Hopf's Type Estimates near Singular Boundary Points

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Abstract

Subject of this thesis is the behaviour of the solution of the Laplace-Poisson equation under zero Dirichlet boundary condition near non-regular boundary points. The first part gives an example of a domain where Hopf's Boundary Point Lemma holds true pointwise but not uniformly, therefore the solution operator is not strongly positive. A second result addresses a sharp replacement of Hopf's estimate near the boundary whenever such boundary has a conical point. As a consequence one is able to prove an optimal anti-maximum type result for domains with conical shapes. The last part is concerned with the behaviour of the solution at points where an interface reaches the boundary. At the interface the Poisson equation is not satisfied but instead a jump condition for the normal derivatives appears.

Zusammenfassung

In der vorliegenden Doktorarbeit wird das Verhalten von Lösungen der Laplace-Poisson-Gleichung unter der Null Dirichlet-Randbedingung in der Nähe von nichtregulären Randpunkten untersucht. Im ersten Teil der Arbeit wird ein Gebiet konstruiert, bei dem Hopfs Randpunktlemma zwar punktweise gilt, aber nicht uniform, weswegen der Lösungsoperator nicht stark positiv sein kann. Das zweite Resultat zeigt, wie man Hopfs Ungleichung bei einem Punkt auf dem Rand, bei dem das Gebiet kegelförmig ist, durch eine scharfe Abschätzung ersetzen kann. Als Konsequenz kann ein optimales Anti-Maximum-Resultat für Gebiete mit kegelförmigen Teilen bewiesen werden. Im letzten Teil handelt es sich um das Verhalten der Lösungen an Punkten, indenen ein Interface den Rand trifft. Bei diesem Interface ist die Poisson-Gleichung nicht erfüllt und man erhält stattdessen eine Sprungbedingung für die Richtungsableitungen in der Normalenrichtung.

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Chapter 1

Introduction

The *Laplace equation*, $\Delta u = 0$, and its inhomogeneous form, *Poisson's equation* $-\Delta u = f$, are among the most important of all partial differential equations. We can find applications of them to problems in gravitation, elastic membranes, electrostatics, fluid flow, steady-state heat conduction and many other topics in both pure and applied mathematics. Poisson's boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in domains with smooth boundary have been widely investigated over the past two hundreds years as Poisson's and Laplace's equations are basic models of linear elliptic equations.

For the problem (1.1) with $\Omega \subset \mathbb{R}^n$ bounded and open whose boundary is at least of class C^2, there are a couple of powerful tools which have been developed and used to investigate the qualitative and quantitative properties of the solution u.

Above all, *regularity results* provide the information about the existence, the uniqueness and the smoothness of u depending on the smoothness of the right hand side function f. While the existence and the uniqueness of *the weak solution* of (1.1) are directly obtained from Riesz Representation Theorem, the existence and the uniqueness of *the classical solution* and *the strong solution* depend on the smoothness of the boundary of the domain Ω and the smoothness and the summability of f. For example if $f \in L^p(\Omega)$ then the weak solution u of (1.1) lies also in $W^{2,p}(\Omega)$ for $1 < p < \infty$ provided that $\partial\Omega \in C^{1,1}$, see [1]. On the other hand, when $f \gneqq 0$ the *maximum principle* implies that $u \geq 0$ in Ω and *Hopf's boundary point Lemma* states for a twice differentiable solution u of (1.1) that if an interior sphere condition holds at $x_0 \in \partial\Omega$, then the

normal derivative of u at x_0 is strictly positive (i.e. positive and non-zero). The regularity and Hopf's Lemma together establish a sharp estimate for the solution u as follows:
There exist positive constants $c, C > 0$ such that

$$cd(x, \Omega) \leq u(x) \leq Cd(x, \Omega) \text{ for all } x \in \Omega, \tag{1.2}$$

where $d(x, \Omega)$ is the distance of x to boundary. The question that may arise, is: *How can one describe the behaviour of the solution of Poisson's problem on nonsmooth domains?* In order to answer this question, we need to find or construct appropriate tools depending on the type of singularity of the singular points of the boundary. The existence and the uniqueness of the weak solution of (1.1) in this case are still attained by Riesz Representation Theorem, but we are interested in finding more than just the weak solution. In other words, we would like to see when and how we can have more information about the features of the solution(s) of (1.1) near the singular points of the boundary. This is the main purpose of this thesis. We will show that the weak solution of (1.1) with certain f, depending on the properties of the boundary around the singular point, can be considered also in Hölder spaces. This is done by building up a Hopf's type estimate for the weak solution near the singular point.

We first provide the fundamental definitions and theorems that are used in the other chapters in Chapter 2.

In Chapter 3 we represent an example of a planar domain whose boundary is of the class C^0 but not of the class C^1 near the origin. We show that on our domain the Hopf's Lemma holds at all boundary points but it does not hold uniformly. We prove that this pointwise Hopf's Lemma is not sufficient for the solution operator of the problem (1.1) to be strongly positive. Consequently, the standard Krein-Rutman Theorem can not be used. This counterexample is important since it demonstrates the notable role of uniformity for Hopf's Lemma result.

In Chapter 4 we consider bounded domains in general dimension that have smooth boundary with the exception of a vertex where the domain looks like a cone. The main purpose of this chapter and Chapters 5 and 6 is to investigate the behaviour of solution u of (1.1) near such a conical point. These problems also are of great interest in application areas. Rectangular shapes are used in physical or electrical models for example, to achieve a high density of component along the edges. In spite of the maximum principle that still holds true on a domain Ω containing conical points, the conditions on the smoothness of $\partial\Omega$ can not be relaxed without losing the full regularity results or Hopf's Lemma in general. In [26] Kondratiev established a theorem concerning the regularity of the solution of the general elliptic boundary value problem with constant coefficients of order $2m$ such as

$$\begin{cases} Lu = f & \text{in } \Omega, \\ Bu = g & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

He considered the solution u in weighted spaces, namely the space of functions whose derivatives are summable with respect to a weight. His work was followed by Grisvard [20], Dauge [12], Kozlov et al. [27], Nazarov and Plamenevsky [36] and Maz'ya et al. [32].
Kondratiev also obtained an asymptotic form for the solution u of (1.3) in a neighborhood of the conical point as follows:

$$\sum_{Im\lambda>h} \sum_{k=0}^{k_j} r^{-i\lambda_j} (\ln r)^s \psi_{jk}(\omega),$$

where $\omega = (\omega_1 \cdots \omega_{n-1}) \in \mathbb{S}^{n-1}$ (by $\mathbb{S}^{n-1}$ we mean the unit sphere in $\mathbb{R}^n$ centered at origin), ψ_{jk} are infinitely differentiable functions and λ_j, k_j and h are determined by the operator L. In the case of Laplacian problem for $f \in L^p(\Omega)$, u will have the form

$$u = \sum_{\gamma_j<h} |x|^{\gamma_j} \psi_j\left(\frac{x}{|x|}\right) \text{ for } x \in \Omega, \tag{1.4}$$

if we fix the conical point at 0, see [21]. The functions ψ_j and the numbers γ_j are determined by the Laplace-Beltrami operators and h depends on n and p. The main aspect of Chapter 4 is to find out if and how the radial type (1.4) provides a Hopf's type estimate for the solution u. More precisely, in a sufficient small neighborhood of the cone the leading term of (1.4), namely $|x|^{\gamma_1}\psi_1(\frac{x}{|x|})$ can be used to describe the growth rate of u. This leads to a pointwise Hopf's type estimate near singular points of boundary. This pointwise behaviour of the weak solution give the description of properties of the weak solution in Hölder type regularity way.

Chapter 5 is devoted to introducing the weighted spaces by Kondratiev [26] and Nazarov and Plamenevsky [36] and the corresponding results on such spaces.

In Chapter 6 an *anti-maximum* result for the solution of

$$\begin{cases} -\Delta u = \mu u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

on a domain containing conical points is established. Let $\mu_1 > 0$ be the first eigenvalue of the Laplace problem with zero Dirichlet boundary condition. It is proven in [10] that if Ω is a smooth bounded domain in $\mathbb{R}^n$, for $\mu_1 < \mu < \mu_1 + \varepsilon_f$ and for $0 < f$ smooth enough, the solution u of (1.5) is negative. This result is known as an anti-maximum principle which for the non-smooth domains does not hold in general. In this chapter we prove that when Ω contains conical points, an anti-maximum result holds for the solution u under a certain assumption on the growth rate of f near conical points. This result can be served as an application of the Hopf's type estimates achieved in Chapter 4.

And finally, Chapter 7 treats an *interface problem*. Here, by an interface problem we mean a second order elliptic problem with a discontinuous coefficient for the second order derivatives. We consider the corresponding boundary value problem equipped with zero boundary condition on a smooth planar domain. Despite the smoothness of the domain, Hopf's Lemma may fail at the boundary points where the coefficient of the second derivatives is not continuous. We refer to these points as *interface points*. We study the behaviour of the solution of this interface problem in the neighborhood of the interface points to provide a Hopf's type estimate.

Chapter 2

Preliminaries

In this chapter we present some basic definitions and theorems that will be used in the rest of this thesis. Some facts are standard and well-known. For the results which are not clear, proofs are given.

2.1 The Laplace Equation

Let Ω be a bounded open connected subset of $\mathbb{R}^n$ and u be a real function on Ω. The **Laplacian** of u is defined as

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \nabla \cdot \nabla u,$$

where ∇ is the gradient and $\nabla\cdot$ is the divergence. A twice differentiable function u satisfying

$$\Delta u = 0 \quad \text{in } \Omega \tag{2.1}$$

is called ***harmonic*** in Ω.
The equation

$$-\Delta u = f \quad \text{in } \Omega \tag{2.2}$$

for given $f : \Omega \to \mathbb{R}$ is known as **Poisson's equation**.
The weak form of (2.2) is

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \tag{2.3}$$

in which $v \in \mathring{W}^{1,2}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$ is an arbitrary test function.

The two most common types of boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$ are the following.
Dirichlet Condition: For a given function $g : \partial\Omega \to \mathbb{R}$ the condition

$$u(x) = g(x) \quad \text{for} \quad x \in \partial\Omega,$$

is called the Dirichlet condition.
Neumann Condition: We assume that $\partial\Omega \in C^1$. For a given function $g : \partial\Omega \to \mathbb{R}$ the Neumann condition is as follows;

$$\frac{\partial u}{\partial \nu}(x) = g(x) \quad \text{for} \quad x \in \partial\Omega,$$

in which ν is the outward normal of $\partial\Omega$.
In this thesis we only work with the Dirichlet boundary condition.

Definition 2.1.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Consider the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

a) *A function $u \in \mathring{W}^{1,2}(\Omega)$ is called a **weak solution** of (2.4) if u satisfies (2.3) for all test functions $v \in \mathring{W}^{1,2}(\Omega)$.*

b) *A **classical solution** of (2.4) is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the equation (2.4) for all $x \in \Omega$.*

c) *A function $u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ satisfying (2.4) is called a **strong solution**.*

The existence and uniqueness of the weak solution in $\mathring{W}^{1,2}(\Omega)$ of Poisson's problem with the Dirichlet boundary condition (2.4) come out from the Riesz Representation Theorem, see [15, Section 6.2]. By Green's formula, a strong solution u satisfies (2.3) which implies that u is a weak solution. On the other hand, under suitable hypotheses on the smoothness of the function f and the boundary of Ω, our weak solution is, in fact, a strong solution.

2.2 The Eigenvalue Problem

The **classical eigenvalue problem** for the Laplace operator with Dirichlet boundary condition in an open bounded connected domain $\Omega \subseteq \mathbb{R}^n$ is as follow,

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

with the weak form

$$\int_\Omega \nabla\phi\nabla v dx = \lambda \int_\Omega \phi v \, dx,$$

for all $v \in \mathring{W}^{1,2}(\Omega)$. Let us recall that $\lambda \in \mathbb{R}$ is an **eigenvalue** of the Laplace operator provided that there exists a nontrivial solution ϕ, which is called the corresponding eigenfunction of λ, of (2.5). According to the Fredholm alternative Theorem, see [40, Theorem 7.93 and Theorem 8.21], the set Σ of eigenvalues of the Laplace operator is at most countable. Moreover, we have $\Sigma = \{\lambda_i\}_{i=1,2,\cdots}$ where

$$0 < \lambda_1 < \lambda_2 \leq \cdots$$

and

$$\lim_{i\to\infty} \lambda_i = \infty.$$

The set of eigenfunctions $\{\phi_i\}_{i=1,2,\cdots}$ forms an orthonormal basis for $L^2(\Omega)$, see [15, §6.5.1]. The smallest λ_i, namely the first eigenvalue λ_1, is indeed the infimum of the Rayleigh quotient of Laplace, i.e.

$$\lambda_1 = \inf_{\phi \in \mathring{W}^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \phi^2 dx}$$

and the corresponding eigenfunction $\phi_1 \in \mathring{W}^{1,2}(\Omega)$ is the minimizer of the functional $J(u) := \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$ on $\mathring{W}^{1,2}(\Omega) \setminus \{0\}$. The Poincaré inequality implies that $\lambda_1 > 0$. Indeed, Poincaré's inequality states that there exists a positive constant $C_\Omega > 0$, depending only on the domain Ω, such that for all $u \in \mathring{W}^{1,2}(\Omega)$ we have

$$\int_\Omega u^2 dx \leq C_\Omega \int_\Omega |\nabla u|^2 dx.$$

This follows that $0 < \frac{1}{C_\Omega} \leq \lambda_1$.

On the other hand, the Courant Nodal Domain Theorem states that the first eigenfunction ϕ_1, corresponding to the smallest eigenvalue λ_1, is positive in Ω, see [11, Volume I, Chapter VI, §6 (page 452)]. In fact, ϕ_1 is the only eigenfunction that does not change its sign. By defining the nodal set of an eigenfunction

ϕ_i as the set of all $x \in \Omega$ such that $\phi_i(x) = 0$, we observe that the nodal set makes a division of Ω into subsets where $\phi_i(x) > 0$ or $\phi_i(x) < 0$. The Courant Nodal Domain Theorem also states that ϕ_i for $i \geq 2$ divides the domain Ω into at least two and at most i regions.
The following theorem shows how the method of eigenfunction expansion is used to construct solutions, see [40, Theorem 8.22].

Theorem 2.2.1 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain. If $\mu \neq \lambda_i$ for $i = 1, 2, \cdots$ then for every $f \in L^2(\Omega)$ the unique weak solution of*

$$\begin{cases} -\Delta u - \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

can be written as a convergent sequence in $L^2(\Omega)$ using the set of normalized eigenfunctions $\{\phi_i\}_{i=1,2,\cdots}$, i.e. $\langle \phi_i, \phi_j \rangle_2 = \delta_{i,j}$, as follows:

$$u = \sum_{i=1}^{\infty} \frac{\langle \phi_i, f \rangle_2}{\lambda_i - \mu} \phi_i.$$

Here, $\langle \cdot, \cdot \rangle_2$ denotes the inner product in $L^2(\Omega)$; i.e.

$$\langle \phi_i, f \rangle_2 = \int_\Omega \phi_i f \, dx.$$

2.3 Regularity

As we mentioned before, by posing certain hypotheses on the smoothness of the domain and the function f, a weak solution of (2.4) will have a corresponding smoothness. This is the regularity for the weak solution.

Theorem 2.3.1 *Suppose that $u \in \mathring{W}^{1,2}(\Omega)$ is a weak solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $\partial\Omega \in C^2$ and $f \in L^2(\Omega)$, then $u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ and there exists a positive constant C depending only on Ω such that the following estimate holds for u;

$$\|u\|_{W^{2,2}(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

To find a proof of this theorem, we refer to [19, Theorem 8.12] or [15, Theorem 4 at page 317].

Theorem 2.3.2 *Suppose that $u \in \mathring{W}^{1,2}(\Omega)$ is a weak solution of (2.4). In addition, assume that $\partial\Omega$ is of class C^{m+2} for an integer $m > 0$ and let $f \in W^{m,2}(\Omega)$, then $u \in W^{m+2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ and we have the estimate*

$$\|u\|_{W^{m+2,2}(\Omega)} \leq C\left(\|f\|_{W^{m,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where C is a positive constant depending only on m and Ω.

See [15, Theorem 5, page 323] or [18, Theorem 4.14] for the proof.

For the existence and uniqueness of the strong solution we recall the following theorem [19, Theorem 9.15].

Theorem 2.3.3 *Let Ω be a domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$. Then for $f \in L^p(\Omega)$ with $1 < p < +\infty$ the problem (2.4) has a unique solution $u \in W^{2,p}(\Omega)$.*

Remark 2.3.4 *Theorem 2.3.3 can not be extended to the cases $p = 1$ and $p = +\infty$. See [18, Examples 7.5 and 7.6] for instance.*

The following theorem, recalled from [19, Theorem 6.19], shows that under suitable hypotheses on the smoothness of the boundary of Ω and f the smoothness of the strong solution is improved.

Theorem 2.3.5 *Let Ω be a domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{k+2,\alpha}$ for $k \geq 0$ and $0 < \alpha < 1$ and suppose that $u \in C^2(\Omega) \cap C_0\left(\overline{\Omega}\right)$ satisfies the problem (2.4), where $f \in C^{k,\alpha}\left(\overline{\Omega}\right)$, then $u \in C^{k+2,\alpha}\left(\overline{\Omega}\right) \cap C_0\left(\overline{\Omega}\right)$.*

The regularity is an important tool to get an estimate from above for the solution u by the distance function to the boundary.

Theorem 2.3.6 *Let $\Omega \subset \mathbb{R}^n$ be bounded and $\partial\Omega \in C^2$. Suppose that u satisfies (2.4) with $f \in C\left(\overline{\Omega}\right)$ and $f \geq 0$. Then there exists a positive constant M such that for $x \in \Omega$,*

$$u(x) \leq Md(x),$$

where $d(x) := \inf_{x^* \in \partial\Omega} |x - x^*|$ is the distance function to the boundary of Ω.

Proof. From $f \in C\left(\overline{\Omega}\right)$ it follows that $f \in L^p(\Omega)$ for every $p \geq 1$. Then we can choose $p \in \mathbb{R}$ so large that $p > n$. By Theorem 2.3.3 there exists a unique solution $u \in W^{2,p}(\Omega)$ for the problem (2.4) and $u \in C^{1,\gamma}\left(\overline{\Omega}\right)$ by Sobolev imbedding for $\gamma < 1 - \frac{n}{p}$. For any $x \in \Omega$ there is at least one $x_0 \in \partial\Omega$ such that $|x - x_0| = d(x)$. Now, by mean value theorem and the fact that $u = 0$ on $\partial\Omega$ we find that

$$u(x) = u(x) - u(x_0) = (x - x_0) \cdot \nabla u(y)$$

for some $y = x_0 + t\overrightarrow{\nu_0}$, where $t \in \mathbb{R}$ and $\overrightarrow{\nu_0}$ is the interior normal vector at x_0. It follows from $u \in C^{1,\gamma}\left(\overline{\Omega}\right)$ that $|\nabla u(y)| < M$, for $0 \leq M \in \mathbb{R}$. Then we directly get

$$u(x) \leq Md(x),$$

□

2.4 The Maximum Principle

The maximum principle is an important and strong feature of second order elliptic equations. It can be used to show that solutions of certain equations must be non-negative, which is important for quantities with physical interpretations. The maximum principle also leads to uniqueness of the solution. In addition to its many applications, the maximum principle provides pointwise estimates for the solutions. On the other hand, the maximum principle and its consequent properties make the second order elliptic equations distinguished than elliptic equations of higher order.

Theorem 2.4.1 (The weak maximum principle) *Let $\Omega \subset \mathbb{R}$ be a bounded domain. Suppose that*

$$-\Delta u \geq 0 \ \textit{in}\ \Omega,$$

with $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then the infimum of u in $\overline{\Omega}$ is achieved on $\partial\Omega$, that is

$$\inf_{\Omega} u = \inf_{\partial\Omega} u.$$

Theorem 2.4.2 (The strong maximum principle) *Let $\Omega \subseteq \mathbb{R}$ and*

$$-\Delta u \geq 0 \ \textit{in}\ \Omega.$$

If u achieves its infimum in the interior of Ω, then u is a constant.

2.4.1 The Maximum Principle for Weak Solutions

Theorem 2.4.3 (The weak maximum principle for weak solutions) *Let* $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ *satisfy*

$$\int_\Omega \nabla u \cdot \nabla v dx \geq 0 \text{ in } \Omega \tag{2.6}$$

for all $v \in \mathring{W}^{1,2}(\Omega)$. *Then*

$$\inf_\Omega u \geq \inf_{\partial\Omega} u^-.$$

Proof. By taking $v = u_l^- := \inf\{u - l, 0\}$ where $l = \inf_{\partial\Omega} u$, the inequality (2.6) would be as follows;

$$0 \leq \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \nabla u \cdot \nabla u_l^- dx = -\int_\Omega |\nabla u_l^-|^2 dx.$$

Hence, $|\nabla u_l^-| = 0$ which implies u_l^- is equal to a constant c. If c is a nonzero constant then one can find that $c = u - l < 0$ in $\overline{\Omega}$. But $u - l = c$ in $\overline{\Omega}$ is contradiction to $\inf_{\partial\Omega} u = l$. Therefore $c = 0$ and $u \geq l$ in $\overline{\Omega}$.

□

Theorem 2.4.4 (The strong maximum principle for weak solutions) *Let* $u \in W^{1,2}(\Omega)$ *satisfy*

$$\int_\Omega \nabla u \cdot \nabla v dx \geq 0 \text{ in } \Omega \tag{2.7}$$

for all $v \in \mathring{W}^{1,2}(\Omega)$. *Then if for some ball* $B \subset\subset \Omega$ *we have*

$$\text{ess-inf}_B\, u = \text{ess-inf}_\Omega\, u \geq 0,$$

the function u *must be constant in* Ω.

2.5 Hopf's Boundary Point Lemma

Hopf's Boundary Point Lemma has been stated and proved by Hopf in 1952 in a short note [23].

Theorem 2.5.1 (Hopf's Boundary Point Lemma) *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Then a twice differentiable solution* $u \geq 0$ *satisfying*

$$-\Delta u \geq 0 \text{ in } \Omega, \tag{2.8}$$

with $u(x^*) = 0$ for some $x^* \in \partial\Omega$ where an interior sphere condition is present, satisfies either $u \equiv 0$ or

$$\liminf_{t \downarrow 0} \frac{u\left(x^* - t\, \vec{\nu}\right) - u\left(x^*\right)}{t} \in (0, \infty]\,. \tag{2.9}$$

Here $\overrightarrow{\nu}$ is the outward pointing unit vector at x^*. If u is differentiable at x^*, then

$$-\frac{\partial u}{\partial \vec{\nu}}\left(x^*\right) > 0. \tag{2.10}$$

For two-dimensional domains one may exploit the connection with conformal mapping to find some typical examples.

Example 2.5.2 *We define the sector*

$$S_\varphi = \left\{(x_1, x_2)\,;\ x_1 > \cot\left(\varphi\right)|x_2|\right\}.$$

First let $\varphi \in \left(\frac{1}{2}\pi, \pi\right)$ and take $\Omega \subset S_\varphi$ a domain with smooth boundary except at $(0,0)$ and such that $\Omega \cap B_1(0) = S_\varphi \cap B_1(0)$. See Figure 2.1 on the left. Consider

$$u\left(x_1, x_2\right) = \mathrm{Re}\left(\left(x_1 + ix_2\right)^{\frac{\pi}{2\varphi}}\right). \tag{2.11}$$

This function u is positive and harmonic on S_φ and satisfies $u = 0$ on ∂S_φ. For $\varphi \in \left(\frac{1}{2}\pi, \pi\right)$ the domain satisfies the interior sphere condition and since $u\left(x_1, 0\right) = x_1^{\pi/(2\varphi)}$ one may find at $x^* = (0,0)$ with $\vec{\nu} = (-1, 0)$;

$$\begin{aligned} \liminf_{t\downarrow 0} \tfrac{u(x^* - t\ \vec{\nu}) - u(x^*)}{t} &= \liminf_{t\downarrow 0} \tfrac{u(-t(-1,0)) - u(0,0)}{t} = \\ = \liminf_{t\downarrow 0} \tfrac{u(t,0) - 0}{t} &= \liminf_{t\downarrow 0} \tfrac{t^{\pi/(2\varphi)}}{t} = +\infty. \end{aligned}$$

The statement in (2.9) holds true.

Example 2.5.3 *If $\varphi \in \left(0, \frac{1}{2}\pi\right)$ and we take Ω as in the first example, then Ω does not satisfy an interior sphere condition. See Figure 2.1 on the right. The function in (2.11) is still positive and harmonic on S_φ and satisfies $u = 0$ on ∂S_φ. Here one finds $u \in C^1(\overline{\Omega})$ and*

$$\lim_{\Omega \ni x \to (0,0)} \nabla u\left(x\right) = 0.$$

So any directional derivative at $(0,0)$ is 0.

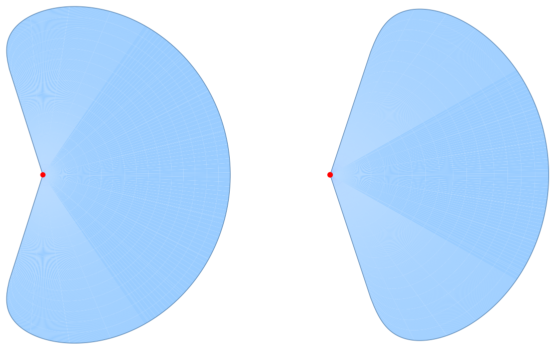


Figure 2.1: *Sketches of domains from Example 2.5.2 and 2.5.3. The dot denotes the origin.*

Since C^2-domains satisfy an interior sphere condition, (2.10) holds true at each point of the boundary. The interior sphere condition is also satisfied for domains with boundaries that consist of several C^2-parts which are non smoothly connected as long as this is done in a reentrant way. For a boundary in $\mathbb{R}^2$ with a reentrant corner the solution in general is not in $C^1(\overline{\Omega})$ but (2.9) still holds. However, the interior sphere condition is sufficient at $x_0 \in \partial\Omega$ for Hopf's Lemma to hold, it is not necessary. It follows from [31, 25] that the optimal hypothesis at $x_0 \in \partial\Omega$ for Hopf's Lemma is the interior Dini-condition. Recalling from [31] this condition is as follows.

