# $\mathbb{Z}$–bases and Hilbert–Poincaré polynomials related to PBW filtrations

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## Zusammenfassung

In dieser Arbeit untersuchen wir irreduzible endlich dimensionale PBW graduierte Höchstgewichtsdarstellungen für komplexe endlich dimensionale einfache Lie Algebren. Dabei ist diese Arbeit in drei Teile gegliedert.

Im ersten Teil konstruieren wir für ausgewählte fundamentale Gewichte und deren Vielfache FFL Basen für die entsprechenden oben genannten Moduln. Zudem geben wir eine explizite Beschreibung für die definierenden Ideale dieser Moduln an. Dabei übertragen wir das Vorgehen von Feigin, Fourier und Littelmann auf die von uns betrachteten Fälle.
Der zweite Teil beinhaltet keine Voraussetzungen an das dominant integrale Höchstgewicht. Wir betrachten monomiale Basen für die oben gennanten Moduln, welche unter Benutzung von bestimmten Differentialoperatoren beschrieben werden können. Ferner stellen wir ein Kriterium für solche Basen zur Verfügung, welches unter anderem auch auf die FFL Basen aus dem ersten Teil dieser Arbeit anwendbar ist. Anhand dieses Kriteriums lässt sich entscheiden, ob die gegebene Basis ebenso eine monomiale Basis liefert, falls der Modul über einen Körper mit beliebiger Charakteristik betrachtet wird.
Im dritten und letzen Teil stellen wir eine allgemeine Formel für den Grad des Hilbert–Poincaré–Polynoms für PBW graduierte Höchstgewichtdarstellungen zur Verfügung. Dabei reicht es den Grad für jedes fundamentale Gewicht zu berechnen, was wir explizit ausführen.
Mit den Resultaten dieser Arbeit verbessern wir in einigen Fällen das Verständnis der Theorie der PBW graduierten Höchstgewichtdarstellungen.



## Abstract

We investigate in this thesis irreducible finite–dimensional PBW graded highest weight representations for complex finite–dimensional simple Lie algebras. The thesis is divided into three parts.

In the first part we construct for several fundamental weights and their multiples FFL bases of the corresponding modules mentioned above. Furthermore, we provide an explicit description of the defining ideals of these modules. We transfer the procedure of Feigin, Fourier and Littelmann to the cases considered by us.
The second part does not contain any assumptions on the dominant integral highest weight. We consider monomial bases for the highest weight representation mentioned above, which can be described by using certain differential operators. Further we provide a criteria for such bases, which can be also applied to the FFL bases from the first part of this thesis. On the basis of this criteria it is possible to decide whether the given basis provides again a monomial basis, if the module is considered over a field of arbitrary characteristic.
In the third and last part we provide a general formula for the degree of the Hilbert–Poincaré polynomials of PBW graded highest weight representations. It is sufficient to calculate the degree for every fundamental, what we do explicitly.
The results of this thesis improve in several cases the understanding of the theory of PBW graded highest weight representations.

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# Contents

# Introduction

One important tool for the investigation of the representation theory of Lie algebras is the Poincaré–Birkhoff–Witt Theorem, also called PBW Theorem. For a Lie algebra $\mathfrak{g}$ this theorem provides an explicit construction for a basis of the universal enveloping algebra $U(\mathfrak{g})$.
The classical idea to understand the representations of a Lie algebra $\mathfrak{g}$ is to investigate the representations of $U(\mathfrak{g})$, because they are the same. In this thesis we consider modules for the associated graded algebra $U^a(\mathfrak{g})$, since it seems likely that there is a strong connection between these modules and the modules for $\mathfrak{g}$. The investigation of these PBW filtered and graded modules for simple finite–dimensional Lie algebras has started in recent years only.

Let us briefly recall the construction of the PBW filtration, which is necessary to understand the resulting PBW graduation. Let us fix a simple complex finite–dimensional Lie algebra $\mathfrak{g}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The degree filtration $U(\mathfrak{n}^-)_s$ on the universal enveloping algebra $U(\mathfrak{n}^-)$ over $\mathfrak{n}^-$ is defined by:

$$U(\mathfrak{n}^-)_s = \mathrm{span}\{x_1 \cdots x_l \mid x_i \in \mathfrak{n}^-,\ l \leq s\}.$$

The associated graded space of $U(\mathfrak{g})$ is given by

$$U^a(\mathfrak{n}^-) = \bigoplus_{s \in \mathbb{Z}_{\geq 0}} U(\mathfrak{n}^-)_s / U(\mathfrak{n}^-)_{s-1},\ U(\mathfrak{n}^-)_{-1} = \{0\}.$$

The definition of $U(\mathfrak{g})$ as quotient of the tensor algebra $T(\mathfrak{g})$ and the ideal $J = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g}\rangle$, implies that $U^a(\mathfrak{n}^-) \cong S(\mathfrak{n}^-)$, since $x \otimes y,\ y \otimes x \in U(\mathfrak{n}^-)_2$ and $[x,y] \in U(\mathfrak{n}^-)_1$ for $x, y \in \mathfrak{g}$.

Let us now consider the irreducible finite–dimensional $\mathfrak{g}$–module $V(\lambda)$ of highest weight $\lambda$ and denote by $v_\lambda$ the highest weight vector. Therefore we consider $V(\lambda) = U(\mathfrak{n}^-)v_\lambda$ and the PBW filtration on $U(\mathfrak{n}^-)$ induces the PBW filtration on $V(\lambda)$, where the $s$–th filtration component is given by $V(\lambda)_s = U(\mathfrak{n}^-)_s v_\lambda$. The associated graded space

$$V^a(\lambda) = \bigoplus_{s \in \mathbb{Z}_{\geq 0}} V(\lambda)_s / V(\lambda)_{s-1},\ V(\lambda)_{-1} = \{0\},$$

is a cyclic $S(\mathfrak{n}^-)$–module, which is called PBW graded module. Thus there is a ideal $I(\lambda) \subseteq S(\mathfrak{n}^-)$, the annihilator of the generating element $v_\lambda$, such that:

$$V^a(\lambda) \cong S(\mathfrak{n}^-)v_\lambda \cong S(\mathfrak{n}^-)/I(\lambda).$$

Note that $V(\lambda)_s$ is a $U(\mathfrak{n}^+)$–module for all $s \in \mathbb{Z}_{\geq 0}$. This induces a $U(\mathfrak{n}^+)$–module structure on $V^a(\lambda)$.

In 2009, E. Feigin started the investigation of PBW graded modules (see [Fei09]), where he defined the PBW filtration for arbitrary Kac–Moody abgebras of finite and affine type. In 2011/12 E. Feigin, G. Fourier and P. Littelmann provided an explicit description of the annihilating ideals $I(\lambda)$, in terms of generators and relations, for the Lie algebras $\mathfrak{sl}_n, \mathfrak{sp}_n$ and arbitrary dominant integral weights $\lambda$ (see [FFL11a, FFL11b]). For certain Demazure modules in the $\mathfrak{sl}_n$–case the explicit descriptions are given in [Fou14a, BF14].

In the first part of this thesis (see Chapter 1), we provide an explicit description of $I(\lambda)$ for PBW graded $\mathfrak{g}$–modules corresponding to special fundamental weights $\omega$ and their multiples (see Table 0.1). Further we provide for these PBW graded modules monomial bases, analogue to [FFL11a, FFL11b, Fou14a, BF14] and [Gor11], where a monomial basis is provided for type $\mathtt{G_2}$. Note that Chapter 1 is a modified version of [BD15] and motivated by [FFL11a] and [FFL11b].
For an arbitrary dominant integral weight we call such a basis a *Feigin–Fourier–Littelmann* or just FFL basis and $V^a(\lambda)$ a FFL module, if the basis of $V^a(m\lambda)$, $m \in \mathbb{Z}_{\geq 0}$ is parametrized by the integer points of a normal polytope $P(m)$ (see Section 1.1). We prove the following result:

**Theorem A** (Backhaus, D.). *Let $\mathfrak{g}$ be a simple complex finite–dimensional Lie algebra and $\lambda = m\omega_i$, $m \in \mathbb{Z}_{\geq 0}$ be a rectangular weight, where $\mathfrak{g}$ is of type $\mathtt{X_n}$ and $\omega_i$ is a corresponding admissible weight (see Table 0.1). Further let $V^a(\lambda) \cong S(\mathfrak{n}^-)/I(\lambda)$. Then we have:*

- $I(\lambda) = S(\mathfrak{n}^-)\left(U(\mathfrak{n}^+) \circ \operatorname{span}\{f_\beta^{\lambda(\beta^\vee)+1} \mid \beta \in \Delta_+\}\right).$
- *$V^a(\lambda)$ is a FFL module.*

*Here we denote by $\Delta_+$ the set of positive roots of $\mathfrak{g}$.*

| *Type of* $\mathfrak{g}$ | *weight* $\omega$ | *Type of* $\mathfrak{g}$ | *weight* $\omega$ |
|---|---|---|---|
| $\mathtt{A_n}$ | $\omega_k,\ 1 \leq k \leq n$ | $\mathtt{E_6}$ | $\omega_1,\ \omega_6$ |
| $\mathtt{B_n}$ | $\omega_1,\ \omega_n$ | $\mathtt{E_7}$ | $\omega_7$ |
| $\mathtt{C_n}$ | $\omega_1$ | $\mathtt{F_4}$ | $\omega_4$ |
| $\mathtt{D_n}$ | $\omega_1,\ \omega_{n-1},\ \omega_n$ | $\mathtt{G_2}$ | $\omega_1$ |

Table 0.1: Admissible weights

**Remark.** *The theorem above implies the existence of a normal polytope $P(m\omega_i)$, such that the integer points $S(m\omega_i)$ parametrize a basis of $V(m\omega_i)$. This polytope is the $m$–th Minkowski sum of the polytope $P(\omega_i)$ corresponding to $V(\omega_i)$. In general this is not true for different fundamental weights, so for $\omega_i \neq \omega_j$ we have $|(P(\omega_i) + P(\omega_j)) \cap \mathbb{Z}^N_{\geq 0}| \neq \dim V(\omega_i + \omega_j)$, because the number of integer points in the Minkowski sum is in general too small. For example in the case of $\mathfrak{g} = \mathfrak{sl}_5$, we have $|(P(\omega_1) + P(w_2) + P(\omega_3) + P(\omega_4)) \cap \mathbb{Z}^N_{\geq 0}| = 1023$ and* $\dim V(\omega_1 + \omega_2 + \omega_3 + \omega_4) = 1024$.

**Remark.** *In the $(\mathtt{C_n}, \omega_1)$ case our bases coincide with the bases obtained in [FFL11b], tough in the $(\mathtt{A_n}, \omega_k)$ case they are different from the bases obtained in [FFL11a], which were conjectured by Vinberg (see [Vin05]). This is due to* *a different choice of the total order on the monomials in $S(\mathfrak{n}^-)$. Nevertheless the induced normal polytopes are isomorphic to the corresponding normal polytopes constructed in [FFL11a, FFL11b]. As consequence in the cases $(\mathtt{A_n}, \omega_k)$ and $(\mathtt{C_n}, \omega_1)$ the corresponding projective toric varieties are isomorphic. In contrast, these are in general not isomorphic to the toric varieties corresponding to Gelfand–Tsetlin polytopes investigated in [GL97] and [KM05].*

Let us briefly explain the methods we used to prove Theorem A. Our main tool is the Hasse diagram $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ of $\mathfrak{g}$ given by the standard partial order on the positive roots of $\mathfrak{g}$ (see Section 1.1). To be more precise $H(\mathfrak{n}_\lambda^-)_\mathfrak{g} := (\Delta_+^\lambda, E)$ is a directed labeled graph, where the set of vertices is indexed by $\Delta_+^\lambda$, a subset of $\Delta_+$ corresponding to a Lie subalgebra $\mathfrak{n}_\lambda^- \subset \mathfrak{n}^-$, and the set of edges $E$ is given as follows:

$$\forall\, 1 \leq i, j \leq N: \ (\beta_i \xrightarrow{k} \beta_j) \in E \Leftrightarrow \exists\ \alpha_k \in \Phi_+: \ \beta_i - \beta_j = \alpha_k,$$

where $\Phi_+$ is the set of simple roots.

**Example.** *The Hasse diagram $H(\mathfrak{n}_{\omega_3}^-)_{\mathfrak{sl}_6}$ and the set $\Delta_+^{\omega_3}$:*

$$\begin{tikzcd}[column sep=small]
 & & \beta_1 \arrow[dl, "1"'] \arrow[dr, "5"] & & \\
 & \beta_2 \arrow[dl, "2"'] \arrow[dr, "5"] & & \beta_3 \arrow[dl, "1"'] \arrow[dr, "4"] & \\
\beta_4 \arrow[dr, "5"] & & \beta_5 \arrow[dl, "2"'] \arrow[dr, "4"] & & \beta_6 \arrow[dl, "1"'] \\
 & \beta_7 \arrow[dr, "4"] & & \beta_8 \arrow[dl, "2"'] & \\
 & & \beta_9 & &
\end{tikzcd}$$

$$\begin{aligned}
\beta_1 &= (1,1,1,1,1),\\
\beta_2 &= (0,1,1,1,1),\\
\beta_3 &= (1,1,1,1,0),\\
\beta_4 &= (0,0,1,1,1),\\
\beta_5 &= (0,1,1,1,0),\\
\beta_6 &= (1,1,1,0,0),\\
\beta_7 &= (0,0,1,1,0),\\
\beta_8 &= (0,1,1,0,0),\\
\beta_9 &= (0,0,1,0,0).
\end{aligned}$$

*For more examples of Hasse diagrams we refer to the Appendix.*

We associate to this directed graph a polytope $P(\lambda) = P(m\omega_i) \subset \mathbb{R}^N_{\geq 0}$ via the directed paths in the diagram and show in Section 1.2 that these polytopes are normal. Further we show in Section 1.3 if given the case the Hasse diagram satisfies certain properties, the set of integer points $S(\lambda) = P(\lambda) \cap \mathbb{Z}^N_{\geq 0}$ parametrizes a spanning set of $V^a(\lambda)$. In fact we will show via induction on $m \in \mathbb{Z}_{\geq 0}$, that this spanning set is a FFL basis of $V^a(\lambda)$ (see 1.4 and Section 1.5).
Note that in the cases $(\mathtt{B_n}, \omega_1)$, $(\mathtt{F_4}, \omega_4)$ and $(\mathtt{G_2}, \omega_1)$ we have to change the Hasse diagram slightly, to be able to apply our procedure. Except for the known cases ($\mathtt{A_n}$, $\mathtt{C_n}$, $\mathtt{G_2}$ and Table 0.1) it is not clear if there exists a polytope which parametrizes a FFL basis.

Let us denote by $\mathfrak{g}^a$ the degenerated Lie algebra $\mathfrak{g}^a = \mathfrak{b} \oplus \mathfrak{n}^{-,a}$, where $\mathfrak{n}^{-,a}$ is $\mathfrak{n}^-$ endowed with the trivial lie bracket. Further there is a vector space isomorphism between the quotient module $\mathfrak{g}/\mathfrak{b}$, which is a $\mathfrak{b}$–module via the adjoint action, and $\mathfrak{n}^{-,a}$, which induces a $\mathfrak{b}$–action on $\mathfrak{n}^{-,a}$. Let $\mathbb{G}^a$, $B$ and $N^{-,a}$ be the corresponding algebraic groups of $\mathfrak{g}^a$, $\mathfrak{b}$ and $\mathfrak{n}^{-,a}$. Then we have $\mathbb{G}^a \cong B \ltimes N^{-,a}$ and we define for the $\mathfrak{g}$–module $V^a(\lambda) \cong S(\mathfrak{n}^-)v_\lambda$ the closure of the orbit $\overline{\mathbb{G}^a.[v_\lambda]} \subset \mathbb{P}(V^a(\lambda))$

to be the *degenerated flag variety* $\mathcal{F}^a_\lambda$.
The authors of [FFL13a] showed that $\mathcal{F}^a_\lambda$ has a lot of nice properties if $V^a(\lambda)$ is a FFL module, e. g. $\mathcal{F}^a_\lambda$ is normal and Cohen–Macaulay. Furthermore, there is an explicit representation theoretical description of the corresponding homogeneous coordinate rings. In addition in recent years it turned out that the PBW theory has many connections to geometric representation theory: Schubert varieties ([CIL14], [CILL15]), degenerated flag varieties ([FFL14], [Fei11], [Fei12],[Hag14]) and quiver Grassmannians [CIFR12].

The work of Feigin, Fourier and Littelmann (see[FFL13b]) also motivated the second part of this thesis (see Chapter 2). We fix an arbitrary simple complex finite–dimensional Lie algebra and choose a Chevalley basis $\mathbb{B}_{\mathrm{Ch}}(\mathfrak{g})$ of $\mathfrak{g}$, then we consider the $\mathbb{Z}$–analogue of our setup.
Let $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}$ be the $\mathbb{Z}$–span of $\mathbb{B}_{\mathrm{Ch}}(\mathfrak{g})$, which is a Lie subalgebra of $\mathfrak{g}$. Analogue we define $\mathfrak{n}^-_\mathbb{Z} \subset \mathfrak{n}^-$. Furthermore, let the *Kostant lattice* $U_\mathbb{Z}(\mathfrak{g})$ be a $\mathbb{Z}$–subalgebra of $U(\mathfrak{g})$ (see for details Section 2.1), with these constructions we define $V_\mathbb{Z}(\lambda) = U_\mathbb{Z}(\mathfrak{n}^-)v_\lambda$. The PBW filtration $U_\mathbb{Z}(\mathfrak{n}^-)_s$, $s \in \mathbb{Z}_{\geq 0}$ on the Kostant lattice induces the PBW filtration $V_\mathbb{Z}(\lambda)_s = U_\mathbb{Z}(\mathfrak{n}^-)_s v_\lambda$. The $\mathbb{Z}$–analogue of the PBW graded module $V^a_\mathbb{Z}(\lambda)$ is defined by

$$V^a_\mathbb{Z}(\lambda) = \bigoplus_{s\in\mathbb{Z}_{\geq 0}} V_\mathbb{Z}(\lambda)_{s+1}/V_\mathbb{Z}(\lambda)_s,\ V_\mathbb{Z}(\lambda)_{-1} = 0,$$
$$V^a_\mathbb{Z}(\lambda) \cong S_\mathbb{Z}(\mathfrak{n}^{-,a})v_\lambda \cong S_\mathbb{Z}(\mathfrak{n}^{-,a})/I_\mathbb{Z}(\lambda),$$

where $S_\mathbb{Z}(\mathfrak{n}^{-,a})$ is a divided power analogue of the symmetric algebra over $\mathfrak{n}^{-,a}_\mathbb{Z}$, the Lie subalgebra $\mathfrak{n}^-_\mathbb{Z}$ endowed with the trivial Lie bracket.
Similar to the complex case the ideal $I_\mathbb{Z}(\lambda)$ is stable under the action of $U_\mathbb{Z}(\mathfrak{n}^+)$, which is induced by the adjoint action. In fact one can see, that these operators in $U_\mathbb{Z}(\mathfrak{n}^+)$ are derivations on $S_\mathbb{Z}(\mathfrak{n}^{-,a})$. These differential operators
$\partial_\gamma = \mathrm{ad}(e_\gamma) \in \mathrm{Der}(S_\mathbb{Z}(\mathfrak{n}^{-,a}))$, $\gamma \in \Delta_+$ can be used to obtain relations in the associated graded module (see [FFL11a, FFL11b, FFL13b] and Chapter 1).
Let $\mathbb{B}(V^a(\lambda))$ be a monomial basis of $V^a(\lambda)$ and $\prec$ a homogenous total order on the monomial in $S_\mathbb{Z}(\mathfrak{n}^{-,a})$. We investigate under which assumptions on $\mathbb{B}(V^a(\lambda))$, the $\mathbb{Z}$–analogue of this basis provides a monomial basis of $V^a_\mathbb{Z}(\lambda)$ (see for more details Section 2.3):

(i) There is a non–empty subset $\mathbf{P}$ of the power set of the positive roots $\mathcal{P}(\Delta_+)$ and each element $\mathbf{p} \in \mathbf{P}$ contains a root $\beta_\mathbf{p} \in \mathbf{p}$, such that for all multi–exponents $\mathbf{m} \in \mathbb{Z}^{|\Delta_+|}_{\geq 0}$, which are supported on $\mathbf{p}$, with $|\mathbf{m}| = \sum_{i=1}^{|\Delta_+|} m_i \geq \lambda(\beta^\vee_\mathbf{p}) + 1$ we have a straightening law

$$c_\mathbf{m} f^\mathbf{m} + \sum_{\mathbf{t}\prec\mathbf{m}} c_\mathbf{t} f^\mathbf{t} \in I_\mathbb{Z}(\lambda),\ c_\mathbf{t} \in \mathbb{C},\ c_\mathbf{m} \in \mathbb{C}^*, \tag{0.0.1}$$

where (0.0.1) is obtained by the action of a sequence of differential operators: $\boldsymbol{\partial}(f^{|\mathbf{m}|}_{\beta_\mathbf{p}}) = \prod_{j=1}^r \partial^{k_j}_{\gamma_j}(f^{|\mathbf{m}|}_{\beta_\mathbf{p}})$.

(ii) All differential operators $\boldsymbol{\partial}$ considered in (i) respect the total order $\prec$.

(iii) The basis $\mathbb{B}(V^a(\lambda))$ is given by

$$\mathbb{B}(V^a(\lambda)) = \{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in \mathbb{Z}_{\geq 0}^{|\Delta_+|},\ \forall\ \mathbf{p} \in \mathbf{P}: \sum_{\beta \in \mathbf{p}} s_\beta \leq \lambda(\beta_{\mathbf{p}}^\vee)\},$$

so $\mathbb{B}(V^a(\lambda))$ is parametrized by the integer points of a polytope $P(\mathbf{P}) \subset \mathbb{R}_{\geq 0}^{|\Delta_+|}$, which depends on the set $\mathbf{P} \subset \mathcal{P}(\Delta_+)$.

Furthermore, we define a certain type of elements in $I_{\mathbb{Z}}(\lambda)$, $\mathbb{Z}$–admissible elements (see Definition 2.2.9).

**Theorem B.** *Let $\mathfrak{g}$ be an arbitrary simple complex finite–dimensional Lie algebra and $\lambda$ an arbitrary dominant integral weight. Further let $\mathbb{B}(V^a(\lambda))$ be a basis of $V^a(\lambda)$ satisfying Property* (i), (ii) *and* (iii) *and let the elements $\boldsymbol{\partial}(f_{\beta_{\mathbf{p}}}^{|\mathbf{m}|})$ be $\mathbb{Z}$–admissible for all $\mathbf{p} \in \mathbf{P}$ and multi–exponents $\mathbf{m}$ described in* (i)*, then*

$$\mathbb{B}(V_{\mathbb{Z}}^a(\lambda)) = \{f^{(\mathbf{s})}.v_\lambda \mid \mathbf{s} \in S(\mathbf{P}) = P(\mathbf{P}) \cap \mathbb{Z}_{\geq 0}^{|\Delta_+|}\}$$

*is a basis of $V_{\mathbb{Z}}^a(\lambda)$ and the ideal $I_{\mathbb{Z}}(\lambda)$ is generated by the subspace*

$$\langle U_{\mathbb{Z}}(\mathfrak{n}^+) \circ \operatorname{span}\{f_{\beta_{\mathbf{p}}}^{(\lambda(\beta_{\mathbf{p}}^\vee)+1)} \mid \mathbf{p} \in \mathbf{P}\}\rangle.$$

Let us now explain in short words the proof of Theorem B. Let $\boldsymbol{\partial}(f_{\beta_{\mathbf{p}}}^{|\mathbf{m}|}) \in I_{\mathbb{Z}}(\lambda)$ be an arbitrary $\mathbb{Z}$–admissible element considered in (i) with maximal monomial $c_{\mathbf{m}} f^{\mathbf{m}}$, $c_{\mathbf{m}} \in \mathbb{Z}$. The crucial point is to show that $c_{\mathbf{m}} = \pm 1$ (see Section 2.2, in particular Lemma 2.2.14). In other words $c_{\mathbf{m}}$ has to be a unit in $\mathbb{Z}$ for all $\mathbb{Z}$–admissible elements considered in (i) to guarantee, in line with the assumed straightening law, that $\mathbb{B}_{\mathbb{Z}}(V^a(\lambda))$ is a spanning set of $V_{\mathbb{Z}}^a(\lambda)$. The linear independence of $\mathbb{B}_{\mathbb{Z}}(V^a(\lambda))$ is a direct implication of the fact that $\mathbb{B}(V^a(\lambda))$ is a basis of $V^a(\lambda)$.

In Section 2.4 we give some applications of Theorem B. Here we explain that our result is an alternative proof of the main result of [FFL13b] and show that all FFL bases constructed in Chapter 1 provide also bases for the corresponding modules over $\mathbb{Z}$.

There are a lot of connections between the PBW theory and combinatorial representation theory. In fact, if we consider again FFL modules, in [FFL13a] is shown, that the describing polytopes can be identified as Newton–Okounkov bodies (see for more details on Newton–Okounkov bodies see [KK12] and [HK13]). A purely combinatorial research on the FFL polytopes can be found in [ABS11]. Furthermore, there are for example connections to Schur functions ([Fou14b]), combinatorics of crystal bases ([Kus13a], [Kus13b]) and Macdonald polynomials ([CF13], [FM14])
Especially we are interested in the Hilbert–Poincaré series of the PBW graded module, often referred to as the *q–dimension of the module*, and denoted by

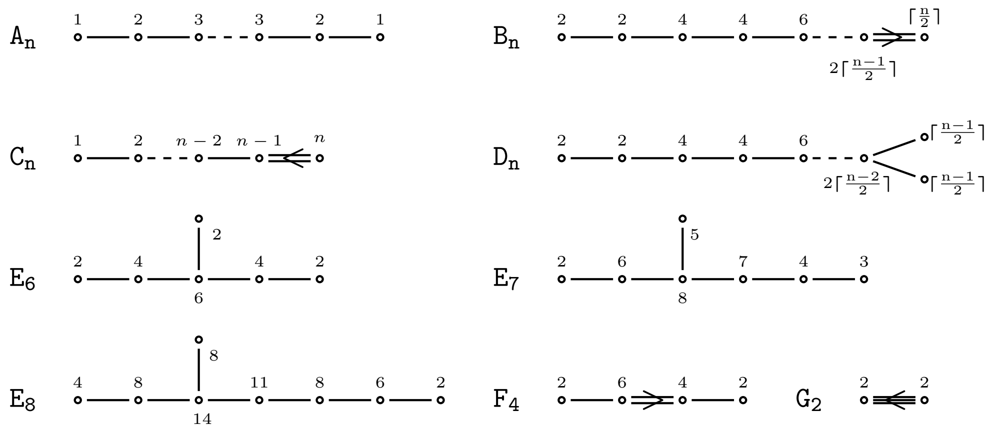
$$p_\lambda(q) = \sum_{s=0}^{\infty} (\dim V(\lambda)_s / V(\lambda)_{s-1})\, q^s.$$

Since $V(\lambda)$ is finite–dimensional, this is obviously a polynomial in $q$. Our main goal, in the third part of this thesis, is to compute the degree of $p_\lambda(q)$. The first step is the following reduction (see [CF13, Theorem 5.3 ii)]).
Let $\lambda_1, \ldots, \lambda_s$ be dominant integral weights and set $\lambda = \lambda_1 + \ldots + \lambda_s$, then

$$\deg p_\lambda(q) = \deg p_{\lambda_1}(q) + \ldots + \deg p_{\lambda_s}(q).$$

Our third main result is the computation of the degree of $p_\lambda(q)$, where $\lambda$ is a fundamental weight. With these degrees it is possible to provide a general formula for the maximal degree of Hilbert–Poincaré polynomial of $V^a(\lambda)$ for arbitrary dominant integral weights $\lambda$ of an arbitrary simple complex finite–dimensional Lie algebra $\mathfrak{g}$ (see Chapter 3):

**Theorem C** (Backhaus, Bossinger, D., Fourier)**.** *The degree of* $p_{\omega_i}(q)$ *is equal to the label of the* $i$*–th node in the following diagrams:*




**Remark.** *Note, that Chapter 3 is a modified version of [BBDF14].*

We provide in Section 3.2 for every fundamental weight a monomial $u \in S(\mathfrak{n}^-)$ of the predicted degree mapping the highest to the lowest weight vector and show that there is no polynomial of smaller degree satisfying this.
In order to prove Theorem C we use the relation between the Hilbert–Poincaré polynomial and the graded Kostant partition function (see Section 3.1) and moreover downward induction on the power of special root vectors contained in the monomials $u \in S(\mathfrak{n}^-)$.

# Preliminaries

Throughout this thesis, unless otherwise stated, we denote by $\mathfrak{g}$ a simple complex finite–dimensional Lie algebra of rank $n$. We provide in the present chapter the necessary notation and recall briefly basic constructions, which are important for this thesis. Note that the details, proofs and precise statements can be found in [Car05] and [Hum72].

We fix a *Cartan subalgebra* $\mathfrak{h} = \langle h_1, \dots, h_n \rangle_{\mathbb{C}}$ of $\mathfrak{g}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The set of *roots*, resp. *positive roots* of $\mathfrak{g}$, is denoted by $\Delta \subset \mathfrak{h}^*$, resp. by $\Delta_+ = \{\beta_1, \dots, \beta_N\} \subset \mathfrak{h}^*$, we denote by $N \in \mathbb{Z}_{\geq 0}$ the cardinality of $\Delta_+$. For the set of *negative roots* we have $\Delta_- = -\Delta_+$ and denote by $\theta$ the *highest root* of $\mathfrak{g}$. Let $\Phi_+ = \{\alpha_1 \dots, \alpha_n\} \subset \Delta_+$, $\omega_i \in \mathfrak{h}^*$, $i = 1, ..., n$ be the *simple roots* and the corresponding *fundamental weights*.
Let $W$ be the *Weyl group* associated to the simple roots and $w_0 \in W$ the *longest element*. For $\beta \in \Delta_+$ we fix a $\mathfrak{sl}_2$ triple $\{e_\beta, f_\beta, h_\beta = [e_\beta, f_\beta]\}$. The *integral weights* and the *dominant integral weights* are denoted $P$ and $P^+$.

Let us denote by $U(\mathfrak{g})$ the *universal enveloping algebra* of $\mathfrak{g}$. This is an associative algebra over $\mathbb{C}$ with $\mathbf{1}$, since $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$, the *Tensor algebra* of $\mathfrak{g}$, and the 2–sided ideal $J \subset T(\mathfrak{g})$, which is generated by the set:

$$\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}.$$

Note that we have a natural linear embedding $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$. Let
$\mathbb{B}(\mathfrak{g}) = \{x_i \mid 1 \leq i \leq D\}$, with $D = \dim \mathfrak{g}$, be a ordered basis of $\mathfrak{g}$, then we know from the PBW Theorem, that

$$\mathbb{B}(U(\mathfrak{g})) = \left\{ \prod_{i=1}^{D} x_i^{r_i} \mid r_i \geq 0,\ \forall\ 1 \leq i \leq D \right\}$$

forms a basis of $U(\mathfrak{g})$. In addition we deal with the following construction. Let $K \subset T(\mathfrak{g})$ be the 2–sided ideal generated by the set:

$$\{x \otimes y - y \otimes x \mid x, y \in \mathfrak{g}\}$$

and set $S(\mathfrak{g}) := T(\mathfrak{g})/K$, then $S(\mathfrak{g})$ is isomorphic to the the polynomial algebra $\mathbb{C}[z_i \mid 1 \leq i \leq n]$, where $n = \mathrm{rg}(\mathfrak{g})$. We call $S(\mathfrak{g})$ the *symmetric algebra* of $\mathfrak{g}$.

**Modules**. For $\lambda \in P^+$ we consider the irreducible finite–dimensional $\mathfrak{g}$–module $V(\lambda)$ with highest weight $\lambda$. Then $V(\lambda)$ admits a decomposition into $\mathfrak{h}$–weight spaces:

$$V(\lambda) = \bigoplus_{\tau \in P} V(\lambda)_\tau,$$

with $V(\lambda)_\lambda$ and $V(\lambda)_{w_0(\lambda)}$, the highest and lowest weight spaces, being one–dimensional. Let us fix a highest weight vector $v_\lambda$ and a lowest weight vector $v_{w_0(\lambda)}$ satisfying

$$e_\beta v_\lambda = 0, \;\; f_\beta v_{w_0(\lambda)} = 0 \;\; \forall\, \beta \in \Delta_+,$$

where $e_\beta \in \mathfrak{n}^+$ and $f_\beta \in \mathfrak{n}^-$. Therefore we obtain the following vector space isomorphisms:

$$U(\mathfrak{n}^-)v_\lambda \cong V(\lambda) \cong U(\mathfrak{n}^+)v_{w_0(\lambda)}.$$

Let $\lambda, \mu \in P^+$ and consider the tensor product of the corresponding highest weight $\mathfrak{g}$–modules $V(\lambda) \otimes V(\mu)$. The comultiplication $(x \mapsto x \otimes 1 + 1 \otimes x)$ provides a $\mathfrak{g}$–module structure on $V(\lambda) \otimes V(\mu)$. This module decomposes into irreducible components, where the *Cartan component* generated by the highest weight vector $v_\lambda \otimes v_\mu$ is isomorphic to $V(\lambda+\mu)$, this fact is important for the application of the main theorem of Chapter 1 and crucial for our procedure in Chapter 3.

**PBW filtration.** Now we introduce the main object that we investigate in this thesis. For $\lambda \in P^+$ we have $V(\lambda) = U(\mathfrak{n}^-)v_\lambda$, further there is a degree filtration $U(\mathfrak{n}^-)_s$ on the universal enveloping algebra of $\mathfrak{n}^-$, defined by:

$$U(\mathfrak{n}^-)_s = \operatorname{span}\{x_1 \cdots x_l \mid x_i \in \mathfrak{n}^-, \; l \leq s\}. \tag{0.0.2}$$

In particular, $U(\mathfrak{n}^-)_0 = \mathbb{C}\mathbf{1}$. Thus we have an increasing chain of subspaces: $U(\mathfrak{n}^-)_0 \subseteq U(\mathfrak{n}^-)_1 \subseteq U(\mathfrak{n}^-)_2 \subseteq \ldots$. The filtration (0.0.2) induces a filtration on $V(\lambda)$, $V(\lambda)_s = U(\mathfrak{n}^-)_s v_\lambda$. This filtration is called the *PBW filtration* on $V(\lambda)$. We consider the *associated graded space* $V^a(\lambda)$ of $V(\lambda)$ defined by:

$$V^a(\lambda) = \bigoplus_{s \in \mathbb{Z}_{\geq 0}} V(\lambda)_s / V(\lambda)_{s-1}, \;\; V(\lambda)_{-1} = \{0\}.$$

From the PBW Theorem we obtain

$$U^a(\mathfrak{n}^-) \cong S(\mathfrak{n}^-) \cong \mathbb{C}[f_{\beta_j} \mid 1 \leq j \leq N].$$

Hence $V^a(\lambda)$ is a cyclic $S(\mathfrak{n}^-)$–module generated by $v_\lambda$, thus there is an ideal $I(\lambda) \subseteq S(\mathfrak{n}^-)$, the annihilator of the generating element, such that:

$$V^a(\lambda) \cong S(\mathfrak{n}^-)v_\lambda \cong S(\mathfrak{n}^-)/I(\lambda). \tag{0.0.3}$$

**Remark 0.0.1.** *We emphasize that:*

$$f_\beta^{\lambda(\beta^\vee)+1} \in I(\lambda), \; \forall\, \beta \in \Delta_+.$$

*This is a very important fact, which we use in Chapter 1 and Chapter 2 for all calculations corresponding to $I(\lambda)$.*

We associate to the multi–exponent $\mathbf{t} = (t_i)_{i=1}^N \in \mathbb{Z}_{\geq 0}^N$ the element

$$f^{\mathbf{t}} = \prod_{i=1}^N f_{\beta_i}^{t_i} \in S(\mathfrak{n}^-),$$

and define the degree of $f^{\mathbf{t}}v_\lambda \neq 0$ in $V^a(\lambda)$ by $\deg(f^{\mathbf{t}}v_\lambda) = \deg(f^{\mathbf{t}}) = \sum_{i=1}^N t_i$, or $\deg(f^{\mathbf{t}}v_\lambda) = 0$ if $f^{\mathbf{t}}v_\lambda = 0$.

**Definition 0.0.2.** *Let $\mathbb{B}(V(\lambda))$ respectively $\mathbb{B}(V^a(\lambda))$ be a basis of $V(\lambda)$ respectively $V^a(\lambda)$. We call $\mathbb{B}(V(\lambda))$ respectively $\mathbb{B}(V^a(\lambda))$ monomial, if there is a finite subset of multi–exponents $\mathbf{T} \in \mathbb{Z}_{\geq 0}^N$, such that*

$$\mathbb{B}(V(\lambda)) = \{f^{\mathbf{t}} v_\lambda \mid \mathbf{t} \in \mathbf{T}\} \subset U(\mathfrak{n}^-)v_\lambda, \text{ resp. } \mathbb{B}(V^a(\lambda)) = \{f^{\mathbf{t}} v_\lambda \mid \mathbf{t} \in \mathbf{T}\} \subset S(\mathfrak{n}^-)v_\lambda.$$

*Throughout this thesis we are only interested in this kind of basis.*

**Remark 0.0.3.** *Because the action of $\mathfrak{n}^+$ on $V(\lambda)$ is induced by the adjoint action, we know that $V(\lambda)_s, s \in \mathbb{Z}_{\geq 0}$ is stable under the action of $\mathfrak{n}^+$. Thus for $e \in \mathfrak{n}^+$ and $\prod_{i=1}^s x_i v_\lambda \in V(\lambda)_s$ we have*

$$e.\prod_{i=1}^s x_i v_\lambda = \sum_{j=1}^s \prod_{i=1}^{j-1} x_i \operatorname{ad}(e)(x_j) \prod_{i=j+1}^s x_i v_\lambda \in V(\lambda)_s.$$

*Hence $V(\lambda)_s$ is a $U(\mathfrak{n}^+)$–module, this implies also a $U(\mathfrak{n}^+)$–module structure on $V^a(\lambda)$. So for $f^{\mathbf{t}} v_\lambda \in V^a(\lambda)$ we have*

$$\deg(uf^{\mathbf{t}} v_\lambda) \in \{0, \deg(f^{\mathbf{t}} v_\lambda)\}, \ \forall\ u \in U(\mathfrak{n}^+).$$

**Remark 0.0.4.** *Let $\circ$ be the action of $U(\mathfrak{n}^+)$ on $S(\mathfrak{g})$ induced by the adjoint action of $\mathfrak{n}^+$ on $\mathfrak{g}$. Via the vector space isomorphism*

$$S(\mathfrak{n}^-) \cong S(\mathfrak{g})/S(\mathfrak{g})(S^+(\mathfrak{n}^+ \oplus \mathfrak{h}))$$

*we obtain an action on $S(\mathfrak{n}^-)$, where $S^+(\mathfrak{n}^+ \oplus \mathfrak{h}) \subset S(\mathfrak{n}^+ \oplus \mathfrak{h})$ be the maximal homogeneous ideal of polynomials without constant term, the augmentation ideal. We denote this action again by $\circ$. Since the action of $U(\mathfrak{n}^+)$ on $V^a(\lambda)$ is induced by the action of $U(\mathfrak{n}^+)$ on $V(\lambda)$ (which is again induced by the adjoint action), we obtain for all $e \in U(\mathfrak{n}^+), f \in S(\mathfrak{n}^-)$*

$$e(fv_\lambda) = (e \circ f)v_\lambda.$$

*Hence $S(\mathfrak{n}^-)$ is a $U(\mathfrak{n}^+)$–module.*
*This fact implies in line with the fundamental Theorem for modules, that the ideal $I(\lambda)$ (see (0.0.3)) carries a $U(\mathfrak{n}^+)$–module structure.*

The next Lemma is devoted to get a better understanding of the module $V^a(\lambda)$, but we do not need it to prove our main statements.

