Tame matrix problems in Lie theory and commutative algebra

Inaugural - Dissertation
zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
Wassilij Gnedin
aus Moskau

Berichterstatter: Prof. Dr. Igor Burban
Prof. Dr. Peter Littelmann

Tag der mündlichen Prüfung: 27.11.2015

Abstract

In this thesis we solve several classification problems from Lie theory and commutative algebra.

In the main part of this thesis, we study Harish-Chandra modules of some classical Lie groups of real rank one.
In the case of the Gelfand quiver, which is related to Harish-Chandra modules over the Lie group $\mathrm{SL}(2, \mathbb{R})$, we construct all indecomposable nilpotent representations. We compute their projective resolutions, contragredient duals and basic homological invariants. We give further an explicit description of the Auslander-Reiten translation on the derived category of the Gelfand quiver.
For Khoroshkin quivers, which correspond to Lorentz groups of type $\mathrm{SO}(n, 1)$ or $\mathrm{SO}_e(n, 1)$, we give an intrinsic description of the derived Auslander-Reiten translation τ and characterize the τ-periodic objects in the derived category.
In the more general setup of nodal orders, we give a homological characterization of the indecomposable objects in the derived category. At last, we reduce the problem to classify the indecomposable modules over any nodal order to a matrix problem.

In the shorter part of this thesis, we study Cohen-Macaulay modules over some non-reduced curve singularities.
We prove that the rings $\mathbb{k}[\![x, y, z]\!]/(x\,y, y^n - z^2)$ have tame Cohen-Macaulay representation type. For the singularity $\mathbb{k}[\![x, y, z]\!]/(x\,y, z^2)$ we give an explicit description of all indecomposable Cohen-Macaulay modules and apply the obtained classification to construct families of indecomposable matrix factorizations of the potential $x^2\,y^2 \in \mathbb{k}[\![x, y]\!]$.

Zusammenfassung

In der vorliegenden Dissertation lösen wir diverse Klassifikationsprobleme der Lie-Theorie und kommutativen Algebra.

Im Hauptteil dieser Dissertation untersuchen wir Harish-Chandra Moduln von klassischen Lie-Gruppen von reellem Rang eins.
Im Fall des Gelfand-Köchers, der in Verbindung zu Harish-Chandra Moduln der Lie-Gruppe $\mathrm{SL}(2, \mathbb{R})$ steht, konstruieren wir alle unzerlegbaren nilpotenten Darstellungen. Wir berechnen ihre projektiven Auflösungen, kontragradienten Duale und ihre wese- ntlichen homologischen Invarianten. Weiterhin geben wir eine explizite Beschreibung der Auslander-Reiten Translation in der abgeleiteten Kategorie des Gelfand-Köchers.
Für Khoroshkin-Köcher, die zu Lorentz-Gruppen vom Typ $\mathrm{SO}(n, 1)$ oder $\mathrm{SO}_e(n, 1)$ korrespondieren, geben wir eine intrinsische Beschreibung der abgeleiteten Auslander-Reiten Translation τ und charakterisieren die τ-periodischen Objekte in der abgeleiteten Kategorie.
In der allgemeineren Situation der nodalen Ordnungen geben wir eine homologische Charakterisierung der unzerlegbaren Objekte in der abgeleiteten Kategorie. Zuletzt reduzieren wir das Problem der Klassifikation aller unzerlegbaren Moduln einer beliebigen nodalen Ordnung auf ein Matrixproblem.

Im kürzeren Teil dieser Dissertation untersuchen wir Cohen-Macaulay Moduln über nicht-reduzierten Kurven-Singularitäten.
Wir beweisen, dass die Ringe $\mathbb{k}[\![x, y, z]\!]/(x\,y, y^n - z^2)$ zahmen Cohen-Macaulay Darstellungstyp haben. Für die Singularität $\mathbb{k}[\![x, y, z]\!]/(x\,y, z^2)$ geben wir eine explizite Beschreibung aller unzerlegbaren Cohen-Macaulay Moduln und benutzen die dadurch gewonnene Klassifikation um Familien von unzerlegbaren Matrix- Faktorisierungen des Potentials $x^2\,y^2 \in \mathbb{k}[\![x, y]\!]$ zu konstruieren.

Introduction

This thesis belongs to representation theory and is divided into two parts. In the main part of this thesis, we investigate Harish-Chandra modules over real Lie groups of rank one. In the short part, we consider Cohen-Macaulay modules over non-reduced curve singularities. In both parts, we classify the indecomposable representations of some one-dimensional order using the technique of matrix problems.

The study of Harish-Chandra modules is motivated by their close relationship to analytic representations of Lie groups. In general, non-compact Lie groups have *indecomposable* representations which are neither irreducible nor unitary. Such representations are related to unstable particles in physics and have attracted a lot of interest from the mathematical as well as the physics community (see [**Can90**] and references therein).

The investigation of *indecomposable Harish-Chandra modules* was initiated by Zhelobenko, Gelfand and Ponomarev, and evolved as follows:

- In [**Zhe59**] Zhelobenko reduced the study of Harish-Chandra modules over the *Lorentz group* to the study of nilpotent representations of the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop left, "\mathsf{y}"] \arrow[r, bend left, "\mathsf{x}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{z}"] \end{tikzcd} \qquad \mathsf{x}\,\mathsf{y} = \mathsf{y}\,\mathsf{z} = 0.$$

- The classification problem of Zhelobenko's quiver was solved by Gelfand and Ponomarev in [**GP68**]. Their work had a big impact on the representation theory of quivers and was also used by physicists.

- At the International Congress of Mathematicians in 1972 I. Gelfand raised the problem to classify the indecomposable nilpotent representations of the quiver

$$\begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-,$$

  stating that the problem "*is apparently solvable but leads to considerable difficulties*" [**Gel71**].

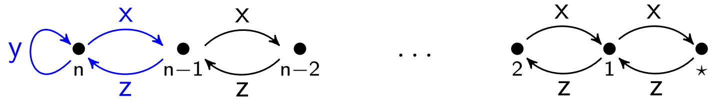
  The study of the Gelfand quiver is motivated by the fact that the category of its nilpotent representations is equivalent to any non-trivial block of Harish-Chandra modules over the Lie group $\mathrm{SL}(2,\mathbb{R})$.

- Shortly afterwards, Nazarova and Roiter reduced Gelfand's problem further to a *matrix problem*, that is, a certain problem of linear algebra, and proved that this problem is tame [**NR73**].

- In [**Kho80**] and [**Kho81**] Khoroshkin obtained similar quiver descriptions for the principal blocks of Harish-Chandra modules over the Lie groups $\mathrm{SU}(n,1)$, $\mathrm{SO}(n,1)$ and their identity components $\mathrm{SO}_e(n,1)$.

(1) The quiver associated to the connected Lorentz group $\mathrm{SO}_e(2n+1,1)$ is *gentle*:

$(Q,I)_{2n+1} =$

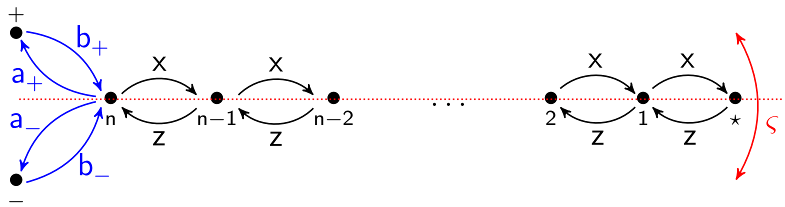


with relations $\mathsf{x}\,\mathsf{y} = \mathsf{y}\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0$.

Their representations can be understood by similar methods as in the work of Gelfand and Ponomarev. In particular, there are two types of indecomposable objects: *usual strings* and *bands*. The same holds true for Khoroshkin quivers associated to the disconnected Lorentz groups $\mathrm{SO}(n,1)$ for any $n \in \mathbb{N}^+$.

(2) The quiver for the Lorentz group $\mathrm{SO}_e(2n+2,1)$ is *skew-gentle*:

$(Q,I)_{2n+2} =$



with relations $\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$, $\mathsf{x}\,\mathsf{b}_\pm = \mathsf{a}_\pm\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0$.

The even series of Khoroshkin quivers includes the Gelfand quiver as the simplest case and possesses a special involution ς. By Khoroshkin's results these quivers are also tame.

(3) The quivers of the Lie groups $\mathrm{SU}(n,1)$ have wild representation type for any $n \geq 2$. The same is true for any real Lie group with rank greater than one.

- Bondarenko [**Bon88**] and Crawley-Boevey [**CB89**] were the first to describe explicit bases for the indecomposable representations of the Gelfand quiver using completely different methods. Other solutions of Gelfand's problem were obtained by Deng [**Den00**] and Iyama [**Iya05**].
- In [**Dro91**] Drozd defined *nodal orders* which include the completed path algebras of Khoroshkin quivers. He proved that nodal orders are the *only* orders which have *tame* categories of finite-dimensional modules.
- Burban and Drozd proved that nodal orders are even *derived-tame* [**BD04**]. They have shown that the indecomposable objects of the derived category of any nodal order are given by four classes: *usual*, *special* and *bispecial strings*, and *bands*. These classes are defined in purely combinatorial terms.

In spite of all aforementioned results, the homological and functorial properties of the indecomposable representations of the Gelfand quiver as well as the Khoroshkin quivers remained to be clarified.

In the following we describe the main results of this thesis in a compact form. A more detailed presentation of these results can be found in the summary following this introduction.

In the main part of this thesis we obtain the following results on the Gelfand quiver, the Khoroshkin quivers and nodal orders:

1. We give an explicit description of the indecomposable nilpotent representations of the Gelfand quiver, their projective resolutions, their contragredient duals and their homological invariants (namely the projective and injective dimension, Jordan-Hölder-multiplicities, Euler characteristic, top and socle). These statements are described in Theorems 5.1.8, 5.4.7 and 5.2.7 and 5.3.6.

2. We compute the derived Auslander-Reiten translation on strings and bands of the derived category of the Gelfand quiver. As an application of this result, we show that there is only one generalized spherical autoequivalence of the derived category. These results are stated in Theorem 4.3.10 and Corollary 4.3.26.

3. We study the derived Auslander-Reiten theory of Khoroshkin quivers. In particular, we give an intrinsic description of the derived Auslander-Reiten translation τ, and show that bands are τ-invariant, bispecial strings have τ-period two, while special or usual strings are not τ-periodic. These statements can be found in Theorem 2.2.3 and Corollary 3.6.4.

4. We give an intrinsic characterization of the four classes in the derived category of finite-dimensional modules over any nodal order. Namely, we introduce the *defect* and show that this homological invariant is *vanishing* for bands and bispecial strings, *one* for special strings and *two* for usual strings. Furthermore, there is a natural *involution* functor which preserves any band, but no bispecial string. In the case of Khoroshkin quivers of connected Lorentz groups, the defect and the involution have a Lie-theoretic interpretation. The characterization of the four classes is described in Theorem 3.6.2.

We study the classification problem of indecomposable modules over any nodal order. In an equivalent formulation, we identify the strings and bands which correspond to *projective presentations* in the derived category of the nodal order. We give two descriptions of the *projective resolutions* in the derived category. The result on projective presentations is given by Theorem 3.5.18, and the last two descriptions by Propositions 3.3.1 and 3.3.11.

In the short part of this thesis, we are concerned with Cohen-Macaulay modules over some curve singularities.

Cohen-Macaulay modules over Cohen-Macaulay rings continue to be a thriving subject. They constitute the commutative realm of the representation theory of orders, are related to algebraic geometry (McKay correspondence) and have applications in physics (matrix factorizations).

So far, the following is known about the Cohen-Macaulay representation type of curve singularities:

- In [**DG92**] Drozd and Greuel characterized all *reduced* curve singularities which are Cohen-Macaulay tame. A family of such singularities is given by the rings

$$\mathsf{P}_{pq} = \Bbbk[\![x, y, z]\!]/(x\,y, x^p + y^q - z^2), \qquad \text{where } p, q \in \mathbb{N}_{\geq 2}. \tag{$*$}$$

- Recent work [**BD**] of Burban and Drozd provides a family of *non-reduced* curve singularities of tame Cohen-Macaulay type. The maximal degeneration of this family is given by the ring $\mathsf{T}_{\infty\infty} = \mathbb{k}[\![x, y]\!]/(x^2 y^2)$. The explicit form of its indecomposable Cohen-Macaulay modules or matrix factorizations is yet unknown.

In the short part of this thesis, we study natural degenerations of the family of curve singularities in $(*)$:

$$\mathsf{P}_{\infty q} = \mathbb{k}[\![x, y, z]\!]/(x\,y, y^q - z^2) \qquad \text{and} \qquad \mathsf{P}_{\infty\infty} = \mathbb{k}[\![x, y, z]\!]/(x\,y, z^2)$$

5. We prove that the non-reduced rings $\mathsf{P}_{\infty q}$ and $P_{\infty\infty}$ have tame Cohen-Macaulay type. We give an explicit description of the indecomposable Cohen-Macaulay modules over the rings $\mathsf{P}_{\infty\infty}$, $\mathsf{P}_{\infty q}$ and P_{pq} for all *odd* values of p and q. These results are contained in Theorems 1.2.1, 1.3.16 and Remark 1.3.21.

As a Corollary, we obtain a partial, but concrete classification of the indecomposable Cohen-Macaulay modules of type $\mathsf{T}_{\infty\infty}$. This application is described in Remark 1.3.23.

The results on Cohen-Macaulay modules are available as a preprint in a slightly different form [**BG**].

In both parts our approach to classification problems is based on the *category of triples* developed by Burban and Drozd [**BD04, BD**] together with Bondarenko's combinatorics of matrix problems over *bunches of semichains* [**Bon88, Bon91**].

Acknowledgement

I would like to thank my advisor Prof. Dr. Igor Burban for his guidance, patience and his contributions to our joint work.

I thank Prof. Dr. Peter Littelmann for refereeing my thesis.

I am grateful to my colleagues Dr. Lennart Galinat, Dr. Andreas Hochenegger and Christian Reinecke for interesting discussions and helpful remarks.

At last, I would like to thank Sebastian Opper and Dr. Bea Schumann who have read large parts of this thesis and improved its readability at numerous points.

This work was supported by the DFG project Bu-1866/2-1.

Summary

In the following we give an extended introduction to this thesis in which we describe our main results in more detail. The description of the structure of this thesis is depicted on page 31.

In the summary below we describe first some Lie-theoretic background behind the main part of this thesis.

Then we present the five results of the short introduction in more detail.

At last, we describe the method unifying both parts and the structure of this thesis.

From Lie groups to quivers

Since the original motivation to study the Gelfand quiver came from the analytic representation theory of Lie groups, we give a brief review of Harish-Chandra's correspondence.

In [**HC53**] Harish-Chandra developed an algebraic theory to study *admissible representations* of any real reductive Lie group G.

The class of admissible representations is big: it contains all irreducible unitary representations and any other "natural" analytic representation of the Lie group G. The algebraic properties of admissible representations can be studied by their *Harish-Chandra modules.*

The category $\mathcal{H}(G)$ of Harish-Chandra modules of the Lie group G is good: it is abelian, has finite global dimension and admits a decomposition $\mathcal{H}(G) = \bigoplus_\chi \mathcal{H}_\chi(G)$ into smaller blocks. Many of these blocks are equivalent, and every block has only finitely many simple objects. Moreover, any Harish-Chandra module (as well as any admissible representation) has finite length.

The analytic and the algebraic category above are related by an exact, full and dense functor

$$\mathrm{H} : \mathrm{admrep}(G) \longrightarrow \mathcal{H}(G).$$

A Harish-Chandra module V can have infinitely many *globalizations*, that is, preimages under the functor H. Nevertheless, any globalization of V has the *same Jordan-Hölder filtrations* as the Harish-Chandra module V.

Moreover, there are several analytic subcategories of the category of admissible representations of G which are even *equivalent* to the category $\mathcal{H}(G)$ of Harish-Chandra modules:

(1) The category of *smooth Fréchet globalizations* by Casselman and Wallach [**Cas89, Wal83**].

(2) The category of *minimal* and the category of *maximal globalizations* by Kashiwara and Schmid [**KS94**].

The blocks of Harish-Chandra modules have a further convenient description. By a theorem of Bernstein, Braverman and Gaitsgory [**BBG97**], any block $\mathcal{H}_\chi(G)$ of

Harish-Chandra modules is equivalent to the category of finite-dimensional modules over some ring $\Lambda_\chi = \Lambda_\chi(G)$:

$$\mathcal{H}_\chi(G) \xrightarrow{\sim} \Lambda_\chi \text{ - fd. mod} \qquad \text{with} \qquad \text{center}(\Lambda_\chi) \cong \mathbb{C}[\![x_1, \dots, x_d]\!].$$

More precisely, the ring Λ_χ is an *order*. The Krull dimension d of its center is given by the *real rank* of the Lie group G.

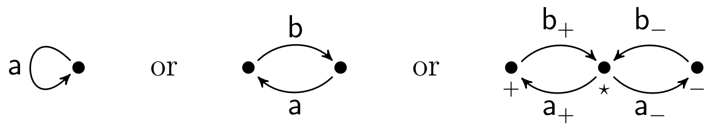
Let us assume that the block $\mathcal{H}_\chi(G)$ of Harish-Chandra modules has *tame* representation type. In this case, the real Lie group G has rank *one* and the order Λ_χ must be *nodal*. Moreover, there is some quiver $(Q, I)_\chi$ with oriented cycles and an equivalence of the categories

$$\Lambda_\chi \text{ - fd. mod} \xrightarrow{\sim} \text{nil. rep}(Q, I)_\chi,$$

where $\text{nil. rep}(Q, I)_\chi$ is the category of nilpotent representations of the quiver $(Q, I)_\chi$. More precisely , the order Λ_χ is the arrow ideal completion of the path algebra of the quiver $(Q, I)_\chi$.

In general, it is difficult to determine the ring Λ_χ or the quiver $(Q, I)_\chi$ of a block. The quivers are known for all blocks $\mathcal{H}_\chi(G)$ of Harish-Chandra modules for the following Lie groups:

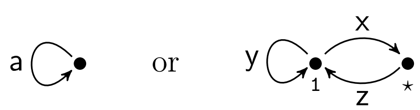
- For $G = \mathrm{SL}(2, \mathbb{R})$ the quivers $(Q, I)_\chi$ are given by the one-cycle, the two-cycle or the Gelfand quiver:



$\mathsf{b}_+ \, \mathsf{a}_+ = \mathsf{b}_- \, \mathsf{a}_-.$

  The representation theory of the first two quivers is completely understood.

- For the Lorentz group $G = \mathrm{SO}_e(3, 1)$, the quivers $(Q, I)_\chi$ are given by the one-cycle quiver or the Zhelobenko quiver:



$\mathsf{x}\,\mathsf{y} = \mathsf{y}\,\mathsf{z} = 0.$

In the cases above, the most interesting block is given by the block $\mathcal{H}_0(G)$ containing the trivial Harish-Chandra module $\mathbb{C}$, which corresponds to the simple quiver representation $S_\star$ at vertex $\star$.

It holds that the principal block $\mathcal{H}_0(G)$ has *maximal* global dimension among all blocks $\mathcal{H}_\chi(G)$ of Harish-Chandra modules for any real Lie group G in general. An explicit description of the principal block $\mathcal{H}_0(G)$ is known for the following real Lie groups:

- For the Lie groups $G = \mathrm{SO}_e(n, 1), \mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ the quiver of principal blocks were determined by Khoroshkin.

- There is a description of the order Λ of the principal block for $G = \mathrm{SL}(3, \mathbb{R})$ due to Bernstein, I. Gelfand and S. Gelfand [**BG78**].

The Gelfand quiver appears as a special case of Khoroshkin quivers because of the following equivalence of blocks and isomorphism of Lie groups:

$$\mathcal{H}_0(\mathrm{SL}(2,\mathbb{R})) \xrightarrow{\sim} \mathcal{H}_0(\mathrm{PSL}(2,\mathbb{R})) \quad \text{and} \quad \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_e(2,1).$$

A similar statement holds for $G = \mathrm{SL}(2,\mathbb{C})$ and the Lorentz group $\mathrm{SO}_e(3,1)$.

The quivers and orders above seem to be the only explicit descriptions of blocks of Harish-Chandra modules over a real Lie group.

Summarized, the relationship between the analytic representation theory of one of the aforementioned Lie groups G (for example $G = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SO}_e(n,1)$) and the quiver of its principal block is given by the following diagram of categories and functors:

$$\begin{tikzcd} \mathrm{admrep}(G) \arrow[r, "\mathrm{H}", "\text{dense}"'] & \mathcal{H}(G) = \bigoplus_\chi \mathcal{H}_\chi(G) & & \\ & \mathcal{H}_0(G) \arrow[u, hook] \arrow[r, "\sim"] & \Lambda_0\text{ - fd. mod} \arrow[r, "\sim"] & \text{nil. rep}(Q,I)_0. \end{tikzcd}$$

Nilpotent representations of the Gelfand quiver

In Chapter 5 of this thesis we study the nilpotent representations of the Gelfand quiver. For simplicity, we set $\Bbbk = \mathbb{C}$ in the following, but the statements below can be adapted to any base field $\Bbbk$.

A nilpotent representation of the Gelfand quiver is given by finite-dimensional vector spaces and matrices of the form

$$\begin{tikzcd} \mathbb{C}^{\mathtt{n}_+} \arrow[r, bend left, "B_+"] & \mathbb{C}^{\mathtt{n}_\star} \arrow[l, bend left, "A_+"] \arrow[r, bend right, "A_-"'] & \mathbb{C}^{\mathtt{n}_-} \arrow[l, bend right, "B_-"'] \end{tikzcd} \quad \text{where} \quad \begin{array}{l} \mathtt{n}_+,\ \mathtt{n}_\star,\ \mathtt{n}_- \ \in \mathbb{N}_0, \\ B_+A_+ = B_-A_- \quad \text{and} \\ (B_+A_+)^m = 0 \quad \text{for some } m \in \mathbb{N}^+ . \end{array}$$

In the following we describe our results on the abelian category of the Gelfand quiver.

1. The indecomposable nilpotent representations of the Gelfand quiver are given by two types: *strings* and *bands*. Moreover, there are three classes of strings: *usual, special* and *bispecial.*

To exemplify the combinatorial flavor of strings and bands, let us give a rough form of their definitions:

(1) A *band* (ω, m, λ) is given by some sequence

$$\omega = \big(\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_{2k-1}, \mathtt{n}_{2k}\big) \quad \text{of natural numbers } \mathtt{n}_1, \mathtt{n}_2 \ldots \mathtt{n}_{2k} \in \mathbb{N}^+,$$

a “multiplicity” $m \in \mathbb{N}^+$ and an “eigenvalue” $\lambda \in \mathbb{C}\backslash\Delta$.
The sequence ω satisfies some additional condition and Δ has only one or two elements.

(2) A word ω is given by some sequence

$$\omega = (\alpha,\ \mathtt{n}_1, \mathtt{n}_2,\ \ldots\ \mathtt{n}_{k-1}, \mathtt{n}_k,\ \beta)$$

of natural numbers $\mathtt{n}_1, \ldots, \mathtt{n}_\mathtt{k} \in \mathbb{N}^+$, and two formal ends $\alpha, \beta \in \{\mathsf{x}_\star, \mathsf{x}_\diamond, \mathsf{y}_\star, \mathsf{y}_\diamond, \mathsf{z}_\star\}$. An end of type $\mathsf{x}_\diamond$ or $\mathsf{y}_\diamond$ is called *special*.

Any *string* has a word with two ends as main datum:

(a) A *usual string* is given by some word ω without special ends.

(b) A *special string* (ω, ε) is given by any word ω with exactly one special end and one sign $\varepsilon \in \{ +, - \}$.

(c) A *bispecial string* $(\omega, m, \varepsilon_1, \varepsilon_2)$ is given by some word ω with two special ends, a multiplicity $m \in \mathbb{N}^+$ and two signs $\varepsilon_1, \varepsilon_2 \in \{ +, - \}$.

Words of usual and bispecial strings have to be asymmetric in a certain sense.

The main point is that strings are defined in terms of *discrete* parameters while bands contain also a *continuous* parameter.

There is a notion when two strings or bands are *equivalent*. The definitions of strings and bands are motivated by the following result:

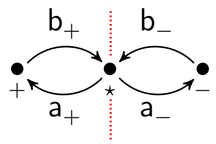
Theorem 1 (Theorem 5.1.8)**.** *There is a bijection between the equivalence classes of strings and bands and the isomorphism classes of indecomposable nilpotent representations of the Gelfand quiver*:

$$\big[\ \text{STRINGS } \textit{and} \text{ BANDS}\ \big] \xleftrightarrow{1:1} \operatorname{ind}\big[\,\mathrm{nil.\,rep}(Q, I)\,\big].$$

In other words, strings and bands parametrize the indecomposable representations of the Gelfand quiver.

2. We compute basic *homological invariants* and describe the *functorial properties* of indecomposable representations of the Gelfand quiver.

We recall that the Gelfand quiver has a special symmetry:



$$\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-.$$

The completed path algebra Λ of the Gelfand quiver has global dimension *two*.

We study the following three Lie-theoretic notions for any nilpotent representation V of the Gelfand quiver:

- Let $S_\star$ be the simple module at vertex $\star$ of the Gelfand quiver. The *defect* of V is given by

$$\delta(V) := \sum_{j=0}^{2} \dim \operatorname{Ext}^j(V, S_\star) = \sum_{j=0}^{2} \dim \mathbf{H}^j_{\mathfrak{g},K}(V).$$

  If V is considered as a Harish-Chandra module, the defect is equal to the total dimension of its *relative Lie algebra cohomology*.

- The involution $\sigma(V)$ and contragredient dual $\mathbb{L}(V)$ of V are given as follows:

V	$\sigma(V)$	$\mathbb{L}(V)$
$\begin{tikzcd} \mathbb{C}^{\mathtt{n}_+} \arrow[r, bend left, "B_+"] & \mathbb{C}^{\mathtt{n}_\star} \arrow[l, bend left, "A_+"] \arrow[r, bend right, "A_-"'] & \mathbb{C}^{\mathtt{n}_-} \arrow[l, bend right, "B_-"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{C}^{\mathtt{n}_-} \arrow[r, bend left, "B_-"] & \mathbb{C}^{\mathtt{n}_\star} \arrow[l, bend left, "A_-"] \arrow[r, bend right, "A_+"'] & \mathbb{C}^{\mathtt{n}_+} \arrow[l, bend right, "B_+"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{C}^{\mathtt{n}_-} \arrow[r, bend left, "A_-^T"] & \mathbb{C}^{\mathtt{n}_\star} \arrow[l, bend left, "B_-^T"] \arrow[r, bend right, "B_+^T"'] & \mathbb{C}^{\mathtt{n}_+} \arrow[l, bend right, "A_+^T"'] \end{tikzcd}$

  In other words, the representation $\sigma(V)$ is given by a vertical flip of V, while the dual $\mathbb{L}(V)$ is given by the flip and transposition of matrices.

  The involution σ is induced by the complex conjugation on the Lie group $G = \mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$. The functor $\mathbb{L}$ corresponds to *contragredient duality of admissible Hilbert representations* of G.

- Let $\underline{\dim}(V) = (\mathtt{n}_+, \mathtt{n}_\star, \mathtt{n}_-)$ denote the dimension vector of V. Then the Euler characteristic of the representation V can be described by

$$\chi(V) := \sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}^i(V,V) = (\mathtt{n}_\star - \mathtt{n}_+)^2 + (\mathtt{n}_\star - \mathtt{n}_-)^2.$$

  The Euler characteristic gives a rough description of the Jordan-Hölder-multiplicities of a Harish-Chandra module.

The second main result on the representation theory of the Gelfand quiver is the following statement:

Theorem 2 (Theorems 5.2.7 and 5.3.6)**.** *Let Ω be a string or band, and let $V = V(\Omega)$ be the corresponding representation of the Gelfand quiver.*

(1) We describe the involution $\sigma(V)$ and the contragredient dual $\mathbb{L}(V)$ in terms of strings and bands.

(2) We compute the defect $\delta(V)$, the projective and injective dimension of V, the dimension vector $\underline{\dim}(V)$, the Euler characteristic $\chi(V)$, the top and the socle of the indecomposable representation V.

More precisely, there are formulas for the homological invariants above in terms of string and band parameters. These formulas are quite simple for the case of bands:

Example 1. *Let $\Omega = (\omega, \mathtt{m}, \lambda)$ be some band. In particular, ω is given by some sequence $(\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_{2k-1}, \mathtt{n}_{2k})$ of $2\mathtt{k}$ natural numbers, $\mathtt{m} \in \mathbb{N}^+$ and $\lambda \in \mathbb{C}\backslash\Delta$. Let $V = V(\Omega)$ be the corresponding band representation of the Gelfand quiver.*

(1) Let us set $\mathtt{n} = \sum_{\mathtt{i}=1}^{2\mathtt{k}} \mathtt{n}_\mathtt{i}$. It holds that

$$\underline{\dim}(V) = (\mathtt{m}\,\mathtt{n},\ \mathtt{m}\,\mathtt{n},\ \mathtt{m}\,\mathtt{n}) \quad \textit{and} \quad \underline{\dim}(\operatorname{top} V) = \underline{\dim}(\operatorname{soc} V) = (\mathtt{m}\,\mathtt{k},\ \mathtt{0},\ \mathtt{m}\,\mathtt{k}).$$

(2) The contragredient dual $\mathbb{L}(V)$ is isomorphic to the band representation of the band $\mathbb{L}(\Omega) = (\mathbb{L}(\omega), \mathtt{m}, \lambda)$ with $\mathbb{L}(\omega) = (\mathtt{n}_{2k}, \mathtt{n}_{2k-1}, \ldots, \mathtt{n}_2, \mathtt{n}_1)$. In particular, $\mathbb{L}(V) \cong V$ if and only if ω is symmetric.

In addition, it turns out that all possible dimension vectors $\underline{\dim}(V)$ of indecomposable representations V of the Gelfand quiver are *completely described* by the possible values of the Euler characteristic $\chi(V)$.

The following Table summarizes these values together with the defect and the involution for each of the four classes:

string or band	Ω	$\sigma(\Omega)$	$\sigma(V)$	$\delta(V)$	$\chi(V)$
usual string	ω	ω	$\sigma(V) \cong V$	2	0 or 2
special string	(ω, ε_1)	$(\omega, \overline{\varepsilon}_1)$	$\sigma(V) \ncong V$	1	1
bispecial string	$(\omega, m, \varepsilon_1, \varepsilon_2)$	$(\omega, m, \overline{\varepsilon}_1, \overline{\varepsilon}_2)$	$\sigma(V) \ncong V$	0	0 or 2
band	(ω, m, λ)	(ω, m, λ)	$\sigma(V) \cong V$	0	0
		where $\overline{\varepsilon} = \mp$ if $\varepsilon = \pm$			

(*)

These results have the following consequences:

(1) The four combinatorial classes of strings and bands can be characterized by the Lie-theoretic notions of defect δ and the involution σ.

(2) We can determine all possible values for *Jordan-Hölder multiplicities* of indecomposable admissible representations over $\mathrm{SL}(2, \mathbb{R})$.

More precisely, these values correspond to the possible dimension vectors of indecomposable representations of the one-cycle, the two-cycle or the Gelfand quiver.

3. At last, we obtain the following answer to Gelfand's original problem:

Theorem 3 (Theorems 5.1.11 and 5.4.7)**.** *Let Ω be a string or band and $V = V(\Omega)$ be the corresponding representation. We give an explicit description of the projective resolution of V and the quiver representation of V.*

In the following we give some examples of the theorem above in terms of the Gelfand order. We recall that the Gelfand order Λ is the order $\Lambda = \Lambda_0(G)$ associated to the principal block $\mathcal{H}_0(G)$ of Harish-Chandra modules over the Lie group $G = \mathrm{SL}(2, \mathbb{R})$.

The Gelfand order Λ has the following *normalization* Γ:

$$\begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \Lambda = \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \hookrightarrow \begin{array}{c} \begin{array}{ccc} \widetilde{P}_\star & \widetilde{P}_\diamond & \widetilde{P}_\diamond \end{array} \\ \Gamma = \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \end{bmatrix} \end{array} \qquad \begin{array}{l} \mathbf{R} = \mathbb{C}[\![x]\!], \\ \mathfrak{m} = (x). \end{array}$$

The projective Λ-modules are given by the columns of their matrix algebras. Their morphisms are described by embeddings ι and multiplications $\cdot x^n$, where $n \in \mathbb{N}^+$. The same is true for the projective modules over the order Γ.

Since Γ is a hereditary order, the category $\mathrm{D}^{\mathrm{b}}(\Gamma)$ has *discrete* representation type and is completely understood.

Roughly speaking, any indecomposable projective resolution of the tame derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ can be *glued* from some projective complexes of the category $\mathrm{D}^{\mathrm{b}}(\Gamma)$.

Let us consider two gluing examples:

Example 2. *Let Ω be the special string $(\omega, +)$ with $\omega = (\mathsf{x}_\star\, \mathsf{1}, \mathsf{1}\ \mathsf{x}_\diamond)$. The special string Ω represents a gluing diagram with a complex $\widetilde{P}_\bullet$ from $\mathrm{D}^{\mathrm{b}}(\Gamma)$:*

gluing diagram of Ω *projective resolution $P_\bullet = P_\bullet(\Omega)$*

$$\begin{tikzcd} \widetilde{P}_\diamond \arrow[r, "\iota"] \arrow[d, dashed, no head] & \widetilde{P}_\star \\ \widetilde{P}_\diamond \arrow[r, "\cdot x"] & \underset{+}{\widetilde{P}_\diamond} \end{tikzcd} \quad \overset{gluing}{\Rightarrow} \quad \begin{tikzcd} P_+ \arrow[r, "\iota"] \arrow[d, dashed] & P_\star \\ P_- \arrow[r, "\cdot x"] \arrow[ur, "\iota"] & P_+ \end{tikzcd} \quad = \quad P_+ \oplus P_- \xrightarrow{\left[\begin{smallmatrix} \iota & \iota \\ 0 & \cdot x \end{smallmatrix}\right]} P_\star \oplus P_+.$$

The application of certain gluing rules *to the gluing diagram yields the projective resolution $P_\bullet$ associated to the string Ω. By computing the homology of $P_\bullet$ we obtain the quiver representation V of the string Ω :*

$$V = \mathbf{H}_0(P_\bullet) \cong \begin{tikzcd} \mathbb{C}^2 \arrow[r, bend left, "\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]"] & \mathbb{C}^2 \arrow[l, bend left, "\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]"] \arrow[r, bend right, "\left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]"'] & \mathbb{C}. \arrow[l, bend right, "\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]"'] \end{tikzcd}$$

Example 3. *Let Ω be the band (ω, m, λ) where $\omega = (\mathsf{1}, \mathsf{1})$, $m = 1$ and $\lambda \in \mathbb{C}\backslash\{0, 1\}$. In this case, the gluing diagram of Ω is given by a "closed" diagram:*

gluing diagram of Ω *projective resolution $P_\bullet = P_\bullet(\Omega)$*

$$\begin{tikzcd} \overset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "\cdot x"] \arrow[d, dashed, no head] & \widetilde{P}_\diamond \arrow[d, dashed, no head] \\ \widetilde{P}_\diamond \arrow[r, "\cdot x"] & \widetilde{P}_\diamond \end{tikzcd} \quad \overset{gluing}{\Rightarrow} \quad \begin{tikzcd} P_- \arrow[r, "\cdot(1-\lambda)x"] \arrow[dr, "\cdot x"] & P_+ \arrow[d, dashed] \\ P_+ \arrow[u, dashed] \arrow[r, "\cdot x"'] \arrow[ur, "\cdot x"] & P_- \end{tikzcd} \quad = P_- \oplus P_+ \xrightarrow{\left[\begin{smallmatrix} \cdot(1-\lambda)\,x & \cdot x \\ \cdot x & \cdot x \end{smallmatrix}\right]} P_+ \oplus P_-$$

The quiver representation V of the band Ω is given by the homology of $P_\bullet$:

$$V = \mathbf{H}_0(P_\bullet) \cong \begin{tikzcd} \mathbb{C}^2 \arrow[r, bend left, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]"] & \mathbb{C}^2 \arrow[l, bend left, "\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]"] \arrow[r, bend right, "\left[\begin{smallmatrix} 1 & \lambda^* \\ 0 & 0 \end{smallmatrix}\right]"'] & \mathbb{C}^2, \arrow[l, bend right, "\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]"'] \end{tikzcd} \quad \textit{where} \ \ \lambda^* = (1-\lambda)^{-1}.$$

The representation above has smallest possible dimension among all band representations of the Gelfand quiver. Moreover, the representation V is self-dual, that is, $\mathbb{L}(V) \cong V$.

Let us note that the above construction of projective resolutions translates into *gluing rules* for cyclic Λ-modules, or *cyclic representations* of the Gelfand quiver. The last formulation does not use notions from the language of derived categories.

Combinatorics of the derived category of the Gelfand order

In Chapter 4 we study the bounded derived category $\mathrm{D^b}(\Lambda)$ of finitely generated modules of the Gelfand order Λ by combinatorial methods.

The category $\mathrm{D^b}(\Lambda)$ is bigger than the derived category $\mathrm{D^b}(\mathcal{H}_0)$ of the principal block $\mathcal{H}_0 = \mathcal{H}_0(\mathrm{SL}(2,\mathbb{R}))$. More precisely, the category $\mathrm{D^b}(\mathcal{H}_0)$ can be identified with the full subcategory $\mathrm{D^b_{fd}}(\Lambda)$ of $\mathrm{D^b}(\Lambda)$ given by complexes with finite-dimensional homology:

$$\mathrm{nil.\,rep}(Q,I) \xrightarrow{\sim} \mathcal{H}_0 \hookrightarrow \mathrm{D^b}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D^b_{fd}}(\Lambda) \hookrightarrow \mathrm{D^b}(\Lambda).$$

Many properties of the abelian category $\mathrm{nil.\,rep}(Q,I)$ of the Gelfand quiver are also valid in the derived category $\mathrm{D^b}(\Lambda)$ of the Gelfand order:

- The indecomposable objects of $\mathrm{D^b}(\Lambda)$ are given by four combinatorial classes of strings and bands.
- There is a natural generalization of the involution functor σ and the defect δ to the derived setup.

However, the derived category $\mathrm{D^b}(\Lambda)$ has some interesting additional structure: since the order Λ has *finite* global dimension, results of van den Bergh [**vdB04**] and Iyama-Reiten [**IR08**] imply that the category $\mathrm{D^b}(\Lambda)$ has an *Auslander-Reiten translation*. More precisely, there is an autoequivalence τ of $\mathrm{D^b}(\Lambda)$ such that

$$\mathrm{Hom}(X_\bullet, Y_\bullet) \cong \mathbb{D}\,\mathrm{Ext}^1(Y_\bullet, \tau(X_\bullet)) \qquad \text{for any} \quad X_\bullet, Y_\bullet \in \mathrm{D^b}(\mathcal{H}_0).$$

This means that the main structure of the category $\mathrm{D^b}(\mathcal{H}_0)$ is described by its *Auslander-Reiten quiver*.

In more detail, the Auslander-Reiten translation $\tau : \mathrm{D^b}(\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\Lambda)$ acts on the projective modules and their morphisms as follows:

$$\Lambda = \begin{array}{c} \begin{matrix} P_\star & P_- & P_+ \end{matrix} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \qquad \begin{matrix} P_\star \xrightarrow{\cdot x} P_\pm & \xmapsto{\tau} & I_\star \xrightarrow{\cdot x} P_\mp \\ P_\pm \xrightarrow{\iota} P_\star & \xmapsto{\tau} & P_\mp \xrightarrow{\iota} I_\star \end{matrix} \qquad \begin{array}{c} I_\star \\ \begin{bmatrix} \mathfrak{m} \\ \mathbf{R} \\ \mathbf{R} \end{bmatrix} \end{array}$$

where the module $I_\star$ is the radical of the projective module $P_\star$.

The main result of Chapter 4 is given as follows:

Theorem 4 (Theorem 4.3.10). *We describe the action of the Auslander-Reiten translation τ on strings and bands of the derived category $\mathrm{D^b}(\Lambda)$ of the Gelfand order Λ.*

The proof uses the explicit description of indecomposable complexes in the derived category $\mathrm{D^b}(\Lambda)$ by Burban and Drozd [**BD04**]. Given a string complex $P_\bullet$, we show that the complex $\tau(P_\bullet)$ is given essentially by the gluing diagram of another string.

Let us consider a basic example:

Example 4. *Let* ω *be the usual string* $(\mathtt{x}_\star\ \mathtt{n}\ \mathtt{y}_\star)$ *for some* $\mathtt{n} \in \mathbb{N}^+$*. By Theorem 4.3.10 it holds that* $\tau(\omega) = (\mathtt{y}_\star\ \mathtt{1}\ \mathtt{n}\ \mathtt{z}_\star)$*. The complexes of these strings are given as follows*:

$P_\bullet(\omega)$:

$$P_\star \xrightarrow{\cdot x^n} P_\star$$

$\tau(P_\bullet(\omega))$:

$$I_\star \xrightarrow{\cdot x^n} I_\star$$

$P_\bullet(\tau(\omega))$:

$$\begin{tikzcd}
P_\star \arrow[r, "\cdot x"] \arrow[dr, "\cdot(-x)"'] & P_+ \arrow[dr, "\cdot x^n"] & \\
& P_- \arrow[u, dashed] \arrow[r, "\cdot x^n"] & P_+ \arrow[d, dashed] \\
& P_\star \arrow[ur, "\cdot(-x)"] \arrow[r, "\cdot x"] & P_-
\end{tikzcd}$$

It can be checked, that there is indeed an isomorphism $\tau(P_\bullet(\omega)) \cong P_\bullet(\tau(\omega))$*.*

In other terms, the Auslander-Reiten translation has a convenient description in terms of gluing diagrams. This shows that the combinatorics of strings and bands is suitable to study their functorial properties.

In the remaining part of Chapter 4, we study *autoequivalences* and a natural subcategory of the derived category $\mathrm{D^b}(\Lambda)$.

A natural source of autoequivalences is given by *twist functors* associated to spherical objects or spherical collections. It is not hard to check that the simple module $S_\star$ is spherical.

As the first application of Theorem 4.3.10 we show the following statement:

Corollary 1 (Corollary 4.3.26)**.** *The spherical object* $S_\star$ *is the only* generalized spherical collection *in* $\mathrm{D^b}(\Lambda)$*. In other words, the twist functor* $\mathbb{T}_{S_\star}$ *is the* only *generalized spherical autoequivalence of* $\mathrm{D^b}(\Lambda)$*.*

As the second application of Theorem 4.3.10 we study the thick subcategory $\langle S_\star \rangle$ of the category $\mathrm{D^b}(\Lambda)$ generated by the 2-spherical object $S_\star$:

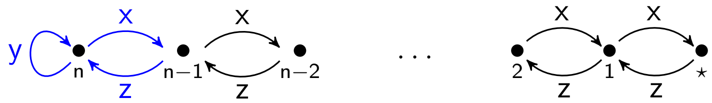
Corollary 2 (Corollary 4.3.29)**.** *We give an explicit description of all indecomposable objects in the subcategory* $\langle S_\star \rangle$*. For a complex* $P_\bullet \in \mathrm{D^b}(\Lambda)$ *it holds that* $P_\bullet \in \langle S_\star \rangle$ *if and only if* $\tau(P_\bullet) \cong P_\bullet[1]$*.*

In different terms, the category $\langle S_\star \rangle$ is given exactly by the 2-Calabi-Yau objects of the derived category $\mathrm{D^b}(\Lambda)$.

Derived Auslander-Reiten theory of Khoroshkin orders

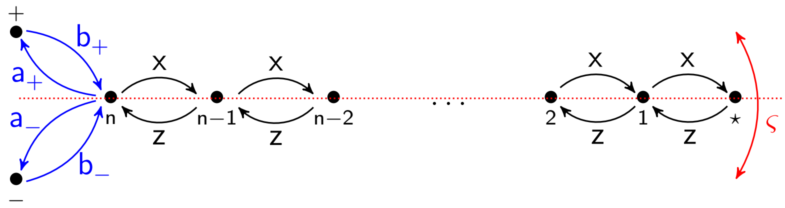
Some of the results obtained on the derived category of the Gelfand order hold in bigger generality for any Khoroshkin order.

We recall that the Khoroshkin quivers of the odd series were given by the gentle quivers

$(Q, I)_{2n+1} =$ 

with relations $\mathsf{x}\,\mathsf{y} = \mathsf{y}\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0.$

In contrast to the case above any Khoroshkin quiver of the even series has a distinguished symmetry:

$(Q, I)_{2n+2} =$ 

with relations $\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$, $\mathsf{x}\,\mathsf{b}_\pm = \mathsf{a}_\pm\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0.$

These quivers are *skew-gentle*: identifying the two special vertices $+$ and $-$ would result in a gentle quiver.

For any $n \in \mathbb{N}^+$ the Khoroshkin quiver $(Q, I)_n$ describes the principal block $\mathcal{H}_0$ of the connected Lorentz group $G_n = \mathrm{SO}_e(n, 1)$:

$$\mathcal{H}_0 = \mathcal{H}_0(G_n) \xrightarrow{\ \sim\ } \mathrm{nil.\,rep}_{\Bbbk}(Q, I)_n \xrightarrow{\ \sim\ } \Lambda_n\text{ - fd. mod}$$

Moreover, the completed path algebra Λ_n of a Khoroshkin quiver is an order over $\mathbf{R}$ of global dimension n.

The main notions for the Gelfand quiver have a natural generalization for the Khoroshkin quivers:

(1) There is a natural involution $\sigma : \mathrm{D^b}(\mathcal{H}_0) \xrightarrow{\ \sim\ } \mathrm{D^b}(\mathcal{H}_0)$ such that $\sigma^2 \cong \mathrm{id}$.

- If n is odd, we set $\sigma = \mathrm{id}$,
- If n is even, σ is induced by the special symmetry ς of the quiver $(Q, I)_n$.

(2) Note that each Khoroshkin quiver $(Q, I)_n$ has a distinguished vertex $\star$. For any complex $P_\bullet$ of $\mathrm{D^b}(\mathcal{H}_0)$ the defect is given by $\delta(P_\bullet) = \sum_{j\in\mathbb{Z}} \dim \mathrm{Ext}^j_{\mathcal{H}_0}(P_\bullet, S_\star)$.

As in the case of the Gelfand quiver, the notions σ and δ have Lie-theoretic interpretations.

(3) The derived category $\mathrm{D^b}(\mathcal{H}_0)$ admits also an Auslander-Reiten translation τ.

In Chapter 2 we prove the following result without using combinatorial methods:

Theorem 5 (Theorem 2.2.3)**.** *As above, let $\mathcal{H}_0 = \mathcal{H}_0(G_n)$ for some $n \geq 2$.*

*(1) The simple module $S_\star$ is an n-*spherical object *in $\mathrm{D^b}(\mathcal{H}_0)$.*

In other terms, the twist functor $\mathbb{T}_{S_\star}$ of $S_\star$ is an autoequivalence *of $\mathrm{D^b}(\mathcal{H}_0)$.*

(2) The Auslander-Reiten translation admits the following factorization:

$$\tau \cong \sigma \circ \mathbb{T}_{S_\star}^{-1} \cong \mathbb{T}_{S_\star}^{-1} \circ \sigma : \quad \mathrm{D^b}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D^b}(\mathcal{H}_0)\,.$$

(3) For any $P_\bullet \in \mathrm{D^b}(\mathcal{H}_0)$ the following conditions are equivalent:

(a) $\delta(P_\bullet) = 0$, (b) $\tau^m(P_\bullet) \cong P_\bullet$ *for some* $m \in \mathbb{N}^+$,

(c) $\tau(P_\bullet) \cong \sigma(P_\bullet)$, (d) $\tau^2(P_\bullet) \cong P_\bullet$,

(e) $\mathbb{T}_{S_\star}^{-1}(P_\bullet) \cong P_\bullet$.

Let us comment on some statements of the theorem above:

- The second statement of the theorem gives an "intrinsic" description of the Auslander-Reiten translation, which can be transferred to the derived category of any analytic category equivalent to the principal block $\mathcal{H}_0$.
- By the third statement, the defect of $P_\bullet$ is *vanishing* if and only if $P_\bullet$ lies in a *regular* component of the Auslander-Reiten quiver of the category $\mathrm{D^b}(\mathcal{H}_0)$. This explains the name "defect": this notion models the classical defect from the representation theory of hereditary algebras.
- Furthermore, the third statement implies that the homogeneous tubes in the Auslander-Reiten quiver of the category $\mathrm{D^b}(\mathcal{H}_0)$ have either rank *one* or *two*.

Let us remark that Theorem 5 is also generalized to the principal block of Harish-Chandra modules over any *disconnected* Lorentz group $\mathrm{SO}(n,1)$ with $n \geq 2$.

Strings and bands of nodal orders

Khoroshkin orders give Lie theoretic examples of nodal orders. These orders were introduced by Drozd in [**Dro91**] and can be viewed as non-commutative generalizations of the nodal singularity $\Bbbk[\![x,y]\!]/(x\,y)$.

From now on let Λ denote *any nodal order*. For simplicity of notation let us also assume that the order Λ has finite global dimension.

In [**BD04**] Burban and Drozd reduced the problem to classify the indecomposable objects in the derived category $\mathrm{D^b}(\Lambda)$ to a tame *matrix problem*. The matrix problem of any nodal order Λ is formalized by a structure $\mathfrak{B} = \mathfrak{B}(\Lambda)$ called a *bunch of semichains*. The canonical forms of any bunch of semichains have been described by Bondarenko in [**Bon88, Bon91**]. His work is the origin of the notions of usual, special and bispecial strings and bands.

Summarized, there is a bijection of isomorphism classes of indecomposable objects in $\mathrm{D^b}(\Lambda)$ and equivalence classes of strings and bands of $\mathfrak{B}$:

$$\mathrm{ind}[\mathrm{D^b}(\Lambda)] \xleftrightarrow{\quad 1:1 \quad} [\ \textsc{strings} \text{ and } \textsc{bands} \text{ of } \mathfrak{B}\ ]. \qquad (**)$$

Moreover, for any string or band Ω the work [**BD04**] describes the construction of the corresponding indecomposable complex $P_\bullet(\Omega)$ in the derived category $\mathrm{D^b}(\Lambda)$.

In Chapter 3 we define the defect δ and the involution σ in the setup of any nodal order Λ. It turns out that the four combinatorial classes of string and band complexes admit the following intrinsic characterization:

Theorem 6 (Theorem 3.6.2)**.** *Let Λ be a nodal order. Let $P_\bullet$ be an indecomposable complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ with finite-dimensional homology. According to bijection (**) there is some usual, special or bispecial string, or band Ω of $\mathfrak{B}$ such that $P_\bullet \cong P_\bullet(\Omega)$. In the setup above, the following statements hold*:

$$\begin{array}{lll} (1) & \Omega \textit{ is a usual string } \omega & \Leftrightarrow \ \delta(P_\bullet) = 2, \\ (2) & \Omega \textit{ is a special string } (\omega, \varepsilon_1) & \Leftrightarrow \ \delta(P_\bullet) = 1, \\ (3) & \Omega \textit{ is a bispecial string } (\omega, m, \varepsilon_1, \varepsilon_2) & \Leftrightarrow \ \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \ncong P_\bullet, \\ (4) & \Omega \textit{ is a band } (\omega, m, \lambda) & \Leftrightarrow \ \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \cong P_\bullet. \end{array}$$

We have seen a special case of this characterization already for nilpotent representations of the Gelfand quiver in Table (*) on page 17.

Theorems 5 and 6 yield the following statement for principal blocks of Lorentz groups:

Corollary 3. *Let G be the Lie group $\mathrm{SO}_e(n,1)$ or $\mathrm{SO}(n,1)$ for some $n \geq 2$. Let $\mathcal{H}_0 = \mathcal{H}_0(G)$ be the principal block of Harish-Chandra modules over G. Let $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ be an indecomposable complex, and let Ω be the string or band such that $P_\bullet \cong P_\bullet(\Omega)$. Then the following statements hold*:

$$\begin{array}{lll} (1) & \Omega \textit{ is a usual string } \omega & \Leftrightarrow \ \delta(P_\bullet) = 2, \\ & & \Leftrightarrow \ P_\bullet \textit{ is not } \tau\textit{-periodic and } \sigma(P_\bullet) \cong P_\bullet. \\ (2) & \Omega \textit{ is a special string } (\omega, \varepsilon_1) & \Leftrightarrow \ \delta(P_\bullet) = 1, \\ & & \Leftrightarrow \ P_\bullet \textit{ is not } \tau\textit{-periodic and } \sigma(P_\bullet) \ncong P_\bullet. \\ (3) & \Omega \textit{ is a bispecial string } (\omega, m, \varepsilon_1, \varepsilon_2) & \Leftrightarrow \ \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \ncong P_\bullet, \\ & & \Leftrightarrow \ \tau^2(P_\bullet) \cong P_\bullet \textit{ and } \tau(P_\bullet) \ncong P_\bullet. \\ (4) & \Omega \textit{ is a band } (\omega, m, \lambda) & \Leftrightarrow \ \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \cong P_\bullet, \\ & & \Leftrightarrow \ \tau(P_\bullet) \cong P_\bullet. \end{array}$$

In particular, the homogeneous tubes of the Auslander-Reiten quiver of $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ of rank one are given by bands, and those of rank two by bispecial strings.

In the case of any connected Lorentz group $\mathrm{SO}_e(n,1)$, the above statement gives a characterization of the four combinatorial classes by purely *Lie-theoretic notions* (the defect δ and the involution σ) or in terms of *functors* (Auslander-Reiten translation τ and involution σ).

At last, we study the problem to classify the indecomposable Λ modules of any nodal order Λ. In analogy to the result of Burban and Drozd, we reduce the classification problem to a matrix problem:

Theorem 7 (Corollary 3.5.19)**.** *Let Λ be any nodal order and $\mathfrak{B}$ be the bunch of semichains of its derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$. We define a bunch of semichains $\mathfrak{B}_0$ by "truncation" of $\mathfrak{B}$. There is a bijection between the isomorphism classes of indecomposable finite-dimensional Λ-modules and equivalence classes of strings and bands of*

$\check{\mathfrak{B}}_0$:

$$\mathrm{ind}[\Lambda\text{ - fd. mod}] \xleftrightarrow{1:1} [\ \text{STRINGS } \textit{and} \text{ BANDS } \textit{of}\ \check{\mathfrak{B}}_0\ ]. \qquad (***)$$

Theorem 1 follows from the Theorem above.

Let us give a few remarks on the bijection above:

- There is a similar result for the category of finitely generated Λ-modules of a nodal order Λ.
- The strings and bands of $\check{\mathfrak{B}}_0$ correspond to *projective presentations* of indecomposable finite-dimensional Λ-modules.
- There are two methods to describe the *projective resolutions* of indecomposable Λ-modules:
  (1) There is some technique to pass from the projective presentations mentioned above to projective resolutions.
  (2) There is an intermediate link in the bijection (**) given by the *category of triples:*

  $$\mathrm{ind}[\mathrm{D^b}(\Lambda)] \xleftrightarrow{1:1} [\mathrm{ind\,Tri}(\Lambda)] \xleftrightarrow{1:1} [\ \text{STRINGS and BANDS of } \mathfrak{B}\ ]. \qquad (0.0.1)$$

  We give also a characterization of triples which correspond to projective resolutions in $\mathrm{D^b}(\Lambda)$.

Both techniques are exemplified on the case of the Gelfand order and yield the combinatorial results at the beginning of this summary.

Let us note that the representation theory of Khoroshkin orders has two levels of complexity:

Lie group G	Khoroshkin quiver	σ	matrix problem $\check{\mathfrak{B}}_0$ or $\mathfrak{B}$	indecomposable objects of $\mathcal{H}_0(G)$ or $\mathrm{D^b}(\mathcal{H}_0(G))$
$\mathrm{SO}(n,1)$ or $\mathrm{SO}_e(2n+1,1)$	gentle	$\sigma = \mathrm{id}$	bunch of chains	bands and usual strings
$\mathrm{SO}_e(2n,1)$	skew-gentle	$\sigma \not\equiv \mathrm{id}$	proper bunch of semichains	bands and usual, special and bispecial strings

For the odd or disconnected series of the Lorentz groups G above, the Khoroshkin quiver is *gentle* and the involution σ is *trivial.* Moreover, the principal block $\mathcal{H}_0(G)$ as well as its derived category $\mathrm{D^b}(\mathcal{H}_0(G))$ have *only two* classes of indecomposable objects. This has to do with the fact that the corresponding matrix problems are *simpler* than in the case of the even series of connected Lorentz groups above.

We will see a similar pattern for the classification problems of the next subsection.

Cohen-Macaulay modules over curve singularities of type P

In Chapter 1 of this thesis we apply the technique of matrix problems to solve some classification problems from commutative algebra.

1. Our guiding example is the curve singularity $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(x\,y, z^2)$, where $\Bbbk$ is an algebraically closed field of arbitrary characteristic. The ring P is not reduced, that is, P has non-trivial nilpotent elements.

Cohen-Macaulay modules of P can be described as submodules of free modules:

$$\mathrm{CM}(\mathsf{P}) = \{\ L \in \mathsf{P}\text{ - mod} \mid \iota : L \hookrightarrow \mathsf{P}^n \text{ for some } n \in \mathbb{N}^+\ \}.$$

The same description of Cohen-Macaulay is true for all other curve singularities of this subsection. Simple examples of Cohen-Macaulay modules are given by ideals.

The curve singularity P has tame Cohen-Macaulay type. The main combinatorial result of Chapter 1 is given by the following statement:

Theorem 8 (Theorem 1.2.1)**.** *We give an explicit classification of the indecomposable Cohen-Macaulay modules over* P*.*

The classification method is very similar to the case of the Gelfand quiver:

- The curve singularity P has a normalization S which is given by a product of curve singularities of type $\mathbb{A}_\infty$:

  $$\mathsf{P} = \Bbbk[\![x, y, z]\!]/(x\,y, z^2) \hookrightarrow \mathsf{S} = \Bbbk[\![x, u]\!]/(u^2) \times \Bbbk[\![y, v]\!]/(v^2).$$

  The normalization S has *discrete* Cohen-Macaulay type.
- Informally speaking, the indecomposable Cohen-Macaulay modules over P can be *glued* from Cohen-Macaulay modules of the normalization S.
- In this context, there is a *category of triples* construction due to Burban and Drozd [**BD**]. It allows to reduce the classification problem of Cohen-Macaulay modules over P to a matrix problem $\mathfrak{B}$.

In contrast to the case of the Gelfand quiver, the matrix problem $\mathfrak{B}$ associated to the category $\mathrm{CM}(\mathsf{P})$ has type of a *bunch of chains.* This means that there are only two types of indecomposable Cohen-Macaulay P-modules: *usual strings* and *bands.*

In particular, the gluing rules for the indecomposable modules are much simpler than in the case of the Gelfand quiver. Let us consider two simple examples:

Example 5. *Let $i, j \in \mathbb{N}^+$. We consider two ideals $I_1 = (x^i, u)$ and $I_2 = (y^j, v)$ of the normalization* S *introduced above.*

(1) The ideals I_1 and I_2 can be glued into the string ideal (x^i, z, y^j) *of the ring* P.

(2) For any $\lambda \in \Bbbk^*$ the ideals I_1 and I_2 can also be glued into the band ideal $(x^i + \lambda\, y^j, z)$ of the ring P*.*

2. The above curve singularity P belongs to a bigger family:

$$\mathsf{P}_{pq} = \Bbbk[\![x, y, z]\!]/(x\,y, x^p + y^q - z^2), \qquad \text{where } p, q \in \mathbb{N}^+ \cup \{\infty\}.$$

If p or q is infinite, we set $x^\infty := 0$ respectively $y^\infty := 0$ above. In any case it holds that $\mathsf{P}_{pq} \cong \mathsf{P}_{qp}$.

The following result complements the tameness result by Drozd and Greuel [**DG92**] for the reduced curve singularities P_{pq}, where p and q are finite.

Theorem 9 (Theorem 1.3.16)**.** *The non-reduced curve singularity* $\mathsf{P}_{\infty q}$ *has tame Cohen-Macaulay representation type for any* $q \geq 2$*.*

As the Khoroshkin orders, curve singularities of type P can be divided into two families:

curve singularity A	matrix problem $\mathfrak{B}$	indecomposable objects of $\mathrm{CM}(A)$
$\mathsf{P}_{\infty\infty}$, $\mathsf{P}_{\infty,2q+1}$ or $\mathsf{P}_{2p+1,2q+1}$	bunch of chains	bands and usual strings
$\mathsf{P}_{\infty,2q}$ or $\mathsf{P}_{2p,2q}$	proper bunch of semichains	bands and usual, special and bispecial strings

3. The classification result for the curve singularity $\mathsf{P}_{\infty\infty}$ yields some additional results:

Corollary 4 (Remarks 1.3.21 and 1.3.23)**.** *(1) We obtain a complete description of the indecomposable Cohen-Macaulay modules over any curve singularity* $\mathsf{P}_{\infty\infty}$*,* $\mathsf{P}_{\infty,2q+1}$ *or* $\mathsf{P}_{2p+1,2q+1}$*, where* $p, q \in \mathbb{N}^+$*.*

(2) We obtain a concrete, but partial classification of the indecomposable Cohen-Macaulay modules of the non-reduced curve singularity $\mathsf{T}_{\infty\infty} = \Bbbk[\![a, b]\!]/(a^2\, b^2)$*.*

More precisely, there are the following injections on isomorphism classes of indecomposable objects of the following categories:

$$\mathrm{ind}[\mathrm{CM}(\mathsf{P}_{2p+1,2q+1})] \hookrightarrow \mathrm{ind}[\mathrm{CM}(\mathsf{P}_{\infty\infty})] \hookrightarrow \mathrm{ind}[\mathrm{CM}(\mathsf{T}_{\infty\infty})], \qquad p, q \in \mathbb{N}^+ \cup \{\infty\}.$$

Cohen-Macaulay modules over $\mathsf{T}_{\infty\infty}$ can be viewed also as matrix factorizations of the potential $a^2\, b^2 \in \Bbbk[\![a, b]\!]$.

Let us consider some examples of these translations:

Example 6. *Let* $i, j \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^*$*.*

(1) The ideal (x^i, z, y^j) *of* $\mathsf{P}_{\infty\infty}$ *is also an indecomposable ideal of* $\mathsf{P}_{2p+1,2q+1}$ *for any* $p \geq i$ *and* $q \geq j$*. The same is true for the band ideal* $(x^i + \lambda\, y^j, z)$ *of* $\mathsf{P}_{\infty\infty}$*.*

(2) The band ideal $(x^i + \lambda\, y^j, z)$ *translates into the ideal* $I = (a^{i+2} + \lambda\, b^{j+2}, a^2\, b + a\, b^2)$ *of* $\mathsf{T}_{\infty\infty}$*. The ideal* I *corresponds to the following matrix factorization* (ϕ, ψ) *of the potential* $a^2\, b^2$ *:*

$$\phi = \begin{bmatrix} a\, b & 0 \\ -\lambda\, b^{j+1} - a^{i+1} & a\, b \end{bmatrix} \qquad \psi = \begin{bmatrix} a\, b & 0 \\ \lambda\, b^{j+1} + a^{i+1} & a\, b \end{bmatrix}.$$

In other words, $\phi \cdot \psi = \psi \cdot \phi = \mathrm{diag}(a^2\, b^2)$.

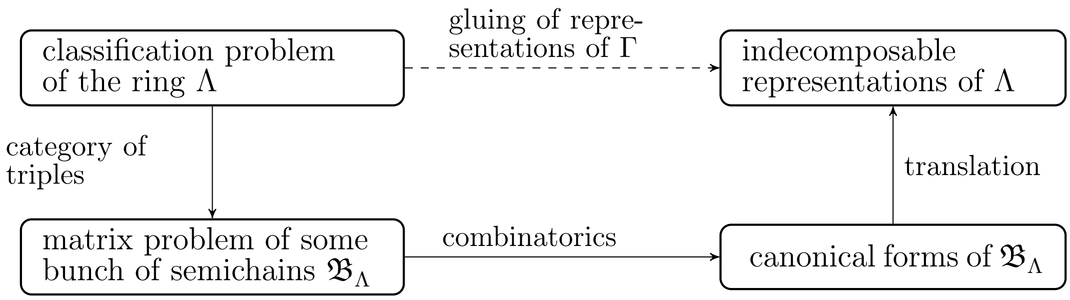
Matrix problems and the gluing method for one-dimensional orders

Next, we explain the main method to obtain classification results in both parts of this thesis.

In both parts, we encounter the following situation:
We want to classify the indecomposable representations of some ring Λ which embeds naturally into a ring Γ. The representation theory of the ring Γ is well-known. Informally speaking, we obtain a method *how to glue* any indecomposable representation of the ring Λ from several representations of its “normalization” Γ:

$$\Lambda \hookrightarrow \Gamma \qquad \operatorname{ind}[\operatorname{Rep}(\Lambda)] \xleftarrow{\text{gluing}} \operatorname{Rep}(\Gamma)$$

We derive gluing rules for representations of Λ in three steps which are shown in the following diagram:



(1) First, we reduce the classification problem of Λ to a matrix problem. The main tool in this step is a categorical construction called *category of triples* associated to the pair of rings Λ and Γ. This construction is due to [**BD**] and [**BD04**]. In our situations the emerging matrix problem is given by some *bunch of semichains* $\mathfrak{B}_\Lambda$.

(2) We apply Bondarenko’s combinatorics [**Bon91**] to determine the strings and bands of the matrix problem $\mathfrak{B}_\Lambda$ and to construct their canonical forms.

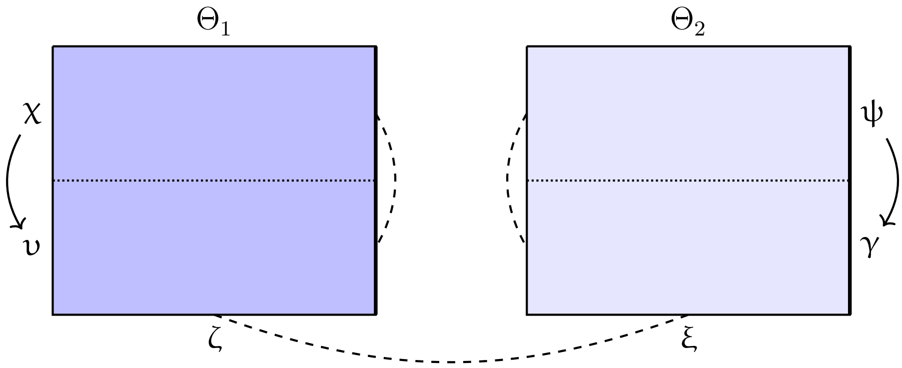
(3) Going the path of reductions backwards, we translate the canonical form of any string or band of $\mathfrak{B}_\Lambda$ back into an indecomposable representation of the ring Λ.

At the end, the resulting representations are formulated without notions of the category of triples or matrix problems.

Let us consider two examples of typical matrix problems which come up in this thesis.

1. The first example concerns the curve singularity $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(x\,y, z^2)$.

A *representative* part of the matrix problem (which is essentially equivalent to the classification problem of Cohen-Macaulay modules over P) is given as follows:



We are given two matrices Θ_1 and Θ_2 with entries from $\Bbbk$ subject to the following conditions:

- The matrices Θ_1 and Θ_2 should have the *same* number of columns. Moreover, both matrices should have full row rank.

- The rows of the matrix Θ_1 are divided into two horizontal stripes labeled by χ and υ, Both stripes are required to have the same number of rows.

- Similarly, the rows of the matrix Θ_2 are divided into the two horizontal stripes ψ and γ with the same number of rows.

The following transformations of the matrices Θ_1 and Θ_2 will be *admissible*:

- We may carry out any elementary transformation of columns in the matrix Θ_1 together with the *same* transformation of columns in Θ_2.

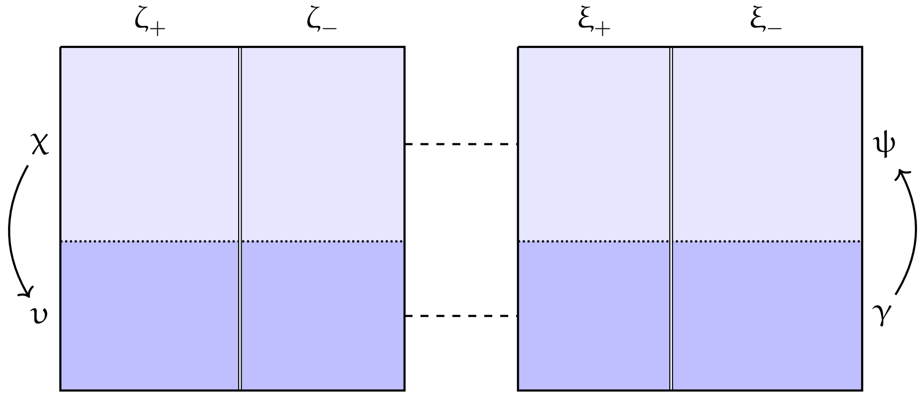
- We may perform any elementary transformation of rows in the horizontal stripe χ together with the *same* transformation of rows in the horizontal stripe υ of the matrix Θ_1. Similary, we may perform any *simultaneous elementary transformation* of rows in the horizontal stripes ψ and γ in Θ_2.

- We may add a scalar multiple of any row of the horizontal stripe χ to any row of a horizontal stripe χ in the matrix Θ_1. Similarly, we may a add a scalar multiple of any row from the stripe ψ to any row of the stripe γ in Θ_2.

The *matrix problem* is the problem to find canonical forms for the matrices Θ_1 and Θ_2 using *only admissible* transformations.

The above matrix problem has type *bunch of chains*.

2. Next, we consider a typical part of the matrix problem which is related to the classification problem of nilpotent representations of the Gelfand quiver:



We are given two *regular*, partitioned matrices Θ_1 and Θ_2 of the following form:

- The columns of Θ_1 are divided into two vertical stripes ζ_+ and ζ_-. Similarly, the columns of Θ_2 have two vertical stripes ξ_+ and ξ_-.
- As in the matrix problem above, the rows of Θ_1 are divided into two horizontal stripes χ and υ, and the matrix Θ_2 has two horizontal stripes ψ and γ.
- In this matrix problem, the horizontal stripes χ and ψ in Θ_1 have the same number of rows. The same holds for the pair of horizontal stripes υ and γ

Now only the following transformations of the matrices Θ_1 and Θ_2 are *admissible*:

- We may perform *any* independent elementary transformation of columns inside *one* of the four vertical stripes ξ_+, ξ_-, ζ_+ or ζ_-.
- We can perform *simultaneous elementary transformations* of rows in the horizontal stripes χ and ψ, and similarly for the pair of stripes υ and γ.
- We are allowed to add a scalar multiple of any row of stripe χ to any row of stripe υ in the matrix Θ_1.
- In the second matrix, we may add a scalar multiple of any row of stripe γ to any row of stripe ψ in Θ_2.

As above, the matrix problem is to describe the canonical forms of both matrices Θ_1 and Θ_2 using only admissible transformations.

This matrix problem has type over a *proper bunch of semichains*.

The essential difference between the two matrix problems is given by the following remark:

- We may *not* add any scalar multiple of any column of the vertical stripe ζ_+ to any column of the vertical stripe ζ_- in the same matrix Θ_1, or vice versa. Transformations of this form are also forbidden for the pair of vertical stripes ξ_+ and ξ_- in the other matrix Θ_2.

This restriction makes the matrix problem considerably more difficult.

Structure of the thesis

The structure of this thesis is illustrated in the figure on page 31. This thesis is organized as follows:

- This thesis has two independent parts, which are unequal in size but similar in spirit.
- The five chapters of this thesis are essentially independent from each other.
- There are three appendices. Appendix A is relevant for both parts of this thesis while Appendices B and C are important only for Part 2.
- Appendix A forms the technical heart of the classification results obtained in this thesis. In particular, it contains Bondarenko's description of canonical forms for matrix problems over *bunches of chains* and *proper bunches of semichains*.

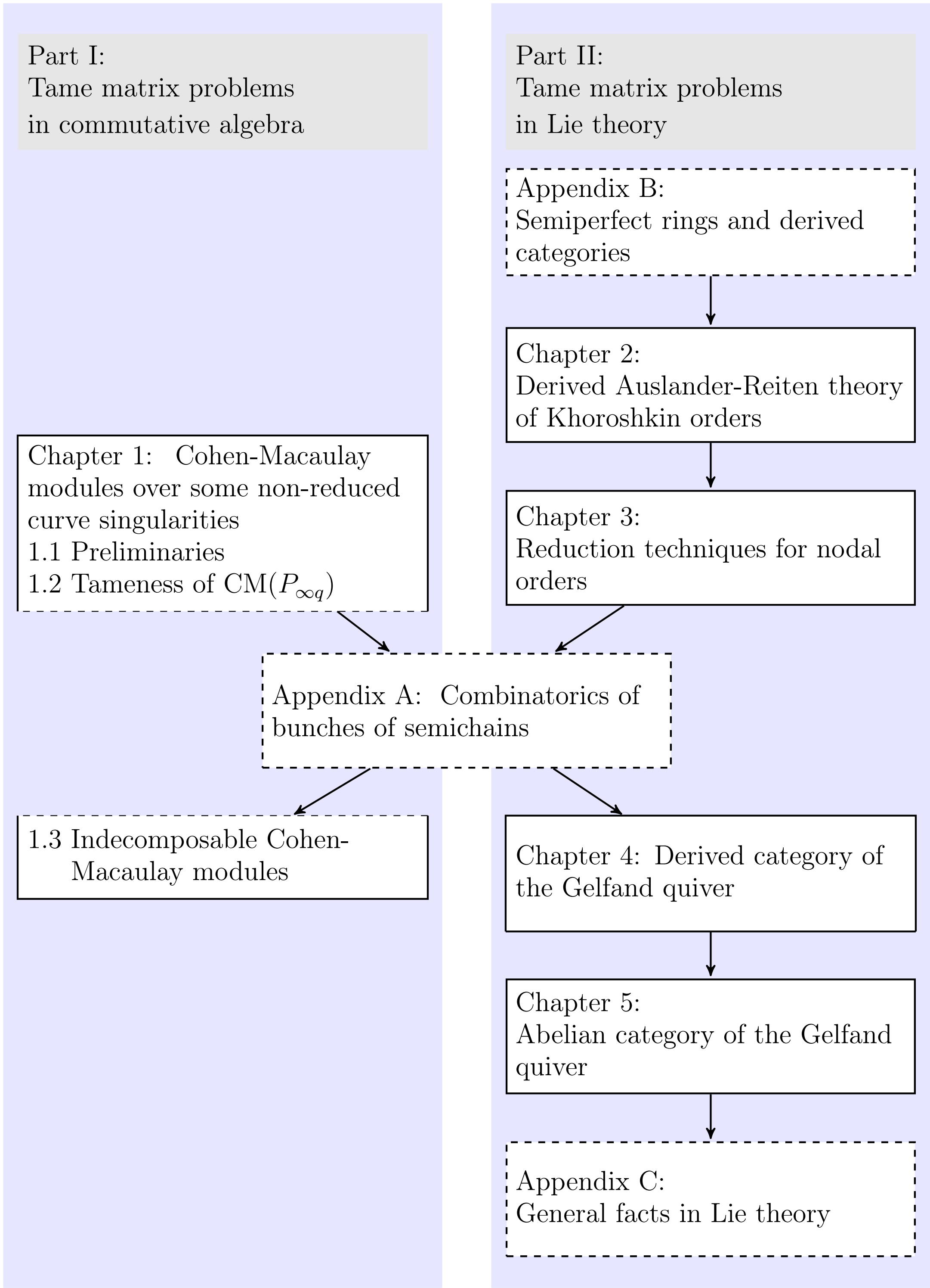
(1) Part 1 is the short part of this thesis. It consists of a single chapter and deals with matrix problems in commutative algebra.

  - The classification results of Section 1.3 rely on the combinatorics of *bunches of chains*, which are presented in Sections A.1, A.2, and Subsection A.3.1 Let us note that the case of bunches of chains is technically simpler than the case of bunches of proper semichains. This explains why the thesis starts with this topic.

(2) Part 2 is the main part of this thesis. It deals with tame matrix problems motivated by Lie theory.

  - Appendix B contains the preliminaries for all chapters of Part 2.
  - The Chapters 2 and 3 contain all "category-theoretic" results.
  - The Chapters 4 and 5 describe the combinatorics of certain categories. These classification results are based on canonical forms for some *proper bunch of semichains* which are described in Appendix A in Sections A.1, A.2, and Subsection A.3.2.
  - Appendix C describes the Lie-theoretic background of the Khoroshkin orders. The relations to Lie theory are the original motivation to study the Gelfand quiver. In particular, Appendix C may be read before Chapter 5, at the beginning or at the end of Part 2.
  - Chapter 5 contains results on indecomposable representations of the Gelfand quiver. This chapter may also be read first and does not assume familiarity with derived categories or matrix problems.

Thesis structure



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Part 1

Tame matrix problems in commutative algebra

CHAPTER 1

Cohen-Macaulay modules over some non-reduced curve singularities

Cohen-Macaulay modules over Cohen-Macaulay rings have been intensively studied in recent years. They appear in the literature in various incarnations like matrix factorizations, objects of the triangulated category of singularities or lattices over orders.

Our interest to Cohen-Macaulay modules is representation theoretic. In the case of a *reduced* curve singularity, the behavior of the representation type of the category of Cohen-Macaulay modules $\mathrm{CM}(A)$ is completely understood. Assume, for simplicity, that A is an algebra over an algebraically closed field $\Bbbk$ of characteristic zero.

- According to Drozd and Roiter [**DR67**], Jacobinski [**Jac67**] and Greuel and Knörrer [**GK85**], $\mathrm{CM}(A)$ is representation finite if and only if A dominates a simple curve singularity. See also the expositions in the monographs [**LW12**] and [**Yos90**].
- Drozd and Greuel have proven in [**DG93**] that if $\mathrm{CM}(A)$ is tame then A dominates a singularity of type

  $$\mathsf{T}_{pq}(\lambda) = \Bbbk[\![x, y]\!]/(x^{p-2} - y^2)(x^2 - \lambda y^{q-2}) \qquad \begin{cases} \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, & \lambda \in \Bbbk\backslash\{0, 1\}, \\ \frac{1}{p} + \frac{1}{q} < \frac{1}{2}, & \lambda = 1. \end{cases}$$

  In particular, they have shown that the following singularities

  $$\mathsf{P}_{pq} = \Bbbk[\![x, y, z]\!]/(xy, x^p + y^q - z^2), \qquad \text{where } p, q \in \mathbb{N}_{\geq 2},$$

  are Cohen-Macaulay tame.
- A reduced curve singularity which neither dominates a simple nor a $\mathsf{T}_{pq}(\lambda)$ singularity has wild Cohen-Macaulay representation type [**DG92**].
- There are also other approaches to establish tameness of $\mathrm{CM}\big(\mathsf{T}_{pq}(\lambda)\big)$: one using the generalized geometric McKay Correspondence [**Kah89, DGK03**] and another via cluster-tilting theory [**BIKR08**].

The following results about the representation type of a *non-reduced* curve singularity are known so far.

- By a theorem of Auslander [**Aus99**], a non-reduced curve singularity always has infinite Cohen-Macaulay representation type.
- Buchweitz, Greuel and Schreyer have shown in [**BGS87**] that the singularities $\mathsf{A}_\infty = \Bbbk[\![x, y]\!]/(y^2)$ and $\mathsf{D}_\infty = \Bbbk[\![x, y]\!]/(xy^2)$ have discrete Cohen-Macaulay representation type.

- Leuschke and Wiegand have proven in [**LW05**] that the only curve singularities of bounded but infinite Cohen-Macaulay type are A_∞, D_∞ and $\Bbbk[\![x,y,z]\!]/(xy,yz,z^2)$.
- Burban and Drozd have proven in [**BD**] that the hypersurface singularities $\mathsf{T}_{\infty q} = \Bbbk[\![x,y]\!]/(x^2y^2-y^q)$, where $q \in \mathbb{N}_{\geq 3}$, (respectively $\mathsf{T}_{\infty\infty} = \Bbbk[\![x,y]\!]/(x^2y^2)$) are Cohen-Macaulay tame (where $\mathsf{char}(\Bbbk)=0$, respectively $\mathsf{char}(\Bbbk) \neq 2$). However, an explicit description of the corresponding indecomposable matrix factorizations is still not known.

In this chapter, we obtain the following results.

1. First, we prove (see Theorem 1.2.1) that the curve singularities

$$\mathsf{P}_{\infty q} = \Bbbk[\![x,y,z]\!]/(xy, y^q - z^2) \quad \text{and} \quad \mathsf{P}_{\infty\infty} = \Bbbk[\![x,y,z]\!]/(xy, z^2)$$

are Cohen-Macaulay tame for any algebraically closed field $\Bbbk$ of any characteristic (in the case $\mathsf{char}(\Bbbk)=2$ the definition of $\mathsf{P}_{\infty q}$ has to be modified, see Remark 1.2.5). The method of the proof extends the approach of Drozd and Greuel [**DG93**] to the case of non-reduced curve singularities and is based on Bondarenko's work on representations of *bunches of semichains* [**Bon91**] (presented in Appendix A). Our approach can be summarized by the following diagram of categories and functors:

$$\mathrm{CM}(\mathsf{P}) \xhookleftarrow{\;\mathbf{I}\;} \mathrm{CM}(\mathsf{R}) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Tri}(\mathsf{R}) \xrightarrow{\;\mathbf{H}\;} \mathrm{Rep}(\mathfrak{B}).$$

We start with a singularity $\mathsf{P} = \mathsf{P}_{\infty q}$ or $\mathsf{P}_{\infty\infty}$ and replace it by its *minimal overring* R. The forgetful functor $\mathbf{I}$ embeds $\mathrm{CM}(\mathsf{R})$ into $\mathrm{CM}(\mathsf{P})$ as a full subcategory. By a result of Bass [**Bas63**], the “difference” between $\mathrm{CM}(\mathsf{R})$ and $\mathrm{CM}(\mathsf{P})$ is very small. The *category of triples* $\mathrm{Tri}(\mathsf{R})$ plays a key role in our approach. According to [**BD**], the functors $\mathbf{F}$ and $\mathbf{G}$ are quasi-inverse equivalences of categories. Finally, $\mathrm{Rep}(\mathfrak{B})$ is a certain *bimodule category* in the sense of [**Dro72**]. The functor $\mathbf{H}$ preserves isomorphism classes and indecomposability of objects. We prove that $\mathrm{Rep}(\mathfrak{B})$ is the category of representations of some *bunch of semichains*. According to Theorem A.2.21 by Bondarenko, $\mathrm{Rep}(\mathfrak{B})$ is representation tame. This implies tameness of $\mathrm{CM}(\mathsf{P})$.

2. Next, we show how to pass from canonical forms describing indecomposable objects of $\mathrm{Rep}(\mathfrak{B})$ to a concrete description of the corresponding indecomposable Cohen-Macaulay P-modules. We illustrate this technique giving an explicit description of the indecomposable Cohen-Macaulay modules over $\mathsf{P}_{\infty\infty}$. They are described in terms of combinatorial data called *strings* and *bands*, see Theorem 1.3.19. The obtained classification is suitable to identify those Cohen-Macaulay modules which are *locally free on the punctured spectrum*, see Remark 1.3.17.

3. At last, we construct explicit families of indecomposable matrix factorizations of $x^2y^2 \in \Bbbk[\![x,y]\!]$. In this context, there is the following diagram of categories:

$$\mathrm{CM}(\mathsf{R}) \xhookrightarrow{\;\mathbf{J}\;} \mathrm{CM}(\mathsf{T}) \longrightarrow \underline{\mathsf{MF}}(x^2y^2).$$

Here, R is the minimal overring of $\mathsf{P}_{\infty\infty}$, the functor $\mathbf{J}$ is a fully faithful embedding, $\mathsf{T} = \Bbbk[\![x,y]\!]/(x^2y^2)$, and $\underline{\mathsf{MF}}(x^2y^2)$ is the homotopy category of matrix factorizations of x^2y^2 (which is equivalent to the stable category $\underline{\mathrm{CM}}(\mathsf{T})$ by a result of Eisenbud

[**Eis80**]). Results of this chapter provide a partial classification of the indecomposable objects of $\underline{\mathsf{MF}}(x^2y^2)$ as well as an equivalent category $\underline{\mathsf{MF}}(x^2y^2+uv)$.

1.1 Preliminaries on Cohen-Macaulay modules over curve singularities

In this section we collect definitions and some known facts on Cohen-Macaulay modules over curve singularities. The proofs of the mentioned statements can be found in the monographs [**BH93, LW12, Yos90**], see also the survey article [**BD08**].

1.1.1 Basic properties and Bass' rejection lemma

Let $(A,\mathfrak{m})$ be a local Noetherian ring of Krull dimension one (a curve singularity), $\Bbbk = A/\mathfrak{m}$ its residue field and $Q = Q(A)$ its total ring of fractions.

Definition 1.1.1. *A curve singularity A is*

- Cohen-Macaulay *if and only if* $\mathrm{Hom}_A(\Bbbk, A) = 0$ *(equivalently, A contains a regular element).*
- Gorenstein *if and only if it is Cohen-Macaulay and* $\mathrm{Ext}^1_A(\Bbbk, A) \cong \Bbbk$ *(equivalently,* $\mathsf{inj.dim}_A(A) = 1$*).*

Note that a *reduced* curve singularity is automatically Cohen-Macaulay. However, in this chapter we mainly focus on non-reduced ones.

Lemma 1.1.2. *Let A be a Cohen-Macaulay curve singularity. Then Q is an Artinian ring. Moreover, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ is the set of minimal prime ideals of A then there exists a ring isomorphism $\zeta : Q \longrightarrow A_{\mathfrak{p}_1} \times \cdots \times A_{\mathfrak{p}_t}$ making the following diagram*

$$\begin{tikzcd} & Q \arrow[dd, "\zeta"] \\ A \arrow[ur, "can_1"] \arrow[dr, "can_2"'] & \\ & A_{\mathfrak{p}_1} \times \cdots \times A_{\mathfrak{p}_t} \end{tikzcd}$$

commutative, where can_1 and can_2 are canonical morphisms.

PROOF. Since A is Cohen-Macaulay, the associator of A coincides with $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. By [**Bou98**, Chapter IV, Proposition 2.5.10] Q is Artinian and its maximal ideals are $\mathfrak{p}_1 Q, \dots, \mathfrak{p}_n Q$. Hence, $Q \cong Q_{\mathfrak{p}_1 Q} \times \cdots \times Q_{\mathfrak{p}_n Q}$. Since $Q_{\mathfrak{p}_i Q} = A_{\mathfrak{p}_i}$ for $1 \le i \le n$, the result follows. $\square$

Definition 1.1.3. *For an A-module M we set*

$$\Gamma_{\mathfrak{m}}(M) := \left\{ \, x \in M \;\middle|\; \mathfrak{m}^t x = 0 \;\; \textit{for some} \;\; t \in \mathbb{N}^+ \, \right\}.$$

The following result can be easily deduced from Lemma 1.1.2.

Lemma 1.1.4. *Let A be a Cohen-Macaulay curve singularity. For a Noetherian A-module M we have*:

$$\Gamma_{\mathfrak{m}}(M) = \ker\bigl(M \longrightarrow Q \otimes_A M\bigr) =: \mathsf{tor}(M).$$

Moreover, the following statements are equivalent:

- $\mathrm{Hom}_A(\Bbbk, M) = 0$.
- *M is* torsion free*, that is,* $\mathsf{tor}(M) = 0$.

Definition 1.1.5. *A Noetherian module M satisfying the conditions of Lemma 1.1.4 is called* maximal Cohen-Macaulay*. In what follows we just say that M is Cohen-Macaulay. In this case, the Q-module $Q(M)$ is called the* rational envelope *of M. More generally, a Noetherian module N over a Noetherian ring S (say, of Krull dimension one) is (maximal) Cohen-Macaulay if for any maximal ideal $\mathfrak{n}$ in S the localization $N_{\mathfrak{n}}$ is Cohen-Macaulay. In what follows,* $\mathrm{CM}(S)$ *denotes the category of Cohen-Macaulay S-modules.*

Lemma 1.1.6. *Assume that a Cohen-Macaulay curve singularity A is Gorenstein in codimension zero (that is, Q is self-injective). Then for a Noetherian A-module M, the following conditions are equivalent:*

- *M is Cohen-Macaulay.*
- *M embeds into a finitely generated free A-module.*

A proof of this Lemma can be found in [**LW12**, Appendix A, Corollary 15].

Remark 1.1.7. *The statement of Lemma 1.1.6 is not true for an arbitrary Cohen-Macaulay curve singularity. For example, let $A = \Bbbk[\![x, y, z]\!]/(x^2, xy, y^2)$ and K be a canonical A-module. Then K does not embed into a free A-module.*

Definition 1.1.8. *A ring R is an* overring *of A if $A \subseteq R \subset Q$ and the ring extension $A \subseteq R$ is finite. We also say that R* birationally dominates *A.*

Proposition 1.1.9. *Let A be a Cohen-Macaulay curve singularity and R an overring of A. Then the following results are true.*

(1) R is Cohen-Macaulay.

(2) We have an adjoint pair $\bigl(R \boxtimes_A -, \mathbf{I}(-)\bigr)$, where $\mathbf{I}\colon \mathrm{CM}(R) \longrightarrow \mathrm{CM}(A)$ is the restriction (or forgetful) functor and $R \boxtimes_A - : \mathrm{CM}(A) \longrightarrow \mathrm{CM}(R)$ sends a Cohen-Macaulay module M to $R \otimes_A M/\mathsf{tor}(R \otimes_A M)$.

(3) $\mathbf{I}$ is fully faithful.

(4) If $M = \langle w_1, \dots, w_t\rangle_A \subset Q^n$, then $R \boxtimes_A M \cong R \cdot M := \langle w_1, \dots, w_t\rangle_R \subset Q^n$.

Proof. The first statement follows from the fact that $\mathsf{depth}_A(R) = 1 = \mathsf{depth}_R(R)$. The second result follows from the functorial isomorphisms

$$\mathrm{Hom}_R\bigl(R \boxtimes_A M, N\bigr) \cong \mathrm{Hom}_R\bigl(R \otimes_A M, N\bigr) \cong \mathrm{Hom}_A\bigl(M, \mathbf{I}(N)\bigr).$$

For a proof of the third statement, see for example [**LW12**, Lemma 4.14]. The fourth result follows from the fact that the kernel of the natural morphism $R \otimes_A M \longrightarrow R \cdot M$ is $\mathsf{tor}(R \otimes_A M)$. $\square$

Corollary 1.1.10. *Let A be a Cohen-Macaulay curve singularity and R be an overring of A. Then the following statements are true.*

(1) Let N_1 and N_2 be Cohen-Macaulay R-modules. Then $N_1 \cong N_2$ if and only if $\mathbf{I}(N_1) \cong \mathbf{I}(N_2)$ in $\mathrm{CM}(A)$.

(2) A Cohen-Macaulay R-module N is indecomposable if and only if N is indecomposable viewed as an A-module.

The following result is due to Bass [**Bas63**, Proposition 7.2], see also [**LW12**, Lemma 4.9].

Theorem 1.1.11. *(Bass) Let $(A, \mathfrak{m})$ be a Gorenstein curve singularity and let $R = \mathrm{End}_A(\mathfrak{m})$. Then the following results are true.*

(1) $R \cong \{r \in Q \mid r\,\mathfrak{m} \subseteq \mathfrak{m}\}$. In particular, R is an overring of A.

(2) If A is not regular, we have an exact sequence of A-modules

$$0 \longrightarrow A \xrightarrow{\imath} R \longrightarrow \Bbbk \longrightarrow 0,$$

where $\imath$ is the canonical inclusion. This short exact sequence defines a generator of the A-module $\mathrm{Ext}^1_A(\Bbbk, A) \cong \Bbbk$.

(3) In the latter case, let S be any other proper overring of A. Then S contains R. In other words, R is the minimal overring *of the curve singularity A.*

(4) Let M be a Cohen-Macaulay A-module without free direct summands. Then there exists a Cohen-Macaulay R-module N such that $M = \mathbf{I}(N)$.

Remark 1.1.12. *Theorem 1.1.11 gives a precise measure of the representation theoretic difference between the categories $\mathrm{CM}(A)$ and $\mathrm{CM}(R)$. Namely, an indecomposable Cohen-Macaulay A-module M is either regular or is the restriction of an indecomposable Cohen-Macaulay R-module. In more concrete terms, assume that $M = \langle w_1, \dots, w_t \rangle_A \subset Q^n$ contains no free direct summands (according to Lemma 1.1.6, any Cohen-Macaulay A-module admits such embedding). Then $M = \langle w_1, \dots, w_t \rangle_R$.*

Proposition 1.1.13. *In the setup of Theorem 1.1.11, assume that $N = \langle w_1, \dots, w_t \rangle_R \subset Q^n$ is an indecomposable Cohen-Macaulay R-module. Then either $N \cong R$ or $N = \langle w_1, \dots, w_t \rangle_A$.*

PROOF. Pose $M := \langle w_1, \dots, w_t \rangle_A$. Obviously, we have: $N = R \cdot M$. If M contains a free direct summand, that is, $M \cong M' \oplus A^m$, then $N = R \cdot M \cong R \cdot M' \oplus R^m$. As N is assumed to be indecomposable, $N \cong R$. If $N \ncong R$, then M has no free direct summands. Hence, by Theorem 1.1.11 and Remark 1.1.12 we have: $R \cdot M = M$. $\square$

Definition 1.1.14. *A Cohen-Macaulay A-module M is* locally free on the punctured spectrum *if for any minimal prime ideal $\mathfrak{p}$ in A the localization $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$.*

Remark 1.1.15. *According to Lemma 1.1.2, a Cohen-Macaulay A-module M is locally free on the punctured spectrum if and only if its rational envelope $Q(M)$ is projective over Q.*

In what follows, $\mathrm{CM}^{\mathsf{lf}}(A)$ denotes the category of Cohen-Macaulay A-modules which are locally free on the punctured spectrum.

1.1.2 Category of triples for Cohen-Macaulay modules

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay curve singularity and let S be an overring of R. Next, we define a category $\mathrm{Tri}(R)$ which is equivalent to the category $\mathrm{CM}(R)$.

Let I be the *conductor ideal* of the rings R and S:

$$I = \mathrm{ann}_R(S/R) = \left\{\, r \in R \mid r\,S \subseteq R \,\right\}.$$

The next result is straightforward, see for example [**BD**, Lemma 12.1].

Lemma 1.1.16. *The following statements are true.*

(1) $I = IR = IS$. In other words, I is an ideal both in R and in S. Moreover, I is the biggest ideal having this property.

(2) The rings R/I and S/I are Artinian.

The data R, S and I gives rise to a commutative diagram called the *conductor square*:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads] & S \arrow[d, two heads, dotted] \\ R/I \arrow[r, hook, dotted] & S/I \end{tikzcd}$$

In this diagram, the ring R is realized as the pullback of the dotted arrows.

The conductor square induces the following diagram of categories and functors:

$$\begin{tikzcd} \mathrm{CM}(R) \arrow[r, "S \boxtimes_R -"] \arrow[d, "R/I \otimes_R -"'] & \mathrm{CM}(S) \arrow[d, "S/I \otimes_S -"] \\ R/I\text{ - mod} \arrow[r, "S/I \otimes_{R/I} -"'] & S/I\text{ - mod} \end{tikzcd}
\qquad
\begin{tikzcd} M \arrow[rrr, mapsto] \arrow[d, mapsto] & & & S \underset{R}{\boxtimes} M \arrow[d, mapsto] \\ R/I \underset{R}{\otimes} M \arrow[r, mapsto] & S/I \underset{R/I}{\otimes} R/I \underset{R}{\otimes} M \arrow[rr, dashed, "\widetilde{\mu}_M"'] & & S/I \underset{S}{\otimes} S \underset{R}{\boxtimes} M \end{tikzcd}$$

We note that for any Cohen-Macaulay R-module M there is a natural map $\widetilde{\mu}_M$ of S/I-modules.

The *rough idea* of the category of triples is that the Cohen-Macaulay R-module M can be reconstructed from the map $\widetilde{\mu}_M$. This statement will be made precise next.

By [**BD**, Lemma 12.2] the map above has the following properties:

Lemma 1.1.17. *For any Cohen-Macaulay R-module M the following results are true:*

- *the canonical map* $\widetilde{\mu}_M \colon S/I \otimes_{R/I} R/I \otimes_R M \longrightarrow S/I \otimes_S S \boxtimes_R M$ *is surjective,*

- *its adjoint map* $\mu_M \colon R/I \otimes_R M \longrightarrow S/I \otimes_S S \boxtimes_R M$ *is injective.*

Definition 1.1.18. *Let R be a Cohen-Macaulay curve singularity and S an overring of R. The* category of triples $\mathrm{Tri}(R) = \mathrm{Tri}(R, S, I)$ *is defined as follows*:

(1) An object of $\mathrm{Tri}(R)$ *is given by a triple (V, N, ϑ) where*

  (a) V is a Noetherian S/I-module,

  (b) N is a Cohen-Macaulay S-module, and

(c) $\vartheta\colon S/I \otimes_{R/I} V \longrightarrow S/I \otimes_S N$ *is an epimorphism of* S/I*-modules such that the adjoint morphism* $\vartheta\colon V \longrightarrow S/I \otimes_S N$ *is a monomorphism.*

(2) A morphism $(V', N', \vartheta') \longrightarrow (V'', N'', \vartheta'')$ *between objects in* $\mathrm{Tri}(R)$ *is given by a tuple* (ϕ, ψ) *where*

(a) $\phi\colon V' \longrightarrow V'$ *is a morphism of* R/I*-modules and*

(b) $\psi\colon N' \longrightarrow N''$ *is a morphism of* S*-modules*

such that the following diagram is commutative:

$$\begin{array}{ccc} S/I \underset{R/I}{\otimes} V' & \xrightarrow{\vartheta'} & S/I \underset{S}{\otimes} N' \\ {\scriptstyle S/I \underset{R/I}{\otimes} \phi}\big\downarrow & & \big\downarrow{\scriptstyle S/I \underset{S}{\otimes} \psi} \\ S/I \underset{R/I}{\otimes} V'' & \xrightarrow{\vartheta''} & S/I \underset{\Gamma}{\otimes} N'' \end{array}$$

There is a natural notion of the composition of morphisms and the direct sum of triples in $\mathrm{Tri}(R)$.

Definition 1.1.19. *We define a functor* $\mathbf{F}\colon \mathrm{CM}(R) \longrightarrow \mathrm{Tri}(R)$ *as follows.*

(1) For any Cohen-Macaulay R*-module* M *we set* $\mathbf{F}(M) = (R/I \otimes_R M, S \boxtimes_R M, \widetilde{\mu}_M)$ *where* $\widetilde{\mu}_M\colon S/I \otimes_{R/I} R/I \otimes_R M \longrightarrow S/I \otimes_S S \boxtimes_R M$ *is the natural map.*

(2) For a morphism $\varrho\colon M' \longrightarrow M''$ *in* $\mathrm{CM}(R)$ *we set* $\mathbf{F}(\varrho) = (R/I \otimes_R \varrho,\ S \boxtimes_R \varrho)$.

Definition 1.1.20. *The "gluing functor"* $\mathbf{G}\colon \mathrm{Tri}(R) \longrightarrow \mathrm{CM}(R)$ *is defined as follows.*

(1) Let $T = (V, N, \vartheta)$ *be an object of* $\mathrm{Tri}(R)$*, and let* $\pi : N \longrightarrow S/I \otimes_S N$ *be the canonical projection. Then* $\mathbf{G}(T) := \pi^{-1}\big(\mathrm{im}(\vartheta)\big) \subseteq N$*. In other words, the module* $\mathbf{G}(T)$ *is defined as the pullback with respect to the maps* ϑ *and* π *in the category of* R*-modules:*

$$\begin{array}{ccccccccc} 0 & \dashrightarrow & IN & \dashrightarrow & \mathbf{G}(T) & \dashrightarrow & V & \dashrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle\vartheta} & & \\ 0 & \longrightarrow & IN & \longrightarrow & N & \xrightarrow{\pi} & S/I \underset{S}{\otimes} N & \longrightarrow & 0 \end{array} \tag{1.1.1}$$

(2) There is a natural extension of the definition of $\mathbf{G}$ *to morphisms in* $\mathrm{Tri}(R)$.

Definition 1.1.18 is motivated by the following theorem, see [**BD**, Theorem 12.5].

Theorem 1.1.21. *The functors* $\mathbb{F}$ *and* $\mathbb{G}$ *are well-defined quasi-inverse equivalences of categories:*

$$\mathrm{CM}(R) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathrm{Tri}(R)$$

In the case of a radical embedding, that is, if $\operatorname{rad} R = \mathfrak{m} = \operatorname{rad} S$, Theorem 1.1.21 provides an efficient tool to reduce the classification of indecomposable objects of $\mathrm{CM}(R)$ to a certain problem of linear algebra (a *matrix problem*).

Remark 1.1.22. *There are several variations of the construction appearing in Theorem 1.1.21, see* [**BD**, Appendix A] *for an account of them. There is also an analogue of the category of triples in the context of derived categories (see Subsection 3.2)*

1.1.3 Cohen-Macaulay modules over simple curve singularities of type A

Let $\Bbbk$ be an algebraically closed field. For simplicity, let us additionally assume that $\mathsf{char}(\Bbbk) \neq 2$, see however Remark 1.1.24. For any $m \in \mathbb{N}$, denote

$$S = \mathsf{A}_m := \Bbbk[\![x, u]\!]/(x^{m+1} - u^2) \tag{1.1.2}$$

the corresponding simple curve singularity of type A_m. The following is essentially due to Bass [**Bas63**], see also [**LW12, Yos90**].

Theorem 1.1.23. *The indecomposable Cohen-Macaulay S-modules have the following description.*

- *Assume $m = 2n$, $n \in \mathbb{N}^+$. For any $1 \leq i \leq n$ consider the ideal $X_i := (x^i, u)$. Then $X_0 = (1) = S, X_1, \dots, X_n$ is the complete list of indecomposable objects of* $\mathrm{CM}(S)$*. Moreover, the Auslander-Reiten quiver of* $\mathrm{CM}(S)$ *has the form*

$$\begin{tikzcd} X_0 \arrow[r, shift left, "\cdot x"] & X_1 \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & X_2 \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & \cdots \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & X_n \arrow[l, shift left, "\iota"] \arrow[loop right, "\pi"] \end{tikzcd} \tag{1.1.3}$$

  Here, $\imath$ denotes the inclusion of ideals and $\cdot x$ is the multiplication by x. The endomorphism $\pi \in \operatorname{End}_S(X_n)$ is defined as follows: $\pi(x^n) = u$ and $\pi(u) = x^{n+1}$.

- *Assume $m = 2n+1$, $n \in \mathbb{N}_0$. Again, for any $1 \leq i \leq n$ consider $X_i := (x^i, u) \subset S$. Additionally, denote $X_{n+1}^{\pm} := (x^{n+1} \pm u)$. Then the indecomposable Cohen-Macaulay S-modules are $X_0 = (1) = S, X_1, \dots, X_n, X_{n+1}^+$ and X_{n+1}^-. Moreover, the Auslander-Reiten quiver of* $\mathrm{CM}(S)$ *is in this case*

$$\begin{tikzcd} & & & & & X_{n+1}^+ \arrow[dl, shift left, "\iota^+"] \\ X_0 \arrow[r, shift left, "\cdot x"] & X_1 \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & X_2 \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & \cdots \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\cdot x"] & X_n \arrow[l, shift left, "\iota"] \arrow[ur, shift left, "\pi^+"] \arrow[dr, shift left, "\pi^-"] & \\ & & & & & X_{n+1}^- \arrow[ul, shift left, "\iota^-"] \end{tikzcd} \tag{1.1.4}$$

  Here, ι and $\iota^{\pm}$ denote inclusions of ideals, $x\cdot$ stands for multiplication by x. The maps $\pi^{\pm} \colon X_n \longrightarrow X_{n+1}^{\pm}$ are defined as follows: $\pi^{\pm}(x^n) = (x^{n+1} \pm u)$ and $\pi^{\pm}(u) = x(x^{n+1} \pm u)$.

Remark 1.1.24. *In the case $\mathsf{char}(\Bbbk) = 2$ there are the following subtleties in defining simple curve singularities of type A_m.*

- *For $m = 2n+1$ one should take the ring $\mathsf{A}_{2n+1} = \Bbbk[\![x, u]\!]/(u(u - x^{n+1}))$ (the ring defined by (1.1.2) is no longer reduced). In this case, one should pose $X_{n+1}^+ := (u)$*

and $X_{n+1}^{-} := (u - x^{n+1})$. *Then the indecomposable Cohen-Macaulay modules are* $X_0, \dots, X_n, X_{n+1}^{\pm}$, *where* X_i *has the same definition as in the case* $\mathsf{char}(\Bbbk) \neq 2$ *for* $0 \leq i \leq n$.

- *For* $m = 2n$ *there are more simple singularities than in the case* $\mathsf{char}(\Bbbk) \neq 2$. *Namely, for* $1 \leq s \leq n-1$ *consider the ring* $\mathsf{A}_{2n}^{s} = \Bbbk[\![x,u]\!]/(u^2 + x^{2n+1} + ux^{n+s})$. *Then* $\mathsf{A}_{2n}^{s} \not\cong \mathsf{A}_{2n}^{t}$ *for any* $1 \leq s \neq t \leq n-1$. *Moreover,* $\mathsf{A}_{2n}^{s} \not\cong \mathsf{A}_{2n}$ *for any* $1 \leq s \leq n-1$. *However, the description of indecomposable Cohen-Macaulay modules over* A_{2n}^{s} *is essentially the same as over* A_{2n}, *see* [**Bas63**] *and* [**KS85**]. *In particular, the Auslander-Reiten quivers of* A_{2n}^{s} *and* A_{2n} *coincide.*

The following result is due to Buchweitz, Greuel and Schreyer [**BGS87**, Section 4.1].

Theorem 1.1.25. *For an algebraically closed field* $\Bbbk$ *(of arbitrary characteristic) let* $S = \mathsf{A}_{\infty} := \Bbbk[\![x,u]\!]/(u^2)$. *Then the indecomposable Cohen-Macaulay* S*-modules are* $X_0, X_1, \dots, X_{\infty}$, *where* $X_0 = (1) = S, X_{\infty} = (u)$ *and* $X_i = (x^i, u)$ *for* $i \in \mathbb{N}$. *In particular,* X_{∞} *is the only indecomposable Cohen-Macaulay* S*-module which is not locally free on the punctured spectrum of* S. *The Auslander-Reiten quiver of the category* $\mathrm{CM}^{\mathsf{lf}}(S)$ *has the form*

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_i \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \tag{1.1.5}$$

Remark 1.1.26. *It is natural to extend the quiver (1.1.5) with the remaining indecomposable Cohen-Macaulay* S*-module* X_{∞}. *Moreover, for any* $i \in \mathbb{N}_0$ *denote* $\pi_i \colon X_i \longrightarrow X_{\infty}$ *the map sending* x^i *to* u *and* u *to* 0. *Of course,* $\pi_{i+1} = x \cdot \pi_i$ *for any* $i \in \mathbb{N}_0$. *The entire structure of the category* $\mathrm{CM}(S)$ *can be visualized by the diagram:*

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_i \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\pi}{\rightleftarrows}} X_{\infty} \tag{1.1.6}$$

Definition 1.1.27. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay curve singularity. Consider the category* $\overline{\mathrm{CM}}(A)$ *defined as follows:*

- $\mathrm{Ob}\big(\overline{\mathrm{CM}}(A)\big) = \mathrm{Ob}\big(\mathrm{CM}(A)\big)$.
- *For* $L, M \in \mathrm{Ob}\big(\overline{\mathrm{CM}}(A)\big)$ *we set*

$$\overline{\mathrm{Hom}}_A(L, M) = \mathrm{Im}\Big(\mathrm{Hom}_A(L, M) \longrightarrow \mathrm{Hom}_{\Bbbk}\big(\Bbbk \otimes_A L, \Bbbk \otimes_A M\big)\Big).$$

- *The composition of morphisms in* $\overline{\mathrm{CM}}(A)$ *is induced by the composition of morphisms in* $\mathrm{CM}(A)$.

The following result is straightforward.

Lemma 1.1.28. *The canonical projection functor* $\mathrm{CM}(A) \to \overline{\mathrm{CM}}(A)$ *is full and respects isomorphism classes of objects. Moreover, if* S *is a curve singularity of type* A_m *for some* $m \in \mathbb{N}^+ \cup \{\infty\}$, *then* $\overline{\mathrm{CM}}(S)$ *is equivalent to the additive closure of the path algebra category of the corresponding Auslander-Reiten quivers (1.1.3), (1.1.4) respectively (1.1.6) subject to the following zero relations:*

- $(\cdot x) \circ \imath = \imath \circ (\cdot x) = 0$.
- *The inclusion* $\imath \colon X_1 \to X_0$ *is zero in* $\overline{\mathrm{CM}}(S)$.

- $\begin{cases} \pi^2 = 0, & \textit{if } m \textit{ is even,} \\ \pi^{\pm} \circ \imath^{+} = \pi^{\pm} \circ \imath^{-} = \imath^{+} \circ \pi^{+} + \imath^{-} \circ \pi^{-} = 0, & \textit{if } m \textit{ is odd,} \\ \pi \circ \iota = 0, & \textit{if } m = \infty. \end{cases}$

1.2 Tameness of Cohen-Macaulay modules over $\mathsf{P}_{\infty q}$

Let $\Bbbk$ be an algebraically closed field such that $\mathsf{char}(\Bbbk) \neq 2$ and $p, q \in \mathbb{N}_{\geq 2}$. Consider the curve singularity

$$\mathsf{P}_{pq} := \Bbbk[\![x, y, z]\!]/(xy, x^p + y^q - z^2). \tag{1.2.1}$$

By a result of Drozd and Greuel [**DG93**, Section 3], the category $\mathrm{CM}(\mathsf{P}_{pq})$ is representation tame. For any $q \in \mathbb{N}_{\geq 2}$ consider the limiting non-reduced singularity

$$\mathsf{P}_{\infty q} := \Bbbk[\![x, y, z]\!]/(xy, y^q - z^2) \tag{1.2.2}$$

as well as the "largest degeneration" $\mathsf{P}_{\infty\infty} := \Bbbk[\![x, y, z]\!]/(xy, z^2)$ of the family (1.2.1).

The first major result of this chapter is the following.

Theorem 1.2.1. *The non-reduced curve singularities $\mathsf{P}_{\infty q}$ have tame Cohen-Macaulay representation type for any $q \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.*

Proof. 1. Since $\mathsf{P} := \mathsf{P}_{\infty q}$ is a complete intersection, it is Gorenstein. Let $\mathsf{R} = \mathrm{End}_{\mathsf{P}}(\mathfrak{m})$ be the minimal overring of P. The first statement of Theorem 1.1.11 implies that

$$\mathsf{R} \cong \Bbbk[\![x, y, u, v]\!]/(xy, yu, uv, vx, u^2, y^q - v^2), \tag{1.2.3}$$

where the canonical inclusion $\imath\colon \mathsf{P} \longrightarrow \mathsf{R}$ maps x to x, y to y and z to $u + v$ (in what follows, the generator $y^q - v^2$ has to be replaced by v^2 for $q = \infty$). Note that $u = \dfrac{xz}{x+y}$ and $v = \dfrac{yz}{x+y}$ if we regard R as a subring of the total ring of fractions $\mathsf{Q} = \mathsf{Q}(\mathsf{P})$. According to Theorem 1.1.11, any non-regular indecomposable Cohen-Macaulay P-module is a restriction of some indecomposable Cohen-Macaulay R-module. Thus, it has to be shown that the category $\mathrm{CM}(\mathsf{R})$ has tame representation type.

2. Next, note that $\mathsf{S} = \Bbbk[\![x, u]\!]/(u^2) \times \Bbbk[\![y, v]\!]/(y^q - v^2)$ is an overring of R. Indeed, we have an inclusion $\mathsf{S} \subset \mathsf{Q}$, where both idempotents of S can be expressed as follows:

$$e_x := (1, 0) = \frac{x}{x+y} \quad \text{and} \quad e_y := (0, 1) = \frac{y}{x+y}.$$

The reason to pass from P to its overring R is explained by the following observation: the conductor ideal $I := \mathsf{ann}_{\mathsf{R}}(\mathsf{S}/\mathsf{R})$ coincides with the maximal ideal $\mathfrak{m} = (x, y, u, v)_{\mathsf{R}}$. Hence, the rings $\mathsf{R}/I \cong \Bbbk$ and $\mathsf{S}/I \cong \Bbbk_x \times \Bbbk_y = \Bbbk \times \Bbbk$ are semisimple. Under this identification, the canonical inclusion $\mathsf{R}/I \to \mathsf{S}/I$ is identified with the diagonal embedding.

According to Theorem 1.1.21, the category $\mathrm{CM}(\mathsf{R})$ is equivalent to the category of triples $\mathrm{Tri}(\mathsf{R})$. Thus, we have to show tameness of $\mathrm{Tri}(\mathsf{R})$. Let $T = (V, N, \vartheta)$ be a triple of R. Then the following facts are true.

- Since V is just a module over $\mathsf{R}/I \cong \Bbbk$, we have: $V \cong \Bbbk^t$ for some $t \in \mathbb{N}_0$.
- Denote $\mathsf{S}_x = \Bbbk[\![x, u]\!]/(u^2)$ and $\mathsf{S}_y = \Bbbk[\![y, v]\!]/(y^q - v^2)$. Then $\mathsf{S} = \mathsf{S}_x \times \mathsf{S}_y$ and $N \cong N_x \oplus N_y$, where $N_x \in \mathrm{CM}(\mathsf{S}_x)$ and $N_y \in \mathrm{CM}(\mathsf{S}_y)$. According to Theorem 1.1.23 and Theorem 1.1.25, the Cohen-Macaulay modules N_x and N_y split into a direct sum of ideals
  - $X_0 = (e_x)_{\mathsf{S}} \cong \mathsf{S}_x$, $X_i = (x^i, u)_{\mathsf{S}}$, for $i \in \mathbb{N}^+$ and $X_\infty = (u)$.
  - $Y_0 = (e_y)_{\mathsf{S}} \cong \mathsf{S}_y$, $Y_j = (y^j, v)$ for $1 \le j \le s - 1$ and $Y_s^\pm = (y^s \pm v)_{\mathsf{S}}$ if $q = 2s$ is even, respectively $Y_s = (y^s, v)$ if $q = 2s + 1$ is odd.
- We have: $\mathsf{S}/I \otimes_{\mathsf{S}} N_x \cong \Bbbk_x^m$ and $\mathsf{S}/I \otimes_{\mathsf{S}} N_y \cong \Bbbk_y^n$ for some $m, n \in \mathbb{N}_0$. In what follows, we choose bases of $\mathsf{S}/I \otimes_{\mathsf{S}} N_x$ and $\mathsf{S}/I \otimes_{\mathsf{S}} N_y$ induced by the distinguished generators of the ideals which occur in a direct sum decomposition of N_x and N_y. Thus, the gluing map $\vartheta \colon \mathsf{S}/I \otimes_{\mathsf{R}/I} V \longrightarrow \mathsf{S}/I \otimes_{\mathsf{S}} N$ is given by a pair of matrices $(\Theta_x, \Theta_y) \in \mathrm{Mat}_{m \times t}(\Bbbk) \times \mathrm{Mat}_{n \times t}(\Bbbk)$.
- The condition that the morphism of S/I-modules ϑ is surjective just means that both matrices Θ_x and Θ_y have full row rank. The condition that ϑ is injective is equivalent to say that the matrix $\Theta|_V := \begin{bmatrix} \Theta_x \\ \Theta_y \end{bmatrix}$ has full column rank.

3. Let us now describe the matrix problem underlying a description of isomorphism classes of objects of $\mathrm{Tri}(\mathsf{R})$. If two triples $T = (V, N, \vartheta)$ and $T' = (V', N', \vartheta')$ are isomorphic, then $N \cong N'$ and $V \cong V'$. Hence, we may without loss of generality pose:

- $V' = V = \Bbbk^t$ for certain $t \in \mathbb{N}_0$.
- $N' = N = N_x \oplus N_y = \bigl(\bigoplus_i X_i^{\oplus m_i}\bigr) \oplus \bigl(\bigoplus_j Y_j^{\oplus n_j}\bigr)$ for some $m_i, n_j \in \mathbb{N}_0$.

Then the following is true: we have an isomorphism

$$\bigl(V, N, (\Theta_x, \Theta_y)\bigr) \cong \bigl(V, N, (\Theta'_x, \Theta'_y)\bigr)$$

in the category $\mathrm{Tri}(\mathsf{R})$ if any only if there exist automorphisms $\Phi \in \mathrm{GL}_t(\Bbbk)$, $\Psi_x \in \mathrm{Aut}_{\mathsf{S}_x}(N_x)$ and $\Psi_y \in \mathrm{Aut}_{\mathsf{S}_y}(N_y)$ such that

$$\Theta'_x = \bar{\Psi}_x \, \Theta_x \, \Phi^{-1} \quad \text{and} \quad \Theta'_y = \bar{\Psi}_y \, \Theta_y \, \Phi^{-1}, \tag{1.2.4}$$

where $\bar{\Psi}_x \in \mathrm{GL}_m(\Bbbk)$ (respectively $\bar{\Psi}_y \in \mathrm{GL}_n(\Bbbk)$) is the induced automorphisms of $\mathsf{S}/I \otimes_{\mathsf{S}} N_x \cong \Bbbk^m$ (respectively $\mathsf{S}/I \otimes_{\mathsf{S}} N_y \cong \Bbbk^n$). The transformation rule (1.2.4) leads to the following problem of linear algebra (a matrix problem).

- We have two matrices Θ_x and Θ_y over $\Bbbk$ with the same number of columns. The number of rows of Θ_x and Θ_y can be different. In particular, it can be zero for one of these matrices.
- Rows of Θ_x are divided into horizontal blocks indexed by elements of the linearly ordered set

$$\mathfrak{R}_x = \{\, \chi_0 < \chi_1 < \cdots < \chi_i < \cdots < \upsilon_\infty < \cdots < \upsilon_i < \cdots < \upsilon_1 \,\}.$$

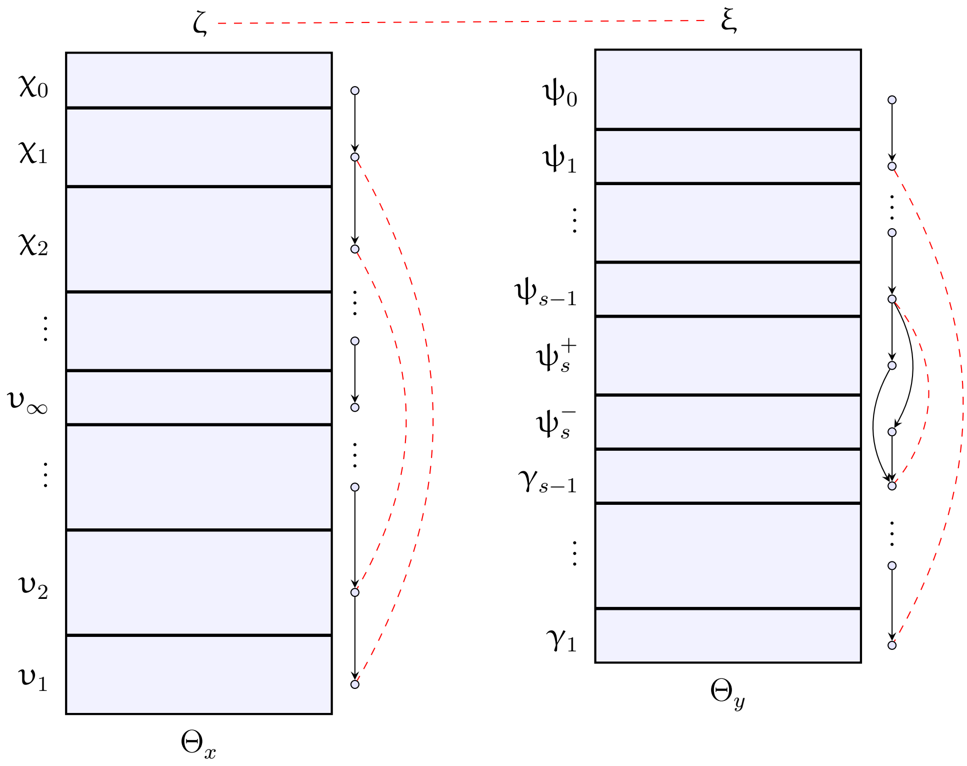


Figure 1.2.1. **Matrix problem for the case** $q = 2s$

The role of the ordering $<$ will be explained below.