Definition 2.5.4 *We say that domain Ω has a local C^1-representation at $x_0 \in \partial\Omega$ if there is*

1. *a similarity transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, i.e. there exist $s \in \mathbb{R}^+$, M an orthogonal matrix and $v_0 \in \mathbb{R}^n$ such that $Tx = sMx + v_0$, with $T(0) = x_0$, and*

2. *a C^1 function $h : \mathbb{R}^{n-1} \longrightarrow (-1, 1)$ with $h(0) = 0$, such that*

$$\Omega \cap T\left(B_1'(0) \times [-1, 1]\right) = \left\{T(x', x_n);\ h(x') < x_n < 1 \text{ and } x' \in B_1'(0)\right\},$$

where $B_1'(0) = \{x' \in \mathbb{R}^{n-1}; |x'| < 1\}$.

Definition 2.5.5 *We say that Ω is **Dini** at x_0 if the following holds. There exists a local C^1-representation h at $x_0 \in \partial\Omega$ as in Definition 2.5.4 with $Dh(0) = 0$ and moreover, there is a function $\omega \in C[0, 2]$ such that*

(a) *ω is increasing;*

(b) *$\omega(0) = 0$;*

(c) $\int_0^2 \frac{\omega(s)}{s} ds$ *is finite;*

(d) $|Dh(x') - Dh(y')| \leq \omega(|x' - y'|)$ *for* $|x'|, |y'| \in B_1'(0)$.

We say that Ω satisfies an **interior Dini condition** at $x_0 \in \partial\Omega$ if there is an open set $\Omega' \subset \Omega$ such that $x_0 \in \partial\Omega'$ and Ω' is Dini at x_0. The interior sphere condition implies the interior Dini condition but the inverse does not hold.

Example 2.5.6 *Recalling from [19, page 35], set* $u(x_1, x_2) := -Re(\frac{(x_1+ix_2)}{\log(x_1+ix_2)})$. *Then* u *is a strictly positive and harmonic function on*

$$\Omega = \left\{(x_1, x_2); x_1 \in \left(0, \tfrac{1}{2}\right) \text{ and } u(x_1, x_2) > 0\right\},$$

which satisfies $u_{x_1}(0,0) = 0$ *so the Hopf type result does not hold at* $(0,0)$. *For* Ω *see Figure 2.2 on the left. Near* 0 *the boundary is* C^1 *but the boundary is not Dini-smooth and certainly does not satisfy an interior sphere condition. One may show that the boundary near* 0 *is given by* $x = \frac{\pi}{2}\frac{|y|}{-\ln|y|}\left(1 + \mathcal{O}\left(\frac{1}{-\ln|y|}\right)\right)$. *Indeed, the function* $y \mapsto \frac{|y|}{-\ln|y|}$ *(with* 0 *in* 0*) is* $C^{1,0}$ *but not Dini-smooth.*

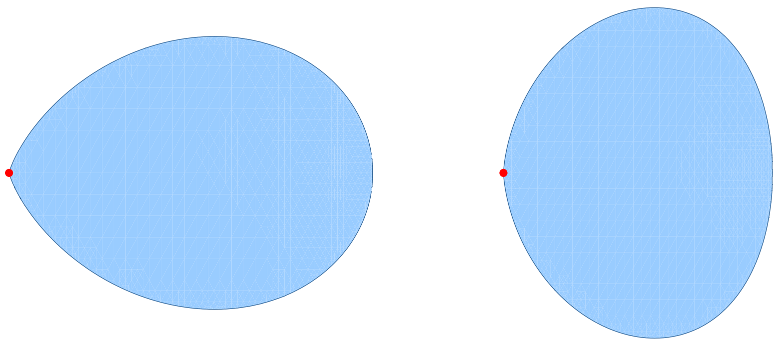


Figure 2.2: *Sketches of the domains from Example 2.5.6 and 2.5.7.*

Example 2.5.7 *Set* $u(x_1, x_2) := -Re\left((x_1 + ix_2)\left(1 - \frac{1}{\log(x_1+ix_2)}\right)\right)$. *Then* u *is strictly positive and harmonic on*

$$\Omega = \left\{(x_1, x_2); x_1 \in \left(0, \tfrac{1}{2}\right) \text{ and } u(x_1, x_2) > 0\right\},$$

which satisfies $u_{x_1}(0,0) = 1$ *and hence the Hopf type result holds for* u. *Near* 0 *the boundary is Dini-smooth but not* $C^{1,\gamma}$, *nor does it satisfy an interior sphere condition. Here one finds that the boundary near* 0 *is given by* $x = \frac{\pi}{2}\frac{|y|}{(\ln|y|)^2}\left(1 + \mathcal{O}\left(\frac{1}{-\ln|y|}\right)\right)$. *The function* $y \mapsto \frac{|y|}{(\ln|y|)^2}$ *(with* 0 *in* 0*) is Dini-smooth but not Hölder-smooth. See Figure 2.2 on the right.*

2.5.1 An estimate from below by Hopf's Lemma

We finish this chapter with a useful application of Hopf's Lemma in getting an estimate from below for the solution of (2.4).

Theorem 2.5.8 *Suppose that* $\Omega \subset \mathbb{R}^n$ *is bounded and* $\partial\Omega \in C^2$. *If* $u \in C^2(\Omega)$ *satisfies*

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

and u *is not identical zero* $(u \not\equiv 0)$ *then there is a constant* $C > 0$ *such that*

$$Cd(x) \leq u(x) \quad \text{for all } x \in \Omega.$$

Where $d(x)$ *is the distance of* x *to* $\partial\Omega$.

Proof. Assume that $x_0 \in \partial\Omega$. By Hopf's Lemma we find $\frac{\partial u}{\partial \overrightarrow{\nu}}(x_0) < 0$. First we show that there is a positive constant $c \in \mathbb{R}$ such that for all $x_0 \in \partial\Omega$,

$$\frac{\partial u}{\partial \overrightarrow{\nu}}(x_0) \leq -c < 0. \tag{2.12}$$

Suppose (2.12) does not hold true, i.e. there exists a sequence $\{x_n^*\}$ of boundary points such that $\frac{\partial u}{\partial \overrightarrow{\nu}}(x_n^*) < 0$ and $\frac{\partial u}{\partial \overrightarrow{\nu}}(x_n^*) \to 0$ (that means $\nabla u(x_n^*) \cdot \overrightarrow{\nu}(x_n^*) \longrightarrow 0$). From $\partial\Omega \in C^2$ it follows directly that the mapping $x \mapsto \overrightarrow{\nu}(x)$ is a C^1 function and since $u \in C^2(\Omega)$ we find $\nabla u(x) \in C\left(\overline{\Omega}, \mathbb{R}^n\right)$. Consequently, $\frac{\partial u}{\partial \overrightarrow{\nu}}(\cdot) = \nabla u(\cdot) \cdot \overrightarrow{\nu}(\cdot)$ is a continuous mapping.
On the other hand, $\partial\Omega$ is compact so it follows that there is a convergent subsequence $\left\{x_{n_k}^*\right\}$ of $\{x_n^*\}$. So there is an $x^* \in \partial\Omega$ such that $x_{n_k}^* \longrightarrow x^*$ and $\nabla u\left(x_{n_k}^*\right) \cdot \overrightarrow{\nu}\left(x_{n_k}^*\right) \longrightarrow 0$. Consequently, one finds

$$\frac{\partial u}{\partial \overrightarrow{\nu}}(x^*) = \lim_{k\to\infty} \frac{\partial u}{\partial \overrightarrow{\nu}}\left(x_{n_k}^*\right) = \lim_{k\to\infty} \nabla u\left(x_{n_k}^*\right) \cdot \overrightarrow{\nu}\left(x_{n_k}^*\right) = 0$$

which is a contradiction to Hopf's Lemma. Thus (2.12) holds true for all points x on the boundary.
Furthermore, since $\partial\Omega \in C^2$, there is an $\varepsilon > 0$ and the neighborhood

$$A_\varepsilon = \{x - \varrho \overrightarrow{\nu}(x) \,|x \in \partial\Omega, 0 < \varrho < \varepsilon\} \subset \Omega$$

such that $\overrightarrow{\nu}(y)$ is well-defined for $y \in A_\varepsilon$. It follows from $\nabla u \in C\left(\overline{\Omega}\right)$ that for $y \in A_\varepsilon$,

$$-\frac{\partial u}{\partial \overrightarrow{\nu}}(y) \geq \tfrac{1}{2}c > 0.$$

Suppose $y \in A_\varepsilon$ and $x_0 \in \partial\Omega$ satisfies $d(y) = |y - x_0|$. Then one finds

$$u(y) = u(y) - u(x_0) = (y - x_0) \cdot \nabla u(\theta), \tag{2.13}$$

for some θ lying on the line $\overline{yx_0}$. The right hand side of (2.13) is as follows;

$$\begin{aligned} (y - x_0) \cdot \nabla u(\theta) &= -|y - x_0| \tfrac{\partial u}{\partial \overline{\nu}}(\theta) \\ &\geq \tfrac{1}{2} c |y - x_0| = \tfrac{1}{2} c d(y). \end{aligned}$$

Hence, for all $y \in A_\varepsilon$ one finds $u(y) \geq \frac{c}{2} d(y)$. Note that the maximum principle guarantees that $u(x) \geq 0$ for $x \in \overline{\Omega}$. Now, we set $\overline{\Omega_\varepsilon} = \Omega \setminus A_\varepsilon$ which is a closed set, because A_ε is open, and set

$$u_{\min} := \inf_{x \in \overline{\Omega_\varepsilon}} u(x) = \min_{x \in \overline{\Omega_\varepsilon}} u(x).$$

Thus, we find for $x \in \overline{\Omega_\varepsilon}$

$$u(x) \geq u_{\min} \geq u_{\min} \frac{d(x)}{d(\Omega)}$$

where $d(\Omega)$ is the diameter of Ω. By setting

$$C := \min\left(\frac{c}{2}, \frac{u_{\min}}{d(\Omega)}\right)$$

we find for all $x \in \Omega$

$$u(x) \geq C d(x).$$

$\square$

Chapter 3

Hopf's Lemma and the Krein-Rutman Theorem

The existence, positivity and simplicity of the first eigenvalue of the Dirichlet Laplacian on general domains follow directly from variational arguments as given in [11]. This method can be used even for self-adjoint divergence form operators with bounded and measurable coefficients. However, for more general elliptic operators without a symmetric bilinear form, the Rayleigh quotient approach fails. In such cases, where the methods of the calculus variations do not work, the Krein-Rutman Theorem and De Pagter's generalisation for compact irreducible operators are useful. When $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, the combination of the Krein-Rutman Theorem and Hopf's boundary point lemma is a strong tool for second order elliptic boundary value problems. In this chapter, first we have a brief look at how this combination is used by Amann in [2]. Then we give an example that illustrates that a uniform Hopf's lemma is necessary in this combination. The main result stated in this chapter has been published in [5].

3.1 The Krein-Rutman Theorem

Definition 3.1.1 *If E is an ordered Banach space then the* ***positive cone*** $P = \{u \in E; u \geq 0\}$ *is called* ***total*** *if* $\overline{P - P} = E$.

For a Banach space E, we denote by $\mathcal{L}(E)$ the space of all bounded linear operators on E. Recalling from [29] the classical Krein-Rutman Theorem is as follows.

Theorem 3.1.2 (Krein-Rutman Theorem) *Let (E,P) be an ordered Banach space with total positive cone. Suppose that $T \in \mathcal{L}(E)$ is a compact and positive operator with a strictly positive spectral radius $r(T) = \limsup_{n\to\infty} \sqrt[n]{\|T^n\|}$. Then $r(T)$ is an eigenvalue of T and of the dual operator T^* with eigenvectors in P and in P^*, .*

Let us remark that an eigenvalue λ of a linear operator T is called **simple** if

$$\dim\left(\bigcup_{k=1}^{\infty} \ker(\lambda I - T)^k\right) = 1.$$

Where I denotes the identity operator.

The following theorem is the combination of the Krein-Rutman Theorem and an important result of De Pagter [39] that replaces the positivity of the spectral radius of T by irreducibility. First we recall the notion of the irreducible operator on a Banach lattice from [41].

Definition 3.1.3 *A real vector space with a partial ordering, say $(E, \geqslant)$ is called a **vector lattice** if $f, g \in E$ implies that $\sup(f,g) \in E$.*
*With a norm supplied $(E, \geqslant, \|\cdot\|)$ is called a **Banach lattice** if $(E, \|\cdot\|)$ is a Banach space and if $(E, \geqslant)$ is a vector lattice such that $|f| \leqslant |g|$ implies $\|f\| \leq \|g\|$, where $|f| := \sup(f, -f)$.*
*The set $A \subseteq E$ is called a **lattice ideal** if $|f| \leq |g|$ and $g \in A$ imply $f \in A$.*
*A positive operator $S \in L(E)$, i.e. if $x \geq 0$ then $Sx \geq 0$, is called **irreducible** if $\{0\}$ and E are the only closed lattice ideals that are invariant under S.*

Theorem 3.1.4 (Krein-Rutman-De Pagter Theorem) *[8] Let E be a Banach lattice with $\dim(E) \geq 2$ and let $T \in \mathcal{L}(E)$ be positive, compact and irreducible. Then the spectral radius of T is an eigenvalue of T and the corresponding eigenfunction is unique. Moreover, the spectral radius of T is the largest (in absolute value) eigenvalue of T.*

Amann in [2] used the Krein-Rutman Theorem for proving the existence and positivity of the first eigenvalue and corresponding eigenfunction of the second order elliptic boundary value problem

$$\begin{cases} Lu = \lambda m u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, and

$$L := -\sum_{i,k=1}^{n} a_{ik} D_i D_k + \sum_{i=1}^{n} a_i D_i + a \tag{3.2}$$

and $a_{ik}, a_i, a \in C^{\mu}\left(\overline{\Omega}\right)$ such that $a \geq 0$ and there exists a constant $\gamma > 0$ such that

$$\sum_{i,k=1}^{n} a_{ik} \zeta_i \zeta_k \geq \gamma |\zeta|^2$$

for all $x \in \overline{\Omega}$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n$. Moreover, it is supposed that $m \in C^{\mu}\left(\overline{\Omega}\right)$ and $m(x) > 0$ for almost all $x \in \Omega$. Since the solution operators $K : C^{\mu}\left(\overline{\Omega}\right) \longrightarrow C^{\mu+2}\left(\overline{\Omega}\right)\left(\hookrightarrow C^{\mu}\left(\overline{\Omega}\right)\right)$ and $T : L^p(\Omega) \longrightarrow W^{2,p}(\Omega)\left(\hookrightarrow L^p(\Omega)\right)$ for

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

do not fulfill all the hypotheses of the standard Krein-Rutman Theorem, Amann considered the solution operator G_Ω in $C_e\left(\overline{\Omega}\right)$, which is defined as

$$C_e\left(\overline{\Omega}\right) = \left\{ u \in C_0\left(\overline{\Omega}\right) \,\middle|\, \|u\|_e := \sup_{x \in \Omega} \frac{|u(x)|}{e(x)} < \infty \right\}$$

and e is the unique solution of

$$\begin{cases} Le = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

As the boundary of Ω is smooth, one finds that for any $0 \lneqq f \in C\left(\overline{\Omega}\right)$ there are positive constants $C_f, c_f > 0$ such that

$$c_f e \leq G_\Omega f \leq C_f e.$$

Now, we assume that L is simply the Laplacian operator, however, the following argument is valid for general second order elliptic operators with the form of (3.2) with sufficient smooth coefficients. For $\partial\Omega \in C^2$, Hopf's Lemma and regularity imply that there are $C, c > 0$ such that

$$cd \leq e \leq Cd,$$

where d is the distance function to the boundary of Ω, see Theorems (2.3.6) and (2.5.8).

Definition 3.1.5 *The solution operator* $G_\Omega : C(\overline{\Omega}) \to C_e(\overline{\Omega})$ *for* (3.3) *with* $L = -\Delta$ *is said to be* ***strongly positive*** *whenever* $0 \lneqq f \in C(\overline{\Omega})$ *implies that for some* $c_f > 0$ *and for all* $x \in \Omega$ *one finds that*

$$u(x) \geq c_f \; e(x). \tag{3.4}$$

Remark 3.1.6 *For* Ω *bounded and* $\partial\Omega \in C^2$ *the definition of strongly positive (see e.g. [28]) implies that* $G_\Omega \in L\left(C_e(\overline{\Omega})\right)$ *has a positive spectral radius* $r\left(G_\Omega\right) \geq \|G_\Omega e\|_e$.

It may be misunderstood that if Hopf's boundary point Lemma holds on a domain Ω, then the solution operator G_Ω is strongly positive. We show by a counter example that a pointwise Hopf's Lemma is not sufficient. We will construct a special domain Ω for which there is an interior sphere condition at each point and hence, Hopf's boundary point Lemma holds true at each point. Nevertheless, the solution operator is not strongly positive in the sense of Definition 3.1.5. The solutions on Ω will in general not be in $C^1(\overline{\Omega})$ although the normal derivative is expected to exist at each boundary point. This illustrates that it is necessary that Hopf's Lemma holds uniformly for the solution operator G_Ω to be strongly positive.

3.2 Main Result

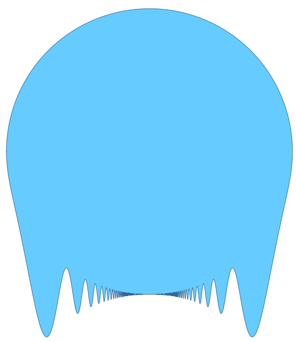


Figure 3.1: *Our example domain* Ω ©Springer Basel

The domain $\Omega \subset \mathbb{R}^2$ (see Figure 3.1) is defined as follows. Let $q > 1$ and set

$$\phi_q\left(t\right) = \begin{cases} 0 & \text{for } t = 0, \\ \left(\sqrt{1-t^2}-1\right)\cos\left(\pi\left|t\right|^{-q}\right) & \text{for } 0 < |t| < 1. \end{cases}$$

We define

$$\Omega := \left\{ (x_1, x_2) \in \mathbb{R}^2 | \quad |x_1| < 1 \text{ and } \phi_q(x_1) < x_2 < 1 + \sqrt{1 - x_1^2} \right\}. \tag{3.5}$$

Theorem 3.2.1 *Consider Ω as in (3.5). Let $f \in C(\overline{\Omega})$ and suppose that $u \in C_0(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ with $p > 1$ solves*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

If $f \gneqq 0$ holds true, then u satisfies Hopf's boundary point lemma at each point of $\partial\Omega$. Nevertheless, for such u there is no $c > 0$ such that u satisfies (3.4) with e replaced by d_Ω.

Remark 3.2.2 *By Arendt and Bénilan [3, Lemma 2.2], using results from [13], one finds for bounded domains $\Omega \subset \mathbb{R}^n$ that $u \in C_0(\overline{\Omega})$, with $\Delta u \in L^p(\Omega)$ and $p > n$, is in $\mathring{W}^{1,2}(\Omega)$. Hence the solution in Theorem 3.2 coincides with the weak solution that one obtains through the Riesz' Representation Theorem. See e.g. [19]. The reverse is in general not true. A punctured disk shows that the weak solution does not have to be in $C_0(\overline{\Omega})$. In higher dimensions there are even simply connected domains, for example with a Lebesgue Thorn, that may serve as a counterexample.*

In order to prove the Theorem , first we prove some lemmas as follows.

3.2.1 Auxiliary Lemmas

We fix domains $A \subset \Omega \subset B$ as follows:

$$A = \left\{ (x_1, x_2) ; x_1^2 + (x_2 - 1)^2 \leq 1 \right\}$$

and with

$$\tilde{\phi}_q(t) = \begin{cases} \sqrt{1 - t^2} - 1 & \text{for } 0 \leq |t|^q \leq \frac{1}{2}, \\ \left(\sqrt{1 - t^2} - 1\right) \cos\left(\pi |t|^{-q}\right) & \text{for } \frac{1}{2} < |t|^q < 1, \end{cases}$$

we set

$$B = \left\{ (x_1, x_2) ; |x_1| < 1 \text{ and } \tilde{\phi}_q(x_1) < x_2 < 1 + \sqrt{1 - x_1^2} \right\}.$$

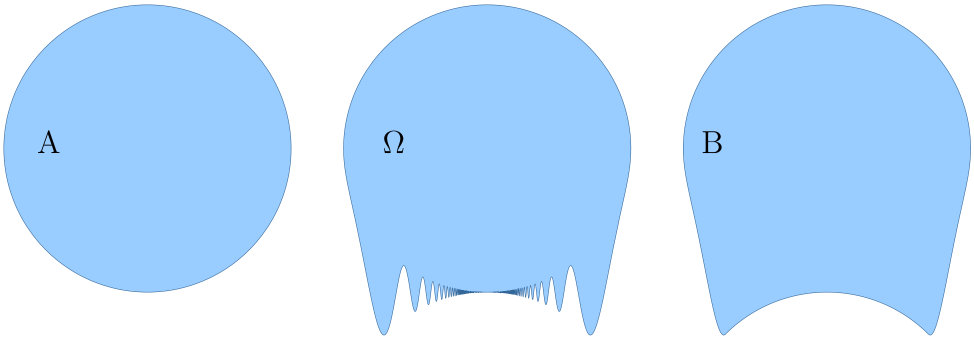


Figure 3.2: $A \subset \Omega \subset B$ ©Springer Basel

See Figure 3.2. For $D \in \{A, \Omega, B\}$ we then define u_A, u_Ω and u_B in $C_0\left(\bar{\Omega}\right) \cap W^{2,p}_{loc}(\Omega)$ to be the unique solution of

$$\begin{cases} -\Delta u = 1 & \text{inside } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{3.7}$$

The Maximum Principle shows that these functions are positive inside their domains.

Lemma 3.2.3 *Let u_A, u_B and u_Ω be as above, the solutions of (3.7) on the corresponding domains, then*

$$0 < u_A < u_\Omega \text{ on } A, \text{ and } 0 < u_\Omega < u_B \text{ on } \Omega \tag{3.8}$$

and with $\nu = \binom{0}{-1}$, the external normal vector at the origin,

$$0 < -\frac{\partial u_A}{\partial \nu}(0) \leq \liminf_{\varepsilon \downarrow 0} \frac{u_\Omega(-\varepsilon\nu)}{\varepsilon} \leq \limsup_{\varepsilon \downarrow 0} \frac{u_\Omega(-\varepsilon\nu)}{\varepsilon} < -\frac{\partial u_B}{\partial \nu}(0). \tag{3.9}$$

Proof. Let u_A, u_Ω and u_B be as before. The function $u_\Omega - u_A$ satisfies the equation

$$\begin{cases} -\Delta(u_\Omega - u_A) = 0 & \text{on } A, \\ u_\Omega - u_A \geq 0 & \text{on } \partial A, \end{cases}$$

and by the Maximum Principle we get $u_\Omega - u_A > 0$. The same argument applies to $u_\Omega - u_B$ on Ω. For the second part of Lemma, since A is a smooth domain, by Hopf's boundary point Lemma at $0 \in \partial A$ we have $-\frac{\partial}{\partial \nu} u_A(0) > 0$. Using this and (3.8), the inequalities in (3.9) follow. □

Lemma 3.2.4 *Let u_A, u_B and u_Ω be as in Lemma 3.2.3. Then there exists $C^* \in \mathbb{R}$ such that for $x = (x_1, x_2) \in \Omega \cap B_{1/2}(0)$ the following holds true:*

$$0 < u_\Omega(x) < u_B(x) < C^* d(x, B). \tag{3.10}$$

Here $d(x, B) = dist(x, \partial B)$.

