,j'_0} - F_{i,j'} = \sum_{k=j'}^{j'_0} \underbrace{1+1+\ldots+1}_{m_{k,i}} - \left(\sum_{k=j'}^{j'_0-1} \underbrace{1+1+\ldots+1}_{m_{k,i-1}} \right) \leq 0.$$

Thus the rightmost "-" appearing in the i-signature must belong to a direct summand of M isomorphic to a $V(l, i)$ with $l \leq j'$. This contradicts the assumption that the rightmost "-" belongs to $V(j'_0, i)$.

We conclude that j'_0 is maximal such that $F_{i,j}$ is maximal.

□

We now prove Theorem 3.58.

Proof. Clearly $\mathrm{wt}(\vartheta(b_{[M]}) = \mathrm{wt}(b_{[M]})$. The fact that $\varepsilon_i(\vartheta(b_{[M]})) = \varepsilon_i(b_{[M]})$ is shown in Lemma 3.59 and the equalities $\tilde{f}_i\vartheta(b_{[M]}) = \vartheta(\tilde{f}_i b_{[M]})$, $\tilde{e}_i\vartheta(b_{[M]}) = \vartheta(\tilde{e}_i b_{[M]})$ and $\varphi_i(\vartheta(b_{[M]})) = \varphi_i(b_{[M]})$ are straightforward using the constructions. □

RELATION TO YOUNG TABLEAUX

In [24] Savage uses the combinatorial description of the crystal structure on the quiver varieties recalled in Section 3.4 to prove an isomorphism to the Young Tableaux model of $B(\infty)$ for type A_n given in [7] and [14]. Let us recall this realization briefly.

Definition 3.60 A **semistandard Young Tableau** is a filling of a Young diagram such that the entries are weakly increasing from left to right along each row and strictly increasing down columns. Such a tableau is called **large** if it consists of n rows, and for $1 \leq i \leq n$, the number of i-entries in the i-th row is strictly greater than the number of all boxes in the $i+1$-th row.

Let T_1 and T_2 be two large tableaux with n non-empty rows. Then they are called **related** if for the i-th row of T_1 and T_2, the numbers of entries $j > i$ are equal for all $1 \leq i \leq n$. We write $T_1 \sim T_2$ for two related tableaux and note that this is an equivalence relation. We denote the set of equivalence classes of large tableaux by $T(\infty)$.

The actions of the Kashiwara operators on such a tableau are defined as follows. First read of the entries in far eastern reading, i.e. read each column from top to bottom starting from the rightmost column, continuing to the left. Then write it in the tensor product form.

Example 12*:*

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} = \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}$$

Then the action of the Kashiwara operators can be determined via the i-signature.

Definition 3.61 Let $\bar{T} \in T(\infty)$ be an equivalence class of large tableaux and let T be a representative of $\bar{T}$.

1. Under each tensor component with entry i of T write a "+" and under each tensor component with entry $i+1$ of T write a "-".

2. Cancel all $(+,-)$-pairs until a (possibly empty) sequence of "-"'s followed by a (possibly empty) sequence of "+"'s is left over, called the i-signature.

3. To apply $\tilde{f}_i$ to the whole product, apply it to the leftmost "+" in the i-signature (i.e. change the i-entry belonging to this "+" to an $i+1$-entry).

Let T' be the tableau obtained through this procedure. If T' is already large, define $\tilde{f}_i T := \bar{T}'$. If T' is not large then we define $\tilde{f}_i T$ to be the class of large tableau such that a representative is obtained by inserting one column consisting of i rows to the left of the box of T which $\tilde{f}_i$ acted upon. The added column should have a k-box at the k-th row for $1 \leq k \leq i$.

4. Similarly, to apply $\tilde{e}_i$, apply it to the rightmost "-" in the i-signature. Let T'' be the tableau obtained through this procedure and define $\tilde{e}_i T := \overline{T''}$.If there is no remaining "-", then $\tilde{e}_i$ acts by 0.

Remark 3.62 The condition large ensures, there is always a remaining "+" in the i-signature, i.e. the action $\tilde{f}_i$ given in Definition 3.61 is well-defined (see [14, Lemma 3.2]).

Example 13: We determine the action of $\tilde{f}_2$ on the tableau T defined in Example 12. The 2-signature of T only consists of one "+" belonging to the 2-entry circled below:

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

Then the tableau T' of Definition 3.61 step 3 reads

$$T' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

This tableau is not large. We thus have to add a column consisting of 2 rows as described in Definition 3.61. We get as a representative of $\tilde{f}_2 \overline{T}$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array}$$

Theorem 3.63 ([14, Theorem 3.4, Theorem 4.3]) *A crystal structure on $T(\infty)$ is defined with the Kashiwara-operators given in Definition 3.61, and, for $\overline{T} \in T(\infty)$, let the representatives of $\overline{T}$ have b^i_j-many j's ($i < j \leq n$) in the i-th row. Then*

$$\begin{aligned} \mathrm{wt}(\overline{T}) &= \sum_{j=1}^{n} \left(\sum_{k=j+1}^{n+1} b^1_k + \sum_{k=j+1}^{n+1} b^2_k + \ldots + \sum_{k=j+1}^{n+1} b^n_k \right) \alpha_j; \\ \varepsilon_i(\overline{T}) &= \#\{\text{"} - \text{"}\text{'s in the } i\text{-signature of } \overline{T}\}; \\ \varphi_i(\overline{T}) &= \varepsilon_i(\overline{T}) + \langle h_i, \mathrm{wt}(\overline{T}) \rangle . \end{aligned}$$

Moreover, this is isomorphic to the crystal $B(\infty)$.

Thus there must exist an isomorphism between the geometric realization $B^g(\infty)$ and the combinatorial realization $T(\infty)$ of $B(\infty)$. With the notation introduced in section 3.4 Savage describes the isomorphism explicitly in the following theorem.