- The block labeled by χ_0 has m_0 rows, the block labeled by υ_∞ has m_∞ rows. The blocks labeled by χ_i and υ_i both have m_i rows. Thus, the matrix Θ_x has $m = m_0 + m_\infty + 2(m_1 + \cdots + m_i + \dots)$ rows.
- The row division of Θ_y depends on the parity of the parameter q.
  - For $q = \infty$ the horizontal blocks of Θ_y are marked with the symbols of the linearly ordered set
  $$\mathfrak{R}_y = \mathfrak{R}_y^\infty = \{\, \psi_0 < \psi_1 < \cdots < \psi_j < \cdots < \gamma_\infty < \cdots < \gamma_j < \cdots < \gamma_1 \,\},$$
  completely analogously as it is done for Θ_x.
  - For $q = 2s + 1$, the labels are elements of the chain
  $$\mathfrak{R}_y = \mathfrak{R}_y^q = \{\, \psi_0 < \psi_1 < \cdots < \psi_s < \gamma_s < \cdots < \gamma_1 \,\}.$$
  For any $1 \le j \le s$, the number of rows in blocks marked by ψ_j and γ_j is the same (and equal to n_j).
  - For $q = 2s$, the labels are elements of the semichain
  $$\mathfrak{R}_y = \mathfrak{R}_y^q = \{\, \psi_0 < \psi_1 < \cdots < \psi_{s-1} < \psi_s^+, \psi_s^- < \gamma_{s-1} < \cdots < \gamma_1 \,\}$$
  as shown in Figure 1.2.1 (the elements ψ_s^+ and ψ_s^- are incomparable). Again, the number of rows in blocks ψ_j and γ_j is the same for $1 \le j \le s - 1$.
- We can perform any *simultaneous* elementary transformation of columns of Θ_x and Θ_y.

- Transformations of rows of Θ_x are of three types.
  - We can add any multiple of any row with lower weight to any row with higher weight.
  - For any $i \in \mathbb{N}^+$ we can perform any *simultaneous* elementary transformation of rows within blocks marked by χ_i and υ_i.
  - We can make any elementary transformation of rows in block χ_0 or υ_∞.
- The transformation rules for rows of Θ_y *depend on the parity* of q.
  - Let us take the case q is even (the most complicated one, see Figure 1.2.1).
    - We can add any multiple of any row with lower weight to any row with higher weight.
    - For any $1 \leq j \leq s-1$ we may perform any *simultaneous* elementary transformation of rows within blocks marked by ψ_j and γ_j.
    - We can make any (independent) elementary transformation of rows in the block ψ_0 or $\psi_s^{\pm}$.
  - For $q = \infty$, the transformation rules for Θ_y are analogous to those listed above for Θ_x (The matrix problem for this case will be studied in much detail in Subsection 1.3.1).
  - For odd $q = 2s+1$, the transformation rules of Θ_y are the same as for $q = \infty$. The only difference between these cases lies in the absence of certain symbols in $\mathfrak{R}_y$.

4. For any $q \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ the described matrix problem is an example of *representations of a bunch of semichains*. Tameness of the latter class of problems has been shown by Bondarenko in [**Bon91**] (Theorem A.2.21). This implies the tameness of the category of triples $\mathrm{Tri}(\mathsf{R})$. Tameness of $\mathrm{CM}(\mathsf{R})$ follows from Theorem 1.1.21. □

Remark 1.2.2. *Consider the following combinatorial data:*

- *The index set* $I = \{x, y\}$.
- *Let* $\mathfrak{C}_x = \{\zeta\}$, $\mathfrak{C}_y = \{\xi\}$, $\mathfrak{C} = \mathfrak{C}_x \cup \mathfrak{C}_y$.
- *Let* $\mathfrak{R}_x$ *and* $\mathfrak{R}_y = \mathfrak{R}_y^q$ *be as above,* $\mathfrak{R} = \mathfrak{R}_x \cup \mathfrak{R}_y$.
- *In the set* $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C}$ *consider an equivalence relation* $\approx$ *defined as follows:*

$$\zeta \approx \xi, \qquad \chi_i \approx \upsilon_i \quad \textit{for } i \in \mathbb{N}^+,$$

$$\psi_j \approx \gamma_j \quad \textit{for } \begin{cases} 1 \leq j \leq s & \textit{if } q = 2s \textit{ or } q = 2s+1, \\ j \in \mathbb{N}^+ & \textit{if } q = \infty. \end{cases}$$

The entire data is an example of a bunch of semichains in the sense of Definition A.1.3, and defines a certain bimodule category $\mathrm{Rep}(\mathfrak{B})$, *see* [**Dro72**] *or* [**BD**] *for more details. The description of isomorphism classes of objects of* $\mathrm{Rep}(\mathfrak{B})$ *reduces precisely to the matrix problem described above. In our particular case, the category* $\mathrm{Rep}(\mathfrak{B})$ *admits the following intrinsic description: it is the* comma category *of the* *following diagram of categories and functors*

$$\overline{\mathrm{CM}}(\mathsf{S}_x) \times \overline{\mathrm{CM}}(\mathsf{S}_y) \xrightarrow{\mathsf{For}} (\Bbbk \times \Bbbk)\text{ -mod} \xleftarrow{(\Bbbk\times\Bbbk)\otimes_{\Bbbk} -} \Bbbk\text{ -mod},$$

where $\overline{\mathrm{CM}}(\mathsf{S}_x)$ *and* $\overline{\mathrm{CM}}(\mathsf{S}_y)$ *have been defined in Definition 1.1.27 (see also Lemma 1.1.28 for their explicit description) and* For *is the forgetful functor. According to Bondarenko* [**Bon91**] *(A.2), there are the following types of indecomposable objects in* $\mathrm{Rep}(\mathfrak{B})$*: bands (continuous series) and (bispecial, special and usual) strings (discrete series). The precise combinatorics of the discrete series is rather complicated.*

Definition 1.2.3. *The forgetful functor* $\mathbf{H}\colon \mathrm{Tri}(\mathsf{R}) \longrightarrow \mathrm{Rep}(\mathfrak{B})$ *assigns to a triple* $\big(V, N, (\Theta_x, \Theta_y)\big)$ *the pair of partitioned matrices* (Θ_x, Θ_y)*. To be more precise, we recall that*

- $V = \Bbbk^t$ *for some* $t \in \mathbb{N}_0$.
- $N = N_x \oplus N_y = \big(\bigoplus_i X_i^{\oplus m_i}\big) \oplus \big(\bigoplus_j Y_j^{\oplus n_j}\big)$ *where* $m_i, n_j \in \mathbb{N}_0$.
- $\Theta_x\colon V = \Bbbk^t \longrightarrow \mathsf{S}/I \otimes_{\mathsf{S}} N_x \cong \Bbbk^m$,
  $\Theta_y\colon V = \Bbbk^t \longrightarrow \mathsf{S}/I \otimes_{\mathsf{S}} N_y \cong \Bbbk^n$ *for certain* $m, n \in \mathbb{N}_0$.

At this point, we have chosen bases for $\mathsf{S}/I \otimes_{\mathsf{S}} N_x$ *respectively* $\mathsf{S}/I \otimes_{\mathsf{S}} N_y$ *which are induced by the distinguished generators of the indecomposable modules over* R_x *and* R_y *as in the body of the proof of Theorem 1.2.1. These generators correspond exactly to the elements of the bunch of semichains* $\mathfrak{B}$ *indexing the horizontal stripes of* Θ_x *and* Θ_y.

Remark 1.2.4. *The functor* $\mathbf{H}$ *has the following properties.*

(1) $\mathbf{H}$ *is additive, full and preserves isomorphism classes of objects.*

(2) The essential image of $\mathbf{H}$ *consists of all pairs of matrices* (A, B) *in* $\mathrm{Rep}(\mathfrak{B})$ *such that* A *and* B *have both full row rank and* $\begin{bmatrix} A \\ B \end{bmatrix}$ *has full column rank.*

(3) $\mathbf{H}$ *is not faithful.*

Remark 1.2.5. *Let* $\mathsf{char}(\Bbbk) = 2$*. Then the simple curve singularities of type* A *have to be redefined according to Remark 1.1.24. It follows that the equation of* $\mathsf{P}_{\infty,2s}$ *should be* $\Bbbk[\![x, y, z]\!]/(xy, z(y^s - z))$*. Moreover, there are more singularities of type* $\mathsf{P}_{\infty,2s+1}$*, namely*

$$\mathsf{P}^t_{\infty,2s+1} := \Bbbk[\![x, y, z]\!]/(xy, y^{2s+1} + y^{s+t}z - z^2), \qquad 1 \le t \le s-1.$$

Nevertheless, they are all tame and the proof of Theorem 1.2.1 applies literally to this case as well.

Remark 1.2.6. *For any* $q \in \mathbb{N}^+ \cup \{\infty\}$ *consider the hypersurface singularity*

$$\mathsf{T} = \mathsf{T}_{\infty,q+2} = \Bbbk[\![a, b]\!]/\big(b^2(a^2 - b^q)\big).$$

Observe that R *is an overring of* T *via the embedding*

$$\mathsf{T} \longrightarrow \mathsf{R}, \qquad a \longmapsto x + v, \quad b \longmapsto y + u$$

where R *is the ring defined by (1.2.3). It was shown in* [**BD**, Section 11.1] *that* $\mathrm{CM}(\mathsf{T})$ *has tame representation type (under the additional assumption* $\mathsf{char}(\Bbbk) = 0$*). This gives another argument that* $\mathrm{CM}(\mathsf{R})$*, and hence* $\mathrm{CM}(\mathsf{P})$*, has either tame or* *discrete Cohen-Macaulay type. The latter case does also occur: if $q = 1$, then $\mathsf{T}_{\infty 3}$ is representation tame whereas*

$$\mathsf{P}_{\infty 1} = \Bbbk[\![x, y, z]\!]/(xy, y - z^2) \cong \Bbbk[\![x, z]\!]/(xz^2) =: \mathsf{D}_\infty$$

is representation discrete [**BGS87**].

1.3 Indecomposable Cohen-Macaulay modules over $\mathsf{P}_{\infty\infty}$ and $\mathsf{T}_{\infty\infty}$

In this section we shall explain that the technique of matrix problems, introduced in the course of the proof of Theorem 1.2.1, leads to a completely explicit description of indecomposable Cohen-Macaulay modules over $\mathsf{P} = \mathsf{P}_{\infty\infty} = \Bbbk[\![x, y, z]\!]/(xy, z^2)$. Although P is the "maximal degeneration" of the family (1.2.1), the combinatorics of the indecomposable objects in $\mathrm{CM}(\mathsf{P})$ are more transparent than for the less degenerate singularities $\mathsf{P}_{\infty,2r}$. The reason is that the underlying matrix problem has the type *representations of bunches of chains* and not of *semichains* as for $\mathsf{P}_{\infty,2r}$. Another motivation to study Cohen-Macaulay modules over P is that it allows to construct interesting examples of Cohen-Macaulay modules over the hypersurface singularity $\mathsf{T} = \mathsf{T}_{\infty\infty} = \Bbbk[\![a, b]\!]/(a^2b^2)$. Moreover, the explicit classification of indecomposable Cohen-Macaulay modules over P yields a classification for any curve singularity of type $\mathsf{P}_{\infty,2s+1}$ or $\mathsf{P}_{2r+1,2s+1}$.

Until the end of this section we keep the following notation:

- $\mathsf{R} = \Bbbk[\![x, y, u, v]\!]/(xy, yu, uv, vx, u^2, v^2)$ is the minimal overring of P. The embedding $\mathsf{P} \longrightarrow \mathsf{R}$ sends z to $u + v$.
- Let $\mathsf{S} = \mathsf{S}_x \times \mathsf{S}_y = \Bbbk[\![x, u]\!]/(u^2) \times \Bbbk[\![y, v]\!]/(v^2)$.
- For any $l \in \mathbb{N}^+$ we denote $X_l = (u, x^l)_{\mathsf{S}}$ and $Y_l = (v, y^l)_{\mathsf{S}}$. Next, we pose $X_0 = \mathsf{S}_x$, $Y_0 = \mathsf{S}_y$, $X_\infty = (u)_{\mathsf{S}}$ and $Y_\infty = (v)_{\mathsf{S}}$.
- Let $\mathsf{Q} = \Bbbk(\!(x)\!)[u]/(u^2) \times \Bbbk(\!(y)\!)[v]/(v^2)$.
- Denote $\mathfrak{m} = (x, y, u, v)_{\mathsf{R}}$. Recall that $\mathfrak{m} = \mathsf{ann}_{\mathsf{R}}(\mathsf{S}/\mathsf{R}) = \mathsf{rad}(\mathsf{S})$.

Observe that Q is the common total ring of fractions of P, R and S. In particular, we have the following equalities in Q:

$$u = \frac{xz}{x+y}, \quad v = \frac{yz}{x+y}, \quad e_x := (1, 0) = \frac{x}{x+y} \quad \text{and} \quad e_y := (0, 1) = \frac{y}{x+y}. \tag{1.3.1}$$

Next, note that R is also an overring of T. Indeed, we have an injective ring homomorphism $\jmath\colon \mathsf{T} \longrightarrow \mathsf{R}$ given by $a \mapsto x + v$ and $b \mapsto y + u$. It is also not difficult to see that $\jmath$ induces an isomorphism of the total rings of fractions $\mathsf{Q}(\mathsf{T}) \longrightarrow \mathsf{Q}(\mathsf{R}) = \mathsf{Q}$.

Summing up, we have the following diagram of categories and functors:

$$\begin{tikzcd}
\mathrm{CM}(\mathsf{P}) & & & \\
& \mathrm{CM}(\mathsf{R}) \arrow[ul, hook', "\mathbf{I}"'] \arrow[dl, hook, "\mathbf{J}"] \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\sim"] & \mathrm{Tri}(\mathsf{R}) \arrow[l, shift left, "\mathbf{G}"] \arrow[r, "\mathbf{H}"] & \mathrm{Rep}(\mathfrak{B}) \\
\mathrm{CM}(\mathsf{T}) & & &
\end{tikzcd} \qquad (1.3.2)$$

- $\mathbf{I}$ and $\mathbf{J}$ denote restriction functors. According to Proposition 1.1.9, they are both fully faithful. Moreover, by Theorem 1.1.11 $\mathrm{ind}[\mathrm{CM}(\mathsf{P})] = \{\mathsf{P}\} \cup \mathrm{ind}[\mathrm{CM}(\mathsf{R})]$.
- $\mathbf{F}$ and $\mathbf{G}$ are quasi-inverse equivalences of categories from Theorem 1.1.21.
- The functor $\mathbf{H}$ assigns to a triple $\bigl(V, N, (\Theta_x, \Theta_y)\bigr)$ the pair of matrices (Θ_x, Θ_y), whose rows are equipped with some additional "weights". $\mathbf{H}$ preserves isomorphism classes of objects as well as their indecomposability. However, $\mathbf{H}$ is not essentially surjective because Θ_x and Θ_y obey some additional constraints, see (1.3.3).

The goal of this section is to show how one can translate the combinatorics of indecomposable objects of $\mathrm{Rep}(\mathfrak{B})$ into an explicit description of indecomposable objects of $\mathrm{CM}(\mathsf{P})$ and $\mathrm{CM}(\mathsf{T})$.

1.3.1 Indecomposable objects of $\mathrm{Rep}(\mathfrak{B})$

According to Theorem 1.2.1, the matrix problem corresponding to a description of isomorphism classes of objects in $\mathrm{Rep}(\mathfrak{B})$ is as follows.

We are given two matrices Θ_x and Θ_y as depicted in Figure 1.3.1 with entries from an algebraically closed field $\Bbbk$ and the same number of columns. The rows of Θ_x and Θ_y are both divided into horizontal blocks. Any two horizontal blocks in Θ_x (respectively Θ_y) connected by a dotted line have the same number of rows.

Transformation rules.
The following transformations of columns and rows of Θ_x and Θ_y are admissible:

- any *simultaneous* elementary transformation of columns of Θ_x and Θ_y.
- addition of any multiple of any row of Θ_x (respectively, Θ_y) with lower weight to any row of Θ_x (respectively, Θ_y) with higher weight.
- any *simultaneous* elementary transformation of rows within horizontal blocks of Θ_x (respectively, Θ_y) connected by a dotted line.
- any elementary transformation of rows in the horizontal block of Θ_x (respectively, Θ_y) which is not connected to any other block by a dotted line.

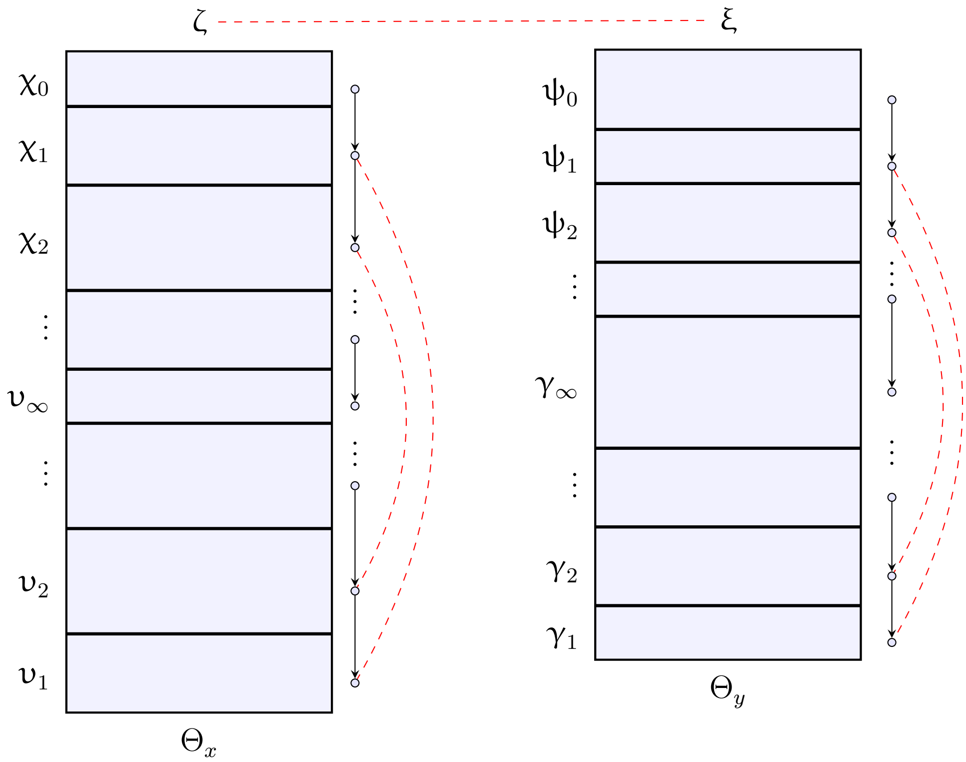


FIGURE 1.3.1. **Matrix problem for the case $q = \infty$**

Additionally, there are the following *regularity constraints* on Θ_x and Θ_y:

$$\begin{gathered} \Theta_x \text{ and } \Theta_y \text{ have both full row rank,} \\ \text{the matrix } \begin{bmatrix} \Theta_x \\ \Theta_y \end{bmatrix} \text{ has full column rank.} \end{gathered} \tag{1.3.3}$$

In this subsection we are going to apply Bondarenko's result [**Bon91**], which is also summarized in Appendix A, to describe the canonical forms of the matrix problem above.

Definition 1.3.1. *Consider the following data The data:*

- *The index set $I = \{x, y\}$.*
- *The set of column symbols $\mathfrak{C} = \mathfrak{C}_x \cup \mathfrak{C}_y$, where $\mathfrak{C}_x = \{\zeta\}$, $\mathfrak{C}_y = \{\xi\}$.*
- *The set of row symbols $\mathfrak{R} = \mathfrak{R}_x \cup \mathfrak{R}_y$, where*

$$\mathfrak{R}_x = \{ \chi_0 < \chi_1 < \cdots < \chi_i < \cdots < \upsilon_\infty < \cdots < \upsilon_i < \cdots < \upsilon_1 \},$$
$$\mathfrak{R}_y = \{ \psi_0 < \psi_1 < \cdots < \psi_j < \cdots < \gamma_\infty < \cdots < \gamma_j < \cdots < \gamma_1 \}.$$

- *The set $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C}$ is equipped with an equivalence relation $\sim$ defined as follows:*

$$\zeta \sim \xi, \qquad \chi_l \sim \upsilon_l \quad \textit{and} \quad \psi_l \sim \gamma_l \quad \textit{for } l \in \mathbb{N}^+.$$

In addition, we introduce another symmetric relation $-$ on the set $\mathfrak{B}$ as follows:

$$\zeta - \varrho_x \textit{ for any } \varrho_x \in \mathfrak{R}_x \qquad \textit{and} \qquad \xi - \varrho_y \textit{ for any } \varrho_y \in \mathfrak{R}_y.$$

The data $\mathfrak{B} = (\mathfrak{R}, \mathfrak{C}, \sim, -)$ is the alphabet of a *bunch of chains* in the sense of Definition A.1.3.

The problem to classify the indecomposable objects of the category Rep($\mathfrak{B}$) up to isomorphism is exactly the matrix problem above without the "regularity conditions" in (1.3.3).

Now, we define *strings* and *bands* of the bunch of chains $\mathfrak{B}$ following Section A.2. They describe the invariants of the indecomposable representations in Rep($\mathfrak{B}$).

Definition 1.3.2. *Let $\mathfrak{B}$ be the bunch of chains from Definition 1.3.1.*

(1) A finite word *w of $\mathfrak{B}$ is a sequence*

$$w = \alpha_1\rho_1\alpha_2\rho_2 \dots \alpha_{k-1}\rho_{k-1}\alpha_k$$

of symbols $\alpha_j \in \mathfrak{B}$ and relations $\rho_j \in \{\sim, -\}$ subject to the following conditions:

- *the relation $\alpha_j\rho_j\alpha_{j+1}$ holds in $\mathfrak{B}$ for $1 \le j \le k-1$.*
- *the sequence of relations alternates, that is, $\rho_j \neq \rho_{j+1}$ for $1 \le j \le k-2$.*
- *either $\alpha_1 \in \{\chi_0, \upsilon_\infty, \psi_0, \gamma_\infty\}$ or ρ_1 is $\sim$.*
- *either $\alpha_k \in \{\chi_0, \upsilon_\infty, \psi_0, \gamma_\infty\}$ or ρ_{k-1} is $\sim$.*

(2) The reversed word *of a finite word w as above is defined by*

$$w^{\mathrm{rev}} = \alpha_k\rho_{k-1}\alpha_{k-1}\rho_{k-2} \dots \alpha_2\rho_1\alpha_1.$$

(3) A (usual) string *of $\mathfrak{B}$ is given by any finite word w.*

(4) Two strings w and v are equivalent *if and only if $v = w$ or $v = w^{\mathrm{rev}}$.*

Definition 1.3.3. *Let $\mathfrak{B}$ be the bunch of chains from Definition 1.3.1.*

(1) A cyclic word *w is given by a sequence*

$$w = \alpha_1 \sim \alpha_2 - \alpha_3 \sim \alpha_4 - \quad \dots \quad \alpha_{2k-1} \sim \alpha_{2k} - \tag{1.3.4}$$

of an even number of letters $\alpha_j \in \mathfrak{A}_{\mathfrak{B}}$, where $1 \le j \le 2k$, subject to the following conditions:

- *$\alpha_{2j-1} \sim \alpha_{2j}$ holds in $\mathfrak{B}$ for any $1 \le j \le k$,*
- *$\alpha_{2j} - \alpha_{2j+1}$, where $1 \le j < k$, and also $\alpha_{2k} - \alpha_1$ hold in $\mathfrak{B}$.*

(2) Let $0 \le j < 2k$ be even. *The j*-th rotation *$w^{[j]}$ of w is defined as*

$$w^{[j]} = \alpha_{j+1} \sim \alpha_{j+2} - \alpha_{j+3} \sim \alpha_{j+4} - \quad \dots \quad \alpha_{j+2k-1} \sim \alpha_{j+2k} -$$

where all indices are considered modulo $2k$.

(3) The cyclic word w is periodic *if $w = w^{[j]}$ for some non-trivial even index $2 \le j < 2k$.*

(4) The reversed word *w^{rev} of w is defined as*

$$w^{rev} = \alpha_{2k} \sim \alpha_{2k-1} - \quad \dots \quad \alpha_4 \sim \alpha_3 - \alpha_2 \sim \alpha_1 - .$$

(5) A band *(w, m, λ) of $\mathfrak{B}$ consists of a non-periodic cyclic word w, a multiplicity parameter $m \in \mathbb{N}^+$ and a continuous parameter $\lambda \in \mathbb{k}^*$.*

(6) *Two bands* (w, m, λ) *and* (v, n, μ) *are* equivalent *if and only if both words* w *and* v *have the same length* n *and if* (v, n, μ) *is given by* $(w^{\mathrm{rev}}, m, \lambda)$ *or* $(w^{[4j]}, m, \lambda)$ *or* $(w^{[4j+2]}, m, \lambda^{-1})$ *for some* $j \in \mathbb{Z}_{2k}$.

The above definitions are motivated by the following result, which is a special case of Theorem A.2.21 by Bondarenko [**Bon91**].

Theorem 1.3.4. *There is a bijection between the equivalence classes of strings and bands of the bunch of chains* $\mathfrak{B}$ *and the isomorphism classes of indecomposable objects in the category* $\mathrm{Rep}(\mathfrak{B})$:

$$[\text{STRINGS } \textit{and} \text{ BANDS } \textit{of}\ \mathfrak{B}] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Rep}(\mathfrak{B})]$$

Now we explain Bondarenko's construction of indecomposable objects in $\mathrm{Rep}(\mathfrak{B})$ corresponding to a string or band, which is also stated in Subsection A.3.1.

1. Let w be a string of $\mathfrak{B}$. The corresponding object $\Theta(w)$ of $\mathrm{Rep}(\mathfrak{B})$ is given by a pair of matrices $\Theta_x(w)$ and $\Theta_y(w)$ defined as follows:

(1) Let t be the number of times the symbol $\mathfrak{z}$ (or $\mathfrak{x}$) occurs as a letter in w. Then both matrices $\Theta_x(w)$ and $\Theta_y(w)$ have t columns.

(2) For each $\varrho \in \mathfrak{R}$, let m_ϱ be the number of times the symbol ϱ occurs as a letter in w. Then the horizontal block ϱ in $\Theta_x(w)$ (respectively $\Theta_y(w)$) has m_ϱ rows.

(3) Next, we assign to every letter α_j in w the number of times the letter α_j occurred in the subword $\alpha_1\rho_1 \dots \rho_{j-1}\alpha_j$. In other words, we number every letter in w by the time it occurs in w.

(4) Every appearance of the relation $-$ in w contributes to a non-zero entry in $\Theta(w)$ in the following way. Let $\varrho - \zeta$ or $\zeta - \varrho$ be a subword in w such that $\varrho \in \mathfrak{R}$ and $\zeta \in \mathfrak{C}$. Let i be the occurrence number of ϱ and j be the occurrence number of ζ. We fill the entries of Θ_x respectively Θ_y according to the following rule.

   - If $\zeta = \mathfrak{z}$ (respectively $\zeta = \mathfrak{x}$), the (i, j)-th entry of the ϱ-th horizontal block of Θ_x (respectively Θ_y) is set to be 1. This rule is applied for every relation $-$ in w.
   - All remaining entries of Θ_x and Θ_y are set to be 0.

2. Let (w, m, λ) be a band. The corresponding object $\Theta(w, m, \lambda)$ of $\mathrm{Rep}(\mathfrak{B})$ is given by a pair of matrices $\big(\Theta_x(w, m, \lambda), \Theta_y(w, m, \lambda)\big)$ defined as follows.

(1) First construct the canonical form $\Theta(w)$, where w is viewed as a string.

(2) Replace any zero entry in $\Theta_x(w)$ (respectively $\Theta_y(w)$) by the zero matrix of size m and any identity entry in $\Theta_x(w)$ (respectively $\Theta_y(w)$) by the identity matrix I of size m.

(3) We consider the last relation $\rho_{2k} = -$ in w. Then $\alpha_{2k}\rho_{2k}\alpha_1$ in $\mathfrak{B}$ and either α_1 or $\alpha_{2k} \in \mathfrak{R}$. Replace the zero matrix at the intersection of the first m rows of the horizontal block indexed by α_1 (respectively α_{2k}) and its last m columns by the Jordan block $J_m(\lambda)$ with eigenvalue λ and size m.

Remark 1.3.5. *Let Θ be the object of* $\mathrm{Rep}(\mathfrak{B})$ *associated to a string or band. It is straightforward to show, that Θ satisfies the regularity constraints (1.3.3) if and only if the following two conditions are satisfied:*

- *either w is cyclic or w begins and ends with symbols from $\mathfrak{C}$.*
- *w is neither equal to $\zeta \sim \xi$, nor equal to $\xi \sim \zeta$.*

Next, we give some examples of canonical forms of bands and strings.

Example 1.3.6. *Consider a band (w, m, λ) where w is given by the following cyclic word:*

$$w = \underset{\#1}{\xi} \sim \underset{\#1}{\zeta} - \underset{\#1}{\chi_i} \sim \underset{\#1}{\upsilon_i} - \underset{\#2}{\zeta} \sim \underset{\#2}{\xi} - \underset{\#1}{\psi_{j_1}} \sim \underset{\#1}{\gamma_{j_1}} - \underset{\#3}{\xi} \sim \underset{\#3}{\zeta} - \underset{\#2}{\upsilon_i} \sim \underset{\#2}{\chi_i} - \underset{\#4}{\zeta} \sim \underset{\#4}{\xi} - \underset{\#1}{\psi_{j_2}} \sim \underset{\#1}{\gamma_{j_2}} -$$

Then the corresponding canonical forms $\big(\Theta_x(w, m, \lambda), \Theta_y(w, m, \lambda)\big)$ are the following:

$$\begin{array}{c} \\ \chi_i \\ \\ \upsilon_i \\ \\ \end{array}\overset{\zeta}{\left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right]} \qquad \overset{\xi}{\left[\begin{array}{cccc} 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ \hline 0 & 0 & I & 0 \\ \hline J & 0 & 0 & 0 \end{array}\right]}\begin{array}{c} \psi_{j_1} \\ \psi_{j_2} \\ \gamma_{j_1} \\ \gamma_{j_2} \end{array}$$

Here, I is the identity matrix of size m and $J = J_m(\lambda)$ is the Jordan block of size m with eigenvalue $\lambda \in \mathbb{k}^*$.

Example 1.3.7. *Consider the string given by the word*

$$w = \xi \sim \zeta - \chi_i \sim \upsilon_i - \zeta \sim \xi - \psi_j \sim \gamma_j - \xi \sim \zeta.$$

Then the corresponding canonical forms $\big(\Theta_x(w), \Theta_y(w)\big)$ are the following:

$$\begin{array}{c} \chi_i \\ \upsilon_i \end{array}\overset{\zeta}{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]} \qquad \overset{\xi}{\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}\begin{array}{c} \psi_j \\ \gamma_j \end{array}$$

Example 1.3.8. *Consider the string given by the word*

$$w = \chi_0 - \zeta \sim \xi - \psi_{j_1} \sim \gamma_{j_1} - \xi \sim \zeta - \chi_i \sim \upsilon_i - \zeta \sim \xi - \gamma_{j_2} \sim \psi_{j_2} - \xi \sim \zeta$$

Then the corresponding canonical forms $\big(\Theta_x(w), \Theta_y(w)\big)$ are the following:

$$\begin{array}{c} \chi_0 \\ \chi_i \\ \upsilon_i \end{array}\overset{\zeta}{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]} \qquad \overset{\xi}{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]}\begin{array}{c} \psi_{j_1} \\ \psi_{j_2} \\ \gamma_{j_1} \\ \gamma_{j_2} \end{array}$$

Example 1.3.9. *Consider the string given by the word*

$$w = \upsilon_\infty - \zeta \sim \xi - \gamma_j \sim \psi_j - \xi \sim \zeta - \chi_i \sim \upsilon_i - \zeta \sim \xi - \gamma_\infty.$$

Then the corresponding canonical forms $\big((\Theta_x(w), \Theta_y(w)\big)$ *are the following:*

$$\begin{array}{cc} & \zeta \\ \chi_i & \\ \upsilon_\infty & \left|\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right| \\ \upsilon_i & \end{array} \qquad \begin{array}{cc} \xi & \\ & \psi_j \\ \left|\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right| & \gamma_\infty \\ & \gamma_j \end{array}$$

1.3.2 Indecomposable Cohen-Macaulay modules over $\mathsf{P}_{\infty\infty}$

Consider an object $T = \big(V, N, (\Theta_x, \Theta_y)\big)$ of the category $\mathrm{Tri}(\mathsf{R})$. Then we may assume that

- $V = \Bbbk^t$ for some $t \in \mathbb{N}_0$.
- $N = X_{i_1} \oplus \cdots \oplus X_{i_k} \oplus Y_{j_1} \oplus \cdots \oplus Y_{j_l} \subseteq \mathsf{S}^{k+l}$ for certain indices $i_1, \ldots, i_k, j_1, \ldots, j_l \in \mathbb{N}_0 \cup \{\infty\}$, see the beginning of Section 1.3 for the definition of S-modules X_i and Y_i for $i \in \mathbb{N}_0 \cup \{\infty\}$.

According to Theorem 1.1.21, the corresponding Cohen-Macaulay R-module $M = \mathbf{G}(T)$ is determined by the following commutative diagram in R - mod:

$$\begin{tikzcd} 0 \arrow[r, dashed] & \mathfrak{m}N \arrow[r, dashed] \arrow[d, equal] & M \arrow[r, dashed, "\sigma"] \arrow[d, hook, dashed] & \Bbbk^t \arrow[r, dashed] \arrow[d, hook, "\vartheta"] & 0 \\ 0 \arrow[r] & \mathfrak{m}N \arrow[r] & N \arrow[r, "\pi"] & N/\mathfrak{m}N \arrow[r] & 0 \end{tikzcd} \tag{1.3.5}$$

Lemma 1.3.10. *Let* $\{e_1, \ldots, e_t\}$ *be the standard basis of* $\Bbbk^t$. *For any* $1 \le i \le t$ *choose* $w_i \in N$ *such that* $\pi(w_i) = \vartheta(e_i)$. *Then we have:*

$$M = \langle w_1, \ldots, w_t \rangle_{\mathsf{R}} \subseteq \mathsf{S}^{k+l} \tag{1.3.6}$$

and t is the minimal number of generators of M.

PROOF. By definition of M, for any $1 \le i \le t$ we have: $w_i \in M$. Next, the induced map $\bar{\sigma} : M/\mathfrak{m}M \longrightarrow \Bbbk^t$ is an isomorphism (see the proof of [**BD**, Theorem 12.5]) and $\bar{\sigma}(w_i) = e_i$. Hence, $\{\bar{w}_1, \ldots, \bar{w}_t\}$ is a basis of $M/\mathfrak{m}M$ and (1.3.6) follows from Nakayama's Lemma. □

Lemma 1.3.11. *Let* $T = \big(\Bbbk^t, N, (\Theta_x, \Theta_y)\big)$ *be an indecomposable object of* $\mathrm{Tri}(\mathsf{R})$ *as above and* $M = \mathbf{G}(T) = \langle w_1, \ldots, w_t \rangle_{\mathsf{R}} \subseteq \mathsf{S}^{k+l}$. *Then the following results are true.*

(1) We have either

- $T \cong \Big(\Bbbk, \mathsf{S}, \big((1), (1)\big)\Big)$, *in this case* $M \cong \mathsf{R}$, *or*
- $\mathbf{I}(M) = \langle w_1, \ldots, w_t \rangle_{\mathsf{P}} \subseteq \mathsf{S}^{k+l}$.

In both cases we have:

$$\mathbf{J}(M) = \big\langle w_1, \dots, w_t, vw_1, \dots, vw_t, uw_1, \dots, uw_t \big\rangle_{\mathsf{T}} \subseteq \mathsf{S}^{k+l}.$$

(2) The Cohen-Macaulay module M, respectively $\mathbf{I}(M)$ and $\mathbf{J}(M)$, is locally free on the punctured spectrum of R, respectively P and T, if and only if N contains no direct summands isomorphic to X_∞ or Y_∞.

PROOF. (1) The statement about $\mathbf{I}(M)$ is a corollary of Proposition 1.1.13. The statement about $\mathbf{J}(M)$ follows from the fact that $\mathbf{J}(\mathsf{T}) = \langle 1, u, v\rangle_{\mathsf{R}} \subset Q$ in R - mod.

(2) The Cohen-Macaulay module M is locally free on the punctured spectrum of R if and only $M \cong \mathsf{S} \boxtimes_{\mathsf{R}} M$ is locally free on the punctured spectrum of S, see Remark 1.1.15. The latter is equivalent that N contains no direct summands isomorphic to X_∞ and Y_∞. Since the rational envelopes of M, $\mathbf{I}(M)$ and $\mathbf{J}(M)$ are the same, the result follows. $\square$

Now we can state the final classification of the indecomposable Cohen-Macaulay P-modules. For any $l \in \mathbb{N}^+$ introduce the letters $\mathbf{x}_l^{\pm}$ and $\mathbf{y}_l^{\pm}$ as well as $\mathbf{x}_0, \mathbf{x}_\infty, \mathbf{y}_0$ and $\mathbf{y}_\infty$.

Definition 1.3.12. Band modules $B = B(\omega, m, \lambda)$ *have the following combinatorics:*

- $\omega = \mathbf{x}_{i_1}^{\sigma_1}\mathbf{y}_{j_1}^{\tau_1} \dots \mathbf{x}_{i_n}^{\sigma_n}\mathbf{y}_{j_n}^{\tau_n}$ *is a* non-periodic *word, where* $\sigma_k, \tau_k \in \{+, -\}$ *and* $i_k, j_k \in \mathbb{N}^+$ *for* $1 \le k \le n$.
- $m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^*$.

Consider the following Cohen-Macaulay S*-module*

$$N := X_{i_1}^{\oplus m} \oplus Y_{j_1}^{\oplus m} \oplus \dots \oplus X_{i_n}^{\oplus m} \oplus Y_{j_n}^{\oplus m} \subseteq \mathsf{S}^{2mn}.$$

Then B is the following R*-submodule of N:*

$$B := \left[\begin{pmatrix} f''_{i_1} I \\ g'_{j_1} I \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g''_{j_1} I \\ f'_{i_2} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ f''_{i_n} I \\ g'_{j_n} I \end{pmatrix}, \begin{pmatrix} f'_{i_1} J \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ g''_{j_n} I \end{pmatrix} \right]_{\mathsf{R}} \subseteq N, \quad (1.3.7)$$

where for any $1 \le k \le n$ the elements $f'_{i_k}, f''_{i_k}, g'_{j_k}$ g''_{j_k} are defined by the tables:

σ_k	f'_{i_k}	f''_{i_k}
$+$	u	x^{i_k}
$-$	x^{i_k}	u

τ_k	g'_{j_k}	g''_{j_k}
$+$	v	y^{j_k}
$-$	y^{j_k}	v

(1.3.8)

Definition 1.3.13. *A* string module $S = S(\omega)$ *is defined by a word ω of the following form. ω has a beginning, an intermediate part and an end. The beginning as well as the end may consist of zero, one or two letters. The following table lists all* *possible beginnings and ends for ω (any beginning from the first column can match any ending from the last column):*

$\hat{\mathbf{z}}_{i_0}\check{\mathbf{z}}_{j_0}^{\tau_0}$	*intermediate part*	$\hat{\mathbf{z}}_{i_n}^{\sigma_n}\check{\mathbf{z}}_{j_n}$
void		*void*
$\mathbf{y}_0$		$\mathbf{x}_0$
$\mathbf{y}_\infty$	$\mathbf{x}_{i_1}^{\sigma_1}\mathbf{y}_{j_1}^{\tau_1}\ldots\mathbf{x}_{i_{n-1}}^{\sigma_{n-1}}\mathbf{y}_{j_{n-1}}^{\tau_{n-1}}$	$\mathbf{x}_\infty$
$\mathbf{y}_{j_0}^{\tau_0}$		$\mathbf{x}_{i_n}^{\sigma_n}$
$\mathbf{x}_0\mathbf{y}_{j_0}^{\tau_0}$		$\mathbf{x}_{i_n}^{\sigma_n}\mathbf{y}_0$
$\mathbf{x}_\infty\mathbf{y}_{j_0}^{\tau_0}$		$\mathbf{x}_{i_n}^{\sigma_n}\mathbf{y}_\infty$

where

- $n \in \mathbb{N}^+$. *For $n = 1$ the intermediate part of ω is void.*
- *For any $0 \leq k \leq n$ we have: $i_k, j_k \in \mathbb{N}^+$ and $\sigma_k, \tau_k \in \{+, -\}$.*

In other words, a string word ω is given by an alternating sequence of letters $\mathbf{x}_i$ or $\mathbf{y}_j$ such that a letter of the form $\mathbf{x}_0, \mathbf{x}_\infty, \mathbf{y}_0$ or $\mathbf{y}_\infty$ may only occur as the first or last letter of ω.

Consider the Cohen-Macaulay S-module

$$N = \hat{Z}_{i_0} \oplus \check{Z}_{j_0} \oplus X_{i_1} \oplus Y_{j_1} \oplus \cdots \oplus X_{i_{n-1}} \oplus Y_{j_{n-1}} \oplus \hat{Z}_{i_n} \oplus \check{Z}_{j_n},$$

where for each $i \in \{i_0, i_n\}$ and each $j \in \{j_0, j_n\}$, we set

$$\hat{Z}_i = \begin{cases} X_i & \text{if } \mathbf{z}_i = \mathbf{x}_i, \\ 0 & \text{if } \mathbf{z}_i \text{ is void.} \end{cases} \qquad \check{Z}_j = \begin{cases} Y_j & \text{if } \mathbf{z}_j = \mathbf{y}_j, \\ 0 & \text{if } \mathbf{z}_j \text{ is void.} \end{cases}$$

Then $S = S(\omega)$ is the following R-submodule of N:

$$S := \left[\begin{pmatrix} f''_{i_0} \\ g'_{j_0} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g''_{j_0} \\ f'_{i_1} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g''_{j_{n-1}} \\ f'_{i_n} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f''_{i_n} \\ g'_{j_n} \end{pmatrix} \right]_{\mathsf{R}} \subseteq N \qquad (1.3.9)$$

where for any $1 \leq k \leq n$, the elements $f'_{i_k}, f''_{i_k}, g'_{j_k}$ and g''_{j_k} are defined by the tables:

σ_k	f'_{i_k}	f''_{i_k}
$+$	ux	x^{i_k+1}
$-$	x^{i_k+1}	ux

τ_k	g'_{j_k}	g''_{j_k}
$+$	vy	y^{j_k+1}
$-$	y^{j_k+1}	vy

(1.3.10)

The remaining entries are defined as follows:

	f''_{i_0}
$i_0 = 0$	x
$i_0 = \infty$	ux
$\hat{\mathbf{z}}_{i_0}$ *is void*	0

	g'_{j_n}
$j_n = 0$	y
$j_n = \infty$	vy
$\check{\mathbf{z}}_{j_n}$ *is void*	0

(1.3.11)

	σ_n	f'_{i_n}	f''_{i_n}
$i_n = 0$		x	0
$i_n \in \mathbb{N}^+$	$+$	ux	x^{j_n+1}
$i_n \in \mathbb{N}^+$	$-$	x^{j_n+1}	ux
$i_n = \infty$		ux	0

	τ_0	g'_{j_0}	g''_{j_0}
$j_0 = 0$		0	y
$j_0 \in \mathbb{N}^+$	$+$	vy	y^{j_0+1}
$j_0 \in \mathbb{N}^+$	$-$	y^{j_0+1}	vy
$j_0 = \infty$		0	vy

Remark 1.3.14. *If the string parameter ω contains neither $\mathbf{x}_0$ nor $\mathbf{y}_0$, there is a better presentation of the module $S(\omega)$: we divide all entries of type f'_i or f''_i by x and all entries of type g'_j or g''_j by y.*

Remark 1.3.15. *Any string module S in* (1.3.9) *has a more compact presentation by "merging" every odd row with its subsequent row:*

$$S \cong \left[\begin{pmatrix} f''_{i_0}+g'_{j_0} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g''_{j_0} \\ f'_{i_1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f''_{i_1}+g'_{j_1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g''_{j_{n-1}} \\ f'_{i_n} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f''_{i_n}+g'_{j_n} \end{pmatrix}\right]_{\mathsf{R}}$$

The same can be done with the horizontal stripes of any band module B in (1.3.7)*:*

$$B \cong \left[\begin{pmatrix} (f''_{i_1}+g'_{j_1})I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} g''_{j_1}I \\ f'_{i_2}I \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (f''_{i_n}+g'_{j_n})I \end{pmatrix}, \begin{pmatrix} f'_{i_1}J \\ 0 \\ \vdots \\ g''_{j_n}I \end{pmatrix}\right]_{\mathsf{R}}$$

Theorem 1.3.16. *For the ring $\mathsf{R} = \Bbbk[\![x,y,u,v]\!]/(xy,xv,yu,uv,u^2,v^2)$ the classification of the indecomposable objects of* $\mathrm{CM}(\mathsf{R})$ *is the following.*

- *The modules $B(\omega,m,\lambda)$ and $S(\omega)$ are indecomposable. Moreover, any indecomposable Cohen-Macaulay R-module is isomorphic to some band or some string module.*
- *$B(\omega,m,\lambda) \not\cong S(\check{\upsilon})$ for any choice of parameters $\omega, \check{\upsilon}, m$ and λ.*
- *$S(\omega) \cong S(\check{\upsilon})$ if and only if $\check{\upsilon} = \omega$ or $\check{\upsilon} = \omega^{\mathrm{rev}}$, where ω^{rev} is the reversed word.*
- *$B(\omega,m,\lambda) \cong B(\check{\upsilon},n,\mu)$ if and only if $m = n$, $\lambda = \mu$ and $\check{\upsilon}$ is given by a (possibly trivial) cyclic shift on all letters of ω.*

Proof. According to Theorem 1.2.1 the classification problem of indecomposable objects of $\mathrm{CM}(\mathsf{R})$ is equivalent to the matrix problem over the bunch of chains $\mathfrak{B}$

from Definition 1.3.1. More precisely, we had a diagram of categories and functors

$$\mathrm{CM}(\mathsf{R}) \underset{\mathbf{G}}{\overset{\mathbf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Tri}(\mathsf{R}) \xrightarrow{\mathbf{H}} \mathrm{Rep}(\mathfrak{B})$$

where $\mathbf{F}$ and $\mathbf{G}$ are mutually inverse equivalences of categories and $\mathbf{H}$ is a full functor preserving isomorphism classes and indecomposability of objects.

1. Indecomposable objects of the category $\mathrm{Rep}(\mathfrak{B})$ are classified by strings and bands according to Theorem 1.3.4. The construction of the canonical forms described below Theorem 1.3.4 follows from Theorem A.3.8 by Bondarenko. Moreover, the indecomposable objects of $\mathrm{Rep}(\mathfrak{B})$ lying in the essential image of $\mathbf{H}$ are described by Remark 1.3.5.

2. Let w be the word of a string or band in $\mathfrak{B}$, see Definition 1.3.2 and Definition 1.3.3. Note that w is uniquely determined by the symbols in $\mathfrak{R}$ it contains. It follows that we may delete all subwords of the form $\zeta \sim \xi$ or $\xi \sim \zeta$ and relations $-$ *without loss of information.* Now we can translate the remaining subwords as follows:

χ_0	$\upsilon_i \sim \chi_i$	$\chi_i \sim \upsilon_i$	υ_∞	ψ_0	$\gamma_j \sim \psi_j$	$\psi_j \sim \gamma_j$	γ_∞
$\mathbf{x}_0$	$\mathbf{x}_i^+$	$\mathbf{x}_i^-$	$\mathbf{x}_\infty$	$\mathbf{y}_0$	$\mathbf{y}_j^+$	$\mathbf{y}_j^-$	$\mathbf{y}_\infty$

This table allows to pass from a word w to a word ω as in Definitions 1.3.12 and 1.3.13.

3. Consider a string w (obeying the constraint from Remark 1.3.5) or a band (w, m, λ). In Subsection 1.3.1 we explained the construction of the corresponding indecomposable object $\Theta = \big(\Theta_x, \Theta_y\big)$ of $\mathrm{Rep}(\mathfrak{B})$. Now we give the construction of a triple $T = (V, N, \Theta)$ in $\mathrm{Tri}(\mathsf{R})$ such that $\mathbf{H}(T) = \Theta$. Let m_0, $m_i^\pm$, m_∞, n_0, $n_j^\pm$ respectively n_∞ be the number of times the letter $\mathbf{x}_0$, $\mathbf{x}_i^\pm$, $\mathbf{x}_\infty$, $\mathbf{y}_0$, $\mathbf{y}_j^\pm$ respectively $\mathbf{y}_\infty$ occurs in w. Let t be the number of times ζ (or ξ) occurs in w. Then $T = (V, N, \Theta)$, where

- $V = \Bbbk^t$,
- $N = X_0^{m_0} \oplus \bigoplus_{i\in\mathbb{N}^+} X_i^{m_i^+ + m_i^-} \oplus X_\infty^{m_\infty} \oplus Y_0^{n_0} \oplus \bigoplus_{j\in\mathbb{N}^+} Y_j^{n_j^+ + n_j^-} \oplus Y_\infty^{n_\infty}$,
- $\Theta = \big(\Theta_x, \Theta_y\big)$.

4. Now recall the construction of the indecomposable Cohen-Macaulay R-module $M = \mathbf{G}(T)$. Consider a basis of a $\Bbbk$-vector space $N/\mathfrak{m}N$ given by the images of the distinguished generators of the indecomposable direct summands of N. Let $\pi \colon N \to N/\mathfrak{m}N$ be the canonical projection and $\Theta|_V := \begin{bmatrix} \Theta_x \\ \Theta_y \end{bmatrix} \colon V \to N/\mathfrak{m}N$. By Theorem 1.1.21 we have:

$$M := \mathbf{G}(T) = \pi^{-1}\big(\mathrm{im}(\Theta|_V)\big) \subseteq N.$$

To compute M we do the following:

(1) We multiply each entry of $\Theta|_V$ with its horizontal weight.

(2) For each subword $\varrho' \sim \varrho''$ in w such that $\varrho', \varrho'' \in \mathfrak{R}$ we merge the two corresponding rows in $\Theta|_V$ and add their entries to each other.

(3) We translate all entries of the new matrix $\Theta|_V$ as follows:

χ_0	χ_k	υ_∞	υ_k	ψ_0	ψ_k	γ_∞	γ_k
e_1	x^k	u	u	e_2	y^k	v	v

$k \in \mathbb{N}^+$.

Lemma 1.3.11 implies that M is generated by the columns of the modified matrix $\Theta|_V$. Finally, observe that $M \cong (x+y)M$ in R - mod. Thus, if w is a string word containing one of the symbols χ_0 or ψ_0, one has to multiply the columns of $\Theta|_V$ with $(x+y)$ to obtain entries which lie in R. After permutation of rows in $\Theta|_V$, one obtains exactly the presentations (1.3.7) respectively (1.3.9) for the words in Definition 1.3.12 or 1.3.13.

5. The statement about the isomorphism classes of string modules in $\mathrm{CM}(\mathsf{R})$ is a direct translation of the corresponding result for the category $\mathrm{Rep}(\mathfrak{B})$ stated in Theorem A.3.8. Considering all pairwise non-equivalent bands (w, m, λ), we may assume that the last letter of w is ζ or ξ by the equivalence conditions in Definition 1.3.3. Then Theorem A.3.8 by Bondarenko yields the isomorphism conditions for band modules $B(w, m, \lambda)$ as stated above.

6. Summing up, the key point of the proof of Theorem 1.3.16 is that *by construction* we have the following isomorphisms in the category $\mathrm{Rep}(\mathfrak{B})$:

$$\mathbf{H} \circ \Big(\mathbf{F}\big(B(\omega, m, \lambda)\big)\Big) \cong \big(\Theta_x(w, m, \lambda), \Theta_x(w, m, \lambda)\big)$$

for a band module $B(\omega, m, \lambda)$ from Definition 1.3.12 and

$$\mathbf{H} \circ \Big(\mathbf{F}\big(S(\omega)\big)\Big) \cong \big(\Theta_x(w), \Theta_x(w)\big).$$

for a string module $S(\omega)$ from Definition 1.3.13. $\square$

Remark 1.3.17. *By Lemma 1.3.11 any band module $B(\omega, m, \lambda)$ is locally free on the punctured spectrum. A string module $S(\omega)$ is* not *locally free on the punctured spectrum if and only if ω contains a letter $\mathbf{x}_\infty$ or $\mathbf{y}_\infty$.*

Example 1.3.18. *In the following examples we translate the canonical forms of the preceding subsection into indecomposable Cohen-Macaulay modules over R as described in the proof of Theorem 1.3.16.*

(1) Let (ω, m, λ) be the band from Example 1.3.6.
Then ω corresponds to $\omega = \mathbf{x}_i^- \mathbf{y}_{j_1}^- \mathbf{x}_i^+ \mathbf{y}_{j_2}^-$ and its band module is given by

$$B(\omega, m, \lambda) \cong \left[\begin{pmatrix} x^i I \\ 0 \\ 0 \\ vJ \end{pmatrix}, \begin{pmatrix} uI \\ 0 \\ y^{j_1} I \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ uI \\ vI \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^i I \\ 0 \\ y^{j_2} I \end{pmatrix} \right]_{\mathsf{R}}$$

$$\cong \left[\begin{pmatrix} x^i I \\ vJ \end{pmatrix}, \begin{pmatrix} (u + y^{j_1}) I \\ 0 \end{pmatrix}, \begin{pmatrix} vI \\ uI \end{pmatrix}, \begin{pmatrix} 0 \\ (x^i + y^{j_2}) I \end{pmatrix} \right]_{\mathsf{R}}$$

Here, J denotes the Jordan block with eigenvalue λ and I the identity matrix, both of size m.

(2) *Let w be the string from Example 1.3.7.*
Then w corresponds to $\omega = \mathbf{x}_i^- \mathbf{y}_j^-$ and its string module is given by

$$S(\omega) \cong \left[\begin{pmatrix} x^i \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ y^j \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix} \right]_{\mathsf{R}} \cong (x^i, u + y^j, v)_{\mathsf{R}}$$

(3) *Let w be the string from Example 1.3.8.*
Then w corresponds to $\omega = \mathbf{x}_0 \mathbf{y}_{j_1}^- \mathbf{x}_i^- \mathbf{y}_{j_2}^+$ and its string module is given by

$$\begin{aligned} S(\omega) &\cong \left[\begin{pmatrix} x \\ 0 \\ y^{j_1+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{i+1} \\ vy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ ux \\ 0 \\ vy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^{j_2+1} \end{pmatrix} \right]_{\mathsf{R}} \\ &\cong \left[\begin{pmatrix} x + y^{j_1+1} \\ 0 \end{pmatrix}, \begin{pmatrix} vy \\ x^{i+1} \end{pmatrix}, \begin{pmatrix} 0 \\ ux + vy \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j_2+1} \end{pmatrix} \right]_{\mathsf{R}} \end{aligned}$$

(4) *Let w be the string from Example 1.3.9.*
Then w corresponds to $\omega = \mathbf{x}_\infty \mathbf{y}_j^+ \mathbf{x}_i^- \mathbf{y}_\infty$ and its string module is given by

$$S(\omega) \cong \left[\begin{pmatrix} u \\ 0 \\ v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^i \\ y^j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u \\ 0 \\ v \end{pmatrix} \right]_{\mathsf{R}} \cong \left[\begin{pmatrix} u + v \\ 0 \end{pmatrix}, \begin{pmatrix} y^j \\ x^i \end{pmatrix}, \begin{pmatrix} 0 \\ u + v \end{pmatrix} \right]_{\mathsf{R}}$$

Our original motivation was to describe indecomposable Cohen-Macaulay modules over the ring $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(xy, z^2)$. Theorem 1.1.11, Lemma 1.3.11 and Theorem 1.3.16 yield a complete classification of indecomposable objects of $\mathrm{CM}(\mathsf{P})$.

Theorem 1.3.19. *An indecomposable Cohen-Macaulay P-module is either P, or one of the band modules (1.3.7) respectively string modules (1.3.9). Moreover, in the formulae (1.3.7) and (1.3.9), the generation over R can be replaced by the generation over P ()with the only exception of $S(\mathbf{x}_0\mathbf{y}_0)$).*

Remark 1.3.20. *Any string or band module M over R can be translated into a Cohen-Macaulay module $\mathbf{I}(M)$ over P as follows.*

(1) *Assume that $M = B(\omega, m, \lambda)$ or $M = S(\omega)$ where the string ω does not contain $\mathbf{x}_0$ or $\mathbf{y}_0$. We translate the entries of M as follows:*

x^i	u	y^j	v
x^{i+1}	xz	y^{j+1}	yz

(2) If $M = S(\omega) \ncong S(\mathbf{x}_0\mathbf{y}_0)$ is a string module such that the string ω contains $\mathbf{x}_0$ or $\mathbf{y}_0$, we translate all entries of M as follows:

x^i	ux	y^j	vy
x^i	xz	y^j	yz

Remark 1.3.21. *Theorem 1.3.19 remains valid for the curve singularity $\mathsf{P}_{2r+1,2s+1}$, where $r, s \in \mathbb{N}_0 \cup \{\infty\}$, but string and band modules have to be redefined in the following way:*

(1) The band and string modules over $\mathsf{P}_{2r+1,\infty}$ are given by the Definitions 1.3.12 and 1.3.13, but their string and band words ω may only contain letters $\mathbf{x}_i$ such that $0 \le i \le r$ or $\mathbf{y}_j$, where $j \in \mathbb{N}_0 \cup \{\infty\}$.

(2) Bands and strings over $\mathsf{P}_{2r+1,2s+1}$, where $r, s \in \mathbb{N}_0$, may only contain the letters $\mathbf{x}_i$ such that $0 \le i \le r$ or $\mathbf{y}_j$ such that $0 \le j \le s$.

The method of this section can also be generalized using Bondarenko's work [**Bon91**] *(Subsection A.3.2) to obtain an explicit classification of the indecomposable Cohen-Macaulay modules over the remaining curve singularities $\mathsf{P}_{2r,q}$, where $r \in \mathbb{N}^+$ and $q \in \mathbb{N}^+ \cup \{\infty\}$.*

Example 1.3.22. *In the following we apply Remark 1.3.20 to translate the string and band modules over* R *from 1.3.18 into indecomposable Cohen-Macaulay modules over* P.

(1) Let (ω, m, λ) be a band with $\omega = \mathbf{x}_i^-\mathbf{y}_{j_1}^-\mathbf{x}_i^+\mathbf{y}_{j_2}^-$. Then its band module $B = B(\omega, m, \lambda)$ translates over P *into*

$$\mathbf{I}(B) \cong \left[\begin{pmatrix} x^{i+1}\, I \\ yz\, J \end{pmatrix}, \begin{pmatrix} (xz + y^{j_1+1})\, I \\ 0 \end{pmatrix}, \begin{pmatrix} xz\, I \\ yz\, I \end{pmatrix}, \begin{pmatrix} 0 \\ (x^{i+1} + y^{j_2+1})\, I \end{pmatrix} \right]_{\mathsf{P}}$$

(2) Let $\omega = \mathbf{x}_i^-\mathbf{y}_j^-$. Then the string module $S = S(\omega)$ translates over P *into*

$$\mathbf{I}(S) \cong \left(x^{i+1}, xz + y^{j+1}, yz\right)_{\mathsf{P}}$$

(3) Let $\omega = \mathbf{x}_0\mathbf{y}_{j_1}^-\mathbf{x}_i^-\mathbf{y}_{j_2}^+$. Then the string module $S = S(\omega)$ translates over P *into*

$$\mathbf{I}(S) \cong \left[\begin{pmatrix} x + y^{j_1+1} \\ 0 \end{pmatrix}, \begin{pmatrix} yz \\ x^{i+1} \end{pmatrix}, \begin{pmatrix} 0 \\ (x+y)\, z \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j_2+1} \end{pmatrix} \right]_{\mathsf{P}}$$

(4) Let $\omega = \mathbf{x}_\infty\mathbf{y}_j^+\mathbf{x}_i^-\mathbf{y}_\infty$. Then the string module $S = S(\omega)$ translates over P *into*

$$\mathbf{I}(S) \cong \left[\begin{pmatrix} xz \\ 0 \\ yz \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^i \\ y^j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xz \\ 0 \\ yz \end{pmatrix} \right]_{\mathsf{P}} \cong \left[\begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} y^j \\ x^i \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right]_{\mathsf{P}}$$

1.3.3 Restricted Cohen-Macaulay modules over $\mathsf{T}_{\infty\infty}$

Our next motivation was to study Cohen-Macaulay modules over the hypersurface singularity $\mathsf{T} = \Bbbk[\![a,b]\!]/(a^2b^2)$. At the beginning of Section 1.3 we have constructed a fully faithful functor $\mathbf{J}\colon \mathrm{CM}(\mathsf{R}) \hookrightarrow \mathrm{CM}(\mathsf{T})$. Its explicit description, adapted to the combinatorics of bands and strings, was explained in Lemma 1.3.11:
if $M = \langle w_1, \dots, w_t \rangle_{\mathsf{R}} \subseteq \mathsf{S}^{k+l}$ then

$$\mathbf{J}(M) = \langle w_1, \dots, w_t, uw_1, \dots, uw_t, vw_1, \dots, vw_t \rangle_{\mathsf{T}} \subseteq \mathsf{S}^{k+l}.$$

Remark 1.3.23. *Any string or band module M over R can be translated into a Cohen-Macaulay module $\mathbf{J}(M)$ over T as follows.*

- *Let M be a band module or a string module $S(\omega)$ such that ω does not contain $\mathbf{x}_0$ or $\mathbf{y}_0$. Then we replace all entries in every generator of $\mathbf{J}(M)$ by the table:*

x^i	u	y^j	v
a^{i+2}	a^2b	b^{j+2}	ab^2

- *Let $M = S(\omega)$ such that ω contains $\mathbf{x}_0$ or $\mathbf{y}_0$. In this case, we translate all entries of generators of $\mathbf{J}(M)$ into elements of T using the table:*

x^i	ux	y^j	vy
a^{i+1}	a^2b	b^{j+1}	ab^2

Example 1.3.24. *Now we translate the string and band modules over R from 1.3.18 into indecomposable Cohen-Macaulay modules over T using Remark 1.3.23.*

(1) Let (ω, m, λ) be a band with $\omega = \mathbf{x}_i^- \mathbf{y}_{j_1}^- \mathbf{x}_i^+ \mathbf{y}_{j_2}^-$. Then its band module $B = B(\omega, m, \lambda)$ translates over T into

$$\mathbf{J}(B) \cong \left[\begin{pmatrix} a^{i+2}\, I \\ ab^2\, J \end{pmatrix}, \begin{pmatrix} (a^2b + b^{j_1+2})\, I \\ 0 \end{pmatrix}, \begin{pmatrix} ab^2\, I \\ a^2b\, I \end{pmatrix}, \begin{pmatrix} 0 \\ (a^{i+2} + b^{j_2+2})\, I \end{pmatrix} \right]_{\mathsf{T}}$$

(2) Let $\omega = \mathbf{x}_i^- \mathbf{y}_j^-$. Then the string module $S = S(\omega)$ translates over T into

$$\mathbf{J}(S) \cong \left(a^{i+2}, a^2b + y^{j+2}, ab^2\right)_{\mathsf{T}}$$

(3) Let $\omega = \mathbf{x}_0 \mathbf{y}_{j_1}^- \mathbf{x}_i^- \mathbf{y}_{j_2}^+$. Then the string module $S = S(\omega)$ translates over P into

$$\mathbf{J}(S) \cong \left[\begin{pmatrix} a^2 + b^{j_1+2} \\ 0 \end{pmatrix}, \begin{pmatrix} ab^2 \\ a^{i+2} \end{pmatrix}, \begin{pmatrix} 0 \\ a^2b + ab^2 \end{pmatrix}, \begin{pmatrix} 0 \\ b^{j_2+2} \end{pmatrix}, \begin{pmatrix} a^2b \\ 0 \end{pmatrix} \right]_{\mathsf{T}}$$

(4) Let $\omega = \mathbf{x}_\infty \mathbf{y}_j^+ \mathbf{x}_i^- \mathbf{y}_\infty$. Then the string module $S = S(\omega)$ translates over P into

$$\mathbf{J}(S) \cong \left[\begin{pmatrix} a^2b + ab^2 \\ 0 \end{pmatrix}, \begin{pmatrix} b^{j+2} \\ a^{i+2} \end{pmatrix}, \begin{pmatrix} 0 \\ a^2b + ab^2 \end{pmatrix} \right]_{\mathsf{T}}$$

$$\cong \left[\begin{pmatrix} ab \\ 0 \end{pmatrix}, \begin{pmatrix} b^j \\ a^i \end{pmatrix}, \begin{pmatrix} 0 \\ ab \end{pmatrix} \right]_{\mathsf{T}}$$

Remark 1.3.25. *The ring* $\mathsf{R} = \Bbbk[\![x, y, u, v]\!]/(xy, xv, yu, uv, u^2, v^2)$ *has an involution* σ *which interchanges* x *and* y, u *and* v. *Restricted to* $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(xy, z^2) \subset \mathsf{R}$, σ *is still an involution such that* $\sigma(z) = z$. *The restriction of* σ *to* $\mathsf{T} = \Bbbk[\![a, b]\!]/(a^2b^2) \subset A$ *interchanges* a *and* b. *Overall,* σ *induces an involution on the category of Cohen-Macaulay modules over* R, P *or* T. *The corresponding action of* σ *on words* ω *of strings or bands of* $\mathrm{CM}(\mathsf{R})$ *is given by interchanging* $\mathbf{x}$ *and* $\mathbf{y}$ *in* ω.

In the table below, we list all indecomposable Cohen-Macaulay ideals of R and P (except P itself) and the corresponding ideals of T up to isomorphism and involution σ. For all bands the multiplicity parameter m is set to 1.

TABLE 1.3.1. **Indecomposable ideals in R, P and T**

string or band of $\mathfrak{B}$	ideal in R	ideal in P	ideal in T
$\mathbf{x}_0$	(x)	(x)	(a^2)
$\mathbf{x}_\infty$	(u)	(xz)	(a^2b)
$\mathbf{x}_i^-$	(x^i, u)	(x^{i+1}, xz)	(a^{i+2}, a^2b)
$\mathbf{x}_0\mathbf{y}_0$	(1)	$(x+y, xz)$	(a^2+b^2, a^2b, ab^2)
$\mathbf{x}_i^+\mathbf{y}_0$	$(ux, x^{i+1}+y)$	$(xz, x^{i+1}+y)$	$(a^2b, a^{i+2}+b^2, ab^2)$
$\mathbf{x}_i^+\mathbf{y}_j^-$	(u, x^i+y^j, v)	$(xz, x^{i+1}+y^{j+1}, yz)$	$(a^2b, a^{i+2}+b^{j+2}, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_0$	$(x^{i+1}, ux+y)$	$(x^{i+1}, xz+y)$	$i=1$: (a^3, a^2b+b^2) $i \geq 2$: $(a^{i+2}, a^2b+b^2, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_j^-$	$(x^i, u+y^j, v)$	$(x^{i+1}, xz+y^{j+1}, yz)$	$(a^{i+2}, a^2b+b^{j+2}, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_j^+$	$(x^i, u+v, y^j)$	(x^i, z, y^j)	$(a^{i+2}, a^2b+ab^2, b^{j+2})$
$\mathbf{x}_\infty\mathbf{y}_0$	$(ux+y)$	$(xz+y)$	(a^2b+b^2, ab^2)
$\mathbf{x}_\infty\mathbf{y}_j^-$	$(u+y^j, v)$	$(xz+y^{j+1}, yz)$	(a^2b+b^{j+2}, ab^2)
$\mathbf{x}_\infty\mathbf{y}_j^+$	$(u+v, y^j)$	(z, y^j)	(a^2b+ab^2, b^{j+2})
$\mathbf{x}_\infty\mathbf{y}_\infty$	$(u+v)$	(z)	(ab)
$(\mathbf{x}_i^-\mathbf{y}_j^-, 1, \lambda)$	$(x^i+\lambda v, y^j+u)$	$(x^{i+1}+\lambda yz,$ $y^{j+1}+xz)$	$(a^{i+2}+\lambda ab^2, b^{j+2}+a^2b)$ except if $i=j=\lambda=1$: $(a^3+ab^2, b^3+a^2b, ab^3)$
$(\mathbf{x}_i^-\mathbf{y}_j^+, 1, \lambda)$	$(x^i+\lambda y^j, u+v)$	$(x^i+\lambda y^j, z)$	$(a^{i+2}+\lambda b^{j+2}, a^2b+ab^2)$
			where $i, j \in \mathbb{N}^+$ and $\lambda \in \Bbbk^*$

Remark 1.3.26. *The above list does not contain all indecomposable ideals of* T. *For example, the ideal* $(a^2+\lambda\, b^2, a^2b+ab^2)$ *is not a restriction of an ideal in* R *for any* $\lambda \in \Bbbk^*$.

Let $\underline{\mathsf{MF}}(a^2b^2)$ be the homotopy category of matrix factorizations of a^2b^2, and let $\underline{\mathrm{CM}}(\mathsf{T})$ denote the stable category of Cohen-Macaulay modules over T. There is an equivalence of triangulated categories $\underline{\mathrm{CM}}(\mathsf{T}) \xrightarrow{\sim} \underline{\mathsf{MF}}(a^2b^2)$ by a result of Eisenbud [**Eis80**]. In the table below, we list the matrix factorizations of a^2b^2 which originate from an indecomposable ideal in R (up to isomorphism and involution).

TABLE 1.3.2. **Indecomposable matrix factorizations of a^2b^2 of low rank**

ideal in $\Bbbk[\![a,b]\!]/(a^2b^2)$	matrix factorization (ϕ,ψ) of a^2b^2
(a^2)	$(b^2)\,(a^2)$
(a^2b)	$(b)\,(a^2b)$
(a^{i+2}, a^2b)	$\begin{bmatrix} b & 0 \\ -a^i & b \end{bmatrix} \begin{bmatrix} a^2b & 0 \\ a^{i+2} & a^2b \end{bmatrix}$
$(a^{i+1}+b^{j+1}, a^2b, ab^2)$	$\begin{bmatrix} ab & 0 & 0 \\ -a^i & b & 0 \\ -b^j & 0 & a \end{bmatrix} \begin{bmatrix} ab & 0 & 0 \\ a^{i+1} & a^2b & 0 \\ b^{j+1} & 0 & ab^2 \end{bmatrix}$
(a^3, a^2b+b^2)	$\begin{bmatrix} ab & 0 \\ -a^2 & a^2b \end{bmatrix} \begin{bmatrix} ab & 0 \\ -a & b \end{bmatrix}$
$(a^{i+3}, a^2b+b^2, ab^2)$	$\begin{bmatrix} b & 0 & 0 \\ -a^{i+1} & ab & 0 \\ 0 & -b & a \end{bmatrix} \begin{bmatrix} a^2b & 0 & 0 \\ a^{i+2} & ab & 0 \\ a^{i+1}b & b^2 & ab^2 \end{bmatrix}$
$(a^{i+2}, a^2b+b^{j+2}, ab^2)$	$\begin{bmatrix} b & 0 & 0 \\ -a^i & ab & 0 \\ a^{i-1}b^j & -b^{j+1} & a \end{bmatrix} \begin{bmatrix} a^2b & 0 & 0 \\ a^{i+1} & ab & 0 \\ 0 & b^{j+2} & ab^2 \end{bmatrix}$
$(a^{i+2}, b^{j+2}, a^2b+ab^2)$	$\begin{bmatrix} b & 0 & 0 \\ 0 & a & 0 \\ -a^i & -b^j & ab \end{bmatrix} \begin{bmatrix} a^2b & 0 & 0 \\ 0 & ab^2 & 0 \\ a^{i+1} & b^{j+1} & ab \end{bmatrix}$
(a^2b+b^{j+1}, ab^2)	$\begin{bmatrix} ab & 0 \\ -b^j & a \end{bmatrix} \begin{bmatrix} ab & 0 \\ b^{j+1} & ab^2 \end{bmatrix}$
(b^{j+2}, a^2b+ab^2)	$\begin{bmatrix} a & 0 \\ -b^j & ab \end{bmatrix} \begin{bmatrix} ab^2 & 0 \\ b^{j+1} & ab \end{bmatrix}$
(ab)	$(ab)\,(ab)$
$(a^{i+2}+\lambda ab^2, a^2b+b^{j+2})$ where i or j or $\lambda \neq 1$	$\begin{bmatrix} ab & -b^{j+1} \\ -a^{i+1} & \lambda ab \end{bmatrix} \begin{bmatrix} u\,ab & \lambda^{-1}\,u\,b^{j+1} \\ \lambda^{-1}\,u\,a^{i+1} & \lambda^{-1}\,u\,ab \end{bmatrix}$ where u is the unit $\left(1-\lambda^{-1}a^{i-1}b^{j-1}\right)^{-1}$
	where $i,j \in \mathbb{N}^+$ and $\lambda \in \Bbbk^*$

TABLE 1.3.2. **Indecomposable matrix factorizations of a^2b^2 of low rank**

ideal in $\Bbbk[\![a,b]\!]/(a^2b^2)$	matrix factorization (ϕ,ψ) of a^2b^2
$(a^3+ab^2, a^2b+b^3, ab^3)$	$\begin{bmatrix} b & 0 & 0 \\ -a & ab & 0 \\ 0 & -b & a \end{bmatrix} \begin{bmatrix} a^2b & 0 & 0 \\ a^2 & ab & 0 \\ ab & b^2 & ab^2 \end{bmatrix}$
$(a^{i+2}+\lambda b^{j+2}, a^2b+ab^2)$	$\begin{bmatrix} ab & 0 \\ -\lambda b^{j+1}-a^{i+1} & ab \end{bmatrix} \begin{bmatrix} ab & 0 \\ \lambda b^{j+1}+a^{i+1} & ab \end{bmatrix}$
	where $i,j\in\mathbb{N}^+$ and $\lambda\in\Bbbk^*$

Remark 1.3.27. *By Knörrer's periodicity* [**Knö87**] *the functor*

$$\begin{array}{ccc} \underline{\mathsf{MF}}(a^2b^2) & \xrightarrow{\sim} & \underline{\mathsf{MF}}(a^2b^2+uv) \\ (\phi,\psi) & \longmapsto & \begin{bmatrix} \phi & -u\cdot I \\ v\cdot I & \psi \end{bmatrix} \begin{bmatrix} \psi & v\cdot I \\ -u\cdot I & \phi \end{bmatrix} \end{array}$$

is an equivalence of triangulated categories. It allows to get explicit families of matrix factorizations of any potential of type

$$a^2b^2+u_1v_1+\ldots+u_dv_d\in\Bbbk[\![a,b,u_1,\ldots,u_d,v_1,\ldots,v_d]\!].$$

Remark 1.3.28. *Let* $\mathsf{char}(\Bbbk)\neq 2$*. Then there is a ring isomorphism*

$$\Bbbk[\![a,b,c]\!]/(a^2b^2-c^2)\cong\Bbbk[\![x,y,z]\!]/(z^2-xyz)=:\mathsf{T}_{\infty\infty2}.$$

The indecomposable Cohen-Macaulay modules over the surface singularity $\mathsf{T}_{\infty\infty2}$ *have been classified in* [**BD**]. *On the other hand, Knörrer's correspondence* [**Knö87**] *relates* $\mathsf{T}_{\infty\infty2}$ *to* $\mathsf{T}_{\infty\infty}$ *by a restriction functor*

$$\underline{\mathsf{MF}}(a^2b^2-c^2)\longrightarrow\underline{\mathsf{MF}}(a^2b^2),$$

such that every indecomposable matrix factorization of a^2b^2 appears as a direct summand *of the restriction of some indecomposable matrix factorization of $a^2b^2-c^2$. With some efforts, one can compute the matrix factorizations of a^2b^2 corresponding to Cohen-Macaulay $\mathsf{T}_{\infty\infty2}$-modules of small rank. However, it is not straightforward to derive all indecomposable matrix factorizations of a^2b^2 by this approach.*

Remark 1.3.29. *The approach to classify indecomposable Cohen-Macaulay modules using the technique of tame matrix problems is close in spirit to the study of torsion free sheaves on degenerations of elliptic curves. See* [**BBDG06**] *for a survey of the corresponding results and methods.*

1.3.4 Some remarks on the stable category of Cohen-Macaulay modules

Let $(A, \mathfrak{m})$ be a Gorenstein singularity (of any Krull dimension d). By a result of Buchweitz [**Buc87**], the natural functor

$$\underline{\mathrm{CM}}(A) \longrightarrow \mathrm{D}_{sg}(A) := \frac{\mathrm{D}^{\mathrm{b}}\big(A\text{ - mod}\big)}{\mathsf{Perf}(A)}$$

is an equivalence of triangulated categories. If the singularity A is not isolated, then $\underline{\mathrm{CM}}(A)$ is Hom-infinite [**Aus78**]. On the other hand, the stable category of Cohen-Macaulay modules $\underline{\mathrm{CM}}^{\mathsf{lf}}(A)$ is always a Hom-finite triangulated subcategory of $\underline{\mathrm{CM}}(A)$. By a result of Auslander [**Aus78**], the category $\underline{\mathrm{CM}}^{\mathsf{lf}}(A)$ is $(d-1)$-Calabi-Yau. This means that for any objects M_1 and M_2 of $\underline{\mathrm{CM}}^{\mathsf{lf}}(A)$ we have an isomorphism

$$\underline{\mathrm{Hom}}_A(M_1, M_2) \cong \mathbb{D}\big(\underline{\mathrm{Hom}}_A(M_2, \Sigma^{d-1}(M_1))\big),$$

functorial in both arguments M_1 and M_2, where $\mathbb{D}$ is the Matlis duality functor and $\Sigma = \Omega^{-1}$ is the suspension functor. In particular, if A is a Gorenstein curve singularity, then for any $M \in \underline{\mathrm{CM}}^{\mathsf{lf}}(A)$ the algebra $\underline{\mathrm{End}}_A(M)$ is Frobenius. Thus, Theorem 1.2.1 gives a family of examples of representation tame 0-Calabi-Yau triangulated categories and Theorem 1.3.19 provides a complete and explicit description of indecomposable objects in one of such categories $\underline{\mathrm{CM}}^{\mathsf{lf}}(\mathsf{P})$ for $\mathsf{P} = \mathbb{k}[\![x, y, z]\!]/(xy, z^2)$.

Part 2

Tame matrix problems in Lie theory

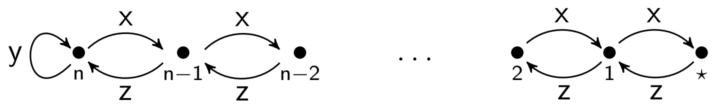
CHAPTER 2

Derived Auslander-Reiten theory of Khoroshkin quivers

The *Khoroshkin quivers* of *connected* Lorentz groups are given by the following two series:

(1) Any Khoroshkin quiver of the odd series is "gentle with oriented cycles":

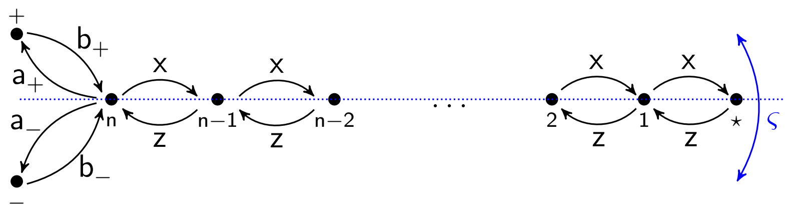
$$(Q, I)_{2n+1} =$$



with relations $\mathsf{x}\,\mathsf{y} = \mathsf{y}\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0$.

(2) Any Khoroshkin quiver of even type is "skew-gentle with oriented cycles":

$$(Q, I)_{2n+2} =$$



(∗)

with relations $\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$, $\mathsf{x}\,\mathsf{b}_\pm = \mathsf{a}_\pm\,\mathsf{z} = 0$ and $\mathsf{x}^2 = \mathsf{z}^2 = 0$.

In particular, such a quiver has an involution ς interchanging $+$ and $-$.

The goal of this chapter is to study the derived category of any Khoroshkin quiver without using combinatorial methods. As mentioned in the introduction, the original motivation to study these quivers came from Lie theory.
More precisely, for any $n \in \mathbb{N}^+$ let

- $G_n = \mathrm{SO}_e(n, 1)$ be the identity component of the Lie group $\mathrm{SO}(n, 1)$,
- $\mathcal{H}_0 = \mathcal{H}_0(G_n)$ be the principal block of Harish-Chandra modules over G, and
- Λ_n be the arrow ideal completion of the path algebra of the quiver $(Q, I)_n$

By a Theorem of Khoroshkin [**Kho81**] there is an equivalence between the principal block of Harish-Chandra modules of the Lorentz group G_n and the category of nilpotent representations of the quiver $(Q, I)_n$:

$$\mathcal{H}_0 = \mathcal{H}_0(G_n) \underset{\text{if } \mathbb{k}=\mathbb{C}}{\overset{\sim}{\dashrightarrow}} \mathrm{nil.\,rep}_{\mathbb{k}}(Q, I)_n \xrightarrow{\ \sim\ } \Lambda_n\text{ - fd. mod}$$

Moreover, the nilpotent representations of the quiver $(Q, I)_n$ may be viewed as finite-dimensional modules of the *order* Λ_n. The main result of this chapter gives a relationship between the following three notions on the derived category $\mathrm{D^b}(\mathcal{H}_0)$:

(1) There is an autoequivalence $\sigma : \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ such that $\sigma^2 \cong \mathrm{Id}$ defined as follows:

- if $\mathcal{H}_0 = \mathcal{H}_0(G_{2n+1})$, we set $\sigma = \mathrm{Id}$ for simplicity of notation.
- if $\mathcal{H}_0 = \mathcal{H}_0(G_{2n+2})$, the functor σ is induced by the symmetry ς in $(*)$.

In both cases, the involution σ is related to an automorphism of the Lie group $G^{\#} = \mathrm{SO}(n,1)$ (see Subsection C.3).

(2) Let $S_\star$ be the simple Λ_n-module associated to the vertex $\star$ in the quiver $(Q,I)_n$. For any complex $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ we define the *defect* of $P_\bullet$ by

$$\delta(P_\bullet) = \sum_{i\in\mathbb{Z}} \delta^{(i)}(P_\bullet) = \sum_{i\in\mathbb{Z}} \dim \mathrm{Ext}^i_{\mathcal{H}_0}(P_\bullet, S_\star) = \sum_{i\in\mathbb{Z}} \dim \mathbf{H}^i_{\mathfrak{g},K}(P_\bullet).$$

In Lie-theoretic terms, the module $S_\star$ corresponds to the *trivial Harish-Chandra module* $\mathbb{C}$ while the defect corresponds to the total dimension of the *relative Lie algebra cohomology* of the complex $P_\bullet$.

(3) Since the order Λ_n has *finite* global dimension, results of van den Bergh **[vdB04]** and Iyama-Reiten **[IR08]** imply that the category $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ has an *Auslander-Reiten translation*. More precisely, there is an autoequivalence τ of $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ such that there is a functorial isomorphism:

$$\mathrm{Hom}(X_\bullet, Y_\bullet) \cong \mathbb{D}\,\mathrm{Ext}^1(Y_\bullet, \tau(X_\bullet)) \qquad \text{for any} \quad X_\bullet, Y_\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0).$$

In other terms, the category $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ admits *Auslander-Reiten triangles.*

The main result can be stated in terms of the category $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ as follows:

Theorem (Theorem 2.2.3)**.** *As above, let $\mathcal{H}_0 = \mathcal{H}_0(G_n)$ for some $n \geq 2$.*

(1) The simple module $S_\star$ is an n-spherical object *in* $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$.

In other terms, the module $S_\star$ induces an autoequivalence $\mathbb{T}_{S_\star}$ *of* $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$.

(2) The Auslander-Reiten translation admits the following factorization:

$$\tau \cong \sigma \circ \mathbb{T}^{-1}_{S_\star} \cong \mathbb{T}^{-1}_{S_\star} \circ \sigma : \quad \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$$

(3) For any $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ the following conditions are equivalent:

(a) $\delta(P_\bullet) = 0$	(b) $\tau^m(P_\bullet) \cong P_\bullet$ *for some* $m \in \mathbb{N}^+$
(c) $\tau(P_\bullet) \cong \sigma(P_\bullet)$	(d) $\tau^2(P_\bullet) \cong P_\bullet$
(e) $\mathbb{T}^{-1}_{S_\star}(P_\bullet) \cong P_\bullet$	

Let us comment on the statements of the theorem:

- The second statement of the theorem gives an "intrinsic" description of the Auslander-Reiten translation, which can be transferred to analytic categories equivalent to $\mathcal{H}_0$.
- The third statement implies that the homogeneous tubes in the Auslander-Reiten quiver of the category $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ have either rank *one* or *two*.

Section 2.2 of this chapter contains the following additional results:

(1) The subcategory $\mathcal{T}_{\mathrm{fd}}$ of τ-periodic complexes in $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ has several additional descriptions (Corollary 2.2.9). In particular, there is a *Gorenstein order* Λ^e of *infinite* global dimension and an equivalence of categories

$$\mathcal{T}_{\mathrm{fd}} \xrightarrow{\sim} \mathsf{Perf}_{\mathrm{fd}}(\Lambda^e)$$

where $\mathsf{Perf}_{\mathrm{fd}}(\Lambda^e)$ is the category of perfect complexes with finite-dimensional homology.

(2) The projective and injective dimension of any Harish-Chandra module V depend only on the defect numbers $\delta^{(i)}(V)$ (Proposition 2.2.17).

(3) Anticipating some notions of the next chapter, we divide the indecomposable objects of $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ into the following *four classes*:

$P_\bullet \in \operatorname{ind} \mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$	$\delta(P_\bullet) > 0$, or $\tau^2(P_\bullet) \ncong P_\bullet$	$\delta(P_\bullet) = 0$, or $\tau^2(P_\bullet) \cong P_\bullet$
$\sigma(P_\bullet) \cong P_\bullet$	"usual string"	"band"
$\sigma(P_\bullet) \ncong P_\bullet$	"special string"	"bispecial string"

We describe some simple examples and basic properties of the four classes (Example 2.2.22 and Lemma 2.2.25).

Finally let us note that all results presented in this summary have analogues for the Khoroshkin quivers $(Q, I)_n^{\#}$ of the disconnected Lorentz groups $\mathrm{SO}(n, 1)$.

2.1 Main properties of Khoroshkin orders

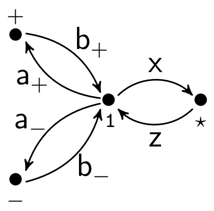
In this section we introduce the Khoroshkin quivers and their orders. Our first goal is to express the Auslander-Reiten translation on the derived category in terms of a spherical twist functor.

2.1.1 A representative example

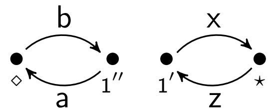
To see what will be going on in the future sections, we consider a basic example first. In this example we consider the Khoroshkin order Λ associated to the connected Lorentz group $\mathrm{SO}_e(4,1)$. This case shows all phenomena which will appear for any Khoroshkin order Λ_n associated to the Lie group $\mathrm{SO}_e(n,1)$, where $n \geq 2$.

2.1.1.1 The quiver and its normalization

Let (Q,I) and $\widetilde{Q}$ denote the following quivers:

$(Q,I) =$ 

$$\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_- \qquad \mathsf{x}\,\mathsf{b}_\pm = \mathsf{a}_\pm\,\mathsf{z} = 0$$

$\widetilde{Q} =$ 

We will say that the quiver (Q,I) can be obtained from the quiver $\widetilde{Q}$ by *gluing* the vertices $1'$ and $1''$ of $\widetilde{Q}$, and by *blowing up* vertex 2 in $\widetilde{Q}$.
This motivates the following terminology for the vertices of the quiver (Q,I):

(1) vertex 1 will be called a *glued* vertex,

(2) the vertices $+$ and $-$ form a pair of *special* vertices, and

(3) the vertex $\star$ is a *neutral* vertex.

Note that the quiver (Q,I) has an involution which interchanges the special vertices.

2.1.1.2 The orders and their matrix algebras

The completed path algebra Λ of the quiver (Q,I) is defined as follows:

- Let $A = \Bbbk Q/I$ be the path algebra of the quiver (Q,I),
- Let $\mathfrak{a}$ denote the arrow ideal of A,
- Let $\Lambda = \widehat{A}$ be the $\mathfrak{a}$-adic completion of the path algebra A.

It can be checked that the algebra Λ is isomorphic to the following matrix algebra:

$$\Lambda \cong \begin{array}{c} \begin{array}{cccc} P_\star & P_1 & P_+ & P_- \end{array} \\ \begin{bmatrix} \Bbbk[\![x]\!] & (x) & 0 & 0 \\ \Bbbk[\![x]\!] & \Bbbk[\![x,y]\!]/(x\,y) & (y) & (y) \\ 0 & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] & (y) \\ 0 & \Bbbk[\![y]\!] & (y) & \Bbbk[\![y]\!] \end{bmatrix} \end{array} \subset \mathrm{Mat}_{4\times 4}(\Bbbk[\![x]\!] \times \Bbbk[\![y]\!])$$

The matrix algebra Λ is actually a *one-dimensional order* via the embedding

$$\mathbf{R} = \Bbbk[\![t]\!] \hookrightarrow \Lambda \qquad t \longmapsto \begin{bmatrix} x & & & \\ & x+y & & \\ & & y & \\ & & & y \end{bmatrix}$$

Similarly, let $\Gamma = (\Bbbk\widetilde{Q})\widehat{\ }$ be the completed path algebra of the quiver $\widetilde{Q}$. Then

$$\Gamma \cong \begin{array}{c} \begin{array}{cc} \widetilde{P}_\star & \widetilde{P}_{1'} \end{array} \\ \begin{bmatrix} \Bbbk[\![x]\!] & (x) \\ \Bbbk[\![x]\!] & \Bbbk[\![x]\!] \end{bmatrix} \end{array} \times \begin{array}{c} \begin{array}{ccc} \widetilde{P}_{1''} & \widetilde{P}_\diamond & \widetilde{P}_\diamond \end{array} \\ \begin{bmatrix} \Bbbk[\![y]\!] & (y) & (y) \\ \Bbbk[\![y]\!] & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] \\ \Bbbk[\![y]\!] & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] \end{bmatrix} \end{array} \subset \mathrm{Mat}_{2\times 2}(\Bbbk[\![x]\!]) \times \mathrm{Mat}_{3\times 3}(\Bbbk[\![y]\!])$$

The algebra Γ is also an order over $\mathbf{R}$. In particular, there is an embedding $\Lambda \subset \Gamma$ of orders. We will call Γ a "normalization" of Λ.

The following Lemma is a straightforward computation.

Lemma 2.1.1. *It holds that* $\operatorname{gldim}\Lambda = \operatorname{pd} S_\star = 4$.

In the following let $\mathrm{D^b}(\Lambda)$ denote the bounded derived category of finitely generated Λ-modules.

2.1.1.3 Canonical bimodule and Auslander-Reiten translation

Next, we consider the Auslander-Reiten translation on the derived category $\mathrm{D^b}(\Lambda)$:

- Let $(\_)^\vee$ denote the functor $(\_)^\vee = \mathrm{Hom}_{\mathbf{R}}(\_, \mathbf{R})$.
- The *canonical bimodule* ω of Λ is defined as $\omega := \Lambda^\vee = \mathrm{Hom}_{\mathbf{R}}(\Lambda, \mathbf{R})$.
- According to Subsection B.2.3 the Auslander-Reiten translation τ is given by the left-derived tensor product

$$\tau = \omega \underset{\Lambda}{\otimes} \_ : \mathrm{D^b}(\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\Lambda).$$

For the order Λ above, its canonical bimodule ω can be computed as follows:

$$\omega \cong \begin{bmatrix} \Bbbk[\![x]\!]^\vee & \Bbbk[\![x]\!]^\vee & 0 & 0 \\ (x)^\vee & \big(\Bbbk[\![x,y]\!]/(x\,y)\big)^\vee & \Bbbk[\![y]\!]^\vee & \Bbbk[\![y]\!]^\vee \\ 0 & (y)^\vee & \Bbbk[\![y]\!]^\vee & (y)^\vee \\ 0 & (y)^\vee & (y)^\vee & \Bbbk[\![y]\!]^\vee \end{bmatrix}$$

$$\cong \begin{bmatrix} \Bbbk[\![x]\!] & \Bbbk[\![x]\!] & 0 & 0 \\ (x^{-1}) & ((x+y)^{-1}) & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] \\ 0 & (y^{-1}) & \Bbbk[\![y]\!] & (y^{-1}) \\ 0 & (y^{-1}) & (y^{-1}) & \Bbbk[\![y]\!] \end{bmatrix} \xrightarrow[\sim]{\cdot t} \begin{array}{c} \begin{array}{cccc} rP_\star & P_1 & P_- & P_+ \end{array} \\ \begin{bmatrix} (x) & (x) & 0 & 0 \\ \Bbbk[\![x]\!] & \Bbbk[\![x,y]\!]/(x\,y) & (y) & (y) \\ 0 & \Bbbk[\![y]\!] & (y) & \Bbbk[\![y]\!] \\ 0 & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] & (y) \end{bmatrix} \end{array}$$

In particular, the AR-translation τ acts on projective Λ-modules as follows:

(1) τ sends the projective of the neutral vertex to its radical: $\tau(P_\star) \cong rP_\star$,
(2) τ preserves the projective of the glued vertex: $\tau(P_1) \cong P_1$,
(3) τ interchanges the projectives of special vertices: $\tau(P_\pm) \cong P_\mp$.

We will see in the next subsection, that these statements hold for any Khoroshkin order.

2.1.1.4 Twist functor of the neutral simple module

Let $S_\star$ be the simple module at the neutral vertex $\star$ of (Q, I). We will prove the following Lemma in a wider context in the next Section:

Lemma 2.1.2. *The module $S_\star$ is* 4-spherical *in the sense of Definition B.3.2.*

Let $\mathbb{T}^\vee_{S_\star}$ be *dual twist functor* associated to $S_\star$ (as defined in Subsection B.3). Since $S_\star$ is spherical, the functor $\mathbb{T}^\vee_{S_\star}$ is an *autoequivalence* of $\mathrm{D}^{\mathrm{b}}(\Lambda)$. The dual spherical twist $\mathbb{T}^\vee_{S_\star}$ can be described as a *standard functor*, that is, as a left-derived tensor product with a bimodule, as follows.
Let $\pi : \Lambda \twoheadrightarrow S_\star$ be the canonical projection. Since Λ has finite global dimension, the module $S_\star$ is simple and spherical, Theorem B.3.12 and Corollary B.3.12 imply that there are isomorphisms of functors

$$\mathbb{T}^\vee_{S_\star}(\_) \cong \mathbb{T}^\vee_{S_\star}(\Lambda) \underset{\Lambda}{\otimes} \_ \cong \ker\pi \underset{\Lambda}{\otimes} \_ : \quad \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$$

$$\text{where } \ker\pi \cong \begin{array}{c} \begin{array}{cccc} rP_\star & P_1 & P_+ & P_- \end{array} \\ \begin{bmatrix} (x) & (x) & 0 & 0 \\ \Bbbk[\![x]\!] & \Bbbk[\![x,y]\!]/(x\,y) & (y) & (y) \\ 0 & \Bbbk[\![y]\!] & \Bbbk[\![y]\!] & (y) \\ 0 & \Bbbk[\![y]\!] & (y) & \Bbbk[\![y]\!] \end{bmatrix} \end{array}.$$

2.1.1.5 The Auslander-Reiten translation as a spherical twist

So far, we have seen the following three autoequivalences on $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

(1) the involution σ interchanging P_+ and P_-,

(2) the Auslander-Reiten translation $\tau = \omega \otimes \_$, and

(3) the dual spherical twist $\mathbb{T}^{\vee}_{S_\star} \cong \ker \pi \otimes \_$.

Lemma 2.1.3. *There are the following isomorphisms of functors:*

$$\tau \cong \sigma \circ \mathbb{T}^{\vee}_{S_\star} \cong \mathbb{T}^{\vee}_{S_\star} \circ \sigma : \quad \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda) \tag{2.1.1}$$

Proof. The statement follows from the computations of the bimodules ω and $\ker \pi$:

$$\omega \cong \left[\begin{array}{c|c|c|c} rP_\star & P_1 & P_- & P_+ \end{array} \right] \quad \text{and} \quad \ker(\pi) \cong \left[\begin{array}{c|c|c|c} rP_\star & P_1 & P_+ & P_- \end{array} \right].$$

□

2.1.2 Khoroshkin orders of the connected Lorentz groups $\mathrm{SO}_e(n,1)$

In this subsection we introduce the Khoroshkin orders Λ_n which are related to the connected Lorentz groups $\mathrm{SO}_e(n,1)$, where $n \in \mathbb{N}^+$. The goal of the present subsection is to give a description of the Auslander-Reiten translation τ on the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda_n)$ similar to statement (2.1.1).

2.1.2.1 Khoroshkin quivers and their normalizations

We begin with the description of Khoroshkin quivers $(Q,I)_n$, where $n \in \mathbb{N}^+$.

(1) For $n \in \mathbb{N}_0$ a *Khoroshkin quiver of odd type* is given by the quiver

$$(Q,I)_{2n+1} = \begin{tikzcd} \underset{\mathsf{n}}{\bullet} \arrow[loop left, "\mathsf{a_n}"] \arrow[r, bend left, "\mathsf{b_{n-1}}"] & \underset{\mathsf{n-1}}{\bullet} \arrow[l, bend left, "\mathsf{a_{n-1}}"] \arrow[r, bend left, "\mathsf{b_{n-2}}"] & \underset{\mathsf{n-2}}{\bullet} \arrow[l, bend left, "\mathsf{a_{n-2}}"] & \cdots & \underset{\mathsf{2}}{\bullet} \arrow[r, bend left, "\mathsf{b_1}"] & \underset{\mathsf{1}}{\bullet} \arrow[l, bend left, "\mathsf{a_1}"] \arrow[r, bend left, "\mathsf{b_0}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a_0}"] \end{tikzcd}$$

with relations $\quad \mathsf{b_{n-1}\,a_n} = \mathsf{a_n\,a_{n-1}} = 0 \quad$ and $\quad \mathsf{b_{i-1}\,b_i} = \mathsf{a_i\,a_{i-1}} = 0, \quad (\mathsf{0 < i < n})$.

The quiver above is closely related to the union of the following quivers:

$$\widetilde{Q}_{2n+1} = \begin{tikzcd} \underset{\mathsf{n''}}{\bullet} \arrow[loop left, "\mathsf{a_n}"] & \underset{\mathsf{n'}}{\bullet} \arrow[r, bend left, "\mathsf{b_{n-1}}"] & \underset{\mathsf{n-1''}}{\bullet} \arrow[l, bend left, "\mathsf{a_{n-1}}"] & \underset{\mathsf{n-1'}}{\bullet} \arrow[r, bend left, "\mathsf{b_{n-2}}"] & \underset{\mathsf{n-2''}}{\bullet} \arrow[l, bend left, "\mathsf{a_{n-2}}"] & \cdots & \underset{\mathsf{2'}}{\bullet} \arrow[r, bend left, "\mathsf{b_1}"] & \underset{\mathsf{1''}}{\bullet} \arrow[l, bend left, "\mathsf{a_1}"] & \underset{\mathsf{1'}}{\bullet} \arrow[r, bend left, "\mathsf{b_0}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a_0}"] \end{tikzcd}$$

We will say that the quiver $(Q,I)_{2n+1}$ has one *neutral* vertex and n *glued* vertices.

(2) For $n \in \mathbb{N}_0$ a *Khoroshkin quiver of even type* is given by the quiver

$$
(Q,I)_{2n+2} = \begin{tikzcd}
\overset{+}{\bullet} \arrow[dr, bend left, "\mathsf{b}_+"] & & & & & & & \\
& \underset{n}{\bullet} \arrow[ul, bend left, "\mathsf{a}_+"] \arrow[dl, bend left, "\mathsf{a}_-"] \arrow[r, bend left, "\mathsf{b}_{n-1}"] & \underset{n-1}{\bullet} \arrow[l, bend left, "\mathsf{a}_{n-1}"] \arrow[r, bend left, "\mathsf{b}_{n-2}"] & \underset{n-2}{\bullet} \arrow[l, bend left, "\mathsf{a}_{n-2}"] & \cdots & \underset{2}{\bullet} \arrow[r, bend left, "\mathsf{b}_1"] & \underset{1}{\bullet} \arrow[l, bend left, "\mathsf{a}_1"] \arrow[r, bend left, "\mathsf{b}_0"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_0"] \\
\underset{-}{\bullet} \arrow[ur, bend left, "\mathsf{b}_-"] & & & & & & &
\end{tikzcd}
$$

with relations $\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-, \quad \mathsf{b}_{n-1}\,\mathsf{b}_\pm = \mathsf{a}_\pm\,\mathsf{a}_{n-1} = 0,$ and

$$\mathsf{b}_{i-1}\,\mathsf{b}_i = \mathsf{a}_i\,\mathsf{a}_{i-1} = 0, \quad (0 < \mathsf{i} < \mathsf{n}).$$

The quiver above can be “embedded” into the union of the following quivers:

$$
\widetilde{Q}_{2n+2} = \begin{tikzcd}
\underset{\diamond}{\bullet} \arrow[r, bend left, "\mathsf{b}"] & \underset{n''}{\bullet} \arrow[l, bend left, "\mathsf{a}"] & \underset{n'}{\bullet} \arrow[r, bend left, "\mathsf{b}_{n-1}"] & \underset{n-1''}{\bullet} \arrow[l, bend left, "\mathsf{a}_{n-1}"] & \underset{n-1'}{\bullet} \arrow[r, bend left, "\mathsf{b}_{n-2}"] & \underset{n-2''}{\bullet} \arrow[l, bend left, "\mathsf{a}_{n-2}"] & \cdots & \underset{2'}{\bullet} \arrow[r, bend left, "\mathsf{b}_1"] & \underset{1''}{\bullet} \arrow[l, bend left, "\mathsf{a}_1"] & \underset{1'}{\bullet} \arrow[r, bend left, "\mathsf{b}_0"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_0"]
\end{tikzcd}
$$

The quiver $(Q,I)_{2n+2}$ has one *neutral* vertex, n *glued* and one pair of *special* vertices. In particular, there is an involution of the quiver interchanging $+$ and $-$ in vertices and arrows.

Remark 2.1.4. *(1) Any quiver of odd type $(Q,I)_{2n+1}$ satisfies the conditions of a* gentle *quiver with relations in the sense of* [**ASS06**, Definition IX.6.1] - *except that it has* oriented cycles*.*

(2) Similarly, any quiver of even type $(Q,I)_{2n+2}$ is skew-gentle *in the sense of* [**GdlP99**] *but with oriented cycles.*

2.1.2.2 Khoroshkin orders and their overorders

Let $(Q,I)_n$ be a Khoroshkin quiver, we define

- $\widehat{(\Bbbk Q/I)}_n$ to be the completed path algebra of the quiver $(Q,I)_n$, where the completion is taken with respect to the arrow ideal of $(\Bbbk Q/I)_n$.

In this subsection, we give a matrix description and consider the global dimension of these completed path algebras.

Let us fix the following notation:

$$\mathbf{R}_i = \Bbbk[\![x_i]\!] \qquad \mathfrak{m}_i = (x_i) \qquad \mathbf{R}_{i-1,i} = \Bbbk[\![x_{i-1}, x_i]\!]/(x_{i-1}\,x_i) \qquad (1 \le i \le n)$$

(1) The *Khoroshkin order of odd type* is defined as the matrix algebra

$$\Lambda_{2n+1} = \begin{array}{cc} & \begin{array}{cccccccc} P_\star & P_1 & P_2 & \cdots & P_{n-2} & P_{n-1} & P_n \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ n-2 \\ n-1 \\ n \end{array} & \begin{bmatrix} \mathbf{R}_0 & \mathfrak{m}_0 & 0 & & & & \\ \mathbf{R}_0 & \mathbf{R}_{0,1} & \mathfrak{m}_1 & & & & \\ 0 & \mathbf{R}_1 & \mathbf{R}_{1,2} & \ddots & & & \\ & & \ddots & \ddots & & & \\ & & & & \mathbf{R}_{n-3,n-2} & \mathfrak{m}_{n-2} & 0 \\ & & & & \mathbf{R}_{n-2} & \mathbf{R}_{n-2,n-1} & \mathfrak{m}_{n-1} \\ & & & & 0 & \mathbf{R}_{n-1} & \mathbf{R}_{n-1,n} \end{bmatrix} \end{array}$$

The order Λ_{2n+1} can be embedded into the order

$$\Gamma_{2n+1} = \begin{array}{c} \begin{array}{cc} \widetilde{P}_\star & \widetilde{P}_{1'} \end{array} \\ \begin{bmatrix} \mathbf{R}_0 & \mathfrak{m}_0 \\ \mathbf{R}_0 & \mathbf{R}_0 \end{bmatrix} \end{array} \times \prod_{i=1}^{n-1} \begin{array}{c} \begin{array}{cc} \widetilde{P}_{i''} & \widetilde{P}_{i'} \end{array} \\ \begin{bmatrix} \mathbf{R}_i & \mathfrak{m}_i \\ \mathbf{R}_i & \mathbf{R}_i \end{bmatrix} \end{array} \times \begin{array}{c} \widetilde{P}_{n''} \\ \begin{bmatrix} \mathbf{R}_n \end{bmatrix} \end{array}$$

The orders Λ_{2n+1} and Γ_{2n+1} are the completed path algebras of the quivers $(Q,I)_{2n+1}$ respectively $\widetilde{Q}_{2n+1}$:

- There is a $\Bbbk$-algebra isomorphism $(\Bbbk Q/I)\widehat{\ }_{2n+1} \xrightarrow{\sim} \Lambda_{2n+1}$ given by

  $\mathsf{b}_i \longmapsto x_i\,\delta_{i-1,i}, \quad \mathsf{a}_i \longmapsto \delta_{i,i-1} \quad (0 \leq i < n) \quad$ and $\quad \mathsf{a}_n \longmapsto x_n\,\delta_{n,n}.$

  where $\delta_{i,j}$ is given by the matrix with 1 at entry (i,j) and zero elsewhere.
- Similarly, Γ_{2n+1} is isomorphic to the completed path algebra $(\Bbbk\widetilde{Q})\widehat{\ }_{2n+1}$.

(2) The *Khoroshkin order of even type* is defined as

$$\Lambda_{2n+2} = \begin{array}{cc} & \begin{array}{cccccccc} P_\star & P_1 & P_2 & \cdots & P_{n-1} & P_n & P_+ & P_- \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \\ + \\ - \end{array} & \begin{bmatrix} \mathbf{R}_0 & \mathfrak{m}_0 & 0 & & & & & \\ \mathbf{R}_0 & \mathbf{R}_{0,1} & \mathfrak{m}_1 & & & & & \\ 0 & \mathbf{R}_1 & \mathbf{R}_{1,2} & \ddots & & & & \\ & & \ddots & \ddots & & & & \\ & & & & \mathbf{R}_{n-2,n-1} & \mathfrak{m}_{n-1} & 0 & 0 \\ & & & & \mathbf{R}_{n-1} & \mathbf{R}_{n-1,n} & \mathfrak{m}_n & \mathfrak{m}_n \\ & & & & 0 & \mathbf{R}_n & \mathbf{R}_n & \mathfrak{m}_n \\ & & & & 0 & \mathbf{R}_n & \mathfrak{m}_n & \mathbf{R}_n \end{bmatrix} \end{array}$$

The order Λ_{2n+2} can be embedded into the order

$$\Gamma_{2n+2} \cong \begin{array}{c} \begin{array}{cc} \widetilde{P}_\star & \widetilde{P}_{1'} \end{array} \\ \begin{bmatrix} \mathbf{R}_0 & \mathfrak{m}_0 \\ \mathbf{R}_0 & \mathbf{R}_0 \end{bmatrix} \end{array} \times \prod_{i=1}^{n-1} \begin{array}{c} \begin{array}{cc} \widetilde{P}_{i''} & \widetilde{P}_{i'} \end{array} \\ \begin{bmatrix} \mathbf{R}_i & \mathfrak{m}_i \\ \mathbf{R}_i & \mathbf{R}_i \end{bmatrix} \end{array} \times \begin{array}{c} \begin{array}{ccc} \widetilde{P}_{n''} & \widetilde{P}_\diamond & \widetilde{P}_\diamond \end{array} \\ \begin{bmatrix} \mathbf{R}_n & \mathfrak{m}_n & \mathfrak{m}_n \\ \mathbf{R}_n & \mathbf{R}_n & \mathbf{R}_n \\ \mathbf{R}_n & \mathbf{R}_n & \mathbf{R}_n \end{bmatrix} \end{array}$$

As in the odd case, the orders Λ_{2n+2} and Γ_{2n+2} are completed path algebras:

- There is a $\Bbbk$-algebra isomorphism $\widehat{(\Bbbk Q/I)}_{2n+2} \xrightarrow{\sim} \Lambda_{2n+2}$ given by

$$\begin{aligned} \mathsf{b_i} &\longmapsto x_i\,\delta_{i-1,i}, & \mathsf{a_i} &\longmapsto \delta_{i,i-1} \quad (\mathsf{0 \le i < n}), \\ \mathsf{b_\pm} &\longmapsto x_{n-1}\,\delta_{n-1,\pm}, & \mathsf{a_\pm} &\longmapsto \delta_{\pm,n-1}. \end{aligned}$$

- Γ_{2n+2} is isomorphic to the completed path algebra of the quiver $\widetilde{Q}_{2n+2}$.

For any $n \in \mathbb{N}^+$ let

- $\operatorname{nil.rep}_\Bbbk(Q,I)_n$ denote the category of nilpotent representations of $(Q,I)_n$ in the sense of Definition B.1.7,
- Λ_n - fd. mod be the category of finite-dimensional Λ_n-modules,
- $\mathcal{H}_0(G_n)$ be the principal block of Harish-Chandra modules over the connected Lie group $G_n = \mathrm{SO}_e(n,1)$.

By Corollary B.1.8 and Theorem C.3.1 there are the following equivalences of categories:

$$\operatorname{nil.rep}_\Bbbk(Q,I)_n \xrightarrow{\sim} \Lambda_n \text{ - fd. mod} \underset{\text{if } \Bbbk=\mathbb{C}}{\dashrightarrow^{\sim}} \mathcal{H}_0(G_n)$$

Lemma 2.1.5. *Let $n \in \mathbb{N}^+$.*

(1) It holds that $\operatorname{gldim}\Lambda_n = \operatorname{pd} S_\star = n$*.*

(2) For any simple Λ_n-module S it holds that

$$\operatorname{syz}(S) \in \operatorname{add}\{\ \operatorname{syz}^j(S_\star) \mid 1 \le j \le n\ \}$$

PROOF. Let $\Lambda = \Lambda_n$ for some $n \in \mathbb{N}^+$. We recall that $\operatorname{gldim}\Lambda = \operatorname{pd}(\operatorname{top}\Lambda)$ by Proposition B.1.14. The *projective resolutions of simple* Λ*-modules* are given as follows:

(1) If $\Lambda = \Lambda_{2n+1}$ for some $n \in \mathbb{N}_0$, the minimal projective resolution of the simple module $S_\star$ is given by

$$0 \longrightarrow \underset{2n+1}{P_\star} \xrightarrow{\cdot \mathsf{b_0}} \underset{2n}{P_\mathsf{1}} \xrightarrow{\cdot \mathsf{b_1}} \underset{2n-1}{P_\mathsf{2}} \longrightarrow \cdots \qquad \underset{n+3}{P_\mathsf{n-2}} \xrightarrow{\cdot \mathsf{b_{n-2}}} \underset{n+2}{P_\mathsf{n-1}} \xrightarrow{\cdot \mathsf{b_{n-1}}} \underset{n+1}{P_\mathsf{n}} \xrightarrow{\cdot \mathsf{a_n}} \underset{n}{P_\mathsf{n}}$$

$$\overset{\cdot \mathsf{a_n}}{\dashrightarrow} \underset{n}{P_\mathsf{n}} \xrightarrow{\cdot \mathsf{a_{n-1}}} \underset{n-1}{P_\mathsf{n-1}} \xrightarrow{\cdot \mathsf{a_{n-2}}} \underset{n-2}{P_\mathsf{n-2}} \longrightarrow \cdots \qquad \underset{2}{P_\mathsf{2}} \xrightarrow{\cdot \mathsf{a_1}} \underset{1}{P_\mathsf{1}} \xrightarrow{\cdot \mathsf{a_0}} \underset{0}{P_\star}$$

For the other simple modules we note that

$$\operatorname{syz}(S_\mathsf{i}) = rP_\mathsf{i} = (\mathsf{a_i}) \oplus (\mathsf{b_{i-1}}) = \operatorname{syz}^{i+1}(S_\star) \oplus \operatorname{syz}^{2n+2-i}(S_\star) \qquad (\mathsf{1 \le i \le n}) \quad (2.1.2)$$

In particular, the projective resolution of S_i is given by some parts of the projective resolution of $S_\star$. We obtain that

$$\operatorname{pd} S_\mathsf{i} = 2n + 1 - i = \operatorname{pd} S_\star - i \qquad (\mathsf{1 \le i \le n})$$

(2) If $\Lambda = \Lambda_{2n+2}$ for some $n \in \mathbb{N}_0$, the minimal projective resolution of the module $S_\star$ is given by

$$0 \longrightarrow \underset{2n+2}{P_\star} \xrightarrow{\cdot \mathsf{b}_0} \underset{2n+1}{P_{\mathsf{1}}} \xrightarrow{\cdot \mathsf{b}_1} \underset{2n}{P_{\mathsf{2}}} \xrightarrow{\cdot \mathsf{b}_2} \dots \qquad \underset{n+3}{P_{\mathsf{n-1}}} \xrightarrow{\cdot \mathsf{b}_{\mathsf{n-1}}} \underset{n+2}{P_{\mathsf{n}}} \xrightarrow{\begin{bmatrix} \cdot \mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}} \underset{n+1}{P_+ \oplus P_-}$$

$$\overset{\begin{bmatrix} \cdot \mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}}{\dashrightarrow} \underset{n+1}{P_+ \oplus P_-} \xrightarrow{[\cdot \mathsf{a}_+ \ \cdot \mathsf{a}_-]} \underset{n}{P_{\mathsf{n}}} \xrightarrow{\cdot \mathsf{a}_{\mathsf{n-1}}} \underset{n-1}{P_{\mathsf{n-1}}} \xrightarrow{\cdot \mathsf{a}_{\mathsf{n-2}}} \dots \qquad \underset{2}{P_{\mathsf{2}}} \xrightarrow{\cdot \mathsf{a}_1} \underset{1}{P_{\mathsf{1}}} \xrightarrow{\cdot \mathsf{a}_0} \underset{0}{P_\star}$$

Similarly to the odd case, the following holds:

$$\begin{aligned} \operatorname{syz}(S_{\mathsf{i}}) &= rP_{\mathsf{i}} = \operatorname{syz}^{i+1}(S_\star) \oplus \operatorname{syz}^{2n+3-i}(S_\star) \qquad (\mathsf{1} \leq \mathsf{i} \leq \mathsf{n}) \\ \operatorname{syz}(S_\pm) &= rP_\pm = (\mathsf{b}_\pm) \cong \operatorname{syz}^{n+2}(S_\star) \end{aligned} \tag{2.1.3}$$

In particular, we obtain that

$$\operatorname{pd} S_\pm = n + 1 = \frac{1}{2} \operatorname{pd} S_\star \quad \text{and} \quad \operatorname{pd} S_{\mathsf{i}} = 2n + 2 - i = \operatorname{pd} S_\star - i \qquad (\mathsf{1} \leq \mathsf{i} \leq \mathsf{n}).$$

For any $n \in \mathbb{N}^+$ and any simple Λ_n-module S we obtain that $\operatorname{pd} S \leq \operatorname{pd} S_\star = n$. This shows the first claim. The second claim follows from (2.1.2) and (2.1.3) above. □

2.1.2.3 Canonical bimodule and Auslander-Reiten translation

Let us denote by

- $\mathrm{D^b}(\Lambda_n)$ the bounded derived category of finitely generated Λ_n-modules,
- $\mathrm{D^b_{fd}}(\Lambda_n)$ its full subcategory of complexes with finite-dimensional homology.

The next Remark summarizes some technical details on these categories:

Remark 2.1.6. *(1) Let $\mathrm{K^b}(\Lambda_n \text{ -proj})$ be the homotopy category of projective complexes. There is an equivalence of categories $\mathrm{K^b}(\Lambda_n \text{ -proj}) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_n)$.*

(2) Let us note that there are the following equivalences of categories:

$$\mathrm{K^b_{fd}}(\Lambda_n \text{ -proj}) \xrightarrow{\sim} \mathrm{D^b_{fd}}(\Lambda_n) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_n \text{ -fd. mod}) \underset{\textit{if } \Bbbk = \mathbb{C}}{\overset{\sim}{\dashrightarrow}} \mathrm{D^b}(\mathcal{H}_0(G_n))$$

In particular, the category $\mathrm{D^b}(\mathcal{H}_0(G_n))$ is a full subcategory of $\mathrm{D^b}(\Lambda)$.

We note that any ring Λ_n is an order over $\mathbf{R} = \Bbbk[\![t]\!]$. The canonical bimodule of Λ_n is defined as $\omega_n = \operatorname{Hom}_{\mathbf{R}}(\Lambda_n, \mathbf{R})$. This bimodule induces the Auslander-Reiten translation

$$\tau = \omega_n \underset{\Lambda_n}{\otimes} \_ : \mathrm{D^b}(\Lambda_n) \longrightarrow \mathrm{D^b}(\Lambda_n)$$

By Lemma 2.1.5 the global dimension of Λ_n is finite. In this case, Theorem B.2.12 of van-den-Bergh and Iyama-Reiten implies that the Auslander-Reiten translation τ has the following properties:

(1) τ is an autoequivalence of $\mathrm{D^b}(\Lambda_n)$, which preserves the subcategory $\mathrm{D^b_{fd}}(\Lambda_n)$,

(2) for any $X_\bullet, Y_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda_n)$ with $X_\bullet$ or $Y_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda_n)$ there is a functorial isomorphism

$$\mathrm{Hom}(X_\bullet, Y_\bullet) \cong \mathbb{D}\,\mathrm{Ext}^1(Y_\bullet, \tau(X_\bullet)) = \mathbb{D}\,\mathrm{Hom}(Y_\bullet, \tau(X_\bullet)[1])$$

where $\mathbb{D} = \mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$ is the standard duality.

In different terms, $\mathrm{D}^{\mathrm{b}}(\Lambda_n)$ has a *relative Serre functor* $\mathbb{S} = \tau \circ [1]$.
Next, we determine the action of τ on projective Λ_n-modules.

(1) It is straightforward to compute the *canonical bimodule of* Λ_{2n+1} :

$$\omega_{2n+1} \cong \begin{array}{c} \\ \star \\ 1 \\ 2 \\ \vdots \\ \mathsf{n-2} \\ \mathsf{n-1} \\ \mathsf{n} \end{array} \begin{array}{c} \begin{array}{ccccccc} rP_\star & P_1 & P_2 & \cdots & P_{\mathsf{n-2}} & P_{\mathsf{n-1}} & P_{\mathsf{n}} \end{array} \\ \begin{bmatrix} \mathfrak{m}_0 & \mathfrak{m}_0 & 0 & & & & \\ \mathbf{R}_0 & \mathbf{R}_{0,1} & \mathfrak{m}_1 & & & & \\ 0 & \mathbf{R}_1 & \mathbf{R}_{1,2} & \ddots & & & \\ & & \ddots & \ddots & & & \\ & & & & \mathbf{R}_{\mathsf{n-3,n-2}} & \mathfrak{m}_{\mathsf{n-2}} & 0 \\ & & & & \mathbf{R}_{\mathsf{n-2}} & \mathbf{R}_{\mathsf{n-2,n-1}} & \mathfrak{m}_{\mathsf{n-1}} \\ & & & & 0 & \mathbf{R}_{\mathsf{n-1}} & \mathbf{R}_{\mathsf{n-1,n}} \end{bmatrix} \end{array}$$

In particular, the AR translation τ preserves all projectives except $P_\star$:

$$\tau(P_\star) \cong rP_\star \quad \text{and} \quad \tau(P_{\mathsf{i}}) \cong P_{\mathsf{i}} \qquad (1 \le \mathsf{i} \le \mathsf{n}).$$

(2) Computing the *canonical bimodule* of Λ_{2n+2} one obtains

$$\omega_{2n+2} \cong \begin{array}{c} \\ \star \\ 1 \\ 2 \\ \vdots \\ \mathsf{n-1} \\ \mathsf{n} \\ + \\ - \end{array} \begin{array}{c} \begin{array}{cccccccc} rP_\star & P_1 & P_2 & \cdots & P_{\mathsf{n-1}} & P_{\mathsf{n}} & P_- & P_+ \end{array} \\ \begin{bmatrix} \mathfrak{m}_0 & \mathfrak{m}_0 & 0 & & & & & \\ \mathbf{R}_0 & \mathbf{R}_{0,1} & \mathfrak{m}_1 & & & & & \\ 0 & \mathbf{R}_1 & \mathbf{R}_{1,2} & \ddots & & & & \\ & & \ddots & \ddots & & & & \\ & & & & \mathbf{R}_{\mathsf{n-2,n-1}} & \mathfrak{m}_{\mathsf{n-1}} & 0 & 0 \\ & & & & \mathbf{R}_{\mathsf{n-1}} & \mathbf{R}_{\mathsf{n-1,n}} & \mathfrak{m}_{\mathsf{n}} & \mathfrak{m}_{\mathsf{n}} \\ & & & & 0 & \mathbf{R}_{\mathsf{n}} & \mathfrak{m}_{\mathsf{n}} & \mathbf{R}_{\mathsf{n}} \\ & & & & 0 & \mathbf{R}_{\mathsf{n}} & \mathbf{R}_{\mathsf{n}} & \mathfrak{m}_{\mathsf{n}} \end{bmatrix} \end{array}$$

Now the AR translation τ preserves all projectives except $P_\star, P_+$ and P_- :

$$\tau(P_\star) \cong rP_\star, \quad \tau(P_\pm) \cong P_\mp, \quad \text{and} \quad \tau(P_{\mathsf{i}}) \cong P_{\mathsf{i}} \qquad (1 \le \mathsf{i} < \mathsf{n}).$$

Let us note that for any Khoroshkin order Λ_n, the Auslander-Reiten translation τ *preserves* the projectives of *glued* vertices, *interchanges* the *special* projectives P_+ and P_- and maps the *neutral* projective $P_\star$ to its *radical.*

2.1.2.4 The spherical twist $\mathbb{T}_{S_\star}$

Lemma 2.1.7. *For any $n \in \mathbb{N}^+$ the simple Λ_n-module $S_\star$ is* n-spherical.