Proof. The second inequality in (3.10) follows from $\Omega \subset B$ and the Maximum Principle. The third one from standard regularity (see Theorem (2.3.6)) near a smooth boundary ∂B near 0. □

Now we consider the boundary $\partial\Omega$ in a neighborhood of 0. This boundary oscillates by the definition of ϕ_q. See Figure 3.2. The positive local minima of ϕ_q are denoted by the decreasing sequence $\{b_k\}_{k=1}^{\infty}$ and $\{a_k, c_k\}_{k=1}^{\infty}$ will denote the zeroes of ϕ_q, that is:

$$0 < \cdots < c_3 < b_3 < a_3 < c_2 < b_2 < a_2 < c_1 < b_1 < a_1.$$

For each fixed k, we define the ball $B_k := B_{r_k}(b_k, -r_k)$ with $r_k = -\frac{1}{2}\phi_q(b_k)$. See Figure 3.3. We write for the minimum points, the centers of these balls,

$$x_k = (b_k, \phi_q(b_k)) \text{ and } m_k = \left(b_k, \tfrac{1}{2}\phi_q(b_k)\right). \tag{3.11}$$

Next we define the function φ_k on ∂B_k as follows:

$$\varphi_k(x) = \begin{cases} 2r_k & \partial B_k \cap \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Let u_k be the solution of the following boundary value problem:

$$\begin{cases} -\Delta u = 1 & \text{in } B_k, \\ u = \varphi_k & \text{on } \partial B_k. \end{cases} \tag{3.13}$$

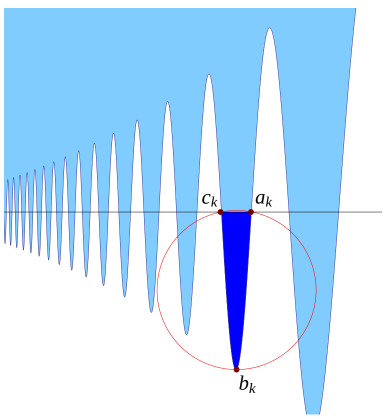


Figure 3.3: The disk B_k ©Springer Basel

Lemma 3.2.5 *Let C^* be as in (3.10). On $B_k \cap \Omega$ we have*

$$u_\Omega(x) < C^* u_k(x) \tag{3.14}$$

and

$$-\frac{\partial u_\Omega}{\partial \nu}(x_k) < -C^* \frac{\partial u_k}{\partial \nu}(x_k). \tag{3.15}$$

Proof. Both inequalities follow from the Maximum Principle and Hopf's Boundary Point Lemma on $B_k \cap \Omega$. □

Lemma 3.2.6 *Let u_k be as in (3.13). Then for $q > 1$ we get*

$$-\frac{\partial u_k}{\partial \nu}(x_k) \to 0. \tag{3.16}$$

Proof. The lowest points of each bump, $x_k = (b_k, \phi_q(b_k))$, $k = 1, 2, \ldots$ is computed to be

$$b_k^q = \frac{1}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

We may approximate the width of $\partial B_k \cap \Omega$ as follows:

$$a_k - c_k = \sqrt[q]{\frac{1}{2k - 1/2}} - \sqrt[q]{\frac{1}{2k + 1/2}} = \frac{1}{q}\left(\frac{1}{2k}\right)^{\frac{1}{q}+1}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right) \tag{3.17}$$

The diameter of the ball B_k is

$$\phi_q(b_k) = 1 - \sqrt{1 - b_k^2} = \tfrac{1}{2}b_k^2 + \mathcal{O}\left(b_k^4\right) = \tfrac{1}{2}\left(\frac{1}{2k}\right)^{2/q}\left(1 + \mathcal{O}\left(\frac{1}{k}\right) + \mathcal{O}\left(\frac{1}{k}\right)^{2/q}\right).$$

So, the ratio of the width of $\partial B_k \cap \Omega$ to the radius of B_k is approximately

$$\begin{aligned}\frac{a_k - c_k}{\frac{1}{2}\phi_q(b_k)} &= \frac{\frac{1}{q}\left(\frac{1}{2k}\right)^{1/q+1}}{\frac{1}{4}\left(\frac{1}{2k}\right)^{2/q}}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)^{\min(1,2/q)}\right)\\ &= \frac{4}{q}\left(\frac{1}{2k}\right)^{1-\frac{1}{q}}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)^{\min(1,2/q)}\right).\end{aligned}$$

When $k \to \infty$, the ratio goes to zero for $q > 1$.

When the ratio of the width of $\partial B_k \cap \Omega$ to the radius of B_k vanishes for $k \to \infty$, it follows from the Poisson formula

$$u_k(x) = \frac{r_k^2 - |x - m_k|^2}{2\pi r_k}\int_{|y|=r_k}\frac{\varphi_k(y)}{|x-y|^2}d\sigma_y + \tfrac{1}{4}\left(r_k^2 - |x - m_k|^2\right)$$

that $\frac{\partial u_k}{\partial x_2}(x_k) \to 0$ for $k \to \infty$. Indeed, the derivative of the second term goes to 0 since

$$\frac{\partial}{\partial x_2}\left(\tfrac{1}{4}\left(r_k^2 - |x-m_k|^2\right)\right)_{|x=x_k} = -\tfrac{1}{2}\phi_q(b_k) \to 0.$$

The derivative of the first term goes to zero, since $|\partial B_k \cap \Omega|/r_k \to 0$:

$$\frac{\partial}{\partial x_2}\left(\frac{r_k^2 - |x-m_k|^2}{2\pi r_k}\int_{|y|=r_k}\frac{\varphi_k(y)}{|x-y|^2}d\sigma_y\right)|_{x=x_k}$$
$$= \frac{1}{\pi}\int_{|y|=r_k}\frac{\varphi_k(y)}{|x_k-y|^2}d\sigma_y \le \frac{2r_k\,|\partial B_k \cap \Omega|}{\pi r_k^2} \to 0.$$

With (3.15) the claim in (3.16) follows. □

Assume that G_Ω is the solution operator for (3.7) for $D = \Omega$. The following proposition shows that the standard argument for a Krein-Rutman result cannot be used.

Proposition 3.2.7 *G_Ω is not strongly positive in the sense of (3.4) for $e = d_\Omega$.*

Proof. Suppose that G_Ω is strongly positive. Then by taking $f = 1$ there exists $c > 0$ such that

$$u(x) = (G_\Omega f)(x) > c d_\Omega(x).$$

However, since we have a sequence of boundary points $\{x_k\}_{k\in\mathbb{N}}$ such that $\frac{\partial u}{\partial \nu}(x_k)$ exists and $\frac{\partial u}{\partial \nu}(x_k) \to 0$ for $k \to \infty$, there is an k_0 such that $-\frac{\partial u}{\partial \nu}(x_k) < \frac{1}{2}c$ for all $k > k_0$, and by the mean value theorem the contradiction follows for $x = x_k - \varepsilon\nu$ and ε sufficiently small. □

3.2.2 Proof of the Main Theorem

Now we are able to prove the main result of this chapter, namely, Theorem (3.2).

Proof of Theorem 3.2. Except in 0 the boundary is $C^{1,1}$ and the positive outside derivative follows from the classical version of Hopf's Lemma. Lemma 3.2.3 covers 0. Proposition 3.2.7 shows that (3.4) does not hold. □

3.3 Eigenfunction and Eigenvalue

Although the Krein-Rutman Theorem can not be used for the solution G_Ω described in Theorem (3.2), nevertheless, the first positive eigenvalue and the corresponding positive first eigenfunction of G_Ω exist. Indeed, the Krein-Rutman-De Pagter Theorem is helpful. However, although the existence and the positivity of the first eigenvalue of Laplace operator is a direct consequence of the Rayleigh quotient, we show how the Krein-Rutman-De Pagter Theorem is used.

Assume that Ω is as defined by (3.5), consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

For any bounded domain the Riesz Representation Theorem supplies us with a solution operator $G_\Omega : L^2(\Omega) \to \mathring{W}^{1,2}(\Omega)$ that gives us a weak solution of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

For Ω defined in (3.5) the boundary is not smooth enough and so we do not get strong positivity as in Definition 3.1.5. Hence the condition that the spectral radius is strictly positive, which is needed in a Krein-Rutman Theorem as in [2], is not clear. The version above however will save our day. First we will need the following result.

Lemma 3.3.1 *For Ω as above, there is a solution operator $G_\Omega \in L\left(C^0(\overline{\Omega})\right)$.*

Proof. The boundary of Ω is not smooth as required for a standard procedure as in the introduction. To bypass this difficulty, let $\Omega' \subset\subset \overline{\Omega} \setminus \{0\}$. By [19] Theorem 9.13, we get $G_\Omega f \in W^{2,p}(\overline{\Omega'})$, if $f \in L^2(\Omega)$. A Sobolev imbedding gives $W^{2,p}(\Omega') \hookrightarrow C\left(\overline{\Omega'}\right)$. So we find $u = G_\Omega f \in C(\overline{\Omega} \setminus \{0\})$. On the other hand, u is continuous at 0. Indeed, since $\Omega \subset B$, we find by the Maximum Principle that

$$|u(x)| = |(G_\Omega f)(x)| \le |(G_\Omega \|f\|_\infty)(x)| \le |(G_B \|f\|_\infty)(x)|, \tag{3.20}$$

and hence, if x approaches 0 we find

$$\lim_{x\to 0} |u(x)| = \lim_{x \to 0} (G_B \|f\|_\infty)(x) = 0.$$

So $u = G_\Omega f \in C^0(\overline{\Omega})$. Obviously G_Ω is linear. To show that the operator $G_\Omega : C(\overline{\Omega}) \to C(\overline{\Omega})$ is continuous, we use (3.20) to show that G_B is bounded:

$$\|G_\Omega f\|_\infty = \sup_{x\in\Omega} |(G_\Omega f)(x)| \le \sup_{x\in\Omega} |(G_B \|f\|_\infty)(x)| \le c_B \|f\|_\infty.$$

So $G_\Omega \in L\left(C^0(\overline{\Omega})\right)$. □

Theorem 3.3.2 *Let Ω and $G_\Omega \in L\left(C^0(\overline{\Omega})\right)$ be as above. Then there is a unique positive eigenfunction $\psi \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$ for (3.18) and the corresponding eigenvalue λ is strictly positive.*

Proof. We will show that the linear operator G_Ω satisfies the conditions of the Krein-Rutman-De Pagter Theorem. G_Ω is positive by the Maximum Principle. To show that G_Ω is compact, consider $\{f_n\} \subset C(\overline{\Omega})$. Then all the restricted functions $\left\{u_n|_{\overline{\Omega}\setminus B_\varepsilon(0)} = G_\Omega f_n|_{\overline{\Omega}\setminus B_\varepsilon(0)}\right\} \subset C(\overline{\Omega})$ are differentiable, hence $\left\{u_n|_{\overline{\Omega}\setminus B_\varepsilon(0)}\right\} \subset C^1\left(\overline{\Omega} \setminus B_\varepsilon(0)\right)$ for $\varepsilon > 0$. Thus by the Arzela-Ascoli Theorem [40], the restricted operator $f_n \mapsto (Gf_n)|_{\overline{\Omega}\setminus B_{\frac{1}{m}}(0)}$ is compact for each $m \in \mathbb{N}$. So for $m = 1$ the sequence $\left\{u_n = Gf_n|_{\overline{\Omega}\setminus B_1(0)}\right\}$ has a convergent subsequence which we denote it by $\mathcal{U}_1 = \{u_{k_i}\}$. For $m = 2$ there is a subsubsequence $\mathcal{U}_2 = \{u_{k_{ii}}\}$ of $\mathcal{U}_1$. By iterating this we find subsequences $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \mathcal{U}_3 \supset \ldots$ and by using a diagonal argument we get a subsequence $\{u_{k_1}, u_{k_{22}}, u_{k_{333}}, \ldots\}$ which is convergent in $C\left(\overline{\Omega} \setminus \{0\}\right)$. Since all the functions $u_{k_{i\times i}}$ are zero at $x = 0$, also $\left\{u_{k_{i\times i}}\right\}$ converges in $C(\overline{\Omega})$. Consequently, $G_\Omega \in L\left(C(\overline{\Omega})\right)$ is compact.

The Strong Maximum Principle shows that $G_\Omega : C(\overline{\Omega}) \to C(\overline{\Omega})$ is irreducible. Indeed, $\left(C(\overline{\Omega}), \leq, \|.\|_\infty\right)$ is a Banach lattice under the canonical ordering defined by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \overline{\Omega}$. The Strong Maximum Principle implies for $f \in C(\overline{\Omega})$ with $f \geq 0$ that either $G_\Omega f(x) > 0$ in Ω when $f \not\equiv 0$ or $G_\Omega f(x) = 0$ in Ω when $f \equiv 0$. So $C(\overline{\Omega})$ and $\{0\}$ are the only closed lattice ideals in $C(\overline{\Omega})$ which are invariant under G_Ω. The closed lattice ideals in $\overline{\Omega}$ are sets $\{g \in C(\overline{\Omega}) \mid g = 0 \text{ for } x \in K\}$ with K a closed set in Ω, see [41]. Hence, by the Krein-Rutman-De Pagter, G_Ω has a unique positive eigenfunction ψ in $C(\overline{\Omega})$, that is, $G_\Omega \psi = \lambda\psi$ with $\psi \in C(\overline{\Omega})^+$ and $\lambda \in \mathbb{R}^+$. □

Chapter 4

Poisson's Problem on Cone Shaped Domains

It has been demonstrated in Theorems (2.3.6) and (2.5.8) in Chapter 2, that for a domain Ω with boundary of class C^2 and a positive $f \in C(\overline{\Omega})$, the solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

satisfies the estimate

$$cd(x) \leq u(x) \leq Cd(x) \text{ for } x \in \Omega, \tag{4.2}$$

where $d(x)$ is the distance function to the boundary and c and C are positive real numbers. The regularity results imply the estimate from above in (4.2) and the Hopf's boundary point Lemma is the main tool to get the estimate from below.

In this chapter, we consider bounded domains $\Omega \subset \mathbb{R}^n$ which are smooth with the exception of a vertex, we fix this vertex at zero, satisfying the following condition:

Condition 4.0.1 *$\Omega \subset \mathbb{R}^n$ is such that $\partial\Omega \setminus \{0\} \in C^\infty$ and there exists $\rho > 0$ such that*

$$\Omega \cap B_\rho(0) = \mathcal{C}_{\rho,S} := \{r\theta | \ \ 0 < r < \rho \text{ and } \theta \in S\}, \tag{4.3}$$

where S is a smooth proper subdomain of $\mathbb{S}^{n-1}$. See Figure 4.1.

The following examples show that for Ω satisfying Condition 4.0.1, there is no guarantee for regularity and for Hopf's Lemma to hold true.

Example 4.0.2 *Let Ω be the infinite planar sector*

$$\mathcal{C}_{\infty,S} = \left\{ (x_1, x_2) \in \mathbb{R}^2 | \quad x_1 > |x_2| \cot \frac{\omega}{2} \right\}$$

with fixed $0 < \omega < \frac{\pi}{2}$. Here, $S \in \mathbb{S}^1$ is the interval $S = \left(-\frac{\omega}{2}, \frac{\omega}{2}\right)$. The function $u(x_1, x_2) = x_1^2 \tan\left(\frac{\omega}{2}\right) - x_2^2$ satisfies the following boundary value problem:

$$\begin{cases} -\Delta u = 2\left(1 - \tan\left(\frac{\omega}{2}\right)\right) & \text{in } \mathcal{C}_{\infty,S}, \\ u = 0 & \text{on } \partial\mathcal{C}_{\infty,S}. \end{cases}$$

One observes that $-\Delta u$ is a positive constant but

$$\frac{\partial}{\partial\nu} u(0,0) = 0,$$

which implies that Hopf's boundary point Lemma fails at $(0,0)$.

Example 4.0.3 *Let $\Omega := \left\{ (r,\theta) | \ r > 0, 0 < \theta < \frac{\pi}{2} \right\}$ and set*

$$u(r,\theta) = r^2 \left(\pi^{-1} \left(\sin 2\theta \ln r + \theta \cos 2\theta \right) + \frac{(\sin\theta)^2}{2} \right).$$

Then we observe that $-\Delta u = 1 \in C^\infty\left(\overline{\Omega}\right)$ and $u = 0$ on $\partial\Omega$ but $u \notin W^{2,2}(\Omega)$. That means the derivatives of u may have singularities even with the right hand side function $f \in C^\infty\left(\overline{\Omega}\right)$.

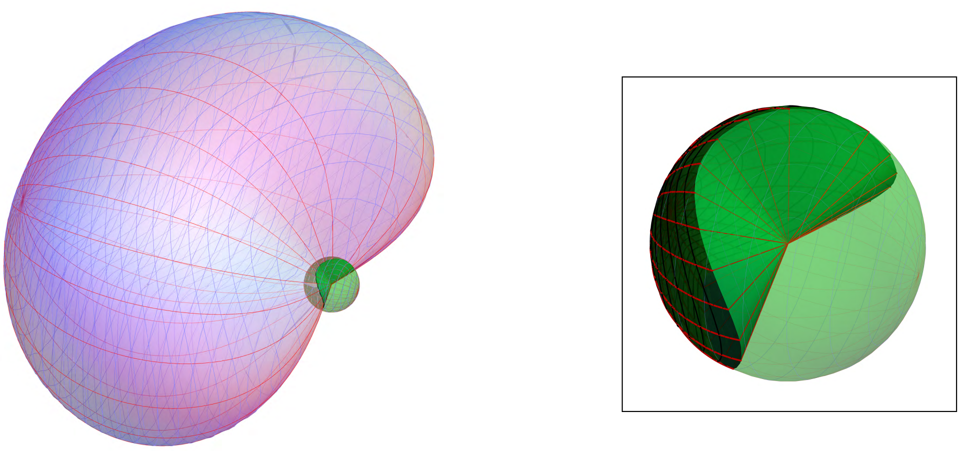


Figure 4.1: *On the left a domain $\Omega \subset \mathbb{R}^3$ locally resembling a cone as in Condition 4.0.1 and $B_\rho(0)$; on the right the enlarged cone inside $B_\rho(0)$.*

In this chapter we will derive an estimate for the solutions of (4.1) with Ω satisfying the Condition 4.0.1, in a neighborhood of the conical point. This estimate will be a replacement for (4.2). Indeed, under the appropriate hypotheses on the behaviour of f, we will find a sharp power-type estimate for u.

Kondratiev [26] and Grisvard [21] assessed the regularity near a conical point. They proved that the regularity is ruled by a power-type function. We are interested in whether and when such a power-type function also determines a Hopf's type result for (4.1), when f is nonnegative. In this work we are looking for an alternative function for $d(x)$ in (4.2) for the solution of (4.1) when Ω is a domain containing conical points. The results of this chapter have been published in [6].

4.1 The Eigenfunction

Assume $\Omega \subset \mathbb{R}^n$ satisfies Condition 4.0.1 with a smooth domain $S \subsetneq \mathbb{S}^{n-1}$. The eigenfunction of the Dirichlet Laplacian in a cone-shape domain has an important role to approximate the solution of (4.1), see [20, 21, 26, 32, 36]. So, first we consider the eigenvalue problem

$$\begin{cases} -\Delta\Psi = \mu\Psi & \text{in } \mathcal{C}_{1,S}, \\ \Psi = 0 & \text{on } \partial\mathcal{C}_{1,S}. \end{cases} \tag{4.4}$$

We are going to compare the first eigenfunction of (4.4) and the lowest order power type solution $w(r,\theta) = r^\alpha\psi(\theta)$ with $\alpha > 0$ of

$$\begin{cases} -\Delta w = 0 & \text{in } \mathcal{C}_{\infty,S}, \\ w = 0 & \text{on } \partial\mathcal{C}_{\infty,S}, \end{cases} \tag{4.5}$$

on a neighborhood of zero.

Let $(\psi_1, \lambda_{LB,1})$ be the first eigenfunction/eigenvalue of

$$\begin{cases} -\Delta_{LB}\psi = \lambda\,\psi & \text{in } S, \\ \psi = 0 & \text{on } \partial S, \end{cases} \tag{4.6}$$

where Δ_{LB} denotes the Laplace-Beltrami operator on the unit sphere.

We may write the fundamental solution of (4.5) by $w(r,\theta) = r^\alpha\psi_1(\theta)$, see [26]. Since the Laplace operator in the spherical coordinates takes the form

$$r^{1-n}\tfrac{\partial}{\partial r}r^{n-1}\tfrac{\partial}{\partial r} + r^{-2}\Delta_{LB}, \tag{4.7}$$

by substituting and calculating, w solves (4.5), provided that

$$\begin{aligned} -\Delta w = -r^{1-n}\tfrac{\partial}{\partial r}r^{n-1}\tfrac{\partial}{\partial r}\left(r^{\alpha}\psi_1\left(\theta\right)\right) + r^{-2}\Delta_{LB}\left(r^{\alpha}\psi_1\left(\theta\right)\right) &= \\ -\alpha\left(n+\alpha-2\right)r^{\alpha-2}\psi_1\left(\theta\right) + \lambda_{LB,1}r^{\alpha-2}\psi_1\left(\theta\right) &= 0. \end{aligned}$$

The quadratic equation

$$-\alpha\left(n+\alpha-2\right) + \lambda_{LB,1} = 0,$$

has the positive root

$$\alpha_1 = \sqrt{\lambda_{LB,1} + \left(\tfrac{n-2}{2}\right)^2} - \tfrac{n-2}{2}. \tag{4.8}$$

We get $w(r,\theta) = r^{\alpha_1}\psi_1(\theta)$.
We are looking for the first eigenfunction Ψ of (4.4) of the form $\Psi\left(r,\theta\right) = v\left(r\right)\psi_1\left(\theta\right)$ satisfying

$$-\Delta\Psi = -r^{1-n}\tfrac{\partial}{\partial r}r^{n-1}\tfrac{\partial}{\partial r}(v\left(r\right)\psi_1\left(\theta\right)) + r^{-2}\Delta_{LB}(v\left(r\right)\psi_1\left(\theta\right)) = \mu v\left(r\right)\psi_1\left(\theta\right).$$

Then one finds

$$-\left((n-1)r^{-1}v'(r) + v''(r)\right)\psi_1(\theta) + \lambda_{LB,1}r^{-2}v(r)\psi_1(\theta) = \mu v(r)\psi_1(\theta).$$

As $\psi_1(\theta)$ is a positive function and nonzero in S, we only need to solve the following ordinary differential equation

$$-v''\left(r\right) - \tfrac{n-1}{r}v'\left(r\right) + r^{-2}\lambda_{LB,1}v\left(r\right) = \mu v\left(r\right). \tag{4.9}$$

Consequently, the function $\Psi\left(r,\theta\right) = v\left(r\right)\psi_1\left(\theta\right)$ solves (4.4) if there exists a nontrivial solution of (4.9).
After the transformation $v\left(r\right) = \left(\sqrt{\mu}r\right)^{1-n/2} g\left(\sqrt{\mu}r\right)$ and $s = \sqrt{\mu}r$, one finds

$$\begin{gathered} -\left(\tfrac{2-n}{2}\right)\tfrac{-n}{2}g\left(s\right) - 2s\left(\tfrac{2-n}{2}\right)g'\left(s\right) - s^2g''\left(s\right) + \\ -\left(n-1\right)\left(\tfrac{2-n}{2}\right)g\left(s\right) - \left(n-1\right)sg'\left(s\right) + \lambda_{LB,1}g\left(s\right) = s^2g\left(s\right), \end{gathered}$$

or, in other words

$$s^2g''\left(s\right) + sg'\left(s\right) + \left(s^2 - \left(\lambda_{LB,1} + \left(\tfrac{n-2}{2}\right)^2\right)\right)g\left(s\right) = 0. \tag{4.10}$$

Equation (4.10) is the well-known Bessel equation of order $\lambda_{LB,1}+\left(\frac{n-2}{2}\right)^2$ and and the solutions, which are bounded in 0, are given by multiples of the Bessel function

$$g\left(s\right)=J_{\beta_1}\left(s\right)\ \text{ with } \beta_1:=\sqrt{\lambda_{LB,1}+\left(\tfrac{n-2}{2}\right)^2}$$

where

$$J_{\beta_1}\left(s\right)=\left(\tfrac{s}{2}\right)^{\beta_1}\sum_{m=0}^{\infty}\tfrac{(-1)^m}{m!\Gamma(m+\beta_1+1)}\left(\tfrac{s}{2}\right)^{2m}.$$

So one can take μ such that $\sqrt{\mu}=\rho_{\beta_1,1}$ is the first positive zero of $J_{\beta_1}\left(\cdot\right)$. We observe that the growth rate of w is as follows:

$$\alpha_1=\beta_1+1-\tfrac{n}{2}=\sqrt{\lambda_{LB,1}+\left(\tfrac{n-2}{2}\right)^2}-\tfrac{n-2}{2}.$$

In conclusion, we have proved:

Lemma 4.1.1 *Let α_1 and β_1 be as above. The solutions of (4.4) are as follows:*

$$\Psi\left(r,\theta\right)=r^{\alpha_1}\psi_1\left(\theta\right)\sum_{m=0}^{\infty}\tfrac{(-1)^m}{m!\Gamma(m+\beta_1+1)}\left(\tfrac{\rho_{\beta_1,1}r}{2}\right)^{2m}. \tag{4.11}$$

Note that Ψ and w have the same growth rate near 0.

4.2 Growth rate of the solution

For the sake of simple statements we will use the following notation.

Notation 4.2.1 *Let $u,v:A\mapsto\mathbb{R}^+$ be two positive functions. We write '$v\left(x\right)\preceq u\left(x\right)$ for $x\in A$', if there exists a constant $c>0$ such that $v\left(x\right)\leq cu\left(x\right)$ for all $x\in A$. If $v\left(x\right)\preceq u\left(x\right)$ and $u\left(x\right)\preceq v\left(x\right)$ for $x\in A$, we write '$v\left(x\right)\simeq u\left(x\right)$ for $x\in A$'.*

Let $\Omega\subset\mathbb{R}^n$ be as in Condition 4.0.1. By the standard Maximum Principle on smooth domains (or on general domains near smooth boundary parts) for the solution of (4.1) one finds

$$u(x)\geq cd(x,\partial\Omega)\text{ for }x\in\Omega\setminus B_\varepsilon(0)$$

and with regularity for the estimate from above we find hence that

$$u(x)\simeq\psi_1\left(\tfrac{x}{|x|}\right)\text{ for }x\in\Omega\cap\partial B_\varepsilon(0).$$

So we are left to find estimates for u on $\Omega \cap B_\varepsilon(0)$ starting from

$$\begin{cases} -\Delta u = f \gneqq 0 & \text{in } \Omega \cap B_\varepsilon(0), \\ u = 0 & \text{on } \partial\Omega \cap B_\varepsilon(0), \\ u \simeq \psi_1\left(\frac{\cdot}{\varepsilon}\right) & \text{on } \Omega \cap \partial B_\varepsilon(0). \end{cases} \tag{4.12}$$

Theorem 4.2.2 *Let $\mathcal{C}_{1,S}$, $\lambda_{LB,1}$, ψ_1 and α_1 and w be as defined in (4.3), (4.6), (4.8) and (4.5), then for each nontrivial solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ of (4.1) with $f \geq 0$ there exists $c_f > 0$ such that*

$$c_f w(x) \leq u(x) \text{ on } \mathcal{C}_{1,S}. \tag{4.13}$$

Theorem 4.2.2 provides an estimate from below for u on $\mathcal{C}_{1,S}$. One can find from (4.11) and (4.13) that $\Psi_1 \preceq u$ on $\mathcal{C}_{1,S}$ for all $f \geq 0$. In general an estimate from above by w for u does not hold. For example, the function $u(x_1, x_2) = x_1^2 \tan\left(\frac{\omega}{2}\right) - x_2^2$ in Example 4.0.2 satisfies

$$w = |x|^{\frac{\pi}{\omega}} \cos\left(\frac{\pi}{\omega}\frac{x}{|x|}\right) \preceq u(x),$$

but the estimate from above does not hold.