Theorem 3.64 ([24, Theorem 5.1]) *For $T \in T(\infty)$, let $\gamma(T)$ be a function from the set of pairs $\{(k,l) \mid 1 \leq k \leq l \leq n\}$ to $\mathbb{Z}_{\geq 0}$ such that $\gamma(T)(k,l)$ is equal to the number of $(l+1)$-entries in the $k-$th row of T, for $1 \leq k \leq l \leq n$. Then the map $T(\infty) \to B^g(\infty)$, given by $T \mapsto X_{\gamma(T)}$ is a crystal isomorphism.*

As a direct consequence of Theorem 3.58, we get

Theorem 3.65 *The map $T(\infty) \to B^{\mathscr{H}}(\infty)$ given by $T \mapsto b_{[M(\gamma(T),\mathbb{C})]}$ for $T \in T(\infty)$ and $M(\gamma(T), \mathbb{C})$ is a crystal isomorphism.*

Example 14: The tableau T defined in Example 12 corresponds to $b_{[M]}$ for $M \cong V(1,1) \oplus V(2,2)$. Then $\tilde{f}_2 b_{[M]} = b_{[M']}$, where $M' \cong V(1,1) \oplus V(2,2) \oplus V(2,2)$ corresponding to the class of large tableau with a representative given by

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 2 \\
\hline
2 & 2 & 3 & 3 & \multicolumn{2}{c}{} \\
\cline{1-4}
3 & \multicolumn{5}{c}{} \\
\cline{1-1}
\end{array}\ .$$

This is exactly the element $\tilde{f}_2\overline{T}$ of $T(\infty)$.

4 CRYSTAL GRAPHS OF REPRESENTATIONS

Let $V(\lambda)$ be the irreducible finite dimensional $U_v(\mathfrak{g})$-module of highest weight λ and $B(\lambda)$ its crystal graph. In this section, we focus on the realization of $B(\lambda)$ as a full subgraph of $B(\infty)$.

4.1 $B^{\mathscr{H}}(\lambda)$ in $B^{\mathscr{H}}(\infty)$

To describe the embedding of the crystal graph $B(\lambda)$ of the irreducible $\mathfrak{g}$-module of highest weight $\lambda \in P^+$ into $B(\infty)$, Reineke uses Theorem 2.2. We first describe the Kashiwara-involution in the Hall algebra setting. Let $*$ be the $\mathbb{Q}(v)$-linear antiautomorphism of $U_v(\mathfrak{n}^-)$ fixing the generator F_i. Then using the isomorphism $\nu : U_v(\mathfrak{n}^-) \to \mathscr{H}(\mathrm{Q})$ which is sending the generator F_i to the generator $u_{[S(i)]}$ for $i \in I$, $*$ reads as follows

$$\begin{aligned} * : \mathscr{H}(\mathrm{Q}) &\to \mathscr{H}(\mathrm{Q})^{op} \\ u_{[S(i)]} &\mapsto u_{[S(i)]}. \end{aligned}$$

Since $\mathscr{H}(\mathrm{Q})^{op} \cong \mathscr{H}(\mathrm{Q}^*)$, we have the following description of the Kashiwara involution $*$ on $B^{\mathscr{H}}(\infty)$

$$\begin{array}{rcl} * : B^{\mathscr{H}}(\infty) & \to & B^{\mathscr{H}}(\infty) \\ b_{[M]} & \mapsto & b_{M(\gamma_{\mathrm{D}\,M},k)}, \end{array}$$

where $\mathrm{D}\,M \in (k\mathrm{Q})^{\mathrm{op}} - \mathrm{mod}$ is obtained via the standard duality described in Remark 2.5.

To describe the function ε_i^* in the language of quiver representations, we thus have to dualize the notions of Section 3.1. This yields an additional condition on the orientation Ω of the quiver Q.

Definition 4.1 A quiver Q is called **cospecial** if $\dim \mathrm{Hom}_{k\mathrm{Q}}(S(i), X) \leq 1$ for all $i \in I$ and all indecomposable kQ-modules X.

For the rest of this section we make the following assumption:

Q is a fixed special and cospecial Dynkin quiver.

Note that a quiver Q is cospecial if and only if Q^* is special. For the non-exceptional types we can thus find such an orientation (see (4)).

We dualize the notion of the posets $\mathscr{P}_i(\mathrm{Q})$ and $\mathscr{S}_i(\mathrm{Q})$:

Definition 4.2 We define the poset

$$\mathscr{P}_i^{\vee}(\mathrm{Q}) := \{X \in \mathbb{C}\mathrm{Q}{-}\mathrm{mod} \mid X \text{ is indecomposable and } \dim \mathrm{Hom}_{k\mathrm{Q}}(S(i), X) \neq 0\}$$

together with the relation $\preceq$ given in (2),i.e.

$$X \preceq Y \iff \mathrm{Hom}_{k\mathrm{Q}}(X, Y) \neq 0.$$

Furthermore we define the poset

$$\mathscr{S}_i^{\vee}(\mathrm{Q}) := \{V = \bigoplus_{j=1}^{k} X_j \mid \text{the set of elements } X_j \text{ form an antichain in} \mathscr{P}_i^{\vee}(\mathrm{Q})\}$$

together with the relation $\trianglelefteq^{\vee}$ given by

$$V \trianglelefteq^{\vee} V' \iff \dim \mathrm{Hom}_{k\mathrm{Q}}(V, B) \neq 0$$

for each indecomposable direct summand B of V'.

Example 15: 1. *For* $\mathrm{Q} = 1 \leftarrow 2 \leftarrow 3$*, the poset* $\mathscr{P}_3^\vee(\mathrm{Q})$ *is the union of all framed modules:*

$$\begin{tikzcd}
 & & \boxed{\mathbf{111}} \arrow[dr] & & \\
 & \boxed{\mathbf{011}} \arrow[ur] \arrow[dr] & & 110 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
\boxed{\mathbf{001}} \arrow[ur] & & 010 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & 100 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

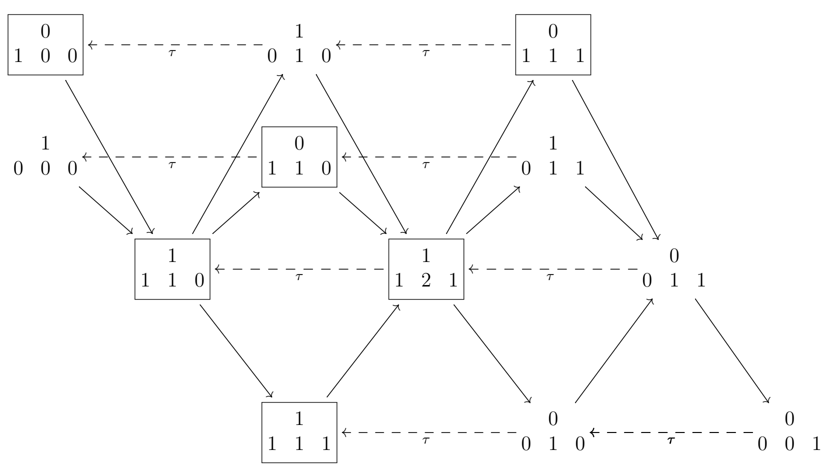
We have that $\mathscr{S}_3^\vee(\mathrm{Q}) = \mathscr{P}_3^\vee(\mathrm{Q})$*. The elements can be ordered as follows:*

$$001 \trianglelefteq^\vee 011 \trianglelefteq^\vee 111.$$

2. *Let* Q *be the following quiver*

$$\begin{tikzcd}
 & 2 & \\
1 & 3 \arrow[l] \arrow[u] & 4. \arrow[l]
\end{tikzcd}$$