PROOF. Let $\Lambda = \Lambda_n$ for some $n \in \mathbb{N}^+$. By Definition B.3.2 of n-spherical objects we have to show that $S_\star$ is n-*Calabi-Yau* and n-*spherelike*.

(1) $S_\star$ is n-*Calabi-Yau*:
It is sufficient to show that $\mathbb{S}(S_\star) \cong S_\star[n]$, or $\tau(S_\star) \cong S_\star[n-1]$. There is the following morphism ϕ of complexes:

$$\begin{tikzcd}
S_\star[n-1] \arrow[d, "\phi"] & \ldots \arrow[r] & P_2 \arrow[r, "\cdot\mathsf{a}_1"] \arrow[d] & P_1 \arrow[r, "\cdot\mathsf{a}_0"] \arrow[d, two heads, "\cdot\mathsf{a}_0"] & P_\star \arrow[r] \arrow[d, "\cdot\mathsf{b}_1"] & 0 \arrow[r] \arrow[d] & \ldots \arrow[r] & 0 \arrow[d] \\
\tau(S_\star) & \ldots \arrow[r] & \underset{n+1}{0} \arrow[r] & \underset{n}{rP_\star} \arrow[r, "\cdot\mathsf{b}_1"] & \underset{n-1}{P_1} \arrow[r, "\cdot\mathsf{b}_2"] & \underset{n-2}{P_2} \arrow[r] & \ldots \arrow[r] & \underset{0}{rP_\star}
\end{tikzcd}$$

Let us note that for $n = 2$ the above picture specializes as follows:

$$\begin{tikzcd}
S_\star[1] \arrow[d, "\phi"] & P_\star \arrow[r, "d_2"] \arrow[d] & P_+ \oplus P_- \arrow[r, "d_1"] \arrow[d, "d_1"] & P_\star \arrow[r] \arrow[d, "d_2"] & 0 \arrow[d] \\
\tau(S_\star) & 0 \arrow[r] & rP_\star \arrow[r, "d_2"] & P_+ \oplus P_- \arrow[r, two heads, "d_1"] & rP_\star
\end{tikzcd}
\quad \text{where} \quad d_2 = \begin{bmatrix} \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix} \quad \text{and} \quad d_1 = [\, \cdot\mathsf{a}_+ \;\; \cdot\mathsf{a}_- \,]$$

It can be verified that $\mathbf{H}_j(\phi)$ is an isomorphism for any $j \in \mathbb{Z}$. Equivalently, $\tau(S_\star) \cong S_\star[n-1]$ in $\mathrm{D^b}(\Lambda_n)$. So $S_\star$ is n-Calabi-Yau.

(2) $S_\star$ is also n-*spherelike*:
Since $\operatorname{pd} S_\star = n$ and $S_\star$ is n-Calabi-Yau we obtain that

$$\mathrm{Ext}^i_\Lambda(S_\star, S_\star) \cong \begin{cases} \Bbbk & \text{if } i = 0 \text{ or } n, \\ 0 & \text{if } i < 0 \text{ or } i > n. \end{cases}$$

Let $0 < i < n$. Let $P_\bullet$ denote the projective resolution of $S_\star$. Note that the projective in $P_\bullet$ at degree i does not contain a direct summand of type $P_\star$. This implies that $\mathrm{Ext}^i_\Lambda(S_\star, S_\star) = \mathbf{H}_i(\mathrm{Hom}_\Lambda(P_\bullet, S_\star)) = 0$. □

Let us fix some $n \in \mathbb{N}^+$. Let $\mathbb{T}^\vee_{S_\star} : \mathrm{D^b}(\Lambda_n) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_n)$ be the *dual twist functor* associated to $S_\star$ as defined in Subsection B.3. Since $S_\star$ is spherical, the functor $\mathbb{T}^\vee_{S_\star}$ is an autoequivalence by Proposition B.3.3. Next, we want to express $\mathbb{T}^\vee_{S_\star}$ as a *standard* functor.

Let $\pi_n : \Lambda_n \twoheadrightarrow S_\star$ be the natural projection considered as a morphism of Λ_n-bimodules. By Theorem B.3.12 and Corollary B.3.13 the spherical twist $\mathbb{T}^\vee_{S_\star}$ can be expressed as follows:

$$\mathbb{T}^\vee_{S_\star}(\_) \cong \mathbb{T}^\vee_{S_\star}(\Lambda_n) \underset{\Lambda}{\otimes} \_ \cong \ker(\pi_n) \underset{\Lambda}{\otimes} \_$$

The bimodule $\ker(\pi_n)$ is closely related to the canonical bimodule ω_n:

(1) For $n \in \mathbb{N}_0$ we have that

$$\ker(\pi_{2n+1}) \cong \left[\begin{array}{c|c|c|c|c|c} rP_\star & P_1 & P_2 & \ldots & P_{\mathsf{n}-1} & P_{\mathsf{n}} \end{array}\right] \cong \omega_{2n+1} \tag{2.1.4}$$

(2) For $n \in \mathbb{N}_0$ we obtain that

$$\ker(\pi_{2n+2}) \cong \left[\begin{array}{c|c|c|c|c|c|c} rP_\star & P_1 & P_2 & \dots & P_n & P_+ & P_- \end{array}\right] \tag{2.1.5}$$

Remark 2.1.8. *Let* $\mathbb{T}_{S_\star}$ *denote the (usual)* twist functor *associated to the spherical object* $S_\star$*. Its quasi-inverse is given by the dual twist functor of* $S_\star$ *:* $\mathbb{T}_{S_\star}^{-1} \cong \mathbb{T}_{S_\star}^{\vee}$.

2.1.2.5 The Auslander-Reiten translation as a spherical twist

Definition 2.1.9. *Let* $\Lambda = \Lambda_{2n+1}$ *or* Λ_{2n+2} *for some* $n \in \mathbb{N}_0$*. We define a functor* $\sigma : \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$ *such that* $\sigma^2 \cong \mathrm{Id}$ *as follows*:

- *if* $\Lambda = \Lambda_{2n+1}$*, we set* $\sigma = \mathrm{Id}$.
- *if* $\Lambda = \Lambda_{2n+2}$*, the functor* σ *is given by interchanging* $+$ *and* $-$ *in projectives and their differentials*:

$$\begin{array}{lcllcl} P_n \xrightarrow{\cdot b_\pm} P_\pm & \xmapsto{\sigma} & P_n \xrightarrow{\cdot b_\mp} P_\mp & \qquad P_i \xrightarrow{d} P_j & \xmapsto{\sigma} & P_i \xrightarrow{d} P_j \\ P_\pm \xrightarrow{\cdot a_\pm} P_n & \xmapsto{\sigma} & P_\mp \xrightarrow{\cdot a_\mp} P_n & & & \textit{if } i,j \notin \{+,-\} \end{array}$$

In other words, the functor σ is induced from the involution of the quiver $(Q, I)_n$.

The following statement is the main result of this subsection:

Proposition 2.1.10. *For any* $n \in \mathbb{N}^+$ *there are isomorphisms of the following functors*:

$$\tau \cong \mathbb{T}_{S_\star}^{\vee} \circ \sigma \cong \sigma \circ \mathbb{T}_{S_\star}^{\vee} : \quad \mathrm{D}^{\mathrm{b}}(\Lambda_n) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda_n) \tag{2.1.6}$$

Proof. Let $\Lambda = \Lambda_n$ for some $n \in \mathbb{N}^+$. The Auslander-Reiten translation τ and the dual spherical twist $\mathbb{T}_{S_\star}$ can both be expressed as standard functors:

$$\tau = \omega_n \underset{\Lambda}{\otimes} \_ \quad \text{and} \quad \mathbb{T}_{S_\star}^{\vee} \cong \ker(\pi_n) \underset{\Lambda}{\otimes} \_$$

Now, the statement follows from (2.1.4) respectively (2.1.5):

$$\mathbb{T}_{S_\star}^{\vee} \cong \ker(\pi_n) \underset{\Lambda}{\otimes} \_ \cong \omega_n \underset{\Lambda}{\otimes} \_ \circ \sigma = \tau \circ \sigma \cong \sigma \circ \tau$$

where the last isomorphism holds because the Auslander-Reiten translation commutes with any auto-equivalence of $\mathrm{D}^{\mathrm{b}}(\Lambda)$. □

2.1.2.6 Cohen-Macaulay modules

By Lemma B.1.24 the Cohen-Macaulay modules of any Khoroshkin order Λ_n are exactly the submodules of projective Λ_n-modules. The indecomposable Cohen-Macaulay modules of Khoroshkin orders Λ_n are given as follows:

Proposition 2.1.11. *Let* $n \in \mathbb{N}_0$.

(1) The indecomposable Cohen-Macaulay modules of Λ_{2n+1} *are given by*

- *the* n *projective-injective Cohen-Macaulay modules* $P_1, P_2, \dots, P_n$,

- *and the $2n+1$ ideals*

$$(\mathsf{b}_0),\ (\mathsf{b}_1),\ \ldots,\ (\mathsf{b}_{\mathsf{n}-1}),\ (\mathsf{b}_\mathsf{n}), (\mathsf{a}_{\mathsf{n}-1}),\ \ldots,\ (\mathsf{a}_1),\ (\mathsf{a}_0).$$

(2) The indecomposable Cohen-Macaulay modules of Λ_{2n+2} are given by

- *the $n+2$ projective-injective Cohen-Macaulay modules $P_1, P_2, \ldots, P_\mathsf{n}, P_+, P_-$,*
- *and the $2n+2$ ideals*

$$(\mathsf{b}_0),\ \ldots,\ (\mathsf{b}_{\mathsf{n}-1}),\ (\mathsf{b}_\pm),\ (\mathsf{a}_+, \mathsf{a}_-),\ (\mathsf{a}_{\mathsf{n}-1}),\ \ldots,\ (\mathsf{a}_0),$$

We note that $(\mathsf{b}_+) \cong (\mathsf{b}_-)$. The non-projective-injective Cohen-Macaulay module are given by

$$P_\star \cong \begin{cases} (\mathsf{b}_0) & \text{if } n > 0 \\ (\mathsf{b}_\pm) & \text{if } n = 0 \end{cases} \qquad \text{and} \qquad rP_\star = \begin{cases} (\mathsf{a}_0) & \text{if } n > 0 \\ (\mathsf{a}_+, \mathsf{a}_-) & \text{if } n = 0 \end{cases}$$

COMMENTS ON THE PROOF. This statement can be derived using the classification method for lattices over one-dimensional orders by [**RR79**] or [**GR78**]. This method is based on a non-commutative analogue of the category of triples of Subsection 1.1.2. Alternatively, the classification can be obtained using Auslander-Reiten theory [**Aus99**, Chapter VI, Section 7]. □

Let us note that the simple module $S_\star$ also "knows" all Cohen-Macaulay modules:

Corollary 2.1.12. *Let $n \in \mathbb{N}^+$. Any indecomposable Cohen-Macaulay module over Λ_n is projective or isomorphic to some syzygy of $S_\star$:*

$$\mathrm{ind}[\ \mathrm{CM}(\Lambda_n)\ ] = \ \mathrm{ind}[\ \mathrm{proj}(\Lambda_n)\ ]\ \oplus\ [\ \mathrm{syz}^j(S_\star)\ |\ 1 \le j < n\ ]$$

2.1.3 Khoroshkin orders of the disconnected Lorentz groups $\mathrm{SO}(n,1)$

At last, we will briefly discuss the orders $\Lambda_n^\#$ associated to the disconnected Lorentz groups $G_n^\# = \mathrm{SO}(n,1)$. Again, our main goal is to describe a factorization of the Auslander-Reiten translation. The difference to the previous cases is only technical. More precisely, we have to use a *generalized* twist functor associated to a spherical *collection* of several simple modules. We will skip the computational proofs in this section for brevity.

Khoroshkin quivers of disconnected Lorentz groups

Let us introduce the quivers $(Q,I)_n^\#$, where $n \in \mathbb{N}^+$:

(1) For any $n \in \mathbb{N}_0$ we set $\quad (Q,I)_{2n+1}^\# = (Q,I)_{2n+1} \times (Q,I)_{2n+1}$.

(2) For any $n \in \mathbb{N}_0$ we have

$$(Q,I)^{\#}_{2n+2} = \begin{tikzcd}
 & n^+ \arrow[r, bend left, "b^+_{n-1}"] \arrow[dl, bend left, "a^+_n"] & n{-}1^+ \arrow[l, bend left, "a^+_{n-1}"] \arrow[r, bend left, "b^+_{n-2}"] & n{-}2^+ \arrow[l, bend left, "a^+_{n-2}"] & \cdots & 2^+ \arrow[r, bend left, "b^+_1"] & 1^+ \arrow[l, bend left, "a^+_1"] \arrow[r, bend left, "b^+_0"] & \star^+ \arrow[l, bend left, "a^+_0"] \\
n{+}1 \arrow[ur, bend left, "b^+_n"] \arrow[dr, bend left, "a^-_n"] & & & & & & & \\
 & n^- \arrow[ul, bend left, "b^-_n"] \arrow[r, bend left, "a^-_{n-1}"] & n{-}1^- \arrow[l, bend left, "b^-_{n-1}"] \arrow[r, bend left, "a^-_{n-2}"] & n{-}2^- \arrow[l, bend left, "b^-_{n-2}"] & \cdots & 2^- \arrow[r, bend left, "a^-_1"] & 1^- \arrow[l, bend left, "b^-_1"] \arrow[r, bend left, "a^-_0"] & \star^- \arrow[l, bend left, "b^-_0"]
\end{tikzcd} \tag{2.1.7}$$

$$\text{with relations} \quad b^+_n\, b^-_n = a^-_n\, a^+_n = 0 \quad \text{and} \quad b^{\pm}_i\, b^{\pm}_{i+1} = a^{\pm}_{i+1}\, a^{\pm}_i = 0 \quad (1 \leq i \leq n). \tag{2.1.8}$$

This quiver has *two* neutral vertices $\star^+$ and $\star^-$, and $2n+1$ glued vertices.

Remark 2.1.13. *For any $n \in \mathbb{N}^+$ the Khoroshkin quiver $(Q,I)^{\#}_n$ is "gentle with oriented cycles".*

Khoroshkin orders of disconnected Lorentz groups

For any $n \in \mathbb{N}^+$ let $\Lambda^{\#}_n$ be the completed path algebra of the quiver $(Q,I)^{\#}_n$.

(1) For any $n \in \mathbb{N}_0$ it holds that $\Lambda^{\#}_{2n+1} = \Lambda_{2n+1} \times \Lambda_{2n+1}$.
In this case there are two neutral simple $\Lambda^{\#}_{2n+1}$-modules which we will denote by $S_{\star^+} = S_\star \times 0$ and $S_{\star^-} = 0 \times S_\star$.

(2) For any $n \in \mathbb{N}_0$ there is an isomorphism

$$\Lambda^{\#}_{2n+2} \cong \begin{array}{cc}
 & \begin{array}{ccccccccc} P_{\star^+} & P_{1^+} & P_{2^+} & \cdots & P_{n+1} & \cdots & P_{2^-} & P_{1^-} & P_{\star^-} \end{array} \\
\begin{array}{c} \star^+ \\ 1^+ \\ 2^+ \\ \vdots \\ n{+}1 \\ \vdots \\ 2^- \\ 1^- \\ \star^- \end{array} &
\begin{bmatrix}
\mathbf{R}_{0^+} & \mathfrak{m}_{0^+} & 0 & & & & & & \\
\mathbf{R}_{0^+} & \mathbf{R}_{0^+,1^+} & \mathfrak{m}_{1^+} & & & & & & \\
0 & \mathbf{R}_{1^+} & \mathbf{R}_{1^+,2^+} & \ddots & & & & & \\
 & & \ddots & \ddots & \mathfrak{m}_n & & & & \\
 & & & & \mathbf{R}_{n,n+1} & & & & \\
 & & & & \mathbf{R}_{n+1} & \ddots & \ddots & & \\
 & & & & & \ddots & \mathbf{R}_{2^-,1^-} & \mathfrak{m}_{1^-} & 0 \\
 & & & & & & \mathbf{R}_{1^-} & \mathbf{R}_{1^-,0^-} & \mathfrak{m}_{0^-} \\
 & & & & & & 0 & \mathbf{R}_{0^-} & \mathbf{R}_{0^-}
\end{bmatrix}
\end{array}$$

The minimal projective resolution of the simple $\Lambda^{\#}_{2n+2}$-module $S_{\star^{\pm}}$ is given as follows:

$$0 \longrightarrow \underset{2n+2}{P_{\star^{\mp}}} \xrightarrow{\cdot \mathsf{b}_0^{\mp}} \underset{2n+1}{P_{\mathsf{1}^{\mp}}} \xrightarrow{\cdot \mathsf{b}_1^{\mp}} \underset{2n}{P_{\mathsf{2}^{\mp}}} \xrightarrow{\cdot \mathsf{b}_2^{\mp}} \cdots \qquad \underset{n+3}{P_{\mathsf{n-1}^{\mp}}} \xrightarrow{\cdot \mathsf{b}_{\mathsf{n-1}}^{\mp}} \underset{n+2}{P_{\mathsf{n}^{\mp}}} \xrightarrow{\cdot \mathsf{b}_{\mathsf{n}}^{\mp}} \underset{n+1}{P_{n+1^{\pm}}}$$

$$\overset{\cdot \mathsf{b}_{\mathsf{n}}^{\mp}}{\dashrightarrow} \underset{n+1}{P_{\mathsf{n+1}^{\pm}}} \xrightarrow{\mathsf{a}_{\mathsf{n}}^{\pm}} \underset{n}{P_{\mathsf{n}^{\pm}}} \xrightarrow{\cdot \mathsf{a}_{\mathsf{n-1}}^{\pm}} \underset{n-1}{P_{\mathsf{n-1}^{\pm}}} \xrightarrow{\cdot \mathsf{a}_{\mathsf{n-2}}^{\pm}} \cdots \qquad \underset{2}{P_{\mathsf{2}^{\pm}}} \xrightarrow{\cdot \mathsf{a}_1^{\pm}} \underset{1}{P_{\mathsf{1}^{\pm}}} \xrightarrow{\cdot \mathsf{a}_0^{\pm}} \underset{0}{P_{\star^{\pm}}}$$

For any $n \in \mathbb{N}^+$ we define a semi-simple $\Lambda^{\#}_n$-module by $S_\star = S_{\star^+} \oplus S_{\star^-}$.

Lemma 2.1.14. *Let $n \in \mathbb{N}^+$.*

(1) It holds that $\operatorname{gldim} \Lambda^{\#}_n = \operatorname{pd} S_{\star^{\pm}} = n$.

(2) For any simple $\Lambda^{\#}_n$-module S it holds that

$$\operatorname{syz}(S) \in \operatorname{add}\{\ \operatorname{syz}^j(S_\star) \mid 1 \le j \le n\ \}$$

As in the previous section, Corollary B.1.8 and Theorem C.3.1 yield the following equivalences of categories for any $n \in \mathbb{N}^+$:

$$\operatorname{nil.rep}_{\Bbbk}(Q, I)^{\#}_n \xrightarrow{\sim} \Lambda^{\#}_n\text{-}\operatorname{fd.mod} \underset{\text{if } \Bbbk = \mathbb{C}}{\overset{\sim}{\dashrightarrow}} \mathcal{H}_0(G^{\#}_n)$$

where $\mathcal{H}_0(G^{\#}_n)$ is the principal block of Harish-Chandra modules over the disconnected Lorentz group $G^{\#}_n = \mathrm{SO}(n, 1)$.

Canonical bimodule and Auslander-Reiten translation

In the following let $n \in \mathbb{N}_0$.

(1) The canonical bimodule of $\Lambda^{\#}_{2n+1}$ is the direct product $\omega^{\#}_{2n+1} = \omega^{\times 2}_{2n+1}$.

(2) The canonical bimodule of $\Lambda^{\#}_{2n+2}$ is given by

$$\omega^{\#}_{2n+2} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} rP_{\star^+} & P_{1^+} & P_{2^+} & \cdots & P_{n+1} & \cdots & P_{2^-} & P_{1^-} & rP_{\star^-} \end{array}\right]$$

In both cases, the Auslander-Reiten translation τ preserves the projectives of glued vertices, and maps each neutral projective to its radical.

The spherical twist $\mathbb{T}_{S_\star}$

Again, let $n \in \mathbb{N}_0$. For simplicity of the following notation let $n \in \mathbb{N}^+$.

(1) if n is *odd*, the order $\Lambda^{\#}_n$ has the two n-spherical modules $S_{\star^+}$ and $S_{\star^-}$. In the following, we will consider the dual twist functor $\mathbb{T}^{\vee}_{S_\star}$ associated to the semisimple module $S_\star = S_{\star^+} \oplus S_{\star^-}$. This functor is a direct sum of autoequivalences:

$$\mathbb{T}^{\vee}_{S_\star} \cong \mathbb{T}^{\vee}_{S_{\star^+}} \oplus \mathbb{T}^{\vee}_{S_{\star^-}} : \quad \mathrm{D^b}(\Lambda^{\#}_n) \xrightarrow{\sim} \mathrm{D^b}(\Lambda^{\#}_n) \qquad \text{where } \Lambda^{\#}_n = \Lambda^{\times 2}_n.$$

(2) if n is *even*, there is a new phenomenon:

Lemma 2.1.15. *As above, let $n \in \mathbb{N}^+$ be even. The collection $\mathcal{S}_\star = \{S_{\star^+}, S_{\star^-}\}$ is an n-*spherical collection *in $\mathrm{D^b}(\Lambda_n^\#)$ in the sense of Definition B.3.9.*

In the following let

- $\mathbb{T}^\vee_{\mathcal{S}_\star}$ be the *dual twist functor* of the spherical collection $\mathcal{S}_\star$, and
- $\pi_n : \Lambda_n^\# \twoheadrightarrow S_{\star^+} \oplus S_{\star^-}$ be the natural projection.

According to Section B.3.3 the dual twist $\mathbb{T}^\vee_{\mathcal{S}_\star}$ yields also an autoequivalence, and can also be expressed as a standard functor:

$$\mathbb{T}^\vee_{\mathcal{S}_\star}(\_) \cong \ker(\pi_n) \otimes \_ : \qquad \pi_n : \mathrm{D^b}(\Lambda_n^\#) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_n^\#)$$

Remark 2.1.16. *The dual twist of the collection is not isomorphic to the dual twist of the direct sum:* $\mathbb{T}^\vee_{\mathcal{S}_\star} \ncong \mathbb{T}^\vee_{S_{\star^+} \oplus S_{\star^-}} \cong \mathbb{T}^\vee_{S_{\star^+}} \oplus \mathbb{T}^\vee_{S_{\star^-}}$

Auslander-Reiten translation as a generalized spherical twist

Proposition 2.1.17. *Let $n \in \mathbb{N}_0$. There is an isomorphism of functors*:

$$\tau \cong \mathbb{T}^\vee_{\mathcal{S}_\star} : \quad \mathrm{D^b}(\Lambda_n^\# \text{ -mod}) \xrightarrow{\sim} \mathrm{D^b}(\Lambda_n^\# \text{ -mod})$$

Proof. Let $\Lambda^\# = \Lambda^\#_{2n+1}$ or $\Lambda^\#_{2n+2}$ for some $n \in \mathbb{N}^+$.

(1) if $\Lambda^\# = \Lambda^\#_{2n+1}$, the statement follows from the odd case of Proposition 2.1.10

(2) if $\Lambda^\# = \Lambda^\#_{2n+2}$, there is a bimodule isomorphism $\omega_{2n+2} \cong \ker(\pi_{2n+2})$. □

Remark 2.1.18. *Despite the fact that for any $n \in \mathbb{N}^+$ the Khoroshkin quiver $(Q, I)^\#_n$ has a natural involution (which corresponds to involution functor κ in Section C.3), it does* not *occur in the factorization of the Auslander-Reiten translation. This is related to the fact that such a quiver has* no *special vertices.*

Cohen-Macaulay modules

Proposition 2.1.19. *For any $n \in \mathbb{N}_0$ the indecomposable Cohen-Macaulay modules of $\Lambda^\#_{2n+2}$ are given by*

- *the $2n + 1$ projective-injective Cohen-Macaulay modules $P_{\mathsf{1}^\pm}, P_{\mathsf{2}^\pm}, \ldots, P_{\mathsf{n}^\pm}, P_{\mathsf{n+1}}$,*
- *and the $4n + 4$ ideals*

$$(\mathsf{b}_0^\pm), (\mathsf{b}_1^\pm), \ldots, \ (\mathsf{b}_\mathsf{n}^\pm), (\mathsf{a}_\mathsf{n}^\pm), \ \ldots, \ (\mathsf{a}_1^\pm), (\mathsf{a}_0^\pm).$$

Corollary 2.1.20. *Any indecomposable Cohen-Macaulay module over Λ_{2n+2} is either a projective-injective Cohen-Macaulay module or isomorphic to some syzygy of $S_\star = S_{\star^+} \oplus S_{\star^-}$:*

$$\mathrm{ind}[\mathrm{CM}(\Lambda_{2n+2})] = [\ \mathrm{syz}^j(S_\star) \mid 1 \leq j \leq 2n+2\ ] \ \oplus \ \mathrm{ind}[\mathrm{proj}\,.\,\mathrm{inj}\,\mathrm{CM}(\Lambda)]$$

The precise relationship between any two Khoroshkin orders $\Lambda^\#_n$ and Λ_n is described in Subsection C.3.

2.2 Derived Auslander-Reiten Theory of Khoroshkin orders

This section contains all results of this chapter. In the first subsection, we introduce a homological invariant called the *defect* for any Khoroshkin order Λ. The main result shows that objects of the derived category $\mathrm{D^b}(\Lambda)$ with *vanishing* defect are exactly the *τ-periodic* objects.
Then we show that τ-periodic objects can be related to some category of a certain suborder $\Lambda^e \subset \Lambda$ of infinite global dimension. It turns out that the defect numbers also determine the *homological dimensions* of finite-dimensional Λ-modules.
At last, we divide the indecomposable objects in the derived category into *four classes* which will be characterized by combinatorial terms in Chapter 3.

2.2.1 The defect, the involution and the Auslander-Reiten translation

Definition 2.2.1. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ be a Khoroshkin order for some $n \in \mathbb{N}^+$.*

- *if $\Lambda = \Lambda_n$, let $S_\star$ be the simple Λ-module corresponding to vertex $\star$ in $(Q, I)_n$,*
- *if $\Lambda = \Lambda_n^{\#}$, let $S_\star = S_{\star^+} \oplus S_{\star^-}$ be the direct sum of simple Λ-modules corresponding to the two neutral vertices in $(Q, I)_n^{\#}$.*

For $P_\bullet \in \mathrm{D^b}(\Lambda)$ and any $i \in \mathbb{Z}$ we set

$$\delta^{(i)}(P_\bullet) = \dim_{\Bbbk} \mathrm{Ext}^i_\Lambda(P_\bullet, S_\star), \qquad \text{and} \qquad \delta(P_\bullet) = \sum_{i\in\mathbb{Z}} \delta^{(i)}(P_\bullet)$$

The invariants $\delta^{(i)}(P_\bullet)$ will be called the defect numbers, *and $\delta(P_\bullet)$ the* defect of $P_\bullet$.

Remark 2.2.2. *Let $P_\bullet \in \mathrm{D^b}(\Lambda)$. Without loss of generality, we may assume that $P_\bullet$ is a minimal projective complex. By Lemma B.2.17 the defect $\delta(P_\bullet)$ counts the* number of projectives of type $P_\star$ respectively $P_{\star^\pm}$ in $P_\bullet$.

The following statement is the main result of this chapter:

Theorem 2.2.3. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ be a Khoroshkin order for some $n \in \mathbb{N}^+$. Let $\mathcal{S}_\star$ be a collection of objects in $\mathrm{D^b}(\Lambda)$ defined as follows:*

$$\mathcal{S}_\star = \begin{cases} \{\ S_\star\ \} & \text{if } \Lambda = \Lambda_n \text{ for some } n \in \mathbb{N}^+ \\ \{S_{\star^+}\ ,\ S_{\star^-}\} & \text{if } \Lambda = \Lambda_{2n}^{\#} \text{ for some } n \in \mathbb{N}^+ \\ \{S_{\star^+} \oplus S_{\star^-}\} & \text{if } \Lambda = \Lambda_{2n-1}^{\#} \text{ for some } n \in \mathbb{N}^+ \end{cases}$$

Let $\mathbb{T}^\vee_{\mathcal{S}_\star}$ be the dual twist functor associated to the collection $\mathcal{S}_\star$.

(1) The dual twist functor $\mathbb{T}^\vee_{\mathcal{S}}$ is an autoequivalence of $\mathrm{D^b}(\Lambda)$.

(2) The Auslander-Reiten translation on $\mathrm{D^b}(\Lambda)$ has the following factorization:

$$\tau \cong \sigma \circ \mathbb{T}^\vee_{\mathcal{S}_\star} \cong \mathbb{T}^\vee_{\mathcal{S}_\star} \circ \sigma : \quad \mathrm{D^b}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D^b}(\mathcal{H}_0)$$

where the involution functor σ was given as follows:

- *$\sigma = \mathrm{Id}$, if $\Lambda = \Lambda_{2n-1}$ or $\Lambda = \Lambda_n^{\#}$ for some $n \in \mathbb{N}^+$, or*
- *σ is the involution interchanging $+$ and $-$, if $\Lambda = \Lambda_{2n}$ for some $n \in \mathbb{N}^+$.*

(3) Assume that $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$.
For any $P_\bullet \in \mathrm{D}^\mathrm{b}(\Lambda)$ the following conditions are equivalent:

(a)	$\delta(P_\bullet) = 0$	(b)	$\tau^m(P_\bullet) \cong P_\bullet$ *for some* $m \in \mathbb{N}^+$
(c)	$\tau(P_\bullet) \cong \sigma(P_\bullet)$	(d)	$\tau^2(P_\bullet) \cong P_\bullet$
(e)	$\mathbb{T}^\vee_{\mathcal{S}_\star}(P_\bullet) \cong P_\bullet$		

Proof. (1) The first statement follows from Lemma 2.1.7 respectively Lemma 2.1.15.

(2) The second statement collects Propositions 2.1.10 and 2.1.17.

(3) Only the third statement of the Theorem is new.

- $(a) \Leftrightarrow (e)$: This is stated in Lemma B.3.14.
- $(e) \Rightarrow (c)$: This follows from statement (2) of the Theorem.
- The implications $(c) \Rightarrow (d) \Rightarrow (b)$ are obvious.
- It remains to show that $(b) \Rightarrow (e)$: Let $P_\bullet$ be τ-periodic for some $m \in \mathbb{N}^+$. Then $\tau^{2m}(P_\bullet) \cong P_\bullet$. By (2) it follows that

$$P_\bullet \cong \tau^{2m}(P_\bullet) \cong (\sigma \circ \mathbb{T}^\vee_{\mathcal{S}_\star})^{2m}(P_\bullet) \cong \sigma^{2m} \circ (\mathbb{T}^\vee_{\mathcal{S}_\star})^{2m}(P_\bullet) \cong (\mathbb{T}^\vee_{\mathcal{S}_\star})^{2m}(P_\bullet)$$

By Lemma B.3.14 it follows that $\mathbb{T}^\vee_{\mathcal{S}_\star}(P_\bullet) \cong P_\bullet$. □

The theorem above has the following consequence:

Corollary 2.2.4. *For an order Λ as above, the homogeneous tubes in the Auslander-Reiten quiver of $\mathrm{D}^\mathrm{b}_{\mathrm{fd}}(\Lambda)$ have either rank* one *or* two.

Remark 2.2.5. *Let us give a few remarks on the Theorem above*:

(1) The defect δ and involution σ have certain Lie-theoretic interpretations for orders of type Λ_n (see Section C.3). In particular, Theorem 2.2.3 gives a relationship between these Lie-theoretic notions and the Auslander-Reiten translation.

(2) Theorem 2.2.3 is slightly more general than its formulation in the introduction (Theorem 2), since $\mathrm{D}^\mathrm{b}(\mathcal{H}_0) \xrightarrow{\sim} \mathrm{D}^\mathrm{b}_{\mathrm{fd}}(\Lambda) \hookrightarrow \mathrm{D}^\mathrm{b}(\Lambda)$.

(3) For any $P_\bullet \in \mathrm{D}^\mathrm{b}_{\mathrm{fd}}(\Lambda)$, we have $\delta(P_\bullet) = 0$ if and only if the complex $P_\bullet$ is contained in some regular *component of the Auslander-Reiten quiver of $\mathrm{D}^\mathrm{b}_{\mathrm{fd}}(\Lambda)$. In particular, the name "defect" is motivated by the analogous notion in the representation theory of hereditary algebras (see Remark A.3.3 for an example).*

(4) Theorem 2.2.3 does not hold for $n = 1$. For example, if $\Lambda = \Lambda_1 = \Bbbk[\![t]\!]$, it holds that $\delta(P_\bullet) > 0$ and $\tau(P_\bullet) \cong P_\bullet$ for any $P_\bullet \in \mathrm{D}^\mathrm{b}(\Lambda_1)$.

2.2.2 The subcategory of τ-periodic objects

As usual, let Λ be some Khoroshkin order. Next, we study the category $\mathcal{T}$ of τ-periodic objects in $\mathrm{D}^\mathrm{b}(\Lambda)$ in detail. In the first part of this section we consider a Gorenstein suborder Λ^e of infinite global dimension of Λ. In the second part we show that the category $\mathcal{T}$ is equivalent to the category $\mathsf{Perf}(\Lambda^e)$ of perfect complexes of Λ^e. In particular, $\mathcal{T}$ can be viewed as the "singular part" of the category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

2.2.2.1 The restricted Khoroshkin orders

First, let us introduce certain subrings of Khoroshkin orders:

- Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ be a Khoroshkin order for some $n \geq 2$,
- let $(Q, I) = (Q, I)_n$ or $(Q, I)_n^{\#}$ be the corresponding quiver.
- we denote by $e_\star$ the following idempotent of Λ :
  - if $\Lambda = \Lambda_n$, let $e_\star$ be the idempotent corresponding to the neutral vertex $\star$,
  - if $\Lambda = \Lambda_n^{\#}$, let $e_\star = e_{\star^+} + e_{\star^-}$ be the sum of idempotents of the neutral vertices,
- Let $e = 1 - e_\star$ and Λ^e denote the subalgebra $\Lambda^e = e\Lambda e$ of Λ.
- let $(Q, I)^e$ be the subquiver of (Q, I) on the non-neutral vertices.

The subalgebra Λ^e is isomorphic to the completed path algebra of the subquiver $(Q, I)^e$ and there is an equivalence of categories

$$\Lambda^e\text{ - fd. mod} \xrightarrow{\sim} \text{nil. rep}(Q, I)^e$$

In the case of *Khoroshkin orders* of *connected* Lorentz groups, the subquivers $(Q, I)^e$ have the following form:

(1) (a) assume that $\Lambda = \Lambda_{2n+1}$ for some $n \geq 2$. Then

$$(Q, I)^e_{2n+1} = \begin{tikzcd} n \arrow[loop left, "\mathsf{x}"] \arrow[r, bend left, "\mathsf{b}"] & n-1 \arrow[l, bend left, "\mathsf{a}"] \arrow[r, bend left, "\mathsf{b}"] & n-2 \arrow[l, bend left, "\mathsf{a}"] & \cdots & 3 \arrow[r, bend left, "\mathsf{b}"] & 2 \arrow[l, bend left, "\mathsf{a}"] \arrow[r, bend left, "\mathsf{b}"] & 1 \arrow[l, bend left, "\mathsf{a}"] \arrow[loop right, "\mathsf{y}"] \end{tikzcd}$$

with relations $\quad \mathsf{b\,x} = \mathsf{x\,a} = \mathsf{b}^2 = \mathsf{a}^2 = \mathsf{y\,b} = \mathsf{a\,y} = 0. \qquad (2.2.1)$

(b) in the special case $\Lambda = \Lambda_3$, let us note that $\Lambda^e \cong \Bbbk[\![x, y]\!]/(x\,y)$.

For any $n \geq 1$, the quiver $(Q, I)^e_{2n+1}$ has only *glued* vertices.

(2) (a) Let $\Lambda = \Lambda_{2n+2}$ for some $n \geq 1$. Then

$$(Q, I)^e_{2n+2} = \begin{tikzcd} + \arrow[dr, bend left, "\mathsf{b}_+"] & & & & & & \\ & n \arrow[ul, bend left, "\mathsf{a}_+"] \arrow[dl, bend left, "\mathsf{a}_-"] \arrow[r, bend left, "\mathsf{b}"] & n-1 \arrow[l, bend left, "\mathsf{a}"] \arrow[r, bend left, "\mathsf{b}"] & n-2 \arrow[l, bend left, "\mathsf{a}"] & \cdots & 3 \arrow[r, bend left, "\mathsf{b}"] & 2 \arrow[l, bend left, "\mathsf{a}"] \arrow[r, bend left, "\mathsf{b}"] & 1 \arrow[l, bend left, "\mathsf{a}"] \arrow[loop right, "\mathsf{y}"] \\ - \arrow[ur, bend left, "\mathsf{b}_-"] & & & & & & \end{tikzcd}$$

with relations $\quad \mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-, \quad$ and $\quad \mathsf{b\,b}_\pm = \mathsf{a}_\pm\,\mathsf{a} = \mathsf{b}^2 = \mathsf{a}^2 = \mathsf{y\,b} = \mathsf{a\,y} = 0.$

(b) in the case of the Gelfand order $\Lambda = \Lambda_2$ we have

$$\Lambda_2^e = \begin{bmatrix} \mathbf{R} & \mathfrak{m} \\ \mathfrak{m} & \mathbf{R} \end{bmatrix} \quad \text{and} \quad (Q,I)_2^e = \begin{tikzcd} \underset{+}{\bullet} \arrow[loop left, "\mathsf{x}_+"] \arrow[r, bend left, "\mathsf{y}_+"] & \underset{-}{\bullet} \arrow[loop right, "\mathsf{x}_-"] \arrow[l, bend left, "\mathsf{y}_-"] \end{tikzcd} \qquad \begin{array}{l} \mathsf{x}_\pm^2 = \mathsf{y}_\mp \mathsf{y}_\pm \\ \mathsf{x}_\pm \mathsf{y}_\mp = \mathsf{y}_\mp \mathsf{x}_\mp . \end{array}$$

For any $n \geq 0$ the quiver $(Q,I)_{2n+2}^e$ has only *special* and *glued* vertices.

For any $n \in \mathbb{N}^+$, the subquiver $(Q,I)_n^e$ has *no neutral* vertices.

Remark 2.2.6. *(1) Let $\Lambda = \Lambda_n^\#$ be some Khoroshkin order for some $n \geq 2$. In this case, the subquiver of Λ^e is given by quivers of the form* (2.2.1)*. More precisely, it holds that*

$$(Q,I)_{2n+1}^{\#,e} = (Q,I)_{2n+1}^e \times (Q,I)_{2n+1}^e \quad \textit{and} \quad (Q,I)_{2n+2}^{\#,e} = (Q,I)_{4n+3}^e.$$

In particular, all statements of this subsection hold also for Khoroshkin orders $\Lambda_n^\#$.

(2) The quiver $(Q,I)_{2n+1}^e$ in (2.2.1) *has another Lie-theoretic interpretation. It has appeared in the study of* cuspidal representations of $\mathfrak{sl}(n+1)$ [**GS10**].

Remark 2.2.7. *Let the characteristic of the base field $\Bbbk$ be different from two. Then the ring Λ_2^e is Morita equivalent to a skew group ring of the nodal singularity*:

$$\Lambda_2^e\text{-mod} \xrightarrow{\sim} \mathbb{Z}_2 \# \mathbf{R}_{\mathsf{xy}}\text{-mod}, \quad \textit{where } \mathbf{R}_{\mathsf{xy}} = \Bbbk[\![x,y]\!]/(x\,y).$$

It is well-known that the category $\mathbf{R}_{\mathsf{xy}}$-fd. mod is tame [**GP68**]*. By results of* [**Kho81**] *or* [**RR85**] *it follows that Λ_2^e-fd. mod is also tame.*
Similar arguments show that the category Λ_n^e-fd. mod is tame for any $n \in \mathbb{N}^+$.

Lemma 2.2.8. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^\#$ for some $n \geq 2$. Its subring $\Lambda^e = e\Lambda e$ has the following properties*:

(1) The ring Λ^e is a Gorenstein order *of* infinite *global dimension.*

(2) The canonical bimodule of Λ^e is given by $\omega^e \cong \sigma(\Lambda^e)$.
In particular, the Auslander-Reiten translation $\tau : \mathrm{D}^-(\Lambda^e) \xrightarrow{\sim} \mathrm{D}^-(\Lambda^e)$ is given by the involution σ and satisfies $\tau^2 \cong \mathrm{Id}$.

(3) There is the following double centralizer property:

$$\Lambda \cong \mathrm{End}_{e\Lambda e}(\Lambda\, e), \quad \textit{where } \Lambda\, e \textit{ is viewed as a right } e\Lambda e\textit{-module.}$$

Proof. These statements follow from straightforward computations. □

Since the suborder Λ^e has *infinite* global dimension whereas the order Λ has global dimension n, the restricted Khoroshkin order Λ^e is “more singular” than Λ.

2.2.2.2 Further characterization of τ-periodic complexes

Next, we consider the following categories:

- As before, let $\mathcal{T}$ be the subcategory of τ-periodic complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

We recall that a full subcategory in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is *thick* if it is closed under cones, the shift functor and direct summands.

- Let $\langle S_\star\rangle$ be the smallest thick subcategory of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ containing $S_\star$.
- The left and right-perpendicular categories of $\langle S_\star\rangle$ are defined by:

$$\begin{aligned} {}^\perp\langle S_\star\rangle &= \{\ P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda) \mid \mathrm{Ext}^i_\Lambda(P_\bullet, Y_\bullet) = 0 \text{ for any } i \in \mathbb{Z} \text{ and any } Y_\bullet \in \langle S_\star\rangle\ \} \\ \langle S_\star\rangle^\perp &= \{\ P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda) \mid \mathrm{Ext}^i_\Lambda(X_\bullet, P_\bullet) = 0 \text{ for any } i \in \mathbb{Z} \text{ and any } X_\bullet \in \langle S_\star\rangle\ \} \end{aligned}$$

- Let $\mathrm{K}^{\mathrm{b}}(\mathrm{add}\,\Lambda\, e)$ denote the bounded homotopy category of projective complexes of Λ-modules without projectives of type $P_\star$,
- Let $\mathsf{Perf}(\Lambda^e) = \mathrm{K}^{\mathrm{b}}(\Lambda^e\text{ - proj})$ be the category of perfect complexes of Λ^e-modules.

Corollary 2.2.9. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$. In the notations above there are the following equalities and equivalences of categories:*

$$\mathcal{T} = {}^\perp\langle S_\star\rangle = \langle S_\star\rangle^\perp \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\mathrm{add}\,\Lambda\, e) \xrightarrow{\sim} \mathsf{Perf}(\Lambda^e) \qquad (2.2.2)$$

This statement restricts to subcategories of objects with finite-dimensional homology. In particular, the categories $\mathcal{T}$ and $\mathcal{T}_{\mathrm{fd}}$ are thick subcategories of $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

PROOF. By Lemma B.2.16 it holds that $\langle S_\star\rangle^\perp = S_\star^\perp$ and ${}^\perp\langle S_\star\rangle = {}^\perp S_\star$ are thick subcategories.

(1) The main point is that $\mathcal{T} = {}^\perp S_\star$ which is stated in Theorem 2.2.3 (3), the equivalence of conditions (a) and (b).

(2) Since $S_\star$ is n-Calabi-Yau, it follows that ${}^\perp S_\star = S_\star^\perp$.

(3) $\mathcal{T} \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\mathrm{add}\,\Lambda e)$:
Let $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$. We may assume that $P_\bullet$ is a minimal projective complex. Then $P_\bullet \in \mathrm{K}^{\mathrm{b}}(\mathrm{add}\,\Lambda e)$ if and only if $P_\bullet$ has no projective modules of type $P_\star$. By Lemma B.2.17 this is equivalent to $\delta(P_\bullet) = 0$.

(4) At last, let us note that there is an equivalence of additive categories:

$$\Lambda e \underset{e\Lambda e}{\otimes} \_ : e\Lambda e\text{ - proj} \xrightarrow{\sim} \mathrm{add}(\Lambda\, e)$$

This functor yields an equivalence of the homotopy categories:

$$\Lambda e \underset{e\Lambda e}{\otimes} \_ : \quad \mathsf{Perf}(\Lambda^e) = \mathrm{K}^{\mathrm{b}}(e\Lambda e\text{ - proj}) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\mathrm{add}\,\Lambda e)\,.$$

□

2.2.3 Projective and injective dimension in terms of defect numbers

In Lemmas 2.1.5 and 2.1.14 we have seen that the global dimension of any Khoroshkin order Λ is given by the projective dimension of the module $S_\star$. Next, we want to show that $S_\star$ determines even the projective and injective dimension of any Λ-module.

Lemma 2.2.10. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$ and let $M \in \Lambda\text{-mod}$. Then*

$$\operatorname{id} M = \max\Big\{\ 1,\ \max_{1\leq j\leq n}\{\ \operatorname{Ext}^j_\Lambda(S_\star, M) \neq 0\ \}\ \Big\} \tag{2.2.3}$$

$$= \max\Big\{\ 1,\ n - \min_{0\leq j<n}\{\ \operatorname{Ext}^j_\Lambda(M, S_\star) \neq 0\ \}\ \Big\}. \tag{2.2.4}$$

Proof. For simplicity of notation, let us assume that $\Lambda = \Lambda_n$ for some $n \geq 2$. Let $M \in \Lambda\text{-mod}$. By Proposition B.1.15 we have

$$\operatorname{id} M = \sup_{j\in\mathbb{N}_0}\{\ \operatorname{Ext}^j_\Lambda(\operatorname{top}\Lambda, M) \neq 0\ \} \geq \sup_{j\in\mathbb{N}_0}\{\ \operatorname{Ext}^j_\Lambda(S_\star, M) \neq 0\ \}. \tag{2.2.5}$$

- Assume that $\operatorname{id} M \geq 2$. We have to show the opposite inequality in (2.2.5). By assumption there is some simple Λ-module S and some $j \geq 2$ such that $\operatorname{Ext}^j_\Lambda(S, M) \cong \operatorname{Ext}^{j-1}(\operatorname{syz}(S), M) \neq 0$. We recall that by the statements (2.1.2) or (2.1.3) in the proof of Lemma 2.1.5 there is some $k \in \mathbb{N}^+$ respectively there are some $k, l \in \mathbb{N}^+$ such that

$$\operatorname{syz}(S) \cong \operatorname{syz}^k(S_\star) \quad \text{respectively} \quad \operatorname{syz}(S) \cong \operatorname{syz}^k(S_\star) \oplus \operatorname{syz}^l(S_\star). \tag{2.2.6}$$

  In the second, more general, case it follows that

$$\begin{aligned}\operatorname{Ext}^j_\Lambda(S, M) \cong \operatorname{Ext}^{j-1}_\Lambda(\operatorname{syz}(S), M) &\cong \operatorname{Ext}^{j-1}_\Lambda(\operatorname{syz}^k(S_\star) \oplus \operatorname{syz}^l(S_\star), M)\\ &\cong \operatorname{Ext}^{j+k-1}_\Lambda(S_\star, M) \oplus \operatorname{Ext}^{j+l-1}_\Lambda(S_\star, M)\end{aligned}$$

  Since $j+k-1 \geq j$ and $j+l-1 \geq j$, we obtain the inequality

$$\operatorname{id} M = \sup_{j\in\mathbb{N}_0}\{\ \operatorname{Ext}^j_\Lambda(\operatorname{top}\Lambda, M) \neq 0\ \} \leq \sup_{j\in\mathbb{N}_0}\{\ \operatorname{Ext}^j_\Lambda(S_\star, M) \neq 0\ \}.$$

  Since $\operatorname{gldim}\Lambda = n$ and $S_\star$ is n-Calabi-Yau, we obtain finally

$$\begin{aligned}\operatorname{id} M &= \max_{1\leq j\leq n}\{\ \operatorname{Ext}^j_\Lambda(S_\star, M) \neq 0\ \}\\ &= \max_{1\leq j\leq n}\{\ \operatorname{Ext}^{n-j}_\Lambda(M, S_\star) \neq 0\ \} = n - \min_{0\leq j<n}\{\ \operatorname{Ext}^j_\Lambda(M, S_\star) \neq 0\ \}.\end{aligned} \tag{2.2.7}$$

  This proves the equalities (2.2.3) and (2.2.4) in the case that $\operatorname{id} M \geq 2$.

- if $\operatorname{id} M \leq 1$, we have $\operatorname{id} M = 1$ by Remark B.1.16. In this case, the equality (2.2.3) holds automatically, and (2.2.4) follows by (2.2.7).

The proof above can be adapted directly for Khoroshkin orders of type $\Lambda_n^{\#}$, where $n \geq 2$, with slightly different notation at (2.2.6). □

Example 2.2.11. *Let $\Lambda = \Lambda_n$ for some $n \geq 2$. Let P be an indecomposable projective Λ-module. Then* $\operatorname{id} P = \begin{cases} n & \text{if } P \cong P_\star \\ 1 & \text{if } P \ncong P_\star \end{cases}$

To show an analogous formula for the projective dimension, we need to introduce the *twisted Matlis duality.* Note that any Khoroshkin quiver (Q, I) can be identified with its opposite quiver $(Q, I)^{op}$ by switching the arrows in two-cycles. In other words, there is a ring isomorphism $\psi : \Lambda \xrightarrow{\sim} \Lambda^{op}$ such that $\psi(e) = e$ for any idempotent $e \in \Lambda$.

Remark 2.2.12. *More precisely, the isomorphism $\psi : \Lambda \xrightarrow{\sim} \Lambda^{op}$ is given as follows.*

Λ_{2n+1}		Λ_{2n+2}		$\Lambda^{\#}_{2n+2}$		$n \in \mathbb{N}_0$
α	$\psi(\alpha)$	α	$\psi(\alpha)$	α	$\psi(\alpha)$	
$\mathsf{a_i}$	$\mathsf{b_i}$	$\mathsf{a_i}$	$\mathsf{b_i}$	$\mathsf{a_i^{\pm}}$	$\mathsf{b_i^{\mp}}$	$0 \leq \mathsf{i} < \mathsf{n}$
$\mathsf{b_i}$	$\mathsf{a_i}$	$\mathsf{b_i}$	$\mathsf{a_i}$	$\mathsf{b_i^{\pm}}$	$\mathsf{a_i^{\mp}}$	
$\mathsf{a_n}$	$\mathsf{a_n}$	$\mathsf{a_{\pm}}$	$\mathsf{b_{\pm}}$	$\mathsf{a_n^{\pm}}$	$\mathsf{b_n^{\mp}}$	
		$\mathsf{b_{\pm}}$	$\mathsf{a_{\pm}}$	$\mathsf{b_n^{\pm}}$	$\mathsf{a_n^{\mp}}$	

The case that $\Lambda = \Lambda^{\#}_{2n+1}$ for some $n \in \mathbb{N}_0$ is similar, since $\Lambda^{\#}_{2n+1} = \Lambda^{\times 2}_{2n+1}$.

Definition 2.2.13. *Let $\psi : \Lambda \xrightarrow{\sim} \Lambda^{op}$ be the ring isomorphism which fixes any idempotent. Let $\psi^* : \Lambda_n^{op}\text{-fd. mod} \xrightarrow{\sim} \Lambda_n\text{-fd. mod}$ denote the equivalence of categories induced by the isomorphism ψ^*. The* twisted Matlis duality $\widetilde{\mathbb{D}}$ *is defined as the composition*

$$\widetilde{\mathbb{D}} := \psi^* \circ \mathbb{D} : \quad \Lambda_n\text{-fd. mod} \xrightarrow{\sim} \Lambda_n^{op}\text{-fd. mod} \xrightarrow{\sim} \Lambda_n\text{-fd. mod},$$

where $\mathbb{D} = \mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$ is the standard duality.

Remark 2.2.14. *(1) Let us note that the standard duality $\mathbb{D}$ is related to the Matlis duality of orders by Proposition B.1.17.*

(2) The twisted Matlis duality $\widetilde{\mathbb{D}}$ is a contravariant endofunctor such that

$$\widetilde{\mathbb{D}}^2 \cong \mathrm{Id} \qquad \textit{and} \qquad \widetilde{\mathbb{D}}(S) \cong S \quad \textit{for any simple } \Lambda_n\textit{-module } S. \tag{2.2.8}$$

Lemma 2.2.15. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$. For any finite-dimensional Λ-module M it holds that*

$$\mathrm{pd}\, M = \max\left\{\ 1,\ \max_{1\leq j\leq n}\{\ \mathrm{Ext}^j_\Lambda(M, S_\star) \neq 0\ \}\ \right\}.$$

PROOF. By (2.2.8) it follows that

$$\mathrm{pd}(M) = \mathrm{id}(\widetilde{\mathbb{D}}(M)) = \max\{\ 1, \max_{1\leq j\leq n}\{\ \mathrm{Ext}^j_\Lambda(S_\star, \widetilde{\mathbb{D}}(M)) \neq 0\ \}\ \}.$$

and also that $\mathrm{Ext}^j_\Lambda(S_\star, \widetilde{\mathbb{D}}(M)) \cong \mathrm{Ext}^j_\Lambda(M, S_\star)$ for any $j \in \mathbb{Z}$. ☐

Example 2.2.16. *For any simple Λ-module S it holds that* $\mathrm{pd}\, S = \mathrm{id}\, \widetilde{\mathbb{D}}(S) = \mathrm{id}(S)$.

Let us recall that the defect numbers and the defect of $M \in \Lambda\text{-mod}$ were given by

$$\delta^{(j)}(M) = \dim \mathrm{Ext}^j_\Lambda(M, S_\star) \quad \text{and} \quad \delta(M) = \sum_{1\leq j\leq n} \delta^{(j)}(M).$$

We may summarize Lemmas 2.2.15 and 2.2.10 as follows:

Proposition 2.2.17. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ be a Khoroshkin order for some $n \geq 2$. For any finite-dimensional Λ-module M the following statements hold:*

- *if $\delta(M) = 0$, then* $\mathrm{pd}\, M = \mathrm{id}\, M = 1$.
- *if $\delta(M) \neq 0$, then*

$$\mathrm{pd}\, M = \max_{0<j\leq n}\{\ \delta^{(j)}(M) \neq 0\ \} \quad \textit{and} \quad \mathrm{id}\, M = n - \min_{0\leq j<n}\{\ \delta^{(j)}(M) \neq 0\ \}$$

At last, let us characterize modules of maximal projective or injective dimension:

Corollary 2.2.18. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$ and let M be a finite-dimensional Λ-module.*

(1) $\operatorname{id} M = n$ *if and only if* $\delta^{(0)}(M) \neq 0$.

(2) $\operatorname{pd} M = n$ *if and only if* $\delta^{(n)}(M) \neq 0$.

Moreover, the invariants above have the following interpretations:

$$\delta^{(0)}(M) := \dim \operatorname{Hom}_\Lambda(M, S_\star) = \dim \operatorname{Ext}^n_\Lambda(S_\star, M) = [\operatorname{top} M : S_\star]$$
$$\delta^{(n)}(M) := \dim \operatorname{Ext}^n_\Lambda(M, S_\star) = \dim \operatorname{Hom}_\Lambda(S_\star, M) = [\operatorname{soc} M : S_\star]$$

Remark 2.2.19. *Lemma 2.2.15, Proposition 2.2.17 and the Corollary 2.2.18 can be extended to all non-projective* finitely generated *Λ-modules using the classification of Cohen-Macaulay Λ-modules stated in Propositions 2.1.11 and 2.1.19.*

2.2.4 Four classes of indecomposable objects

As before, let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ be the Khoroshkin order for some $n \geq 2$. In this subsection we will divide the indecomposable objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ into four classes.

2.2.4.1 Lie-theoretic definition and examples

Definition 2.2.20. *Let $P_\bullet$ be any* indecomposable *object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Then*

$$P_\bullet \textit{ is a } \begin{cases} \text{usual string} & \textit{if and only if} \quad \delta(P_\bullet) > 0 \quad \textit{and} \quad \sigma(P_\bullet) \cong P_\bullet, \\ \text{special string} & \textit{if and only if} \quad \delta(P_\bullet) > 0 \quad \textit{and} \quad \sigma(P_\bullet) \ncong P_\bullet, \\ \text{bispecial string} & \textit{if and only if} \quad \delta(P_\bullet) = 0 \quad \textit{and} \quad \sigma(P_\bullet) \ncong P_\bullet, \\ \text{band} & \textit{if and only if} \quad \delta(P_\bullet) = 0 \quad \textit{and} \quad \sigma(P_\bullet) \cong P_\bullet. \end{cases}$$

Obviously, any indecomposable object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ belongs to one of the four disjoint classes. The same division applies to indecomposable finite-dimensional Λ-modules.

Let us note that the above terminology was originally motivated by a combinatorial characterization of the four classes which will be given in Theorem 3.6.2 in the next chapter.

Remark 2.2.21. *Let $\Lambda = \Lambda_{2n+1}$ or $\Lambda_n^{\#}$ for some $n \in \mathbb{N}^+$. In this case, $\sigma = \operatorname{id}$ and the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ has only* two *classes of indecomposable objects:* usual strings *and* bands.

Let us describe some simple examples of strings and bands:

Example 2.2.22 (String and band modules of the Gelfand quiver). *Let $\Lambda = \Lambda_2$ be the Gelfand order and $(Q, I) = (Q, I)_2$ be the corresponding quiver*:

$$\Lambda_2 = \begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \qquad (Q,I)_2 = \begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$$

The following are some of the simplest examples of string and band modules:

V	*quiver representation*	*projective resolution*	$\delta(V)$	$\sigma(V)$	*class*
$S_\star$	$\begin{tikzcd} 0 \arrow[r, bend left] & \Bbbk \arrow[l, bend left] \arrow[r, bend right] & 0 \arrow[l, bend right] \end{tikzcd}$	$P_\star \xrightarrow{\left[\begin{smallmatrix} \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{smallmatrix}\right]} \bigoplus_{i \in \{+,-\}} P_i \xrightarrow{[\,\cdot\mathsf{a}_+ \ \cdot\mathsf{a}_-\,]} P_\star$	2	$S_\star$	*usual string*
S_+	$\begin{tikzcd} \Bbbk \arrow[r, bend left] & 0 \arrow[l, bend left] \arrow[r, bend right] & 0 \arrow[l, bend right] \end{tikzcd}$	$P_\star \xrightarrow{\cdot\mathsf{b}_+} P_+$	1	S_-	*special string*
S_-	$\begin{tikzcd} 0 \arrow[r, bend left] & 0 \arrow[l, bend left] \arrow[r, bend right] & \Bbbk \arrow[l, bend right] \end{tikzcd}$	$P_\star \xrightarrow{\cdot\mathsf{b}_-} P_-$	1	S_+	*special string*
B_{+-}	$\begin{tikzcd} 0 \arrow[r, bend left] & \Bbbk \arrow[l, bend left] \arrow[r, bend right, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"'] & \Bbbk^2 \arrow[l, bend right, "{[1\,0]}"'] \end{tikzcd}$	$P_+ \xrightarrow{\cdot\mathsf{a}_+\mathsf{b}_-} P_-$	0	B_{-+}	*bispecial string*
B_{-+}	$\begin{tikzcd} \Bbbk^2 \arrow[r, bend left, "{[1\,0]}"] & \Bbbk \arrow[l, bend left, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"] \arrow[r, bend right] & 0 \arrow[l, bend right] \end{tikzcd}$	$P_- \xrightarrow{\cdot\mathsf{a}_-\mathsf{b}_+} P_+$	0	B_{+-}	*bispecial string*
V_λ	$\begin{tikzcd} \Bbbk^2 \arrow[r, bend left, "{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]}"] & \Bbbk^2 \arrow[l, bend left, "{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]}"] \arrow[r, bend right, "{\left[\begin{smallmatrix} 1 & \lambda \\ 0 & 0 \end{smallmatrix}\right]}"'] & \Bbbk^2 \arrow[l, bend right, "{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]}"'] \end{tikzcd}$	$P_- \oplus P_+ \xrightarrow{\left[\begin{smallmatrix} \cdot\lambda\,\mathsf{a}_-\mathsf{b}_+ & \cdot\mathsf{a}_+\mathsf{b}_+ \\ \cdot\mathsf{a}_-\mathsf{b}_- & \cdot\mathsf{a}_+\mathsf{b}_- \end{smallmatrix}\right]} P_+ \oplus P_-$	0	V_λ	*band*
where $\lambda \in \Bbbk$ such that $\lambda \neq 0$ or 1					

We refer to Tables C.2.2 and C.2.3 for further examples of string representations of the Gelfand quiver.

At last let us consider the simple modules of some Khoroshkin orders:

Example 2.2.23. *Let $\Lambda = \Lambda_{2n+1}$ or Λ_{2n+2} for some $n \in \mathbb{N}_0$.*

- *the simple Λ-modules $S_\star, S_1, \dots S_n$ are usual strings of defect 2.*
- *if $\Lambda = \Lambda_{2n+2}$, the simple modules S_+ and S_- are special strings of defect 1.*

2.2.4.2 Functorial characterization and basic properties

Theorem 2.2.3 provides the following "functorial" characterization of the four classes:

Corollary 2.2.24. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^{\#}$ for some $n \geq 2$. Let $P_\bullet$ be an indecomposable object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Then*

$$P_\bullet \textit{ is a } \begin{cases} \text{usual string} & \Leftrightarrow \quad \tau^2(P_\bullet) \ncong P_\bullet \textit{ and } \sigma(P_\bullet) \cong P_\bullet \\ \text{special string} & \Leftrightarrow \quad \tau^2(P_\bullet) \ncong P_\bullet \textit{ and } \sigma(P_\bullet) \ncong P_\bullet \\ \text{bispecial string} & \Leftrightarrow \quad \tau^2(P_\bullet) \cong P_\bullet \textit{ and } \tau(P_\bullet) \cong \sigma(P_\bullet) \ncong P_\bullet \\ \text{band} & \Leftrightarrow \quad \tau(P_\bullet) \cong \sigma(P_\bullet) \cong P_\bullet \end{cases}$$

The proof of the following Lemma is straightforward.

Lemma 2.2.25. *Let $\mathcal{C}$ be the additive closure of one of the four classes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Let $\mathbb{F}$ be any auto-equivalence of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ such that*

$$\mathbb{F}(S_\star) \cong S_\star[k] \textit{ for some } k \in \mathbb{Z}, \qquad \textit{and} \qquad \mathbb{F} \circ \sigma \cong \sigma \circ \mathbb{F} \textit{ on } \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda).$$

Then $\mathbb{F}$ preserves the subcategory $\mathcal{C}$.
In particular, the additive closure of one of the four classes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ is preserved by the auto-equivalences τ, $[1]$, $\mathbb{S}$, σ or $\widetilde{\mathbb{D}}$.

Remark 2.2.26. *Let us recall that there are the following two thick subcategories in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$:*

- *the full subcategory $\mathcal{T}_{\mathrm{fd}}$ of τ-periodic complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$,*
- *the subcategory $\langle S_\star \rangle$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ generated by the spherical object $S_\star$.*

The Auslander-Reiten quiver *of these categories has the following* components:

(1) We note that $\mathcal{T}_{\mathrm{fd}}$ is exactly the additive closure of bispecial strings *and* bands. *In particular, the Auslander-Reiten quiver of $\mathcal{T}_{\mathrm{fd}}$ is given by* homogeneous tubes of rank one and two. *By Remark 2.2.7 the category $\mathcal{T}_{\mathrm{fd}}$ is representation-tame.*

(2) Assume that $\Lambda = \Lambda_n$ for some $n \geq 2$. In this case, Theorem B.3.4 by Keller, Yang and Zhou **[KYZ09]** *and Theorem B.3.5 by Jorgensen* **[Jor04]** *imply that*

$$\textit{the AR-quiver of } \langle S_\star \rangle \textit{ is given by } n-1 \textit{ components of type } \mathbb{Z}\mathbb{A}_\infty.$$

In particular, the category $\langle S_\star \rangle$ has discrete representation type.

For $\Lambda = \Lambda_2$ we give an explicit description of the category $\langle S_\star \rangle$ in Subsection 4.3.5.

Remark 2.2.27. *Let $\mathcal{C}$ be the additive closure of one, two or three of the four classes - except "bispecial strings and bands" or just "bands". Then $\mathcal{C}$ is* not *a thick subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.*

CHAPTER 3

Reduction techniques for nodal orders

In [**Dro91**] Drozd characterized the orders Λ such that the category of finite length modules over Λ has *tame* representation type. Such orders are the *nodal* orders. We will not define nodal orders in this introduction but note some of their properties.

First of all, any Khoroshkin order of the previous chapter gives a Lie-theoretic example of a nodal order. Second, nodal orders can be considered as non-commutative generalizations of the nodal singularity $\Bbbk[\![x,y]\!]/(x\,y)$. At last, nodal orders can be viewed as infinite-dimensional versions of *gentle* or *skew-gentle* algebras in the representation theory of quivers.

The goal of the present chapter is to study the classification problem of indecomposable Λ-modules for any nodal order Λ.

In [**BD04**] Burban and Drozd have shown that any nodal order Λ is even *derived-tame*. For simplicity of notation, let us assume that Λ is a nodal order of *finite* global dimension. The order Λ embeds naturally into some ring Γ, which yields a *radical embedding*, that is,

$$\Lambda \hookrightarrow \Gamma \qquad \text{and} \qquad \operatorname{rad}\Lambda = \operatorname{rad}\Gamma \tag{$*$}$$

Moreover, the order Γ is a hereditary order. In particular, the derived category $\mathrm{D^b}(\Gamma)$ has only *countably* many indecomposable objects. The work of [**BD04**] provides a technique to *glue* any indecomposable projective complex in $\mathrm{D^b}(\Lambda)$ from several complexes of the well-understood category $\mathrm{D^b}(\Gamma)$.

To state the main results of this chapter, we need to recall the approach of [**BD04**] in more detail. Their approach relies on the following diagram of categories and functors:

$$\mathrm{D^b}(\Lambda) \xrightarrow{\mathbf{F}} \mathrm{Tri}(\Lambda) \xrightarrow{\mathbf{H}} \mathrm{Rep}(\mathfrak{B})$$

In particular, there is a notion of the category of triples $\mathrm{Tri}(\Lambda)$ for derived categories. Any triple of Λ can be considered as "gluing data" for a complex in the category $\mathrm{D^b}(\Lambda)$. The functor $\mathbf{F}$ in the above diagram is dense and preserves indecomposability and isomorphism classes of objects.

Furthermore, there is a bimodule category $\mathrm{Rep}(\mathfrak{B})$ associated to the category of triples $\mathrm{Tri}(\Lambda)$. An object of the category $\mathrm{Rep}(\mathfrak{B})$ is given by a family of certain matrices.

The functor $\mathbf{H}$ is full and dense. Its essential image is given by the category $\mathrm{Rep}^*(\mathfrak{B})$ of so-called *regular* representations in the category $\mathrm{Rep}(\mathfrak{B})$. The kernel of $\mathbf{H}$ is given by the derived category $\mathrm{D^b}(\Gamma_\star)$ for some explicit hereditary subalgebra $\Gamma_\star$ of Γ.

In other words, the representation-theoretic difference between the category of triples $\mathrm{Tri}(\Lambda)$ and the bimodule category $\mathrm{Rep}(\mathfrak{B})$ is insignificant.

Summarized, the above functors yield bijections between isomorphism classes of indecomposable objects in the following categories:

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}}^{1:1} \mathrm{ind}[\mathrm{Tri}(\Lambda)] \underset{\mathbf{I}}{\overset{\mathbf{H}}{\rightleftarrows}}^{1:1} \mathrm{ind}[\mathrm{Rep}^{*}(\mathfrak{B})] \cup \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Gamma_{\star})]$$

Via this path of bijections, the classification problem of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is reduced to a *matrix problem* over some bunch of semichains $\mathfrak{B}$.

At this point, Bondarenko's combinatorics (Appendix A) can be used to construct the canonical forms of the category $\mathrm{Rep}^{*}(\mathfrak{B})$. These canonical forms can be translated back via the maps **I** and the gluing map **G** to an indecomposable projective complex of the nodal order Λ.

In this chapter we study the classification problem of the indecomposable modules of any nodal order Λ by two approaches.

In the first approach, we aim at a description of the *projective presentations* of indecomposable Λ-modules.

In Theorem 3.5.18 we define a "truncated" bunch of semichains $\mathfrak{B}_0$ and show that there are bijections of isomorphism classes of indecomposable objects in the following categories:

$$\mathrm{ind}[\Lambda\text{ - fd. mod}] \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}}^{1:1} \mathrm{ind}[\mathrm{Tri}_0(\Lambda)] \underset{\mathbf{I}}{\overset{\mathbf{H}}{\rightleftarrows}}^{1:1} \mathrm{ind}[\mathrm{Rep}^{\circ}(\mathfrak{B}_0)] \cup \mathrm{ind}[\Gamma_{\star}\text{ - fd. mod}]$$

Here the category $\mathrm{Tri}_0(\Lambda)$ can be viewed as a "truncated" version of the whole category $\mathrm{Tri}(\Lambda)$ of triples. The indecomposable triples in this category correspond exactly to minimal indecomposable projective presentations. The category $\mathrm{Tri}_0(\Lambda)$ can be defined in the setup of any radical embedding $(*)$. There is also a method to pass from a projective presentation obtained in this way, to a projective resolution. This approach relies on a simple observation and is summarized in Proposition 3.3.1 and Remark 3.3.6

On the other hand, in Proposition 3.3.11 we obtain a direct characterization of triples in $\mathrm{Tri}(\Lambda)$ which correspond to projective resolutions in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

Both methods are applied in Chapter 5 to describe the indecomposable objects in the abelian category of the Gelfand quiver.

In this chapter we also introduce the notions of defect and involution in the setup of any nodal order Λ. We give an intrinsic characterization of the four classes of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ in Theorem 3.6.2.

In the first part of this chapter we deal with the category of triples $\mathrm{Tri}(\Lambda)$. From Section 3.4 until the end of this chapter we focus on the category of matrix representations $\mathrm{Rep}^{*}(\mathfrak{B})$.

3.1 Introduction to nodal orders

In this subsection, we introduce nodal orders and discuss their representation type. However let us remark that the methods presented in the later Sections can be applied to a more general class of rings than the nodal orders.

Throughout this section, let $\mathbf{R} = \Bbbk[\![t]\!]$ be the ring of formal power series. We refer to Definition B.1.12 for the notion of orders.

Definition 3.1.1. *An* $\mathbf{R}$*-order* Λ *is* nodal *if and only if the following conditions hold*:

(1) the order $\Gamma := \mathrm{End}_\Lambda(\mathrm{rad}\,\Lambda)$ *is hereditary,*

(2) the natural embedding $\Lambda \subset \Gamma$ *is a radical embedding, that is,* $\mathrm{rad}\,\Lambda = \mathrm{rad}\,\Gamma$,

(3) for any simple left Λ*-module* S *the* Λ*-module* ${}_\Lambda\Gamma \otimes_\Lambda S$ *has at most length two.*

We will call the overorder Γ *the* normalization *of* Λ.

Example 3.1.2. *(1) The nodal singularity* $\Lambda = \Bbbk[\![x, y]\!]/(x\,y)$ *is the only* commutative *nodal order. In this case,* $\Gamma = \Bbbk[\![x]\!] \times \Bbbk[\![y]\!]$ *and* Γ *is the normalization of* Λ *in the sense of commutative algebra.*

(2) Any Khoroshkin order Λ_n *or* $\Lambda_n^\#$ *from Subsection 2.1.2 or in 2.1.3 is a nodal order with finite global dimension. The restricted Khoroshkin orders defined in Subsection 2.2.2.1 gives examples of nodal orders with infinite global dimension.*

Remark 3.1.3. *The following two properties of nodal orders have been shown in* [**DZ13**, Proposition 1.3, Corollary 1.10]:

(1) Any nodal order is Morita equivalent to some basic *nodal order.*

(2) The third condition in the definition of nodal order is equivalent to the same condition with right simple modules.

The class of nodal orders is distinguished by the following result:

Theorem 3.1.4 ([**Dro90**])**.** *Let* $\Bbbk$ *be an algebraically closed field. Let* Λ *be an order over* $\Bbbk[\![x_1, \ldots, x_d]\!]$ *for some* $d \in \mathbb{N}^+$.
Then the category $\Lambda\text{-}\mathrm{fl}\,\mathrm{mod}$ *of finite length modules over* Λ *has* tame *representation type if and only if* $d = 1$ *and* Λ *is a* nodal order*.*

In other words, for "most" orders the classification problem of finite length modules is *wild*, which is considered hopelessly difficult. Nodal orders are the only orders for which this classification problem seems feasible.

For simplicity of notation we will consider only nodal orders of *finite* global dimension in the following.

Surprisingly, nodal orders are even derived-tame:

Theorem 3.1.5. [**BD04**, Theorem 5.2] *For any nodal order* Λ *the description of the indecomposable objects in the derived category* $\mathrm{D}^\mathrm{b}(\Lambda)$ *can be reduced to a matrix problem of some* bunch of semichains*. In other words, the category* $\mathrm{D}^\mathrm{b}(\Lambda)$ *is "tame in a pragmatic sense".*

The reduction method of this theorem is the topic of the present section.

Remark 3.1.6. *For any nodal order Λ the category $\mathrm{CM}(\Lambda)$ of Cohen-Macaulay modules has finite representation type. This follows from* [**RR79**] *or* [**GR78**].

Definition 3.1.7. *Let Λ be a nodal order and S a simple Λ-module.*

(1) S is neutral *if and only if ${}_\Lambda\Gamma \otimes_\Lambda S \cong S$.*

(2) S is glued *if and only if ${}_\Lambda\Gamma \otimes_\Lambda S \cong S \oplus S$.*

(3) A pair of simple Λ-modules S_+ and S_- is special *if and only if ${}_\Lambda\Gamma \otimes_\Lambda S_+ \cong {}_\Lambda\Gamma \otimes_\Lambda S_- \cong S_+ \oplus S_-$.*

A primitive idempotent $e_i \in \Lambda$ will be called neutral *or* glued *if $S = \operatorname{top} \Lambda e_i$ is a neutral or glued simple module. Similarly, a pair of primitive idempotents is* special *if the two corresponding simple modules form a special pair.*

Let us note that we have already used the above notions in the context of Khoroshkin orders in Chapter 2.

For any nodal order Λ, the classification of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ will be formulated in terms of the *defect* and the *involution*:

Definition 3.1.8. *Let Λ be a nodal order.*

(1) Let $S_\star$ be the direct sum of all neutral simple Λ-modules (where each neutral simple module occurs with multiplicity one). For $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ and any $i \in \mathbb{Z}$ we define the defect numbers *and the* defect *of $P_\bullet$ by*

$$\delta^{(i)}(P_\bullet) = \dim_{\Bbbk} \operatorname{Ext}^i(P_\bullet, S_\star) \quad \text{and} \quad \delta(P_\bullet) = \sum_{i \in \mathbb{Z}} \delta^{(i)}(P_\bullet)$$

(2) Let ς be the automorphism of the ring Λ which interchanges the two idempotents in any special pair of primitive idempotents of Λ. Let σ be the involution functor $\sigma\colon \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$ induced by ring involution ς.

Remark 3.1.9. *Let Λ be a nodal order.*

(1) It is possible that $S_\star = 0$, that is, there are no neutral simple modules. In this case the defect is always vanishing. Examples of this kind are given by the restricted Khoroshkin orders of Subsection 2.2.2.1.

(2) Is also possible that the involution σ is trivial, that is, there are no pairs of special simple modules. This is for example the case for any Khoroshkin order Λ_{2n-1} or $\Lambda_n^{\#}$, where $n \geq 1$ (see Subsections 2.1.2 and 2.1.3).

For later use let us note the following two Lemmas:

Lemma 3.1.10. *Let Λ be a nodal order and Γ its normalization. Let e be the sum of all neutral idempotents of Λ. Then the conductor ideal is given by*

$$I := \operatorname{ann}_\Lambda(\Gamma/\Lambda) = \operatorname{rad} \Lambda + \Lambda\, e\, \Lambda = \operatorname{rad} \Gamma + \Gamma\, e\, \Gamma.$$

In particular, a simple Λ-module S is neutral if and only if $\Lambda/I \otimes_\Lambda S = 0$.

Lemma 3.1.11. *Let Λ be a basic nodal order, Γ its normalization, I the conductor ideal and σ the involution. For any Λ/I-module V it holds that*

$$\underline{\dim}_{\,\Lambda/I}(\Gamma/I \otimes_{\Lambda/I} V) = \underline{\dim}\, V + \underline{\dim}\, \sigma(V) \quad \text{and} \quad \dim \Gamma/I \otimes_{\Lambda/I} V = 2 \dim V.$$

Proof. It is sufficient to prove the claim for simple Λ/I-modules S. Note that S cannot be neutral by Lemma 3.1.10. In particular, one of the following cases occurs:

- if S is glued, then ${}_{\Lambda/I}\Gamma/I \otimes_{\Lambda/I} S \cong S^{\oplus 2}$. Since $S \cong \sigma(S)$ the claim follows.
- if S is a special module, then ${}_{\Lambda/I}\Gamma/I \otimes_{\Lambda/I} S \cong S \oplus \sigma(S)$. □

3.2 Category of triples for the derived category

Throughout this section let $\Lambda \subset \Gamma$ be a *radical embedding*:

$$\Lambda \text{ and } \Gamma \text{ be semiperfect } \Bbbk\text{-algebras such that } \quad \Lambda \subset \Gamma \quad \text{and} \quad \operatorname{rad}\Lambda = \operatorname{rad}\Gamma \qquad (3.2.1)$$

In particular, Λ could be any nodal order but also the finite-dimensional algebra of any gentle or skew-gentle quiver. As usual, we assume that Λ and Γ have finite global dimension for simplicity of the notation.

In this section we define a category $\operatorname{Tri}(\Lambda)$ which is representation equivalent to the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ following [**BD04**].

3.2.1 Definition of the category of triples and main Theorem

For two algebras Λ and Γ as in (3.2.1) we consider the *conductor ideal*

$$I := \operatorname{ann}_{\Lambda}(\Gamma/\Lambda) = \{\ a \in \Lambda \mid a\Gamma \subseteq \Lambda\ \}.$$

It can be shown that the ideal I is the biggest common two-sided ideal of Λ and Γ. Since $I \supseteq \operatorname{rad}\Lambda = \operatorname{rad}\Gamma$, the $\Bbbk$-algebras Λ/I and Γ/I are both semi-simple.

The data Λ, Γ and I gives rise to a commutative diagram called the *conductor square*:

$$\begin{tikzcd}
\Lambda \arrow[r, hook] \arrow[d, two heads] & \Gamma \arrow[d, two heads, dotted] \\
\Lambda/I \arrow[r, hook, dotted] & \Gamma/I
\end{tikzcd}$$

In this diagram, the ring Λ is realized as the pullback of the dotted arrows.

The conductor square induces the following diagram of bounded derived categories and left-derived functors:

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\Lambda) \arrow[r, "\Gamma \otimes_{\Lambda} -"] \arrow[d, "\Lambda/I \otimes_{\Lambda} -"'] & \mathrm{D}^{\mathrm{b}}(\Gamma) \arrow[d, "\Gamma/I \otimes_{\Gamma} -"] \\
\mathrm{D}^{\mathrm{b}}(\Lambda/I) \arrow[r, "\Gamma/I \otimes_{\Lambda/I} -"'] & \mathrm{D}^{\mathrm{b}}(\Gamma/I)
\end{tikzcd}
\qquad
\begin{tikzcd}
P_\bullet \arrow[rrr, mapsto] \arrow[d, mapsto] & & & \Gamma \otimes_{\Lambda} P_\bullet \arrow[d, mapsto] \\
\Lambda/I \otimes_{\Lambda} P_\bullet \arrow[r, mapsto] & \Gamma/I \otimes_{\Lambda/I} \Lambda/I \otimes_{\Lambda} P_\bullet \arrow[rr, dashed, "\sim", "\widetilde{\mu}_{P_\bullet}"'] & & \Gamma/I \otimes_{\Gamma} \Gamma \otimes_{\Lambda} P_\bullet
\end{tikzcd}$$

We note that for any complex $P_\bullet$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ there is a natural isomorphism $\widetilde{\mu}_{P_\bullet}$ in $\mathrm{D}^{\mathrm{b}}(\Gamma/I)$. Roughly speaking, the *main idea* of the category of triples is that the complex $P_\bullet$ can be recovered from the morphism $\widetilde{\mu}_{P_\bullet}$ in $\mathrm{D}^{\mathrm{b}}(\Gamma/I)$. This statement will be made precise below.

Definition 3.2.1. *Let* $\Lambda \subset \Gamma$ *be a radical embedding as in* (3.2.1) *and* I *their conductor ideal. The* category of triples $\mathrm{Tri}(\Lambda) = \mathrm{Tri}(\Lambda, \Gamma, I)$ *is defined as follows*:

(1) An object of $\mathrm{Tri}(\Lambda)$ *is given by a triple* $(V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ *where*

(a) $V_\bullet$ *is a complex from* $\mathrm{D}^{\mathrm{b}}(\Lambda/I)$*,*

(b) $\widetilde{P}_\bullet$ *is a complex from* $\mathrm{D}^{\mathrm{b}}(\Gamma)$*, and*

(c) $\widetilde{\vartheta}\colon \Gamma/I \underset{\Lambda/I}{\otimes} V_\bullet \xrightarrow{\sim} \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet$ *is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\Gamma/I)$*.*

(2) A morphism $(V'_\bullet, \widetilde{P}'_\bullet, \widetilde{\vartheta}') \longrightarrow (V''_\bullet, \widetilde{P}''_\bullet, \widetilde{\vartheta}'')$ *between objects in* $\mathrm{Tri}(\Lambda)$ *is given by a tuple* (ϕ, ψ) *where*

(a) $\phi\colon V'_\bullet \longrightarrow V'_\bullet$ *is a morphism in* $\mathrm{D}^{\mathrm{b}}(\Lambda/I)$*, and*

(b) $\psi\colon \widetilde{P}'_\bullet \longrightarrow \widetilde{P}''_\bullet$ *is a morphism in* $\mathrm{D}^{\mathrm{b}}(\Gamma)$*,*

such that the following diagram is commutative:

$$\begin{tikzcd} \Gamma/I \underset{\Lambda/I}{\otimes} V'_\bullet \arrow[r, "\widetilde{\vartheta}'", "\sim"'] \arrow[d, "\Gamma/I \underset{\Lambda/I}{\otimes} \phi"'] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}'_\bullet \arrow[d, "\Gamma/I \underset{\Gamma}{\otimes} \psi"] \\ \Gamma/I \underset{\Lambda/I}{\otimes} V''_\bullet \arrow[r, "\widetilde{\vartheta}''", "\sim"'] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}''_\bullet \end{tikzcd}$$

There is a natural notion of the composition of morphisms and the direct sum of triples in $\mathrm{Tri}(\Lambda)$*.*

Remark 3.2.2. *The category of triples is a full subcategory of the* comma-category *of the additive functors* $\Gamma/I \otimes_{\Lambda/I} \_$ *and* $\Gamma/I \otimes_\Gamma \_$ *in the sense of* [**ML98**, Chapter II.6]*. In particular,* $\mathrm{Tri}(\Lambda)$ *is an additive* $\Bbbk$*-linear category.*

Definition 3.2.3. *We define a functor* $\mathbf{F}\colon \mathrm{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathrm{Tri}(\Lambda)$ *as follows.*

(1) For any complex $P_\bullet$ *in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *we set* $\mathbf{F}(P_\bullet) = (\Lambda/I \otimes_\Lambda P_\bullet, \Gamma \otimes_\Lambda P_\bullet, \widetilde{\mu}_{P_\bullet})$ *where* $\widetilde{\mu}_{P_\bullet}\colon \Gamma/I \otimes_{\Lambda/I} \Lambda/I \otimes_\Lambda P_\bullet \xrightarrow{\sim} \Gamma/I \otimes_\Gamma \Gamma \otimes_\Lambda P_\bullet$ *is the natural isomorphism.*

(2) For a morphism $\varrho\colon P'_\bullet \longrightarrow P''_\bullet$ *in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *we set* $\mathbf{F}(\varrho) = (\Lambda/I \otimes_\Lambda \varrho,\ \Gamma \otimes_\Lambda \varrho)$*.*

The definition of the category of triples is motivated by the following Theorem:

Theorem 3.2.4. [**BD04**, Theorem 2.4]

The functor $\mathbf{F}\colon \mathrm{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathrm{Tri}(\Lambda)$ *is dense, full and respects isomorphism classes of objects.*

Remark 3.2.5. *The functor* $\mathbf{F}$ *is not faithful. More precisely,* $\ker \mathbf{F} = \ker(\Gamma \otimes_\Lambda \_)$*.*

Corollary 3.2.6. *The functor* $\mathbf{F}$ *induces a bijection between isomorphism classes of indecomposable complexes in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *and isomorphism classes of indecomposable triples in* $\mathrm{Tri}(\Lambda)$:

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Tri}(\Lambda)] \tag{3.2.2}$$

In particular, $\mathbf{F}$ is a *representation equivalence.*

Our next task is to describe the inverse map $\mathbf{G}$ of this bijection. Before we define $\mathbf{G}$ let us give two technical remarks.

Remark 3.2.7. *Let* $T_\bullet = (V_\bullet, P_\bullet, \widetilde{\vartheta})$ *be any triple from* $\mathrm{Tri}(\Lambda)$. *The isomorphism* $\widetilde{\vartheta}$ *in* $\mathrm{D}^{\mathrm{b}}(\Gamma/I)$ *has an adjoint morphism* $\vartheta : V_\bullet \longrightarrow \Gamma/I \otimes_{\Lambda/I} \widetilde{P}_\bullet$ *in* $\mathrm{D}^{\mathrm{b}}(\Lambda/I)$. *This morphism is given by the composition* $V_\bullet \longrightarrow \Gamma/I \otimes_{\Lambda/I} V_\bullet \xrightarrow{\widetilde{\vartheta}} \Gamma/I \otimes_{\Lambda/I} \widetilde{P}_\bullet$ *.*
In particular, the adjoint morphism ϑ *is a* monomorphism.

Let us recall that a projective complex $(P_\bullet, d_\bullet) \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ is *minimal* if $\mathrm{im}\, d_j \subseteq \mathrm{rad}\, P_j$ for any $j \in \mathbb{Z}$. Since Λ is semiperfect, any projective complex is isomorphic to a minimal complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ (Remark B.2.2).

The corresponding notion in the category of triples is given by the following one:

Definition 3.2.8. *Let* $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ *be a triple of* $\mathrm{Tri}(\Lambda)$. *We will call* $T_\bullet$ minimal *if the following conditions hold*:

(1) $V_\bullet$ *is a complex with zero differentials, and*

(2) $\widetilde{P}_\bullet$ *is a minimal projective complex.*

(3) $\Gamma/I \otimes_{\Lambda/I} V_\bullet$ *and* $\Gamma/I \otimes_{\Gamma} \widetilde{P}_\bullet$ *are also complexes with zero differentials, and* $\widetilde{\vartheta}$ *is an isomorphism of bounded complexes of* Γ/I*-modules.*

We will show in the next subsection that minimal triples correspond to minimal projective complexes. For now let us only note the following:

Remark 3.2.9. *(1) The first two conditions in Definition 3.2.8 imply the third condition.*

(2) Any triple of $\mathrm{Tri}(\Lambda)$ *is isomorphic to some minimal triple.*

3.2.2 Definition of the complex of a triple

Let $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a triple in $\mathrm{Tri}(\Lambda)$. We will define a complex $\mathbf{G}(T_\bullet)$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ such that there is an isomorphism $\mathbf{F}\,\mathbf{G}(T_\bullet) \cong T_\bullet$ in $\mathrm{Tri}(\Lambda)$ in three steps. In the following let ϑ denote the adjoint morphism of the isomorphism $\widetilde{\vartheta}$.

(1) We may assume that $T_\bullet$ is a minimal triple by Remark 3.2.9.

(2) The complex $\widetilde{P}_\bullet$ gives rise to the short exact sequence

$$0 \longrightarrow I\widetilde{P}_\bullet \longrightarrow \widetilde{P}_\bullet \xrightarrow{\widetilde{\pi}} \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet \longrightarrow 0 \tag{3.2.3}$$

of complexes of Γ-modules.

(3) $\mathbf{G}(T_\bullet)$ is defined as the pull-back of π and the adjoint morphism ϑ in the abelian category of complexes of Λ-modules:

$$\begin{tikzcd} 0 \arrow[r, dashed] & I\widetilde{P}_\bullet \arrow[r, dashed] \arrow[d, equal] & \mathbf{G}(T_\bullet) \arrow[r, dashed, "\alpha"] \arrow[d, dashed, hook, "\beta"] & V_\bullet \arrow[r, dashed] \arrow[d, hook, "\vartheta"] & 0 \\ 0 \arrow[r] & I\widetilde{P}_\bullet \arrow[r] & \widetilde{P}_\bullet \arrow[r, "\widetilde{\pi}"] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet \arrow[r] & 0 \end{tikzcd} \tag{3.2.4}$$

Equivalently, the complex $\mathbf{G}(T)$ may be defined as the kernel of the following short exact sequence:

$$0 \dashrightarrow \mathbf{G}(T) \dashrightarrow V_\bullet \oplus \widetilde{P}_\bullet \xrightarrow{[\vartheta,\ \widetilde{\pi}]} \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet \longrightarrow 0 \tag{3.2.5}$$

At last, $\mathbf{G}(T_\bullet)$ is viewed as an object $P_\bullet$ in the derived category $\mathrm{D^b}(\Lambda)$.

Remark 3.2.10. *The short exact sequence* (3.2.5) *gives rise to the following long exact sequence of* Λ*-modules*:

$$\ldots \mathbf{H}_j(P_\bullet) \longrightarrow \mathbf{H}_j(V_\bullet) \oplus \mathbf{H}_j(\widetilde{P}_\bullet) \longrightarrow \mathbf{H}_j(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet) \longrightarrow \mathbf{H}_{j-1}(P_\bullet) \ldots \tag{3.2.6}$$

This sequence will be used frequently to study the homology of $P_\bullet = \mathbf{G}(T_\bullet)$.

The above procedure defines a map $\mathbf{G} :\ \mathrm{Ob}\,\mathrm{Tri}(\Lambda) \longrightarrow \mathrm{Ob}\,\mathrm{D^b}(\Lambda)$ on objects of categories. It has the following properties:

Proposition 3.2.11. [**BD04**, Proof of Theorem 2.4] *Let* $T_\bullet \in \mathrm{Tri}(\Lambda)$ *be minimal triple. Then the following statements hold*:

(1) $\mathbf{G}(T_\bullet)$ *is a complex of* projective Λ*-modules.*

(2) There are isomorphisms $\mathbf{F}\,\mathbf{G}(T_\bullet) \cong T_\bullet$ *in* $\mathrm{Tri}(\Lambda)$ *and* $\mathbf{G}\,\mathbf{F}(P_\bullet) \cong P_\bullet$ *in* $\mathrm{D^b}(\Lambda)$ *for any* $P_\bullet \in \mathrm{D^b}(\Lambda)$.

Summarized, any projective complex $P_\bullet \in \mathrm{D^b}(\Lambda)$ is the pull-back $\mathbf{G}(T_\bullet)$ for some triple $T_\bullet \in \mathrm{Tri}(\Lambda)$.

In the following we will say that a complex $P_\bullet$ has *length* n for some $n \in \mathbb{N}^+$, if $P_\bullet$ is minimal and $P_i \neq 0$ if and only if $0 \leq i \leq n$.

Lemma 3.2.12. *Let* $T = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta}) \in \mathrm{Tri}(\Lambda)$ *and let* $P_\bullet = \mathbf{G}(T)$.
Then $P_j = 0$ *if and only if* $\widetilde{P}_j = 0$ *for any* $j \in \mathbb{Z}$. *In particular,* $P_\bullet$ *has length* n *for some* $n \in \mathbb{N}^+$ *if and only if* $\widetilde{P}_\bullet$ *has length* n.

In the remaining part of this section we prove some technical statements on triples which will be needed in the sequel.

3.2.3 Gluing the complex of a triple

Let $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a minimal triple from $\mathrm{Tri}(\Lambda)$. In this subsection we describe an explicit construction of the corresponding complex $P_\bullet = \mathbf{G}(T_\bullet)$ in $\mathrm{D^b}(\Lambda)$.

Let us fix some degree $j \in \mathbb{Z}$. The projective module P_j can be identified with a submodule of $\widetilde{P}_j$ in a natural way. The difficulty is to describe explicitly the differential $d_j : P_j \longrightarrow P_{j-1}$ between the projectives.

(1) The differential is defined by the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & P_j \arrow[r, "{\left[\begin{smallmatrix}-\alpha_j \\ \beta_j\end{smallmatrix}\right]}"] \arrow[d, dashed, "d_j"] & V_j \oplus \widetilde{P}_j \arrow[r, "{[\vartheta_j, \widetilde{\pi}_j]}"] \arrow[d, "{\left[\begin{smallmatrix}0 & 0 \\ 0 & \widetilde{d}_j\end{smallmatrix}\right]}"] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_j \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & P_{j-1} \arrow[r, "{\left[\begin{smallmatrix}-\alpha_{j-1} \\ \beta_{j-1}\end{smallmatrix}\right]}"'] & V_{j-1} \oplus \widetilde{P}_{j-1} \arrow[r, "{[\vartheta_{j-1}, \widetilde{\pi}_{j-1}]}"'] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_{j-1} \arrow[r] & 0
\end{tikzcd} \tag{3.2.7}$$

(2) The isomorphism $(\phi, \psi) : \mathbf{F}(P_\bullet) \xrightarrow{\sim} T_\bullet$ of Proposition 3.2.11 is given by the following morphisms:

$$\phi_j : \Lambda/I \underset{\Lambda}{\otimes} P_j \xrightarrow{\Gamma/I \underset{\Lambda}{\otimes} \alpha_j} \Lambda/I \underset{\Lambda}{\otimes} V_j \xrightarrow{\bar{\varepsilon}_j} V_j \qquad \bar{a} \otimes p \longmapsto \bar{a} \otimes \alpha_j(p) \longmapsto \bar{a} \cdot \alpha_j(p)$$

$$\psi_j : \Gamma \underset{\Lambda}{\otimes} P_j \xrightarrow{\Gamma/I \underset{\Lambda}{\otimes} \beta_j} \Gamma \underset{\Lambda}{\otimes} \widetilde{P}_j \xrightarrow{\widetilde{\varepsilon}_j} \widetilde{P}_j \qquad b \otimes p \longmapsto b \otimes \beta_j(p) \longmapsto b \cdot \beta_j(p) \tag{3.2.8}$$

It is straightforward to check that $\Gamma \otimes d_j = \psi_{j-1}^{-1} \cdot \widetilde{d}_j \cdot \psi_j$.

(3) Now we claim that each map ψ_j is "essentially" given by the gluing map ϑ_j.

Namely, we have the commutative diagram

$$\begin{tikzcd}
\Gamma/I \underset{\Lambda/I}{\otimes} \Lambda/I \underset{\Lambda}{\otimes} P_j \arrow[r, "\mu_j", "\sim"'] \arrow[d, "\Gamma/I \underset{\Lambda/I}{\otimes} \phi_j"', "\wr"] & \Gamma/I \underset{\Gamma}{\otimes} \Gamma \underset{\Lambda}{\otimes} P_j \arrow[d, "\Gamma/I \underset{\Gamma}{\otimes} \psi_j", "\wr"'] \\
\Gamma/I \underset{\Lambda/I}{\otimes} V_j \arrow[r, "\widetilde{\vartheta}_j", "\sim"'] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_j
\end{tikzcd}$$

That is, $\Gamma/I \otimes \psi_j = \widetilde{\vartheta}_j \cdot (\Gamma/I \otimes \phi_j) \cdot \mu_j^{-1}$.

(4) Let us recall that Γ/I is a semisimple $\Bbbk$-algebra. Now we may identify both modules of the top row with $\Gamma/I \otimes_\Lambda P_j$. Then we may choose such a basis of $\Gamma/I \otimes_\Lambda P_j$ that the map $\Gamma/I \otimes \phi_j$ becomes the identity matrix. In other words, we may achieve that $(\Gamma/I \otimes \phi_j) \cdot \mu_j^{-1}$ is the identity matrix – without changing the map ϑ_j. Let Θ_j be the matrix of $\widetilde{\vartheta}_j$ in some basis. Then $\Gamma/I \otimes \psi_j = \Theta_j$.

(5) We may extend the matrix Θ_j to the following matrix $\breve{\Theta}_j$:

$$\begin{tikzcd}
0 \arrow[r] & I(\Gamma \underset{\Lambda}{\otimes} P_j)/\operatorname{rad}\Gamma \underset{\Lambda}{\otimes} P_j \arrow[r] \arrow[d, "\mathrm{Id}"] & \operatorname{top}\Gamma \underset{\Lambda}{\otimes} P_j \arrow[r] \arrow[d, "\breve{\Theta}_j = \left[\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & \Theta_j \end{smallmatrix}\right]"] & \Gamma/I \underset{\Gamma}{\otimes} \Gamma \underset{\Lambda}{\otimes} P_j \arrow[r] \arrow[d, "\Theta_j", "\wr"'] & 0 \\
0 \arrow[r] & I\widetilde{P}_j/\operatorname{rad}\widetilde{P}_j \arrow[r] & \operatorname{top}\widetilde{P}_j \arrow[r] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_j \arrow[r] & 0
\end{tikzcd}$$

(6) Finally, we may regard the matrix $\breve{\Theta}_j$ as a map $\breve{\Theta}_j : \Gamma \otimes_\Lambda P_j \longrightarrow \widetilde{P}_j$.

Summarized we obtain that the differential is given as follows:

$$d_j = \breve{\Theta}_{j-1}^{-1} \cdot \widetilde{d}_j \cdot \breve{\Theta}_j \tag{3.2.9}$$

In other terms, the differential d_j in the complex $P_\bullet$ is given by some column and row transformations of the differential $\widetilde{d}_j$ in the complex $\widetilde{P}_\bullet$.

3.2.4 Minimal triples with finite-dimensional homology

In this subsection we show that the bijection between projective complexes and triples in (3.2.2) can be restricted to *minimal* objects with *finite-dimensional homology.*

Lemma 3.2.13. *Let* $V \in \Lambda/I$ -mod, $\widetilde{P} \in \Gamma$ -mod, *and* $\vartheta : V \hookrightarrow \Gamma/I \otimes_\Gamma \widetilde{P}$ *be a monomorphism of* Λ/I*-modules. Define* P *to be the following pullback in* Λ -mod*:*

$$\begin{tikzcd}
0 \arrow[r, dashed] & I\widetilde{P} \arrow[r, dashed] \arrow[d, equal] & P \arrow[r, dashed] \arrow[d, hook, dashed, "\beta"] & V \arrow[r, dashed] \arrow[d, hook, "\vartheta"] & 0 \\
0 \arrow[r] & I\widetilde{P} \arrow[r] & \widetilde{P} \arrow[r] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P} \arrow[r] & 0
\end{tikzcd}$$

Then the monomorphism $\beta : P \hookrightarrow \widetilde{P}$ *restricts to an isomorphism*

$\beta' : JP \xrightarrow{\sim} J\widetilde{P}$ *for any subideal* J *of* I.

Proof. Let $\eta : P \longrightarrow \Gamma \otimes_\Lambda P$ be given by $p \longmapsto 1 \otimes p$ for any $p \in P$, and let $\psi : \Gamma \otimes_\Lambda P \longrightarrow \widetilde{P}$ be defined via $b \otimes p \longmapsto b\,\beta(p)$ for any $b \in \Gamma$ and $p \in P$. Then $\beta = \psi \circ \eta$ and the map η is injective. As mentioned in (3.2.8) the map ψ is actually an isomorphism of Λ-modules.

Restricting all three maps we obtain that $\beta' = \psi' \circ \eta' :\ JP \hookrightarrow J\Gamma \otimes_\Lambda P \xrightarrow{\sim} J\widetilde{P}$ is the composition of an injective with a bijective map.

We claim that η' is surjective. Let $a \in J$, $b \in \Gamma$ and $p \in P$. Then

$$a\,b \in J\Gamma = J \subseteq I \subseteq \Lambda \quad \text{and} \quad a\,b \otimes_\Lambda p = 1 \otimes_\Lambda (ab)\,p = \eta(a\,b\,p).$$

This shows that the maps η' and β' are bijective. $\square$

Lemma 3.2.14. *The following statements hold*:

(1) For any minimal projective complex $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$*, the triple* $\mathbf{F}(P_\bullet)$ *is minimal.*

(2) For any minimal triple $T \in \mathrm{Tri}(\Lambda)$*, the projective complex* $\mathbf{G}(T)$ *is minimal.*

PROOF. (1) Let $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ and let $j \in \mathbb{Z}$. Since $P_\bullet$ is minimal, we have that $\operatorname{im} d_j \subseteq \operatorname{rad} P_{j-1}$.

(a) Since $I \supseteq \operatorname{rad}\Lambda$ we obtain $\Lambda/I \otimes_\Lambda d = 0$. So, $\Lambda/I \otimes_\Lambda P_\bullet$ has zero differentials.

(b) Moreover, $\operatorname{im} \Gamma \otimes_\Lambda d_j = \Gamma \otimes_\Lambda \operatorname{im} d_j \subseteq \Gamma \otimes_\Lambda \operatorname{rad} P_{j-1} = \operatorname{rad}(\Gamma \otimes_\Lambda P_{j-1})$. In particular, $\Gamma \otimes_\Lambda P_\bullet$ is a minimal projective complex.

These two statements imply that $\mathbf{F}(P_\bullet)$ is a minimal triple.

(2) Let $T = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a minimal triple and let $j \in \mathbb{Z}$. Let us set $P_\bullet = \mathbf{G}(T_\bullet)$. The differential d_j in $P_\bullet$ is defined by diagram (3.2.7). Since $\widetilde{P}_\bullet$ is minimal, we have that $\operatorname{im} \widetilde{d}_j \subseteq \operatorname{rad} \widetilde{P}_{j-1}$. It follows that

$$\beta_{j-1}(\operatorname{im} d_j) = \operatorname{im}(\beta_{j-1}\, d_j) = \operatorname{im}(\widetilde{d}_j\, \beta_j) \subseteq \operatorname{rad} \widetilde{P}_{j-1}.$$

By Lemma 3.2.13, $\beta_{j-1}^{-1}(\operatorname{rad} \widetilde{P}_{j-1}) \subseteq \operatorname{rad} P_{j-1}$. This shows that $\operatorname{im} d_j \subseteq \operatorname{rad} P_{j-1}$. It follows that $P_\bullet$ is a minimal projective complex. □

Eventually, we shall be interested in the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ given by projective complexes with finite-dimensional homology:

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) = \{\ P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda) \mid \dim \mathbf{H}_j(P_\bullet) < \infty \ \text{ for any } j \in \mathbb{Z}\ \}.$$

Its counterpart in the category of triples is defined as follows:

$$\mathrm{Tri}_{\mathrm{fd}}(\Lambda) = \{\ (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta}) \in \mathrm{Tri}(\Lambda) \mid \widetilde{P}_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma)\ \}.$$

Lemma 3.2.15. *The following statements hold*: *Let* $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ *be a triple from* $\mathrm{Tri}(\Lambda)$ *and* $P_\bullet = \mathbf{G}(T_\bullet)$*. For any* $j \in \mathbb{Z}$ *it holds that*

$$\mathbf{H}_j(P_\bullet) \text{ is finite-dimensional if and only if } \mathbf{H}_j(\widetilde{P}_\bullet) \text{ is finite-dimensional.} \tag{3.2.10}$$

In particular, the following statements hold:

(1) For any triple $T_\bullet \in \mathrm{Tri}_{\mathrm{fd}}(\Lambda)$ *it holds that* $\mathbf{G}(T_\bullet) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

(2) For any complex $P_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ *it holds that* $\mathbf{F}(P_\bullet) \in \mathrm{Tri}_{\mathrm{fd}}(\Lambda)$.

PROOF. Let us recall the following basic fact. Whenever we have an exact sequence of vector spaces $V_{j-1} \longrightarrow V_j \longrightarrow V_{j+1}$ such that V_{j-1} and V_{j+1} are finite-dimensional, then also V_j is finite-dimensional.
In our setup, there is the following long exact homology sequence from (3.2.6):

$$\dots \mathbf{H}_{j+1}(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet) \longrightarrow \mathbf{H}_j(P_\bullet) \longrightarrow \mathbf{H}_j(V_\bullet) \oplus \mathbf{H}_j(\widetilde{P}_\bullet) \longrightarrow \mathbf{H}_j(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet) \dots$$

Since Λ/I and Γ/I are semisimple $\Bbbk$-algebras, the vector spaces $\mathbf{H}_{j+1}(\Gamma/I \otimes_\Gamma \widetilde{P}_\bullet)$, $\mathbf{H}_j(V_\bullet)$ and $\mathbf{H}_j(\Gamma/I \otimes_\Gamma \widetilde{P}_\bullet)$ are finite-dimensional. This implies (3.2.10). □

Corollary 3.2.16. *The bijection in* (3.2.2) *restricts to isomorphism classes of minimal objects with finite-dimensional homology on both sides*:

$$\operatorname{ind}[\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)] \xleftrightarrow{1:1} \operatorname{ind}[\mathrm{Tri}_{\mathrm{fd}}(\Lambda)]$$

Proof. This follows from Lemmas 3.2.14 and 3.2.15 □

3.3 Triples corresponding to modules

Let $\Lambda \subseteq \Gamma$ be a *radical embedding* of semiperfect rings:

$$\Lambda \text{ and } \Gamma \text{ be semiperfect } \Bbbk\text{-algebras such that } \Lambda \subset \Gamma \text{ and } \operatorname{rad}\Lambda = \operatorname{rad}\Gamma \quad (3.3.1)$$

In this case we may consider the category of triples $\mathrm{Tri}(\Lambda)$ which is representation equivalent to the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

In this section we describe modules over Λ in terms of triples by two methods.

The first method is based on the fact that Λ-modules can be represented by *minimal projective presentations.* The second method uses projective resolutions of Λ-modules.

3.3.1 Projective presentations via triples

The statements of this subsection hold for the setup of any *radical embedding* (3.3.1). In this subsection we describe the triples which correspond to the indecomposable minimal presentations:

$$\begin{tikzcd} \operatorname{ind}[\Lambda\text{-}\mathrm{mod}] \arrow[r, hook] & \mathrm{D}^{\mathrm{b}}(\Lambda) \arrow[r, shift left, "\mathbf{F}"] & \mathrm{Tri}(\Lambda) \arrow[l, dashed, shift left, "\mathbf{G}", "1:1"'] \\ M \arrow[r, mapsto] & (P_1 \longrightarrow P_0) & ? \arrow[l, dashed, mapsto] \end{tikzcd}$$

Let us recall that by Subsection B.2.2 we may view "minimal projective presentations" of Λ-modules as a subcategory $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Its objects are given as follows:

$$\Lambda\text{-}\overrightarrow{\mathrm{proj}} = \Big\{\; P_\bullet = P_1 \xrightarrow{d_1} \underset{\neq 0}{P_0} \;\text{ such that }\; \ker d_1 \subseteq \operatorname{rad} P_1 \;\text{ and }\; \operatorname{im} d_1 \subseteq \operatorname{rad} P_0 \Big\}.$$

Proposition 3.3.1. *Let* $\Lambda \subseteq \Gamma$ *be a radical embedding as in* (3.3.1)*. Let* $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ *be an indecomposable minimal triple of* $\mathrm{Tri}(\Lambda)$*.*

(1) Then $\mathbf{G}(T_\bullet)$ *is a minimal presentation in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$*, that is, an indecomposable object in the category* $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$*, if and only if the complex* $\widetilde{P}_\bullet$ *is given by some complex* $\widetilde{P}_1 \xrightarrow{\widetilde{d}_1} \underset{\neq 0}{\widetilde{P}_0}$ *of length at most one.*

Assume that $\widetilde{P}_\bullet$ satisfies the conditions above. In particular, $P_\bullet$ is the minimal presentation of some indecomposable Λ-module M. Then the following holds:

(2) The module M is finite-dimensional if and only if $\mathbf{H}_0(\widetilde{P}_\bullet)$ is finite-dimensional.

(3) There is an isomorphism $\operatorname{syz}^2(M) \xrightarrow{\sim} \ker \widetilde{d}_1 \cap I\widetilde{P}_1 =: \widetilde{K}_1$ of Λ-modules.

PROOF. (1) Let us set $P_\bullet = \mathbf{G}(T_\bullet)$. By Corollary B.2.10 an indecomposable presentation in $\Lambda\text{-}\overrightarrow{\text{proj}}$ is the same as a "minimal indecomposable projective complex of length at most one". By Lemma 3.2.14 and Theorem 3.2.4 $P_\bullet$ is a minimal indecomposable projective complex. By Lemma 3.2.12 the complexes $P_\bullet$ and $\widetilde{P}_\bullet$ have equal length.

(2) This follows from Lemma 3.2.15.

(3) Let $\widetilde{P}_\bullet$ be a complex of length at most one. The minimal presentation $P_\bullet$ is defined by the two bottom rows in the following commutative diagram of Λ-modules:

$$\begin{tikzcd}
 & \widetilde{K}_1 \arrow[d, hook, dotted, "\iota"] \arrow[dr, hook, dotted, "{\left[\begin{smallmatrix}\widetilde{\iota}\\ 0\end{smallmatrix}\right]}"] & & \\
0 \arrow[r, dashed] & \ker d_1 \arrow[r, dashed, "{\left[\begin{smallmatrix}-\alpha_1\\ \beta_1\end{smallmatrix}\right]}"'] \arrow[d, hook] & V_1 \oplus \ker \widetilde{d}_1 \arrow[r, dashed, "{[\vartheta_1, \widetilde{\pi}_1]}"'] \arrow[d, hook] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_1 \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & P_1 \arrow[r, "{\left[\begin{smallmatrix}-\alpha_1\\ \beta_1\end{smallmatrix}\right]}"'] \arrow[d, "d_1"] & V_1 \oplus \widetilde{P}_1 \arrow[r, "{[\vartheta_1, \widetilde{\pi}_1]}"'] \arrow[d, "{\left[\begin{smallmatrix}0 & 0\\ 0 & \widetilde{d}_1\end{smallmatrix}\right]}"] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_1 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & P_0 \arrow[r] & V_0 \oplus \widetilde{P}_0 \arrow[r] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_0 \arrow[r] & 0
\end{tikzcd}$$

The two bottom rows induce the row with dashed maps. Let us note that $\operatorname{syz}^2(M) = \ker d_1$ and recall that $\widetilde{K}_1 = \ker \widetilde{d}_1 \cap I\widetilde{P}_1$.

We claim that the inclusion β_1 restricts to an isomorphism $\ker d_1 \xrightarrow{\sim} \widetilde{K}_1$. Equivalently, we have to show that $\beta(\ker d_1) = \widetilde{K}_1$.

- $\supseteq$: The dotted map $\widetilde{\iota} : \widetilde{K}_1 \hookrightarrow \ker \widetilde{d}_1$ in the diagram is the natural inclusion. Since $\widetilde{K}_1 \subseteq I\widetilde{P}_1 = \ker \widetilde{\pi}_1$, there is a unique injective map $\iota : \widetilde{K}_1 \hookrightarrow \ker d_1$ such that $\beta\,\iota = \widetilde{\iota}$. It follows that $\widetilde{K}_1 = \widetilde{\iota}(\widetilde{K}_1) = \beta\,\iota(\widetilde{K}_1) \subseteq \beta(\ker d_1)$.
- $\subseteq$: Obviously, $\beta_1(\ker d_1) \subseteq \ker \widetilde{d}_1$. Since $P_\bullet$ is indecomposable, Lemma B.2.9 implies that $\ker d_1 \subseteq \operatorname{rad} P_1 \subseteq IP_1$. By Lemma 3.2.13 the map β_1 restricts to an isomorphism $IP_1 \cong I\widetilde{P}_1$. It follows that $\beta(\ker d_1) \subseteq \beta(IP_1) = I\widetilde{P}_1$. Thus, $\beta(\ker d_1) \subseteq \ker \widetilde{d}_1 \cap I\widetilde{P}_1$. □

Remark 3.3.2. *In the notations of Proposition 3.3.1 assume that the overring Γ is also hereditary. Then $\widetilde{K}_1 = \ker \widetilde{d}_1 \cap I\widetilde{P}_1 = I \ker \widetilde{d}_1$. This will be shown in Lemma 3.3.9.*

This first statement of Proposition 3.3.1 already shows the main advantage of the category of presentations $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$, the characterization of indecomposable presentations in terms of triples is very simple.

In the following, let us use the following notation:

- $\Lambda\text{-}\mathrm{mod}$ be the category of finitely generated Λ-modules, and
- $\Lambda\text{-}\mathrm{fd.mod}$ denote the category of finite-dimensional modules over Λ.

The above categories of Λ-modules are related to the following categories:

- Let $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ denote the full subcategory given by presentations in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ with finite-dimensional homology at degree zero.
- Let $\mathrm{Tri}_{0,\infty}(\Lambda)$ denote the full subcategory of minimal triples $(V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ in $\mathrm{Tri}(\Lambda)$ such that $\widetilde{P}_\bullet = \widetilde{P}_1 \longrightarrow \underset{\neq 0}{\widetilde{P}_0}$ is a projective complex of length at most one.
- Let $\mathrm{Tri}_0(\Lambda)$ be the full subcategory of triples $(V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ in $\mathrm{Tri}_{0,\infty}(\Lambda)$ such that the homology $\mathbf{H}_0(\widetilde{P}_\bullet)$ is finite-dimensional.

The categories $\mathrm{Tri}_{0,\infty}(\Lambda)$ and $\mathrm{Tri}_0(\Lambda)$ yield an analogue of Corollaries 3.2.6 and 3.2.16 for the abelian module categories:

Corollary 3.3.3. *With the notations above, there are bijections between the isomorphism classes of indecomposable objects in the following categories*:

$$\mathrm{ind}[\Lambda\text{-}\mathrm{mod}] \xleftrightarrow{1:1} \mathrm{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Tri}_{0,\infty}(\Lambda)]$$

This bijection restricts to the isomorphism classes of indecomposable objects in the following categories:

$$\mathrm{ind}[\Lambda\text{-}\mathrm{fd.mod}] \xleftrightarrow{1:1} \mathrm{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Tri}_0(\Lambda)]$$

Proof. This follows from Proposition 3.3.1 and Corollary B.2.8. □

Corollary 3.3.4. *Under the assumptions above let $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be some minimal indecomposable triple from* $\mathrm{Tri}(\Lambda)$ *such that $\widetilde{P}_\bullet$ is a projective resolution of length one. Then $P_\bullet = \mathbf{G}(T_\bullet)$ is also a projective resolution of length one.*

Proof. In the notations above, we have that $\ker d_1 \cong \ker \widetilde{d}_1 \cap I\widetilde{P}_1 = 0$. □

Corollary 3.3.5. *Let $M \in \Lambda\text{-}\mathrm{mod}$. Then there is some Γ-module $\widetilde{K}$ such that* $\mathrm{syz}^2(M) \cong \widetilde{K}$ *in* $\Lambda\text{-}\mathrm{mod}$.

In other words, for any radical embedding $\Lambda \subseteq \Gamma$ the second syzygies of Λ-modules are given by certain modules of Γ.

Remark 3.3.6. *Proposition 3.3.1 suggests the following approach to describe the projective resolutions of all indecomposable Λ-modules*:

(1) *First, we need to classify all* indecomposable *minimal triples $T = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ in* $\mathrm{Tri}(\Lambda)$ *such that $\widetilde{P}_\bullet$ has length one. This problem is equivalent to some matrix problem.*

(2) Given a triple $T_\bullet$ as above we have to compute $\widetilde{K}_1 = \ker \widetilde{d}_1 \cap I \widetilde{P}_1$ and its projective resolution in the category of Λ-modules.

(3) The result is a projective resolution $P_\bullet$ of an indecomposable Λ-module.

Note that this approach works in the setup of any radical embedding.

3.3.2 Projective resolutions via triples

As above, let $\Lambda \subset \Gamma$ be a radical embedding of semiperfect rings as in (3.2.1). In this section we are going to characterize those triples in $\mathrm{Tri}(\Lambda)$ which correspond to projective resolutions in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

$$\begin{array}{ccccc} \Lambda\text{ - mod} & \hookrightarrow & \mathrm{D}^{\mathrm{b}}(\Lambda) & \underset{\mathbf{G}}{\overset{\mathbf{F}}{\underset{\leftarrow\,1:1\,-}{\longrightarrow}}} & \mathrm{Tri}(\Lambda) \\ M & \longmapsto & P_\bullet^M & \leftarrow\text{-----}\dashv & ? \end{array}$$

We will need to prove three Lemmas before we can characterize the projective resolutions in terms of triples.

We need to fix some notation. Let $T = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a triple in $\mathrm{Tri}(\Lambda)$.

- Let $\vartheta : V_\bullet \hookrightarrow \Gamma/I \otimes_\Gamma \widetilde{P}_\bullet$ be the adjoint morphism of $\widetilde{\vartheta}$.
- Let $\widetilde{\pi} : \widetilde{P}_\bullet \twoheadrightarrow \Gamma/I \otimes_\Gamma \widetilde{P}_\bullet$ be the natural projection.
- For each degree $j \in \mathbb{Z}$ these two morphisms give rise to the map

$$\gamma_j = [\mathbf{H}_j(\vartheta), \mathbf{H}_j(\widetilde{\pi})] : \quad \mathbf{H}_j(V_\bullet) \oplus \mathbf{H}_j(\widetilde{P}_\bullet) \longrightarrow \mathbf{H}_j(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet).$$

The maps γ_j determine the homology of $\mathbf{G}(T_\bullet)$ in the following way:

Lemma 3.3.7. *Let $T = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a triple in* $\mathrm{Tri}(\Lambda)$. *Let $P_\bullet = \mathbf{G}(T_\bullet)$. Then*

$$\mathbf{H}_j(P_\bullet) = 0 \textit{ for all } j \geq 1 \quad \textit{if and only if} \quad \begin{cases} \gamma_1 \textit{ is injective,} & \textit{and} \\ \gamma_j \textit{ is bijective} & \textit{for all } j \geq 2. \end{cases}$$

Proof. In the notations above there is the long exact homology sequence (3.2.6):

$$\ldots \mathbf{H}_j(P_\bullet) \longrightarrow \mathbf{H}_j(V_\bullet) \oplus \mathbf{H}_j(\widetilde{P}_\bullet) \xrightarrow{\gamma_j} \mathbf{H}_j(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet) \longrightarrow \mathbf{H}_{j-1}(P_\bullet) \ldots$$

This implies the statement. □

Next, we are going to study when the maps γ_j are injective.

Lemma 3.3.8. *Let $T = \big(V_\bullet, (\widetilde{P}_\bullet, \widetilde{d}), \widetilde{\vartheta}\big)$ be a minimal triple from* $\mathrm{Tri}(\Lambda)$. *Let $P_\bullet = \mathbf{G}(T)$ be the corresponding complex in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *and let $j \in \mathbb{Z}$. Let $d_j : P_j \longrightarrow P_{j-1}$ be the differential of the complex $P_\bullet$ at degree j. Then the following three conditions are equivalent:*

(1) γ_j is injective,

(2) $\ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$ *and* $\operatorname{im} \vartheta_j \cap \ker \widetilde{d}_j / I\widetilde{P}_j = 0$,

(3) $\ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$ *and* $\ker d_j \subseteq IP_j$.