**Lemma 0.0.5.** *Let $f^{\mathbf{m}} \in S(\mathfrak{n}^-)$ with $f^{\mathbf{m}} v_\lambda \neq 0$ in $V^a(\lambda)$ and weight $\operatorname{wt}(f^{\mathbf{m}}) = \lambda - w_0(\lambda)$. Then*

$$\deg(f^{\mathbf{n}}) \leq \deg(f^{\mathbf{m}}), \ \forall f^{\mathbf{n}} v_\lambda \neq 0 \in V^a(\lambda).$$

*Proof.* Let $v_{w_0(\lambda)}$ be a lowest weight vector such that:

$$V(\lambda) = U(\mathfrak{n}^+)v_{w_0(\lambda)}.$$

Hence we can interpret $V(\lambda)$ as a lowest weight module. The lowest weight $\omega_0(\lambda)$ is in the Weyl group orbit of $\lambda$, thus $\dim V(\lambda)_{w_0(\lambda)} = 1 = \dim V(\lambda)_\lambda$. Thus there is a minimal $s \in \mathbb{Z}_{\geq 0}$, such that: $V(\lambda)_{w_0(\lambda)} \subseteq V(\lambda)_s$. Furthermore, there exists a scalar $c \in \mathbb{C}$ with $f^{\mathbf{m}} v_\lambda = cv_{w_0(\lambda)}$.
For an arbitrary element $f^{\mathbf{n}} v_\lambda \neq 0 \in V^a(\lambda)$ we fix the order of the factors to obtain $f^{\mathbf{n}} v_\lambda \in V(\lambda)$. Then there exists an element $x \in U(\mathfrak{n}^+)$ such that: $f^{\mathbf{n}} v_\lambda = x(f^{\mathbf{m}} v_\lambda)$. This implies with Remark 0.0.3: $\deg(f^{\mathbf{n}}) \leq \deg(f^{\mathbf{m}})$. ∎

In the following we define two important tools for the considerations in this thesis.

**Differential operators.** Let $\beta_j, \beta_i \in \Delta_+$ and $e_{\beta_i} \in \mathfrak{n}^+, f_{\beta_j} \in \mathfrak{n}^-$ be corresponding root vectors. Then we define the *differential operator*

$$\partial_{\beta_i}\left(f_{\beta_j}\right) := \begin{cases} f_{\beta_j - \beta_i}, & \text{if } \beta_j - \beta_i \in \Delta_+, \\ 0, & \text{otherwise.} \end{cases} \tag{0.0.4}$$

The differential operator satisfies

$$\partial_{\beta_i}\left(f_{\beta_j}\right) = \left(c_{\beta_i,-\beta_j}\right)^{-1} \mathrm{ad}(e_{\beta_i})\left(f_{\beta_j}\right),$$

where $c_{\beta_i,-\beta_j} \in \mathbb{C}^*$ is the corresponding *structure constant*. Thus, if $\beta_j = \beta_i$ or if the root vectors commute, then $\partial_{\beta_i}(f_{\beta_j}) = 0$.
Since the adjoint action satisfies the properties of a derivation and the $U(\mathfrak{n}^+)$–module structure on $S(\mathfrak{n}^-)$ is induced by the adjoint action (see Remark 0.0.4), we define differential operators on $S(\mathfrak{n}^-)$: Let $k, m \in \mathbb{Z}_{\geq 0}$:

$$\partial_{\beta_i}^k\left(f_{\beta_j}^m\right) = \underbrace{\partial_{\beta_i}\cdots\partial_{\beta_i}}_{\text{k–times}}\left(f_{\beta_j}^m\right), \tag{0.0.5}$$

$$\text{and } \partial_{\beta_i}\left(f_{\beta_j}^m\right) = \sum_{\ell=1}^{m} f_{\beta_j}^{\ell-1}\partial_{\beta_i}\left(f_{\beta_j}\right)f_{\beta_j}^{m-\ell} = m\partial_{\beta_i}\left(f_{\beta_j}\right)f_{\beta_j}^{m-1}.$$

**Remark 0.0.6.** *Remark 0.0.4 implies that the ideal $I(\lambda)$ is stable under the $U(\mathfrak{n}^+)$–action on $S(\mathfrak{n}^-)$. Thus, for an arbitrary sequence of differential operators and an arbitrary element $v \in I(\lambda)$ we have*

$$\prod_{l=1}^{r} \partial_{\beta_{i_l}}^{k_l}(v) \in I(\lambda),$$

*where $k_l \in \mathbb{Z}_{\geq 0}$ and $\beta_{i_l} \in \Delta_+$. Especially in Chapter 1 and Chapter 2 we use this fact in order to obtain relations in $V^a(\lambda)$ and with these we are able to describe the ideal $I(\lambda)$ explicitly.*

**Abstract paths.** Let $\mathbf{p}$, be an element of the power set of the positive roots $\mathcal{P}(\Delta_+)$. We call such an element an *abstract path* in $\Delta_+$. In addition we say that a multi–exponent $\mathbf{t} \in \mathbb{Z}_{\geq 0}^N$ is supported on the abstract path $\mathbf{p}$, if

$$t_i = 0, \ \forall\ \beta_i \notin \mathbf{p}.$$

Furthermore, we call a subset $\mathbf{P}$ of $\mathcal{P}(\Delta_+)$ a *set of abstract paths* in $\Delta_+$. Notice that the elements in $\mathbf{P}$ do not have to have the same cardinality.

**Remark.** *We use in this thesis abstract paths to translate our representation theoretical questions into combinatorial problems. Note, that in Chapter 1 we call the abstract paths Dyck paths, in order to be consistent with [FFL11a, FFL11b].*

# 1 Feigin-Fourier-Littelmann modules via Hasse diagrams

We emphasize that the present chapter is a modified version of [BD15]. Throughout this chapter we focus on selected rectangular weights $\lambda = m\omega_i$, $m \in \mathbb{Z}_{\geq 0}$ (see Table 1.1). Moreover we provide special monomial bases, so called FFL bases, for the corresponding irreducible finite–dimensional PBW graded highest weight representation $V^a(\lambda)$. Furthermore, we assume the notation of the Preliminaries.

## 1.1 Hasse diagrams and Dyck paths

In this section we define and consider the Hasse diagram $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$, which is the most important combinatorial tool for the procedure of this chapter. To do so we introduce the Lie subalgebra $\mathfrak{n}_\lambda^-$ of $\mathfrak{n}^-$ and provide, analogue to the Preliminaries, definitions and facts in order to work with this Lie subalgebra.

Let $\mathfrak{g}$ be as usual and fix a rectangular weight $\lambda = m\omega_i$, with $m \in \mathbb{Z}_{\geq 0}$ and $1 \leq i \leq n$, further let $\lambda(\beta^\vee) = \frac{2(\lambda,\beta)}{(\beta,\beta)}$, where $\beta^\vee = \frac{2\beta}{(\beta,\beta)}$ is the coroot of $\beta$ and $(\cdot,\cdot)$ is the Killing form. Then we define

$$\mathfrak{n}_\lambda^- := \operatorname{span}\{f_\beta \mid \lambda(\beta^\vee) \geq 1\} \subset \mathfrak{n}^-.$$

Let $\beta = \sum_{j=1}^n n_j\alpha_j,\ n_j \in \mathbb{Z}_{\geq 0}$ be a positive root with $n_i \geq 1$. Then we have for the coroot $\beta^\vee = \sum_{j=1}^n n_j^\vee \alpha_j^\vee$: $n_i^\vee \geq 1$. Conversely starting with a coroot $\beta^\vee$, with $n_i^\vee \geq 1$ we have for the corresponding positive root $\beta$: $n_i \geq 1$. Hence, independent of the choice of $m \geq 1$: $\mathfrak{n}_{\omega_i}^- = \mathfrak{n}_{m\omega_i}^- \subset \mathfrak{n}^-$ is the Lie subalgebra spanned by those root vectors $f_\beta$, where $\alpha_i$ is a summand of $\beta$.
From the PBW–Theorem we get

$$U^a(\mathfrak{n}_\lambda^-) \cong S(\mathfrak{n}_\lambda^-) \cong \mathbb{C}[f_\beta \mid \lambda(\beta^\vee) \geq 1,\ \beta \in \Delta_+],$$

where $S(\mathfrak{n}_\lambda^-)$ is the symmetric algebra over $\mathfrak{n}_\lambda^-$.

**Remark 1.1.1.** *(i) We have $V(\lambda) = U(\mathfrak{n}_\lambda^-)v_\lambda$. The action of $U(\mathfrak{n}_\lambda^-)$ on $V(\lambda)$ induces the structure of a $S(\mathfrak{n}_\lambda^-)$–module on $V^a(\lambda)$ and*

$$V^a(\lambda) \cong S(\mathfrak{n}^-)v_\lambda \cong S(\mathfrak{n}_\lambda^-)v_\lambda. \tag{1.1.1}$$

*(ii) The action of $U(\mathfrak{n}^+)$ on $V(\lambda)$ induces the structure of a $U(\mathfrak{n}^+)$–module on $V^a(\lambda)$. Note for $e_\alpha \in \mathfrak{n}^+ \hookrightarrow U(\mathfrak{n}^+), f_\beta \in \mathfrak{n}_\lambda^- \hookrightarrow S(\mathfrak{n}_\lambda^-)$, $[e_\alpha, f_\beta]$ is not in general an element of $S(\mathfrak{n}_\lambda^-)$, but for $f_\nu \in S(\mathfrak{n}^-) \setminus S(\mathfrak{n}_\lambda^-)$ we have $f_\nu v_\lambda = 0$. That follows from the well known description (see [Hum72]) of $V(\lambda)$:*

$$V(\lambda) = U(\mathfrak{n}^-)/\langle f_\beta^{\lambda(\beta^\vee)+1} \mid \beta \in \Delta_+\rangle. \tag{1.1.2}$$

Equation (1.1.1) shows that $V^a(\lambda)$ is a cyclic $S(\mathfrak{n}_\lambda^-)$–module and hence there is an ideal $I_\lambda \subseteq S(\mathfrak{n}_\lambda^-)$ such that $V^a(\lambda) \simeq S(\mathfrak{n}_\lambda^-)/I_\lambda$, where $I_\lambda$ is the annihilating ideal of $v_\lambda$. We have therefore the following projections:

$$S(\mathfrak{n}^-) \to S(\mathfrak{n}^-)/\left\langle f_\beta \mid \lambda(\beta^\vee) = 0 \right\rangle = S(\mathfrak{n}_\lambda^-) \to S(\mathfrak{n}_\lambda^-)/I_\lambda.$$

Hence, although we work with $\mathfrak{n}_\lambda^-$, we actually consider $\mathfrak{n}^-$–modules. Therefore, our aims in this chapter are

- To describe $V^a(\lambda)$ as a $S(\mathfrak{n}_\lambda^-)$–module, i. e. describe explicitly generators of the ideal $I_\lambda$.
- To find a basis of $V^a(\lambda)$ parametrized by integer points of a normal polytope $P(\lambda)$ (see (1.1.6)).

To achieve these goals we have to introduce further terminology. We denote the set of positive roots associated to $\mathfrak{n}_\lambda^-$ by

$$\Delta_+^\lambda = \{\beta \in \Delta_+ \mid \lambda(\beta^\vee) \geq 1\} =: \{\beta_{i_1}, \ldots, \beta_{i_{N_\lambda}}\} \subseteq \Delta_+, \ |\Delta_+^\lambda| = N_\lambda \leq N.$$

Since we deal in the present chapter only with $\mathfrak{n}_\lambda^-$ we denote, by abuse of notation, for $1 \leq j \leq N_\lambda$ $i_j = j$ and $N_\lambda = N$. Therefore, we have $\Delta_+^\lambda = \{\beta_1, \ldots, \beta_N\}$, $|\Delta_+^\lambda| = N \in \mathbb{Z}_{\geq 0}$.

**Example 1.1.2.** *We write $(r_1, r_2, \ldots, r_n)$ for the sum: $\sum_{k=1}^n r_k \alpha_k$. Let $\mathfrak{g}$ be of type $\mathtt{A}_4$ and $\lambda = \omega_3$, the third fundamental weight. Then we have:*

$$\begin{aligned}\Delta_+^{\omega_3} = \{&\beta_1 = (1,1,1,1), \beta_2 = (0,1,1,1), \beta_3 = (1,1,1,0),\\ &\beta_4 = (0,0,1,1), \beta_5 = (0,1,1,0), \beta_6 = (0,0,1,0)\} \subset \Delta_+.\end{aligned}$$

We choose a total order $\prec$ on $\Delta_+^\lambda$:

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_{N-1} \prec \beta_N. \tag{1.1.3}$$

We assume that this order satisfies the following conditions:

(i) Let $\geq$ be the standard partial order on the positive roots, then

$$\beta_i > \beta_j \Rightarrow \beta_i \prec \beta_j.$$

(ii) Let $\beta_i = (r_1, \ldots, r_n), \beta_j = (t_1, \ldots, t_n)$ and we define the height as the sum over these entries: $\mathrm{ht}(\beta_i) = \sum_{i=1}^n r_i, \mathrm{ht}(\beta_j) = \sum_{i=1}^n t_i$. Then

$$\mathrm{ht}(\beta_i) > \mathrm{ht}(\beta_j) \Rightarrow \beta_i \prec \beta_j.$$

(iii) If $\beta_i$ and $\beta_j$ are not comparable in the sense of $(i)$ and $(ii)$, then $\beta_i \prec \beta_j \Leftrightarrow \beta_i$ is greater than $\beta_j$ lexicographically, i.e. there exists $1 \leq k \leq n$, such that $r_k > t_k$ and $r_i = t_i$ for $1 \leq i < k$.

**Remark 1.1.3.** *The explicit order of the roots depends on the Lie algebra and the chosen weight, see Section 1.4. But in all cases considered in this chapter we have $\beta_1 = \theta$, the highest root of $\mathfrak{g}$ and $\beta_N$ is the simple root $\alpha_i$.*

In order to make our equations more readable we write for $1 \leq i \leq N$: $f_i = f_{\beta_i}$ and $s_i = s_{\beta_i}$. We associate to the multi–exponent $\mathbf{s} = (s_i)_{i=1}^N \in \mathbb{Z}_{\geq 0}^N$ the element

$$f^{\mathbf{s}} = \prod_{i=1}^{N} f_i^{s_i} \in S(\mathfrak{n}_\lambda^-),$$

and define the degree of $f^{\mathbf{s}} v_\lambda \neq 0$ in $V^a(\lambda)$ by $\deg(f^{\mathbf{s}} v_\lambda) = \deg(f^{\mathbf{s}}) = \sum_{i=1}^N s_i$, or $\deg(f^{\mathbf{s}} v_\lambda) = 0$ if $f^{\mathbf{s}} v_\lambda = 0$. We extend $\prec$ to the homogeneous lexicographical total order on the monomials of $S(\mathfrak{n}_\lambda^-)$ (resp. multi–exponents).
Let $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_{\geq 0}^N$ be two multi–exponents. We say $f^{\mathbf{s}} \succ f^{\mathbf{t}}$ or $\mathbf{s} \succ \mathbf{t}$ if

- $\deg(f^{\mathbf{s}}) > \deg(f^{\mathbf{t}})$ or
- $\deg(f^{\mathbf{s}}) = \deg(f^{\mathbf{t}})$ and $\exists\, 1 \leq k \leq N : (s_k > t_k) \wedge \forall\, k < j \leq N : (s_j = t_j)$.

For example: $f_1^1 f_2^2 f_3^0 \prec f_1^2 f_2^0 f_3^1 \prec f_1^1 f_2^0 f_3^2$.

Associated to $\mathfrak{n}_\lambda^-$ we define a directed graph $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}} := (\Delta_+^\lambda, E)$. The set of vertices is given by $\Delta_+^\lambda$ and the set of edges $E$ is constructed as follows:

$$\forall\ 1 \leq i, j \leq N:\ (\beta_i \xrightarrow{k} \beta_j) \in E \Leftrightarrow \exists\ \alpha_k \in \Phi_+ :\ \beta_i - \beta_j = \alpha_k.$$

We call this directed graph *Hasse diagram* of $\mathfrak{g}$ associated to $\lambda$. For the considerations in this chapter will $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ be the most important tool.

**Example 1.1.4.** *The Hasse diagram $H(\mathfrak{n}_{\omega_3}^-)_{\mathfrak{sl}_5}$ is given by:*

$$\begin{tikzcd}
 & & \beta_1 \arrow[dl, "1"'] \arrow[dr, "4"] & \\
 & \beta_2 \arrow[dl, "2"'] \arrow[dr, "4"] & & \beta_3 \arrow[dl, "1"'] \\
\beta_4 \arrow[dr, "4"] & & \beta_5 \arrow[dl, "2"'] & \\
 & \beta_6 & &
\end{tikzcd}
\qquad
\begin{array}{l}
\beta_1 = (1,1,1,1) \\
\beta_2 = (0,1,1,1) \\
\beta_3 = (1,1,1,0) \\
\beta_4 = (0,0,1,1) \\
\beta_5 = (0,1,1,0) \\
\beta_6 = (0,0,1,0)
\end{array}$$

We define an ordered sequence of roots in $\Delta_+^\lambda$: $(\beta_{i_1}, \ldots, \beta_{i_r})$ with $\beta_{i_j} \prec \beta_{i_{j+1}}$ to be a *directed path* from $\beta_{i_1}$ to $\beta_{i_r}$.

**Remark 1.1.5.** *For our purposes we allow the trivial path $(\emptyset)$ and any ordered subsequence of a directed path to be a directed path again. Therefore, in Example 1.1.4 $(\beta_1, \beta_2, \beta_4, \beta_6)$ and $(\beta_1, \beta_2, \beta_6)$ are two possible directed paths.*

In general it is possible that two edges in $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$, one ending in a root $\beta$ and one starting in $\beta$, have the same label:

$$\gamma \xrightarrow{k} \beta \xrightarrow{k} \delta.$$

We call this construction a *$k$–chain (of length 2).*

Associated to $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ we construct two subsets $D_\lambda, \overline{D}_\lambda \subset \mathcal{P}(\Delta_+^\lambda)$ of the power set of $\Delta_+^\lambda$: For $\mathbf{p} \in \mathcal{P}(\Delta_+^\lambda)$ we define

$$\mathbf{p} \in D_\lambda :\Leftrightarrow \mathbf{p} = \{\beta_{i_1}, \ldots, \beta_{i_r}\}, \tag{1.1.4}$$

for a directed path $(\beta_{i_1}, \ldots, \beta_{i_r})$ in $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$. Therefore, from now on by (1.1.4) we interpret $\mathbf{p} \in D_\lambda$ as a directed path in $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$.

**Remark 1.1.6.** *Let $\beta_i, \beta_j \in \Delta_+^\lambda$ be arbitrary. Then there exist a $\mathbf{p} \in D_\lambda$ with $\beta_i, \beta_j \in \mathbf{p}$ if and only if $\beta_i - \beta_j$ or $\beta_j - \beta_i$ is a non–negative linear combination of simple roots.*

**Remark 1.1.7.** *A staircase walk from (0,0) to (n,n) beyond the diagonal in a $n \times n$–lattice is a called Dyck path. In the general $\mathtt{A_n}$–case ([FFL11a]) the constructed directed paths are Dyck paths in this sense. To be consistent with their notation we call our directed paths $D_\lambda$ also Dyck paths.*

Further we define the set of *co-chains* by

$$\overline{D}_\lambda := \{\overline{\mathbf{p}} \in \mathcal{P}(\Delta_+^\lambda) \mid |\overline{\mathbf{p}} \cap \mathbf{p}| \leq 1, \forall\ \mathbf{p} \in D_\lambda\}. \tag{1.1.5}$$

If necessary we use an additional index $\overline{D}_\lambda^{\text{type of } \mathfrak{g}}$, to distinguish which type of $\mathfrak{g}$ we consider. We consider the integral points of a polytope which is connected to $D_\lambda$ in a very natural way. Fix $\lambda = m\omega_i$, with $m \in \mathbb{Z}_{\geq 0}$. Let

$$P(m\omega_i) = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^N \mid \sum_{\beta_j \in \mathbf{p}} x_j \leq m,\ \ \forall\ \mathbf{p} \in D_{\omega_i}\}, \tag{1.1.6}$$

be the *associated polytope* to $D_{\omega_i}$. Denote by $S(m\omega_i)$ the integer points in $P(m\omega_i)$: $S(m\omega_i) = P(m\omega_i) \cap \mathbb{Z}_{\geq 0}^N$. We define the map

$$\operatorname{supp}_1 : S(\omega_i) \to \mathcal{P}(\Delta_+^{\omega_i}),\ \operatorname{supp}_1(\mathbf{s}) = \{\beta_j \mid s_j > 0\}.$$

For $\mathbf{s} \in S(\omega_i)$ we have with (1.1.5) immediately $\operatorname{supp}_1(\mathbf{s}) \in \overline{D}_{\omega_i}$. Conversely every $\overline{\mathbf{p}} \in \overline{D}_{\omega_i}$ has a non–empty pre–image. With $\mathbf{s} \in \{0,1\}^N$ we conclude that $\operatorname{supp}_1$ is injective and that we have the immediate proposition:

**Proposition 1.1.8.** *The map $\operatorname{supp}_1 : S(\omega_i) \to \overline{D}_{\omega_i}$ is a bijection.* □

Hence in Section 1.4 it is sufficient to determine the co–chains in $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ to find the elements in $S(\omega_i)$. Now we are able to formulate our main statements.

**Main statements.** Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra and $\lambda = m\omega_i$ be a rectangular weight, with $\omega_i(\theta^\vee) = 1$ and $m \in \mathbb{Z}_{\geq 0}$, where $\theta$ is the highest root of $\mathfrak{g}$. Further we assume that $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ has no $k$–chains of length 2. In the following table we list up all cases where these assumptions are satisfied. Additionally in the cases $(\mathtt{B_n}, \omega_1), (\mathtt{F_4}, \omega_4)$ and $(\mathtt{G_2}, \omega_1)$, we can rewrite $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ in a diagram without $k$–chains of length 2:

| Type of $\mathfrak{g}$ | weight $\omega_i$ | Type of $\mathfrak{g}$ | weight $\omega_i$ |
|---|---|---|---|
| $\mathtt{A_n}$ | $\omega_k,\ 1 \leq k \leq n$ | $\mathtt{E_6}$ | $\omega_1,\ \omega_6$ |
| $\mathtt{B_n}$ | $\omega_1,\ \omega_n$ | $\mathtt{E_7}$ | $\omega_7$ |
| $\mathtt{C_n}$ | $\omega_1$ | $\mathtt{F_4}$ | $\omega_4$ |
| $\mathtt{D_n}$ | $\omega_1,\ \omega_{n-1},\ \omega_n$ | $\mathtt{G_2}$ | $\omega_1$ |

Table 1.1: Admissible weights

Let $I(m\omega_i) \subset S(\mathfrak{n}^-)$ be the ideal such that $V^a(m\omega_i) = S(\mathfrak{n}^-)/I(m\omega_i)$.

**Theorem 1.1.9.**

$$I(m\omega_i) = S(\mathfrak{n}^-)\left(U(\mathfrak{n}^+) \circ \operatorname{span}\{f_\beta^{m\omega_i(\beta^\vee)+1} \mid \beta \in \Delta_+\}\right).$$

*Proof.* This statement follows by Theorem 1.5.4. □

**Theorem 1.1.10.** *$\mathbb{B}_{m\omega_i} = \{f^{\mathbf{s}} v_{m\omega_i} \mid \mathbf{s} \in S(m\omega_i)\}$ is a FFL basis of $V^a(m\omega_i)$.*

*Proof.* In Section 1.2 we show that the polytope $P(m\omega_i)$ is normal. By Theorem 1.3.4 we conclude that $\mathbb{B}_{m\omega_i}$ is a spanning set for $V^a(m\omega_i)$. After fixing the order of the factors, with Theorem 1.5.2 we have a FFL basis of $V(m\omega_i)$. Because this basis is monomial and $V(m\omega_i) \cong V^a(m\omega_i)$ as vector spaces, we conclude that $\mathbb{B}_{m\omega_i}$ is a FFL basis of $V^a(m\omega_i)$. □

**Applications.**To state an important consequence of Theorem A and Theorem B we give the definitions of*essential monomials* due to Vinberg (see [Vin05], [Gor11]) and *Feigin–Fourier–Littelmann* (FFL) modules due to [FFL13a]. Let $\lambda$ be a dominant integral weight. Recall that we have a homogeneous lexicographical total order $\prec$ on the set of multi–exponents induced by the order on $\Delta_+^\lambda$:

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_N.$$

In the following we fix a ordering on the factors in a vector

$$f^{\mathbf{p}} v_\lambda = f_N^{p_N} f_{N-1}^{p_{N-1}} \dots f_1^{p_1} v_\lambda. \tag{1.1.7}$$

**Definition 1.1.11.** *(i) We call a multi–exponent $\mathbf{p} \in \mathbb{Z}_{\geq 0}^N$ essential if*

$$f^{\mathbf{p}} v_\lambda \notin \operatorname{span}\{f^{\mathbf{q}} v_\lambda \mid \mathbf{q} \prec \mathbf{p}\}.$$

*(ii) Define $\operatorname{es}(V(\lambda)) \subset \mathbb{Z}_{\geq 0}^N$ to be the set of essential multi–exponents.*

By [FFL13a, Section 1] $\{f^{\mathbf{p}} v_\lambda \mid \mathbf{p} \in \operatorname{es}(V(\lambda))\}$ is a basis of $V^a(\lambda)$ and of $V(\lambda)$.

Let $M = U(\mathfrak{n}^-)v_M$ and $M' = U(\mathfrak{n}^-)v_{M'}$ be two cyclic modules. Then we denote with $M \odot M' := U(\mathfrak{n}^-)(v_M \otimes v_{M'}) \subset M \otimes M'$ the *Cartan component* and we write $M^{\odot n} := M \odot \cdots \odot M$ (n-times).

**Definition 1.1.12.** *We call a cyclic module $M$ a FFL module if:*

*(i) There exists a normal polytope $P(M)$ such that $\operatorname{es}(M) = S(M)$, where $S(M)$ is the set of lattice points in $P(M)$.*

*(ii) $\forall n \in \mathbb{N} : \dim M^{\odot n} = |nS(M)|$, where $nS(M)$ is the n-fold Minkowski sum of $S(M)$.*

**Corollary 1.1.13.** *For the cases of Table 1.1 $V(m\omega_i)$ is a FFL module.*

*Proof.* Proposition 1.2.8 shows that $P(m\omega_i)$ is a normal polytope. By Theorem B a basis of $V(m\omega_i)$ is given by $\mathbb{B}_{m\omega_i}$, hence with Lemma 1.5.1 we have $S(m\omega_i) = \mathrm{es}(V(m\omega_i))$.
Let $n \in \mathbb{N}$ be arbitrary, then $\dim V(m\omega_i)^{\odot n} = \dim V(nm\omega_i)$. Again by Theorem B we have $\dim V(nm\omega_i)) = |S(nm\omega_i))|$. Because $P(nm\omega_i))$ is a normal polytope and therefore satisfies the Minkowski sum property, we conclude $|S(nm\omega_i))| = |nS(m\omega_i))|$. □

**Remark 1.1.14.** *Note that in [FFL13a] the FFL modules are called favourable modules.*

## 1.2 Normal polytopes

Our goal in this section is to show, that the polytopes defined in (1.1.6) are normal. A convex lattice polytope $P \subset \mathbb{R}^K, K \in \mathbb{Z}_{\geq 0}$, i.e. P is the convex hull of finitely many integer points, is called normal, if the set of integer points in the m-th dilation $mP$ is the m-fold Minkowski sum of the integer points in $P$.
To achieve our goal we prove the normality condition for a larger class of polytopes in a more abstract setting than in Section 1.1.

**General setting.** Let $\Delta = \{z_1, z_2, \ldots, z_K\}$ be a finite, non-empty set with a total order: $z_1 \succ z_2 \succ \cdots \succ z_K$. We extend $\succ$ to the (non-homogeneous) lexicographic order on $\mathcal{P}(\Delta)$, the power set of $\Delta$. Let $D = \{\mathbf{p}_1, \ldots, \mathbf{p}_t\} \subset \mathcal{P}(\Delta)$ be an arbitrary subset.

**Remark 1.2.1.** *(i) To illustrate this non-homogeneous lexicographical order we give for $K \geq 3$ an example:*

$$\{z_1, z_2\} \succ \{z_1\} \succ \{z_2, z_3\}$$

*(ii) Let $\mathbf{p} = \{z_{i_1}, \ldots, z_{i_r}\} \in \mathcal{P}(\Delta)$ be an arbitrary set. We always assume without loss of generality (wlog): $z_{i_1} \succ \cdots \succ z_{i_r}$.*

We can associate a collection of polytopes to $D$ in a natural way:

$$P(m) = \{\mathbf{x} \in \mathbb{R}^K_{\geq 0} \mid \sum_{z_j \in \mathbf{p}} x_j \leq m, \ \ \forall \mathbf{p} \in D\}, \, m \in \mathbb{Z}_{\geq 0}. \tag{1.2.1}$$

To work with these polytope, in particular with the elements in $D$, we define the following.

**Definition 1.2.2.**

(1) *For $\mathbf{p} \in \mathcal{P}(\Delta)$ define $\mathbf{p}_{\min} = \min_{\succ}\{z \in \mathbf{p}\}$ and $\mathbf{p}_{\max}$ analogously.*

(2) *Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}(\Delta)$, $\mathbf{p} = \{z_{i_1}, \ldots, z_{i_r}\}$, $\mathbf{q} = \{z_{j_1}, \ldots, z_{j_s}\}$ with $\mathbf{p}_{\min} = \mathbf{q}_{\max}$. Then we define the concatenation of $\mathbf{p}$ and $\mathbf{q}$ by*

$$\mathbf{p} \cup \mathbf{q} = \{z_{i_1}, z_{i_2} \ldots, z_{i_r} = z_{j_1}, z_{j_2}, \ldots, z_{j_s}\} \in \mathcal{P}(\Delta).$$

**Normality condition.**

**Definition 1.2.3.** *Assume $D \subset \mathcal{P}(\Delta)$ has the following properties:*

1. *Subsets of elements in $D$ are again in $D$:*
$$\forall A \subset \mathbf{p} \in D : A \in D.$$
2. *Every $z \in \Delta$ lies at least in one element of $D$:*
$$\bigcup_{\mathbf{p} \in D} \mathbf{p} = \Delta$$
3. *The concatenation of two elements in $D$, if possible, lies again in $D$:*
$$\forall \mathbf{p}, \mathbf{q} \in D \text{ with } \mathbf{p}_{\min} = \mathbf{q}_{\max}\text{: } \mathbf{p} \cup \mathbf{q} \in D.$$

*Then we call $D \subset \mathcal{P}(\Delta)$ a set of Dyck paths.*

We define for $m \in \mathbb{Z}_{\geq 0}$, $\mathrm{supp_m} : S(m) \to \mathcal{P}(\Delta)$, by
$$\mathbf{t} = (t_z)_{z\in\Delta} \mapsto \mathrm{supp_m}(\mathbf{t}) = \{z \in \Delta \mid \ t_z > 0\}.$$

Note that the map $\mathrm{supp_m}$ is in general not injective. Furthermore, we have $\mathrm{supp}_1(S(1)) \subseteq \mathrm{supp_m}(S(m))$, because of $S(1) \subseteq S(m)$ and $\mathrm{supp_m}|_{S(1)} = \mathrm{supp}_1$.

**Remark 1.2.4.** *Let $D \subset \mathcal{P}(\Delta)$ be a set of Dyck paths, then $P(m)$ defined in (1.2.1) is a bounded convex polytope for all $m \in \mathbb{Z}_{\geq 0}$.*
*By the definition of $P(m)$ and the second property of $D$, which guarantees that each $z \in \Delta$ lies in at least one Dyck path, we have $t_z \in \{0, 1\}, \forall z \in \Delta$, for $\mathbf{t} \in S(1)$. Hence $\mathrm{supp}_1$ is an injective map and we get an induced (non-homogeneous) total order on $S(1)$.*

Now we give a characterization of the image of $\mathrm{supp}_1$.

**Remark 1.2.5.** *Let $D \subset \mathcal{P}(\Delta)$ be a set of Dyck paths, then*
$$\mathrm{supp}_1(S(1)) = \{A \in \mathcal{P}(\Delta) \mid |A \cap \mathbf{p}| \leq 1, \forall \mathbf{p} \in D\} =: \Gamma.$$

*"$\subseteq$": Assume there is an element $\mathbf{t} \in S(1)$ with $\mathrm{supp}_1(\mathbf{t}) = A \in \mathcal{P}(\Delta)$ and $|A \cap \mathbf{p}| > 1$ for some $\mathbf{p} \in D$. Then we have $\sum_{z\in A\cap\mathbf{p}} t_z > 1$, since $t_z > 0$, $\forall z \in A$. And so we have: $\sum_{z\in\mathbf{p}} t_z > 1$. But this is a contradiction to the assumption $\mathbf{t} \in S(1)$.*

*"$\supseteq$": Let $B \in \Gamma$ be arbitrary. Associated to $B$ we define $\mathbf{q}^B \in \mathbb{Z}^K_{\geq 0}$ by $q^B_z = 1$ if $z \in B$ and $q^B_z = 0$ else. By the definition of $\Gamma$ we have for every Dyck path $\mathbf{p} \in D$: $\sum_{z\in\mathbf{p}} q^B_z \leq 1$. Hence $\mathbf{q}^B \in S(1)$ with $\mathrm{supp}_1(\mathbf{q}^B) = B$.*

Let $\mathbf{s} \in S(m), m \in \mathbb{Z}_{\geq 0}, \mathbf{s} \neq 0$ be an arbitrary non-zero element. Consider $\mathrm{supp_m}(\mathbf{s}) \in \mathcal{P}(\Delta)$, we have $\mathcal{P}(\mathrm{supp_m}(\mathbf{s})) \subseteq \mathcal{P}(\Delta)$. Let
$$\nabla = (\mathrm{supp}_1(S(1)) \cap \mathcal{P}(\mathrm{supp}_m(\mathbf{s})) \subseteq \mathcal{P}(\Delta). \tag{1.2.2}$$

Note that $\nabla$ is a total ordered, non-empty set, because $S(1)$ contains all unit vectors and $\mathbf{s} \neq 0$ by assumption. Therefore, there is a unique maximal element (with respect to $\succ$), denoted by $M_{\mathbf{s}} \in \nabla$.

**Lemma 1.2.6.** *Let $D$ be a set of Dyck paths, $\mathbf{s} \in S(m)$ non-zero and $\mu \in M_{\mathbf{s}}$. Then we have $s_\nu = 0$ for all $\nu \in \Delta$ such that ($\nu \succ \mu$ and $\exists \mathbf{q} \in D\colon \nu, \mu \in \mathbf{q}$).*

*Proof.* We assume the contrary. That implies there exists $\nu \in \Delta$ with $\nu \succ \mu$, $s_\nu \neq 0$ and a Dyck path $\mathbf{p} \in D$ such that $\nu, \mu \in \mathbf{p}$. Define

$$V := \{\tau \in M_{\mathbf{s}} \mid \exists \mathbf{q} \in D : \nu, \tau \in \mathbf{q}, \nu \succ \tau\} \subset M_{\mathbf{s}}$$

and $M'_{\mathbf{s}} := (\{\nu\} \cup M_{\mathbf{s}}) \setminus V$. By assumption we have $\mu \in V$ and so $|V| \geq 1$. Further we have $M'_{\mathbf{s}} \in \mathcal{P}(\mathrm{supp}_m(\mathbf{s}))$ and we show that $M'_{\mathbf{s}} \in \mathrm{supp}_1(S(1))$.
We assume that this is not the case. Therefore, there exists some $\mathbf{b} \in D$ such that $|M'_{\mathbf{s}} \cap \mathbf{b}| > 1$. By the definition of V this can only happen, if there exists a $\alpha \in M_{\mathbf{s}}$ with $\alpha \succ \nu$ and $\alpha, \nu \in \mathbf{b}$. The following picture is intended to give a better understanding of the foregoing situation.

$$\begin{tikzcd}
 & & & & \tau_1 & & \\
 & & \cdot \arrow[r] & \nu \arrow[r] & \cdot \arrow[u] \arrow[r] & \cdot \arrow[r, "\mathbf{p}"] \arrow[d] & \mu\ , \quad \tau_1, \tau_2 \in V. \\
\alpha \arrow[r] & \cdot \arrow[r] & \cdot \arrow[u, "\mathbf{b}"'] & & & \tau_2 &
\end{tikzcd}$$

We can assume wlog that $\mathbf{b}_{\min} = \nu$ and $\mathbf{p}_{\max} = \nu$, because subsets of Dyck paths are again Dyck paths. Therefore, the concatenation $\mathbf{b} \cup \mathbf{p} \in D$ is defined and we have $\alpha, \nu \in \mathbf{b} \cup \mathbf{p}$. But then, because of $\alpha, \nu \in M_{\mathbf{s}}$: $|M_{\mathbf{s}} \cap \mathbf{b}| > 1$, which is a contradiction to $M_{\mathbf{s}} \in \mathrm{supp}_1(S(1))$.
Therefore, for all $\mathbf{q} \in D$ we have $|M'_{\mathbf{s}} \cap \mathbf{q}| \leq 1$. By that and with $M'_{\mathbf{s}} \in \mathcal{P}(\Delta)$ we conclude $M'_{\mathbf{s}} \in \mathrm{supp}_1(S(1))$. Therefore $M'_{\mathbf{s}} \in \nabla$ and by construction, because $\succ$ is a lexicographic order, $M'_{\mathbf{s}} \succ M_{\mathbf{s}}$, which is a contradiction to the maximality of $M_{\mathbf{s}}$. Therefore, the assumption on the existence of $\nu$ was wrong, which proves the Lemma. □

**Proposition 1.2.7.** *Let $D \subset \mathcal{P}(\Delta)$ be a set of Dyck paths, then we have for the integer points $S(m)$ of the polytopes $P(m)$ associated to $D$:*

$$S(m-1) + S(1) = S(m), \ \forall m \in \mathbb{Z}_{\geq 1}, \tag{1.2.3}$$

*where the left-hand side* (lhs) *of* (1.2.3) *is the Minkowski sum of $S(m-1)$ and $S(1)$.*

*Proof.* Let $m \geq 1$. From the definition of $P(m)$ and of the Minkowski sum follows $S(m-1) + S(1) \subset S(m)$. Therefore, it is sufficient to show that

$$S(m-1) + S(1) \supset S(m)$$

holds. For that let $\mathbf{s} = (s_z)_{z \in \Delta} \in S(m) \setminus S(m-1)$ be an arbitrary element. We show that there exists an integer point $\mathbf{t}^1 \in S(1) \setminus \{0\}$ such that: $\mathbf{s} - \mathbf{t}^1 \in S(m-1)$. We define for $M_{\mathbf{s}}$ defined as in (1.2.2):

$$\mathbf{t}^1 := \mathrm{supp}_1^{-1}(M_{\mathbf{s}}) \in S(1) \setminus \{0\}.$$

This element is unique because of the injectivity of $\mathrm{supp}_1$. Now we consider the integer point $\mathbf{s} - \mathbf{t}^1$. We know that there are no negative entries, because $s_z = 0$

implies for all $A \in \nabla : z \notin A$ and so $t^1_z = 0$. Hence $\mathbf{s} - \mathbf{t}^1 \in S(m)$ and so the second step is to show that $\mathbf{s} - \mathbf{t}^1$ lies already in $S(m-1)$.
To achieve that we assume contrary $\mathbf{s} - \mathbf{t}^1 \in S(m) \setminus S(m-1)$, i.e. that there is a Dyck path $\mathbf{p} \in D$ such that:

$$\sum_{z \in \mathbf{p}} (s_z - t^1_z) = m.$$

Since $\mathbf{s} \in S(m)$ we have:

$$m = \sum_{z \in \mathbf{p}} (s_z - t^1_z) = \underbrace{\sum_{z \in \mathbf{p}} s_z}_{\leq m} - \underbrace{\sum_{z \in \mathbf{p}} t^1_z}_{\geq 0} \Rightarrow \sum_{z \in \mathbf{p}} s_z = m \text{ and } \sum_{z \in \mathbf{p}} t^1_z = 0.$$

We construct another Dyck path $\overline{\mathbf{p}} \in D$ such that $\sum_{z \in \overline{\mathbf{p}}} s_z > m$.
Let $\beta \in \Delta$ be maximal with the property $\beta \in \mathbf{p} \wedge s_\beta > 0$. In particular, since $\sum_{z \in \mathbf{p}} (s_z - t^1_z) = m$ we have $\mathbf{p} \cap M_{\mathbf{s}} = \emptyset$ and so $\beta \notin M_{\mathbf{s}}$. We define

$$\mathbf{p}' = \mathbf{p} \setminus \{\gamma \in \mathbf{p} \mid \gamma \succ \beta\},$$

which is an element of $D$ since subsets of Dyck paths are again Dyck paths. By construction we have

$$\sum_{z \in \mathbf{p}'} s_z = m = \sum_{z \in \mathbf{p}} s_z.$$

There are two possibilities to extend the path $\mathbf{p}'$ with a further Dyck path $\mathbf{p}'' \in D$:

$$(i)\, \mathbf{p}''_{\min} = \beta \text{ or } (ii)\, \mathbf{p}''_{\max} = \mathbf{p}_{\min}.$$

To obtain a path $\overline{\mathbf{p}} = \mathbf{p}'' \cup \mathbf{p}'$ (respectively $\overline{\mathbf{p}} = \mathbf{p}' \cup \mathbf{p}''$) with $\sum_{z \in \overline{\mathbf{p}}} s_z > m$, the extension $\mathbf{p}''$ has to satisfy the following condition: $\mathbf{p}'' \cap M_{\mathbf{s}} \neq \emptyset$.