In view of Proposition 3.3 and Lemma 3.4 in [35], one may conclude the following proposition.

Proposition 4.2.3 *Let $\Omega \subset \mathbb{R}^n$ satisfy Condition 4.0.1 and α_1 correspond to S as in (4.8). Problem (4.1) with $f \in L^2(\Omega)$ has a unique solution $u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ if and only if*

$$1 - \alpha_1 < -1 + \tfrac{n}{2} < n - 1 + \alpha_1. \tag{4.14}$$

In [21, Theorem 4.6], Grisvard presented a formula for the solution of (4.12) as follows.

Theorem 4.2.4 *Let Ω satisfy Condition 4.0.1 and assume that $(\lambda_{LB,j}, \psi_j)$, for $j = 1, 2, \cdots$ are the eigenvalues/eigenfunctions of (4.6) for the corresponding $S \subsetneqq \mathbb{S}^{n-1}$, and furthermore, assume that $\lambda_{LB,j} \neq \left(2 - \frac{n}{p}\right)\left(n - \frac{n}{p}\right)$ for $j = 1, 2, \cdots$. Consider $u \in \mathring{W}^{1,2}(\Omega)$ to be the solution of (4.1), for $f \in L^p(\Omega)$, $p \geq 2$. Then there exist constants a_j such that*

$$u = u_f + \sum_{\lambda_{LB,j} < \left(2-\frac{n}{p}\right)\left(n-\frac{n}{p}\right)} a_j |x|^{1-\frac{n}{2}+\beta_j} \psi_j\left(\frac{x}{|x|}\right) \tag{4.15}$$

with $u_f \in W^{2,p}(\mathcal{C}_{1,S})$ and $\beta_j = \sqrt{\left(\frac{n}{2} - 1\right)^2 + \lambda_{LB,j}}$.

Remark 4.2.5 *Note that due to formula (4.15), when the first eigenvalue of Laplace Beltrami operator* $\lambda_{LB,1}$ *is larger than* $\left(2-\frac{n}{p}\right)\left(n-\frac{n}{p}\right)$*, then the power type part of (4.15) is cancelled and* $u = u_f \in W^{2,p}(\mathcal{C}_{1,S})$*. The estimates of the behaviour of* u *when* $\lambda_{LB,1} < \left(2-\frac{n}{p}\right)\left(n-\frac{n}{p}\right)$ *are the main subject of the rest of this chapter.*

Proposition 4.2.6 *Let* Ω *satisfy Condition 4.0.1 and* $\lambda_{LB,1}$*,* ψ_1 *and* α_1 *and* w *be as defined in (4.6), (4.8) and (4.5). Assume* $u \in \mathring{W}^{1,2}(\Omega)$ *is the solution of (4.1) for* $0 \lneqq f \in L^p(\Omega)$*,* $p \geq n$ *and suppose that*

$$\lambda_{LB,1} < \left(2-\tfrac{n}{p}\right)\left(n-\tfrac{n}{p}\right). \tag{4.16}$$

Then, if $\alpha_1 < 1$*, the solution* u *satisfies*

$$u \simeq |x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right), \text{ for } x \in \mathcal{C}_{1,S}. \tag{4.17}$$

Proof. Following formula (4.15), the solution u has the form

$$u(x) = u_f(x) + a_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right) + \cdots \tag{4.18}$$

and u_f lies in $W^{2,p}(\Omega)$ which imbeds in $C^{0,\alpha_1}(\overline{\Omega})$ by setting $p > \frac{n}{2-\alpha_1}$. Hence, one finds for some $c_1 > 0$;

$$|u_f(x)| \leq c_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right). \tag{4.19}$$

Note that ψ_1 appears in right hand side of (4.19) because $u_f(x) = 0$ for $x \in \partial\Omega$. By (4.19) and using the fact that α_1 is the smallest power in the power-type part of right hand side of (4.18), one can find that there is a positive constant a such that

$$u(x) = u_f(x) + a_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right) + \cdots \leq c_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right) + a|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right),$$

and consequently, we find

$$u \preceq |x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right).$$

The estimate from below is given by Theorem 4.2.2, so the result (4.21) is achieved. $\square$

Proposition 4.2.7 *Let Ω satisfy Condition 4.0.1, $\lambda_{LB,1}$, ψ_1, α_1 and w be as defined in (4.6), (4.8) and (4.5). Assume $u \in \mathring{W}^{1,2}(\mathcal{C}_{1,S})$ is the solution of (4.1) for $0 \lneqq f \in L^p(\mathcal{C}_{1,S})$, $p \geq n$ and suppose that*

$$\lambda_{LB,1} < \left(2 - \tfrac{n}{p}\right)\left(n - \tfrac{n}{p}\right). \tag{4.20}$$

Then, if $1 < \alpha_1 < 2$, for p large enough, the solution u satisfies

$$u \simeq |x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right), \quad \textit{for } x \in \mathcal{C}_{1,S}. \tag{4.21}$$

Proof. For $p > n$ the Sobolev space $W^{2,p}(\Omega)$ can be imbedded in $C^{1,\alpha_1-1}(\overline{\Omega})$ when $2 - \frac{n}{p} > \alpha_1 = 1 + (\alpha_1 - 1) > 0$. Thus the function u_f in the formula (4.15) lies in $C^{1,\alpha_1-1}(\overline{\Omega})$. This implies that ∇u_f is continuous on $\overline{\Omega}$. Since $0 \in \partial\Omega$ is the singular point and ∇u_f is continuous at 0, one finds $\nabla u_f(0) = 0$, which implies for some positive constant c_1,

$$|\nabla u_f(x)| \leq c|x|^{\alpha_1-1}, \tag{4.22}$$

for all $x \in \overline{\mathcal{C}_{1,S}}$. From the fact that $u_f(x) = 0$ for $x \in \partial\Omega$ and the estimate (4.22), it follows that

$$|u_f| \leq c_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right),$$

for some positive real c and $x \in \overline{\mathcal{C}_{1,S}}$. The estimate from above for u is now directly concluded as follows:

$$u = u_f + a_1|x|^{1-\frac{n}{2}+\beta_1}\psi_1\left(\frac{x}{|x|}\right) + \cdots \leq c_1|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right) + c_2|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right).$$

The estimate from below is established by Theorem 4.2.2.

□

Theorem 4.2.8 *Let $(\lambda_{LB,1}, \psi_1)$, α_1 and $\mathcal{C}_{1,S}$ be as defined above and let Ω satisfy Condition 4.0.1. Assume that $u \in \mathring{W}^{1,2}(\Omega)$ is the solution of (4.1) with $0 \lneqq f \in C^1(\overline{\Omega})$ and $f(0) > 0$. Furthermore, suppose that $\alpha_1 > 2$. Then the solution u can be estimated as follows:*

$$u \simeq |x|^2\psi_1\left(\frac{x}{|x|}\right), \tag{4.23}$$

for all $x \in \mathcal{C}_{1,S}$.

Proof. To achieve the estimate from above, we set $M := \max_{x \in \overline{\Omega}} f(x)$, which is strictly positive since $f(0) > 0$. Let v be the solution of

$$\begin{cases} -\Delta_{LB} v - 2nv = 1 & \text{in } S, \\ v = 0 & \text{on } \partial S. \end{cases} \tag{4.24}$$

Under the assumption $\alpha_1 > 2$ we have $\lambda_{LB,1} > 2n$ which implies that v, the solution of (4.24), exists and is positive on S and moreover, since S is smooth, $v \simeq \psi_1$ holds. Then the function

$$u_p := M|x|^2 v\left(\frac{x}{|x|}\right)$$

satisfies the following boundary value problem;

$$\begin{cases} -\Delta u_p = M & \text{in } \mathcal{C}_{1,S}, \\ u_p = 0 & \text{on } \partial\mathcal{C}_{1,S} \cap \partial\Omega, \\ u_p = Mv\left(\frac{x}{|x|}\right) & \text{on } S. \end{cases}$$

Since $f \preceq -\Delta u_p = M$ on $\mathcal{C}_{1,S}$ and $u \simeq u_p$ on $\partial\mathcal{C}_{1,S}$, by the maximum principle one finds $u \preceq u_p$ on $\mathcal{C}_{1,S}$ and consequently, the estimate $u(x) \preceq |x|^2\psi_1\left(\frac{x}{|x|}\right)$ on $\mathcal{C}_{1,S}$ is achieved.
To get the estimate from below, we set

$$\Omega^+ := \{x \in \Omega | f(x) > 0\} \subset \Omega$$

which is nonempty since $f(0) > 0$ and hence $0 \in \Omega^+$. Let $m := \min_{x \in \Omega^+} f(x) > 0$ and set $u_q := m|x|^2 v^+\left(\frac{x}{|x|}\right)$ where v^+ is the solution of

$$\begin{cases} -\Delta_{LB} v^+ - 2nv^+ = \chi_{\Omega^+ \cap S}\left(\frac{x}{|x|}\right) & \text{in } S, \\ v^+ = 0 & \text{on } \partial S. \end{cases} \tag{4.25}$$

Here, χ_A is the characteristic function of the set A. Similar to v, one finds that v^+ exists and is positive on S and $v^+ \simeq \psi_1$. Then one finds that u_q satisfies

$$\begin{cases} -\Delta u_q = m\chi_{\Omega^+} & \text{in } \mathcal{C}_{1,S}, \\ u_p = 0 & \text{on } \partial\mathcal{C}_{1,S} \cap \partial\Omega, \\ u_p \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{on } S. \end{cases}$$

Consequently, $-\Delta u_q \preceq f$ in $\mathcal{C}_{1,S}$ and $u \simeq u_q$ on $\partial\mathcal{C}_{1,S}$. Thus the estimate from below $|x|^2\psi_1\left(\frac{x}{|x|}\right) \simeq u_q \preceq u$ in $\mathcal{C}_{1,S}$ is proved.

□

Remark 4.2.9 *One can observe in the proofs above that the estimate from below and the estimate from above are independently achieved. They have been combined to get a short statement, which directly shows that the estimate is optimal. This is still true for the estimates in our general results presented in the next section.*

4.3 General Results

In [43] one finds that the solution of (4.1) on a concave corner domain $\Omega = \mathcal{C}_{1,S}$ in $\mathbb{R}^2$ has the same growth rate as the eigenfunction Ψ_1 when $0 \le f \in L^p(\Omega) \cap C(\Omega)$ with $p > 2$. For a convex sector one finds $u \simeq \Psi_1$ as long as $f \in C(\Omega)$ and $0 \le f(x) \preceq |x|^\vartheta$ near the corner point for appropriate $\vartheta \in \mathbb{R}$, see [43, Corollary 6].

For domains containing conical points in general dimension we present the following result.

Theorem 4.3.1 *Suppose that $\Omega \subset \mathbb{R}^n$ satisfies Condition 4.0.1 with S a smooth subdomain of $\mathbb{S}^{n-1}$. Let $\mathcal{C}_{1,S}$, $(\lambda_{LB,1}, \psi_1)$ and α_1 be as defined in (4.3), (4.6) and (4.8) . Suppose that $0 \lneqq f \in W^{-1,2}(\Omega) \cap C\left(\bar{\Omega} \setminus \{0\}\right)$.*

- *Then the weak solution $u \in \mathring{W}^{1,2}(\Omega)$ of (4.1) is positive and there exists $C = C(f, \Omega) > 0$ such that*

$$C \ \ |x|^{\alpha_1} \psi_1\left(\tfrac{x}{|x|}\right) \le u(x) \text{ for all } x \in \mathcal{C}_{\rho,S}. \tag{4.26}$$

- *Moreover, let $m > -2$, let $S_0 \subset \mathbb{S}^{n-1}$ be an open and nonempty subset of S and assume that $0 < \rho < 1$. Then there are $C_1 = C_1(\rho, S_0, \Omega, m)$ and $C_2 = C_2(\Omega, m) \in \mathbb{R}^+$ such that the following holds.*

If f satisfies for some $c_1, c_2 \ge 0$

$$f(x) \le c_2\, |x|^m \qquad \text{for all } x \in \Omega, \text{ and} \tag{4.27}$$

$$c_1\, |x|^m \le f(x) \qquad \text{for all } x \in \mathcal{C}_{\rho,S_0}, \tag{4.28}$$

then for $c_1' = C_1\, c_1$ and $c_2' = C_2\, c_2$ the solution u of (4.1) satisfies:

1. *If $m < \alpha_1 - 2$, then for all $x \in \mathcal{C}_{\rho,S}$*

$$c_1'\, |x|^{m+2}\, \psi_1\left(\tfrac{x}{|x|}\right) \le u(x) \le c_2'\, |x|^{m+2}\, \psi_1\left(\tfrac{x}{|x|}\right). \tag{4.29}$$

2. *If $m = \alpha_1 - 2$, then for all $x \in \mathcal{C}_{\rho,S}$*

$$c_1'\, |x|^{\alpha_1} \ln\left(\tfrac{1}{|x|}\right)\ \psi_1\left(\tfrac{x}{|x|}\right) \le u(x) \le c_2'\, |x|^{\alpha_1} \ln\left(\tfrac{1}{|x|}\right)\ \psi_1\left(\tfrac{x}{|x|}\right). \tag{4.30}$$

3. If $m > \alpha_1 - 2$, then for all $x \in \mathcal{C}_{\rho,S}$

$$c_1' \, |x|^{\alpha_1} \, \psi_1 \left(\tfrac{x}{|x|} \right) \leq u(x) \leq c_2' \, |x|^{\alpha_1} \, \psi_1 \left(\tfrac{x}{|x|} \right). \tag{4.31}$$

Remark 4.3.2 *In the above theorem, the bound from below for f is more general than the one stated in [43, Theorem 5 and Lemma 7].*

Proof. Note that it will be sufficient to prove the estimates from above for the solutions $\overline{u}_m$ of

$$-\Delta u = |x|^m \quad \text{in } \Omega \quad \text{with } u = 0 \text{ on } \partial\Omega, \tag{4.32}$$

and the estimate from below for the solutions $\underline{u}_m$ of

$$-\Delta u = \chi_{\mathcal{C}_{\rho,S_0}}(x) \; |x|^m \quad \text{in } \Omega \quad \text{with } u = 0 \text{ on } \partial\Omega. \tag{4.33}$$

The function $\chi_{\mathcal{C}_{\rho,S_0}}$ denotes the characteristic function for $\mathcal{C}_{\rho,S_0}$, that is

$$\chi_{\mathcal{C}_{\rho,S_0}}(x) = 1 \text{ for } x \in \mathcal{C}_{\rho,S_0} \text{ and } \chi_{\mathcal{C}_{\rho,S_0}}(x) = 0 \text{ elsewhere.}$$

Away from the cone the domain is smooth and the right hand side is bounded. Hence, we may use the maximum principle and Hopf's boundary point lemma to find that there is a $c_1 > 0$ such that

$$c_1 d(x, \partial\Omega) \leq \underline{u}_m(x) \quad \text{for } x \in \Omega \setminus \mathcal{C}_{\rho,S}. \tag{4.34}$$

We may use regularity theory to find that there exist $c_2 > 0$ such that

$$\overline{u}_m(x) \leq c_2 d(x, \partial\Omega) \quad \text{for } x \in \Omega \setminus \mathcal{C}_{\rho,S}. \tag{4.35}$$

This leaves us to find estimates in $\mathcal{C}_{\rho,S}$ for

$$\begin{cases} -\Delta u = \chi_{\mathcal{C}_{\rho,S_0}}(x) \; |x|^m & \text{in } \mathcal{C}_{\rho,S}, \\ \quad u = \underline{u}_m(x) & \text{on } \partial\mathcal{C}_{\rho,S}, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = |x|^m & \text{in } \mathcal{C}_{\rho,S}, \\ \quad u = \overline{u}_m(x) & \text{on } \partial\mathcal{C}_{\rho,S}, \end{cases}$$

knowing the estimates in (4.34) and (4.35) on $\partial\mathcal{C}_{\rho,S} \cap \Omega$. Note that $B_\rho(0) \cap \Omega \subset \rho\mathbb{S}^{n-1}$ has a smooth boundary $B_\rho(0) \cap \partial\Omega$ which implies that the first eigenfunction ψ_1 of the Laplace-Beltrami problem on S is such that

$$\psi_1\left(\frac{x}{|x|}\right) \simeq d(x, \partial\Omega) \quad \text{for all } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega.$$

Let α_1 be as in (4.8) and let $\lambda_{LB,1}$ be the first eigenvalue of (4.6).

• **The case $m < \alpha_1 - 2$ and the estimate from above.** We set $\mu := -(m+2)(m+n)$ and since $m+2 < \alpha_1$ is equivalent to

$$(m+2)(m+n) < \lambda_{LB,1},$$

we find that

$$\begin{cases} -\Delta_{LB} v_\mu + \mu v_\mu = 1 & \text{in } S, \\ v_\mu = 0 & \text{on } \partial S, \end{cases} \tag{4.36}$$

has a unique positive solution v_μ, which satisfies, using regularity for the estimate from above and Hopf's boundary point lemma for the estimate from below:

$$\psi_1(\theta) \simeq v_\mu(\theta) \ \text{ for all } \theta \in S.$$

Taking

$$u_A(x) := |x|^{m+2} v_\mu\left(\frac{x}{|x|}\right)$$

we find that u_A satisfies

$$\begin{cases} -\Delta u_A(x) = |x|^m & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_A = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_A(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega. \end{cases}$$

Hence one finds by the maximum principle that $\overline{u}_m(x) \preceq u_A(x)$ on $\mathcal{C}_{\rho,S}$.

• **The case $m < \alpha_1 - 2$ and the estimate from below.** We take μ as before but instead of (4.36) we consider the unique positive solution w_μ of

$$\begin{cases} -\Delta_{LB} w_\mu + \mu w_\mu = \chi_{S_0} & \text{in } S, \\ w_\mu = 0 & \text{on } \partial S, \end{cases} \tag{4.37}$$

and set

$$u_B(x) := |x|^{m+2} w_\mu\left(\frac{x}{|x|}\right)$$

to find that u_B satisfies

$$\begin{cases} -\Delta u_B(x) = |x|^m \chi_{S_0}\left(\frac{x}{|x|}\right) & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_B = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_B(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega. \end{cases}$$

It follows that $\underline{u}_m(x) \succeq u_B(x)$ on $\mathcal{C}_{\rho,S}$.

• **The case $m = \alpha_1 - 2$ and the estimate from above.** Let $v_0(\theta)$ be the solution of

$$\begin{cases} -\Delta_{LB} v_0 = 1 & \text{in } S, \\ v_0 = 0 & \text{on } \partial S, \end{cases} \tag{4.38}$$

and set

$$u_C(x) := |x|^{m+2} \ln\left(\frac{1}{|x|}\right)\psi_1\left(\frac{x}{|x|}\right) + \kappa |x|^{m+2} v_0\left(\frac{x}{|x|}\right),$$

which is positive on $\mathcal{C}_{1,S}$ for $\kappa \geq 0$. We have

$$\begin{aligned} -\Delta\left(|x|^{m+2}\ln\left(\frac{1}{|x|}\right)\psi_1\left(\frac{x}{|x|}\right)\right) &= (2m+n+2)\,|x|^m\,\psi_1\left(\frac{x}{|x|}\right), \\ -\Delta\left(|x|^{m+2} v_0\left(\frac{x}{|x|}\right)\right) &= |x|^m\left(-(m+2)(m+n)\,v_0\left(\frac{x}{|x|}\right)+1\right). \end{aligned} \tag{4.39}$$

Since $\psi(\theta) \simeq v_0(\theta)$ for $\theta \in S$ holds true, we may take $\kappa > 0$ small enough to find

$$(2m+n+2)\,\psi_1(\theta) \geq \kappa\,(m+2)(m+n)\,v_0(\theta) \text{ for all } \theta \in S.$$

For such $\kappa > 0$ one finds that u_C satisfies

$$\begin{cases} -\Delta u_C(x) = \kappa\,|x|^m & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_C = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_C(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega. \end{cases}$$

and hence $\overline{u}_m(x) \preceq u_C(x)$ on $\mathcal{C}_{\rho,S}$.

• **The case $m = \alpha_1 - 2$ and the estimate from below.** Let $w_0(\theta)$ be the solution of

$$\begin{cases} -\Delta_{LB} w_0 = \chi_{S_0} & \text{in } S, \\ w_0 = 0 & \text{on } \partial S, \end{cases}$$

and consider

$$u_D(x) = |x|^{m+2} w_0\left(\frac{x}{|x|}\right) + \kappa\,|x|^{m+2}\ln\left(\frac{1}{|x|}\right)\psi_1\left(\frac{x}{|x|}\right).$$

We find

$$\begin{aligned} -\Delta\left(|x|^{m+2} w_0\left(\frac{x}{|x|}\right)\right) &= -(m+2)(m+n)\,|x|^m\,w_0\left(\frac{x}{|x|}\right) + |x|^m\,\chi_{S_0}\left(\frac{x}{|x|}\right) \\ -\Delta\left(\kappa\,|x|^{m+2}\ln\left(\frac{1}{|x|}\right)\psi_1\left(\frac{x}{|x|}\right)\right) &= \kappa\,(2m+n+2)\,|x|^m\,\psi_1\left(\frac{x}{|x|}\right). \end{aligned}$$

Since $w_0(\theta) \simeq \psi_1(\theta)$ for $\theta \in S$ holds true, we may take $\kappa > 0$ but small enough to find

$$\kappa\,(2m+n+2)\,\psi_1(\theta) \leq (m+2)(m+n)\,w_0(\theta) \text{ for } \theta \in S$$

and hence it follows that u_D satisfies

$$\begin{cases} -\Delta u_D(x) \le |x|^m \chi_{S_0}\left(\frac{x}{|x|}\right) & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_D = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_D(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega, \end{cases}$$

and hence $\underline{u}_m(x) \succeq u_D(x)$ on $\mathcal{C}_{\rho,S}$.

• **The case $m > \alpha_1 - 2$ and the estimate from above.** Let $v_0(\theta)$ be the solution of (4.38) and set

$$u_E(x) := \left(|x|^{\alpha_1} - |x|^{m+2}\right)\psi_1\left(\frac{x}{|x|}\right) + \kappa\,|x|^{m+2} v_0\left(\frac{x}{|x|}\right),$$

which is positive on $\mathcal{C}_{1,S}$ for $\kappa \ge 0$ since $\alpha_1 < m+2$. We have

$$-\Delta\left(|x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right)\right) = 0, \tag{4.40}$$

$$-\Delta\left(-|x|^{m+2}\psi_1\left(\frac{x}{|x|}\right)\right) = |x|^m\left((m+2)(m+n) - \lambda_{LB,1}\right)\psi_1\left(\frac{x}{|x|}\right), \tag{4.41}$$

$$-\Delta\left(|x|^{m+2} v_0\left(\frac{x}{|x|}\right)\right) = |x|^m\left(-(m+2)(m+n)\,v_0\left(\frac{x}{|x|}\right) + 1\right), \tag{4.42}$$

the last one as in (4.39). Moreover, $m+2 > \alpha_1$ implies that $(m+2)(m+n) > \lambda_{LB,1}$. Since $\psi(\theta) \simeq v_0(\theta)$ for $\theta \in S$ holds true, we may take $\kappa > 0$ small enough to find

$$\left((m+2)(m+n) - \lambda_{LB,1}\right)\psi_1(\theta) \ge \kappa\,(m+2)(m+n)\,v_0(\theta) \ \text{ for } \theta \in S.$$

For such κ we have

$$\begin{cases} -\Delta u_E(x) \ge \kappa\,|x|^m & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_E = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_E(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega, \end{cases}$$

and hence $\overline{u}_m(x) \preceq u_E(x)$ on $\mathcal{C}_{\rho,S}$.

• **The case $m > \alpha_1 - 2$ and the estimate from below.** One takes

$$u_F(x) := |x|^{\alpha_1}\psi_1\left(\frac{x}{|x|}\right),$$

to find that it satisfies

$$\begin{cases} -\Delta u_F(x) = 0 & \text{for } x \in \mathcal{C}_{\rho,S}, \\ u_F = 0 & \text{on } \partial\mathcal{C}_{\rho,S} \cap \partial\Omega, \\ u_F(x) \simeq \psi_1\left(\frac{x}{|x|}\right) & \text{for } x \in \partial\mathcal{C}_{\rho,S} \cap \Omega, \end{cases}$$

and hence $\underline{u}_m(x) \succeq u_F(x)$ on $\mathcal{C}_{\rho,S}$. $\square$

Next we consider a variant of (4.1), namely

$$\begin{cases} -\Delta u = \mu u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.43}$$

with $f \geq 0$. Let $(\Phi_{1,\Omega}, \mu_{1,\Omega})$ be the first eigenfunction, eigenvalue of the coresponding eigenvalue problem

$$\begin{cases} -\Delta \Phi = \mu \Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.44}$$

It is well known that the first eigenvalue $\mu_{1,\Omega}$ for (4.44) is positive and that the corresponding eigenfunction $\Phi_{1,\Omega}$ can be taken positive. Moreover, (4.43) is positivity preserving, i.e. $f \gneqq 0 \implies u > 0$, if and only if $\mu < \mu_{1,\Omega}$. Since $\Phi_{1,\Omega} \in \mathring{W}^{1,2}(\Omega)$ holds, we find by a Sobolev imbedding that $\Phi_{1,\Omega} \in L^\infty(\Omega)$ for $n \geq 3$. For $n = 2$ we find that $\Phi_{1,\Omega} \in L^p(\Omega)$ for all $p \in [1, \infty)$ and hence by [19, Theorem 8.30] we find $\Phi_{1,\Omega} \in L^\infty(\Omega)$. Hence, due to Theorem 4.3.1, $\Phi_{1,\Omega}$ necessarily satisfies (4.31) in a neighbourhood of 0.