Proof. • (1) ⇔ (2) :
Let us consider the map

$$\xi_j = \mathbf{H}_j(\widetilde{\pi}): \quad \mathbf{H}_j(\widetilde{P}_\bullet) = \ker \widetilde{d}_j / \operatorname{im} \widetilde{d}_{j+1} \longrightarrow \mathbf{H}_j(\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet) = \widetilde{P}_j / I\widetilde{P}_j$$

which is given by $\xi_j(\widetilde{p} + \operatorname{im} \widetilde{d}_{j+1}) = \widetilde{p} + I\widetilde{P}_j$ for any $\widetilde{p} \in \ker \widetilde{d}_j$. It holds that

$$\ker \xi_j = (\ker \widetilde{d}_j \cap I\widetilde{P}_j)/(\operatorname{im} \widetilde{d}_{j+1}) \quad \text{and} \quad \operatorname{im} \xi_j = \ker \widetilde{d}_j / I\widetilde{P}_j. \tag{3.3.2}$$

Since T is minimal, we have that $\gamma_j = [\mathbf{H}_j(\widetilde{\pi}), \mathbf{H}_j(\vartheta)] = [\xi_j, \vartheta_j]$. It holds that

$$\ker \gamma_j = 0 \quad \text{if and only if} \quad \ker \vartheta_j = \ker \xi_j = \operatorname{im} \vartheta_j \cap \operatorname{im} \xi_j = 0.$$

By assumption ϑ_j is injective. It follows that (1) ⇔ (2) by (3.3.2).

Before we show the equivalence of (1) and (3) let us note the following. Since $\mathbf{F}\,\mathbf{G}(T_\bullet) = \mathbf{F}(P_\bullet) \cong T_\bullet$, there is the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P_\bullet \arrow[r, "\left[\begin{smallmatrix}\pi\\ \eta\end{smallmatrix}\right]"] \arrow[d, equal] & \Lambda/I \underset{\Lambda}{\otimes} P_\bullet \oplus \Gamma \underset{\Lambda}{\otimes} P_\bullet \arrow[r, "{[\mu\ \widetilde{\varepsilon}]}"] \arrow[d, "\wr", "\left[\begin{smallmatrix}\phi & 0\\ 0 & \psi\end{smallmatrix}\right]"'] & \Gamma/I \underset{\Gamma}{\otimes} \Gamma \underset{\Lambda}{\otimes} P_\bullet \arrow[r] \arrow[d, "\wr"', "\Gamma/I\otimes_\Gamma \psi"] & 0 \\
0 \arrow[r] & P_\bullet \arrow[r, "\left[\begin{smallmatrix}\alpha\\ -\beta\end{smallmatrix}\right]"] & V_\bullet \oplus \widetilde{P}_\bullet \arrow[r, "{[\vartheta\ \ \widetilde{\pi}]}"] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet \arrow[r] & 0
\end{tikzcd}$$

Taking homology at degree j we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbf{H}_j(P_\bullet) \arrow[r, "\left[\begin{smallmatrix}\mathbf{H}_j(\pi)\\ \mathbf{H}_j(\eta)\end{smallmatrix}\right]"] \arrow[d, equal] & \Lambda/I \underset{\Lambda}{\otimes} P_j \oplus \mathbf{H}_j(\Gamma \underset{\Lambda}{\otimes} P_\bullet) \arrow[r, "{[\mu_j\ \mathbf{H}_j(\widetilde{\varepsilon})]}"] \arrow[d, "\wr", "\left[\begin{smallmatrix}\mathbf{H}_j(\phi) & 0\\ 0 & \mathbf{H}_j(\psi)\end{smallmatrix}\right]"'] & \Gamma/I \underset{\Gamma}{\otimes} \Gamma \underset{\Lambda}{\otimes} P_j \arrow[d, "\wr"', "\Gamma/I\otimes_\Gamma \psi_j"] \\
\mathbf{H}_j(P_\bullet) \arrow[r, "\left[\begin{smallmatrix}\mathbf{H}_j(\alpha)\\ -\mathbf{H}_j(\beta)\end{smallmatrix}\right]"] & V_j \oplus \mathbf{H}_j(\widetilde{P}_\bullet) \arrow[r, "{\gamma_j=[\vartheta_j\ \xi_j]}"] & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_j
\end{tikzcd} \tag{3.3.3}$$

In particular, γ_j is injective if and only if $\mathbf{H}_j(\alpha) = \mathbf{H}_j(\beta) = 0$. Let us note that

$$\mathbf{H}_j(\alpha) = 0 \quad \Leftrightarrow \quad \mathbf{H}_j(\pi) = 0 \quad \Leftrightarrow \quad \ker d_j \subseteq IP_j. \tag{3.3.4}$$

since $\operatorname{im} \mathbf{H}_j(\pi) = \ker d_j / IP_j$ similarly to (3.3.2).

• (1) ⇒ (3) :
Let γ_j be injective. Then $\ker d_j \subseteq IP_j$ by (3.3.4). Since (1) ⇒ (2) has been shown, it holds that $\ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$.

• (3) ⇒ (1) :
Vice versa, let $\ker d_j \subseteq IP_j$ and $\ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$. The first assumption is equivalent to $\mathbf{H}_j(\alpha) = 0$ by (3.3.4). The second assumption is equivalent to the injectivity of ξ_j by (3.3.2). Since $\xi_j\,\mathbf{H}_j(\beta) = \vartheta_j\,\mathbf{H}_j(\alpha) = 0$ it follows that $\mathbf{H}_j(\beta) = 0$. This implies that γ_j is injective. □

Any of the conditions in Lemma 3.3.8 imply that the homology of $\widetilde{P}_\bullet$ has to be *semisimple* in positive degrees. More precisely, the following holds:

Lemma 3.3.9. *Let $(\widetilde{P}_\bullet, \widetilde{d})$ be a complex in $\mathrm{D}^{\mathrm{b}}(\Gamma)$ and let $j \in \mathbb{Z}$.*

(1) If $\ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$, *then* $\mathbf{H}_j(\widetilde{P}_\bullet) \in \Gamma/I\text{ - mod}$.

(2) Assume that Γ is hereditary. Then the converse of (1) is also true.

Proof. (1) For the first statement let us note that $I \ker \widetilde{d}_j \subseteq \ker \widetilde{d}_j \cap I\widetilde{P}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$. This implies that $I\,\mathbf{H}_j(\widetilde{P}_\bullet) = I\,(\ker \widetilde{d}_j / \operatorname{im} \widetilde{d}_{j+1}) = 0$, so $\mathbf{H}_j(\widetilde{P}_\bullet) \in \Gamma/I\text{ - mod}$.

(2) Let Γ be hereditary and $\mathbf{H}_j(\widetilde{P}_\bullet) \in \Gamma/I\text{ - mod}$. Equivalently, $I \ker \widetilde{d}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$. Consider the following short exact sequence of Γ-modules:

$$0 \longrightarrow \ker \widetilde{d}_j \longrightarrow \widetilde{P}_j \underset{\iota}{\overset{\widetilde{d}'_j}{\rightleftarrows}} \operatorname{im} \widetilde{d}_j \longrightarrow 0$$

Since Γ is hereditary, $\operatorname{im} \widetilde{d}_j$ is projective and so $\widetilde{d}'_j$ is a split epimorphism. Equivalently, there is an embedding $\iota : \operatorname{im} \widetilde{d}_j \hookrightarrow \widetilde{P}_j$ such that $\widetilde{d}'_j \cdot \iota = \mathrm{id}$.

It can be shown that $\widetilde{P}_j$ is *equal* to $\ker \widetilde{d}_j \oplus \operatorname{im} \iota$.

This implies that $\ker \widetilde{d}_j \cap I \operatorname{im} \iota \subseteq \ker \widetilde{d}_j \cap \operatorname{im} \iota = 0$. Moreover, we obtain

$$\ker \widetilde{d}_j \cap I\widetilde{P}_j = I \ker \widetilde{d}_j \oplus (\ker \widetilde{d}_j \cap I \operatorname{im} \iota) = I \ker \widetilde{d}_j. \tag{3.3.5}$$

By assumption $I \ker \widetilde{d}_j \subseteq \operatorname{im} \widetilde{d}_{j+1}$. This shows the second statement. □

Remark 3.3.10. *In general, the converse of Lemma 3.3.9 (1) is not true :*

Let Γ be the path algebra of the quiver $\underset{3}{\bullet} \xrightarrow{\cdot \mathsf{a}} \underset{2}{\bullet} \xrightarrow{\cdot \mathsf{b}} \underset{1}{\bullet}$ *with the relation* $\mathsf{b}\,\mathsf{a} = 0$.

Let $I = \operatorname{rad} \Gamma$ and let $\widetilde{P}_\bullet$ be the complex $\widetilde{P}_2 \xrightarrow{\cdot \mathsf{a}} \widetilde{P}_3$ *at degrees 1 and 0.*

Then $\mathbf{H}_1(\widetilde{P}_\bullet) = \langle\, \mathsf{b} \,\rangle \cong \operatorname{top} \widetilde{P}_2 \in \Gamma/I\text{ - mod}$ *and* $\ker \widetilde{d}_1 \cap I\widetilde{P}_1 = \langle\, \mathsf{b} \,\rangle \not\subset 0 = \operatorname{im} \widetilde{d}_2$.

The following Proposition characterizes projective resolutions of nodal orders via triples:

Proposition 3.3.11. *Let Λ be a basic nodal order, Γ its normalization and I the conductor ideal. Let $T_\bullet = \big(V_\bullet, \widetilde{P}_\bullet, \vartheta\big)$ be a minimal triple in $\mathrm{Tri}(\Lambda)$ and let $(P_\bullet, d) = \mathbf{G}(T_\bullet)$ be the corresponding minimal projective complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Let us set $N_j = \mathbf{H}_j(\widetilde{P}_\bullet)$ for any $j \in \mathbb{Z}$. Then the following statements hold:*

(1) The complex $P_\bullet$ is a projective resolution *in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ if and only if the following conditions are satisfied:*

$$\begin{cases} \text{(a)} \quad \widetilde{P}_0 \neq 0 \quad \textit{and} \quad \widetilde{P}_{-j} &= \; 0 \qquad\qquad \textit{for any } j \geq 1, \textit{ and} \\ \text{(b)} \qquad\qquad\qquad\qquad N_j &\in \; \Gamma/I\text{ - mod} \quad \textit{for any } j \geq 1, \textit{ and} \\ \text{(c)} \qquad\qquad\qquad \ker d_j &\subseteq \; IP_j \qquad\qquad \textit{for any } j \geq 1, \textit{ and} \\ \text{(d)} \qquad\qquad\quad 2 \dim N_j &= \; \dim \widetilde{P}_j / I\widetilde{P}_j \quad \textit{for any } j \geq 2. \end{cases} \tag{3.3.6}$$

In the following let $P_\bullet$ be a projective resolution and set $M = \mathbf{H}_0(P_\bullet)$.

(2) The Λ-module M is finite-dimensional if and only if N_0 is finite-dimensional.

Proof. (1) Let us first note that by Lemma 3.2.12

$$P_0 \neq 0 \iff \widetilde{P}_0 \neq 0 \quad \text{and} \quad P_{-j} = 0 \iff \widetilde{P}_{-j} = 0 \quad \text{for any } j \geq 1.$$

We recall that the homology of $P_\bullet$ was determined by the maps

$$\gamma_j = [\vartheta_j, \mathbf{H}_j(\widetilde{\pi})] : \quad V_j \oplus N_j \longrightarrow \widetilde{P}_j/I\widetilde{P}_j. \tag{3.3.7}$$

The following conditions are equivalent by Lemma 3.3.7:

$$\mathbf{H}_j(P_\bullet) = 0 \text{ for any } j \geq 1 \iff \begin{cases} \gamma_j \text{ is injective} & \text{for any } j \geq 1, \\ \gamma_j \text{ is surjective} & \text{for any } j \geq 2. \end{cases}$$

$$\iff \begin{cases} \gamma_j \text{ is injective} & \text{for all } j \geq 1, \\ \dim V_j + \dim N_j = \dim \widetilde{P}_j/I\widetilde{P}_j & \text{for all } j \geq 2. \end{cases} \tag{3.3.8}$$

The last equivalence follows because the modules in (3.3.7) are *finite-dimensional* $\Bbbk$-vector spaces if γ_j is injective.

- The injectivity of the maps γ_j for any $j \geq 1$ is equivalent to the conditions (b) and (c) in (3.3.6) by Lemmas 3.3.8 and 3.3.9.
- Let $j \geq 2$ and assume that the conditions (b) and (c) hold. By Lemma 3.1.11 we have $\dim \Gamma \otimes_\Lambda S = 2 \dim S$ for any simple Λ/I-module S. Since $\widetilde{\vartheta} : \Gamma/I \otimes_{\Lambda/I} V_j \xrightarrow{\sim} \widetilde{P}_j/I\widetilde{P}_j$ it follows that $\dim \widetilde{P}_j/I\widetilde{P}_j = \dim \Gamma/I \otimes V_j = 2 \dim V_j$. This implies that

$$\dim V_j + \dim N_j = \dim \widetilde{P}_j/I\widetilde{P}_j \quad \text{if and only if} \quad 2 \dim N_j = \dim \widetilde{P}_j/I\widetilde{P}_j. \tag{3.3.9}$$

It follows that the conditions in (3.3.8) are equivalent to conditions (b), (c) and (d) in (3.3.6).

Summarized, we obtain that $P_\bullet$ is a projective resolution if and only if the four conditions in (3.3.6) are satisfied.

(2) The last statement follows from Lemma 3.2.15. □

Corollary 3.3.12. *In the notations of Proposition 3.3.11 $P_\bullet$ is a projective resolution of length one if and only if $\widetilde{P}_\bullet$ is a projective resolution of length one.*

Proof. • $\Rightarrow$:
Let $P_\bullet$ be a projective resolution of length one. Then $\widetilde{P}_\bullet$ is a complex in $\mathrm{D}^{\mathrm{b}}(\Gamma)$ of length one by Lemma 3.2.12. On the one hand, the module $N_1 = \ker \widetilde{d}_1$ is a submodule of projective module. Since Γ is hereditary, N_1 is projective. On the other hand, condition (b) holds which implies that N_1 is semi-simple. Since Γ is an order, it follows that $N_1 = 0$.

• $\Leftarrow$:
Let $\widetilde{P}_\bullet$ be a projective resolution of length one. Then $N_1 = \ker \widetilde{d}_1 = 0$ and conditions (a), (b) and (d) are satisfied. It remains to show that condition (c) holds for $j = 1$. It holds also that $\operatorname{im} \vartheta_1 \cap \ker \widetilde{d}_1/I\widetilde{P}_1 = 0$. In particular, condition (2) of Lemma 3.3.8 is satisfied. By Lemma 3.3.8 this implies that $\ker d_1 \subseteq IP_1$. □

Remark 3.3.13. *In the notations above, the following holds for any nodal order Λ :*

If T is a minimal, indecomposable *triple in* $\mathrm{Tri}(\Lambda)$ *satisfying conditions* (a), (b) *and* (d) *in* (3.3.6), *then condition* (c) *holds also.*

This means that indecomposable triples of projective resolutions are completely determined by the normalization complex $\widetilde{P}_\bullet$. This statement will be shown in the proof of Theorem 5.1.11 for the Gelfand order using the classification of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

3.3.3 Homological invariants via triples

Next, we want to express the defect and the involution which were introduced for nodal orders in Definition 3.1.8 via triples.

Definition 3.3.14. *Let Λ be a nodal order, Γ its normalization and let $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be some triple from* $\mathrm{Tri}(\Lambda)$. *Let $S_\star$ denote the "neutral semi-simple module" of Λ according to Definition 3.1.8. Set $\widetilde{S}_\star = \Gamma \otimes S_\star$. Moreover, for any $j \in \mathbb{Z}$ we set*

$$\delta^{(j)}(T_\bullet) = \delta^{(j)}(\widetilde{P}_\bullet) = \dim \mathrm{Ext}^j_\Gamma(\widetilde{P}_\bullet, \widetilde{S}_\star) \quad \text{and} \quad \delta(T_\bullet) = \delta(\widetilde{P}_\bullet) = \sum_{j\in\mathbb{Z}} \delta^{(j)}(\widetilde{P}_\bullet)$$

Lemma 3.3.15. *Let Λ be a nodal order and $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$.*
It holds that $\delta^{(j)}(\mathbf{F}(P_\bullet)) = \delta^{(j)}(P_\bullet)$ for any $j \in \mathbb{Z}$. Hence, $\delta(\mathbf{F}(P_\bullet)) = \delta(P_\bullet)$.

Proof. We recall that the derived functor $\Gamma \otimes_\Lambda \_ \ : \ \mathrm{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathrm{D}^{\mathrm{b}}(\Gamma)$ is left adjoint to the derived forgetful functor ${}_\Lambda(\_)$. Since $S_\star$ is neutral, we have that ${}_\Lambda\widetilde{S}_\star = {}_\Lambda\Gamma \otimes_\Lambda S_\star = S_\star$. For any $j \in \mathbb{Z}$ we obtain that

$$\mathrm{Ext}^j_\Lambda(P_\bullet, S_\star) = \mathrm{Ext}^j_\Lambda(P_\bullet, {}_\Lambda\widetilde{S}_\star) \cong \mathrm{Ext}^j_\Gamma(\Gamma \underset{\Lambda}{\otimes} P_\bullet, \widetilde{S}_\star) = \mathrm{Ext}^j_\Gamma(\widetilde{P}_\bullet, \widetilde{S}_\star).$$

This shows that $\delta^{(j)}(P_\bullet) = \delta^{(j)}(\widetilde{P}_\bullet)$ for any $j \in \mathbb{Z}$, and $\delta(P_\bullet) = \delta(\widetilde{P}_\bullet) = \delta(\mathbf{F}(P_\bullet))$. □

Having characterized projective resolutions via triples, it is natural to describe homological invariants of modules in terms of triple data.

Lemma 3.3.16. *Let Λ be a basic nodal order. Let $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be a minimal triple in $\mathrm{Tri}_{\mathrm{fd}}(\Lambda)$ such that $P_\bullet = \mathbf{G}(T_\bullet)$ is a projective resolution in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Let us set $M = \mathbf{H}_0(P_\bullet)$ and $N_j = \mathbf{H}_j(\widetilde{P}_\bullet)$ for $j = 0$ or 1. Then the following statements hold:*

(1) $\mathrm{pd}(M) =$ *length of* $\widetilde{P}_\bullet$.

(2) $\mathrm{top}(M) \cong V_0 \oplus I\widetilde{P}_0/\operatorname{rad}\widetilde{P}_0$.

(3) The Jordan-Hölder multiplicities of M are given by

$$\underline{\dim}(M) = \underline{\dim}\,\big(N_0 \oplus \sigma(V_1)\big) - \underline{\dim}\,\big(N_1 \oplus \sigma(V_0)\big), \tag{3.3.10}$$

where N_0 and N_1 are considered as Λ-modules.

Proof. (1) This follows from Lemma 3.2.12.

(2) We have that $\mathrm{top}(M) \cong \mathrm{top}(P_0) \cong (\Lambda/I \otimes_\Lambda P_0) \oplus (IP_0)/\operatorname{rad} P_0$. Since $T_\bullet \cong \mathbf{F}(P_\bullet)$ it follows that $V_0 \cong \Lambda/I \otimes_\Lambda P_0$. By Lemma 3.2.13 the monomorphism $\beta : P_0 \hookrightarrow \widetilde{P}_0$ restricts to isomorphisms $IP_0 \cong I\widetilde{P}_0$ and $\operatorname{rad} P_0 \cong \operatorname{rad} \widetilde{P}_0$. It follows that $(IP_0)/\operatorname{rad} P_0 \cong I\widetilde{P}_0/\operatorname{rad} \widetilde{P}_0$.

(3) Because $\mathbf{H}_1(P_\bullet) = 0$ and $T_\bullet$ is minimal, the long exact homology sequence (3.2.6) of Λ-modules yields an exact sequence

$$0 \longrightarrow V_1 \oplus N_1 \longrightarrow \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_1 \longrightarrow M \longrightarrow V_0 \oplus N_0 \longrightarrow \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_0 \longrightarrow 0$$

Since $\Gamma/I \otimes_\Gamma \widetilde{P}_j \cong \Gamma/I \otimes_{\Lambda/I} V_j$ for $j = 0$ or 1 we obtain the dimension formula

$$\underline{\dim}(M) = \underline{\dim}\big(V_0 \oplus N_0 \oplus (\Gamma/I \underset{\Lambda/I}{\otimes} V_1)\big) - \underline{\dim}\big(V_1 \oplus N_1 \oplus (\Gamma/I \underset{\Lambda/I}{\otimes} V_0)\big).$$

By Lemma 3.1.11 we have $\underline{\dim}\, \Gamma/I \underset{\Lambda/I}{\otimes} V_j = \underline{\dim}\, V_j + \underline{\dim}\, \sigma(V_j)$ for $j = 0$ or 1. This implies the claim. □

Remark 3.3.17. *In the notations above the homological invariants of the module M do* not *depend on the gluing map $\widetilde{\vartheta}$ of the corresponding triple $T_\bullet$.*

3.4 The matrix problem of the abelian category of the Gelfand quiver

In Subsection 3.2, we defined the category of triples $\mathrm{Tri}(\Lambda)$ which is representation equivalent to the derived category $\mathrm{D^b}(\Lambda)$ for any nodal order Λ. For simplicity of notation we still assume that Λ is a nodal order of finite global dimension.

In the next step, we define a bunch of semichains $\mathfrak{B} = \mathfrak{B}(\Lambda)$ and a bimodule category $\mathrm{Rep}^*(\mathfrak{B})$ which is closely related to the category $\mathrm{Tri}(\Lambda)$:

$$\mathrm{ind}[\mathrm{D^b}(\Lambda)] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Tri}(\Lambda)] \twoheadrightarrow \mathrm{ind}[\mathrm{Rep}^*(\mathfrak{B})]$$

By this construction, the classification problem of the derived category $\mathrm{D^b}(\Lambda)$ can be reduced to a matrix problem of the bunch of semichains $\mathfrak{B}$.

Our main goal is to restrict this bijection to the abelian category of a nodal order. In this section, we give an example for the case of the Gelfand order.

3.4.1 Category of triples for the derived category of the Gelfand order

Let (Q,I) be the Gelfand quiver and $\widetilde{Q}$ its normalization:

$$(Q,I) = \begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \widetilde{Q} = \begin{tikzcd} \underset{\diamond}{\bullet} \arrow[r, bend left, "\mathsf{b}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}"] \end{tikzcd}$$

$$\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$$

Let Λ and Γ be the arrow ideal completions of the path algebras of (Q,I) respectively $\widetilde{Q}$. Then the conductor ideal of the Gelfand order Λ is given as follows:

$$I = \operatorname{ann}_\Lambda(\Gamma/\Lambda) = \{\ a \in \Lambda \mid a\,\Gamma \subseteq \Lambda\ \} = \Lambda\, e_\star\, \Lambda = \Gamma\, e_\star\, \Gamma.$$

The data Λ, Γ and I gives rise to the following conductor square:

$$\begin{tikzcd} \Lambda = \begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \arrow[r, hook] \arrow[d, two heads] & \Gamma = \begin{array}{c} \begin{array}{ccc} \widetilde{P}_\star & \widetilde{P}_\diamond & \widetilde{P}_\diamond \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \end{bmatrix} \end{array} \arrow[d, two heads] \\ \Lambda/I = \begin{array}{c} \begin{array}{cc} S_+ & S_- \end{array} \\ \begin{bmatrix} \Bbbk & 0 \\ 0 & \Bbbk \end{bmatrix} \end{array} \arrow[r, hook] & \Gamma/I = \begin{array}{c} \begin{array}{cc} \widetilde{S}_\diamond & \widetilde{S}_\diamond \end{array} \\ \begin{bmatrix} \Bbbk & \Bbbk \\ \Bbbk & \Bbbk \end{bmatrix} \end{array} \end{tikzcd}$$

Since Γ is hereditary, the indecomposable objects of the derived category $\mathrm{D}^{\mathrm{b}}(\Gamma)$ are given by shifts of projective resolutions:

$$\mathrm{D}^{\mathrm{b}}(\Gamma) = \prod_{d\in\mathbb{Z}} \operatorname{add}\Big\{\{\ \underset{d+1}{\widetilde{P}_{i_1}} \xrightarrow{\ \widehat{\mathsf{n}}\ } \underset{d}{\widetilde{P}_{i_2}} \mid i_1, i_2 \in \widetilde{Q}_0, \mathsf{n} \in \mathbb{N}^+\ \} \ \oplus\ \{\underset{d}{\widetilde{P}_i} \mid i \in \widetilde{Q}_0\}\Big\} \qquad (3.4.1)$$

In the following, we denote the path in the differential of an indecomposable resolution $\widetilde{P}_\bullet \in \mathrm{D}^{\mathrm{b}}(\Gamma)$ by its length $\widehat{\mathsf{n}} = 2\mathsf{n}$ (if $i_1 = i_2$) respectively $2\mathsf{n} - 1$ (if $i_1 \neq i_2$):

$$\widetilde{P}_{i_1} \xrightarrow{\ \widehat{\mathsf{n}}\ } \widetilde{P}_{i_2} = \begin{cases} \widetilde{P}_\diamond \xrightarrow{\ 2\mathsf{n}\ } \widetilde{P}_\diamond \\ \widetilde{P}_\star \xrightarrow{\ 2\mathsf{n}-1\ } \widetilde{P}_\diamond \\ \widetilde{P}_\diamond \xrightarrow{\ 2\mathsf{n}-1\ } \widetilde{P}_\star \\ \widetilde{P}_\star \xrightarrow{\ 2\mathsf{n}\ } \widetilde{P}_\star \end{cases} = \begin{cases} \widetilde{P}_\diamond \xrightarrow{\ \cdot(\mathsf{a}\,\mathsf{b})^{\mathsf{n}}\ } \widetilde{P}_\diamond & \text{if } i_1 = i_2 = \diamond \\ \widetilde{P}_\star \xrightarrow{\ \cdot\mathsf{b}(\mathsf{a}\,\mathsf{b})^{\mathsf{n}-1}\ } \widetilde{P}_\diamond & \text{if } i_1 = \star \ \text{ and } \ i_2 = \diamond \\ \widetilde{P}_\diamond \xrightarrow{\ \cdot(\mathsf{a}\,\mathsf{b})^{\mathsf{n}-1}\mathsf{a}\ } \widetilde{P}_\star & \text{if } i_1 = \diamond \ \text{ and } \ i_2 = \star \\ \widetilde{P}_\star \xrightarrow{\ \cdot(\mathsf{b}\,\mathsf{a})^{\mathsf{n}-1}\ } \widetilde{P}_\star & \text{if } i_1 = i_2 = \star \end{cases}$$

Let $\mathrm{Tri}(\Lambda)$ be the category of triples of the Gelfand order Λ (as introduced in Definition 3.2.1). We are going to describe the objects of $\mathrm{Tri}(\Lambda)$ in detail. Namely, we may assume that any object from the category $\mathrm{Tri}(\Lambda)$ is given by a minimal triple $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ of the following form:

(1) $V_\bullet = (V_j, d_j)_{j\in\mathbb{Z}} = (\Bbbk^{m_j^+} \times \Bbbk^{m_j^-}\ ,\ 0)_{j\in\mathbb{Z}}$ is a bounded complex of pairs of $\Bbbk$-vector spaces of some dimensions m_j^+ and $m_j^- \in \mathbb{N}_0$ and with zero differentials.

(2) $\widetilde{P}_\bullet$ is a direct sum of shifted projective resolutions in $\mathrm{D^b}(\Gamma)$ as described in (3.4.1)

(3) $\widetilde{\vartheta} = (\,\vartheta_j : \Gamma/I \otimes_{\Lambda/I} V_j \xrightarrow{\ \sim\ } \Gamma/I \otimes_\Gamma \widetilde{P}_j\,)_{j\in\mathbb{Z}}$ is given by isomorphisms of $\mathrm{Mat}_{2\times 2}(\mathbb{k})$-modules. Let us note that $\Gamma/I \otimes_{\Lambda/I} V_j \cong \widetilde{S}_\diamond^{m_j}$, where $m_j = m_j^+ + m_j^-$ for each $j \in \mathbb{Z}$. Since $\mathrm{Mat}_{2\times 2}(\mathbb{k})$ is Morita equivalent to $\mathbb{k}$, we may actually assume that the maps $(\vartheta_j)_{j\in\mathbb{Z}}$ are given by *regular matrices* $\Theta_j \in \mathrm{GL}_{m_j}(\mathbb{k})$.

3.4.2 From the classification problem of triples to a matrix problem

Next, we consider the classification problem of indecomposable objects in the category $\mathrm{Tri}(\Lambda)$ in a general setup.

First, we need to clarify when two triples are isomorphic.

(1) Let $T'_\bullet = (V'_\bullet, \widetilde{P}'_\bullet, \widetilde{\vartheta}')$ and $T''_\bullet = (V''_\bullet, \widetilde{P}''_\bullet, \widetilde{\vartheta}'')$ be two minimal triples in $\mathrm{Tri}(\Lambda)$. By definition $T'_\bullet$ is isomorphic to $T''_\bullet$ if and only if there are isomorphisms $\phi\colon V'_\bullet \xrightarrow{\ \sim\ } V''_\bullet$ in $\mathrm{D^b}(\Lambda/I)$ and $\psi\colon \widetilde{P}'_\bullet \xrightarrow{\ \sim\ } \widetilde{P}''_\bullet$ in $\mathrm{D^b}(\Gamma)$ such that for each degree $j \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} \Gamma/I \underset{\Lambda/I}{\otimes} V'_j & \xrightarrow[\sim]{\vartheta'_j} & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}'_j \\ {\scriptstyle \Gamma/I \underset{\Lambda/I}{\otimes} \phi_j}\big\downarrow \wr & & \wr\big\downarrow {\scriptstyle \Gamma/I \underset{\Gamma}{\otimes} \psi_j} \\ \Gamma/I \underset{\Lambda/I}{\otimes} V''_j & \xrightarrow[\sim]{\vartheta''_j} & \Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}''_j \end{array} \qquad \vartheta''_j = (\Gamma/I \underset{\Gamma}{\otimes} \psi_j) \cdot \vartheta'_j \cdot (\Gamma/I \underset{\Lambda/I}{\otimes} \phi_j)^{-1} \tag{3.4.2}$$

(2) Without loss of generality we may identify $V'_\bullet$ with $V''_\bullet$ and $\widetilde{P}'_\bullet$ with $\widetilde{P}''_\bullet$.

(3) Since Γ/I is a semisimple $\mathbb{k}$-algebra, it is Morita equivalent to some product of the field $\mathbb{k}$. In particular, we may view the modules in diagram (3.4.2) as $\mathbb{k}$-linear *vector spaces* and the maps between them as *matrices*.

(4) In this context, we view for each degree $j \in \mathbb{Z}$ the operation

- $\vartheta'_j \longmapsto \vartheta'_j \cdot (\Gamma/I \otimes_{\Lambda/I} \phi_j)^{-1}$ as *column transformations* of ϑ'_j, and

- $\vartheta'_j \longmapsto (\Gamma/I \otimes_\Gamma \psi_j) \cdot \vartheta'_j$ as *row transformations* of ϑ'_j.

Summarized, the triples $T'_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta}')$ and $T''_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta}'')$ are isomorphic if and only if for each degree $j \in \mathbb{Z}$ the matrix ϑ''_j can be obtained from ϑ'_j by column or row transformations which were "induced" from any automorphism ϕ of $V_\bullet$ or any automorphism ψ of $\widetilde{P}_\bullet$.

Let us also note that a triple $(V, \widetilde{P}_\bullet, \vartheta)$ is *decomposable* if and only if it is isomorphic to some direct sum $(V' \oplus V'', \widetilde{P}'_\bullet \oplus \widetilde{P}''_\bullet, \vartheta' \oplus \vartheta'')$ for two other triples $(V', \widetilde{P}'_\bullet, \vartheta') \neq 0$ and $(V'', \widetilde{P}''_\bullet, \vartheta'') \neq 0$. In particular, the decomposability of the triple $(V, \widetilde{P}_\bullet, \vartheta)$

is equivalent to decomposability of the matrices $(\vartheta_j)_{j\in\mathbb{Z}}$ via the "induced" row or column transformations described above.

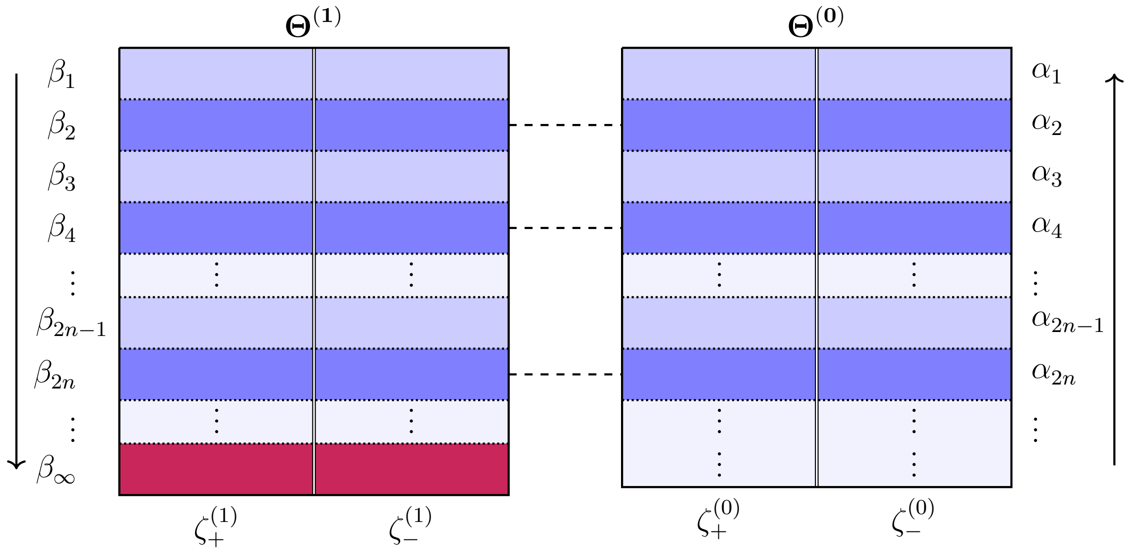
The main conclusion is that the problem to classify indecomposable triples is equivalent to the problem to find canonical forms of certain matrices using certain special column or row transformations.

3.4.3 Matrix problem of Gelfand quiver representations

In this subsection we describe the matrix problem which is essentially equivalent to the classification problem of nilpotent representations of the Gelfand quiver.

In the matrix problem of the Gelfand quiver representations, we begin with two, regular partitioned matrices $\Theta^{(0)} \in \mathrm{GL}_{m_0}(\mathbb{k})$ and $\Theta^{(1)} \in \mathrm{GL}_{m_1}(\mathbb{k})$, where $m_0, m_1 \in \mathbb{N}^+$. These matrices are depicted in Figure 3.4.1 and have the following properties:

Figure 3.4.1. **Matrix problem of the Gelfand quiver**



- the columns of the matrix $\Theta^{(1)}$ are divided into two vertical stripes labeled by $\zeta_+^{(1)}$ and $\zeta_-^{(1)}$. Similarly, the columns of $\Theta^{(0)}$ are split into vertical stripes $\zeta_+^{(0)}$ and $\zeta_-^{(0)}$.
- the rows of the matrix $\Theta^{(0)}$ are divided into horizontal stripes which are labeled by α_n, where $n \in \mathbb{N}^+$. The rows of $\Theta^{(1)}$ are split into horizontal stripes β_n, where $n \in \mathbb{N}^+$ and an additional horizontal stripe β_∞. In both matrices, only finitely many horizontal stripes have a non-zero number of rows.
- for every $n \in \mathbb{N}^+$ the even-numbered horizontal stripes α_{2n} in $\Theta^{(0)}$ and β_{2n} in $\Theta^{(1)}$ are required to have the *same* number of rows.

There is an additional constraint for technical reasons:

- there is some $n \in \mathbb{N}^+$ such that the horizontal stripe β_n in the matrix $\Theta^{(1)}$ or the horizontal stripe α_n in the matrix $\Theta^{(0)}$ is *not* empty. (3.4.3)

Let us note that any pair of vertical stripes or any pair of horizontal stripes of the matrices in Figure 3.4.1 may have *different* numbers of columns respectively rows – *except* any pair of horizontal stripes connected by a dashed edge.

The additional constraint (3.4.3) states that a regular representation may not consist of the bottom horizontal stripe β_∞ of the matrix $\Theta^{(1)}$ alone.

The following transformations of the block matrices $\Theta^{(1)}$ and $\Theta^{(0)}$ are *admissible*:

(1) elementary transformations inside stripes:

- We may perform *any elementary transformation of columns* in one of the vertical stripes $\zeta_+^{(1)}, \zeta_-^{(1)}, \zeta_+^{(0)}$ or $\zeta_-^{(0)}$.
- we may perform *any elementary transformation of rows* of the horizontal stripe β_∞ or β_{2n-1} in $\Theta^{(1)}$ for any $n \in \mathbb{N}^+$, or any elementary transformations of rows of the horizontal stripe α_{2n-1} in $\Theta^{(0)}$ for any $n \in \mathbb{N}^+$.

(2) simultaneous elementary transformations of different stripes:

- we may carry out any elementary transformation of rows in the horizontal stripe with label α_{2n} *together with the same transformation* of rows in the horizontal stripe β_{2n} for any $n \in \mathbb{N}^+$.

(3) row transformations of different stripes inside one matrix:

- in the first matrix $\Theta^{(1)}$ we may *add a scalar multiple* of any row of stripe β_m to any row of β_n with $m < n$, or to any row of stripe β_∞.
- in the second matrix $\Theta^{(0)}$ we may *add a scalar multiple* of any row of stripe α_n to any row of stripe α_m for any $m, n \in \mathbb{N}^+$ such that $m > n$.

In Figure 3.4.1. these transformations are indicated by the arrows on the left and on the right.

The **matrix problem** is the problem to find *canonical forms* for the matrices $\Theta^{(1)}$ and $\Theta^{(0)}$ using *only admissible transformations.*

Let us make a few remarks on the formal framework of the matrix problem above:

- The problem above is a matrix problem over some *bunch of semichains* $\mathfrak{B}_0$ in the sense of Definition A.1.3. The bunch of semichains $\mathfrak{B}_0$ is described below.
- A *regular representation* of the bunch of semichains $\mathfrak{B}_0$ is given by any two matrices $\Theta^{(0)}$ and $\Theta^{(1)}$ satisfying the conditions described above – without the technical constraint in (3.4.3).
- The regular representations of $\mathfrak{B}_0$ form the objects of a *bimodule category* which will be denoted by $\mathrm{Rep}^*(\mathfrak{B}_0)$.
- There is a natural notion of morphisms in $\mathrm{Rep}^*(\mathfrak{B}_0)$. However, we will only need to know that *isomorphisms* of objects in $\mathrm{Rep}^*(\mathfrak{B}_0)$ are given exactly by the *admissible transformations* of matrices.
- The category $\mathrm{Rep}^*(\mathfrak{B}_0)$ is additive and has the Krull-Remak-Schmidt-property.

For the complete definition of the bimodule category $\operatorname{Rep}^{*}(\mathfrak{B}_0)$ and its properties we refer to [**BD04**, Appendix B] or [**BD**, Section 6].

Let us consider the bunch of semichains $\mathfrak{B}_0$ in more detail.

Remark 3.4.1. *The structure $\mathfrak{B} = \big(\mathfrak{C}, \mathfrak{R}, \approx\big)$ corresponding to the matrix problem in 3.4.1 is defined by the following data:*

(1) The set of column labels $\mathfrak{C}$ is given by $\mathfrak{C} = \mathfrak{C}^{(0)} \cup \mathfrak{C}^{(1)}$ where $\mathfrak{C}^{(d)} = \{\zeta_+^{(d)}, \zeta_-^{(d)}\}$ is a partially ordered set with two incomparable elements for $d = 0$ or 1.

(2) The set of row labels $\mathfrak{R}$ is given by $\mathfrak{R} = \mathfrak{R}^{(0)} \cup \mathfrak{R}^{(1)}$ with the totally ordered sets

$$\mathfrak{R}^{(0)} = \{\ \alpha_1 > \alpha_2 > \ldots > \alpha_n > \alpha_{n+1} > \ldots\ \} \quad \textit{and}$$
$$\mathfrak{R}^{(1)} = \{\ \beta_1 < \beta_2 < \ldots < \beta_n < \beta_{n+1} < \ldots < \beta_\infty\ \}.$$

The underlying set of the bunch of semichains is also denoted by $\mathfrak{B}$ and given by the set of all labels $\mathfrak{B} = \mathfrak{C} \cup \mathfrak{R}$.

(3) There is an equivalence relation on $\mathfrak{B}$ given by $\alpha_{2n} \approx \beta_{2n}$ for any $n \in \mathbb{N}^+$.

This data defines a bunch of semichains $\mathfrak{B}$ *in the sense of Definition A.1.3.*

In the following, we will denote the full subcategory of regular representations of $\mathfrak{B}$ satisfying the additional condition (3.4.3) *by $\operatorname{Rep}^{\circ}(\mathfrak{B}_0)$. The indecomposable objects of $\operatorname{Rep}^{\circ}(\mathfrak{B}_0)$ will be also called canonical forms of $\mathfrak{B}_0$.*

Next, we are going to describe the relationship between the bunch of semichains $\mathfrak{B}_0$ and the Gelfand quiver (Q, I).

As usual, by a nilpotent representation of the Gelfand quiver we will mean a $\Bbbk$-linear, nilpotent and finite-dimensional representation.

For any $\mathtt{n} \in \mathbb{N}^+$ let $W_{\mathtt{n}}$ be the following nilpotent representation of the Gelfand quiver:

$$W_{\mathtt{n}} = \begin{tikzcd} \Bbbk^{\mathtt{n}} \arrow[r, bend left, "\mathsf{J}"] & \Bbbk^{\mathtt{n}} \arrow[l, bend left, "\mathsf{Id}"] \arrow[r, bend right, "\mathsf{Id}"'] & \Bbbk^{\mathtt{n}} \arrow[l, bend right, "\mathsf{J}"'] \end{tikzcd} \qquad \text{where } \mathtt{n} \in \mathbb{N}^+ \tag{3.4.4}$$

In this context, the matrix J denotes the nilpotent Jordan block of size n.

With the terminology above we can formulate one of the main statements of this chapter:

Proposition 3.4.2. *Let (Q, I) be the Gelfand quiver and $\mathfrak{B}_0$ be the bunch of semichains of Remark 3.4.1.*
There is a bijection between the isomorphism classes of indecomposable nilpotent representations of the Gelfand quiver and the isomorphism classes of canonical forms over $\mathfrak{B}_0$ together with isomorphism classes of nilpotent representations W_n of (3.4.4):

$$\operatorname{ind}[\ \text{nil.}\operatorname{rep}(Q, I)\ ] \xleftrightarrow{1:1} \operatorname{ind}[\ \operatorname{Rep}^{\circ}(\mathfrak{B}_0)\ ] \cup \operatorname{ind}[\ W_{\mathtt{n}} \mid \mathtt{n} \in \mathbb{N}^+\ ]$$

Summarized, the classification problem of nilpotent representations over the Gelfand quiver is essentially equivalent to the matrix problem in Figure 3.4.1.

The proof of this statement is one of the goals of the present section. Actually, we will prove this statement in the setup of an arbitrary nodal order.

Remark 3.4.3. *The first formulation of the classification problem of the Gelfand quiver in terms of some bunch of semichains is due to Nazarova and Roiter* [**NR73**].

3.5 Matrix problems of a nodal order

Next, we describe the bunch of semichains $\mathfrak{B}$ associated to the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of any nodal order. The matrix problem of $\mathfrak{B}$ is essentially equivalent to the classification problem of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

In subsection 3.5.5 we describe also the matrix problem corresponding to the classification problem of indecomposable finite-dimensional Λ-modules.

3.5.1 The bunch of semichains for the derived category of a nodal order

The construction of the bunch of semichains $\mathfrak{B}_\Lambda$ can be found in Section 5 of the paper [**BD04**].

For the sake of completeness, we recall this construction in an equivalent terminology.

Let us assume for simplicity that Λ is a *basic* nodal order. We fix the following notation:

- Let Γ be the normalization of Λ and $I = \mathrm{ann}_\Lambda(\Gamma/\Lambda)$ be the conductor ideal.
- Let $\widetilde{R}$ denote an index set for the isomorphism classes of minimal indecomposable complexes in the derived category $\mathrm{D}^{\mathrm{b}}(\Gamma)$.

Since Γ is a hereditary order, its derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is representation-discrete and the set $\widetilde{R}$ is countable. Let us also recall that the quotient algebras Λ/I and Γ/I are semi-simple in our setup.

- Let C be the set of isomorphism classes of simple Λ/I-modules.
- Similarly, let $\widetilde{C}$ be an index set for the simple Γ/I-modules.
- Let $\widetilde{Q}_0$ denote the set of isomorphism classes of all simple Γ-modules.

For the general definition of bunches of semichains we refer to Definition A.1.3.

The bunch of semichains $\mathfrak{B} = \mathfrak{B}(\Lambda)$ of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is given by the data $\mathfrak{B} = (\mathfrak{C}, \mathfrak{R}, \approx)$ which is defined by the following steps:

(1) The index set I of $\mathfrak{B}$ is given by $I = \mathbb{Z} \times \widetilde{C}$. This means that the set of row labels $\mathfrak{C}$ and the set of column labels $\mathfrak{R}$ are given by the unions

$$\mathfrak{C} = \bigcup_{d\in\mathbb{Z}} \mathfrak{C}^{(d)} = \bigcup_{d\in\mathbb{Z}} \bigcup_{\iota\in\widetilde{C}} \mathfrak{C}^{(d)}_\iota \quad \text{and} \quad \mathfrak{R} = \bigcup_{d\in\mathbb{Z}} \mathfrak{R}^{(d)} = \bigcup_{d\in\mathbb{Z}} \bigcup_{\iota\in\widetilde{C}} \mathfrak{R}^{(d)}_\iota,$$

where the sets $\mathfrak{C}^{(d)}_\iota$ and $\mathfrak{R}^{(d)}_\iota$ have yet to be defined. In addition, we also have to introduce an equivalence relation $\approx$ on the set of all labels $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C}$.

(2) The sets of column labels $\mathfrak{C}^{(d)}_\iota$ are defined by the following procedure.
Let $i \in C$. There are two cases for the corresponding simple module S_i:

- The module S_i is *glued*, that is $\Gamma/I \otimes S_i \cong \widetilde{S}_{i'} \oplus \widetilde{S}_{i''}$ for some pair $i', i'' \in \widetilde{C}$. In this case, we set $\mathfrak{C}^{(d)}_{i'} = \{ \zeta^{(d)}_{i'} \}$, $\mathfrak{C}^{(d)}_{i''} = \{ \zeta^{(d)}_{i''} \}$, and $\zeta^{(d)}_{i'} \approx \zeta^{(d)}_{i''}$ for any $d \in \mathbb{Z}$.
- The module S_i is *special*. Let us denote $j^+ = i$. Then there is some index $j^- \in C$ such that $\Gamma/I \otimes S_{j^+} \cong \Gamma/I \otimes S_{j^-} \cong \widetilde{S}_j$ for some index $j \in \widetilde{C}$.
In this case we define a semichain $\mathfrak{C}^{(d)}_j = \{ \zeta^{(d)}_{j^+}, \zeta^{(d)}_{j^-} \}$ for any $d \in \mathbb{Z}$. In other words, the row labels $\zeta^{(d)}_{j^+}$ and $\zeta^{(d)}_{j^-}$ are not comparable.

This procedure is carried out for each vertex $i \in C$.

In this way, we obtain a bijection between the isomorphism classes of simple Λ/I-modules and the equivalence classes of $\mathfrak{C}^{(d)}$ for each $d \in \mathbb{Z}$:

$$\mathfrak{C}^{(d)}/_{\approx} \xleftrightarrow{1:1} C \xleftrightarrow{1:1} \operatorname{ind}[\Lambda/I\text{ - mod}] \tag{3.5.1}$$

(3) Next, we build the sets of row labels $\mathfrak{R}^{(d)}_\iota$ for each $d \in \mathbb{Z}$ and $\iota \in \widetilde{C}$.
Let $\varrho \in \widetilde{R}$ and $\widetilde{P}_\bullet[\varrho]$ denote the corresponding indecomposable complex in $\mathrm{D}^{\mathrm{b}}(\Gamma)$.
There are two cases concerning the homology of $\widetilde{P}_\bullet[\varrho]$:

(a) If $\widetilde{P}_\bullet \notin \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$, then $\widetilde{P}_\bullet[\varrho] = \underset{d}{\widetilde{P}_\iota}$ is given by the indecomposable projective Γ-module of some vertex $\iota \in \widetilde{Q}_0$ and shift $d \in \mathbb{Z}$.

In this case, the projective $\widetilde{P}_\bullet[\varrho]$ contributes one or zero row labels to the set $\mathfrak{R}$:

- if $\iota \in \widetilde{C}$, then $\Gamma/I \otimes \widetilde{P}_\bullet = \widetilde{S}_\iota$ and we include a symbol $\beta^{(d)}_{\iota,\infty}$ into the row label set $\mathfrak{R}^{(d)}_\iota$ with vertex ι and degree d.
- if $\iota \notin \widetilde{C}$, then $\Gamma/I \otimes \widetilde{P}_\bullet[\varrho] = 0$ and there is no contribution to the set $\mathfrak{R}$.

(b) If $\widetilde{P}_\bullet[\varrho] \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma)$, then $\widetilde{P}_\bullet[\varrho] = \underset{d+1}{\widetilde{P}_{\iota_1}} \xrightarrow{\phi} \underset{d}{\widetilde{P}_{\iota_2}}$ is given by a projective resolution for some degree $d \in \mathbb{Z}$, vertices $\iota_1, \iota_2 \in \widetilde{Q}_0$ and some differential ϕ.

In this case, the resolution $\widetilde{P}_{\bullet}[\varrho]$ contributes the following row labels to :

	$\Gamma/I \otimes \widetilde{P}_{\bullet}[\varrho]$	row labels in $\mathfrak{R}$	$\delta(\widetilde{P}_{\bullet}[\varrho])$
• if $\iota_1 \in \widetilde{C}, \iota_2 \in \widetilde{C}$:	$\widetilde{S}_{\iota_1} \xrightarrow{0} \widetilde{S}_{\iota_2}$	$\beta^{(d+1)}_{\iota_1,\varrho} \approx \alpha^{(d)}_{\iota_2,\varrho}$	0
• if $\iota_1 \in \widetilde{C}, \iota_2 \notin \widetilde{C}$:	$\widetilde{S}_{\iota_1} \longrightarrow 0$	$\beta^{(d+1)}_{\iota_1,\varrho}$	1
• if $\iota_1 \in \widetilde{C}, \iota_2 \in \widetilde{C}$:	$0 \longrightarrow \widetilde{S}_{\iota_2}$	$\alpha^{(d)}_{\iota_2,\varrho}$	1
• if $\iota_1 \notin \widetilde{C}, \iota_2 \notin \widetilde{C}$:	$0 \longrightarrow 0$	(none)	2

(3.5.2)

More precisely, any row label of the form $\beta^{(d+1)}_{\iota_1,\varrho}$ above is included into the label set $\mathfrak{R}^{(d+1)}_{\iota_1}$, while any label of type $\alpha^{(d)}_{\iota_2,\varrho}$ is included into $\mathfrak{R}^{(d)}_{\iota_2}$. In the first case, we set up an equivalence relation between the two elements.

For later use, we also keep track of the defect δ of $P_{\bullet}[\varrho]$ in (3.5.2).

This procedure is carried out for every index $\varrho \in \widetilde{R}$.

In other words, there is the following bijection between row labels and isomorphism classes of "induced" simple Γ/I-modules:

$$\mathfrak{R} \xleftrightarrow{1:1} \{ \text{ simple summands of } \Gamma/I \otimes \widetilde{P}_{\bullet}[\varrho] \mid \varrho \in \widetilde{R} \ \}. \tag{3.5.3}$$

Moreover, the procedure above sets up a bijection between the equivalence classes of $\mathfrak{R}$ and isomorphism classes of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Gamma)$:

$$\mathfrak{R}/_{\approx} \xleftrightarrow{1:1} \widetilde{R} \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Gamma)] \tag{3.5.4}$$

(4) At last, we need to introduce an *order relation* on the row label set $\mathfrak{R}$. This relation is induced by the morphisms of the category $\mathrm{D}^{\mathrm{b}}(\Gamma)$ in the following way.

Let ψ be a morphism of two non-isomorphic complexes in $\mathrm{D}^{\mathrm{b}}(\Gamma)$ such that $\Gamma/I \otimes \psi \neq 0$.

(a) In this case, there are some indices $\varrho', \varrho'' \in \widetilde{R}$ such that $\varrho' \neq \varrho''$, and some degree $d \in \mathbb{Z}$ such that $\psi : \widetilde{P}_{\bullet}[\varrho'] \longrightarrow \widetilde{P}_{\bullet}[\varrho'']$ and $\Gamma/I \otimes \psi_d \neq 0$.

(b) Moreover, there is some label $\gamma \in \mathfrak{R}^{(d)}$ which is the index of the simple module $(\Gamma/I \otimes \widetilde{P}_{\bullet}[\varrho'])_d$ by the bijection (3.5.3). Similarly, there is some row label $\delta \in \mathfrak{C}^{(d)}$ which corresponds to the simple module $(\Gamma/I \otimes \widetilde{P}_{\bullet}[\varrho''])_d$.

(c) With the notation above, we set $\gamma < \delta$.

This procedure is carried out for every morphism ψ of two non-isomorphic indecomposable complexes such that $\Gamma/I \otimes \psi \neq 0$.

The outcome is the bunch of semichains $\mathfrak{B} = \mathfrak{B}(\Lambda)$ associated to derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of the nodal order Λ.

Remark 3.5.1. *For any nodal Λ the bunch of semichains $\mathfrak{B}(\Lambda)$ has the following properties:*

- *For any $d \in \mathbb{Z}$ the set of row labels $\mathfrak{R}^{(d)}$ is the disjoint union of three subsets*

$$\mathfrak{R}^{(d)} = \mathfrak{R}^{(d)}_{\beta} \cup \mathfrak{R}^{(d)}_{\infty} \cup \mathfrak{R}^{(d)}_{\alpha}, \tag{3.5.5}$$

where $\mathfrak{R}^{(d)}_{\beta}$ *is given by labels of type* $\beta^{(d)}_{\iota,\varrho}$*, the set* $\mathfrak{R}^{(d)}_{\infty}$ *by labels of type* $\beta_{\infty,\varrho}$ *and similarly* $\mathfrak{R}^{(d)}_{\iota,\alpha}$ *contains only labels* $\alpha^{(d)}_{\iota,\varrho}$ *for certain* $\iota \in \widetilde{C}$ *and* $\varrho \in \widetilde{R}$*.*

- *More precisely, for any* $d \in \mathbb{Z}$ *and* $\iota \in \mathbb{Z}$ *each row label set* $\mathfrak{R}^{(d)}_{\iota}$ *has a partition*

$$\mathfrak{R}^{(d)}_{\iota} = \mathfrak{R}^{(d)}_{\iota,\beta} \cup \mathfrak{R}^{(d)}_{\iota,\infty} \cup \mathfrak{R}^{(d)}_{\iota,\alpha},$$

into three sets of labels of different types. The set $\mathfrak{R}^{(d)}_{\iota,\infty}$ *is given by one single element* $\beta^{(d)}_{\iota,\infty}$*. Since* Γ *is a hereditary order, the row label set* $\mathfrak{R}^{(d)}_{\iota}$ *is a* chain *such that* $\beta^{(d)}_{\iota,\varrho'} < \beta^{(d)}_{\iota,\infty} < \alpha^{(d)}_{\iota,\varrho''}$ *for any triple of labels from the three disjoint sets above.*

For later use, we define also a slightly smaller bunch of semichains $\mathfrak{B}_{\mathrm{fd}} = \mathfrak{B}_{\mathrm{fd}}(\Lambda)$. This will be the bunch of semichains associated to the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ of complexes with finite-dimensional homology.

- The bunch of semichains $\mathfrak{B}_{\mathrm{fd}}$ is obtained from the bunch of semichains $\mathfrak{B}$ by deleting of all labels of type $\beta_{\infty,\varrho}$.

Equivalently, the bunch of semichains $\mathfrak{B}_{\mathrm{fd}}$ is constructed by the same rules as the bunch of semichains $\mathfrak{B}$, but using only complexes $\widetilde{P}_{\bullet}[\varrho]$ from $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma)$ with finite-dimensional homology.

Remark 3.5.2. *In the terminology of Subsection A.2.1 a label* α *of a bunch of semichains is* free *if* α *is only equivalent to itself, and* α *is not contained in the two-point link of some semichain. Let* $\mathfrak{B}_{\mathrm{fd}} = \mathfrak{B}_{\mathrm{fd}}(\Lambda)$ *be the bunch of semichains of a derived category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ *of a nodal order* Λ*. Then the* free *labels in* $\mathfrak{B}$ *arise only in the second and the third case of* (3.5.2)*. That is, any indecomposable object* $\widetilde{P}_{\bullet}[\varrho] \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma)$ *of defect* one *induces a free label:*

$$\{ \textit{ free labels in } \mathfrak{B}_{\mathrm{fd}} \ \} \xleftrightarrow{1:1} \{ \ \widetilde{P}_{\bullet}[\varrho] \ \ \textit{has defect} \text{ one } \ \mid \varrho \in \widetilde{R} \ \}. \tag{3.5.6}$$

This observation will play a role in the characterization of strings and bands by functorial terms.

Next, we consider some examples of the construction above.

Example 3.5.3. *Let* Λ *be the Gelfand order. Its normalization* Γ *is Morita equivalent to the* two-cycle quiver*. An explicit description of the indecomposable objects of the derived category* $\mathrm{D}^{\mathrm{b}}(\Gamma)$ *and their morphisms can be found in Section B.2.5.*

Using this description, the bunch of semichains $\mathfrak{B} = \big(\mathfrak{C}, \mathfrak{R}, \approx\big)$ *associated to the derived category* $\mathrm{D}^{\mathrm{b}}(\Lambda)$: *can be determined as follows*:

(1) The set of column labels $\mathfrak{C}$ *is given by* $\mathfrak{C} = \bigcup_{d\in\mathbb{Z}} \mathfrak{C}^{(d)}$ *where* $\mathfrak{C}^{(d)} = \{\zeta^{(d)}_{+}, \zeta^{(d)}_{-}\}$ *is a semichain for each* $d \in \mathbb{Z}$*.*

(2) The row labels $\mathfrak{R}$ *are given by* $\mathfrak{R} = \bigcup_{d\in\mathbb{Z}} \mathfrak{R}^{(d)}$ *where for each* $d \in \mathbb{Z}$ *the set of row labels* $\mathfrak{R}^{(d)}$ *is the chain*

$$\mathfrak{R}^{(d)} = \{ \ \beta^{(d)}_{1} < \ldots < \beta^{(d)}_{n} < \beta^{(d)}_{n+1} < \ldots < \beta^{(d)}_{\infty} < \ldots < \alpha^{(d)}_{n+1} < \alpha^{(d)}_{n} < \ldots < \alpha^{(d)}_{1} \ \}$$

(3) The equivalence relation on $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C}$ *is given by* $\alpha^{(d)}_{2n} \approx \beta^{(d+1)}_{2n}$ *for any* $n \in \mathbb{N}^{+}$ *and any* $d \in \mathbb{Z}$*.*

The matrix problem over the bunch of semichains $\mathfrak{B}$ is depicted in Figure 3.5.1.

The bunch of semichains $\mathfrak{B}_{\mathrm{fd}}$ of the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ is obtained from the bunch $\mathfrak{B}$ above by deleting the elements $\beta_{\infty}^{(d)}$ for all $d \in \mathbb{Z}$.

Remark 3.5.4. *The matrix problem of the nilpotent representations of the Gelfand quiver in Figure 3.4.1 can be viewed as a subproblem of the matrix problem of the derived category of the Gelfand order Λ in Figure 3.5.1.*

In other words, the bunch of semichains $\mathfrak{B}_0$ for the Gelfand quiver is a truncation *of the bunch of semichains $\mathfrak{B}$ for the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.*

We will see in Theorem 3.5.18 that this holds for any nodal order Λ.

Example 3.5.5. *Next, we consider the bunches of semichains associated to the derived category of some Khoroshkin order.*

(1) *Let Λ_{2m+2} be some Khoroshkin order for some $m \in \mathbb{N}_0$ as defined in Subsection 2.1.2.*

   Let $\mathfrak{B}_{2m+2} = \big(\mathfrak{C}, \mathfrak{R}, \approx\big)$ be the bunch of semichains of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda_{2m+2})$. It is given as follows:

   (a) *The set of column labels $\mathfrak{C}$ is given by*

$$\mathfrak{C} = \bigcup_{d\in\mathbb{Z}} \bigcup_{i=1}^{2m+1} \mathfrak{C}_i^{(d)} \text{ where } \quad \mathfrak{C}_i^{(d)} = \{\zeta_i^{(d)}\} \text{ for any } 1 \leq i \leq 2m$$

   and $\mathfrak{C}_{2m+1}^{(d)} = \{\zeta_+^{(d)}, \zeta_-^{(d)}\}$ is a semichain for any $d \in \mathbb{Z}$.

   (b) *The set of row labels $\mathfrak{R}$ are given by*

$$\mathfrak{R} = \bigcup_{d\in\mathbb{Z}} \bigcup_{i=1}^{2m+1} \mathfrak{R}_i^{(d)} \text{ where } \mathfrak{R}_i^{(d)} \text{ is the chain}$$

$$\mathfrak{R}_i^{(d)} = \{\ \beta_{i,1}^{(d)} < \ldots < \beta_{i,n}^{(d)} < \beta_{i,n+1}^{(d)} < \ldots < \beta_{i,\infty}^{(d)} < \ldots < \alpha_{i,n+1}^{(d)} < \alpha_{i,n}^{(d)} < \ldots < \alpha_{i,1}^{(d)}\ \}$$

   for each $d \in \mathbb{Z}$ and $1 \leq i \leq 2m+1$.

   (c) *The equivalence relation on $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C}$ is given by $\alpha_{i,2n}^{(d)} \approx \beta_{i,2n}^{(d+1)}$ for any $1 \leq i \leq 2m+1$ and $d \in \mathbb{Z}$, and by*

$$\zeta_{2i-1}^{(d)} \approx \zeta_{2i}^{(d)}, \quad \alpha_{2i,2n-1}^{(d)} \approx \beta_{2i+1,2n-1}^{(d+1)} \text{ and } \alpha_{2i+1,2n-1}^{(d)} \approx \beta_{2i,2n-1}^{(d+1)}$$

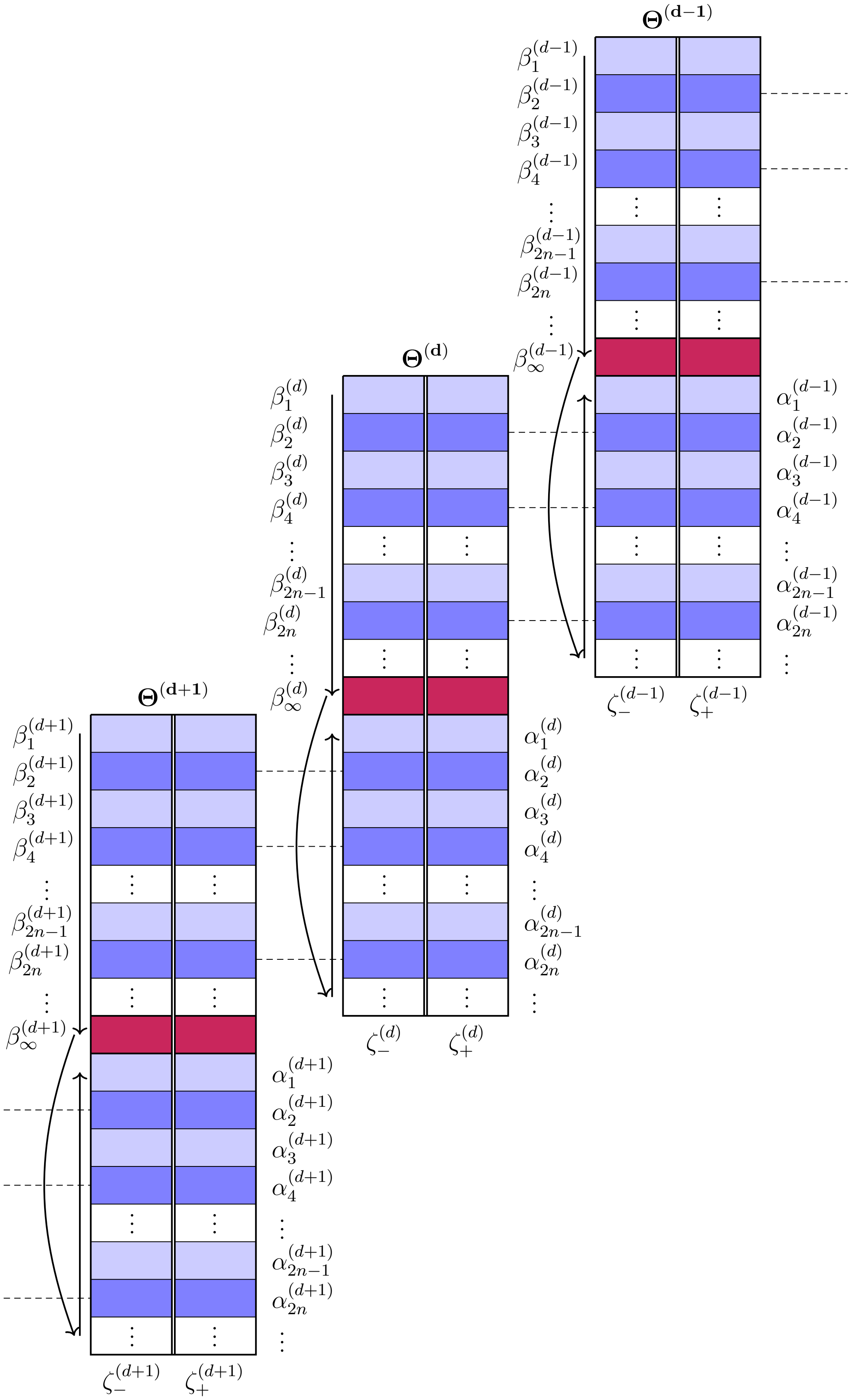
   for any $1 \leq i \leq m$, $n \in \mathbb{N}^+$ and $d \in \mathbb{Z}$.

   Since any column label set $\mathfrak{C}_{2m+1}^{(d)}$ is a semichain, the bunch $\mathfrak{B}_{2m+2}$ is a proper *bunch of semichains.*

(2) *Let Λ_{2m+1} be the Khoroshkin order for some $m \in \mathbb{N}_0$. The bunch of semichains $\mathfrak{B}_{2m+1}$ of its derived category can be obtained from $\mathfrak{B}_{2m+2}$ by deleting the sets $\mathfrak{C}_{2m+1}^{(d)}$ and $\mathfrak{R}_{2m+1}^{(d)}$ and setting $\alpha_{2m,n}^{(d)} \approx \beta_{2m,n}^{(d+1)}$ for any $n \in \mathbb{N}^+$ and $d \in \mathbb{Z}$.*

   In particular, $\mathfrak{B}_{2m+1}$ is actually a bunch of chains.

FIGURE 3.5.1. **Matrix problem for the derived category** $\mathrm{D}^{\mathrm{b}}(\Lambda)$



3.5.2 Projective complexes, triples and matrices

Next, we describe the relationship between the nodal order Λ and its associated bunches of semichains $\mathfrak{B}$ and $\mathfrak{B}_{\mathrm{fd}}$.

More precisely, this is a relationship between the following three categories:

- the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of the nodal order Λ,
- the category of triples $\mathrm{Tri}(\Lambda)$, and
- the category $\mathrm{Rep}^{*}(\mathfrak{B})$ of regular representations of the bunch of semichains $\mathfrak{B}$.

Definition 3.5.6. *The* homology functor *on triples* $\mathbf{H} : \mathrm{Tri}(\Lambda) \longrightarrow \mathrm{Rep}^{*}(\mathfrak{B})$ *is defined as follows*:

(1) for any triple $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta}) \in \mathrm{Tri}(\Lambda)$ *we set* $\mathbf{H}(T_\bullet) = \big(\mathbf{H}_j(\widetilde{\vartheta})\big)_{j\in\mathbb{Z}}$,

(2) for any morphism $(\phi, \psi) : (V'_\bullet, \widetilde{P}'_\bullet, \widetilde{\vartheta}') \longrightarrow (V''_\bullet, \widetilde{P}''_\bullet, \widetilde{\vartheta}'')$ *of triples in* $\mathrm{Tri}(\Lambda)$ *we set* $\mathbf{H}(\phi, \psi) = \big(\mathbf{H}_j(\Gamma/I \otimes_{\Lambda/I} \phi), \mathbf{H}_j(\Gamma/I \otimes_{\Gamma/I} \psi)\big)_{j\in\mathbb{Z}}$.

Remark 3.5.7. *If* T *is a minimal triple, then* $\mathbf{H}_j(\widetilde{\vartheta}) = \widetilde{\vartheta}_j$ *for any* $j \in \mathbb{Z}$.

In other words, the homology functor maps triples to partitioned matrices, which are representations of the bunch of semichains $\mathfrak{B}$.

The relationship between the category of triples $\mathrm{Tri}(\Lambda)$ and the category $\mathrm{Rep}^{*}(\mathfrak{B})$ of matrix representations is described by the following Proposition.

Proposition 3.5.8. [**BD04**] *The functor* $\mathbf{H} : \mathrm{Tri}(\Lambda) \longrightarrow \mathrm{Rep}^{*}(\mathfrak{B})$ *is additive, full and dense.*

The approach of [**BD04**] to the classification problem for $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is based on the following diagram of categories and functors

$$\mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\mathbf{F}} \mathrm{Tri}(\Lambda) \xrightarrow{\mathbf{H}} \mathrm{Rep}^{*}(\mathfrak{B})$$

Since the composition of functors $\mathbf{H} \circ \mathbf{F}$ is dense and full, there is an equivalence of categories:

$$\mathrm{D}^{\mathrm{b}}(\Lambda)/\ker(\mathbf{H} \circ \mathbf{F}) \xrightarrow{\sim} \mathrm{Rep}^{*}(\mathfrak{B})$$

In other words, the canonical forms of the matrix problem over $\mathfrak{B}$ correspond to all indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ - except those contained in the kernel of the composition $\mathbf{H} \circ \mathbf{F}$. To describe this kernel we need the following notation:

- Let $e_\star \in \Lambda$ be the idempotent of the *neutral semi-simple* module $S_\star$. The module $S_\star$ was introduced in Definition 3.1.8).

In particular, the conductor ideal is given by $I = \mathrm{rad}\,\Lambda + \Lambda e_\star \Lambda = \mathrm{rad}\,\Gamma + \Gamma e_\star \Gamma$.

- We set $\Lambda_\star = e_\star \Lambda e_\star$ and $\Gamma_\star = e_\star \Gamma e_\star$.

Let us note that the ring $\Lambda_\star$ can be identified with $\Gamma_\star$, and both rings are hereditary orders. In particular, the category $\mathrm{D}^{\mathrm{b}}(\Lambda_\star)$ is a *representation-discrete* subcategory of the tame category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ for any nodal order Λ.

The kernels of the functors $\mathbf{H}$ and $\mathbf{H}\circ\mathbf{F}$ are given as follows:

Lemma 3.5.9. *In the notations above, there are equalities and equivalences of the following categories*:

$$\begin{tikzcd} \ker \mathbf{H}\circ\mathbf{F} \arrow[r, equal] \arrow[d, "\wr"] & \ker \Lambda/I \underset{\Lambda}{\otimes} \_ \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{K}^{\mathrm{b}}(\operatorname{add} P_\star) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{D}^{\mathrm{b}}(\Lambda_\star) \arrow[d, "\wr"] \\ \ker \mathbf{H} \arrow[r, "\sim"] & \ker \Gamma/I \underset{\Gamma}{\otimes} \_ \arrow[r, "\sim"] & \mathrm{K}^{\mathrm{b}}(\operatorname{add} \widetilde{P}_\star) \arrow[r, "\sim"] & \mathrm{D}^{\mathrm{b}}(\Gamma_\star) \end{tikzcd}$$

In particular, any category in the above diagram is representation-discrete*.*

The proof of the lemma above is simple but rather formal.

Example 3.5.10. *Let Λ be the Gelfand order. In this case, $\Lambda_\star = \Gamma_\star = \Bbbk[\![t]\!]$ is the formal power series ring, Moreover, the indecomposable objects in the kernel of the composition $\mathbf{H}\circ\mathbf{F}$ are given by the following complexes*:

$$\ker \mathbf{H}\circ\mathbf{F} \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\operatorname{add} P_\star) = \prod_{d\in\mathbb{Z}} \operatorname{add}\big\{\{\underset{\mathtt{d+1}}{P_\star} \xrightarrow{\mathtt{2n}} \underset{\mathtt{d}}{P_\star}, \mid \mathtt{n}\in\mathbb{N}^+\} \oplus \{\underset{\mathtt{d}}{P_\star}\}\big\}$$

As indicated in the Lemma above, there is only a formal difference between the kernels of the functors $\mathbf{H}\circ\mathbf{F}$ and $\mathbf{H}$:

$$\ker \mathbf{H} \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\operatorname{add} \widetilde{P}_\star) = \prod_{d\in\mathbb{Z}} \operatorname{add}\big\{\{\underset{\mathtt{d+1}}{\widetilde{P}_\star} \xrightarrow{\mathtt{2n}} \underset{\mathtt{d}}{\widetilde{P}_\star}, \mid \mathtt{n}\in\mathbb{N}^+\} \oplus \{\underset{\mathtt{d}}{\widetilde{P}_\star}\}\big\}$$

Remark 3.5.11. *There are nodal orders Λ without neutral simple modules. In this case $\Lambda_\star = \Gamma_\star = 0$.*

The following Theorem was shown by Burban and Drozd [**BD04**] in a different formulation:

Theorem 3.5.12 ([**BD04**])**.** *Let Λ be a nodal order, Γ its normalization and $\mathfrak{B}$ be the bunch of semichains associated to the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.*
There is a bijection between the "isomorphism classes of indecomposable objects in the category $\mathrm{D}^{\mathrm{b}}(\Lambda)$" and the "isomorphism classes of canonical forms in $\operatorname{Rep}^*(\mathfrak{B})$" together with the "isomorphism classes of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Gamma_\star)$":

$$\operatorname{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \xleftrightarrow{1:1} \operatorname{ind}[\operatorname{Rep}^*(\mathfrak{B})] \cup \operatorname{ind}[\mathrm{D}^{\mathrm{b}}(\Gamma_\star)] \tag{3.5.7}$$

Corollary 3.5.13. *The bijection in* (3.5.7) *restricts to isomorphism classes of objects with finite-dimensional homology and the between the isomorphism classes of indecomposable objects with finite-dimensional homology and canonical forms of the bunch of semichains $\mathfrak{B}_{\mathrm{fd}}$*

$$\operatorname{ind}[\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)] \xleftrightarrow{1:1} \operatorname{ind}[\mathrm{Tri}_{\mathrm{fd}}(\Lambda)] \xleftrightarrow{1:1} \operatorname{ind}[\operatorname{Rep}^*(\mathfrak{B}_{\mathrm{fd}})] \cup \operatorname{ind}[\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma_\star)] \tag{3.5.8}$$

PROOF. This follows from Theorem 3.5.12 together with Corollary 3.2.16. $\square$

The canonical forms of bunches of semichains have been described by Bondarenko in [**Bon88, Bon91**]. We give a summary of his results in Appendix A. In particular, there is the following consequence:

Corollary 3.5.14. *There is a bijection between the isomorphism classes of indecomposable objects in* $\mathrm{D^b}(\Lambda)$ *and regular strings and bands of* $\mathfrak{B}$ *together with the isomorphism classes of indecomposable objects in* $\mathrm{D^b}(\Gamma_\star)$*:*

$$\mathrm{ind}[\mathrm{D^b}(\Lambda)] \xleftrightarrow{1:1} [\text{STRINGS}^* \textit{ and } \text{BANDS } \textit{of } \mathfrak{B}] \cup \mathrm{ind}[\mathrm{D^b}(\Gamma_\star)]$$

3.5.3 From matrices to triples

Let $\Theta = (\Theta^{(d)})_{d\in\mathbb{Z}}$ be a family of matrices from $\mathrm{Rep}^*(\mathfrak{B})$. We define a triple $\mathbf{I}(M) = (V_\bullet, P_\bullet, \vartheta) \in \mathrm{Tri}(\Lambda)$ in the following way:

(1) the complex $V_\bullet \in \mathrm{D^b}(\Lambda/I)$:

- Let $d \in \mathbb{Z}$. We define the semi-simple module V_d as follows.

  For each column label $\zeta \in \mathfrak{C}^{(d)}$ let $m_{\zeta,d}$ be the number of columns of the vertical stripe labeled by ζ in the matrix $\Theta^{(d)}$.

  For every equivalence class $[\zeta]$ of $\overline{\mathfrak{C}} = \mathfrak{C}/\approx$ let $S[\zeta]$ denote the corresponding simple Λ/I-module according to the bijection (3.5.1).

  We include the module $S[\zeta]$ with multiplicity $m_{\zeta,d}$ into the semi-simple Λ/I-module V_d :

$$V_d = \bigoplus_{[\zeta]\in\overline{\mathfrak{C}}} S[\zeta]^{m_{\zeta,d}}$$

  This is well-defined, since $m_{\zeta',d} = m_{\zeta'',d}$ whenever $\zeta' \approx \zeta''$ in $\mathfrak{B}$ by Definition of the representation Θ.

The above construction is carried out for every degree $d \in \mathbb{Z}$.

- We set the differentials in $V_\bullet$ to be zero. That is, $V_\bullet$ is the complex $V_\bullet = \big((V_d)_{d\in\mathbb{Z}}, 0\big)$.

(2) the complex $\widetilde{P}_\bullet \in \mathrm{D^b}(\Gamma)$ is defined in a similar way as the complex $V_\bullet$:

For every degree $d \in \mathbb{Z}$ and every row label $\varrho \in \mathfrak{R}^{(d)}$ let m_ϱ be the number of rows of the horizontal stripe labeled by ϱ in the matrix $\Theta^{(d)}$.

For every equivalence class $[\varrho]$ of $\overline{\mathfrak{R}} = \mathfrak{R}/\approx$ let $\widetilde{P}_\bullet[\varrho]$ be the corresponding indecomposable projective complex via the bijection (3.5.4). We may assume that the complex $\widetilde{P}_\bullet[\varrho]$ is minimal.

The projective complex $\widetilde{P}_\bullet$ is defined as the following direct sum

$$\widetilde{P}_\bullet = \bigoplus_{[\varrho]\in\overline{\mathfrak{R}}} \widetilde{P}_\bullet[\varrho]^{m_\varrho}$$

As above, this complex is well-defined, since $m_{\varrho'} = m_{\varrho''}$ for any pair of equivalent row labels ϱ', ϱ'' in $\mathfrak{R}$.

(3) the isomorphism $\widetilde{\vartheta}$ in $\mathrm{D^b}(\Gamma/I)$:
For each degree $d \in \mathbb{Z}$ we may view the matrix Θ_d as a map $\Theta_d \colon \Gamma/I \otimes_\Lambda V_d \xrightarrow{\sim} \Gamma/I \otimes_\Gamma \widetilde{P}_d$ of simple Γ/I-modules. Finally, we set $\widetilde{\vartheta} = (\Theta_d)_{d\in\mathbb{Z}}$.

In this way, we obtain a *minimal* triple $\mathbf{I}(\Theta) \in \mathrm{Tri}(\Lambda)$. It is not hard to check that $\mathbf{H}(\mathbf{I}(\Theta)) = \Theta$.

In the following we consider an explicit example for the construction of a triple from a matrix representation.

3.5.4 Example for the Gelfand order

Let Λ be the Gelfand order and $\mathfrak{B}$ the bunch of semichains of the derived category $\mathrm{D^b}(\Lambda)$.

Let $\Theta = (\Theta_d)_{d\in\mathbb{Z}}$ be a family of regular matrices from $\mathrm{Rep}^*(\mathfrak{B})$. The triple $\mathbf{I}(M) = \big(V_\bullet, P_\bullet, \vartheta\big) \in \mathrm{Tri}(\Lambda)$ is constructed as follows:

(1) the complex $V_\bullet \in \mathrm{D^b}(\Lambda/I)$:
For each $d \in \mathbb{Z}$ let $m_d^\pm$ be the number of columns of the horizontal stripe $\zeta_\pm^{(d)}$ in the matrix Θ_d and $m_d = m_d^+ + m_d^-$. We set $V_d = S_+^{m_d^+} \oplus S_-^{m_d^-}$ for each $d \in \mathbb{Z}$ and $(V_\bullet, d) = ((V_d)_{d\in\mathbb{Z}}, 0)$.

(2) the projective complex $\widetilde{P}_\bullet \in \mathrm{D^b}(\Gamma)$:
For each $d \in \mathbb{Z}$ and each row label $\varrho \in \mathfrak{R}_d$ let m_ϱ be the number of rows of the vertical stripe indexed by ϱ in the matrix Θ_d. For each equivalence class $[\varrho] \in \overline{\mathfrak{R}} = \mathfrak{R}/\approx$ let $\widetilde{P}_\bullet[\varrho]$ denote the indecomposable complex in $\mathrm{D^b}(\Gamma)$ according to the following table:

equivalence class $[\varrho]$ in $\overline{\mathfrak{R}}$	complex $\widetilde{P}_\bullet[\varrho] \in \mathrm{ind}[\mathrm{D^b}(\Gamma)]$	$\Gamma/I \underset{\Gamma}{\otimes} \widetilde{P}_\bullet[\varrho]$
$[\beta_\infty^{(d)}]$	$\widetilde{P}_\diamond$	$\widetilde{S}$
$[\alpha_{2n-1}^{(d)}]$	$\widetilde{P}_\star \xrightarrow{2n-1} \widetilde{P}_\diamond$	$0 \longrightarrow \widetilde{S}$
$[\beta_{2n}^{(d+1)}] = [\alpha_{2n}^{(d)}]$	$\widetilde{P}_\diamond \xrightarrow{2n} \widetilde{P}_\diamond$	$\widetilde{S} \xrightarrow{0} \widetilde{S}$
$[\beta_{2n-1}^{(d+1)}]$	$\underset{d+1}{\widetilde{P}_\diamond} \xrightarrow{2n-1} \underset{d}{\widetilde{P}_\star}$	$\underset{d+1}{\widetilde{S}} \longrightarrow \underset{d}{0}$
		$n\in\mathbb{N}^+$ and $d\in\mathbb{Z}$

The minimal complex $\widetilde{P}_\bullet$ of $\mathrm{D^b}(\Gamma)$ is given by

$$\widetilde{P}_\bullet = \bigoplus_{[\varrho]\in\overline{\mathfrak{R}}} \widetilde{P}_\bullet[\varrho]^{\oplus m_\varrho}$$

(3) the isomorphism $\widetilde{\vartheta}$ in $\mathrm{D^b}(\Gamma/I)$:
For each $d \in \mathbb{Z}$ we view Θ_d as a map $\Theta_d \colon \Gamma/I \otimes_\Lambda V_d \xrightarrow{\sim} \Gamma/I \otimes_\Gamma \widetilde{P}_d$ and set $\widetilde{\vartheta} = (\Theta_d)_{d\in\mathbb{Z}}$.

This completes the construction of the *minimal* triple $\mathbf{I}(\Theta) \in \mathrm{Tri}(\Lambda)$ from a regular representation Θ of $\mathrm{Rep}^{*}(\mathfrak{B})$ in the case of the Gelfand order Λ.

3.5.5 The bunch of semichains for the abelian category of a nodal order

Let Λ be any nodal order and $\mathfrak{B} = \mathfrak{B}_\Lambda$ be the bunch of semichains of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ as defined in Subsection 3.5.1.

In the following we will define two "truncations" of the bunch of semichains $\mathfrak{B}$ which will be related to

- the category Λ - mod of *finitely generated* Λ-modules, and
- the category Λ - fd. mod of *finite-dimensional* Λ-modules.

Let us recall that the bunch of semichains $\mathfrak{B}$ of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is given by $\mathfrak{B} = (\mathfrak{C}, \mathfrak{R} \approx)$ where the set of column labels as well as the set of row labels admit partitions

$$\mathfrak{C} = \bigcup_{d\in\mathbb{Z}} \mathfrak{C}^{(d)} \quad \text{and} \quad \mathfrak{R} = \bigcup_{d\in\mathbb{Z}} \mathfrak{R}^{(d)}$$

Moreover, according to (3.5.5) for any $d \in \mathbb{Z}$ the row label set $\mathfrak{R}^{(d)}$ can be written as disjoint union of three sets:

$$\mathfrak{R}^{(d)} = \mathfrak{R}^{(d)}_\beta \cup \mathfrak{R}^{(d)}_\infty \cup \mathfrak{R}^{(d)}_\alpha$$

where each set contains labels only of one particular type.

Definition 3.5.15. *Let Λ be a nodal order and $\mathfrak{B}$ be the bunch of semichains of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$. In the notations above, the bunches of semichains of the abelian categories of the order Λ are defined as follows*:

(1) The bunch of semichains $\mathfrak{B}_0$ of finitely-dimensional Λ-modules *is given by* $\mathfrak{B}_0 = (\mathfrak{C}_0, \mathfrak{R}_0, \approx)$ *with the sets of column and row labels set to*

$$\mathfrak{C}_0 = \mathfrak{C}^{(1)} \;\cup\; \mathfrak{C}^{(0)} \quad \textit{and} \quad \mathfrak{R}_0 = \mathfrak{R}^{(1)}_\beta \;\cup\; \mathfrak{R}^{(1)}_\infty \;\cup\; \mathfrak{R}^{(0)}_\alpha.$$

The equivalence and the order relations in these sets remain the same as in the bunch of semichains $\mathfrak{B}$.

(2) The bunch of semichains $\mathfrak{B}_0$ of finitely generated Λ-modules *is given by* $\mathfrak{B}_0^\infty = (\mathfrak{C}_0^\infty, \mathfrak{R}_0^\infty, \approx)$ *with column and row label sets*

$$\mathfrak{C}_0^\infty = \mathfrak{C}^{(1)} \;\cup\; \mathfrak{C}^{(0)} \quad \textit{and} \quad \mathfrak{R}_0^\infty = \mathfrak{R}^{(1)}_\beta \;\cup\; \mathfrak{R}^{(1)}_\infty \;\cup\; \mathfrak{R}^{(0)}_\infty \cup \mathfrak{R}^{(0)}_\alpha.$$

Again, the all relations are inherited from the bunch of semichains $\mathfrak{B}$.

Example 3.5.16. *Let Λ be the Gelfand order and $\mathfrak{B}$ the bunch of semichains of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$. The bunch of semichains $\mathfrak{B}$ was described in Example 3.5.3.*

(1) Then $\mathfrak{B}_0^\infty = \big(\mathfrak{E}_0^\infty, \mathfrak{R}_0^\infty, \approx\big)$ *where*

$$\mathfrak{E}_0^\infty = \mathfrak{E}^{(1)} \cup \mathfrak{E}^{(0)} = \{\zeta_+^{(1)}, \zeta_-^{(1)}\} \cup \{\zeta_+^{(0)}, \zeta_-^{(0)}\},$$

$$\mathfrak{R}_0^\infty = \mathfrak{R}_\beta^{(1)} \cup \mathfrak{R}_\infty^{(1)} \cup \mathfrak{R}_\infty^{(0)} \cup \mathfrak{R}_\alpha^{(0)}, \quad \textit{where}$$

$$\mathfrak{R}_\beta^{(1)} \cup \mathfrak{R}_\infty^{(1)} = \{\ \beta_1^{(1)} < \ldots < \beta_n^{(1)} < \beta_{n+1}^{(1)} < \ldots < \beta_\infty^{(1)}\ \}, \ \textit{and}$$

$$\mathfrak{R}_\infty^{(0)} \cup \mathfrak{R}_\alpha^{(0)} = \{\ \beta_\infty^{(0)} < \ldots < \alpha_{n+1}^{(0)} < \alpha_n^{(0)} < \ldots < \alpha_1^{(0)}\ \}.$$

The equivalence relation on $\mathfrak{B}_0^\infty$ *is given by* $\alpha_{2n}^{(0)} \approx \beta_{2n}^{(1)}$ *for any* $n \in \mathbb{N}^+$.

(2) The bunch of semichains $\mathfrak{B}_0$ *is obtained from* $\mathfrak{B}_0^\infty$ *by deleting the row label* $\beta_\infty^{(0)}$. *Note that* $\mathfrak{B}_0$ *is given exactly by the bunch of semichains of nilpotent representations of the Gelfand quiver in Remark 3.4.1 (if we drop the superscripts in the notation).*

Definition 3.5.17. *Let* $\operatorname{Rep}^*(\mathfrak{B}_0)$ *be the category of regular representations of* $\mathfrak{B}_0$.

Let $\operatorname{Rep}^\circ(\mathfrak{B}_0)$ *be the full subcategory of the category* $\operatorname{Rep}^*(\mathfrak{B}_0)$ *given by a pair of regular matrices* $(\Theta^{(1)}, \Theta^{(0)})$ *satisfying the following additional condition*:

- *there is some row label* $\varrho \in \mathfrak{R}_\beta^{(1)} \cup \mathfrak{R}_\alpha^{(0)}$ *such that the horizontal stripe* (3.5.9) *with label* ϱ *in the matrix* $\Theta^{(1)}$ *respectively* $\Theta^{(0)}$ *is* not empty.

In other words, any representation Θ *given by the empty matrix* $\Theta^{(0)}$ *and a regular matrix* $\Theta^{(1)}$ *where all non-empty horizontal stripes are labeled by elements of* $\mathfrak{R}_\infty^{(1)}$*, is not an object of* $\operatorname{Rep}^\circ(\mathfrak{B}_0)$.

The category $\operatorname{Rep}^\circ(\mathfrak{B}_0^\infty)$ *for the bunch of semichains* $\mathfrak{B}_0^\infty$ *is defined in* exactly *the same way.*

Theorem 3.5.18. *(1) There is a bijection between the* isomorphism classes of indecomposable finitely generated Λ-modules *and* the isomorphism classes of canonical forms in $\operatorname{Rep}^\circ(\mathfrak{B}_0^\infty)$ *together with the* isomorphism classes of indecomposable $\Gamma_\star$-modules:

$$\operatorname{ind}[\Lambda\text{ -mod}] \xleftrightarrow{1:1} \operatorname{ind}[\operatorname{Rep}^\circ(\mathfrak{B}_0^\infty)] \cup \operatorname{ind}[\Gamma_\star\text{ -mod}]$$

(2) This bijection restricts to isomorphism classes of finite-dimensional Λ*-modules and canonical forms in* $\operatorname{Rep}^\circ(\mathfrak{B}_0)$ *for the* bunch of semichains $\mathfrak{B}_0$ *together with isomorphism classes of indecomposable* finite-dimensional $\Gamma_\star$*-modules*:

$$\operatorname{ind}[\Lambda\text{ -fd. mod}] \xleftrightarrow{1:1} \operatorname{ind}[\operatorname{Rep}^\circ(\mathfrak{B}_0)] \cup \operatorname{ind}[\Gamma_\star\text{ -fd. mod}]$$

PROOF. 1. By Theorem 3.5.12, Corollaries 3.2.6 and 3.3.3 there are the following maps on isomorphism classes of indecomposable objects in the following categories:

$$\begin{tikzcd}
\operatorname{ind}[\mathrm{D^b}(\Lambda)] \arrow[r, shift left, "\mathbf{F}"] & \operatorname{ind}[\operatorname{Tri}(\Lambda)] \arrow[l, shift left, "1:1", "\mathbf{G}"'] \arrow[r, shift left, two heads, "\mathbf{H}"] & \operatorname{ind}[\operatorname{Rep}^*(\mathfrak{B})] \cup \operatorname{ind}[\mathrm{D^b}(\Lambda_\star)] \arrow[l, shift left, "\mathbf{I}"] \\
\operatorname{ind}[\Lambda\text{ -mod}] \arrow[u, hook] \arrow[r, shift left, "\mathbf{F}"] & \operatorname{ind}[\operatorname{Tri}_{0,\infty}(\Lambda)] \arrow[u, hook] \arrow[l, shift left, "1:1", "\mathbf{G}"'] \arrow[r, shift left, dashed, "\mathbf{H}'"] & \operatorname{ind}[\operatorname{Rep}^\circ(\mathfrak{B}_0^\infty)] \cup \operatorname{ind}[\Lambda_\star\text{ -mod}] \arrow[u, hook] \arrow[l, shift left, dashed, "\mathbf{I}'"]
\end{tikzcd} \tag{3.5.10}$$

To show the first statement we have to show that the restricted maps $\mathbf{H}'$ and $\mathbf{I}'$ are well-defined.

(1) Let $T_\bullet \in \mathrm{ind}[\mathrm{Tri}_{0,\infty}(\Lambda)]$. This means that $T_\bullet = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ is given by a triple of the following form:

(a) The complex $V_\bullet = V_1 \xrightarrow{0} V_0$ has at most length one.

(b) The projective complex $\widetilde{P}_\bullet$ is given by some minimal complex $\widetilde{P}_\bullet = \widetilde{P}_1 \longrightarrow \underset{\neq 0}{\widetilde{P}_0}$ of length at most one.

(c) The map $\widetilde{\vartheta}$ is given by two matrices $\Theta^{(1)}$ and $\Theta^{(0)}$.

- If $\mathbf{H}'(T_\bullet) = 0$, then $T_\bullet \cong (0, \widetilde{P}_\bullet, 0)$ with $\widetilde{P}_\bullet \in \mathrm{D}^{\mathrm{b}}(\Gamma_\star)$. Since $T_\bullet$ is indecomposable, the complex $\widetilde{P}_\bullet$ is the projective resolution of some indecomposable $\Gamma_\star$-module. The application of functor $\mathbf{G}$ to the triple $T_\bullet$ yields a projective resolution of some indecomposable $\Lambda_\star$-module. Vice versa, any $\Lambda_\star$-module is mapped by $\mathbf{F}$ to a triple in the kernel of $\mathbf{H}'$. In other words, there is a bijection:

$$\mathrm{ind}[\Lambda_\star\text{ - mod}] \xleftrightarrow{1:1} \mathrm{ind}[\ker \mathbf{H}'] \xleftrightarrow{1:1} \mathrm{ind}[\Gamma_\star\text{ - mod}]$$

- Assume that $\mathbf{H}'(T_\bullet) \neq 0$. In this case, $\mathbf{H}'(T_\bullet)$ is given by pair of matrices $(\Theta^{(1)}, \Theta^{(0)})$. Since $\widetilde{P}_\bullet$ has at most length one, and $\widetilde{P}_0 \neq 0$ the translation in (3.5.2) yields that there is some label ϱ from $\mathfrak{R}^{(1)}_\beta$ or $\mathfrak{R}^{(0)}_\alpha$ such that the horizontal stripe ϱ has a non-zero number of rows. This shows that $\mathbf{H}'$ is well-defined.

(2) Vice versa, let $\Theta = (\Theta^{(1)}, \Theta^{(0)})$ be a canonical form of $\mathfrak{B}_0^\infty$ satisfying the constraint (3.5.9). By the construction in Subsection 3.5.3 the corresponding triple $T_\bullet = \mathbf{I}'(\Theta)$ satisfies (a) and (c) above. The complex $\widetilde{P}_\bullet$ of $T_\bullet$ has also the form $\widetilde{P}_\bullet = \widetilde{P}_1 \longrightarrow \underset{\neq 0}{\widetilde{P}_0}$. Now the constraint (3.5.9) implies that $\widetilde{P}_0 \neq 0$.

Summarized, we have shown that the bijections in diagram (3.5.10) of the "derived level" restrict to the "abelian level".

2. To show the second statement, we restrict the bijection of the bottom row in (3.5.10) further:

$$\mathrm{ind}[\Lambda\text{ - fd. mod}] \underset{1:1}{\dashleftarrow\dashrightarrow} \mathrm{ind}[\mathrm{Tri}_0(\Lambda)] \underset{1:1}{\dashleftarrow\dashrightarrow} \mathrm{ind}[\mathrm{Rep}^\circ(\mathfrak{B}_0)] \cup \mathrm{ind}[\Lambda_\star\text{ - fd. mod}]$$

The bijection on the left follows from Corollary 3.3.3 and the bijection on the right by Corollary 3.5.13. □

Corollary 3.5.19. *For any nodal order Λ there is a bijection between isomorphism classes of finite-dimensional Λ-modules and regular strings and bands of $\mathfrak{B}_0$ together with isomorphism classes of finite-dimensional $\Gamma_\star$-modules*

$$\mathrm{ind}[\Lambda\text{ - fd. mod}] \xleftrightarrow{\quad 1:1 \quad} [\ \text{STRINGS}^*\ \textit{and}\ \text{BANDS}\ \textit{of}\ \mathfrak{B}_0\ ] \cup [\mathrm{ind}(\Gamma_\star\text{ - fd. mod})]. \tag{3.5.11}$$

3.6 String and band complexes over nodal orders

The precise definitions of strings and bands are given in Appendix A. In the present subsection it will we sufficient to note that string and band complexes can be characterized completely via the defect δ and the involution σ :

Definition 3.6.1. *Let Θ be a regular representation of $\mathfrak{B}$.*

(1) The defect $\delta(\Theta)$ *is given by the total number of rows of all horizontal stripes labeled by* free *labels of $\mathfrak{B}$.*

(2) The involution $\sigma(\Theta)$ *is given by interchanging is given by interchanging the labels of vertical stripes which are labeled by pairs of* special labels*.*

The next theorem relates the combinatorial description of the indecomposable objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ to the defect δ and the involution σ of any nodal order.

Theorem 3.6.2. *Let Λ be a nodal order. Let $\mathfrak{B}_{\mathrm{fd}}$ be the bunch of semichains associated to the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Let $P_\bullet$ be an indecomposable complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ with finite-dimensional homology such that $P_\bullet \notin \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda_\star)$. According to Theorem 3.5.12 there is some usual, special or bispecial string, or band Ω of $\mathfrak{B}$ such that $P_\bullet \cong P_\bullet(\Omega)$.*
In the setup above, the following statements hold:

$$\begin{array}{lll}
(1) & \Omega \textit{ is a usual string } \omega & \Leftrightarrow \delta(P_\bullet) = 2, \\
(2) & \Omega \textit{ is a special string } (\omega, \varepsilon_1) & \Leftrightarrow \delta(P_\bullet) = 1, \\
(3) & \Omega \textit{ is a bispecial string } (\omega, m, \varepsilon_1, \varepsilon_2) & \Leftrightarrow \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \ncong P_\bullet, \\
(4) & \Omega \textit{ is a band } (\omega, m, \lambda) & \Leftrightarrow \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \cong P_\bullet.
\end{array}$$

PROOF. In the following we assume that Ω is a regular string or a band of $\mathfrak{B}$. This means that the representation $\Theta(\Omega)$ is given by regular matrices. Let $P_\bullet \cong P_\bullet(\Omega)$ in the notations above. Let us recall that the complex $P_\bullet(\Omega)$ is constructed by the following operations:

$$\begin{array}{ccccccc}
\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] & \underset{\mathbf{G}}{\overset{\mathbf{F}}{\underset{1:1}{\rightleftarrows}}} & \mathrm{ind}[\mathrm{Tri}(\Lambda)] & \underset{\mathbf{I}}{\overset{\mathbf{H}}{\rightleftarrows}} & \mathrm{ind}[\mathrm{Rep}^*(\mathfrak{B})] & \xleftarrow[1:1]{} & \left[\begin{array}{c}\text{STRINGS}^* \text{ and} \\ \text{BANDS of } \mathfrak{B}\end{array}\right] \\
P_\bullet(\Omega) & \longleftarrow\!\shortmid & T_\bullet(\Omega) & \longleftarrow\!\shortmid & \Theta(\Omega) & \longleftarrow\!\shortmid & \Omega
\end{array} \tag{3.6.1}$$

We claim that these operations preserve the defect and commute with the involution:

$$\delta(P_\bullet(\Omega)) = \delta(\Omega) \quad \text{and} \quad \sigma(P_\bullet(\Omega)) \cong P_\bullet(\sigma(\Omega))$$

(1) It holds that $\delta(\Omega) = \delta(\Theta(\Omega))$ by Lemma A.4.4. The number $\delta(\Theta(\Omega))$ is the total number of rows labeled by free elements in $\mathfrak{B}_0$.

To construct $\mathbf{I}(\Theta(\Omega)) = T_\bullet(\Omega)$ and Remark 3.5.2 every row of a *free* element gives rise to *one* direct summand of $\widetilde{P}_\bullet$ with defect *one*. Let $\widetilde{P}_\bullet(\Omega)$ be the normalization complex in the triple $T_\bullet(\Omega)$.

Then we obtain that $\delta(\Theta(\Omega)) = \delta(\widetilde{P}_\bullet(\Omega))$ which is $\delta(T_\bullet(\Omega))$ by definition of the defect for triples.

Finally, Lemma 3.3.15 yields that $\delta(\mathbf{G}(T_\bullet(\Omega))) = \delta(P_\bullet(\Omega))$.

In other words, all operations preserve the defect.

(2) Concerning the involution we have that $\Theta(\sigma(\Omega)) \cong \sigma(\Theta(\Omega))$ by Lemma A.4.2, that is the involution σ commutes the construction of canonical forms.

On the other hand it is not hard to check that $\mathbf{H}\big(\mathbf{F}(\sigma(P_\bullet))\big) \cong \sigma(\mathbf{H}(\mathbf{F}(P_\bullet)))$.

Now the statement follows from the characterization of the four classes of regular representations of $\mathfrak{B}$ in Proposition A.4.7. □

Remark 3.6.3. *Let $P_\bullet$ be an indecomposable complex from $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda_\star)$. In this case, $\delta(P_\bullet) = 2$ and $\sigma(P_\bullet) \cong P_\bullet$. In particular, we will view such complexes as* usual strings.

Corollary 3.6.4. *Let $\Lambda = \Lambda_n$ or $\Lambda_n^\#$ be a Khoroshkin order for some $n \geq 2$. Let $\mathfrak{B}$ be the bunch of semichains associated to Λ. Let Ω be a string or band of $\mathfrak{B}_{\mathrm{fd}}$ and $P_\bullet = P_\bullet(\Omega)$ be the corresponding indecomposable complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Then the following statements hold:*

$$
\begin{array}{lll}
(1) & \Omega \textit{ is a usual string } \omega & \Leftrightarrow \delta(P_\bullet) > 0 \textit{ and } \sigma(P_\bullet) \cong P_\bullet \\
 & & \Leftrightarrow \delta(P_\bullet) = 2 \\
 & & \Leftrightarrow P_\bullet \textit{ is not } \tau\textit{-periodic and } \sigma(P_\bullet) \cong P_\bullet \\
(2) & \Omega \textit{ is a special string } (\omega, \varepsilon_1) & \Leftrightarrow \delta(P_\bullet) > 0 \textit{ and } \sigma(P_\bullet) \ncong P_\bullet \\
 & & \Leftrightarrow \delta(P_\bullet) = 1 \\
 & & \Leftrightarrow P_\bullet \textit{ is not } \tau\textit{-periodic and } \sigma(P_\bullet) \ncong P_\bullet \\
(3) & \Omega \textit{ is a bispecial string } (\omega, m, \varepsilon_1, \varepsilon_2) & \Leftrightarrow \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \ncong P_\bullet \\
 & & \Leftrightarrow \tau^2(P_\bullet) \cong P_\bullet \textit{ and } \tau(P_\bullet) \ncong P_\bullet \\
(4) & \Omega \textit{ is a band } (\omega, m, \lambda) & \Leftrightarrow \delta(P_\bullet) = 0 \textit{ and } \sigma(P_\bullet) \cong P_\bullet \\
 & & \Leftrightarrow \tau(P_\bullet) \cong P_\bullet
\end{array}
$$

where $\varepsilon_1, \varepsilon_2 \in \{+,-\}, m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk \setminus \Delta$

PROOF. The theorem follows directly from Theorems 3.6.2 and 2.2.3. □

The above theorem states that the four classes of indecomposable objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ can be characterized in combinatorial terms (strings and bands of $\mathfrak{B}_{\mathrm{fd}}$), by Lie-theoretic notions (the defect δ and the involution σ) or via some natural functors (Auslander-Reiten translation τ and involution σ).

CHAPTER 4

Derived Category of the Gelfand quiver

In the present and the next chapter, we focus on the combinatorics of the Gelfand quiver

$$\begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-.$$

From now on, we will denote by Λ always the *Gelfand order*, that is, the arrow ideal completion of the path algebra of the Gelfand quiver. The present chapter deals with the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of finitely generated Λ-modules.

At the beginning of this chapter, we recover the classification of indecomposable objects in the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ by Burban and Drozd [**BD04**]. More precisely, the indecomposable complexes are parametrized by usual, special and bispecial strings and bands of a matrix problem of type *bunch of semichains*. We give a proof of the gluing rules for string and band complexes of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ in Subsection 4.2.3.

The explicit description of the indecomposable complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is central for the study of their functorial properties. There are two natural autoequivalences on the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

(1) The involution functor σ interchanging $+$ and $-$ in projectives and their morphisms.

(2) The Auslander-Reiten translation τ which is described as follows:

$$\begin{array}{cc} & \begin{array}{ccc} P_\star & P_- & P_+ \end{array} \\ \Lambda = & \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \qquad \begin{array}{ccc} P_\star \xrightarrow{\cdot \mathsf{b}_\pm} P_\pm & \xmapsto{\ \tau\ } & rP_\star \xrightarrow{\cdot \mathsf{b}_\mp} P_\mp \\ P_\pm \xrightarrow{\cdot \mathsf{a}_\pm} P_\star & \xmapsto{\ \tau\ } & P_\mp \xrightarrow{\cdot \mathsf{a}_\mp} rP_\star \end{array} \qquad \begin{array}{c} rP_\star \\ \begin{bmatrix} \mathfrak{m} \\ \mathbf{R} \\ \mathbf{R} \end{bmatrix} \end{array}$$

A description of the involution functor on strings and band complexes can be derived from Bondarenko's combinatorics of bunches of semichains (Lemma 4.3.3).

The main result of this chapter is a description of the Auslander-Reiten translation τ in terms of strings and bands (Theorem 4.3.10).

As an application of this result, we determine all fractionally Calabi-Yau objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ (Proposition 4.3.22), show that $S_\star$ is the only "generalized" spherical object (Corollary 4.3.26) and give an explicit description of the thick category $\langle S_\star \rangle$ generated by the simple module $S_\star$ (Corollary 4.3.29).
In Subsection 4.3.6 we summarize these statements together with the main results of Chapter 2 for the case of the Gelfand order.

4.1 Strings and bands of indecomposable projective complexes

In this section we recall the description of indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ due to **[BD04]**.

4.1.1 Notation for projective complexes of the Gelfand order

Let $\mathbf{R} = \Bbbk[\![x]\!]$ be the ring of formal power series and $\mathfrak{m} = (x)$ its unique maximal ideal. Throughout this chapter, let (Q, I) be the *Gelfand quiver* and let Λ be the arrow ideal completion of the path algebra $\Bbbk Q/I$:

$$(Q,I) = \begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \Lambda = \begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array}$$

$$\mathsf{b}_+ \, \mathsf{a}_+ = \mathsf{b}_- \, \mathsf{a}_- =: \mathsf{c}$$

The ring Λ is an $\mathbf{R}$-order (in the sense of Definition B.1.12) and will be called the *Gelfand order.*

Remark 4.1.1. *By Corollary B.1.8 there is an equivalence of categories*:

$$\mathrm{nil.\,rep}_{\Bbbk}(Q, I) \xrightarrow{\ \sim\ } \Lambda \text{ - fd. mod}$$

That is, we may study finite-dimensional Λ-modules instead of nilpotent representations of the Gelfand quiver.

In this chapter we will study the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of finitely generated modules over the Gelfand order.

First, we need to fix some notation for the projective complexes of $\mathrm{D}^{\mathrm{b}}(\Lambda)$. This notation uses the following properties of the Gelfand quiver:

- Let $i, j \in \{ +, \star, - \}$ be two vertices of the Gelfand quiver and $\mathtt{n} \in \mathbb{N}^+$. Then there is *exactly one* path p_{ij} from vertex j to vertex i of length $\widehat{\mathtt{n}}$, where
$$\widehat{\mathtt{n}} = \begin{cases} 2\mathtt{n} & \text{if } \ i = j = \star \quad \text{or} \quad i, j \in \{+, -\} \\ 2\mathtt{n} - 1 & \text{if } \ i = \star \text{ and } j = \pm \quad \text{or} \quad i = \pm \text{ and } j = \star \end{cases}$$
- let P_i and P_j be the corresponding indecomposable projectives. Then any map $d : P_i \longrightarrow P_j$ is actually given by right multiplication $d = \cdot\lambda\, p_{ij}$ for some scalar $\lambda \in \Bbbk$ and some path p_{ij} in the Gelfand quiver. We will abbreviate d by $\widehat{n}_{ij}$.

In more detail, we will use the following notation:

Table 4.1.1. **Notation for differentials of projective complexes**

$(\widehat{\mathtt{n}})_{++} = (2\mathtt{n})_{++} \quad = \cdot\, \mathsf{a}_+\mathsf{c}^{\mathtt{n}-1}\mathsf{b}_+$	$(\widehat{\mathtt{n}})_{\star+} = (2\mathtt{n}-1)_{\star+} = \cdot\, \mathsf{c}^{\mathtt{n}-1}\mathsf{b}_+$	$(\widehat{\mathtt{n}})_{-+} = (2\mathtt{n})_{-+} \quad = \cdot\, \mathsf{a}_-\mathsf{c}^{\mathtt{n}-1}\mathsf{b}_+$
$(\widehat{\mathtt{n}})_{+\star} = (2\mathtt{n}-1)_{+\star} = \cdot\, \mathsf{a}_+\mathsf{c}^{\mathtt{n}-1}$	$(\widehat{\mathtt{n}})_{\star\star} = (2\mathtt{n})_{\star\star} \quad = \cdot\, \mathsf{c}^{\mathtt{n}}$	$(\widehat{\mathtt{n}})_{-\star} = (2\mathtt{n}-1)_{-\star} = \cdot\, \mathsf{a}_-\mathsf{c}^{\mathtt{n}-1}$
$(\widehat{\mathtt{n}})_{+-} = (2\mathtt{n})_{+-} \quad = \cdot\, \mathsf{a}_+\mathsf{c}^{\mathtt{n}-1}\mathsf{b}_-$	$(\widehat{\mathtt{n}})_{\star-} = (2\mathtt{n}-1)_{\star-} = \cdot\, \mathsf{c}^{\mathtt{n}-1}\mathsf{b}_-$	$(\widehat{\mathtt{n}})_{--} = (2\mathtt{n})_{--} \quad = \cdot\, \mathsf{a}_-\mathsf{c}^{\mathtt{n}-1}\mathsf{b}_-$

We will also skip the subscripts in differentials if they are clear from the context. For example, the projective resolutions of simple Λ-modules translate to the new notation as follows:

$$
\begin{array}{ccc}
\underset{1}{P_\star} \xrightarrow{\cdot \mathsf{b}_\pm} \underset{0}{P_\pm} \twoheadrightarrow S_\pm & \qquad & \underset{2}{P_\star} \xrightarrow{\left[\begin{smallmatrix} \cdot \mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{smallmatrix}\right]} \underset{1}{P_+ \oplus P_-} \xrightarrow{[\cdot \mathsf{a}_+ \ \cdot \mathsf{a}_-]} \underset{0}{P_\star} \twoheadrightarrow S_\star \\
\Downarrow & & \Downarrow \\
\underset{1}{P_\star} \xrightarrow{1} \underset{0}{P_\pm} \twoheadrightarrow S_\pm & & \underset{2}{P_\star} \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]} \underset{1}{P_+ \oplus P_-} \xrightarrow{[1\ 1]} \underset{0}{P_\star} \twoheadrightarrow S_\star
\end{array}
$$

In particular, the Gelfand order has global dimension *two*.

4.1.2 The normalization of the Gelfand order

The *normalization* of the Gelfand order Λ is given by the ring $\Gamma = \mathrm{End}_\Lambda(\mathrm{rad}\,\Lambda)$. We recall that the Gelfand order is *nodal* in the sense of Definition 3.1.1 and there is an embedding

$$
\Lambda = \begin{array}{c} \begin{matrix} P_\star & P_+ & P_- \end{matrix} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array} \hookrightarrow \Gamma = \begin{array}{c} \begin{matrix} \widetilde{P}_\star & \widetilde{P}_\diamond & \widetilde{P}_\diamond \end{matrix} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} & \mathbf{R} \end{bmatrix} \end{array}
$$

The ring Γ is Morita equivalent to the arrow ideal completion of the path algebra $\Bbbk\widetilde{Q}$ of the two-cycle quiver $\widetilde{Q}$:

$$
(Q, I) = \begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \widetilde{Q} = \begin{tikzcd} \underset{\diamond}{\bullet} \arrow[r, bend left, "\mathsf{b}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}"] \end{tikzcd}
$$

$$\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$$

Informally speaking, the method of **[BD04]** tells us "how to *glue* any indecomposable projective complex in the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ from a direct sum of complexes from the derived category $\mathrm{D}^{\mathrm{b}}(\Gamma)$."

The crucial point here is that the representation theory of $\mathrm{D}^{\mathrm{b}}(\Gamma)$ is well-known. Since Γ is a *hereditary* ring, Theorem B.2.5 implies that the indecomposable objects of the derived category $\mathrm{D^b}(\Gamma)$ are given by shifts of projective resolutions:

$$\mathrm{D^b}(\Gamma) = \prod_{\mathsf{d}\in\mathbb{Z}} \mathrm{add}\,\big\{\{\ \underset{\mathsf{d}+1}{\widetilde{P}_j} \xrightarrow{\ \widehat{\mathsf{n}}\ } \underset{\mathsf{d}}{\widetilde{P}_i}\ \mid i,j\in\widetilde{Q}_0, \mathsf{n}\in\mathbb{N}^+\ \}\ \oplus\ \{\underset{\mathsf{d}}{\widetilde{P}_i} \mid i\in\widetilde{Q}_0\}\big\} \qquad (4.1.1)$$

$$\text{where } \widetilde{P}_{i_1} \xrightarrow{\ \widehat{\mathsf{n}}\ } \widetilde{P}_{i_2} = \begin{cases} \widetilde{P}_\diamond \xrightarrow{\ 2\mathsf{n}\ } \widetilde{P}_\diamond \\ \widetilde{P}_\star \xrightarrow{\ 2\mathsf{n}-1\ } \widetilde{P}_\diamond \\ \widetilde{P}_\diamond \xrightarrow{\ 2\mathsf{n}-1\ } \widetilde{P}_\star \\ \widetilde{P}_\star \xrightarrow{\ 2\mathsf{n}\ } \widetilde{P}_\star \end{cases} = \begin{cases} \widetilde{P}_\diamond \xrightarrow{\ \cdot(\mathsf{a}\,\mathsf{b})^{\mathsf{n}}\ } \widetilde{P}_\diamond \\ \widetilde{P}_\star \xrightarrow{\ \cdot\mathsf{b}(\mathsf{a}\,\mathsf{b})^{\mathsf{n}-1}\ } \widetilde{P}_\diamond \\ \widetilde{P}_\diamond \xrightarrow{\ \cdot(\mathsf{a}\,\mathsf{b})^{\mathsf{n}-1}\mathsf{a}\ } \widetilde{P}_\star \\ \widetilde{P}_\star \xrightarrow{\ \cdot(\mathsf{b}\,\mathsf{a})^{\mathsf{n}-1}\ } \widetilde{P}_\star \end{cases}$$

As in the case of the Gelfand order, we will denote the differential $\cdot p_{ji}$ of an indecomposable resolution $\widetilde{P}_\bullet = \widetilde{P}_j \xrightarrow{\cdot p_{ji}} \widetilde{P}_i$ from $\mathrm{D^b}(\Gamma)$ by its *length* $\widehat{\mathsf{n}} = 2\mathsf{n}$ (if $j = i$) respectively $2\mathsf{n}-1$ (if $j \neq i$).

4.1.3 Words and their gluing diagrams

In this subsection we introduce some formal expressions called “words of $\check{\mathfrak{B}}$ or $\check{\mathfrak{B}}_{\mathrm{fd}}$”. These words will be visualized by *gluing diagrams*.

A gluing diagram is given by some complex $\widetilde{P}_\bullet \in \mathrm{D^b}(\Gamma)$, some “gluing edges” between direct summands of $\widetilde{P}_\bullet$ and some additional parameters which will be specified below.

4.1.3.1 Gluing diagrams of cyclic words

Definition 4.1.2. *A* cyclic word ω *of* $\check{\mathfrak{B}}$ *is given by some sequence*

$$\omega = ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \dots \mathtt{n}_{2\mathtt{k}-1}^{[\mathtt{d}_{2\mathtt{k}-1}]}\mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}_{2\mathtt{k}}]})$$

of an even number of natural numbers $\mathtt{n_j} \in \mathbb{N}^+$*, where* $1 \leq \mathtt{j} \leq 2\mathtt{k}$ *and* $\mathtt{k} \geq 1$*, and degrees* $\mathtt{d_j} \in \mathbb{Z}$*, where* $0 \leq \mathtt{j} \leq 2\mathtt{k}$*, such that* $\mathtt{d_0} = \mathtt{d_k}$ *and*

- $\mathtt{d_{j+1}} = \mathtt{d_j} - 1$ *or* $\mathtt{d_{j+1}} = \mathtt{d_j} + 1$ *for any* $0 \leq \mathtt{j} < 2\mathtt{k}$.

Let us consider an example how some cyclic word translates into a gluing diagram:

Example 4.1.3. *Let* $\omega = (4^{[1]}3^{[0]}5^{[1]}5^{[2]})$. *Then the gluing diagram of* ω *has the form*:

$$\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "8"] \arrow[ddd, dash, dashed] & \widetilde{P}_\diamond \arrow[d, dash, dashed] & \\
& \widetilde{P}_\diamond \arrow[r, "6"] & \widetilde{P}_\diamond \arrow[d, dash, dashed] \\
& \widetilde{P}_\diamond \arrow[r, "10"] \arrow[d, dash, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond \arrow[r, "10"] & \widetilde{P}_\diamond & \\
2 & 1 & 0
\end{tikzcd}$$

In general, cyclic words are translated into gluing diagrams in the following way.

Let ω be some cyclic word of length $\mathtt{2k}$:

$$\omega = ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \dots \mathtt{n}_{2\mathtt{k}-1}^{[\mathtt{d}_{2\mathtt{k}-1}]}\mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}_{2\mathtt{k}}]})$$

The *gluing diagram* of ω is obtained as follows:

(1) *transformed word*:
First, we define a new word

$$\widehat{\omega} = ({}^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \dots, \widehat{\mathtt{n}}_{2\mathtt{k}-1}^{[\mathtt{d}_{2\mathtt{k}-1}]}, \widehat{\mathtt{n}}_{2\mathtt{k}}^{[\mathtt{d}_{2\mathtt{k}}]}), \text{ where } \widehat{\mathtt{n}}_\mathtt{j} = 2\mathtt{n}_\mathtt{j} \text{ for any } 1 \le \mathtt{j} \le 2\mathtt{k}. \quad (4.1.2)$$

(2) the *normalization complex*:
We define a complex $\widetilde{P}_\bullet = \bigoplus_{\mathtt{j}=1}^{2\mathtt{k}} \widetilde{P}_\bullet^{(\mathtt{j})} \in \mathrm{D}^\mathrm{b}(\Gamma)$ such that for any $1 \le \mathtt{j} \le 2\mathtt{k}$:

$$\widetilde{P}_\bullet^{(\mathtt{j}+1)} = \begin{cases} \begin{array}{ccc} & \widetilde{P}_\diamond \xrightarrow{\widehat{\mathtt{n}}_{\mathtt{j}+1}} & \widetilde{P}_\diamond \end{array} & \text{if } \mathtt{d}_{\mathtt{j}+1} = \mathtt{d}_\mathtt{j} - 1 \\ \begin{array}{ccc} \underset{\mathtt{d}_\mathtt{j}+1}{\widetilde{P}_\diamond} \xrightarrow{\widehat{\mathtt{n}}_{\mathtt{j}+1}} & \underset{\mathtt{d}_\mathtt{j}}{\widetilde{P}_\diamond} & \underset{\mathtt{d}_\mathtt{j}-1}{} \end{array} & \text{if } \mathtt{d}_{\mathtt{j}+1} = \mathtt{d}_\mathtt{j} + 1 \end{cases}$$

Each complex $\widetilde{P}_\bullet^{(\mathtt{j}+1)}$ is uniquely determined by the subword ${}^{[\mathtt{d}_\mathtt{j}]}\widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]}$ of ω.

(3) *gluing edges*:
For each index $1 \le \mathtt{j} \le 2\mathtt{k}$ the projective module of $\widetilde{P}_\bullet^{(\mathtt{j}+1)}$ at degree $\mathtt{d}_{\mathtt{j}+1}$ is connected to the projective module of $\widetilde{P}_\bullet^{(\mathtt{j}+2)}$ at the same degree by an edge.

Note that there is also a gluing edge connecting the last complex to the first complex in the gluing diagram of any cyclic word.