The poset $\mathscr{P}_1^\vee(\mathrm{Q})$ *is again the union of all framed modules:*



We have

$$\mathscr{S}_1^\vee(\mathrm{Q}) = \left\{ \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \right\}.$$

We have two chains of maximal length in $\mathscr{S}_i^\vee(\mathrm{Q})$*:*

$$\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 1 & \\ 1 & 1 & 0, \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix}$$

and

$$\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix} \trianglelefteq^{\vee} \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix}$$

We furthermore have a dual version of the sum $F_i(M, V)$ given in Definition 3.7:

Definition 4.3 Fix $i \in I$. For a kQ–module M and an element $V \in \mathscr{S}_i^{\vee}(\mathrm{Q})$ define

$$F_i^{\vee}(M, V) := \sum_{B \in \mathcal{P}_i^{\vee}(\mathrm{Q});\ V \trianglelefteq^{\vee} B} \mu_B(M) - \mu_{\tau^{-1}B}(M). \tag{25}$$

Then the embedding of the crystal graph of $B^{\mathscr{H}}(\lambda)$ into the crystal graph of $B^{\mathscr{H}}(\infty)$ can be described as:

Theorem 4.4 ([19, Proposition 7.4]) *For $\lambda \in P^+$, $\lambda = \sum_{i \in I} \lambda_i \omega_i$, the crystal graph of $B^{\mathscr{H}}(\lambda)$ is the full subgraph of $B^{\mathscr{H}}(\infty)$ with vertices given as*

$$B^{\mathscr{H}}(\lambda) = \{b_{[M]} \mid F_i^{\vee}(M, V) \leq \lambda_i \text{ for all } i \in I \text{ and for all } V \in \mathscr{S}_i^{\vee}(\mathrm{Q})\}.$$

Example 16*: Consider once more the following quiver*

$$\mathrm{Q} = 1 \leftarrow 2.$$

Recall the Auslander-Reiten quiver of Q*:*

$$\begin{tikzcd} & 11 \arrow[dr] & \\ 10 \arrow[ur] & & 01 \arrow[ll, dashed, "\tau"] \end{tikzcd}$$

Fix $\lambda = 2\omega_1 \in P^+$. The vertices of the crystal graph of $B^{\mathscr{H}}(\lambda)$ are given by the isomorphism classes of kQ-modules M such that

$$M \cong V(1,1)^{\gamma(1,1)} \oplus V(1,2)^{\gamma(1,2)} \oplus V(2,2)^{\gamma(2,2)}$$

with

$$F_2^{\vee}(M, 01) = \gamma(2,2) \leq 0;$$
$$F_1^{\vee}(M, 11) = \gamma(1,2) \leq 2;$$
$$F_1^{\vee}(M, 10) = \gamma(1,2) + \gamma(1,1) - \gamma(2,2) \leq 2.$$

Thus the crystal graph of $B^{\mathscr{H}}(\lambda)$ is the following full subgraph of $B^{\mathscr{H}}(\infty)$:

$$\begin{tikzcd}
b_{[0]} \arrow[r, "1"] & b_{[10]} \arrow[r, "1"] \arrow[d, "2"] & b_{[10\oplus 10]} \arrow[d, "2"] \\
& b_{[11]} \arrow[r, "1"] & b_{[11\oplus 10]} \arrow[d, "2"] \\
& & b_{[11\oplus 11]}
\end{tikzcd}$$

4.2 $B^g(\lambda)$ in $B^g(\infty)$

NAKAJIMA'S QUIVER VARIETY

In [23], Saito gives a realization of the crystal $B(\lambda)$ via Nakajima's quiver varieties. In this section we recall the definition of those spaces.

To define his quiver variety Nakajima considers a framing the double quiver $\overline{\mathrm{Q}}$ by adding an extra vertex i' and an extra arrow $t_i : i \to i'$ for all $i \in I$.

Example 17*: We give an example of the extended double quiver for the Dynkin graph of type A_3:*

$$\begin{tikzcd}
1 \arrow[r, shift left, "\overline{h}_1"] \arrow[d, "t_1"] & 2 \arrow[l, shift left, "h_1"] \arrow[r, shift left, "\overline{h}_2"] \arrow[d, "t_2"] & 3 \arrow[l, shift left, "h_2"] \arrow[d, "t_3"] \\
1' & 2' & 3'
\end{tikzcd}$$

For $v, \lambda \in \mathbb{Z}^I_{\geq 0}$, we choose I-graded vector spaces V and W of graded dimension v, λ, respectively and define

$$\Lambda \equiv \Lambda(v, \lambda) := \Lambda_V \times \bigoplus_{i \in I} \mathrm{Hom}(V_i, W_i).$$

Then the action of the group $G_v = \prod_{i \in I} GL(V_i)$ can be extended on Λ via

$$g(x,t) := (g_i)_{i\in I}((x_h)_{h\in H}, (t_i)_{i\in I}) = ((g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)})_{h\in H}, (t_i g_i^{-1})_{i\in I})$$

fo $g \in G_v$ and $x \in \Lambda$.

To state the definition of Nakajima's quiver variety we have to restrict Λ to the **subset of stable points** Λ^{st} defined as

$$\Lambda^{st} := \{(x,t) \in \Lambda \mid \bigcap_{\mathrm{out}(h)=i} (\ker x_h \cap \ker t_i) = 0\}.$$

Remark 4.5 This Definition is equivalent to the one given in [18] stating that there is no non-trivial x-stable subspace of V contained in the kernel of t, see [3, Lemma 3.4].

The subset Λ^{st} is open in Λ and we clearly have an action of the group G_v on Λ^{st}. We further have the following.