Corollary 4.3.3 *Let Ω be as in Theorem 4.3.1. For $\mu < \mu_{1,\Omega}$, similar results for the solution of (4.43) as in Theorem 4.3.1 hold with C, C_1, C_2 depending additionally on μ.*

Proof. Let Ω be a bounded domain with a Lipschitz boundary. Then for all $\mu < \mu_{1,\Omega}$ the Green functions $G_\mu(\cdot,\cdot) : \bar{\Omega} \times \bar{\Omega} \to [0, \infty]$ of (4.43) have a similar behaviour, that is, there exists $c_{\mu,\Omega}, C_{\mu,\Omega} > 0$ such that

$$c_{\mu,\Omega} G_0(x,y) \leq G_\mu(x,y) \leq C_{\mu,\Omega} G_0(x,y) \text{ for all } x, y \in \Omega. \tag{4.45}$$

See [24, Theorem 7.22]. The assumption that $(D, q) := (\Omega, \mu)$ is gaugeable follows for example from [24, Theorem 4.19, iii], since for $\mu < \mu_{1,\Omega}$ the solution of (4.43) with $f \equiv 1$ is bounded. As a consequence of (4.45) it follows that the behaviour of the solution u_0 for (4.1) and u_μ for (4.43) near the conical point with the same $f \gneqq 0$, are similar, namely

$$c_{\mu,\Omega} u_0(x) \leq u_\mu(x) \leq C_{\mu,\Omega} u_0(x) \text{ for all } x \in \Omega.$$

So the estimates in Theorem 4.3.1 hold with the obvious modification due to the μ-dependance. $\square$

Chapter 5

Weighted Spaces

In [26] Kondratiev considers the solution of a boundary value problem on domains containing conical points in special spaces of functions. Indeed, he introduces the spaces containing functions whose derivatives are summable with respect to a weight. These spaces capture very well the main characteristic of the solutions of such problems. In fact, the solution is smooth everywhere, except at the conical points and on approach to the conical points the derivatives have pole singularities. Following Kondratiev, these spaces have been used in [20, 27, 32, 36].

5.1 Weighted Sobolev Spaces

Definition 5.1.1 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain that $\partial\Omega \setminus \{0\}$ is smooth and $\Omega \cap B_\epsilon(0) = \mathcal{C}_{\epsilon,S}$, see Condition 4.0.1 in Chapter 4. Let $\beta \in \mathbb{R}$ and $l = 0, 1, 2, \ldots$.*

*For $p \in (1, \infty)$ the **weighted Sobolev space** $V_\beta^{l,p}(\Omega)$ is defined as the completion of $C_c^\infty(\overline{\Omega} \setminus 0)$ with respect to the norm*

$$\|u\|_{V_\beta^{l,p}(\Omega)} = \left(\sum_{|j|=0}^{l} \int_\Omega \left| |x|^{\beta - l + |j|} D_x^j u(x) \right|^p dx \right)^{\frac{1}{p}}. \tag{5.1}$$

Remark 5.1.2 *One defines $\mathring{V}_\beta^{l,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm (5.1). They will supply the appropriate spaces for functions with zero Dirichlet boundary condition.*

To demonstrate the relation between the Sobolev spaces $W^{l,p}(\Omega)$ and the weighted spaces $V_\beta^{l,p}(\Omega)$, we recall the following lemma from [17].

Lemma 5.1.3 *Let $\beta \in \mathbb{R}$ and $l = 0, 1, 2, \ldots$. Then*

$$V_{\beta}^{l,p}(\Omega) \subset W^{l,p}(\Omega) \text{ if and only if } \beta \leq 0,$$

and

$$W^{l,p}(\Omega) \subset V_{\beta}^{l,p}(\Omega) \text{ if and only if } \beta \geq l.$$

The following theorem follows from the general result by Nazarov and Plamenevsky applied to the Laplace operator. See [36, Theorem 6.10, Chapter 3, page 82].

Let $\Upsilon \in C^{\infty}(\mathbb{R})$ be such that

$$\begin{cases} \Upsilon(t) = 1 & \text{for } t \leq \frac{1}{2}, \\ \Upsilon(t) \in [0,1] & \text{for } \frac{1}{2} < t < 1, \\ \Upsilon(t) = 0 & \text{for } t \geq 1. \end{cases} \tag{5.2}$$

Theorem 5.1.4 (Nazarov-Plamenevsky) *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain that $\partial\Omega \setminus \{0\}$ is smooth and $\Omega \cap B_{\epsilon}(0) = \mathcal{C}_{\epsilon,S}$ and let $\{\lambda_{LB,j}\}$ be the eigenvalues of Laplace-Beltrami operator on S. Let $V_{\beta}^{l,p}(\Omega)$ and $\mathring{V}_{\beta-l-1}^{1,p}(\Omega)$ be as in Definition 5.1.1. Then the operator $\mathcal{A} = -\Delta$ of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

considered as the mapping

$$\mathcal{A} : V_{\beta}^{l+2,p}(\Omega) \cap \mathring{V}_{\beta-l-1}^{1,p}(\Omega) \longrightarrow V_{\beta}^{l,p}(\Omega)$$

is an isomorphism if and only if

$$\left| l + \tfrac{n+2}{2} - \beta - \frac{n}{p} \right| \neq \sqrt{\lambda_{LB,j} + \left(\tfrac{n-2}{2}\right)^2}, \text{ for } j = 1, 2, \ldots \tag{5.4}$$

So for the parameters satisfying (5.4) there exists for each $f \in V_{\beta}^{l,p}(\Omega)$ a unique solution $u \in V_{\beta}^{l+2,p}(\Omega) \cap \mathring{V}_{\beta-l-1}^{1,p}(\Omega)$ of (5.3). Moreover there exists $C = C_{l,p,\beta,\Omega} > 0$ such that for all $u \in V_{\beta}^{l+2,p}(\Omega) \cap \mathring{V}_{\beta-l-1}^{1,p}(\Omega)$ with $f = \mathcal{A}u$ the following holds:

$$\|u\|_{V_{\beta}^{l+2,p}(\Omega)} \leq C \|f\|_{V_{\beta}^{l,p}(\Omega)}. \tag{5.5}$$

Since $\beta_1 < \beta_2$ implies that $V^{l,p}_{\beta_1}(\Omega) \subset V^{l,p}_{\beta_2}(\Omega)$, Theorem 5.1.4 might give for $f \in V^{l,p}_{\beta_1}(\Omega)$ a solution $u_1 \in V^{l+2,p}_{\beta_1}(\Omega) \cap \mathring{V}^{1,p}_{\beta_1-l-1}(\Omega)$ as well as a solution $u_2 \in V^{l+2,p}_{\beta_2}(\Omega) \cap \mathring{V}^{1,p}_{\beta_2-l-1}(\Omega)$. From Kondratiev [26] one knows for $p = 2$ that if both sets of parameters lie in the same interval defined by (5.4), then $u_1 = u_2$. The corresponding result was proven by Maz'ya and Plamenevski for $p \in (1, \infty)$. See [32, Theorem 3.3.2 page 107]. For the present case it leads to the following result, since the first eigenvalue $\lambda_{LB,1}$ is simple both algebraically and geometrically.

Corollary 5.1.5 *Suppose that $\beta_* < \beta$ are such that*

$$\left| l + \tfrac{n+2}{2} - \beta - \frac{n}{p} \right| < \sqrt{\lambda_{LB,1} + \left(\tfrac{n-2}{2}\right)^2}, \tag{5.6}$$

$$\sqrt{\lambda_{LB,1} + \left(\tfrac{n-2}{2}\right)^2} < l + \tfrac{n+2}{2} - \beta_* - \frac{n}{p} < \sqrt{\lambda_{LB,2} + \left(\tfrac{n-2}{2}\right)^2}. \tag{5.7}$$

Then there exists $C > 0$ such that for all $f \in V^{l,p}_{\beta_*}(\Omega)$ $(\subset V^{l,p}_{\beta}(\Omega))$ the solution $u \in V^{l+2,p}_{\beta}(\Omega) \cap \mathring{V}^{1,p}_{\beta-l-1}(\Omega)$ of (5.3) can be written as

$$u(x) = c_f \ \Upsilon\left(|x|/\rho\right) \ |x|^{\alpha_1} \psi_1\left(\frac{x}{|x|}\right) + w(x) \ \textit{ for all } x \in \Omega$$

for some $0 < \rho < 1$, $c_f \in \mathbb{R}$ and $w \in V^{l+2,p}_{\beta_*}(\Omega)$ with

$$|c_f| + \|w\|_{V^{l+2,p}_{\beta_*}(\Omega)} \leq C \|f\|_{V^{l,p}_{\beta_*}(\Omega)},$$

where α_1 is as defined in (4.8).

Remark 5.1.6 *The conditions in (5.6), (5.7), can be rewritten as*

$$-\alpha_1 - (n-2) < l + 2 - \beta - \frac{n}{p} < \alpha_1,$$

$$\alpha_1 < l + 2 - \beta_* - \frac{n}{p} < \alpha_2 = \sqrt{\lambda_{LB,2} + \left(\tfrac{n-2}{2}\right)^2} - \frac{n-2}{2}.$$

5.2 Weighted Hölder Spaces

Definition 5.2.1 *For $\sigma \in (0,1)$ the* ***weighted Hölder space*** *$\Lambda^{l,\sigma}_{\beta}(\Omega)$ is defined as the completion of $C^\infty_c(\overline{\Omega} \setminus 0)$ with respect to the norm*

$$\|u\|_{\Lambda^{l,\sigma}_{\beta}(\Omega)} = \sup_{x\in\Omega} \sum_{|j|=0}^{l} |x|^{\beta-l-\sigma+|j|} \left|D^j_x u(x)\right| + \sup_{x,y\in\Omega} \sum_{|j|=l} \frac{\left||x|^\beta D^j_x u(x) - |y|^\beta D^j_y u(y)\right|}{|x-y|^\sigma}. \tag{5.8}$$

The space $\Lambda^{l,\sigma}_{\beta}(\partial\Omega)$ is defined as consisting of the traces on $\partial\Omega$ of functions in $\Lambda^{l,\sigma}_{\beta}(\Omega)$ with the norm

$$\|u\|_{\Lambda^{l,\sigma}_{\beta}(\partial\Omega)} = \inf\left\{ \|v\|_{\Lambda^{l,\sigma}_{\beta}(\Omega)} \,|v = u \text{ on } \partial\Omega \right\}. \tag{5.9}$$

Remark 5.2.2 *One defines $\mathring{\Lambda}^{l,\sigma}_{\beta}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm (5.8). They will supply the appropriate spaces for functions with zero Dirichlet boundary condition.*

Remark 5.2.3 *Let $C^{l,\sigma}_{\beta}(\Omega)$ be defined as the completion of $C_c^\infty(\overline{\Omega} \setminus 0)$ with respect to the norm*

$$\|u\|_{C^{l,\sigma}_{\beta}(\Omega)} = \sup_{x\in\Omega} \sum_{|j|=0}^{l} |x|^\beta |D^j_x u(x)| + \sup_{x,y\in\Omega} \sum_{|j|=l} \frac{\left| |x|^\beta D^j_x u(x) - |y|^\beta D^j_y u(y) \right|}{|x-y|^\sigma}. \tag{5.10}$$

The spaces $C^{l,\sigma}_{\beta}(\Omega)$ and $\Lambda^{l,\sigma}_{\beta}(\Omega)$ coincide if $\beta \notin [0, l+\sigma]$.

Let us compare our estimate with the estimate following through Hölder-regularity in domains with cones as stated in [36, Theorem 6.11, page 82].

Theorem 5.2.4 *For $(f,g) \in \Lambda^{l,\sigma}_{\beta}(\mathcal{C}_{\infty,S}) \times \Lambda^{l+2,\sigma}_{\beta}(\partial\mathcal{C}_{\infty,S})$ there exists a unique solution $u \in \Lambda^{l+2,\sigma}_{\beta}(\mathcal{C}_{\infty,S})$ of*

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{C}_{\infty,S}, \\ u = g & \text{on } \partial\mathcal{C}_{\infty,S}, \end{cases} \tag{5.11}$$

provided that

$$\beta - l - 2 - \sigma \neq \frac{n}{2} - 1 \pm \sqrt{\left(\tfrac{n-2}{2}\right)^2 + \lambda_{LB,j}} \quad \text{for} \quad j = 1, 2, \dots \tag{5.12}$$

Assume that $g \equiv 0$, $0 \le f \in C(\overline{\Omega})$ and $f \simeq r^m$, $m > 0$ and $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary except at the conical point 0 where $\Omega \cap B_\epsilon(0) = \mathcal{C}_{\epsilon,S}$ for some smooth $S \subsetneqq \mathbb{S}^{n-1}$. Let us decompose m as $m = \eta + \sigma$ where $\eta := m - \sigma$ and $0 < \sigma < 1$. Note that if m is not an integer we may consider η as the integer part of m. By definition there are $c_1, c_2 > 0$ such that

$$c_1 r^\sigma < \frac{f(r,\theta)}{r^\eta} < c_2 r^\sigma,$$

which implies $f \in \Lambda^{0,\sigma}_{-\eta}(\Omega)$. So there is a solution $u \in \Lambda^{2,\sigma}_{-\eta}(\Omega) \cap \mathring{\Lambda}^{0,\sigma}_{-\eta-2}(\Omega)$ for the Poisson equation (5.11) on Ω if

$$\tfrac{n}{2} - 1 - \sqrt{\left(\tfrac{n-2}{2}\right)^2 + \lambda_{LB,1}} < -\eta - 2 - \sigma < \tfrac{n}{2} - 1 + \sqrt{\left(\tfrac{n-2}{2}\right)^2 + \lambda_{LB,1}}. \tag{5.13}$$

Note that (5.13) is equivalent to $2-n-\alpha_1<m+2<\alpha_1$. For $u\in\Lambda^{2,\sigma}_{-\eta}(\Omega)\cap\mathring{\Lambda}^{0,\sigma}_{-\eta-2}(\Omega)$ we can estimate $u\,(r\theta)$ by integrating along a curve $\tau\mapsto r\theta\,(\tau)$ with constant distance to 0 starting from a boundary point $r\theta_0$ to find for some $c,c'>0$ that

$$u\,(r\theta)=\int_{[\theta_0,\theta]}\left|\nabla u\,(r\theta\,(\tau))\right|d\tau\leq c\;r\left|\theta-\theta_0\right|r^{\eta+1+\sigma}\leq c'\;\psi_1\,(\theta)\,r^{m+2}.$$

By our present estimates, see Theorem 4.3.1, we get $u\simeq r^{m+2}\psi_1(\theta)$, since $m+2<\alpha_1$. So the estimates from above are identical in these two approaches.

Chapter 6

An Anti-Maximum Type Result

Let $\{(\mu_i, \Phi_i)\}_{i=1}^{\infty}$ denote the eigenvalues and corresponding eigenfunctions for

$$\begin{cases} -\Delta\Phi = \mu\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

For $\mu < \mu_1$, it is well known that the problem

$$\begin{cases} -\Delta u = \mu u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

is positivity preserving, meaning $f \geq 0$ implies $u \geq 0$. Clément and Peletier [10] were the first to notice that, if $\partial\Omega \in C^2$, for $\mu > \mu_1$ but near μ_1, the problem is sign-reversing for sufficiently regular $f \gneqq 0$. This seemingly surprising behaviour attracted a lot of attention and became known as the anti-maximum principle. For anti-maximum type results on non-smooth domains see [7] and [4]. In this chapter, we prove an anti-maximum principle for the solution of Poisson's problem on a domain containing conical points. This chapter closely follows [6].

6.1 Anti-maximum Principle on Cones

In [10], Clément and Peletier established the anti-maximum principle for general second order elliptic operators on domains with smooth boundary. Here, we present it for the Laplace operator as follow.

Theorem 6.1.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let μ_1 be the first eigenvalue of (6.1). Suppose that $f \in L^p(\Omega)$, $p > n$, such* *that* $f > 0$ *and suppose* u *satisfies*

$$\begin{cases} -\Delta u - \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a $\delta > 0$*, which depends on* f*, such that if* $\mu_1 < \mu < \mu_1 + \delta$ *the following hold,*

(i) $u(x) < 0$ *for all* $x \in \Omega$*,*

(ii) $\frac{\partial u}{\partial \nu}(x) > 0$ *at each point* $x \in \partial\Omega$*.*

The rough explanation for this result is as follows.
Normalizing by $\int_\Omega \Phi_1^2 dx = 1$ and setting

$$P_1 f = (f, \Phi_1)\, \Phi_1 = \left(\int_\Omega f \Phi_1 dx \right) \Phi_1, \tag{6.3}$$

one writes $f = P_1 f + (\mathcal{I} - P_1)\, f$. Defining the weak solution operator $\mathcal{G}_\mu$ for (6.2) that exists for $\mu \neq \mu_i$, i.e.

$$\mathcal{G}_\mu := (-\Delta - \mu)_0^{-1} : W^{-1,2}(\Omega) \to \mathring{W}^{1,2}(\Omega) \tag{6.4}$$

the anti-maximum result follows, whenever one shows

$$|\mathcal{G}_\mu (\mathcal{I} - P_1)\, f| \leq c_f \Phi_1 \text{ for all } \mu \in (\mu_1 - \varepsilon, \mu_1 + \varepsilon) \tag{6.5}$$

and compares with

$$\mathcal{G}_\mu P_1 f = \frac{1}{\mu_1 - \mu} P_1 f = \frac{(f, \Phi_1)}{\mu_1 - \mu} \Phi_1 \text{ for all } \mu \neq \mu_1.$$

Indeed, in such a setting the sign of the solution

$$u = \frac{(f, \Phi_1)}{\mu_1 - \mu} \Phi_1 + \mathcal{G}_\mu (\mathcal{I} - P_1)\, f$$

is determined by $\mathcal{G}_\mu P_1 f$ for $0 \neq |\mu - \mu_1|$ but small.

As Birindelli [7, Proposition 3.2] noticed, such a result does not hold in full generality for the square $\Omega = (0,1)^2$. The estimates in Theorem 4.3.1 and Corollary 5.1.5 allow us to formulate a corresponding result for domains with cones.

Theorem 6.1.2 (Anti-maximum principle on domains with cones) *Suppose that* $\Omega \subset \mathbb{R}^n$ *satisfies Condition 4.0.1 with* S *a smooth subdomain of* $\mathbb{S}^{n-1}$*. Let*

$\mathcal{C}_{1,S}$, $(\lambda_{LB,1},\psi_1)$ and α_1 be as defined in (4.3), (4.6) and (4.8) in Chapter 4. Let $0 \leq f \in L^p(\Omega)$ with $p > n$ and suppose that for some $m > \alpha_1 - 2$

$$f(x) \preceq |x|^m \text{ for } x \in \mathcal{C}_{\rho,S}. \tag{6.6}$$

Then there exists $\varepsilon_f > 0$ such that for all $\mu \in (\mu_1, \mu_1 + \varepsilon_f)$ the solution of (6.2) satisfies

$$u(x) \leq 0 \text{ for all } x \in \Omega.$$

Remark 6.1.3 *By Corollary 4.3.3 one notices that for $\mu < \mu_1$ the singular behaviour of the solution near 0 for a positive right hand side f does change in size with μ but does not change in type. This even holds true for $\mu \in (\mu_1, \mu_2)$. Since the type of behaviour of the first eigenfunction only depends on the cone, the assumption $m > \alpha_1 - 2$ is sharp.*

Proof. **Step 1.** First we derive some properties for f. With the assumptions on f in the theorem it follows that $f \in V^{0,p}_{1-\alpha_1}(\Omega)$ for some $p > n$. Indeed, whenever

$$p(1 - \alpha_1 + m) + n > 0 \tag{6.7}$$

holds, then

$$\begin{aligned}\int_\Omega \left| |x|^{1-\alpha_1} f(x)\right|^p dx &\leq \int_{\mathcal{C}_{\rho,S}} \left| |x|^{1-\alpha_1} f(x)\right|^p dx + c_\rho \int_{\Omega\setminus\mathcal{C}_{\rho,S}} |f(x)|^p dx \\ &\leq \tilde{c} \int_{r=0}^{\rho} r^{p(1-\alpha_1+m)} r^{n-1} dr + c_\rho |f|^p_{L^p(\Omega)} < \infty.\end{aligned}$$

By assumption we have $m > \alpha_1 - 2$ and hence $\alpha_1 - m - 1 < 1$. We define $n_* \in (n, \infty]$ by

$$n_* = \begin{cases} \infty & \text{if } \alpha_1 - m - 1 \leq 0, \\ \frac{n}{\alpha_1 - m - 1} & \text{if } \alpha_1 - m - 1 \in (0,1). \end{cases}$$

The estimate in (6.7) holds true for all $p \in (n, n_*)$.

Note that $V^{0,p}_{1-\alpha_1}(\Omega)$ with $p > n$ is imbedded in $W^{-1,2}(\Omega)$. Indeed, for $\alpha_1 \geq 1$ we find $V^{0,p}_{1-\alpha_1}(\Omega) \subset L^p(\Omega) \subset W^{-1,2}(\Omega)$. For $\alpha_1 \in (0,1)$ we proceed as follows. For $f \in V^{0,2}_1(\Omega)$ we find by Hardy's inequality, namely

$$\int_\Omega |\varphi(x)|^2 |x|^{-2} dx \leq C_H \int_\Omega |\nabla\varphi|^2 dx \text{ for all } \varphi \in \mathring{W}^{1,2}(\Omega),$$

and Cauchy-Schwarz, that

$$\begin{aligned}|f|_{W^{-1,2}(\Omega)} &= \sup\left\{\langle f,\varphi\rangle \mid |\varphi|_{\mathring{W}^{1,2}(\Omega)} \leq 1\right\} \\ &\leq \sup\left\{\langle f,\varphi\rangle \mid \int_\Omega |\varphi(x)|^2 |x|^{-2}\, dx \leq C_H\right\} \\ &\leq \left(C_H \int_\Omega |f(x)|^2 |x|^2\, dx\right)^{1/2} \\ &\leq C_H^{1/2}\left(\int_\Omega \left||x|\, f(x)\right|^2 dx\right)^{1/2} = C_H^{1/2}\, |f|_{V_1^{0,2}(\Omega)}\,.\end{aligned}$$

For $p > 2$ there are $c_\Omega, c'_\Omega \in \mathbb{R}^+$ such that

$$\int_\Omega \left||x|\, f(x)\right|^2 dx \leq c_\Omega \int_\Omega \left||x|^{1-\alpha_1} f(x)\right|^2 dx \leq c'_\Omega \left(\int_\Omega \left||x|^{1-\alpha_1} f(x)\right|^p dx\right)^{2/p},$$

and hence we find that $V^{0,p}_{1-\alpha_1}(\Omega) \subset W^{-1,2}(\Omega)$.

Step 2. Next we show that for $f \in L^p(\Omega)$ with $p > n$ satisfying (6.6), one finds

$$|\mathcal{G}_0 f(x)| \preceq \Phi_1(x) \ \text{ for } x \in \Omega,$$

with $\mathcal{G}_0$ defined in (6.4). Indeed, by the assumption on f and from the first step we find $0 \leq f \in W^{-1,2}(\Omega)$ and $f(x) \preceq |x|^m$ for $x \in \mathcal{C}_{\rho,S}$ with $m > \alpha_1 - 2$, which allows us to use Theorem 4.3.1 to get

$$|\mathcal{G}_0 f(x)| \preceq |x|^{\alpha_1}\, \psi_1\left(\frac{x}{|x|}\right) \ \text{ for } x \in \mathcal{C}_{\rho,S}.$$

Notice that $\Phi_1(x) \simeq |x|^{\alpha_1}\, \psi_1\left(\frac{x}{|x|}\right)$ for $x \in \mathcal{C}_{\rho,S}$ and hence, since $\partial\Omega \setminus B_\rho(0)$ is smooth,

$$|\mathcal{G}_0 f(x)| \preceq \Phi_1(x) \ \text{ for } x \in \Omega. \tag{6.8}$$

Step 3. The crucial step in an anti-maximum type result is to split the right hand f in a Φ_1 -component and a remainder. Since Φ_1 lies in $\mathring{W}^{1,2}(\Omega)$ the definition of the projection in (6.3) can be extended to $f \in W^{-1,2}(\Omega)$:

$$P_1 f = \langle \Phi_1, f\rangle\, \Phi_1,$$

where $\langle\cdot,\cdot\rangle : \mathring{W}^{1,2}(\Omega) \times W^{-1,2}(\Omega) \to \mathbb{R}$ is the duality relation. We consider separately the two components u_1 and u_2 of the solution u to (6.2) defined by

$$u_1 = \frac{1}{\mu_1 - \mu} P_1 f \ \text{ and } \ u_2 = \mathcal{G}_\mu (\mathcal{I} - P_1)\, f.$$

Since the first eigenvalue has (algebraic) multiplicity 1, the operator $\mathcal{G}_\mu(\mathcal{I} - P_1)$ is well-defined for all $\mu < \mu_2$. Moreover, for all $\mu < \mu_2$ one finds $C_\mu \in \mathbb{R}^+$ such that

$$|\mathcal{G}_\mu (\mathcal{I} - P_1)\, f|_{\mathring{W}^{1,2}(\Omega)} \leq C_\mu\, |(\mathcal{I} - P_1)\, f|_{W^{-1,2}(\Omega)}\,.$$

Since the constant C_μ can be taken continuously dependent on μ, for $\delta > 0$ there exist $C_\delta \in \mathbb{R}^+$ such that for all $\mu \in [0, \mu_2 - \delta]$

$$|\mathcal{G}_\mu (\mathcal{I} - P_1) f|_{\mathring{W}^{1,2}(\Omega)} \leq C_\delta |(\mathcal{I} - P_1) f|_{W^{-1,2}(\Omega)} \text{ for all } f \in W^{-1,2}(\Omega). \quad (6.9)$$

Since $\mathcal{G}_0 (\mathcal{I} - P_1) f = \mathcal{G}_0 f - \frac{1}{\mu_1} \langle \Phi_1, f \rangle \Phi_1$ one finds as in (6.8) that

$$|\mathcal{G}_0 (\mathcal{I} - P_1) f(x)| \preceq \Phi_1(x) \text{ for } x \in \Omega. \quad (6.10)$$

Step 4. We now show the estimate in (6.10) for $u_2 = \mathcal{G}_\mu (\mathcal{I} - P_1) f$. Notice that by iterating the process we get

$$u_2 = \mathcal{G}_0 ((\mathcal{I} - P_1) f + \mu u_2) = \mathcal{G}_0 \sum_{k=0}^{k_*} (\mu \mathcal{G}_0)^k (\mathcal{I} - P_1) f + \mu^{k_*+1} \mathcal{G}_0^{k_*+1} u_2 \quad (6.11)$$

for any $k^* \geq 0$. Since (6.10) holds, the results in Theorem 4.3.1 show that for any fixed $k^* \in \mathbb{N}$

$$\left| \mathcal{G}_0 \sum_{k=0}^{k_*} (\mu \mathcal{G}_0)^k (\mathcal{I} - P_1) f(x) \right| \preceq \Phi_1(x) \text{ for } x \in \Omega.$$

So we are left with showing $\left| \mathcal{G}_0^{k_*+1} u_2(x) \right| \preceq \Phi_1(x)$ for $x \in \Omega$ for some fixed $k_* \in \mathbb{N}$. We know that $\mathcal{G}_0 u_2 \in \mathring{W}^{1,2}(\Omega)$.