4.1.3.2 Gluing diagrams of finite words

Definition 4.1.4. *Let* $\mathfrak{E}$ *denote the following set of formal symbols*:

$$\mathfrak{E} = \{\ \mathtt{p}_\star,\ \mathtt{p}_\diamond,\ \mathtt{p}_\infty\ \}.$$

A finite word ω *of* $\check{\mathfrak{B}}$ *is given by a sequence*

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta)$$

consisting of two ends $\mathtt{p}_\alpha, \mathtt{p}_\beta \in \mathfrak{E}$*, a sequence of natural numbers* $\mathtt{n}_\mathtt{j} \in \mathbb{N}^+$ *and a sequence of degrees* $\mathtt{d}_\mathtt{j} \in \mathbb{Z}$ *, where* $\mathtt{1} \leq \mathtt{j} \leq \mathtt{k}$ *and* $\mathtt{k} \geq \mathtt{1}$*, such that*

- $\mathtt{d}_{\mathtt{j}+\mathtt{1}} = \mathtt{d}_\mathtt{j} - \mathtt{1}$ *or* $\mathtt{d}_{\mathtt{j}+\mathtt{1}} = \mathtt{d}_\mathtt{j} + \mathtt{1}$ *for each* $\mathtt{0} \leq \mathtt{j} \leq \mathtt{k} - \mathtt{1}$.

There are also words of length zero *which are given by* $\mathtt{p}_\star^{[\mathtt{d}]}$ *and* $\mathtt{p}_\diamond^{[\mathtt{d}]}$ *for some* $\mathtt{d} \in \mathbb{Z}$.

Example 4.1.5. *Let* $\omega = (\mathtt{p}_\diamond^{[1]}3^{[0]}2^{[1]}3^{[2]}\mathtt{p}_\star)$. *Then the corresponding gluing diagram has the following form*:

$$\begin{tikzcd}
 & \widetilde{P}_\diamond \arrow[r, "6"] & \widetilde{P}_\diamond \arrow[d, dash, dashed] \\
 & \widetilde{P}_\diamond \arrow[r, "4"] \arrow[d, dash, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\star \arrow[r, "5"] & \widetilde{P}_\diamond & \\
\mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}$$

Let ω be a finite word of $\check{\mathfrak{B}}$,

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta),$$

such that the length $\mathtt{k}$ is not zero and $\omega \neq \mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{n}^{[\mathtt{d}_1]}\mathtt{p}_\star$ for any $\mathtt{d}_0, \mathtt{d}_1 \in \mathbb{Z}$ and $\mathtt{n} \in \mathbb{N}^+$. Next, we consider the translation of finite words into gluing diagrams:

(1) *transformed word*:
The transformed word of ω is given by

$$\widehat{\omega} = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_2]}\dots\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta), \qquad \text{where } \widehat{\mathtt{n}}_\mathtt{j} = \mathtt{2n}_\mathtt{j} \text{ for any } \mathtt{1} < \mathtt{j} < \mathtt{2k} \tag{4.1.3}$$

$$\widehat{\mathtt{n}}_1 = \begin{cases} \mathtt{2n}_1 - \mathtt{1} & \text{if } \mathtt{p}_\alpha = \mathtt{p}_\star, \\ \mathtt{2n}_1 & \text{if } \mathtt{p}_\alpha = \mathtt{p}_\diamond, \end{cases} \quad \text{and} \quad \widehat{\mathtt{n}}_\mathtt{k} = \begin{cases} \mathtt{2n}_\mathtt{k} - \mathtt{1} & \text{if } \mathtt{p}_\beta = \mathtt{p}_\star, \\ \mathtt{2n}_\mathtt{k} & \text{if } \mathtt{p}_\beta = \mathtt{p}_\diamond. \end{cases} \tag{4.1.4}$$

(2) the *normalization complex*:
We define a complex $\widetilde{P}_\bullet = \bigoplus_{\mathtt{j}=1}^{\mathtt{k}} \widetilde{P}_\bullet^{(\mathtt{j})} \in \mathrm{D}^\mathrm{b}(\Gamma)$ as follows.

(a) The first summand is determined by the beginning $\mathtt{p}_\alpha^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}$ of ω. We set

$$\widetilde{P}_\bullet^{(1)} = \begin{cases} \begin{array}{cccccl} & & \widetilde{P}_\alpha & \xrightarrow{\widehat{\mathtt{n}}_1} & \widetilde{P}_\diamond & \quad \text{if } \mathtt{d}_1 = \mathtt{d}_0 - \mathtt{1} \\ \widetilde{P}_\diamond & \xrightarrow{\widehat{\mathtt{n}}_1} & \widetilde{P}_\alpha & & & \quad \text{if } \mathtt{d}_1 = \mathtt{d}_0 + \mathtt{1} \\ \mathtt{d}_0+\mathtt{1} & & \mathtt{d}_0 & & \mathtt{d}_0-\mathtt{1} & \end{array} \end{cases}$$

(b) For each index $\mathtt{j}$ with $\mathtt{1} \leq \mathtt{j} < \mathtt{k}$ the intermediate subword ${}^{[\mathtt{d}_\mathtt{j}]}\widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]}$ of ω defines a projective complex in the same way as for cyclic words:

$$\widetilde{P}_\bullet^{(\mathtt{j}+1)} = \underset{\mathtt{d}+\mathtt{1}}{\widetilde{P}_\diamond} \xrightarrow{\widehat{\mathtt{n}}_{\mathtt{j}+1}} \underset{\mathtt{d}}{\widetilde{P}_\diamond} \qquad \text{where } \mathtt{d} = \min\{\mathtt{d}_\mathtt{j}, \mathtt{d}_{\mathtt{j}+1}\}.$$

(c) The last summand is defined by the ending ${}^{[\mathtt{d}_{k-1}]}\mathtt{n}_{k}^{[\mathtt{d}_k]}\mathtt{p}_\beta$ of ω by setting

$$\widetilde{P}_\bullet^{(k)} = \begin{cases} \quad\quad\quad \widetilde{P}_\beta \xrightarrow{\widehat{\mathtt{n}}_k} \widetilde{P}_\diamond & \text{if } \mathtt{d}_{k-1} = \mathtt{d}_k - 1 \\ \underset{\mathtt{d}_k+1}{\widetilde{P}_\diamond} \xrightarrow{\widehat{\mathtt{n}}_k} \underset{\mathtt{d}_k}{\widetilde{P}_\beta} \quad \underset{\mathtt{d}_k-1}{} & \text{if } \mathtt{d}_{k-1} = \mathtt{d}_k + 1 \end{cases}$$

(3) *gluing edges*:
For each index $1 \leq j < k$ the projective module of $\widetilde{P}_\bullet^{(j)}$ at degree $\mathtt{d}_j$ is connected to the projective module of $\widetilde{P}_\bullet^{(j+1)}$ at the same degree by an edge.

In contrast to the cyclic case, the last complex is not connected the first complex in the gluing diagram of any finite word.

Remark 4.1.6. *The following finite words were not covered by the translation above*:

word ω	$\mathtt{p}_\star^{[\mathtt{d}_0]}$	$\mathtt{p}_\diamond^{[\mathtt{d}_0]}$	$\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{n}^{[\mathtt{d}_1]}\mathtt{p}_\star$	*where* $\mathtt{d}_0,\mathtt{d}_1\in\mathbb{Z},\mathtt{n}\in\mathbb{N}^+$
gluing diagram	$\underset{\mathtt{d}_0}{\widetilde{P}_\star}$	$\underset{\mathtt{d}_0}{\widetilde{P}_\diamond}$	$\underset{\mathtt{d}+1}{\widetilde{P}_\star} \xrightarrow{2\mathtt{n}} \underset{\mathtt{d}}{\widetilde{P}_\star}$	*and* $\mathtt{d}=\min\{\mathtt{d}_0,\mathtt{d}_1\}$

4.1.4 Strings and bands

In this subsection we introduce the invariants which parametrize the *indecomposable* projective complexes in $\mathrm{D^b}(\Lambda)$.

Definition 4.1.7. *Let ω be a cyclic word of $\check{\mathfrak{B}}$ of length $2k$ as in Definition 4.1.2*:

$$\omega = ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ldots \mathtt{n}_{2k-1}^{[\mathtt{d}_{2k-1}]}\mathtt{n}_{2k}^{[\mathtt{d}_{2k}]})$$

(1) The reversed word ω^{rev} *of ω is defined as*

$$\omega^{\mathrm{rev}} = ({}^{[\mathtt{d}_{2k}]}\mathtt{n}_{2k}^{[\mathtt{d}_{2k-1}]}\mathtt{n}_{2k-1} \ldots {}^{[\mathtt{d}_2]}\mathtt{n}_2^{[\mathtt{d}_1]}\mathtt{n}_1^{[\mathtt{d}_0]}).$$

(2) Let $0 \leq j < 2k$. The rotation *of ω by j is defined as*

$$\omega^{[j]} = ({}^{[\mathtt{d}_j]}\mathtt{n}_{j+1}^{[\mathtt{d}_{j+1}]}\mathtt{n}_{j+2}^{[\mathtt{d}_{j+2}]} \ldots \mathtt{n}_{j+2k-1}^{[\mathtt{d}_{j+2k-1}]}\mathtt{n}_{j+2k}^{[\mathtt{d}_{j+2k}]}).$$

where all indices are considered modulo the length $2k$.

(3) The word ω is periodic *if $\omega = \omega^{[j]}$ for some non-trivial rotation $1 \leq j < 2k$.*

(4) The word ω is symmetric *if $\omega^{\mathrm{rev}} = \omega^{[j]}$ for some rotation $0 \leq j < 2k$.*

Example 4.1.8. *Let us note the following examples of cyclic words*:

non-periodic, symmetric :	$({}^{[0]}1^{[1]}1^{[0]})$	$({}^{[1]}3^{[2]}3^{[1]}4^{[0]}4^{[1]})$
	$({}^{[0]}1^{[1]}2^{[2]}3^{[3]}3^{[2]}2^{[1]}1^{[0]})$	$({}^{[1]}2^{[2]}3^{[3]}3^{[2]}2^{[1]}1^{[0]}1^{[1]})$
periodic, non-symmetric :	$({}^{[0]}1^{[1]}2^{[0]}1^{[1]}2^{[0]}1^{[1]}2^{[0]})$	$({}^{[1]}3^{[2]}4^{[1]}1^{[0]}2^{[1]}3^{[2]}4^{[1]}1^{[0]}2^{[1]})$
non-periodic, non-symmetric :	$({}^{[0]}1^{[1]}2^{[0]}2^{[1]}1^{[0]}1^{[1]}2^{[0]})$	$({}^{[1]}1^{[2]}2^{[1]}3^{[2]}1^{[1]}2^{[2]}3^{[1]})$

Definition 4.1.9. *Let $\Bbbk$ be an algebraically closed field.*

(1) A band word *is given by any cyclic and non-periodic word ω of $\check{\mathfrak{B}}$.*

(2) A band (ω, m, λ) of $\check{\mathfrak{B}}$ is given by a band word ω, some "multiplicity" $m \in \mathbb{N}$ and some "eigenvalue" $\lambda \in \Bbbk^*$ such that $\lambda \neq (-1)^{\mathtt{k}+1}$ if ω is symmetric with length $2\mathtt{k}$.

(3) Two bands (ω, m, λ) and (υ, n, μ) are equivalent *if $m = n$, ω and υ have the same length $2\mathtt{k}$, and there is some index $0 \leq \mathtt{j} < 2\mathtt{k}$ such that*

- $\upsilon = \omega^{[\mathtt{j}]}$ *and* $\mu = \lambda$, *or*
- $\upsilon^{\mathrm{rev}} = \omega^{[\mathtt{j}]}$ *and* $\mu = \lambda^{-1}$.

Next, we are going to define three classes of strings.

Definition 4.1.10. *Let ω be a finite word of $\check{\mathfrak{B}}$ of length $k \geq 0$ as in Definition 4.1.4:*

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ldots \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta)$$

(1) The reversed word ω^{rev} *of ω is defined as*

$$\omega^{\mathrm{rev}} = (\mathtt{p}_\beta^{[\mathtt{d}_\mathtt{k}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-2}]} \ldots \mathtt{n}_2^{[\mathtt{d}_1]}\mathtt{n}_1^{[\mathtt{d}_0]}\mathtt{p}_\alpha).$$

(2) The word ω is symmetric *if $\omega = \omega^{\mathrm{rev}}$.*

(3) The left end $\mathtt{p}_\alpha$ *of ω is* special *if $\alpha = \diamond$. Similarly, the* right end $\mathtt{p}_\beta$ *of ω is* special *if $\beta = \diamond$.*

(4) Assume that both ends of ω are special. Set $\breve{\omega} = (\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ldots {}^{[\mathtt{d}_{\mathtt{k}-2}]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k})$. The word ω is quasi-symmetric *if there is some subword $\upsilon = (\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ldots {}^{[\mathtt{d}_{\mathtt{j}-1}]}\mathtt{n}_\mathtt{j})$ of ω of shorter length $1 \leq \mathtt{j} < \mathtt{k}$ such that $\mathtt{d}_\mathtt{j} = \mathtt{d}_\mathtt{k}$ and*

$$\breve{\omega} \;=\; \upsilon \;^{[\mathtt{d}_\mathtt{j}]}\; \upsilon^{\mathrm{rev}} \;^{[\mathtt{d}_0]} \;\ldots\; \upsilon \;^{[\mathtt{d}_\mathtt{j}]}\; \upsilon^{\mathrm{rev}} \;^{[\mathtt{d}_0]}\; \upsilon$$

where υ appears an odd number and υ^{rev} an even number of times.

Words of zero length will be considered as non-symmetric.

Example 4.1.11. *Let us note the following examples of finite words*:

symmetric :	$\mathtt{p}_\star^{[1]}2^{[2]}1^{[3]}3^{[4]}3^{[3]}1^{[2]}2^{[1]}\mathtt{p}_\star$	$\mathtt{p}_\diamond^{[1]}2^{[0]}1^{[1]}3^{[2]}3^{[1]}1^{[0]}2^{[1]}\mathtt{p}_\diamond$
non-symmetric :	$\mathtt{p}_\star^{[1]}2^{[2]}1^{[1]}3^{[0]}3^{[1]}1^{[0]}2^{[1]}\mathtt{p}_\star$	$\mathtt{p}_\star^{[1]}2^{[0]}1^{[1]}3^{[2]}3^{[1]}1^{[0]}2^{[1]}\mathtt{p}_\diamond$
quasi-symmetric :	$\mathtt{p}_\diamond^{[1]}2^{[0]}1^{[1]}1^{[0]}2^{[1]}2^{[0]}1^{[1]}\mathtt{p}_\diamond$	$\mathtt{p}_\diamond^{[1]}1^{[0]}2^{[1]}3^{[2]}3^{[1]}2^{[0]}1^{[1]}1^{[0]}2^{[1]}3^{[2]}\mathtt{p}_\diamond$
not (quasi-)symmetric :	$\mathtt{p}_\diamond^{[1]}2^{[0]}1^{[1]}2^{[0]}1^{[1]}2^{[0]}1^{[1]}\mathtt{p}_\diamond$	$\mathtt{p}_\diamond^{[1]}2^{[0]}1^{[1]}3^{[0]}1^{[1]}2^{[0]}\mathtt{p}_\diamond$

Definition 4.1.12. *In the following let ω be a finite word of $\check{\mathfrak{B}}$.*

(1) (a) A finite word ω is usual *if ω has no special ends and is not symmetric.*

(b) A usual string *is given by any usual word ω.*

(c) Two usual strings υ and ω are equivalent *if $\upsilon = \omega$ or $\upsilon = \omega^{\mathrm{rev}}$.*

(2) (a) A finite word ω is special *if ω has exactly one special end. (if ω has length zero, the only end of ω counts as only* one *end).*

(b) A special string *(ω, ε) is given by a special word ω and one sign $\varepsilon \in \{+, -\}$.*

(c) *Two special strings* (υ, δ) *and* (ω, ε) *are* equivalent *if* $(\upsilon, \delta) = (\omega, \varepsilon)$ *or* $(\upsilon, \delta) = (\omega^{\mathrm{rev}}, \varepsilon)$.

(3) (a) *A finite word* ω *is* bispecial *if* ω *has two special ends and* ω *is neither symmetric nor quasi-symmetric.*

(b) *A* bispecial string $(\omega, m, \varepsilon_1, \varepsilon_2)$ *is given by any bispecial word* ω*, some "multiplicity"* $m \in \mathbb{N}^+$ *and two signs* $\varepsilon_1, \varepsilon_2 \in \{+, -\}$.

(c) *Two bispecial strings* (υ, n) *and* (ω, m) *are* equivalent *if* $(\upsilon, n, \delta_1, \delta_2) = (\omega, m, \varepsilon_1, \varepsilon_2)$ *or* $(\upsilon, n, \delta_1, \delta_2) = (\omega^{rev}, m, \varepsilon_2, \varepsilon_1)$.

A string *of* $\check{\mathfrak{B}}$ *is given by any usual, special or bispecial string. By a* string word *we will mean any usual, special or bispecial word. A string of one of three types cannot be equivalent to any band, or any string of another of the three types.*

Definition 4.1.13. *Strings and bands of* $\check{\mathfrak{B}}_{\mathrm{fd}}$ *are defined as follows*:

(1) *A string of* $\check{\mathfrak{B}}_{\mathrm{fd}}$ *is given by any string of* $\check{\mathfrak{B}}$ *such that its string word* ω *has* non-zero *length and does* not *contain an end of type* $\mathtt{p}_\infty$.

(2) *A band of* $\check{\mathfrak{B}}_{\mathrm{fd}}$ *is the same as a band of* $\check{\mathfrak{B}}$.

The definitions of strings and bands are motivated by the following result:

Theorem 4.1.14 ([**BD04**])**.** *There is a bijection between the equivalence classes of strings and bands of* $\check{\mathfrak{B}}$ *and the isomorphism classes of indecomposable complexes in* $\mathrm{D^b}(\Lambda)$:

$$[\text{STRINGS } \textit{and} \text{ BANDS } \textit{of } \check{\mathfrak{B}}] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{D^b}(\Lambda)]$$

This bijection restricts to strings and bands of $\check{\mathfrak{B}}_{\mathrm{fd}}$ *and indecomposable complexes in* $\mathrm{D^b_{fd}}(\Lambda)$.

Since our formulation of strings and bands is different from the formulation in [**BD04**] we give a proof of Theorem 4.1.14 in the next subsection.

4.1.5 Abbreviation of strings and bands of $\mathfrak{B}$

In this subsection we apply the results on combinatorics of bunches of semichains from Appendix A to prove Theorem 4.1.14.

Let $\mathfrak{B}$ be the bunch of semichains of the derived category $\mathrm{D^b}(\Lambda)$ as described in Example 3.5.3. As noted in (3.6.1) there is a bijection between the isomorphism classes of indecomposable complexes in $\mathrm{D^b}(\Lambda)$ and equivalence classes of strings and bands of $\check{\mathfrak{B}}$.

$$\mathrm{ind}[\mathrm{D^b}(\Lambda)] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{Rep}^*(\mathfrak{B})] \cup \mathrm{ind}[\mathrm{D^b}(\Lambda_\star)] \xleftrightarrow{1:1} \begin{bmatrix} \text{STRINGS and} \\ \text{BANDS of } \check{\mathfrak{B}} \end{bmatrix}$$

The definitions of strings and bands for any bunch of semichains are stated in Appendix A, Section A.2. Next, we will describe how to translate strings and bands of the bunch of semichains $\mathfrak{B}$ into strings and bands of $\check{\mathfrak{B}}$ as defined in 4.1.12 and 4.1.9.

4.1.5.1 The alphabet $\mathfrak{A}_{\mathfrak{B}}$ of the bunch of semichains $\mathfrak{B}$

First, we need to consider the *alphabet* $\mathfrak{A}_{\mathfrak{B}}$ associated to the bunch $\mathfrak{B}$. According to Definition A.2.2 the alphabet $\mathfrak{A}_{\mathfrak{B}} = \big(\overline{\mathfrak{C}}, \overline{\mathfrak{R}}, \sim, -\big)$ of $\mathfrak{B}$ is given as follows:

(1) The column letters $\overline{\mathfrak{C}}$ are given by $\overline{\mathfrak{C}} = \bigcup\limits_{d\in\mathbb{Z}} \overline{\mathfrak{C}}_d$ where $\overline{\mathfrak{C}}_d = \{\zeta^{(d)}\}$ for each $d \in \mathbb{Z}$.

(2) The row letters $\overline{\mathfrak{R}}$ are given by $\overline{\mathfrak{R}} = \bigcup\limits_{d\in\mathbb{Z}}$ where for each $d \in \mathbb{Z}$ we have a chain

$$\overline{\mathfrak{R}}_d = \{\ \beta_1^{(d)} < \ldots < \beta_n^{(d)} < \beta_{n+1}^{(d)} < \ldots < \gamma^{(d)} < \ldots < \alpha_{n+1}^{(d)} < \alpha_n^{(d)} < \ldots < \alpha_1^{(d)}\ \}$$

(3) There are two symmetric relations $-$ and $\sim$ on the set $\mathfrak{A}_{\mathfrak{B}} = \overline{\mathfrak{R}} \cup \overline{\mathfrak{C}}$ given by
   - $\varrho^{(d)} - \zeta^{(d)}$ for any $\varrho^{(d)} \in \overline{\mathfrak{R}}_d$ and any $d \in \mathbb{Z}$,
   - $\zeta^{(d)} \sim \zeta^{(d)}$ for any $d \in \mathbb{Z}$, and $\alpha_{2n}^{(d)} \sim \beta_{2n}^{(d+1)}$ for any $n \in \mathbb{N}^+$ and any $d \in \mathbb{Z}$.

A string or band of $\mathfrak{B}$ has some *finite* respectively *cyclic word* of the alphabet $\mathfrak{A}_{\mathfrak{B}}$ as main datum. Words of $\mathfrak{A}_{\mathfrak{B}}$ will be defined next. For the sake of brevity, we will not recall all notions from Section A.2 such as *(quasi-)symmetry* of finite words, or *periodicity* of cyclic words. These definitions will only play a technical role in the proof of Theorem 4.1.14.

4.1.5.2 Cyclic words

The following Definition is an adaptation of Definition A.2.11 of cyclic words to the particular case of the alphabet $\mathfrak{A}_{\mathfrak{B}}$:

Definition 4.1.15. *A* cyclic word w *of the alphabet* $\mathfrak{A}_{\mathfrak{B}}$ *is given by a sequence*

$$w = \alpha_1 \sim \alpha_2 - \alpha_3 \sim \alpha_4 - \quad \ldots \quad \alpha_{2k-1} \sim \alpha_{2k} - \tag{4.1.5}$$

of letters $\alpha_j \in \mathfrak{A}_{\mathfrak{B}}$*, where* $1 \leq j \leq 2k$*, subject to the following conditions*:

- $\alpha_{2j-1} \sim \alpha_{2j}$ *holds in* $\mathfrak{A}_{\mathfrak{B}}$ *for any* $1 \leq j \leq k$*,*
- $\alpha_{2j} - \alpha_{2j+1}$*, where* $1 \leq j < k$*, and also* $\alpha_{2k} - \alpha_1$ *hold in* $\mathfrak{A}_{\mathfrak{B}}$*.*

Let w be a cyclic word of $\mathfrak{A}_{\mathfrak{B}}$. Let us note that w has length $\mathtt{8k}$ for some $\mathtt{k} \in \mathbb{N}^+$. The cyclic word w of $\mathfrak{A}_{\mathfrak{B}}$ is translated into a cyclic word ω of $\check{\mathfrak{B}}$ as follows:

(1) we erase all relations of type $-$ in w

(2) we translate the remaining subwords $\alpha' \sim \alpha''$ of w by the following table:

subword of w	$\zeta^{(d)} \sim \zeta^{(d)}$	$\beta_{2n}^{(d)} \sim \alpha_{2n}^{(d-1)}$	$\alpha_{2n}^{(d)} \sim \beta_{2n}^{(d+1)}$
parameter in ω	$^{[\mathtt{d}]}$	$\mathtt{n}$	$\mathtt{n}$

$d \in \mathbb{Z}, \quad n \in \mathbb{N}^+.$

(3) if the new word w' starts (respectively, ends) with $^{[\mathtt{d}]}$ for some $\mathtt{d} \in \mathbb{Z}$ we add another symbol $^{[\mathtt{d}]}$ after the right end (respectively, before the left end) of w'.

The outcome is some cyclic word ω of $\check{\mathfrak{B}}$:

$$\omega = ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2 \dots {}^{[\mathtt{d}_{2\mathtt{j}}]}\mathtt{n}_{2\mathtt{j}+1}^{[\mathtt{d}_{2\mathtt{j}+1}]}\mathtt{n}_{2\mathtt{j}+2} \dots {}^{[\mathtt{d}_{2\mathtt{k}-2}]}\mathtt{n}_{2\mathtt{k}-1}^{[\mathtt{d}_{2\mathtt{k}-1}]}\mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}_{2\mathtt{k}}]})$$

with $\mathtt{n}_1, \mathtt{n}_2, \dots, \mathtt{n}_{2\mathtt{k}} \in \mathbb{N}^+$ and certain degrees $\mathtt{d}_0, \mathtt{d}_1, \dots, \mathtt{d}_{2\mathtt{k}} \in \mathbb{Z}$, where $\mathtt{d}_0 = \mathtt{d}_{2\mathtt{k}}$.

Example 4.1.16. *Let w be the following cyclic word of $\mathfrak{A}_{\mathfrak{B}}$:*

$$\begin{aligned} w =& \zeta^{(2)} \sim \zeta^{(2)} - \beta_8^{(2)} \sim \alpha_8^{(1)} - \zeta^{(1)} \sim \zeta^{(1)} - \beta_6^{(1)} \sim \alpha_6^{(0)} - \\ & \zeta^{(0)} \sim \zeta^{(0)} - \alpha_{10}^{(1)} \sim \beta_{10}^{(2)} - \zeta^{(1)} \sim \zeta^{(1)} - \alpha_{10}^{(1)} \sim \beta_{10}^{(2)} - \end{aligned}$$

Then w translates into the cyclic word $\omega = ({}^{[2]}4^{[1]}3^{[0]}5^{[1]}5^{[2]})$ from Example 4.1.3.

Remark 4.1.17. *In the notations above, the cyclic word w of $\mathfrak{A}_{\mathfrak{B}}$ is periodic (respectively symmetric) in the sense of Definition A.2.11 (respectively Definition A.2.16) if and only if the word ω of $\check{\mathfrak{B}}$ is periodic (respectively symmetric) .*

4.1.5.3 Finite regular words

The next Definition is a specialization of Definition A.2.4 of finite regular words to the alphabet $\mathfrak{A}_{\mathfrak{B}}$:

Definition 4.1.18. *Let $\mathfrak{A}_{\mathfrak{B}}$ be the alphabet defined above.*

(1) A finite regular word *w of $\mathfrak{A}_{\mathfrak{B}}$ with length $k \in \mathbb{N}^+$ is given by a sequence*

$$w = \alpha_1 - \alpha_2 \sim \alpha_3 - \alpha_4 \sim \quad \dots \quad \alpha_{2k-1} - \alpha_{2k}$$

of letters $\alpha_j \in \mathfrak{A}_{\mathfrak{B}}$, where $1 \le j \le 2k$, subject to the following conditions:

- *the ends of w satisfy $\alpha_1, \alpha_{2k} \in \{\alpha_{2n-1}^{(d)}, \beta_{2n-1}^{(d)}, \gamma^{(d)}, \zeta^{(d)} \mid d \in \mathbb{Z}, n \in \mathbb{N}^+ \}$,*
- *the relation $\alpha_{2j-1} - \alpha_{2j}$ holds in $\mathfrak{A}_{\mathfrak{B}}$ for any $1 \le j \le k$,*
- *the relation $\alpha_{2j} \sim \alpha_{2j+1}$ holds in $\mathfrak{A}_{\mathfrak{B}}$ for any $1 \le j < k$.*

*(2) The left end α_1 of w is special if $\alpha_1 = \zeta^{(d)}$ for some $d \in \mathbb{Z}$.
Similarly, the right end α_{2k} of w is special if $\alpha_{2k} \in \mathfrak{C}$.*

The word w of $\mathfrak{A}_{\mathfrak{B}}$ is translated into a word ω of $\check{\mathfrak{B}}$ as follows:

(1) we erase all relations of type $-$ in w

(2) the left end of w is abbreviated as follows

left end of w	$\beta_{2n-1}^{(d)}$	$\alpha_{2n-1}^{(d)}$	$\zeta^{(d)}$	$\gamma^{(d)}$
beginning of ω	$\mathtt{p}_\star^{[\mathtt{d}-1]}\mathtt{n}^{[\mathtt{d}]}$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}$	$\mathtt{p}_\diamond^{[\mathtt{d}]}$	$\mathtt{p}_\infty^{[\mathtt{d}]}$

$d \in \mathbb{Z}, \quad n \in \mathbb{N}^+.$

(3) the intermediate part of w is translated by the same rules as for cyclic words:

subword of w	$\zeta^{(d)} \sim \zeta^{(d)}$	$\beta_{2n}^{(d)} \sim \alpha_{2n}^{(d-1)}$	$\alpha_{2n}^{(d)} \sim \beta_{2n}^{(d+1)}$
parameter in ω	$^{[\mathtt{d}]}$	$\mathtt{n}$	$\mathtt{n}$

$d \in \mathbb{Z}, \quad n \in \mathbb{N}^+.$

(4) finally, the right end of w is abbreviated similarly to the rules of the left end:

right end of w	$\beta^{(d)}_{2n-1}$	$\alpha^{(d)}_{2n-1}$	$\zeta^{(d)}$	$\gamma^{(d)}$
ending of ω	${}^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\star$	${}^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}+1]}\mathtt{p}_\star$	${}^{[\mathtt{d}]}\mathtt{p}_\diamond$	${}^{[\mathtt{d}]}\mathtt{p}_\infty$

$d \in \mathbb{Z}, \quad n \in \mathbb{N}^+.$

The outcome is some finite word ω of $\check{\mathfrak{B}}$:

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta), \qquad \text{where} \quad \alpha,\beta \in \{\, \star, \diamond, \infty \,\}$$

Example 4.1.19. *In the following we abbreviate some finite regular words of* $\mathfrak{A}_{\mathfrak{B}}$ *which are actually* string words *in the sense of Definition A.2.9.*

(1) Let w be the usual word:

$$w = \alpha_1^{(0)} - \zeta^{(0)} \sim \zeta^{(0)} - \alpha_4^{(0)} \sim \beta_4^{(1)} - \zeta^{(1)} \sim \zeta^{(1)} - \alpha_6^{(1)} \sim \beta_6^{(2)} - \zeta^{(2)} \sim \zeta^{(2)} - \beta_1^{(3)}$$

Then its translation is given by the word $\omega = (\mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{2}^{[1]}\mathtt{p}_\star)$.

(2) Let w be the special word:

$$w = \zeta^{(1)} - \beta_6^{(1)} \sim \alpha_6^{(0)} - \zeta^{(0)} \sim \zeta^{(0)} - \alpha_4^{(0)} \sim \beta_4^{(1)} - \zeta^{(1)} \sim \zeta^{(1)} - \alpha_5^{(1)}.$$

Then $\omega = (\mathtt{p}_\diamond^{[1]}\mathtt{3}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{p}_\star)$*, the word from Example 4.1.5.*

(3) Let w be the bispecial word:

$$w = \zeta^{(1)} - \beta_2^{(1)} \sim \alpha_2^{(0)} - \zeta^{(0)} \sim \zeta^{(0)} - \alpha_4^{(0)} \sim \beta_4^{(1)} - \\ \zeta^{(1)} \sim \zeta^{(1)} - \alpha_6^{(1)} \sim \beta_6^{(2)} - \zeta^{(2)} \sim \zeta^{(2)} - \beta_4^{(2)} \sim \alpha_4^{(1)} - \zeta^{(1)}$$

Then w translates into $\omega = (\mathtt{p}_\diamond^{[1]}\,\mathtt{1}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{2}^{[1]}\,\mathtt{p}_\diamond)$.

Remark 4.1.20. *Let w be some word of* $\mathfrak{B}$ *and ω be the translated word of* $\check{\mathfrak{B}}$.

(1) The words w and ω have the same *number of special ends.*

(2) The word w is (quasi-)symmetric in the sense of Definition A.2.8) if and only if ω is (quasi-)symmetric in the sense of Definition of 4.1.10).

- By Remark A.3.20 each end of w is given by some letter of type $\alpha^{(d)}_{2n-1}$ or $\beta^{(d)}_{2n-1}$ for some $n \in \mathbb{N}^+$ and $d \in \mathbb{Z}$.

4.1.5.4 Proof of Theorem 4.1.14 on strings and bands of the derived category

At last, we define a map on the equivalence classes of strings and bands of $\mathfrak{B}$:

$$(\,\check{}\,): \qquad [\text{STRINGS and BANDS of } \mathfrak{B}] \longrightarrow [\text{STRINGS and BANDS of } \check{\mathfrak{B}}]$$

Namely, we abbreviate only the words in strings and bands of $\mathfrak{B}$ according to Subsections 4.1.5.2 and 4.1.5.3:

Ω	w	(w, ε_1)	$(w, m, \varepsilon_1, \varepsilon_2)$	(w, m, λ)
$\check{\Omega}$	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)

$\varepsilon_1,\varepsilon_2 \in \{+,-\}, \quad m \in \mathbb{N}^+,\ \lambda \in \Bbbk^* \setminus \Delta$ (4.1.6)

For the sake of completeness, we give a proof of Theorem 4.1.14:

PROOF. The claim is that there is a bijection between the equivalence classes of strings and bands of $\mathfrak{B}$ and the isomorphism classes of indecomposable complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \overset{1:1}{\dashleftarrow\dashrightarrow} [\text{STRINGS and BANDS of } \check{\mathfrak{B}}]$$

By 3.5.14 there is the bijection

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda_\star)] \cup [\text{STRINGS and BANDS of } \mathfrak{B}]$$

Therefore it is sufficient to show that there are the following two bijections of equivalence respectively isomorphism classes:

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda_\star)] \cup [\text{STRINGS of } \mathfrak{B}] \overset{1:1}{\dashleftarrow\dashrightarrow} [\text{STRINGS of } \check{\mathfrak{B}}] \tag{4.1.7}$$

$$[\text{BANDS of } \mathfrak{B}] \overset{1:1}{\dashleftarrow\dashrightarrow} [\text{BANDS of } \check{\mathfrak{B}}] \tag{4.1.8}$$

(1) By Lemma 3.5.9 the indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Lambda_\star)$ are given by $\underset{\mathtt{d+1}}{P_\star} \xrightarrow{\mathtt{2n}} \underset{\mathtt{d}}{P_\star}$ and $\underset{\mathtt{d}}{P_\star}$ for some $\mathtt{n} \in \mathbb{N}^+$ and $\mathtt{d} \in \mathbb{Z}$. By Remark 4.1.6 these objects are represented by the usual strings $\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\star$ and $\mathtt{p}_\star^{[\mathtt{d}]}$ of $\check{\mathfrak{B}}$. Let us call these *strings of* $\check{\mathfrak{B}}_\star$. So we have a bijection:

$$\mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda_\star)] \xleftrightarrow{1:1} [\text{STRINGS of } \check{\mathfrak{B}}_\star]$$

(2) We consider the abbreviation from (4.1.6) on *strings*:

$$(\,\check{}\,): \ [\text{STRINGS of } \mathfrak{B}] \longrightarrow [\text{STRINGS of } \check{\mathfrak{B}}] \tag{4.1.9}$$

(a) Let Ω be a string of $\mathfrak{B}$. Then $\check{\Omega}$ is a string of $\check{\mathfrak{B}}$ by Remark 4.1.20. Note that for any regular finite word w of $\mathfrak{A}_{\mathfrak{B}}$ it holds that $(w^{\mathrm{rev}})\check{} = \omega^{\mathrm{rev}}$. This implies that two strings Ω', Ω'' of $\mathfrak{B}$ are equivalent if and only if $\check{\Omega}'$ and $\check{\Omega}''$ are equivalent. That is, the map $(\,\check{}\,)$ in (4.1.9) is injective.

(b) Let υ be some finite word of $\check{\mathfrak{B}}$ which is not a string of $\check{\mathfrak{B}}_\star$. The main observation is that there is some *unique* finite regular word w of $\mathfrak{A}_{\mathfrak{B}}$ such that $\omega = \upsilon$. In other words, a word w of $\mathfrak{A}_{\mathfrak{B}}$ is *uniquely determined* by the sequence of natural numbers of its row letters and degrees of its column letters.

By the arguments above there is a well-defined bijection between the equivalence classes of the following strings:

$$[\text{STRINGS of } \check{\mathfrak{B}}_\star \text{ or } \mathfrak{B}] \xrightarrow{1:1} [\text{STRINGS of } \check{\mathfrak{B}}]$$

Together with step (1) of the proof this yields the first bijection of (4.1.7).

(3) It remains to consider the abbreviation on bands:

$$(\,\check{}\,): \ [\text{BANDS of } \mathfrak{B}] \longrightarrow [\text{BANDS of } \check{\mathfrak{B}}] \tag{4.1.10}$$

Note that any band of $\mathfrak{B}$ is equivalent to some band (w, m, λ) of $\mathfrak{B}$ such that *w begins with some column letter $\zeta \in \mathfrak{C}$*. In the following we will consider only representatives of this type in the equivalence classes of bands of $\mathfrak{B}$.

(a) Let $\Omega = (w, m, \lambda)$ be some band of $\mathfrak{B}$. We claim that $\check{\Omega} = (\omega, m, \lambda)$ is a band of $\check{\mathfrak{B}}$. The word ω of $\check{\mathfrak{B}}$ is non-periodic by Remark 4.1.17. Let $2k$ denote the length of ω for some $k \in \mathbb{N}^+$. Then w has length $8k$. Moreover, since Ω is a band, it holds that $\lambda \neq (-1)^{\varsigma(w)+1}$, where $\varsigma(w)$ is given by Definition A.2.16. Let us recall that $\varsigma(w) = \frac{1}{2} \cdot \widetilde{\varsigma}(w)$, where

$$\widetilde{\varsigma}(w) = \#\{2 \leq i < 2k \text{ even} \mid \alpha_{i-1} \neq \alpha_i \text{ and } \alpha_{i-1} \sim \alpha_i \text{ is a parallel subword } \}.$$

For the bunch of semichains $\mathfrak{B}$ any subword of the form $\alpha' \sim \alpha''$ is parallel. It follows that $\varsigma(w) = k$. This shows the claim.

(b) Let υ be some cyclic word of $\check{\mathfrak{B}}$. As in the case of finite words, the main point is that there is a *unique* cyclic word w of $\mathfrak{A}_{\mathfrak{B}}$ such that w begins with some column letter $\zeta \in \mathfrak{C}$ and $\omega = \upsilon$.

(c) Next, we derive the equivalence conditions in Definition 4.1.9 for bands of $\check{\mathfrak{B}}$ from the equivalence conditions of bands in Definition A.2.18 for $\mathfrak{B}$. Let (υ, m, λ) be some band of $\check{\mathfrak{B}}$. Let $2k$ denote the length of the cyclic word υ. and let w be the cyclic word of $\mathfrak{A}_{\mathfrak{B}}$ beginning with a column letter such that $\omega = \upsilon$.

(i) Let $0 \leq j < 2k$. It holds that $\upsilon^{[j]} = (w^{[4j]})\check{}$. By Definition A.2.18 the band (w, m, λ) of $\mathfrak{A}_{\mathfrak{B}}$ is equivalent to $(w^{[4j]}, m, \lambda^{\xi(4j,w)})$, where $\xi(4j, w) := (-1)^{\nu(4j,w)}$ and

$$\nu(4j, w) := \#\{\ 2 \leq i \leq 4j \text{ even} \mid \alpha_{i-1} \sim \alpha_i \text{ is a parallel subword in } w\ \}.$$

We note that $\nu(4j, w) = 2j$, hence $\xi(4j, w) = 1$. This implies that the bands (υ, m, λ) and $(w^{[j]}, m, \lambda)$ of $\check{\mathfrak{B}}$ are equivalent.

(ii) For the second equivalence condition, we note that $\upsilon^{\text{rev}} = \big((w^{[2]})^{\text{rev}}\big)\check{}$. Since $\nu(2, w) = 1$ it holds that $\xi(2, w) = -1$. It follows that the band (w, m, λ) is equivalent to $(w^{[2]}, m, \lambda^{-1})$ and $\big((w^{[2]})^{\text{rev}}, m, \lambda^{-1}\big)$. This shows that the bands (υ, m, λ) and $(\upsilon^{\text{rev}}, m, \lambda^{-1})$ of $\check{\mathfrak{B}}$ are equivalent.

Finally let us note that there are no further equivalence conditions for bands of $\check{\mathfrak{B}}$, or bands of $\mathfrak{B}$ in which the cyclic words begin with a column letter.

It follows that the abbreviation in (4.1.10) is a well-defined bijection and proves the second bijection of (4.1.8).

Summarized, the abbreviation of strings and bands of $\mathfrak{B}$ of the preceding subsections together with the strings of $\check{\mathfrak{B}}_\star$ yields exactly Definitions 4.1.12 and 4.1.9 of strings and bands of $\mathfrak{B}$. $\square$

4.2 Gluing of indecomposable projective complexes

Our next goal is to describe the construction of indecomposable projective complexes in $D^b(\Lambda)$ from strings or bands of $\check{\mathfrak{B}}$.

As described in Subsection 4.1.3 strings and bands can be viewed by gluing diagrams with some additional parameters.

4.2.1 Gluing arrows in words

To pass from gluing diagrams to indecomposable complexes, we need to assign to *every gluing edge* in the gluing diagram an *orientation* first.

Let ω be a string word of length $\mathtt{k}$, or a band word of even length $\mathtt{k}$. Let $\widehat{\omega}$ denote the word ω with altered numbers $\widehat{\mathtt{n}}_{\mathtt{j}}$ as defined in (4.1.2) respectively (4.1.3). In particular, $\widehat{\omega}$ has the form

$$\widehat{\omega} = \begin{cases} (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_2]}\ldots{}^{[\mathtt{d}_{k-2}]}\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{k-1}]}\widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d}_k]}\mathtt{p}_\beta) & \text{if } \omega \text{ is a string word} \\ ({}^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_2]}\ldots{}^{[\mathtt{d}_{k-2}]}\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{k-1}]}\widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d}_k]}) & \text{where } \mathtt{k} \in 2\mathbb{N}^+, \text{ if } \omega \text{ is a band word.} \end{cases}$$

In this section, we are going to transform $\widehat{\omega}$ into a new word of the form

$$\overset{\leftrightarrow}{\omega} = \begin{cases} (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]} \updownarrow \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \updownarrow \ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{k-1}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d}_k]}\mathtt{p}_\beta) & \text{if } \omega \text{ is a string word} \\ (\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]} \updownarrow \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \updownarrow \ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{k-1}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d}_k]} \updownarrow) & \text{if } \omega \text{ is a band word} \end{cases}$$

where each symbol $\updownarrow$ stands either for an oriented arrow $\uparrow$ or $\downarrow$.

Let $\mathtt{j}$ be any index with $\mathtt{1} \leq \mathtt{j} < \mathtt{k} - \mathtt{1}$ for string words, respectively $\mathtt{1} \leq \mathtt{j} \leq \mathtt{k}$ for band words. To determine the orientation in $\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d}_j]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{j+1}]}$ we have to distinguish between two cases:

(1) the case of a "*non-symmetric subword*":
Assume that $\mathtt{d}_{\mathtt{j}-1} \neq \mathtt{d}_{\mathtt{j}+1}$ or $\widehat{\mathtt{n}}_{\mathtt{j}} \neq \widehat{\mathtt{n}}_{\mathtt{j}+1}$ in the word $\widehat{\omega}$.
This is the simpler case. We recall that $\mathtt{d}_{\mathtt{j}-1}, \mathtt{d}_{\mathtt{j}+1} \in \{\mathtt{d}_{\mathtt{j}} - \mathtt{1}, \mathtt{d}_{\mathtt{j}} + \mathtt{1}\}$. The orientation at degree $\mathtt{d}_{\mathtt{j}}$ is set up as follows:

$$\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d}_j]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{j+1}]} = \begin{cases} \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d}_j]} \uparrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{j+1}]} & \text{if } \mathtt{d}_{\mathtt{j}-1} > \mathtt{d}_{\mathtt{j}} > \mathtt{d}_{\mathtt{j}+1}, \\ & \text{or } \mathtt{d}_{\mathtt{j}-1} > \mathtt{d}_{\mathtt{j}} < \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j}} < \widehat{\mathtt{n}}_{\mathtt{j}+1}, \\ & \text{or } \mathtt{d}_{\mathtt{j}-1} < \mathtt{d}_{\mathtt{j}} > \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j}} > \widehat{\mathtt{n}}_{\mathtt{j}+1}. \\ \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d}_j]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{j+1}]} & \text{in any other case such that} \\ & \mathtt{d}_{\mathtt{j}-1} \neq \mathtt{d}_{\mathtt{j}+1} \text{ or } \mathtt{n}_{\mathtt{j}} \neq \mathtt{n}_{\mathtt{j}+1}. \end{cases} \tag{4.2.1}$$

In terms of gluing diagrams, the gluing edges are oriented as follows:

$\begin{tikzcd} \widetilde{P}' \arrow[r, "\mathtt{n_j}"] & \widetilde{P}_\diamond & \\ & \widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}'' \end{tikzcd}$	$\begin{tikzcd} \widetilde{P}' \arrow[r, "\mathtt{n_j}"] & \widetilde{P}_\diamond \\ \widetilde{P}'' \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}_\diamond \arrow[u, dashed] \end{tikzcd}$ if $\mathtt{n_j} < \mathtt{n_{j+1}}$	$\begin{tikzcd} \widetilde{P}_\diamond \arrow[r, "\mathtt{n_j}"] & \widetilde{P}' \\ \widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}'' \end{tikzcd}$ if $\mathtt{n_j} > \mathtt{n_{j+1}}$
$\begin{tikzcd} & \widetilde{P}_\diamond \arrow[r, "\mathtt{n_j}"] \arrow[d, dashed] & \widetilde{P}' \\ \widetilde{P}'' \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}_\diamond & \end{tikzcd}$	$\begin{tikzcd} \widetilde{P}' \arrow[r, "\mathtt{n_j}"] & \widetilde{P}_\diamond \arrow[d, dashed] \\ \widetilde{P}'' \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}_\diamond \end{tikzcd}$ if $\mathtt{n_j} > \mathtt{n_{j+1}}$	$\begin{tikzcd} \widetilde{P}_\diamond \arrow[r, "\mathtt{n_j}"] \arrow[d, dashed] & \widetilde{P}' \\ \widetilde{P}_\diamond \arrow[r, "\mathtt{n_{j+1}}"] & \widetilde{P}'' \end{tikzcd}$ if $\mathtt{n_j} < \mathtt{n_{j+1}}$

(2) the case of a "*symmetric subword*":
Assume that $\mathtt{d_{j-1}} = \mathtt{d_{j+1}}$ and $\mathtt{n_j} = \mathtt{n_{j+1}}$ in the word ω.
This case is more technical.

(a) If ω is not cyclic, let us denote by $\breve{\omega}$ and $\breve{\omega}^{\mathrm{rev}}$ the subwords

$$\breve{\omega} = \widehat{\mathtt{n}}_1^{[\mathtt{d_1}]}\widehat{\mathtt{n}}_2^{[\mathtt{d_2}]}\dots{}^{[\mathtt{d_{k-2}}]}\widehat{\mathtt{n}}_{\mathtt{k-1}}^{[\mathtt{d_{k-1}}]}\widehat{\mathtt{n}}_{\mathtt{k}}, \qquad \text{and} \qquad \breve{\omega}^{\mathrm{rev}} = \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d_{k-1}}]}\widehat{\mathtt{n}}_{\mathtt{k-1}}^{[\mathtt{d_{k-2}}]}\dots{}^{[\mathtt{d_2}]}\widehat{\mathtt{n}}_2^{[\mathtt{d_1}]}\mathtt{n_1}.$$

We define an "ambient word" $\overline{\omega}$ as follows:

$$\overline{\omega} = \begin{cases} \omega & \text{if } \omega \text{ is a band word} \\ {}^{[\mathtt{d_0}]}\ \breve{\omega}\ {}^{[\mathtt{d_k}]} & \text{if } \omega \text{ is a usual word} \\ {}^{[\mathtt{d_k}]}\ \breve{\omega}^{\mathrm{rev}}\ {}^{[\mathtt{d_0}]}\ \breve{\omega}\ {}^{[\mathtt{d_k}]} & \text{if } \omega \text{ is a special word with special left end} \\ {}^{[\mathtt{d_0}]}\ \breve{\omega}\ {}^{[\mathtt{d_k}]}\ \breve{\omega}^{\mathrm{rev}}\ {}^{[\mathtt{d_0}]} & \text{if } \omega \text{ is a special word with special right end} \\ {}^{[\mathtt{d_k}]}\ \breve{\omega}^{\mathrm{rev}}\ {}^{[\mathtt{d_0}]}\ \breve{\omega}\ {}^{[\mathtt{d_k}]}\ \breve{\omega}^{\mathrm{rev}}\ {}^{[\mathtt{d_0}]} & \text{if } \omega \text{ is a bispecial word} \end{cases}$$

In the following we view the subword $\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]}\widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]}$ of $\widehat{\omega}$ as a subword of $\overline{\omega}$.

(b) Let Υ be the *maximal symmetric* subword of $\overline{\omega}$ with $\widehat{\mathtt{n}}_{\mathtt{j}}{}^{[\mathtt{d_j}]}\widehat{\mathtt{n}}_{\mathtt{j+1}}$ in the middle and with degrees as ends. In particular, $\Upsilon = (\upsilon\,{}^{[\mathtt{d_j}]}\,\upsilon^{\mathrm{rev}})$ for some subword υ of $\overline{\omega}$:

$$\overline{\omega} = \dots{}^{[\mathtt{d_{j-m-1}}]}\widehat{\mathtt{n}}_{\mathtt{j-m}}\underbrace{\overbrace{{}^{[\mathtt{d_{j-m}}]}\widehat{\mathtt{n}}_{\mathtt{j-m+1}}\dots{}^{[\mathtt{d_{j-2}}]}\widehat{\mathtt{n}}_{\mathtt{j-1}}^{[\mathtt{d_{j-1}}]}\widehat{\mathtt{n}}_{\mathtt{j}}}^{\upsilon}\ {}^{[\mathtt{d_j}]}\ \overbrace{\widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]}\widehat{\mathtt{n}}_{\mathtt{j+2}}^{[\mathtt{d_{j+2}}]}\quad\dots\quad\widehat{\mathtt{n}}_{\mathtt{j+m}}^{[\mathtt{d_{j+m}}]}}^{\upsilon^{\mathrm{rev}}}}_{\Upsilon}\widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d_{j+m+1}}]}\dots \tag{4.2.2}$$

(c) There are two cases concerning the maximal symmetric subword Υ :

- Assume that $\Upsilon = \overline{\omega}$. In this rather special case, the word ω must be a *symmetric band word* such that $\widehat{\mathtt{n}}_{\mathtt{j-i}} = \widehat{\mathtt{n}}_{\mathtt{j+i+1}}$ and $\mathtt{d_{j-i}} = \mathtt{d_{j+i}}$ for any $\mathtt{0} \leq \mathtt{i} < \mathtt{2k}$. In this case, we set $\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} = \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]}$.
- Let $\Upsilon \neq \overline{\omega}$. Let $\mathtt{2m}$ denote the length of Υ. In this more common case we consider the values of ${}^{[\mathtt{d_{j-m-1}}]}\widehat{\mathtt{n}}_{\mathtt{j-m}}^{[\mathtt{d_{j-m}}]}$ and $\widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d_{j+m+1}}]}$. Since Υ is symmetric, it holds that $\mathtt{d_{j+m}} = \mathtt{d_{j-m}}$. By the maximality of Υ, we have that

$\mathtt{n}_{\mathtt{j-m}} \neq \mathtt{n}_{\mathtt{j+m+1}}$ or $\mathtt{d}_{\mathtt{j-m-1}} \neq \mathtt{d}_{\mathtt{j+m+1}}$. Now we set

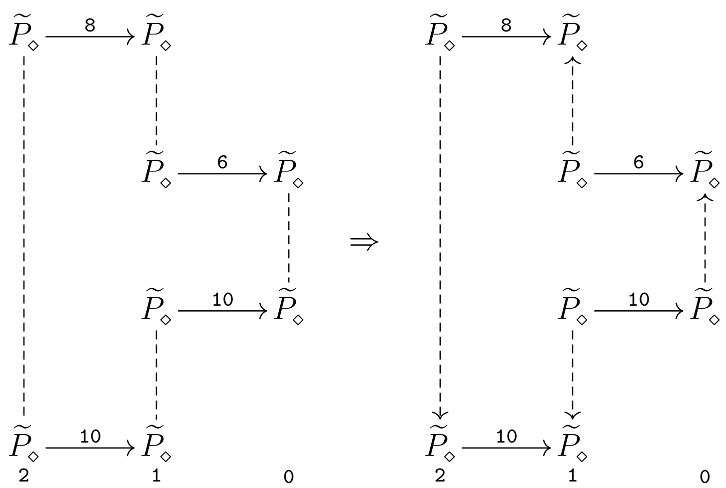
$$\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} = \begin{cases} \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \uparrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} & \text{if } \mathtt{d}_{\mathtt{j-m-1}} > \mathtt{d}_{\mathtt{j\pm m}} > \mathtt{d}_{\mathtt{j+m+1}}, \\ & \text{or } \mathtt{d}_{\mathtt{j-m-1}} > \mathtt{d}_{\mathtt{j\pm m}} < \mathtt{d}_{\mathtt{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} < \widehat{\mathtt{n}}_{\mathtt{j+m+1}}, \\ & \text{or } \mathtt{d}_{\mathtt{j-m-1}} < \mathtt{d}_{\mathtt{j\pm m}} > \mathtt{d}_{\mathtt{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} > \widehat{\mathtt{n}}_{\mathtt{j+m+1}}. \\ \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} & \text{otherwise.} \end{cases} \tag{4.2.3}$$

In other terms, the orientation in $\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]}$ is the same as the orientation in $\widehat{\mathtt{n}}_{\mathtt{j-m}}^{[\mathtt{d_{j-m}}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d_{j+m+1}}]}$ which is determined by the rules (4.2.1) for non-symmetric subwords.

The rules above are applied for any index $\mathtt{j}$ such that $\mathtt{1} \leq \mathtt{j} \leq \mathtt{k} - \mathtt{1}$ for string words respectively $\mathtt{1} \leq \mathtt{j} \leq \mathtt{k}$ for band words.

The new word $\overset{\leftrightarrow}{\omega}$ translates directly into a gluing diagram with oriented edges.

Example 4.2.1. *(1) a* band word *without symmetric subwords*:
Let $\omega = ({}^{[2]}4^{[1]}3^{[0]}5^{[1]}5^{[2]})$. *Then* $\overset{\leftrightarrow}{\omega} = (8^{[1]} \uparrow 6^{[0]} \uparrow 10^{[1]} \downarrow 10^{[2]} \uparrow)$:



(2) a symmetric band word:
Let $\omega = ({}^{[2]}3^{[1]}2^{[0]}2^{[1]}3^{[2]})$. *Then* $\widehat{\omega} = ({}^{[2]}6^{[1]}4^{[0]}4^{[1]}6^{[2]})$.
Using (4.2.1) *we get* $6^{[1]} \uparrow 4^{[0]}$ *and* $4^{[1]} \downarrow 6^{[2]}$ *in* $\overset{\leftrightarrow}{\omega}$. *The remaining gluing arrows are at symmetry axes, so* $\overset{\leftrightarrow}{\omega} = (6^{[1]} \uparrow 4^{[0]} \downarrow 4^{[1]} \downarrow 6^{[2]} \downarrow)$.

(3) a usual string *without symmetric subwords*:
Let $\omega = (\mathtt{p}_\star^{[1]}1^{[0]}2^{[1]}3^{[2]}2^{[1]}\mathtt{p}_\star)$ *be the usual string of* $\check{\mathfrak{B}}$ *from Example 4.1.19. Then its gluing word is given by* $\widehat{\omega} = (\mathtt{p}_\star^{[1]}1^{[0]} \uparrow 4^{[1]} \downarrow 6^{[2]} \uparrow 3^{[1]}\mathtt{p}_\star)$ *according to* (4.2.1).

(4) a special word *without symmetric subwords*:
Let $\omega = (\mathtt{p}_\diamond^{[1]}\mathtt{3}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{p}_\star)$. *Then* $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_\diamond^{[1]}\mathtt{6}^{[0]} \downarrow \mathtt{4}^{[1]} \downarrow \mathtt{5}^{[2]}\mathtt{p}_\star)$:

$$\begin{tikzcd}
 & & \widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{3}}"] & \widetilde{P}_\diamond \arrow[d, dash, dashed] & & & \widetilde{P}_\diamond \arrow[r, "\mathtt{6}"] & \widetilde{P}_\diamond \arrow[d, dashed] \\
 & & \widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{2}}"] \arrow[d, dash, dashed] & \widetilde{P}_\diamond & \Rightarrow & & \widetilde{P}_\diamond \arrow[r, "\mathtt{4}"] \arrow[d, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\star \arrow[rr, "\widehat{\mathtt{3}}"] & & \widetilde{P}_\diamond & & & \widetilde{P}_\star \arrow[r, "\mathtt{5}"] & \widetilde{P}_\diamond & \\
\mathtt{2} & & \mathtt{1} & \mathtt{0} & & \mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}$$

(5) a special word *with one symmetric subword*:
Let $\omega = (\mathtt{p}_\star^{[1]}\mathtt{2}^{[2]}\mathtt{2}^{[1]}\mathtt{2}^{[0]}\mathtt{1}^{[1]}\mathtt{1}^{[0]}\mathtt{p}_\diamond)$. *Then* $\widehat{\omega} = (\mathtt{p}_\star^{[1]}\mathtt{3}^{[2]}\mathtt{4}^{[1]}\mathtt{4}^{[0]}\mathtt{2}^{[1]}\mathtt{2}^{[0]}\mathtt{p}_\diamond)$.
By (4.2.1) *we have* $\overset{\leftrightarrow}{\omega} = (\mathtt{3}^{[2]} \downarrow \mathtt{4}^{[1]} \uparrow \mathtt{4}^{[0]} \downarrow \mathtt{2}^{[1]} \updownarrow \mathtt{2}^{[0]})$. *To determine the last gluing arrow in* $\overset{\leftrightarrow}{\omega}$ *we consider the ambient word*

$$\overline{\omega} = \left({}^{[1]}\ \breve{\omega}\ {}^{[0]}\ \breve{\omega}^{\mathrm{rev}}\ {}^{[1]}\right) = \left({}^{[1]}\ \underbrace{\mathtt{3}^{[2]}\,\mathtt{4}^{[1]}\,\mathtt{4}^{[0]}\,\mathtt{2}^{[1]}\mathtt{2}}_{\breve{\omega}}\ {}^{[0]}\ \underbrace{\mathtt{2}^{[1]}\,\mathtt{2}^{[0]}\,\mathtt{4}^{[1]}\,\mathtt{4}^{[2]}\,\mathtt{3}}_{\breve{\omega}^{\mathrm{rev}}}\ {}^{[1]}\right)$$

$$\overbrace{\mathtt{4}^{[1]}\,\mathtt{4}^{[0]}\,\mathtt{2}^{[1]}\mathtt{2}\ {}^{[0]}\ \mathtt{2}^{[1]}}^{{}^{[1]}\mathtt{4}^{[0]} \downarrow \mathtt{2}^{[1]}}$$

In the notations of (4.2.2) *the maximal symmetric subword is given by* $\Upsilon = {}^{[0]}\mathtt{2}^{[1]}\mathtt{2}^{[0]}$. *In particular,* ${}^{[\mathtt{d}_{\mathtt{j-m-1}}]}\widehat{\mathtt{n}}_{\mathtt{j-m}}^{[\mathtt{d}_{\mathtt{j-m}}]} = {}^{[1]}\mathtt{4}^{[0]}$ *and* $\widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d}_{\mathtt{j+m+1}}]} = \mathtt{2}^{[1]}$. *By* (4.2.3) *it follows that* $\mathtt{2}^{[1]} \downarrow \mathtt{2}^{[0]}$ *in* $\overset{\leftrightarrow}{\omega}$.

(6) *a bispecial word with two symmetric subwords*:
Let $\omega = (\mathtt{p}_\diamond^{[1]}\mathtt{3}^{[0]}\mathtt{1}^{[1]}\mathtt{1}^{[0]}\mathtt{3}^{[1]}\mathtt{3}^{[0]}\mathtt{2}^{[1]}\mathtt{p}_\diamond)$. *Then* $\widehat{\omega} = \left(\mathtt{p}_\diamond^{[1]}\mathtt{6}^{[0]}\mathtt{2}^{[1]}\mathtt{2}^{[0]}\mathtt{6}^{[1]}\mathtt{6}^{[0]}\mathtt{4}^{[1]}\mathtt{p}_\diamond\right)$. *By* (4.2.1) *we obtain that* $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_\diamond{}^{[1]}\mathtt{6}^{[0]} \downarrow \mathtt{2}^{[1]} \updownarrow \mathtt{2}^{[0]} \uparrow \mathtt{6}^{[1]} \updownarrow \mathtt{6}^{[0]} \downarrow \mathtt{4}^{[1]}\mathtt{p}_\diamond)$. *To orient the remaining arrows we consider the ambient word of* ω :

$$\overline{\omega} = \left({}^{[1]}\,\breve{\omega}^{\mathrm{rev}}\,{}^{[1]}\,\breve{\omega}\,{}^{[1]}\,\breve{\omega}^{\mathrm{rev}}\,{}^{[1]}\right) = \left({}^{[1]}\,\underbrace{\mathtt{4}^{[0]}\ldots\mathtt{2}^{[1]}\,\mathtt{2}^{[0]}\,\mathtt{6}}_{\dot{\omega}^{\mathrm{rev}}}\ {}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{2}\ \underbrace{{}^{[1]}\mathtt{2}^{[0]}\,\mathtt{6}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{4}\ {}^{[1]}}_{{}^{[1]}\mathtt{2}^{[0]} \uparrow \mathtt{4}^{[1]}}\ \underbrace{\mathtt{4}^{[0]}\ldots\mathtt{6}}_{\dot{\omega}^{\mathrm{rev}}}\ {}^{[1]}\right)$$

$$\overbrace{\mathtt{2}^{[1]}\,\mathtt{2}^{[0]}\,\mathtt{6}\ {}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{2}\ {}^{[1]}\mathtt{2}^{[0]}\,\mathtt{6}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{4}\ {}^{[1]}}^{{}^{[1]}\mathtt{2}^{[0]} \uparrow \mathtt{4}^{[1]}}$$

By (4.2.3) *it follows that* $\mathtt{2}^{[1]} \uparrow \mathtt{2}^{[0]}$ *and* $\mathtt{6}^{[1]} \uparrow \mathtt{6}^{[0]}$ *in* $\overset{\leftrightarrow}{\omega}$.

4.2.2 Gluing rules for string and band complexes

In this subsection, we pass from the gluing word $\overset{\leftrightarrow}{\omega}$ in a string or band of $\breve{\mathfrak{B}}$ to an indecomposable projective complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

$$\begin{tikzcd}
\text{string or band } \Omega \text{ of } \breve{\mathfrak{B}} \arrow[rr, mapsto] \arrow[dr, squiggly, "\text{gluing diagram with}"'] & & P_\bullet(\Omega) \in \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \\
 & \widetilde{P}_\bullet \in \mathrm{D}^{\mathrm{b}}(\Gamma) \arrow[ur, squiggly, "\text{gluing rules}"'] &
\end{tikzcd}$$

4.2.2.1 Usual, special and bispecial strings with trivial multiplicity

In the following let ω be some finite word of $\check{\mathfrak{B}}$:

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta), \qquad \text{where} \qquad \alpha,\beta \in \{\ \star, \diamond\ \}.$$

Let Ω be one the following strings of $\check{\mathfrak{B}}$:

- a usual string $\Omega = \omega$. In this case, ω is non-symmetric and $\alpha = \beta = \star$.
- a special string $\Omega = (\omega, \varepsilon)$, where $\varepsilon \in \{\ +, -\ \}$. In this case, either α or $\beta = \diamond$.
- a bispecial string $\Omega = (\omega, 1, \varepsilon_1, \varepsilon_2)$, where $\varepsilon_1, \varepsilon_2 \in \{\ +, -\ \}$, with *trivial multiplicity.* In this case, the word ω is neither symmetric nor quasi-symmetric and $\alpha = \beta = \diamond$.

We refer to Definitions 4.1.10 and 4.1.12 for the notions of strings of $\check{\mathfrak{B}}$.

Let $\overset{\leftrightarrow}{\omega}$ denote the word of ω with gluing arrows as defined in Subsection 4.2.1.

The string complex $P_\bullet(\Omega)$ is obtained in two steps from the gluing diagram of $\overset{\leftrightarrow}{\omega}$:

(1) *new differentials*:
Whenever we have one of the following two situations in the gluing diagram of $\overset{\leftrightarrow}{\omega}$ we apply the following rules:

first gluing rule	second gluing rule
$\begin{tikzcd} \widetilde{P}_\diamond \arrow[r, "\widetilde{d}"] \arrow[d, dashed] & \widetilde{P}_i \\ \widetilde{P}_\diamond & \end{tikzcd} \Rightarrow \begin{tikzcd} \widetilde{P}_\diamond \arrow[r, "\widetilde{d}"] \arrow[d, dashed] & \widetilde{P}_i \\ \widetilde{P}_\diamond \arrow[ur, "\widetilde{d}"'] & \end{tikzcd}$	$\begin{tikzcd} \widetilde{P}_i \arrow[r, "\widetilde{d}"] & \widetilde{P}_\diamond \\ & \widetilde{P}_\diamond \arrow[u, dashed] \end{tikzcd} \Rightarrow \begin{tikzcd} \widetilde{P}_i \arrow[r, "\widetilde{d}"] \arrow[dr, "-\widetilde{d}"'] & \widetilde{P}_\diamond \\ & \widetilde{P}_\diamond \arrow[u, dashed] \end{tikzcd}$

$i \in \{\star, \diamond\}$

In particular, we obtain additional differentials in the gluing diagram.
After application of these rules, there can be new subdiagrams of the above form. In this case both gluing rules are applied again.

(2) *new projectives*:

(a) The ends of the gluing diagram are changed into projective Λ-modules as follows:

- if $\Omega = \omega$ is usual, both projectives $\widetilde{P}_\star$ at the ends of the gluing diagram are replaced by $P_\star$.
- if $\Omega = (\omega, \varepsilon)$ is special, then the end of type $\widetilde{P}_\star$ in the gluing diagram is changed to $P_\star$ while the other end $\widetilde{P}_\diamond$ becomes the projective P_ε.
- if $\Omega = (\omega, m, \varepsilon_1, \varepsilon_2)$ is bispecial, the first end $\widetilde{P}_\diamond$ is changed into $\widetilde{P}_{\varepsilon_2}$ while the last end of type $\widetilde{P}_\diamond$ obtains the sign P_{ε_2}.

(b) alternating *signs* at gluing edges:
At this point, we ignore the orientations of the gluing edges. We replace any projective $\widetilde{P}_\diamond$ at the top of a gluing edge by P_+, while any projective $\widetilde{P}_\diamond$ at the bottom of an edge becomes P_-.

We note that after this step, we obtain *uniquely determined* differentials of projective Λ-modules, where we use the notation from Table 4.1.1.

The outcome is some projective complex $P_\bullet(\Omega)$ of $\mathrm{D^b}(\Lambda)$.

Example 4.2.2. *Let* $\omega = (\mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{2}^{[1]}\mathtt{p}_\star)$ *be the usual string of* $\check{\mathfrak{B}}$ *from Example 4.2.1. Then* $\widehat{\omega} = (\mathtt{p}_\star^{[1]}\mathtt{1}^{[0]} \uparrow \mathtt{4}^{[1]} \downarrow \mathtt{6}^{[2]} \uparrow \mathtt{3}^{[1]}\mathtt{p}_\star)$ *and the gluing diagram is obtained as follows*:

$$\begin{tikzcd}
 & \widetilde{P}_\star \arrow[r, "\mathtt{1}"] & \widetilde{P}_\diamond \\
 & \widetilde{P}_\diamond \arrow[r, "\mathtt{4}"] \arrow[d, dashed] & \widetilde{P}_\diamond \arrow[u, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\mathtt{6}"] & \widetilde{P}_\diamond & \\
\widetilde{P}_\diamond \arrow[r, "\mathtt{3}"] \arrow[u, dashed] & \widetilde{P}_\star & \\
\mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & P_\star \arrow[r, "\mathtt{1}"] \arrow[dr, "-\mathtt{1}"'] & P_+ \\
 & P_+ \arrow[r, "\mathtt{4}"] \arrow[d, dashed] & P_- \arrow[u, dashed] \\
P_+ \arrow[r, "\mathtt{6}"] \arrow[ur, "-\mathtt{6}"] \arrow[dr, "\mathtt{3}"] & P_- \arrow[ur, "\mathtt{4}"'] & \\
P_- \arrow[r, "\mathtt{3}"] \arrow[u, dashed] & P_\star & \\
\mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}$$

Example 4.2.3. *Let* Ω *be the special string* $(\omega, +)$ *of* $\check{\mathfrak{B}}$ *where* $\omega = (\mathtt{p}_\diamond^{[1]}\mathtt{3}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{p}_\star)$. *According to Example 4.2.1 its gluing word is given by* $\widehat{\omega} = (\mathtt{p}_\diamond^{[1]}\mathtt{6}^{[0]} \downarrow \mathtt{4}^{[1]} \downarrow \mathtt{5}^{[2]}\mathtt{p}_\star)$ *and the gluing diagram has the following form*:

$$\begin{tikzcd}
 & \overset{+}{\widetilde{P}}_\diamond \arrow[r, "\mathtt{6}"] & \widetilde{P}_\diamond \arrow[d, dashed] \\
 & \widetilde{P}_\diamond \arrow[r, "\mathtt{4}"] \arrow[d, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\star \arrow[r, "\mathtt{5}"] & \widetilde{P}_\diamond & \\
\mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & P_+ \arrow[r, "\mathtt{6}"] & P_+ \arrow[d, dashed] \\
 & P_+ \arrow[r, "\mathtt{4}"] \arrow[ur, "-\mathtt{4}"] \arrow[d, dashed] & P_- \\
P_\star \arrow[r, "\mathtt{5}"] \arrow[ur, "-\mathtt{5}"] & P_- \arrow[ur, "\mathtt{4}"'] \arrow[uur, "-\mathtt{4}"'] & \\
\mathtt{2} & \mathtt{1} & \mathtt{0}
\end{tikzcd}$$

In this case the gluing rules have been applied twice.

4.2.2.2 Bands

Let $\Omega = (\omega, m, \lambda)$ be a band of $\check{\mathfrak{B}}$ in the sense of Definition 4.1.7. More precisely, let

$$\omega = ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \dots \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[\mathtt{d}_{\mathtt{k}}]})$$

be a non-periodic cyclic word of some length $\mathtt{k} \in 2\mathbb{N}^+$, $m \in \mathbb{N}^+$, and $\lambda \in \Bbbk^*$ such that $\lambda \neq (-1)^{\frac{\mathtt{k}}{2}+1}$ if ω is symmetric.

Let $\overset{\leftrightarrow}{\omega}$ denote the word with gluing arrows as defined in Subsection 4.2.1.

We are going to define the corresponding band complex $P_\bullet(\Omega)$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.
For the shortest possible band of length 2, we refer to Remark 4.2.5. In the following we assume that ω has even length $\mathtt{k} \geq 4$.

(1) *new differentials* with eigenvalue λ :
The projective at degree $\mathtt{d}_0$ of the first complex in the gluing diagram of $\overset{\leftrightarrow}{\omega}$ is decorated by the eigenvalue λ. At this gluing edge the gluing rules are modified as follows:

$$
\begin{tikzcd}
\overset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "\tilde d"] \arrow[d, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "\tilde d"] \arrow[d, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond \arrow[ur, "\lambda \tilde d"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "\tilde d"] & \overset{\lambda}{\widetilde{P}_\diamond} \\
& \widetilde{P}_\diamond \arrow[u, dashed]
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "\tilde d"] \arrow[dr, "-\lambda^{-1} \tilde d"'] & \widetilde{P}_\diamond \\
& \widetilde{P}_\diamond \arrow[u, dashed]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overset{\lambda}{\widetilde{P}_\diamond} & \\
\widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "\tilde d"] & \widetilde{P}_\diamond
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\widetilde{P}_\diamond \arrow[dr, "\lambda^{-1} \tilde d"] & \\
\widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "\tilde d"] & \widetilde{P}_\diamond
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overset{\lambda}{\widetilde{P}_\diamond} \arrow[d, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\tilde d"] & \widetilde{P}_\diamond
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
& \widetilde{P}_\diamond \arrow[d, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\tilde d"] \arrow[ur, "-\lambda \tilde d"] & \widetilde{P}_\diamond
\end{tikzcd}
\tag{4.2.4}
$$

(2) Next, we carry out the *same steps* as described for *strings* in Subsection 4.2.2.1.

(3) *Blowing up*:
We replace every projective P_+ by its m-fold direct sum P_+^m and every projective P_- by P_-^m. Every differential of type $\pm(\widehat{\mathtt{n}})$ in the gluing diagram is replaced by the square matrix $\pm(\widehat{\mathtt{n}})\,\mathrm{Id}_m$ of size m. Finally, every differential $\pm\lambda^{\pm 1}(\widehat{\mathtt{n}})$ is replaced by the square matrix $\pm(\widehat{\mathtt{n}})J_m(\lambda^{\pm 1})$, where $J_m(\lambda^{\pm 1})$ denotes the Jordan block with eigenvalue λ.

Example 4.2.4. *Let $\Omega = (\omega, 2, \lambda)$ be the band of $\check{\mathfrak{B}}$ with $\omega = (4^{[1]} \uparrow 3^{[0]} \uparrow 5^{[1]} \downarrow 5^{[2]})$ and $\lambda \in \Bbbk^*$. According to Example 4.2.1 the gluing word of ω is given by $\widehat{\omega} = (8^{[1]} \uparrow 6^{[0]} \uparrow 10^{[1]} \downarrow 10^{[2]} \uparrow)$. The band complex $P_\bullet(\Omega)$ has the following form:*

$$
\begin{tikzcd}
\overset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "8"] \arrow[ddd, dashed] & \widetilde{P}_\diamond & \\
& \widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "6"] & \widetilde{P}_\diamond \\
& \widetilde{P}_\diamond \arrow[r, "10"] \arrow[d, dashed] & \widetilde{P}_\diamond \arrow[u, dashed] \\
\widetilde{P}_\diamond \arrow[r, "10"] & \widetilde{P}_\diamond & \\
2 & 1 & 0
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
P_-^2 \arrow[r, "8\,\mathrm{Id}_2"] \arrow[ddd, dashed] \arrow[dr, "-8\,\mathrm{Id}_2"'] & P_+^2 \arrow[dr, "6\,\mathrm{Id}_2"] \arrow[ddr, "-6\,\mathrm{Id}_2"] & \\
& P_-^2 \arrow[u, dashed] \arrow[r, "6\,\mathrm{Id}_2"] \arrow[dr, "-6\,\mathrm{Id}_2"'] & P_+^2 \\
& P_+^2 \arrow[r, "10\,\mathrm{Id}_2"] \arrow[d, dashed] & P_-^2 \arrow[u, dashed] \\
P_+^2 \arrow[r, "10\,\mathrm{Id}_2"] \arrow[uuur, "8\,\mathrm{J}_2(\lambda)"] \arrow[ur, "-10\,\mathrm{Id}_2"] & P_-^2 \arrow[ur, "10\,\mathrm{Id}_2"'] & \\
2 & 1 & 0
\end{tikzcd}
$$

Remark 4.2.5 (Shortest band)**.** *Let $\Omega = (\omega, m, \lambda)$ be a band of $\check{\mathfrak{B}}$ such that ω is a cyclic word of length 2.*

- *By passing to an equivalent band we may assume that* $\omega = ({}^{[\mathtt{d+1}]}\mathtt{n}_1^{[\mathtt{d}]}\mathtt{n}_2^{[\mathtt{d+1}]})$ *for some* $\mathtt{d} \in \mathbb{Z}$ *and* $\mathtt{n}_1, \mathtt{n}_2 \in \mathbb{N}^+$ *with* $\mathtt{n}_1 \leq \mathtt{n}_2$.
- *The remaining data of* Ω *is given by some multiplicity* $m \in \mathbb{N}^+$ *and an eigenvalue* $\lambda \in \Bbbk^*$ *with* $\lambda \neq 1$ *if* $\mathtt{n}_1 = \mathtt{n}_2$.
- *The gluing word of* ω *is given by* $\overset{\leftrightarrow}{\omega} = (\widehat{\mathtt{n}}_1^{[\mathtt{d+1}]} \downarrow \widehat{\mathtt{n}}_2^{[\mathtt{d}]} \downarrow)$.
- *In the following we skip the identity matrix of size* m *on differentials and indicate only the Jordan block* $\mathrm{J} = \mathrm{J}_m(-\lambda)$.

The gluing diagram and the band complex of Ω *are given as follows*:

$$\begin{tikzcd}
\overset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_1"] & \widetilde{P}_\diamond \arrow[d, dashed] & & P_-^m \arrow[r, "\mathrm{J}\,\widehat{\mathtt{n}}_1"] \arrow[r, bend left, "\widehat{\mathtt{n}}_2"] \arrow[dr, "\widehat{\mathtt{n}}_2"] & P_+^m \arrow[d, dashed] & & \\
\underset{\mathtt{d+1}}{\widetilde{P}_\diamond} \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2"] & \underset{\mathtt{d}}{\widetilde{P}_\diamond} & \Rightarrow & \underset{\mathtt{d+1}}{P_+^m} \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[ur, "\widehat{\mathtt{n}}_2"] & \underset{\mathtt{d}}{P_-^m} & \Rightarrow & P_-^m \oplus P_+^m \xrightarrow{\ d\ } P_+^m \oplus P_-^m ,
\end{tikzcd}$$

where the differential is given by

$$d = \begin{bmatrix} \mathrm{J}^*_{-+} & (\widehat{\mathtt{n}}_2)_{++}\ \mathrm{Id} \\ (\widehat{\mathtt{n}}_2)_{--}\ \mathrm{Id} & (\widehat{\mathtt{n}}_2)_{+-}\ \mathrm{Id} \end{bmatrix} = \begin{bmatrix} (\mathsf{a}_-\mathsf{c}^{\mathsf{n}_1-1}\mathsf{b}_+)\,\mathsf{J} + (\mathsf{a}_-\mathsf{c}^{\mathsf{n}_2-1}\mathsf{b}_+)\ \mathrm{Id} & (\mathsf{a}_+\mathsf{c}^{\mathsf{n}_2-1}\mathsf{b}_+)\ \mathrm{Id} \\ (\mathsf{a}_-\mathsf{c}^{\mathsf{n}_2-1}\mathsf{b}_-)\ \mathrm{Id} & (\mathsf{a}_+\mathsf{c}^{\mathsf{n}_2-1}\mathsf{b}_-)\ \mathrm{Id} \end{bmatrix}$$

Remark 4.2.6. *As a special case of Remark A.3.13 there is one exceptional case in the construction of symmetric bands of* $\mathfrak{B}$. *Let* $\Omega = (\omega, m, \lambda)$ *be some band of* $\mathfrak{B}$ *with symmetric band word* ω. *Let* $\overset{\leftrightarrow}{\omega}$ *be the word with gluing arrows. Then* Ω *is equivalent to* $\Omega' = (\omega^{\mathrm{rev}}, m, \lambda^{-1})$. *In the construction of* $P_\bullet(\Omega')$ *all gluing arrows in the word* $\overset{\leftrightarrow}{\omega}^{\mathrm{rev}}$ *are set to be the opposite arrows of* $\overset{\leftrightarrow}{\omega}$. *This convention ensures that there is an isomorphism* $P_\bullet(\omega, m, \lambda) \cong P_\bullet(\omega^{\mathrm{rev}}, m, \lambda^{-1})$ *of the glued complexes for symmetric bands.*

4.2.2.3 Bispecial strings with any multiplicity

Let $\Omega = (\omega, m, \varepsilon_1, \varepsilon_2)$ be a bispecial string of $\check{\mathfrak{B}}$ as defined in 4.1.12. More precisely, the word ω is neither symmetric nor quasi-symmetric and has the form

$$\omega = (\mathtt{p}_\diamond^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ldots \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\diamond),$$

$m \in \mathbb{N}^+$, and $\varepsilon_1, \varepsilon_2 \in \{ +, - \}$. As usual, let $\overset{\leftrightarrow}{\omega}$ denote the word with gluing arrows (Subsection 4.2.1).

In the following we assume that ω has length $\mathtt{k} \geq 2$. For the case that $\mathtt{k} = 1$, we refer to Remark 4.2.8.

We will need the following notation:

- Let $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$ denote the opposite signs of ε_1 respectively ε_2.
- Let δ_{ij} denote the square matrix of size m with 1 at entry (i, j) and zero elsewhere.

- Let A_m and B_m denote the following square matrices of size m :

$$\mathrm{A}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i,2i+1} = \begin{bmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \quad \mathrm{B}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i-1,2i} = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}$$

The bispecial string complex $P_\bullet(\Omega)$ is obtained as follows:

(1) First, we perform the *same steps* as for *strings* described in Subsection 4.2.2.1.

(2) *Blowing up*:
Similarly as for bands, every module P_+ at the top of a gluing edge is replaced by P_+^m, while every module P_- at the bottom of an edge is changed into P_-^m. Any differential $\pm(\widehat{\mathtt{n}}_\mathtt{j})$ is replaced by the matrix $\pm(\widehat{\mathtt{n}}_\mathtt{j})\,\mathrm{Id}_m$ for each $\mathtt{2} \leq \mathtt{j} \leq \mathtt{k}-\mathtt{1}$.

(3) *Twisting the beginning*:
The projective $P_\diamond$ at the beginning of the gluing diagram is replaced by

$$P_{\varepsilon_1}^* = \bigoplus_{i=1}^{m} P_i, \qquad \text{where } P_i = \begin{cases} P_{\varepsilon_1} & \text{if } i \text{ is odd,} \\ P_{\overline{\varepsilon}_1} & \text{if } i \text{ is even.} \end{cases} \tag{4.2.5}$$

The next rule depends on whether $P_{\varepsilon_1}^*$ is a source or a sink in the gluing diagram:

- if $\mathtt{d}_1 = \mathtt{d}_0 - \mathtt{1}$, every differential $\pm\widehat{\mathtt{n}}_1$ starting in $P_{\varepsilon_1}^*$ is replaced by $\pm\widehat{\mathtt{n}}_1\ \mathrm{A}_m$.
- if $\mathtt{d}_1 = \mathtt{d}_0 + \mathtt{1}$, every differential $\pm\widehat{\mathtt{n}}_1$ ending in $P_{\varepsilon_1}^*$ is changed into $\pm\widehat{\mathtt{n}}_1\ \mathrm{A}_m^{-1}$.

(4) *Twisting the end*:
Similarly, the last projective $P_\diamond$ at the end of the gluing diagram is replaced by

$$P_{\varepsilon_2}^* = \bigoplus_{i=1}^{m} P_i, \qquad \text{where } P_i = \begin{cases} P_{\varepsilon_2} & \text{if } i \text{ is odd,} \\ P_{\overline{\varepsilon}_2} & \text{if } i \text{ is even.} \end{cases} \tag{4.2.6}$$

The next rule depends on whether $P_{\varepsilon_2}^*$ is a source or a sink in the gluing diagram:

- if $\mathtt{d}_{\mathtt{k}-1} = \mathtt{d}_\mathtt{k} - \mathtt{1}$, every differential $\pm\widehat{\mathtt{n}}_\mathtt{k}$ starting in $P_{\varepsilon_2}^*$ becomes $\pm\widehat{\mathtt{n}}_\mathtt{k}\ \mathrm{B}_m$.
- otherwise, every differential $\pm\widehat{\mathtt{n}}_\mathtt{k}$ ending in $P_{\varepsilon_2}^*$ is renewed by $\pm\widehat{\mathtt{n}}_1\ \mathrm{B}_m^{-1}$.

The outcome is the bispecial string complex $P_\bullet(\Omega)$.

If $m = 1$ for the multiplicity of Ω, the above algorithm yields the same complex as the construction of Subsection 4.2.2.1.

Example 4.2.7. *Let $\Omega = (\omega, 3, +, -)$ be a bispecial string of $\check{\mathfrak{B}}$, where the word is given by $\omega = (\mathtt{p}_\diamond^{[1]}\,\mathtt{1}^{[0]}\mathtt{2}^{[1]}\mathtt{3}^{[2]}\mathtt{2}^{[1]}\,\mathtt{p}_\diamond)$. Then $\widehat{\omega} = (\mathtt{p}_\diamond^{[1]}\,\mathtt{2}^{[0]} \uparrow \mathtt{4}^{[1]} \downarrow \mathtt{6}^{[2]} \uparrow \mathtt{4}^{[1]}\,\mathtt{p}_\diamond)$, and the* *bispecial string complex $P_\bullet(\Omega)$ is given as follows*:

$$
\begin{tikzcd}
 & \overset{\varepsilon_1=+}{\widetilde{P}_\diamond} \arrow[r, "2"] & \widetilde{P}_\diamond \\
 & \widetilde{P}_\diamond \arrow[r, "4"] \arrow[d, dashed] & \widetilde{P}_\diamond \arrow[u, dashed] \\
\widetilde{P}_\diamond \arrow[r, "6"] & \widetilde{P}_\diamond & \\
\widetilde{P}_\diamond \arrow[r, "4"] \arrow[u, dashed] & \underset{\varepsilon_2=-}{\widetilde{P}_\diamond} &
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & & P^*_{\varepsilon_1} \arrow[r, "2\ \mathrm{A}_3"] \arrow[dr, "-2\ \mathrm{A}_3"] & P^3_+ \\
 & & P^3_+ \arrow[r, "4\,\mathrm{Id}_3"] \arrow[d, dashed] & P^3_- \arrow[u, dashed] \\
 & P^3_+ \arrow[r, "6\,\mathrm{Id}_3"] \arrow[ur, "-6\,\mathrm{Id}_3"] \arrow[dr, "4\ \mathrm{B}_3^{-1}"] & P^3_- \arrow[ur, "4\,\mathrm{Id}_3"'] & \\
 & P^3_- \arrow[r, "4\ \mathrm{B}_3^{-1}"] \arrow[u, dashed] & P^*_{\varepsilon_2} &
\end{tikzcd}
$$

In the right diagram the ends are given by $P^*_{\varepsilon_1} = P_+ \oplus P_- \oplus P_+$ and $P^*_{\varepsilon_2} = P_- \oplus P_+ \oplus P_-$.

Remark 4.2.8 (Shortest bispecial string)**.**
Let $\Omega = (\omega, m, \varepsilon_1, \varepsilon_2)$ be a bispecial string of $\check{\mathfrak{B}}$ such that the word ω has length one. Without loss of generality we may assume that $\omega = \mathtt{p}_\diamond^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}\mathtt{p}_\diamond$ for some $\mathtt{d} \in \mathbb{Z}$ and $n \in \mathbb{N}^+$. Let $P^*_{\varepsilon_1}$ and $P^*_{\varepsilon_2}$ be defined as in (4.2.5) *and* (4.2.6)*. Then the bispecial string complex of Ω is given by*

$$P_\bullet(\Omega) = \underset{\mathtt{d}+1}{P^*_{\varepsilon_1}} \xrightarrow{\ d\ } \underset{\mathtt{d}}{P^*_{\varepsilon_2}} \qquad \textit{with differential} \quad d = (2\mathtt{n})\ \mathrm{A}_m\ \mathrm{B}_m^{-1}\,.$$

4.2.3 The gluing theorem by Burban and Drozd

Theorem 4.2.9 ([**BD04**])**.** *Let Λ be the Gelfand order and let $P_\bullet$ be any indecomposable indecomposable complex from $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Then there is some string or band Ω of $\check{\mathfrak{B}}$ such that $P_\bullet$ is isomorphic to the string or band complex $P_\bullet(\Omega)$ as constructed in Subsection 4.2.2.*

The following remark lists the main properties of the "glued complexes" in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

Remark 4.2.10. *Let $P_\bullet = P_\bullet(\Omega)$ be the indecomposable complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ for some string or band Ω of $\check{\mathfrak{B}}$. Let ω be the string or band word in Ω. Let $\mathtt{k}$ be the length of ω and $\mathtt{d}_0, \mathtt{d}_1, \ldots \mathtt{d}_\mathtt{k}$ denote all degree integers in ω. Then the following statements hold*:

(1) The complex $P_\bullet$ is minimal.

(2) The complex $P_\bullet$ has length n for some $n \in \mathbb{N}_0$ if and only if $\max_{0\le j\le k} \mathtt{d}_\mathtt{j} = n$ and $\min_{0\le j\le k} \mathtt{d}_\mathtt{j} = 0$ [1] *.*

(3) The following numbers are equal:

$$\textit{number of projectives of type } P_\star \textit{ in } P_\bullet = \textit{number of ends of type } \mathtt{p}_\star \textit{ in } \omega \tag{4.2.7}$$

[1]We refer to Subsection B.2.2 for the notions of minimality and length of a complex.

For the sake of completeness, we give a proof of the Theorem above.

PROOF. By Theorem 4.1.14 there is a bijection of equivalence classes of strings and bands of $\check{\mathfrak{B}}$ and indecomposable complexes of $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

$$[\textsc{strings} \text{ and } \textsc{bands} \text{ of } \check{\mathfrak{B}}] \xleftrightarrow{1:1} \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)]$$

In more detail, this bijection is given by the composition of the solid arrows in the following diagram:

$$\begin{tikzcd}
{[\textsc{strings} \text{ and } \textsc{bands} \text{ of } \check{\mathfrak{B}}]} \arrow[r, dashed, leftrightarrow] \arrow[d, "\Theta"] & \text{gluing diagrams} \arrow[r, dashed, "\text{gluing rules}"'] \arrow[d, dotted] & \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \arrow[d, equal] \\
\mathrm{ind}[\mathrm{Rep}^{*}(\mathfrak{B})] \cup \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda_{\star})] \arrow[r, "\mathbf{I}"] & \mathrm{ind}[\mathrm{Tri}(\Lambda)] \arrow[r, "\mathbf{G}"] & \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)]
\end{tikzcd} \tag{4.2.8}$$

Let $\check{\Omega}$ be a string or band of $\check{\mathfrak{B}}$. We will recall the abbreviation Θ, the construction of a triple $\mathbf{I}$, and the gluing functor $\mathbf{G}$ in detail below.

We claim that the translation of $\check{\Omega}$ into a gluing diagram and the application of gluing rules yields *the same* indecomposable complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ as the application of the operations Θ, $\mathbf{I}$ and $\mathbf{G}$ to $\check{\Omega}$:

$$P_{\bullet}(\check{\Omega}) \cong \mathbf{G}(\mathbf{I}(\Theta(\check{\Omega})))$$

In other words, we have to show that the above diagram commutes.

A. Let us first deal with the simpler case. Assume that $\check{\Omega}$ does not correspond to an object of the category $\mathrm{Rep}^{*}(\mathfrak{B})$. This means that $\check{\Omega}$ corresponds to some complex $\widetilde{P}_{\bullet} = \Theta(\check{\Omega})$ of $\mathrm{D}^{\mathrm{b}}(\Gamma_{\star})$.

- On the one hand, let $T_{\bullet} = \mathbf{I}(\widetilde{P}_{\bullet})$ be the triple constructed from $\check{\Omega}$. Since $\widetilde{P}_{\bullet} \in \mathrm{D}^{\mathrm{b}}(\mathrm{add}\,\widetilde{P}_{\star})$, the triple $T_{\bullet}$ is given by $(0, \widetilde{P}_{\bullet}, 0)$. The application of the functor $\mathbf{G}$ to $\widetilde{P}_{\bullet}$ yields a complex $P_{\bullet} \in \mathrm{D}^{\mathrm{b}}(\mathrm{add}\,P_{\star})$. The complex $P_{\bullet}$ is given by replacing any projective module $\widetilde{P}_{\star}$ in $\widetilde{P}_{\bullet}$ by $P_{\star}$.
- On the other hand, the gluing diagram of $\check{\Omega}$ is given by the same complex $\widetilde{P}_{\bullet}$ from $\mathrm{D}^{\mathrm{b}}(\mathrm{add}\,\widetilde{P}_{\star})$. The application of gluing rules is trivial. That is, $P_{\bullet}(\check{\Omega})$ is given by the similar complex $P_{\bullet}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{add}\,P_{\star})$.

This yields that $\mathbf{G}(T_{\bullet}) \cong P_{\bullet}(\check{\Omega})$.

B. In the main case, the datum $\check{\Omega}$ corresponds to some canonical form in $\mathrm{Rep}^{*}(\mathfrak{B})$.

We will prove the commutativity of the diagram (4.2.8) for a *representative* case and give some remarks on the general case at the end of the proof.

Let ω denote the string respectively band word in $\check{\Omega}$. We will consider a subword of the following type in ω :

$$\omega = (\ldots\ {}^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\mathtt{n}_3^{[0]}\mathtt{n}_4^{[1]} \ldots) \qquad \text{where } \mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3, \mathtt{n}_4 \in \mathbb{N}^+$$

(1) The translation Θ of $\check{\Omega}$ is given by the following steps:

By Subsection 4.1.5 the datum $\check{\Omega}$ is just a compact notation of some string or band Ω of the bunch of semichains $\mathfrak{B}$. The datum Ω of $\mathfrak{B}$ corresponds to some regular representation $\Theta(\Omega)$ in $\mathrm{Rep}^*(\mathfrak{B})$. The representation $\Theta(\Omega)$ is given by the *canonical forms* constructed by the algorithm of Subsection A.3.2.

Next, we apply the translation Θ to the representative subword of ω. In the case above, the subword of ω is the abbreviation of

$$w = (\dots \zeta^{(1)} - \beta^{(1)}_{\widehat{n}_1} \sim \alpha^{(0)}_{\widehat{n}_1} - \zeta^{(0)} \sim \zeta^{(0)} - \alpha^{(0)}_{\widehat{n}_2} \sim \beta^{(1)}_{\widehat{n}_2} - \zeta^{(1)} \sim \zeta^{(1)} \\ - \beta^{(1)}_{\widehat{n}_3} \sim \alpha^{(0)}_{\widehat{n}_3} - \zeta^{(0)} \sim \zeta^{(0)} - \alpha^{(0)}_{\widehat{n}_4} \sim \beta^{(1)}_{\widehat{n}_4} - \zeta^{(1)})$$

To construct the representation $\Theta(\Omega)$ we pass first to the word $\overset{\leftrightarrow}{w}$ with arrows and signs:

$$\overset{\leftrightarrow}{w} = (\dots \zeta^{(1)}_- - \beta^{(1)}_{\widehat{n}_1} \sim \alpha^{(0)}_{\widehat{n}_1} - \overleftrightarrow{\zeta^{(0)}_+ \sim \zeta^{(0)}_-} - \alpha^{(0)}_{\widehat{n}_2} \sim \beta^{(1)}_{\widehat{n}_2} - \overleftrightarrow{\zeta^{(1)}_+ \sim \zeta^{(1)}_-} \\ -\beta^{(1)}_{\widehat{n}_3} \sim \alpha^{(0)}_{\widehat{n}_3} - \overleftrightarrow{\zeta^{(0)}_+ \sim \zeta^{(0)}_-} - \alpha^{(0)}_{\widehat{n}_4} \sim \beta^{(1)}_{\widehat{n}_4} - \zeta^{(1)}_- \dots)$$

The orientations of the arrows are determined by the rules described in Subsection A.3.2.1. The compact notation of the word $\overset{\leftrightarrow}{w}$ is given by

$$\overset{\leftrightarrow}{\omega} = (\dots\ \widehat{\mathtt{n}}^{[0]}_1 \updownarrow \widehat{\mathtt{n}}^{[1]}_2 \updownarrow \widehat{\mathtt{n}}^{[0]}_3 \updownarrow \widehat{\mathtt{n}}^{[1]}_4 \dots\ )$$

It is straightforward to check that the rules for the gluing arrows in $\overset{\leftrightarrow}{\omega}$ described in Subsection 4.2.1 are exactly the translation of the rules for $\overset{\leftrightarrow}{\omega}$ of Subsection A.3.2.1.

More precisely, *upward* arrows in $\overset{\leftrightarrow}{\omega}$ correspond to *right* arrows in $\overset{\leftrightarrow}{w}$, for example it holds that

$$\widehat{\mathtt{n}}^{[0]}_1 \uparrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \quad \text{if and only if} \quad \alpha^{(0)}_{\widehat{n}_1} - \overrightarrow{\zeta^{(0)} \sim \zeta^{(0)}} - \alpha^{(0)}_{\widehat{n}_2} \text{ in } \overset{\leftrightarrow}{w}, \text{ and}$$
$$\widehat{\mathtt{n}}^{[0]}_1 \downarrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \quad \text{if and only if} \quad \alpha^{(0)}_{\widehat{n}_1} - \overleftarrow{\zeta^{(0)} \sim \zeta^{(0)}} - \alpha^{(0)}_{\widehat{n}_2} \text{ in } \overset{\leftrightarrow}{w}\,.$$

In the following we will use the following δ-symbols to keep track of the gluing arrows:

$$\delta^{\uparrow}_1 = \begin{cases} 1 & \text{if } \widehat{\mathtt{n}}^{[0]}_1 \uparrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \\ 0 & \text{if } \widehat{\mathtt{n}}^{[0]}_1 \downarrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \end{cases} \qquad \text{and} \qquad \delta^{\downarrow}_1 = \begin{cases} 0 & \text{if } \widehat{\mathtt{n}}^{[0]}_1 \uparrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \\ 1 & \text{if } \widehat{\mathtt{n}}^{[0]}_1 \downarrow \widehat{\mathtt{n}}^{[1]}_2 \text{ in } \overset{\leftrightarrow}{\omega} \end{cases}$$

We will assume the same definitions for $\delta^{\uparrow}_2, \delta^{\downarrow}_2, \delta^{\uparrow}_3$ and $\delta^{\downarrow}_3$.

Finally, the construction of canonical forms in Subsection A.3.2 yields the following two matrices in the representation $\Theta(\Omega) = (\Theta^{(d)})_{d\in\mathbb{Z}}$ of $\mathrm{Rep}^*(\mathfrak{B})$:

$$\Theta^{(1)} = \begin{array}{c} \\ \beta^{(1)}_{\widehat{n}_1} \\ \beta^{(1)}_{\widehat{n}_2} \\ \beta^{(1)}_{\widehat{n}_3} \\ \beta^{(1)}_{\widehat{n}_4} \end{array} \begin{array}{c} \begin{array}{cccc} \zeta^{(1)}_- & \zeta^{(1)}_+ & \zeta^{(1)}_- & \zeta^{(1)}_+ \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta^{\downarrow}_2 & 0 \\ 0 & \delta^{\uparrow}_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad \Theta^{(0)} = \begin{array}{c} \\ \alpha^{(0)}_{\widehat{n}_1} \\ \alpha^{(0)}_{\widehat{n}_2} \\ \alpha^{(0)}_{\widehat{n}_3} \\ \alpha^{(0)}_{\widehat{n}_4} \end{array} \begin{array}{c} \begin{array}{cccc} \zeta^{(0)}_+ & \zeta^{(0)}_- & \zeta^{(0)}_+ & \zeta^{(0)}_- \end{array} \\ \begin{bmatrix} 1 & \delta^{\downarrow}_1 & 0 & 0 \\ \delta^{\uparrow}_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta^{\downarrow}_3 \\ 0 & 0 & \delta^{\uparrow}_3 & 1 \end{bmatrix} \end{array} \tag{4.2.9}$$

(2) The representation $\Theta(\Omega)$ gives rise to some triple $T_\bullet(\Omega) = \mathbf{I}(\Theta(\Omega))$ according to the definition of the functor $\mathbf{I}$ in Subsection 3.5.4.

In the case above, the triple is given by $T_\bullet(\Omega) = (V_\bullet(\Omega), \widetilde{P}_\bullet(\Omega), \Theta(\Omega))$, where

- the complex $V_\bullet(\Omega)$ of simple Λ/I-modules is

$$\dots S_- \oplus S_+ \underset{1}{\oplus} S_- \oplus S_+ \xrightarrow{0} S_+ \oplus S_- \underset{0}{\oplus} S_+ \oplus S_- \dots$$

- the projective complex $\widetilde{P}_\bullet(\Omega)$ of the category $\mathrm{D^b}(\Gamma)$ the minimal complex

$$\dots \underset{1}{\widetilde{P}^4_\diamond} \xrightarrow{\widetilde{d}_1} \underset{0}{\widetilde{P}^4_\diamond} \dots \qquad \text{where} \quad \widetilde{d}_1 = \begin{array}{c c} & \begin{array}{cccc} \beta^{(1)}_{\widehat{n}_1} & \beta^{(1)}_{\widehat{n}_2} & \beta^{(1)}_{\widehat{n}_3} & \beta^{(1)}_{\widehat{n}_4} \end{array} \\ \begin{array}{c} \alpha^{(0)}_{\widehat{n}_1} \\ \alpha^{(0)}_{\widehat{n}_2} \\ \alpha^{(0)}_{\widehat{n}_3} \\ \alpha^{(0)}_{\widehat{n}_4} \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_1 & 0 & 0 & 0 \\ 0 & \widehat{\mathtt{n}}_2 & 0 & 0 \\ 0 & 0 & \widehat{\mathtt{n}}_3 & 0 \\ 0 & 0 & 0 & \widehat{\mathtt{n}}_4 \end{bmatrix} \end{array} \tag{4.2.10}$$

- the gluing map $\Theta(\Omega)$ in $\mathrm{D^b}(\Gamma/I)$ is given by the matrices above. More precisely, these matrices yield the isomorphisms

$$\Theta^{(1)} : \Gamma/I \otimes_{\Lambda/I} V_1 \xrightarrow{\sim} \Gamma/I \otimes_\Gamma \widetilde{P}_1 \quad \text{and} \quad \Theta^{(0)} : \Gamma/I \otimes_{\Lambda/I} V_0 \xrightarrow{\sim} \Gamma/I \otimes_\Gamma \widetilde{P}_0$$

In the matrix of the differential (4.2.10), the rows and columns are indexed by the corresponding basis vectors of $\Gamma/I \otimes \widetilde{P}_\bullet$.

So far, we have translated the datum $\check{\Omega}$ into a triple $T_\bullet(\check{\Omega})$. We want to establish first the commutativity of the left square in (4.2.8) as an intermediate step.

(3) Going the other way, we may translate the datum $\check{\Omega}$ first into a gluing diagram.

More precisely, we translate the word ω in $\check{\Omega}$ into a gluing diagram as described in Subsection 4.1.3.

In our main example, the gluing diagram of ω is simply given by the left diagram of (4.2.14).

We may decorate any gluing diagram with the additional signs, multiplicity or eigenvalue of the datum $\check{\Omega}$. This yields a bijection between "decorated" gluing diagrams and string and bands of $\check{\mathfrak{B}}$. In other words, gluing diagrams are just a visual way to write strings and bands of $\check{\mathfrak{B}}$.

Next, we may define the dotted arrow in (4.2.8), that is, the passage from gluing diagrams to triples simply by translating the decorated gluing diagram into a string and band of $\mathfrak{B}$ and then applying Θ and $\mathbf{I}$. This yields commutativity of the left square in (4.2.8) and also a bijection between gluing diagrams and indecomposable triples.

As above, let $T_\bullet = T_\bullet(\check{\Omega})$ be the triple of the gluing diagram of $\check{\Omega}$.

In the final step, we have to show that the complex $\mathbf{G}(T_\bullet)$ in $\mathrm{D^b}(\Lambda)$ can be also obtain by application of gluing rules to the gluing diagram of $\check{\Omega}$.

(4) Next, we apply the gluing operation $\mathbf{G}$.

The complex $P_\bullet(\check{\Omega}) = \mathbf{G}(T_\bullet)$ is constructed according to the description in Subsection 3.2.3.

In our guiding example, the projective modules of $P_\bullet(\Omega)$ are given by "lifting" the simple modules of $V_\bullet(\Omega)$:

$$\ldots P_- \oplus P_+ \underset{1}{\oplus} P_- \oplus P_+ \xrightarrow{d_1} P_+ \oplus P_- \underset{0}{\oplus} P_+ \oplus P_- \ldots \tag{4.2.11}$$

The main point is that the differential of the complex $P_\bullet$ is given by the formula (3.2.9). In our particular case, this formula boils down to the expression

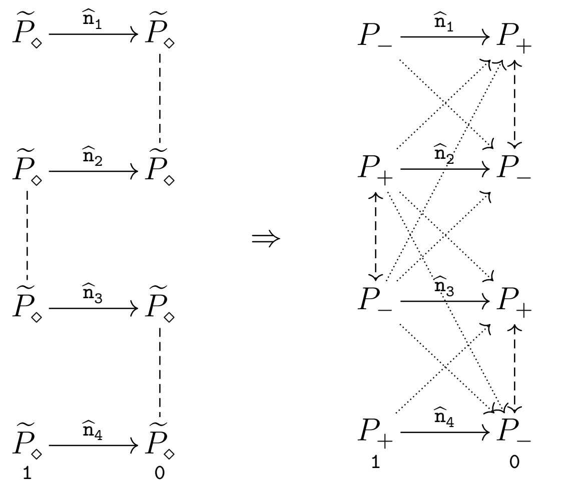
$$d_1 = \left(\Theta^{(0)}\right)^{-1} \cdot \widetilde{d}_1 \cdot \Theta^{(1)} \tag{4.2.12}$$

Using the canonical forms (4.2.9) and the differential (4.2.10) of the complex $\widetilde{P}_\bullet$ for the three matrices in question we obtain the differential

$$d_1 = \begin{array}{c} \\ P_+ \\ P_- \\ P_+ \\ P_- \end{array} \begin{array}{c} \begin{array}{cccc} P_- \quad & \quad P_+ \quad & \quad P_- \quad & \quad P_+ \end{array} \\ \left[\begin{array}{cccc} \widehat{\mathtt{n}}_1 & -\delta_1^{\downarrow}\,\widehat{\mathtt{n}}_2 & -\delta_1^{\downarrow}\,\delta_2^{\downarrow}\,\widehat{\mathtt{n}}_2 & 0 \\ -\delta_1^{\uparrow}\,\widehat{\mathtt{n}}_1 & \widehat{\mathtt{n}}_2 & \delta_2^{\downarrow}\,\widehat{\mathtt{n}}_2 & 0 \\ 0 & \delta_2^{\uparrow}\,\widehat{\mathtt{n}}_3 & \widehat{\mathtt{n}}_3 & -\delta_3^{\downarrow}\,\widehat{\mathtt{n}}_4 \\ 0 & -\delta_2^{\uparrow}\,\delta_3^{\uparrow}\,\widehat{\mathtt{n}}_3 & -\delta_3^{\uparrow}\widehat{\mathtt{n}}_3 & \widehat{\mathtt{n}}_4 \end{array}\right] \end{array} \tag{4.2.13}$$

(5) At last, we apply the gluing rules of Subsection 4.2.2 to the gluing diagram of $\check{\Omega}$.

For the word ω as above we obtain the following picture:



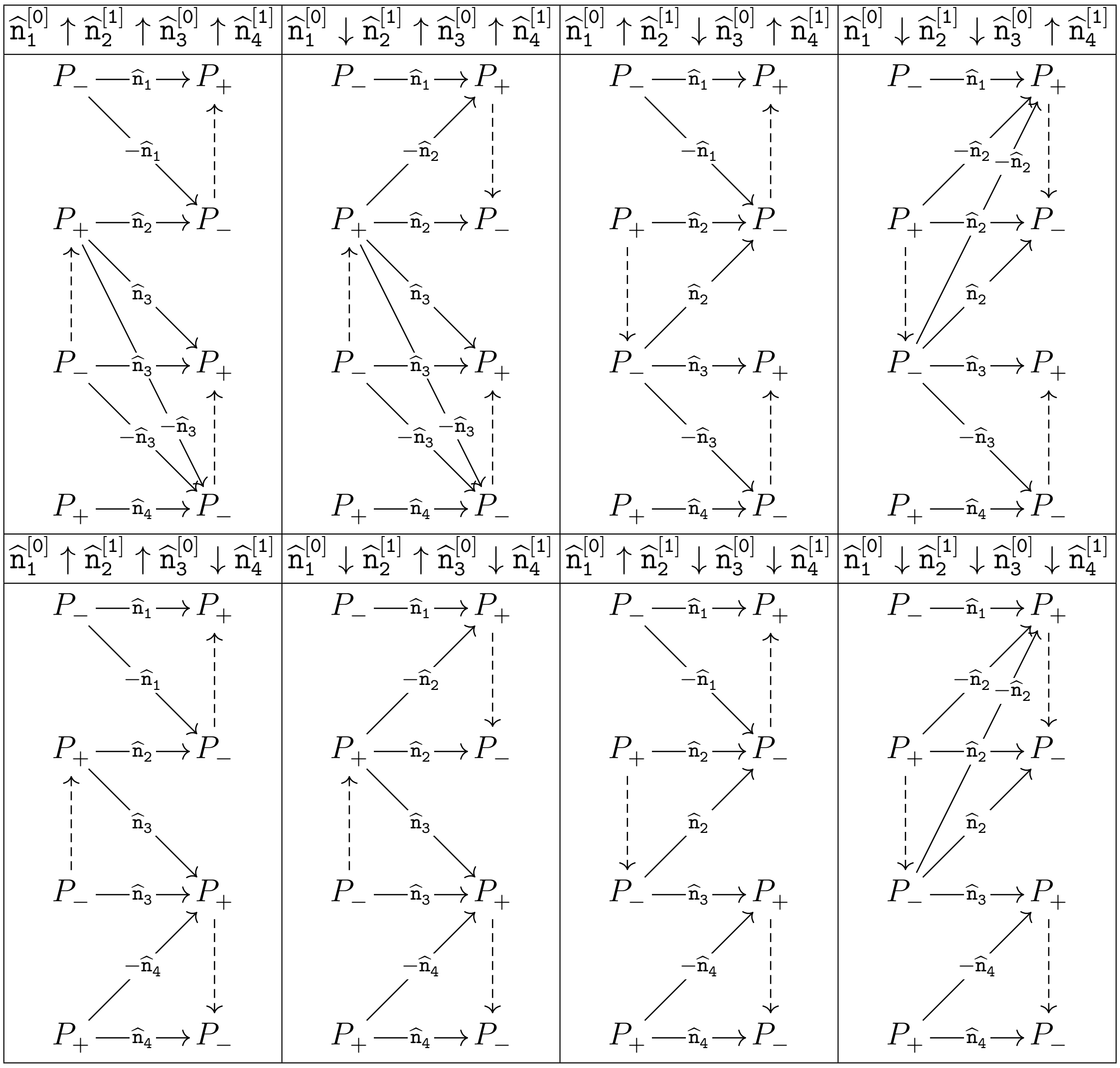
(4.2.14)

All eight possibilities for the glued diagram on the right are displayed in Figure 4.2.1. In all eight cases the gluing rules yield exactly the differential of the form (4.2.13).

It remains to discuss why the above example was representative:

- For any string or band $\check{\Omega}$ of $\check{\mathfrak{B}}$ the canonical forms $\Theta(\check{\Omega})$ are *sparse* (see Remarks A.3.20 and A.3.14). For the particular case of the bunch of semichains $\check{\mathfrak{B}}$, any column or any row of any canonical matrix of $\Theta(\check{\Omega})$ has one or two non-zero entries

FIGURE 4.2.1. **Column-complete part of gluing diagrams**



(respectively one or two non-zero blocks of size m if $\check{\Omega}$ is a bispecial string or band with multiplicity m).

- By the formula (4.2.12) the row or column of any differential of the complex $\mathbf{G}(T(\check{\Omega}))$ has at most four non-zero entries (respectively four non-zero blocks of size m).
- In particular, the second and third column of the "local" differential matrix in (4.2.13) contains all non-zero entries of the whole differential.

By these arguments the main example above was the "local case" in the proof.

There are some modifications necessary in the following cases:

- For usual and special strings $\check{\Omega}$ there are projectives of type $\widetilde{P}_\star$ in the complex $\widetilde{P}_\bullet$ of the triple $T_\bullet(\check{\Omega})$. In this case, the formula (4.2.12) has to be modified according to (3.2.9). Any projective module $\widetilde{P}_\star$ translates directly into a projective module $P_\star$ in $P_\bullet(\check{\Omega})$.

- In the construction of bands it can happen that formula (4.2.12) yields an inverse Jordan block $\mathrm{J}(\lambda)^{-1}$ at some entry of the differential. By the combinatorics of bunches of semichains, such blocks may be replaced by $\mathrm{J}(\lambda^{-1})$ in the canonical forms. This explains the additional gluing rules for bands.

Summarized, the above arguments give a proof of the gluing rules of Subsection 4.2.2. □

The sparsity argument of the proof above translates into the following statement for gluing diagrams:

Remark 4.2.11. *For any string or band $\check{\Omega}$ of $\check{\mathfrak{B}}$ its gluing diagram has at most four arrows starting or ending in each vertex.*

In particular, any *continuation of the gluing diagram in Figure 4.2.1 will not have additional arrows starting in the second left and the third left projective.*

4.3 Functorial properties of string and band complexes

In the previous section, we have seen that the indecomposable complexes of the category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ are parametrized by strings and bands of $\check{\mathfrak{B}}$. Moreover, we described how to construct the string and band complexes.
In this section, we consider two natural autoequivalences of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$: the involution functor σ which is induced by an involution of the ring Λ and the Auslander-Reiten translation τ. We describe the action of these two functors in terms of strings and bands of $\check{\mathfrak{B}}$:

$$\begin{tikzcd} \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \arrow[r, leftrightarrow, "1:1"] \arrow[d, "\sigma \text{ or } \tau"] & {[\text{STRINGS and BANDS of } \check{\mathfrak{B}}]} \arrow[d, dashed, "\sigma \text{ or } \tau"] \\ \mathrm{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \arrow[r, leftrightarrow, "1:1"] & {[\text{STRINGS and BANDS of } \check{\mathfrak{B}}]} \end{tikzcd}$$

The description of the Auslander-Reiten translation is the main result of this section.

4.3.1 The defect and the involution on strings and bands

In this subsection, we describe the defect δ and the involution σ on strings and bands of $\check{\mathfrak{B}}$. Let us recall these two notions:

Definition 4.3.1. *Let Λ be the Gelfand order and $S_\star = \operatorname{top} P_\star$.*

(1) For any $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ and any $i \in \mathbb{Z}$ we set

$$\delta^{(i)}(P_\bullet) = \dim_{\Bbbk} \operatorname{Ext}^i(P_\bullet, S_\star), \qquad \textit{and} \qquad \delta(P_\bullet) = \sum_{i\in\mathbb{Z}} \delta^{(i)}(P_\bullet)$$

The invariants $\delta^{(i)}(P_\bullet)$ are the defect numbers*, and $\delta(P_\bullet)$ is the* defect *of $P_\bullet$.*

(2) Let $\sigma : \mathrm{D^b}(\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\Lambda)$ *be the* involution *which interchanges + and − in projectives and their differentials:*

$$\Lambda = \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \atop \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \qquad \begin{array}{ccc} P_\star \xrightarrow{\cdot \mathsf{b}_\pm} P_\pm & \xmapsto{\sigma} & P_\star \xrightarrow{\cdot \mathsf{b}_\mp} P_\mp \\ P_\pm \xrightarrow{\cdot \mathsf{a}_\pm} P_\star & \xmapsto{\sigma} & P_\mp \xrightarrow{\cdot \mathsf{a}_\mp} P_\star \end{array}$$

Remark 4.3.2. *Let* $P_\bullet$ *be any complex from* $\mathrm{D^b}(\Lambda)$.

(1) By 3.2.9 we may assume that $P_\bullet$ *is minimal. In this case, the defect* $\delta(P_\bullet)$ *counts the* number of projectives of type $P_\star$ in $P_\bullet$ *(according to Lemma B.2.17).*

In particular, there is a simple description of the defect for string or band complexes:

(2) Let Ω *be a usual string* ω, *a special string* (ω, ε), *a bispecial string* $(\omega, m, \varepsilon_1, \varepsilon_2)$ *or a band* (ω, m, λ) *of* $\check{\mathfrak{B}}$. *Let* $P_\bullet = P_\bullet(\Omega)$ *be the corresponding indecomposable complex in* $\mathrm{D^b}(\Lambda)$. *Then the defect of* $P_\bullet$ *is given by*

$$\begin{aligned} \delta(P_\bullet) &= \textit{number of projectives of type } P_\star \textit{ in the complex } P_\bullet \\ &= \textit{number of ends of type } \mathtt{p}_\star \textit{ of the word } \omega \end{aligned} \qquad (4.3.1)$$

(3) In particular, for any indecomposable *complex* $P_\bullet \in \mathrm{D^b}(\Lambda)$ *it holds that the defect* $\delta(P_\bullet)$ *is* zero, one *or* two.

Lemma 4.3.3. *Let* Ω *be a string or band of* $\check{\mathfrak{B}}$ *and* $P_\bullet(\Omega)$ *be the corresponding complex in* $\mathrm{D^b}(\Lambda)$. *In the following, for any sign* $\varepsilon = \pm$ *we denote by* $\overline{\varepsilon} = \mp$ *the opposite sign. Let* $\sigma(\Omega)$ *be defined by changing the sign data of* Ω :

	usual string	*special string*	*bispecial string*	*band*
Ω	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)
$\sigma(\Omega)$	ω	$(\omega, \overline{\varepsilon}_1)$	$(\omega, m, \overline{\varepsilon}_1, \overline{\varepsilon}_2)$	(ω, m, λ)

(4.3.2)

(*where* $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, $m \in \mathbb{N}^+$ *and* $\lambda \in \mathbb{k}^* \setminus \Delta$)

Let $P_\bullet(\sigma(\Omega))$ *be the complex corresponding to the string respectively band* $\sigma(\Omega)$. *Then* $\sigma(P_\bullet(\Omega)) \cong P_\bullet(\sigma(\Omega))$. *In other words, the involution* σ *changes the signs in any special or bispecial string, and preserves any usual string or band.*

PROOF. The claim of the Lemma is obviously true for strings Ω such that $P(\Omega) \in \mathrm{D^b}(\Lambda_\star)$. For all other strings and bands of $\check{\mathfrak{B}}$ the claim follows from Lemma A.4.2 by the translation of strings and bands in Subsection 4.1.5. □

As noted in Subsection 2.2.4 the defect and the involution can be used to characterize string and band complexes in $\mathrm{D^b_{fd}}(\Lambda)$. We will recall this characterization in the summary in Subsection 4.3.6

4.3.2 Auslander-Reiten translation on strings and bands

Next, we describe the Auslander-Reiten translation on strings and bands of $\check{\mathfrak{B}}$.

Let us recall that the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ has an Auslander-Reiten translation according to Theorem B.2.12 by [**vdB04**] and [**IR08**], since Λ is an $\mathbf{R}$-order of *finite* global dimension.
More precisely, let $\omega = \mathrm{Hom}_{\mathbf{R}}(\Lambda, \mathbf{R})$ be the canonical bimodule of Λ. Then the left-derived tensor product $\tau = \omega \otimes \_ : \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$ defines an autoequivalence such that

$$\mathrm{Hom}_\Lambda(X_\bullet, Y_\bullet) \cong \mathbb{D}\,\mathrm{Ext}^1_\Lambda(Y_\bullet, \tau(X_\bullet)) \quad \text{for any } X_\bullet, Y_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$$
$$\text{such that } X_\bullet \text{ or } Y_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda).$$

In different terms, the category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ admits a *relative Serre functor* $\mathbb{S} = \tau \circ [1]$.
For the Gelfand order Λ, the canonical bimodule ω and the action of the Auslander-Reiten translation τ on projectives and their morphisms are given as follows:

$$\omega \cong \begin{array}{c} \begin{array}{ccc} rP_\star & P_- & P_+ \end{array} \\ \begin{bmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \end{bmatrix} \end{array} \qquad \begin{array}{ccc} P_\star \xrightarrow{\cdot \mathtt{b}_\pm} P_\pm & \xmapsto{\tau} & rP_\star \xrightarrow{\cdot \mathtt{b}_\mp} P_\mp \\ P_\pm \xrightarrow{\cdot \mathtt{a}_\pm} P_\star & \xmapsto{\tau} & P_\mp \xrightarrow{\cdot \mathtt{a}_\mp} rP_\star \end{array} \tag{4.3.3}$$

where $rP_\star = \mathrm{rad}\, P_\star$ denotes the radical of the projective $P_\star$.

In particular, the Auslander-Reiten translation is given by the involution on any complex with *vanishing* defect:

Remark 4.3.4. *Let $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ such that $\delta(P_\bullet) = 0$. Then $\tau(P_\bullet) \cong \sigma(P_\bullet)$. In particular, this holds if $P_\bullet \in \mathrm{K}^{\mathrm{b}}(\mathrm{add}\, P_+ \oplus P_-)$, or $P_\bullet \cong P_\bullet(\Omega)$ for some bispecial string or band Ω of $\check{\mathfrak{B}}$.*

Before we define the Auslander-Reiten translation on arbitrary strings let us consider some basic examples. We recall that the subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ admits Auslander-Reiten triangles.