Assume we are in the case $(ii)$. Then there exists $\tau \in \mathbf{p}'' \cap M_{\mathbf{s}}$ with $s_\tau > 0$. Further we have $s_\beta > 0$ and $\tau, \beta \in \mathbf{p}' \cup \mathbf{p}'' = \overline{\mathbf{p}} \in D$. By construction we have $\beta \prec \tau$ and so Lemma 1.2.6 implies that $s_\beta = 0$. This is a contradiction to $s_\beta > 0$.

Therefore, we show the existence of a path $\mathbf{p}'' \in D$ with condition $(i)$ and $\mathbf{p}'' \cap M_{\mathbf{s}} \neq \emptyset$. We assume contrary there is no such Dyck path $\mathbf{p}''$:

$$\forall \mathbf{q} \in D \text{ with } \mathbf{q}_{\min} = \beta : \mathbf{q} \cap M_{\mathbf{s}} = \emptyset. \tag{1.2.4}$$

Under this assumption and by using Lemma 1.2.6 we show:

$$\forall \mathbf{q} \in D \text{ with } \beta \in \mathbf{q} : \mathbf{q} \cap M_{\mathbf{s}} = \emptyset. \tag{1.2.5}$$

Assume (1.2.5) is not true, so there is some $\beta \neq \tau \in \mathbf{q} \cap M_{\mathbf{s}}$ for $\mathbf{q} \in D$ with $\beta \in \mathbf{q}$. Then we have two cases.
Let $\tau \succ \beta$, then $\tau$ and $\beta$ lie in $\mathbf{q}$. Now the path from $\tau$ to $\beta$ is again a Dyck path. But this is a contradiction to Assumption (1.2.4).
Let $\beta \succ \tau$, by $\tau \in \mathbf{q} \cap M_{\mathbf{s}}$ we have $t^1_\tau \neq 0$. Then Lemma 1.2.6 implies $s_\beta = 0$,

which is a contradiction to the choice of $\beta$.
Therefore (1.2.5) holds. Recall the properties of $M_{\mathbf{s}}$. We have

$$M_{\mathbf{s}} = \mathrm{supp}_1(\mathbf{t}^1) \in \mathcal{P}(\Delta) \text{ with } |M_{\mathbf{s}} \cap \mathbf{q}| \leq 1, \forall \mathbf{q} \in D.$$

Now consider $M'_{\mathbf{s}} := M_{\mathbf{s}} \cup \{\beta\} \in \mathcal{P}(\mathrm{supp}_{\mathrm{m}}(\mathbf{s}))$. We show that $M'_{\mathbf{s}} \in \mathrm{supp}_1(S(1))$.
For $\mathbf{q} \in D$ with $\beta \in \mathbf{q}$ we have $|M'_{\mathbf{s}} \cap \mathbf{q}| = 1$ by (1.2.5).
For $\mathbf{q} \in D$ with $\beta \notin \mathbf{q}$ we have $|M'_{\mathbf{s}} \cap \mathbf{q}| \leq 1$ by $|M_{\mathbf{s}} \cap \mathbf{q}| \leq 1$.
We conclude $M'_{\mathbf{s}} \in \mathrm{supp}_1(S(1))$ and so

$$M'_{\mathbf{s}} \in \nabla = \mathrm{supp}_1(S(1)) \cap \mathcal{P}(\mathrm{supp}_{\mathrm{m}}(\mathbf{s})).$$

But with $M'_{\mathbf{s}} \succ M_{\mathbf{s}}$ we get a contradiction to the maximality of $M_{\mathbf{s}}$.
Therefore, Assumption (1.2.4) was wrong and there exists

$$\mathbf{p}'' \in D \text{ with } \mathbf{p}''_{\mathrm{min}} = \beta : \mathbf{p}'' \cap M_{\mathbf{s}} \neq \emptyset.$$

We recall that $\beta \notin M_{\mathbf{s}}$ and therefore $\tilde{\mathbf{p}} \neq \{\beta\}$. Define the concatenation of $\mathbf{p}''$ and $\mathbf{p}'$ in $\beta$ as $\overline{\mathbf{p}} := \mathbf{p}'' \cup \mathbf{p}' \in D$ which is indeed defined because $\mathbf{p}''_{\mathrm{min}} = \beta = \mathbf{p}'_{\mathrm{max}}$. From Definition 1.2.3(3) we know that $\overline{\mathbf{p}}$ is a Dyck path. Now by construction we conclude

$$\sum_{z \in \overline{\mathbf{p}}} s_z = \underbrace{\sum_{z \in \mathbf{p}''} s_z}_{>0} + \underbrace{\sum_{z \in \mathbf{p}'} s_z}_{=m} > m.$$

But this is a contradiction to the choice of $\mathbf{s} \in S(m)$ and the assumption $\sum_{z \in \mathbf{p}}(s_z - t_z^1) = m$ was wrong. We conclude $\mathbf{s} - \mathbf{t}^1 \in S(m-1)$ and with $\mathbf{t}^1 \in S(1)$ we have $\mathbf{s} \in S(m-1) + S(1)$. Finally we get $S(m) \subset S(m-1) + S(1)$. □

**Consequences.** We recall the construction of the Hasse diagram and the Dyck paths from Section 1.1 and show that we can apply Proposition 1.2.7 to this setup. Let $\lambda = m\omega_i$ as before and we set $\Delta = \Delta_+^{\omega_i}, D = D_{\omega_i}$. Then we have for the associated polytopes:

$$P(m) = P(m\omega_i).$$

For $\Delta_+^{\lambda} = \{\beta_1, \ldots, \beta_N\}$ we chose in Section 1.1 the order $\beta_1 \prec \cdots \prec \beta_N$. To apply Proposition 1.2.7 we can use the same order on the positive roots and extend this order to the (non-homogeneous) lexicographical order on $\mathcal{P}(\Delta_+^{\omega_i})$ as before. We show that the Dyck paths defined in Section 1.1 are Dyck paths in the sense of Definition 1.2.3.
(1) Every $\mathbf{p}' \subset \mathbf{p} \in D_{\omega_i}$ is again a Dyck path: We saw that any ordered subset of a directed path in $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ is again a Dyck path.
(2) For each $\beta \in \Delta_+^{\omega_i}$ there is at least one $\mathbf{p} \in D_{\omega_i}$ such that $\beta \in \mathbf{p}$: The set of vertices in $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ is exactly $\Delta_+^{\omega_i}$. By construction we allow paths of cardinality one, so for example the path $(\beta)$ contains $\beta$.
(3) Let $\mathbf{p}, \mathbf{p}' \in D_{\omega_i}$ be two Dyck paths, such that $\mathbf{p}_{\mathrm{min}} = \mathbf{p}'_{\mathrm{max}}$. Then there are directed paths $W, W'$ in $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ realizing $\mathbf{p}$ and $\mathbf{p}'$ such that the end point of $W$ is equal to the starting point of $W'$. We consider the directed path, which we obtain by the concatenation of the directed paths $W$ and $W'$. This directed path realizes $\mathbf{p} \cup \mathbf{p}'$. Hence $\mathbf{p} \cup \mathbf{p}'$ lies in $D_{\omega_i}$.
With Proposition 1.2.7 we get immediately for $S(m\omega_i) = P(m\omega_i) \cap \mathbb{Z}_{\geq 0}^N, m \in \mathbb{Z}_{\geq 0}$ :

**Proposition 1.2.8.** $S(m\omega_i) = S((m-1)\omega_i) + S(\omega_i), m \in \mathbb{Z}_{\geq 1}$. $\square$

Finally we conclude that the polytopes constructed in (1.1.6) are normal convex lattice polytopes.

## 1.3 Spanning Property

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, $\lambda = m\omega$, with $m \in \mathbb{Z}_{\geq 0}$, be a rectangular dominant integral weight such that $\omega(\theta^\vee) = 1$. In this section we show that $\mathbb{B}_\lambda = \{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ is a spanning set for $V^a(\lambda)$. Recall that we have

$$V^a(\lambda) \cong S(\mathfrak{n}_\lambda^-)/I_\lambda,$$

where $I_\lambda$ is the annihilating ideal of $v_\lambda$. We know that $f_\alpha^{\lambda(\alpha^\vee)+1} v_\lambda$ is zero in $V(\lambda)$ (see (1.1.2)). Hence $f_\alpha^{\lambda(\alpha^\vee)+1} v_\lambda = 0$ in $V^a(\lambda)$. By the action of $U(\mathfrak{n}^+)$ on $V^a(\lambda)$ we obtain further relations. We will see that these relations are enough to rewrite every element as a linear combination of $f^{\mathbf{s}} v_\lambda, \mathbf{s} \in S(\lambda)$.
In our proof it is essential to have a Hasse diagram $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ without $k$–chains. A Dyck path is defined as before to be the set of roots corresponding to a directed path in $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$.
Analogue to Remark 0.0.4 we explain the $U(\mathfrak{n}^+)$—module structure on $S(\mathfrak{n}_\lambda^-)$. Let $\circ$ be the action of $U(\mathfrak{n}^+)$ on $S(\mathfrak{g})$ induced by the adjoint action of $\mathfrak{n}^+$ on $\mathfrak{g}$. Via the isomorphism $S(\mathfrak{n}^-) \cong S(\mathfrak{g})/S(\mathfrak{g})(S_+(\mathfrak{n}^+ \oplus \mathfrak{h}))$ we obtain an action on $S(\mathfrak{n}^-)$, where $S^+(\mathfrak{n}^+ \oplus \mathfrak{h}) \subset S(\mathfrak{n}^+ \oplus \mathfrak{h})$ is the augmentation ideal. By

$$S(\mathfrak{n}_\lambda^-) \cong S(\mathfrak{n}^-)/S(\mathfrak{n}^-)(\operatorname{span}\{f_\beta \mid \beta \in \Delta_+ \setminus \Delta_+^\lambda\})$$

we get an action on $S(\mathfrak{n}_\lambda^-)$. We denote this action again by $\circ$. Since the action of $U(\mathfrak{n}^+)$ on $V^a(\lambda)$ is induced by the action of $U(\mathfrak{n}^+)$ on $V(\lambda)$ (which is again induced by the adjoint action), we obtain that for all $e \in U(\mathfrak{n}^+), f \in S(\mathfrak{n}_\lambda^-)$

$$e(f v_\lambda) = (e \circ f) v_\lambda, \tag{1.3.1}$$

holds. Therefore we can restrict our further discussion on the $U(\mathfrak{n}^+)$–module $S(\mathfrak{n}_\lambda^-)$. Equation (1.3.1) and $U(\mathfrak{n}^+)(f v_\lambda) = U(\mathfrak{n}^+)(0) = \{0\}$ for all $f \in I_\lambda$ imply that $I_\lambda$ is stable under $\circ$. Furthermore, by Remark 0.0.3 the total degree of a monomial in $S(\mathfrak{n}_\lambda^-)/I_\lambda$ is invariant or it is zero under $\circ$. We denote as before $\Delta_+^\lambda = \{\beta_1, \ldots, \beta_N\}$ and use the same total order $\prec$ on the multi–exponents (resp. monomials) as defined in Section 1.1, which is induced by $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_N$.

Analogue to (0.0.4) we define differential operators; for $\alpha, \beta \in \Delta_+$ let

$$\partial_\alpha f_\beta := \begin{cases} f_{\beta-\alpha}, & \text{if } \beta - \alpha \in \Delta_+^\lambda \\ 0, & \text{else.} \end{cases}$$

The operators satisfy

$$\partial_\alpha f_\beta = (c_{\alpha,\beta})^{-1} \operatorname{ad}(e_\alpha)(f_\beta),$$

for the structure constants $c_{\alpha,\beta} \in \mathbb{C}^*$. Therefore, instead of using $\circ$ we can work with these differential operators. We point out that we need the differential operators for arbitrary roots in $\Delta_+$.

**Remark 1.3.1.** *Here we illustrate the problem which occurs if we allow $k$–chains in our Hasse diagram. Let $\gamma \prec \beta \prec \delta$ the roots of a $k$–chain $\gamma \xrightarrow{k} \beta \xrightarrow{k} \delta$ and consider for $\ell \geq 2$:*

$$\partial_k^2 f_\gamma^\ell = \partial_k(\ell f_\beta^1 f_\gamma^{\ell-1}) = \underbrace{c_0 \ell f_\delta^1 f_\beta^0 f_\gamma^{\ell-1}}_{\text{maximal monomial}} + c_1 \ell(\ell-1) f_\beta^2 f_\gamma^{\ell-2}, \tag{1.3.2}$$

*with $c_0 = c_{\gamma,\alpha_k} c_{\beta,\alpha_k}$ and $c_1 = c_{\gamma,\alpha_k}^2$ where $c_{\gamma,\alpha_k}, c_{\beta,\alpha_k}$ are the structure constants corresponding to $[e_{\alpha_k}, f_\beta]$ and $[e_{\alpha_k}, f_\gamma]$ respectively. Therefore, it is more involved to find a relation which contains $\beta$ and $\delta$.*

The next Lemma describes the action of the differential operators and gives an explicit characterization of the maximal monomial of $\partial_\nu f^{\mathbf{s}}$ for certain $\nu \in \Delta_+$ and $\mathbf{s} \in \mathbb{Z}_{\geq 0}^N$.

**Lemma 1.3.2.** *Assume $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ has no $k$–chains.*
*(i) Let $\mathbf{p} = \{\beta_{i_1}, \ldots, \beta_{i_r}\} \in D_\lambda$ with $\beta_{i_1} \prec \cdots \prec \beta_{i_r}$ and $\nu \in \Delta_+$. Further let $\beta_{i_k}, k \leq r$ be maximal such that $\partial_\nu f_{\beta_{i_k}} \neq 0$. Let $\mathbf{s} \in \mathbb{Z}_{\geq 0}^N$ be a multi–exponent supported on $\mathbf{p}$, i.e. $s_\beta = 0$ for $\beta \notin \mathbf{p}$. Then the maximal monomial in $\partial_\nu^l f^{\mathbf{s}} = \partial_\nu^l (f_{i_1}^{s_1} \ldots f_{i_r}^{s_r})$, $l \leq s_k$, is given by*

$$f_{i_1}^{s_1} \ldots f_{i_{k-1}}^{s_{k-1}} (f_{i_k - \nu}^l f_{i_k}^{s_k - l}) f_{i_{k+1}}^{s_{k+1}} \ldots f_{i_r}^{s_r}.$$

*(ii) Let $\sum_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} c_\mathbf{u} f^\mathbf{u} \in S(\mathfrak{n}^-)$ and $\nu \in \Delta_+$. Let $\mathbf{h} = \max_{\prec}\{\mathbf{u} \mid \partial_\nu f^\mathbf{u} \neq 0, c_\mathbf{u} \neq 0\}$. Further let $\beta_k = \max_{\prec}\{\beta \mid f_\beta \text{ is a factor of } f^\mathbf{u}, \partial_\nu f_\beta \neq 0, c_\mathbf{u} \neq 0\}$ and assume $h_{\beta_k} > 0$. Then for $l \leq h_{\beta_k}$ the maximal monomial in*

$$\partial_\nu^l \sum_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} c_\mathbf{u} f^\mathbf{u} = \sum_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} c_\mathbf{u} \partial_\nu^l f^\mathbf{u}$$

*appears in $\partial_\nu^l f^\mathbf{h}$.*

*Proof.* $(i)$ Assume we have two roots $\beta_i, \beta_j \in \Delta_+^\lambda$ with $\beta_i \prec \beta_j$ and $\beta_i - \nu$ and $\beta_j - \nu$ are again roots in $\Delta_+^\lambda$. For $\beta_{i_l} - \nu \notin \Delta_+^\lambda$ we have $\partial_\nu f_{\beta_{i_l}} = 0$, so we do not need to consider such roots $\beta_{i_l} \in \Delta_+^\lambda$. So in order to prove $(i)$, because our monomial order is lexicographic, it is sufficient to show that

$$\beta_i \prec \beta_j \Rightarrow \beta_i - \nu \prec \beta_j - \nu. \tag{1.3.3}$$

If $\beta_i > \beta_j$ with respect to the standard partial order we have $\beta_i - \nu > \beta_j - \nu$ and therefore $\beta_i - \nu \prec \beta_j - \nu$, by the choice of the total order (1.1.3) on $\Delta_+^\lambda$.
If the roots are not comparable with respect to the standard partial order, the second step is to compare the heights of the roots. Thus if $\mathrm{ht}(\beta_i) > \mathrm{ht}(\beta_j)$ then $\mathrm{ht}(\beta_i - \nu) > \mathrm{ht}(\beta_j - \nu)$ and again $\beta_i - \nu \prec \beta_j - \nu$.
If $\mathrm{ht}(\beta_i) = \mathrm{ht}(\beta_j)$, we have to consider $\beta_i = (s_1, \ldots, s_n)$ and $\beta_j = (t_1, \ldots, t_n)$ in terms of the fixed basis of the simple roots (see Remark 1.1.3). Then there is a $1 \leq k \leq n$, such that $s_k > t_k$ and $s_i = t_i$ for all $1 \leq i < k$. Let $\nu = (u_1, \ldots, u_n)$, then $\beta_i - \nu = (s_1 - u_1, \ldots, s_n - u_n)$ is lexicographically greater than $\beta_j - \nu =$

$(t_1 - u_1, \ldots, t_n - u_n)$. Thus $\beta_i - \nu \prec \beta_j - \nu$ and (1.3.3) holds.

$(ii)$ We only have to consider the multi–exponents $\mathbf{s} \in \mathbb{Z}_{\geq 0}^N$ such that $\partial_\nu f^{\mathbf{s}} \neq 0$. Now let $\mathbf{t}$ be the maximal multi–exponent with this property and let $l \leq t_{\beta_k}$. Then we have $\partial_\nu^l f^{\mathbf{t}} \neq 0$ and by (i) the maximal monomial appearing in $\partial_\nu^l f^{\mathbf{t}}$ is

$$f_{\beta_k - \nu}^l f_{\beta_k}^{t_{\beta_k} - l} \prod_{\substack{\beta \in \Delta_+^\lambda, \beta \neq \beta_k \\ \beta \neq \beta_k - \nu}} f_\beta^{t_\beta}. \tag{1.3.4}$$

The observation (1.3.3) tells us that $f_{\beta_k - \nu} = \max\{f_{\beta - \nu} \mid \partial_\nu f_\beta \neq 0, s_\beta > 0\}$. Thus by the choice of $\mathbf{t}$ and because our order is lexicographic, the element (1.3.4) is the maximal monomial in $\sum_{\mathbf{s} \in \mathbb{Z}_{\geq 0}^N} c_{\mathbf{s}} \partial_\nu^l f^{\mathbf{s}}$. □

**Proposition 1.3.3.** *Assume $H(\mathfrak{n}_\lambda^-)_{\mathfrak{g}}$ has no $k$–chains and let $\mathbf{p} \in D_\lambda$ be a Dyck path, $\mathbf{s} \in \mathbb{Z}_{\geq 0}^N$ be a multi–exponent supported on $\mathbf{p}$. Suppose further $\langle \lambda, \theta^\vee \rangle = m$ and $\sum\limits_{\alpha \in \mathbf{p}} s_\alpha > m$. Then there exist constants $c_{\mathbf{t}} \in \mathbb{C}$, $\mathbf{t} \in \mathbb{Z}_{\geq 0}^N$ such that:*

$$f^{\mathbf{s}} + \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} f^{\mathbf{t}} \in I_\lambda.$$

We follow an idea of [FFL11a, FFL11b] who showed a similar statement in the cases $\mathfrak{sl}_n$ and $\mathfrak{sp}_n$ for arbitrary dominant integral weights.

*Proof.* Let $\mathbf{p} = \{\tau_0, \tau_1, \ldots, \tau_r\} \in D_\lambda$ be an arbitrary Dyck path. By construction we have for $1 \leq i \leq r$: $\tau_{i-1} \prec \tau_i$. Because $\sum_{i=0}^r s_{\tau_i} > m$ we have

$$f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} \in I_\lambda.$$

By the construction of the Hasse diagram there is a Dyck path $\mathbf{p}' \in D_\lambda$ with $\mathbf{p} \subset \mathbf{p}'$, such that there is no path $\mathbf{p}''$ with $\mathbf{p}' \subsetneq \mathbf{p}''$. Hence we can assume wlog

$$\mathbf{p} = \{\tau_0 = \theta, \tau_1, \ldots, \tau_{r-1}, \tau_r = \beta_N\}.$$

Let $\nu_1, \ldots, \nu_r \in \Delta_+$, with $\nu_i \neq \nu_{i+1}$ be the labels at the edges of $\mathbf{p}$. We consider $f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} \in I_\lambda$. Because $I_\lambda$ is stable under $\circ$, we have for arbitrary $x_1, \ldots, x_l \in \Delta_+$ and $f^{\mathbf{t}} \in I_\lambda$:

$$\partial_{x_1} \ldots \partial_{x_l} f^{\mathbf{t}} \in I_\lambda.$$

We define

$$A := \partial_{\nu_r}^{s_{\tau_r}} \ldots \partial_{\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} \partial_{\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} \in I_\lambda.$$

**Claim:** There exist constants $c_{\mathbf{s}} \neq 0, c_{\mathbf{t}} \in \mathbb{C}, \mathbf{t} \in \mathbb{Z}_{\geq 0}^N$ with $\mathbf{t} \prec \mathbf{s}$, such that:

$$A = c_{\mathbf{s}} f^{\mathbf{s}} + \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} f^{\mathbf{t}} \in I_\lambda$$

If the claim holds the Proposition is proven.

*Proof of the claim.* Now we need the explicit description of the Dyck paths given by the Hasse diagram. Above we defined $\nu_1$ to be the label at the edge $\theta \xrightarrow{\nu_1} \tau_1$ in $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$. Because we assumed that $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$ has no $\nu_1$–chains of length 2, there is no edge labeled by $\nu_1$ starting in the vertex $\theta - \nu_1 = \tau_1$. That implies $\partial_{\nu_1} f_{\theta-\nu_1} = 0$. Therefore we obtain

$$\partial_{\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} = a_0 \, f_\theta^{s_{\tau_0}} \, f_{\theta-\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} \in I_\lambda$$

for some constant $a_0 \in \mathbb{C} \setminus \{0\}$. Now $\nu_2$ is the label at the edge between the vertices $\tau_1$ and $\tau_2$. Again there is no $\nu_2$–chain in $H(\mathfrak{n}_\lambda^-)_\mathfrak{g}$, so $\partial_{\nu_2} f_{\theta-\nu_1-\nu_2} = 0$ and $\partial_{\nu_2} f_{\theta-\nu_2} = 0$, so we have for $k = \min\{s_{\tau_0}, s_{\tau_2} + \cdots + s_{\tau_r}\}$, $b_q \in \mathbb{C} \setminus \{0\}$:

$$\begin{aligned} &\partial_{\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} a_0 f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} = \\ &b_0 \, f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1}} f_{\theta-\nu_1-\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} + \sum_{q=1}^{k} b_q f_\theta^{s_{\tau_0} - q} f_{\theta-\nu_1}^{s_{\tau_1} + q} f_{\theta-\nu_1-\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r} - q} f_{\theta-\nu_2}^{q}. \end{aligned} \tag{1.3.5}$$

For our purposes, we do not need to pay attention to the scalars unless they are zero. We also notice that the terms of the sum are only non-zero, if $\theta - \nu_2 \in \Delta_+^\lambda$. The first part of Lemma 1.3.2 implies, that the monomial $f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1}} f_{\theta-\nu_1-\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}}$ is the largest (with respect to $\prec$) in (1.3.5), because $\theta \prec \theta - \nu_1 \prec \theta - \nu_1 - \nu_2$. By construction $\partial_{\nu_{i+1}} f_{\theta-\nu_1-\nu_2-\cdots-\nu_i} \neq 0$, because $\theta - \nu_1 - \nu_2 - \cdots - \nu_i - \nu_{i+1}$ is an element of $\Delta_+^\lambda$, for $i < r$. Thus the second statement of Lemma 1.3.2 implies that the largest element is obtained by acting in each step on the largest root vector. To be more precise, we consider the following equations:

$$\begin{aligned} \partial_{\nu_r}^{s_{\tau_r}} \ldots \partial_{\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} \partial_{\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} &= \\ a_0 \, \partial_{\nu_r}^{s_{\tau_r}} \ldots \partial_{\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} &= \\ b_0 \, \partial_{\nu_r}^{s_{\tau_r}} \ldots \partial_{\nu_3}^{s_{\tau_3} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1}} f_{\theta-\nu_1-\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} &+ \sum \text{smaller monomials} = \\ &\vdots \\ b_0' \, f_\theta^{s_{\tau_0}} f_{\theta-\nu_1}^{s_{\tau_1}} f_{\theta-\nu_1-\nu_2}^{s_{\tau_2}} \ldots f_{\theta-\nu_1-\nu_2-\cdots-\nu_r}^{s_{\tau_r}} &+ \sum \text{smaller monomials} \in I_\lambda. \end{aligned}$$

for some $b_0' \in \mathbb{C} \setminus \{0\}$. But the last term is exactly what we wanted to obtain, so for constants $c_\mathbf{t} \in \mathbb{C}$, $c_\mathbf{s} \in \mathbb{C} \setminus \{0\}$ we have by assumption that $s_\alpha = 0$ if $\alpha \notin \mathbf{p}$:

$$\begin{aligned} \partial_{\nu_r}^{s_{\tau_r}} \ldots \partial_{\nu_2}^{s_{\tau_2} + \cdots + s_{\tau_r}} \partial_{\nu_1}^{s_{\tau_1} + \cdots + s_{\tau_r}} f_\theta^{s_{\tau_0} + \cdots + s_{\tau_r}} &= \\ c_\mathbf{s} f_\theta^{s_{\tau_0}} f_{\tau_1}^{s_{\tau_1}} f_{\tau_2}^{s_{\tau_2}} \ldots f_{\tau_r}^{s_{\tau_r}} + \sum_{\mathbf{t} \prec \mathbf{s}} c_\mathbf{t} f^\mathbf{t} &= \\ c_\mathbf{s} f^\mathbf{s} + \sum_{\mathbf{t} \prec \mathbf{s}} c_\mathbf{t} f^\mathbf{t} &\in I_\lambda. \end{aligned}$$

□

**Theorem 1.3.4.** *The set $\{f^\mathbf{s} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ spans the module $V(\lambda)^a$.*

*Proof.* Let $m \in \mathbb{Z}_{\geq 0}$ and $\mathbf{t} \in \mathbb{Z}_{\geq 0}^N$ with $\mathbf{t} \notin S(\lambda)$. Therefore, there exists a Dyck path $\mathbf{p} \in D_\lambda$ such that $\sum\limits_{\beta \in \mathbf{p}} t_\beta > m$. Define a new multi–exponent $\mathbf{t}'$ by

$$t'_\beta := \begin{cases} t_\beta, & \text{if } \beta \in \mathbf{p}, \\ 0, & \text{else.} \end{cases}$$

Because of $\sum\limits_{\beta \in \mathbf{p}} t'_\beta = \sum\limits_{\beta \in \mathbf{p}} t_\beta > m$ we can apply Proposition 1.3.3 to $\mathbf{t}'$ and get

$$f^{\mathbf{t}'} = \sum_{\mathbf{s}' \prec \mathbf{t}'} c_{\mathbf{s}'} f^{\mathbf{s}'} \in S(\mathfrak{n}_\lambda^-)/I_\lambda,$$

for some $c_{\mathbf{s}'} \in \mathbb{C}$. Because the order of the factors of $f^{\mathbf{t}} \in S(\mathfrak{n}_\lambda^-)$ is arbitrary and since we have a monomial order, we get

$$f^{\mathbf{t}} = f^{\mathbf{t}'} \prod_{\beta \notin \mathbf{p}} f_\beta^{t_\beta} = \sum_{\mathbf{s} \prec \mathbf{t}} c_{\mathbf{s}} f^{\mathbf{s}} \in S(\mathfrak{n}_\lambda^-)/I_\lambda, \tag{1.3.6}$$

where $c_{\mathbf{s}} = c_{\mathbf{s}'}$ and $f^{\mathbf{s}} = f^{\mathbf{s}'} \prod_{\beta \notin \mathbf{p}} f_\beta^{s_\beta}$. Equation (1.3.6) shows that we can express an arbitrary multi–exponent as a sum of strictly smaller multi–exponents. We repeat this procedure until all multi–exponents in the sum lie in $S(\lambda)$. There are only finitely many multi–exponents of a fixed degree and the degree is invariant or zero under the action $\circ$. So after a finite number of steps, we can express $\mathbf{t}$ in terms of $\mathbf{r} \in S(\lambda)$ for some $c_{\mathbf{r}} \in \mathbb{C}$:

$$f^{\mathbf{t}} = \sum_{\mathbf{r} \in S(\lambda)} c_{\mathbf{r}} f^{\mathbf{r}} \in S(\mathfrak{n}_\lambda^-)/I_\lambda.$$

□

**Corollary 1.3.5.** *Fix for every $\mathbf{s} \in S(\lambda)$ an arbitrary ordering of the factors $f_\beta$ in the product $\prod_{\beta > 0} f_\beta^{s_\beta} \in S(\mathfrak{n}_\lambda^-)$. Let $f^{\mathbf{s}} = \prod_{\beta > 0} f_\beta^{s_\beta} \in U(\mathfrak{n}^-)$ be the ordered product. Then the elements $f^{\mathbf{s}} v_\omega, \mathbf{s} \in S(\lambda)$ span the module $V(\lambda)$.*

*Proof.* Let $f^{\mathbf{t}} v_\lambda \in V(\lambda)$ with $\mathbf{t} \in \mathbb{Z}_{\geq}^N$ arbitrary. We consider $f^{\mathbf{t}} v_\lambda$ as an element in $V^a(\lambda)$. By Theorem 1.3.4 we get

$$f^{\mathbf{t}} v_\lambda = \sum_{\mathbf{s} \in S(\lambda)} c_{\mathbf{s}} f^{\mathbf{s}} v_\lambda \text{ in } V^a(\lambda).$$

The ordering of the factors in a product in $S(\mathfrak{n}_\lambda^-)$ is irrelevant, so we can adjust the ordering of the factors to the fixed ordering and get an induced linear combination:

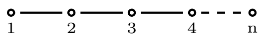
$$f^{\mathbf{t}} v_\lambda = \sum_{\mathbf{s} \in S(\lambda)} c_{\mathbf{s}} f^{\mathbf{s}} v_\lambda \text{ in } V(\lambda).$$

□

## 1.4 FFL Basis of $V(\omega)$

Throughout this section we refer to the definitions in Section 1.1. In this section we calculate explicit FFL bases of the highest weight modules $V(\omega)$, where $\omega$ occurs in Table 1.1. We do this by giving characterizations of the co–chains $\overline{\mathbf{p}} \in \overline{D}_\omega$ (see (1.1.5)) and using the one-to-one correspondence between $\overline{D}_\omega$ and $S(\omega)$ (see Proposition 1.1.8).
The results of this section, i.e. $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a FFL basis of $V(\omega)$, provide the start of an inductive procedure in the proof of Theorem 1.5.2. With Proposition 1.2.7 we will be able to give an explicit basis of $V(m\omega)$, $m \in \mathbb{Z}_{\geq 0}$, parametrized by the $m$-th Minkowski sum of $S(\omega)$.

**Type $\mathtt{A_n}$.** Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{A_n}$ with $n \geq 1$ and the associated Dynkin diagram

$\mathtt{A_n}$ 


The highest root is of the form $\theta = \sum_{i=1}^n \alpha_i$. Since a Lie algebra $\mathfrak{g}$ of type $\mathtt{A_n}$ is simply laced we have $\theta^\vee = \sum_{i=1}^n \alpha_i^\vee$ and so $\omega(\theta^\vee) = 1 \Leftrightarrow \omega \in \{\omega_k \mid \ 1 \leq k \leq n\}$. The positive roots of $\mathfrak{g}$ are described by: $\Delta_+ = \{\alpha_{i,j} = \sum_{l=i}^j \alpha_l \mid 1 \leq i \leq j \leq n\}$. Therefore, for the roots corresponding to $\mathfrak{n}^-_{\omega_k}$ we have:

$$\Delta_+^{\omega_k} = \{\alpha_{i,j} \in \Delta_+ | \ 1 \leq i \leq k \leq j \leq n\} \subset \Delta_+. \tag{1.4.1}$$

Before we define the total order on $\Delta_+^{\omega_k}$, we define a total order on $\Delta_+$:

$$\begin{gathered} \beta_1 = \alpha_{1,n}, \\ \beta_2 = \alpha_{2,n}, \ \beta_3 = \alpha_{1,n-1}, \\ \beta_4 = \alpha_{3,n}, \ \beta_5 = \alpha_{2,n-1}, \ \beta_6 = \alpha_{1,n-2}, \\ \cdots, \\ \beta_{n(n-1)/2+1} = \alpha_n, \ \beta_{n(n-1)/2+2} = \alpha_{n-1}, \cdots, \ \beta_{n(n+1)/2} = \alpha_1. \end{gathered}$$

Now we delete every root $\beta_i \in \Delta_+ \setminus \Delta_+^{\omega_k}$ and relabel the remaining roots. For an example of this procedure see Appendix, Figure 3.2 and Example 1.1.4. In the following it is more convenient to use the description $\alpha_{i,j}$ instead of $\beta_k$. First we give a characterization of the co–chains $\overline{\mathbf{p}} \in \overline{D}_{\omega_k} \subset \mathcal{P}(\Delta_+^{\omega_k})$.

**Proposition 1.4.1.** *Let be $\overline{\mathbf{p}} = \{\alpha_{i_1,j_1}, \dots, \alpha_{i_s,j_s}\} \in \mathcal{P}(\Delta_+^{\omega_k})$ arbitrary, then:*

$$\overline{\mathbf{p}} \in \overline{D}_{\omega_k} \Leftrightarrow \forall \alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}, \ i_l \leq i_m : i_l < i_m \leq k \leq j_l < j_m. \tag{1.4.2}$$

*Further we have:* $\overline{\mathbf{p}} \in \overline{D}_{\omega_k} \Rightarrow s \leq \min\{k, n+1-k\}$.

*Proof.* First we prove (1.4.2): "$\Leftarrow$": Let $\overline{\mathbf{p}} = \{\alpha_{i_1,j_1}, \dots, \alpha_{i_s,j_s}\} \in \mathcal{P}(\Delta_+^{\omega_k})$ be an element with the properties of the right-hand side (rhs) of (1.4.2). Let $\alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}$, with $i_l < i_m$. Consider now:

$$\alpha_{i_l,j_l} - \alpha_{i_m,j_m} = \sum_{r=i_l}^{j_l} \alpha_r - \sum_{r=i_m}^{j_m} \alpha_r = \sum_{r=i_l}^{i_m-1} \alpha_r - \sum_{r=j_l+1}^{j_m} \alpha_r.$$

Since $j_l < j_m$ holds, Remark 1.1.6 implies that there is no Dyck path $\mathbf{q} \in D_{\omega_k}$ such that $\alpha_{i_m,j_m}$ and $\alpha_{i_l,j_l}$ are contained in $\mathbf{q}$.

"$\Rightarrow$": Let be $\overline{\mathbf{p}} \in \overline{D}_{\omega_k}$ and $\alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}$ with $\alpha_{i_l,j_l} \neq \alpha_{i_m,j_m}$. Further we have $i_l \leq j_l, i_m \leq j_m$. Assume wlog $i_m = j_m$, then $\alpha_{i_m,j_m} = \alpha_k$ and $i_l < j_l$. Hence

$$\alpha_{i_l,j_l} - \alpha_k = \sum_{r=i_l}^{k-1} \alpha_r + \sum_{r=k+1}^{j_l} \alpha_r,$$

which is a contradiction to $\overline{\mathbf{p}} \in \overline{D}_{\omega_k}$ by Remark 1.1.6. Therefore, $i_l < j_l, i_m < j_m$ and we assume wlog $i_l \leq i_m$.

**1. Step:** $i_l = i_m =: y$. Set $x = \min\{j_l, j_m\}$ and $\overline{x} = \max\{j_l, j_m\}$:

$$\alpha_{y,\overline{x}} - \alpha_{y,x} = \sum_{r=y}^{\overline{x}} \alpha_r - \sum_{r=y}^{x} \alpha_r = \sum_{r=x+1}^{\overline{x}} \alpha_r.$$

Again this contradicts to $\overline{\mathbf{p}} \in \overline{D}_{\omega_k}$. Hence we have: $i_l < i_m$.

**2. Step:** $(i_l < i_m) \wedge (j_l = j_m =: x)$:

$$\alpha_{i_l,x} - \alpha_{i_m,x} = \sum_{r=i_l}^{x} \alpha_r - \sum_{r=i_m}^{x} \alpha_r = \sum_{r=i_l}^{i_m-1} \alpha_r.$$

We conclude: $j_l \neq j_m$.

**3. Step:** $(i_l < i_m < j_m) \wedge (i_l < j_l)$. Therefore, there are three possible cases:

(a) $i_l < j_l < i_m < j_m$, (b) $i_l < i_m < j_l < j_m$ and (c) $i_l < i_m < j_m < j_l$.

The case (a) can not occur because $k \leq j_l < i_m \leq k$ is a contradiction. Therefore, let us assume $\alpha_{i_l,j_l}, \alpha_{i_m,j_m}$ satisfy the case (c), then we have:

$$\alpha_{i_l,j_l} - \alpha_{i_m,j_m} = \sum_{r=i_l}^{j_l} \alpha_r - \sum_{r=i_m}^{j_m} \alpha_r = \sum_{r=i_l}^{i_m-1} \alpha_r + \sum_{r=j_m}^{j_l} \alpha_r.$$

Finally we conclude that for two arbitrary roots $\alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}} \in \overline{D}_{\omega_k}$ with $i_l \leq i_m$ we have: $i_l < i_m < j_l < j_m$.

It remains to show that the cardinality $s$ of $\overline{\mathbf{p}}$ is bounded by $\min\{k, n+1-k\}$:

**1. Case:** $\min\{k, n+1-k\} = k$. Let $\alpha_{i_r,j_r} \in \overline{\mathbf{p}}$ be an arbitrary root in $\overline{\mathbf{p}}$. Then we know from (1.4.1) $1 \leq i_r \leq k$. But we also know that for any two roots $\alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}$ we have $i_l \neq i_m$. Therefore, there are at most $k$ different roots in $\overline{\mathbf{p}}$.

**2. Case:** $\min\{k, n+1-k\} = n+1-k$. For two roots $\alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}$ we have $j_l \neq j_m$ and $k \leq j_l, j_m \leq n$. Therefore, the number of different roots in $\overline{\mathbf{p}}$ is bounded by $n+1-k$.

Finally we conclude: $|\overline{\mathbf{p}}| = s \leq \min\{k, n+1-k\}$. □

**Remark 1.4.2.** *Let* $\overline{\mathbf{p}} = \{\alpha_{i_1,j_1}, \dots, \alpha_{i_s,j_s}\} \in \overline{D}_{\omega_k}$ *then* (1.4.2) *implies*

$$i_1 < i_2 < \dots < i_s \leq k \leq j_1 < j_2 < \dots < j_s.$$

*Assume wlog* $k = j_1 = j_2$, *then there is Dyck path containing* $\alpha_{i_1,j_1}$ *and* $\alpha_{i_2,j_2}$, *because* $\alpha_{i_1,j_1} - \alpha_{i_2,j_2} = \alpha_{i_1,i_2-1} \in \Delta_+$.

Because of Corollary 1.3.5 we know that the elements $\{f^{\mathbf{s}} v_{\omega_k} \mid \mathbf{s} \in S(\omega_k)\}$ span $V(\omega_k)$ and by Proposition 1.1.8 there is a bijection between $S(\omega_k)$ and $\overline{D}_{\omega_k}$. We show that these elements are linear independent. To achieve that we show that $|\overline{D}_{\omega_k}| = \dim V(\omega_k)$. To be more explicit:

**Proposition 1.4.3.** *For all* $1 \leq k \leq n$ *we have:* $|\overline{D}_{\omega_k}| = \dim V(\omega_k) = \binom{n+1}{k}$.

*Proof.* Let $V(\omega_1)$ be the vector representation with basis $\{e_1, e_2, \dots, e_{n+1}\}$. Then $\bigwedge^k V(\omega_1)$ is a $U(\mathfrak{g})$-representation with $v_{\omega_k} = e_1 \wedge e_2 \wedge \dots \wedge e_k$:

$$f_{\alpha_{i_1,j_1}} v_{\omega_k} = e_1 \wedge \dots \wedge e_{i_1-1} \wedge e_{j_1+1} \wedge e_{i_1+1} \wedge \dots \wedge e_k, \tag{1.4.3}$$

and we have $\bigwedge^k V(\omega_1) \cong V(\omega_k)$. We define $f_{\overline{\mathbf{p}}} v_{\omega_k} := f_{\alpha_{i_1,j_1}} f_{\alpha_{i_2,j_2}} \cdots f_{\alpha_{i_m,j_m}} v_{\omega_k}$ for $\overline{\mathbf{p}} = \{\alpha_{i_1,j_1}, \alpha_{i_2,j_2}, \dots, \alpha_{i_m,j_m}\} \in \overline{D}_{\omega_k}$ and claim that the set $\{f_{\overline{\mathbf{p}}} v_{\omega_k} \mid \overline{\mathbf{p}} \in \overline{D}_{\omega_k}\}$ is linear independent in $\bigwedge^k V(\omega_1)$. If the claim holds we have $|\overline{D}_{\omega_k}| \leq \dim V(\omega_k)$ and with Corollary 1.3.5 we conclude that $|\overline{D}_{\omega_k}| = \dim V(\omega_k) = \binom{n+1}{k}$.