If $\alpha_1 \leq 1$, then $\mathring{W}^{1,2}(\Omega) \subset L^p(\Omega) \subset V^{0,p}_{1-\alpha_1}(\Omega)$ for $p \in (2, p^*)$ with

$$p^* = \frac{2n}{n-2} \text{ for } n > 2 \text{ and } p^* = \infty \text{ for } n = 2.$$

For such p one finds

$$-\alpha_1 - (n-2) < \alpha_1 + 1 - \frac{n}{p} < \alpha_1,$$

which means that the first condition of Theorem 5.1.4 is satisfied and we find $\mathcal{G}_0^2 u_2 \in V^{2,p}_{1-\alpha_1}(\Omega)$. For $n = 2, 3$ we are done since $p^* > n$. For $n \geq 4$ one uses that $V^{2,p}_{1-\alpha_1}(\Omega) \subset V^{0,\tilde{p}}_{1-\alpha_1}(\Omega)$ for $\tilde{p} \in (2, p^{**})$ with

$$p^{**} = \frac{2n}{n-6} \text{ for } n > 6 \text{ and } p^* = \infty \text{ for } n \leq 6.$$

Indeed, for $g \in V^{2,p}_{1-\alpha_1}(\Omega)$ one can find $|x|^{1-\alpha_1} g \in W^{2,p}(\Omega)$. By Sobolev imbedding, $W^{2,p}(\Omega) \hookrightarrow L^{\tilde{p}}(\Omega)$ provided that

$$2 - \tfrac{n}{p} > -\tfrac{n}{\tilde{p}}. \quad (6.12)$$

Since $p < \frac{2n}{n-2}$, we find $\tilde{p} < \frac{2n}{n-6}$ for $n > 6$. If $n \leq 6$ we observe that (6.12) satisfies for all $\tilde{p} > 2$. Now, $|x|^{1-\alpha_1} g \in L^{\tilde{p}}(\Omega)$ is equivalent to $g \in V^{0,\tilde{p}}_{1-\alpha_1}(\Omega)$ which implies $V^{2,p}_{1-\alpha_1}(\Omega) \subset V^{0,\tilde{p}}_{1-\alpha_1}(\Omega)$ for such $\tilde{p}$.
After at most $k^* = \left[\frac{n}{4}\right] + 1$ steps we find $\mathcal{G}_0^{k^*} u_2 \in V^{0,p}_{1-\alpha_1}(\Omega)$ with $p > n$ and

$$\left|\mathcal{G}_0^{k^*+1} u_2(x)\right| \preceq \Phi_1(x) \text{ for } x \in \Omega.$$

If $\alpha_1 > 1$ then we use $\mathring{W}^{1,2}(\Omega) \subset L^p(\Omega)$ for $p \in (2, p^*)$ with

$$p^* = \frac{2n}{n-2} \text{ for } n > 2 \text{ and } p^* = \infty \text{ for } n = 2.$$

For all $p > 2$ the condition

$$-\alpha_1 - (n-2) < 1 - \frac{n}{p} < \alpha_1$$

is satisfied, which allows us to use Theorem 5.1.4 that shows $\mathcal{G}_0 u_2 \in V^{2,p}_0(\Omega)$. We proceed as in the case $\alpha_1 \leq 1$ to find after at most $k^* = \left[\frac{n}{4}\right] + 1$ steps that $\mathcal{G}_0^{k^*} u_2 \in V^{0,p}_0(\Omega)$ with $p > n$ and hence that

$$\left|\mathcal{G}_0^{k^*+1} u_2(x)\right| \preceq 1 \text{ for } x \in \Omega.$$

By (6.11) and Theorem 4.3.1 one finds that,

1. if $\alpha_1 < 2$, then $|u_2(x)| \preceq \Phi_1(x)$ for $x \in \Omega$,
2. if $\alpha_1 = 2$, then $|u_2(x)| \preceq |x|^{-\varepsilon} \Phi_1(x)$ for $x \in \Omega$ for any $\varepsilon > 0$, which compensates the logarithmic term, and
3. if $\alpha_1 > 2$, then $|u_2(x)| \preceq |x|^{2-\alpha_1} \Phi_1(x)$ for $x \in \Omega$.

We are done when $\alpha_1 < 2$. If $\alpha_1 \geq 2$ we go back to the formula

$$u_2 = \mathcal{G}_0(\mathcal{I} - P_1) f + \mu \mathcal{G}_0 u_2.$$

Theorem 4.3.1 shows that

1. if $t < 2$, then $|u_2(x)| \preceq |x|^{-t} \Phi_1(x)$ for $x \in \Omega$ implies that $|u_2(x)| \preceq \Phi_1(x)$ for $x \in \Omega$.
2. if $t > 2$, then $|u_2(x)| \preceq |x|^{-t} \Phi_1(x)$ for $x \in \Omega$ implies that $|u_2(x)| \preceq |x|^{2-t} \Phi_1(x)$ for $x \in \Omega$.

We are done after at most finitely many steps. $\square$

Chapter 7

An Interface Problem

A boundary value problem may have singularities in two ways; first, it may have discontinuous coefficients, secondly, the domain contains (singular) conical points. In Chapters 4, 5 and 6 we investigated extensively the behaviour of the solution of the Poisson problem on the domains containing conical points and similar techniques can be used here.
Let

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} \tag{7.1}$$

be a second order elliptic operator, i.e. $\sum_{i,j} a_{ij}\xi_i\xi_j \geq c\,|\xi|^2$ for some $c > 0$ and all $\xi \in \mathbb{R}^n$. Suppose that u is a twice differentiable solution of

$$\begin{cases} Lu \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{7.2}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$, then the maximum principle holds while the coefficients a_{ij} and b_i are just bounded, see [19, Theorem 8.1]. But as we will see in this chapter, for Hopf's Lemma it is required that a_{ij} and b_i be continuous. A boundary value problem such as (7.2) with discontinuous coefficients appears when studying a so-called interface or transmission problem. Such a problem is modeled formally by

$$\begin{cases} -\nabla \cdot \sigma \nabla u = \sigma f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.3}$$

where σ is piecewise constant with jumps. Such problems arise in a number of applications, for example, at the interface between tow materials with different diffusion parameters in steady state heat diffusion or electrostatic problems.

See for example [30] for these applications. The aim of this chapter is to present a Hopf type estimate for the solution of (7.3) at the boundary points where σ is discontinuous. We will see that estimates similar to Theorem 4.3.1 for the solution of (7.3) holds. The results of this chapter have been collected in a separate manuscript, which has been submitted.

7.1 The Setting of the Problem

Consider $\Omega \subset \mathbb{R}^2$ to be bounded and smooth and suppose it consists of k subdomains, i.e. $\overline{\Omega} = \bigcup_{i=1}^{k} \overline{\Omega}_i$ such that $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$. See Figure 7.1. Assume that $\sigma : \overline{\Omega} \to \mathbb{R}^+$ is a piecewise constant positive function defined by

$$\sigma(x) = \sigma_i \text{ for } x \in \Omega_i \text{ with } \sigma_i \in \mathbb{R}^+. \tag{7.4}$$

and $\sigma_i \neq \sigma_{i+1}$ for $i = 1, \cdots, k$. By this setting, the solution of (7.3) can not be considered in the classical sense. Therefore, we consider weak solutions, that is, a function $u \in \mathring{W}^{1,2}(\Omega)$ satisfying

$$\int_{\Omega} \sigma \left(\nabla u \cdot \nabla \varphi - f\varphi\right) dx = 0 \text{ for all } \varphi \in \mathring{W}^{1,2}(\Omega). \tag{7.5}$$

Indeed, (7.5) is the Euler-Lagrange equation for

$$J(u) = \int_{\Omega} \sigma \left(\tfrac{1}{2} |\nabla u|^2 - fu\right) dx. \tag{7.6}$$

The reason, that we put σ not only just for the gradient term, but also for f, is that it simplifies some notations and does not alter the problem for $f \in L^2(\Omega)$. The existence and uniqueness of the weak solution $u \in \mathring{W}^{1,2}(\Omega)$ satisfying (7.5) is guaranteed by the Riesz representation theorem. Assuming that the subdomains meet at $\partial\Omega$ in cone-like way, Nicaise and Sändig [38] could show that $u_i := u|_{\Omega_i}$ can be written as $u_i = \tilde{u}_i + h_i$, where $\tilde{u}_i \in W^{2,2}(\Omega_i)$ and h_i is harmonic on Ω_i. Moreover, if one considers $p_0 \in \partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$ for some $i \neq j$, that is, a boundary point where at least two subdomains meet, then, although the solution u has a non-smooth behaviour in a neighborhood of p_0, this behaviour is similar to the one for corners studied by Kondratiev [26]. In [38] one finds that for $f \in L^2(\Omega)$ the solution u has the following decomposition near such $p_0 = 0$:

$$u(x) = \tilde{u}(x) + \eta(|x|) \sum_{0<\mu_k<1} c_j |x|^{\sqrt{\mu_k}} \phi_k\left(\tfrac{x}{|x|}\right). \tag{7.7}$$

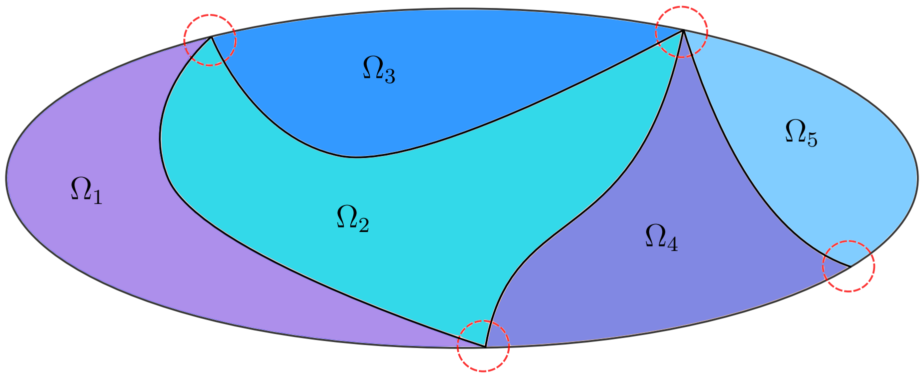


Figure 7.1: *A domain* Ω with *five subdomains and four singular points.*

Where $\tilde{u}|_{\Omega_i} \in W^{2,2}(\Omega_i)$, η is an appropriate radially symmetric smooth cut-off function equal to 1 in a neighborhood of $p_0 = 0$, the c_j are real constants and (μ_j, ϕ_j) are eigenvalues/eigenfunctions of the weighted Laplace Beltrami operator on $\frac{1}{\rho}\Omega \cap \partial B_1(0)$. Indeed, $x \mapsto |x|^{\sqrt{\mu_j}} \phi_j\left(\frac{x}{|x|}\right)$ are singular functions independent of f and are harmonic on $\frac{1}{\rho}\Omega \cap B_1(0)$. We define $\frac{1}{\rho}\Omega$ as follows;

$$\frac{1}{\rho}\Omega := \{(x_1, x_2) \in \mathbb{R}^n \mid (\rho x_1, \rho x_2) \in \Omega\}.$$

For polygonal interface problems see also [37].

Lemma 7.1.1 *Suppose that the domain* Ω *is the union of subdomains* Ω_i *with* $i = 1, \ldots, k$*, that is* $\overline{\Omega} = \bigcup_{i=1}^k \overline{\Omega}_i$ *and* $\Omega_i \cap \Omega_j = \emptyset$ *whenever* $i \neq j$*, and is such that* $\partial\Omega, \partial\Omega_i \cap \partial\Omega_j \in C^2$*. Suppose also that the weight function* $\sigma : \overline{\Omega} \to \mathbb{R}$ *is a piecewise constant positive function defined by*

$$\sigma(x) = \sigma_i \text{ for } x \in \Omega_i \text{ with } \sigma_i \in \mathbb{R}^+.$$

Suppose that $u \in W^{1,2}(\Omega)$ *and* $u_i := u|_{\Omega_i} \in W^{2,2}(\Omega_i)$*. Then the following is equivalent:*

1. *u is such that*

$$\begin{cases} -\Delta u_i = f & \text{in } \Omega_i, \\ \left.\begin{array}{c} u_i = u_j \\ \sigma_i \frac{\partial u_i}{\partial \nu_i} = -\sigma_j \frac{\partial u_j}{\partial \nu_j} \end{array}\right\} & \text{as traces on } \partial\Omega_i \cap \partial\Omega_j. \end{cases} \tag{7.8}$$

2. *u satisfies*

$$\int_\Omega \sigma \left(\nabla u \cdot \nabla \varphi - f\varphi\right) dx = 0 \quad \text{for all } \varphi \in \mathring{W}^{1,2}(\Omega). \tag{7.9}$$

Proof. One directly finds for $\varphi \in \mathring{W}^{1,2}(\Omega)$ that

$$\begin{aligned}
\int_\Omega \sigma\ (\nabla u \cdot \nabla\varphi - f\varphi)\,dx &= \sum_{i=1}^{k} \int_{\Omega_i} \sigma_i\ (\nabla u_i \cdot \nabla\varphi - f\varphi)\,dx \\
&= \sum_{i=1}^{k} \left(\int_{\partial\Omega_i} \sigma_i\ \frac{\partial u_i}{\partial \nu_i}\ \varphi\ dx + \int_{\Omega_i} \sigma_i\ (-\Delta u_i - f)\,\varphi dx \right) \\
&= \sum_{i,j=1}^{k} \left(\int_{\partial\Omega_i \cap \partial\Omega_j} \left(\sigma_i\ \frac{\partial u_i}{\partial \nu_i} + \sigma_j\ \frac{\partial u_j}{\partial \nu_j} \right)\ \varphi\ dx + \int_{\Omega_i} \sigma_i\ (-\Delta u_i - f)\,\varphi dx \right).
\end{aligned} \tag{7.10}$$

By the assumption that $u_i \in W^{2,2}(\Omega_i)$ these integrals are well-defined. Note that the boundary integral over $\partial\Omega$ drops out since $\varphi = 0$ as trace on $\partial\Omega$. So (7.8) implies (7.9).
Assuming (7.9) and testing with $\varphi \in C_0^\infty(\Omega_i)$ gives $-\Delta u_i - f = 0$. The condition $u_i = u_j$ on $\partial\Omega_i \cap \partial\Omega_j$ follows from $u \in W^{1,2}(\Omega)$. By taking testfunctions with support intersecting $\partial\Omega_i \cap \partial\Omega_j$ one establishes the jump condition in the normal derivatives. $\square$

By Lemma 7.1.1, the function $\tilde{u}$ in (7.7) satisfies the boundary value problem

$$\begin{cases} -\Delta u_i = f & \text{in } \Omega_i, \\ \left.\begin{array}{c} u_i = u_j \\ \sigma_i \frac{\partial u_i}{\partial \nu_i} = -\sigma_j \frac{\partial u_j}{\partial \nu_j} \end{array}\right\} & \text{on } \partial\Omega_i \cap \partial\Omega_j, \\ u_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega, \end{cases} \tag{7.11}$$

where $u_i = u|_{\Omega_i}$ and ν_i is the outward normal with respect to Ω_i. The power-type part of (7.7), which consists of harmonic functions on Ω_i, satisfies the boundary conditions on $\partial\Omega_i \cap \partial\Omega_j$ in (7.11) by construction. Indeed, it is a pointwise defined function and by the properties of ϕ_k (which is explained in Section 7.2), the jump conditions on $\partial\Omega_i \cap \partial\Omega_j$ are satisfied. Consequently, the problem of finding a minimizer $u \in \mathring{W}^{1,2}(\Omega)$ for the energy functional (7.6) with given $f \in L^p(\Omega)$ and σ as in (7.4) leads to the boundary value problem (7.11).

We will restrict ourselves mainly to the 2-dimensional case. Regularity for the 2-dimensional case was also focused upon by Mercier in [33]. The problem was also studied in [22], but it seems that this paper did not consider the appropriate power type functions in a decomposition as in (7.7).

In two dimensions multiple boundaries meet in a point and to assume that at such a point the domains look like a sector seems quite natural and simplifies

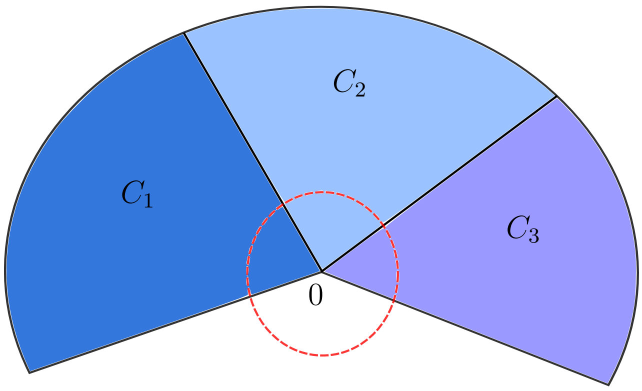


Figure 7.2: *The domain* Ω *and its subdomains are shaped like cones near the vertex* $p_0 = 0$

the arguments, that is, after translation and rotation we will assume, that near such a point the domain and the subdomains are as follows. See also Figure 7.2.

Condition 7.1.2 *Let* $0 = \theta_0 < \theta_1 < \cdots < \theta_k < 2\pi$. *The domain* $\Omega \subset \mathbb{R}^2$ *is such that for some* $\rho \in (0,1)$

$$\left(\tfrac{1}{2\rho}\Omega\right) \cap B_1(0) = C := \{(r,\theta)|0 < r < 1,\ 0 < \theta < \theta_k\}, \tag{7.12}$$

with the subdomains Ω_i, $i = 1, 2, ..., k$ *of* Ω *such that*

$$\left(\tfrac{1}{2\rho}\Omega_i\right) \cap B_1(0) = C_i := \{(r,\theta)|0 < r < 1,\ \theta_{i-1} < \theta < \theta_i\}. \tag{7.13}$$

We write

$$\Gamma_i = \{(r,\theta)|0 < r < 1,\ \theta = \theta_i\}. \tag{7.14}$$

A domain Ω will in general have several points where interfaces meet at the boundary and we will call these $\{p_0 = 0, p_1, \ldots, p_m\}$. Since our result is mainly based on a local analysis, it is sufficient to consider only the behaviour near $p_0 = 0$, namely, only on C as in (7.12). The remaining p_i with $i \in \{1, \ldots, m\}$ may even lie in the interior.

After a rescaling near a boundary point, where interfaces meet, the problem in (7.11) leads to the following boundary value on a sector C as in (7.12):

$$\begin{cases} -\Delta u_i = f_i := f|_{C_i} & \text{in } C_i, \quad i = 1, ..., k, \\ u_1 = 0 & \text{on } \Gamma_0, \\ \left.\begin{array}{r} u_i = u_{i+1} \\ \sigma_i \frac{\partial u_i}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{i+1}}{\partial \theta} \end{array}\right\} & \text{on } \Gamma_i, i = 1, ..., k-1, \\ u_k = 0 & \text{on } \Gamma_k, \\ u_i = w & \text{on } \partial C \cap \partial B_1(0), \end{cases} \tag{7.15}$$

where w is some given nonnegative function. The fourth line in (7.15) displays the jump conditions. The problem in (7.15) is closely related to the study of elliptic equations near corners as can be found in [26], [20], [21], [27], [32]. In the sequel we will show that similar to the results in Chapter 4, an estimate can be inferred for the solution of (7.15) by the use of the eigenvalue/eigenfunction (μ_1, ϕ_1).

7.2 Eigenvalue problem

Let $\Omega \subset \mathbb{R}^2$ satisfy Condition 7.1.2. The Laplace-Beltrami operator in $\mathbb{R}^2$ is the second derivative with respect to θ where $\theta \in \mathbb{S}^1$. Hence the eigenvalue problem for Laplace-Beltrami on the intersection of Ω and the unit sphere is as follows:

$$\begin{cases} -\phi''_{m,i} = \mu_m \phi_{m,i} \text{ on } (\theta_{i-1}, \theta_i) \text{ with } i \in \{1, \ldots, k\}, \\ \phi_{m,1}(0) = 0 \\ \left.\begin{array}{l} \phi_{m,i}(\theta_i) = \phi_{m,i+1}(\theta_i) \\ \sigma_i \phi'_{m,i}(\theta_i) = \sigma_{i+1} \phi'_{m,i+1}(\theta_i) \end{array}\right\} \text{ for } i \in \{1, \ldots, k-1\}, \\ \phi_{m,k}(\theta_k) = 0; \end{cases} \tag{7.16}$$

with $\phi_{m,i} = \phi_m|_{(\theta_{i-1},\theta_i)}$. Let (μ_m, ϕ_m), for $m = 1, 2, \ldots$ be the m-th eigenvalue/eigenfunction of (7.16) with $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$. These eigenfunctions form the singular power type part of (7.7). In Section 7.4 we will show that the first eigenvalue/eigenfunction (μ_1, ϕ_1) is used to describe the behaviour of the solution of (7.15). The Rayleigh quotient for which the first eigenfunction is a minimizer is as follows:

$$R_\sigma(\phi) = \frac{\int_0^{\theta_k} \tilde{\sigma}(\theta) \ \phi'(\theta)^2 \ d\theta}{\int_0^{\theta_k} \tilde{\sigma}(\theta) \ \phi(\theta)^2 \ d\theta}, \tag{7.17}$$

where $\phi \in \mathring{W}^{1,2}(0, \theta_k) \setminus \{0\}$ and $\tilde{\sigma}(\theta) = \sigma_i \in \mathbb{R}^+$ for $\theta \in (\theta_{i-1}, \theta_i)$.