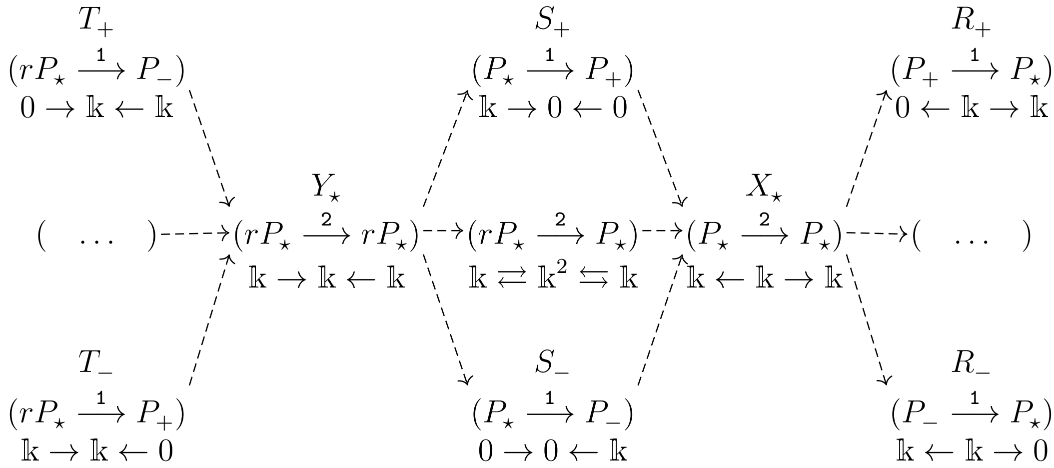
Example 4.3.5. *Figure 4.3.1 shows the Auslander-Reiten triangles of the Gelfand quiver representations which correspond to some* classical representations of $\mathrm{SL}(2, \mathbb{R})$ *from Tables C.2.2 and C.2.3.*

Example 4.3.6 (Some short usual strings)**.** *The Auslander-Reiten translations of the indecomposable complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda_\star)$ are given as follows*:

(1) Let $\omega = \mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}\mathtt{p}_\star$ for some $\mathtt{d} \in \mathbb{Z}$ and $\mathtt{n} \in \mathbb{N}^+$. Then $P_\bullet(\omega) = \underset{\mathtt{d}+1}{P_\star} \xrightarrow{2\mathtt{n}} \underset{\mathtt{d}}{P_\star}$ and there are isomorphisms

$$\tau(P_\bullet(\omega)) \cong (\underset{\mathtt{d}+1}{rP_\star} \xrightarrow{2\mathtt{n}} \underset{\mathtt{d}}{rP_\star}) \cong \begin{tikzcd} P_\star \arrow[r, "1"] \arrow[dr, "-1"'] & P_+ \arrow[dr, "2\mathtt{n}_1"] & \\ & P_- \arrow[u, dashed] \arrow[r, "2\mathtt{n}_1"] & P_+ \arrow[d, dashed] \\ & P_\star \arrow[ur, "-1"] \arrow[r, "1"] & P_- \end{tikzcd}$$

FIGURE 4.3.1. **Auslander-Reiten triangles of standard representations**

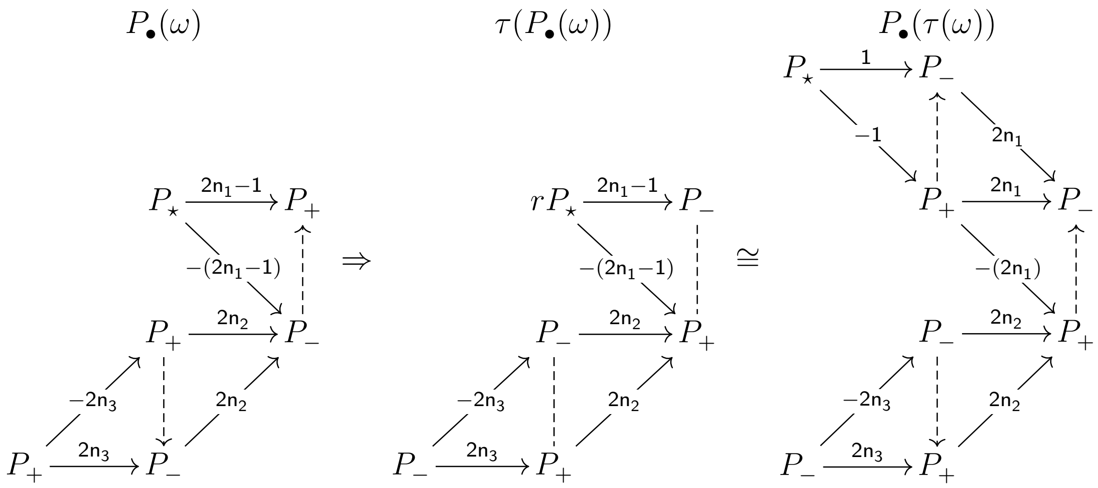


Note that the new complex $\tau(P_\bullet(\omega))$ is isomorphic to some string complex *in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$. *More precisely, it holds that $\tau(P_\bullet(\omega)) \cong P_\bullet(\tau(\omega))$ with $\tau(\omega) = \mathtt{p}_\star^{[\mathtt{d}+2]}\mathtt{1}^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}+1]}\mathtt{p}_\star$.*

(2) Let $\omega = \mathtt{p}_\star^{[\mathtt{d}]}$ for some $\mathtt{d} \in \mathbb{Z}$. Then $P_\bullet(\omega) = P_\star[\mathtt{d}]$ and the same arguments as above yield that $\tau(P_\bullet(\omega)) \cong P_\bullet(\tau(\omega))$ where $\tau(\omega) = \mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{p}_\star$.

Next, let us consider the Auslander-Reiten translation of some "arbitrary" string complex:

Example 4.3.7. *Let $\omega = (\mathtt{p}_\star^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\mathtt{n}_3^{[2]}\mathtt{p}_\diamond)$ for some $\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3 \in \mathbb{N}^+$ with $\mathtt{n}_1 < \mathtt{n}_2$. Then ω is a usual string of $\breve{\mathfrak{B}}$ and $\overleftrightarrow{\omega} = (\mathtt{p}_\star^{[1]}\widehat{\mathtt{n}}_1^{[0]} \uparrow \widehat{\mathtt{n}}_2^{[1]} \downarrow \widehat{\mathtt{n}}_3^{[2]}\mathtt{p}_\diamond)$. Let $P_\bullet(\omega)$ be the corresponding string complex. According to* (4.3.3) *the complex $\tau(P_\bullet(\omega))$ is given as follows*:



It can be checked that $\tau(P_\bullet(\omega))$ is isomorphic to the string complex $P_\bullet(\tau(\omega))$ where $\tau(\omega) := (\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\mathtt{n}_3^{[2]}\mathtt{p}_\diamond)^{\mathrm{rev}}$.

In the example above the Auslander-Reiten translation has changed *only the ends* of the gluing diagram. We will see that this holds in general.

Definition 4.3.8. *The Auslander-Reiten translation of finite words of* $\check{\mathfrak{B}}$ *is defined as follows*:

(1) Let ω *be some finite word of* $\check{\mathfrak{B}}$ *of some length* $\mathtt{k} \in \mathbb{N}^+$:

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\mathtt{n}_3^{[\mathtt{d}_3]}\dots\mathtt{n}_{\mathtt{k}-2}^{[\mathtt{d}_{\mathtt{k}-2}]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta)$$

- *Let us assume that* $\mathtt{k} \geq 2$ *and that* ω *is not equivalent to the usual string* $\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{1}^{[\mathtt{d}_0+1]}\mathtt{n}^{[\mathtt{d}_2]}\mathtt{p}_\star$ *for any* $\mathtt{d}_0 \in \mathbb{Z}$, $\mathtt{d}_2 \in \{\mathtt{d}_0, \mathtt{d}_0+2\}$ *and any* $\mathtt{n} \in \mathbb{N}^+$. *For the definition of* $\tau(\omega)$ *in these special cases we refer to Remark 4.3.9 below.*

We define a new word $\tau(\omega)$ *by changing only the first and the last part of* ω.

(a) The first part of ω *determines the first part of* $\tau(\omega)$ *by one of the following six cases*:

$\omega =$	$\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_0-1]}\dots$	$\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_0+1]}\dots$ *with* $\mathtt{n}_1 \neq 1$
$\tau(\omega) =$	$\mathtt{p}_\star^{[\mathtt{d}_0+1]}\mathtt{1}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_0-1]}\dots$	$\mathtt{p}_\star^{[\mathtt{d}_0+1]}\mathtt{1}^{[\mathtt{d}_0]}(\mathtt{n}_1-1)^{[\mathtt{d}_0+1]}\dots$

$\omega =$	$\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{1}^{[\mathtt{d}_0+1]}\mathtt{n}_2^{[\mathtt{d}_0]}\dots$	$\mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{1}^{[\mathtt{d}_0+1]}\mathtt{n}_2^{[\mathtt{d}_0+2]}\dots$	$\mathtt{p}_\diamond^{[\mathtt{d}_0]}\dots$	$\mathtt{p}_\infty^{[\mathtt{d}_0]}\dots$
$\tau(\omega) =$	$\mathtt{p}_\star^{[\mathtt{d}_0+1]}(\mathtt{n}_2+1)^{[\mathtt{d}_0]}\dots$	$\mathtt{p}_\star^{[\mathtt{d}_0+1]}\mathtt{n}_2^{[\mathtt{d}_0+2]}\dots$	$\mathtt{p}_\diamond^{[\mathtt{d}_0]}\dots$	$\mathtt{p}_\infty^{[\mathtt{d}_0]}\dots$

(b) The last part is defined similarly to the first part:

$\omega =$	$\dots{}^{[\mathtt{d}_\mathtt{k}-1]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\star$ *with* $\mathtt{n}_\mathtt{k} \neq 1$
$\tau(\omega) =$	$\dots{}^{[\mathtt{d}_\mathtt{k}-1]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{1}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}+1]}(\mathtt{n}_\mathtt{k}-1)^{[\mathtt{d}_\mathtt{k}]}\mathtt{1}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{p}_\star$

$\omega =$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{1}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}+2]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{1}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\diamond$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\infty$
$\tau(\omega) =$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}(\mathtt{n}_{\mathtt{k}-1}+1)^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}+2]}\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_\mathtt{k}+1]}\mathtt{p}_\star$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\diamond$	$\dots{}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\infty$

(c) The remaining intermediate part of ω *yields the intermediate part of* $\tau(\omega)$.

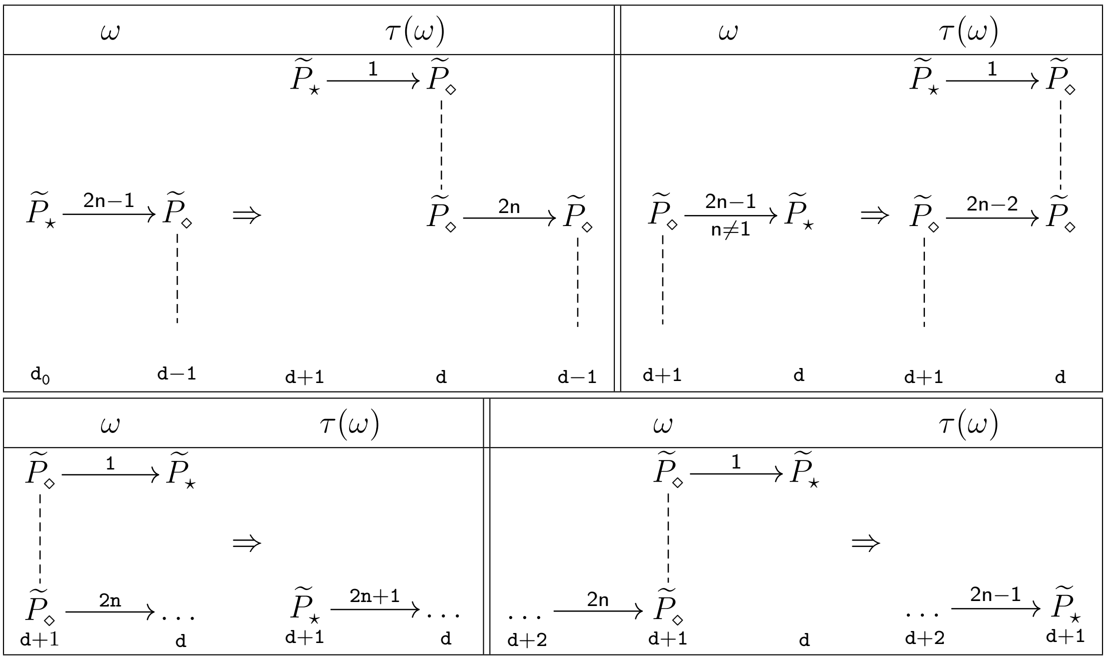
Let us note that $\tau(\omega) = \omega$ *if both ends of* ω *are given by* $\mathtt{p}_\diamond$ *or* $\mathtt{p}_\infty$.
The gluing diagrams for the cases with usual ends are shown in Figure 4.3.2.

Remark 4.3.9. *The action of* τ *on* usual and special strings of length one *and* usual strings of length two *which are not covered by Definition 4.3.8 can be computed as follows*:

ω	$\mathtt{p}_\star^{[\mathtt{d}]}$	$\mathtt{p}_\diamond^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\star$	$\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\star$
$\tau(\omega)$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{p}_\infty$	$\begin{cases}\mathtt{p}_\diamond^{[\mathtt{d}]}(\mathtt{n}-1)^{[\mathtt{d}-1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{p}_\star & \text{if } \mathtt{n}>1,\\ \mathtt{p}_\star^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}-1]}\mathtt{p}_\diamond & \text{if } \mathtt{n}=1.\end{cases}$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{p}_\star$
ω	$\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\diamond$	$\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}\mathtt{p}_\star$	$\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}+2]}\mathtt{p}_\star$
$\tau(\omega)$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{1}^{[\mathtt{d}]}\mathtt{n}^{[\mathtt{d}-1]}\mathtt{p}_\diamond$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}+1]}\mathtt{p}_\star$	$\mathtt{p}_\star^{[\mathtt{d}+1]}\mathtt{n}^{[\mathtt{d}+2]}\mathtt{1}^{[\mathtt{d}+3]}\mathtt{p}_\star$

The following theorem is the main result of this section:

FIGURE 4.3.2. **Auslander-Reiten translation on gluing diagrams**



Theorem 4.3.10. *Let Ω be a string or band of $\check{\mathfrak{B}}$ and let $P_\bullet(\Omega)$ be its glued complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Let $\tau(\Omega)$ be defined by application of τ to the word in Ω :*

	usual string	*special string*	*bispecial string*	*band*
Ω	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)
$\tau(\Omega)$	$\tau(\omega)$	$(\tau(\omega), \overline{\varepsilon}_1)$	$(\omega, m, \overline{\varepsilon}_1, \varepsilon_2)$	(ω, m, λ)

(4.3.4)

(*where* $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, $m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^* \backslash \Delta$)

where $\tau(\omega)$ is given by Definition 4.3.8 respectively Remark 4.3.9.
Then $\tau(P_\bullet(\Omega)) \cong P_\bullet(\tau(\Omega))$, where $P_\bullet(\tau(\Omega))$ is the glued complex of the string respectively band $\tau(\Omega)$.

In different terms, the theorem states that the Auslander-Reiten translation changes *only the ends* of the gluing diagram of any string complex and *preserves* its intermediate part. We will give a proof of Theorem 4.3.10 in the next subsection.

Let us note the following consequences of the theorem above:

Remark 4.3.11. *Let Ω be a usual string ω or a special string (ω, ε) of $\check{\mathfrak{B}}_{\mathrm{fd}}$. In this case, the word ω has an end of type $\mathtt{p}_\star^{[\mathtt{d}]}$ for some $\mathtt{d} \in \mathbb{Z}$. Then $\tau^m(\omega)$ has an end of type $\mathtt{p}_\star^{[\mathtt{d+m}]}$. In particular, $\tau^m(\omega)$ cannot be equivalent to ω.*
In other words, usual and special strings of $\check{\mathfrak{B}}_{\mathrm{fd}}$ are not *τ-periodic.*

Remark 4.3.12. *Let Ω be a usual string ω or a special string (ω, ε) of $\check{\mathfrak{B}}$, but not of $\check{\mathfrak{B}}_{\mathrm{fd}}$. In particular, the corresponding complex $P_\bullet = P_\bullet(\Omega)$ of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ has* infinite-dimensional *homology. Then one of the following three cases occurs*:

(1) *Ω is a usual string given by a non-symmetric word ω with both ends of type $\mathtt{p}_\infty$. In this case, the complex $P_\bullet$ satisfies $\tau(P_\bullet) \cong \sigma(P_\bullet) \cong P_\bullet$ and $\delta(P_\bullet) = 0$.*

(2) Ω is a usual string given by any word ω with ends of type $\mathtt{p}_\infty$ and also $\mathtt{p}_\star$. This case occurs if and only if $\sigma(P_\bullet) \cong P_\bullet$ and $\delta(P_\bullet) = 1$. In this case, the complex $P_\bullet$ is not *τ-periodic.*

(3) Ω is given by a special string (ω, ε) with any word ω with ends of type $\mathtt{p}_\infty$ and $\mathtt{p}_\diamond$. In this case, it holds that $\tau(P_\bullet) \cong \sigma(P_\bullet) \ncong P_\bullet$ and $\delta(P_\bullet) = 0$. In particular, the complex $P_\bullet$ has τ-period two.

This means, that in the big derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ there are τ-invariant usual strings, there is a functorial characterization of usual strings of type (2), and there are special strings with τ-period two (3).

The two remarks above yield the following description of τ-periodic objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

Corollary 4.3.13. *Let Ω be any string or band of $\mathfrak{B}$ and let $P_\bullet = P_\bullet(\Omega)$ be the corresponding complex. Then $\delta(P_\bullet) = 0$ if and only if $P_\bullet$ is τ-periodic.*
In this case, $\tau(P_\bullet) \cong \sigma(P_\bullet)$ and $P_\bullet$ is τ-invariant or has τ-period two.

This statement was also shown without combinatorial methods in Theorem 2.2.3.

4.3.3 Proof of Theorem 4.3.10 on the Auslander-Reiten translation

Let $\mathfrak{B}$ be the bunch of semichains for the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of the Gelfand order. To prove Theorem 4.3.10 we need to define the action of the Auslander-Reiten translation τ strings of the bunch of semichains $\mathfrak{B}$.

We recall that a string of $\mathfrak{B}$ has a word of the associated alphabet $\mathfrak{A}_{\mathfrak{B}}$ of $\mathfrak{B}$ as main datum.

As stated in Subsection 4.1.5, the alphabet $\mathfrak{A}_{\mathfrak{B}} = \left(\overline{\mathfrak{C}}, \overline{\mathfrak{R}}, \sim, -\right)$ of $\mathfrak{B}$ is given as follows:

(1) The column letters $\overline{\mathfrak{C}}$ are given by $\overline{\mathfrak{C}} = \bigcup\limits_{d \in \mathbb{Z}} \overline{\mathfrak{C}}_d$ where $\overline{\mathfrak{C}}_d = \{\zeta^{(d)}\}$ for each $d \in \mathbb{Z}$.

(2) The row letters $\overline{\mathfrak{R}}$ are given by $\overline{\mathfrak{R}} = \bigcup\limits_{d \in \mathbb{Z}} \overline{\mathfrak{R}}_d$ where for each $d \in \mathbb{Z}$ we have a chain

$$\overline{\mathfrak{R}}_d = \{\ \beta_1^{(d)} < \ldots < \beta_n^{(d)} < \beta_{n+1}^{(d)} < \ldots < \gamma^{(d)} < \ldots < \alpha_{n+1}^{(d)} < \alpha_n^{(d)} < \ldots < \alpha_1^{(d)}\ \}$$

(3) There are two symmetric relations $-$ and $\sim$ on the set $\mathfrak{A}_{\mathfrak{B}} = \overline{\mathfrak{R}} \cup \overline{\mathfrak{C}}$ given by

- $\varrho^{(d)} - \zeta^{(d)}$ for any $\varrho^{(d)} \in \overline{\mathfrak{R}}_d$ and any $d \in \mathbb{Z}$,
- $\zeta^{(d)} \sim \zeta^{(d)}$ for any $d \in \mathbb{Z}$, and $\alpha_{2n}^{(d)} \sim \beta_{2n}^{(d+1)}$ for any $n \in \mathbb{N}^+$ and any $d \in \mathbb{Z}$.

In addition, we will need the following notation. For any $d \in \mathbb{Z}$ we define

- $\mathrm{sub}(\alpha_n^{(d)}) = \alpha_{n+1}^{(d)})$ and $\mathrm{sub}(\beta_{n+1}^{(d)}) = \beta_n^{(d)})$ for any $n \in \mathbb{N}^+$.

Remark 4.3.14. *For any $d \in \mathbb{Z}$ let us note the following statements*:

(1) The chain $\mathfrak{R}^{(d)}$ has the maximum element $\alpha_1^{(d)}$ and the minimum element $\beta_1^{(d)}$.

(2) For any $\varrho', \varrho'' \in \mathfrak{R}^d \backslash \{\beta_1^{(d)}\}$ *it holds that* $\varrho' < \varrho''$ *if and only if* $\mathrm{sub}(\varrho') < \mathrm{sub}(\varrho'')$.

Definition 4.3.15. *Let* w *be some finite word of* $\mathfrak{A}_{\mathfrak{B}}$. *We define a new word* $\tau(w)$ *by changing the first and the last part of* w *as follows.*

(1) The first part of $\tau(w)$ *is defined as follows*:

$$\begin{array}{|r|r|}
\hline
w = & \alpha_{2n-1}^{(d-1)} \dots \\
\hline
\tau(w) = & \alpha_1^{(d)} - \zeta^{(d)} \sim \zeta^{(d)} - \beta_{2n}^{(d)} \sim \alpha_{2n}^{(d-1)} \dots \\
\hline
w = & \beta_{2n+1}^{(d+1)} \dots \\
\hline
\tau(w) = & \alpha_1^{(d)} - \zeta^{(d)} \sim \zeta^{(d)} - \alpha_{2n}^{(d)} \sim \beta_{2n}^{(d+1)} \dots \\
\hline
w = & \beta_1^{(d+1)} - \zeta^{(d+1)} \sim \zeta^{(d+1)} - \beta_{2n}^{(d+1)} \sim \alpha_{2n}^{(d)} \dots \\
\hline
\tau(w) = & \alpha_{2n+1}^{(d)} \dots \\
\hline
w = & \beta_1^{(d+1)} - \zeta^{(d+1)} \sim \zeta^{(d+1)} - \alpha_{2n}^{(d+1)} \sim \beta_{2n}^{(d+2)} \dots \\
\hline
\tau(w) = & \beta_{2n-1}^{(d+2)} \dots \\
\hline
\end{array} \tag{4.3.5}$$

Finally, if w *begins with* $\zeta^{(d)}$ *or* $\gamma^{(d)}$, *then* $\tau(w)$ *begins with the same end.*

(2) The last part of $\tau(w)$ *is defined analogously.*

(3) The remaining intermediate part of w *defines the intermediate part of* $\tau(w)$.

Let us remark that the translation of Definition 4.3.15 via Subsection 4.1.5 above yields Definition 4.3.8.

Remark 4.3.16. *Let us consider the four cases in* (4.3.5).

- *In the first two cases the first letter* ϱ *of* w *is replaced by the letter* $\mathrm{sub}(\varrho)$ *in* $\tau(w)$.
- *In the last two cases the first letter of* $\tau(w)$ *is given by* $\mathrm{sub}(\varrho)$ *for some letter* ϱ *at the same position in* w.

This will play a role in the proof of the next lemma.

Lemma 4.3.17. *Let* w *be a usual or special word of* $\mathfrak{A}_{\mathfrak{B}}$. *Let* $\overset{\leftrightarrow}{w}$ *and* $\overset{\leftrightarrow}{\tau(w)}$ *be the words with oriented arrows of* w *respectively* $\tau(w)$ *as defined in Subsection A.3.2.1. Then the arrows in the common part of the words* $\overset{\leftrightarrow}{w}$ *and* $\overset{\leftrightarrow}{\tau(w)}$ *have the* same *orientations.*

PROOF. In the following we will use some terminology from Subsection A.3.2.1.

(1) First, we need to fix some notation:

- Let $\zeta^{(d)} \sim \zeta^{(d)}$ be any common subword of w and $\tau(w)$.
- Let $\overline{w}$ be the *ambient word* of w. Let Υ be the maximal symmetric subword of $\overline{w}$ with the subword $\zeta^{(d)} \sim \zeta^{(d)}$ in the middle. Let γ_l denote the predecessor of the first letter of Υ in $\overline{w}$ and γ_r the successor of the last letter of Υ in $\overline{w}$.

Since w is a regular word, any end of w is either special or free. In particular, no end of Υ can be an end of $\overline{w}$. It follows that there is some $d \in \mathbb{Z}$ such that γ_l and $\gamma_r \in \overline{\mathfrak{R}}_d$ and either $\gamma_l < \gamma_r$ or $\gamma_l > \gamma_r$.

By (A.3.3) it holds that $\overleftarrow{\zeta^{(d)} \sim \zeta^{(d)}}$ in $\overset{\leftrightarrow}{w}$ if $\gamma_l < \gamma_r$ and that $\overrightarrow{\zeta^{(d)} \sim \zeta^{(d)}}$ in $\overset{\leftrightarrow}{w}$ if $\gamma_l > \gamma_r$.

- Let $\overline{\tau(w)}$ be the ambient word of $\tau(w)$. Let Υ^τ be the maximal symmetric subword of $\overline{\tau(w)}$ with the same subword $\zeta^{(d)} \sim \zeta^{(d)}$ in the middle. Let γ_l^τ and γ_r^τ denote the predecessor of Υ^τ respectively the successor of Υ^τ in $\tau(\overset{\leftrightarrow}{w})$.

Let us note that $\tau(w)$ is symmetric if and only if w is symmetric. In particular, since w is a string word, also the word $\tau(w)$ is a string word. As above, it follows that there is some $d^\tau \in \mathbb{Z}$ such that γ_l^τ and $\gamma_r^\tau \in \overline{\mathfrak{R}}_{d^\tau}$ and both symbols are comparable but not equal.

(2) To show that the common subword $\zeta^{(d)} \sim \zeta^{(d)}$ has the same orientation in $\overset{\leftrightarrow}{w}$ as in $\tau(\overset{\leftrightarrow}{w})$, it is sufficient to show the following claim:

$$\gamma_l < \gamma_r \qquad \text{if and only if} \qquad \gamma_l^\tau < \gamma_r^\tau \tag{4.3.6}$$

Let us note that there are the following three cases for the letter γ_l^τ :

- either (a) $\gamma_l^\tau = \gamma_l$, or (b) $\gamma_l^\tau = \mathrm{sub}(\gamma_l)$, or
  (c) γ_l^τ is the left end of $\tau(w)$ and γ_l is the left end of w.

The same is true for the letter γ_r^τ :

- either (a) $\gamma_r^\tau = \gamma_r$, or (b) $\gamma_r^\tau = \mathrm{sub}(\gamma_r)$, or
  (c)γ_r^τ is the right end of $\tau(w)$ and γ_r is the right end of w.

Next, we show claim (4.3.6) case-by-case:

- cases (aa) and (bb): in these cases the claim is obviously true.
- case (ab): let $\gamma_l^\tau = \gamma_l$ and $\gamma_r^\tau = \mathrm{sub}(\gamma_r)$.
  - if $\gamma_l < \gamma_r$, then $\gamma_l^\tau = \gamma_l \leq \mathrm{sub}(\gamma_r) = \gamma_r^\tau$. But $\gamma_l^\tau \neq \gamma_r^\tau$ since $\tau(w)$ is a string.
  - if $\gamma_l > \gamma_r$, then of course $\gamma_l^\tau = \gamma_l > \mathrm{sub}(\gamma_r) = \gamma_r^\tau$.
- the case (ba) with $\gamma_l^\tau = \mathrm{sub}(\gamma_l)$ and $\gamma_r^\tau = \gamma_r$ is dual to the previous one.
- in the cases (ca) and (cb) we assume that γ_l and γ_l^τ are the left ends in their respective words and that $\gamma_r^\tau = \gamma_r$ or $\gamma_r^\tau = \mathrm{sub}(\gamma_r)$. Then one of the following two cases occurs:
  - Let $\gamma_l^\tau = \alpha_1^{(d)}$ for some $d \in \mathbb{Z}$. Note that this case is only possible if $\mathrm{sub}(\gamma_l) = \gamma_r$. In particular, $\gamma_l > \gamma_r$. Since γ_l^τ is the maximum element it follows that $\gamma_l^\tau > \gamma_r^\tau$.
  - Let $\gamma_l = \beta_1^{(d)}$ for some $d \in \mathbb{Z}$. In this case it holds that $\gamma_l^\tau = \mathrm{sub}(\gamma_r^\tau)$. It follows that $\gamma_l^\tau < \gamma_r^\tau$ as well as $\gamma_l < \gamma_r$, since γ_l is minimum.
- The cases (ac) and (bc) are dual to the previous two.
- in the case (cc) there are again two possibilities:

– Let $\gamma_l^\tau = \alpha_1^{(d)}$ for some $d \in \mathbb{Z}$. We have that either $\gamma_r^\tau = \alpha_1^{(d)}$ or $\gamma_r = \beta_1^{(j)}$ for some $j \in \mathbb{Z}$. Since $\gamma_l^\tau \neq \gamma_l^\tau$, it follows that γ_r is minimum. In particular, we obtain that $\gamma_l > \gamma_r$ as well as $\gamma_l^\tau > \gamma_r^\tau$.

– The case that $\gamma_l = \beta_1^{(d)}$ for some $d \in \mathbb{Z}$ is similar. We have that $\gamma_r^\tau = \alpha_1^{(j)}$ for some $j \in \mathbb{Z}$, because $\gamma_l \neq \gamma_r$. It follows that $\gamma_l < \gamma_r$ and $\gamma_l^\tau < \gamma_r^\tau$.

This shows claim (4.3.6) and proves the Lemma. $\square$

We may reformulate Lemma 4.3.17 as follows:

Lemma 4.3.18. *Let w be a usual or special word of $\check{\mathfrak{B}}$. Let $\overset{\leftrightarrow}{w}$ and $\overset{\leftrightarrow}{\tau(w)}$ be the words with oriented arrows of w respectively $\tau(w)$ as defined in Subsection A.3.2.1. Then the arrows in the common part of the words $\overset{\leftrightarrow}{w}$ and $\overset{\leftrightarrow}{\tau(w)}$ have the* same *orientations.*

PROOF. If w is a usual string of $\check{\mathfrak{B}}$ such that $P_\bullet(w) \in \mathrm{D^b}(\Lambda_\star)$ there is nothing to show. In any other case, the statement follows from Lemma 4.3.17 by the abbreviation of strings as described in Subsection 4.1.5. $\square$

Finally, let us prove the theorem on the Auslander-Reiten translation of strings:

PROOF OF THEOREM 4.3.10. Let Ω be some string or band of $\check{\mathfrak{B}}$ and let $P_\bullet(\Omega)$ be the corresponding complex in $\mathrm{D^b}(\Lambda)$. We have to show that

$$\tau(P_\bullet(\Omega)) \cong P_\bullet(\tau(\Omega)),$$

where $\tau(P_\bullet(\Omega))$ is obtained via the rules (4.3.3), while $P_\bullet(\tau(\Omega))$ is the glued complex of $\tau(\Omega)$ which was defined in (4.3.4).

- The main case is that Ω is some usual string ω or special string (ω, ε) of $\check{\mathfrak{B}}$ such that $P_\bullet(\Omega)$ is not a complex from $\mathrm{D^b}(\Lambda_\star)$.

  (1) By (4.3.3) the complex $\tau(P_\bullet(\Omega))$ is given by interchanging signs and replacing $P_\star$ by $rP_\star$ in $P_\bullet(\Omega)$. There is a minimal projective complex $\tau(P_\bullet(\Omega))'$ isomorphic to $\tau(P_\bullet)$. To obtain the complex $\tau(P_\bullet(\Omega))'$, one has to resolve the radical $rP_\star$ "locally" in the diagram of $\tau(P_\bullet(\Omega))$ (as shown in Example 4.3.7).

  (2) The main point is that the complex $\tau(P_\bullet(\Omega))'$ can be viewed as the *glued complex* of some finite word of $\check{\mathfrak{B}}$, that is, by a gluing diagram. It is straightforward to check that the gluing diagrams of the usual string $\tau(\omega)^{\mathrm{rev}}$ respectively the special string $(\tau(\omega)^{\mathrm{rev}}, \overline{\varepsilon})$, and the complex $\tau(P_\bullet(\Omega))'$ coincide if we ignore the orientations of the gluing edges. By passing to an equivalent string we may replace $\tau(\omega)^{\mathrm{rev}}$ by $\tau(\omega)$ in the following.

  It remains to show that the gluing diagrams of $\tau(P_\bullet(\Omega))'$ and $\tau(\Omega)$ have the same gluing arrows.

  (3) The resolution of any $rP_\star$ in $\tau(P_\bullet(\Omega))'$ corresponds to an end of type $\mathtt{p}_\star^{[\mathtt{d}]}\mathtt{1}^{[\mathtt{d}-1]}\uparrow$ in $\overset{\leftrightarrow}{\tau(\Omega)}$. Such ends have a canonical orientation in both gluing diagrams. Note that the gluing edges of the common part of the gluing diagram of $P_\bullet(\Omega)$ and the gluing diagram of $\tau(P_\bullet(\Omega))'$ have the *same* orientation. By Lemma 4.3.18 we have that the arrows in the common part of the decorated words $\overset{\leftrightarrow}{\omega}$ and $\overset{\leftrightarrow}{\tau(\omega)}$ are also the same. This implies that the intermediate parts of the

gluing diagrams of $P_\bullet(\Omega)$, $\tau(P_\bullet(\Omega))'$ and $P_\bullet(\tau(\Omega))$ coincide. It follows that the complex $\tau(P_\bullet(\Omega))'$ is exactly the glued complex of the string $\tau(\Omega)$.

Summarized, we obtain that $\tau(P_\bullet(\Omega)) \cong \tau(P_\bullet(\Omega))' \cong P_\bullet(\tau(\Omega))$.

- If Ω is a bispecial string or a band, the complex $P_\bullet(\Omega)$ has vanishing defect. In this case, Remark 4.3.4 and Lemma 4.3.3 imply that $\tau(P_\bullet(\Omega)) = \sigma(P_\bullet(\Omega)) \cong P_\bullet(\sigma(\Omega)) = P_\bullet(\tau(\Omega))$.
- Let Ω be some usual string of $\check{\mathfrak{B}}$ such that $P_\bullet(\Omega) \in \mathrm{D^b}(\Lambda_\star)$.
  Then $\tau(P_\bullet(\Omega)) \cong P_\bullet(\tau(\Omega))$ by the computations in Example 4.3.6. $\square$

4.3.4 Fractionally Calabi-Yau objects and spherical autoequivalences

Next, we will classify all indecomposable fractionally Calabi-Yau objects in $\mathrm{D^b}(\Lambda)$.

First, let us recall some terminology. Let $P_\bullet \in \mathrm{D^b}(\Lambda)$ be any complex.

- For any $n \in \mathbb{N}^+$ the complex $P_\bullet$ is *n-Calabi-Yau* if $\mathbb{S}(P_\bullet) \cong P_\bullet[n]$, where $\mathbb{S} = \tau \circ [1]$ is the relative Serre functor on $\mathrm{D^b}(\Lambda)$.
- For any $m, n \in \mathbb{N}^+$ the complex $P_\bullet$ is *(n, m)-Calabi-Yau* if $\mathbb{S}^m(P_\bullet) \cong P_\bullet[n]$.

Let us note that the complex $P_\bullet$ is *(n, m)-Calabi-Yau* if and only if $\tau^m(P_\bullet) \cong P_\bullet[n - m]$.

Remark 4.3.12 and Corollary 4.3.13 can be reformulated now as follows:

Remark 4.3.19. *Let Ω be a string or band of $\check{\mathfrak{B}}$ and $P_\bullet = P_\bullet(\Omega)$ be the corresponding complex in $\mathrm{D^b}(\Lambda)$. Then the complex $P_\bullet$ is (m, m)-Calabi-Yau for some $m \in \mathbb{N}^+$ if and only if $\delta(P_\bullet) = 0$. More precisely, the following holds:*

(1) The complex $P_\bullet$ is 1-Calabi-Yau if and only if Ω is a band or Ω is a usual string ω with both ends of type $\mathtt{p}_\infty$.

(2) The complex $P_\bullet$ is $(2, 2)$-Calabi-Yau (but not 1-Calabi-Yau) if and only if Ω is a bispecial string or Ω is a special string (ω, ε) such that ω has one end of type $\mathtt{p}_\infty$.

Let us introduce the following usual strings of $\check{\mathfrak{B}}_{\mathrm{fd}}$.

- For any $\mathtt{t} \in \mathbb{N}^+$ and $\mathtt{d} \in \mathbb{Z}$ let $\mathtt{c}_\star^{\mathtt{t}}[d]$ denote the usual string

$$\mathtt{c}_\star^{\mathtt{t}}[\mathtt{d}] = \mathtt{p}_\star^{[\mathtt{d+t+1}]} \mathtt{1}^{[\mathtt{d+t}]} \mathtt{1}^{[\mathtt{d+t-1}]} \ldots \mathtt{1}^{[\mathtt{d+2}]} \mathtt{1}^{[\mathtt{d+1}]} \mathtt{1}^{[\mathtt{d}]} \mathtt{p}_\star$$

  The usual string $\mathtt{c}_\star^{\mathtt{t}}[\mathtt{d}]$ has length $\mathtt{t+1} \geq 2$.
- For $\mathtt{d} = \mathtt{0}$ the corresponding string complex $C_\star^{\mathtt{t}} = P_\bullet(\mathtt{c}_\star^{\mathtt{t}}[\mathtt{0}])$ has the form

$$C_\star^{\mathtt{t}} = \underset{\mathtt{t+1}}{P_\star} \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]} \underset{\mathtt{t}}{P_+ \oplus P_-} \xrightarrow{\left[\begin{smallmatrix} 2 & 2 \\ -2 & -2 \end{smallmatrix}\right]} \ldots \underset{\mathtt{2}}{P_+ \oplus P_-} \xrightarrow{\left[\begin{smallmatrix} 2 & 2 \\ -2 & -2 \end{smallmatrix}\right]} \underset{\mathtt{1}}{P_+ \oplus P_-} \xrightarrow{[1\ 1]} \underset{\mathtt{0}}{P_\star} \tag{4.3.7}$$

The homology of this complex is given by $\mathbf{H}_j(C_\star^{\mathtt{t}}) \cong \begin{cases} S_\star & \text{if } \mathtt{0} \leq \mathtt{j} \leq \mathtt{t}-\mathtt{1} \\ 0 & \text{otherwise} \end{cases}$

Let us note that the complex $C_\star^1$ is the projective resolution of $S_\star$.

Theorem 4.3.10 or a direct computation implies the following statement:

Remark 4.3.20. *For any* $\mathtt{t} \in \mathbb{N}^+$ *and any* $\mathtt{d} \in \mathbb{Z}$ *it holds that* $\tau(C_\star^{\mathtt{t}}[d]) \cong C_\star^{\mathtt{t}}[d+1]$.

Example 4.3.21. *To make the proof of the next proposition more accessible, let us consider a few iterated applications of the Auslander-Reiten translation* τ *to some usual string* ω *of* $\check{\mathfrak{B}}$:

ω	$\mathtt{p}_\star^{[1]}\mathtt{1}^{[2]}\mathtt{5}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{8}^{[4]}\mathtt{1}^{[3]}\mathtt{1}^{[2]}\mathtt{1}^{[1]}\mathtt{p}_\star$
$\tau(\omega)$	$\mathtt{p}_\star^{[2]}\mathtt{5}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{8}^{[4]}\mathtt{1}^{[3]}\mathtt{1}^{[2]}\mathtt{p}_\star$
$\tau^2(\omega)$	$\mathtt{p}_\star^{[3]}\mathtt{1}^{[2]}\mathtt{4}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{8}^{[4]}\mathtt{1}^{[3]}\mathtt{p}_\star$
$\tau^3(\omega)$	$\mathtt{p}_\star^{[4]}\mathtt{1}^{[3]}\mathtt{1}^{[2]}\mathtt{4}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{9}^{[4]}\mathtt{p}_\star$
$\tau^4(\omega)$	$\mathtt{p}_\star^{[5]}\mathtt{1}^{[4]}\mathtt{1}^{[3]}\mathtt{1}^{[2]}\mathtt{4}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{9}^{[4]}\mathtt{1}^{[5]}\mathtt{p}_\star$
$\tau^5(\omega)$	$\mathtt{p}_\star^{[6]}\mathtt{1}^{[5]}\mathtt{1}^{[4]}\mathtt{1}^{[3]}\mathtt{1}^{[2]}\mathtt{4}^{[3]}\mathtt{n}_1^{[2]}\mathtt{n}_2^{[\mathtt{d}_4]}\dots\mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_{\mathtt{k}}^{[3]}\mathtt{9}^{[4]}\mathtt{1}^{[5]}\mathtt{1}^{[6]}\mathtt{p}_\star$

$\mathtt{n}_1, \mathtt{n}_2 \dots \mathtt{n}_{\mathtt{k}} \in \mathbb{N}^+$,
$\mathtt{d}_3 \dots \mathtt{d}_{\mathtt{k}-1} \in \mathbb{Z}$.

In particular, the usual string ω *has a "big" intermediate part which is invariant under* τ. *Moreover, the length of the complex* $\tau^m(P_\bullet(\omega))$ *is increasing for* $m \gg 0$.

Proposition 4.3.22. *Let* Ω *be some string or band of* $\check{\mathfrak{B}}$ *such that the corresponding complex* $P_\bullet = P_\bullet(\Omega)$ *is* (n,m)*-Calabi-Yau for some* $n, m \in \mathbb{N}^+$. *Then one of the following two cases occurs*:

(1) $P_\bullet$ *is* 1*-Calabi-Yau or* $(2,2)$*-Calabi-Yau and* Ω *is a band or a string as described in Remark 4.3.19, or*

(2) $P_\bullet$ *is* 2*-Calabi-Yau and* Ω *is equivalent to some usual string* $\mathtt{c}_\star^{\mathtt{t}}[\mathtt{d}]$ *for some* $\mathtt{t} \in \mathbb{N}^+$ *and* $\mathtt{d} \in \mathbb{Z}$.

Let us note that this proposition gives a description of all indecomposable fractionally Calabi-Yau objects of the category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

PROOF. In view of Remark 4.3.19 we may assume that Ω is given by some usual string ω or some special string (ω, ε) of $\check{\mathfrak{B}}_{\mathrm{fd}}$. Without loss of generality let

$$\omega = \mathtt{p}_\star^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_{\mathtt{k}}^{[\mathtt{d}_{\mathtt{k}}]}\mathtt{p}_\beta, \quad \text{where} \quad \mathtt{p}_\beta = \mathtt{p}_\star \text{ or } \mathtt{p}_\diamond.$$

The main idea is to consider the following invariant $l(P_\bullet)$ of the string complex $P_\bullet$:

$$\begin{aligned} l(P_\bullet) &= |\max\{\, i \in \mathbb{Z} \mid P_i \neq 0 \,\} - \min\{\, i \in \mathbb{Z} \mid P_i \neq 0 \,\}| \\ &= |\max\{\, \mathtt{0} \leq \mathtt{j} \leq \mathtt{k} \mid \mathtt{d}_{\mathtt{j}} \neq \mathtt{0} \,\} - \min\{\, \mathtt{0} \leq \mathtt{j} \leq \mathtt{k} \mid \mathtt{d}_{\mathtt{j}} \neq \mathtt{0} \,\}| \end{aligned}$$

Since $P_\bullet$ is (n,m)-Calabi-Yau, it holds that $\tau^m(P_\bullet) \cong P_\bullet[n-m]$. In particular, this implies that

$$l(\tau^{r\,m}(P_\bullet)) = l(P_\bullet[r\,n - r\,m]) = l(P_\bullet) \qquad \text{for any} \quad r \in \mathbb{Z}. \tag{4.3.8}$$

In other words, τ^m *preserves* the invariant $l(P_\bullet)$ of the fractionally Calabi-Yau object $P_\bullet$.

- Assume that Ω is a special string (ω, ε).In this case, $\mathtt{p}_\beta = \mathtt{p}_\diamond$. Then for any $r \in \mathbb{N}^+$ it holds that $\tau^{r\,m}(\omega) = \mathtt{p}_\star^{[\mathtt{d}_0+\mathtt{r}\,\mathtt{m}]} \ldots {}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\diamond$. For $r \gg 0$ it follows that $l(\tau^{r\,m}(P_\bullet)) = l(P_\bullet) + r\,m > l(P_\bullet)$, a contradiction to (4.3.8).

- So ω must be a usual string, that is, $\mathtt{p}_\beta = \mathtt{p}_\star$.

  (1) We have that $\tau^m(\omega) = \mathtt{p}_\star^{[\mathtt{d}_0+\mathtt{m}]} \ldots {}^{[\mathtt{d}_\mathtt{k}+\mathtt{m}]}\mathtt{p}_\star$. Since $\tau^m(P_\bullet) \cong P_\bullet[n-m]$ it holds that the string $\tau^m(\omega)$ or $(\tau^m(\omega))^{\mathrm{rev}}$ is equal to $\mathtt{p}_\star^{[\mathtt{d}_0+\mathtt{n}-\mathtt{m}]} \ldots {}^{[\mathtt{d}_\mathtt{k}+\mathtt{n}-\mathtt{m}]}\mathtt{p}_\star$. In both cases it follows that $n = 2m$. In particular, $\tau^m(P_\bullet) \cong P_\bullet[m]$ and $P_\bullet$ must be $(2m, m)$-Calabi-Yau.

  (2) Condition (4.3.8) and Theorem 4.3.10 imply that ω or ω^{rev} is equal to the string

  $$\upsilon = \mathtt{p}_\star^{[\mathtt{d}]}\mathtt{n}_1^{[\mathtt{d}-1]}\mathtt{n}_2^{[\mathtt{d}-2]} \ldots \mathtt{n}_{\mathtt{k}-1-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+1+\mathtt{m}]}\mathtt{n}_{\mathtt{k}-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+\mathtt{m}]}\mathtt{1}^{[\mathtt{d}-\mathtt{k}-1+\mathtt{m}]} \ldots \mathtt{1}^{[\mathtt{d}-\mathtt{k}+1]}\mathtt{1}^{[\mathtt{d}-\mathtt{k}]}\mathtt{p}_\star.$$

  for some $\mathtt{d} \in \mathbb{Z}$ and $\mathtt{n}_1, \ldots, \mathtt{n}_{\mathtt{k}-\mathtt{m}} \in \mathbb{N}^+$. In other words, any other string complex has some *τ-invariant homology* (as indicated in Example (4.3.21)). This is the crucial point in the proof.

  (3) Without loss of generality let $\omega = \upsilon$. Then

  $$\tau^m(\omega) = \mathtt{p}_\star^{[\mathtt{d}+\mathtt{m}]}\mathtt{1}^{[\mathtt{d}-1+\mathtt{m}]}\mathtt{1}^{[\mathtt{d}-2+\mathtt{m}]} \ldots \mathtt{1}^{[\mathtt{d}]}\mathtt{n}_1^{[\mathtt{d}-1]}\mathtt{n}_2^{[\mathtt{d}-2]} \ldots \mathtt{n}_{\mathtt{k}-1-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+1+\mathtt{m}]}\mathtt{n}_{\mathtt{k}-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+\mathtt{m}]}\mathtt{p}_\star.$$

  On the other hand, the complex $P_\bullet[m]$ is the string complex of

  $$\omega[m] = \mathtt{p}_\star^{[\mathtt{d}+\mathtt{m}]}\mathtt{n}_1^{[\mathtt{d}-1+\mathtt{m}]} \ldots \mathtt{n}_{\mathtt{k}-1-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+1+2\mathtt{m}]}\mathtt{n}_{\mathtt{k}-\mathtt{m}}^{[\mathtt{d}-\mathtt{k}+2\mathtt{m}]}\mathtt{1}^{[\mathtt{d}-\mathtt{k}-1+2\mathtt{m}]} \ldots \mathtt{1}^{[\mathtt{d}-\mathtt{k}-1]}\mathtt{1}^{[\mathtt{d}-\mathtt{k}]}\mathtt{p}_\star.$$

  Since $\tau^m(P_\bullet) \cong P_\bullet[m]$ the strings $\tau^m(\omega)$ and $\omega[m]$ must be equivalent. Note that $\tau^m(\omega) \neq \omega[m]^{\mathrm{rev}}$. So $\tau^m(\omega) = \omega[m]$. Then $\mathtt{1} = \mathtt{n}_1 = \mathtt{n}_2 = \ldots = \mathtt{n}_{\mathtt{k}-\mathtt{m}}$. In other words, ω is given by the usual string $\mathtt{c}_\star^{\mathtt{k}-1}[\mathtt{d}-\mathtt{k}]$. □

Next, we consider spherical objects and generalized spherical collections in the derived category $\mathrm{D^b}(\Lambda)$.

The simple module $S_\star$ is also distinguished by the following result:

Corollary 4.3.23. *The module $S_\star$ is the* only *spherical object in* $\mathrm{D^b}(\Lambda)$ *up to shift. In other terms, the twist functor $\mathbb{T}_{S_\star}$ is the only spherical autoequivalence of* $\mathrm{D^b}(\Lambda)$.

PROOF. Let $P_\bullet$ be an n-spherical object in $\mathrm{D^b}(\Lambda)$ for some $n \in \mathbb{N}_0$. By Proposition 4.3.22 it holds that either $n = 1$ or $n = 2$.

- Assume that $n = 1$. Then $P_\bullet$ is 1-Calabi-Yau and $P_\bullet \cong P_\bullet(\Omega)$ for some band Ω of $\check{\mathfrak{B}}_{\mathrm{fd}}$ or some string ω of $\check{\mathfrak{B}}$ with both ends of type $\mathtt{p}_\infty$. We may assume that the minimal complex $P_\bullet$ has length l for some $l \in \mathbb{N}_0$, that is, $P_j \neq 0$ if and only if $0 \leq j \leq l$. By construction of the complex $P_\bullet$ both projectives P_l and P_0 have a direct summand $P_+ \oplus P_-$. It follows that $\dim \mathrm{Ext}^l_\Lambda(P_\bullet, P_\bullet) \geq 2$, so $P_\bullet$ cannot be 1-spherelike, a contradiction.

- Assume that $n = 2$. Then $P_\bullet$ is 2-Calabi-Yau and $P_\bullet \cong C_\star^{\mathtt{t}}[d]$ for some $\mathtt{t} \in \mathbb{N}^+$ and some shift $d \in \mathbb{Z}$. We may assume that $d = 0$. Then $P_\bullet$ has length $\mathtt{t}+\mathtt{1}$ and the projectives P_{t+1} and P_0 have both a direct summand of type $P_\star$. It follows that $\mathrm{Ext}^{t+1}_\Lambda(P_\bullet, P_\bullet) \neq 0$. Since $P_\bullet$ is 2-spherelike, it follows that $\mathtt{t} = \mathtt{1}$ and $P_\bullet = S_\star^1$

which is the projective resolution of $S_\star$. It is straightforward to check that the module $S_\star$ is indeed 2-spherical (see also Lemma 2.1.7). □

In the last part of this subsection, we want to show that Corollary 4.3.23 can be generalized to *spherical collections* in the sense of Definition B.3.9.

For $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ let us recall the following notion:

- The complex $P_\bullet$ is *exceptional* if $\mathrm{End}_\Lambda(P_\bullet) \cong \Bbbk$ and $\mathrm{Ext}^j_\Lambda(P_\bullet, P_\bullet) = 0$ for any $j \in \mathbb{Z}^*$.

Remark 4.3.24. *Let $C^{\mathtt{t}}_\star$ be the 2-Calabi-Yau complex for some $\mathtt{t} \in \mathbb{N}^+$. The complex $C^{\mathtt{t}}_\star$ has length $\mathtt{t}+1$. The projectives $C^{\mathtt{t}}_{t+1}$ and $C^{\mathtt{t}}_0$ are both given by $P_\star$. This implies that $\mathrm{Ext}^{t+1}_\Lambda(C^{\mathtt{t}}_\star, C^{\mathtt{t}}_\star) \neq 0$, so $C^{\mathtt{t}}_\star$ is* not *exceptional. In the following, such an argument will be frequently used to show that an object is* not *exceptional.*

To generalize Corollary 4.3.23 we need the following Lemma:

Lemma 4.3.25. *Let Ω be string or band of $\check{\mathfrak{B}}$ such that its complex $P_\bullet = P_\bullet(\Omega)$ has* vanishing *defect. Then $P_\bullet$ is* not *exceptional.*

PROOF. Since $\delta(P_\bullet) = 0$, we have that $P_\bullet \in \mathrm{K}^{\mathrm{b}}(\mathrm{add}\, P_+ \oplus P_-)$. Assume that the complex $P_\bullet$ is exceptional. We may assume that the minimal complex $P_\bullet$ has length $l \in \mathbb{N}_0$.

(1) We claim that $l = 1$. If $l = 0$, then $P_\bullet$ is given by some projective Λ-module $P_\pm$, which is not exceptional. So $l \geq 1$. Since $\mathrm{Ext}^l_\Lambda(P_\bullet, P_\bullet) = 0$, it holds that $P_l \in \mathrm{add}\, P_\pm$ and $P_0 \in \mathrm{add}\, P_\mp$. This implies also that $P_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ has finite-dimensional homology. It follows that $\mathrm{Ext}^l_\Lambda(P_\bullet, \sigma(P_\bullet)) \neq 0$. Since $\tau(P_\bullet) \cong \sigma(P_\bullet)$, the Auslander-Reiten formula yields that $\mathrm{Ext}^{1-l}_\Lambda(P_\bullet, P_\bullet) \cong \mathbb{D}\,\mathrm{Ext}^l_\Lambda(P_\bullet, \sigma(P_\bullet)) \neq 0$. Since $P_\bullet$ is exceptional, we obtain that $l = 1$.

(2) By the classification of indecomposable complexes in Section 4.2, there is only one series of complexes of length one with $P_1 \in \mathrm{add}\, P_\pm$ and $P_0 \in \mathrm{add}\, P_\mp$, that is, $P_\bullet = P_\pm \xrightarrow{\mathtt{2n}} P_\mp$ for some $\mathtt{n} \in \mathbb{N}^+$. Since $\mathrm{Ext}^1_\Lambda(P_\bullet, P_\bullet) = 0$, it follows that $\mathtt{n} = \mathtt{1}$. It remains to note that $\mathrm{End}_\Lambda(P_\bullet) \cong \Bbbk^2$, so $P_\bullet$ cannot be exceptional. □

Corollary 4.3.26. *Let $\mathcal{S} = (S_1, S_2, \ldots S_d)$ be a generalized n-spherical collection of objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ for some $d \in \mathbb{N}^+$ and $n \in \mathbb{N}_0$.*
Then $d = 1$ and $\mathcal{S}$ is given by the spherical object $S_\star$ up to shift.

PROOF. • Assume that $d \geq 2$. As noted in Remark B.3.11, the object S_j must be exceptional and fractionally Calabi-Yau for any $1 \leq j \leq d$. However, Proposition 4.3.22, Remark 4.3.24 and Lemma 4.3.25 imply that there are *no* exceptional fractionally Calabi-Yau objects in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

- It follows that $d = 1$. In this case $\mathcal{S}$ is given by a single n-spherical object. The statement follows now from Corollary 4.3.23. □

Remark 4.3.27. *There is a further generalization of spherical collections given by* exceptional cycles *in the sense of* **[BPP13]**. *Corollary 4.3.26 generalizes also to exceptional cycles.*

Summarized, the above results state there is *only one generalized spherical autoequivalence* on $\mathrm{D}^{\mathrm{b}}(\Lambda)$, that is, only one autoequivalence given by some generalized twist functor $\mathbb{T}_{\mathcal{S}}$ associated to some spherical collection $\mathcal{S}$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. This only autoequivalence is given by the twist $\mathbb{T}_{S_\star}$ associated to the simple module $S_\star$.

4.3.5 The subcategory generated by the spherical object $S_\star$

Let $\langle S_\star \rangle$ be the thick subcategory of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ generated by the 2-spherical object $S_\star$. In other words, $\langle S_\star \rangle$ is the smallest subcategory in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ which contains $S_\star$ and is closed under cones, shifts and direct summands. In this subsection, we give an explicit description of the subcategory $\langle S_\star \rangle$.

A special case of Theorem B.3.4 by Keller, Yang and Zhou together Theorem B.3.5 by Jorgensen yields the following statement:

Proposition 4.3.28 ([**KYZ09**], [**Jor04**])**.** *The Auslander-Reiten quiver of the category* $\langle S_\star \rangle$ *is given by* one component of type $\mathbb{Z}\mathbb{A}_\infty$.

Using the Proposition above and Remark 4.3.20 we can derive the following description of the category $\langle S_\star \rangle$.

Corollary 4.3.29. *(1) Any indecomposable object in* $\langle S_\star \rangle$ *is isomorphic to the complex* $C_\star^{\mathtt{t}}[d]$ *for some* $\mathtt{t} \in \mathbb{N}^+$ *and* $d \in \mathbb{Z}$ *as defined in* (4.3.7).

(2) The Auslander-Reiten triangles in $\langle S_\star \rangle$ *are given up to shift as follows*:

- *The Auslander-Reiten triangle at* $C_\star^1 = S_\star$ *is given by*

$$S_\star[1] \longrightarrow C_\star^2 \longrightarrow S_\star \longrightarrow S_\star[2] \tag{4.3.9}$$

- *For any* $\mathtt{t} \geq 2$ *the Auslander-Reiten triangle at* $S_\star^{\mathtt{t}}$ *is given by*

$$C_\star^{\mathtt{t}}[1] \longrightarrow C_\star^{\mathtt{t+1}} \oplus C_\star^{\mathtt{t-1}}[1] \longrightarrow C_\star^{\mathtt{t}} \longrightarrow C_\star^{\mathtt{t}}[2] \tag{4.3.10}$$

In particular, a complex $P_\bullet \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ *is* 2*-Calabi-Yau if and only if* $P_\bullet \in \langle S_\star \rangle$.

PROOF. (2) First, we show the second statement by induction on $\mathtt{t} \in \mathbb{N}^+$. Remark 4.3.20 states that any object $S_\star^{\mathtt{t}}$ is 2-Calabi-Yau, that is, $\tau(S_\star^{\mathtt{t}}) \cong S_\star^{\mathtt{t}}[1]$ for any $\mathtt{t} \in \mathbb{N}^+$.

- Let $\mathtt{t} = \mathtt{1}$. There is a short exact sequence in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of the following form:

$$0 \longrightarrow S_\star[1] \longrightarrow C_\star^2 \longrightarrow S_\star \longrightarrow 0$$

  This short exact sequence gives rise to a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ of the form (4.3.9). Since $\mathrm{Ext}^1_\Lambda(S_\star, \tau(S_\star)) \cong \mathbb{D}\,\mathrm{End}_\Lambda(S_\star) \cong \mathbb{k}$, this distinguished triangle must be the *Auslander-Reiten triangle* ending in $S_\star$.

- Assume that the Auslander-Reiten triangle ending in $C_\star^{\mathtt{t}}$ is given by (4.3.10) for some $\mathtt{t} \in \mathbb{N}^+$. Then the Auslander-Reiten triangle ending in $C_\star^{\mathtt{t+1}}$ is given by

$$C_\star^{\mathtt{t+1}}[1] \longrightarrow P_\bullet \oplus C_\star^{\mathtt{t}}[1] \longrightarrow C_\star^{\mathtt{t+1}} \longrightarrow C_\star^{\mathtt{t+1}}[2] \tag{4.3.11}$$

for some complex $P_\bullet \in \mathrm{D^b}(\Lambda)$. We claim that $P_\bullet \cong S_\star^{\mathtt{t}+1}$ in $\mathrm{D^b}(\Lambda)$. Proposition 4.3.28 implies that the middle term in the Auslander-Reiten triangle (4.3.11) is given by *two* indecomposable objects, that is, the complex $P_\bullet$ must be indecomposable. The long exact homology sequence of the triangle (4.3.11) yields that

$$\mathbf{H}_j(P_\bullet) \cong \mathbf{H}_j(C_\star^{\mathtt{t}+1}) \cong \begin{cases} S_\star & \text{if } \mathtt{0} \leq \mathtt{j} \leq \mathtt{t} \\ 0 & \text{otherwise} \end{cases}$$

Using the classification of indecomposable complexes in $\mathrm{D^b}(\Lambda)$ of Section 4.2 it can be shown that the complex $P_\bullet$ is uniquely determined by its homology. It follows that $P_\bullet \cong C_\star^{\mathtt{t}+1}$ in $\mathrm{D^b}(\Lambda)$. This shows the induction step.

(1) The second statement yields that $C_\star^{\mathtt{t}} \in \langle S_\star \rangle$ for any $d \in \mathbb{Z}$ and $\mathtt{t} \in \mathbb{N}^+$. By Proposition 4.3.28 the Auslander-Reiten quiver of $\langle S_\star \rangle$ has only one component. So the Auslander-Reiten triangles described above contain all indecomposable objects of $\langle S_\star \rangle$ up to shift. This shows the first statement. □

4.3.6 Summary on the derived category of the Gelfand order

At last, we summarize the results of Section 4.3 together with the main results from Chapter 2 specialized to the Gelfand order.

First, string and band complexes can be characterized without combinatorics:

Theorem 4.3.30. *Let Ω be a string or band of $\check{\mathfrak{B}}_{\mathrm{fd}}$ and $P_\bullet = P_\bullet(\Omega)$ be the corresponding indecomposable complex in $\mathrm{D^b_{fd}}(\Lambda)$. Then the following statements hold*:

(1)	Ω *is a usual string* ω	$\Leftrightarrow$	$\delta(P_\bullet) > 0$ *and* $\sigma(P_\bullet) \cong P_\bullet$ $\Leftrightarrow$ $\delta(P_\bullet) = 2$
		$\Leftrightarrow$	$P_\bullet$ *is not* τ*-periodic and* $\sigma(P_\bullet) \cong P_\bullet$
(2)	Ω *is a special string* ω_{ε_1}	$\Leftrightarrow$	$\delta(P_\bullet) > 0$ *and* $\sigma(P_\bullet) \ncong P_\bullet$ $\Leftrightarrow$ $\delta(P_\bullet) = 1$
		$\Leftrightarrow$	$P_\bullet$ *is not* τ*-periodic and* $\sigma(P_\bullet) \ncong P_\bullet$
(3)	Ω *is a bispecial string* $\omega^m_{\varepsilon_1,\varepsilon_2}$	$\Leftrightarrow$	$\delta(P_\bullet) = 0$ *and* $\sigma(P_\bullet) \ncong P_\bullet$
		$\Leftrightarrow$	$\tau^2(P_\bullet) \cong P_\bullet$ *and* $\tau(P_\bullet) \ncong P_\bullet$
(4)	Ω *is a band* ω^m_λ	$\Leftrightarrow$	$\delta(P_\bullet) = 0$ *and* $\sigma(P_\bullet) \cong P_\bullet$
		$\Leftrightarrow$	$\tau(P_\bullet) \cong P_\bullet$

where $\varepsilon_1, \varepsilon_2 \in \{+,-\}$ *and* $\lambda \in \Bbbk \setminus \Delta$

This statement was shown in Corollary 3.6.4 for any Khoroshkin order.

PROOF. Let us remark that the Theorem above follows also from the purely combinatorial statements about the defect in (4.3.1), the involution in Lemma 4.3.3 and Corollary 4.3.13. □

Remark 4.3.31. *Theorem 4.3.30 is not true for the whole derived category* $\mathrm{D^b}(\Lambda)$, *that is, for all string complexes with infinite-dimensional homology. See Remark 4.3.12.*

Second, the main *combinatorial results* on the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ are given by the following:

(1) There is an explicit classification of the indecomposable complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ by Theorem 4.2.9 of Burban and Drozd. These are parametrized by strings and bands (Theorem 4.1.14).

(2) There is an explicit description of the Auslander-Reiten translation on strings and bands (Theorem 4.3.10).

Third, let us summarize what we know so far about the *Auslander-Reiten quiver* of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$:

(1) The subcategory $\mathcal{T}_{\mathrm{fd}}$ of τ-periodic complexes is given by homogeneous tubes of rank *one* (bands) or rank *two* (bispecial strings).

(2) It is possible to compute the Auslander-Reiten translation of any string (or band) of $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

(3) The subcategory $\langle S_\star \rangle$ generated by the 2-spherical object $S_\star$ is given exactly by the 2-Calabi-Yau objects of $\mathrm{D}^{\mathrm{b}}(\Lambda)$. The Auslander-Reiten quiver of $\langle S_\star \rangle$ is given by one component of type $\mathbb{Z}\mathbb{A}_\infty$.

(4) The thick categories $\mathcal{T}_{\mathrm{fd}}$ and $\langle S_\star \rangle$ contain all fractionally Calabi-Yau objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

(5) As shown in Figure 4.3.1 there are Auslander-Reiten triangles with *three* middle terms in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

Let us also mention some results of Chapter 2 on the derived category of the Gelfand order.

The Auslander-Reiten translation τ of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ has the following intrinsic description: The simple module $S_\star$ of the Gelfand order is distinguished by the following result:

Proposition 4.3.32. *(1) The simple module $S_\star$ is* 2-spherical. *In particular, the dual twist functor $\mathbb{T}^{\vee}_{S_\star}$ associated to $S_\star$ is an autoequivalence of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$..

(2) There are isomorphisms of functors

$$\tau \cong \sigma \circ \mathbb{T}^{\vee}_{S_\star} \cong \mathbb{T}^{\vee}_{S_\star} \circ \sigma : \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$$

(3) Any generalized spherical collection in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *is given by the* 2*-spherical object $S_\star$. That is, the twist $\mathbb{T}_{S_\star}$ is the only generalized spherical autoequivalence of* $\mathrm{D}^{\mathrm{b}}(\Lambda)$ *or* $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$

This isomorphism restricts to the subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ *in* $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

Let us comment on these statements:

- Taking the definition of the defect into account, the spherical module $S_\star$ "knows" the main homological properties of objects in the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

- The second statement yields an intrinsic description of the Auslander-Reiten translation. This description can be transferred to the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{H}_0)$ of the principal block of Harish-Chandra modules over $\mathrm{SL}(2,\mathbb{R})$ or any equivalent analytic category.

- By the third statement the derived category seems to have no further "symmetries" than the twist $\mathbb{T}_{S_\star}$ and the involution σ.

  Since we can compute the involution σ and the Auslander-Reiten translation τ, hence, also the twist $\mathbb{T}_{S_\star}$ all known autoequivalences of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ are well-understood.

Fourth, the category $\mathcal{T}_{\mathrm{fd}}$ of τ-periodic objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ admits various interpretations. To state them we need the following notations:

- The left and right-perpendicular categories of $\langle S_\star\rangle$ are given as follows:

$$\begin{aligned} {}^{\perp}\langle S_\star\rangle &= \{\ P_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \mid \mathrm{Ext}^i_\Lambda(P_\bullet, X_\bullet) = 0 \text{ for any } X_\bullet \in \langle S_\star\rangle \text{ and any } i\in\mathbb{Z}\ \} \\ \langle S_\star\rangle^{\perp} &= \{\ P_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \mid \mathrm{Ext}^i_\Lambda(X_\bullet, P_\bullet) = 0 \text{ for any } X_\bullet \in \langle S_\star\rangle \text{ and any } i\in\mathbb{Z}\ \} \end{aligned}$$

- Let $\mathrm{K}^{\mathrm{b}}_{\mathrm{fd}}(\operatorname{add} P_+ \oplus P_-)$ denote the bounded homotopy category of projective complexes of Λ-modules with finite-dimensional homology and without projectives of type $P_\star$.
- Let $\Lambda^e = e\,\Lambda\,e$ where $e = e_+ + e_-$. In particular, the subalgebra Λ^e is isomorphic to the arrow ideal completion of the path algebra of the following quiver:

$$\Lambda^e = \begin{bmatrix} \mathbf{R} & \mathfrak{m} \\ \mathfrak{m} & \mathbf{R} \end{bmatrix} \quad \text{and} \quad (Q,I)^e = \begin{tikzcd} \underset{+}{\bullet} \arrow[loop left, "\mathsf{x}_+"] \arrow[r, bend left, "\mathsf{y}_+"] & \underset{-}{\bullet} \arrow[l, bend left, "\mathsf{y}_-"] \arrow[loop right, "\mathsf{x}_-"] \end{tikzcd} \qquad \begin{matrix} \mathsf{x}_\pm^2 = \mathsf{y}_\mp\,\mathsf{y}_\pm \\ \mathsf{x}_\pm\,\mathsf{y}_\mp = \mathsf{y}_\mp\,\mathsf{x}_\mp. \end{matrix}$$

  The subring Λ^e is a nodal *Gorenstein* order and has *infinite* global dimension.
- Let $\mathsf{Perf}_{\mathrm{fd}}(\Lambda^e)$ denote the category of perfect complexes of Λ^e-modules with finite-dimensional homology.

The subcategory $\mathcal{T}_{\mathrm{fd}}$ of τ-periodic complexes in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ has the following descriptions:

$$\mathcal{T}_{\mathrm{fd}} = \langle S_\star\rangle^{\perp} = {}^{\perp}\langle S_\star\rangle \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}_{\mathrm{fd}}(\operatorname{add} P_+ \oplus P_-) \xrightarrow{\sim} \mathsf{Perf}_{\mathrm{fd}}(\Lambda^e)$$

In other words, the additive category $\mathcal{T}_{\mathrm{fd}}$ of bispecial string and bands is orthogonal to the category generated by the 2-spherical object $S_\star$. Moreover, the category $\mathcal{T}_{\mathrm{fd}}$ can be described in terms of a "very singular" of the Gelfand order Λ.

Since the category $\mathcal{T}_{\mathrm{fd}}$ contains all continuous series of indecomposable objects in the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$, the subquiver $(Q,I)^e$ views the "tame part" of the Gelfand quiver.

CHAPTER 5

Abelian Category of the Gelfand quiver

In this final chapter we apply the techniques of the preceding chapter to obtain an explicit description of the nilpotent representations of the Gelfand quiver, their homological invariants and their functorial properties.

In the last section we describe an algorithm to glue any indecomposable representation of the Gelfand quiver from *cyclic representations*.

5.1 Strings and bands of nilpotent representations

5.1.1 Main theorem

In this section we introduce the invariants which parametrize the indecomposable nilpotent representations of the Gelfand quiver. These will be called *strings* and *bands* of $\mathfrak{B}_0$, or just strings and bands of the Gelfand quiver. We begin with the definitions for bands since they are less technical.

Definition 5.1.1. *Let $\Bbbk$ be an algebraically closed field.*

(1) A cyclic word ω *of $\mathfrak{B}_0$ is given by a sequence*

$$\omega = \left({}^{[\mathtt{d}]}\ \mathtt{n}_1, \mathtt{n}_2,\ \dots\ \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}\ {}^{[\mathtt{d}]}\right)$$

of one parameter $\mathtt{d} \in \{0,1\}$ and a sequence of $2\mathtt{k}$ natural numbers $\mathtt{n}_\mathtt{j} \in \mathbb{N}^+$, where $1 \leq \mathtt{j} \leq 2\mathtt{k}$ and $\mathtt{k} \geq 1$.
Given a cyclic word as above, we set $\mathtt{n}_{\mathtt{j}+\mathtt{i}\cdot 2\mathtt{k}} = \mathtt{n}_\mathtt{j}$ for all $\mathtt{i} \in \mathbb{Z}$ and $1 \leq \mathtt{j} \leq 2\mathtt{k}$.

In the following, let ω be a cyclic word of length $2\mathtt{k}$ as above.

(2) Let $0 \leq \mathtt{j} < 2\mathtt{k}$. The $\mathtt{j}$-th rotation $\omega^{[\mathtt{j}]}$ *of ω is defined as*

$$\omega^{[\mathtt{j}]} = \left({}^{[\mathtt{d}_*]}\ \mathtt{n}_{\mathtt{j}+1}, \mathtt{n}_{\mathtt{j}+2},\ \dots\ \mathtt{n}_{\mathtt{j}+2\mathtt{k}-1}, \mathtt{n}_{\mathtt{j}+2\mathtt{k}}\ {}^{[\mathtt{d}_*]}\right), \quad \textit{where}\ \ \mathtt{d}_* = \begin{cases} 1-\mathtt{d} & \textit{if } \mathtt{j} \textit{ is odd,} \\ \mathtt{d} & \textit{if } \mathtt{j} \textit{ is even.} \end{cases}$$

(3) The reversed word ω^{rev} *of ω is defined as*

$$\omega^{\mathrm{rev}} = \left({}^{[\mathtt{d}]}\ \mathtt{n}_{2\mathtt{k}}, \mathtt{n}_{2\mathtt{k}-1},\ \dots\ \mathtt{n}_2, \mathtt{n}_1\ {}^{[\mathtt{d}]}\right).$$

(4) The word ω is periodic *if $\omega = \omega^{[\mathtt{j}]}$ for some non-trivial rotation $1 \leq \mathtt{j} < 2\mathtt{k}$.*

(5) The word ω is symmetric *if $\omega^{\text{rev}} = \omega^{[\mathtt{j}]}$ for some rotation $0 \le \mathtt{j} < 2\mathtt{k}$.*

Example 5.1.2. *Let us note the following examples*:

non-periodic, symmetric:	$\left(^{[0]}\ 1,1\ ^{[0]}\right)$,	$\left(^{[1]}\ 3,2,2,3\ ^{[1]}\right)$,
	$\left(^{[0]}\ 1,2,3,3,2,1\ ^{[0]}\right)$,	$\left(^{[1]}\ 2,3,3,1,1,1\ ^{[1]}\right)$.
periodic, non-symmetric:	$\left(^{[0]}\ 1,2,1,2,1,2\ ^{[0]}\right)$,	$\left(^{[1]}\ 3,4,1,2,3,4,1,2\ ^{[1]}\right)$.
non-periodic, non-symmetric:	$\left(^{[0]}\ 1,2,2,1,1,2\ ^{[0]}\right)$,	$\left(^{[1]}\ 1,2,3,1,2,3\ ^{[1]}\right)$.

Definition 5.1.3. *Let $\Bbbk$ be an algebraically closed field.*

(1) A band word *of $\mathfrak{B}_0$ is given by any cyclic and non-periodic word ω of $\mathfrak{B}_0$.*

(2) A band $\big(\omega, m, \lambda\big)$ *of $\mathfrak{B}_0$ is given by some band word*

$$\omega = \left(^{[\mathtt{d}]}\ \mathtt{n}_1, \mathtt{n}_2, \ \dots\ \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}\ ^{[\mathtt{d}]}\right),$$

some "multiplicity" $m \in \mathbb{N}^+$ and some "eigenvalue" $\lambda \in \Bbbk^*$ such that $\lambda \neq (-1)^{\mathtt{k}+1}$ if ω is symmetric and has length $2\mathtt{k}$.

(3) Two bands (ω, m, λ) and (υ, n, μ) are equivalent *if $m = n$, the words ω and υ have the same length $2\mathtt{k}$, and there is some $0 \le \mathtt{j} < 2\mathtt{k}$ such that*

- $\upsilon = \omega^{[\mathtt{j}]}$ *and* $\mu = \lambda$, *or*
- $\upsilon^{\text{rev}} = \omega^{[\mathtt{j}]}$ *and* $\mu = \lambda^{-1}$.

There is only one series of **bands of the Gelfand quiver** up to equivalence:

$$\begin{aligned} (\omega, m, \lambda) \quad \text{where}\ \ \omega = \left(^{[1]}\ \mathtt{n}_1, \mathtt{n}_2, \ \dots\ \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}\ ^{[1]}\right) \text{ is non-periodic} \\ m \in \mathbb{N}^+,\ \text{and}\ \ \lambda \in \Bbbk^* \text{ with } \lambda \neq (-1)^{\mathtt{k}+1} \text{ if } \omega = \omega^{\text{rev}}. \end{aligned} \tag{5.1.1}$$

Next, we give the necessary definitions for strings:

Definition 5.1.4. *(1) Let $\mathfrak{E}_\alpha$ and $\mathfrak{E}_\beta$ denote the following sets of formal symbols*:

$$\begin{aligned} \mathfrak{E}_\alpha &= \left\{\ \mathtt{p}_\star^{[0]},\ \ \mathtt{p}_\diamond^{[0]},\ \ \mathtt{p}_\star^{[1]},\ \ \mathtt{p}_\diamond^{[1]},\ \ \mathtt{p}_\star^{[2]}\ \right\}, \\ \mathfrak{E}_\beta &= \left\{\ ^{[0]}\mathtt{p}_\star\ ,\ ^{[0]}\mathtt{p}_\diamond\ ,\ ^{[1]}\mathtt{p}_\star\ ,\ ^{[1]}\mathtt{p}_\diamond\ ,\ ^{[2]}\mathtt{p}_\star\ \right\}. \end{aligned}$$

(2) A finite word *ω of $\mathfrak{B}_0$ is given by a sequence*

$$\omega = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\ \mathtt{n}_1, \mathtt{n}_2, \ \dots\ \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}\ ^{[\mathtt{d}_\beta]}\mathtt{p}_\beta\right) \tag{5.1.2}$$

consisting of two ends $\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]} \in \mathfrak{E}_\alpha$ and $^{[\mathtt{d}_\beta]}\mathtt{p}_\beta \in \mathfrak{E}_\beta$ and $\mathtt{k}$ natural numbers $\mathtt{n}_\mathtt{j} \in \mathbb{N}^+$, where $1 \le \mathtt{j} \le \mathtt{k}$ and $\mathtt{k} \ge 1$, such that $\mathtt{k}$ has the following parity:

$$\begin{cases} \mathtt{k} \textit{ is even} & \textit{if } \mathtt{d}_\alpha = \mathtt{d}_\beta = 0, \textit{ or } \mathtt{d}_\alpha, \mathtt{d}_\beta \in \{\ 1, 2\ \}, \\ \mathtt{k} \textit{ is odd} & \textit{otherwise.} \end{cases}$$

In other terms, a finite word ω of $\mathfrak{B}_0$ can be divided into three parts. The following Table lists all possibilities for these:

first part	*intermediate part*	*last part*
$\mathtt{p}_\star^{[0]}$		${}^{[0]}\mathtt{p}_\star$
$\mathtt{p}_\diamond^{[0]}$		${}^{[0]}\mathtt{p}_\diamond$
$\mathtt{p}_\star^{[1]}\ \mathtt{n}_0,$	$\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3 \ldots, \mathtt{n}_{2k-2}, \mathtt{n}_{2k}$	$\mathtt{n}_{2k+1}\ {}^{[1]}\mathtt{p}_\star$
$\mathtt{p}_\diamond^{[1]}\ \mathtt{n}_0,$		$\mathtt{n}_{2k+1}\ {}^{[1]}\mathtt{p}_\diamond$
$\mathtt{p}_\star^{[2]}\ \mathtt{n}_0,$		$\mathtt{n}_{2k+1}\ {}^{[2]}\mathtt{p}_\star$

$\mathtt{n}_0, \mathtt{n}_1, \ldots, \mathtt{n}_{2k} \in \mathbb{N}^+$

In the word ω any of the five beginnings can match any of the five ends of the Table. In the following let ω denote a finite word of the form (5.1.2)*.*

(3) The reversed word ω^{rev} *of ω is defined as*

$$\omega^{\mathrm{rev}} = \Big(\mathtt{p}_\beta^{[\mathtt{d}_\beta]},\ \mathtt{n}_k, \mathtt{n}_{k-1},\ \ldots\ \mathtt{n}_2, \mathtt{n}_1,\ {}^{[\mathtt{d}_\alpha]}\mathtt{p}_\alpha\Big).$$

(4) The word ω is symmetric if $\omega = \omega^{\mathrm{rev}}$.

(5) The left end $\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}$ *of ω is* special *if $\alpha = \diamond$.*
Similarly, the right end ${}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta$ *of ω is* special *if $\beta = \diamond$.*

(6) Assume that both ends of ω are special. The word ω is quasi-symmetric *if there is some subsequence $\upsilon = (\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_j)$ of ω with $1 \leq j < k$ such that*

$$\breve{\omega} := (\mathtt{n}_1, \mathtt{n}_2, \quad \ldots \quad, \mathtt{n}_k) = \big(\underbrace{\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_j}_{\upsilon},\ \underbrace{\mathtt{n}_j, \ldots, \mathtt{n}_2, \mathtt{n}_1}_{\upsilon^{\mathrm{rev}}}, \quad \ldots$$
$$\ldots\quad \underbrace{\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_j}_{\upsilon},\ \underbrace{\mathtt{n}_j, \ldots, \mathtt{n}_2, \mathtt{n}_1}_{\upsilon^{\mathrm{rev}}},\ \underbrace{\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_j}_{\upsilon}\big).$$

On the right hand side, the sequence υ appears an odd number, while υ^{rev} appears an even number of times.

Example 5.1.5. *Let us note the following examples of finite words*:

symmetric:	$\mathtt{p}_\star^{[1]}\ 1,2,3,3,2,1\ {}^{[1]}\mathtt{p}_\star,\quad \mathtt{p}_\diamond^{[0]}\ 2,1,3,3,1,2\ {}^{[0]}\mathtt{p}_\diamond.$
non-symmetric:	$\mathtt{p}_\star^{[1]}\ 1,2,3,3,2,1\ {}^{[2]}\mathtt{p}_\star,\quad \mathtt{p}_\star^{[1]}\ 2,1,3,3,1,2\ {}^{[1]}\mathtt{p}_\diamond.$
quasi-symmetric:	$\mathtt{p}_\diamond^{[1]}\ 2,1,1,2,2,1\ {}^{[1]}\mathtt{p}_\diamond,\quad \mathtt{p}_\diamond^{[1]}\ 1,2,3,3,2,1,1,2,3\ {}^{[0]}\mathtt{p}_\diamond.$
not (quasi-)symmetric:	$\mathtt{p}_\diamond^{[1]}\ 2,1,2,1,2,1\ {}^{[1]}\mathtt{p}_\diamond,\quad \mathtt{p}_\diamond^{[1]}\ 2,1,3,1,2\ {}^{[0]}\mathtt{p}_\diamond.$

Remark 5.1.6. *Any symmetric word ω has ends of same type and* even *length.*

Next, strings of $\mathfrak{B}_0$ are defined in a similar way as strings for the derived category of the Gelfand quiver (Definition 4.1.12):

Definition 5.1.7. *In the following let ω be a finite word of $\mathfrak{B}_0$.*

(1) (a) A finite word ω is usual *if ω has no special ends and is not symmetric.*

(b) A usual string *is given by any usual word ω.*

(c) Two usual strings υ and ω are equivalent *if $\upsilon = \omega$ or $\upsilon = \omega^{\mathrm{rev}}$.*

(2) *(a)* *A finite word ω is* special *if ω has exactly one special end.* (*if ω has length zero, the only end of ω counts as only* one *end*).

(b) *A* special string (ω, ε) *is given by a special word ω and one sign $\varepsilon \in \{+, -\}$.*

(c) *Two special strings (υ, δ) and (ω, ε) are* equivalent *if $(\upsilon, \delta) = (\omega, \varepsilon)$ or $(\upsilon, \delta) = (\omega^{\mathrm{rev}}, \varepsilon)$.*

(3) *(a)* *A finite word ω is* bispecial *if ω has two special ends and ω is neither symmetric nor quasi-symmetric.*

(b) *A* bispecial string $(\omega, m, \varepsilon_1, \varepsilon_2)$ *is given by any bispecial word ω, some "multiplicity" $m \in \mathbb{N}^+$ and two signs $\varepsilon_1, \varepsilon_2 \in \{+, -\}$.*

(c) *Two bispecial strings $(\upsilon, n, \delta_1, \delta_2)$ and $(\omega, m, \varepsilon_1, \varepsilon_2)$ are* equivalent *if $(\upsilon, n, \delta_1, \delta_2) = (\omega, m, \varepsilon_1, \varepsilon_2)$ or $(\upsilon, n, \delta_1, \delta_2) = (\omega^{rev}, m, \varepsilon_2, \varepsilon_1)$.*

A string *of $\mathfrak{B}_0$ is given by any usual, special or bispecial string. By a* string word *we will mean any usual, special or bispecial word. A string of one of three types cannot be equivalent to any band, or any string of another of the three types.*

Table 5.1.1 lists all possible strings of nilpotent representations of $\mathfrak{B}_0$ up to equivalence.

TABLE 5.1.1. **Strings of the Gelfand quiver**

Usual strings ω	**Special strings** (ω, ε)	**Bispecial strings** $(\omega, m, \varepsilon_1, \varepsilon_2)$
$(\mathtt{p}_\star^{[0]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[0]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[0]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[0]}\mathtt{p}_\diamond, m, \pm, \pm)$
$(\mathtt{p}_\star^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[1]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[1]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[0]}\mathtt{p}_\diamond, m, +, -)$
$(\mathtt{p}_\star^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[2]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[2]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[1]}\mathtt{p}_\diamond, m, \pm, \pm)$
$(\mathtt{p}_\star^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[1]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[1]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[0]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[0]}, \mathtt{n}_1, \dots \mathtt{n}_{2k-1}, {}^{[1]}\mathtt{p}_\diamond, m, \pm, \mp)$
$(\mathtt{p}_\star^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[2]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[1]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[1]}\mathtt{p}_\diamond, m, \pm, \pm)$
$(\mathtt{p}_\star^{[2]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[2]}\mathtt{p}_\star)$	$(\mathtt{p}_\diamond^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[2]}\mathtt{p}_\star, \pm)$	$(\mathtt{p}_\diamond^{[1]}, \mathtt{n}_1, \dots\ \mathtt{n}_{2k}, {}^{[1]}\mathtt{p}_\diamond, m, +, -)$
$\omega \neq \omega^{\mathrm{rev}}$	(no further conditions)	$\omega \neq \omega^{\mathrm{rev}}$ and $\breve{\omega} \neq \upsilon\, \upsilon^{\mathrm{rev}} \dots \upsilon$
		$\mathtt{n}_1, \mathtt{n}_2 \dots, \mathtt{n}_{2k} \in \mathbb{N}^+, m \in \mathbb{N}^+$

The definitions of strings and bands of $\mathfrak{B}_0$ are motivated by the following result:

Theorem 5.1.8. *As above, let $\Bbbk$ be an algebraically closed field.*
There is a bijection between the equivalence classes of strings and bands and the isomorphism classes of indecomposable finite-dimensional nilpotent $\Bbbk$-linear representations of the Gelfand quiver:

$$\big[\text{STRINGS } \textit{and} \text{ BANDS } \textit{of } \mathfrak{B}_0\big] \xleftrightarrow{1:1} \operatorname{ind}\big[\,\mathrm{nil.\,rep}_{\Bbbk}(Q, I)\,\big]$$

In other words, strings and bands defined in this section parametrize the indecomposable representations of the Gelfand quiver. We will give a proof of this theorem in the next subsection. In section 5.4 the construction of the indecomposable nilpotent representation of the Gelfand quiver from any string or band.

Remark 5.1.9. *There are generalizations of Theorem 5.1.8 to the following setups*:

(1) The base field $\Bbbk$ does not have to be algebraically closed. In this setup, bands and their equivalence conditions have to be redefined according to Remark A.2.20.

(2) We recall that nilpotent representations of the Gelfand quiver correspond to finite-dimensional modules of the Gelfand order Λ :

$$\mathrm{nil.\,rep}_{\Bbbk}(Q,I) \xrightarrow{\ \sim\ } \Lambda\text{ - fd. mod} \qquad \Lambda = \begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array}$$

There is an analogue of Theorem 5.3.6 also for indecomposable finitely generated *Λ-modules. The additive subcategory of finitely generated but* infinite-dimensional *Λ-modules has* discrete *representation type.*

5.1.2 String and bands of projective resolutions

Let Ω be a string or band of the Gelfand quiver. Then Ω corresponds to some nilpotent representation, or finite-dimensional Λ-module over the Gelfand order Λ.

In this subsection, we translate Ω into a string respectively band $\Omega^{[*]}$ of the bunch of semichains $\breve{\mathfrak{B}}$. The datum $\Omega^{[*]}$ corresponds to a *projective resolution* in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

In particular, the present subsection links the present chapter to Chapter 4 on the derived category of the Gelfand quiver.

Within the formalism of derived categories, we give two proofs of Theorem 5.1.8.

5.1.2.1 Expansion of strings and bands of nilpotent representations

Let ω be a cyclic or finite word of the Gelfand quiver.

We translate the word ω into a new word $\omega^{[*]}$ as follows:

- Let ω be a cyclic word of the Gelfand quiver (in the sense of Definition 5.1.1):

$$\omega = \Big({}^{[\mathtt{d}]}\ \mathtt{n}_1, \mathtt{n}_2,\ \ldots\ \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}\ {}^{[\mathtt{d}]}\Big), \qquad \text{where } \mathtt{d} \in \{\mathtt{0},\mathtt{1}\},$$

  We define $2\mathtt{k}+\mathtt{1}$ parameters $\mathtt{d}_0, \mathtt{d}_1, \ldots, \mathtt{d}_{2\mathtt{k}} \in \{\mathtt{0},\mathtt{1}\}$ as follows:

$$\begin{cases} \mathtt{d}_{2\mathtt{j}-\mathtt{1}} = \mathtt{1}-\mathtt{d} & \text{for any index } \ \mathtt{0} < 2\mathtt{j}-\mathtt{1} < 2\mathtt{k}, \\ \mathtt{d}_{2\mathtt{j}} \quad = \mathtt{d} & \text{for any index } \ \mathtt{0} \le 2\mathtt{j} \le 2\mathtt{k}. \end{cases}$$

The word with degrees is denoted by

$$\omega^{[*]} = \left({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}, \mathtt{n}_2^{[\mathtt{d}_2]}, \ \ldots \ \mathtt{n}_{2\mathtt{k}-1}^{[\mathtt{d}_{2\mathtt{k}-1}]}, \mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}_{2\mathtt{k}}]}\right).$$

- Now, let ω be a finite word of the Gelfand quiver (in the sense of Definition 5.1.4):

$$\omega = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}, \ \mathtt{n}_1, \mathtt{n}_2, \ \ldots \ \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}, \ {}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta\right), \qquad \text{where } \mathtt{d}_\alpha, \mathtt{d}_\beta \in \{0, 1, 2\}.$$

We define $\mathtt{k}-1$ parameters $\mathtt{d}_1, \mathtt{d}_2, \ldots, \mathtt{d}_{\mathtt{k}-1} \in \{0, 1\}$ as follows. For any $\mathtt{j} \in \mathbb{N}^+$ such that $1 \leq 2\mathtt{j}-1 \leq \mathtt{k}-1$ respectively $1 < 2\mathtt{j} \leq \mathtt{k}-1$ we set

$$\mathtt{d}_{2\mathtt{j}-1} = \begin{cases} 1 & \text{if } \mathtt{d}_\alpha = 0 \\ 0 & \text{if } \mathtt{d}_\alpha \neq 0 \end{cases} \qquad \text{and} \qquad \mathtt{d}_{2\mathtt{j}} = \begin{cases} 0 & \text{if } \mathtt{d}_\alpha = 0, \\ 1 & \text{if } \mathtt{d}_\alpha \neq 0. \end{cases}$$

The outcome is denoted by

$$\omega^{[*]} = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\mathtt{n}_1^{[\mathtt{d}_1]}, \mathtt{n}_2^{[\mathtt{d}_2]} \ldots, \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}, \mathtt{n}_\mathtt{k}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta\right)$$

Moreover, we expand the ends $\omega^{[*]}$ in the following cases:

– if $\mathtt{d}_\alpha = 2$, we change the beginning $\mathtt{p}_\star^{[2]}\mathtt{n}_1$ into $\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{n}_1$.

– similarly, if $\mathtt{d}_\beta = 2$, we replace the end $\mathtt{n}_\mathtt{k}^{[2]}\mathtt{p}_\star$ by $\mathtt{n}_\mathtt{k}^{[1]}\mathtt{1}^{[2]}\mathtt{p}_\star$.

In both cases, the word $\omega^{[*]}$ is a cyclic or finite word in the sense of Definition 4.1.2 respectively Definition 4.1.4.

The following table lists all expanded finite words of $\mathfrak{B}_0$:

first part	intermediate part	last part
$\mathtt{p}_\star^{[0]}$ $\mathtt{p}_\diamond^{[0]}$ $\mathtt{p}_\star^{[1]}\mathtt{n}_0^{[0]}$ $\mathtt{p}_\diamond^{[1]}\mathtt{n}_0^{[0]}$ $\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{n}_0^{[0]}$	$\mathtt{n}_1^{[1]}\,\mathtt{n}_2^{[0]}\,\mathtt{n}_3^{[1]}\ldots\mathtt{n}_{2\mathtt{k}-2}^{[1]}\,\mathtt{n}_{2\mathtt{k}-1}$	${}^{[0]}\mathtt{p}_\star$ ${}^{[0]}\mathtt{p}_\diamond$ ${}^{[0]}\mathtt{n}_{2\mathtt{k}}{}^{[1]}\mathtt{p}_\star$ ${}^{[0]}\mathtt{n}_{2\mathtt{k}}{}^{[1]}\mathtt{p}_\diamond$ ${}^{[0]}\mathtt{n}_{2\mathtt{k}}{}^{[1]}\mathtt{1}^{[2]}\mathtt{p}_\star$

$\mathtt{n}_0, \mathtt{n}_1, \ldots \mathtt{n}_{2\mathtt{k}} \in \mathbb{N}^+$

(5.1.3)

For any string or band Ω of the Gelfand quiver, we define $\Omega^{[*]}$ by expansion of the word in Ω :

	usual string	special string	bispecial string	band
Ω	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)
$\Omega^{[*]}$	$\omega^{[*]}$	$(\omega^{[*]}, \varepsilon_1)$	$(\omega^{[*]}, m, \varepsilon_1, \varepsilon_2)$	$(\omega^{[*]}, m, \lambda)$

(5.1.4)

(where $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, $m \in \mathbb{N}^+$ and $\lambda \in \mathbb{k}^* \setminus \Delta$)

Note that $\Omega^{[*]}$ is a string or band of $\check{\mathfrak{B}}_{\text{fd}}$ (in the sense of Definition 4.1.12 or 4.1.7).

For example, the series of bands Ω from (5.1.5) is translated to

$$\Omega^{[*]} = (\omega^{[*]}, m, \lambda) \quad \text{where } \omega^{[*]} = \left({}^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]} \ \ldots \ \mathtt{n}_{2\mathtt{k}-1}^{[0]}\mathtt{n}_{2\mathtt{k}}^{[1]}\right) \text{ is non-periodic} \tag{5.1.5}$$

$$m \in \mathbb{N}^+, \text{ and } \lambda \in \mathbb{k}^* \text{ with } \lambda \neq (-1)^{\mathtt{k}+1} \text{ if } \omega = \omega^{\text{rev}}.$$

Let us note that the datum $\Omega^{[*]}$ corresponds to an indecomposable projective complex in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ with finite-dimensional homology.

5.1.2.2 Indecomposable projective resolutions of the Gelfand order

The goal of this subsection is to prove Theorem 5.1.8.

More precisely, we will characterize the strings and bands of $\check{\mathfrak{B}}$ which correspond to indecomposable *projective resolutions* in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Then we will restrict to projective resolutions with finite-dimensional homology.

Let $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ be the full subcategory of *minimal presentations* in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ as defined in Subsection B.2.2.

According to Corollary B.2.10 the indecomposable objects in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ are given exactly by the "indecomposable projective complexes of length at most one":

$$\operatorname{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}] = \big\{P_\bullet \in \operatorname{ind}[\mathrm{D}^{\mathrm{b}}(\Lambda)] \text{ with } P_\bullet = P_1 \xrightarrow{\ d\ } \underset{\neq 0}{P_0} \text{ and } \operatorname{im} d \subseteq \operatorname{rad} P_0\big\}. \tag{5.1.6}$$

Let $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ denote the full subcategory of $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ given by projective presentations with finite-dimensional homology in degree zero.

As a first step, we describe the strings and bands of the indecomposable objects in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ and $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$:

Theorem 5.1.10. *Let Ω be a usual string ω, a special string (ω, ε), a bispecial string $(\omega, m, \varepsilon_1, \varepsilon_2)$ or a band (ω, m, λ) of $\mathfrak{B}$. Let ω be given by some string respectively band word of length $\mathtt{k} \in \mathbb{N}^+$:*

$$\omega = \begin{cases} (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta) & \textit{with } \mathtt{p}_\alpha, \mathtt{p}_\beta \in \{\mathtt{p}_\diamond, \mathtt{p}_\star, \mathtt{p}_\infty\}, \textit{ if } \omega \textit{ is a string word} \\ ({}^{[\mathtt{d}_0]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]}\dots\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}) & \textit{with } \mathtt{k} \in 2\mathbb{N}^+, \textit{ if } \omega \textit{ is a band word.} \end{cases}$$

Let $P_\bullet = P_\bullet(\Omega)$ be the corresponding indecomposable complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.
Then the following statements hold:

(1) the complex $P_\bullet$ is an indecomposable object in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ if and only if $\mathtt{d}_\mathtt{j} \in \{0, 1\}$ for any $0 \leq \mathtt{j} \leq \mathtt{k}$.

(2) it holds that $P_\bullet \in \Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ if and only if ω has no end of type $\mathtt{p}_\infty^{[0]}$ or ${}^{[0]}\mathtt{p}_\infty$

Assume that $P_\bullet$ is an indecomposable object of $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$. Let ω' be the word obtained from ω as follows:

- *if $\mathtt{p}_\alpha = \mathtt{p}_\infty$ and $\mathtt{d}_0 = 1$, replace the left end $\mathtt{p}_\infty^{[1]}$ of ω by $\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}$,*
- *if $\mathtt{p}_\beta = \mathtt{p}_\infty$ and $\mathtt{d}_\mathtt{k} = 1$, replace the right end ${}^{[1]}\mathtt{p}_\infty$ of ω by ${}^{[1]}\mathtt{1}^{[2]}\mathtt{p}_\star$,*
- *if ω is a band word, then $\omega' = \omega$.*

Let Ω' be the datum with the new word ω' :

	usual string	*special string*	*bispecial string*	*band*
Ω	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)
Ω'	ω'	(ω', ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)

Let $V = \mathbf{H}_0(P_\bullet)$ be the homology of the minimal presentation $P_\bullet$. Then the following statements hold:

(3) The datum Ω' is a string or band of $\mathfrak{B}$.

(4) The module V is finite-dimensional if and only if Ω' is a string or band of $\mathfrak{B}_{\mathrm{fd}}$.

(5) The glued complex $P_\bullet(\Omega')$ of Ω' is the projective resolution of V.

The fifth statement is the main statement of the theorem. It states that strings and bands of projective presentations can be canonically transformed into strings and bands of projective resolutions of the Gelfand quiver.

Proof. (1) According to (5.1.6) we have to characterize strings and bands Ω such that the complex $P_\bullet(\Omega)$ has length at most one. Now the first statement of the theorem follows from the second statement of Remark 4.2.10.

(2) By Theorem 4.1.14 the complex $P_\bullet(\Omega)$ has infinite-dimensional homology if and only if Ω is a string of $\mathfrak{B}$ but not of $\mathfrak{B}_{\mathrm{fd}}$ if and only if ω has an end of type $\mathtt{p}_\infty$.

(3) Let Ω be a usual string. The new word ω' is symmetric if and only if ω is symmetric. It follows that Ω' is also a usual string of $\mathfrak{B}$. In the other cases there is nothing to check.

From now on we assume that the word ω satisfies the condition of the first statement of the Theorem.

(4) The homology V is finite-dimensional if and only if ω has no end of type $\mathtt{p}_\infty^{[0]}$ or ${}^{[0]}\mathtt{p}_\infty$.

(5) Let $\widetilde{P}_\bullet = \widetilde{P}_\bullet(\Omega) = \widetilde{P}_1 \xrightarrow{\widetilde{d}_1} \widetilde{P}_0$ be the projective complex in $\mathrm{D}^{\mathrm{b}}(\Gamma)$ associated to Ω according to the definition of gluing diagrams in Subsection 4.1.3.

By Proposition 3.3.1 and Lemma 3.3.9 the second syzygy of V is isomorphic to $I \ker \widetilde{d}_1$ in the category of Λ-modules, where $I = (e_\star)$ is the conductor ideal.

Let m_∞ be the number of ends of type $\mathtt{p}_\infty^{[1]}$ or ${}^{[1]}\mathtt{p}_\infty$ in ω. By construction of the complex $\widetilde{P}_\bullet$ it holds that $\ker \widetilde{d}_1 = \widetilde{P}_\diamond^{m_\infty}$. In particular, $\mathrm{syz}^2(V) \cong I \ker \widetilde{d}_1 = P_\star^{m_\infty}$.

- If $m_\infty = 0$, then $\mathrm{syz}^2(V) = 0$, which means that $P_\bullet = P_\bullet(\Omega)$ is already the projective resolution of V. Since $m_\infty = 0$, it holds also that $\Omega = \Omega'$ and the claim follows.
- Let $m_\infty = 1$. Then $\mathrm{syz}^2(V) \cong P_\star$. In this case $\Omega = (\omega, \varepsilon)$ and $\Omega' = (\omega', \varepsilon)$ are special strings. Let $P''_\bullet$ be the minimal projective resolution of V. Let Ω'' be the string or band of $\mathfrak{B}$ such that $P''_\bullet \cong P_\bullet(\Omega'')$. We have to show that Ω'' is also a special string and that Ω' is equivalent to Ω''.

Since $\mathrm{syz}^2(V)$ is projective, the module V has projective dimension two. This implies that the gluing diagram of Ω'' is given by the gluing diagram of Ω at degree zero and one, and by $P_\star$ at degree two.

Let ω'' be the word in Ω''. By the shape of the gluing diagram of Ω'' it follows that ω'' has exactly one end of type $\mathtt{p}_\star^{[2]}$, and contains the number and degree sequence of the word ω. Moreover, Ω'' must be a special string with the same sign as Ω'.

Without loss of generality we may assume that ω begins with $\mathtt{p}_\infty^{[1]}\,\mathtt{n}_1^{[0]}$, that is,

$$\omega = (\mathtt{p}_\infty^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\dots\mathtt{n}_\mathtt{k}^{[\mathtt{d_k}]}\mathtt{p}_\diamond). \qquad \mathtt{d_k}\in\{0,1\}$$

Then ω'' must have the form

$$\omega'' = (\mathtt{p}_\star^{[2]}\mathtt{m}^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\dots\mathtt{n}_\mathtt{k}^{[\mathtt{d_k}]}\mathtt{p}_\diamond) \qquad \text{for some } \mathtt{m}\in\mathbb{N}^+.$$

Let us consider the glued complex $P_\bullet(\Omega'')$ of Ω'' :

$$\begin{tikzcd}
\widetilde{P}_\star \arrow[r, "\mathtt{m}"] & \widetilde{P}_\diamond & & & P_\star \arrow[r, "\mathtt{m}"] \arrow[dr, "-\mathtt{m}"'] & P_+ \arrow[dr, "\widehat{\mathtt{n}}_1"] & \\
& & & \Rightarrow & & & \\
& \widetilde{P}_\diamond \arrow[uu, dashed] \arrow[r, "\widehat{\mathtt{n}}_1"] & P_\diamond & & & P_- \arrow[uu, dashed] \arrow[r, "\widehat{\mathtt{n}}_1"] & P_+ \\
& & \vdots & & & & \vdots
\end{tikzcd}$$

The main observation of the proof is that $\left\langle\begin{bmatrix}(1)_{\star+}\\(1)_{\star-}\end{bmatrix}\right\rangle \subseteq \ker d_1$ for the first differential d_1 in $P_\bullet(\Omega'')$. Since $P_\bullet(\Omega'')$ is a projective resolution it follows that $\mathtt{m}=1$.

Finally, we obtain that $\omega' = \omega''$ and $\Omega' = \Omega''$.

- The same arguments as above apply to the case $m_\infty = 2$. □

FIRST PROOF OF THEOREM 5.1.8. The following Table shows all possible finite words ω' in strings of projective presentations of Gelfand quiver representations.

first part	intermediate part	last part
$\mathtt{p}_\star^{[0]}$		${}^{[0]}\mathtt{p}_\star$
$\mathtt{p}_\diamond^{[0]}$		${}^{[0]}\mathtt{p}_\diamond$
$\mathtt{p}_\star^{[1]}\mathtt{n}_0^{[0]}$	$\mathtt{n}_1^{[1]}\,\mathtt{n}_2^{[0]}\,\mathtt{n}_3^{[1]}\dots\mathtt{n}_{2k-2}^{[1]}\,\mathtt{n}_{2k-1}$	${}^{[0]}\mathtt{n}_{2k}{}^{[1]}\mathtt{p}_\star$
$\mathtt{p}_\diamond^{[1]}\mathtt{n}_0^{[0]}$		${}^{[0]}\mathtt{n}_{2k}{}^{[1]}\mathtt{p}_\diamond$
$\mathtt{p}_\infty^{[1]}\mathtt{n}_0^{[0]}$		${}^{[0]}\mathtt{n}_{2k}{}^{[1]}\mathtt{p}_\infty$

$\mathtt{n}_0, \mathtt{n}_1, \dots \mathtt{n}_{2k} \in \mathbb{N}^+$

By the definition of the word ω' for the strings of projective resolutions in Theorem 5.1.10 we obtain exactly the list of words in 5.1.3.

Bands of projective presentations are already identical to bands of Gelfand quiver representations. □

Theorem 5.1.11. *Let Ω be a usual string ω, a special string (ω, ε), a bispecial string $(\omega, m, \varepsilon_1, \varepsilon_2)$ or a band (ω, m, λ) of $\mathfrak{B}_{\mathrm{fd}}$. Let ω be given by some string respectively band word of length $\mathtt{k} \in \mathbb{N}^+$:*

$$\omega = \begin{cases} (\mathtt{p}_\alpha^{[\mathtt{d}_0]} \mathtt{n}_1^{[\mathtt{d}_1]} \mathtt{n}_2^{[\mathtt{d}_2]} \dots \mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]} \mathtt{p}_\beta) & \text{with } \mathtt{p}_\alpha, \mathtt{p}_\beta \in \{\mathtt{p}_\diamond, \mathtt{p}_\star\}, \text{ if } \omega \text{ is a string word} \\ ({}^{[\mathtt{d}_0]} \mathtt{n}_1^{[\mathtt{d}_1]} \mathtt{n}_2^{[\mathtt{d}_2]} \dots \mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}) & \text{with } \mathtt{k} \in 2\mathbb{N}^+, \text{ if } \omega \text{ is a band word.} \end{cases}$$

Let $T_\bullet = T_\bullet(\Omega) = (V_\bullet, \widetilde{P}_\bullet, \widetilde{\vartheta})$ be the indecomposable triple in $\mathrm{Tri}_{\mathrm{fd}}(\Lambda)$ corresponding to Ω, and let $P_\bullet = \mathbf{G}(T_\bullet)$ be the corresponding complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. Then the following conditions are equivalent:

(1) the complex $P_\bullet$ is an indecomposable projective resolution *in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$,*

(2) the complex $\widetilde{P}_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Gamma)$ has length two and $\mathbf{H}_1(\widetilde{P}_\bullet) \in \Gamma/I$ -mod. Equivalently,

$$\widetilde{P}_\bullet \in \operatorname{add} \{ \{ \underset{2}{\widetilde{P}_\star} \xrightarrow{\;1\;} \underset{1}{\widetilde{P}_\diamond} \} \oplus \{ \underset{1}{\widetilde{P}_{i_1}} \xrightarrow{\;\hat{\mathtt{n}}\;} \underset{0}{\widetilde{P}_{i_2}} \mid i_1, i_2 \in \widetilde{Q}_0, \mathtt{n} \in \mathbb{N}^+ \} \tag{5.1.7}$$

(3) The word of the string respectively the band Ω : satisfies the following conditions:

- *it Ω is a band, then $\mathtt{d}_1, \dots, \mathtt{d}_\mathtt{k} \in \{\mathtt{0}, \mathtt{1}\}$,*
- *if Ω is a string, then $\mathtt{d}_1, \dots, \mathtt{d}_\mathtt{k} \in \{\mathtt{0}, \mathtt{1}\}$, and $\mathtt{d}_\alpha, \mathtt{d}_\beta \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$.*

  Moreover, there are the following additional constraints:

  – *if $\mathtt{d}_\alpha = \mathtt{2}$, then $\mathtt{p}_\alpha = \mathtt{p}_\star$, $\mathtt{n}_1 = \mathtt{1}$ and $\mathtt{d}_1 = \mathtt{1}$*

  – *if $\mathtt{d}_\beta = \mathtt{2}$, then $\mathtt{p}_\beta = \mathtt{p}_\star$, $\mathtt{n}_\mathtt{k} = \mathtt{1}$ and $\mathtt{d}_{\mathtt{k}-1} = \mathtt{1}$.*

This Theorem describes all strings and bands of projective resolutions in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

PROOF. We note that by the assumptions, $T_\bullet$ is a minimal indecomposable triple and $P_\bullet$ a minimal indecomposable complex.

- $(1) \Rightarrow (2)$:
  Let $P_\bullet$ be a projective resolution in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. By Proposition 3.3.11, condition (b) of 3.3.6 it holds that $\mathbf{H}_1(\widetilde{P}_\bullet) \in \Gamma/I$ -mod. Since $\operatorname{gldim} \Lambda = 2$, the length of $\widetilde{P}_\bullet$ is at most two. This implies that $\mathbf{H}_2(\widetilde{P}_\bullet) = 0$ and yields the list in (5.1.7).
- $(2) \Rightarrow (3)$:
  Let $\widetilde{P}_\bullet$ have only direct summands from the list in (5.1.7). Then the translation of words to gluing diagrams in Subsection 4.1.3 yields the following statements:

  – if the length of $\widetilde{P}_\bullet$ is one, then $d_1, \dots, d_k \in \{0, 1\}$.

  – if the length of $\widetilde{P}_\bullet$ is two, then word ω is usual or special. Moreover ω may only begin with $p_\star^{[2]} 1^{[1]}$ or $p_\pm^{[d_\alpha]}$, contain as subwords only ${}^{[1]}n^{[0]}$ or ${}^{[0]}n^{[1]}$, and end only with ${}^{[1]}1^{[2]}p_\star$ or ${}^{[d_\beta]}p_\pm$, where $d_\alpha, d_\beta \in \{0, 1\}$.

  Note that the restrictions in (3) describe *all possible* gluing diagrams built from these subwords.

- (3) $\Rightarrow$ (1) :
  This is the main point. Obviously, conditions (a), (b) and (d) of Proposition 3.3.11 are satisfied. Let n be the length of $P_\bullet$ and let $1 \leq j \leq n$. We have to show that $\ker d_j \subseteq IP_j$.
  - assume that $j = n$. Since $P_\bullet$ is a minimal indecomposable projective complex of length j, Lemma B.2.9 implies that $\ker d_j \subseteq \operatorname{rad} P_j \subseteq IP_j$.
  - let $j = 1 < n = 2$. For the last step, we consider the brutal truncation

$$tr(P_\bullet) := (\ldots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0 \ldots)$$

We claim that $tr(P_\bullet)$ is indecomposable.
Similar as above, let $tr(T_\bullet) = (tr(V_\bullet), tr(\widetilde{P}_\bullet), tr(\widetilde{\vartheta}))$ be the truncated triple.
Since $n = 2$, the datum Ω is given by some usual or special string ω with $\mathtt{d}_\alpha = 2$ or $\mathtt{d}_\beta = 2$. Let $tr(\omega)$ denote the truncated word defined as follows:

ω	$(\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{n}_2^{[0]}\ldots\mathtt{n}_\mathtt{k}^{[1]}\mathtt{p}_\beta)$	$(\mathtt{p}_\alpha^{[1]}\mathtt{n}_1^{[0]}\ldots\mathtt{n}_{\mathtt{k}-1}^{[1]}\mathtt{1}^{[2]}\mathtt{p}_\star)$	$(\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{n}_2^{[0]}\ldots\mathtt{n}_{\mathtt{k}-1}^{[1]}\mathtt{1}^{[2]}\mathtt{p}_\beta)$
$tr(\omega)$	$(\mathtt{p}_\infty^{[1]}\mathtt{n}_2^{[0]}\ldots\mathtt{n}_\mathtt{k}^{[1]}\mathtt{p}_\beta)$	$(\mathtt{p}_\alpha^{[1]}\mathtt{n}_1^{[0]}\ldots\mathtt{n}_{\mathtt{k}-1}^{[1]}\mathtt{p}_\infty)$	$(\mathtt{p}_\infty^{[1]}\mathtt{n}_2^{[0]}\ldots\mathtt{n}_{\mathtt{k}-1}^{[1]}\mathtt{p}_\infty)$

Note that $tr(\omega)$ is a symmetric word if and only if ω is symmetric. Since ω is non-symmetric, it follows that $tr(\omega)$ defines a usual or special string of the bunch of semichains $\mathfrak{B}$. The string $tr(\omega)$ corresponds to the truncated triple $tr(T_\bullet)$. In particular, $tr(T_\bullet)$ is an indecomposable triple. It follows that $\mathbf{G}(tr(T_\bullet)) = tr(\mathbf{G}(T_\bullet)) = tr(P_\bullet)$ is indeed indecomposable.
At last, Lemma B.2.9 implies that $\ker d_1 \subseteq \operatorname{rad} P_1 \subseteq IP_1$. □

Remark 5.1.12. *The proof of Theorem 5.1.11 shows that condition (c) in Proposition 3.3.11 is automatically satisfied if we assume the triple to be indecomposable. In particular, the indecomposable triples corresponding to projective resolutions can be described essentially in terms of the normalized complex $\widetilde{P}_\bullet$.*

SECOND PROOF OF THEOREM 5.1.8. The strings and bands described in Theorem 5.1.11 are exactly expanded strings and bands of the Gelfand quiver of (5.1.3). □

In the next section we will describe how to construct an indecomposable nilpotent representation of the Gelfand quiver from any string or band.

The gluing diagrams of strings and bands of projective resolutions of the Gelfand quiver are depicted in Figure 5.1.1 and Figure 5.1.2.

5.1.3 String and band words with gluing arrows

Let ω be a usual, special or bispecial string word or band word of the Gelfand quiver. In Subsection 5.1.2.1 we defined the word $\omega^{[*]}$ with degrees and expanded ends. In the present subsection we convert $\omega^{[*]}$ into a new word $\overset{\leftrightarrow}{\omega}$ which contains an altered number sequence and additional arrows. This step is necessary for the construction of projective resolutions.

In fact, we have described this procedure for the construction of projective complexes in Subsection 4.2.1. The reader familiar with the latter subsection may skip the present one.

FIGURE 5.1.1. **Gluing diagrams of projective resolutions I: Bands**

$$\left({}^{[1]}\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3, \mathtt{n}_4, \mathtt{n}_5, \mathtt{n}_6, \quad \dots \quad, \mathtt{n}_{2\mathtt{k}-3}, \mathtt{n}_{2\mathtt{k}-2}, \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}^{[1]}\right)$$

$$\begin{tikzcd}
\overset{\lambda}{P_-} \arrow[r, "2\mathtt{n}_1"] \arrow[ddddddddddd, dash, dashed, bend right] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_2"] \arrow[d, dash, dashed] & P_- \\
P_- \arrow[r, "2\mathtt{n}_3"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_4"] \arrow[d, dash, dashed] & P_- \\
P_- \arrow[r, "2\mathtt{n}_5"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_6"] \arrow[d, dash, dashed] & P_- \\
\vdots \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{2\mathtt{k}-3}"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_{2\mathtt{k}-2}"] \arrow[d, dash, dashed] & P_- \\
P_- \arrow[r, "2\mathtt{n}_{2\mathtt{k}-1}"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_{2\mathtt{k}}"] & P_-
\end{tikzcd}$$

Let $\mathtt{k}$ denote the length of $\omega^{[*]}$ and assume that $\mathtt{k} \geq 2$.

$$\omega^{[*]} = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ \dots \ \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta\right) \quad \text{or} \quad \left({}^{[\mathtt{d}]}\mathtt{n}_1^{[\mathtt{d}_1]}\mathtt{n}_2^{[\mathtt{d}_2]} \ \dots \ \mathtt{n}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\right)$$

Note that $\mathtt{n}_1^{[\mathtt{d}_1]} = \mathtt{1}^{[1]}$ if $\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]} = \mathtt{p}_\star^{[2]}$ and ${}^{[\mathtt{d}_{\mathtt{k}-1}]}\mathtt{n}_\mathtt{k} = {}^{[1]}\mathtt{1}$ if ${}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta = {}^{[2]}\mathtt{p}_\star$.

FIGURE 5.1.2. **Gluing diagrams of projective resolutions II: Strings**

Beginning of the gluing diagram (5 types):

$\mathtt{p}_\star^{[2]}\mathtt{n}_1, \mathtt{n}_2 \ldots$

$$\begin{tikzcd}
P_\star \arrow[r, "1"] & P_+ & \\
& P_- \arrow[u, dashed] \arrow[r, "2\mathtt{n}_1"] & P_+ \arrow[d, dash, dashed] \\
& P_+ \arrow[r, "2\mathtt{n}_2"] \arrow[d, dash, dashed] & P_- \\
& {} &
\end{tikzcd}$$

$\mathtt{p}_\diamond^{[1]}\mathtt{n}_1, \mathtt{n}_2 \ldots$

$$\begin{tikzcd}
P_\diamond \arrow[r, "2\mathtt{n}_1"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_2"] \arrow[d, dash, dashed] & P_- \\
{} &
\end{tikzcd}$$

$\mathtt{p}_\star^{[1]}\mathtt{n}_1, \mathtt{n}_2 \ldots$

$$\begin{tikzcd}
P_\star \arrow[r, "2\mathtt{n}_1-1"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_2"] \arrow[d, dash, dashed] & P_- \\
{} &
\end{tikzcd}$$

$\mathtt{p}_\diamond^{[0]}\mathtt{n}_2 \ldots$

$$\begin{tikzcd}
P_+ \arrow[r, "2\mathtt{n}_2"] \arrow[d, dash, dashed] & P_\diamond \\
{} &
\end{tikzcd}$$

$\mathtt{p}_\star^{[0]}\mathtt{n}_2 \ldots$

$$\begin{tikzcd}
P_+ \arrow[r, "2\mathtt{n}_2-1"] \arrow[d, dash, dashed] & P_\star \\
{} &
\end{tikzcd}$$

the intermediate part is given by an iteration of the following diagram:

$\ldots \mathtt{n}_{2\mathtt{j}+1}, \mathtt{n}_{2\mathtt{j}+2}, \mathtt{n}_{2\mathtt{j}+3}, \mathtt{n}_{2\mathtt{j}+4}, \ldots$ where $3 \leq 2\mathtt{j}+1 \leq \mathtt{k}-5$

$$\begin{tikzcd}
{} \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{2\mathtt{j}+1}"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_{2\mathtt{j}+2}"] \arrow[d, dash, dashed] & P_- \\
P_- \arrow[r, "2\mathtt{n}_{2\mathtt{j}+3}"] & P_+ \arrow[d, dash, dashed] \\
P_+ \arrow[r, "2\mathtt{n}_{2\mathtt{j}+4}"] \arrow[d, dash, dashed] & P_- \\
{} &
\end{tikzcd}$$

Endings of the gluing diagram (5 types)

$\ldots \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}^{[2]}\mathtt{p}_\star$

$$\begin{tikzcd}
& {} \arrow[d, dash, dashed] & \\
& P_- \arrow[r, "2\mathtt{n}_{\mathtt{k}-1}"] & P_+ \arrow[d, dash, dashed] \\
& P_+ \arrow[r, "2\mathtt{n}_\mathtt{k}"] \arrow[d, dashed] & P_- \\
P_\star \arrow[r, "1"] & P_- &
\end{tikzcd}$$

$\ldots \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}^{[1]}\mathtt{p}_\diamond$

$$\begin{tikzcd}
{} \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{\mathtt{k}-1}"] & P_+ \arrow[d, dash, dashed] \\
P_\diamond \arrow[r, "2\mathtt{n}_\mathtt{k}"] & P_-
\end{tikzcd}$$

$\ldots \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}^{[1]}\mathtt{p}_\star$

$$\begin{tikzcd}
{} \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{\mathtt{k}-1}"] & P_+ \arrow[d, dash, dashed] \\
P_\star \arrow[r, "2\mathtt{n}_\mathtt{k}-1"] & P_-
\end{tikzcd}$$

$\ldots \mathtt{n}_{\mathtt{k}-1}^{[0]}\mathtt{p}_\diamond$

$$\begin{tikzcd}
{} \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{\mathtt{k}-1}"] & P_\diamond
\end{tikzcd}$$

$\ldots \mathtt{n}_{\mathtt{k}-1}^{[0]}\mathtt{p}_\star$

$$\begin{tikzcd}
{} \arrow[d, dash, dashed] & \\
P_- \arrow[r, "2\mathtt{n}_{\mathtt{k}-1}-1"] & P_\star
\end{tikzcd}$$

First we transform $\omega^{[*]}$ into a new word

$$\widehat{\omega} = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}, \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \ldots, \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}, \widehat{\mathtt{n}}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta\right) \quad \text{respectively} \quad \left({}^{[\mathtt{d}]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}, \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \ldots, \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}, \widehat{\mathtt{n}}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\right)$$

with new values $\widehat{\mathtt{n}}_1, \widehat{\mathtt{n}}_2, \ldots, \widehat{\mathtt{n}}_{\mathtt{k}-1}, \widehat{\mathtt{n}}_\mathtt{k} \in \mathbb{N}^+$.