Lemma 4.6 ([18, Lemma 3.1]) *The action of G_v on Λ^{st} is free and Λ^{st} is a non-singular subvariety of Λ.*

Using [13], Nakajima shows in [18, Section 3.ii] that this is a stability condition in the sense of Mumford. We define **Nakajima's quiver variety** to be the geometric quotient of Λ^{st} by G_v:

$$\mathfrak{L} \equiv \mathfrak{L}(v,\lambda) := \Lambda(\lambda, w)^{st} \big/ G_v .$$

THE SUBGRAPH

For brevity, we denote by $\operatorname{Irr}\mathfrak{L}$ the **set of irreducible components of $\mathfrak{L}$** and by $\operatorname{Irr}\Lambda$ the **set of irreducible components of Λ**. Lemma 4.6 then yields the following identification:

$$\operatorname{Irr}\mathfrak{L}(v,\lambda) \cong \{Y \in \operatorname{Irr}\Lambda(v,\lambda) \mid Y \cap \Lambda(v,\lambda)^{st} \neq \varnothing\}.$$

We define

$$Y_{[M]} := \left(\left(\overline{\mathcal{C}_{[M]}} \times \bigoplus_{i\in I} \operatorname{Hom}(V_i, W_i) \right) \cap \Lambda(v,\lambda)^{st} \right) \big/ G_v$$

and note that

$$\operatorname{Irr}\mathfrak{L}(v,\lambda) = \{Y_{[M]} \mid M \in \mathbb{C}\mathrm{Q}-\mathrm{mod} \text{ and } Y_{[M]} \neq \varnothing\}. \tag{26}$$

We conclude that the irreducible components of $\mathfrak{L}(v,\lambda)$ are in one-to-one correspondence to the irreducible components of $\Lambda(v,\lambda)$ that contain a stable point.

In [23] Saito describes a crystal structure on $\operatorname{Irr}\mathfrak{L}$ using similar arguments as in [12]. The key point for our approach is the following lemma:

Lemma 4.7 ([23, Lemma 4.6.2, Lemma 4.6.3]) *There exists an injective map $\mathfrak{i} : \operatorname{Irr}\mathfrak{L}(v,\lambda) \to B(v,\infty)$ which commutes with the crystal operators. For $M \in \mathbb{C}\mathrm{Q}-\mathrm{mod}$ a representative of $[M]$ and $Y_{[M]} \neq \varnothing$ it is given by $\mathfrak{i}(Y_{[M]}) = \overline{\mathcal{C}_{[M]}}$.*

Hence we have the following result.

Corollary 4.8 *The crystal graph of $B^g(\lambda)$ is the full subgraph of $B^g(\infty)$ with vertices the irreducible components*

$$\{\overline{\mathcal{C}_{[M]}} \in B^g(\infty) \mid Y_{[M]} \neq \varnothing\}.$$

4.3 COMPARISON

To describe the connection of the two constructions, we first need a better description of the irreducible components of $\Lambda(v,\lambda)$, that contain stable points.

Lemma 4.9 *The irreducible components of $\Lambda(v,\lambda)$ that contain a stable point are precisely those components that contain a point $(x,s) \in \Lambda_V \times \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i)$ such that $\dim(\bigcap_{\mathrm{out}(h)=i} \ker x_h) \le w_i$.*

Proof. Considering $(x,t) \in \Lambda^{st}$, the condition $\bigcap_{\mathrm{out}(h)=i} (\ker x_h \cap \ker t_i) = 0$ forces the existence of an isomorphism $\bigcap_{\mathrm{out}(h)=i} \ker x_h \cong \widetilde{V}$ for a vector subspace $\widetilde{V}$ of $V/\ker t_i$. But $V/\ker t_i \cong \mathrm{Im}\, t_i$ and $\dim \mathrm{Im}\, t_i \le \dim W_i$. $\square$

Definition 4.10 We define

$$\varepsilon_i^*(x) := \dim(\bigcap_{\mathrm{out}(h)=i} \ker x_h)$$

and for $X \in \mathrm{Irr}\, \Lambda_V$, we define

$$\varepsilon_i^*(X) := \min_{x\in X} \varepsilon_i^*(x).$$

In [12, Section 5.3], Kashiwara and Saito translate the standard duality described in Remark 2.5 to their setting. By choosing an isomorphism between V_i and it's dual for every $i \in I$, they obtain an involution

$$* : B^g(v;\infty) \to B^g(v;\infty),$$

via

$$\left(\overline{\mathcal{C}_{[M]}}\right)^* = \{x^{\mathrm{t}} \mid x \in \overline{\mathcal{C}_{[M]}}\}$$

where x^{t} is the transpose of the linear map x.

Note that since Λ_V is G_v-invariant, this does not depend on the choice of vector space isomorphism.

Let us describe the image of $*$ more precisely. Consider the following projection (compare with (9)):

$$\begin{aligned} \tilde{\mathrm{pr}} : \quad \Lambda_V &\to E_{V,\bar{\Omega}}, \\ (x_h)_{h\in H} &\mapsto (x_h)_{h\in\bar{\Omega}}. \end{aligned}$$

Let y be in $E_{V,\bar{\Omega}}$. Then clearly $\overline{\tilde{\mathrm{pr}}^{-1}(G_v y)}$ is an irreducible component of Λ_V as the closure of a conormal bundle of a G_v-orbit in $E_{V,\overline{\Omega}}$. Fix $M = (V,x) \in \mathbb{C}Q - \mathrm{mod}$, then

$$(\overline{\mathcal{C}_{[M]}})^* = \overline{\tilde{pr}^{-1}(GL_v x^{\mathrm{t}})}.$$

We get immediately:

$$\varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) = \varepsilon_i((\overline{\mathcal{C}_{[M]}})^*). \tag{27}$$

Remark 4.11 Recall from Remark 2.6 that the representations of $(\mathbb{C}Q)^{\mathrm{op}}$ can be identified with the representations of $\mathbb{C}Q^*$. We may choose $\bar{\Omega}$ as a orientation of the Dynkin diagram. Denote the resulting quiver variety by Λ_V^*. It is clearly isomorphic to Λ_V and the irreducible component $\overline{\mathcal{C}_{[M^*]}}$ of Λ_V is mapped to the irreducible component $\overline{\mathcal{C}_{[\mathrm{D}\,M]}}$ of Λ_V^* under this isomorphism. Further

$$\varepsilon_i((\overline{\mathcal{C}_{[M]}})^*) = \varepsilon_i(\overline{\mathcal{C}_{[\mathrm{D}\,M]}}).$$