Lemma 7.2.1 *Let R_σ be as defined in (7.17). Then the following holds.*

1. *R_σ attains its infimum μ_1 for some $\phi_1 \in \mathring{W}^{1,2}(0,\theta_k) \setminus \{0\}$ and $\mu_1 \geq \frac{1}{4}\frac{\min \sigma_i}{\max \sigma_i} > 0$.*

2. *The minimizing function ϕ_1 is unique up to multiplication, has a fixed sign and, after normalizing by*

$$\max \{\phi_1(\theta); 0 < \theta < \theta_k\} = 1, \tag{7.18}$$

satisfies for some $C_\sigma, c_\sigma > 0$:

$$c_\sigma \sin\left(\frac{\pi}{\theta_k}\theta\right) \leq \phi_1(\theta) \leq C_\sigma \sin\left(\frac{\pi}{\theta_k}\theta\right) \text{ for all } \theta \in [0,\theta_k]. \tag{7.19}$$

3. *ϕ_1 is the unique first eigenfunction, in the sense that $\phi_{1,i} := \phi_1|_{[\theta_{i-1},\theta_i]} \in C^2[\theta_{i-1},\theta_i]$ satisfies (7.16) and there is no other, independent, eigenfunction for $\mu \leq \mu_1$.*

Proof. By [19, Section 8.12] one finds that the minimizer $\phi_1 \in \mathring{W}^{1,2}(0,\theta_k)$ of (7.17), that we may normalize by (7.18), exists, is unique and is of fixed sign. Let μ_1 be the minimum value of (7.17). Since for $\phi \not\equiv 0$ one has

$$\frac{\int_0^{\theta_k} \tilde{\sigma}(\theta)\ \phi'(\theta)^2\, d\theta}{\int_0^{\theta_k} \tilde{\sigma}(\theta)\ \phi(\theta)^2\, d\theta} \geq \frac{\min \sigma_i}{\max \sigma_i}\frac{\int_0^{\theta_k} \phi'(\theta)^2\, d\theta}{\int_0^{\theta_k} \phi(\theta)^2\, d\theta} \geq \frac{1}{4}\frac{\min \sigma_i}{\max \sigma_i},$$

and one finds $\mu_1 \geq \frac{1}{4}\frac{\min \sigma_i}{\max \sigma_i}$. The function ϕ_1 satisfies the weak Euler-Lagrange equation

$$\int_0^{\theta_k} \sigma\left(\phi_1' w' - \mu_1 \phi_1 w\right) dx = 0 \text{ for all } w \in \mathring{W}^{1,2}(0,\theta_k).$$

Taking testfunctions with support in (θ_{i-1},θ_i) one finds that

$$\phi_{1,i} := \phi_1|_{[\theta_{i-1},\theta_i]} \in W^{2,2}(\theta_{i-1},\theta_i) \tag{7.20}$$

satisfies $-\phi_{1,i}'' = \mu_1 \phi_{1,i}$ on (θ_{i-1},θ_i) and even that $\phi_{1,i} \in C^\infty[\theta_{i-1},\theta_i]$. Since $\phi_1 \in \mathring{W}^{1,2}(0,\theta_k)$ holds, the functions $\phi_{1,i}$ satisfy the continuity equation $\phi_{1,i}(\theta_i) = \phi_{1,i+1}(\theta_i)$ and boundary conditions $\phi_{1,1}(0) = \phi_{1,k}(\theta_k) = 0$. The jump condition $\sigma_i \phi_{1,i}'(\theta_i) = \sigma_{i+1}\phi_{1,i+1}'(\theta_i)$ follows by taking testfunctions in the weak Euler-Lagrange equation with support near θ_i. Assuming $\phi_1 \geq 0$ holds, the strict positivity, with $\phi_1'(0) > 0$ and $\phi_1'(\theta_k) < 0$, follows from the unique continuation. Indeed, if $\phi_{1,i}'(\theta^*) = 0 = \phi_{1,i}(\theta^*)$ for some i and some $\theta^* \in [\theta_{i-1},\theta_i]$, then $\phi_1 \equiv 0$. The estimate in (7.19) is a direct consequence. □

The eigenfunction ϕ_m is as follows:

$$\phi_m(\theta) = \begin{cases} \phi_{m,1}(\theta) = \frac{\sin(\sqrt{\mu_m}\theta)}{\sin(\sqrt{\mu_m}\theta_1)}; & 0 < \theta \leq \theta_1 \\ \phi_{m,2}(\theta) = \frac{\sin\left(\sqrt{\mu_m}(\theta-\theta_1)+\alpha_2(\mu_m)\right)}{\sin(\alpha_2(\mu_m))}; & \theta_1 < \theta \leq \theta_2 \\ \vdots \\ \phi_{m,k}(\theta) = \phi_{m,k-1}(\theta_{k-1})\frac{\sin\left(\sqrt{\mu_m}(\theta-\theta_{k-1})+\alpha_k(\mu_m)\right)}{\sin(\alpha_k(\mu_m))}; & \theta_{k-1} < \theta \leq \theta_k; \end{cases} \tag{7.21}$$

in which for $j \leq k-1$,

$$\begin{aligned} \alpha_1(\mu) &= 0 \\ \alpha_2(\mu) &= \operatorname{arccot}\left(\frac{\sigma_1 \cot(\sqrt{\mu}\theta_1)}{\sigma_2}\right) \\ &\vdots \\ \alpha_{j+1}(\mu) &= \operatorname{arccot}\left(\frac{\sigma_j \cot(\sqrt{\mu}(\theta_j-\theta_{j-1}))}{\sigma_{j+1}} + \alpha_j(\mu)\right) \end{aligned}$$

And the eigenvalues μ_m, $m = 1, 2, \ldots$ are the solutions of

$$\sqrt{\mu_m}(\theta_k - \theta_{k-1}) + \alpha_k(\mu_m) = m\pi. \tag{7.22}$$

Note that μ_m is a function of all θ_is and σ_is for $i = 1, \ldots, k$. These formula can be extended for any positive integer k. By (7.22) we observe that in the case $k = 3$

$$\operatorname{arccot}\left(\frac{\sigma_2 \cot\left(\sqrt{\mu_m}(\theta_2-\theta_1)+\operatorname{arccot}\left(\frac{\sigma_1\cot(\sqrt{\mu_m}\theta_1)}{\sigma_2}\right)\right)}{\sigma_3}\right) = m\pi - \sqrt{\mu_m}(\theta_3 - \theta_2) \tag{7.23}$$

and by the fact that the codomain of $\operatorname{arccot}(\cdot)$ is the interval $(0, \pi)$, we have

$$\frac{(m-1)\pi}{\theta_3 - \theta_2} < \sqrt{\mu_m} < \frac{m\pi}{\theta_3 - \theta_2}. \tag{7.24}$$

For general k we need

$$\frac{(m-1)\pi}{\theta_k - \theta_{k-1}} < \sqrt{\mu_m} < \frac{m\pi}{\theta_k - \theta_{k-1}} \quad m = 1, 2, \cdots \tag{7.25}$$

Remark 7.2.2 *Note that in the case $k = 2$, when $\theta_2 = 2\theta_1$, the first eigenvalue μ_1 (or in general, any eigenvalue μ_m) is independent of σ_1 and σ_2. Indeed, the equation (7.22) has the following form*

$$\operatorname{arccot}\left(\frac{\sigma_1 \cot(\sqrt{\mu 1}\theta_1)}{\sigma_2}\right) = \pi - \sqrt{\mu_1}\theta_1,$$

for $0 < \sqrt{\mu_1} < \frac{\pi}{\theta_1}$ *we get*

$$\begin{aligned} \tfrac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1) &= \cot(\pi - \sqrt{\mu_1}\theta_1) \\ &= -\cot(\sqrt{\mu_1}\theta_1). \end{aligned}$$

The only solution of the above equation is

$$\sqrt{\mu_1} = \frac{\pi}{2\theta_1},$$

which is obviously independent of σ_1 *and* σ_2*.*

Remark 7.2.3 *By the above notations, the first eigenfunction of the problem*

$$\left\{ \begin{array}{ll} -\Delta\Phi_{1,i} = f_i := \lambda_1\Phi_{1,i} & in\ C_i, \quad i = 1, ..., k, \\ \Phi_{1,1} = 0 & on\ \Gamma_0, \\ \left. \begin{array}{c} \Phi_{1,i} = \Phi_{1,i+1} \\ \sigma_i\frac{\partial\Phi_{1,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial\Phi_{1,i+1}}{\partial\theta} \end{array} \right\} & on\ \Gamma_i, i = 1, ..., k-1, \\ \Phi_{1,k} = 0 & on\ \Gamma_k, \\ \Phi_{1,i} = w & on\ \partial C \cap \partial B_1(0), \end{array} \right. \tag{7.26}$$

has the form $\Phi_1(r, \theta) = \phi_1(\theta)J_{\sqrt{\mu_1}}(\rho_{\sqrt{\mu_1},1}r)$ *where* $J_\beta(\cdot)$ *is the Bessel function with the first positive zero* $\rho_{\beta,1}$*.*

7.3 Special Cases

As an special case for the interface problem we consider σ_i to be very large for some $i = 1, \cdots, k$ when $k = 2$ and $k = 3$. In the case $k = 2$, since the ordering of subdomains is compatible, it is enough to consider only one of the following states of $\sigma_1 \to \infty$ and $\sigma_2 \to \infty$. For the case $k = 3$, we consider first $\sigma_1 \to +\infty$, which is similar to $\sigma_3 \to +\infty$, and then the case $\sigma_2 \to +\infty$.

Lemma 7.3.1 *Let* $k = 2$*. The first eigenfunction* ϕ_1 *of Problem (7.16), given by Formula (7.21) for* $m = 1$*, is positive and has exactly one maximum point in* $(0, \theta_2)$*. Furthermore,*

1. *if* $\theta_1 > \theta_2 - \theta_1$ *then* $\phi_{1,1}$ *achieves the maximum point* θ_M*, i.e.* $\theta_M \in (0, \theta_1]$*;*

2. *if* $\theta_1 < \theta_2 - \theta_1$ *then* $\phi_{1,2}$ *reaches the maximum point* θ_M *and* $\theta_M \in [\theta_1, \theta_2]$*;*

3. *if* $\theta_1 = \theta_2 - \theta_1$*, due to the Remark (7.2.2),* $\theta_M = \theta_1$*.*

Proof. Due to the Lemma 7.2.1, the first eigenfunction is positive, by the maximum principle, ϕ_1 has minima at the boundary points 0 and θ_2 and has its maximum inside the interval $(0, \theta_2)$. We observe from the jump condition

$$\sigma_1 \frac{\partial \phi_{1,1}}{\partial \theta}(\theta_1) = \sigma_2 \frac{\partial \phi_{1,2}}{\partial \theta}(\theta_1),$$

that the derivatives $\frac{\partial \phi_{1,i}}{\partial \theta}$, $i = 1, 2$ have the same sign at the point θ_1. Now, if $\phi_{1,1}$ reaches the maximum we observe that

$$\frac{\partial \phi_{1,1}}{\partial \theta}(\theta_{M1}) = \sqrt{\mu_1} \frac{\cos(\sqrt{\mu_1}\theta_{M1})}{\sin(\sqrt{\mu_1}\theta_1)} = 0$$

has the solution at $\theta_{M1} = \frac{\pi/2}{\sqrt{\mu_1}}$, if and only if

$$\theta_{M1} < \theta_1. \tag{7.27}$$

And from $\frac{\partial \phi_{1,2}}{\partial \theta}(\theta_{M2}) = 0$ we observe that

$$\frac{\cos\left(\sqrt{\mu_1}(\theta_{M2} - \theta_1) + \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1)\right)\right)}{\sin\left(\operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1)\right)\right)} = 0$$

has the solution

$$\theta_{M2} = \frac{\pi/2 + \sqrt{\mu_1}\theta_1 - \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1)\right)}{\sqrt{\mu_1}}, \tag{7.28}$$

as long as

$$\theta_1 \leq \theta_{M2} \leq \theta_2. \tag{7.29}$$

By comparing (7.22) and (7.28) one finds

$$\theta_{M2} = \theta_2 - \tfrac{\pi/2}{\sqrt{\mu_1}}.$$

Note that (7.27) and (7.29) do not hold at the same time. Indeed, if we let (7.27) and (7.29) both hold simultaneously, then from (7.29) we get

$$\tfrac{\pi/2}{\sqrt{\mu_1}} < \theta_2 - \theta_1 \text{ or } \tfrac{\pi}{2} < \sqrt{\mu_1}(\theta_2 - \theta_1).$$

Applying this to formula (7.22) one finds

$$\operatorname{arccot}\left(\tfrac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1)\right) < \tfrac{\pi}{2},$$

then $\frac{\sigma_1}{\sigma_2}\cot(\sqrt{\mu_1}\theta_1) < 0$ and since $\sigma_1, \sigma_2 > 0$, it follows $\sqrt{\mu_1}\theta_1 < \frac{\pi}{2}$, which contradicts (7.27).

Next we consider two cases as follows.

1. Suppose $\theta_1 > \theta_2 - \theta_1$ and assume (7.29) is satisfied. Then from

$$\theta_1 < \theta_2 - \frac{\pi/2}{\sqrt{\mu_1}} < \theta_2,$$

it follows that

$$\theta_1 < \frac{\pi/2}{\sqrt{\mu_1}} < \theta_2 - \theta_1$$

which is a contradiction. Thus, (7.27) holds true and $\phi_{1,1}$ achieves the maximum point.

2. Suppose that $\theta_1 < \theta_2 - \theta_1$ and (7.27) holds, then $\frac{\pi/2}{\sqrt{\mu_1}} < \theta_1$ along with $\theta_2 - \frac{\pi/2}{\sqrt{\mu_1}} < \theta_1$ gives the contradiction $\theta_1 > \theta_2 - \theta_1$.

The proof is complete. □

Lemma 7.3.2 *Let $k = 2$. Assume that $0 < \sigma_1$ is finite. If $\sigma_2 \gg \sigma_1$ (or equivalently $\frac{\sigma_2}{\sigma_1} \to +\infty$) then the first eigenvalue μ_1 of Problem (7.16) which is the solution of the Formula (7.22) for $m = 1$, satisfies;*

1. *if $\theta_1 < \theta_2 - \theta_1$ then $\mu_1 \longrightarrow \left(\frac{\pi/2}{\theta_2 - \theta_1}\right)^2$,*

2. *if $\theta_2 - \theta_1 < \theta_1 \leq 2(\theta_2 - \theta_1)$ then $\mu_1 \longrightarrow \left(\frac{\pi/2}{\theta_2 - \theta_1}\right)^2$,*

3. *if $2(\theta_2 - \theta_1) \leq \theta_1$ then $\mu_1 \longrightarrow \left(\frac{\pi}{\theta_1}\right)^2$.*

Remark 7.3.3 *Let us recall from Remark 7.2.2 that for the case $\theta_1 = \theta_2 - \theta_1$, μ_1 is always fixed and independent of σ_1 and σ_2.*

Proof. It follows from (7.22) that

$$\begin{aligned} \sigma_1 \cot\left(\sqrt{\mu_1}\theta_1\right) &= \sigma_2 \cot\left(\pi - \sqrt{\mu_1}\,(\theta_2 - \theta_1)\right) \\ &= -\sigma_2 \cot\left(\sqrt{\mu_1}\,(\theta_2 - \theta_1)\right). \end{aligned} \tag{7.30}$$

So, as $\sigma_2 \longrightarrow +\infty$, then either $\cot\left(\sqrt{\mu_1}\,(\theta_2 - \theta_1)\right) \longrightarrow 0$ or $\cot\left(\sqrt{\mu_1}\theta_1\right) \longrightarrow \pm\infty$.
First, we show that $\cot(\sqrt{\mu_1}\theta_1) \longrightarrow +\infty$ does not hold. Indeed, if $\cot(\sqrt{\mu_1}\theta_1) \longrightarrow +\infty$ then $\sqrt{\mu_1}\theta_1 \downarrow 0$ which implies $\sqrt{\mu_1} \downarrow 0$, it follows $\cot(\sqrt{\mu_1}(\theta_2 - \theta_1)) \longrightarrow +\infty$ and it results in different signs on both sides of (7.30). So we are left with the two other possibilities. Hence, we consider three cases as follows.

1. If $\theta_1 < \theta_2 - \theta_1$, then Lemma 7.3.1 states that $\phi_{1,2}$ attains the maximum which implies that

$$\frac{\partial \phi_{1,1}}{\partial \theta}(\theta_1) = \sqrt{\mu_1}\cot(\sqrt{\mu_1}\theta_1) \geq 0.$$

Hence, the righthand side of (7.30) must be nonnegative, and $\cot(\sqrt{\mu_1}(\theta_2-\theta_1)) \downarrow 0$ i.e. $\sqrt{\mu_1}(\theta_2-\theta_1) \uparrow \pi/2$ or $\mu_1 \uparrow \left(\frac{\pi/2}{\theta_2-\theta_1}\right)^2$.

2. If $\theta_2 - \theta_1 < \theta_1 \leq 2(\theta_2-\theta_1)$, then by Lemma 7.3.1 $\phi_{1,1}$ attains the maximum and

$$\frac{\partial \phi_{1,1}}{\partial \theta}(\theta_1) = \sqrt{\mu_1}\cot\left(\sqrt{\mu_1}\theta_1\right) \leq 0.$$

First, we show that $\cot(\sqrt{\mu_1}\theta_1) \nrightarrow -\infty$. Assume that $\cot(\sqrt{\mu_1}\theta_1) \longrightarrow -\infty$, it follows $\sqrt{\mu_1}\theta_1 \uparrow \pi$. On the other hand, $\cot(\sqrt{\mu_1}(\theta_2-\theta_1)) = \cot(\frac{\theta_2-\theta_1}{\theta_1}\pi)$. And since $\theta_1 \leq 2(\theta_2-\theta_1)$, one finds $\cot(\frac{\theta_2-\theta_1}{\theta_1}\pi) < 0$. This is a contradiction because it leads to different signs on both sides of (7.30). Hence, $\cot\left(\sqrt{\mu_1}\left(\theta_2-\theta_1\right)\right) \downarrow 0$ must be satisfied and it follows $\sqrt{\mu_1} \uparrow \frac{\pi/2}{\theta_2-\theta_1}$.

3. If $2(\theta_2-\theta_1) \leq \theta_1$, similar to the previous case, $\cot(\sqrt{\mu_1}\theta_1) \leq 0$. But in this case $\cot(\sqrt{\mu_1}\theta_1) \longrightarrow -\infty$ holds and $\cot(\sqrt{\mu_1}(\theta_2-\theta_1)) \longrightarrow 0$ fails to happen, because of the sign of (7.30). It follows that $\sqrt{\mu_1} \uparrow \frac{\pi}{\theta_1}$.

□

Lemma 7.3.4 *If $k = 3$ and $\sigma_2, \sigma_3 \in \mathbb{R}^+$ are fixed, then $\sigma_1 \to \infty$ implies $\mu_1 \to \left(\frac{\pi/2}{\theta_1}\right)^2$.*

Proof. With Formula (7.23), for $m = 1$, namely

$$\frac{\sigma_1}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right) = \cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)\right) - \sqrt{\mu_1}(\theta_2-\theta_1)\right). \tag{7.31}$$

one finds that when $\sigma_1 \to +\infty$ then either $\cot\left(\sqrt{\mu_1}\theta_1\right) \to 0$ or

$$\cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right) - \sqrt{\mu_1}(\theta_2-\theta_1)\right)\right) \to \pm\infty.$$

To verify $\cot\left(\sqrt{\mu_1}\theta_1\right) \to 0$ or equivalently $\sqrt{\mu_1} \to \frac{\pi/2}{\theta_1}$, we need to show that the right side of (7.31) can not be infinite. Indeed, if the right side is infinite, so is the left. Hence, $\frac{\sigma_1}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right) \to \pm\infty$ implies that

$$\sin\left(\operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)\right) \longrightarrow 0.$$

This makes the eigenfunction

$$\phi_{1,2}(\theta) = \frac{\sin\left(\sqrt{\mu_m}(\theta-\theta_1) + \operatorname{arccot}\left(\frac{\sigma_1\cot(\sqrt{\mu_m}\theta_1)}{\sigma_2}\right)\right)}{\sin\left(\operatorname{arccot}\left(\frac{\sigma_1\cot(\sqrt{\mu_m}\theta_1)}{\sigma_2}\right)\right)}$$

undefined. Therefore, we can only have

$$\cot\left(\sqrt{\mu_1}\theta_1\right) \longrightarrow 0.$$

The proof is complete. $\square$

Lemma 7.3.5 *If $k = 3$ and $\sigma_1, \sigma_3 \in \mathbb{R}^+$ are fixed, then $\sigma_2 \to \infty$ implies $\mu_1 \downarrow 0$.*

Proof. Consider the Formula (7.31). We shall show that when σ_1 and σ_3 are finite and $\sigma_2 \to +\infty$, the only possibility is that $\cot\left(\sqrt{\mu_1}\theta_1\right) \to +\infty$. Indeed, if $\cot\left(\sqrt{\mu_1}\theta_1\right)$ is finite and if $\cot\left(\sqrt{\mu_1}\theta_1\right) \to -\infty$, we will find contradictions. First, assume that $\cot\left(\sqrt{\mu_1}\theta_1\right) < \infty$. Then since $\frac{\sigma_1}{\sigma_2}$ vanishes, the right side of (7.31) goes to zero, i.e.

$$\cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)\right) - \sqrt{\mu_1}(\theta_2-\theta_1)\right) = 0.$$

This implies

$$\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)\right) = \pi/2 + \sqrt{\mu_1}(\theta_2-\theta_1), \tag{7.32}$$

and by the definition of $\operatorname{arccot}(\cdot)$ one finds that $\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)$ can not be positive. Thus $\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)$ must be positive. In the case that $\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)$ is finite, since $\frac{\sigma_3}{\sigma_2}$ goes to zero, we get the contradiction from (7.32) as follows

$$\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)\right) \to \pi/2 = \pi/2 + \sqrt{\mu_1}(\theta_2-\theta_1).$$

If we let $\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)$ go to infinity, since it must be positive, we find $\sqrt{\mu_1} \to 0$, which is a contradiction with $\cot\left(\sqrt{\mu_1}\theta_1\right) < \infty$. So we conclude that $\cot\left(\sqrt{\mu_1}\theta_1\right)$ can not be finite.
Second, if $\cot\left(\sqrt{\mu_1}\theta_1\right) \to -\infty$ then we find $\sqrt{\mu_1} \to \frac{\pi}{\theta_1}$. Assume that $\theta_1 \neq \theta_3 - \theta_2$, then the right-hand side of (7.31) is positive, namely,

$$\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)\right)=\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\frac{\pi(\theta_3-\theta_2)}{\theta_1}\right)\right)\to\pi/2,$$

since $\frac{\sigma_3}{\sigma_2}\to 0$. Hence, we find

$$\cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}(\theta_3-\theta_2)\right)-\sqrt{\mu_1}(\theta_2-\theta_1)\right)\right)$$
$$\to\cot\left(\pi/2-\sqrt{\mu_1}(\theta_2-\theta_1)\right)\geq 0.$$

But the left side of (7.31) is negative, a contradiction appears.
In the special case $\theta_1=\theta_3-\theta_2$, the Formula (7.31) has the following form;

$$\sigma_1\cot\left(\sqrt{\mu_1}\theta_1\right)=\sigma_2\cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)-\sqrt{\mu_1}(\theta_2-\theta_1)\right).$$

When $\sigma_2\to+\infty$ and $\cot\left(\sqrt{\mu_1}\theta_1\right)\to-\infty$, then while $\sigma_1\cot\left(\sqrt{\mu_1}\theta_1\right)$ is negative, the right hand side is positive, since

$$\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)\leq\pi/2,$$

and so

$$\cot\left(\operatorname{arccot}\left(\frac{-\sigma_3}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)-\sqrt{\mu_1}(\theta_2-\theta_1)\right)$$
$$\geq\cot\left(\pi/2-\sqrt{\mu_1}(\theta_2-\theta_1)\right)\geq 0.$$

We conclude that $\cot\left(\sqrt{\mu_1}\theta_1\right)\to+\infty$ and consequently $\mu_1\downarrow 0$. □

7.3.1 Some examples

Example 7.3.6 *We first consider the simplest case, namely*

$$\begin{aligned}\Omega_1\cap B_\rho(0)&=\left\{(r\cos\theta,r\sin\theta)\,|0<r<\rho\text{ and }\theta\in(0,\theta_1)\right\},\\ \Omega_2\cap B_\rho(0)&=\left\{(r\cos\theta,r\sin\theta)\,|0<r<\rho\text{ and }\theta\in(\theta_1,\pi)\right\},\end{aligned}$$

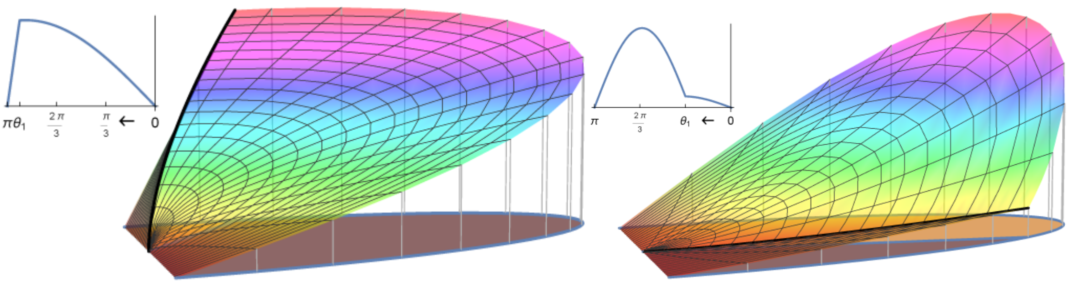
and $\overline{\Omega}=\overline{\Omega_1\cup\Omega_2}$. *Then the first eigenfunction of (7.16) is given by*

$$\phi_1(\theta)=\begin{cases}\sin\left(\sqrt{\mu_1}\theta\right) & \text{for }\theta\in[0,\theta_1],\\ \frac{\sin\left(\sqrt{\mu_1}\theta_1\right)}{\sin(\alpha_1(\mu_1))}\sin\left(\sqrt{\mu_1}(\theta-\theta_1)+\alpha_1(\mu_1)\right) & \text{for }\theta\in(\theta_1,\pi],\end{cases}$$

with $\alpha_1(\mu_1) = \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)$ *and* $\mu_1 > 0$ *the smallest value such that* $\phi_1(\pi) = 0$.