*Proof of the claim.* Assume we have $\overline{\mathbf{p}}_1 = \{\alpha_{i_1,j_1}, \alpha_{i_2,j_2}, \dots, \alpha_{i_m,j_m}\}$ and $\overline{\mathbf{p}}_2 = \{\alpha_{s_1,t_1}, \alpha_{s_2,t_2}, \dots, \alpha_{s_\ell,t_\ell}\}$ in $\overline{D}_{\omega_k}$ with linear dependent images under the action (1.4.3), i. e. $f_{\overline{\mathbf{p}}_1} v_{\omega_k} = \pm f_{\overline{\mathbf{p}}_2} v_{\omega_k}$. Then we have $m = \ell$, $\{j_1, \dots, j_m\} = \{t_1, \dots, t_\ell\}$ and we can assume wlog: $m = k = \ell$. Hence: $f_{\overline{\mathbf{p}}_1} v_{\omega_k} = e_{j_1} \wedge \dots \wedge e_{j_m} = \pm f_{\overline{\mathbf{p}}_2} v_{\omega_k}$, with Remark 1.4.2 we conclude $\overline{\mathbf{p}}_1 = \overline{\mathbf{p}}_2$. □

**Example 1.4.4.** *The non-redundant inequalities of the polytope* $P(m\omega_3)$ *in the case* $\mathfrak{g} = \mathfrak{sl}_5$ *are:*

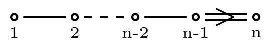
$$P(m\omega_3) = \left\{ \mathbf{x} \in \mathbb{R}^6_{\geq 0} \mid \begin{array}{r} x_1 + x_2 + x_4 + x_6 \leq m \\ x_1 + x_2 + x_5 + x_6 \leq m \\ x_1 + x_3 + x_5 + x_6 \leq m \end{array} \right\}.$$

*Example 1.1.4 shows the corresponding Hasse diagram* $H(\mathfrak{n}^-_{\omega_3})_{\mathfrak{sl}_5}$.

Proposition 1.4.3 implies immediately for $1 \leq k \leq n$:

**Proposition 1.4.5.** *The vectors* $f^{\mathbf{s}} v_{\omega_k}, \mathbf{s} \in S(\omega_k)$ *are a FFL basis of* $V(\omega_k)$. □

**Type $\mathtt{B_n}$.** Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{B_n}$, $n \geq 2$ with associated Dynkin diagram

$\mathtt{B_n}$ 


The highest root for a Lie algebra of type $\mathtt{B_n}$ is of the form $\theta = \alpha_1 + 2\sum_{i=2}^n \alpha_i$. Thus we have $\theta^\vee = \alpha_1^\vee + 2\sum_{i=2}^{n-1} \alpha_i^\vee + \alpha_n^\vee$ and $\omega(\theta^\vee) = 1 \Leftrightarrow \omega \in \{\omega_1, \omega_n\}$.

First we consider the case $\omega = \omega_1$. We consider the case $\mathtt{B}_2, w_1$ separately. Because there are not enough roots, this case does not fit in our general description of $\mathtt{B_n}, w_1$. We claim that the following polytope parametrizes a FFL basis of $V(m\omega_1), m \in \mathbb{Z}_{\geq 0}$:

$$P(m\omega_1) = \left\{ \mathbf{x} \in \mathbb{R}^3_{\geq 0} \mid \begin{array}{c} x_2 + x_1 \leq m \\ x_2 + x_3 \leq m \end{array} \right\}.$$

We fix $\beta_1 = (2,1), \beta_2 = (1,1), \beta_3 = (1,0)$ and the order $\beta_2 \prec \beta_1 \prec \beta_3$. Then with Proposition 1.2.7 it is immediate that this polytope is normal. The following actions of the differential operators imply the spanning property in the sense of Section 1.3 Proposition 1.3.3.

$$\begin{aligned} \partial^{s_1}_{\alpha_2} f_1^{s_1+s_2} &= c_0 f_1^{s_1} f_2^{s_2} + \text{smaller terms} \in I_\lambda \\ \partial^{s_2+2s_3}_{\alpha_1} f_1^{s_2+s_3} &= c_1 f_2^{s_2} f_3^{s_3} + \text{smaller terms} \in I_\lambda,\ c_i \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

We conclude that $\{f^{\mathbf{s}} v_{\omega_1} \mid \mathbf{s} \in S(m\omega_1)\} = \{v_{\omega_1}, f_1 v_{\omega_1}, f_2 v_{\omega_1}, f_3 v_{\omega_1}, f_1 f_3 v_{\omega_1},\}$ is a spanning set of $V(\omega_1)$.

Now we consider the case $n \geq 3$. If we construct $H(\mathfrak{n}^-_{\omega_1})_{\mathfrak{g}}$ as in Section 1.1 we get a $n$–chain of length 2. Therefore we choose a new order on the roots and change our Hasse diagram slightly to obtain a diagram without $k$–chains of length 2. We illustrate this procedure for $\mathfrak{g}$ of type $\mathtt{B}_3$. Then the roots $\Delta^{\omega_1}_+$ are given by

| | | | | |
|---|---|---|---|---|
| $\beta_1 = (1,2,2)$ | $\beta_2 = (1,1,2)$ | $\beta_3 = (1,1,1)$ | $\beta_4 = (1,1,0)$ | $\beta_5 = (1,0,0)$ |

We choose a new order

$$\beta_1 \prec \beta_2 \prec \beta_4 \prec \beta_5 \prec \beta_3,$$

and change the Hasse diagram

$$\begin{tikzcd} \beta_1 \arrow[r, "2"] & \beta_2 \arrow[r, "3"] & \beta_3 \arrow[r, "3"] & \beta_4 \arrow[r, "2"] & \beta_5 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & & \beta_2 \arrow[dr, "012"] & \\ \beta_1 \arrow[r, "011"] & \beta_3 \arrow[ur, "2"] \arrow[dr, "012"'] & & \beta_5. \\ & & \beta_4 \arrow[ur, "2"'] & \end{tikzcd}$$

First we check, if the new diagram has no $k$–chains. The first edge is labeled by $\alpha_2 + \alpha_3 = 011$ and we have $\beta_3 - (\alpha_2 + \alpha_3) = \beta_5$. If we have a monomial $f_1^{k_1} f_3^{k_2} \in S(\mathfrak{n}^-_{\omega_1}), k_1, k_2 \geq 1$ and we act by $\partial_{\alpha_2+\alpha_3}$ we get:

$$c_0 f_1^{k_1-1} f_3^{k_2+1} + c_1 f_1^{k_1} f_3^{k_2-1} f_5,\ c_i \in \mathbb{C}.$$

By the change of order $\beta_3$ is larger than $\beta_5$ and so $f_1^{k_1-1} f_3^{k_2+1} \succ f_1^{k_1} f_3^{k_2-1} f_5$. Therefore we can neglect the edge between $\beta_3$ and $\beta_5$.
Now we consider $\partial^{k_3}_{\alpha_2} f_1^{k_1} f_3^{k_2}$. Because of $\partial_{\alpha_2} f_3, \partial_{\alpha_2} f_2 = 0$ we get $f_1^{k_1-k_3} f_3^{k_2} f_2^{k_3}$, for $k_3 \leq k_1$. Thus instead of drawing an edge directly from $\beta_1$ to $\beta_2$, we can draw an edge, labeled by 2, from $\beta_3$ to $\beta_2$. Similar, because of $\beta_1 - \alpha_2 - 2\alpha_3 = \beta_4$, we can draw an edge labeled by 012 from $\beta_3$ to $\beta_4$. The other edges do not cause any problems.

The second step is to show that the paths in the new diagram, define the actions by differential operators and the corresponding maximal elements like in Section 1.3 Proposition 1.3.3. By the choice of order we get the following equalities:

$$\begin{aligned}
\partial^{s5}_{\alpha_2+2\alpha_3}\partial^{s2}_{2}\partial^{s3}_{\alpha_2+\alpha_3} f_1^{s1+s3+s2+s5} &= c_0 f_1^{s1} f_3^{s3} f_2^{s2} f_5^{s5} + \text{smaller terms} \in I_\lambda \\
\partial^{s5}_{\alpha_2}\partial^{s4}_{\alpha_2+2\alpha_3}\partial^{s3}_{\alpha_2+\alpha_3} f_1^{s1+s3+s4+s5} &= c_1 f_1^{s1} f_3^{s3} f_4^{s4} f_5^{s5} + \text{smaller terms} \in I_\lambda,
\end{aligned}$$

with $c_i \in \mathbb{C} \setminus \{0\}$. In the general case, for arbitrary $n > 3$, we have $N = 2n - 1$. Let $r := \lceil N/2 \rceil$, then $\Delta_+^{\omega_1}$ is given by:

| | | | |
|---|---|---|---|
| $\beta_1 = (1,2,2,\ldots,2)$ | $\beta_2 \;\; = (1,1,2,\ldots,2,2)$ | $\ldots$ | $\beta_{r-1} = (1,1,\ldots,1,2)$ |
| $\beta_r = (1,1,1,\ldots,1)$ | $\beta_{r+1} = (1,1,1,\ldots,1,0)$ | $\ldots$ | $\beta_N \;\; = (1,0,\ldots,0,0)$ |

Then the only $n$–chain has the following form $\beta_{r-1} \xrightarrow{n} \beta_r \xrightarrow{n} \beta_{r+1}$ We change the order from $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_N$ to

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_{r-1} \prec \beta_{r+2} \preceq \cdots \preceq \beta_{N-1} \preceq \beta_{r+1} \prec \beta_N \prec \beta_r.$$

The modifications of the diagram are similar to them in the case of $\mathtt{B_3}$, so the Hasse diagram for a Lie algebra of type $\mathtt{B_n}$ has the following shape

$$\begin{tikzcd}
 & & & & & & & & & \beta_2 \arrow[dr, "012\ldots2"] & \\
\beta_1 \arrow[r, "0110\ldots0"] & \beta_3 \arrow[r, "4"] & \beta_4 \arrow[r, "5"] & \cdots \arrow[r, "n"] & \beta_r \arrow[r, "001\ldots12"] & \beta_{r+1} \arrow[r, "n-1"] & \beta_{r+2} \arrow[r, "n-2"] & \cdots \arrow[r, "4"] & \beta_{N-2} \arrow[ur, "2"] \arrow[dr, "012\ldots2"'] & & \beta_N. \\
 & & & & & & & & & \beta_{N-1} \arrow[ur, "2"'] &
\end{tikzcd}$$

Associated to the diagrams we get the following polytope for $m \in \mathbb{Z}_{\geq 0}$:

$$P(m\omega_1) = \left\{ \mathbf{x} \in \mathbb{R}^N_{\geq 0} \mid \begin{array}{c} x_1 + x_2 + \cdots + x_{N-2} + x_N \leq m \\ x_1 + x_3 + \cdots + x_{N-1} + x_N \leq m \end{array} \right\}.$$

By Section 1.3, Corollary 1.3.5 the elements

$$v_{\omega_1}, f_1 v_{\omega_1}, f_2 v_{\omega_1}, \ldots, f_N v_{\omega_1}, f_2 f_{N-1} v_{\omega_1}$$

span $V(\omega)$ and with [Car05, p. 276] we have $\dim V(\omega_1) = 2n + 1$.

**Proposition 1.4.6.** *The vectors $f^{\mathbf{s}} v_{\omega_1}, \mathbf{s} \in S(\omega_1)$ are a FFL basis of $V(\omega_1)$.* □

*Proof.* The previous observations imply that $\{f^{\mathbf{s}} v_{\omega_1}, \mathbf{s} \in S(\omega_1)\}$ is a basis of $V(\omega_1)$. Thus it remains to show that $P(\omega_1)$ is a normal polytope.
Because we changed the Hasse diagram we have to change the order of the roots to apply Section 1.2. One possible new order is given by:

$$\beta_1 \prec \beta_3 \prec \beta_4 \prec \cdots \prec \beta_{N-2} \prec \beta_2 \prec \beta_{N-1} \prec \beta_N.$$

Using this order we see immediately that $P(\omega_1)$ is a normal polytope. □

Now we consider the case $\omega = \omega_n$. In the following it will be again convenient to describe the roots and fundamental weights of $\mathtt{B_n}$ in terms of an orthogonal basis:

$$\Delta_+^{\omega_n} = \{\varepsilon_{i,j} = \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_k \mid 1 \leq k \leq n\}. \tag{1.4.4}$$

The total order on $\Delta_+^{\omega_n}$ is obtained by considering the Hasse diagram. We begin with $\beta_1 = \theta$ on the top and then labeling from left to right with increasing label on each level of the Hasse diagram, which correspond to the height of the roots in $\Delta_+^{\omega_n}$. For a concrete example see Figure 3.3 in the Appendix. The corresponding polytope is defined as usual, see Table 3.1 for an example. The elements of $\Delta_+^{\omega_n}$ correspond to $\varepsilon_{i,j} = \sum_{r=i}^{j-1} \alpha_r + 2\sum_{r=j}^{n} \alpha_r$ and $\varepsilon_k = \sum_{r=k}^{n} \alpha_r$. The highest weight of $V(\omega_n)$ has the description $\omega_n = \frac{1}{2}\sum_{r=1}^{n} \varepsilon_r$. Further the lowest weight is $-\omega_n = -\frac{1}{2}\sum_{r=1}^{n} \varepsilon_r$. With this observation, the fact that $\omega_n$ is minuscule and (1.4.4) we see that

$$\mathbb{B}_{V(\omega_n)} = \left\{ f_\alpha v_{\omega_n} \mid \alpha = \frac{1}{2}\sum_{r=1}^{n} l_r \varepsilon_r, l_r \in \{-1, 1\},\ \forall 1 \leq r \leq n \right\} \subset V(\omega_n)$$

is a basis. We note that $|\mathbb{B}_{V(\omega_n)}| = 2^n = \dim V(\omega_n)$.

**Remark 1.4.7.** *For an arbitrary element $\overline{\mathbf{p}} \in \overline{D}_{\omega_n}^{\mathtt{B_n}}$ we have at most one root of the form $\varepsilon_k \in \overline{\mathbf{p}}$, because if there are $\varepsilon_{k_1}, \varepsilon_{k_2} \in \overline{\mathbf{p}}$ (wlog $k_1 < k_2$) we have: $\varepsilon_{k_1} - \varepsilon_{k_2} = \sum_{r=k_1}^{k_2-1} \alpha_r$. Thus with Remark 1.1.6 we know that there is a Dyck path $\mathbf{p} \in D_{\omega_n}$ with $\varepsilon_{k_1}, \varepsilon_{k_2} \in \mathbf{p}$. This observation implies that the elements $\overline{\mathbf{p}} \in \overline{D}_{\omega_n}^{\mathtt{B_n}}$ have two possible forms:*

$$(B_1)\ \overline{\mathbf{p}} = \{\varepsilon_k, \varepsilon_{i_2,j_2}, \ldots, \varepsilon_{i_r,j_r}\} \quad or \quad (B_2)\ \overline{\mathbf{p}} = \{\varepsilon_{i_1,j_1}, \ldots, \varepsilon_{i_t,j_t}\}.$$

Thus we can characterize the elements $\overline{\mathbf{p}} \in \overline{D}_{\omega_n}^{\mathtt{B_n}}$ as follows.

**Proposition 1.4.8.** *For $\overline{\mathbf{p}} \in \mathcal{P}(\Delta_+^{\omega_n})$ arbitrary we have:*

$$\overline{\mathbf{p}} \in \overline{D}_{\omega_n}^{\mathtt{B_n}} \Leftrightarrow \begin{cases} \overline{\mathbf{p}} \text{ is of the form } (B_1), \text{ with } (a) \text{ and } (b), \\ \overline{\mathbf{p}} \text{ is of the form } (B_2), \text{ with } (b). \end{cases} \tag{1.4.5}$$

*In addition:* $\overline{\mathbf{p}} \in \overline{D}_{\omega_n}^{\mathtt{B_n}} \Rightarrow \begin{cases} s \leq \lceil \frac{n}{2} \rceil, \ \overline{\mathbf{p}} \text{ is of the form } (B_1), \\ s \leq \lfloor \frac{n}{2} \rfloor, \ \overline{\mathbf{p}} \text{ is of the form } (B_2), \end{cases}$
*with $s = |\overline{\mathbf{p}}|$. The properties (a) and (b) are defined by*

*(a)* $\forall\, 1 \leq l \leq s: \ k < i_l < j_l,$

*(b)* $\forall\, \alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}, i_l \leq i_m : i_l < i_m < j_m < j_l.$

*Proof.* First we prove (1.4.5): "$\Leftarrow$": Let $\mathbf{p} = \{\varepsilon_k, \varepsilon_{i_2,j_2}, \ldots, \varepsilon_{i_s,j_s}\}$ be an element of form $(B_1)$ with the properties $(a)$ and $(b)$. Assume there are two roots $x, y \in \mathbf{p}$ such that there exists a Dyck path $\mathbf{q} \in D_{\omega_n}$ containing them.

**1. Case:** $x = \varepsilon_k$ and $y = \varepsilon_{i_m,j_m}$, for $1 \leq m \leq s$. Then we have

$$\varepsilon_{i_m,j_m} - \varepsilon_k = \sum_{r=i_m}^{j_m-1} \alpha_r + 2\sum_{r=j_m}^{n} \alpha_r - \sum_{r=k}^{n} \alpha_r = -\sum_{r=k}^{i_m-1} \alpha_r + \sum_{r=j_m}^{n} \alpha_r.$$

Hence there is no Dyck path $\mathbf{q} \in D_{\omega_n}$ such that $x$ and $y$ are contained in $\mathbf{q}$. This is a contradiction to the assumption.

**2. Case:** $x = \varepsilon_{i_m,j_m}$ and $y = \varepsilon_{i_l,j_l}$, wlog $i_l < i_m$. Then we have

$$\varepsilon_{i_l,j_l} - \varepsilon_{i_m,j_m} = \sum_{r=i_l}^{j_l-1} \alpha_r + 2\sum_{r=j_l}^{n} \alpha_r - \sum_{r=i_m}^{j_m-1} \alpha_r - 2\sum_{r=j_m}^{n} \alpha_r = \sum_{r=i_l}^{i_m-1} \alpha_r - \sum_{r=j_m}^{j_l-1} \alpha_r.$$

This is a contradiction to our assumption and hence: $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$.

Let $\mathbf{p}$ be of form $(B_2)$ with property $(b)$, and assume there are two roots $x, y \in \mathbf{p}$ such that there exists a Dyck path $\mathbf{q} \in D_{\omega_n}$ containing them. Like in the second case of our previous consideration the assumption is false and therefore: $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$.

"$\Rightarrow$": Let $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$. Then we know from Remark 1.4.7 that $\mathbf{p}$ is of the form $(B_1)$ or $(B_2)$. Let $\mathbf{p} = \{\varepsilon_k, \varepsilon_{i_1,j_1}, \dots, \varepsilon_{i_s,j_s}\}$ be of form $(B_1)$, with $i_l < j_l$ for all $1 \leq l \leq s$.

**1. Step:** Assume $\exists\ 1 \leq m \leq s:\ k > i_m$. Then we have:

$$\varepsilon_{i_m,j_m} - \varepsilon_k = \sum_{r=i_m}^{j_m-1} \alpha_r + 2\sum_{r=j_m}^{n} \alpha_r - \sum_{r=k}^{n} \alpha_r = \sum_{r=i_m}^{k-1} \alpha_r + \sum_{r=j_m}^{n} \alpha_r.$$

Thus by Remark 1.1.6 this contradicts $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$. Hence: $k < i_m$ for all $1 \leq m \leq s$.

Let $\varepsilon_{i_l,j_l}, \varepsilon_{i_m,j_m} \in \mathbf{p}$ be two roots with $\varepsilon_{i_l,j_l} \neq \varepsilon_{i_m,j_m}$. We assume wlog $i_l \leq i_m$.

**2. Step:** Assume $i_l = i_m =: y$. Set $x = \min\{j_l, j_m\}$ and $\bar{x} = \max\{j_l, j_m\}$:

$$\varepsilon_{y,x} - \varepsilon_{y,\bar{x}} = \sum_{r=y}^{x-1} \alpha_r + 2\sum_{r=x}^{n} \alpha_r - \sum_{r=y}^{\bar{x}-1} \alpha_r - 2\sum_{r=\bar{x}}^{n} \alpha_r = \sum_{r=x}^{\bar{x}} \alpha_r.$$

Again by Remark 1.1.6 this contradicts $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$ and we have: $i_l < i_m$.

**3. Step:** Let $i_l < i_m$ and assume $j_l = j_m =: x$, we consider:

$$\varepsilon_{i_l,x} - \varepsilon_{i_m,x} = \sum_{r=i_l}^{x} \alpha_r + 2\sum_{r=x}^{n} \alpha_r - \sum_{r=i_m}^{x} \alpha_r - 2\sum_{r=x}^{n} \alpha_r = \sum_{r=i_l}^{i_m-1} \alpha_r.$$

This contradicts $\mathbf{p} \in \overline{D}_{\omega_n}^{\mathsf{B_n}}$ by Remark 1.1.6, so: $j_l \neq j_m$.

**4. Step:** $(i_l < i_m < j_m) \wedge (i_l < j_l)$. Thus there are three possible cases:

(a) $i_l < j_l < i_m < j_m$, (b) $i_l < i_m < j_l < j_m$ and (c) $i_l < i_m < j_m < j_l$.

Let us assume $\varepsilon_{i_l,j_l}$ and $\varepsilon_{i_m,j_m}$ have the property of case (a):

$$\varepsilon_{i_l,j_l}-\varepsilon_{i_m,j_m} = \sum_{r=i_l}^{j_l-1}\alpha_r+2\sum_{r=j_l}^{n}\alpha_r-\sum_{r=i_m}^{j_m-1}\alpha_r-2\sum_{r=j_m}^{n}\alpha_r = \sum_{r=i_l}^{j_m-1}\alpha_r+2\sum_{r=j_l}^{i_m-1}\alpha_r+\sum_{r=i_m}^{j_m-1}\alpha_r.$$

This contradicts $\mathbf{p}\in\overline{D}^{\mathsf{B_n}}_{\omega_n}$ by Remark 1.1.6. We assume now that $\varepsilon_{i_l,j_l}$ and $\varepsilon_{i_m,j_m}$ have the property of case (b):

$$\varepsilon_{i_l,j_l}-\varepsilon_{i_m,j_m} = \sum_{r=i_l}^{j_l-1}\alpha_r+2\sum_{r=j_l}^{n}\alpha_r-\sum_{r=i_m}^{j_m-1}\alpha_r-2\sum_{r=j_m}^{n}\alpha_r = \sum_{r=i_l}^{i_m-1}\alpha_r+\sum_{r=j_l}^{j_m-1}\alpha_r.$$

Again by Remark 1.1.6 this contradicts $\mathbf{p}\in\overline{D}^{\mathsf{B_n}}_{\omega_n}$. Finally we conclude that two roots $\varepsilon_{i_l,j_l},\varepsilon_{i_m,j_m}\in\mathbf{p}$, with $i_l\le i_l$, satisfy (c): $i_l<i_m<j_m<j_l$. To prove this statement for a $\mathbf{p}\in\overline{D}^{\mathsf{B_n}}_{\omega_n}$ of form $(B_2)$ we only have to restrict our consideration to the second, third and fourth step.

It remains to show that the cardinality $s$ of $\mathbf{p}$ is bounded by $\lceil\frac{n}{2}\rceil$ respectively $\lfloor\frac{n}{2}\rfloor$. Again we consider the two possible cases:

**1. Case:** $\mathbf{p}=\{\varepsilon_k,\varepsilon_{i_2,j_2},\dots,\varepsilon_{i_s,j_s}\}$ is of the form $(B_1)$ and we assume $|\mathbf{p}|=s>\lceil\frac{n}{2}\rceil$. Then we know from our previous consideration that after reordering the roots in $\mathbf{p}$ we have a strictly increasing chain of integers:

$$C_{\mathbf{p}}:\ k<i_2<i_3\dots<i_s<j_s<j_{s-1}<\dots<j_3<j_2. \tag{1.4.6}$$

Thus there are $2s-1$ different integers, where each of these correspond to a $\varepsilon_i$ for $1\le i\le n$. By assumption we know $2s-1\ge 2(\lceil\frac{n}{2}\rceil+1)-1\ge n+1$, but there are only $n$ different elements in $\{\varepsilon_r\mid 1\le r\le n\}$. Thus this is a contradiction and hence: $|\mathbf{p}|=s\le\lceil\frac{n}{2}\rceil$.

**2. Case:** $\mathbf{p}=\{\varepsilon_{i_1,j_1},\dots,\varepsilon_{i_s,j_s}\}$ is of the form $(B_2)$ and we assume $|\mathbf{p}|=s>\lfloor\frac{n}{2}\rfloor$. As in the first case we have a strictly increasing chain of integers:

$$C_{\mathbf{p}}:\ i_1<i_2\dots<i_s<j_s<j_{s-1}<\dots<j_2<j_1. \tag{1.4.7}$$

Therefore, we have $2s$ different integers corresponding to at most $n$ different elements in $\{\varepsilon_r\mid 1\le r\le n\}$, but by assumption we have $2s\ge 2(\lfloor\frac{n}{2}\rfloor+1)\ge n+1$. Again we have a contradiction and therefore: $|\mathbf{p}|=s\le\lfloor\frac{n}{2}\rfloor$. $\square$

Because of Corollary 1.3.5 we know that the elements $\{f^{\mathbf{s}}v_{\omega_n}\mid \mathbf{s}\in S(\omega_n)\}$ span $V(\omega_n)$ and by Proposition 1.1.8 there is a bijection between $S(\omega_n)$ and $\overline{D}^{\mathsf{B_n}}_{\omega_n}$. We show that these elements are linear independent. To achieve that we show that $|\overline{D}^{\mathsf{B_n}}_{\omega_n}|=\dim V(\omega_n)$. To be more explicit:

**Proposition 1.4.9.** $|\overline{D}^{\mathsf{B_n}}_{\omega_n}|=\dim V(\omega_n)=2^n$.

*Proof.* We know from (1.4.5) that for an arbitrary element $\overline{\mathbf{p}}\in\overline{D}^{\mathsf{B_n}}_{\omega_n}$ the number of roots $s$ in $\overline{\mathbf{p}}$ is bounded by $\lceil\frac{n}{2}\rceil$ respective by $\lfloor\frac{n}{2}\rfloor$. Therefore, the number of integers occurring in $C_{\overline{\mathbf{p}}}$ (see (1.4.6) and (1.4.7)) is also bounded:

$$|C_{\overline{\mathbf{p}}}|=\begin{cases}2s-1\le 2\lceil\frac{n}{2}\rceil-1\le n, & \overline{\mathbf{p}}\text{ is of the form }(B_1),\\ 2s\le 2\lfloor\frac{n}{2}\rfloor\le n, & \overline{\mathbf{p}}\text{ is of the form }(B_2).\end{cases}$$

In order to simplify our notation, we define $l := |C_{\overline{\mathbf{p}}}|$, so we have for an arbitrary $\overline{\mathbf{p}} \in \overline{D}^{\mathtt{B_n}}_{\omega_n}$: $0 \leq l \leq n$. Further we define the subsets $\overline{D}^{\mathtt{B_n}}_{\omega_n}(l) \subset \overline{D}^{\mathtt{B_n}}_{\omega_n}$:

$$\overline{D}^{\mathtt{B_n}}_{\omega_n}(l) := \{\overline{\mathbf{p}} \in \overline{D}^{\mathtt{B_n}}_{\omega_n} \mid |C_{\overline{\mathbf{p}}}| = l\}, \ \forall \ 0 \leq l \leq n.$$

Therefore, the elements in $\overline{D}^{\mathtt{B_n}}_{\omega_n}(l)$ are parametrized by $l$ totally ordered integers $u_i$ in $\{r \mid 1 \leq r \leq n\}$, $\forall \, 1 \leq i \leq l$. Hence we conclude: $|\overline{D}^{\mathtt{B_n}}_{\omega_n}(l)| \leq \binom{n}{l}$, $\forall \, 1 \leq l \leq n$ and so

$$|\overline{D}^{\mathtt{B_n}}_{\omega_n}| = |\bigcup_{l=1}^{n} \overline{D}^{\mathtt{B_n}}_{\omega_n}(l)| = \sum_{l=0}^{n} |\overline{D}^{\mathtt{B_n}}_{\omega_n}(l)| \leq \sum_{l=0}^{n} \binom{n}{l} = 2^n.$$

We also know from Corollary 1.3.5 that we have $|\overline{D}^{\mathtt{B_n}}_{\omega_n}| \geq \dim V(\omega_n) = \binom{n}{l} = 2^n$. Finally we conclude: $|\overline{D}^{\mathtt{B_n}}_{\omega_n}| = 2^n$. □

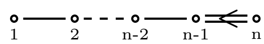
**Example 1.4.10.** *The polytope $P(m\omega_3)$ in the case $\mathfrak{g} = \mathfrak{so}_7$ has the following shape.*

$$P(m\omega_3) = \left\{ \mathbf{x} \in \mathbb{R}^6_{\geq 0} \mid \begin{array}{c} x_1 + x_2 + x_3 + x_5 + x_6 \leq m \\ x_1 + x_2 + x_4 + x_5 + x_6 \leq m \end{array} \right\}.$$

Proposition 1.4.9 implies immediately:

**Proposition 1.4.11.** *The vectors $f^{\mathbf{s}} v_{\omega_n}, \mathbf{s} \in S(\omega_n)$ are a FFL basis of $V(\omega_n)$.* □

**Type $\mathtt{C_n}$.** Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{C_n}$ for $n \geq 3$ with the associated Dynkin diagram

$\mathtt{C_n}$ 

For all fundamental weights $\omega_k$ we have $\omega_k(\theta^\vee) = 1$, where $\theta = (2, 2, \ldots, 2, 1)$ is the highest root and $\theta^\vee = (1, 1, \ldots, 1)$ the corresponding coroot. But only for $\omega_1$ the associated Hasse diagram $H(\mathfrak{n}^-_{\omega_1})_{\mathfrak{g}}$ has no $i$–chains. In fact for $1 \leq k \leq n$, $H(\mathfrak{n}^-_{\omega_k})_{\mathfrak{g}}$ has $k-1$ different $i$–chains, with $1 \leq i \leq k-1$. The following example explains, why we are not able to rewrite the diagram in these cases, with our approach.
For all $\omega_k$ with $k \neq 1$ we have the following 1–chain.

$$\beta_1 \xrightarrow{1} \beta_2 \xrightarrow{1} \beta_3.$$

Here $\beta_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$ is the highest root, $\beta_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ and $\beta_3 = 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. Note that $\beta_1 - \beta_3 = 2\alpha_1$, which is not a root. Further, because $\beta_1$ is the highest root, there are no roots $\gamma \in \Delta_+, \nu \in \Delta^{\omega_k}_+$ with $\partial_\gamma f_\nu = f_3$, except for $\nu = \beta_2$. Hence it is more involved to rewrite the diagram into a diagram without $k$–chains such that there is a path connecting $\beta_1$ and $\beta_3$. Nevertheless, in [FFL11b] similar statements to Theorem A and Theorem B were proven for arbitrary dominant integral weights.

Now we consider $\omega = \omega_1$. Then we have $2n - 1 = N$ and $\Delta^\omega_+$ is given by

| | | | |
|---|---|---|---|
| $\beta_1 \quad = (2, 2, \ldots, 2, 1)$ | $\beta_2 \ = (1, 2, \ldots, 2, 1)$ | $\ldots$ | $\beta_n = (1, 1, \ldots, 1, 1)$ |
| $\beta_{n+1} = (1, 1, \ldots, 1, 0)$ | $\beta_{n+2} = (1, \ldots, 1, 0, 0)$ | $\ldots$ | $\beta_N = (1, 0, \ldots, 0, 0)$ |

The diagram $H(\mathfrak{n}_\omega^-)_\mathfrak{g}$ has the following form.

$$\beta_1 \xrightarrow{1} \beta_2 \xrightarrow{2} \beta_3 \xrightarrow{3} \cdots \xrightarrow{\text{n-2}} \beta_{n-1} \xrightarrow{\text{n-1}} \beta_n \xrightarrow{\text{n}} \beta_{n+1} \xrightarrow{\text{n-1}} \cdots \xrightarrow{2} \beta_N.$$

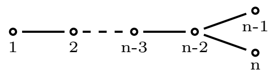
There are no $k$–chains and the associated polytope is given by

$$P(m\omega) = \{\mathbf{x} \in \mathbb{R}^N_{\geq 0} \mid x_1 + x_2 + \cdots + x_N \leq m\}.$$

By Corollary 1.3.5 the elements $v_\omega, f_1 v_\omega, f_2 v_\omega, \ldots, f_N v_\omega$ span $V(\omega)$ and with [Car05, p295] we know $\dim V(\omega) = 2n$. From these observations we get immediately:

**Proposition 1.4.12.** *The set $\mathbb{B}_\omega = \{f^\mathbf{s} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a FFL basis of $V(\omega)$.* $\square$

**Type $\mathtt{D_n}$.** Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{D_n}$ with associated Dynkin diagram

$\mathtt{D_n}$ 


The highest root in type $\mathtt{D_n}$ is of the form $\theta = \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n$. Since $\mathfrak{g}$ is simply-laced we have $\theta^\vee = \alpha_1^\vee + 2\sum_{i=2}^{n-2} \alpha_i^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$. Hence $\omega(\theta^\vee) = 1 \Leftrightarrow \omega \in \{\omega_1, \omega_{n-1}, \omega_n\}$.

First we consider the case $\omega = \omega_1$. Then we have $2n - 2 = N$ and $\Delta_+^{\omega_1}$ has the following form:

| | | | |
|---|---|---|---|
| $\beta_1 = (1, 2, 2 \ldots, 2, 1, 1)$ | $\beta_2 = (1, 1, 2, \ldots, 2, 1, 1)$ | $\ldots$ | $\beta_{n-2} = (1, 1, 1 \ldots, 1, 1, 1)$ |
| $\beta_{n-1} = (1, 1, 1 \ldots, 1, 0, 1)$ | $\beta_n = (1, 1, 1, \ldots, 1, 1, 0)$ | $\ldots$ | $\beta_N = (1, 0, 0 \ldots, 0, 0, 0)$ |

The Hasse diagram has no $k$–chain. In addition in $\overline{D}_{\omega_1}$ there are only co–chains of cardinality at most 1, except for one with cardinality 2.

$$\begin{tikzcd}[column sep=small] & & & & & \beta_{n-1} \arrow[dr, "\text{n}"] & & & & & \\ \beta_1 \arrow[r, "2"] & \beta_2 \arrow[r, "3"] & \beta_3 \arrow[r, "4"] & \cdots \arrow[r, "\text{n-2}"] & \beta_{n-2} \arrow[ur, "\text{n-1}"] \arrow[dr, "\text{n}"'] & & \beta_{n+1} \arrow[r, "\text{n-2}"] & \beta_{n+2} \arrow[r, "\text{n-3}"] & \cdots \arrow[r, "2"] & \beta_N. \\ & & & & & \beta_n \arrow[ur, "\text{n-1}"'] & & & & & \end{tikzcd}$$

Associated to this diagram we get the following polytope for $m \in \mathbb{Z}_{\geq 0}$:

$$P(m\omega) = \left\{ \mathbf{x} \in \mathbb{R}^N_{\geq 0} \mid \begin{array}{l} x_1 + \cdots + x_{n-2} + x_{n-1} + x_{n+1} + \cdots + x_N \leq m \\ x_1 + \cdots + x_{n-2} + x_n \quad + x_{n+1} + \cdots + x_N \leq m \end{array} \right\}.$$

By Corollary 1.3.5 the elements $\mathbb{B}_{\omega_1} = \{v_{\omega_1}, f_1 v_{\omega_1}, f_2 v_{\omega_1}, \ldots, f_N v_{\omega_1}, f_{n-1} f_n v_{\omega_1}\}$ span $V(\omega_1)$ and with [Car05, p. 280] we have $\dim V(\omega_1) = 2n$. From these observations we get immediately.

**Proposition 1.4.13.** *The vectors $f^\mathbf{s} v_{\omega_1}, \mathbf{s} \in S(\omega_1)$ are a FFL basis of $V(\omega_1)$.* $\square$

For most of the proofs of the statements in the case $\omega = \omega_{n-1}, \omega_n$ we refer to the proofs of the corresponding statements for type $\mathtt{B_n}$.

Now we consider the case $\omega = \omega_{n-1}$. For further considerations it will be convenient to describe the roots and fundamental weights of $\mathfrak{g}$ in terms of an orthogonal basis $\{\varepsilon_i \mid 1 \le i \le n\}$. Then $\Delta_+^{\omega_{n-1}}$ is given by

$$\{\varepsilon_{i,j} = \varepsilon_i + \varepsilon_j \mid 1 \le i < j \le n-1\} \cup \{\varepsilon_{k,\overline{n}} = \varepsilon_k - \varepsilon_n \mid 1 \le k \le n-1\}. \quad (1.4.8)$$

The total order on $\Delta_+^{\omega_{n-1}}$ is defined like in the $\mathtt{B_n}$, $\omega_n$-case (see Figure 3.3). The elements of $\Delta_+^{\omega_{n-1}}$ correspond to $\varepsilon_{i,j} = \sum_{r=i}^{j-1} \alpha_r + 2\sum_{r=j}^{n-2} \alpha_r + \alpha_{n-1} + \alpha_n$ and $\varepsilon_{k,\overline{n}} = \sum_{r=k,}^{n-1} \alpha_r$. The highest weight of $V(\omega_{n-1})$ has the description $\omega_{n-1} = \frac{1}{2}\left(\sum_{r=1}^{n-1} \varepsilon_r - \varepsilon_n\right)$. Further the lowest weight is $-\omega_{n-1} = -\frac{1}{2}\left(\sum_{r=1}^{n-1} \varepsilon_r - \varepsilon_n\right)$. With this observation, the fact that $\omega_{n-1}$ is minuscule and (1.4.8) we see that

$$\mathbb{B}_{V(\omega_{n-1})} = \left\{ f_\alpha v_{\omega_{n-1}} \mid \alpha = \frac{1}{2}\sum_{r=1}^{n} l_r \varepsilon_r, l_r = \pm 1, \ \forall 1 \le r \le n, \ 2 \nmid \#\{l_r \mid l_r = -1\} \right\}$$

is a basis of $V(\omega_{n-1})$. We note that $|\mathbb{B}_{V(\omega_{n-1})}| = 2^{n-1} = \dim V(\omega_{n-1})$.

**Remark 1.4.14.** *Similar arguments as in Remark 1.4.7 show that the elements $\overline{\mathbf{p}} \in \overline{D}_{\omega_{n-1}}^{\mathtt{D_n}}$ have two possible forms:*

$$(D_1)\ \overline{\mathbf{p}} = \{\varepsilon_{k,\overline{n}}, \varepsilon_{i_2,j_2}, \ldots, \varepsilon_{i_r,j_r}\} \quad \textit{or} \quad (D_2)\ \overline{\mathbf{p}} = \{\varepsilon_{i_1,j_1}, \ldots, \varepsilon_{i_t,j_t}\}.$$

We denote with $\mathbf{1}_{2\nmid n} : \mathbb{Z}_{\ge 0} \to \{0,1\}$ (respective $\mathbf{1}_{2|n}$) the Indicator function for the odd (respective even) integers, which is defined by $\mathbf{1}_{2\nmid n}(n) = 1$ if $2 \nmid n$ (respective $\mathbf{1}_{2|n}(n) = 1$ if $2 \mid n$) and 0 otherwise. Therefore, we can characterize the elements $\overline{\mathbf{p}} \in \overline{D}_{\omega_{n-1}}^{\mathtt{D_n}}$ as follows

**Proposition 1.4.15.** *For $\overline{\mathbf{p}} \in \mathcal{P}(\Delta_+^{\omega_{n-1}})$ arbitrary we have:*

$$\overline{\mathbf{p}} \in \overline{D}_{\omega_{n-1}}^{\mathtt{D_n}} \Leftrightarrow \begin{cases} \overline{\mathbf{p}} \textit{ is of the form } (D_1), \textit{ with (a) and (b)}, \\ \overline{\mathbf{p}} \textit{ is of the form } (D_2), \textit{ with (b)}. \end{cases} \quad (1.4.9)$$

*In addition:* $\overline{\mathbf{p}} \in \overline{D}_{\omega_{n-1}}^{\mathtt{D_n}} \Rightarrow \begin{cases} s \le \lceil \frac{n}{2} \rceil - \mathbf{1}_{2\nmid n}(n), \ \overline{\mathbf{p}} \textit{ is of the form } (D_1), \\ s \le \lfloor \frac{n}{2} \rfloor - \mathbf{1}_{2|n}(n), \ \overline{\mathbf{p}} \textit{ is of the form } (D_2), \end{cases}$

*with $s = |\overline{\mathbf{p}}|$. The properties (a) and (b) are defined by*

*(a)* $\forall 1 \le l \le s : \ k < i_l < j_l$,

*(b)* $\forall \alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}, i_l \le i_m : i_l < i_m < j_m < j_l$.