Example 18: We consider an example of type A_2. Recall the corresponding double quiver:

$$1 \underset{h_1}{\overset{\overline{h}_1}{\rightleftarrows}} 2$$

Let Ω consist of the arrow h_1, then $\overline{\Omega}$ consists of the arrow $\overline{h}_1$. Fix $V = \mathbb{C} \oplus \mathbb{C}$. Then Λ_V has two irreducible components, namely

$$\Lambda_V = (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C}).$$

We have two isomorphism classes of representations of the quiver

$$\mathrm{Q} = 1 \leftarrow 2$$

with dimension vector $(1,1)$, the isomorphism class of

$$M_1 = \mathbb{C} \xleftarrow{1} \mathbb{C}$$

and the isomorphism class of

$$M_2 = \mathbb{C} \xleftarrow{0} \mathbb{C}.$$

Hence $\mathcal{C}_{[M_1]} = \{\mathbb{C}\backslash\{0\}, 0\}$ and $\mathcal{C}_{[M_2]} = \{0, \mathbb{C}\}$. To determine the image under $*$ of these irreducible components, consider the quiver Q^*:

$$\mathrm{Q}^* = 1 \rightarrow 2.$$

We have the identification

$$\mathrm{D}\,M_1 = \mathbb{C} \xrightarrow{1} \mathbb{C},$$

$$\mathrm{D}\, M_2 = \mathbb{C} \xrightarrow{0} \mathbb{C}.$$

Further

$$(\overline{\mathcal{C}_{[M_1]}})^* = \{(\mathbb{C}, 0)\}^* = \{(0, \mathbb{C})\}$$

and

$$(\overline{\mathcal{C}_{[M_2]}})^* = \{(0, \mathbb{C})\}^* = \{(\mathbb{C}, 0)\}.$$

Hence we have the following identification of irreducible components.

Corollary 4.12

$$\operatorname{Irr} \mathfrak{L}(v, \lambda) = \{Y_{[M]} \mid M \in \mathbb{C}\mathrm{Q} - \mathrm{mod} \ \textit{ and } \varepsilon_i(\overline{\mathcal{C}_{[\mathrm{D}\, M]}}) \leq \lambda_i \ \forall i \in I\}.$$

Moreover, Kashiwara and Saito show that $* : B^g(\infty) \to B^g(\infty)$ is precisely the Kashiwara involution on $B^g(\infty)$ (see [12, Proposition 3.2.3, Proposition 5.3.1]). The remainder of this section makes the connection between Reneike's description of ε_i^* and the description of this function in the geometric setting precise.

Lemma 4.13 *The elements of $\mathscr{S}_i^\vee(\mathrm{Q})$ can be identified with the elements of $\mathscr{S}_i(\mathrm{Q}^*)$ via the standard duality* D. *Moreover, for V and V' in $\mathscr{S}_i^\vee(\mathrm{Q})$, we have*

$$V \trianglelefteq V' \textit{ if and only if } \mathrm{D}\, V \trianglelefteq^\vee \mathrm{D}\, V'.$$

Proof. This is a straightforward consequence of Remark 2.6. □

We thus have:

Corollary 4.14 *Let M be a $k\mathrm{Q}$-module, $\mathrm{D}\, M \in (k\mathrm{Q})^{\mathrm{op}} - \mathrm{mod}$ the dual module of M obtained through the standard duality functor* D *and $V \in \mathscr{S}_i(Q)$, then*

$$F_i(\mathrm{D}\, M, \mathrm{D}\, V) = F_i^\vee(M, V).$$

Lemma 4.15 *Let V_0 be an antichain in $\mathscr{S}_i(Q)^\vee$ such that the maximal value of $F_i^\vee(M, V)$ is reached at V_0. Then*

$$\varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) = F_i^\vee(M, V_0).$$

Proof. Since we have $\varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) = \varepsilon_i(\overline{\mathcal{C}_{[\mathrm{D}\, M]}}) = \max_{\mathrm{D}\, V \in \mathscr{S}_i(\mathrm{Q}^*)} F_i(\mathrm{D}\, M, \mathrm{D}\, V)$, the claim follows from Corollary 4.14. □

We have thus proved:

Theorem 4.16 *Let $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ be in P^+. Then the crystal graph of $B^g(\lambda)$ is isomorphic to the full subgraph of $B^g(\infty)$ with vertices the irreducible components*

$$\{\overline{\mathcal{C}_{[M]}} \mid F_i^\vee(M, V) \leq \lambda_i \textit{ for all } i \in I \textit{ and for all } V \in \mathscr{S}_i(Q)^\vee\}.$$

4.4 NON-EMBEDDED STRUCTURE

In this section we give a self-contained description of $B(\lambda)$. For this we prove the following theorem:

Theorem 4.17 *For $\lambda = \sum_{i=1}^{n} w_i \omega_i$ in P^+, the crystal $B^g(\lambda)$ is isomorphic to the following crystal structure on $\operatorname{Irr} \mathfrak{L}(v,w) \sqcup \{0\}$ as abstract crystals. For $Y_{[M]} \in \operatorname{Irr} \mathfrak{L}(v,w)$, we have:*

$$\operatorname{wt}(Y_{[M]}) = \lambda + \operatorname{wt}(\overline{\mathcal{C}_{[M]}})$$
$$\varepsilon_i(Y_{[M]}) = \varepsilon_i(\overline{\mathcal{C}_{[M]}})$$
$$\varphi_i(Y_{[M]}) = \varepsilon_i(Y_{[M]}) + \langle h_i, \operatorname{wt}(Y_{[M]}) \rangle$$
$$\tilde{f}_i Y_{[M]} = \begin{cases} \left(\left(\tilde{f}_i \overline{\mathcal{C}_{[M]}} \times \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \right) \cap \Lambda(v,w)^{st} \right) / G_v & \text{if } \varphi_i(Y_{[M]}) > 0 \\ 0 & \text{else} \end{cases}$$

$$\tilde{e}_i Y_{[M]} = \begin{cases} \left(\left(\tilde{e}_i \overline{\mathcal{C}_{[M]}} \times \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \right) \cap \Lambda(v,w)^{st} \right) / G_v & \text{if } \varepsilon_i(Y_{[M]}) > 0 \\ 0 & \text{else} \end{cases}$$

Before we are able to prove Theorem 4.17, we need some preparatory work to show that the crystal operators are well-defined. In order to do that, we have to describe the precise relationship between the two function ε_i^* and φ_i on $\operatorname{Irr} \Lambda(v)$.