- If ω is a band word, we set $\widehat{\mathtt{n}}_\mathtt{j} = 2\mathtt{n}_\mathtt{j}$ for any $1 \le \mathtt{j} \le \mathtt{k}$.
- If ω is a string word, we set $\widehat{\mathtt{n}}_\mathtt{j} = 2\mathtt{n}_\mathtt{j}$ for any $1 < \mathtt{j} < \mathtt{k}$ as well as

$$\widehat{\mathtt{n}}_1 = \begin{cases} 2\mathtt{n}_1 - 1 & \text{if } \alpha = \star, \\ 2\mathtt{n}_1 & \text{if } \alpha = \diamond, \end{cases} \qquad \text{and} \qquad \widehat{\mathtt{n}}_\mathtt{k} = \begin{cases} 2\mathtt{n}_\mathtt{k} - 1 & \text{if } \beta = \star, \\ 2\mathtt{n}_\mathtt{k} & \text{if } \beta = \diamond. \end{cases}$$

In the final step, we are going to define a word of the form

$$\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]} \updownarrow \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \updownarrow \ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]} \updownarrow \widehat{\mathtt{n}}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta\right),$$

where each symbol $\updownarrow$ stands either for $\uparrow$ or $\downarrow$.
There are the following special cases for the first and last arrow in $\omega^{[*]}$:

- if $\mathtt{d}_\alpha = 2$ then $\widehat{\mathtt{n}}_1 = 1$ and we set $\widehat{\mathtt{n}}_1 \uparrow \widehat{\mathtt{n}}_2$ in $\widehat{\omega}$,
- similarly, if $\mathtt{d}_\beta = 2$, then $\widehat{\mathtt{n}}_\mathtt{k} = 1$ and we set $\widehat{\mathtt{n}}_{\mathtt{k}-1} \downarrow \widehat{\mathtt{n}}_\mathtt{k}$ in $\widehat{\omega}$.

In the following, we add an arrow $\uparrow$ or $\downarrow$ between every two consecutive numbers $\widehat{\mathtt{n}}_\mathtt{j}^{[\mathtt{d}_\mathtt{j}]}$ and $\widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]}$ in the word $\omega^{[*]}$, where $\mathtt{j} \in \mathbb{N}^+$ such that $1 \le \mathtt{j} < \mathtt{k}$ respectively $1 \le \mathtt{j} \le \mathtt{k}$. We have to distinguish between two cases:

(1) Assume that $\widehat{\mathtt{n}}_\mathtt{j} \neq \widehat{\mathtt{n}}_{\mathtt{j}+1}$. In this case we set

$$\widehat{\mathtt{n}}_\mathtt{j}^{[\mathtt{d}_\mathtt{j}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]} = \begin{cases} \widehat{\mathtt{n}}_\mathtt{j}^{[\mathtt{d}_\mathtt{j}]} \uparrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]} & \text{if } \mathtt{d}_\mathtt{j} < \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_\mathtt{j} < \widehat{\mathtt{n}}_{\mathtt{j}+1}, \\ & \text{or } \mathtt{d}_\mathtt{j} > \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_\mathtt{j} > \widehat{\mathtt{n}}_{\mathtt{j}+1}. \\ \widehat{\mathtt{n}}_\mathtt{j}^{[\mathtt{d}_\mathtt{j}]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j}+1}^{[\mathtt{d}_{\mathtt{j}+1}]} & \text{if } \mathtt{d}_\mathtt{j} < \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_\mathtt{j} > \widehat{\mathtt{n}}_{\mathtt{j}+1}, \\ & \text{or } \mathtt{d}_\mathtt{j} > \mathtt{d}_{\mathtt{j}+1} \text{ and } \widehat{\mathtt{n}}_\mathtt{j} < \widehat{\mathtt{n}}_{\mathtt{j}+1}; \end{cases} \tag{5.1.8}$$

(2) Assume that $\widehat{\mathtt{n}}_\mathtt{j} = \widehat{\mathtt{n}}_{\mathtt{j}+1}$. In this case we need the following notation.
If ω is a string word we denote by $\dot{\omega}$ and $\dot{\omega}^{\text{rev}}$ the following sequences:

$$\dot{\omega} = \left(\widehat{\mathtt{n}}_1^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \ \ldots \ \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]}\widehat{\mathtt{n}}_\mathtt{k}^{[\mathtt{d}_\beta]}\right) \qquad \text{and} \qquad \dot{\omega}^{\text{rev}} = \left(\widehat{\mathtt{n}}_\mathtt{k}^{[\mathtt{d}_{\mathtt{k}-1}]}\widehat{\mathtt{n}}_{\mathtt{k}-1} \ \ldots \ {}^{[\mathtt{d}_2]}\widehat{\mathtt{n}}_2^{[\mathtt{d}_1]}\widehat{\mathtt{n}}_1^{[\mathtt{d}_\alpha]}\right),$$

We define a bigger word $\overline{\omega}$ as follows:

$$\overline{\omega} = \begin{cases} (\omega) & \text{if } \omega \text{ is a band word,} \\ (\dot{\omega}) & \text{if } \omega \text{ is a usual word,} \\ (\dot{\omega}^{\text{rev}}, \dot{\omega}) & \text{if } \omega \text{ is a special word with special left end,} \\ (\dot{\omega}, \ \dot{\omega}^{\text{rev}}) & \text{if } \omega \text{ is a special word with special right end,} \\ (\dot{\omega}^{\text{rev}}, \dot{\omega}, \ \dot{\omega}^{\text{rev}}) & \text{if } \omega \text{ is a bispecial word.} \end{cases}$$

In the following we consider $\dot{\omega}$ as a subsequence of $\overline{\omega}$.
Let υ be the maximal symmetric subword in $\overline{\omega}$ with $\widehat{\mathtt{n}}_\mathtt{j}{}^{[\mathtt{d}_\mathtt{j}]}\widehat{\mathtt{n}}_{\mathtt{j}+1}$ in the middle.

In particular, υ has the form $(\psi^{\mathrm{rev}[\mathtt{d_j}]}\psi)$ for some subsequence ψ of $\overline{\omega}$:

$$\overline{\omega} = \ldots \widehat{\mathtt{n}}_{\mathtt{j-m}}{}^{[\mathtt{d_{j-m}}]} \underbrace{\widehat{\mathtt{n}}_{\mathtt{j-m+1}}^{[\mathtt{d_{j-m+1}}]} \ldots \widehat{\mathtt{n}}_{\mathtt{j-1}}^{[\mathtt{d_{j-1}}]} \widehat{\mathtt{n}}_{\mathtt{j}}}_{\psi^{\mathrm{rev}}} {}^{[\mathtt{d_j}]} \underbrace{\widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} \widehat{\mathtt{n}}_{\mathtt{j+2}}^{[\mathtt{d_{j+2}}]} \ldots \widehat{\mathtt{n}}_{\mathtt{j+m}}}_{\psi} {}^{[\mathtt{d_{j+m}}]} \widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d_{j+m+1}}]} \ldots$$

Let $\mathtt{2m}$ denote the length of the symmetric subword υ. In other words:

$$\mathtt{m} = \min\left\{\mathtt{i} \in \mathbb{N}^+ \mid \widehat{\mathtt{n}}_{\mathtt{j+1}} = \widehat{\mathtt{n}}_{\mathtt{j}}, \quad \widehat{\mathtt{n}}_{\mathtt{j+2}} = \widehat{\mathtt{n}}_{\mathtt{j-1}}, \ \ldots \ \widehat{\mathtt{n}}_{\mathtt{j+i}} = \widehat{\mathtt{n}}_{\mathtt{j-i+1}}, \quad \widehat{\mathtt{n}}_{\mathtt{j+i+1}} \neq \widehat{\mathtt{n}}_{\mathtt{j-i}}\right\}.$$

The number $\mathtt{m}$ is well-defined except in the following case:

(a) assume that ω is a symmetric band word such that $\widehat{\mathtt{n}}_{\mathtt{j+i}} = \widehat{\mathtt{n}}_{\mathtt{j-i-1}}$ for all $\mathtt{i} \in \mathbb{N}_0$. In this case we set $\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]}$.

The word υ begins with the number $\widehat{\mathtt{n}}_{\mathtt{j-m+1}}$ and ends with the number $\widehat{\mathtt{n}}_{\mathtt{j+m}}$. Regarding the position of these numbers in $\overline{\omega}$ only the following case remains:

(b) $\widehat{\mathtt{n}}_{\mathtt{j-m+1}}$ is not the first and $\widehat{\mathtt{n}}_{\mathtt{j+m}}$ is not the last number in $\overline{\omega}$.
In this case we consider the values of $\widehat{\mathtt{n}}_{\mathtt{j-m}}^{[\mathtt{d_{j-m}}]}$ and $\widehat{\mathtt{n}}_{\mathtt{j+m+1}}^{[\mathtt{d_{j+m+1}}]}$. Namely, we set

$$\widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} = \begin{cases} \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \uparrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} & \text{if } \mathtt{d_{j-m}} < \mathtt{d_{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} < \widehat{\mathtt{n}}_{\mathtt{j+m+1}}, \\ & \text{or } \mathtt{d_{j-m}} > \mathtt{d_{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} > \widehat{\mathtt{n}}_{\mathtt{j+m+1}}. \\ \widehat{\mathtt{n}}_{\mathtt{j}}^{[\mathtt{d_j}]} \downarrow \widehat{\mathtt{n}}_{\mathtt{j+1}}^{[\mathtt{d_{j+1}}]} & \text{if } \mathtt{d_{j-m}} < \mathtt{d_{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} > \widehat{\mathtt{n}}_{\mathtt{j+m+1}}, \\ & \text{or } \mathtt{d_{j-m}} > \mathtt{d_{j+m+1}} \text{ and } \widehat{\mathtt{n}}_{\mathtt{j-m}} < \widehat{\mathtt{n}}_{\mathtt{j+m+1}}; \end{cases} \tag{5.1.9}$$

Let us note that $\widehat{\mathtt{n}}_{\mathtt{j-m}} \neq \widehat{\mathtt{n}}_{\mathtt{j+m+1}}$ by definition of the number $\mathtt{m}$.

Summarized we obtain some word of the form

$$\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_{\alpha}^{[\mathtt{d}_{\alpha}]}\widehat{\mathtt{n}}_{1}^{[\mathtt{d_1}]} \updownarrow \widehat{\mathtt{n}}_{2}^{[\mathtt{d_2}]} \updownarrow \ldots \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[\mathtt{d_{k-1}}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d_k}]}\mathtt{p}_{\beta}\right) \text{ or } \left(\widehat{\mathtt{n}}_{1}^{[\mathtt{d_1}]} \updownarrow \widehat{\mathtt{n}}_{2}^{[\mathtt{d_2}]} \updownarrow \ldots \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[\mathtt{d_{k-1}}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d_k}]} \updownarrow\right)$$

with alternating degrees and gluing arrows $\uparrow$ or $\downarrow$ between the numbers.

Let us consider some examples.

Example 5.1.13 (Band words). *Let us consider the following band words of $\mathfrak{B}_0$:*

(1) a non-symmetric band word:
Let $\omega = \left({}^{[1]}2, 1, 2, 3^{[1]}\right)$. *Then* $\widehat{\omega} = \left(4^{[0]}2^{[1]}4^{[0]}6^{[1]}\right)$.
According to (5.1.8) *we obtain* $\overset{\leftrightarrow}{\omega} = \left(4^{[0]} \downarrow 2^{[1]} \downarrow 4^{[0]} \uparrow 6^{[1]} \uparrow\right)$.

(2) a symmetric band word:
$\omega = \left({}^{[1]}3, 2, 2, 3^{[1]}\right)$. *Then* $\widehat{\omega} = \left(6^{[0]}, 4^{[1]}, 4^{[0]}, 6^{[1]}\right)$.
Using (5.1.8) *we get* $\overset{\leftrightarrow}{\omega} = \left(6^{[0]} \downarrow 4^{[1]} \updownarrow 4^{[0]} \uparrow 6^{[1]} \updownarrow\right)$. *The other gluing arrows are given by* : $\overset{\leftrightarrow}{\omega} = \left({}^{[1]}6^{[0]} \downarrow 4^{[1]} \downarrow 4^{[0]} \uparrow 6^{[1]} \downarrow\right)$.

Example 5.1.14 (Usual word). *Let ω be the usual word* $\left(\mathtt{p}_{\star}^{[0]}2, 2, 2^{[1]}\mathtt{p}_{\star}\right)$. *Note that ω is not symmetric. Then* $\widehat{\omega} = \left(\mathtt{p}_{\star}^{[0]}3^{[1]}4^{[0]}3^{[1]}\mathtt{p}_{\star}\right)$. *According to* (5.1.8) $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_{\star}^{[0]}3^{[1]} \downarrow 4^{[0]} \downarrow 3^{[1]}\mathtt{p}_{\star}\right)$.

Example 5.1.15 (Special word). *Let ω be the special word* $\left(\mathtt{p}_\star^{[2]}\mathtt{2},\mathtt{2},\mathtt{1},\mathtt{1}^{[1]}\mathtt{p}_\diamond\right)$. *It holds that* $\widehat{\omega} = \left(\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]}\mathtt{4}^{[0]}\mathtt{4}^{[1]}\mathtt{2}^{[0]}\mathtt{2}^{[1]}\mathtt{p}_\diamond\right)$.
We get $1^{[1]} \uparrow 4^{[0]}$ *by and* $4^{[1]} \uparrow 2^{[0]}$ *in* $\widehat{\omega}$ *by* (5.1.8).
To determine the arrows in $\mathtt{4}^{[0]} \updownarrow \mathtt{4}^{[1]}$ *and* $\mathtt{2}^{[0]} \updownarrow \mathtt{2}^{[1]}$ *in* $\widehat{\omega}$ *we consider the ambient word*

$$\overline{\omega} = \left(\dot{\omega}\ \dot{\omega}^{\mathrm{rev}}\right) = \big(\underbrace{\mathtt{1}^{[1]}\,\mathtt{4}^{[0]}\,\overbrace{\mathtt{4}^{[1]}\,\mathtt{2}^{[0]}}^{\mathtt{4}^{[1]}\uparrow\mathtt{2}^{[0]}}}_{\mathtt{1}^{[1]}\downarrow\mathtt{2}^{[0]}}\,\mathtt{2}^{[1]}\,\underbrace{\mathtt{2}^{[0]}\,\mathtt{2}^{[1]}\,\mathtt{4}^{[0]}\,\mathtt{4}^{[1]}\,\mathtt{1}^{[2]}}_{\dot{\omega}^{\mathrm{rev}}}\big)$$

By (5.1.9) *the gluing word of ω is given by* $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]} \uparrow \mathtt{4}^{[0]} \downarrow \mathtt{4}^{[1]} \uparrow \mathtt{2}^{[0]} \uparrow \mathtt{2}^{[1]}\mathtt{p}_\diamond\right)$.

Example 5.1.16 (Bispecial word). *Let ω be the word* $\left(\mathtt{p}_\diamond{}^{[1]}\mathtt{3},\mathtt{1},\mathtt{1},\mathtt{3},\mathtt{3},\mathtt{2}^{[1]}\mathtt{p}_\diamond\right)$. *In this case,* $\widehat{\omega} = \left(\mathtt{p}_\diamond{}^{[1]}\mathtt{6}^{[0]}\mathtt{2}^{[1]}\mathtt{2}^{[0]}\mathtt{6}^{[1]}\mathtt{6}^{[0]}\mathtt{4}^{[1]}\mathtt{p}_\diamond\right)$.
The ambient word of ω is given by

$$\overline{\omega} = \left(\dot{\omega}^{\mathrm{rev}}\ \dot{\omega}\ \dot{\omega}^{\mathrm{rev}}\right) = \big(\underbrace{\mathtt{4}^{[0]}\ldots\mathtt{2}^{[1]}}_{\dot{\omega}^{\mathrm{rev}}}\,\overbrace{\mathtt{2}^{[0]}\,\mathtt{6}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{2}^{[1]}\,\underbrace{\mathtt{2}^{[0]}\,\mathtt{6}^{[1]}\,\mathtt{6}^{[0]}\,\mathtt{4}^{[1]}}_{\mathtt{2}^{[0]}\uparrow\mathtt{4}^{[1]}}}^{\mathtt{2}^{[0]}\uparrow\mathtt{4}^{[1]}}\,\underbrace{\mathtt{4}^{[0]}\ldots\mathtt{6}^{[1]}}_{\dot{\omega}^{\mathrm{rev}}}\big).$$

We obtain that $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_\diamond{}^{[1]}\mathtt{6}^{[0]} \downarrow \mathtt{2}^{[1]} \uparrow \mathtt{2}^{[0]} \uparrow \mathtt{6}^{[1]} \uparrow \mathtt{6}^{[0]} \downarrow \mathtt{4}^{[1]}\mathtt{p}_\diamond\right)$.

5.2 Functorial properties of indecomposable representations

In this section we describe the action of three natural functors on strings and bands of the Gelfand quiver.

5.2.1 The involution, the contragredient dual and the Matlis dual

In this subsection we study some natural functors on the following categories:

- the category $\operatorname{nil.rep}(Q, I)$ of nilpotent finite-dimensional $\Bbbk$-linear representations of the Gelfand quiver
- the category $\Lambda\text{-}\operatorname{fd.mod}$ of finite-dimensional modules over the Gelfand order Λ, and
- the full category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ of projective complexes with finite-dimensional complexes in the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$

More precisely, we introduce the involution σ, the twisted Matlis duality $\widetilde{\mathbb{D}}$ and the contragredient duality $\mathbb{L}$ on these categories.

One of the goals of this section is to describe the action of these three functors on strings and bands of the Gelfand quiver:

$$
\begin{tikzcd}
\operatorname{ind}[\operatorname{nil.rep}(Q,I)] \arrow[r, leftrightarrow, "1:1"] \arrow[d, "\sigma \text{ or } \widetilde{\mathbb{D}} \text{ or } \mathbb{L}"] & {[\text{STRINGS and BANDS of } \breve{\mathfrak{B}}]} \arrow[d, dashed, "\sigma \text{ or } \widetilde{\mathbb{D}} \text{ or } \mathbb{L}"] \\
\operatorname{ind}[\operatorname{nil.rep}(Q,I)] \arrow[r, leftrightarrow, "1:1"] & {[\text{STRINGS and BANDS of } \breve{\mathfrak{B}}]}
\end{tikzcd}
$$

5.2.1.1 Nilpotent representations

For a matrix A we will denote its transpose by A^T. The $\mathbb{k}$-linear dual of a vector space V will be denoted by V^*.

For any representation $M \in \operatorname{nil.rep}(Q,I)$ we define $\sigma(V)$, $\widetilde{\mathbb{D}}(V)$ and $\mathbb{L}(V)$ as follows:

$$
\begin{tikzcd}
M = & U \arrow[r, bend left, "B"] & V \arrow[l, bend left, "A"] \arrow[r, bend right, "C"'] & W \arrow[l, bend right, "D"'] \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{\mathbb{D}}(M) = & U^* \arrow[r, bend left, "A^T"] & V^* \arrow[l, bend left, "B^T"] \arrow[r, bend right, "D^T"'] & W^* \arrow[l, bend right, "C^T"'] \arrow[d, mapsto] \\
\sigma(M) = & W \arrow[r, bend left, "D"] & V \arrow[l, bend left, "C"] \arrow[r, bend right, "A"'] & U \arrow[l, bend right, "B"'] \arrow[r, mapsto] & \mathbb{L}(M) = & W^* \arrow[r, bend left, "C^T"] & V^* \arrow[l, bend left, "D^T"] \arrow[r, bend right, "B^T"'] & U^* \arrow[l, bend right, "A^T"']
\end{tikzcd}
$$

In other words, the representation $\sigma(V)$ is given by a "vertical flip", while $\widetilde{\mathbb{D}}(V)$ is given by "taking duals". At last, $\mathbb{L}$ is simply the composition of these two operations.

The definition of $\sigma, \widetilde{\mathbb{D}}$ and $\mathbb{L}$ extends naturally to morphisms of representations in $\operatorname{nil.rep}(Q,I)$. In particular, we obtain the following diagram of categories :

$$
\begin{tikzcd}
\operatorname{nil.rep}(Q,I) \arrow[r, "\widetilde{\mathbb{D}}", "\sim"'] \arrow[d, "\sigma", "\wr"'] \arrow[dr, dashed, "\mathbb{L}"] & \operatorname{nil.rep}(Q,I) \arrow[d, "\sigma", "\wr"'] \\
\operatorname{nil.rep}(Q,I) \arrow[r, "\widetilde{\mathbb{D}}", "\sim"'] & \operatorname{nil.rep}(Q,I)
\end{tikzcd}
\qquad \mathbb{L} = \widetilde{\mathbb{D}} \circ \sigma = \sigma \circ \widetilde{\mathbb{D}}
$$

The three functors have the following properties:

- It holds that $\sigma^2 = \widetilde{\mathbb{D}}^2 = \mathbb{L}^2 = \mathrm{Id}$. In particular, the functors $\widetilde{\mathbb{D}}$ and $\mathbb{L}$ are contravariant equivalences, while σ is an autoequivalence of $\operatorname{nil.rep}(Q,I)$.
- The composition of two of the three functors gives the third one:

$$\mathbb{L} = \widetilde{\mathbb{D}} \circ \sigma = \sigma \circ \widetilde{\mathbb{D}}, \quad \widetilde{\mathbb{D}} = \mathbb{L} \circ \sigma = \sigma \circ \mathbb{L} \quad \text{and} \quad \sigma = \mathbb{L} \circ \widetilde{\mathbb{D}} = \widetilde{\mathbb{D}} \circ \mathbb{L}.$$

5.2.1.2 Finite-dimensional modules

Let Λ be the Gelfand order, that is, the arrow ideal completion of the path algebra of the Gelfand quiver. In the following, let $\mathbf{F}$ will denote the equivalence functor

$$\mathbf{F} : \Lambda\text{ - fd. mod} \xrightarrow{\sim} \text{nil. rep}(Q, I) \qquad \mathbf{F}(M) = \begin{tikzcd} e_+M \arrow[r, bend left, "b_+\cdot"] & e_\star M \arrow[l, bend left, "a_+\cdot"] \arrow[r, bend right, "a_-\cdot"'] & e_-M \arrow[l, bend right, "b_-\cdot"'] \end{tikzcd}$$

Next, we will view the three functors $\sigma, \widetilde{\mathbb{D}}$ and $\mathbb{L}$ as endofunctors of Λ - fd. mod.

(1) The following map defines an automorphism of Λ of order two:

$$\varsigma\colon \quad \Lambda \xrightarrow{\sim} \Lambda \qquad \mathsf{e}_\pm \longmapsto \mathsf{e}_\mp \qquad \mathsf{e}_\star \longmapsto \mathsf{e}_\star$$
$$\mathsf{a}_\pm \longmapsto \mathsf{a}_\mp \qquad \mathsf{b}_\pm \longmapsto \mathsf{b}_\mp$$

The ring involution ς gives rise to an equivalence $\varsigma^*\colon \Lambda\text{ - mod} \xrightarrow{\sim} \Lambda\text{ - mod}$. Restricting the equivalence ς^* to finite-dimensional modules, we obtain the following diagram of categories and functors:

$$\begin{tikzcd} \Lambda\text{ - fd. mod} \arrow[r, "\varsigma^*", "\sim"'] \arrow[d, "\mathbf{F}"', "\wr"] & \Lambda\text{ - fd. mod} \arrow[d, "\mathbf{F}"', "\wr"] \\ \text{nil. rep}(Q, I) \arrow[r, "\sigma", "\sim"'] & \text{nil. rep}(Q, I) \end{tikzcd} \qquad \mathbf{F} \circ \varsigma^* \cong \varsigma^* \circ \mathbf{F}.$$

In other words, the functor ς^* corresponds to the involution σ on nilpotent representations. We will denote ς^* also by σ in the following.

(2) Let Λ^{op} be the opposite ring of the Gelfand order Λ. There is the following isomorphism of rings

$$\psi : \Lambda \xrightarrow{\sim} \Lambda^{op} \qquad \text{given by} \qquad \mathsf{a}_\pm \longmapsto \mathsf{b}_\pm \qquad \mathsf{b}_\pm \longmapsto \mathsf{a}_\pm$$

and $\psi(\mathsf{e}) = \mathsf{e}$ for any idempotent e of Λ. The isomorphism ψ gives rise to an equivalence of categories $\psi^* : \Lambda^{op}\text{ - mod} \xrightarrow{\sim} \Lambda\text{ - mod}$. We may restrict and compose the equivalence ψ^* with the standard duality $\mathbb{D} = \mathrm{Hom}_{\mathbb{k}}(\_, \mathbb{k})$:

$$\psi^* \circ \mathbb{D} : \Lambda\text{ - fd. mod} \xrightarrow[\sim]{\mathbb{D}} \Lambda^{op}\text{ - fd. mod} \xrightarrow[\sim]{\psi^*} \Lambda\text{ - fd. mod}$$

In the following we will call the composed functor $\widetilde{\mathbb{D}}$ the *twisted Matlis duality*. As for the involution, there is the following diagram of categories and functors:

$$\begin{tikzcd} \Lambda\text{ - fd. mod} \arrow[r, "\psi^*\circ\mathbb{D}", "\sim"'] \arrow[d, "\mathbf{F}"', "\wr"] & \Lambda\text{ - fd. mod} \arrow[d, "\mathbf{F}"', "\wr"] \\ \text{nil. rep}(Q, I) \arrow[r, "\widetilde{\mathbb{D}}", "\sim"'] & \text{nil. rep}(Q, I) \end{tikzcd} \qquad \mathbf{F} \circ (\psi^* \circ \mathbb{D}) \cong \widetilde{\mathbb{D}} \circ \mathbf{F}.$$

As above, we will identify $\widetilde{\mathbb{D}}$ with $\psi^* \circ \mathbb{D}$. Let us note that $\widetilde{\mathbb{D}}(S) \cong S$ for any simple Λ-module S.

(3) The functor $\mathbb{L} : \Lambda\text{ - fd. mod} \xrightarrow{\sim} \Lambda\text{ - fd. mod}$ is given again by $\mathbb{L} = \widetilde{\mathbb{D}} \circ \sigma$ or $\sigma \circ \widetilde{\mathbb{D}}$.

Remark 5.2.1. *The three functors have the following interpretations*:

(1) The involution σ comes from an involution of the Lie group $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. *This involution is given by the complex conjugation on the Lie group* $\mathrm{SU}(1,1)$.

(2) The standard duality $\mathbb{D}$ on finite-dimensional Λ-modules is isomorphic to the Matlis duality. *This explains the name for $\widetilde{\mathbb{D}}$.*

(3) Finally, the functor $\mathbb{L}$ corresponds to the contragredient duality *of Harish-Chandra modules or admissible Hilbert representations of* $\mathrm{SL}(2,\mathbb{R})$.

We refer to C.3 for Lie-theoretic details.

5.2.1.3 Projective complexes with finite-dimensional homology

Let Λ be the Gelfand order. The indecomposable projective Λ-modules are given by the columns of the matrix algebra Λ :

$$\Lambda = \begin{array}{c} \begin{array}{ccc} P_\star & P_+ & P_- \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array}$$

In the following we will denote by $rP_\star$ the radical of the projective Λ-modules $P_\star$.

For technical reasons we will need to consider the three functors on projective resolutions of finite-dimensional Λ-modules. More generally, we may extend the definitions of $\sigma, \widetilde{\mathbb{D}}$ and $\mathbb{L}$ to the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ as follows. We recall that there is an equivalence of categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda\text{ - proj})$, where $\mathrm{K}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda\text{ - proj})$ is the bounded homotopy category of projective complexes.

(1) The involution σ is given as follows:

$$\begin{array}{ccc} P_\star \xrightarrow{\cdot \mathsf{b}_\pm} P_\pm & \xmapsto{\sigma} & P_\mp \xrightarrow{\cdot \mathsf{b}_\mp} P_\star \\ \underset{\mathsf{d}+1}{P_\star} \xrightarrow{\cdot \mathsf{a}_\pm} \underset{\mathsf{d}}{P_\mp} & \xmapsto{\sigma} & \underset{\mathsf{d}+1}{P_\mp} \xrightarrow{\cdot \mathsf{a}_\mp} \underset{\mathsf{d}}{P_\star} \end{array}$$

(2) The twisted Matlis duality gives rise to a derived functor $\widetilde{\mathbb{D}} :\ \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. According to Lemma B.1.19, this functor admits the following factorization:

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \arrow[r, "\widetilde{\mathbb{D}}", "\sim"'] \arrow[d, "\mathrm{Hom}_\Lambda(\_,\omega)"', "\wr"] & \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda^{op}) \arrow[r, "\psi^*", "\sim"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \arrow[u, "{[1]}"', "\wr"] \end{tikzcd} \qquad \widetilde{\mathbb{D}} \cong [1] \circ \psi^* \circ \mathbb{R}\,\mathrm{Hom}_\Lambda(\_,\omega)$$

Since ω is an injective Cohen-Macaulay module, the functor $\mathrm{Hom}_\Lambda(\_,\omega)$ is exact on projective Λ-modules. In particular, the twisted Matlis duality $\widetilde{\mathbb{D}}$ is given by the following prescriptions:

$$\begin{array}{ccc}
P_{\star} \xrightarrow{\cdot \mathsf{b}_{\pm}} P_{\pm} & \xmapsto{\widetilde{\mathbb{D}}} & P_{\mp} \xrightarrow{\cdot \mathsf{a}_{\mp}} rP_{\star} \\
\underset{\mathtt{d+1}}{P_{\pm}} \xrightarrow{\cdot \mathsf{a}_{\pm}} \underset{\mathtt{d}}{P_{\star}} & \xmapsto{\widetilde{\mathbb{D}}} & \underset{\mathtt{-d+1}}{rP_{\star}} \xrightarrow{\cdot \mathsf{b}_{\mp}} \underset{\mathtt{-d}}{P_{\mp}}
\end{array} \tag{5.2.1}$$

(3) Again, the contragredient duality $\mathbb{L}: \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ is given by the composition of σ and $\widetilde{\mathbb{D}}$:

$$\begin{array}{ccc}
P_{\star} \xrightarrow{\cdot \mathsf{b}_{\pm}} P_{\pm} & \longmapsto & P_{\pm} \xrightarrow{\cdot \mathsf{a}_{\pm}} rP_{\star} \\
\underset{\mathtt{d+1}}{P_{\pm}} \xrightarrow{\cdot \mathsf{a}_{\pm}} \underset{\mathtt{d}}{P_{\star}} & \longmapsto & \underset{\mathtt{-d+1}}{rP_{\star}} \xrightarrow{\cdot \mathsf{b}_{\pm}} \underset{\mathtt{-d}}{P_{\pm}}
\end{array}$$

5.2.2 The involution and the duals of strings and bands

Before we describe the action of the twisted Matlis and the contragredient duality on strings and bands, let us consider some examples:

Example 5.2.2. *Let ω be the usual string of $\mathfrak{B}_0$ given by $(\mathtt{p}_{\star}^{[1]}\mathtt{1}\ \mathtt{n}^{[2]}\mathtt{p}_{\star})$ for some $\mathtt{n} \in \mathbb{N}^{+}$. Then its gluing word is given by $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_{\star}^{[1]}\mathtt{1}^{[0]} \uparrow \mathtt{n}^{[1]} \downarrow \mathtt{1}^{[2]}\mathtt{p}_{\star})$. The string resolution $P_{\bullet} = P_{\bullet}(\omega)$ and its contragredient dual are given as follows*:

$$\begin{tikzcd}[column sep=small]
 & P_{\bullet} & & & & \mathbb{L}(P_{\bullet}) & & & \mathbb{L}(P_{\bullet})' & \\
P_{\star} \arrow[r, "1"] \arrow[dr, "-1"'] & P_{+} \arrow[dr, "2n"] & & & & P_{+} \arrow[r, "1"] \arrow[d, dash, dashed] & rP_{\star} & & & \\
 & P_{-} \arrow[r, "2n"] \arrow[u, dashed] & P_{+} \arrow[d, dashed] & \Rightarrow & P_{+} \arrow[ur, "2n"] \arrow[r, "2n"] \arrow[dr, "1"] \arrow[d, dash, dashed] & P_{-} \arrow[ur, "-1"'] & & \cong & P_{\star} \arrow[r, "2n"] & P_{\star} \\
 & P_{\star} \arrow[r, "1"] \arrow[ur, "1"] & P_{-} & & P_{-} \arrow[r, "1"] & rP_{\star} & & & & \\
2 & 1 & 0 & & 1 & 0 & -1 & & 1 & 0
\end{tikzcd}$$

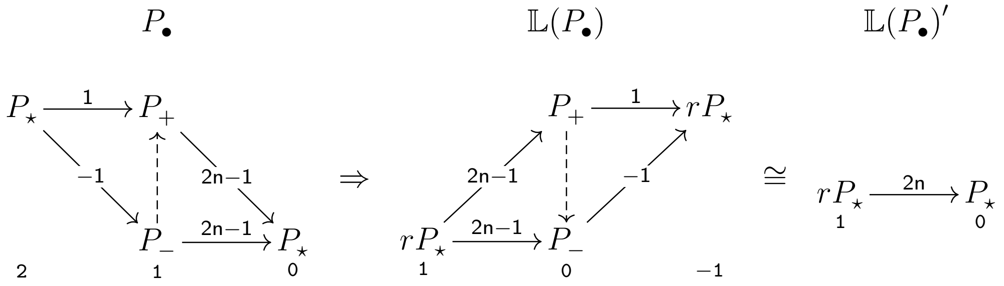
$\mathbb{L}(P_{\bullet})$ is isomorphic to the glued complex of the string $\omega' = (\mathtt{p}_{\star}^{[1]}\mathtt{n}^{[0]}\mathtt{p}_{\star})$, that is, $\mathbb{L}(P_{\bullet}(\omega)) = P_{\bullet}(\omega')$.

Let $V(\omega) = \mathbf{H}_0(P_{\bullet}(\omega))$ be the string representation corresponding to ω. and $V(\omega') = \mathbf{H}_0(P_{\bullet}(\omega'))$ the representation of ω'. These representation can be computed as follows:

$$\begin{tikzcd}
 & V & & & & V(\omega') & \\
\mathbb{k}^{\mathrm{n}} \arrow[r, bend left, "\mathsf{J}"] & \mathbb{k}^{\mathrm{n}} \arrow[l, bend left, "\mathsf{Id}"] \arrow[r, bend right, "\mathsf{Id}"'] & \mathbb{k}^{\mathrm{n}} \arrow[l, bend right, "\mathsf{J}"'] & \arrow[r, mapsto, dashed, "\mathbb{L}"] & \mathbb{k}^{\mathrm{n}} \arrow[r, bend left, "\mathsf{Id}"] & \mathbb{k}^{\mathrm{n}} \arrow[l, bend left, "\mathsf{J}"] \arrow[r, bend right, "\mathsf{J}"'] & \mathbb{k}^{\mathrm{n}} \arrow[l, bend right, "\mathsf{Id}"']
\end{tikzcd}$$

In particular, we obtain that $\mathbb{L}(\omega) = \omega'$.

Example 5.2.3 (Self-dual string)**.**

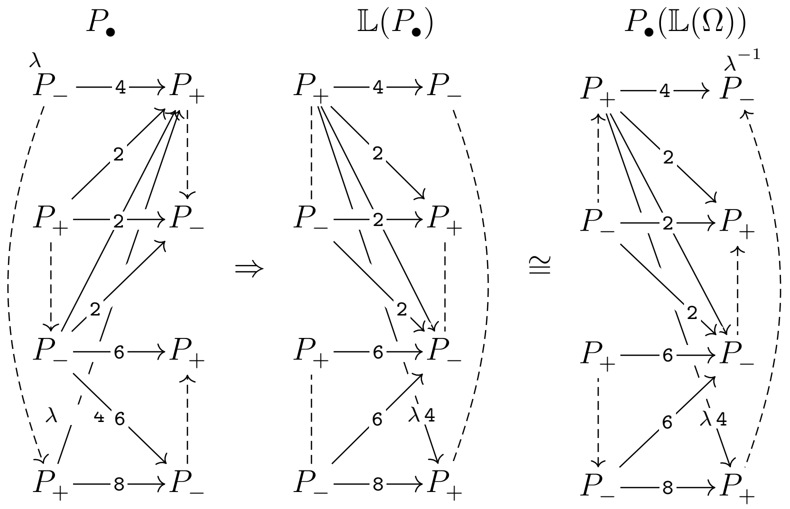


It is straightforward to check that $P_\bullet$ is isomorphic to $\mathbb{L}(P_\bullet)'$. More precisely, both projective resolutions have the following homology:

$$\begin{tikzcd} \Bbbk^{n-1} \arrow[r, bend left, "\mathsf{G}"] & \Bbbk^{n} \arrow[l, bend left, "\mathsf{H}"] \arrow[r, bend right, "\mathsf{H}"'] & \Bbbk^{n-1} \arrow[l, bend right, "\mathsf{G}"'] \end{tikzcd} \quad \textit{where} \quad \mathsf{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \quad \textit{and} \quad \mathsf{H} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & 1 & 0 \end{bmatrix}$$

In other terms, $\mathbb{L}(\omega) = \omega$.

Example 5.2.4. *Let Ω be the band (ω, m, λ) of $\mathfrak{B}_0$ with the word $\omega = \left({}^{[1]}\mathsf{2\ 1\ 3\ 4}^{[1]}\right)$, multiplicity $m = 1$ and some $\lambda \in \Bbbk^*$. Then the gluing word of ω is given by $\overset{\leftrightarrow}{\omega} = \left(\mathsf{4}^{[0]} \downarrow \mathsf{2}^{[1]} \downarrow \mathsf{6}^{[0]} \uparrow \mathsf{8}^{[1]} \uparrow\right)$. The complex $P_\bullet = P_\bullet(\Omega)$ and its contragredient dual are given as follows*:



The complex $\mathbb{L}(P_\bullet)$ can be identified with the glued complex of the band $\mathbb{L}(\Omega) := (\mathbb{L}(\omega), m, \lambda^{-1})$, where $\mathbb{L}(\omega) = \left({}^{[0]}\mathsf{2\ 1\ 3\ 4}^{[0]}\ \right)$.

In the above examples, the contragredient duality $\mathbb{L}$ can be described by the following operations on gluing diagrams:

(1) taking the radical of projectives of type $P_\star$,

(2) inverting all differential and all gluing arrows, and

(3) inverting the eigenvalue in band diagrams.

The Matlis dual σ is given by the same operations together with

(4) interchanging $+$ and $-$ in projectives

We will see that these operations describe the dualities in the general case.

First, we have to define the contragredient dual on words of strings and bands:

Definition 5.2.5. *(1) Let ω be a finite word of $\mathfrak{B}_0$:*

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_0]}\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3 \dots \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}^{[\mathtt{d}_\mathtt{k}]}\mathtt{p}_\beta).$$

If ω has length $\mathtt{k} = 1$, we refer to Remark 5.2.6 for the definition of $\mathbb{L}(\omega)$.

In the following we assume that ω has length $\mathtt{k} \geq 2$. The contragredient dual word $\mathbb{L}(\omega)$ is defined by the following operations on ω :

(a) The beginning of ω is changed by the following rules:

$$\mathtt{p}_\diamond^{[1]} \dots \xleftrightarrow{\mathbb{L}} \mathtt{p}_\diamond^{[0]} \dots \qquad \mathtt{p}_\star^{[2]} \dots \xleftrightarrow{\mathbb{L}} \mathtt{p}_\star^{[0]} \dots \qquad \mathtt{p}_\star^{[1]} \mathtt{1}\,\mathtt{n} \dots \xleftrightarrow{\mathbb{L}} \mathtt{p}_\star^{[1]}(\mathtt{n}+\mathtt{1}) \dots$$

for any $\mathtt{n} \in \mathbb{N}^+$

(b) The ending of ω is altered by similar rules:

$$\dots{}^{[1]}\mathtt{p}_\diamond \xleftrightarrow{\mathbb{L}} \dots{}^{[0]}\mathtt{p}_\diamond \qquad \dots{}^{[2]}\mathtt{p}_\star \xleftrightarrow{\mathbb{L}} \dots{}^{[0]}\mathtt{p}_\star \qquad \dots \mathtt{n}\,\mathtt{1}^{[1]}\mathtt{p}_\star \xleftrightarrow{\mathbb{L}} \dots (\mathtt{n}+\mathtt{1})^{[1]}\mathtt{p}_\star$$

for any $\mathtt{n} \in \mathbb{N}^+$

(c) The remaining intermediate part of ω is reversed.

This yields the word $\mathbb{L}(\omega)$.

(2) Let ω be some cyclic word of $\mathfrak{B}_0$:

$$\omega = ({}^{[\mathtt{d}]}\mathtt{n}_1, \mathtt{n}_2, \dots \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}]}), \quad \textit{where } \mathtt{d} = \mathtt{0} \textit{ or } \mathtt{1}.$$

The contragredient dual of ω is defined as

$$\mathbb{L}(\omega) = ({}^{[\mathtt{d}^*]}\mathtt{n}_1, \mathtt{n}_2, \dots \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}^{[\mathtt{d}^*]}), \quad \textit{where } \mathtt{d}^* = \ \mathtt{1} - \mathtt{d} = \ \mathtt{1} \textit{ or } \mathtt{0}.$$

Remark 5.2.6. *The action of the contragredient dual $\mathbb{L}$ on words of length* one *can be computed as follows:*

$$\mathtt{p}_\diamond^{[1]}\mathtt{n}^{[0]}\mathtt{p}_\star \xleftrightarrow{\mathbb{L}} \mathtt{p}_\diamond^{[0]}\mathtt{n}^{[2]}\mathtt{p}_\star \qquad \mathtt{p}_\star^{[1]}(\mathtt{n}+\mathtt{1})^{[0]}\mathtt{p}_\diamond \xleftrightarrow{\mathbb{L}} \mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{n}^{[1]}\mathtt{p}_\diamond$$

$$\mathtt{p}_\star^{[0]}\mathtt{n}^{[1]}\mathtt{p}_\star \xleftrightarrow{\mathbb{L}} \mathtt{p}_\star^{[2]}\mathtt{n}^{[0]}\mathtt{1}^{[1]}\mathtt{p}_\star \qquad \mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{p}_\diamond \xleftrightarrow{\quad\mathbb{L}\quad} \mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{p}_\diamond$$

The special string $\Omega = (\omega, \pm)$ with $\omega = \mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{p}_\diamond$ corresponds to the simple module $S_\pm$. Since $\mathbb{L}(S_\pm) = S_\mp$ we have that $\mathbb{L}(\Omega) = (\omega, \mp)$. This is an exceptional case for contragredient duals of special strings.

The following Theorem describes the action of the three functors on strings and bands of the Gelfand quiver:

Theorem 5.2.7. *Let Ω be a string or band of $\mathfrak{B}_0$ such that Ω is not equivalent $(\omega, \pm)$, where $\omega = \mathtt{p}_\star^{[1]}\mathtt{1}^{[0]}\mathtt{p}_\diamond$. Let $V(\Omega)$ be the corresponding nilpotent representation of the Gelfand quiver.*

In the following, we will use the following notation for sign data:

- *for any sign $\varepsilon = \pm$ we denote by $\overline{\varepsilon} = \mp$ the opposite sign.*
- *for the signs in bispecial strings $(\omega, m, \varepsilon_1, \varepsilon_2)$ we set for each $j = 1$ or 2*

$$\varepsilon_j^* = \begin{cases} \varepsilon_j & \text{if } m \text{ is odd,} \\ \overline{\varepsilon}_j & \text{if } m \text{ is even,} \end{cases} \qquad \text{and} \qquad \overline{\varepsilon}_j^* = \begin{cases} \overline{\varepsilon}_j & \text{if } m \text{ is odd,} \\ \varepsilon_j & \text{if } m \text{ is even.} \end{cases}$$

Let $\sigma(\Omega)$, $\mathbb{L}(\Omega)$ and $\widetilde{\mathbb{D}}(\Omega)$ be defined by the following table:

	usual string	*special string*	*bispecial string*	*band*
Ω	ω	(ω, ε_1)	$(\omega, m, \varepsilon_1, \varepsilon_2)$	(ω, m, λ)
$\sigma(\Omega)$	ω	$(\omega, \overline{\varepsilon})$	$(\omega, m, \overline{\varepsilon}_1, \overline{\varepsilon}_2)$	(ω, m, λ)
$\mathbb{L}(\Omega)$	$\mathbb{L}(\omega)$	$(\mathbb{L}(\omega), \varepsilon)$	$(\mathbb{L}(\omega), m, \varepsilon_1^*, \varepsilon_2^*)$	$(\mathbb{L}(\omega), m, \lambda^{-1})$
$\widetilde{\mathbb{D}}(\Omega)$	$\mathbb{L}(\omega)$	$(\mathbb{L}(\omega), \overline{\varepsilon})$	$(\mathbb{L}(\omega), m, \overline{\varepsilon}_1^*, \overline{\varepsilon}_2^*)$	$(\mathbb{L}(\omega), m, \lambda^{-1})$

(5.2.2)

(*where* $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, $m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^* \backslash \Delta$)

Then there are the following isomorphisms of glued representations:

$$\sigma(V(\Omega)) \cong V(\sigma(\Omega)) \quad \mathbb{L}(V(\Omega)) \cong V(\mathbb{L}(\Omega)) \quad \text{and} \quad \widetilde{\mathbb{D}}(V(\Omega)) \cong V(\widetilde{\mathbb{D}}(\Omega)).$$

In particular, any of the four classes of indecomposable representations is preserved by the functors σ, $\mathbb{L}$ and $\widetilde{\mathbb{D}}$.

Of course, the action of the involution on strings and bands of the Gelfand quiver is a special case of the corresponding statement for the derived category (Lemma 4.3.3).

We will prove Theorem 5.2.7 in the next subsection.

Next, we determine the self-dual strings and bands.

Corollary 5.2.8. *Let V be an indecomposable representation of the Gelfand quiver. Let Ω be the string or band of $\mathfrak{B}_0$ such that $V \cong V(\Omega)$.*

(1) It holds that $V \cong \mathbb{L}(V)$ if and only if Ω is given by one of the following strings or bands:

- *a usual string $\Omega = \omega$ such that ω or ω^{rev} is given by*

$$\begin{gathered} \mathtt{p}_\star^{[2]} \mathtt{1}^{[1]} \mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}, \mathtt{n}_{\mathtt{k}-1}, \ldots \mathtt{n}_2, \mathtt{n}_1^{[0]} \mathtt{p}_\star \\ \mathtt{p}_\star^{[1]}(\mathtt{n}_1+1), \mathtt{n}_2, \ldots, \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}, \mathtt{n}_{\mathtt{k}-1}, \ldots \mathtt{n}_2, \mathtt{n}_1, \mathtt{1}^{[1]} \mathtt{p}_\star \quad \text{or} \quad \mathtt{p}_\star^{[1]}(\mathtt{n}_1+1), \mathtt{1}^{[1]} \mathtt{p}_\star \end{gathered} \tag{5.2.3}$$

for some $\mathtt{n}_1, \mathtt{n}_2, \ldots \mathtt{n}_\mathtt{k} \in \mathbb{N}^+$ *and* $\mathtt{k} \geq 1$.

- *a bispecial string $\Omega = (\omega, m, \varepsilon_1, \varepsilon_2)$ with a bispecial word ω or ω^{rev} given by*

$$\mathtt{p}_\diamond^{[1]} \mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}, \mathtt{n}_{\mathtt{k}-1}, \ldots \mathtt{n}_2, \mathtt{n}_1^{[0]} \mathtt{p}_\diamond \quad \text{for some } \mathtt{n}_1, \mathtt{n}_2, \ldots \mathtt{n}_\mathtt{k} \in \mathbb{N}^+ \text{ and } \mathtt{k} \geq 1. \tag{5.2.4}$$

any multiplicity $m \in \mathbb{N}^+$, and signs $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 = \varepsilon_2$ if m is odd, and $\varepsilon_1 \neq \varepsilon_2$ if m is even.

- *any symmetric band* Ω*, that is, any band* $\Omega = (\omega, m, \lambda)$ *where* ω *is a symmetric band word,* $m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^*$ *such that* $\lambda \neq (-1)^{\mathtt{k}+1}$*, where* $2\mathtt{k}$ *is the length of* ω*.*

(2) It holds that $V \cong \widetilde{\mathbb{D}}(V)$ *if and only if* Ω *is given by one of the following strings or bands*:

- *any usual string of the series in* (5.2.3) *or any symmetric band.*
- *a bispecial string* $\Omega = (\omega, m, \varepsilon_1, \varepsilon_2)$ *with* ω *or* ω^{rev} *is given by* (5.2.4)*,* $m \in \mathbb{N}^+$ *and signs* $\varepsilon_1, \varepsilon_2$ *such that* $\varepsilon_1 \neq \varepsilon_2$ *if* m *is odd, and* $\varepsilon_1 = \varepsilon_2$ *if* m *is even.*
- *the special string of some simple module* $S_\pm$*.*

In particular, symmetric bands are exactly the self-dual bands with respect to the duality $\mathbb{L}$ *or* $\widetilde{\mathbb{D}}$*:*

$$\text{A band } \Omega \text{ of } \mathfrak{B}_0 \text{ is symmetric if and only if } \mathbb{L}(V) \cong V \text{ or } \widetilde{\mathbb{D}}(V) \cong V.$$

PROOF. These statements follow directly using the equivalence conditions for strings and bands. □

The last statement yields a functorial characterization of symmetric bands.

Remark 5.2.9. *The derived category of the Gelfand order* Λ *has an Auslander-Reiten translation* $\tau : \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$*. However, the Auslander-Reiten translation does not preserve the abelian category* $\Lambda\text{-}\mathrm{mod}$ *of finitely generated* Λ*-modules. More precisely, the following holds*:

(1) Let $M, N \in \Lambda\text{-}\mathrm{mod}$ *such that* $\delta(M) = \delta(N) = 0$*. Then there is a functorial isomorphism*

$$\mathrm{Hom}_\Lambda(M, N) \cong \mathbb{D}\,\mathrm{Ext}^1_\Lambda(N, \tau(M)), \qquad \textit{where } \tau \cong \sigma.$$

In other terms, the abelian subcategory $e\Lambda e\text{-}\mathrm{mod}$*, where* $e = e_+ + e_-$ *has an Auslander-Reiten translation* σ *induced by an involution of the ring* $e\Lambda e$*.*

(2) For any module $M \in \Lambda\text{-}\mathrm{mod}$ *the following holds*:

- $\tau(M) \in \Lambda\text{-}\mathrm{mod}$ *if and only if* $\mathrm{pd}(M) = 1$*.*
- *if* $\delta(M) > 0$*, then* $\tau^3(M) \notin \Lambda\text{-}\mathrm{mod}$*.*

Remark 5.2.10. *Let* V *be a finite-dimensional* Λ*-module of projective dimension one. Let the dimension vector of* V *be denoted by* $\underline{\dim}\, V = (\mathtt{n}_+, \mathtt{n}_\star, \mathtt{n}_-)$*, where* $\mathtt{n}_+, \mathtt{n}_\star, \mathtt{n}_- \in \mathbb{N}_0$*. Then* $\tau(V) \in \Lambda\text{-}\mathrm{fd.mod}$ *and* $\underline{\dim}\,\tau(V) = (n_-, n_+ - n_\star + n_-, n_+)$*. If* $\mathrm{pd}(\tau(V)) = 1$*, then also* $\tau^2(V) \in \Lambda\text{-}\mathrm{fd.mod}$ *and* $\underline{\dim}\,\tau^2(V) = \underline{\dim}(V)$*.*

5.2.3 Proof of Theorem 5.2.7 on dualities of strings and bands

The Matlis duality $\widetilde{\mathbb{D}}$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ can be described as $\widetilde{\mathbb{D}} = \mathrm{Hom}_{\Lambda^{op}}(\_, \Lambda^{op}) \circ \tau$. Next we will define "transpose dual" $(\_)^{tr} = \mathrm{Hom}_{\Lambda^{op}}(\_, \Lambda^{op})$ on words of $\mathfrak{A}_{\mathfrak{B}}$.

Definition 5.2.11. *Let $\omega = x_1 - x_2 \sim \dots x_{k-1} - x_k$ be some word of the alphabet $\mathfrak{A}_{\mathfrak{B}}$. We set $\omega^{tr} = x_1^{tr} - x_2^{tr} \sim \dots x_{k-1}^{tr} - x_k^{tr}$, where the letters are defined as follows:*

letter $x \in \mathfrak{A}_{\mathfrak{B}}$	ζ^d	γ^d	α_n^d	β_n^d
$x^{tr} \in \mathfrak{A}_{\mathfrak{B}}$:	ζ^{-d}	γ^{-d}	β_n^{-d}	α_n^{-d}

In other terms, to obtain ω^{tr} we replace any degree ^{d} by its negative $^{-d}$ and interchange all symbols α and β in ω.

Remark 5.2.12. *Let ϱ', ϱ'' in $\mathfrak{R}^{(d)}$ for some $d \in \mathbb{Z}$. Then $\varrho' < \varrho''$ in $\mathfrak{R}^{(d)}$ if and only if $\varrho'^{tr} > \varrho''^{tr}$ in $\mathfrak{R}^{(-d)}$. That is, the operation $(\_)^{tr}$ reverses the order relation.*

Lemma 5.2.13. *Let ω be a string or band word of $\mathfrak{B}_0$ and ω^{tr} its transpose. Let $\overset{\leftrightarrow}{\omega}$ and $\overset{\leftrightarrow}{\omega}^{tr}$ be the gluing words of ω respectively ω^{tr}.*

(1) The arrows in $\overset{\leftrightarrow}{\omega}^{tr}$ are given by reversing *all arrows in $\overset{\leftrightarrow}{\omega}$.*

PROOF. (1) First, we need to fix some notation:

- Let $x_j \sim x_{j+1} = \zeta^{(d)} \sim \zeta^{(d)}$ be some subword of ω for some even index $2 \leq j \leq k$. Let $x_j^{tr} \sim x_{j+1}^{tr} = \zeta^{(-d)} \sim \zeta^{(-d)}$ be the subword in the word ω^{tr}.

- Let $\overline{\omega}$ be the ambient word of ω. Let Υ be the maximal symmetric subword of $\overline{\omega}$ with the subword $\zeta^{(d)} \sim \zeta^{(d)}$ in the middle. Let γ_l denote the predecessor of the first letter of Υ in $\overline{\omega}$ and γ_r the successor of the last letter of Υ in $\overline{\omega}$.

Since any end of ω special or free, there is some $d \in \mathbb{Z}$ such that γ_l and $\gamma_r \in \overline{\mathfrak{R}}_d$ and either $\gamma_l < \gamma_r$ or $\gamma_l > \gamma_r$. We recall that $\overleftarrow{\zeta^{(d)} \sim \zeta^{(d)}}$ in $\overset{\leftrightarrow}{\omega}$ if $\gamma_l < \gamma_r$ and that $\overrightarrow{\zeta^{(d)} \sim \zeta^{(d)}}$ in $\overset{\leftrightarrow}{\omega}$ if $\gamma_l > \gamma_r$ by (A.3.3).

- Let $\overline{\omega^{tr}}$ be the ambient word of ω^{tr}. Let Υ' be the maximal symmetric subword of $\overline{\omega^{tr}}$ with the subword $x_j^{tr} \sim x_{j+1}^{tr} = \zeta^{(-d)} \sim \zeta^{(-d)}$ in the middle. Let γ_l' and γ_r' denote the predecessor of Υ' respectively the successor of Υ' in $\tau(\overset{\leftrightarrow}{\omega})$.

We note that since ω is a string or band word, also the word ω^{tr} is a string respectively band word. As above, it follows that γ_l' and $\gamma_r' \in \overline{\mathfrak{R}}^{-d}$ and both symbols are comparable but not equal.

(2) To show that the subword $x_j \sim x_{j+1}$ has an orientation in $\overset{\leftrightarrow}{\omega}$ which is *opposite* to the orientation of $x_j^{tr} \sim x_{j+1}^{tr}$ in $\overset{\leftrightarrow}{\omega}^{tr}$, it is sufficient to show that

$$\gamma_l < \gamma_r \qquad \text{if and only if} \qquad \gamma_l^{tr} > \gamma_r^{tr} \tag{5.2.5}$$

Since the operation $(\_)^{tr}$ preserves symmetric words, it follows that $\Upsilon' = \Upsilon^{tr}$. This implies that $\gamma_l' = \gamma_l^{tr}$ and $\gamma_r' = \gamma_r^{tr}$. Now claim (5.2.5) follows from Remark 5.2.12. □

The idea of the proof is similar to the proof of Theorem 4.3.10 which describes the action of the Auslander-Reiten translation on string and band complexes.

PROOF OF THEOREM 5.2.7. Let Ω be some string or band of $\mathfrak{B}_0$ and let $P_\bullet(\Omega)$ be the corresponding resolution in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. We have to show that

$$\widetilde{\mathbb{D}}(P_\bullet(\Omega)) \cong P_\bullet(\widetilde{\mathbb{D}}(\Omega)),$$

where $\widetilde{\mathbb{D}}(P_\bullet(\Omega))$ is described by (5.2.6), while $P_\bullet(\widetilde{\mathbb{D}}(\Omega))$ is the glued complex of the string or band $\widetilde{\mathbb{D}}(\Omega)$, which was defined in (4.3.4). The complex $\widetilde{\mathbb{D}}(P_\bullet(\Omega))$ is given exactly by the application of *Auslander-Reiten translation*, *transposition* and some shift $P_\bullet(\Omega)$.

$$\begin{array}{ccccc}
P_\star \xrightarrow{\cdot \mathtt{b}_\pm} P_\pm & \xmapsto{\tau} & rP_\star \xrightarrow{\cdot \mathtt{b}_\mp} P_\mp & \xmapsto{(_)^{tr}} & P_\mp \xrightarrow{\cdot \mathtt{a}_\mp} rP_\star \\
\underset{\mathtt{d+1}}{P_\pm} \xrightarrow{\cdot \mathtt{a}_\pm} \underset{\mathtt{d}}{P_\star} & \xmapsto{\tau} & \underset{\mathtt{d+1}}{P_\mp} \xrightarrow{\cdot \mathtt{a}_\mp} \underset{\mathtt{d}}{rP_\star} & \xmapsto{(_)^{tr}} & \underset{\mathtt{-d+1}}{rP_\star} \xrightarrow{\cdot \mathtt{b}_\mp} \underset{\mathtt{-d}}{P_\mp}
\end{array} \tag{5.2.6}$$

It is straightforward to check that the gluing diagrams of $\widetilde{\mathbb{D}}(P_\bullet(\Omega))$ and $P_\bullet(\widetilde{\mathbb{D}}(\Omega))$ coincide if we ignore the signs of some differentials. The technical details are given as follows:

- if $\Omega = \omega$ is a usual or special string, then it is straightforward to check that the complex $\widetilde{\mathbb{D}}(P_\bullet(\omega))$ coincides with the string complex of the gluing diagram of $\widetilde{\mathbb{D}}(\omega)$.
- if Ω is a bispecial string or band, the diagram of $\widetilde{\mathbb{D}}(P_\bullet(\omega))$ is isomorphic to the gluing diagram of $P_\bullet(\widetilde{\mathbb{D}}(\Omega))$ after appropriate changes of signs. In the case of bands, the change of the eigenvalue λ into λ^{-1} can be seen by application of $\widetilde{\mathbb{D}}$ to the gluing rules for bands in 4.2.4.

In all cases, it follows that there is an isomorphism $\widetilde{\mathbb{D}}(P_\bullet(\Omega)) \cong P_\bullet(\widetilde{\mathbb{D}}(\Omega))$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$. □

Remark 5.2.14. *In the proof above, we have used that the twisted Matlis duality has factorization into Auslander-Reiten translation, transposition and shift. This factorization can be derived rigorously as follows. By Lemma B.1.19 the twisted Matlis duality* $\widetilde{\mathbb{D}} = \psi^* \circ \mathbb{D}$ *admits the factorization*

$$\widetilde{\mathbb{D}} \cong \psi^* \circ [1] \circ \mathrm{Hom}_\Lambda(\_\Lambda) \circ \tau^{-1} : \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow[\tau^{-1}]{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda^{op}) \xrightarrow[\psi^*]{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$$

where ψ^* *is the induced equivalence of the natural isomorphism* $\psi : \Lambda \xrightarrow{\sim} \Lambda^{op}$. *Since* $\widetilde{\mathbb{D}}^2 \cong \mathrm{Id}$ *it follows that*

$$\widetilde{\mathbb{D}} \cong \widetilde{\mathbb{D}}^{-1} \cong \tau \circ (\_)^{tr} : \quad \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda)$$

$$\text{where } (\_)^{tr} := (\psi^* \circ [1] \circ \mathrm{Hom}_\Lambda(\_, \Lambda))^{-1} \cong [1] \circ (\psi^{-1})^* \cdot \mathrm{Hom}_\Lambda(\_, \Lambda).$$

At last, it can be checked that the functor $(\_)^{tr}$ *is given by the shifted transposition in* (5.2.6).

5.3 Invariants of indecomposable representations

Next, we determine basic homological invariants of string and band representations of the Gelfand quiver.

5.3.1 Homological dimensions of an indecomposable module

The defect of a nilpotent representation V of the Gelfand quiver is defined as

$$\delta(V) = \sum_{j=0}^{2} \delta^{(j)}(V) = \sum_{j=0}^{2} \dim \operatorname{Ext}^j(V, S_\star),$$

where $S_\star$ is the simple quiver representation at the middle vertex $\star$.

In this subsection we will relate the defect of a module to its projective and injective dimension.

Lemma 5.3.1. *Let $V \in \Lambda\text{-mod}$ and let $P_\bullet$ be a minimal projective resolution of V. Let us denote the nilpotent representation of V by*

$$\begin{tikzcd} \mathsf{V}_+ \arrow[r, bend left, "\mathsf{B}_+"] & \mathsf{V}_\star \arrow[l, bend left, "\mathsf{A}_+"] \arrow[r, bend right, "\mathsf{A}_-"'] & \mathsf{V}_- \arrow[l, bend right, "\mathsf{B}_-"'] \end{tikzcd} \qquad \mathsf{B}_+\,\mathsf{A}_+ = \mathsf{B}_-\,\mathsf{A}_- \tag{5.3.1}$$

Let $U_\star := \ker \mathsf{A}_+ \cap \ker \mathsf{A}_-$ and $W_\star := V_\star/(\operatorname{im} \mathsf{B}_+ + \operatorname{im} \mathsf{B}_-)$.
Then the following equalities hold:

$$\delta^{(0)}(V) \overset{1}{=} \dim \operatorname{Hom}_\Lambda(V, S_\star) \overset{2}{=} \dim \operatorname{Ext}^2_\Lambda(S_\star, V) \overset{3}{=} [\operatorname{top} V : S_\star] \overset{4}{=} \dim W_\star \tag{5.3.2}$$

$$\delta^{(2)}(V) = \dim \operatorname{Ext}^2_\Lambda(V, S_\star) = \dim \operatorname{Hom}_\Lambda(S_\star, V) = [\operatorname{soc} V : S_\star] = \dim U_\star. \tag{5.3.3}$$

Proof. (1) The first and third equalities in (5.3.2) follow from the definitions. The second equality follows because $S_\star$ is 2-Calabi-Yau. For the fourth equality, let $f = (0, f_\star, 0) : V = (V_+, V_\star, V_-) \longrightarrow S_\star = (0\ , \Bbbk,\ 0)$ be a morphism of quiver representations. Then f is uniquely determined by some vector $v \in W_\star$ such that $f_\star(v) \neq 0$. It follows that $\dim \operatorname{Hom}(V, S_\star) = \dim W_\star$.

(2) The second formula is proved exactly in the same way. □

Let us note that it is possible to compute the homological invariants $\delta^{(0)}(V)$ and $\delta^{(2)}(V)$ via linear algebra of the quiver representation of V by the formulas (5.3.2) and (5.3.3).

Proposition 5.3.2. *For any V be a nilpotent representation of the Gelfand quiver. Then the following statements hold*:

(1) $\operatorname{pd}(V) = 2$ *if and only if* $\delta^{(2)}(V) \neq 0$.

(2) $\operatorname{id}(V) = 2$ *if and only if* $\delta^{(0)}(V) \neq 0$.

In particular, if $\delta(V) = 0$, then $\operatorname{pd}(V) = \operatorname{id}(V) = 1$.

Proof. These statements follow immediately from the formulas of homological dimensions in Proposition 2.2.17 □

Table 5.3.1. **Homological dimensions of usual and special strings**

Usual string	pd V	id V	**Special string**	pd V	id V
$(\mathtt{p}_\star^{[0]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[0]}\mathtt{p}_\star)$	1	2	$(\mathtt{p}_\pm^{[0]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[0]}\mathtt{p}_\star)$	1	2
$(\mathtt{p}_\star^{[0]},\ \mathtt{n}_1, \dots \mathtt{n}_{2k-1},\ {}^{[1]}\mathtt{p}_\star)$	1	2	$(\mathtt{p}_\pm^{[0]},\ \mathtt{n}_1, \dots \mathtt{n}_{2k-1},\ {}^{[1]}\mathtt{p}_\star)$	1	1
$(\mathtt{p}_\star^{[0]},\ \mathtt{n}_1, \dots \mathtt{n}_{2k-1},\ {}^{[2]}\mathtt{p}_\star)$	2	2	$(\mathtt{p}_\pm^{[0]},\ \mathtt{n}_1, \dots \mathtt{n}_{2k-1},\ {}^{[2]}\mathtt{p}_\star)$	2	1
$(\mathtt{p}_\star^{[1]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[1]}\mathtt{p}_\star)$	1	1	$(\mathtt{p}_\pm^{[1]},\ \mathtt{n}_1, \dots \mathtt{n}_{2k-1},\ {}^{[0]}\mathtt{p}_\star)$	1	2
$(\mathtt{p}_\star^{[1]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[2]}\mathtt{p}_\star)$	2	1	$(\mathtt{p}_\pm^{[1]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[1]}\mathtt{p}_\star)$	1	1
$(\mathtt{p}_\star^{[2]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[2]}\mathtt{p}_\star)$	2	1	$(\mathtt{p}_\pm^{[1]},\ \mathtt{n}_1, \dots\ \mathtt{n}_{2k},\ \ {}^{[2]}\mathtt{p}_\star)$	2	1

5.3.2 The defect via Hom-spaces

In this subsection we will describe the defect without higher extensions.

We will use the following notation:

- for any $M, N \in \Lambda\text{-mod}$ we will denote $|(M,N)| := \dim \operatorname{Hom}_\Lambda(M,N)$.

The following Lemma expresses the first defect number of a module in terms of standard representations which were introduced in C.2.2.

Lemma 5.3.3. *Let $M \in \Lambda\text{-mod}$. Then*

$$\delta_\star^{(1)}(M) = |(S_\star \oplus S_\pm, M)| + |(M, S_\star \oplus S_\mp)| - |(R_\mp, M)| - |(M, T_\pm)| \tag{5.3.4}$$

Proof. We are going to express $\delta^{(1)}(M)$ via dimensions of Hom-spaces. There is the following short exact sequence of standard representations:

$$0 \longrightarrow S_\pm \longrightarrow R_\mp \longrightarrow S_\star \longrightarrow 0. \tag{5.3.5}$$

Let us note that $\operatorname{pd} R_\mp = 1$. Moreover, $\tau(S_\pm) = T_\pm$ and $\tau(R_\mp) = S_\mp$ are both Λ-modules.
Applying $\operatorname{Hom}_\Lambda(\_, M)$ to (5.3.5) yields a long exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}_\Lambda(S_\star, M) & \longrightarrow & \operatorname{Hom}_\Lambda(R_\mp, M) & \longrightarrow & \operatorname{Hom}_\Lambda(S_\pm, M) & \\
& \operatorname{Ext}^1_\Lambda(S_\star, M) & \longrightarrow & \operatorname{Ext}^1_\Lambda(R_\mp, M) & \longrightarrow & \operatorname{Ext}^1_\Lambda(S_\pm, M) & \\
& \operatorname{Ext}^2_\Lambda(S_\star, M) & \longrightarrow & \operatorname{Ext}^2_\Lambda(R_\mp, M) & \longrightarrow & \dots &
\end{array} \tag{5.3.6}$$

By the Auslander-Reiten formula it holds that

$$\dim \operatorname{Ext}^1_\Lambda(L, M) = |(\tau^{-1}(M), L)| = |(M, \tau(L))| \quad \text{for } L = R_\mp \text{ or } L = S_\pm.$$

Concerning higher extensions we have

$$\dim \operatorname{Ext}^2_\Lambda(S_\star, M) = \dim \operatorname{Hom}_\Lambda(M, S_\star) = |(M, S_\star)|, \quad \text{and}$$
$$\operatorname{Ext}^2_\Lambda(R_\mp, M) = 0 \qquad \text{since } \operatorname{pd} R_\mp = 1.$$

For the dimensions of the $\Bbbk$-vector spaces in (5.3.6) it follows that

$$\begin{aligned}0 =&|(S_\star, M)| - |(R_\mp, M)| + |(S_\pm, M)| \\ &- \delta_\star^{(1)}(M) + |(M, \tau(R_\mp))| - |(M, \tau(S_\pm))| + |(M, S_\star)|.\end{aligned}$$

Since $\tau(R_\mp) = S_\mp$ and $\tau(S_\pm) = T_\pm$ this is equivalent to

$$\delta_\star^{(1)}(M) =|(S_\star \oplus S_\pm, M)| + |(M, S_\star \oplus S_\mp)| - |(R_\mp, M)| - |(M, T_\pm)|. \qquad \square$$

For the total defect we obtain the following formula in terms of Hom-spaces:

Lemma 5.3.4. *Let $M \in \Lambda$ -mod. Then the defect of M can be expressed as*

$$\delta(M) = |(S_\star^{\oplus 2} \oplus S_\pm, M)| + |(M, S_\star^{\oplus 2} \oplus S_\mp)| - |(R_\mp, M)| - |(M, T_\pm)|. \tag{5.3.7}$$

Proof. Let M be an Λ-module. Then its defect is given by

$$\begin{aligned}\delta(M) &= \dim \mathrm{Hom}_\Lambda(M, S_\star) + \dim \mathrm{Ext}^1_\Lambda(M, S_\star) + \dim \mathrm{Ext}^2_\Lambda(M, S_\star) \\ &= |(M, S_\star)| + \delta_\star^{(1)}(M) + |(S_\star, M)|.\end{aligned}$$

Combining this expression with (5.3.4) yields (5.3.7). $\square$

5.3.3 Homological invariants of indecomposable modules

Next, we describe some homological invariants like Jordan-Hölder-multiplicities, top and socle of the string and band modules over the Gelfand order.

We will also consider the Euler form of indecomposable modules:

Remark 5.3.5. *Let $M, N \in \Lambda$ -fd. mod. Then the Euler form of M, N is given by*

$$\langle M, N\rangle := \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i_\Lambda(M, N) = (\underline{\dim}\, M)^t \cdot \mathsf{E}_\Lambda \cdot \underline{\dim}\, N$$

$$\textit{with } \mathsf{E}_\Lambda = (\ \langle S_i, S_j\rangle\ )_{i,j\in Q_0} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

In particular, if $\underline{\dim}\, M = (\mathtt{n}_+, \mathtt{n}_\star, \mathtt{n}_-)$, for some $\mathtt{n}_+, \mathtt{n}_\star, \mathtt{n}_- \in \mathbb{N}_0$, then the Euler characteristic of M is given by

$$\chi(M) := \langle M, M\rangle = (\mathtt{n}_+ - \mathtt{n}_\star)^2 + (\mathtt{n}_- - \mathtt{n}_\star)^2 \tag{5.3.8}$$

Theorem 5.3.6. *Let $V \in \Lambda$ -fd. mod be indecomposable. Let Ω be the string or band of the Gelfand quiver such that $V \cong V(\Omega)$.*

(1) The involution $\sigma(V)$, the defect $\delta(V)$ and the possible values for the projective and the injective dimension $\mathrm{pd}(V)$ and $\mathrm{id}(V)$, the Jordan-Hölder-multiplicities $\underline{\dim}(V)$ and the Euler characteristic $\chi(V)$ are given by Table 5.3.2.

In particular, $\delta(V)$ and $\chi(V) \in \{0, 1, 2\}$.

(2) the precise values of the projective *and* injective dimension *of V are listed in Table 5.3.1,*

(3) *the complete description of* Jordan-Hölder multiplicities, *the* top *and the* socle *of indecomposable representations can be found in the tables on page 213, 214 and 215 below.*

Table 5.3.2. **Homological invariants of indecomposable representations**

Ω	$\sigma(V)$	$\delta(V)$	$\mathrm{pd}(V)$	$\mathrm{id}(V)$	$\underline{\dim}\, V$	$\chi(V)$
usual string ω	$\sigma(V) \cong V$	2	1 or 2	1 or 2	$(\mathtt{n}+1, \mathtt{n}+1, \mathtt{n}+1)$	0
					$(\mathtt{n}, \mathtt{n}+1, \mathtt{n})$	2
			1	1	$(\mathtt{n}+1, \mathtt{n}, \mathtt{n}+1)$	
special string (ω, ε_1)	$\sigma(V) \ncong V$	1	1	1	$(\mathtt{n}+1, \mathtt{n}, \mathtt{n})$	1
					$(\mathtt{n}, \mathtt{n}, \mathtt{n}+1)$	
			1 or 2	1 or 2	$(\mathtt{n}, \mathtt{n}+1, \mathtt{n}+1)$	
					$(\mathtt{n}+1, \mathtt{n}+1, \mathtt{n})$	
bispecial string $(\omega, \varepsilon_1, \varepsilon_2)$	$\sigma(V) \ncong V$	0	1	1	$(\mathtt{n}+1, \mathtt{n}+1, \mathtt{n}+1)$	0
					$(\mathtt{n}+2, \mathtt{n}+1, \mathtt{n})$	2
					$(\mathtt{n}, \mathtt{n}+1, \mathtt{n}+2)$	
band (ω, m, λ)	$\sigma(V) \cong V$	0	1	1	$(\mathtt{n}+1, \mathtt{n}+1, \mathtt{n}+1)$	0
					for some $\mathtt{n} \in \mathbb{N}_0$	

Remark 5.3.7. *The possible values for the dimension vector* $\underline{\dim}(V)$ *of any indecomposable representation* V *of the Gelfand quiver are completely described by the restriction that the Euler characteristic* $\chi(V)$ *is zero, one or two.*

Proof. All statements follow from case-by-case computations with the string and band data using the following arguments:

- The functorial behavior with respect to σ and the defect δ and the defect follow directly from Theorem 3.6.2.
- The projective and injective dimensions have been described in 5.3.1.
- The dimension vector follows using the dimension vector formula for triples in Lemma 3.3.16 (3).
- Also the formula for the top follows from Lemma 3.3.16 (2).
- The Euler characteristic $\chi(V)$ is computed by the dimension vector $\underline{\dim}(V)$by (5.3.8).
- The socle can be computing using duality: $\underline{\dim}\,\mathrm{soc}\, V = \underline{\dim}\,\mathrm{top}\, \widetilde{\mathbb{D}}(V)$.

Many computations can be omitted using using Theorem 5.2.7 on the the action of the involution σ and the contragredient duality $\mathbb{L}$ and with the help of Remark 5.2.10

the Auslander-Reiten translation τ according to on the various classes of strings and bands. $\square$

Also the Euler form of any two indecomposable modules is bounded:

Corollary 5.3.8. *Let $M, N \in \Lambda\text{-fd.mod}$ be indecomposable. Then* $|\langle M, N\rangle| \leq 2$.

Bands	
Band	$\big((\mathtt{n_i})_{\mathtt{i=1}}^{\mathtt{2k}}, \mathtt{m}, \lambda\big)$
$\underline{\dim}(V)$	$(\ \mathtt{m\,n},\ \ \mathtt{m\,n},\ \ \mathtt{m\,n}\ )$
$\underline{\dim}(\operatorname{top} V)$	$(\ \mathtt{m\,k},\ \ \mathtt{0},\ \ \mathtt{m\,k}\ )$
$\underline{\dim}(\operatorname{soc} V)$	$(\ \mathtt{m\,k},\ \ \mathtt{0},\ \ \mathtt{m\,k}\ )$
where $\mathtt{n} = \sum_{\mathtt{i=1}}^{\mathtt{2k}} \mathtt{n_i}$	

Usual strings

String	$\left(p_\star^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\star\right)$	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\star\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k}, {}^{[2]}p_\star\right)$
$\underline{\dim}(V)$	$(n-1, n, n-1)$	$(n-1, n-2, n-1)$	$(n-1, n, n-1)$
$\underline{\dim}(\operatorname{top} V)$	$(k-1, 2, k-1)$	$(k, 0, k)$	$(k, 0, k)$
$\underline{\dim}(\operatorname{soc} V)$	$(k, 0, k)$	$(k+1, 0, k+1)$[1]	$(k-1, 2, k-1)$
String	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\star\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\star\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\star\right)$
$\underline{\dim}(V)$	$(n-1, n-1, n-1)$[2]	$(n-1, n-1, n-1)$	$(n-1, n, n-1)$
$\underline{\dim}(\operatorname{top} V)$	$(k-1, 1, k-1)$	$(k, 0, k)$	$(k-1, 1, k-1)$
$\underline{\dim}(\operatorname{soc} V)$	$(k, 0, k)$[3]	$(k, 1, k)$[4]	$(k-1, 1, k-1)$

Special Strings

String	$\left(p_\star^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\diamond, +\right)$	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, -\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, +\right)$
$\underline{\dim}(V)$	$(n, n, n-1)$	$(n-1, n-1, n)$	$(n, n, n-1)$
$\underline{\dim}(\operatorname{top} V)$	$(k, 1, k-1)$	$(k-1, 0, k)$	$(k, 0, k-1)$
$\underline{\dim}(\operatorname{soc} V)$	$(k, 0, k)$	$(k, 0, k)$[5]	$(k-1, 1, k-1)$
String	$\left(p_\star^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\diamond, -\right)$	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, +\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, -\right)$
$\underline{\dim}(V)$	$(n-1, n, n)$	$(n, n-1, n-1)$	$(n-1, n, n)$
$\underline{\dim}(\operatorname{top} V)$	$(k-1, 1, k)$	$(k, 0, k-1)$	$(k-1, 0, k)$
$\underline{\dim}(\operatorname{soc} V)$	$(k, 0, k)$	$(k, 0, k)$[5]	$(k-1, 1, k-1)$
String	$\left(p_\star^{[0]}, (n_i)_{i=1}^{2k-1}, {}^{[1]}p_\diamond, +\right)$	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, -\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, +\right)$
$\underline{\dim}(V)$	$(n-1, n, n)$	$(n, n-1, n-1)$	$(n-1, n, n)$
$\underline{\dim}(\operatorname{top} V)$	$(k-1, 1, k-1)$	$(k, 0, k)$	$(k, 0, k)$
$\underline{\dim}(\operatorname{soc} V)$	$(k-1, 0, k)$	$(k+1, 0, k)$[6]	$(k-1, 1, k)$
String	$\left(p_\star^{[0]}, (n_i)_{i=1}^{2k-1}, {}^{[1]}p_\diamond, -\right)$	$\left(p_\star^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, +\right)$	$\left(p_\star^{[2]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, -\right)$
$\underline{\dim}(V)$	$(n, n, n-1)$	$(n-1, n-1, n)$	$(n, n, n-1)$
$\underline{\dim}(\operatorname{top} V)$	$(k-1, 1, k-1)$	$(k, 0, k)$	$(k, 0, k)$
$\underline{\dim}(\operatorname{soc} V)$	$(k, 0, k-1)$	$(k, 0, k+1)$[7]	$(k, 1, k-1)$

where $n = \sum_{i=1}^{2k-1} n_i$ respectively $n = \sum_{i=1}^{2k} n_i$

[1] $\underline{\dim}\operatorname{soc}(V) = \begin{cases} (k, 0, k) & \text{if either } n_1 = 1 \text{ or } n_{2k} = 1, \\ (k-1, 0, k-1) & \text{if } n_1 = n_{2k} = 1. \end{cases}$

[2] $\underline{\dim}(V) = (n, n, n)$ if $k = 1$.

[3] $\underline{\dim}\operatorname{soc}(V) = (k-1, 0, k-1)$ if $n_1 = 1$ and $k \neq 1$

[4] $\underline{\dim}\operatorname{soc}(V) = (k-1, 1, k-1)$ if $n_1 = 1$

[5] $\underline{\dim}(\operatorname{soc} V) = (k-1, 0, k-1)$ if $n_1 = 1$.

[6] $\underline{\dim}(\operatorname{soc} V) = (k, 0, k-1)$ if $n_1 = 1$.

[7] $\underline{\dim}(\operatorname{soc} V) = (k-1, 0, k)$ if $n_1 = 1$.

	Bispecial Strings $(\omega, m, \varepsilon_1, \varepsilon_2)$	
String	$\left(p_\diamond^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\diamond, m, +, +\right)$	$\left(p_\diamond^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\diamond, m, -, -\right)$
$\underline{\dim}(V)$	$\begin{cases} (mn+1, \ mn, \ mn-1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$	$\begin{cases} (mn-1, \ mn, \ mn+1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$
$\underline{\dim}(\operatorname{top} V)$	$\begin{cases} (mk+1, \ 0, \ mk-1)^1 \\ (mk, \ 0, \ mk)^2 \end{cases}$	$\begin{cases} (mk-1, \ 0, \ mk+1)^1 \\ (mk, \ 0, \ mk)^2 \end{cases}$
$\underline{\dim}(\operatorname{soc} V)$	$(mk, \ 0, \ mk)$	$(mk, \ 0, \ mk)$
String	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, m, +, +\right)$	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, m, -, -\right)$
$\underline{\dim}(V)$	$\begin{cases} (mn-1, \ mn, \ mn+1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$	$\begin{cases} (mn+1, \ mn, \ mn-1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$
$\underline{\dim}(\operatorname{top} V)$	$(mk, \ 0, \ mk)$	$(mk, \ 0, \ mk)$
$\underline{\dim}(\operatorname{soc} V)$	$\begin{cases} (mk-1, \ 0, \ mk+1)^1 \\ (mk, \ 0, \ mk)^2 \end{cases}$	$\begin{cases} (mk+1, \ 0, \ mk-1)^1 \\ (mk, \ 0, \ mk)^2 \end{cases}$
String	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, m, +, -\right)$	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, m, -, +\right)$
$\underline{\dim}(V)$	$\begin{cases} (mn-1, \ mn, \ mn+1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$	$\begin{cases} (mn+1, \ mn, \ mn-1)^1 \\ (mn, \ mn, \ mn)^2 \end{cases}$
$\underline{\dim}(\operatorname{top} V)$	$\begin{cases} (mk-\frac{m+1}{2}, \ 0, \ mk-\frac{m-1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$	$\begin{cases} (mk-\frac{m-1}{2}, \ 0, \ mk-\frac{m+1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$
$\underline{\dim}(\operatorname{soc} V)$	$\begin{cases} (mk-\frac{m+1}{2}, \ 0, \ mk-\frac{m-1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$	$\begin{cases} (mk-\frac{m-1}{2}, \ 0, \ mk-\frac{m+1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$
String	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, m, +, +\right)$	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k-1}, {}^{[0]}p_\diamond, m, -, -\right)$
$\underline{\dim}(V)$	$(mn, \ mn, \ mn)$	$(mn, \ mn, \ mn)$
$\underline{\dim}(\operatorname{top} V)$	$\begin{cases} (mk-\frac{m-1}{2}, \ 0, \ mk-\frac{m+1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$	$\begin{cases} (mk-\frac{m+1}{2}, \ 0, \ mk-\frac{m-1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$
$\underline{\dim}(\operatorname{soc} V)$	$\begin{cases} (mk-\frac{m+1}{2}, \ 0, \ mk-\frac{m-1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$	$\begin{cases} (mk-\frac{m-1}{2}, \ 0, \ mk-\frac{m+1}{2})^1 \\ (mk-\frac{m}{2}, \ 0, \ mk-\frac{m}{2})^2 \end{cases}$
String	$\left(p_\diamond^{[0]}, (n_i)_{i=1}^{2k}, {}^{[0]}p_\diamond, m, +, -\right)$	$\left(p_\diamond^{[1]}, (n_i)_{i=1}^{2k}, {}^{[1]}p_\diamond, m, +, -\right)$
$\underline{\dim}(V)$	$(mn, \ mn, \ mn)$	$(mn, \ mn, \ mn)$
$\underline{\dim}(\operatorname{top} V)$	$(mk, \ 0, \ mk)$	$(mk, \ 0, \ mk)$
$\underline{\dim}(\operatorname{soc} V)$	$(mk, \ 0, \ mk)$	$(mk, \ 0, \ mk)$
		where $m \in \mathbb{N}^+$ and $n = \sum_{i=1}^{2k-1} n_i$ respectively $n = \sum_{i=1}^{2k} n_i$

[1]if m is odd

[2]if m is even

5.4 Indecomposable representations of the Gelfand quiver

In this section we describe all indecomposable nilpotent representations of the Gelfand quiver in terms of generators and relations.

In this construction algorithm, we begin with any string or band Ω of the Gelfand quiver. By Theorem 5.1.8 Ω corresponds to an indecomposable representation $V := V(\Omega)$ of the Gelfand quiver. The representation V is defined in two steps:

(1) We define a direct sum $\check{V} = \bigoplus V_i$ of some cyclic representations. This step defines the *generators* of V.

(2) Then we define some quotient $V = \check{V}/I$. In other words, we describe the relations I of the generators of V.

This process will be called *gluing* of cyclic representations of the Gelfand quiver.

First, we need to fix some notation and list all cyclic representations.

5.4.1 Cyclic representations of the Gelfand quiver

As usual, (Q, I) be the Gelfand quiver and Λ its completed path algebra:

$$(Q,I) = \begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd} \qquad \Lambda = \begin{array}{c} \begin{matrix} P_\star & P_+ & P_- \end{matrix} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix} \end{array},$$

$$\mathsf{b}_+ \mathsf{a}_+ = \mathsf{b}_- \mathsf{a}_- =: \mathsf{c} \qquad \text{where } \mathbf{R} = \Bbbk[\![x]\!] \text{ and } \mathfrak{m} = (x).$$

We will use the following notation for basis vectors of cyclic projective Λ-modules and their quotients:

Table 5.4.1. **Notation for basis vectors of cyclic modules**

P_i	$V_+ = e_+P_i$	$V_\star = e_\star P_i$	$V_- = e_-P_i$
P_+	$v_+^{(1)} = \mathsf{e}_+ = (0)_{++}$ $v_+^{(n+1)} = \mathsf{a}_+\mathsf{c}^{n-1}\mathsf{b}_+ = (2n)_{++}$	$v_\star^{(n)} = \mathsf{c}^{n-1}\mathsf{b}_+ = (2n-1)_{\star+}$	$v_-^{(n)} = \mathsf{a}_-\mathsf{c}^{n-1}\mathsf{b}_+ = (2n)_{-+}$
$P_\star$	$v_+^{(n)} = \mathsf{a}_+\mathsf{c}^{n-1} = (2n-1)_{+\star}$	$v_\star^{(1)} = \mathsf{e}_\star = (0)_{\star\star}$ $v_\star^{(n+1)} = \mathsf{c}^{n} = (2n)_{\star\star}$	$v_-^{(n)} = \mathsf{a}_-\mathsf{c}^{n-1} = (2n-1)_{-\star}$
P_-	$v_+^{(n)} = \mathsf{a}_+\mathsf{c}^{n-1}\mathsf{b}_- = (2n)_{+-}$	$v_\star^{(n)} = \mathsf{c}^{n-1}\mathsf{b}_- = (2n-1)_{\star-}$	$v_-^{(1)} = \mathsf{e}_- = (0)_{--}$ $v_-^{(n+1)} = \mathsf{a}_-\mathsf{c}^{n-1}\mathsf{b}_- = (2n)_{--}$

Cyclic representations of the Gelfand quiver are given by strings with only one parameter:

Lemma 5.4.1. *Let $V \in \operatorname{nil.rep}(Q, I)$ be a cyclic representation of the Gelfand quiver. Then there is some usual, special or bispecial string with trivial multiplicity $\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\mathtt{n}^{[0]}\mathtt{p}_\beta)$ such that $V \cong V(\omega)$, where $\alpha, \beta \in Q_0$, $\mathtt{n} \in \mathbb{N}^+$ and $\mathtt{d}_\alpha = 1$ or 2.*

In the following we will write $\mathtt{n}^{[\mathtt{d}_\alpha]}_{\alpha,\beta} = (\mathtt{p}^{[\mathtt{d}_\alpha]}_\alpha \mathtt{n}^{[0]} \mathtt{p}_\beta)$ for the string of a cyclic representation.

To describe the cyclic quiver representations we need to introduce the following matrices:

- Let Id denote the identity, and J the nilpotent Jordan matrix of some size $\mathtt{n} \in \mathbb{N}^+$.
- Let $\mathsf{G} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 0\;0 & \dots & 0\;0 \\ & \mathrm{Id}_{\mathtt{n}-1} & \end{bmatrix}$ and $\mathsf{H} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_{\mathtt{n}-1} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{matrix} \end{bmatrix}$.

  In particular, $\mathsf{G}\colon \Bbbk^{\mathtt{n}-1} \longrightarrow \Bbbk^{\mathtt{n}}$ and $\mathsf{H}\colon \Bbbk^{\mathtt{n}} \longrightarrow \Bbbk^{\mathtt{n}-1}$ for some $\mathtt{n} \in \mathbb{N}^+$.

The size parameter $\mathtt{n}$ of the matrices will become apparent from the context.

Table 5.4.2 lists the strings, projective resolutions and quiver representations of all cyclic finite-dimensional Λ-modules. In all cyclic representations of the Gelfand quiver - except the two series $\mathtt{n}^{[2]}_{\star\star}$ and $\mathtt{n}^{[1]}_{\star\star}$ - there are one or two distinguished basis vectors, which will be called *gluing vectors*. These gluing vectors will be important in the construction algorithm of indecomposable representations.

TABLE 5.4.2. **Cyclic representations of the Gelfand quiver**

Quotients of $P_\star$			
string ω $\left(\mathtt{p}^{[\mathtt{d}_\alpha]}_\alpha \mathtt{n}^{[\mathtt{d}_\beta]} \mathtt{p}_\beta\right)$	projective resolution $P_\bullet(\omega)$	quiver representation $V(\omega)$	gluing vectors $v(\omega)$
$\mathtt{n}^{[1]}_{\star\star}$	$P_\star \xrightarrow{2\mathtt{n}} P_\star$	$\begin{tikzcd} \Bbbk^{\mathtt{n}} \arrow[r, bend left, "\mathsf{J}"] & \Bbbk^{\mathtt{n}} \arrow[l, bend left, "\mathsf{Id}"] \arrow[r, bend right, "\mathsf{Id}"'] & \Bbbk^{\mathtt{n}} \arrow[l, bend right, "\mathsf{J}"'] \end{tikzcd}$	$\varnothing$
$\mathtt{n}^{[1]}_{+\star}$	$P_+ \xrightarrow{2\mathtt{n}-1} P_\star$	$\begin{tikzcd} \Bbbk^{\mathtt{n}-1} \arrow[r, bend left, "\mathsf{G}"] & \Bbbk^{\mathtt{n}} \arrow[l, bend left, "\mathsf{H}"] \arrow[r, bend right, "\mathsf{Id}"'] & \Bbbk^{\mathtt{n}} \arrow[l, bend right, "\mathsf{J}"'] \end{tikzcd}$	$v^{(\mathtt{n})}_\star$
$\mathtt{n}^{[1]}_{-\star}$	$P_- \xrightarrow{2\mathtt{n}-1} P_\star$	$\begin{tikzcd} \Bbbk^{\mathtt{n}} \arrow[r, bend left, "\mathsf{J}"] & \Bbbk^{\mathtt{n}} \arrow[l, bend left, "\mathsf{Id}"] \arrow[r, bend right, "\mathsf{H}"'] & \Bbbk^{\mathtt{n}-1} \arrow[l, bend right, "\mathsf{G}"'] \end{tikzcd}$	$v^{(\mathtt{n})}_\star$
$\mathtt{n}^{[2]}_{\star\star}$	$P_\star \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} P_+ \oplus P_- \xrightarrow{\begin{bmatrix} \mathtt{2n\text{-}1} & \mathtt{2n\text{-}1} \end{bmatrix}} P_\star$	$\begin{tikzcd} \Bbbk^{\mathtt{n}-1} \arrow[r, bend left, "\mathsf{G}"] & \Bbbk^{\mathtt{n}} \arrow[l, bend left, "\mathsf{H}"] \arrow[r, bend right, "\mathsf{H}"'] & \Bbbk^{\mathtt{n}-1} \arrow[l, bend right, "\mathsf{G}"'] \end{tikzcd}$	$\varnothing$

Quotients of P_+			
ω	$P_\bullet(\omega)$	$V(\omega)$	$v(\omega)$

ω	$P_\bullet(\omega)$	$V(\omega)$	$v(\omega)$
$\mathtt{n}^{[1]}_{\star+}$	$P_\star \xrightarrow{\mathtt{2n-1}} P_+$	$\Bbbk^{\mathtt{n}} \underset{\mathsf{G}}{\overset{\mathsf{H}}{\rightleftarrows}} \Bbbk^{\mathtt{n-1}} \underset{\mathsf{Id}}{\overset{\mathsf{J}}{\leftrightarrows}} \Bbbk^{\mathtt{n-1}}$	$v_+^{(\mathtt{n})},\ v_-^{(\mathtt{n-1})}$
$\mathtt{n}^{[1]}_{++}$	$P_+ \xrightarrow{\mathtt{2n}} P_+$	$\Bbbk^{\mathtt{n}} \underset{\mathsf{J}}{\overset{\mathsf{Id}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{Id}}{\overset{\mathsf{J}}{\leftrightarrows}} \Bbbk^{\mathtt{n}}$	$v_\star^{(\mathtt{n})}$
$\mathtt{n}^{[1]}_{-+}$	$P_- \xrightarrow{\mathtt{2n}} P_+$	$\Bbbk^{\mathtt{n+1}} \underset{\mathsf{G}}{\overset{\mathsf{H}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{H}}{\overset{\mathsf{G}}{\leftrightarrows}} \Bbbk^{\mathtt{n-1}}$	$v_\star^{(\mathtt{n})}$
$\mathtt{n}^{[2]}_{\star+}$ $P_\star \xrightarrow{\begin{bmatrix}\mathtt{1}\\ \mathtt{-1}\end{bmatrix}} P_+ \oplus P_- \xrightarrow{\begin{bmatrix}\mathtt{2n} & \mathtt{2n}\end{bmatrix}} P_+$		$\Bbbk^{\mathtt{n}} \underset{\mathsf{J}}{\overset{\mathsf{Id}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{H}}{\overset{\mathsf{G}}{\leftrightarrows}} \Bbbk^{\mathtt{n-1}}$	$v_\star^{(\mathtt{n})}$

Quotients of P_-

ω	$P_\bullet(\omega)$	$V(\omega)$	$v(\omega)$
$\mathtt{n}^{[1]}_{\star-}$	$P_\star \xrightarrow{\mathtt{2n-1}} P_-$	$\Bbbk^{\mathtt{n-1}} \underset{\mathsf{Id}}{\overset{\mathsf{J}}{\rightleftarrows}} \Bbbk^{\mathtt{n-1}} \underset{\mathsf{G}}{\overset{\mathsf{H}}{\leftrightarrows}} \Bbbk^{\mathtt{n}}$	$v_+^{(\mathtt{n-1})},\ v_-^{(\mathtt{n})}$
$\mathtt{n}^{[1]}_{+-}$	$P_+ \xrightarrow{\mathtt{2n}} P_-$	$\Bbbk^{\mathtt{n-1}} \underset{\mathsf{H}}{\overset{\mathsf{G}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{G}}{\overset{\mathsf{H}}{\leftrightarrows}} \Bbbk^{\mathtt{n+1}}$	$v_\star^{(\mathtt{n})}$
$\mathtt{n}^{[1]}_{--}$	$P_- \xrightarrow{\mathtt{2n}} P_-$	$\Bbbk^{\mathtt{n}} \underset{\mathsf{Id}}{\overset{\mathsf{J}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{J}}{\overset{\mathsf{Id}}{\leftrightarrows}} \Bbbk^{\mathtt{n}}$	$v_\star^{(\mathtt{n})}$
$\mathtt{n}^{[2]}_{\star-}$ $P_\star \xrightarrow{\begin{bmatrix}\mathtt{1}\\ \mathtt{-1}\end{bmatrix}} P_+ \oplus P_- \xrightarrow{\begin{bmatrix}\mathtt{2n} & \mathtt{2n}\end{bmatrix}} P_-$		$\Bbbk^{\mathtt{n-1}} \underset{\mathsf{H}}{\overset{\mathsf{G}}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \underset{\mathsf{J}}{\overset{\mathsf{Id}}{\leftrightarrows}} \Bbbk^{\mathtt{n}}$	$v_\star^{(\mathtt{n})}$

At last, let us consider a basic example of gluing in much detail.

Example 5.4.2 (Gluing of two cyclic quiver representations).
Let $\mathtt{n}_1, \mathtt{n}_2 \in \mathbb{N}^+$ *such that* $\mathtt{n}_1 \leq \mathtt{n}_2$. *Let* $\omega = \left(\mathtt{p}_\star^{[0]}\mathtt{n}_1, \mathtt{n}_2^{[0]}\mathtt{p}_\diamond\right)$ *and* $\Omega = (\omega, +)$ *The representation* $V := V(\Omega)$ *of the special string* Ω *is constructed as follows*:

(1) *According to 4.2.1 the gluing word of* ω *is given by* $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_\star^{[0]}\widehat{\mathtt{n}}_1^{[1]} \downarrow \widehat{\mathtt{n}}_2^{[0]}\mathtt{p}_\diamond\right)$, *where* $\widehat{\mathtt{n}}_1 = \mathtt{2n}_1 - \mathtt{1}$ *and* $\widehat{\mathtt{n}}_2 = \mathtt{2n}_2$.

(2) The gluing word $\overset{\leftrightarrow}{\omega}$ *yields a gluing diagram. Applying the gluing rules to this diagram we obtain the projective resolution* $P_\bullet = P_\bullet(\omega)$ *of the representation* V :

$$\begin{array}{ccc} \widetilde{P}_\diamond \xrightarrow{\widehat{\mathtt{n}}_1} \widetilde{P}_\star \\ \downarrow \\ \widetilde{P}_\diamond \xrightarrow{\widehat{\mathtt{n}}_2} \widetilde{P}_{\diamond_+} \end{array} \Rightarrow \quad P_\bullet = \begin{array}{ccc} P_+ \xrightarrow{\widehat{\mathtt{n}}_1} P_\star \\ \downarrow \quad \nearrow \widehat{\mathtt{n}}_1 \\ P_- \xrightarrow{\widehat{\mathtt{n}}_2} P_+ \end{array} = \quad P_+ \oplus P_- \xrightarrow{\begin{bmatrix} (\widehat{\mathtt{n}}_1)_{+\star} & (\widehat{\mathtt{n}}_1)_{-\star} \\ 0 & (\widehat{\mathtt{n}}_2)_{-+} \end{bmatrix}} P_\star \oplus P_+$$

(3) To compute the representation V *we consider the homology of* $P_\bullet$ *in degree zero*:

$$V = \mathbf{H}_0(P_\bullet) = \big(P_\star \oplus P_+\big)/\operatorname{im} d \qquad \textit{where} \qquad \operatorname{im} d = \Big\langle \begin{bmatrix} (\widehat{\mathtt{n}}_1)_{+\star} \\ 0 \end{bmatrix}, \begin{bmatrix} (\widehat{\mathtt{n}}_1)_{-\star} \\ (\widehat{\mathtt{n}}_2)_{-+} \end{bmatrix} \Big\rangle_\Lambda.$$

$$\cong (P_\star/(\widehat{\mathtt{n}}_1)_{+\star} \oplus P_+) \ / \ \langle \begin{bmatrix} (\widehat{\mathtt{n}}_1)_{-\star} \\ (\widehat{\mathtt{n}}_2)_{-+} \end{bmatrix} \rangle$$

As in the notation of Table 5.4.1 let us set

$$v_\star^{(\mathtt{n}_1,1)} = \begin{bmatrix} (2\mathtt{n}_1-2)_{\star\star} \\ 0 \end{bmatrix}, \quad v_-^{(\mathtt{n}_1,1)} = \begin{bmatrix} (2\mathtt{n}_1-1)_{-\star} \\ 0 \end{bmatrix} \qquad \textit{and} \qquad v_\star^{(\mathtt{n}_2,2)} = \begin{bmatrix} 0 \\ (2\mathtt{n}_2-1)_{\star+} \end{bmatrix}.$$

There are two relations in the module V :

$$A_+ \cdot v_\star^{(\mathtt{n}_1,1)} = A_+ \cdot \begin{bmatrix} (2\mathtt{n}_1-2)_{\star\star} \\ 0 \end{bmatrix} = \begin{bmatrix} (2\mathtt{n}_1-1)_{+\star} \\ 0 \end{bmatrix} = 0, \quad \textit{and}$$

$$A_- \cdot v_\star^{(\mathtt{n}_2,2)} = A_- \cdot \begin{bmatrix} 0 \\ (2\mathtt{n}_2-1)_{\star+} \end{bmatrix} = \begin{bmatrix} 0 \\ (2\mathtt{n}_2)_{-+} \end{bmatrix} = - \begin{bmatrix} (2\mathtt{n}_1-1)_{-\star} \\ 0 \end{bmatrix} = -v_+^{(\mathtt{n}_1,1)}.$$

Summarized, we have obtained an isomorphism:

$$V \cong \check{V}/I \qquad \textit{where} \qquad \check{V} = V(\mathtt{n}_1)_{+\star} \oplus V(\mathtt{n}_2)_{-+},$$
$$\textit{and} \quad I = \langle \mathsf{A}_-\, v_\star^{(\mathtt{n}_2,2)} + v_+^{(\mathtt{n}_1,1)} \rangle_\Lambda.$$

That is, V *is given by a direct sum of cyclic representations modulo one relation with the gluing vector* $v_\star^{(\mathtt{n}_2,2)}$. *The quiver representation of* V *is viewed in Figure 5.4.1.*

5.4.2 Usual, special and bispecial strings with trivial multiplicity

Let ω be a usual string or special string of the Gelfand quiver, or let $\omega = \omega^1$ be a bispecial string with trivial multiplicity. We may assume that ω is given by some finite word of length $\mathtt{k}$:

$$\omega = (\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\ \mathtt{n}_1, \mathtt{n}_2,\ \ldots\ \mathtt{n}_{\mathtt{k}-1}, \mathtt{n}_\mathtt{k}\ {}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta)$$

We recall that the word ω may not be symmetric or quasi-symmetric as defined in A.2.9.

5.4.2.1 Generators of string representations

We define $\mathtt{k}$ cyclic representations $V_1, \ldots, V_\mathtt{k}$ as follows:

FIGURE 5.4.1. **Gluing two cyclic representations**

	$\mathtt{n}_1 \le \mathtt{n}_2$	$\mathtt{n}_1 = \mathtt{n}_2 = 1$
$V(\mathtt{n}_1)_{+\star}$	$\Bbbk^{\mathtt{n}_1-1} \overset{\mathsf{G}}{\underset{\mathsf{H}}{\rightleftarrows}} \Bbbk^{\mathtt{n}_1} \overset{\mathsf{J}}{\underset{\mathsf{Id}}{\leftrightarrows}} \Bbbk^{\mathtt{n}_1}$	$0 \rightleftarrows \Bbbk \overset{0}{\underset{1}{\leftrightarrows}} \Bbbk$
relations:	$v_\star^{(\mathtt{n}_2,2)} \xmapsto{\mathsf{A}_-} -v_-^{(\mathtt{n}_1,1)}$	$v_\star^{(1,2)} \xmapsto{\mathsf{A}_-} -v_-^{(1,1)}$
$V(\mathtt{n}_2)_{-+}$	$\Bbbk^{\mathtt{n}_2+1} \overset{\mathsf{H}}{\underset{\mathsf{G}}{\rightleftarrows}} \Bbbk^{\mathtt{n}_2} \overset{\mathsf{G}}{\underset{\mathsf{H}}{\leftrightarrows}} \Bbbk^{\mathtt{n}_2-1}$	$\Bbbk^2 \overset{[1\ 0]}{\underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\rightleftarrows}} \Bbbk \leftrightarrows 0$
V	$\Bbbk^{\mathtt{n}} \overset{\left[\begin{smallmatrix}\mathsf{G}&0\\0&\mathsf{H}\end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix}\mathsf{H}&0\\0&\mathsf{G}\end{smallmatrix}\right]}{\rightleftarrows}} \Bbbk^{\mathtt{n}} \overset{\left[\begin{smallmatrix}\mathsf{J}&0\\0&\mathsf{G}\end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix}\mathsf{Id}&0\\0&\mathsf{H}^*\end{smallmatrix}\right]}{\leftrightarrows}} \Bbbk^{\mathtt{n}-1}$ $\mathtt{n}=\mathtt{n}_1+\mathtt{n}_2$	$\Bbbk^2 \overset{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]}{\rightleftarrows}} \Bbbk^2 \overset{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}{\underset{[1\ -1]}{\leftrightarrows}} \Bbbk$

- Assume that $\mathtt{d}_\alpha = 0$. Then we set

$$
\begin{aligned}
V_1 &= V(\mathtt{n}_1)^{[1]}_{+\alpha},\\
V_{2\mathtt{j}} &= V(\mathtt{n}_{2\mathtt{j}})^{[1]}_{-+} \qquad \text{for any } 2 \le 2\mathtt{j} \le \mathtt{k}-1,\\
V_{2\mathtt{j}+1} &= V(\mathtt{n}_{2\mathtt{j}+1})^{[1]}_{+-} \qquad \text{for any } 3 \le 2\mathtt{j}+1 \le \mathtt{k}-1, \text{ and}\\
V_{\mathtt{k}} &= \begin{cases} V(\mathtt{n}_{\mathtt{k}})^{[1]}_{-\beta} & \text{if } \mathtt{k} \text{ is even (or, equivalently, } \mathtt{d}_\beta = 0),\\ V(\mathtt{n}_{\mathtt{k}})^{[\mathtt{d}_\beta]}_{\beta-} & \text{if } \mathtt{k} \text{ is odd (or, equivalently, } \mathtt{d}_\beta = 1 \text{ or } 2). \end{cases}
\end{aligned}
$$

- If $\mathtt{d}_\alpha = 1$ or 2, we set:

$$
\begin{aligned}
V_1 &= V(\mathtt{n}_1)^{[\mathtt{d}_\alpha,0]}_{\alpha+},\\
V_{2\mathtt{j}} &= V(\mathtt{n}_{2\mathtt{j}})^{[1]}_{+-} \qquad \text{for any } 2 \le 2\mathtt{j} \le \mathtt{k}-1,\\
V_{2\mathtt{j}+1} &= V(\mathtt{n}_{2\mathtt{j}+1})^{[1]}_{-+} \qquad \text{for any } 3 \le 2\mathtt{j}+1 \le \mathtt{k}-1, \text{ and}\\
V_{\mathtt{k}} &= \begin{cases} V(\mathtt{n}_{\mathtt{k}})^{[\mathtt{d}_\beta,0]}_{\beta-} & \text{if } \mathtt{k} \text{ is even (or, equivalently, } \mathtt{d}_\beta = 1 \text{ or } 2),\\ V(\mathtt{n}_{\mathtt{k}})^{[1,\mathtt{d}_\beta]}_{-\beta} & \text{if } \mathtt{k} \text{ is odd (or, equivalently, } \mathtt{d}_\beta = 0). \end{cases}
\end{aligned}
$$

In both cases we set $\check{V}(\omega) = \bigoplus_{\mathtt{j}=1}^{\mathtt{k}} V_{\mathtt{j}}$.
We need to fix the notation for basis vectors of $\check{V}(\omega)$:

Let $1 \le \mathtt{j} \le \mathtt{k}$, $i \in \{+,\star,-\}$ and $\mathtt{n} \le e_i V_{\mathtt{j}}$. We recall that the basis vectors of the cyclic representation $V_{\mathtt{j}}$ are enumerated according to Table 5.4.1.

For the indices above the vector $v_i^{(\mathtt{n},\mathtt{j})}$ denotes the $\mathtt{n}$-th basis vector of the vector space $e_i V_{\mathtt{j}}$ of the cyclic representation $V_{\mathtt{j}}$ in the direct sum $\check{V}$.

5.4.2.2 Relations of string representations

Let $\overset{\leftrightarrow}{\omega}$ be the gluing word of ω as defined in 4.2.1.

We need to introduce the following δ-symbols:

$$\delta_\alpha^\pm = \begin{cases} -1 & \text{if } \alpha = \diamond \text{ and } \varepsilon_1 = \pm \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta_\beta^\pm = \begin{cases} -1 & \text{if } \beta = \diamond \text{ and } \varepsilon_2 = \pm \\ 0 & \text{otherwise} \end{cases}$$

For any $\mathtt{1} \le \mathtt{j} \le \mathtt{k}$ we set

$$\delta_{\mathtt{j}}^\uparrow = \begin{cases} -1 & \text{if } \widehat{\mathtt{n}}_{\mathtt{j}} \uparrow \widehat{\mathtt{n}}_{\mathtt{j+1}} \text{ in } \overset{\leftrightarrow}{\omega} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta_{\mathtt{j}}^\downarrow = \begin{cases} -1 & \text{if } \widehat{\mathtt{n}}_{\mathtt{j}} \downarrow \widehat{\mathtt{n}}_{\mathtt{j+1}} \text{ in } \overset{\leftrightarrow}{\omega} \\ 0 & \text{otherwise} \end{cases}$$

Moreover let us define

$$\delta_{\mathtt{j},\mathtt{j+1}}^{\uparrow\uparrow} = \begin{cases} -1 & \text{if } \widehat{\mathtt{n}}_{\mathtt{j}} \uparrow \widehat{\mathtt{n}}_{\mathtt{j+1}} \uparrow \widehat{\mathtt{n}}_{\mathtt{j+2}} \text{ in } \overset{\leftrightarrow}{\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Beginning of a string representation

We consider first the beginning of the gluing word $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]}\, \widehat{\mathtt{n}}_1^{[\mathtt{d}_1]} \updownarrow \widehat{\mathtt{n}}_2^{[\mathtt{d}_2]} \ldots)$.
There are three cases to distinguish:

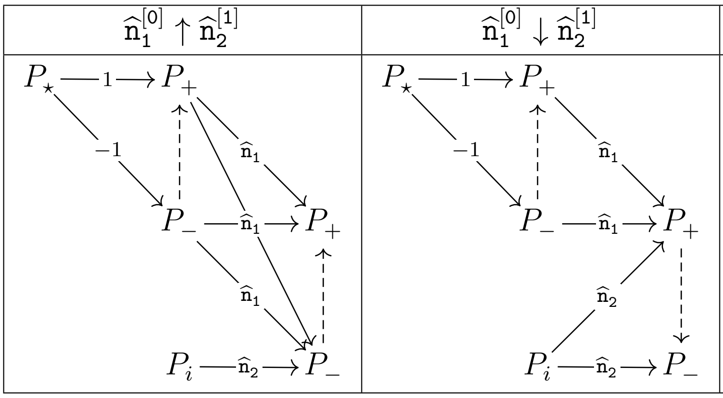
(1) Assume that $\mathtt{d}_\alpha = \mathtt{1}$. In this case, $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_\alpha^{[1]}\, \widehat{\mathtt{n}}_1^{[0]} \updownarrow \widehat{\mathtt{n}}_2^{[1]} \ldots)$.
There are two possible beginnings of the projective resolution $P_\bullet$:

$\widehat{\mathtt{n}}_1^{[0]} \uparrow \widehat{\mathtt{n}}_2^{[1]}$	$\widehat{\mathtt{n}}_1^{[0]} \downarrow \widehat{\mathtt{n}}_2^{[1]}$	d
$\begin{tikzcd} P_\alpha \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[dr, "\widehat{\mathtt{n}}_1"'] & P_+ \\ P_i \arrow[r, "\widehat{\mathtt{n}}_2"] & P_- \arrow[u, dashed] \end{tikzcd}$	$\begin{tikzcd} P_\alpha \arrow[r, "\widehat{\mathtt{n}}_1"] & P_+ \arrow[d, dashed] \\ P_i \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[ur, "\widehat{\mathtt{n}}_2"] & P_- \end{tikzcd}$	$\begin{array}{c c} & \begin{array}{cc} P_\alpha & P_i \end{array} \\ \begin{array}{c} P_+ \\ P_- \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_1 & \delta_1^\downarrow \widehat{\mathtt{n}}_2 \\ \delta_1^\uparrow \widehat{\mathtt{n}}_1 & \widehat{\mathtt{n}}_2 \end{bmatrix} \end{array}$

The relations of the gluing vectors in the first cyclic representation V_1 are given as follows:

- if $\alpha = \star$: $\qquad B_+ v_+^{(\mathtt{n}_1-1,1)} = B_- v_-^{(\mathtt{n}_1,1)} = \delta_1^\uparrow\, v_\star^{(\mathtt{n}_1,2)}$,
- if $\alpha = \diamond$: $\qquad A_{\varepsilon_1} v_\star^{(\mathtt{n}_1,1)} = \delta_1^\uparrow\, v_{\varepsilon_1}^{(\mathtt{n}_1+\delta_\alpha^+,2)}$.

(2) Assume that $\mathtt{d}_\alpha = 2$. In this case $\overset{\leftrightarrow}{\omega}= (\mathtt{p}_\star^{[2]}\, 1^{[1]} \uparrow \widehat{\mathtt{n}}_1^{[0]} \updownarrow \widehat{\mathtt{n}}_2^{[1]}, \ldots)$.



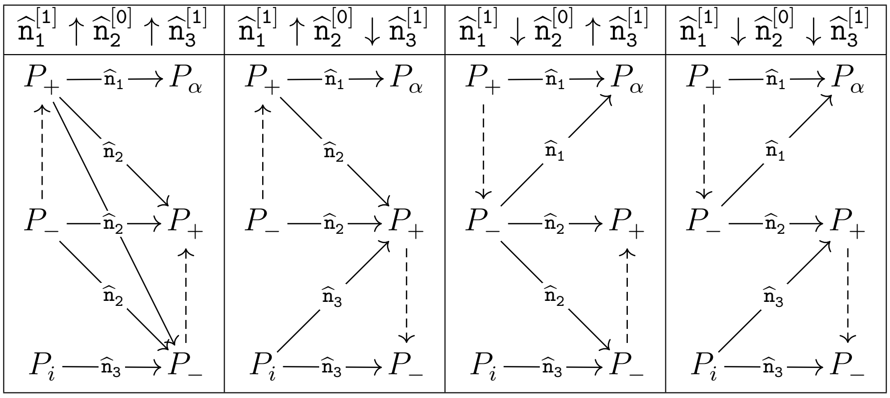
d

$$\begin{array}{c|ccc} & P_+ & P_- & P_i \\ \hline P_+ & \widehat{\mathtt{n}}_1 & \widehat{\mathtt{n}}_1 & \delta_1^{\downarrow}\widehat{\mathtt{n}}_2 \\ P_- & \delta_1^{\uparrow}\widehat{\mathtt{n}}_1 & \delta_1^{\uparrow}\widehat{\mathtt{n}}_1 & \widehat{\mathtt{n}}_2 \end{array}$$

In this case, the relations on the gluing vector in V_1 are given by

$$A_+ v_\star^{(\mathtt{n}_1,1)} = \delta_1^{\uparrow}\, v_+^{(\mathtt{n}_1,2)} \qquad \text{and} \qquad A_- v_\star^{(\mathtt{n}_1,1)} = \delta_1^{\uparrow}\, v_-^{(\mathtt{n}_1+1,2)}.$$

(3) Assume that $\mathtt{d}_\alpha = 0$. For the case that ω has length $\mathtt{k} = 2$, we refer to Example 5.4.3. Let $\mathtt{k} \geq 3$. In this case, we have $\overset{\leftrightarrow}{\omega}= (\mathtt{p}_\alpha^{[0]}, \widehat{\mathtt{n}}_1^{[1]} \updownarrow \widehat{\mathtt{n}}_2^{[0]} \updownarrow \widehat{\mathtt{n}}_3^{[1]} \ldots)$ and $P_\bullet$ begins as follows:



The corresponding part of the first differential is given as follows:

$$d = \begin{array}{c|ccc} & P_+ & P_- & P_i \\ \hline P_\alpha & \widehat{\mathtt{n}}_1 & \delta_1^{\downarrow}\widehat{\mathtt{n}}_1 & 0 \\ P_+ & \delta_1^{\uparrow}\widehat{\mathtt{n}}_2 & \widehat{\mathtt{n}}_2 & \delta_2^{\downarrow}\widehat{\mathtt{n}}_3 \\ P_- & \delta_{12}^{\uparrow\uparrow}\widehat{\mathtt{n}}_2 & \delta_2^{\uparrow}\widehat{\mathtt{n}}_2 & \widehat{\mathtt{n}}_2 \end{array}$$

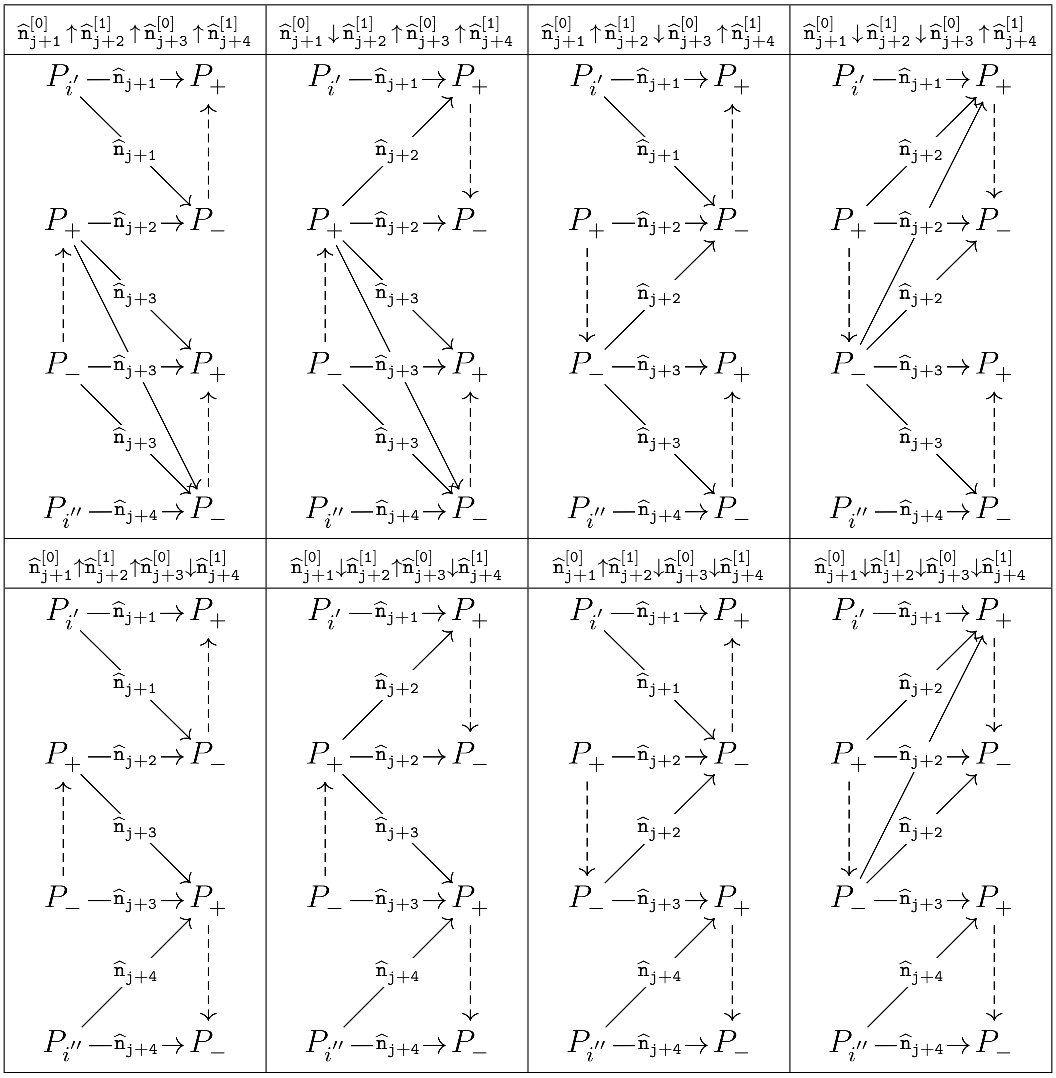
In particular, there are the following relations of the gluing vectors in V_1 and V_2:

$$\begin{aligned} A_+ v_\star^{(\mathtt{n}_2,2)} &= \delta_1^{\downarrow}\, v_+^{(\mathtt{n}_1+\delta_\alpha^+,1)} + \delta_2^{\uparrow}\, v_+^{(\mathtt{n}_2+1,3)}, \quad \text{and} \\ A_- v_\star^{(\mathtt{n}_1,1)} &= \delta_1^{\uparrow}\, v_-^{(\mathtt{n}_2+1,2)} + \delta_{12}^{\uparrow\uparrow}\, v_-^{(\mathtt{n}_2,3)}. \end{aligned}$$

Intermediate part of a string representation

For any $\mathtt{j} \in \{0, \ldots \mathtt{k}-4\}$ such that $\mathtt{d}_\mathtt{j} = 1$ we consider the subword $(\ldots \mathtt{n}_{\mathtt{j}+1}^{[0]} \updownarrow \mathtt{n}_{\mathtt{j}+2}^{[1]} \updownarrow \mathtt{n}_{\mathtt{j}+3}^{[0]} \updownarrow \mathtt{n}_{\mathtt{j}+4}^{[1]} \ldots)$ of $\overset{\leftrightarrow}{\omega}$. There are 8 possible corresponding gluing diagrams which are listed in Figure 5.4.2. In any of the 8 cases, the differential of

Figure 5.4.2. **Intermediate part of gluing diagrams**



$P_\bullet$ is given by

$$d = \begin{array}{c c} & \begin{array}{cccc} P_{i'} & P_+ & P_- & P_{i''} \end{array} \\ \begin{array}{c} P_+ \\ P_- \\ P_+ \\ P_- \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{j+1}} & \delta^{\downarrow}_{\mathtt{j+1}}\,\widehat{\mathtt{n}}_{\mathtt{j+2}} & \delta^{\downarrow\downarrow}_{\mathtt{j+1,j+2}}\,\widehat{\mathtt{n}}_{\mathtt{j+2}} & 0 \\ \delta^{\uparrow}_{\mathtt{j+1}}\,\widehat{\mathtt{n}}_{\mathtt{j+1}} & \widehat{\mathtt{n}}_{\mathtt{j+2}} & \delta^{\downarrow}_{\mathtt{j+2}}\,\widehat{\mathtt{n}}_{\mathtt{j+2}} & 0 \\ 0 & \delta^{\uparrow}_{\mathtt{j+2}}\,\widehat{\mathtt{n}}_{\mathtt{j+3}} & \widehat{\mathtt{n}}_{\mathtt{j+3}} & \delta^{\downarrow}_{\mathtt{j+3}}\,\widehat{\mathtt{n}}_{\mathtt{j+4}} \\ 0 & \delta^{\uparrow\uparrow}_{\mathtt{j+2,j+3}}\,\widehat{\mathtt{n}}_{\mathtt{j+3}} & \delta^{\uparrow}_{\mathtt{j+3}}\widehat{\mathtt{n}}_{\mathtt{j+3}} & \widehat{\mathtt{n}}_{\mathtt{j+4}} \end{bmatrix} \end{array}$$

We consider the two gluing vectors $v_\star^{(\mathtt{n_{j+2}},\mathtt{j+2})}$ and $v_\star^{(n_{j+3},j+3)}$ in V_{j+2} respectively V_{j+3}. Their relations are determined as follows:

$$A_+ v_\star^{(\mathtt{n_{j+2}},\mathtt{j+2})} = \delta^{\downarrow}_{\mathtt{j+1}} \cdot v_+^{(\mathtt{n_{j+2}+1},\mathtt{j+1})} + \delta^{\uparrow}_{\mathtt{j+2}} \cdot v_+^{(\mathtt{n_{j+3}+1},\mathtt{j+3})} + \delta^{\uparrow\uparrow}_{\mathtt{j+2,j+3}} \cdot v_+^{(\mathtt{n_{j+3}},\mathtt{j+4})}$$
$$A_- v_\star^{(\mathtt{n_{j+3}},\mathtt{j+3})} = \delta^{\downarrow\downarrow}_{\mathtt{j+1,j+2}} \cdot v_-^{(\mathtt{n_{j+2}},\mathtt{j+1})} + \delta^{\downarrow}_{\mathtt{j+2}} \cdot v_-^{(\mathtt{n_{j+2}+1},\mathtt{j+2})} + \delta^{\uparrow}_{\mathtt{j+3}} \cdot v_-^{(\mathtt{n_{j+3}+1},\mathtt{j+4})}.$$

This step is carried out for each $\mathtt{0} \le \mathtt{j} \le \mathtt{k} - \mathtt{4}$ such that $\mathtt{d_j} = \mathtt{1}$.