*Proof.* To prove this statement we adapt the idea of Proposition 1.4.8. We use exactly the same approach but we consider $\Delta_+^{\omega_{n-1}}$ of type $\mathtt{D_n}$.
To check that that the cardinality $s$ of an arbitrary element $\overline{\mathbf{p}} \in \overline{D}_{\omega_{n-1}}^{\mathtt{D_n}}$ is bounded, like we claim on the rhs of (1.4.9), we use only fundamental combinatorics, again analogue to the idea of the proof of Proposition 1.4.8. □

Because of Corollary 1.3.5 we know that the elements $\{f^{\mathbf{s}}v_{\omega_{n-1}} \mid \mathbf{s} \in S(\omega_{n-1})\}$ span $V(\omega_{n-1})$ and by Proposition 1.1.8 there is a bijection between $S(\omega_{n-1})$ and $\overline{D}^{\mathtt{D_n}}_{\omega_{n-1}}$. We show that these elements are linear independent. To achieve that we show that $|\overline{D}^{\mathtt{D_n}}_{\omega_{n-1}}| = \dim V(\omega_{n-1})$. To be more explicit:

**Proposition 1.4.16.** $|\overline{D}^{\mathtt{D_n}}_{\omega_{n-1}}| = \dim V(\omega_{n-1}) = 2^{n-1}$.

*Proof.* This is a direct consequence of Lemma 1.4.22 and Proposition 1.4.9. □

Proposition 1.4.16 implies immediately

**Proposition 1.4.17.** $\mathbb{B}_{\omega_{n-1}} = \{f^{\mathbf{s}}v_{\omega_{n-1}} \mid \mathbf{s} \in S(\omega_{n-1})\}$ *is a basis for* $V(\omega_{n-1})$.□

Finally we consider the case $\omega = \omega_n$. For the proofs of the statements in this case we refer to the proofs of the analogous statements in the previous case $\omega = \omega_{n-1}$ and the $\mathtt{B_n}, \omega_n$-case.
The set of roots $\Delta_+^{\omega_n}$, where $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$ is a summand, is given by:

$$\{\varepsilon_{i,j} = \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n-1\} \cup \{\varepsilon_{k,n} = \varepsilon_k + \varepsilon_n \mid 1 \leq k \leq n-1\}.$$

Again the total order on $\Delta_+^{\omega_n}$ is defined like in the $\mathtt{B_n}, \omega_n$-case (see Figure 3.3), where the elements of $\Delta_+^{\omega_n}$ correspond to $\varepsilon_{i,j} = \sum_{r=i}^{j-1} \alpha_r + 2\sum_{r=j}^{n-2} \alpha_r + \alpha_{n-1} + \alpha_n$ and $\varepsilon_{k,n} = \sum_{r=k,\ r\neq n-1}^{n} \alpha_r$. The highest weight of $V(\omega_n)$ has the description $\omega_n = \frac{1}{2}\left(\sum_{r=1}^{n} \varepsilon_r\right)$. Further the lowest weight is $-\omega_n = -\frac{1}{2}\left(\sum_{r=1}^{n} \varepsilon_r\right)$. As before we see that

$$\mathbb{B}_{V(\omega_n)} = \left\{ f_\alpha v_{\omega_n} \mid \alpha = \frac{1}{2}\sum_{r=1}^{n} l_r\varepsilon_r, l_r\{-1,1\},\ \forall 1 \leq r \leq n,\ 2 \mid \#\{l_r \mid l_r = -1\} \right\}$$

is a basis of $V(\omega_n)$. We note that $|\mathbb{B}_{\omega_n}| = 2^{n-1} = \dim V(\omega_n)$.

**Remark 1.4.18.** *Similar arguments as in Remark 1.4.7 show that the elements* $\overline{\mathbf{p}} \in \overline{D}^{\mathtt{D_n}}_{\omega_n}$ *have two possible forms:*

$$(D_1^*)\ \overline{\mathbf{p}} = \{\varepsilon_{k,n}, \varepsilon_{i_2,j_2}, \ldots, \varepsilon_{i_s,j_s}\} \quad \textit{and} \quad (D_2^*)\ \overline{\mathbf{p}} = \{\varepsilon_{i_1,j_1}, \ldots, \varepsilon_{i_s,j_s}\}.$$

Therefore, we can characterize the elements $\overline{\mathbf{p}} \in \overline{D}^{\mathtt{D_n}}_{\omega_n}$ as follows:

**Proposition 1.4.19.** *For* $\overline{\mathbf{p}} \in \mathcal{P}(\Delta_+^{\omega_n})$ *arbitrary we have:*

$$\overline{\mathbf{p}} \in \overline{D}^{\mathtt{D_n}}_{\omega_n} \Leftrightarrow \begin{cases} \overline{\mathbf{p}} \text{ is of the form } (D_1^*), \text{ with (a) and (b)}, \\ \overline{\mathbf{p}} \text{ is of the form } (D_2^*), \text{ with (b)}. \end{cases}$$

*In addition:* $\overline{\mathbf{p}} \in \overline{D}^{\mathtt{D_n}}_{\omega_n} \Rightarrow \begin{cases} s \leq \lceil \frac{n}{2} \rceil - \mathbf{1}_{2\nmid n}(n),\ \overline{\mathbf{p}} \text{ is of the form } (D_1^*), \\ s \leq \lfloor \frac{n}{2} \rfloor - \mathbf{1}_{2\mid n}(n),\ \overline{\mathbf{p}} \text{ is of the form } (D_2^*), \end{cases}$
*with* $s = |\overline{\mathbf{p}}|$. *The properties (a) and (b) are defined by*

*(a)* $\forall\, 1 \leq l \leq s:\ k < i_l < j_l$,

*(b)* $\forall\, \alpha_{i_l,j_l}, \alpha_{i_m,j_m} \in \overline{\mathbf{p}}, i_l \leq i_m : i_l < i_m < j_m < j_l$.

*Proof.* To prove this statement we refer to the proof of Proposition 1.4.15. □

Because of Corollary 1.3.5 we know that the elements of $\overline{D}^{\mathtt{D_n}}_{\omega_n}$ span the highest weight module $V(\omega_n)$. But we still have to show that these elements are linear independent. To achieve that we show:

**Proposition 1.4.20.** $|\overline{D}^{\mathtt{D_n}}_{\omega_n}| = \dim V(\omega_n) = 2^{n-1}$.

*Proof.* This is a direct consequence of Lemma 1.4.22 and Proposition 1.4.9. □

Proposition 1.4.20 implies immediately

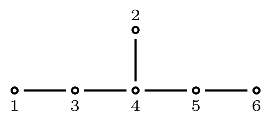
**Proposition 1.4.21.** *The set* $\mathbb{B}_{\omega_n} = \{f^{\mathbf{s}} v_{\omega_n} \mid \mathbf{s} \in S(\omega_n)\}$ *is a basis for* $V(\omega_n)$. □

The following Lemma gives us a very useful connection between the co–chains of $\mathfrak{g}$ of type $\mathtt{B_{n-1}}$ and $\mathtt{D_n}$:

**Lemma 1.4.22.** *We have:* $|\overline{D}^{\mathtt{D_n}}_{\omega_{n-1}}| = |\overline{D}^{\mathtt{B_{n-1}}}_{\omega_{n-1}}|$ *and* $|\overline{D}^{\mathtt{D_n}}_{\omega_n}| = |\overline{D}^{\mathtt{B_{n-1}}}_{\omega_{n-1}}|$.

*Proof.* We only use basic combinatorics to prove this statement. □

**Type $\mathtt{E_6}$.** Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathtt{E_6}$ with associated Dynkin diagram

$\mathtt{E_6}$ 


We have $\omega(\theta^\vee) = 1 \Leftrightarrow \omega = \omega_1, \omega_6$ and first we fix $\omega$ to be $\omega_6$. The set is $\Delta_+^{\omega_6}$ given as follows:

| | |
|---|---|
| $\beta_1 = (1,2,2,3,2,1)$ | $\beta_9 = (1,1,1,1,1,1)$ |
| $\beta_2 = (1,1,2,3,2,1)$ | $\beta_{10} = (0,1,1,1,1,1)$ |
| $\beta_3 = (1,1,2,2,2,1)$ | $\beta_{11} = (1,0,1,1,1,1)$ |
| $\beta_4 = (1,1,1,2,2,1)$ | $\beta_{12} = (0,0,1,1,1,1)$ |
| $\beta_5 = (1,1,2,2,1,1)$ | $\beta_{13} = (0,1,0,1,1,1)$ |
| $\beta_6 = (0,1,1,2,2,1)$ | $\beta_{14} = (0,0,0,1,1,1)$ |
| $\beta_7 = (1,1,1,2,1,1)$ | $\beta_{15} = (0,0,0,0,1,1)$ |
| $\beta_8 = (0,1,1,2,1,1)$ | $\beta_{16} = (0,0,0,0,0,1)$ |

The Hasse diagram $H(\mathfrak{n}^-_{\omega_6})_{\mathtt{E_6}}$ has no $k$–chains and the maximal cardinality of a co–chain of $H(\mathfrak{n}^-_{\omega_6})_{\mathtt{E_6}}$ is two (see Appendix, Figure 3.4). The associated polytope is given for $m \in \mathbb{Z}_{\geq 0}$ by:

$$P(m\omega_6) = \{\mathbf{x} \in \mathbb{R}^{16}_{\geq 0} \mid \sum_{\beta_j \in \mathbf{p}} x_j \leq m, \ \forall \mathbf{p} \in D_{\omega_6}\},$$

in particular see Appendix, Table 3.2 for the non-redundant inequalities.

**Proposition 1.4.23.** *The set* $\mathbb{B}_{\omega_6} = \{f^{\mathbf{s}} v_{\omega_6} \mid \mathbf{s} \in S(\omega_6)\}$ *is a FLL basis of* $V(\omega_6)$.

*Proof.* The co–chains of the Hasse diagram give us immediately:

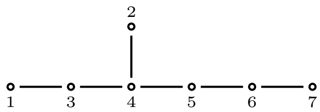
$$\begin{aligned}\mathbb{B}_{\omega_6} = \{&v_{\omega_6}, f_1 v_{\omega_6}, f_2 v_{\omega_6}, \dots, f_{16} v_{\omega_6}, f_4 f_5 v_{\omega_6}, f_5 f_6 v_{\omega_6}, f_6 f_7 v_{\omega_6}, f_6 f_9 v_{\omega_6},\\ &f_8 f_9 v_{\omega_6}, f_8 f_{10} v_{\omega_6}, f_8 f_{11} v_{\omega_6}, f_{10} f_{11} v_{\omega_6}, f_{11} f_{13} v_{\omega_6}, f_{12} f_{13} v_{\omega_6}\}.\end{aligned}$$

Note that there are 27 elements in $\mathbb{B}_{\omega_6}$. By Corollary 1.3.5, we get that $\mathbb{B}_{\omega_6}$ is a spanning set of $V(\omega_6)$. By [Car05, p. 303] we have $\dim V(\omega_6) = 27$ and therefore the claim holds. $\square$

It is shown in Figure 3.4 that the Hasse diagrams $H(\mathfrak{n}^-_{\omega_1})_{\mathtt{E}_6}$ and $H(\mathfrak{n}^-_{\omega_6})_{\mathtt{E}_6}$ have a very similar shape. Therefore, with same arguments as above we conclude:

**Proposition 1.4.24.** *The vectors $f^{\mathbf{s}} v_{\omega_1}$, $\mathbf{s} \in S(\omega_1)$ are a FLL basis of $V(\omega_1)$.* $\square$

**Type $\mathtt{E}_7$.** Let $\mathfrak{g}$ be the simple Lie algebra of type $\mathtt{E}_7$ with associated Dynkin diagram

E7 


In this case $\omega = \omega_7$ is the only fundamental weight satisfying $\omega(\theta^\vee) = 1$.

| | | |
|---|---|---|
| $\beta_1 = (2,2,3,4,3,2,1)$ | $\beta_{10} = (1,1,2,3,2,1,1)$ | $\beta_{19} = (1,1,1,1,1,1,1)$ |
| $\beta_2 = (1,2,3,4,3,2,1)$ | $\beta_{11} = (1,1,1,2,2,2,1)$ | $\beta_{20} = (0,1,1,1,1,1,1)$ |
| $\beta_3 = (1,2,2,4,3,2,1)$ | $\beta_{12} = (1,1,2,2,2,1,1)$ | $\beta_{21} = (1,0,1,1,1,1,1)$ |
| $\beta_4 = (1,2,2,3,3,2,1)$ | $\beta_{13} = (0,1,1,2,2,2,1)$ | $\beta_{22} = (0,0,1,1,1,1,1)$ |
| $\beta_5 = (1,1,2,3,3,2,1)$ | $\beta_{14} = (1,1,1,2,2,1,1)$ | $\beta_{23} = (0,1,0,1,1,1,1)$ |
| $\beta_6 = (1,2,2,3,2,2,1)$ | $\beta_{15} = (1,1,2,2,1,1,1)$ | $\beta_{24} = (0,0,0,1,1,1,1)$ |
| $\beta_7 = (1,1,2,3,2,2,1)$ | $\beta_{16} = (0,1,1,2,2,1,1)$ | $\beta_{25} = (0,0,0,0,1,1,1)$ |
| $\beta_8 = (1,2,2,3,2,1,1)$ | $\beta_{17} = (1,1,1,2,1,1,1)$ | $\beta_{26} = (0,0,0,0,0,1,1)$ |
| $\beta_9 = (1,1,2,2,2,2,1)$ | $\beta_{18} = (0,1,1,2,1,1,1)$ | $\beta_{27} = (0,0,0,0,0,0,1)$ |

As in the $\mathtt{E}_6$-case the Hasse diagram has no $k$–chains. In addition there are only co–chains of cardinality at most 2, except for one with cardinality 3 (see Appendix, Figure 3.5). As before the polytope is defined by the paths in the Hasse diagram. For $m \in \mathbb{Z}_{\geq 0}$ we have:

$$P(m\omega) = \{\mathbf{x} \in \mathbb{R}^{27}_{\geq 0} \mid \sum_{\beta_j \in \mathbf{p}} x_j \leq m,\ \forall \mathbf{p} \in D_\omega\}.$$

Because the polytope is defined by 77 non-redundant inequalities we will not state it explicitly.

**Proposition 1.4.25.** *The set $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a FFL basis of $V(\omega)$.*

*Proof.* The co–chains of the Hasse diagram give us immediately:

$$\begin{aligned}\mathbb{B}_\omega = \{&v_\omega, f_1 v_\omega, f_2 v_\omega, \dots, f_{27} v_\omega, f_5 f_6 v_\omega, f_5 f_8 v_\omega, f_7 f_8 v_\omega, f_8 f_9 v_\omega,\\ &f_9 f_{10} v_\omega, f_8 f_{11} v_\omega, f_{10} f_{11} v_\omega, f_{11} f_{12} v_\omega, f_8 f_{13} v_\omega, f_{10} f_{13} v_\omega,\\ &f_{12} f_{13} v_\omega, f_{13} f_{14} v_\omega, f_{11} f_{15} v_\omega, f_{13} f_{15} v_\omega, f_{14} f_{15} v_\omega, f_{15} f_{16} v_\omega,\\ &f_{13} f_{17} v_\omega, f_{16} f_{17} v_\omega, f_{13} f_{19} v_\omega, f_{16} f_{19} v_\omega, f_{18} f_{19} v_\omega, f_{13} f_{21} v_\omega,\\ &f_{16} f_{21} v_\omega, f_{18} f_{21} v_\omega, f_{20} f_{21} v_\omega, f_{21} f_{23} v_\omega, f_{22} f_{23} v_\omega, f_{13} f_{14} f_{15} v_\omega\}.\end{aligned}$$

Note that there are 56 elements in $\mathbb{B}_\omega$. By Corollary 1.3.5, we get that this is a spanning set of $V(\omega)$. By [Car05, p. 303] we have $\dim V(\omega) = 56$ and therefore that $\mathbb{B}_\omega$ is a basis. $\square$

**Type $\mathtt{F}_4$.** Let $\mathfrak{g}$ be the simple Lie algebra of type $\mathtt{F}_4$ with associated Dynkin diagram

$$\mathtt{F}_4 \qquad \underset{1}{\circ} - \underset{2}{\circ} \Leftarrow \underset{3}{\circ} - \underset{4}{\circ}$$

The highest root is of the form $\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_4 + 2\alpha_4$. And we have $\theta^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$. Thus $\omega(\theta^\vee) = 1 \Leftrightarrow \omega = \omega_4$, so we consider the case $\omega = \omega_4$. If we construct $H(\mathfrak{n}_\omega^-)_{\mathtt{F}_4}$ as in Section 1.1 we get a 3–chain of length 2, but here we are able to solve this problem. Therefore, we change the order of the roots such that we can draw a new diagram without any $k$–chains. As usual we start with the set of roots $\Delta_+^\omega$:

| | | |
|---|---|---|
| $\beta_1 = (2,3,4,2)$ | $\beta_6 = (1,2,3,1)$ | $\beta_{11} = (0,1,2,1)$ |
| $\beta_2 = (1,3,4,2)$ | $\beta_7 = (1,1,2,2)$ | $\beta_{12} = (1,1,1,1)$ |
| $\beta_3 = (1,2,4,2)$ | $\beta_8 = (1,2,2,1)$ | $\beta_{13} = (0,1,1,1)$ |
| $\beta_4 = (1,2,3,2)$ | $\beta_9 = (0,1,2,2)$ | $\beta_{14} = (0,0,1,1)$ |
| $\beta_5 = (1,2,2,2)$ | $\beta_{10} = (1,1,2,1)$ | $\beta_{15} = (0,0,0,1)$ |

Here we have $\beta_i \succ \beta_j \Leftrightarrow i > j$. With this order we are not able to find relations derived from differential operators (see Section 1.3), which include the root vector $f_4$ (see (1.3.2)). In order to find relations including $f_4$ we adjust the order on the roots in this case as follows:

$$\beta_1 \prec \beta_2 \prec \beta_3 \prec \beta_5 \prec \beta_4 \prec \beta_6 \prec \beta_7 \prec \cdots \prec \beta_{15}.$$

Thus we just switched the positions of $\beta_4$ and $\beta_5$. Now we consider our Hasse diagram constructed as in Section 1.1 and the diagram we obtain by changing the order of the roots and by using differential operators corresponding to non-simple roots, see Figure 1.1.

The idea of this adjustment is that we split up the 3–chain by using the non-simple differential operators mentioned above. After this we still want to get as many roots as possible on each path. To do so we use two non-simple differential operators: $\partial_{0110} = \partial_{\alpha_2+\alpha_3}$ and $\partial_{0011} = \partial_{\alpha_3+\alpha_4}$. In the adjusted diagram also occurs a directed edge labeled by $\mathfrak{a}$ from $\beta_2$ to $\beta_5$ and a second labeled by $\mathfrak{b}$ from $\beta_5$ to $\beta_4$. We cannot label the second edge with a differential operator, because there is no element $\gamma \in \Delta_+$ satisfying: $\beta_5 - \gamma = \beta_4$. We use the following observation to explain the existence of these edges and labels. For $a_0, b_0 \in \mathbb{C}\setminus\{0\}$ we have:

$$\begin{aligned}\partial_3^{n_3}\partial_3^{n_2+n_3}\partial_2^{n_2+n_3}\partial_1^{n_1} f_1^{m+1} &= \partial_3^{n_3}\partial_3^{n_2+n_3}(a_0 f_3^{n_2+n_3} f_2^{n_1-n_2-n_3} f_1^{m+1-n_1}) \\ &= b_0 f_5^{n_3} f_4^{n_2} f_2^{n_1-n_2-n_3} f_1^{m+1-n_1} + \text{smaller terms}.\end{aligned}$$

Therefore, we can replace in the path consisting of $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$ the root $\beta_3$ by $\beta_5$. Furthermore the differential operators $\partial_{\alpha_2+\alpha_3}$ and $\partial_{\alpha_3+\alpha_4}$ have no influence on $\beta_5$. That is the reason for the directed edge labeled by $\mathfrak{b}$ from $\beta_5$ to $\beta_4$. The

reason for the edge between $\beta_2$ and $\beta_5$ is that we want to visualize the co–chain which we construct at this point. We label this edge with $\mathfrak{a}$ to prevent confusions about the applied differential operators, where $\mathfrak{a}$ corresponds to $\partial_3^{n_2+2n_3}$. We note that the changed Hasse diagram gives us directly the inequalities of $P(\lambda)$, but in this case it does not describe in general the action of the differential operators.
If we now follow our standard procedure with the adjusted Hasse diagram the next step is to define the polytope associated to the set of Dyck paths $D_\omega$ and $m \in \mathbb{Z}_{\geq 0}$:

$$P(m\omega) = \{\mathbf{x} \in \mathbb{R}^{15}_{\geq 0} \mid \sum_{\beta_j \in \mathbf{p}} x_j \leq m, \ \forall \mathbf{p} \in D_\omega\}.$$

More explicitly: $P(m\omega)$ is the set of all elements $\mathbf{x} \in \mathbb{R}^{15}_{\geq 0}$ such that the 12 inequalities, which can be found in the Appendix, Figure 3.3, are satisfied.
The set $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\} \subset V(\omega)$ is given by:

$$\begin{aligned}\mathbb{B}_\omega = \{ & v_\omega, f_1 v_\omega, f_2 v_\omega, \ldots, f_{15} v_\omega, f_3 f_5 v_\omega, f_4 f_6 v_\omega, f_5 f_6 v_\omega, f_6 f_7 v_\omega, \\ & f_7 f_8 v_\omega, f_6 f_9 v_\omega, f_8 f_9 v_\omega, f_9 f_{10} v_\omega, f_9 f_{12} v_\omega, f_{11} f_{12} v_\omega\}.\end{aligned}$$

**Proposition 1.4.26.** *The set $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a FFL basis of $V(\omega)$.*

*Proof.* By Corollary 1.3.5 we conclude that $\mathbb{B}_\omega$ spans the vector space $V(\omega)$. In addition we know by [Car05, p. 303] that $\dim V(\omega) = 26 = |\mathbb{B}_\omega|$. Hence the set $\mathbb{B}_\omega$ is a basis. $\square$

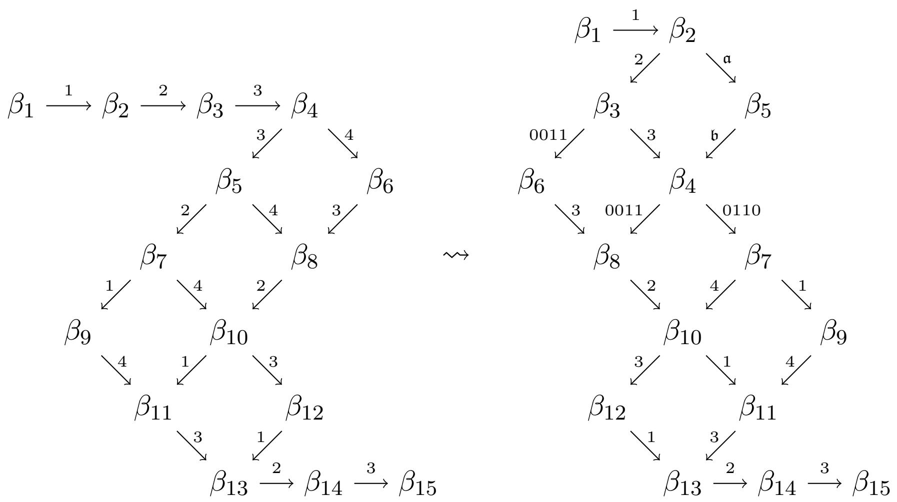
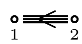



Figure 1.1: $H(\mathfrak{n}^-_\omega)_{\mathtt{F}_4}$

**Type $\mathtt{G}_2$.** Let $\mathfrak{g}$ be the simple Lie algebra of type $\mathtt{G}_2$ with associated Dynkin diagram

$\mathtt{G}_2$ 


For the highest root $\theta = 3\alpha_1 + 2\alpha_2$ we have $\theta^\vee = \alpha_1^\vee + 2\alpha_2^\vee$. Thus we consider $\omega = \omega_1$. In this case the Hasse diagram has one 1–chain. We rewrite $H(\mathfrak{n}^-_\omega)_{\mathtt{G}_2}$ into a diagram without any $k$–chains. Consider the following order on $\Delta^\omega_+$:

$$\beta_1 \prec \beta_2 \prec \beta_4 \prec \beta_5 \prec \beta_3,$$

where

| | | | | |
|---|---|---|---|---|
| $\beta_1 = (3,2)$ | $\beta_2 = (3,1)$ | $\beta_3 = (2,1)$ | $\beta_4 = (1,1)$ | $\beta_5 = (1,0)$ |

Thus we obtain the following diagrams:

$$\beta_1 \xrightarrow{2} \beta_2 \xrightarrow{1} \beta_3 \xrightarrow{1} \beta_4 \xrightarrow{2} \beta_5 \qquad \rightsquigarrow \qquad \begin{tikzcd} & & \beta_2 \arrow[dr, "21"] & \\ \beta_1 \arrow[r, "11"] & \beta_3 \arrow[ur, "2"] \arrow[dr, "21"'] & & \beta_5, \\ & & \beta_4 \arrow[ur, "2"'] & \end{tikzcd}$$

Very similar arguments as in the case of $\mathtt{B}_3, \omega_1$ show that we can apply the results of section 1.3 to the rewritten diagram. We consider the polytope associated to the new diagram for $m \in \mathbb{Z}_{\geq 0}$:

$$P(m\omega) = \left\{ \mathbf{x} \in \mathbb{R}^N_{\geq 0} \mid \begin{array}{l} x_1 + x_2 + x_3 + x_5 \leq m \\ x_1 + x_3 + x_4 + x_5 \leq m \end{array} \right\}.$$

By Section 1.3 the elements $v_\omega$, $f_1 v_\omega$, $f_2 v_\omega$, $f_3 v_\omega$, $f_4 v_\omega$, $f_5 v_\omega$, $f_2 f_4 v_\omega$ span $V(\omega)$ and with [Car05, p. 316] we know $\dim V(\omega) = 7$.

**Proposition 1.4.27.** *The set $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a FFL basis of $V(\omega)$.*$\square$

*Proof.* The previous observations imply that $\{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a basis of $V(\omega)$. It remains to show that $P(\omega)$ is a normal polytope.
Like in the case of $(\mathtt{B_n}, \omega_1)$ we have to change the order of the roots to apply Section 1.2. One possible order is $\beta_1 \prec \beta_3 \prec \beta_4 \prec \beta_2 \prec \beta_5$. Using this order we conclude that $P(\omega)$ is a normal polytope. $\square$

## 1.5 Linear Independence

We refer to the notation of Section 1.1, especially to the definition of essential monomials. Throughout the Section we assume the vectors $f^{\mathbf{p}} v_\lambda \in V(\lambda)$ to be ordered as in (1.1.7) and we fix $\lambda = m\omega$ where $\omega$ appears in Table 1.1.
We investigate the connection between our polytope $P(\lambda)$ and the essential multi–exponents. Via this connection and with the results from Section 1.3 we prove that $\{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ provides a FFL basis of $V(\lambda)$.
Note that one can define essential monomials for an arbitrary total order on $\Delta_+^\lambda$. Hence for the following statements it is very important that we kept in the whole Section 1.1 the same total order.

**Lemma 1.5.1.** *If $\{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ is linear independent in $V(\lambda)$, then*

$$S(\lambda) = \mathrm{es}(V(\lambda)).$$

*Proof.* Let $\mathbf{s} \in \mathrm{es}(V(\lambda)) = \{\mathbf{p} \in \mathbb{Z}^N_{\geq 0} \mid f^{\mathbf{p}} v_\lambda \notin \mathrm{span}\{f^{\mathbf{q}} v_\lambda \mid \mathbf{q} \prec \mathbf{p}\}\}$ and assume $\mathbf{s} \notin S(\lambda)$. By Proposition 1.3.3 we can rewrite $f^{\mathbf{s}}$ such that

$$f^{\mathbf{s}} v_\lambda = \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} f^{\mathbf{t}} v_\lambda, c_{\mathbf{t}} \in \mathbb{C}$$

and we get immediately a contradiction, so $\mathbf{s} \in S(\lambda)$.
Now let $\mathbf{s} \in S(\lambda)$ and $\mathbf{s} \notin \mathrm{es}(V(\lambda))$. Then $f^{\mathbf{s}} v_\lambda \in \mathrm{span}\{f^{\mathbf{q}} v_\lambda \mid \mathbf{q} \prec \mathbf{s}\}$ and so

$$f^{\mathbf{s}} v_\lambda = \sum_{\mathbf{q} \prec \mathbf{s}} c_{\mathbf{q}} f^{\mathbf{q}} v_\lambda,$$

for some $c_{\mathbf{q}} \in \mathbb{C}$. We rewrite each $f^{\mathbf{q}} v_\lambda$ in terms of basis elements $f^{\mathbf{t}} v_\lambda, \mathbf{t} \in S(\lambda)$. Because of the linear independence all coefficients are zero, meaning that $\mathbf{s} = 0$. But this is a contradiction to $\mathbf{s} \notin \mathrm{es}V(\lambda)$. □

**Theorem 1.5.2.** *The elements $\{f^{\mathbf{s}}(v_{\lambda-\omega} \otimes v_\omega) \mid \mathbf{s} \in S(\lambda)\} \subset V(\lambda - \omega) \odot V(\omega)$ are linearly independent and $\mathbb{B}_\lambda = \{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ is a FFL basis of $V(\lambda)$.*

*Proof.* We prove this statement by induction on $m \in \mathbb{Z}_{\geq 1}$. For $m = 1$ we saw in Section 1.4 that $\mathbb{B}_\omega = \{f^{\mathbf{s}} v_\omega \mid \mathbf{s} \in S(\omega)\}$ is a basis for $V(\omega)$ in each type.
Thus let $m \in \mathbb{Z}_{\geq 2}$ be arbitrary and we assume that the claim holds for all $m' < m$.
By induction the set $\mathbb{B}_{\lambda-\omega} = \{f^{\mathbf{s}} v_{\lambda-\omega} \mid \mathbf{s} \in S(\lambda - \omega)\}$ is a basis of $V(\lambda - \omega)$.
Thus we have by Lemma 1.5.1

$$\mathrm{es}(V(\lambda - \omega) = S(\lambda - \omega) \text{ and } \mathrm{es}(V(\omega)) = S(\omega). \tag{1.5.1}$$

But then with [FFL13a, Prop. 1.11]:

$$\mathrm{es}(V(\lambda - \omega) + \mathrm{es}(V(\omega)) \subset \mathrm{es}(V(\lambda - \omega) \odot V(\omega))$$

and so we get the linearly independence of

$$\{f^{\mathbf{s}}(v_{\lambda-\omega} \otimes v_\omega) \mid \mathbf{s} \in \mathrm{es}(V(\lambda - \omega) + \mathrm{es}(V(\omega))\} \subset V(\lambda - \omega) \odot V(\omega)$$

With the equalities in (1.5.1) and Section 1.2 where we proved $S(\lambda - \omega) + S(\omega) = S(\lambda)$, we conclude that the set

$$\{f^{\mathbf{s}}(v_{\lambda-\omega} \otimes v_\omega) \mid \mathbf{s} \in S(\lambda)\} \subset V(\lambda - \omega) \odot V(\omega)$$

is linearly independent. Thus we get $\dim V(\lambda) \geq |S(\lambda)|$ and with the spanning property Corollary 1.3.5 we have $|S(\lambda)| \geq \dim V(\lambda)$. Finally we get

$$|S(\lambda)| = \dim V(\lambda)$$

and that $\mathbb{B}_\lambda$ is a FFL basis of $V(\lambda)$ as claimed. □

**Remark 1.5.3.** *The basis $\mathbb{B}_\lambda$ is a monomial basis, so we get an induced FFL basis of $V^a(\lambda)$.*

**Theorem 1.5.4.** *Let $V^a(\lambda) \cong S(\mathfrak{n}^-)/I(\lambda)$. Then the ideal $I(\lambda)$ is generated by*

$$U(\mathfrak{n}^+) \circ \mathrm{span}\{f_\beta^{\lambda(\beta^\vee)+1} \mid \beta \in \Delta_+\}$$

*as $S(\mathfrak{n}^-)$ ideal.*
*In particular we have that $I(\lambda) = S(\mathfrak{n}^-)(U(\mathfrak{n}^+) \circ \mathrm{span}\{f_\beta, f_\theta^{m+1} \mid \beta \in \Delta_+ \setminus \Delta_+^\lambda\})$.*

*Proof.* Let $I$ be an Ideal generated by $U(\mathfrak{n}^+) \circ \text{span}\{f_\beta^{\lambda(\beta^\vee)+1} \mid \beta \in \Delta_+\}$ as $S(\mathfrak{n}^-)$ ideal. By $Iv_\lambda = \{0\}$ we have $I \subset I(\lambda)$, so there is a canonical projection:

$$\phi : S(\mathfrak{n}^-)/I \to S(\mathfrak{n}^-)/I(\lambda) \cong V^a(\lambda)$$

Let $f^{\mathbf{t}} = 0$ in $S(\mathfrak{n}^-)/I(\lambda)$. Because we have a basis of $V^a(\lambda)$ we can rewrite $f^{\mathbf{t}}$ as follows:

$$f^{\mathbf{t}} = \sum_{\mathbf{s} \in S(\lambda)} c_{\mathbf{s}} f^{\mathbf{s}} \in S(\mathfrak{n}^-)/I(\lambda) \tag{1.5.2}$$

for some $c_{\mathbf{s}} \in \mathbb{C}$. In the proof of Theorem 1.3.4 we already saw that the relations obtained by $I$ are sufficient to achieve (1.5.2). Thus $0 = f^{\mathbf{t}} = \sum_{\mathbf{s} \in S(\lambda)} c_{\mathbf{s}} f^{\mathbf{s}} \in S(\mathfrak{n}^-)/I$. Therefore $\phi$ is injective.

In the proof of Proposition 1.3.3 we do not need powers $f_\beta$ for $\beta \in \Delta_+^\lambda \setminus \{\theta\}$. □

# 2 Monomial bases over $\mathbb{Z}$

Let $\mathbb{B}(V^a(\lambda))$ be a monomial basis of the associated graded space $V^a(\lambda)$. In this chapter we explain under which assumptions the set $\mathbb{B}(V^a(\lambda))$ is also a monomial basis of $V^a_{\mathbb{Z}}(\lambda)$, the $\mathbb{Z}$–analogue of $V^a(\lambda)$ (for a precise definition see Section 2.1). As a consequence we also obtain a monomial basis of $V_k(\lambda) \cong V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$, where $k$ is a field of arbitrary characteristic.

## 2.1 The Kostant Lattice

In the present chapter it is necessary to work with a Chevalley basis of $\mathfrak{g}$. To give the definition of those we need the following general statement for semisimple Lie algebras.

**Proposition 2.1.1.** *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\beta \in \Delta} \mathfrak{g}_\beta$$

*be a Cartan decomposition of $\mathfrak{g}$, where $\Delta \subset \mathfrak{h}^*$ is the corresponding root system. Let $h_\beta \in \mathfrak{h}$ be the co–root corresponding to the root $\beta \in \Delta$. Then is it possible to choose root vectors $x_\beta \in \mathfrak{g}_\beta$ satisfying:*

*(a)* $[x_\beta, x_{-\beta}] = h_\beta$.

*(b) If $\beta, \gamma, \beta + \gamma \in \Delta$ and $[x_\beta, x_\gamma] = c_{\beta,\gamma} x_{\beta+\gamma}$, then $c_{\beta,\gamma} = -c_{-\beta,-\gamma}$.*

*Proof.* See for the proof [Hum72, Proposition 25.2]. □

The constants $c_{\beta,\gamma}$ from Proposition 2.1.1 are called the structure constants.
A basis $\mathbb{B}(\mathfrak{g}) = \{x_\beta, h_j \mid \beta \in \Delta,\ 1 \leq j \leq n\}$ of $\mathfrak{g}$ for which $x_\beta,\ \beta \in \Delta$ satisfy $(a)$ and $(b)$ from Proposition 2.1.1 is called a *Chevalley basis*.
The following theorem is of significant importance for our further considerations.

**Theorem 2.1.2** (Chevalley)**.** *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathbb{B}(\mathfrak{g}) = \{x_\beta, h_j \mid \beta \in \Delta,\ 1 \leq j \leq n\}$ be a Chevalley basis of $\mathfrak{g}$. Then the resulting structure constants are in $\mathbb{Z}$. More precisely:*

*(a)* $[h_i, h_j] = 0$, *for* $1 \leq i, j \leq n$.

*(b)* $[h_i, x_\beta] = c_{\alpha_i,\beta} x_\beta = \alpha_i(\beta^\vee) x_\beta$, *for* $1 \leq i \leq n$ *and* $\beta \in \Delta$.

*(c)* $[x_\beta, x_{-\beta}] = h_\beta$ *is a $\mathbb{Z}$–linear combination of $h_1, \ldots, h_n$.*

*(d) For $\beta, \gamma \in \Delta$, $[x_\beta, x_\gamma] = \pm(p+1)x_{\beta+\gamma}$, if $\beta + \gamma \in \Delta$ and $0$ otherwise, where $p \in \mathbb{Z}_{\geq 0}$ is the greatest integer for which $\gamma - p\beta \in \Delta$.*

*Proof.* See for a proof [Hum72, Theorem 25.2]. □

Let us now consider our previous setup. Again $\mathfrak{g}$ is simple, complex and finite–dimensional Lie algebra. In order to construct irreducible representations of $\mathfrak{g}$ "over $\mathbb{Z}$" we have to introduce the Lie algebra $\mathfrak{g}_{\mathbb{Z}}$.
Let us fix a Chevalley basis of $\mathfrak{g}$

$$\mathbb{B}_{\mathrm{Ch}}(\mathfrak{g}) = \{x_i \mid 1 \leq i \leq D\}, \text{ with } D = \dim \mathfrak{g}.$$

We define

$$\mathfrak{g}_{\mathbb{Z}} := \langle x_i \mid 1 \leq i \leq D\rangle_{\mathbb{Z}} \subset \mathfrak{g}.$$

Thus $\mathfrak{g}_{\mathbb{Z}}$ is a free $\mathbb{Z}$–module. Chevalley's Theorem (see Theorem 2.1.2) implies that $\mathfrak{g}_{\mathbb{Z}}$ is in addition a Lie subalgebra of $\mathfrak{g}$, since all structure constants are in $\mathbb{Z}$. Analogously we define the Lie subalgebras $\mathfrak{n}^+_{\mathbb{Z}} \subset \mathfrak{n}^+$, $\mathfrak{h}_{\mathbb{Z}} \subset \mathfrak{h}$, $\mathfrak{n}^-_{\mathbb{Z}} \subset \mathfrak{n}^-$ and $\mathfrak{b}_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \oplus \mathfrak{n}^+_{\mathbb{Z}} \subset \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Note that we have $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, $\mathfrak{n}^+ \cong \mathfrak{n}^+_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ etc.

Furthermore let $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ be the abelian Lie algebra, which is as vector space equal to $\mathfrak{n}^-_{\mathbb{Z}}$ endowed with the trivial Lie bracket. Note that $\mathfrak{n}^{-,a}_{\mathbb{Z}}$ carries the structure of a $\mathfrak{b}_{\mathbb{Z}}$–module by using the vector space isomorphism $\Psi$ between the quotient module $\mathfrak{g}_{\mathbb{Z}}/\mathfrak{b}_{\mathbb{Z}}$, which is a $\mathfrak{b}_{\mathbb{Z}}$–module via the adjoint action, and $\mathfrak{n}^{-,a}_{\mathbb{Z}}$. To be more explicit we define for $b \in \mathfrak{b}_{\mathbb{Z}}$ and $n \in \mathfrak{n}^{-,a}_{\mathbb{Z}}$: $b.n = \Psi(b.\Psi^{-1}(n))$.