Lemma 4.18 *Let M be in $\mathbb{C}\mathrm{Q} - \mathrm{mod}$. Then we have for all $j \neq i$*

$$\varepsilon_i^*(\tilde{f}_j \overline{\mathcal{C}_{[M]}}) = \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}).$$

Proof. Let x be a generic point of $\overline{\mathcal{C}_{[M]}} \in \operatorname{Irr} \Lambda(v)$ and let y be a generic point of $\tilde{f}_j \overline{\mathcal{C}_{[M]}} \in \operatorname{Irr} \Lambda(v + e^i)$. Then those points correspond to $\Pi(\mathrm{Q})$-representations $\tilde{M} = (V, x)$ and $\tilde{N} = (W, x)$ for some vector spaces V and W of graded dimension v and $v + e^i$. We further have an injective morphism of $\Pi(\mathrm{Q})$-modules:

$$\tilde{M} \hookrightarrow \tilde{N}.$$

Thus

$$\dim \bigcap_{h:\operatorname{out}(h)=i} \ker x_h = \dim \bigcap_{h:\operatorname{out}(h)=i} \ker y_h.$$

□

Lemma 4.19

$$\varepsilon_i^*(\tilde{f}_i\overline{\mathcal{C}_{[M]}}) = \begin{cases} \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) + 1 & \textit{if } \tilde{f}_i\overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[M\oplus S(i)]}} \textit{ and} \\ & \max_{V\in\mathscr{S}_i^\vee} F_i^\vee(M,V) = F_i^\vee(M,S(i)) \\ \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) & \textit{else.} \end{cases}$$

Proof. Let B be an indecomposable $\mathbb{C}$Q-module in $\mathscr{P}_i(\mathrm{Q})\cap\mathscr{P}_i^\vee(\mathrm{Q})$. Then we have the following homomorphisms:

$$B \twoheadrightarrow S(i) \hookrightarrow B.$$

Thus $\mathscr{P}_i(\mathrm{Q})\cap\mathscr{P}_i^\vee(\mathrm{Q}) = \{S(i)\}$. We have two cases.

1. Assume that $\tilde{f}_i\overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[M\oplus S(i)]}}$.

   Hence

   $$F_i^\vee(M,V) = F_i^\vee(M\oplus S(i),V) \text{ for all } V\in\mathscr{S}_i(\mathrm{Q})\backslash\{S(i)\}$$

   and

   $$F_i^\vee(M,S(i)) = F_i^\vee(M\oplus S(i),S(i)) + 1.$$

   We conclude for this case

   $$\varepsilon_i^*(\tilde{f}_i\overline{\mathcal{C}_{[M]}}) = \begin{cases} \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) + 1 & \text{if } \max_{V\in\mathscr{S}_i^\vee} F_i^\vee(M,V) = F_i^\vee(M,S(i)) \\ \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) & \text{else.} \end{cases}$$

2. Consider $\tilde{f}_i\overline{\mathcal{C}_{[M]}} = \overline{\mathcal{C}_{[N]}}$ with $N \ncong M\oplus S(i)$. Then $N\cong M'\oplus V_0$ where $M = M'\oplus U_0$ with U_0 and V_0 as in Definition 3.12. Since there is no indecomposable direct summand of V_0 in $\mathscr{P}_i^\vee(\mathrm{Q})$, we have

   $$\varepsilon_i^*(\tilde{f}_i\overline{\mathcal{C}_{[M]}}) \le \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}).$$

   Assume that there is an indecomposable direct summand C of U_0 in $\mathscr{P}_i^\vee(\mathrm{Q})$. Then, from the definition, there is a $B\in\mathscr{P}_i(\mathrm{Q})$, such that $C = \tau B$ and we have homomorphisms

   $$B \twoheadrightarrow S(i) \hookrightarrow C = \tau B.$$

   Hence $\mathrm{Hom}_{\mathbb{C}\mathrm{Q}}(B,\tau B)\neq 0$. A contradiction.

   An analog argument shows that there cannot be be an indecomposable direct summand B of V_0 such that $\tau^{-1}B\in\mathscr{P}_i^\vee(\mathrm{Q})$.

   This yields

   $$\varepsilon_i^*(\tilde{f}_i\overline{\mathcal{C}_{[M]}}) = \varepsilon_i^*(\overline{\mathcal{C}_{[M]}})$$

   for this case which proves the statement.

$\square$

Proposition 4.20 *For $\lambda = \sum_{i=1}^{n} w_i \omega_i$, let $\overline{\mathcal{C}_{[M]}}$ be an element of $B(\lambda)$. Then*

$$\varepsilon_i^*(\tilde{f}_i \overline{\mathcal{C}_{[M]}}) > w_i \text{ if and only if } \varphi_i(\overline{\mathcal{C}_{[M]}}) = 0.$$

Proof. Assume that

$$\varepsilon_i^*(\tilde{f}_i \overline{\mathcal{C}_{[M]}}) > w_i.$$

Regarding Lemma 4.19, we have

$$\varepsilon_i^*(\overline{\mathcal{C}_{[M]}}) = w_i$$

and

$$\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = F_i(M, S(i)).$$

Recall that the symmetrized Euler form coincides with the Cartan form on the negative root lattice of $\mathfrak{g}$. Hence we have

$$\begin{aligned}
\varphi_i(\overline{\mathcal{C}_{[M]}}) &= \varepsilon_i(\overline{\mathcal{C}_{[M]}}) + \left\langle h_i, \mathrm{wt}(\overline{\mathcal{C}_{[M]}})\right\rangle \\
&= F_i(M, S(i)) + w_i - \big(\dim \mathrm{Hom}_{\mathbb{C}Q}(M, S(i)) - \dim \mathrm{Hom}_{\mathbb{C}Q}(\tau^{-1}M, S(i))\big) \\
&- (\dim \mathrm{Hom}_{\mathbb{C}Q}(S(i), M) - \dim \mathrm{Hom}_{\mathbb{C}Q}(S(i), \tau M)) \\
&= F_i(M, S(i)) + w_i - F_i(M, S(i)) - F_i^{\vee}(M, S(i)) \\
&= 0.
\end{aligned}$$

Conversely, assume that

$$\varphi_i(\overline{\mathcal{C}_{[M]}}) = 0,$$

i.e.

$$0 = \varepsilon_i(\overline{\mathcal{C}_{[M]}}) + \left\langle h_i, \mathrm{wt}(\overline{\mathcal{C}_{[M]}})\right\rangle = \varepsilon_i(\overline{\mathcal{C}_{[M]}}) + w_i - F_i(M, S(i)) - F_i^{\vee}(M, S(i)).$$

Thus

$$\varepsilon_i(\overline{\mathcal{C}_{[M]}}) = F_i(M, S(i))$$

and

$$w_i = F_i^{\vee}(M, S(i)) = \varepsilon_i^*(\overline{\mathcal{C}_{[M]}}).$$

Using Lemma 4.19, we get

$$\varepsilon_i^*(\tilde{f}_i \overline{\mathcal{C}_{[M]}}) > w_i.$$

$\square$

Proof of Theorem 4.17. Recall from Lemma 4.7 that the map

$$\mathfrak{i} : \operatorname{Irr} \mathfrak{L}(v,w) \to \operatorname{Irr} \Lambda(v)$$

given by

$$\mathfrak{i}(Y_{[M]}) = \overline{\mathcal{C}_{[M]}}$$

commutes with the crystal operators.