The behaviour of u*, the solution of (7.11), at* 0 *as in (7.34) and (7.39) is given by* $r^{\sqrt{\mu_1}}\phi_1(\theta)$*. Assuming that* $\sigma_1 > \sigma_2$ *and letting* $\frac{\sigma_1}{\sigma_2} \to \infty$*, one finds the 'extreme' cases for* $\theta_1 \uparrow \pi$ *and for* $\theta_1 = \frac{1}{3}\pi$*. These cases correspond with* $\sqrt{\mu_1} \downarrow \frac{1}{2}$ $\sqrt{\mu_1} \uparrow \frac{3}{2}$*. Sketches with nearby values can be found in Figure 7.3. One may show that for all* $\sigma_1, \sigma_2 \in \mathbb{R}^+$ *and all* $0 = \theta_0 < \theta_1 < \theta_2 = \pi$ *it holds that*

$$\frac{1}{2} < \sqrt{\mu_1} < \frac{3}{2}.$$



$\sigma_1 = 100\sigma_2,\ \theta_1 = \frac{11}{12}\pi,\ \sqrt{\mu_1} = 0.568\ldots$ $\qquad$ $\sigma_1 = 100\sigma_2,\ \theta_1 = \frac{1}{3}\pi,\ \sqrt{\mu_1} = 1.43\ldots$

Figure 7.3: *Plots of* $r^{\sqrt{\mu_1}}\phi_1(\theta)$*, which show the typical behaviour of* u *near a boundary point where* $\partial\Omega$ *is smooth and* σ *has one jump. The inset displays the eigenfunction* ϕ_1*.*

Example 7.3.7 *Also in the next case* Ω *is flat, but now it has three subdomains, such that*

$$\begin{aligned}
\Omega_1 \cap B_\rho(0) &= \{(r\cos\theta, r\sin\theta)\,|0 < r < \rho \text{ and } \theta \in (0,\theta_1)\},\\
\Omega_2 \cap B_\rho(0) &= \{(r\cos\theta, r\sin\theta)\,|0 < r < \rho \text{ and } \theta \in (\theta_1,\theta_2)\},\\
\Omega_3 \cap B_\rho(0) &= \{(r\cos\theta, r\sin\theta)\,|0 < r < \rho \text{ and } \theta \in (\theta_2,\pi)\},
\end{aligned}$$

and $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2 \cup \Omega_3}$*. Then*

$$\phi_1(\theta) = \begin{cases} \sin\left(\sqrt{\mu_1}\theta\right) & \text{for } \theta \in [0,\theta_1],\\ \dfrac{\phi_1(\theta_1)}{\sin(\alpha_1(\mu_1))}\sin\left(\sqrt{\mu_1}(\theta-\theta_1) + \alpha_1(\mu_1)\right) & \text{for } \theta \in (\theta_1,\theta_2],\\ \dfrac{\phi_1(\theta_2)}{\sin(\alpha_2(\mu_1))}\sin\left(\sqrt{\mu_1}(\theta-\theta_2) + \alpha_2(\mu_1)\right) & \text{for } \theta \in (\theta_2,\pi], \end{cases}$$

with

$$\alpha_1(\mu_1) = \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot\left(\sqrt{\mu_1}\theta_1\right)\right)$$

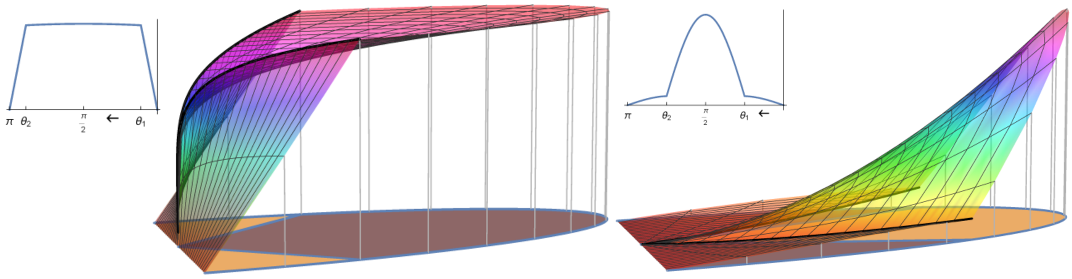
and

$$\alpha_2(\mu_1) = \operatorname{arccot}\left(\frac{\sigma_2}{\sigma_3}\cot\left(\sqrt{\mu_1}(\theta_2-\theta_1)\right)+\alpha_1(\mu_1)\right).$$

Again μ_1 is the smallest positive number such that $\phi_1(\pi)=0$.
Again the behaviour of u, the solution of (7.11), at 0 as in (7.34) and (7.39) is given by $r^{\sqrt{\mu_1}}\phi_1(\theta)$. If $\sigma_1=\sigma_3>\sigma_2$ one finds the extreme cases when $\frac{\sigma_2}{\sigma_1}\to 0$ for $\theta_1=\frac{1}{4}\pi$, $\theta_2=\frac{3}{4}\pi$. For $\sigma_1=\sigma_3<\sigma_2$ and $\frac{\sigma_2}{\sigma_1}\to\infty$ the 'extreme' case appears for $\theta_1=\pi-\theta_2\downarrow 0$. See also Figure 7.4.

For three subdomains as above one may show that for all $\sigma_1,\sigma_2,\sigma_3\in\mathbb{R}^+$ and all $0=\theta_0<\theta_1<\theta_2<\theta_3=\pi$ it holds that

$$0<\sqrt{\mu_1}<2.$$



$\sigma_1=\sigma_3=0.01\sigma_2,\ \theta_1=\frac{1}{9}\pi,\ \theta_2=\frac{8}{9}\pi,\ \sqrt{\mu_1}=0.152\ldots$ $\qquad$ $\sigma_1=\sigma_3=100\sigma_2,\ \theta_1=\frac{1}{4}\pi,\ \theta_2=\frac{3}{4}\pi,\ \sqrt{\mu_1}=1.87\ldots$

Figure 7.4: *Plots of $r^{\sqrt{\mu_1}}\phi_1(\theta)$ are showing the typical behaviour of u near a boundary point where σ has two jumps. The inset displays the corresponding eigenfunction ϕ_1.*

7.4 Hopf Type Estimates

The maximum principle is one of the fundamental tools for our main result. We state the version we use for easy reference.

Theorem 7.4.1 *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Let $\sigma\in L^\infty(\Omega)$ with $\sigma\geq\sigma_0>0$ for some $\sigma_0\in\mathbb{R}^+$. Suppose that $u\in W^{1,2}(\Omega)$ is such that* $\min(u,0)\in\mathring{W}^{1,2}(\Omega)$ *and satisfies*

$$\int_\Omega \sigma\ \nabla u\cdot\nabla\varphi\ dx\geq 0\ \ \text{for all } \varphi\in\mathring{W}^{1,2}(\Omega)\ \text{ with } \varphi\geq 0.$$

Then one finds $u\geq 0$ in Ω.

Proof. With $\varphi = -\min(u,0)$, which lies in $\mathring{W}^{1,2}(\Omega)$ and is nonnegative, we find

$$0 \leq \int_\Omega \sigma\ \nabla u \cdot \nabla\varphi\ dx = -\int_\Omega \sigma\ |\nabla\varphi|^2\ dx \leq 0.$$

Hence $\nabla\varphi = 0$, which implies $\varphi = 0$ and hence $u \geq 0$. $\square$

For the sake of simple statements we recall the following notation.

Notation 7.4.2 *Let $u, v : A \mapsto \mathbb{R}^+$ be two positive functions. We write '$v(x) \preceq u(x)$ for $x \in A$', if there exists a constant $c > 0$ such that $v(x) \leq cu(x)$ for all $x \in A$. If $v(x) \preceq u(x)$ and $u(x) \preceq v(x)$ for $x \in A$, we write '$v(x) \simeq u(x)$ for $x \in A$'.*

Moreover, we will use the function $d : \Omega \to \mathbb{R}^+$ that denotes the distance to the boundary:

$$d(x) = d(x, \partial\Omega) := \inf\{|x - x^*|\,; x^* \in \partial\Omega\}.$$

Assuming Condition 7.1.2 and defining $\tilde{\sigma}(\theta) = \sigma(\rho\theta)$, a crucial role will be played by μ_1, the first eigenvalue of a weighted Laplace-Beltrami operator on $\partial C \cap \partial B_1(0)$ under Dirichlet boundary conditions defined by

$$\mu_1 = \inf_{\phi \in \mathring{W}^{1,2}(0,\theta_k)} R_\sigma(\phi) \tag{7.33}$$

in which $R_\sigma(\phi)$ is as in (7.17).

Theorem 7.4.3 *Suppose that $\Omega \subset \mathbb{R}^2$ is as in Condition 7.1.2 and take C and C_i from there. Assume that $u \in \mathring{W}^{1,2}(\Omega)$ satisfies the boundary value problem (7.11) and $0 \lneqq f \in W^{-1,2}(\Omega) \cap C\left(\bar{\Omega} \setminus \{0\}\right)$. Let μ_1 be as in (7.33). Then the following results hold.*

a) *For all $x \in \Omega \cap B_\rho(0)$ one finds*

$$|x|^{\sqrt{\mu_1}-1}\, d(x) \preceq u(x). \tag{7.34}$$

b) *Moreover, let $m > -2$ and suppose that*

$$f(x) \preceq |x|^m \ \text{for } x \in \Omega, \tag{7.35}$$

and for $\Omega' = \{(r\cos\theta, r\sin\theta) | 0 < r < r_0, \theta_a < \theta < \theta_b\} \subset \Omega$, with some $r_0 > 0$ and $0 \leq \theta_a < \theta_b \leq \theta_k$,

$$|x|^m \preceq f(x) \ \text{for } x \in \Omega'. \tag{7.36}$$

Then we find:

1. *if* $m+2<\sqrt{\mu_1}$, *then*

$$u(x)\simeq |x|^{m+1}\,d\,(x)\ \ \textit{for}\ x\in\Omega\cap B_\rho\,(0)\,, \tag{7.37}$$

2. *if* $m+2=\sqrt{\mu_1}$, *then*

$$u(x)\simeq |x|^{\sqrt{\mu_1}-1}\ln\left(\tfrac{1}{|x|}\right)\ d\,(x)\ \ \textit{for}\ x\in\Omega\cap B_\rho\,(0)\,, \tag{7.38}$$

3. *if* $m+2>\sqrt{\mu_1}$, *then*

$$u(x)\simeq |x|^{\sqrt{\mu_1}-1}\,d\,(x)\ \ \textit{for}\ x\in\Omega\cap B_\rho\,(0)\,. \tag{7.39}$$

Remark 7.4.4 *The items (7.37-7.39) contain both estimates from below and from above. In fact these estimates are independent and only combined in one equivalence relation in order to show the sharpness of the estimate. From the proof one might see, that (7.35) yields the estimates from above and (7.36) the ones from below.*

Remark 7.4.5 *If* Ω *consists near* 0 *of just two subdomains* Ω_1 *and* Ω_2 *such that after a rotation we find that*

$$\begin{aligned}\Omega_1\cap B_\rho\,(0) &= \{(x_1,x_2)\,|x_1>0\ \textit{and}\ x_2>0\}\cap B_\rho\,(0)\,,\\ \Omega_2\cap B_\rho\,(0) &= \{(x_1,x_2)\,|x_1<0\ \textit{and}\ x_2>0\}\cap B_\rho\,(0)\,,\end{aligned}$$

i.e. $\partial\Omega$ *is flat with* Γ_1 *perpendicular, then* $\mu_1=1$ *and (7.34) gives us the classical Hopf Lemma even if* σ_1 *and* σ_2 *are different. For any other angle there is in general no linear growth near the boundary point. See Section 7.3.1.*

Proof. First let us remark that a maximum principle like Theorem 7.4.1 implies that $u\geq 0$ on Ω. Since $u_i\in W^{2,p}\left(\Omega_i\setminus\bigcup_{j=1}^m B_\varepsilon\,(p_j)\right)$ for all $p<\infty$, these u_i are C^1 away from the p_j's. The strong maximum principle implies that on each Ω_i one either has $u_i\equiv 0$ or $u_i>0$. The jump condition at points on $\partial\Omega_i\cap\partial\Omega_j$ implies that $u>0$ on $\partial\Omega_i\cap\partial\Omega_j$. With the classical Hopf's boundary point Lemma at $x\in\partial\Omega\setminus\{p_0,\ldots,p_m\}$ we find that for each $\varepsilon>0$

$$u\,(x)\geq cd\,(x)\ \ \text{for}\ x\in\Omega\setminus\bigcup_{j=1}^m B_\varepsilon\,(p_j)\,.$$

By regularity results we find the reverse inequality on $\Omega\setminus\bigcup_{j=1}^m B_\varepsilon\,(p_j)$ for each $\varepsilon>0$. Note that the constants in the estimate do depend on $\varepsilon>0$ and might blow up when taking $\varepsilon\downarrow 0$.

We are left with proving the estimates near p_j and to do so we restrict ourselves, as stated in the theorem, to the neighborhood of the singular point at 0, where after a scaling the problem appears as in (7.15) and where $w(x)$ on $\partial C \cap \partial B_1(0)$ is a function equivalent the tangential distance along $\partial B_1(0)$ to $\rho^{-1}\partial\Omega$. Here C is as defined in Condition 7.1.2.

Similar as in Theorem 4.3.1 we construct upper and lower barrier functions for the solution of (7.15) with the right hand side $f \simeq |x|^m$. The maximum principle is used to show that barriers form the estimates. The maximum principle that we use is for functions as in (7.7). Such functions can be integrated by part, since $\tilde{u}|_{C_i} \in W^{2,2}(C_i)$ and the power type solutions are $C^1_{\text{piecewise}}$ as a function of θ.
Let $\phi_{1,\sigma}$ be the function in Lemma 7.2.1 normalized by

$$\max\{\phi_1(\theta); 0 < \theta < \theta_k\} = 1.$$

Defining $\Phi : C \to \mathbb{R}$ by

$$\Phi(r\cos\theta, r\sin\theta) = r^{\sqrt{\mu_1}}\phi_1(\theta)$$

with $\Phi_i = \Phi|_{C_i}$, we find that it satisfies

$$\begin{cases} -\Delta\Phi_i = 0 & \text{in } C_i \text{ with } i = 1, ..., k, \\ \Phi_1 = 0 & \text{on } \Gamma_0, \\ \left.\begin{array}{r} \Phi_i = \Phi_{i+1} \\ \sigma_i \frac{\partial}{\partial\theta}\Phi_i = \sigma_{i+1}\frac{\partial}{\partial\theta}\Phi_{i+1} \end{array}\right\} & \text{on } \Gamma_i \text{ with } i = 1, ..., k-1, \\ \Phi_k = 0 & \text{on } \Gamma_k, \\ \Phi_i = \phi_1 & \text{on } \partial C \cap \partial B_1(0). \end{cases} \tag{7.40}$$

Since ϕ_1 satisfies (7.19) and since $d(x, \partial\Omega) = |x|\, d\left(\frac{x}{|x|}, \partial\Omega\right)$ for $x \in \Omega \cap B_\rho(0)$, one finds that

$$\Phi(x) \simeq |x|^{\sqrt{\mu_1}}\, d\left(\frac{x}{|x|}, \Gamma_0 \cup \Gamma_k\right) \simeq |x|^{\sqrt{\mu_1}}\, d\left(\frac{x}{|x|}, \frac{1}{\rho}\partial\Omega\right) \simeq |x|^{\sqrt{\mu_1}-1}\, d(x, \partial\Omega).$$

Indeed, the equivalences follows from $\left(\frac{1}{\rho}\Omega\right) \cap B_2(0) = 2C$ and by scaling. In the remainder the Maximum Principle as in Theorem 7.4.1 is used.

1. Let $m + 2 < \sqrt{\mu_1}$.

- **Estimate from above:** Set υ_κ the solution of

$$\begin{cases} -\upsilon''_{\kappa,i}(\theta)+\kappa\upsilon_{\kappa,i}(\theta)=1 & \text{for } \theta\in[\theta_{i-1},\theta_i] \text{ and } i\in\{1,\dots,k\}\,, \\ \upsilon_{\kappa,i}(\theta_i)=\upsilon_{\kappa,i+1}(\theta_i) & i=1,\dots k, \\ \sigma_i\upsilon'_{\kappa,i}(\theta_i)=\sigma_{i+1}\upsilon'_{\kappa,i+1}(\theta_{i+1}) & i=1,\dots k, \\ \upsilon_\kappa(0)=\upsilon_\kappa(\theta_k)=0. \end{cases} \tag{7.41}$$

with $\kappa=-(m+2)^2$ and the same σ_i as in (7.15). Since $\kappa<\mu_1$, one finds that such a unique solution υ_κ exists, is positive and furthermore, we find

$$\upsilon_\kappa\simeq\phi_1.$$

By taking $u_{1a}:=|x|^{m+2}\upsilon_\kappa\left(\frac{x}{|x|}\right)$, we observe that u_{1a} satisfies the following boundary value problem;

$$\begin{cases} -\Delta u_{1a}|_{C_i}=|x|^m & \text{in } C_i,\ i=1,...,k, \\ u_{1a,i}=u_{1a,i+1} & \text{on } \Gamma_i,\ i=1,...,k-1, \\ \sigma_i\frac{\partial u_{1a,i}}{\partial\theta}=\sigma_{i+1}\frac{\partial u_{1a,i+1}}{\partial\theta} & \text{on } \Gamma_i, i=1,...,k-1, \\ u_{1a}\simeq\phi_1 & \text{on } \ \partial C\cap\partial B_1(0), \\ u_{1a,1}=0 & \text{on } \Gamma_0, \\ u_{1a,k}=0 & \text{on } \Gamma_k. \end{cases}$$

Since $f\preceq|x|^m$ on Ω, it follows by the maximum principle that $u\preceq u_{1a}\simeq|x|^{m+2}\phi_1\left(\frac{x}{|x|}\right)$.

- **Estimate from below:** We take κ as before and we denote by ω_κ the solution of

$$\begin{cases} -\omega''_{\kappa,i}(\theta)+\kappa\omega_{\kappa,i}(\theta)=\chi_{(\theta_a,\theta_b)}(\theta) & \text{for } \theta\in[\theta_{i-1},\theta_i]\,, i\in\{1,\dots,k\}\,, \\ \omega_{\kappa,i}(\theta_i)=\omega_{\kappa,i+1}(\theta_i) & i=1,\dots k, \\ \sigma_i\omega'_{\kappa,i}(\theta_i)=\sigma_{i+1}\omega'_{\kappa,i+1}(\theta_{i+1}) & i=1,\dots k, \\ \omega_\kappa(0)=\omega_\kappa(\theta_k)=0. \end{cases} \tag{7.42}$$

Where χ_A is the characteristic function of a set A. Similar to the previous case, we find $0\le\omega_\kappa\simeq\phi_1$ in $(0,\theta_k)$. By setting

$$u_{1b}:=|x|^{m+2}\omega_\kappa\left(\frac{x}{|x|}\right),$$

one finds that u_{1b} satisfies

$$\begin{cases} -\Delta u_{1b}|_{C_i} = |x|^m \chi_{(\theta_a,\theta_b)}\left(\frac{x}{|x|}\right) & \text{in } C_i,\ i=1,...,k, \\ u_{1b,i} = u_{1b,i+1} & \text{on } \Gamma_i,\ i=1,...,k-1, \\ \sigma_i \frac{\partial u_{1b,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{1b,i+1}}{\partial\theta} & \text{on } \Gamma_i, i=1,...,k-1, \\ u_{1b} \simeq \phi_1 & \text{on } \partial C \cap \partial B_1(0), \\ u_{1b,1} = 0 & \text{on } \Gamma_0, \\ u_{1b,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

Thus, by the maximum principle we find $|x|^{m+2}\phi_1\left(\frac{x}{|x|}\right) \simeq u_{1b} \preceq u$.

2. Let $m+2 = \sqrt{\mu_1}$.

- **Estimate from above:** We denote by υ_0 the solution of

$$\begin{cases} -\upsilon''_{0,i}(\theta) = 1 & \text{for } \theta \in (\theta_{i-1},\theta_i) \text{ and } i \in \{1,\dots,k\}, \\ \upsilon_{0,i}(\theta_i) = \upsilon_{0,i+1}(\theta_i) & i=1,\dots k, \\ \sigma_i\upsilon'_{0,i}(\theta_i) = \sigma_{i+1}\upsilon'_{0,i+1}(\theta_{i+1}) & i=1,\dots k, \\ \upsilon_0(0) = \upsilon_0(\theta_k) = 0, \end{cases} \tag{7.43}$$

which is simply the solution of (7.41) with $\kappa = 0$. Since $\upsilon_0 \simeq \phi_1$ in $(0,\theta_k)$, we can choose a positive constant γ such that

$$\gamma(m+2)\upsilon_0(\theta) \le 2\phi_1(\theta) \quad \text{for all } \theta \in (0,\theta_k).$$

Then by taking

$$u_{2a}(x) := |x|^{m+2}\ln\left(\frac{1}{|x|}\right)\phi_1\left(\frac{x}{|x|}\right) + \gamma|x|^{m+2}\upsilon_0\left(\frac{x}{|x|}\right)$$

one finds that u_{2a} satisfies

$$-\Delta u_{2a} = |x|^m\left(2(m+2)\phi_1\left(\frac{x}{|x|}\right) - \gamma(m+2)^2\upsilon_0\left(\frac{x}{|x|}\right) + \gamma\right)$$

which implies

$$\begin{cases} -\Delta u_{2a}|_{C_i} \simeq |x|^m & \text{in } C_i,\ i=1,...,k, \\ u_{2a,i} = u_{2a,i+1} & \text{on } \Gamma_i,\ i=1,...,k-1 \\ \sigma_i\frac{\partial u_{2a,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{2a,i+1}}{\partial\theta} & \text{on } \Gamma_i, i=1,...,k-1 \\ u_{2a} \simeq \phi_1 & \text{on } \partial C \cap \partial B_1(0), \\ u_{2a,1} = 0 & \text{on } \Gamma_0, \\ u_{2a,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

We observe that $-\Delta u = f(x) \leq -\Delta u_{2a}$ in Ω and $u \simeq u_{2a}$ on $\partial\Omega$. By the maximum principle we get the following estimate from above:

$$u \preceq u_{2a} \simeq |x|^{\sqrt{\mu_1}} \ln\left(\tfrac{1}{|x|}\right)\ \phi_1\left(\tfrac{x}{|x|}\right).$$

- **Estimate from below:** For getting a lower barrier for u in this case, we set

$$u_{2b} := \zeta |x|^{m+2} \ln\left(\frac{1}{|x|}\right) \phi_1\left(\frac{x}{|x|}\right) + |x|^{m+2}\omega_0\left(\frac{x}{|x|}\right),$$

where ω_0 is the solution of (7.42) with $\kappa = 0$ and $\zeta > 0$ is such that

$$2\zeta\phi_1(\theta) \leq (m+2)\omega_0(\theta) \quad \text{for all } \theta \in (0, \theta_k).$$

Then u_{2b} satisfies the following equation for all $x \in C_i$

$$\begin{aligned} -\Delta u_{2b}|_{C_i} = {} & |x|^m \chi_{(\theta_a,\theta_b)}\left(\frac{x}{|x|}\right) + 2(m+2)\zeta|x|^m\phi_1\left(\frac{x}{|x|}\right) \\ & -(m+2)^2|x|^m\omega_0\left(\frac{x}{|x|}\right). \end{aligned}$$

Hence, u_{2b} is a bound from below as follows:

$$\begin{cases} -\Delta u_{2b}|_{C_i} \preceq |x|^m \chi_{(\theta_a,\theta_b)}\left(\frac{x}{|x|}\right) & \text{in } C_i,\ \ i = 1, ..., k, \\ u_{2b,i} = u_{2b,i+1} & \text{on } \Gamma_i,\ \ i = 1, ..., k-1, \\ \sigma_i \frac{\partial u_{2b,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{2b,i+1}}{\partial\theta} & \text{on } \Gamma_i, i = 1, ..., k-1, \\ u_{2b} \simeq \phi_1 & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{2b,1} = 0 & \text{on } \Gamma_0, \\ u_{2b,k} = 0 & \text{on } \Gamma_k, \end{cases}$$

which implies by the maximum principle that $u_{2b} \preceq u$.

3. Let $m + 2 > \sqrt{\mu_1}$

- **Estimate from above:** An upper barrier for u in this case can be taken by

$$u_{3a} := \left(|x|^{\sqrt{\mu_1}} - |x|^{m+2}\right)\phi_1\left(\frac{x}{|x|}\right) + \gamma|x|^{m+2}\upsilon_0\left(\frac{x}{|x|}\right),$$

where υ_0 is the solution of (7.43) and $\gamma > 0$ satisfies

$$\gamma(m+2)^2\upsilon_0(\theta) \leq \left((m+2)^2 - \mu_1\right)\phi_1(\theta).$$

Then u_{3a} satisfies the following equation:

$$-\Delta u_{3a} = |x|^m \left((m+2)^2 - \mu_1\right) \phi_{1,\sigma}\left(\frac{x}{|x|}\right)$$
$$-|x|^m \gamma (m+2)^2 \upsilon_0 \left(\frac{x}{|x|}\right) + \gamma |x|^m$$

for $x \in \Omega$. Thus, one can find that the following holds true,

$$\begin{cases} -\Delta u_{3a}|_{C_i} \simeq |x|^m & \text{in } C_i,\ i = 1, ..., k, \\ u_{3a,i} = u_{3a,i+1} & \text{on } \Gamma_i,\ i = 1, ..., k-1, \\ \sigma_i \frac{\partial u_{3a,i}}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{3a,i+1}}{\partial \theta} & \text{on } \Gamma_i, i = 1, ..., k-1, \\ u_{3a} \simeq \phi_1 & \text{on } \partial C \cap \partial B_1(0), \\ u_{3a,1} = 0 & \text{on } \Gamma_0, \\ u_{3a,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

and this implies that $u \preceq u_{3a} \simeq |x|^{\sqrt{\mu_1}} \phi_1\left(\frac{x}{|x|}\right)$.

- **Estimate from below:** The estimate from below one gets by the harmonic function

$$u_{3b} := |x|^{\sqrt{\mu_1}} \phi_1\left(\frac{x}{|x|}\right)$$

which satisfies

$$\begin{cases} -\Delta u_{3b}|_{C_i} = 0 & \text{in } C_i,\ i = 1, ..., k, \\ u_{3b,i} = u_{3b,i+1} & \text{on } \Gamma_i,\ i = 1, ..., k-1 \\ \sigma_i \frac{\partial u_{3b,i}}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{3b,i+1}}{\partial \theta} & \text{on } \Gamma_i, i = 1, ..., k-1 \\ u_{3b} \simeq \phi_1 & \text{on } \partial C \cap \partial B_1(0), \\ u_{3b,1} = 0 & \text{on } \Gamma_0, \\ u_{3b,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

By the maximum principle we find $u_{3b} \preceq u$ in Ω.

Comparing with the results in Theorem 4.3.1, we observe that the solution of the problem (7.15) has the same form as the solution of the Poisson problem near a conical point but with different type of regularity. □

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Erklärung gemäß §4 der Promotionsordung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angebenen Teilpublikationen- noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Guido Sweers betreut worden.

Teilpublikationen

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