Let us define the basis $\mathbb{B}(\mathfrak{g}_{\mathbb{Z}})$ of $\mathfrak{g}_{\mathbb{Z}}$, which is as set equal to $\mathbb{B}_{\mathrm{Ch}}(\mathfrak{g})$, more explicit

$$\mathbb{B}(\mathfrak{g}_{\mathbb{Z}}) = \{e_\beta, f_\beta, h_j \mid \beta \in \Delta_+,\ 1 \leq j \leq n\} \subset \mathfrak{g}_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \oplus \sum_{\beta \in \Delta} \mathfrak{g}_{\beta,\mathbb{Z}},$$

where the root vector $e_\beta \in \mathfrak{n}^+_{\mathbb{Z}}$ (respectively $f_\beta \in \mathfrak{n}^-_{\mathbb{Z}}$) is an element of the root space $\mathfrak{g}_{\beta,\mathbb{Z}}$ (respectively $\mathfrak{g}_{-\beta,\mathbb{Z}}$).
We write $e^{(m)}_\beta, f^{(m)}_\beta$ for the divided powers $\frac{e^m_\beta}{m!}$ and $\frac{f^m_\beta}{m!}$ in the universal enveloping algebra $U(\mathfrak{g})$. Further we denote by $\binom{h_i}{m}$ the following element in $U(\mathfrak{g})$:

$$\binom{h_i}{m} = \frac{\prod_{k=0}^{m+1}(h_i - k)}{m!}.$$

These notations allow us to define the main object in this section

$$U_{\mathbb{Z}}(\mathfrak{g}) := \left\langle e^{(m)}_{\beta_i}, f^{(m)}_{\beta_i}, \binom{h_i}{m} \mid m \in \mathbb{Z}_{\geq 0} \right\rangle_{\mathbb{Z}} \subset U(\mathfrak{g}),$$

we remark that $U_{\mathbb{Z}}(\mathfrak{g})$ is a $\mathbb{Z}$–subalgebra generated as above, called the *Kostant lattice* in $U(\mathfrak{g})$. Let $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$. For a given multi–exponent $\mathbf{m} \in \mathbb{Z}^N_{\geq 0}$ and a $n$–tuple $b \in \mathbb{Z}^n_{\geq 0}$ we define

$$e^{(\mathbf{m})} := \prod_{\ell=1}^{N} e^{(m_\ell)}_{\beta_\ell},\ \ f^{(\mathbf{m})} := \prod_{\ell=1}^{N} f^{(m_\ell)}_{\beta_\ell} \text{ and } h^{(b)} := \prod_{\ell=1}^{n} \binom{h_\ell}{b_\ell}.$$

**Theorem 2.1.3** (Kostant)**.** *Let $U_{\mathbb{Z}}(\mathfrak{g})$ be the Kostant lattice in $U(\mathfrak{g})$. Then*

$$\mathbb{B}(U_{\mathbb{Z}}(\mathfrak{g})) := \left\{ f^{(\mathbf{m})} h^{(b)} e^{(\mathbf{k})} \mid \mathbf{m}, \mathbf{k} \in \mathbb{Z}^N_{\geq 0},\ b \in \mathbb{Z}^n_{\geq 0} \right\}$$

*forms a $\mathbb{Z}$–basis of $U_{\mathbb{Z}}(\mathfrak{g})$ as free $\mathbb{Z}$–module.*

*Proof.* See for a proof [Hum72, Section 26.4.]. $\square$

**Corollary 2.1.4.** *The $\mathbb{Z}$–subalgebra $U_{\mathbb{Z}}(\mathfrak{n}^+)$ respectively $U_{\mathbb{Z}}(\mathfrak{n}^-)$ admits the ordered monomials*

$$\mathbb{B}(U_{\mathbb{Z}}(\mathfrak{n}^+)) := \{e^{(\mathbf{k})} \mid \mathbf{k} \in \mathbb{Z}^N_{\geq 0}\} \text{ resp. } \mathbb{B}(U_{\mathbb{Z}}(\mathfrak{n}^-)) := \{f^{(\mathbf{m})} \mid \mathbf{m} \in \mathbb{Z}^N_{\geq 0}\}$$

*as $\mathbb{Z}$–basis.*

*Proof.* This is a direct conclusion from Kostant's Theorem. $\square$

Let $U_{\mathbb{Z}}(\mathfrak{n}^-)_s$ be the $\mathbb{Z}$–span of the monomials of degree at most $s$:

$$U_{\mathbb{Z}}(\mathfrak{n}^-)_s = \langle f_{\gamma_1}^{(m_1)} \dots f_{\gamma_\ell}^{(m_\ell)} \mid m_1 + \dots + m_\ell \leq s, \gamma_1, \dots, \gamma_\ell \in \Delta_+ \rangle_{\mathbb{Z}}. \tag{2.1.1}$$

Since changing the ordering is commutative up to terms of smaller degree, the subspaces $U_{\mathbb{Z}}(\mathfrak{n}^-)_s$ define a filtration of the algebra $U_{\mathbb{Z}}(\mathfrak{n}^-)$. By abuse of notation we denote by $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ the associated graded algebra of $\mathfrak{n}_{\mathbb{Z}}^{-,a}$ with respect to the filtration (2.1.1). Note that $\mathfrak{n}_{\mathbb{Z}}^{-,a} \subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. In fact, $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ is a divided power analogue of the symmetric algebra over $\mathfrak{n}_{\mathbb{Z}}^{-,a}$. This algebra can be described as the quotient of a polynomial algebra in infinitely many generators, the "symbols" $\mathfrak{f}_\beta^{(m)}$, modulo the ideal generated by the identities

$$\mathfrak{f}_\beta^{(m)} \mathfrak{f}_\beta^{(k)} = \begin{pmatrix} m+k \\ m \end{pmatrix} \mathfrak{f}_\beta^{(m+k)}. \tag{2.1.2}$$

Thus we have:

$$S_{\mathbb{Z}}(\mathfrak{n}^{-,a}) \simeq \mathbb{Z}[\mathfrak{f}_\beta^{(m)} \mid m \in \mathbb{Z}_{\geq 0}, \beta \in \Delta_+]/\langle \mathfrak{f}_\beta^{(m)} \mathfrak{f}_\beta^{(k)} - \begin{pmatrix} m+k \\ m \end{pmatrix} \mathfrak{f}_\beta^{(m+k)} \rangle.$$

Analogue to Remark 0.0.4, let $U_{\mathbb{Z}}^+(\mathfrak{h} \oplus \mathfrak{n}^+) \subset U_{\mathbb{Z}}(\mathfrak{g})$ be the span of the monomials $h^{(\ell)} e^{(\mathbf{k})}$ such that $\sum_{i=1}^n \ell_i + \sum_{j=1}^N k_j > 0$. The natural map which sends a monomial to its class in the quotient:

$$U_{\mathbb{Z}}(\mathfrak{n}^-) \to U_{\mathbb{Z}}(\mathfrak{g})/U_{\mathbb{Z}}(\mathfrak{g})U_{\mathbb{Z}}^+(\mathfrak{h} \oplus \mathfrak{n}^+), \quad f^{(\mathbf{m})} \to \overline{f^{(\mathbf{m})}},$$

is an isomorphism of free $\mathbb{Z}$–modules. Recall that $U_{\mathbb{Z}}(\mathfrak{g})$ is naturally a $\mathfrak{b}_{\mathbb{Z}}$–module and a $U_{\mathbb{Z}}(\mathfrak{n}^+)$–module via the adjoint action, and $U_{\mathbb{Z}}(\mathfrak{g})U_{\mathbb{Z}}^+(\mathfrak{h} \oplus \mathfrak{n}^+)$ is a proper submodule. Via the identification above, we obtain an induced structure on $U_{\mathbb{Z}}(\mathfrak{n}^-)$ as a $\mathfrak{b}_{\mathbb{Z}}$–module and as a $U_{\mathbb{Z}}(\mathfrak{n}^+)$–module. The filtration of $U_{\mathbb{Z}}(\mathfrak{n}^-)$ by the $U_{\mathbb{Z}}(\mathfrak{n}^-)_s$ is stable under this $\mathfrak{b}_{\mathbb{Z}}$– and $U_{\mathbb{Z}}(\mathfrak{n}^-)_s$–action and hence:

**Lemma 2.1.5.** *The $\mathfrak{b}_{\mathbb{Z}}$–module structure respectively the $U_{\mathbb{Z}}(\mathfrak{n}^+)$–module structure on $U_{\mathbb{Z}}(\mathfrak{n}^-)$ induce a $\mathfrak{b}_{\mathbb{Z}}$–module structure respectively a $U_{\mathbb{Z}}(\mathfrak{n}^+)$–module structure on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$.*

For a dominant integral weight $\lambda$ we fix a highest weight vector $v_\lambda$ and let $V_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}(\mathfrak{g})v_\lambda \subset V(\lambda)$ be the corresponding lattice in $V(\lambda)$. Since $V_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}(\mathfrak{n}^-)v_\lambda$, the filtration (2.1.1) induces an increasing degree filtration $V_{\mathbb{Z}}(\lambda)_s$ on $V_{\mathbb{Z}}(\lambda)$:

$$V_{\mathbb{Z}}(\lambda)_s = U_{\mathbb{Z}}(\mathfrak{n}^-)_s v_\lambda.$$

We denote the associated graded space by $V^a_{\mathbb{Z}}(\lambda)$. Since $\mathfrak{b}_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)_s \subset V_{\mathbb{Z}}(\lambda)_s$, $V^a_{\mathbb{Z}}(\lambda)$ becomes naturally a $\mathfrak{b}_{\mathbb{Z}}$–module. The application by an element $f_\beta^{(m)} \in U_{\mathbb{Z}}(\mathfrak{n}^-)$ provides linear maps for all $s$:

$$\begin{array}{cccc} f_\beta^{(m)}: & V_{\mathbb{Z}}(\lambda)_s & \rightarrow & V_{\mathbb{Z}}(\lambda)_{s+m} \\ & \cup & & \cup \\ & V_{\mathbb{Z}}(\lambda)_{s-1} & \rightarrow & V_{\mathbb{Z}}(\lambda)_{s+m-1} \end{array}$$

and we obtain an induced endomorphism $\psi^a(f_\beta^{(m)}) : V^a_{\mathbb{Z}}(\lambda) \rightarrow V^a_{\mathbb{Z}}(\lambda)$ such that $\psi^a(f_\beta^{(m)})\psi^a(f_\gamma^{(\ell)}) = \psi^a(f_\gamma^{(\ell)})\psi^a(f_\beta^{(m)})$, and hence we obtain an induced representation of the abelian Lie algebra $\mathfrak{n}_{\mathbb{Z}}^{-,a}$ respectively of the algebra $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. Note that $V^a_{\mathbb{Z}}(\lambda)$ is a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$–module:

$$V^a_{\mathbb{Z}}(\lambda) = S_{\mathbb{Z}}(\mathfrak{n}^{-,a})v_\lambda.$$

Thus there is an ideal $I_{\mathbb{Z}}(\lambda) \subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ such that: $V^a_{\mathbb{Z}}(\lambda) \cong S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$. In analogy to the complex case we have: $\left\{ f_{\beta_i}^{(\lambda(\beta_i^\vee)+1)} \mid 1 \leq i \leq N \right\} \subset I_{\mathbb{Z}}(\lambda)$.
The action of $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ on $V^a_{\mathbb{Z}}(\lambda)$ is compatible with the $\mathfrak{b}_{\mathbb{Z}}$–action on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ and on $V^a(\lambda)$, i. e. for arbitrary $b \in \mathfrak{b}_{\mathbb{Z}}$, $s \in S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ is the identity $b.(s.v_\lambda) = (b.s).v_\lambda$ true. Summarizing we have:

**Proposition 2.1.6.** *$V^a_{\mathbb{Z}}(\lambda)$ is a cyclic $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$–module and a $\mathfrak{b}_{\mathbb{Z}}$–module. The $\mathfrak{b}_{\mathbb{Z}}$–action on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ is compatible with the $\mathfrak{b}_{\mathbb{Z}}$–action on $V^a_{\mathbb{Z}}(\lambda) = S_{\mathbb{Z}}(\mathfrak{n}^{-,a})v_\lambda$.*

## 2.2 Differential operators for $\mathfrak{g}_{\mathbb{Z}}$ and $\mathbb{Z}$–admissible elements in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$

Let $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{n}^+_{\mathbb{Z}} \oplus \mathfrak{h}_{\mathbb{Z}} \oplus \mathfrak{n}^-_{\mathbb{Z}}$ be a Cartan decomposition. Further we fix a basis $\mathbb{B}(\mathfrak{g}_{\mathbb{Z}})$ of $\mathfrak{g}_{\mathbb{Z}}$, which is by definition a Chevalley basis of $\mathfrak{g}$. In the following we provide an analogue of the differential operators defined in (0.0.4) for the Lie algebra $\mathfrak{g}_{\mathbb{Z}}$. Let $\beta, \gamma \in \Delta_+$, then we define the differential operator

$$\partial_\gamma (f_\beta) := \begin{cases} |c_{\gamma,-\beta}| f_{\beta-\gamma}, & \text{if } \beta - \gamma \in \Delta_+, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2.1}$$

where $c_{\gamma,-\beta} \in \mathbb{Z}$ is the corresponding structure constant. The differential operator satisfies

$$\partial_\gamma (f_\beta) = \pm \operatorname{ad}(e_\gamma)(f_\beta).$$

Note, that it is no longer possible to multiply by $(c_{\beta,-\gamma})^{-1}$, if $c_{\beta,-\gamma} \neq \pm 1$, since these scalars are not in $\mathbb{Z}$.

Lemma 2.1.5 implies that $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ carries a $U_{\mathbb{Z}}(\mathfrak{n}^+)$–module structure. Thus we define analogue to (0.0.5) differential operators on $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$. For $m, k \in \mathbb{Z}_{\geq 0}$ we define:

$$\partial_\gamma^{(k)} \left( f_\beta^{(m)} \right) = \frac{1}{k!m!} \partial_\gamma^k \left( f_\beta^m \right) = \frac{1}{k!m!} \underbrace{\partial_\gamma \cdots \partial_\gamma}_{k\text{–times}} \left( f_\beta^m \right) \tag{2.2.2}$$

$$\text{and } \partial_\gamma \left( f_\beta^m \right) = m \underbrace{\partial_\gamma (f_\beta)}_{(2.2.1)} f_\beta^{m-1} = m |c_{\gamma,-\beta}| f_{\beta-\gamma} f_\beta^{m-1}.$$

We set $\partial_\gamma^{(k)}(f_\beta^{(m)}) = 0$, if $\beta = \gamma$ or if the root vectors commute. In the following we describe the differential operator $\partial_\gamma^{(k)}(f_\beta^{(m)})$ more explicit:
If $\beta, \gamma$ and $\beta+\gamma$ span a subsystem of type $\mathtt{A}_2$, then

$$\partial_\gamma^{(k)}\left(f_{\beta+\gamma}^{(m)}\right) = \begin{cases} f_\beta^{(k)} f_{\beta+\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise.} \end{cases} \tag{2.2.3}$$

If $\beta, \gamma, \beta+\gamma, \beta+2\gamma \in \Delta_+$ span a subsystem of type $\mathtt{B}_2 = \mathtt{C}_2$, then

$$\partial_\beta^{(k)}\left(f_{\beta+\gamma}^{(m)}\right) = \begin{cases} f_\gamma^{(k)} f_{\beta+\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.4}$$

and

$$\partial_{\beta+\gamma}^{(k)}\left(f_{\beta+2\gamma}^{(m)}\right) = \begin{cases} f_\gamma^{(k)} f_{\beta+2\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.5}$$

and

$$\partial_\gamma^{(k)}\left(f_{\beta+\gamma}^{(m)}\right) = \begin{cases} 2^k f_\beta^{(k)} f_{\beta+\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.6}$$

and

$$\partial_\gamma^{(k)}\left(f_{\beta+2\gamma}^{(m)}\right) = \begin{cases} \sum_{X(k,m)} r_x f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)}, & \text{if } k \leq 2m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.7}$$

where $X(k,m) = \{x = (a,b,c) \in \mathbb{Z}^3_{\geq 0} \mid a+b+c = m,\ 2a+b = k\}$ and $r_x \in \mathbb{Z}$.
If $\beta, \gamma, \beta+\gamma, \beta+2\gamma, \beta+3\gamma, 2\beta+3\gamma \in \Delta_+$ span a subsystem of type $\mathtt{G}_2$, then

$$\partial_{\beta+3\gamma}^{(k)}\left(f_{2\beta+3\gamma}^{(m)}\right) = \begin{cases} f_\beta^{(k)} f_{2\beta+3\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.8}$$

and

$$\partial_{\beta+2\gamma}^{(k)}\left(f_{2\beta+3\gamma}^{(m)}\right) = \begin{cases} f_{\beta+\gamma}^{(k)} f_{2\beta+3\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.9}$$

and

$$\partial_\beta^{(k)}\left(f_{2\beta+3\gamma}^{(m)}\right) = \begin{cases} f_{\beta+3\gamma}^{(k)} f_{2\beta+3\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.10}$$

and

$$\partial_{\beta+\gamma}^{(k)}\left(f_{2\beta+3\gamma}^{(m)}\right) = \begin{cases} \sum_{X(k,m)} r_x f_\gamma^{(a)} f_{\beta+2\gamma}^{(b)} f_{2\beta+3\gamma}^{(c)}, & \text{if } k \leq 2m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.11}$$

and

$$\partial_{\beta+2\gamma}^{(k)}\left(f_{\beta+3\gamma}^{(m)}\right) = \begin{cases} f_\gamma^{(k)} f_{\beta+3\gamma}^{(m-k)}, & \text{if } k \leq m, \\ 0\ , & \text{otherwise} \end{cases} \tag{2.2.12}$$

and

$$\partial_\gamma^{(k)}\left(f_{\beta+3\gamma}^{(m)}\right)=\begin{cases}\sum_{Y(k,m)} r_y f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)} f_{\beta+3\gamma}^{(d)}, & \text{if } k\le 3m,\\ 0\ , & \text{otherwise}\end{cases} \tag{2.2.13}$$

and

$$\partial_{\beta+\gamma}^{(k)}\left(f_{\beta+2\gamma}^{(m)}\right)=\begin{cases}2^k f_\gamma^{(k)} f_{\beta+2\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0\ , & \text{otherwise}\end{cases} \tag{2.2.14}$$

and

$$\partial_\gamma^{(k)}\left(f_{\beta+2\gamma}^{(m)}\right)=\begin{cases}\sum_{X(k,m)} 3^a 2^b \tilde{r}_x f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)}, & \text{if } k\le 2m,\\ 0\ , & \text{otherwise}\end{cases} \tag{2.2.15}$$

and

$$\partial_\gamma^{(k)}\left(f_{\beta+\gamma}^{(m)}\right)=\begin{cases}3^k f_\beta^{(k)} f_{\beta+\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0\ , & \text{otherwise}\end{cases} \tag{2.2.16}$$

and

$$\partial_\beta^{(k)}\left(f_{\beta+\gamma}^{(m)}\right)=\begin{cases}f_\gamma^{(k)} f_{\beta+\gamma}^{(m-k)}, & \text{if } k\le m,\\ 0\ , & \text{otherwise,}\end{cases} \tag{2.2.17}$$

where $r_x, r_y, \tilde{r}_x \in \mathbb{Z}$ and

$$\begin{aligned}X(k,m)&=\{x=(a,b,c)\in\mathbb{Z}^3_{\ge 0}\mid a+b+c=m,\ 2a+b=k\},\\ Y(k,m)&=\{y=(a,b,c,d)\in\mathbb{Z}^4_{\ge 0}\mid a+b+c+d=m,\ 3a+2b+c=k\}.\end{aligned}$$

**Remark 2.2.1.** *Note that in the* $\mathtt{A}_2$– *and* $\mathtt{C}_2$–*case,* (2.2.3)–(2.2.7)*, the coefficient of the* rhs *can be easily checked by applying Chevalley's Theorem (see Theorem 2.1.2). In the* $\mathtt{G}_2$–*case,*(2.2.8)–(2.2.17)*, it is also possible to consider the multiplication table of* $\mathtt{G}_2$ *(see [FH91, Table 22.1]).*

In fact the constants $r_x$ and $r_y$ occurring in (2.2.7), (2.2.11) and (2.2.13) are equal to 1. But before we are able to prove this, we have to introduce another combinatorial tool: The *coefficient graph*.

**The coefficient graph.** We distinguish two types of coefficient graphs: $\mathcal{C}_3(m)$ and $\mathcal{C}_4(m)$ for $m\in\mathbb{Z}_{\ge 0}$. In order to consider these directed graphs we have to define first:

$$\begin{aligned}X(m)&:=\bigcup_{k=1}^{2m} X(k,m)=\{x'=(a,b,c)\in\mathbb{Z}^3_{\ge 0}\mid a+b+c=m\},\\ Y(m)&:=\bigcup_{k=1}^{3m} Y(k,m)=\{y'=(a,b,c,d)\in\mathbb{Z}^4_{\ge 0}\mid a+b+c+d=m\}.\end{aligned}$$

Let $\mathcal{C}_3(m)=(X(m),E)$ be the directed labeled graph, where the set of vertices is indexed by the set $X(m)$ (defined above) and the set of edges $E$ is given by the following rule: For $x'=(a,b,c)\in X(m)$, we have

$$\begin{aligned}((a,b,c)\xrightarrow{2b}(a+1,b-1,c))\in E &:\Leftrightarrow\ b>0,\\ ((a,b,c)\xrightarrow{c}(a,b+1,c-1))\in E &:\Leftrightarrow\ c>0.\end{aligned}$$

The second type is given as follows. Let $\mathcal{C}_4(m) = (Y(m), F)$ be the directed labeled graph, where the set of vertices is indexed by the set $Y(m)$ (defined above) and the set of edges $F$ is given by the following rule: For $y' = (a, b, c, d) \in Y(m)$, we have

$$
\begin{aligned}
((a,b,c,d) &\xrightarrow{3b} (a+1, b-1, c, d)) \in F :\Leftrightarrow \; b > 0,\\
((a,b,c,d) &\xrightarrow{2c} (a, b+1, c-1, d)) \in F :\Leftrightarrow \; c > 0,\\
((a,b,c,d) &\xrightarrow{d} (a, b, c+1, d-1)) \in F :\Leftrightarrow \; d > 0.
\end{aligned}
$$

**Example 2.2.2.** *We consider $\mathcal{C}_3(3)$ and $\mathcal{C}_4(2)$:*

$$
\begin{tikzcd}
(0,0,3) \arrow[d, "3"] & & & \\
(0,1,2) \arrow[r, "2 \cdot 1"] \arrow[d, "2"] & (1,0,2) \arrow[d, "2"] & & \\
(0,2,1) \arrow[r, "2 \cdot 2"] \arrow[d, "1"] & (1,1,1) \arrow[r, "2 \cdot 1"] \arrow[d, "1"] & (2,0,1) \arrow[d, "1"] & \\
(0,3,0) \arrow[r, "2 \cdot 3"] & (1,2,0) \arrow[r, "2 \cdot 2"] & (2,1,0) \arrow[r, "2 \cdot 1"] & (3,0,0)
\end{tikzcd}
$$

Figure 2.1: The coefficient graph $\mathcal{C}_3(3)$.

$$
\begin{tikzcd}
 & & (0,0,0,2) \arrow[dl, "2"] & & & \\
 & (0,0,1,1) \arrow[rrr, "2 \cdot 1"] \arrow[dl, "1"] & & & (0,1,0,1) \arrow[dl, "1"] \arrow[dd, "3 \cdot 1"] & \\
(0,0,2,0) \arrow[rrr, "2 \cdot 2"] & & & (0,1,1,0) \arrow[rr, "2 \cdot 1"] \arrow[dd, "3 \cdot 1"'] & & (0,2,0,0) \arrow[dd, "3 \cdot 2"] \\
 & & & & (1,0,0,1) \arrow[dl, "1"] & \\
 & & & (1,0,1,0) \arrow[rr, "2 \cdot 1"] & & (1,1,0,0) \arrow[d, "3 \cdot 1"] \\
 & & & & & (2,0,0,0)
\end{tikzcd}
$$

Figure 2.2: The coefficient graph $\mathcal{C}_4(2)$.

**Remark 2.2.3.** *The connection between* (2.2.7) *respectively* (2.2.11) *and the coefficient graph* $\mathcal{C}_3(m)$ *can be described as follows:*

*(i) The monomial* $f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)}$*, respectively* $f_\gamma^{(a)} f_{\beta+2\gamma}^{(b)} f_{2\beta+3\gamma}^{(c)}$*, corresponds to the vertex* $(a,b,c)$*.*

*(ii) The actions on a monomial in (i):*

$$f_\beta^{(a)} f_{\beta+\gamma}^{(b)} \left(\partial_\gamma \left(f_{\beta+2\gamma}^{(c)}\right)\right) \text{ and } f_\beta^{(a)} \left(\partial_\gamma \left(f_{\beta+\gamma}^{(b)}\right)\right) f_{\beta+2\gamma}^{(c)},$$
$$\text{resp. } f_\gamma^{(a)} f_{\beta+2\gamma}^{(b)} \left(\partial_{\beta+\gamma} \left(f_{2\beta+3\gamma}^{(c)}\right)\right) \text{ and } f_\gamma^{(a)} \left(\partial_{\beta+\gamma} \left(f_{\beta+2\gamma}^{(b)}\right)\right) f_{2\beta+3\gamma}^{(c)},$$

*correspond to the vertical and horizontal edges.*

*(iii) The labels correspond to the products of the structure constant and the exponent of the related root vector:*

$$c_{\gamma,\beta+2\gamma} \cdot c \text{ and } c_{\gamma,\beta+\gamma} \cdot b, \text{ resp. } c_{\beta+\gamma,2\beta+3\gamma} \cdot c \text{ and } c_{\beta+\gamma,\beta+2\gamma} \cdot b.$$

*The connection between* (2.2.13) *and the coefficient graph* $\mathcal{C}_4(m)$ *can be described analogously.*

Let $\mathbf{q}$ be a directed path in $\mathcal{C}_3(m)$ (respectively $\mathcal{C}_4(m)$) from $(0,0,m)$ to $x' = (a,b,c) \in X(m)$ (respectively from $(0,0,0,m)$ to $y' = (a,b,c,d) \in Y(m)$). We define $c_{\mathbf{q}} \in \mathbb{Z}_{\geq 0}$ to be the product of the labels on the path $\mathbf{q}$ and call $c_{\mathbf{q}}$ the *coefficient of* $\mathbf{q}$.
For an arbitrary vertex $x' \in X(m)$ (respectively $y' \in Y(m)$) we consider $\mathbf{Q}(x') \subseteq \mathcal{C}_3(m)$ (respectively $\mathbf{Q}(y') \subseteq \mathcal{C}_4(m)$) the set of all directed paths from $(0,0,m)$ to $x'$ in $\mathcal{C}_3(m)$ (respectively the set of directed paths from $(0,0,0,m)$ to $y'$ in $\mathcal{C}_4(m)$).

Remark 2.2.3 implies that each path $\mathbf{q} \in \mathbf{Q}(x')$ (respectively $\mathbf{q} \in \mathbf{Q}(y')$) corresponds to exactly one possibility to generate the monomials

$$f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)} \text{ and } f_\gamma^{(a)} f_{\beta+2\gamma}^{(b)} f_{2\beta+3\gamma}^{(c)} \left(\text{resp. } f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)} f_{\beta+3\gamma}^{(d)}\right)$$

in (2.2.7) and (2.2.11) (resp. in (2.2.13)).

Let us consider exemplary (2.2.7) for $k \leq 2m$ in more detail. The definition of the Kostant lattice and (2.2.2) imply:

$$\begin{aligned}
\partial_\gamma^{(k)} \left(f_{\beta+2\gamma}^{(m)}\right) &= \frac{1}{k!m!} \partial_\gamma^k \left(f_{\beta+2\gamma}^{(m)}\right) \\
&= \frac{1}{k!m!} \left(\sum_{X(k,m)} \hat{r}_x f_\beta^a f_{\beta+\gamma}^b f_{\beta+2\gamma}^c\right), \quad \hat{r}_x \in \mathbb{Z} \\
&= \frac{1}{k!m!} \left(\sum_{X(k,m)} \hat{r}_x \frac{a!b!c!}{a!b!c!} f_\beta^a f_{\beta+\gamma}^b f_{\beta+2\gamma}^c\right) \\
&= \sum_{X(k,m)} \hat{r}_x \frac{a!b!c!}{k!m!} f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)} \\
&= \sum_{X(k,m)} r_x f_\beta^{(a)} f_{\beta+\gamma}^{(b)} f_{\beta+2\gamma}^{(c)}.
\end{aligned}$$

The combination of the above considerations and Remark 2.2.3 let us for $x \in X(m) \setminus \{(0,0,m)\}$ and $y \in Y(m) \setminus \{(0,0,0,m)\}$ conclude:

$$r_x = \hat{r}_x \frac{a!b!c!}{k!m!} = \left( \sum_{\mathbf{q} \in \mathbf{Q}(x)} c_{\mathbf{q}} \right) \frac{a!b!c!}{k!m!} \text{ resp. } r_y = \hat{r}_y \frac{a!b!c!d!}{k!m!} = \left( \sum_{\mathbf{q} \in \mathbf{Q}(y)} c_{\mathbf{q}} \right) \frac{a!b!c!d!}{k!m!},$$

where $k = 2a + b$, respectively $k = 3a + 2b + c$.

**Remark 2.2.4.** *The construction of the coefficient graphs $\mathcal{C}_3(m)$ and $\mathcal{C}_4(m)$ implies the following recursive definition for the constants $\hat{r}_x, \hat{r}_y \in \mathbb{Z}$:*

$$\begin{aligned} \hat{r}_{0,0,m} &= \hat{r}_{0,0,0,m} = 1 \\ \hat{r}_x = \hat{r}_{a,b,c} &= 2(b+1)\hat{r}_{a-1,b+1,c} + (c+1)\hat{r}_{a,b-1,c+1}, \\ \hat{r}_y = \hat{r}_{a,b,c,d} &= 3(b+1)\hat{r}_{a-1,b+1,c,d} + 2(c+1)\hat{r}_{a,b-1,c+1,d} + (d+1)\hat{r}_{a,b,c-1,d+1}. \end{aligned}$$

*Set $\hat{r}_x := 0$ respectively $\hat{r}_y := 0$, if at least one of the entries of $x$, respectively $y$, is a negative integer. The following diagrams visualize the recursive definition:*

$$\begin{tikzcd} & (a,b-1,c+1) \arrow[d, "c+1"] \\ (a-1,b+1,c) \arrow[r, "2(b+1)"] & (a,b,c) \end{tikzcd}$$

Figure 2.3: Recursion for $\hat{r}_x$

$$\begin{tikzcd} & (a-1,b+1,c,d) \arrow[d, "3(b+1)"'] & (a,b,c-1,d+1) \arrow[dl, "d+1"'] \\ (a,b-1,c+1,d) \arrow[r, "2(c+1)"] & (a,b,c,d) & \end{tikzcd}$$

Figure 2.4: Recursion for $\hat{r}_y$

**Lemma 2.2.5.** *Let $r_x, r_y \in \mathbb{Z}$ be the coefficients occurring in (2.2.7), (2.2.11) and (2.2.13). Then we have for all $x \in X(m)$ and $y \in Y(m)$: $r_x = 1 = r_y$.*

*Proof.* In order to use induction on $k$ we define a partial order on the set $X(m)$ respectively $Y(m)$ and apply Remark 2.2.4 in the induction step. We split the proof into two parts

$$\text{(i) } \hat{r}_x = \sum_{\mathbf{q} \in \mathbf{Q}(x)} c_{\mathbf{q}} \overset{!}{=} \frac{k!m!}{a!b!c!} \qquad \text{and} \qquad \text{(ii) } \hat{r}_y = \sum_{\mathbf{q} \in \mathbf{Q}(y)} c_{\mathbf{q}} \overset{!}{=} \frac{k!m!}{a!b!c!d!}.$$

To (i): We recall:

$$X(k,m) = \{x = (a,b,c) \in \mathbb{Z}^3_{\geq 0} \mid a+b+c = m,\ 2a+b = k\},$$
$$X(m) = \bigcup_{k=1}^{2m} X(k,m) = \{x = (a,b,c) \in \mathbb{Z}^3_{\geq 0} \mid a+b+c = m\}.$$

We define the partial order $<$ on $X(m)$ via the following rule:
Let $(a_1,b_1,c_1),(a_2,b_2,c_2) \in X(m)$:

$$(a_1,b_1,c_1) > (a_2,b_2,c_2) :\Leftrightarrow 2a_1 + b_1 := k_1 > k_2 := 2a_2 + b_2. \tag{2.2.18}$$

Hence in this order $(0,0,m) \in X(0,m)$ is the minimal and $(m,0,0) \in X(2m,m)$ the maximal element. We use Remark 2.2.4 to verify our claim for $x = (0,0,m)$:

$$\hat{r}_x = \hat{r}_{0,0,m} = 1 = \frac{0!m!}{0!0!m!}.$$

Let us fix $1 \leq k \leq 2m-1$ and assume that the claim holds for all $x \in \bigcup_{j=1}^{k-1} X(j,m)$. Our aim is to prove that the claim also holds for a arbitrary but fixed element $\breve{x} \in X(k,m)$. Note that (2.2.18) implies $x < \breve{x}$ for all $x \in \bigcup_{j=1}^{k-1} X(j,m)$. We consider the following case analysis:

**Case 1**: $\breve{x} = (a,0,c)$, $k = 2a$. Then Remark 2.2.4 and the induction assumption imply:

$$\hat{r}_{\breve{x}} = \hat{r}_{a,0,c} = 2\hat{r}_{a-1,1,c} = 2\frac{(2(a-1)+1)!m!}{(a-1)!1!c!} = \frac{(2a)!m!}{a!0!c!}.$$

**Case 2**: $\breve{x} = (0,b,c)$, $k = b$. Then Remark 2.2.4 and the induction assumption imply:

$$\hat{r}_{\breve{x}} = \hat{r}_{0,b,c} = (c+1)\hat{r}_{0,b-1,c+1} = (c+1)\frac{(b-1)!m!}{0!(b-1)!(c+1)!} = \frac{b!m!}{0!b!c!}.$$

**Case 3**: $\breve{x} = (a,b,c)$, $k = 2a+b$. Then Remark 2.2.4 and the induction assumption imply:

$$\begin{aligned}\hat{r}_{\breve{x}} = \hat{r}_{a,b,c} &= 2(b+1)\hat{r}_{a-1,b+1,c} + (c+1)\hat{r}_{a,b-1,c+1}\\ &= 2(b+1)\frac{(2(a-1)+b+1)!m!}{(a-1)!(b+1)!c!} + (c+1)\frac{(2a+b-1)!m!}{a!(b-1)!(c+1)!}.\\ &= \frac{(2a+b)!m!}{a!b!c!}.\end{aligned}$$

Hence we have $\hat{r}_x = \frac{k!m!}{a!b!c!}$ and therefore $r_x = 1$ for all $x \in X(m)$.

The proof of part (ii) proceeds similarly by using Remark 2.2.4. □

**$\mathbb{Z}$–admissible elements.** Let $k_l \in \mathbb{Z}_{\geq 0}$ and $\beta_{i_l} \in \Delta_+$ for $1 \leq l \leq r \in \mathbb{Z}_{\geq 0}$. Further we consider the differential operator

$$\boldsymbol{\partial} = \prod_{l=1}^{r} \partial_{\beta_{i_l}}^{(k_l)} \in \mathrm{Der}(S_{\mathbb{Z}}(\mathfrak{n}^{-,a})).$$

In the following we consider for $b \in \mathbb{Z}_{\geq 0}$ and $\beta \in \Delta_+$:

$$\boldsymbol{\partial}\left(f_\beta^{(b)}\right) = \prod_{l=1}^{r} \partial_{\beta_{i_l}}^{(k_l)}\left(f_\beta^{(b)}\right) = \sum_{\mathbf{t}\in\mathbb{Z}_{\geq 0}^N} c_{\mathbf{t}} f^{(\mathbf{t})}. \tag{2.2.19}$$

We fix a total order $\prec$ on the positive roots $\Delta_+$ and in addition an induced homogeneous total order on the monomials in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ (by abuse of notation we denote the latter total order also by $\prec$). For an element of the form (2.2.19) we define:

$$\max_{\prec}\left(\boldsymbol{\partial}\left(f_\beta^{(b)}\right)\right) = f^{(\mathbf{m})}, \quad \mathbf{m} := \max_{\prec}\{\mathbf{t}\in\mathbb{Z}_{\geq 0}^N \mid c_{\mathbf{t}} \neq 0\}.$$

Further is an element of the form (2.2.19) said to satisfy the *maximality condition*, if for all $1 \leq v \leq r$:

$$\begin{aligned} c_{\mathbf{m}^v} f^{(\mathbf{m}^v)} &= \max_{\prec}\left(\partial_{\beta_{i_v}}^{(k_v)} \prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{(k_l)}\left(f_\beta^{(b)}\right)\right) \\ &= \max_{\prec}\left(\partial_{\beta_{i_v}}^{(k_v)}\left(\max_{\prec}\left(\prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{(k_l)}\left(f_\beta^{(b)}\right)\right)\right)\right). \end{aligned} \tag{2.2.20}$$

**Remark 2.2.6.** *Note that the maximality condition* (2.2.20)*implies that the maximal monomial in $\prod_{l=1}^{v} \partial_{\beta_{i_l}}^{(k_l)}(f_\beta^{(b)})$ is a summand of $\partial_{\beta_{i_v}}^{(k_v)}$ applied to the maximal monomial of $\prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{(k_l)}(f_\beta^{(b)})$.*

In the following we verify the Leibniz rule for the divided power analogue of $\partial_\gamma^k(f^{\mathbf{m}})$, $\mathbf{m} = (m_1, \ldots, m_N)$, $k \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} \partial_\gamma^{(k)}\left(f^{(\mathbf{m})}\right) &= \partial_\gamma^{(k)}\left(\prod_{j=1}^{N} f_{\beta_j}^{(m_j)}\right) \\ &= \frac{1}{k!m_1!\cdots m_N!}\partial_\gamma^k\left(\prod_{j=1}^{N} f_{\beta_j}^{m_j}\right) \\ &= \frac{1}{k!m_1!\cdots m_N!}\sum_{X_k}\binom{k}{\kappa_1,\ldots,\kappa_N}\prod_{j=1}^{N}\partial_\gamma^{\kappa_j}\left(f_{\beta_j}^{m_j}\right) \\ &= \sum_{X_k}\prod_{j=1}^{N}\partial_\gamma^{(\kappa_j)}\left(f_{\beta_j}^{(m_j)}\right), \end{aligned} \tag{2.2.21}$$

where $X_k = \{(\kappa_1, \ldots, \kappa_N) \in \mathbb{Z}_{\geq 0}^N \mid \sum_{j=1}^N \kappa_j = k\}$ is the set of all partitions of $k \in \mathbb{Z}_{\geq 0}$ of length $N$ and $\binom{k}{\kappa_1,\ldots,\kappa_N}$ is the multimonomialcoefficient defined by:

$$\binom{k}{\kappa_1,\ldots,\kappa_N} := \frac{k!}{\kappa_1!\cdots\kappa_N!} \quad \text{for} \quad k = \sum_{i=1}^{N}\kappa_i, \ \kappa_i \in \mathbb{Z}_{\geq 0}.$$

**Remark 2.2.7.** *Note that a summand of* (2.2.21) *is equal to* 0*, if at least one of the factors in the product is equal to* 0*. More explicit:* $\prod_{j=1}^{N} \partial_\gamma^{(\kappa_j)}(f_{\beta_j}^{(m_j)}) = 0$*, if one of the following statements is true: There is* $1 \leq j' \leq N$*, such that*

- $(\kappa_{j'} > 0) \wedge (m_{j'} = 0)$.
- $(\kappa_{j'} > 0) \wedge (m_{j'} > 0) \wedge (\beta_{j'} + \gamma \notin \Delta_+)$.
- $\kappa_{j'}$ *is greater, than the corresponding bound, given in* (2.2.3)*–*(2.2.17)*.*

Thereby we describe the action of the differential operator $\partial_\gamma^{(k_v)}$ on the maximal monomial more explicit: Let $c_{\mathbf{m}^{v-1}} f^{(\mathbf{m}^{v-1})} = \max\limits_{\prec} (\prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{(k_l)}(f_\beta^{(b)}))$, then:

$$\partial_{\beta_{i_v}}^{(k_v)} \left( c_{\mathbf{m}^{v-1}} f^{(\mathbf{m}^{v-1})} \right) = c_{\mathbf{m}^{v-1}} \sum_{X_{k_v}} \prod_{j=1}^{N} \partial_{\beta_{i_v}}^{(\kappa_j)} \left( f_{\beta_j}^{(m_j^{v-1})} \right) = c_{\mathbf{m}^v} f^{\mathbf{m}^v} + \sum_{\mathbf{t} \prec \mathbf{m}^v} c_{\mathbf{t}} f^{\mathbf{t}},$$

where $X_{k_v} \subset \mathbb{Z}_{\geq 0}^N$ is the set of all partitions of $k_v \in \mathbb{Z}_{\geq 0}$ of length $N$.