We further get from Lemma 4.18 and Proposition 4.20 that $\tilde{f}_i$ is well-defined on $\operatorname{Irr} \mathfrak{L}(v,w)$ for all $i \in I$.

Now the claim follows from Theorem 2.2. $\square$

4.5 TYPE A

In this section we give again an explicit description of $B(\lambda)$ for $\mathfrak{g} = \mathrm{sl}_{n+1}$. We furthermore give a crystal isomorphism to the combinatorial realization via Young Tableaux. In the sequel let us fix once more the following quiver of type A_n:

$$\mathrm{Q} = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$$

Note that Q is special and cospecial. Although our approach is slightly different, this section is again inspired by a result of Savage. In [24] he gives an isomorphism between the Young Tableaux realization of $B(\lambda)$ and the geometric realization via Nakajima's quiver varieties for type A.

HOMOLOGICAL CONSTRUCTION

Recall that for our given quiver, we have $\mathscr{S}_i^\vee(\mathrm{Q}) = \mathscr{P}_i^\vee(\mathrm{Q})$. Recall further that for $1 \le k \le l \le n$, $V(k,l)$ is an indecomposable representation corresponding to the negative root $-\alpha_{k,l} = -\alpha_k - \alpha_{k+1} - \ldots - \alpha_l$. For $1 \le i \le n$, we can describe the poset $\mathscr{P}_i^\vee(\mathrm{Q})$ explicitly

$$\mathscr{P}_i^\vee(\mathrm{Q}) = \{V(i,k) \mid i \le k \le n\}.$$

The ordering $\trianglelefteq$ on $\mathscr{P}_i^\vee(\mathrm{Q})$ reads

$$V(i,k) \trianglelefteq^\vee V(i,k') \iff k \le k'.$$

For $M \in \mathbb{C}\mathrm{Q} - \mathrm{mod}$ let $\gamma(k,l)$ again denote the multiplicity of $V(k,l)$ as a direct summand of M. Definition 4.3 simplifies for $V = V(i,k) \in \mathscr{P}_i^\vee(\mathrm{Q})$ in the following way:

$$F_i^\vee(M,V) = \sum_{k=i}^{n} \gamma(i,k) - \sum_{k=i+1}^{n} \gamma(i+1,k).$$

Corollary 4.21 *Let $\lambda = w_1\omega_1 + w_2\omega_2 + \ldots + w_n\omega_n \in P^+$, then $B^{\mathscr{H}}(\lambda)$ is the full subgraph of $B^{\mathscr{H}}(\infty)$ consisting of all elements $b_{[M]}$ for $M \in \mathbb{C} - \mathrm{mod}$, $M \cong \oplus_{1\leq k\leq l\leq n} V(k,l)^{\gamma(k,l)}$ such that*

$$\sum_{k=i}^{n} \gamma(i,k) - \sum_{k=i+1}^{n} \gamma(i+1,k) \leq w_i \text{ for all } 1 \leq i \leq n.$$

Example 19: Let Q *be the linear oriented quiver of type A_2.*

$$1 \leftarrow 2.$$

Let furthermore $\lambda = \omega_1 + \omega_2$. The crystal graph of $B^{\mathscr{H}}(\lambda)$ is the full subgraph of $B^{\mathscr{H}}(\infty)$ consisting of all elements $b_{[M]}$ such that $M \cong \oplus_{1\leq k\leq l\leq n} V(k,l)^{\gamma(k,l)}$ and

$$\gamma(2,2) \leq 1; \quad \gamma(1,2) \leq 1 \quad \gamma(1,2) + \gamma(1,1) - \gamma(2,2) \leq 1.$$

We thus have the following crystal graph:

$$\begin{tikzcd}
 & & b_{[0]} \arrow[dll, "1"'] \arrow[drr, "2"] & & \\
b_{[V(1,1)]} \arrow[d, "2"] & & & & b_{[V(2,2)]} \arrow[d, "1"] \\
b_{[V(1,2)]} \arrow[d, "2"] & & & & b_{[V(1,1)\oplus V(2,2)]} \arrow[d, "1"] \\
b_{[V(1,1)\oplus V(1,1)\oplus V(2,2)]} \arrow[drr, "2"] & & & & b_{[V(1,2)\oplus V(2,2)]} \arrow[dll, "1"] \\
 & & b_{[V(1,1)\oplus V(1,2)\oplus V(2,2)]} & &
\end{tikzcd}$$

YOUNG TABLEAUX

Fix $\lambda = w_1\omega_1 + w_2\omega_2 + \ldots w_n\omega_n \in P^+$. A semistandard Young Tableau is of **shape** λ if its $i-$th row consists of $w_i + w_{i+1} + \ldots + w_n$ boxes. Denote by $SSYT(\lambda, n+1)$ the **set of all semistandard Young Tableaux of shape** λ where each box is filled with a number in $\{1, 2, \ldots, n+1\}$.

We define the $i-$**signature** of a semistandard Young Tableau T of shape λ analog to Definition 3.61.

Definition 4.22 1. For all $i \in I$, we define operators

$$\tilde{f}_i : SSYT(\lambda, n+1) \longrightarrow SSYT(\lambda, n+1) \sqcup \{0\}$$

via

$$\tilde{f}_i(T) = \begin{cases} 0, & \text{if there is no "+" in the } i-\text{signature of } T \\ T', & \text{else,} \end{cases}$$

where T' is obtained by changing the $i-$entry belonging to the rightmost "$+$" in the $i-$signature of T to an $i+1-$entry.

2. We furthermore define operators for all $i \in I$

$$\tilde{e_i} : SSYT(\lambda, n+1) \longrightarrow SSYT(\lambda, n+1) \sqcup \{0\}$$

via

$$\tilde{e}_i(T) = \begin{cases} 0, & \text{if there is no "-" in the } i-\text{signature of } T \\ T', & \text{else,} \end{cases}$$

where T' is obtained by changing the $i+1-$entry belonging to the leftmost "-" in the $i-$signature of T to an $i-$entry.

Recall that the Cartan subalgebra $\mathfrak{h}$ of sl_{n+1} is the space of traceless diagonal matrices in $M_{n+1,n+1}(\mathbb{C})$ and $\epsilon_i \in \mathfrak{h}^*$ is the functional sending a diagonal matrix in $\mathfrak{h}$ to its i-th diagonal entry.

Theorem 4.23 ([11, Theorem 3.4.2]) *For $\lambda \in P^+$ a crystal structure is defined on $SSYT(\lambda, n+1)$ with Kashiwara operators given in Definition 4.22, and*

$$\begin{aligned} \mathrm{wt}(T) &= \sum_{i=1}^{n+1} \#\{\textit{entries equal to } i \textit{ in } T\}\epsilon_i, \\ \varepsilon_i(T) &= \#\{\textit{"-" in the } i-\textit{signature of } T\}, \\ \varphi_i(T) &= \epsilon_i(T) - \langle h_i, \mathrm{wt}(T)\rangle\,. \end{aligned}$$

Moreover, this crystal structure is isomorphic to $B(\lambda)$.

The remainder of this section is devoted to describing the relationship between Corollary 4.21 and Theorem 4.23. Let Z be the set of all pairs (k,l) such that $1 \le k \le l \le n$ and let $\mathscr{Z}$ be the set of all functions $\gamma : Z \to \mathbb{N}$, s.t. $\sum_{k=i}^{n} \gamma(i,k) - \sum_{k=i+1}^{n} \gamma(i+1,k) \le w_i$ for all $1 \le i \le n$.

Theorem 4.24 *The set of all semistandard Young tableaux of shape λ and entries in $\{1,2,\ldots,n+1\}$ is indexed by the set of functions $\gamma \in \mathscr{Z}$.*

Proof. We define the map

$$\begin{array}{rccc} \Psi : & SSY(\lambda, n+1) & \to & Z \\ & T & \mapsto & \gamma_T \end{array}$$

where $\gamma_T(k,l) = \#\{\text{entries equal to } l+1 \text{ in row } k \text{ of } T\}$.

Note that $w_i = \#\{\text{boxes in row } i\} - \#\{\text{boxes in row } i+1\}$. We get from the semistandardness condition for $i \leq k' \leq n$

$\#\{\text{entries } \geq k' \text{ in row } i \text{ in } T\} - w_i \leq \#\{\text{entries } \geq k'+1 \text{ in row } i+1 \text{ in } T\}$.

Thus

$$\sum_{k=k'}^{n} \gamma_T(i,k) - \sum_{k=k'+1}^{n} \gamma_T(i,k) \leq w_i \quad \forall i \leq k' \leq n.$$

Hence the $\operatorname{Im}\Psi \subset \mathscr{Z}$.

Conversely, define the map

$$\begin{array}{cccc} \Psi': & \mathscr{Z} & \to & SSY(\lambda, n+1) \\ & \gamma & \mapsto & T_\gamma. \end{array}$$

Where the ith row of the tableau T_γ consists of $w_i + w_{i+1} + \ldots + w_n$ boxes which we fill from right to left with $\gamma(i,n)$ many $n+1-$entries, $\gamma(i,n-1)$ many n-entries, $\ldots, \gamma(i,i)$ many $i+1-$entries. Since $\gamma \in \mathscr{Z}$, we have:

$$\sum_{k=i}^{n} \gamma(i,k) \leq w_i + w_{i+1} \ldots + w_n.$$

We fill out the empty $w_i + w_{i+1} \ldots + w_n - \sum_{k=i}^{n} \gamma(i,k)$ boxes on the right of the $i-$th row with $i-$entries. Then T_γ is of shape λ and the entries are weakly increasing in the rows from left to right. Further, the entries are strictly increasing in the columns from top to bottom, since

$$\sum_{k=k'}^{n} \gamma_T(i,k) - \sum_{k=k'+1} \gamma_T(i,k) \leq w_i \quad \forall i \leq k' \leq n.$$

We get that Ψ' is well-defined and clearly the inverse map of Ψ which yields the claim.

$\square$

Theorem 4.25 *The map Ψ is an isomorphism of crystals.*

Proof. Let $T \in SSYT(\lambda, n+1)$. First we show that Φ commutes with the Kashiwara operators. Assume that there is at least one "+" in the $i-$signature of T (resp. at least one "-" in the $i-$signature of T), then it is already shown in 3.58, that $\Psi(\tilde{f}_i(T)) = \tilde{f}_i(\Psi(T))$ (resp. $\Psi(\tilde{e}_i(T)) = \tilde{e}_i(\Psi(T))$). If there is no "+" in the $i-$signature of T (resp. no "-" in the $i-$signature of T), then we get from the definition that $\tilde{f}_i\Psi(T) = 0 = \Psi(\tilde{f}_i T)$.

Let $T' \in SSYT(\lambda, n+1)$ be the tableau corresponding to the highest weight vector, i.e. the tableau with all entries in row k being equal to k. Then T' is mapped by Ψ to the function $\gamma' \in Z$ belonging to the highest weight vector, i.e. $\gamma'(k,l) = 0$ for all $1 \leq k \leq l \leq n$. The equations

1. $\mathrm{wt}(\Psi(T)) = \mathrm{wt}(T)$,
2. $\varepsilon_i(\Psi(T)) = \varepsilon_i(T)$,
3. $\varphi_i(\Psi(T)) = \varphi_i(T)$

follow from the properties of crystals, since $\tilde{f}_i(\Psi(T)) = \Psi(\tilde{f}_i(T))$. □

Example 20*: We continue with Example 19, i.e.* $\lambda = \omega_1 + \omega_2$ *and*

$$\mathrm{Q} = 1 \leftarrow 2.$$

Following Theorem 4.24, we have the following correspondence of elements of $B^{\mathscr{H}}(\lambda)$ *and tableaux in* $SSY(\lambda, 3)$*:*

$$b_{[0]} \mapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad b_{[V(1,1)]} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad b_{[V(2,2)]} \mapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$b_{[V(1,2)]} \mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad b_{[V(1,1)\oplus V(2,2)]} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad b_{[V(1,1)\oplus V(1,1)\oplus V(2,2)]} \mapsto \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$b_{[V(1,2)\oplus V(2,2)]} \mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad b_{[V(1,1)\oplus V(1,2)\oplus V(2,2)]} \mapsto \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

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Erklärung

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Lebenslauf

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