**Remark 2.2.8.** *For simplicity we make the following convention. If* $\partial_\gamma^{(k)}(f_\beta^{(m)})$ *is of the form* (2.2.3)*–*(2.2.5)*,* (2.2.7)*–*(2.2.13) *or* (2.2.16)*–*(2.2.17)*, then we simply write* $\partial_\gamma^{(k)}(f_\beta^{(m)}) \in \mathbb{D}$*. Lemma 2.2.5 and Remark 2.2.1 imply, that* $\mathbb{D}$ *contains exclusively differential operations, which produce only coefficients equal to* 1*.*

Fix a positive root $\gamma$ and let $\beta_j \in \Delta_+$ be such that $\gamma$ and $\beta_j$ are linearly independent roots. We consider the $\gamma$–string through $\beta_j$, to be more precise, we consider for $q, p \in \mathbb{Z}_{\geq 0}$ the following subset of $\Delta_+$:

$$\{\beta_{x_s} = \beta_j + s\gamma \mid -p \leq s \leq q\} = \{\beta_{x_{-p}} = \beta_j - p\gamma, \ldots, \beta_{x_q} = \beta_j + q\gamma\}. \quad (2.2.22)$$

Note, that $\Delta_+$ decomposes in a disjoint union of $\gamma$–strings (see (2.2.26) and (2.2.27)). The next definition is essential for the main statement of the present chapter.

**Definition 2.2.9.** *Let* $\boldsymbol{\partial} = \prod_{l=1}^{r} \partial_{\beta_{i_l}}^{(k_l)} \in \mathrm{Der}(S_{\mathbb{Z}}(\mathfrak{n}^{-,a}))$ *be a sequence of differential operators and assume that* $\boldsymbol{\partial}(f_\beta^{(b)})$ *in* $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ *satisfies the maximality condition* (2.2.20)*. For* $v = 0$*, respectively for* $1 \leq v \leq r$ *we set*

$$c_{\mathbf{m}^0} f^{(\mathbf{m}^0)} := f_\beta^{(b)}, \text{ resp. } c_{\mathbf{m}^v} f^{(\mathbf{m}^v)} := \max_{\prec} \left( \prod_{l=1}^{v} \partial_{\beta_{i_l}}^{(k_l)} \left( f_\beta^{(b)} \right) \right), \quad (2.2.23)$$

*with* $\mathbf{m}^v = (m_1^v, \ldots, m_N^v) \in \mathbb{Z}_{\geq 0}^N$*.*
*We call* $\boldsymbol{\partial}(f_\beta^{(b)})$ $\mathbb{Z}$*–admissible, if for all* $1 \leq v \leq r$*,* $\mathbf{m}^{v-1}$ *satisfies the following assumption:*

*Fix an arbitrary* $\beta_{i_v}$*–string* $\{\beta_{x_{-p}}, \ldots, \beta_{x_q}\}$ *in* $\Delta_+$ *(see* (2.2.22)*).*
*Then we have either*

$$m_{x_s}^v = m_{x_s}^{v-1}, \ \forall \ -p \leq s \leq q, \quad (2.2.24)$$

*where $m^v_{x_s}$ is the entry of $\mathbf{m}^v$ corresponding to the positive root $\beta_{x_s}$.*
*Or there is a integer $-p \leq \underline{s} < q$ minimal with the property: $m^v_{x_{\underline{s}}} \neq m^{v-1}_{x_{\underline{s}}}$ and moreover:*

$$m^{v-1}_{x_s} = 0, \ \ \forall \ \underline{s} \leq s < q. \tag{2.2.25}$$

**Remark 2.2.10.** *Let us explain the assumptions (2.2.24) and (2.2.25):*
*In the $v$–th step of the sequence $\boldsymbol{\partial}$ we apply $\partial^{(k_v)}_{\beta_{i_v}}$ to the maximal monomial of the $(v-1)$–th step: $c_{\mathbf{m}^{v-1}} f^{\mathbf{m}^{(v-1)}}$ (see 2.2.23). In order to describe our assumptions on this action, it is enough to consider the $\beta_{i_v}$–strings in $\Delta_+$. Let us fix now a $\beta_{i_v}$–string (we refer to (2.2.22), if we consider a fixed $\beta_{i_v}$–string in $\Delta_+$):*

***Case 1.*** *All exponents of the root vectors corresponding to the fixed $\beta_{i_v}$–string are not affected by the operator $\partial^{(k_v)}_{\beta_{i_v}}$ (see (2.2.24)). In that case we have no further assumptions on the corresponding exponents.*

***Case 2.*** *The exponents of the root vectors corresponding to the fixed $\beta_{i_v}$–string are affected by the operator $\partial^{(k_v)}_{\beta_{i_v}}$ (see (2.2.24)). In that case there is a integer $-p \leq \underline{s} \leq q$, minimal with the property, that the corresponding exponent $m^{v-1}_{x_{\underline{s}}}$ has been changed by the action of $\partial^{(k_v)}_{\beta_{i_v}}$. Further the exponents corresponding to the fixed $\beta_{i_v}$–string have the following form (see (2.2.25)):*

$$f^{(m^{v-1}_{x_{-p}})}_{\beta_{x_{-p}}} \cdots f^{(m^{v-1}_{x_{\underline{s}-1}})}_{\beta_{x_{\underline{s}-1}}} f^{(m^{v-1}_{x_p})}_{\beta_{x_p}} = f^{(m^{v-1}_{x_{-p}})}_{\beta_{x_{-p}}} \cdots f^{(m^{v-1}_{x_{\underline{s}-1}})}_{\beta_{x_{\underline{s}-1}}} f^{(0)}_{\beta_{x_{\underline{s}}}} \cdots f^{(0)}_{\beta_{x_{p-1}}} f^{(m^{v-1}_{x_p})}_{\beta_{x_p}}.$$

*The second case implies $m^{v-1}_{x_q} > 0$, further note that these assumptions have to be satisfied for all $1 \leq v \leq r$.*

**Example 2.2.11.** *Let $\Delta_+ = \{\beta_1 := \beta, \beta_2 := \gamma, \beta_3 := \beta+\gamma, \beta_4 := \beta + 2\gamma\}$ be the $\mathtt{C}_2$–root system. Further we choose the following total order on $\Delta_+$: $\beta_4 \prec \beta_3 \prec \beta_2 \prec \beta_1$ and choose in addition the induced homogenous lexicographic total order on the monomials.*
*In the following we consider the elements*

$$\boldsymbol{\partial}_1\left(f^{(4)}_{\beta+2\gamma}\right) = \left(\partial^{(2)}_{\beta+\gamma}\partial^{(3)}_{\gamma}\right)\left(f^{(4)}_{\beta+2\gamma}\right) \in S_{\mathbb{Z}}(\mathfrak{n}^{-,a}),$$
$$\boldsymbol{\partial}_2\left(f^{(4)}_{\beta+2\gamma}\right) = \left(\partial^{(3)}_{\gamma}\partial^{(2)}_{\beta+\gamma}\right)\left(f^{(4)}_{\beta+2\gamma}\right) \in S_{\mathbb{Z}}(\mathfrak{n}^{-,a}).$$

*and check if these two elements are $\mathbb{Z}$–admissible. For that we use the definition of the $\mathtt{C}_2$ differential operators (2.2.4)–(2.2.7), the statement of (2.1.2) and the Definition 2.2.9. Further we need the description of $\Delta_+$ as union of $\gamma$–strings and as union of $(\beta+\gamma)$–strings:*

$$\Delta_+ = \{\gamma\} \cup \{\beta, \beta+\gamma, \beta+2\gamma\}. \tag{2.2.26}$$
$$\Delta_+ = \{\beta\} \cup \{\beta+\gamma\} \cup \{\gamma, \beta+2\gamma\}. \tag{2.2.27}$$

*The definition of the differential operators (2.2.1) implies:*
$\partial_\gamma(f_\gamma) = \partial_{\beta+\gamma}(f_\beta) = \partial_{\beta+\gamma}(f_{\beta+\gamma}) = 0.$

*Let us consider* $\boldsymbol{\partial}_1(f^{(4)}_{\beta+2\gamma})$*. First we calculate* $\boldsymbol{\partial}_1(f^{(4)}_{\beta+2\gamma})$ *explicitly:*

$$\begin{aligned}
\boldsymbol{\partial}_1\left(f^{(4)}_{\beta+2\gamma}\right) &= \left(\partial^{(2)}_{\beta+\gamma}\partial^{(3)}_{\gamma}\right)\left(f^{(4)}_{\beta+2\gamma}\right) = \partial^{(2)}_{\beta+\gamma}\left(\partial^{(3)}_{\gamma}\left(f^{(4)}_{\beta+2\gamma}\right)\right) \\
&= \partial^{(2)}_{\beta+\gamma}\left(\sum_{X(3,4)} f^{(a)}_{\beta} f^{(b)}_{\beta+\gamma} f^{(c)}_{\beta+2\gamma}\right) \\
&= \partial^{(2)}_{\beta+\gamma}\left(f^{(1)}_{\beta} f^{(1)}_{\beta+\gamma} f^{(2)}_{\beta+2\gamma} + f^{(0)}_{\beta} f^{(3)}_{\beta+\gamma} f^{(1)}_{\beta+2\gamma}\right) \\
&= f^{(1)}_{\beta} f^{(2)}_{\gamma} f^{(1)}_{\beta+\gamma}.
\end{aligned}$$

*From the above calculation we conclude that* $\boldsymbol{\partial}_1(f^{(4)}_{\beta+2\gamma})$ *satisfies the maximality condition* (2.2.20) *and deduce the maximal monomials:*

$$\begin{aligned}
c_{\mathbf{m}^0} f^{\mathbf{m}^0} &= f^{(0)}_{\beta} f^{(0)}_{\gamma} f^{(0)}_{\beta+\gamma} f^{(4)}_{\beta+2\gamma}, \\
c_{\mathbf{m}^1} f^{\mathbf{m}^1} &= f^{(1)}_{\beta} f^{(0)}_{\gamma} f^{(1)}_{\beta+\gamma} f^{(2)}_{\beta+2\gamma}, \\
c_{\mathbf{m}^2} f^{\mathbf{m}^2} &= f^{(1)}_{\beta} f^{(2)}_{\gamma} f^{(1)}_{\beta+\gamma} f^{(0)}_{\beta+2\gamma}.
\end{aligned}$$

*It is easy to verify that the maximal monomials* $f^{\mathbf{m}^1}$ *and* $f^{\mathbf{m}^2}$ *satisfy* (2.2.24) *and* (2.2.25) *for all* $\gamma$*–strings in* $\Delta_+$ *(see* (2.2.26)*), respectively for all* $(\beta+\gamma)$*–strings in* $\Delta_+$ *(see* (2.2.27)*). Thus,* $\boldsymbol{\partial}_1(f^{(4)}_{\beta+2\gamma})$ *is a* $\mathbb{Z}$*–admissible element.*

*Now we perform the same procedure with* $\boldsymbol{\partial}_2(f^{(4)}_{\beta+2\gamma})$*:*

$$\begin{aligned}
\boldsymbol{\partial}_2\left(f^{(4)}_{\beta+2\gamma}\right) &= \left(\partial^{(3)}_{\gamma}\partial^{(2)}_{\beta+\gamma}\right)\left(f^{(4)}_{\beta+2\gamma}\right) = \partial^{(3)}_{\gamma}\left(\partial^{(2)}_{\beta+\gamma}\left(f^{(4)}_{\beta+2\gamma}\right)\right) \\
&= \partial^{(3)}_{\gamma}\left(f^{(2)}_{\gamma} f^{(2)}_{\beta+2\gamma}\right) = f^{(2)}_{\gamma}\left(\sum_{X(3,2)} f^{(a)}_{\beta} f^{(b)}_{\beta+\gamma} f^{(c)}_{\beta+2\gamma}\right) \\
&= f^{(2)}_{\gamma}\left(f^{(1)}_{\gamma} f^{(1)}_{\beta+\gamma}\right) = 3 f^{(3)}_{\gamma} f^{(1)}_{\beta+\gamma}.
\end{aligned}$$

*Again from the explicit calculation we conclude that* $\boldsymbol{\partial}_2(f^{(4)}_{\beta+2\gamma})$ *satisfies the maximality condition* (2.2.20) *and deduce the maximal monomials:*

$$\begin{aligned}
c_{\mathbf{m}^0} f^{\mathbf{m}^0} &= f^{(0)}_{\beta} f^{(0)}_{\gamma} f^{(0)}_{\beta+\gamma} f^{(4)}_{\beta+2\gamma}, \\
c_{\mathbf{m}^1} f^{\mathbf{m}^1} &= f^{(0)}_{\beta} f^{(2)}_{\gamma} f^{(0)}_{\beta+\gamma} f^{(2)}_{\beta+2\gamma}, \\
c_{\mathbf{m}^2} f^{\mathbf{m}^2} &= 3 f^{(0)}_{\beta} f^{(3)}_{\gamma} f^{(1)}_{\beta+\gamma} f^{(0)}_{\beta+2\gamma}.
\end{aligned}$$

*Again the maximal monomial* $f^{\mathbf{m}^1}$ *satisfies* (2.2.24) *and* (2.2.25) *for all* $(\beta+\gamma)$*–strings in* (2.2.27)*. However the maximal monomial* $f^{\mathbf{m}^2}$ *violates* (2.2.25)*. Thus,* $\boldsymbol{\partial}_2(f^{(4)}_{\beta+2\gamma})$ *is not a* $\mathbb{Z}$*–admissible element.*

Let us now investigate the properties of $\mathbb{Z}$–admissible elements.

**Corollary 2.2.12.** *Let $\boldsymbol{\partial}(f_\beta^{(b)}) = \prod_{l=1}^r \partial_{\beta_{i_l}}^{(k_l)}(f_\beta^{(b)})$ be a $\mathbb{Z}$–admissible element in $S_\mathbb{Z}(\mathfrak{n}^{-,a})$. Then there is for all $1 \leq v \leq r$ a unique partition $\hat{\kappa} \in X_{k_v}$, such that:*

$$c_{\mathbf{m}^v} f^{(\mathbf{m}^v)} \quad \textit{is a summand of} \quad c_{\mathbf{m}^{v-1}} \prod_{j=1}^N \partial_{\beta_{i_v}}^{(\hat{\kappa}_j)}(f_{\beta_j}^{(m_j^{v-1})}). \tag{2.2.28}$$

*Proof.* Assume there are $\hat{\kappa}, \bar{\kappa} \in X_{k_v}$ such that

$$c_{\mathbf{m}_v} f^{(\mathbf{m}_v)} \text{ is a summand of } \prod_{j=1}^N \partial_{\beta_{i_v}}^{(\hat{\kappa}_j)}(f_{\beta_j}^{(m_j^{v-1})}) \text{ and also of } \prod_{j=1}^N \partial_{\beta_{i_v}}^{(\bar{\kappa}_j)}(f_{\beta_j}^{(m_j^{v-1})}).$$

For all $1 \leq j \leq N$ with $m_j^{v-1} = 0$ or $\beta_j - \beta_{i_v} \notin \Delta_+$ we have: $\hat{\kappa}_j = 0 = \bar{\kappa}_j$.
Hence it is enough to consider only $\beta_{i_v}$–strings. For all $\beta_{i_v}$–strings through in $\Delta_+$, where $m_{x_s}^v = m_{x_s}^{v-1}$ for all $-r \leq s \leq q$ we have again $\hat{\kappa}_j = 0 = \bar{\kappa}_j$, because there is no influence via differential operators from inside of the $\beta_{i_v}$–string and there is no possibility to change the exponent from outside of the corresponding string.
Let us now consider a $\beta_{i_v}$–string, which is affected by differential operators, then we know from (2.2.25), that every change of the exponents of root vectors corresponding to this string is the result of one single differential operation $\partial_{\beta_{i_v}}$ to a certain power $\kappa \leq k_v$ applied to the root vector corresponding to $\beta_{x_p}$ (see (2.2.22)).
From this we conclude that there is only one possible choice for the entries of $\hat{\kappa}$ (respectively of $\bar{\kappa}$) corresponding to this $\beta_{i_v}$–string and thus we have for these entries: $\hat{\kappa}_{x_s} = \bar{\kappa}_{x_s}$ for all $-r \leq s \leq q$. This is true for all $\beta_{i_v}$–strings affected by $\hat{\kappa}$ (respectively by $\bar{\kappa}$), hence: $\hat{\kappa} = \bar{\kappa}$. □

**Remark 2.2.13.** *As direct consequence from the assumption (2.2.25), we conclude for the unique partition $\hat{\kappa} \in X_{k_v}$ (see (2.2.28)) that for all $\hat{\kappa}_j \neq 0$:*

$$\partial_{\beta_{i_v}}^{(\hat{\kappa}_j)}\left(f_{\beta_j}^{(m_j^{v-1})}\right) \in \mathbb{D}.$$

*See Remark 2.2.8 for the definition of $\mathbb{D}$.*

Now we are able to state the main advantage of $\mathbb{Z}$–admissible elements.

**Lemma 2.2.14.** *Let $\boldsymbol{\partial}(f_\beta^{(b)}) \neq 0$ be $\mathbb{Z}$–admissible and let $c_\mathbf{m} f^{(\mathbf{m})} = \max_{\prec} (\boldsymbol{\partial}(f_\beta^{(b)}))$. Then we have: $c_\mathbf{m} = 1$.*

*Proof.* Let $\boldsymbol{\partial}(f_\beta^{(b)}) = (\prod_{l=1}^r \partial_{\beta_{i_l}}^{(k_l)})(f_\beta^{(b)})$. In order to prove the statement we prove by induction for all $1 \leq v \leq r$ that the coefficient $c_{\mathbf{m}^v}$ of

$$f^{(\mathbf{m}^v)} = \max_{\prec} \left( \prod_{l=1}^v \partial_{\beta_{i_l}}^{(k_l)} \left( f_\beta^{(b)} \right) \right)$$

is equal to 1. Therefore, let $v = 1$. Then we know from the Remark 2.2.13, that the differential operation of $f^{(\mathbf{m}^1)}$ is in $\mathbb{D}$ and thus $c_{\mathbf{m}^1} = 1$.

Assume the statement holds for a fixed $\bar{v} \leq r-1$ and all $1 \leq v' \leq \bar{v}$. We consider $v = \bar{v}+1$. The induction assumption implies that the coefficient of $f^{(\mathbf{m}^{\bar{v}})}$ is equal to 1, hence

$$\partial_{\beta_{i_v}}^{(k_v)} \left( \max_{\prec} \left( \prod_{l=1}^{\bar{v}} \partial_{\beta_{i_l}}^{(k_l)} (f_\beta^{(b)}) \right) \right) = \partial_{\beta_{i_v}}^{(k_v)} \left( f^{(\mathbf{m}^{\bar{v}})} \right) = \sum_{X_{k_v}} \prod_{j=1}^{N} \partial_{\beta_{i_v}}^{(\kappa_j)} \left( f_{\beta_j}^{(m_j^{\bar{v}})} \right).$$

From Corollary 2.2.12 we know that the multi–exponent $\hat{\kappa} \in X_{k_v}$, where $c_{\mathbf{m}^v} f^{(\mathbf{m}^v)}$ occurs as summand in the corresponding product of operators, is unique. Remark 2.2.13 implies that for all $\hat{\kappa}_j \neq 0$ we have $\partial_{\beta_{i_v}}^{(\hat{\kappa}_j)}(f_{\beta_j}^{(m_j^{\bar{v}})}) \in \mathbb{D}$.
Therefore, the coefficients of these operations are all equal to 1. Hence there is only one other possibility left to influence the coefficient $c_{\mathbf{m}^v}$:
Fix a root $\beta_{x_i}$, $-p \leq i < q$ in a $\beta_{i_v}$–string and assume $m_{x_i}^{\bar{v}} > 0$. Assume further, that the exponent of the corresponding root vector in $f^{(\mathbf{m}^{\bar{v}})}$ is not affected by a differential operator. If in addition the action on $f_{\beta_{x_p}}^{m^{\bar{v}}}$ generates a non–trivial factor $f_{\beta_{x_i}}^{(l)}$, then the multiplication rule (2.1.2) says, that

$$f_{\beta_{x_i}}^{(m_{x_i}^{\bar{v}})} f_{\beta_{x_i}}^{(l)} = \binom{m_{x_i}^{\bar{v}} + l}{m_{x_i}^{\bar{v}}} f_{\beta_{x_i}}^{(m_{x_i}^{\bar{v}}+l)}, \text{ with } \binom{m_{x_i}^{\bar{v}} + l}{m_{x_i}^{\bar{v}}} \geq 1.$$

However (2.2.25) implies, that this situation cannot occur. Therefore, is $c_{\mathbf{m}^v} = 1$ and we conclude $c_{\mathbf{m}} = 1$. □

## 2.3 Bases for $V_{\mathbb{Z}}^a(\lambda)$

Let $\lambda$ be a dominant integral weight and $V^a(\lambda)$ be the associated graded space of the corresponding highest weight $\mathfrak{g}$–module $V(\lambda)$. Further let $\mathbb{B}(V^a(\lambda))$ be a basis of $V^a(\lambda)$ with the following properties:

**Property (1).** There is a set of abstract paths $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_w\} \subset \mathcal{P}(\Delta_+)$ and each abstract path $\mathbf{p}_q$, $1 \leq q \leq w$, contains a root $\beta_{j_q} \in \Delta_+$ such that for every multi–exponent $\mathbf{m} \in \mathbb{Z}_{\geq 0}^N$ with $|\mathbf{m}| \geq \lambda(\beta_{j_q}^\vee)+1$, which is supported on $\mathbf{p}_q$, there is a differential operator $\boldsymbol{\partial}(\mathbf{m}, \mathbf{p}_q)$ depending on $\mathbf{m}$ and $\mathbf{p}_q$ with

$$\boldsymbol{\partial}(\mathbf{m}, \mathbf{p}_q) \left( f_{\beta_{j_q}}^{|\mathbf{m}|} \right) = \prod_{l=1}^{r} \partial_{\beta_{i_l}}^{k_l} \left( f_{\beta_{j_q}}^{|\mathbf{m}|} \right) = c_{\mathbf{m}} f^{\mathbf{m}} + \sum_{\mathbf{t} \prec \mathbf{m}} c_{\mathbf{t}} f^{\mathbf{t}}, \; c_{\mathbf{m}} \neq 0. \tag{2.3.1}$$

We call $\beta_{j_q}$ the *base root* of $\mathbf{p}_q$ and remark that the operators $\partial_{\beta_{i_l}}$, $1 \leq l \leq r$ in (2.3.1) depend on the support of $\mathbf{p}_q$ but they are independent of $|\mathbf{m}|$ different to the exponents $k_l \in \mathbb{Z}_{\geq 0}$.

**Property (2).** For all $1 \leq q \leq w$ and $1 \leq v \leq r$ the element $\boldsymbol{\partial}(\mathbf{m}, \mathbf{p}_q)(f_{\beta_{j_q}}^{|\mathbf{m}|})$ satisfies the complex analogue of the maximality condition (2.2.20):

$$\begin{aligned} c_{\mathbf{m}^v} f^{\mathbf{m}^v} &= \max_{\prec} \left( \partial_{\beta_{i_v}}^{k_v} \prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{k_l} \left( f_\beta^b \right) \right) \\ &= \max_{\prec} \left( \partial_{\beta_{i_v}}^{k_v} \left( \max_{\prec} \left( \prod_{l=1}^{v-1} \partial_{\beta_{i_l}}^{k_l} \left( f_\beta^b \right) \right) \right) \right). \end{aligned}$$

**Property (3).** The basis $\mathbb{B}(V^a(\lambda))$ is parametrized by the integer points $S(\mathbf{P})$ of the polytope

$$P(\mathbf{P}) := \{\mathbf{x} \in \mathbb{R}^N_{\geq 0} \mid \sum_{\beta_i \in \mathbf{p}_q} x_i \leq \lambda(\beta^\vee_{j_q}),\ \forall\ \mathbf{p}_q \in \mathbf{P}\}, \tag{2.3.2}$$

where $\beta_{j_q} \in \Delta_+$ is the base root of $\mathbf{p}_q$. This implies that $\mathbb{B}(V^a(\lambda))$ admits the following description:

$$\mathbb{B}(V^a(\lambda)) = \{f^{\mathbf{s}}v_\lambda \mid \mathbf{s} \in S(\mathbf{P})\},\ \text{where } S(\mathbf{P}) := P(\mathbf{P}) \cap \mathbb{Z}^N_{\geq 0}.$$

**Remark 2.3.1.** *We note that the assumption in Property (1) on the multi–exponent $\mathbf{m} \in \mathbb{Z}^N_{\geq 0}$, $|\mathbf{m}| \geq\ \lambda(\beta^\vee_{j_q}) + 1$, guarantees that the element $f^{|\mathbf{m}|}_{\beta_{j_q}}$ is in the ideal $I(\lambda)$ (see (1.1.2)). Further we know from Remark 0.0.6 that $I(\lambda)$ is invariant under the action of $U(\mathfrak{n}^+)$, so $I(\lambda)$ is also invariant under sequences of differential operators. Hence we conclude that $\boldsymbol{\partial}(\mathbf{m},\mathbf{p}_q)(f^{|\mathbf{m}|}_{\beta_{j_q}})$ is an element of $I(\lambda)$.*

In this section we consider the $\mathbb{Z}$–analogue of the basis $\mathbb{B}(V^a(\lambda))$

$$\mathbb{B}(V^a_{\mathbb{Z}}(\lambda)) := \{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s} \in S(\mathbf{P})\}$$

and show that under an additional assumption $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$ is a basis of $V^a_{\mathbb{Z}}(\lambda)$. Denote by $\mathbf{P}$ the set of abstract paths corresponding to $\mathbb{B}(V^a(\lambda))$ (see Property (1)) and by

$$\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q)\left(f^{(|\mathbf{m}|)}_{\beta_{j_q}}\right) := \prod_{l=1}^{r} \partial^{(k_l)}_{\beta_{i_l}}\left(f^{(|\mathbf{m}|)}_{\beta_{j_q}}\right) \in S_{\mathbb{Z}}(\mathfrak{n}^{-,a}) \tag{2.3.3}$$

the element corresponding to the differential operator
$\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q) \in \operatorname{Der}(S_{\mathbb{Z}}(\mathfrak{n}^{-,a}))$. As above the operator depends on the multi–exponent $\mathbf{m} \in \mathbb{Z}^N_{\geq 0}$ and on the abstract path $\mathbf{p}_q \in \mathbf{P}$, where $\beta_{j_q} \in \Delta_+$ is the base root of $\mathbf{p}_q$.

In the following we show, that if the elements (2.3.3) are $\mathbb{Z}$–admissible, then they satisfy the $\mathbb{Z}$–analogue of (2.3.1). This leads to the fact, that $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$ is a spanning set of $V^a_{\mathbb{Z}}(\lambda)$.

**Proposition 2.3.2.** *Let $\mathbf{p}_q \in \mathbf{P}$ be an abstract path and let $\mathbf{m} \in \mathbb{Z}^N_{\geq 0}$ be a multi–exponent supported in $\mathbf{p}_q$, with $|\mathbf{m}| \geq\ \lambda(\beta^\vee_{j_q})+1$. Further let $\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q) \in$ $\operatorname{Der}(S_{\mathbb{Z}}(\mathfrak{n}^{-,a}))$ satisfy Property (1). Assume that $\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q)$ is $\mathbb{Z}$–admissible, then there exist some constants $c'_{\mathbf{t}} \in \mathbb{Z}$, such that*

$$\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q)(f^{(|\mathbf{m}|)}_{\beta_{j_q}}) = f^{(\mathbf{m})} + \sum_{\mathbf{t} \prec \mathbf{m}} c'_{\mathbf{t}} f^{(\mathbf{t})} \in I_{\mathbb{Z}}(\lambda). \tag{2.3.4}$$

**Remark 2.3.3.** *We refer to (2.3.4) as a straightening law, because it implies*

$$f^{(\mathbf{m})} = -\sum_{\mathbf{t} \prec \mathbf{m}} c'_{\mathbf{t}} f^{(\mathbf{t})} \text{ in } S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda) \cong V^a_{\mathbb{Z}}(\lambda).$$

*The assumption $|\mathbf{m}| \geq\ \lambda(\beta^\vee_{j_q}) + 1$ guarantees, that we consider elements of the ideal $I_{\mathbb{Z}}(\lambda)$.*

*Proof.* Let $\mathbf{p}_q$, $\mathbf{m}$ and $\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q)$ be as assumed above. From the definition of the Kostant lattice we know, that (2.3.3) differs from (2.3.1) only by the constant $c = (k_1! \cdots k_r! |\mathbf{m}|!)^{-1} \in \mathbb{Q}$, hence

$$\begin{aligned}\boldsymbol{\partial}((\mathbf{m}),\mathbf{p}_q)\left(f_{\beta_{j_q}}^{(|\mathbf{m}|)}\right) = c\boldsymbol{\partial}(\mathbf{m},\mathbf{p}_q)\left(f_{\beta_{j_q}}^{|\mathbf{m}|}\right) &= c\left(c_{\mathbf{m}} f^{\mathbf{m}} + \sum_{\mathbf{t}\prec\mathbf{m}} c_{\mathbf{t}} f^{\mathbf{t}}\right)\\ &= c'_{\mathbf{m}} f^{(\mathbf{m})} + \sum_{\mathbf{t}\prec\mathbf{m}} c'_{\mathbf{t}} f^{(\mathbf{t})},\end{aligned}$$

where $c_{\mathbf{m}}, c_{\mathbf{t}}, c'_{\mathbf{m}}, c'_{\mathbf{t}} \in \mathbb{Z}$. Therefore, it is enough to show that: $c'_{\mathbf{m}} = 1$, but from Lemma 2.2.14 we know that the coefficient of the maximal monomial of a $\mathbb{Z}$–admissible element is always equal to 1. This proves the claim. □

**Theorem 2.3.4.** $\mathbb{B}(V_{\mathbb{Z}}^a(\lambda)) = \{f^{(\mathbf{s})}v_\lambda \mid \mathbf{s} \in S(\mathbf{P})\}$ *spans the module* $V_{\mathbb{Z}}^a(\lambda)$.

*Proof.* The idea of the proof is to use the equation (2.3.4) as a straightening algorithm to express $f^{(\mathbf{m})}v_\lambda$, $\mathbf{m} \in \mathbb{Z}_{\geq 0}^N$ arbitrary, as a $\mathbb{Z}$–linear combination of elements in $\mathbb{B}(V_{\mathbb{Z}}^a(\lambda))$.
Let $\mathbf{m}$ be a multi–exponent and suppose $\mathbf{m} \notin S(\mathbf{P})$, then there is an abstract path $\mathbf{p}_q \in \mathbf{P}$ such that $|\mathbf{m}| \geq \lambda(\beta_{j_q}^\vee) + 1$, where $\beta_{j_q}$ is the base root of $\mathbf{p}_q$. We define a new multi–exponent $\mathbf{m}'$ by setting

$$m'_j := \begin{cases} m_j, & \text{if } \beta_j \in \mathbf{p}_q,\\ 0, & \text{otherwise.}\end{cases}$$

This new multi–exponent is supported on $\mathbf{p}_q$ and we have $|\mathbf{m}'| \geq \lambda(\beta_{j_q}^\vee) + 1$. Therefore, we can apply Proposition 2.3.2 to $\mathbf{m}'$ and conclude

$$f^{(\mathbf{m}')} = \sum_{\mathbf{t}'\prec\mathbf{m}'} c_{\mathbf{t}'} f^{(\mathbf{t}')} \text{ in } S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda),$$

where $c_{\mathbf{t}'} \in \mathbb{Z}$. We get $f^{(\mathbf{m})}$ back as $f^{(\mathbf{m})} = f^{(\mathbf{m}')}\prod_{\beta_j\notin\mathbf{p}_q} f_\beta^{(m_j)}$. For a multi–exponent $\mathbf{t}'$ occurring in the sum with $c_{\mathbf{t}'} \neq 0$ let the multi–exponent $\mathbf{t}$ and $c_{\mathbf{t}} \in \mathbb{Z}$ be such that $c_{\mathbf{t}'} f^{(\mathbf{t}')}\prod_{\beta_j\notin\mathbf{p}_q} f_\beta^{(m_j)} = c_{\mathbf{t}} f^{(\mathbf{t})}$. Since we have a monomial order it follows:

$$f^{(\mathbf{m})} = f^{(\mathbf{m}')}\prod_{\beta_j\notin\mathbf{p}_q} f_\beta^{(m_j)} = \sum_{\mathbf{t}\prec\mathbf{m}} c_{\mathbf{t}} f^{(\mathbf{t})} \text{ in } S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda). \tag{2.3.5}$$

The equation (2.3.5) provides an algorithm to express $f^{(\mathbf{m})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ as a sum of elements of the desired form: If some of the $\mathbf{t}$ are not elements of $S(\mathbf{P})$, then we can repeat the procedure and express the $f^{(\mathbf{t})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ as a sum of $f^{(\mathbf{r})}$ with $\mathbf{r} \prec \mathbf{t}$. For the chosen ordering any strictly decreasing sequence of multi–exponents (all of the same total degree) is finite, so after a finite number of steps one obtains an expression of the form $f^{(\mathbf{m})} = \sum c_{\mathbf{s}} f^{(\mathbf{s})}$ in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})/I_{\mathbb{Z}}(\lambda)$ such that $\mathbf{s} \in S(\mathbf{P})$ and $c_{\mathbf{s}} \in \mathbb{Z}$ for all $\mathbf{s}$. □

Now we are able to state the main result of the present chapter:

**Theorem 2.3.5.** *Let $\mathbb{B}(V^a(\lambda))$ be a basis of $V^a(\lambda)$ satisfying Property (1), (2) and (3). Further let $\boldsymbol{\partial}((\mathbf{m}), \mathbf{p}_q)$ be $\mathbb{Z}$–admissible for all $\mathbf{m} \in \mathbb{Z}^N_{\geq 0}$ and $\mathbf{p}_q \in \mathbf{P}$ given as in Property (1), then is $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda)) = \{f^{(\mathbf{s})}.v_\lambda \mid \mathbf{s} \in S(\mathbf{P})\}$ a basis of $V^a_{\mathbb{Z}}(\lambda)$ and the ideal $I_{\mathbb{Z}}(\lambda)$ is generated by the subspace*

$$\langle U_{\mathbb{Z}}(\mathfrak{n}^+) \circ \operatorname{span}\{f_{\beta_{j_q}}^{(\lambda(\beta_{j_q}^\vee)+1)} \mid \mathbf{p}_q \in \mathbf{P}\}\rangle. \tag{2.3.6}$$

*Proof.* We know from Theorem 2.3.4 that the set $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$ spans $V^a_{\mathbb{Z}}(\lambda)$. By assumption, the number $|S(\mathbf{P})|$ is equal to $\dim V(\lambda)$, which implies the linear independence of $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$. By lifting the elements to $V_{\mathbb{Z}}(\lambda)$ we obtain a basis of $V_{\mathbb{Z}}(\lambda)$ which is (by construction) compatible with the PBW filtration: Set

$$S(\mathbf{P})_r := \{\mathbf{s} \in S(\mathbf{P}) \mid \sum_{j=1}^N s_j \leq r\},$$

then the elements $f^{(\mathbf{s})}$ with $\mathbf{s} \in S(\mathbf{P})_r$ span $V_{\mathbb{Z}}(\lambda)_r$.
Let $I \subset S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ be the ideal generated by (2.3.6). By construction we know $I \subseteq I_{\mathbb{Z}}(\lambda)$. But we also know that the relations in $I$ are sufficient to rewrite every element in $V^a_{\mathbb{Z}}(\lambda)$ in terms of the basis elements $f^{(\mathbf{s})}$, $\mathbf{s} \in S(\mathbf{P})$, which implies that the canonical surjective map $S_{\mathbb{Z}}(\mathfrak{n}^-)/I \to S_{\mathbb{Z}}(\mathfrak{n}^-)/I_{\mathbb{Z}}(\lambda) \cong V_{\mathbb{Z}}(\lambda)$ is injective. □

As an immediate consequence we see:

**Corollary 2.3.6.**

*(i) $V^a_{\mathbb{Z}}(\lambda)$ is a free $\mathbb{Z}$–module.*

*(ii) For every $\mathbf{s} \in S(\lambda)$ fix a total order on the set of positive roots and denote by abuse of notation by $f^{(\mathbf{s})} \in U_{\mathbb{Z}}(\mathfrak{n}^-)$ also the corresponding product of divided powers. The set $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$ forms a basis for the module $V_{\mathbb{Z}}(\lambda)$ and for all $s < s'$ we have $V_{\mathbb{Z}}(\lambda)_s$ is a direct summand of $V_{\mathbb{Z}}(\lambda)_{s'}$ as a $\mathbb{Z}$–module.*

*(iii) With the notation above: Let $k$ be a field and denote by $V_k(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$, $U_k(\mathfrak{g}) = U_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} k$, $U_k(\mathfrak{n}^-) = U_{\mathbb{Z}}(\mathfrak{n}^-) \otimes_{\mathbb{Z}} k$ ect. the objects obtained by base change. The set $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda))$ forms a basis for the module $V_k(\lambda)$.*

## 2.4 Applications

**Application (1)**: Let $\mathfrak{g}$ be of type $\mathtt{A_n}$ or $\mathtt{C_n}$. The authors of [FFL13b] provide for an arbitrary $\lambda \in P^+$ a monomial basis of $V^a_{\mathbb{Z}}(\lambda)$ coming from a monomial basis of $V^a(\lambda)$. In fact, this chapter is motivated by the procedure given in [FFL13b]. The authors define also differential operators (see [FFL13b, Section 4 and 7]) for $\mathfrak{g}_{\mathbb{Z}}$ and consider special abstract paths, called Dyck paths (see [FFL11a, FFL11b] and Chapter 1). Furthermore, they also prove that their bases of $V^a(\lambda)$ obtained in [FFL11a] respectively [FFL11b] satisfy Property (1), (2) and (3) given in Section 2.3. Finally, if we consider the proof of the spanning property (see [FFL13b, Section 4 and 7]) carefully, we see, that they use $\mathbb{Z}$–admissible elements only.
Thus our proceed provides an alternative proof of the main statement of [FFL13b].

**Application (2)**: Let $\mathfrak{g}$ be of type $\mathtt{G}_2$. Let $\alpha_1, \alpha_2 \in \Phi_+$ be the simple roots. Then the six positive roots are:

$$\beta_1 = 3\alpha_1 + 2\alpha_2,\ \beta_2 = 3\alpha_1 + \alpha_2,\ \beta_3 = 2\alpha_1 + \alpha_2,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_2,\ \beta_6 = \alpha_1.$$

For $\lambda = m_1\omega_1 + m_2\omega_2$ with $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ define the polytope $P(\lambda)_{\mathtt{G}_2} \subset \mathbb{R}^6_{\geq 0}$ given by the inequalities

$$\begin{aligned}
x_5 &\leq m_2 = \lambda(\beta_5^\vee) \\
x_6 &\leq m_1 = \lambda(\beta_6^\vee) \\
x_2 + x_3 + x_6 &\leq m_1 + m_2 = \lambda(\beta_2^\vee) \\
x_3 + x_4 + x_6 &\leq m_1 + m_2 = \lambda(\beta_2^\vee) \qquad (2.4.1) \\
x_4 + x_5 + x_6 &\leq m_1 + m_2 = \lambda(\beta_2^\vee) \qquad (2.4.2) \\
x_1 + x_2 + x_3 + x_4 + x_5 &\leq m_1 + 2m_2 = \lambda(\beta_1^\vee) \\
x_2 + x_3 + x_4 + x_5 + x_6 &\leq m_1 + 2m_2 = \lambda(\beta_1^\vee). \qquad (2.4.3)
\end{aligned}$$

**Theorem** (Gornitzki)**.** *The set* $\mathbb{B}(V^a(\lambda)) = \{f^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in S(\lambda)_{\mathtt{G}_2} := P(\lambda)_{\mathtt{G}_2} \cap \mathbb{Z}^6_{\geq 0}\}$ *forms a basis of* $V^a(\lambda)$*.*

*Proof.* See for a proof [Gor11]. □

Note that this statement provides just a basis, but not the generators of the ideal $I(\lambda)$. In the following we prove, that $\mathbb{B}(V^a_{\mathbb{Z}}(\lambda)) = \{f^{(\mathbf{s})} v_\lambda \mid f^{\mathbf{s}} v_\lambda \in \mathbb{B}(V^a(\lambda))\}$ is a basis of $V^a_{\mathbb{Z}}(\lambda)$. Thus we have to show that $\mathbb{B}(V^a(\lambda))$ satisfies Property (1), (2) and (3). Moreover, we have to prove that the elements in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$ corresponding to Property (1) are $\mathbb{Z}$–admissible. We use the statement of Gornitzki and the following, in order to prove our claim.

**Theorem** (Backhaus, Kus)**.** *There is a total order* $\prec$ *on the positive roots of* $\mathtt{G}_2$ *and an induced monomial order on* $S(\mathfrak{n}^-)$*, such that for an arbitrary dominant integral weight* $\lambda$ *the ideal* $I(\lambda) \subset S(\mathfrak{n}^-)$*, where* $V^a(\lambda) \cong S(\mathfrak{n}^-)/I(\lambda)$*, is generated by the subspace:*

$$\langle U(\mathfrak{n}^+) \circ \operatorname{span}\{f_{\beta_j}^{\lambda(\beta_j^\vee)+1} \mid j = 1, 2, 5, 6\}\rangle. \qquad (2.4.4)$$

*Proof.* See for the proof [BK15]. □

The total order on $\Delta_+$ considered in the proof the statement above is the following: $\beta_1 \succ \beta_2 \succ \beta_3 \succ \beta_4 \succ \beta_5 \succ \beta_6$. Furthermore, the authors extend this order to the induced reverse lexicographic total order on the monomials in $S_{\mathbb{Z}}(\mathfrak{n}^{-,a})$: Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^6_{\geq 0}$, then

$$\mathbf{a} \succ \mathbf{b} \ :\Leftrightarrow\ \exists\, 1 \leq j \leq 6 :\ (a_j < b_j) \wedge (a_i = b_i,\ \forall\, 1 \leq i < j).$$

We consider the subset $\mathbf{P}_{\mathtt{G}_2}$ of $\mathcal{P}(\Delta)_+$:

$$\begin{aligned}
\mathbf{P}_{\mathtt{G}_2} := \{&\{\beta_5\}, \{\beta_6\}, \{\beta_2, \beta_3, \beta_6\}, \{\beta_2, \beta_3, \beta_4, \beta_5\}, \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}, \\
&\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}, \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}\}.
\end{aligned}$$

The set of abstract paths $\mathbf{P}_{\mathtt{G}_2}$ provides the connection between the calculations in [BK15, Section 7.1] and our general procedure.
The authors obtain in [BK15, Section 7.1], analogue to Proposition 1.3.3, a straightening law for $\mathtt{G}_2$. Moreover they obtain, analogue to Theorem 1.3.4, that the relations in the subspace (2.4.4) of $S(\mathfrak{n}^-)$ are sufficient to rewrite every element in $V^a(\lambda)$ in terms of the elements in $\mathbb{B}(V^a(\lambda))$. Hence with the same argumentation as in the proof of Theorem 1.5.4 they conclude the claim.

These two theorems let us conclude, that the basis of Gornitzki satisfies Property (1) and (3) of Section 2.3. From the proof of the latter theorem it is easy to see that this basis also satisfies Property (2).
Moreover, if we consider the elements $\boldsymbol{\partial}(\mathbf{m}, \mathbf{p}_q)(f_{\beta_{j_q}}^{|\mathbf{m}|})$ from the calculations in [BK15, Section 7.1] it is easy to verify, that these elements are $\mathbb{Z}$–admissible. Hence we can apply Theorem 2.3.5 to the general $\mathtt{G}_2$–case.

**Remark.** *Note that $P(\lambda)_{\mathtt{G}_2}$, contradicts our definition of $P(\mathbf{P})$ (see (2.3.2), (2.4.1), (2.4.2) and (2.4.3)), if we set $\mathbf{P} := \mathbf{P}_{\mathtt{G}_2}$. Nevertheless the polytope $P(\lambda)_{\mathtt{G}_2}$ does satisfy our assumptions. Let us exemplary consider the calculation corresponding (2.4.3):*

$$\partial_{\beta_5}^{m_6}\partial_{\beta_5}^{m_2}\partial_{\beta_4}^{m_3}\partial_{\beta_3}^{m_4+m_6}\partial_{\beta_2}^{m_5}\left(f_{\beta_1}^{m_2+m_3+m_4+m_5+m_6}\right) = f_{\beta_1}^{0}\prod_{i=2}^{6} f_{\beta_i}^{m_i} + \sum_{\mathbf{t}\prec\mathbf{m}} c_{\mathbf{t}} f^{\mathbf{t}}.$$

*Hence we have for every multi–exponent $\mathbf{m} \in \mathbb{Z}_{\geq 0}^6$ described by the operation above $m_1 = 0$, although $\beta_1$ is the base root for the abstract path $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$. Thus in the corresponding inequality in $P(\mathbf{P}) = P(\lambda)_{\mathtt{G}_2}$ we do not sum over the first entry. The analogue calculations for (2.4.1) and (2.4.2) cancel the contradiction to (2.3.2).*

**Application (3)**: Let (Type of $\mathfrak{g}$, $\lambda = \omega_i$) be listed in Table 1.1. We provide in Chapter 1 for $\lambda = m\omega_i$, $m \in \mathbb{Z}_{\geq 0}$, the set $\mathbb{B}_\lambda$ (see Section 1.4 and 1.5),which is a monomial basis of $V^a(\lambda)$.
The straightening law, Proposition 1.3.3 implies, that these basis satisfy Property (1) of Section 2.3. Note, that in order to be consistent with [FFL13a, FFL13b], we call in Chapter 1 the abstract paths, Dyck paths.
Lemma 1.3.2 implies, that the bases $\mathbb{B}_\lambda$ satisfy in addition Property (2). Moreover, the main result of Chapter 1, Theorem 1.5.2, let us conclude, that $\mathbb{B}_\lambda$ satisfies also Property (3).
To be precise: Proposition 1.3.3 and Lemma 1.3.2 imply Property (1) and (2) for all cases listed in Table 1.1, except for the cases $(\mathtt{B_n}, \omega_1)$ and $(\mathtt{G}_2, \omega_1)$. Tough the explicit calculation in these cases (see Section 1.4) show, that Property (1) and (2) is also satisfied in these cases.

Thus it remains to check if the elements corresponding to Property (1) are $\mathbb{Z}$–admissible. For the $\mathtt{A_n}$–,$\mathtt{D_n}$– and $\mathtt{E}_{6,7}$–cases contained in Table 1.1 there is nothing to show, since all structure constants are equal to $\pm 1$ in these cases. Thus the assumptions (2.2.24) and (2.2.25) of Definition 2.2.9 are satisfied trivially, because there are only strings of length at most 2.

In the cases $(\mathtt{B_n}, \omega_1)$, $(\mathtt{B_n}, \omega_n)$, $(\mathtt{C_n}, \omega_1)$ and $(\mathtt{G_2}, \omega_1)$ we have to verify the $\mathbb{Z}$–admissibility by the explicit calculations given in Section 1.4. The only case which provides no monomial basis for $V_{\mathbb{Z}}^a(\lambda)$ is $(\mathtt{F_4}, \omega_4)$. Summarizing we have:

| Type of $\mathfrak{g}$ | weight $\omega_i$ | Type of $\mathfrak{g}$ | weight $\omega_i$ |
|---|---|---|---|
| $\mathtt{A_n}$ | $\omega_k,\ 1 \leq k \leq n$ | $\mathtt{D_n}$ | $\omega_1,\ \omega_{n-1},\ \omega_n$ |
| $\mathtt{B_n}$ | $\omega_1,\ \omega_n$ | $\mathtt{E_6}$ | $\omega_1,\ \omega_6$ |
| $\mathtt{C_n},\ \mathtt{G_2}$ | $\omega_1$ | $\mathtt{E_7}$ | $\omega_7$ |

Table 2.1: Admissible weights over $\mathbb{Z}$

Let us explain, why the $\mathbb{Z}$–admissibility is violated in the $(\mathtt{F_4}, \omega_4)$–case. Note, that there are Dyck paths in the Hasse diagram corresponding to this case (see Figure 1.1), such that $\beta_4 = (1, 2, 3, 2)$ and $\beta_7 = (1, 1, 2, 2)$ are contained (see (3.2.2)–(3.2.7)). In fact, we have

$$\partial_{\alpha_2+\alpha_3}^{(k)} \left( f_{\beta_4}^{(m)} \right) \notin \mathbb{D},$$

since this operation is of the form (2.2.6) (see Remark 2.2.8 for the definition of $\mathbb{D}$). Thus all elements corresponding to these Dyck paths are not $\mathbb{Z}$–admissible.

# 3 The degree of the Hilbert–Poincaré polynomial of PBW graded modules

We emphasize that the present chapter is a modified version of [BBDF14]. All notations and definitions we are using in this chapter, unless they are defined here, can be found in the Preliminaries.

## 3.1 The Hilbert–Poincaré polynomial

In the present chapter we compute the maximal degree of PBW graded modules, i. e. modules which have a grading coming from the PBW filtration, in full generality (for all simple complex Lie algebras), where there have been partial answers in [FFL11a, FFL11b, FFL13b] and Chapter 1 for certain cases (see Table 1.1).

We denote the Hilbert–Poincaré series of the PBW graded module, often referred to as the *q–dimension of the module*, by

$$p_\lambda(q) = \sum_{s=0}^{\infty} (\dim V(\lambda)_s/V(\lambda)_{s-1})\, q^s.$$

Since $V(\lambda)$ is finite–dimensional, this is obviously a polynomial in $q$. In the following we want to study further properties of this polynomial. We see immediately that the constant term of $p_\lambda(q)$ is always 1 and the linear term is equal to

$$\dim(\mathfrak{n}^-) - \dim \operatorname{Ker}\left(\mathfrak{n}^- \longrightarrow \operatorname{End}(V(\lambda))\right).$$

Our main goal is to compute the degree of $p_\lambda(q)$ and the first step is the following reduction [CF13, Theorem 5.3 ii)]:

**Theorem.** *Let $\lambda_1, \ldots, \lambda_s \in P^+$ and set $\lambda = \lambda_1 + \ldots + \lambda_s$. Then*

$$\deg p_\lambda(q) = \deg p_{\lambda_1}(q) + \ldots + \deg p_{\lambda_s}(q).$$

It remains to compute the degree of $p_\lambda(q)$, where $\lambda$ is a fundamental weight. We will do this for all fundamental weights of simple complex finite–dimensional Lie algebras (see Theorem 3.2.1).

**Hilbert–Poincaré series and graded weight spaces.** Let $\mathfrak{g}$, $V(\lambda)$ and $V^a(\lambda)$ be defined as usual in this thesis (see the Preliminaries). The Hilbert–Poincaré series of the PBW graded module $V^a(\lambda) := \bigoplus_{s\geq 0} V(\lambda)_s/V(\lambda)_{s-1}$ is the polynomial

$$\begin{aligned} p_\lambda(q) &= \sum\nolimits_{s\geq 0} \dim(V(\lambda)_s/V(\lambda)_{s-1})q^s \\ &= 1 + \dim(V(\lambda)_1/V(\lambda)_0)q + \dim(V(\lambda)_2/V(\lambda)_1)q^2 + \ldots \end{aligned}$$

and we define the PBW degree of $V(\lambda)$ to be $\deg(p_\lambda(q))$.
Note that we already know from Remark 0.0.3, that the graduation components $V(\lambda)_s$ are $U(\mathfrak{n}^+)$–modules for all $s \in \mathbb{Z}_{\geq 0}$. Let $s_\lambda$ be minimal, such that $v_{w_0(\lambda)} \in V(\lambda)_{s_\lambda}$. Then $V(\lambda) = U(\mathfrak{n}^+)v_{w_0(\lambda)} \subseteq V(\lambda)_{s_\lambda}$ and hence

**Corollary 3.1.1.** *$s_\lambda = \deg(p_\lambda(q))$ and*

$$V(\lambda) = V(\lambda)_{s_\lambda}.$$

The PBW filtration is compatible with the decomposition into $\mathfrak{h}$–weight spaces:

$$\dim V(\lambda)_\tau = \sum_{s \geq 0} \dim \left(V(\lambda)_s / V(\lambda)_{s-1}\right) \cap V(\lambda)_\tau.$$

Therefore we can define for every weight $\tau$ the Hilbert–Poincaré polynomial:

$$p_{\lambda,\tau}(q) = \sum_{s \geq 0} \dim \left(V(\lambda)_s / V(\lambda)_{s-1}\right)_\tau q^s$$

and with this definition

$$p_\lambda(q) = \sum_{\tau \in P} p_{\lambda,\tau}(q).$$

A natural question is, if we can extend our results to these polynomials? If the weight space $V(\lambda)_\tau$ is one–dimensional, then $p_{\lambda,\tau}(q)$ is a power of $q$. For $\tau = \lambda$ this is constant 1, for $\tau = w_0(\lambda)$, the lowest weight, this is $q^{\deg p_\lambda(q)}$ as we have seen in Corollary 3.1.1. A first approach to study these polynomials was taken in [CF13].

**Graded Kostant partition function**. For the readers convenience we recall here the *graded Kostant partition function* (see [Kos59]), which counts the number of decompositions of a fixed weight into a sum of positive roots, and how it is related to our study. We consider the power series

$$\prod_{\beta \in \Delta_+} \frac{1}{(1 - qe^\beta)}$$

and its expansion

$$\sum_{\nu \in P} P_\nu(q) e^\nu.$$

We have immediately

$$\operatorname{char} S(\mathfrak{n}^-) = \sum_{\nu \in P} P_\nu(q) e^{-\nu}.$$

**Remark 3.1.2.** *Can we relate the Hilbert–Poincaré polynomial $p_{\lambda,\nu}(q)$ with the graded Kostant partition function $P_{\lambda-\nu}(q)$? For a polynomial $p(q) = \sum_{i=0}^n a_i q^i$, we denote* $\operatorname{mindeg} p(q)$ *the minimal $j$ such that $a_j \neq 0$. Then we have obviously*

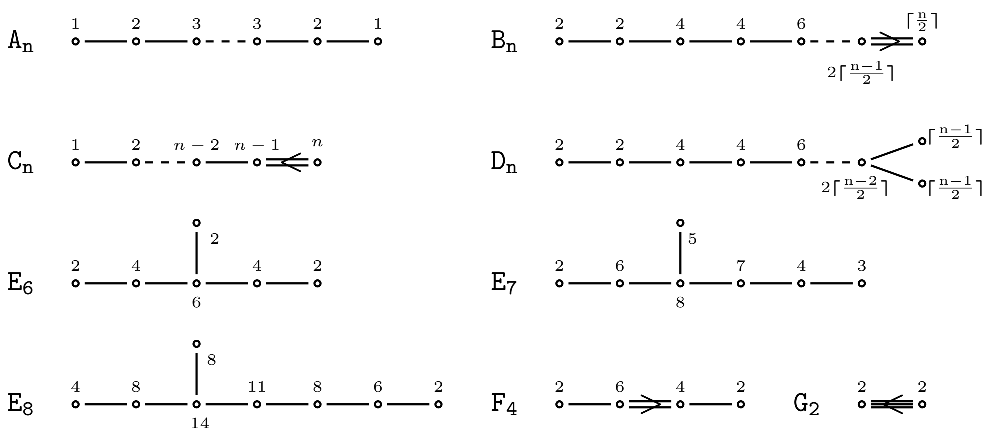
$$\operatorname{mindeg} p_{\lambda,\nu}(q) \geq \operatorname{mindeg} P_{\lambda-\nu}(q). \tag{3.1.1}$$

We use this inequality for the very special case $\nu = w_0(\lambda)$ in the proof of Theorem 3.2.1.
It is shown in [CF13] that apart from the types $\mathtt{A_n}$ and $\mathtt{C_n}$ this inequality might be strict, e.g. $\mathfrak{g}$ of type $B_3$, $\lambda = \omega_1 + \omega_3$ and $\nu = -\omega_1 - \omega_3$. We see from Theorem 3.2.1 that this also happens for $\mathtt{D_n}$ and for all exceptional types.

## 3.2 The degree of $p_{\omega_i}(q)$

In this section we provide a proof of the main statement of this chapter: Theorem 3.2.1. We provide a monomial $u \in S(\mathfrak{n}^-)$ of the predicted degree mapping the highest to the lowest weight vector and show that there is no polynomial of smaller degree satisfying this. To write down these monomials explicitly, let us denote $\theta_{\mathtt{X_n}}$ the highest root of a Lie algebra of type $\mathtt{X_n}$. Further $\mathtt{X_{n-1}}$ denotes the Lie subalgebra generated by the simple roots $\{\alpha_1, \dots, \alpha_n\} \setminus \{\alpha_k\}$, where $\theta_{\mathtt{X_n}} = a_k \omega_k$ or in the $\mathtt{A_n}$–case, $\mathtt{X_{n-2}}$ is generated by the simple roots $\{\alpha_2, \dots, \alpha_{n-1}\}$ (we use the indexing from [Hum72]).

**Theorem 3.2.1.** *The degree of $p_{\omega_i}(q)$ is equal to the label of the $i$–th node in the following diagrams:*




*Proof.* Let $u \in S(\mathfrak{n}^-)$ be one of the monomial in Figure 3.1. These monomials give certainly upper estimates for the degrees since (as they are in fact obtained through the action of the Weyl group):

$$u.v_{\omega_i} = v_{w_0(\omega_i)} \in V(\omega_i).$$

In general the degree of $u$ is bigger than the minimal degree coming from Kostant's graded partition function (3.1.1). For $\mathtt{A_n}, \mathtt{C_n}$ and the even fundamental weights for orthogonal Lie algebras the degrees coincide and hence we are done in these cases.

We prove Theorem 3.2.1 for the remaining cases $\mathtt{X_n}$ by downward induction on the power of $f_{\theta_{\mathtt{X_n}}}$. The maximal non–vanishing power of $f_{\theta_{\mathtt{X_n}}}$ is certainly $a_i^\vee$, where $h_\theta = \sum a_i^\vee h_i$, further $\omega_i - w_0(\omega_i) - a_i^\vee \theta_{\mathtt{X_n}}$ is in the root lattice of a Lie algebra of smaller rank, and we use induction on the rank of $\mathfrak{g}$. Thus it remains to show that if

$$p \in S(\mathfrak{n}^-)_{\omega_i - w_0(\omega_i)}, \deg p < \deg u, p = f^{\ell}_{\theta_{\mathtt{X_n}}} p_1 \text{ with } \ell < a_i^\vee, \tag{3.2.1}$$

then $p.v_{\omega_i} = 0 \in V^a(\omega_i)$.

Let $\mathtt{X_n}$ be of type $\mathtt{B_n}, \mathtt{D_n}$ or exceptional, then $\theta_{\mathtt{X_n}} = \omega_j$ and we denote

$$\Delta_+^k = \{\beta \in \Delta_+ \mid w_j(h_\alpha) = k\},$$

| | | |
|---|---|---|
| $\mathtt{X_n}$ | $\omega_i = \theta_{\mathtt{X_n}}$ | $f^2_{\theta_{\mathtt{X_n}}}$ |
| $\mathtt{A_n}$ | $\omega_i$ | $f_{\theta_{\mathtt{A_n}}} f_{\theta_{\mathtt{A_{n-2}}}} \cdots f_{\theta_{\mathtt{A_{n+2-2\min\{i,n-i\}}}}}$ |
| $\mathtt{C_n}$ | $\omega_i$ | $f_{\theta_{\mathtt{C_n}}} f_{\theta_{\mathtt{C_{n-1}}}} \cdots f_{\theta_{\mathtt{C_{n+1-i}}}}$ |
| $\mathtt{B_n}$ | $\omega_{2i}$ | $f_{\theta_{\mathtt{B_n}}} f_{\theta_{\mathtt{B_{n-1}}}} \cdots f_{\theta_{\mathtt{B_{n+1-2i}}}}$ |
| $\mathtt{B_n}$ | $\omega_{2i+1}$ | $f_{\theta_{\mathtt{B_n}}} f_{\theta_{\mathtt{B_{n-1}}}} \cdots f_{\theta_{\mathtt{B_{n+1-2i}}}} f_{\alpha_{2i-1}}$ |
| $\mathtt{B_n}$ | $n$ even, $\omega_n$ | $f_{\theta_{\mathtt{B_n}}} \cdots f_{\theta_{\mathtt{B_2}}}$ |
| $\mathtt{B_n}$ | $n$ odd, $\omega_n$ | $f_{\theta_{\mathtt{B_n}}} \cdots f_{\theta_{\mathtt{B_2}}} f_{\alpha_n}$ |
| $\mathtt{D_n}$ | $\omega_{2i}$ | $f_{\theta_{\mathtt{D_n}}} f_{\theta_{\mathtt{D_{n-1}}}} \cdots f_{\theta_{\mathtt{D_{n+1-2i}}}}$ |
| $\mathtt{D_n}$ | $\omega_{2i+1}$ | $f_{\theta_{\mathtt{D_n}}} f_{\theta_{\mathtt{D_{n-1}}}} \cdots f_{\theta_{\mathtt{D_{n+1-2i}}}} f_{\alpha_{2i-1}}$ |
| $\mathtt{D_n}$ | $n$ even, $\omega_{n-1}, \omega_n$ | $f_{\theta_{\mathtt{D_n}}} f_{\theta_{\mathtt{D_{n-2}}}} \cdots f_{\theta_{\mathtt{D_4}}} f_{\alpha_{n-1}}$ |
| $\mathtt{D_n}$ | $n$ odd, $\omega_{n-1}, \omega_n$ | $f_{\theta_{\mathtt{D_n}}} f_{\theta_{\mathtt{D_{n-2}}}} \cdots f_{\theta_{\mathtt{D_4}}}$ |
| $\mathtt{E_6}$ | $\omega_1, \omega_6$ | $f_{\theta_{\mathtt{E_6}}} f_{\alpha_2}$ |
| $\mathtt{E_6}$ | $\omega_3, \omega_5$ | $f^2_{\theta_{\mathtt{E_6}}} f_{\theta_{\mathtt{A_5}}} f_{\theta_{\mathtt{A_3}}}$ |
| $\mathtt{E_6}$ | $\omega_4$ | $f^3_{\theta_{\mathtt{E_6}}} f_{\theta_{\mathtt{A_5}}} f_{\theta_{\mathtt{A_3}}} f_{\alpha_4}$ |
| $\mathtt{E_7}$ | $\omega_2$ | $f^2_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{A_4}}} f_{\alpha_2}$ |
| $\mathtt{E_7}$ | $\omega_3$ | $f^3_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{A_3}}} f_{\alpha_3}$ |
| $\mathtt{E_7}$ | $\omega_4$ | $f^4_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}} f^2_{\theta_{\mathtt{D_4}}}$ |
| $\mathtt{E_7}$ | $\omega_5$ | $f^3_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{D_4}}} f_{\alpha_5}$ |
| $\mathtt{E_7}$ | $\omega_6$ | $f^2_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}}$ |
| $\mathtt{E_7}$ | $\omega_7$ | $f_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\alpha_7}$ |
| $\mathtt{E_8}$ | $\omega_1$ | $f^2_{\theta_{\mathtt{E_8}}} f^2_{\theta_{\mathtt{E_7}}}$ |
| $\mathtt{E_8}$ | $\omega_2$ | $f^3_{\theta_{\mathtt{E_8}}} f^2_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{A_4}}} f_{\alpha_2}$ |
| $\mathtt{E_8}$ | $\omega_3$ | $f^4_{\theta_{\mathtt{E_8}}} f^3_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{A_3}}} f_{\alpha_3}$ |
| $\mathtt{E_8}$ | $\omega_4$ | $f^6_{\theta_{\mathtt{E_8}}} f^4_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}} f^2_{\theta_{\mathtt{D_4}}}$ |
| $\mathtt{E_8}$ | $\omega_5$ | $f^5_{\theta_{\mathtt{E_8}}} f^3_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}} f_{\theta_{\mathtt{D_4}}} f_{\alpha_5}$ |
| $\mathtt{E_8}$ | $\omega_6$ | $f^4_{\theta_{\mathtt{E_8}}} f^2_{\theta_{\mathtt{E_7}}} f^2_{\theta_{\mathtt{D_6}}}$ |
| $\mathtt{E_8}$ | $\omega_7$ | $f^3_{\theta_{\mathtt{E_8}}} f_{\theta_{\mathtt{E_7}}} f_{\theta_{\mathtt{D_6}}} f_{\alpha_7}$ |
| $\mathtt{F_4}$ | $\omega_2$ | $f^3_{\theta_{\mathtt{F_4}}} f_{\theta_{\mathtt{C_3}}} f_{\theta_{\mathtt{A_2}}} f_{\alpha_2}$ |
| $\mathtt{F_4}$ | $\omega_3$ | $f^2_{\theta_{\mathtt{F_4}}} f^2_{\theta_{\mathtt{C_3}}}$ |
| $\mathtt{F_4}$ | $\omega_4$ | $f_{\theta_{\mathtt{F_4}}} f_{\theta_{\mathtt{C_3}}}$ |
| $\mathtt{G_2}$ | $\omega_1$ | $f_{\theta_{\mathtt{G_2}}} f_{\alpha_1}$ |

Figure 3.1

Then $\Delta_+^2 = \{\theta_{X_n}\}$ and if $\beta \in \Delta_+^1$ then $\theta_{\mathtt{X_n}} - \beta \in \Delta_+$. Therefore, let $p \in S(\mathfrak{n}^-)$ satisfy (3.2.1), then we have by weight considerations $p = f_{\theta_{\mathtt{X_n}}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1$ for some $\beta_1, \ldots, \beta_{2k} \in \Delta_+^1, p_1 \in S(\mathfrak{n}^-)$. We have to show that $p.v_{\omega_i} = 0 \in V^a(\omega_i)$ and we use induction on $k$ for that:

$$\begin{array}{rcl} 0 = p_1 f_{\theta_{\mathtt{X_n}}}^{a_i^\vee + k}.v_{\omega_i} & = & (e_{\theta_{\mathtt{X_n}} - \beta_1}) \cdots (e_{\theta_{\mathtt{X_n}} - \beta_{2k}}) p_1 f_{\theta_{\mathtt{X_n}}}^{a_i^\vee + k}.v_{\omega_i} \\ & = & c f_{\theta_{\mathtt{X_n}}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1.v_{\omega_i} + \sum_{\ell > 0} f_{\theta_{\mathtt{X_n}}}^{a_i^\vee - k + \ell} q_\ell.v_{\omega_i} \end{array}$$

for some $c \in \mathbb{C}^*, q_\ell \in S(\mathfrak{n}^-)$. By induction the latter terms are equal to zero and so $f_{\theta_{\mathtt{X_n}}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1.v_{\omega_i}$ is also zero.
This proves also that for all $u$ from the list above we have $u.v_{\omega_i} \neq 0$ in $V^a(\omega_i)$. $\square$

# Appendix

Here we present the Hasse diagrams $H(\mathfrak{n}^-_{\omega_6})_{\mathtt{E}_6}$ and $H(\mathfrak{n}^-_{\omega_7})_{\mathtt{E}_7}$ for a better understanding of our work. In addition to convey the ordering of the roots for the classical types $\mathtt{A}_\mathtt{n}$, $\mathtt{B}_\mathtt{n}$ and $\mathtt{D}_\mathtt{n}$ we provide in Figure 3.2 the complete Hasse diagram of $\mathfrak{sl}_4$ and in Figure 3.3 a concrete example of the Hasse diagram in the ($\mathtt{D}_\mathtt{n}$, $\omega_n$)–case, for $n = 5, 6$. We remark that the shape of the Hasse diagram $H(\mathfrak{n}^-_{\omega_{n-1}})_{\mathfrak{so}_{2n}}$ and $H(\mathfrak{n}^-_{\omega_n})_{\mathfrak{so}_{2n}}$ is equal to the shape of $H(\mathfrak{n}^-_{\omega_{n-1}})_{\mathfrak{so}_{2(n-1)+1}}$. Therefore, Figure 3.3 shows also the shape of the Hasse diagrams $H(\mathfrak{n}^-_{\omega_4})_{\mathfrak{so}_{10}}$, $H(\mathfrak{n}^-_{\omega_5})_{\mathfrak{so}_{10}}$ and $H(\mathfrak{n}^-_{\omega_5})_{\mathfrak{so}_{12}}$, $H(\mathfrak{n}^-_{\omega_6})_{\mathfrak{so}_{12}}$. Furthermore, we state the explicit polytopes for $\mathtt{E}_6$ (Table 3.2), $\mathtt{F}_4$ (Table 3.3) and for the special cases: ($\mathtt{B}_4$, $\omega_4$), ($\mathtt{D}_5$, $\omega_4$) and ($\mathtt{D}_5$, $\omega_5$) (Table 3.1).

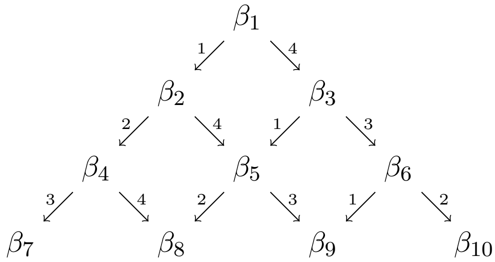


Figure 3.2: Complete Hasse diagram of $\mathfrak{g} = \mathfrak{sl}_5$.

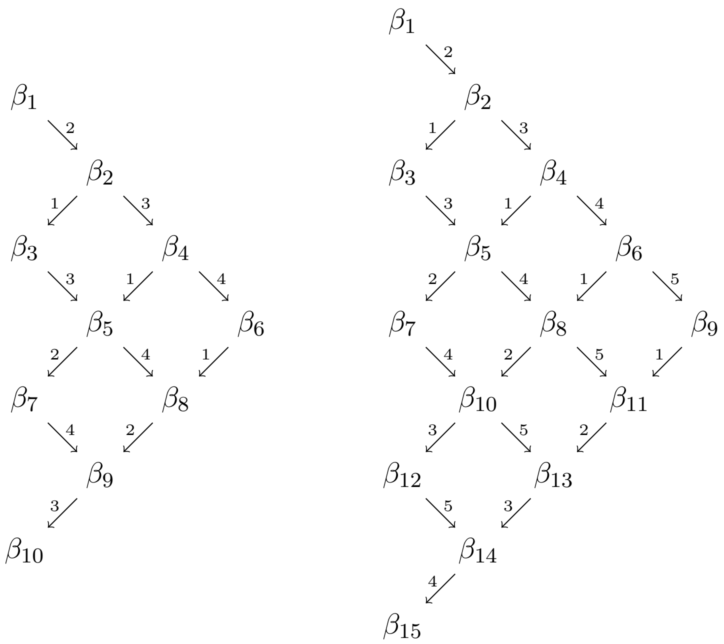


Figure 3.3: $H(\mathfrak{n}^-_{\omega_4})_{\mathfrak{so}_9}$, $H(\mathfrak{n}^-_{\omega_5})_{\mathfrak{so}_{11}}$

$$x_1 + x_2 + x_3 + x_5 + x_7 + x_9 + x_{10} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_8 + x_9 + x_{10} \leq m$$
$$x_1 + x_2 + x_4 + x_5 + x_7 + x_9 + x_{10} \leq m$$
$$x_1 + x_2 + x_4 + x_5 + x_8 + x_9 + x_{10} \leq m$$
$$x_1 + x_2 + x_4 + x_6 + x_8 + x_9 + x_{10} \leq m$$

Table 3.1: Polytope $P(m\omega_4)$ corresponding to $\mathfrak{g} = \mathfrak{so}_9$ and $P(m\omega_4)$, $P(m\omega_5)$ corresponding to $\mathfrak{g} = \mathfrak{so}_{10}$.

$$\begin{tikzcd}
\beta_1 \arrow[r, "2"] & \beta_2 \arrow[r, "4"] & \beta_3 \arrow[d, "3"'] \arrow[dr, "5"] & \\
 & & \beta_4 \arrow[d, "1"'] \arrow[dr, "5"] & \beta_5 \arrow[d, "3"] \\
 & & \beta_6 \arrow[d, "5"'] & \beta_7 \arrow[dl, "1"] \arrow[d, "4"] \\
 & & \beta_8 \arrow[d, "4"'] & \beta_9 \arrow[dl, "1"] \arrow[d, "2"] \\
 & & \beta_{10} \arrow[dl, "3"] \arrow[d, "2"] & \beta_{11} \arrow[dl, "1"] \\
 & \beta_{13} \arrow[dr, "2"] & \beta_{12} \arrow[d, "3"] & \\
\beta_{16} & \beta_{15} \arrow[l, "5"] & \beta_{14} \arrow[l, "4"] &
\end{tikzcd}$$

Figure 3.4: $H(\mathfrak{n}^-_{\omega_6})_{\mathtt{E}_6}$

$$x_1 + x_2 + x_3 + x_4 + x_6 + x_8 + x_{10} + x_{13} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_{10} + x_{13} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_{10} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_9 + x_{10} + x_{13} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_9 + x_{10} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_9 + x_{11} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_7 + x_8 + x_{10} + x_{13} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_7 + x_8 + x_{10} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_7 + x_9 + x_{10} + x_{13} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_7 + x_9 + x_{10} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$
$$x_1 + x_2 + x_3 + x_5 + x_7 + x_9 + x_{11} + x_{12} + x_{14} + x_{15} + x_{16} \leq m$$

Table 3.2: Polytope $P(m)$ corresponding to $\mathtt{E}_6$

$$
\begin{tikzcd}
\beta_1 \arrow[r, "1"] & \beta_2 \arrow[r, "3"] & \beta_3 \arrow[r, "4"] & \beta_4 \arrow[d, "2"'] \arrow[dr, "5"] & & & \\
& & & \beta_5 \arrow[dr, "5"'] & \beta_6 \arrow[d, "2"] \arrow[dr, "6"] & & \\
& & & & \beta_7 \arrow[d, "4"'] \arrow[dr, "6"] & \beta_8 \arrow[d, "2"] & \\
& & & & \beta_9 \arrow[d, "3"'] \arrow[dr, "6"] & \beta_{10} \arrow[d, "4"] & \\
& & & & \beta_{11} \arrow[d, "1"'] \arrow[dr, "6"] & \beta_{12} \arrow[d, "3"] \arrow[dr, "5"] & \\
& & & & \beta_{13} \arrow[d, "6"'] & \beta_{14} \arrow[dl, "1"'] \arrow[d, "5"] & \beta_{15} \arrow[dl, "3"] \\
& & & & \beta_{16} \arrow[d, "5"'] & \beta_{17} \arrow[dl, "1"'] \arrow[d, "4"] & \\
& & & & \beta_{18} \arrow[d, "4"'] & \beta_{19} \arrow[dl, "1"'] \arrow[d, "2"] & \\
& & & & \beta_{20} \arrow[dl, "3"'] \arrow[d, "2"] & \beta_{21} \arrow[dl, "1"] & \\
& & & \beta_{23} \arrow[d, "2"'] & \beta_{22} \arrow[dl, "3"] & & \\
\beta_{27} & \beta_{26} \arrow[l, "6"] & \beta_{25} \arrow[l, "5"] & \beta_{24} \arrow[l, "4"] & & &
\end{tikzcd}
$$

Figure 3.5: $H(\mathfrak{n}^-_{\omega_7})_{\mathtt{E}_7}$

| | | |
|---|---|---|
| $x_1 + x_2 + x_3 + x_4 + x_8 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |
| $x_1 + x_2 + x_3 + x_4 + x_8 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |
| $x_1 + x_2 + x_3 + x_4 + x_7 + x_9 + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.2) |
| $x_1 + x_2 + x_3 + x_4 + x_7 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.3) |
| $x_1 + x_2 + x_3 + x_4 + x_7 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.4) |
| $x_1 + x_2 + x_4 + x_5 + x_8 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |
| $x_1 + x_2 + x_4 + x_5 + x_8 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |
| $x_1 + x_2 + x_4 + x_5 + x_7 + x_9 + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.5) |
| $x_1 + x_2 + x_4 + x_5 + x_7 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.6) |
| $x_1 + x_2 + x_4 + x_5 + x_7 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | (3.2.7) |
| $x_1 + x_2 + x_3 + x_6 + x_8 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |
| $x_1 + x_2 + x_3 + x_6 + x_8 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}$ | $\leq 1$ | |

Table 3.3: Polytope $P(\omega_4)$ corresponding to $\mathtt{F_4}$

# Bibliography


[ABS11] Federico Ardila, Thomas Bliem, and Dido Salazar. Gelfand-Tsetlin polytopes and Feigin-Fourier-Littelmann-Vinberg polytopes as marked poset polytopes. *J. Combin. Theory Ser. A*, 118(8):2454–2462, 2011.

[BBDF14] Teodor Backhaus, Lara Bossinger, Christian Desczyk, and Ghislain Fourier. The degree of the Hilbert-Poincaré polynomial of PBW-graded modules. *C. R. Math. Acad. Sci. Paris*, 352(12):959–963, 2014.

[BD15] T. Backhaus and C. Desczyk. PBW Filtration: Feigin-Fourier-Littelmann modules via Hasse diagrams. *JLT*, 25(3):815–856, 2015.

[BF14] R. Biswal and G. Fourier. PBW-graded Demazure module for rectangular weight, Kogan faces and marked posets. Preprint, 2014.

[BK15] Teodor Backhaus and Deniz Kus. The PBW filtration and convex polytopes in type $\mathtt{B}$. arXiv:1504.06522, 2015.

[Car05] R. W. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2005.

[CF13] M. Cherednik and E. Feigin. Extremal part of the PBW-filtration and E-polynomials. arXiv:1306.3146, 2013.

[CIFR12] Giovanni Cerulli Irelli, Evgeny Feigin, and Markus Reineke. Quiver Grassmannians and degenerate flag varieties. *Algebra Number Theory*, 6(1):165–194, 2012.

[CIL14] G. Cerulli Irelli and M. Lanini. Degenerate flag varieties of type A and C are Schubert varieties. arXiv:1403.2889, 2014.

[CILL15] Giovanni Cerulli Irelli, Martina Lanini, and Peter Littelmann. Degenerate flag varieties and schubert varieties: a characteristic free approach. arXiv:1502.04590, 2015.

[Fei09] Evgeny Feigin. The PBW filtration. *Represent. Theory*, 13:165–181, 2009.

[Fei11] Evgeny Feigin. Degenerate flag varieties and the median Genocchi numbers. *Math. Res. Lett.*, 18(6):1163–1178, 2011.

[Fei12] Evgeny Feigin. $\mathbb{G}_a^M$ degeneration of flag varieties. *Selecta Math. (N.S.)*, 18(3):513–537, 2012.

[FFL11a] E. Feigin, G. Fourier, and P. Littelmann. PBW filtration and bases for irreducible modules in type $\mathbf{A}_n$. *Transform. Groups*, 16(1):71–89, 2011.

[FFL11b] E. Feigin, G. Fourier, and P. Littelmann. PBW filtration and bases for symplectic Lie algebras. *Int. Math. Res. Not. IMRN*, 1(24):5760–5784, 2011.

[FFL13a] E. Feigin, G. Fourier, and P. Littelmann. Favourable modules: Filtrations, polytopes, Newton-Okounkov bodies and flat degenerations. arXiv:1306.1292v3, 2013.

[FFL13b] E. Feigin, G. Fourier, and P. Littelmann. PBW-filtration over $\mathbb{Z}$ and compatible bases for $V(\lambda)$ in type $A_n$ and $C_n$. *Springer Proceedings in Mathematics and Statistics*, 40:35–63, 2013.

[FFL14] Evgeny Feigin, Michael Finkelberg, and Peter Littelmann. Symplectic degenerate flag varieties. *Canad. J. Math.*, 66(6):1250–1286, 2014.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[FM14] Evgeny Feigin and Levgen Makedonskyi. Nonsymmetric macdonald polynomials, demazure modules and PBW filtration. arXiv:1407.6316, 2014.

[Fou14a] G. Fourier. PBW-degenerated Demazure modules and Schubert varieties for triangular elements. arXiv:1408.6939, 2014.

[Fou14b] Ghislain Fourier. New homogeneous ideals for current algebras: filtrations, fusion products and Pieri rules. arXiv:1403.4758, 2014.

[GL97] N. Gonciulea and V. Lakshmibai. Schubert varieties, toric varieties, and ladder determinantal varieties. *Ann. Inst. Fourier (Grenoble)*, 47(4):1013–1064, 1997.

[Gor11] A. Gornitsky. Essential signatures and canonical bases in irreducible representations of the group $G_2$. Diploma thesis, 2011, 2011.

[Hag14] Chuck Hague. Degenerate coordinate rings of flag varieties and Frobenius splitting. *Selecta Math. (N.S.)*, 20(3):823–838, 2014.

[HK13] M. Harada and K. Kaveh. Integrable systems, toric degenerations and okounkov bodies. arXiv:1205.5249, 2013.

[Hum72] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9.

[KK12] Kiumars Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math. (2)*, 176(2):925–978, 2012.

[KM05] Mikhail Kogan and Ezra Miller. Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes. *Adv. Math.*, 193(1):1–17, 2005.

[Kos59] B. Kostant. A formula for the multiplicity of a weight. *Trans. Amer. Math. Soc.*, 93:53–73, 1959.

[Kus13a] Deniz Kus. Kirillov-Reshetikhin crystals, energy function and the combinatorial R-matrix. arXiv:1309.6522, 2013.

[Kus13b] Deniz Kus. Realization of affine type $A$ Kirillov-Reshetikhin crystals via polytopes. *J. Combin. Theory Ser. A*, 120(8):2093–2117, 2013.

[Vin05] Ernest Vinberg. On some canonical bases of representation spaces of simple lie algebras. Conference talk, Bielefeld, 2005.

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Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Peter Littelmann betreut worden.

- PBW Filtration: Feigin–Fourier–Littelmann modules via Hasse diagrams, Journal of Lie Theory, Band 25, Ausgabe 3, Seiten 815–856.
  (Autoren: Teodor Backhaus, Christian Desczyk)
- The degree of the Hilbert–Poincare polynomial of PBW graded modules, Comptes Rendus Mathematique, Band 352, Ausgabe 12, Seiten 959–963.
  (Autoren: Teodor Backhaus, Lara Bossinger, Christian Desczyk, Ghislain Fourier)

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