Ending of a string representation

The cases for the ending are "dual" to the beginning cases.

We consider the ending of the gluing word $\overset{\leftrightarrow}{\omega} = (\ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[\mathtt{d}_{\mathtt{k}-1}]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[\mathtt{d}_\beta]} \mathtt{p}_\beta)$.

(1) Assume that $\mathtt{d}_\beta = \mathtt{1}$. In this case $\overset{\leftrightarrow}{\omega} = (\ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]} \mathtt{p}_\beta)$.
There are two possible ends of the projective resolution $P_\bullet$:

$\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]}$	$\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]}$	d
$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] & P_+ \arrow[d, dashed] \\ P_\beta \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_- \end{tikzcd}$	$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] & P_+ \\ P_\beta \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_- \arrow[u, dashed] \end{tikzcd}$	$\begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{k}-1} & \delta^\downarrow_{\mathtt{k}-1}\widehat{\mathtt{n}}_{\mathtt{k}} \\ \delta^\uparrow_{\mathtt{k}-1}\widehat{\mathtt{n}}_{\mathtt{k}-1} & \widehat{\mathtt{n}}_{\mathtt{k}} \end{bmatrix}$

The relations on the gluing vectors in V_k are given as follows:

- if $\beta = \star$: $\qquad B_+ v_+^{(\mathtt{n}_\mathtt{k}-1,\mathtt{k})} = B_- v_-^{(\mathtt{n}_\mathtt{k},\mathtt{k})} = \delta^\downarrow_{\mathtt{k}-1}\, v_\star^{(\mathtt{n}_\mathtt{k},\mathtt{k}-1)}$,
- if $\beta = \diamond$: $\qquad A_\pm v_\star^{(\mathtt{n}_\mathtt{k},\mathtt{k})} = \delta^\downarrow_{\mathtt{k}-1}\, v_\pm^{(\mathtt{n}_\mathtt{k}+\delta^+_\beta,\mathtt{k}-1)}$.

(2) Assume that $\mathtt{d}_\beta = \mathtt{2}$. In this case $\overset{\leftrightarrow}{\omega} = (\ldots \widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]} \downarrow \mathtt{1}^{[2]} \mathtt{p}_\star)$.

$\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]}$	$\widehat{\mathtt{n}}_{\mathtt{k}-1}^{[0]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[1]}$	d
$\begin{tikzcd} & P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] & P_+ \arrow[d, dashed] \\ & P_+ \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] \arrow[d, dashed] & P_- \\ P_\star \arrow[ur, "-1"] \arrow[r, "1"] & P_- \arrow[uur] \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & \end{tikzcd}$	$\begin{tikzcd} & P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}-1}"] & P_+ \\ & P_+ \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] \arrow[d, dashed] & P_- \arrow[u, dashed] \\ P_\star \arrow[ur, "-1"] \arrow[r, "1"] & P_- \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & \end{tikzcd}$	$\begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{k}-1} & \delta^\downarrow_{\mathtt{k}-1}\,\widehat{\mathtt{n}}_{\mathtt{k}} & \delta^\downarrow_{\mathtt{k}-1}\,\widehat{\mathtt{n}}_{\mathtt{k}} \\ \delta^\uparrow_{\mathtt{k}-1}\,\widehat{\mathtt{n}}_{\mathtt{k}-1} & \widehat{\mathtt{n}}_{\mathtt{k}} & \widehat{\mathtt{n}}_{\mathtt{k}} \end{bmatrix}$

In this case, the relations on the last gluing vector are given by

$$A_- v_\star^{(\mathtt{n}_\mathtt{k},\mathtt{k})} = -\delta^\downarrow_{\mathtt{k}-1}\, v_-^{(\mathtt{n}_\mathtt{k},\mathtt{k}-1)} \qquad \text{and} \qquad A_+ v_\star^{(\mathtt{n}_\mathtt{k},\mathtt{k})} = -\delta^\downarrow_{\mathtt{k}-1}\, v_+^{(\mathtt{n}_\mathtt{k}+1,\mathtt{k}-1)}.$$

(3) Assume that $\mathtt{d}_\beta = \mathtt{0}$. We recall that $\mathtt{k} \geq \mathtt{3}$. In particular, $\overset{\leftrightarrow}{\omega} = (\dots \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[1]} \updownarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[0]} \mathtt{p}_\beta)$. and there 4 possible ends of the gluing diagram of $P_\bullet$:

$\widehat{\mathtt{n}}_{\mathtt{k-2}}^{[0]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[1]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[0]}$	$\widehat{\mathtt{n}}_{\mathtt{k-2}}^{[0]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[1]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[0]}$	$\widehat{\mathtt{n}}_{\mathtt{k-2}}^{[0]} \downarrow \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[1]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[0]}$	$\widehat{\mathtt{n}}_{\mathtt{k-2}}^{[0]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k-1}}^{[1]} \uparrow \widehat{\mathtt{n}}_{\mathtt{k}}^{[0]}$
$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] & P_+ \arrow[d, dashed] \\ P_+ \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[d, dashed] & P_- \\ P_- \arrow[uur] \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_\beta \end{tikzcd}$	$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] & P_+ \\ P_+ \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[d, dashed] & P_- \arrow[u, dashed] \\ P_- \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_\beta \end{tikzcd}$	$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] & P_+ \arrow[d, dashed] \\ P_+ \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_- \\ P_- \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_\beta \end{tikzcd}$	$\begin{tikzcd} P_i \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k-2}}"] & P_+ \\ P_+ \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_- \arrow[u, dashed] \\ P_- \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}"] & P_\beta \end{tikzcd}$

In any case the differential at the last projectives in the gluing diagram is given by

$$
d = \begin{array}{c c} & \begin{array}{ccc} P_i & P_+ & P_- \end{array} \\ \begin{array}{c} P_+ \\ P_- \\ P_\beta \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{k-2}} & \delta^{\downarrow}_{\mathtt{k-2}} \widehat{\mathtt{n}}_{\mathtt{k-1}} & \delta^{\downarrow\downarrow}_{\mathtt{k-2,k-1}} \widehat{\mathtt{n}}_{\mathtt{k-1}} \\ \delta^{\uparrow}_{\mathtt{k-2}} \widehat{\mathtt{n}}_{\mathtt{k-2}} & \widehat{\mathtt{n}}_{\mathtt{k-1}} & \delta^{\downarrow}_{\mathtt{k-1}} \widehat{\mathtt{n}}_{\mathtt{k-1}} \\ 0 & \delta^{\uparrow}_{\mathtt{k-1}} \widehat{\mathtt{n}}_{\mathtt{k}} & \widehat{\mathtt{n}}_{\mathtt{k}} \end{bmatrix} \end{array}
$$

This leads to the following relations on gluing vectors:

$$
\begin{aligned}
A_+ v_\star^{(\mathtt{n_{k-1}},\mathtt{k-1})} &= -\delta^{\downarrow}_{\mathtt{k-2}}\, v_+^{(\mathtt{n_{k-1}+1},\mathtt{k-2})} - \delta^{\uparrow}_{\mathtt{k-1}}\, v_+^{(\mathtt{n_k}+\delta^+_\beta,\mathtt{k})}, \quad \text{and} \\
A_- v_\star^{(\mathtt{n_k},\mathtt{k})} &= -\delta^{\downarrow}_{\mathtt{k-1}}\, v_-^{(\mathtt{n_{k-1}+1},\mathtt{k-1})} - \delta^{\downarrow\downarrow}_{\mathtt{k-2,k-1}}\, v_-^{(\mathtt{n_{k-1}},\mathtt{k-2})}.
\end{aligned}
$$

Example 5.4.3 (Short strings). *Let* $\omega = (\mathtt{p}_\alpha^{[0]}\mathtt{n}_1, \mathtt{n}_2^{[0]}\mathtt{p}_\beta)$ *be a usual or special string, or bispecial string with trivial multiplicity. There are two cases for the gluing diagram*:

$\widehat{\mathtt{n}}_1 \uparrow \widehat{\mathtt{n}}_2$	$\widehat{\mathtt{n}}_1 \downarrow \widehat{\mathtt{n}}_2$	d	$V(\omega) = V/I$
$\begin{tikzcd} P_+ \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[dr, "\widehat{\mathtt{n}}_2"] & P_\alpha \\ P_- \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2"] & P_\beta \end{tikzcd}$	$\begin{tikzcd} P_+ \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[d, dashed] & P_\alpha \\ P_- \arrow[ur, "\widehat{\mathtt{n}}_1"] \arrow[r, "\widehat{\mathtt{n}}_2"] & P_\beta \end{tikzcd}$	$\begin{bmatrix} \widehat{\mathtt{n}}_1 & \delta^{\downarrow}_1 \widehat{\mathtt{n}}_1 \\ \delta^{\uparrow}_1 \widehat{\mathtt{n}}_2 & \widehat{\mathtt{n}}_2 \end{bmatrix}$	$\check{V} = V(\mathtt{n}_1)^{[1]}_{+\alpha} \oplus V(\mathtt{n}_2)^{[1]}_{-\beta}$ *I is generated by the relations* $\delta^{\uparrow}_1\, v_+^{(\mathtt{n}_2+\delta^+_\beta,2)}$ *and* $A_- v_\star^{(\mathtt{n}_2,2)} = -\delta^{\downarrow}_1\, v_-^{(\mathtt{n}_1+\delta^-_\alpha,1)}$

5.4.3 Bands

Let $\Omega = \big(\omega^m, \lambda)$ be some band of the Gelfand quiver. In particular, ω is given by some cyclic word

$$
\omega = \left({}^{[\mathtt{d}]}\ \mathtt{n}_1, \mathtt{n}_2, \ \dots\ \mathtt{n}_{2\mathtt{k}-1}, \mathtt{n}_{2\mathtt{k}}\ {}^{[\mathtt{d}]} \right) \qquad \text{for some} \quad \mathtt{d} \in \{\mathtt{0}, \mathtt{1}\},
$$

some multiplicity $\mathtt{m} \in \mathbb{N}^+$ and some eigenvalue $\lambda \in \Bbbk \backslash \Delta$, where Δ is some finite set. We note that ω has to satisfy the conditions in Definition 4.1.7 of band words.

Let us recall that the band $(\omega^{\mathtt{m}}, \lambda)$ is equivalent to the shifted band $((\omega^{[1]})^{\mathtt{m}}, \lambda)$. In particular, we may assume without loss of generality that $\mathtt{d} = \mathtt{1}$.

5.4.3.1 Generators of band representations

For each $\mathtt{j} \in \mathbb{N}^{+}$ such that $\mathtt{1} \leq \mathtt{2j} - \mathtt{1} < \mathtt{2k}$ respectively $\mathtt{1} < \mathtt{2j} \leq \mathtt{2k}$ we define a cyclic representation by

$$V_{\mathtt{2j}-\mathtt{1}} = \begin{cases} V(\mathtt{n}_{\mathtt{2j}-\mathtt{1}})^{[\mathtt{1}]}_{+-} & \text{if } \mathtt{d} = \mathtt{0}, \\ V(\mathtt{n}_{\mathtt{2j}-\mathtt{1}})^{[\mathtt{1}]}_{-+} & \text{if } \mathtt{d} = \mathtt{1}, \end{cases} \quad \text{respectively} \quad V_{\mathtt{2j}} = \begin{cases} V(\mathtt{n}_{\mathtt{2j}})^{[\mathtt{1}]}_{-+} & \text{if } \mathtt{d} = \mathtt{0}, \\ V(\mathtt{n}_{\mathtt{2j}})^{[\mathtt{1}]}_{+-} & \text{if } \mathtt{d} = \mathtt{1}. \end{cases}$$

For any $\mathtt{j} \in \{1, \ldots, k\}$ and any $\mathtt{l} \in \{1, \ldots, m\}$ we set $V_{\mathtt{j},\mathtt{l}} = V_{\mathtt{j}}$.

Then $\check{V}$ is defined as the following direct sum of $\mathtt{2\,k\,m}$ cyclic representations:

$$\check{V}(\omega^{\mathtt{m}}, \lambda) := \bigoplus_{\mathtt{j}=\mathtt{1}}^{\mathtt{2k}} \bigoplus_{\mathtt{l}=\mathtt{1}}^{\mathtt{m}} V_{\mathtt{j},\mathtt{l}} = \bigoplus_{\mathtt{j}=\mathtt{1}}^{\mathtt{2k}} V_{\mathtt{j}}^{\oplus \mathtt{m}}$$

The basis vectors of $\check{V}(\omega^{\mathtt{m}}, \lambda)$ are indexed as follows:

For any $\mathtt{j} \in \{\mathtt{1}, \ldots, \mathtt{2k}\}$, $\mathtt{l} \in \{\mathtt{1}, \ldots, \mathtt{m}\}$, $i \in \{+, \star, -\}$ and $\mathtt{n} \leq \dim e_i V_j$, the vector $v_i^{(\mathtt{j},\mathtt{l},\mathtt{n})}$ denotes the $\mathtt{n}$-th basis vector of the vector space $e_i V_{\mathtt{j}}$ in the l-th copy $V_{\mathtt{j},\mathtt{l}}$ of the direct sum $V_{\mathtt{j}}^{\oplus \mathtt{m}}$ in $\check{V}$.

5.4.3.2 Relations of band representations

Intermediate part of a band representation

For any even number $\mathtt{j} \in \{\mathtt{0}, \ldots \mathtt{2k} - \mathtt{4}\}$ we look at the subword
$(\ldots \mathtt{n}^{[\mathtt{0}]}_{\mathtt{j}+\mathtt{1}} \updownarrow \mathtt{n}^{[\mathtt{1}]}_{\mathtt{j}+\mathtt{2}} \updownarrow \mathtt{n}^{[\mathtt{0}]}_{\mathtt{j}+\mathtt{3}} \updownarrow \mathtt{n}^{[\mathtt{1}]}_{\mathtt{j}+\mathtt{4}} \ldots)$ of $\overset{\leftrightarrow}{\omega}$

We consider the two gluing vectors $v_\star^{(n_{j+2}, j+2, l)}$ and $v_\star^{(n_{j+3}, j+3, l)}$ in V_{j+2} respectively V_{j+3}.

Relations:

$$\begin{aligned} A_+ v_\star^{(\mathtt{n}_{\mathtt{j}+\mathtt{2}},\mathtt{j}+\mathtt{2},\mathtt{l})} &= -\delta^{\downarrow}_{\mathtt{j}+\mathtt{1}} \cdot v_+^{(\mathtt{n}_{\mathtt{j}+\mathtt{2}}+\mathtt{1},\mathtt{j}+\mathtt{1},\mathtt{l})} - \delta^{\uparrow}_{\mathtt{j}+\mathtt{2}} \cdot v_+^{(\mathtt{n}_{\mathtt{j}+\mathtt{3}}+\mathtt{1},\mathtt{j}+\mathtt{3},\mathtt{l})} - \delta^{\uparrow\uparrow}_{\mathtt{j}+\mathtt{2},\mathtt{j}+\mathtt{3}} \cdot v_+^{(\mathtt{n}_{\mathtt{j}+\mathtt{3}},\mathtt{j}+\mathtt{4},\mathtt{l})} \\ A_- v_\star^{(\mathtt{n}_{\mathtt{j}+\mathtt{3}},\mathtt{j}+\mathtt{3},\mathtt{l})} &= -\delta^{\downarrow\downarrow}_{\mathtt{j}+\mathtt{1},\mathtt{j}+\mathtt{2}} \cdot v_-^{(\mathtt{n}_{\mathtt{j}+\mathtt{2}},\mathtt{j}+\mathtt{1},\mathtt{l})} - \delta^{\downarrow}_{\mathtt{j}+\mathtt{2}} \cdot v_-^{(\mathtt{n}_{\mathtt{j}+\mathtt{2}}+\mathtt{1},\mathtt{j}+\mathtt{2},\mathtt{l})} - \delta^{\uparrow}_{\mathtt{j}+\mathtt{3}} \cdot v_-^{(\mathtt{n}_{\mathtt{j}+\mathtt{3}}+\mathtt{1},\mathtt{j}+\mathtt{4},\mathtt{l})}. \end{aligned}$$

This step is carried out for each $\mathtt{0} \leq \mathtt{j} \leq \mathtt{k} - \mathtt{4}$ such that $\mathtt{d}_{\mathtt{j}} = \mathtt{1}$.

Gluing the end and the beginning of a band representation

It remains to consider the part $(\ldots \mathtt{n}^{[\mathtt{0}]}_{\mathtt{2k}-\mathtt{1}} \updownarrow \mathtt{n}^{[\mathtt{1}]}_{\mathtt{2k}} \updownarrow \mathtt{n}^{[\mathtt{0}]}_{\mathtt{1}} \updownarrow \mathtt{n}^{[\mathtt{1}]}_{\mathtt{2}} \ldots)$ of $\overset{\leftrightarrow}{\omega}$.

differential

$$d = \begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{2k}-\mathtt{1}} & \delta^{\downarrow}_{\mathtt{2k}-\mathtt{1}}\, \widehat{\mathtt{n}}_{\mathtt{2k}} & \delta^{\downarrow\downarrow}_{\mathtt{2k}-\mathtt{1},\mathtt{2k}}\, \lambda\, \widehat{\mathtt{n}}_{\mathtt{2k}} & 0 \\ \delta^{\uparrow}_{\mathtt{2k}-\mathtt{1}}\, \widehat{\mathtt{n}}_{\mathtt{2k}-\mathtt{1}} & \widehat{\mathtt{n}}_{\mathtt{2k}} & \delta^{\downarrow}_{\mathtt{2k}}\, \lambda\, \widehat{\mathtt{n}}_{\mathtt{2k}} & 0 \\ 0 & \delta^{\uparrow}_{\mathtt{2k}}\, \lambda^{-1}\, \widehat{\mathtt{n}}_{\mathtt{1}} & \widehat{\mathtt{n}}_{\mathtt{1}} & \delta^{\downarrow}_{\mathtt{1}}\, \widehat{\mathtt{n}}_{\mathtt{2}} \\ 0 & \delta^{\uparrow\uparrow}_{\mathtt{2k},\mathtt{1}}\, \lambda^{-1}\, \widehat{\mathtt{n}}_{\mathtt{1}} & \delta^{\uparrow}_{\mathtt{1}}\, \widehat{\mathtt{n}}_{\mathtt{1}} & \widehat{\mathtt{n}}_{\mathtt{2}} \end{bmatrix}$$

We consider the two gluing vectors $v_\star^{(n_{2k},2k,l)}$ and $v_\star^{(n_1,1,l)}$ in V_{2k} respectively V_1. Relations:

$$\begin{aligned}
A_+ v_\star^{(\mathtt{n}_{2k},2k,1)} &= -\,\delta^{\downarrow}_{2k-1} \cdot v_+^{(\mathtt{n}_{2k}+1,2k-1,1)} - \delta^{\uparrow}_{2k} \cdot \left(\lambda^{-1}\, v_+^{(\mathtt{n}_1+1,1,1)} + v_+^{(\mathtt{n}_1+1,1,1-1)}\right)\\
&\quad -\,\delta^{\uparrow\uparrow}_{2k,1} \cdot \left(\lambda^{-1}\, v_+^{(\mathtt{n}_1,2,1)} + v_+^{(\mathtt{n}_1,2,1-1)}\right)\\
A_- v_\star^{(\mathtt{n}_1,1,1)} &= -\,\delta^{\downarrow\downarrow}_{2k-1,2k} \cdot \left(\lambda\, v_-^{(\mathtt{n}_{2k},2k-1,1)} + v_-^{(\mathtt{n}_{2k},2k-1,1-1)}\right)\\
&\quad -\,\delta^{\downarrow}_{2k} \cdot \left(\lambda\, v_-^{(\mathtt{n}_{2k}+1,2k,1)} + v_-^{(\mathtt{n}_{2k}+1,2k,1-1)}\right) - \delta^{\uparrow}_1 \cdot v_-^{(\mathtt{n}_1+1,2,1)}.
\end{aligned}$$

Example 5.4.4 (Shortest band).
Let ω be the cyclic word $({}^{[1]}\mathtt{n}_1, \mathtt{n}_2^{[1]})$ where $\mathtt{n}_1 \leq \mathtt{n}_2$. We consider the band $(\omega^{(\mathtt{m})}, \lambda)$ with some multiplicity $\mathtt{m} \in \mathbb{N}^+$ and some eigenvalue $\lambda \in \Bbbk$ with $\lambda \neq 0$ respectively $\lambda \neq 0, 1$ if $\mathtt{n}_1 = \mathtt{n}_2$. Its gluing word is given by $\overset{\leftrightarrow}{\omega} = (\widehat{\mathtt{n}}_1^{[1]} \downarrow \widehat{\mathtt{n}}_2^{[0]} \downarrow \widehat{\mathtt{n}}_1^{[1]})$. In the following let us set $\overline{\lambda} := -\lambda$.

$$\begin{tikzcd}
\overset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_1"] & \widetilde{P}_\diamond \arrow[d, dashed] \\
\widetilde{P}_\diamond \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2"] & \widetilde{P}_\diamond
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{\lambda}{P_-^{\mathtt{m}}} \arrow[r, "\mathsf{J}_{\overline{\lambda}}\widehat{\mathtt{n}}_1"] \arrow[r, bend left, "\widehat{\mathtt{n}}_2"] \arrow[dr, "\widehat{\mathtt{n}}_2"] & P_+^{\mathtt{m}} \arrow[d, dashed] \\
P_+^{\mathtt{m}} \arrow[u, dashed] \arrow[ur, "\widehat{\mathtt{n}}_2"] \arrow[r, "\widehat{\mathtt{n}}_2"] & P_-^{\mathtt{m}}
\end{tikzcd}
\qquad
d = \begin{bmatrix} \mathsf{J}^*_{-+} & (\widehat{\mathtt{n}}_2)_{++} \\ (\widehat{\mathtt{n}}_2)_{--} & (\widehat{\mathtt{n}}_2)_{+-} \end{bmatrix}, \ \textit{where } \mathsf{J}^*_{-+} = \mathsf{J}_{\overline{\lambda}}\,(\widehat{\mathtt{n}}_1)_{-+} + \mathrm{Id}_{\mathtt{m}}\,(\widehat{\mathtt{n}}_2)_{-+}.$$

The Λ-module is given by

$$V(\omega^{(\mathtt{m})}, \lambda) \cong \check{V}/I \qquad \textit{where} \qquad \check{V} = (V(\mathtt{n}_1)^{[1]}_{-+})^{\mathtt{m}} \oplus (V(\mathtt{n}_2)^{[1]}_{+-})^{\mathtt{m}},$$

and I is generated by the relations

$$\begin{aligned}
A_-\, v_\star^{(\mathtt{n}_1,1,1)} &= -\lambda^{-1}\left(v_-^{(\mathtt{n}_2+1,2,1)} + v_-^{(\mathtt{n}_2+1,1,1)} + v_-^{(\mathtt{n}_1,1,1-1)}\right)\\
A_+\, v_\star^{(\mathtt{n}_2,2,1)} &= -v_+^{(\mathtt{n}_2+1,1,1)}
\end{aligned}$$

Remark 5.4.5. *Similarly to the derived setup (Remark 4.2.6), there is an exception concerning the symmetric bands of $\mathfrak{B}_0$. Let $\Omega = (\omega, m, \lambda)$ be some band of $\mathfrak{B}$. Let $\overset{\leftrightarrow}{\omega}$ be the word with gluing arrows. Then Ω is equivalent to $\Omega' = (\omega^{\mathrm{rev}}, m, \lambda^{-1})$. In the construction of the representation $V(\Omega')$ all orientations of the arrows in the word $\overset{\leftrightarrow}{\omega}^{\mathrm{rev}}$ have to be opposite to the corresponding arrows in the word $\overset{\leftrightarrow}{\omega}$. This convention ensures that there is an isomorphism of glued representations $V(\omega, m, \lambda) \cong V(\omega^{\mathrm{rev}}, m, \lambda^{-1})$ for symmetric bands.*

5.4.4 Bispecial strings with higher multiplicity

Let $\omega^{(\mathtt{m})}$ be a bispecial string with some multiplicity $m \in \mathbb{N}^{\geq 2}$.

Let α denote the left and β denote the right end of the bispecial word ω. Then ω is some bispecial word

$$\omega = \left(\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]},\ \mathtt{n}_1, \mathtt{n}_2,\ \ldots\ \mathtt{n}_{k-1}, \mathtt{n}_k,\ {}^{[\mathtt{d}_\beta]}\mathtt{p}_\beta\right) \qquad \text{where} \qquad \alpha, \beta \in \{+, -\}.$$

Let $\overline{\alpha}$ and $\overline{\beta}$ denote the opposite signs of α respectively β.

5.4.4.1 Generators of bispecial string representations

We define cyclic representations $V_{\mathtt{j},\mathtt{l}}$, where $\mathtt{1} \leq \mathtt{j} \leq \mathtt{k}$ and $\mathtt{1} \leq \mathtt{l} \leq \mathtt{m}$ as follows:

(1) For any $\mathtt{l} \in \mathbb{N}^+$ such that $\mathtt{1} \leq \mathtt{2l} - \mathtt{1} \leq \mathtt{m}$ respectively $\mathtt{1} < \mathtt{2l} \leq \mathtt{m}$ we set:

$$V_{\mathtt{1},\mathtt{2l}-\mathtt{1}} = \begin{cases} V(\mathtt{n}_\mathtt{1})^{[\mathtt{1}]}_{\alpha+}) & \text{if } \mathtt{d}_\alpha = \mathtt{1}, \\ V(\mathtt{n}_\mathtt{1})^{[\mathtt{1}]}_{+\alpha}) & \text{if } \mathtt{d}_\alpha = \mathtt{0}, \end{cases} \quad \text{respectively} \quad V_{\mathtt{1},\mathtt{2l}} = \begin{cases} V(\mathtt{n}_\mathtt{1})^{[\mathtt{1}]}_{\overline{\alpha}+}) & \text{if } \mathtt{d}_\alpha = \mathtt{1}, \\ V(\mathtt{n}_\mathtt{1})^{[\mathtt{1}]}_{+\overline{\alpha}}) & \text{if } \mathtt{d}_\alpha = \mathtt{0}, \end{cases}$$

(2) For any $\mathtt{l} \in \{\mathtt{1}, \dots, \mathtt{m}\}$ and any $\mathtt{j} \in \mathbb{N}^+$ such that $\mathtt{1} \leq \mathtt{2j} - \mathtt{1} \leq \mathtt{k}$ respectively $\mathtt{1} < \mathtt{2j} \leq \mathtt{k}$ we set:

$$V_{\mathtt{2j},\mathtt{l}} = \begin{cases} V(\mathtt{n}_{\mathtt{2j}})^{[\mathtt{1}]}_{-+} & \text{if } \mathtt{d}_\alpha = \mathtt{0}, \\ V(\mathtt{n}_{\mathtt{2j}})^{[\mathtt{1}]}_{+-} & \text{if } \mathtt{d}_\alpha = \mathtt{1}, \end{cases} \quad \text{respectively} \quad V_{\mathtt{2j}+\mathtt{1},\mathtt{l}} = \begin{cases} V(\mathtt{n}_{\mathtt{2j}+\mathtt{1}})^{[\mathtt{1}]}_{+-} & \text{if } \mathtt{d}_\alpha = \mathtt{0}, \\ V(\mathtt{n}_{\mathtt{2j}+\mathtt{1}})^{[\mathtt{1}]}_{-+} & \text{if } \mathtt{d}_\alpha = \mathtt{1}, \end{cases}$$

(3) At last for any $\mathtt{l} \in \mathbb{N}^+$ such that $\mathtt{1} \leq \mathtt{2l} - \mathtt{1} \leq \mathtt{m}$ respectively $\mathtt{1} < \mathtt{2l} \leq \mathtt{m}$ we set:

$$V_{\mathtt{k},\mathtt{2l}-\mathtt{1}} = \begin{cases} V(\mathtt{n}_\mathtt{k})^{[\mathtt{1}]}_{\beta-}) & \text{if } \mathtt{d}_\beta = \mathtt{1}, \\ V(\mathtt{n}_\mathtt{k})^{[\mathtt{1}]}_{-\beta}) & \text{if } \mathtt{d}_\beta = \mathtt{0}, \end{cases} \quad \text{respectively} \quad V_{\mathtt{k},\mathtt{2l}} = \begin{cases} V(\mathtt{n}_\mathtt{k})^{[\mathtt{1}]}_{\overline{\beta}+} & \text{if } \mathtt{d}_\beta = \mathtt{1}, \\ V(\mathtt{n}_\mathtt{k})^{[\mathtt{1}]}_{+\overline{\beta}} & \text{if } \mathtt{d}_\beta = \mathtt{0}. \end{cases}$$

Finally, we define $\check{V}(\omega^{(\mathtt{m})})$ to be the direct sum

$$\check{V}(\omega^{(\mathtt{m})}) := \bigoplus_{\mathtt{j}=\mathtt{1}}^{\mathtt{k}} \bigoplus_{\mathtt{l}=\mathtt{1}}^{\mathtt{m}} V_{\mathtt{j},\mathtt{l}} = \bigoplus_{\mathtt{l}=\mathtt{1}}^{\mathtt{m}} V_{\mathtt{1},\mathtt{l}} \oplus \bigoplus_{\mathtt{j}=\mathtt{2}}^{\mathtt{k}-\mathtt{1}} V_{\mathtt{j}}^{\oplus \mathtt{m}} \oplus \bigoplus_{\mathtt{l}=\mathtt{1}}^{\mathtt{m}} V_{\mathtt{k},\mathtt{l}}$$

As in the case of bands, the basis vectors of $\check{V}(\omega^{(\mathtt{m})})$ will be denoted by

$$v_i^{(\mathtt{j},\mathtt{l},\mathtt{n})}, \qquad \text{where} \quad \mathtt{1} \leq \mathtt{j} \leq \mathtt{n}, \quad \mathtt{1} \leq \mathtt{l} \leq \mathtt{m}, \quad i \in \{+, \star, -\} \text{ and } \mathtt{n} \leq \dim e_i V_{\mathtt{j},\mathtt{l}}.$$

5.4.4.2 Relations of bispecial string representations

We use the following notation:

- Let $\overline{\alpha}$ and $\overline{\beta}$ denote the opposite signs of α respectively β.
- Let δ_{ij} denote the square matrix of size m with 1 at entry (i, j) and zero elsewhere.

Let A_m and B_m denote the following square matrices of size m :

$$\mathrm{A}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i,2i+1} = \begin{bmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \quad \mathrm{B}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i-1,2i} = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}$$

$$P_\alpha^* = \bigoplus_{i=1}^{m} P_i, \qquad \text{where } P_i = \begin{cases} P_\alpha & \text{if } i \text{ is odd,} \\ P_{\overline{\alpha}} & \text{if } i \text{ is even.} \end{cases} \tag{5.4.1}$$

$$P_\beta^* = \bigoplus_{i=1}^{m} P_i, \qquad \text{where } P_i = \begin{cases} P_\beta & \text{if } i \text{ is odd,} \\ P_{\overline{\beta}} & \text{if } i \text{ is even.} \end{cases} \tag{5.4.2}$$

(1) Twisting the beginning.

- If $\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]} = \mathtt{p}_\pm^{[1]}$, then

$\widehat{\mathtt{n}}_1^{[0]} \uparrow \widehat{\mathtt{n}}_2^{[1]}$	$\widehat{\mathtt{n}}_1^{[0]} \downarrow \widehat{\mathtt{n}}_2^{[1]}$	d
$\begin{tikzcd} P^*_\alpha \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}"] \arrow[dr, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}"] & P^m_+ \\ P^m_i \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_- \arrow[u, dashed] \end{tikzcd}$	$\begin{tikzcd} P^*_\alpha \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}"] & P^m_+ \arrow[d, dashed] \\ P^m_i \arrow[ur, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_- \end{tikzcd}$	$\begin{array}{c c} & \begin{array}{cc} P^*_\alpha & P^m_i \end{array} \\ \begin{array}{c} P^m_+ \\ P^m_- \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_1\, \mathrm{A}_m & \delta_1^\downarrow\, \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} \\ \delta_1^\uparrow\, \widehat{\mathtt{n}}_1\, \mathrm{A}_m & \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} \end{bmatrix} \end{array}$

Relations:

$$
\begin{aligned}
A_\pm v_\star^{(\mathtt{n}_1,1,1)} &= -\delta_1^\uparrow\, v_\pm^{(\mathtt{n}_1+\delta_\alpha^-,2,1)} \\
A_\mp v_\star^{(\mathtt{n}_1,1,2\mathtt{i})} &= -\delta_1^\uparrow\, v_\mp^{(\mathtt{n}_1+\delta_\alpha^-,2,2\mathtt{i})} \\
A_\pm v_\star^{(\mathtt{n}_1,1,2\mathtt{i}-1)} &= -v_\pm^{(\mathtt{n}_1+\delta_\alpha^+,1,2\mathtt{i}-2)} - \delta_1^\uparrow\, (v_\pm^{(\mathtt{n}_1+\delta_\alpha^-,2,2\mathtt{i}-1)} - v_\pm^{(\mathtt{n}_1+\delta_\alpha^-,2,2\mathtt{i}-2)})
\end{aligned}
$$

- If $\mathtt{p}_\alpha^{[\mathtt{d}_\alpha]} = \mathtt{p}_\pm^{[0]}$, then

$\widehat{\mathtt{n}}_1^{[1]} \uparrow \widehat{\mathtt{n}}_2^{[0]} \uparrow \widehat{\mathtt{n}}_3^{[1]}$	$\widehat{\mathtt{n}}_1^{[1]} \uparrow \widehat{\mathtt{n}}_2^{[0]} \downarrow \widehat{\mathtt{n}}_3^{[1]}$	$\widehat{\mathtt{n}}_1^{[1]} \downarrow \widehat{\mathtt{n}}_2^{[0]} \uparrow \widehat{\mathtt{n}}_3^{[1]}$	$\widehat{\mathtt{n}}_1^{[1]} \downarrow \widehat{\mathtt{n}}_2^{[0]} \downarrow \widehat{\mathtt{n}}_3^{[1]}$
$\begin{tikzcd} P^m_+ \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[dr, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] \arrow[ddr] & P^*_\alpha \\ P^m_- \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] \arrow[dr, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_+ \\ P^m_i \arrow[r, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] & P^m_- \arrow[u, dashed] \end{tikzcd}$	$\begin{tikzcd} P^m_+ \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[dr, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^*_\alpha \\ P^m_- \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_+ \arrow[d, dashed] \\ P^m_i \arrow[ur, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] \arrow[r, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] & P^m_- \end{tikzcd}$	$\begin{tikzcd} P^m_+ \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[d, dashed] & P^*_\alpha \\ P^m_- \arrow[ur, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] \arrow[dr, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_+ \\ P^m_i \arrow[r, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] & P^m_- \arrow[u, dashed] \end{tikzcd}$	$\begin{tikzcd} P^m_+ \arrow[r, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[d, dashed] & P^*_\alpha \\ P^m_- \arrow[ur, "\widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1}"] \arrow[r, "\widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m}"] & P^m_+ \arrow[d, dashed] \\ P^m_i \arrow[ur, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] \arrow[r, "\widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m}"] & P^m_- \end{tikzcd}$

$$
d = \begin{array}{c c} & \begin{array}{ccc} P^m_+ & P^m_- & P^m_i \end{array} \\ \begin{array}{c} P^*_\alpha \\ P^m_+ \\ P^m_- \end{array} & \begin{bmatrix} \widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1} & \delta_1^\downarrow\, \widehat{\mathtt{n}}_1\ \mathrm{A}_\mathtt{m}^{-1} & 0 \\ \delta_1^\uparrow\, \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} & \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} & \delta_2^\downarrow\, \widehat{\mathtt{n}}_3\ \mathrm{Id}_\mathtt{m} \\ \delta_{12}^{\uparrow\uparrow}\, \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} & \delta_2^\uparrow\, \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} & \widehat{\mathtt{n}}_2\ \mathrm{Id}_\mathtt{m} \end{bmatrix} \end{array}
$$

Relations:

$$
\begin{aligned}
A_+ v_\star^{(\mathtt{n}_1,1,2\mathtt{i}-1)} &= -v_+^{(\mathtt{n}_1+\delta_\alpha^-,1,2\mathtt{i}-2)} - \delta_1^\uparrow\, v_+^{(\mathtt{n}_2+1,2,2\mathtt{i}-1)} - \delta_{1,2}^{\uparrow\uparrow}\, v_+^{(\mathtt{n}_2,3,2\mathtt{i}-1)}, \\
A_+ v_\star^{(\mathtt{n}_1,1,2\mathtt{i})} &= -\delta_1^\uparrow\, v_+^{(\mathtt{n}_2+1,2,2\mathtt{i})} - \delta_{1,2}^{\uparrow\uparrow}\, v_+^{(\mathtt{n}_3,3,2\mathtt{i})}, \\
A_- v_\star^{(\mathtt{n}_2,2,2\mathtt{i}-1)} &= -\delta_1^\downarrow\, v_-^{(\mathtt{n}_1+\delta_\alpha^-,1,2\mathtt{i}-1)} - \delta_1^\downarrow\, v_-^{(\mathtt{n}_1+\delta_\alpha^+,1,2\mathtt{i}-2)} - \delta_2^\uparrow\, v_-^{(\mathtt{n}_2+1,3,2\mathtt{i}-1)} \\
A_- v_\star^{(\mathtt{n}_2,2,2\mathtt{i})} &= -\delta_1^\downarrow\, v_-^{(\mathtt{n}_1+\delta_\alpha^+,1,2\mathtt{i})} - \delta_2^\uparrow\, v_-^{(\mathtt{n}_2+1,3,2\mathtt{i})}
\end{aligned}
$$

(2) Intermediate part:
This step is the same as for bands.

(3) Twisting the end:

- If $\mathtt{p}^{[1]}_{\beta} = \mathtt{p}^{[1]}_{\pm}$, then

$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-1}} \uparrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k}}$	$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-1}} \downarrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k}}$	d
$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \arrow[d, dashed] \\ P_\beta^* \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}"] & P_-^m \end{tikzcd}$	$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \\ P_\beta^* \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}"] & P_-^m \arrow[u, dashed] \end{tikzcd}$	$\begin{bmatrix} \widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} & \delta^{\downarrow}_{\mathtt{k-1}}\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}} \\ \delta^{\uparrow}_{\mathtt{k-1}}\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} & \widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}} \end{bmatrix}$

$$
\begin{aligned}
A_{\mp} v_{\star}^{(\mathtt{n_k},\mathtt{k},\mathtt{2i})} &= -v_{\mp}^{(\mathtt{n_k}+\delta^+_\beta,\mathtt{k},\mathtt{2i-1})} - \delta^{\downarrow}_{\mathtt{k-1}}\,(v_{\mp}^{(\mathtt{n_k}+\delta^-_\beta,\mathtt{k-1},\mathtt{2i})} + v_{\mp}^{(\mathtt{n_k}+\delta^-_\beta,\mathtt{k-1},\mathtt{2i-1})}) \\
A_{\pm} v_{\star}^{(\mathtt{n_k},\mathtt{k},\mathtt{2i-1})} &= -\delta^{\downarrow}_{\mathtt{k-1}}\, v_{\pm}^{(\mathtt{n_k}+\delta^+_\beta,\mathtt{k-1},\mathtt{2i-1})}
\end{aligned}
$$

- If $\mathtt{p}^{[0]}_{\beta} = \mathtt{p}^{[0]}_{\pm}$, then

$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-2}} \downarrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k-1}} \downarrow \widehat{\mathtt{n}}^{[0]}_{\mathtt{k}}$	$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-2}} \uparrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k-1}} \downarrow \widehat{\mathtt{n}}^{[0]}_{\mathtt{k}}$	$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-2}} \downarrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k-1}} \uparrow \widehat{\mathtt{n}}^{[0]}_{\mathtt{k}}$	$\widehat{\mathtt{n}}^{[0]}_{\mathtt{k-2}} \uparrow \widehat{\mathtt{n}}^{[1]}_{\mathtt{k-1}} \uparrow \widehat{\mathtt{n}}^{[0]}_{\mathtt{k}}$
$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \arrow[d, dashed] \\ P_+^m \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[d, dashed] & P_-^m \\ P_-^m \arrow[uur] \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_\beta^* \end{tikzcd}$	$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \\ P_+^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[d, dashed] & P_-^m \arrow[u, dashed] \\ P_-^m \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_\beta^* \end{tikzcd}$	$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \arrow[d, dashed] \\ P_+^m \arrow[ur, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_-^m \\ P_-^m \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_\beta^* \end{tikzcd}$	$\begin{tikzcd} P_i^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}}"] & P_+^m \\ P_+^m \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}}"] \arrow[dr, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_-^m \arrow[u, dashed] \\ P_-^m \arrow[u, dashed] \arrow[r, "\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}"] & P_\beta^* \end{tikzcd}$

$$
\begin{array}{cc}
 & \begin{array}{ccc} P_i & P_+ & P_- \end{array} \\
\begin{array}{c} P_+ \\ P_- \\ P_\beta \end{array} &
\begin{bmatrix}
\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}} & \delta^{\downarrow}_{\mathtt{k-2}}\,\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} & \delta^{\downarrow\downarrow}_{\mathtt{k-2},\mathtt{k-1}}\,\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} \\
\delta^{\uparrow}_{\mathtt{k-2}}\,\widehat{\mathtt{n}}_{\mathtt{k-2}}\ \mathrm{Id}_{\mathtt{m}} & \widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} & \delta^{\downarrow}_{\mathtt{k-1}}\,\widehat{\mathtt{n}}_{\mathtt{k-1}}\ \mathrm{Id}_{\mathtt{m}} \\
0 & \delta^{\uparrow}_{\mathtt{k-1}}\,\widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1} & \widehat{\mathtt{n}}_{\mathtt{k}}\ \mathrm{B}_{\mathtt{m}}^{-1}
\end{bmatrix}
\end{array}
$$

$$
\begin{aligned}
A_{-} v_{\star}^{(\mathtt{n_k},\mathtt{k},\mathtt{2i-1})} &= -\delta^{\downarrow}_{\mathtt{k-1}}\, v_{-}^{(\mathtt{n_{k-1}}+1,\mathtt{k-1},\mathtt{2i-1})} - \delta^{\downarrow\downarrow}_{\mathtt{k-2},\mathtt{k-1}}\, v_{-}^{(\mathtt{n_{k-1}},\mathtt{k-2},\mathtt{2i-1})}, \\
A_{-} v_{\star}^{(\mathtt{n_k},\mathtt{k},\mathtt{2i})} &= -v_{-}^{(\mathtt{n_k}+\delta^-_\beta,\mathtt{k},\mathtt{2i-1})} - \delta^{\downarrow}_{\mathtt{k-1}}\, v_{-}^{(\mathtt{n_{k-1}}+1,\mathtt{k-1},\mathtt{2i})} - \delta^{\downarrow\downarrow}_{\mathtt{k-2},\mathtt{k-1}}\, v_{-}^{(\mathtt{n_{k-1}},\mathtt{k-2},\mathtt{2i})}
\end{aligned}
$$

and for the gluing vectors in $V_{k-1}^{\oplus m}$

$$
\begin{aligned}
A_{+} v_{\star}^{(\mathtt{n_{k-1}},\mathtt{k-1},\mathtt{2i-1})} &= -\delta^{\uparrow}_{\mathtt{k-1}}\, v_{+}^{(\mathtt{n_k}+\delta^+_\beta,\mathtt{k},\mathtt{2i-1})} - \delta^{\downarrow}_{\mathtt{k-2}}\, v_{+}^{(\mathtt{n_{k-1}}+1,\mathtt{k-2},\mathtt{2i-1})} \\
A_{+} v_{\star}^{(\mathtt{n_{k-1}},\mathtt{k-1},\mathtt{2i})} &= -\delta^{\downarrow}_{\mathtt{k-2}}\, v_{+}^{(\mathtt{n_{k-1}}+1,\mathtt{k-2},\mathtt{2i})} - \delta^{\uparrow}_{\mathtt{k-1}}\,(v_{+}^{(\mathtt{n_k}+\delta^-_\beta,\mathtt{k},\mathtt{2i})} - v_{+}^{(\mathtt{n_k}+\delta^+_\beta,\mathtt{k},\mathtt{2i-1})})
\end{aligned}
$$

Example 5.4.6 (Shortest bispecial string)**.** *Let* $\omega = \mathtt{p}^{[1]}_{\alpha}\mathtt{n}^{[0]}\mathtt{p}_{\beta}$ *with* $\alpha, \beta \in \{\pm\}$, $\mathtt{d} \in \mathbb{Z}$ *and* $n \in \mathbb{N}^+$. *Let* $m \in \mathbb{N}^+$ *and* $\Omega = \omega^m$. *Let* P^*_α *and* P^*_β *be defined as in*

(5.4.1) *and* (5.4.2). *Then* $P_\bullet(\Omega) = \underset{\mathtt{1}}{P^*_\alpha} \xrightarrow{\mathtt{C_m}} \underset{\mathtt{0}}{P^*_\beta}$ *where*

$$\mathtt{C_m} = \widehat{\mathtt{n}}_1 \ \mathrm{A_m} \ \mathrm{B_m^{-1}} = \begin{bmatrix} (\widehat{\mathtt{n}})_{\alpha\beta} & -(\widehat{\mathtt{n}})_{\overline{\alpha}\beta} & -(\widehat{\mathtt{n}})_{\alpha\beta} & 0 & 0 & \dots \\ 0 & (\widehat{\mathtt{n}})_{\overline{\alpha}\overline{\beta}} & (\widehat{\mathtt{n}})_{\alpha\overline{\beta}} & 0 & 0 & \\ 0 & 0 & (\widehat{\mathtt{n}})_{\alpha\beta} & -(\widehat{\mathtt{n}})_{\overline{\alpha}\beta} & -(\widehat{\mathtt{n}})_{\alpha\beta} & \\ 0 & 0 & 0 & (\widehat{\mathtt{n}})_{\overline{\alpha}\overline{\beta}} & (\widehat{\mathtt{n}})_{\alpha\overline{\beta}} & \\ 0 & 0 & 0 & 0 & (\widehat{\mathtt{n}})_{\alpha\beta} & \\ \vdots & & & & & \ddots \end{bmatrix}$$

$$\delta = \begin{cases} 0 & \textit{if } \alpha \neq \beta \\ 1 & \textit{if } \alpha = \beta \end{cases} \qquad \textit{and} \qquad \overline{\delta} = \begin{cases} 1 & \textit{if } \alpha \neq \beta \\ 0 & \textit{if } \alpha = \beta \end{cases}$$

Set

$$V_{2i-1} = V(\mathtt{n})^{[1]}_{\alpha\beta}, \quad \textit{where } 1 \leq 2i-1 \leq m,$$
$$V_{2i} = V(\mathtt{n})^{[1]}_{\overline{\alpha}\overline{\beta}}, \quad \textit{where } 1 \leq 2i \leq m.$$

Finally, we set $V = \left(\bigoplus_{i=1}^m V_i\right)/I$, *where* I *is generated by the following relations*:

$$A_\alpha v_\star^{(\mathtt{n},2\mathtt{i}-1)} = v_\alpha^{(\mathtt{n}+\delta,2\mathtt{i}-1)} - v_\alpha^{(\mathtt{n}+\overline{\delta},2\mathtt{i}-2)}, \quad \textit{where } 1 \leq 2i-1 \leq m \textit{ and } v_\alpha^{(\mathtt{n}+\overline{\delta},0)} := 0,$$
$$A_{\overline{\alpha}} v_\star^{(\mathtt{n},2\mathtt{i})} = v_{\overline{\alpha}}^{(\mathtt{n}+\overline{\delta},2\mathtt{i}-1)}, \quad \textit{where } 2 \leq 2i \leq m.$$

5.4.5 Main result

Theorem 5.4.7. *Let* V *be an indecomposable nilpotent representation of the Gelfand quiver. Then there is some string or band datum* Ω *such that* V *is isomorphic to the string or band representation* $V(\Omega)$ *constructed above.*

Proof. Let V be an indecomposable object from $\mathrm{nil.\,rep}(Q, I)$. Then there is some string or band Ω of $\mathfrak{B}_0$ such that the indecomposable projective resolution $P_\bullet(\Omega)$ has homology $\mathbf{H}_0(P_\bullet(\Omega)) \cong V$. The generators and relations of V follow by computation of the homology $\mathbf{H}_0(P_\bullet)$. $\square$

5.4.6 Examples of string and band representations

Example 5.4.8 (A usual string). *Let* $\omega = \left(\mathtt{p}_\star^{[0]}\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3^{[1]}\mathtt{p}_\star\right)$, *where* $\mathtt{n}_1 \leq \mathtt{n}_2 \geq \mathtt{n}_3$. *Its gluing word is given by* $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_\star^{[0]}\widehat{\mathtt{n}}_1^{[1]} \downarrow \widehat{\mathtt{n}}_2^{[0]} \downarrow \widehat{\mathtt{n}}_3^{[1]}\mathtt{p}_\star\right)$. *The gluing diagram and the projective resolution have the following form:*

gluing diagram

$$\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[d, dashed] & \widetilde{P}_\star \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_2"] & \widetilde{P}_\diamond \arrow[d, dashed] \\
\widetilde{P}_\star \arrow[r, "\widehat{\mathtt{n}}_3"] & \widetilde{P}_\diamond
\end{tikzcd}$$

$P_\bullet(\omega)$

$$\begin{tikzcd}
P_+ \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[d, dashed] & P_\star \\
P_- \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[ur, "\widehat{\mathtt{n}}_1"] & P_+ \arrow[d, dashed] \\
P_\star \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[ur, "\widehat{\mathtt{n}}_3"] & P_-
\end{tikzcd}$$

$$V(\omega) \cong \check{V}/I$$

$$\check{V} = V(2)_{+\star}^{[1]} \oplus V(2)_{-+}^{[1]} \oplus V(2)_{\star -}^{[1]},$$

I is generated by the relations

$$A_-\, v_\star^{(\mathtt{n}_1,1)} = 0,$$
$$A_+\, v_\star^{(\mathtt{n}_2,2)} = -v_+^{(\mathtt{n}_1,1)}, \quad \text{and}$$
$$B_+\, v_+^{(\mathtt{n}_3-1,3)} = B_-\, v_-^{(\mathtt{n}_3,3)} = -v_\star^{(\mathtt{n}_3,2)}.$$

The quiver representation $V(\omega)$ is depicted in Figure 5.4.3.

Example 5.4.9 (A special string). *Let* $\omega = (\mathtt{p}_\star^{[2]}\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3, \mathtt{n}_4^{[1]}\mathtt{p}_-)$, *where* $\mathtt{n}_1 \leq \mathtt{n}_2 > \mathtt{n}_3 \leq \mathtt{n}_4$. *Then* $\overset{\leftrightarrow}{\omega} = (\mathtt{p}_\star^{[2]}\mathtt{1}^{[1]} \uparrow \mathtt{n}_1^{[0]} \uparrow \mathtt{n}_2^{[1]} \uparrow \mathtt{n}_3^{[0]} \uparrow \mathtt{n}_4^{[1]}\mathtt{p}_-)$. *The gluing diagram and the projective resolution have the form:*

gluing diagram

$$\begin{tikzcd}
\widetilde{P}_\star \arrow[r, "\mathtt{1}"] & \widetilde{P}_\diamond & \\
 & \widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[u, dashed] & \widetilde{P}_\diamond \\
 & \widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_2"] & \widetilde{P}_\diamond \arrow[u, dashed] \\
 & \widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[u, dashed] & \widetilde{P}_\diamond \\
 & \underset{-}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_4"] & \widetilde{P}_\diamond \arrow[u, dashed]
\end{tikzcd}$$

$P_\bullet(\omega)$

$$\begin{tikzcd}
P_\star \arrow[r, "\mathtt{1}"] \arrow[dr, "-\mathtt{1}"] & P_+ \arrow[dr, "\widehat{\mathtt{n}}_1"] \arrow[ddr] & \\
 & P_- \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[u, dashed] \arrow[dr, "\widehat{\mathtt{n}}_1"] & P_+ \\
 & P_+ \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[dr, "\widehat{\mathtt{n}}_3"] \arrow[ddr] & P_- \arrow[u, dashed] \\
 & P_- \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[u, dashed] \arrow[dr, "\widehat{\mathtt{n}}_3"] & P_+ \\
 & P_- \arrow[r, "\widehat{\mathtt{n}}_4"] & P_- \arrow[u, dashed]
\end{tikzcd}$$

differential

$$\begin{bmatrix}
(\widehat{n}_1)_{++} & (\widehat{n}_1)_{-+} & 0 & 0 & 0 \\
(\widehat{n}_1)_{+-} & (\widehat{n}_1)_{--} & (\widehat{n}_2)_{+-} & 0 & 0 \\
0 & 0 & (\widehat{n}_3)_{++} & (\widehat{n}_3)_{-+} & 0 \\
0 & 0 & (\widehat{n}_3)_{+-} & (\widehat{n}_3)_{--} & (\widehat{n}_4)_{--}
\end{bmatrix}$$

$$V(\omega) \cong \check{V}/I$$

$$\check{V} = V(\mathtt{n}_1)_{\star +}^{[2]} \oplus V(\mathtt{n}_2)_{+-}^{[1]} \oplus V(\mathtt{n}_3)_{-+}^{[1]} \oplus V(\mathtt{n}_4)_{--}^{[1]},$$

I is generated by the relations

$$A_+ v_\star^{(\mathtt{n}_1,1)} = -v_+^{(\mathtt{n}_1,2)},$$
$$A_- v_\star^{(\mathtt{n}_1,1)} = -v_-^{(\mathtt{n}_1+1,2)},$$
$$A_+\, v_\star^{(\mathtt{n}_2,2)} = -v_+^{(\mathtt{n}_3+1,3)} - v_+^{(\mathtt{n}_3,4)},$$
$$A_-\, v_\star^{(\mathtt{n}_3,3)} = -v_-^{(\mathtt{n}_3+1,4)}, \quad \text{and}$$
$$A_- v_\star^{(\mathtt{n}_4,4)} = 0.$$

The quiver representation is viewed in Figure 5.4.4.

Example 5.4.10 (A bispecial string with trivial multiplicity).
Let $\omega = (\mathtt{p}_+^{[1]}\mathtt{n}_1, \mathtt{n}_2, \mathtt{n}_3, \mathtt{n}_4, \mathtt{n}_5, \mathtt{n}_6^{[1]}\mathtt{p}_-)$, *where* $\mathtt{n}_4 \geq \mathtt{n}_1 > \mathtt{n}_2 \geq \mathtt{n}_3 < \mathtt{n}_5 \geq \mathtt{n}_5 > \mathtt{n}_6 > \mathtt{n}_2$. *Then* $\overset{\leftrightarrow}{\omega} = \left(\mathtt{p}_+{}^{[1]}\widehat{\mathtt{n}}_1^{[0]} \downarrow \widehat{\mathtt{n}}_2^{[1]} \uparrow \widehat{\mathtt{n}}_3^{[0]} \uparrow \widehat{\mathtt{n}}_4^{[1]} \uparrow \widehat{\mathtt{n}}_5^{[0]} \downarrow \widehat{\mathtt{n}}_6^{[1]}\mathtt{p}_-\right)$. *The gluing diagram and the*

projective resolution of ω *are given as follows*:

gluing diagram

$$
\begin{tikzcd}
\overset{+}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_1"] & \widetilde{P}_\diamond \arrow[d, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_2"] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[u, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_4"] & \widetilde{P}_\diamond \arrow[u, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_5"] \arrow[u, dashed] & \widetilde{P}_\diamond \arrow[d, dashed] \\
\underset{-}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_6"] & \widetilde{P}_\diamond
\end{tikzcd}
$$

$P_\bullet(\omega)$

$$
\begin{tikzcd}
P_+ \arrow[r, "\widehat{\mathtt{n}}_1"] & P_+ \arrow[d, dashed] \\
P_+ \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[ur, "\widehat{\mathtt{n}}_2"] \arrow[dr, "\widehat{\mathtt{n}}_3"] \arrow[ddr] & P_- \\
P_- \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[u, dashed] \arrow[dr, "\widehat{\mathtt{n}}_3"] & P_+ \\
P_+ \arrow[r, "\widehat{\mathtt{n}}_4"] \arrow[dr, "\widehat{\mathtt{n}}_5"] & P_- \arrow[u, dashed] \\
P_- \arrow[r, "\widehat{\mathtt{n}}_5"] \arrow[u, dashed] & P_+ \arrow[d, dashed] \\
P_- \arrow[r, "\widehat{\mathtt{n}}_6"] \arrow[ur, "\widehat{\mathtt{n}}_6"] & P_-
\end{tikzcd}
$$

differential

$$
\begin{bmatrix}
(\widehat{\mathtt{n}}_1)_{++} & (\widehat{\mathtt{n}}_2)_{++} & 0 & 0 & 0 & 0 \\
0 & (\widehat{\mathtt{n}}_2)_{+-} & 0 & 0 & 0 & 0 \\
0 & (\widehat{\mathtt{n}}_3)_{++} & (\widehat{\mathtt{n}}_3)_{-+} & 0 & 0 & 0 \\
0 & (\widehat{\mathtt{n}}_3)_{+-} & (\widehat{\mathtt{n}}_3)_{--} & (\widehat{\mathtt{n}}_4)_{+-} & 0 & 0 \\
0 & 0 & 0 & (\widehat{\mathtt{n}}_5)_{++} & (\widehat{\mathtt{n}}_5)_{-+} & (\widehat{\mathtt{n}}_6)_{-+} \\
0 & 0 & 0 & 0 & 0 & (\widehat{\mathtt{n}}_6)_{--}
\end{bmatrix}
$$

$V(\omega) = V/I$

$$
\check{V} = V(\mathtt{n}_1)^{[2]}_{\star+} \oplus V(\mathtt{n}_2)^{[1]}_{+-} \oplus V(\mathtt{n}_3)^{[1]}_{-+} \oplus V(\mathtt{n}_4)^{[1]}_{--},
$$

I *is generated by the relations*

$$
\begin{aligned}
A_+\, v_\star^{(\mathtt{n}_2,2)} &= -v_+^{(\mathtt{n}_2+1,1)} - v_+^{(\mathtt{n}_3+1,3)} - v_+^{(\mathtt{n}_3,4)}, \\
A_-\, v_\star^{(\mathtt{n}_3,3)} &= -v_-^{(\mathtt{n}_3+1,4)}, \\
A_+\, v_\star^{(\mathtt{n}_4,4)} &= -v_+^{(\mathtt{n}_5+1,5)}, \qquad \textit{and} \\
A_-\, v_\star^{(\mathtt{n}_6,6)} &= -v_-^{(\mathtt{n}_6,5)}.
\end{aligned}
$$

The quiver representation is shown in Figure 5.4.5.

Example 5.4.11 (A typical Band). *Let* ω *be the cyclic word* $\left({}^{[1]}\mathtt{n}_1^{[0]}\mathtt{n}_2^{[1]}\mathtt{n}_3^{[0]}\mathtt{n}_4^{[1]}\right)$, *where*
$\mathtt{n}_1 > \mathtt{n}_2 < \mathtt{n}_3 < \mathtt{n}_4 > \mathtt{n}_1$. *We consider the band* $(\omega^{(\mathtt{m})}, \lambda)$ *for some multiplicity* $\mathtt{m} \in \mathbb{N}^+$ *and some eigenvalue* $\lambda \in \Bbbk$ *with* $\lambda \neq 0$. *Then* $\overset{\leftrightarrow}{\omega} = \left(\widehat{\mathtt{n}}_1^{[0]} \downarrow \widehat{\mathtt{n}}_2^{[1]} \downarrow \widehat{\mathtt{n}}_3^{[0]} \uparrow \widehat{\mathtt{n}}_4^{[1]} \uparrow\right)$. *The gluing diagram and projective resolution have the following form*:

gluing diagram

$$
\begin{tikzcd}
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[ddd, dashed, bend right] & \widetilde{P}_\diamond \arrow[d, dashed] \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[d, dashed] & \widetilde{P}_\diamond \\
\widetilde{P}_\diamond \arrow[r, "\widehat{\mathtt{n}}_3"] & \widetilde{P}_\diamond \\
\underset{\lambda}{\widetilde{P}_\diamond} \arrow[r, "\widehat{\mathtt{n}}_4"] & \widetilde{P}_\diamond \arrow[u, dashed]
\end{tikzcd}
$$

$P_\bullet$

$$
\begin{tikzcd}
P_-^{\mathtt{m}} \arrow[r, "\widehat{\mathtt{n}}_1"] \arrow[ddd, dashed, bend right] & P_+^{\mathtt{m}} \arrow[d, dashed] \\
P_+^{\mathtt{m}} \arrow[r, "\widehat{\mathtt{n}}_2"] \arrow[ur, "\widehat{\mathtt{n}}_2"] \arrow[d, dashed] & P_-^{\mathtt{m}} \\
P_-^{\mathtt{m}} \arrow[r, "\widehat{\mathtt{n}}_3"] \arrow[uur] \arrow[ur, "\widehat{\mathtt{n}}_2"] \arrow[dr, "\widehat{\mathtt{n}}_3"] & P_+^{\mathtt{m}} \\
P_+^{\mathtt{m}} \arrow[r, "\widehat{\mathtt{n}}_4"] \arrow[uuur, "\mathsf{J}_\lambda\widehat{\mathtt{n}}_1"] & P_-^{\mathtt{m}} \arrow[u, dashed]
\end{tikzcd}
$$

differential

$$
\begin{bmatrix}
(\widehat{\mathtt{n}}_1)_{-+}\,\mathrm{Id}_{\mathtt{m}} & (\widehat{\mathtt{n}}_2)_{++}\,\mathrm{Id}_{\mathtt{m}} & (\widehat{\mathtt{n}}_2)_{-+}\,\mathrm{Id}_{\mathtt{m}} & \mathsf{J}_\lambda\,(\widehat{\mathtt{n}}_1)_{++} \\
0 & (\widehat{\mathtt{n}}_2)_{+-}\,\mathrm{Id}_{\mathtt{m}} & (\widehat{\mathtt{n}}_2)_{--}\,\mathrm{Id}_{\mathtt{m}} & 0 \\
0 & 0 & (\widehat{\mathtt{n}}_3)_{-+}\,\mathrm{Id}_{\mathtt{m}} & 0 \\
0 & 0 & (\widehat{\mathtt{n}}_3)_{--}\,\mathrm{Id}_{\mathtt{m}} & (\widehat{\mathtt{n}}_4)_{+-}\,\mathrm{Id}_{\mathtt{m}}
\end{bmatrix}
$$

$V(\omega) = \check{V}/I$

$$
\check{V} = V(\mathtt{n}_1)^{[1]}_{-+} \oplus V(\mathtt{n}_2)^{[1]}_{+-} \oplus V(\mathtt{n}_3)^{[1]}_{-+} \oplus V(\mathtt{n}_4)^{[1]}_{+-},
$$

I *is generated by the relations*

$$
\begin{aligned}
A_+ v_\star^{(\mathtt{n}_2,2,1)} &= -v_+^{(\mathtt{n}_2+1,1,1)} \\
A_- v_\star^{(\mathtt{n}_3,3,1)} &= -v_-^{(\mathtt{n}_2,1,1)} - v_-^{(\mathtt{n}_2+1,2,1)} - v_-^{(\mathtt{n}_3+1,4,1)} \\
A_+ v_\star^{(\mathtt{n}_4,4,1)} &= -\lambda \cdot v_+^{(\mathtt{n}_1+1,1,1)} - v_+^{(\mathtt{n}_1+1,1,1-1)} \\
A_- v_\star^{(\mathtt{n}_1,1,1)} &= 0
\end{aligned}
$$

The quiver representation is shown in Figure 5.4.6.

FIGURE 5.4.3. **A usual string representation**

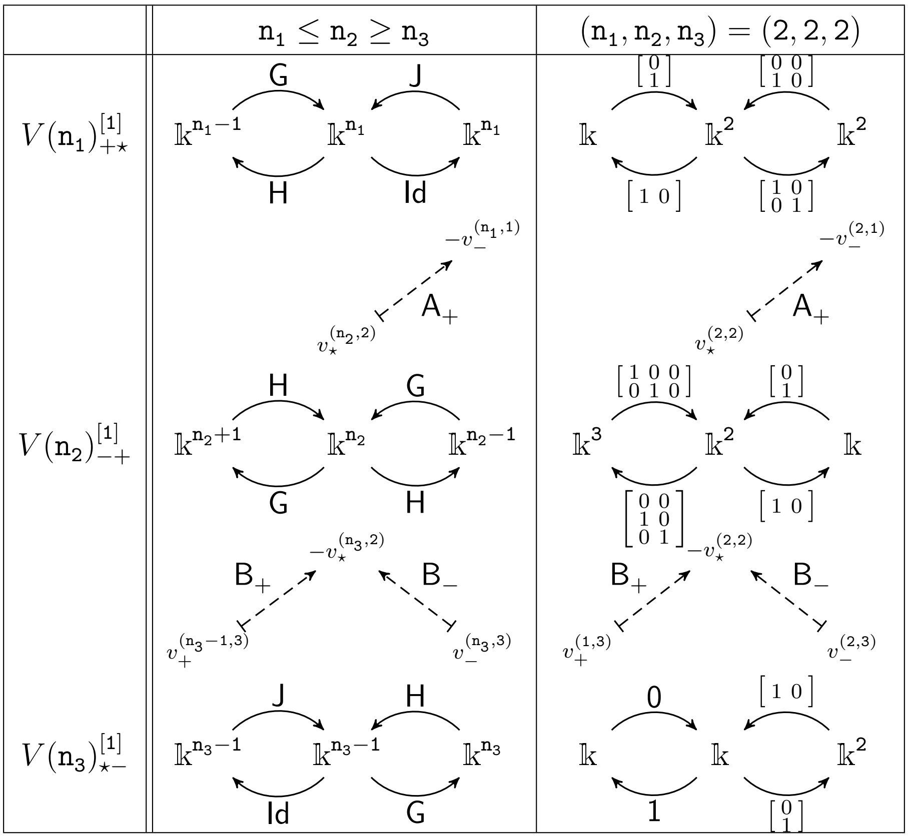


Figure 5.4.4. **A special string representation**

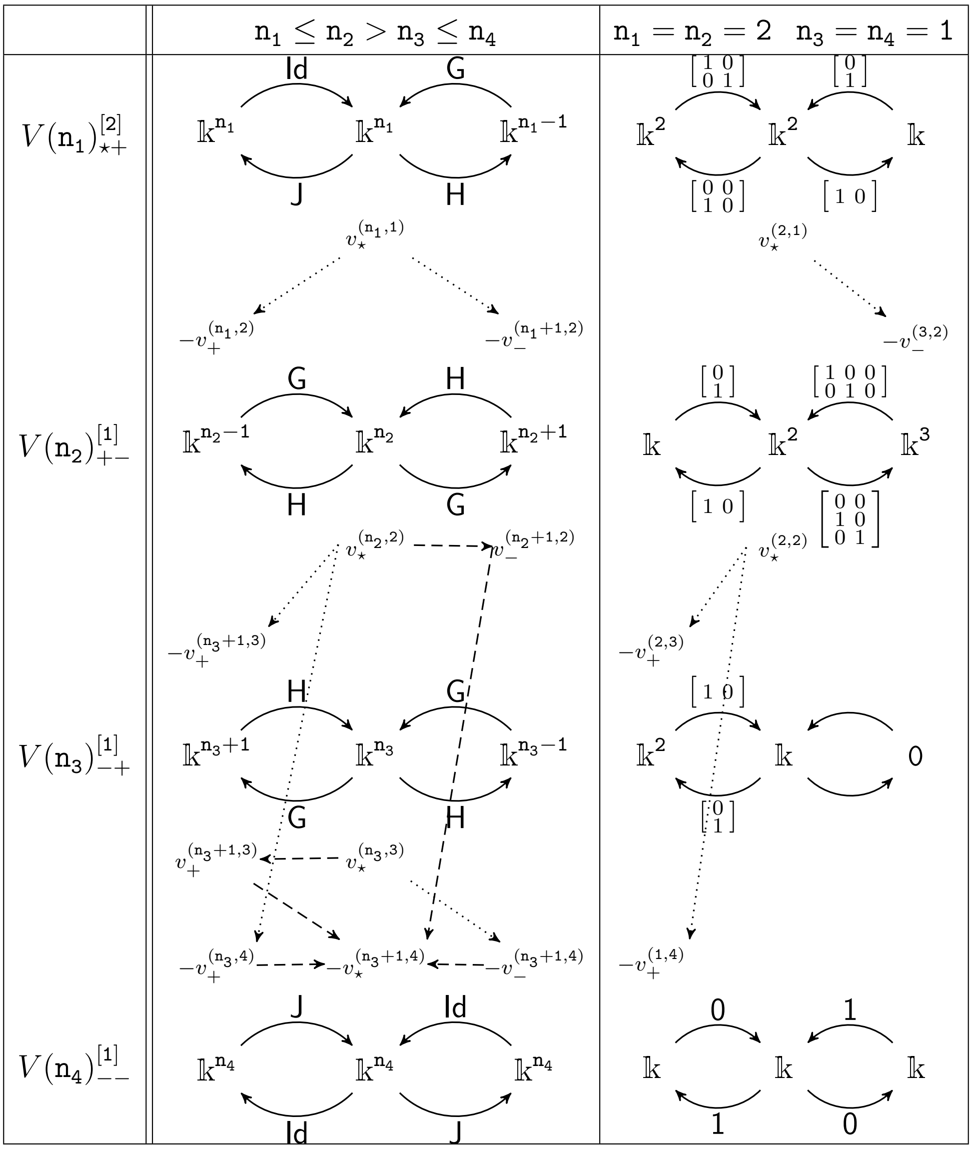


FIGURE 5.4.5. **A bispecial string representation with trivial multiplicity**

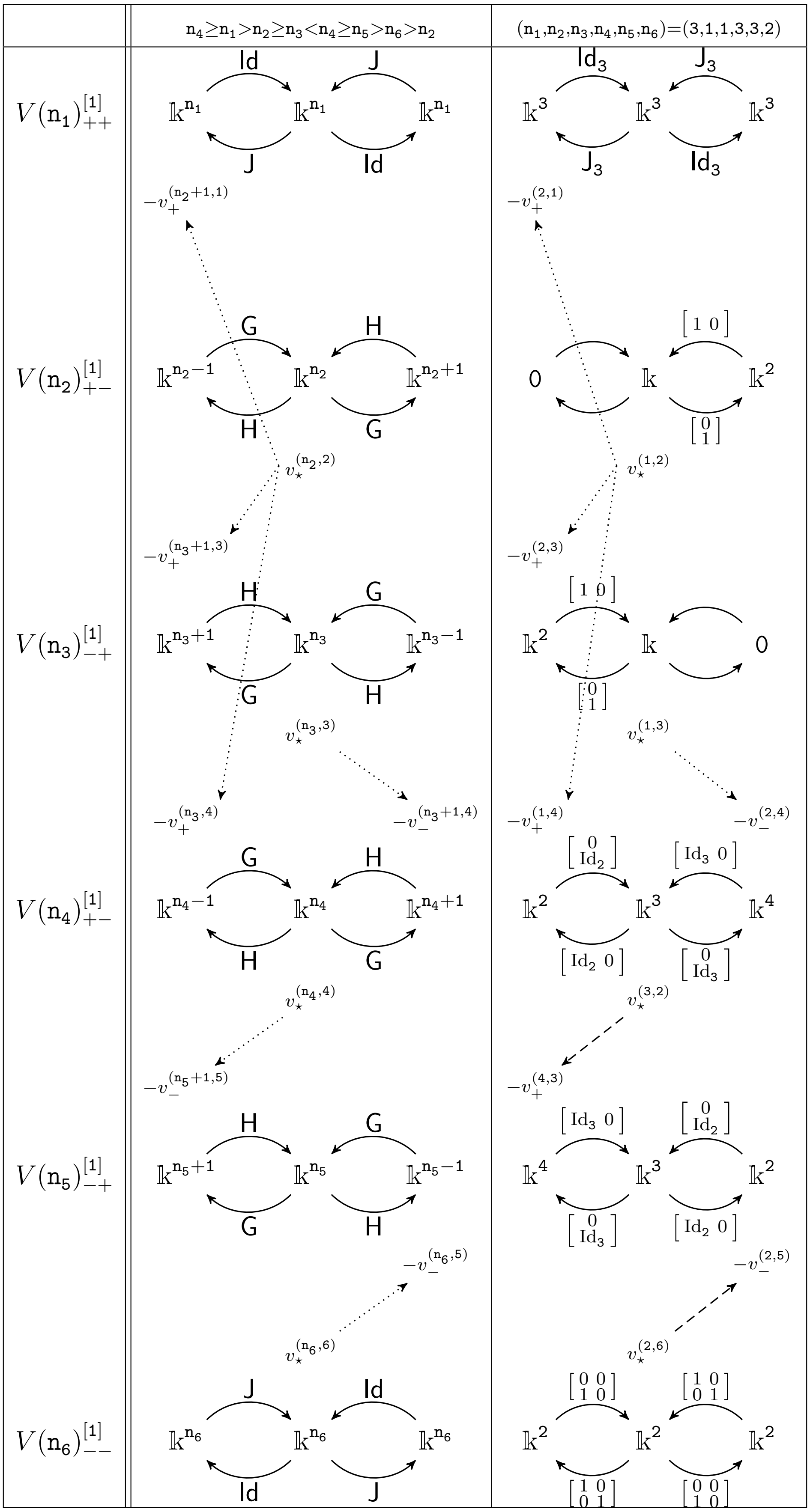
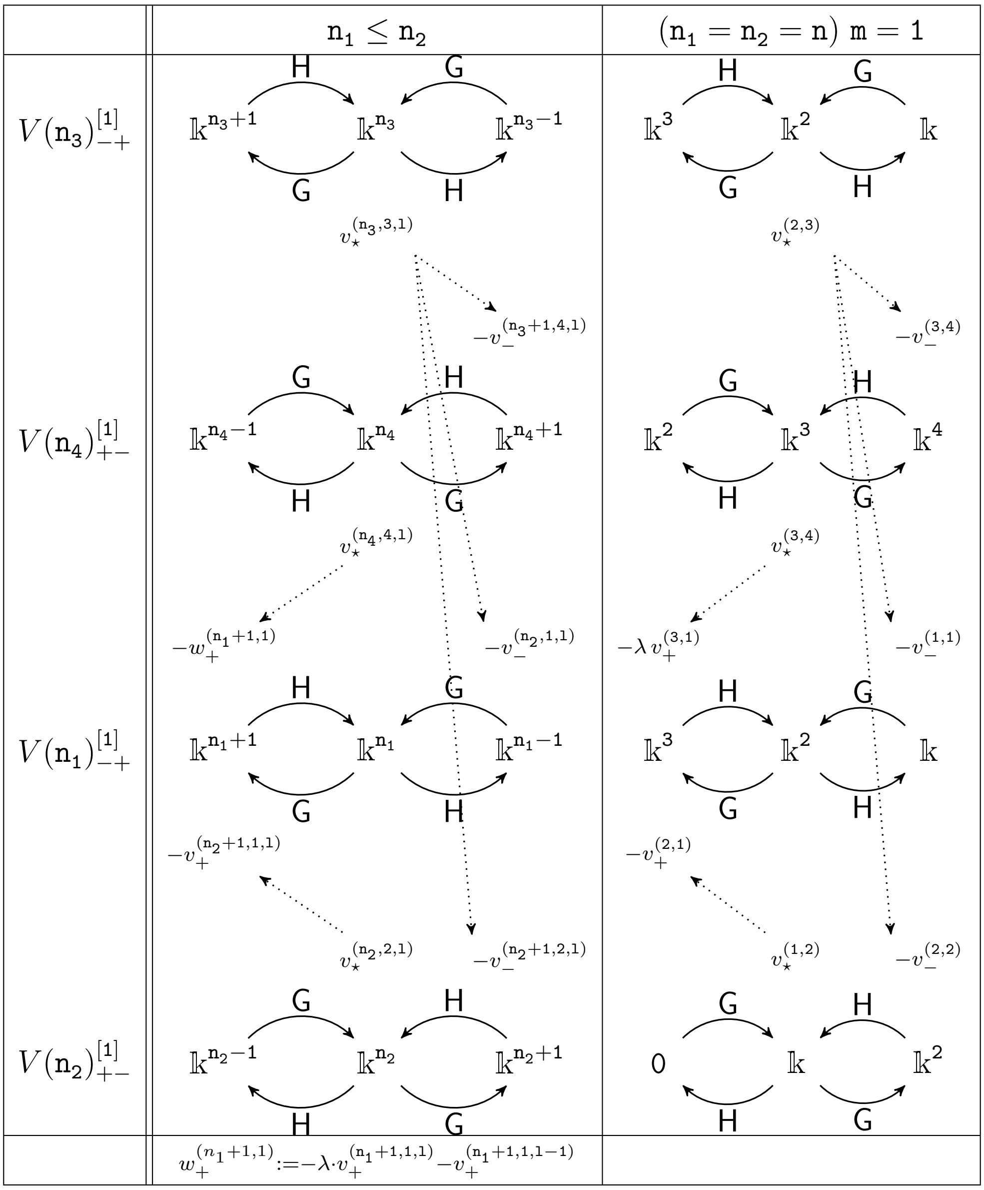


Figure 5.4.6. **A band representation**



Appendices

APPENDIX A

Combinatorics of bunches of semichains

In this appendix we recall first the definition of bunches of semichains and their representations. The problem to classify the indecomposable representations of a bunch of semichains is a tame matrix problem. This problem has been solved by Bondarenko in [**Bon88**]. The indecomposable representations of a bunch of semichains are parametrized by so-called *strings* and *bands*. At last, we present Bondarenkos construction of the canonical forms for bunches of semichains from strings or bands.

A.1 Matrix problems of bunches of semichains

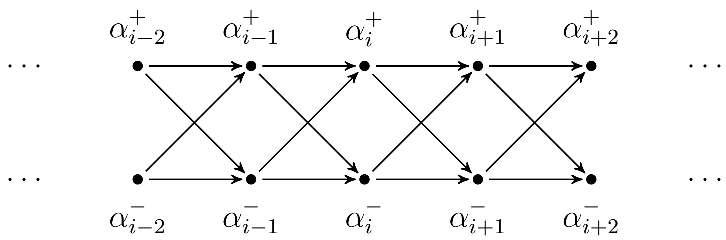
To describe the matrix problems we are interested in, we need to recall the definitions of bunches of semichains.

Definition A.1.1. *A partially ordered set (poset)* $\mathfrak{S}$ *is a* semichain *if* $\mathfrak{S} = \bigcup_{i\in I} \mathfrak{S}_i$ *for some* $I \subseteq \mathbb{Z}$ *subject to the following conditions*:

- *For any* $i \in I$ *the set* $\mathfrak{S}_i$ *is given by one element or two incomparable elements*
- *For any* $i, j \in I$ *such that* $i < j$ *it holds that* $s_i < s_j$ *for any* $s_i \in \mathfrak{S}_i$ *and* $s_j \in \mathfrak{S}_j$.

Remark A.1.2. *Equivalently, a poset* $\mathfrak{S}$ *is a semichain if* $\mathfrak{S}$ *is any full subposet of the maximum semichain which is depicted in Figure A.1.1.*

FIGURE A.1.1. **Hasse Diagram of the maximum semichain**



Roughly speaking, a bunch of semichains is given by two collections of semichains and an equivalence relation on some elements of these collections.

Definition A.1.3. *A* bunch of semichains $\mathfrak{B} = ((\mathfrak{C}_i)_{i\in I}, (\mathfrak{R}_i)_{i\in I}, \approx)$ *is given by the following data*:

(1) A finite or countable index set I,

(2) For each $i \in I$ there are pairwise disjoint semichains $\mathfrak{C}_i$ *and* $\mathfrak{R}_i$, *that is,*

$$\mathfrak{C}_i \cap \mathfrak{R}_i = \emptyset \qquad \text{for all } i \in I, \text{ and also}$$
$$\mathfrak{C}_i \cap \mathfrak{C}_j = \mathfrak{C}_i \cap \mathfrak{R}_j = \mathfrak{R}_i \cap \mathfrak{R}_j = \emptyset \qquad \text{for all } i, j \in I \text{ with } i \neq j.$$

Before we proceed with the third datum, we make the following definitions:

- *We define the* set of column labels $\mathfrak{C}$, *the* set of row labels $\mathfrak{R}$ *and* the set of labels $\mathfrak{B}$ *of the bunch of semichains by setting*

$$\mathfrak{C} = \bigcup_{i \in I} \mathfrak{C}_i, \qquad \text{and} \quad \mathfrak{R} = \bigcup_{i \in I} \mathfrak{R}_i, \qquad \mathfrak{B} = \mathfrak{C} \cup \mathfrak{R}.$$

  In particular, we denote the bunch of semichains $((\mathfrak{C}_i)_{i\in I}, (\mathfrak{R}_i)_{i\in I}, \approx)$ as well as its set of labels $\mathfrak{C} \cup \mathfrak{R}$ by $\mathfrak{B}$.

- *Let $\alpha \in \mathfrak{B}$. Let $i \in I$ be the index such that $\alpha \in \mathfrak{C}_i$ or $\alpha \in \mathfrak{R}_i$. The element α will be called* special *if there is some $\beta \in \mathfrak{C}_i$ (respectively $\beta \in \mathfrak{R}_i$) such that α and β are* not *comparable to each other with respect to the partial order in $\mathfrak{C}_i$ (respectively in $\mathfrak{R}_i$).*

As third datum of the bunch of semichains we have a "stratification" on $\mathfrak{B}$:

(3) There is an equivalence relation $\approx$ on $\mathfrak{B}$ subject to the following conditions:

   (a) Each equivalence class has only one or two elements,

   (b) Any special element of $\mathfrak{B}$ is only equivalent to itself.

Moreover, we will distinguish between the following types of bunches of semichains:

Definition A.1.4. *Let $\mathfrak{B}$ be any bunch of semichains.*

- $\mathfrak{B}$ *is* separated *if for any $\zeta \in \mathfrak{C}$ and any $\varrho \in \mathfrak{R}$ we have $\zeta \not\approx \varrho$.*
- $\mathfrak{B}$ *is a* bunch of chains *if for each $i \in I$ the sets $\mathfrak{C}_i$ and $\mathfrak{R}_i$ are totally ordered.*
- *Otherwise, if $\mathfrak{B}$ contains a special element, $\mathfrak{B}$ will be called a* proper *bunch of semichains.*

The following examples of bunches of semichains are related to some well-known quivers.

Example A.1.5. *(1) Let $\mathfrak{B}^{(1)} = \big(\ (\mathfrak{C}_1, \mathfrak{C}_2), (\mathfrak{R}_1, \mathfrak{R}_2), \approx\ \big)$, where the sets of labels are given by the one-point sets*

$$\mathfrak{C}_1 = \{\ \zeta_1\ \}, \quad \mathfrak{C}_2 = \{\ \zeta_2\ \}, \quad \mathfrak{R}_1 = \{\ \varrho_1\ \} \quad \text{and} \quad \mathfrak{R}_2 = \{\ \varrho_2\ \},$$

and the stratification is given by $\zeta_1 \approx \zeta_2$ and $\varrho_1 \approx \varrho_2$.
We will see below that this bunch of chains is related to the Kronecker quiver.

(2) Let $\mathfrak{B}^{(2)} = \big((\mathfrak{C}_1, \mathfrak{C}_2), (\mathfrak{R}_1, \mathfrak{R}_2), \approx\big)$, where the sets of column labels are two-point semichains and the sets of row labels are one-point sets:

$$\mathfrak{C}_1 = \{\ \zeta_1^+,\ \zeta_1^-\ \}, \quad \mathfrak{C}_2 = \{\ \zeta_2^+,\ \zeta_2^-\ \}, \quad \mathfrak{R}_1 = \{\ \varrho_1\ \} \quad \text{and} \quad \mathfrak{R}_2 = \{\ \varrho_2\ \},$$

The stratification on $\mathfrak{B}^{(2)}$ is given by $\varrho_1 \approx \varrho_2$.
$\mathfrak{B}^{(2)}$ is a proper bunch of semichains and related to the four subspace quiver.

Both bunches of semichains $\mathfrak{B}^{(1)}$ *and* $\mathfrak{B}^{(2)}$ *are separated.*

From now on, we fix a bunch of semichains $\mathfrak{B}$ and a field $\Bbbk$. Next we will define representations of the bunch $\mathfrak{B}$ and say when two representations are isomorphic.

Simply put, a representation of $\mathfrak{B}$ is given by a collection of partitioned matrices, and the partitioning of each matrix is determined by the column and row label sets of $\mathfrak{B}$.

Definition A.1.6. *A* $\Bbbk$*-linear representation* M *of a bunch of semichains* $\mathfrak{B}$ *is given by block matrices* $M = (M_i)_{i \in I}$ *of the following form:*

- *For each* $i \in I$ *we have a block matrix* M_i *with entries from* $\Bbbk$ *and finitely many rows and columns.*
- *The vertical stripes of* M_i *are indexed by column labels* $\zeta \in \mathfrak{C}_i$*, while the horizontal stripes of* M_i *are indexed by row labels* $\varrho \in \mathfrak{R}_i$*.*

For every $i \in I$ *and every* $\zeta \in \mathfrak{C}_i$ *let* $n_\zeta \in \mathbb{N}_0$ *denote the number of columns in the horizontal stripe labeled by* ζ*. Similarly, for every* $i \in I$ *and every* $\varrho \in \mathfrak{R}_i$ *let* $n_\varrho \in \mathbb{N}_0$ *be the number of rows in the vertical stripe with label* ϱ*. The size numbers* $(n_\alpha)_{\alpha \in \mathfrak{B}}$ *have to fulfill the following conditions:*

- *For any* $\alpha, \beta \in \mathfrak{B}$ *such that* $\alpha \approx \beta$ *we have* $n_\alpha = n_\beta$*,*
- *There are only finitely many* $\alpha \in \mathfrak{B}$ *such that* $n_\alpha \neq 0$*.*

In other words, stripes labeled by equivalent elements of the label set $\mathfrak{B}$ *should have the same amount of rows respectively columns. Moreover, the total number of rows and columns of the block matrix* M_i *is finite for every* $i \in I$*.*

Example A.1.7. *(1) A representation of* $\mathfrak{B}^{(1)}$ *is given by two matrices* $A, B \in \mathrm{Mat}_{m \times n}(\Bbbk)$ *with the same number of rows and columns.*

(2) A representation of $\mathfrak{B}^{(2)}$ *is given by two matrices with two horizontal stripes and the same number of rows.*

An isomorphism of two representations of the bunch of semichains $\mathfrak{B}$ is given by certain transformations of matrices:

Definition A.1.8. *Let* M *be a representation of* $\mathfrak{B}$*. An* admissible transformation *of* $M = (M_i)_{i \in I}$ *is a transformation of one of the following three types:*

(1) Independent transformations:
Let $\alpha \in \mathfrak{B}$ *be such that* $\alpha \not\approx \beta$ *for any* $\beta \in \mathfrak{B}$ *with* $\alpha \neq \beta$*.*

- *Assume that* $\alpha \in \mathfrak{C}$ *is a column label. Let* $i \in I$ *such that* $\alpha \in \mathfrak{C}_i$*. In this case, we may perform any elementary transformation of columns* inside *the vertical stripe labeled by* α *in the matrix* M_i*.*
- *If* $\alpha \in \mathfrak{R}$ *is a row label, there is an* $i \in I$ *such that* $\alpha \in \mathfrak{R}_i$*. Similarly, we may perform any elementary transformation of rows* inside *the horizontal stripe with label* α *in the matrix* M_i*.*

(2) Simultaneous transformations:
Let $\alpha, \beta \in \mathfrak{B}$ *such that* $\alpha \approx \beta$ *and* $\alpha \neq \beta$*.*

- *Assume that* $\alpha \in \mathfrak{R}_i$ *for some* $i \in I$ *and* $\beta \in \mathfrak{R}_j$ *for some* $j \in I$. *Then we may perform any elementary transformation of rows in the horizontal stripe labeled by* α *in the matrix* M_i together with the same transformation *of rows in the horizontal stripe with label* β *in the matrix* M_j.
- *Similarly, if* $\alpha \in \mathfrak{C}_i$ *for some* $i \in I$ *and* $\beta \in \mathfrak{C}_j$ *for some* $j \in I$, *we may perform any* simultaneous *elementary transformation of columns in the vertical stripe* α *of the matrix* M_i *and the vertical stripe* β *in the matrix* M_j.

The next case can only occur if $\mathfrak{B}$ *is not separated.*

- *Assume that* $\alpha \in \mathfrak{R}_i$ *for some* $i \in I$ *and* $\beta \in \mathfrak{C}_j$ *for some* $j \in I$. *Then we may perform any elementary transformation of rows in the horizontal stripe with label* α *in the matrix* M_i *together with the* contragredient *transformation of columns in the vertical stripe labeled by* β *in* M_j.

(3) Transformations between stripes*:*

- *Let* $\alpha, \beta \in \mathfrak{B}$ *be such that* $\alpha, \beta \in \mathfrak{C}_i$ *for some* $i \in I$ *and* $\alpha < \beta$ *in the partial order of* $\mathfrak{C}_i$. *We may* add a scalar multiple *of any column of the vertical stripe with label* α *to any column of the vertical stripe labeled by* β *in the matrix* M_i.
- *Similarly, if* $\alpha, \beta \in \mathfrak{R}_i$ *for some* $i \in I$ *and* $\alpha < \beta$, *we may add a scalar multiple of any row in the horizontal stripe* α *to any row in the horizontal stripe* β *of the matrix* M_i.

Definition A.1.9. *Let* $M' = (M'_i)_{i \in I}$ *and* $M'' = (M''_i)_{i \in I}$ *be two representations of* $\mathfrak{B}$.

(1) The representations M' *and* M'' *are* isomorphic *if the representation* M'' *can be obtained from* M' *by any composition of admissible transformations.*

(2) The direct sum *of* M' *and* M'' *is defined componentwise by* $M' \oplus M'' := (M'_i \oplus M''_i)_{i \in I}$

(3) A representation M *is* indecomposable *if* M *is not isomorphic to some direct sum of two non-trivial representations of* $\mathfrak{B}$.

With these notions we may formulate the *classification problem* which is the topic of the present appendix:

Let $\mathfrak{B}$ be any bunch of semichains.

Classify the indecomposable representations of $\mathfrak{B}$ up to isomorphism.

The classification problem for a bunch of semichains is an example of a *matrix problem.* Indecomposable representations of $\mathfrak{B}$ will also be called *canonical forms.* Let us return to the examples above:

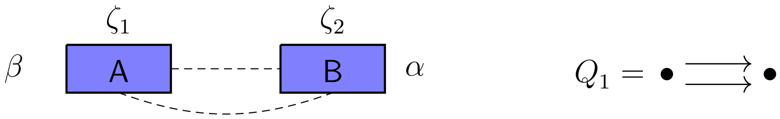
Example A.1.10. *Let* $\mathfrak{B}^{(1)}$ *and* $\mathfrak{B}^{(2)}$ *be the bunches of semichains from Example A.1.5.*

(1) Let $(A, B) \in \mathrm{Mat}_{m \times n}(\Bbbk)^{\times 2}$ *be a representation of the bunch of chains* $\mathfrak{B}^{(1)}$. *Admissible transformations of* (A, B) *are given by*

$$(A, B) \longmapsto (S \cdot A \cdot T^{-1}, S \cdot B \cdot T^{-1}),$$

where $S \in \mathrm{GL}_n(\Bbbk)$ *and* $T \in \mathrm{GL}_m(\Bbbk)$. *The classification problem of* $\mathfrak{B}^{(1)}$ *is the*

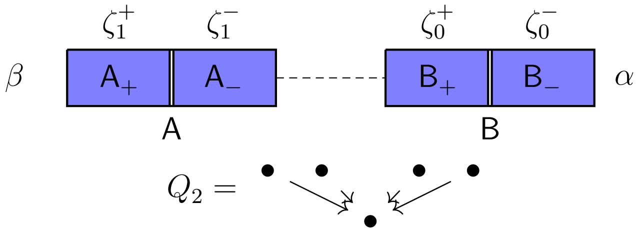
FIGURE A.1.2. **Matrix problem of the Kronecker quiver**



problem to describe canonical forms for the matrices (A, B) using only admissible transformations. In this case, this problem is equivalent to the classification problem of the Kronecker quiver Q_1.

(2) The matrix problem of $\mathfrak{B}^{(2)}$ is depicted in Figure A.1.3. In particular, the matrix

FIGURE A.1.3. **Matrix problem of the four subspace quiver**



problem of $\mathfrak{B}^{(2)}$ is equivalent to the classification problem of the four subspace quiver Q_2.

Remark A.1.11. *(1) By* **[BD04]** *any bunch of semichains $\mathfrak{B}$ gives rise to a* bimodule category $\mathrm{Rep}(\mathfrak{B})$*. The objects of* $\mathrm{Rep}(\mathfrak{B})$ *are given by representations of $\mathfrak{B}$ as defined above.*

(2) It is well-known that the category $\mathrm{Rep}(\mathfrak{B})$ *is an additive $\Bbbk$-linear category with the* Krull-Remak-Schmidt property*.*

Example A.1.12. *In each of the examples above there is an equivalences of categories* $\mathrm{Rep}(\mathfrak{B}^{(i)}) \xrightarrow{\sim} \mathrm{Rep}(Q_i)$ *, where $i = 1$ or 2.*

A.2 Strings and bands

Let $\mathfrak{B}$ be any bunch of semichains. Our goal is to construct the indecomposable objects in $\mathrm{Rep}(\mathfrak{B})$. In the next step, we describe the invariants that parametrize the canonical forms of $\mathfrak{B}$. These are divided into *strings* and *bands*. Strings are given by certain discrete parameters, while the definition of bands involves a continuous parameter from the base field $\Bbbk$.

We will distinguish between the case that $\mathfrak{B}$ is a bunch of chains and the case that $\mathfrak{B}$ is a proper bunch of semichains. The first case already shows what is going on, while the latter case is more technical. The definitions and examples which are relevant only for proper bunches of semichains are marked by the symbol $*$.

A.2.1 The alphabet of a bunch of semichains

Definition* A.2.1. *Let $\mathfrak{S}$ be a semichain. Then $\mathfrak{S} = \bigcup_{i\in I} \mathfrak{S}_i$ for some index set $I \subseteq \mathbb{Z}$, and some one-point or two-point links $\mathfrak{S}_i$. We define the* associated chain $\overline{\mathfrak{S}}$ *by setting*

$$\overline{\mathfrak{S}} = \bigcup_{i\in I} \overline{\mathfrak{S}}_i, \qquad \text{where} \quad \overline{\mathfrak{S}}_i = \begin{cases} \mathfrak{S}_i & \text{if } \mathfrak{S}_i = \{\ \alpha_i\ \}, \\ \{\alpha_i\} & \text{if } \mathfrak{S}_i = \{\ \alpha_i^+, \alpha_i^-\ \} \end{cases}$$

Moreover, for any $i, j \in I$ such that $i < j$ and any $\alpha_i \in S_i$ and any $\alpha_j \in S_j$ we set $\alpha_i < \alpha_j$.
In other words, to define $\overline{\mathfrak{S}}$ we replace any two incomparable elements in $\overline{\mathfrak{S}}$ by one new element with the same relations. The new elements in $\overline{\mathfrak{S}}$ will be called special.

Definition A.2.2. *Let $\mathfrak{B} = \big((\mathfrak{C}_i)_{i\in I}, (\mathfrak{R}_i)_{i\in I}, \approx\big)$ be any bunch of semichains. We define the alphabet $\mathfrak{A}_\mathfrak{B} = \big((\overline{\mathfrak{C}}_i)_{i\in I}, (\overline{\mathfrak{R}}_i)_{i\in I}, \sim, -\big)$ associated to $\mathfrak{B}$ as follows*:

(1) • *If $\mathfrak{B}$ is a bunch of chains, we set $\overline{\mathfrak{R}}_i = \mathfrak{R}_i$ and $\overline{\mathfrak{C}}_i = \mathfrak{C}_i$ for any $i \in I$.*

$*$ *If $\mathfrak{B}$ is a proper bunch of semichains, the set of column letters $\overline{\mathfrak{C}}_i$ is given by the associated chain of $\mathfrak{C}_i$ for each $i \in I$. Similarly, for any $i \in I$ the set of row letters $\overline{\mathfrak{R}}_i$ is the associated chain of $\mathfrak{R}_i$.*

(2) We define the set of letters $\mathfrak{A}_\mathfrak{B}$ by setting

$$\overline{\mathfrak{C}} = \bigcup_{i\in I} \overline{\mathfrak{C}}_i, \qquad \overline{\mathfrak{R}} = \bigcup_{i\in I} \overline{\mathfrak{R}}_i, \qquad \text{and} \quad \mathfrak{A}_\mathfrak{B} = \overline{\mathfrak{C}} \cup \overline{\mathfrak{R}}.$$

(3) we introduce two new symmetric relations $\sim$ and $-$ on $\mathfrak{A}_\mathfrak{B}$. For any $\alpha, \beta \in \mathfrak{A}_\mathfrak{B}$ we set

- *$\alpha - \beta$ if there is some $i \in I$ such that either $\alpha \in \overline{\mathfrak{C}}_i$ and $\beta \in \overline{\mathfrak{R}}_i$, or $\alpha \in \overline{\mathfrak{R}}_i$ and $\beta \in \overline{\mathfrak{C}}_i$.*
- *$\alpha \sim \beta$ if $\alpha \approx \beta$ in $\mathfrak{B}$ and $\alpha \neq \beta$;*
  in this case, the letter α as well as the letter β will be called tied.

The next definition is only relevant for proper bunches of semichains. We set

$*$ *$\alpha \sim \alpha$ if $\alpha \in \mathfrak{A}_\mathfrak{B}$ is special.*

At last, a letter $\alpha \in \mathfrak{A}_\mathfrak{B}$ will be called free *if α is neither tied nor special.*

Example A.2.3. *The alphabets of the bunches of semichains $\mathfrak{B}^{(1)}$ and $\mathfrak{B}^{(2)}$ from Example A.1.5 are given as follows:*

(1) $\mathfrak{A}^{(1)}_\mathfrak{B} = \big(\ (\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_2), (\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2), \sim, -\big)$, where the sets of letters are given by one-point sets

$$\overline{\mathfrak{C}}_1 = \{\ \zeta_1\ \}, \quad \overline{\mathfrak{C}}_2 = \{\ \zeta_2\ \}, \quad \overline{\mathfrak{R}}_1 = \{\ \varrho_1\ \} \quad \text{and} \quad \overline{\mathfrak{R}}_2 = \{\ \varrho_2\ \}, \tag{A.2.1}$$

with relations $\zeta_1 \sim \zeta_2$ *and* $\varrho_1 \sim \varrho_2$, $\zeta_1 - \varrho_1$ *and* $\zeta_2 - \varrho_2$. *All letters of* $\mathfrak{A}^{(1)}_{\mathfrak{B}}$ *are tied.*

(2) $\mathfrak{A}^{(2)}_{\mathfrak{B}} = \big((\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_2), (\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2), \sim, - \big)$ *with the same sets of letters as in* (A.2.1) *but the relations are given by* $\zeta_1 \sim \zeta_1$, $\zeta_2 \sim \zeta_2$ *and* $\varrho_1 \sim \varrho_2$, $\zeta_1 - \varrho_1$ *and* $\zeta_2 - \varrho_2$. *In particular, the letters* ζ_1 *and* ζ_2 *are special, while* ϱ_1 *and* ϱ_2 *are tied.*

A.2.2 Strings

Definition A.2.4. *Let* $\mathfrak{B}$ *be any bunch of semichains and* $\mathfrak{A}_{\mathfrak{B}}$ *be its alphabet.*

(1) A finite word w *of* $\mathfrak{A}_{\mathfrak{B}}$ *with length* $k \in \mathbb{N}^+$ *is given by a sequence*

$$w = \alpha_1\rho_1\alpha_2\rho_2\alpha_3\rho_3\alpha_4\rho_4 \quad \ldots \quad \alpha_{k-1}\rho_{k-1}\alpha_k$$

of letters $\alpha_j \in \mathfrak{A}_{\mathfrak{B}}$, *where* $1 \leq j \leq k$, *and relations* $\rho_j \in \{ \sim, - \}$, *where* $1 \leq j < k$, *subject to the following conditions*:

- *The relation* $\alpha_j\rho_j\alpha_{j+1}$ *holds in* $\mathfrak{A}_{\mathfrak{B}}$ *for any* $1 \leq j < k$,
- *The relations alternate, that is,* $\rho_j \neq \rho_{j-1}$ *for any* $1 < j < k$,
- *If the first letter* α_1 *is tied, then the first relation* ρ_1 *is equal to* $\sim$,
- *Similarly, if the last letter* α_k *is tied, then the last relation* ρ_{k-1} *is* $\sim$.

(2) For a finite word w *as above the* reversed *word* w^{rev} *is given by*

$$w^{rev} = \alpha_k\rho_{k-1}\alpha_{k-1} \ldots \alpha_2\rho_1\alpha_1.$$

Example A.2.5. *(1) A finite word of* $\mathfrak{A}^{(1)}_{\mathfrak{B}}$ *is given by*

$$w = \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 - \varrho_1 \sim \varrho_2$$

(2) A finite word of $\mathfrak{A}^{(2)}_{\mathfrak{B}}$ *is given by*

$$w = \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_1 - \varrho_1 \sim \varrho_2$$

Definition A.2.6. *Let* $\mathfrak{B}$ *be a bunch of chains.*

(1) A usual string *of* $\mathfrak{B}$ *is given by any finite word* w.

(2) Two usual strings v *and* w *are* equivalent *if either* $v = w$ *or* $v = w^{rev}$.

Example A.2.7. *There are four series of* usual strings of the bunch of chains $\mathfrak{B}^{(1)}$ *up to equivalence*:

$$\begin{array}{l} \zeta_2 \sim \zeta_1 - (\varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 -)^n \varrho_1 \sim \varrho_2 \\ \zeta_1 \sim \zeta_2 \ (-\varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2)^n - \varrho_2 \sim \varrho_1 \\ \qquad\qquad (\varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 -)^n \varrho_1 \sim \varrho_2 \\ \zeta_1 \sim \zeta_2 \ (-\varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2)^n \end{array} \qquad (n \in \mathbb{N}_0)$$

We will translate these strings into representations of the Kronecker quiver in the next subsection.

Definition* A.2.8. *Let* $\mathfrak{B}$ *be a proper bunch of semichains and* $\mathfrak{A}_{\mathfrak{B}}$ *be its alphabet. Let* w *be a finite word of* $\mathfrak{A}_{\mathfrak{B}}$ *with length* k.

(1) *The word w is* symmetric *if $w = w^{rev}$.*
Equivalently, there is some proper subword υ of w such that

$$w = \upsilon \sim \upsilon^{rev} \sim \upsilon \sim \upsilon^{rev} \sim \ \ldots \ \upsilon \sim \upsilon^{rev}$$

(2) *The word w is* quasi-symmetric *if there is some proper subword υ of w such that*

$$w = \upsilon \sim \upsilon^{rev} \sim \upsilon \sim \upsilon^{rev} \sim \ \ldots \ \upsilon \sim \upsilon^{rev} \sim \upsilon$$

(3) *The left end α_1 of w is a* special end *if α_1 is a special letter and ρ_1 is $-$.*

(4) *Similarly, the right end α_k of w is a* special end *if α_k is special and ρ_{k-1} is $-$.*

Definition* A.2.9. *Let $\mathfrak{B}$ be a proper bunch of semichains.*

(1) (a) *A finite word w is* usual *if w is not symmetric and has no special ends.*

(b) *A* usual string *is given by any usual word w.*

(c) *Two usual strings υ and w are* equivalent *if either $\upsilon = w$ or $\upsilon = w^{rev}$.*

(2) (a) *A finite word w is* special *if w has exactly one special end;*
(*if w has length one, the only letter of w counts as only one end*).

(b) *A* special string (w, ε) *is given by a special word w and one sign $\varepsilon \in \{+, -\}$.*

(c) *Two special strings (υ, δ) and (w, ε) are equivalent if $(\upsilon, \delta) = (w, \varepsilon)$ or $(\upsilon, \delta) = (w^{\mathrm{rev}}, \varepsilon)$.*

(3) (a) *A finite word w is* bispecial *if w has two special ends and is neither symmetric nor quasi-symmetric.*

(b) *A* bispecial string $(w, m, \varepsilon_1, \varepsilon_2)$ *is given by a bispecial word, a "multiplicity" $m \in \mathbb{N}$ and two signs $\varepsilon_1, \varepsilon_2 \in \{+, -\}$.*

(c) *Two bispecial strings $(\upsilon, n, \delta_1, \delta_2)$ and $(w, m, \varepsilon_1, \varepsilon_2)$ are* equivalent
if $(\upsilon, n, \delta_1, \delta_2) = (w, m, \varepsilon_1, \varepsilon_2)$ or $(\upsilon, n, \delta_1, \delta_2) = (w^{\mathrm{rev}}, m, \varepsilon_2, \varepsilon_1)$.

A string *of $\mathfrak{B}$ is given by any usual, special or bispecial string. By a* string word *we will mean any usual, special or bispecial word. A string of one of the three types cannot be equivalent to any string of another of the three types.*

Example A.2.10. *Let us consider some strings of the bunch of semichains $\mathfrak{B}^{(2)}$.*

(1) *The finite word $w = \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_1 - \varrho_1 \sim \varrho_2$ from Example A.2.7 is a usual string of $\mathfrak{B}^{(2)}$.*

(2) *The finite word $w = \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_1$ is special. In particular, $(w, -)$ is a special string of $\mathfrak{B}^{(2)}$.*

(3) *The finite word $w = \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2$ is bispecial.*
In fact, any bispecial string of $\mathfrak{B}^{(2)}$ is equivalent to the bispecial string $\Omega = (w, m, \varepsilon_1, \varepsilon_2)$ for the bispecial word w above, some $m \in \mathbb{N}^+$ and $\varepsilon_1, \varepsilon_2 \in \{+, -\}$.

A.2.3 Bands

Definition A.2.11. *Let $\mathfrak{B}$ be a bunch of semichains.*

(1) A cyclic word *w is given by a sequence*

$$w = \alpha_1 \sim \alpha_2 - \alpha_3 \sim \alpha_4 - \quad \ldots \quad \alpha_{2k-1} \sim \alpha_{2k} - \tag{A.2.2}$$

of an even number of letters $\alpha_j \in \mathfrak{A}_{\mathfrak{B}}$, where $1 \le j \le 2k$, subject to the following conditions:

- *$\alpha_{2j-1} \sim \alpha_{2j}$ holds in $\mathfrak{A}_{\mathfrak{B}}$ for any $1 \le j \le k$,*
- *$\alpha_{2j} - \alpha_{2j+1}$, where $1 \le j < k$, and also $\alpha_{2k} - \alpha_1$ hold in $\mathfrak{A}_{\mathfrak{B}}$.*

In the following, let w be a cyclic word of length $2k$ as above. For any $1 \le j \le 2k$ and any $i \in \mathbb{Z}$ we set $\alpha_{j+2k\cdot i} = \alpha_j$ and $\rho_{j+2k\cdot i} = \rho_j$, that is, we consider the indices in w modulo the length $2k$.

(2) Let $0 \le j < 2k$ be even. *The j*-th rotation *$w^{[j]}$ of w is defined as*

$$w^{[j]} = \alpha_{j+1} \sim \alpha_{j+2} - \alpha_{j+3} \sim \alpha_{j+4} - \quad \ldots \quad \alpha_{j+2k-1} \sim \alpha_{j+2k} - .$$

(3) The cyclic word w is periodic *if $w = w^{[j]}$ for some non-trivial even index $2 \le j < 2k$.*
Equivalently, there is some proper subword υ of w such that $w = \upsilon\,\upsilon \ldots \upsilon$.

(4) The reversed word *w^{rev} of w is defined as*

$$w^{rev} = \alpha_{2k} \sim \alpha_{2k-1} - \quad \ldots \quad \alpha_4 \sim \alpha_3 - \alpha_2 \sim \alpha_1 - .$$

Example A.2.12. *Let us consider some cyclic words of the alphabet $\mathfrak{A}^{(1)}_{\mathfrak{B}}$.*

(1) Let $w = \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2 -$. Obviously, w is a non-periodic word of $\mathfrak{A}^{(1)}_{\mathfrak{B}}$.

(2) Another cyclic word of $\mathfrak{A}^{(1)}_{\mathfrak{B}}$ is given by

$$w = \zeta_1 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2 - \varrho_2 \sim \varrho_1 -$$

Since $w^{[4]} = w$, this word is periodic. Actually, any cyclic word of $\mathfrak{A}^{(1)}_{\mathfrak{B}}$ with length bigger than four is periodic.

Definition A.2.13. *Let w be a cyclic of word of $\mathfrak{B}$ with length $2k$:*

$$w = \alpha_1 \sim \alpha_2 - \alpha_3 \sim \alpha_4 - \quad \ldots \quad \alpha_{2k-1} \sim \alpha_{2k} -$$

Let us fix some even index $2 \le j \le 2k$.

(1) The subword $\alpha_{j-1} \sim \alpha_j$ of w is called parallel *if Either α_{j-1} and $\alpha_j \in \mathfrak{C}$, or α_{j-1} and $\alpha_j \in \mathfrak{R}$.*

(2) Let $\nu(j, w)$ be the number of parallel subwords in w between indices 1 and j :

$$\nu(j, w) = \#\{\ 2 \le i \le j \text{ even} \mid \alpha_{i-1} \sim \alpha_i \text{ is a parallel subword }\}.$$

We define a sign parameter $\xi(j, w) = (-1)^{\nu(j,w)}$.

Remark A.2.14. *For $j = 0$ we set $\nu(0, w) := \nu(2k, w)$ which is even. Then $\xi(0, w) = 1$.*

Example A.2.15. *Let w be the non-periodic word $\varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2 -$ of $\mathfrak{A}^{(1)}_{\mathfrak{B}}$. Then $\nu(2, w) = 1$ and $\xi(2, w) = -1$.*

Definition* A.2.16. *Let $\mathfrak{B}$ be a proper bunch of semichains and w a cyclic word of $\mathfrak{A}_{\mathfrak{B}}$ with length $2k$.*

(1) w is symmetric *if $w^{rev} = w^{[j]}$ for some, possibly trivial, even index $0 \le j < 2k$.*

(2) We set $\varsigma(w) = \frac{1}{2} \cdot \widetilde{\varsigma}(w)$, where

$$\widetilde{\varsigma}(w) = \#\{2 \le i < 2k \text{ even} \mid \alpha_{i-1} \ne \alpha_i \text{ and } \alpha_{i-1} \sim \alpha_i \text{ is a parallel subword }\}$$

Example* A.2.17. *A cyclic word of the alphabet $\mathfrak{A}_{\mathfrak{B}}^{(2)}$ is given by*

$$w = \zeta_1 \sim \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - .$$

Note that w is symmetric, non-periodic and $\varsigma(w) = 1$.

Definition A.2.18. *Let $\mathfrak{B}$ be a bunch of semichains and let $\Bbbk$ be an algebraically closed field.*

(1) A cyclic word w is a band word *if w is not periodic.*

(2) A band *(w, m, λ) of $\mathfrak{B}$ is given by a band word w, a "multiplicity" $m \in \mathbb{N}^+$ and an "eigenvalue" $\lambda \in \Bbbk^*$ subject to the condition:*

- *If $\mathfrak{B}$ is a proper bunch of semichains and w is a symmetric word, then also $\lambda \ne (-1)^{\varsigma(w)+1}$.*

(3) Two bands (w, m, λ) and (υ, n, μ) are equivalent *if $m = n$, the band words w and υ have the same length $2k$ and (υ, n, μ) can be obtained from (w, m, λ) by some sequence of the following operations :*

(a) We may replace (w, m, λ) by (w^{rev}, m, λ)

(b) For any even index $0 \le j < 2k$ We may replace (w, m, λ) by $(w^{[j]}, m, \lambda^{\xi(j,w)})$.

A string of $\mathfrak{B}$ cannot be equivalent to any band of $\mathfrak{B}$.

Example A.2.19. *(1) Any band of the bunch of chains $\mathfrak{B}^{(1)}$ is equivalent to some band*

$$(w, m, \lambda), \quad \text{where} \quad w = \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2 -, \quad m \in \mathbb{N}^+ \quad \text{and} \quad \lambda \in \Bbbk^*.$$

(2) Any band of $\mathfrak{B}^{(2)}$ is given up to equivalence by (w, m, λ), where

$$w = \zeta_1 \sim \zeta_1 - \varrho_1 - \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 -, \quad m \in \mathbb{N}^+ \quad \text{and} \quad \lambda \in \Bbbk^* \setminus \{1\}.$$

Remark A.2.20. *Let $\mathfrak{B}$ be any bunch of semichains and let $\Bbbk$ be* any field. *In this general setup, Definition A.2.18 has to be changed as follows:*

(1) Instead of (w, m, λ), a band is given by (w, m, f), where $f \in \Bbbk[x]$ is any monic irreducible polynomial *such that $f(x) \ne x$ and also $f(x) \ne x + (-1)^{\varsigma(w)}$ if w is symmetric.*

(2) In the equivalence conditions for bands we have to replace (w, m, λ^{-1}) by $(w, m, \widetilde{f})$ with $\widetilde{f}(x) = x^{\deg f} \cdot f(x^{-1}) \cdot f(0)^{-1}$.

The definitions of strings and bands are motivated by the following theorem:

Theorem A.2.21 ([**Bon88**]). *Let $\mathfrak{B}$ be a bunch of semichains. Then there is a bijection between the equivalence classes of strings and bands of $\mathfrak{B}$ and isomorphism classes of indecomposable objects in* $\operatorname{Rep}(\mathfrak{B})$:

$$[\text{STRINGS } \textit{and} \text{ BANDS } \textit{of } \mathfrak{B}] \xleftrightarrow{\;1:1\;} \operatorname{ind}[\operatorname{Rep}(\mathfrak{B})]$$

A.3 Construction of canonical forms

In this section we are going to describe how to construct an indecomposable representation from a string or band. First we deal with the case of bunches of chains. Then we give the general construction for bunches of semichains.

A.3.1 Canonical forms for bunches of chains

Throughout this subsection, let $\mathfrak{B}$ be a bunch of chains and let $\mathfrak{A}_{\mathfrak{B}}$ denote its alphabet.

A.3.1.1 String representations

Let w be some finite word of $\mathfrak{A}_{\mathfrak{B}}$ in the sense of Definition A.2.4:

$$w = \alpha_1\rho_1\alpha_2\rho_2 \dots \alpha_{k-1}\rho_{k-1}\alpha_k$$

We are going to define an indecomposable representation $M(w)$, associated to the usual string w. To define the representation $M(w)$ of $\mathfrak{B}$ we have to describe a collection of block matrices $(M_i)_{i\in I}$ such that the vertical stripes of the block matrix M_i are labeled by the column labels $\zeta \in \mathfrak{C}_i$, while the horizontal stripes of M_i are labeled by row labels $\varrho \in \mathfrak{R}_i$ for each $i \in I$.

(1) For every letter $\alpha \in \mathfrak{A}_{\mathfrak{B}}$ let n_α denote the total number of times α appears as a letter in the word w.

(2) For every $i \in I$ the horizontal and vertical stripes of the block matrix M_i of M have the following size:

- For each $\zeta \in \mathfrak{C}_i$ the vertical stripe labeled by ζ of the block matrix M_i has exactly n_ζ columns.
- Similarly, for each $\varrho \in \mathfrak{R}_i$, the horizontal stripe ϱ of the block matrix M_i has n_ϱ rows.

(3) For each letter α_j of w, where $1 \leq j \leq k$, we count its *occurrence number*, that is, the number of times α_j appears as a letter in the subword $\alpha_1 \dots \alpha_j$ of length j in w.

(4) Every subword $\alpha - \beta$ of w, where $\alpha, \beta \in \mathfrak{A}_{\mathfrak{B}}$, contributes a non-zero entry in some block matrix M_i of M in the following way:

(a) For any index $1 \leq j < k$ such that $\rho_j = -$ we consider the subword $\alpha_j - \alpha_{j+1}$ in w. Then either $\alpha_j - \alpha_{j+1} = \zeta - \varrho$ or $\varrho - \zeta$ for some $\varrho \in \mathfrak{R}$ and $\zeta \in \mathfrak{C}$. In both cases, there is some index $i \in I$ such that $\zeta \in \mathfrak{C}_i$ and $\varrho \in \mathfrak{R}_i$.

Next, we consider the matrix $M_{\varrho\zeta}$ in the intersection of the horizontal stripe ζ and the vertical stripe ϱ of the block matrix M_i.

(b) Let p be the occurrence number of the letter ϱ and q be the occurrence number of ζ. Then the (p, q)-th entry of the matrix $M_{\varrho\zeta}$ of the block matrix M_i is set to be 1.

The two steps above are carried out for every subword $\alpha - \beta$ in w.

(5) All remaining entries of the block matrices $(M_i)_{i \in I}$ are set to be zero.

The resulting block matrices $(M_i)_{i \in I}$ constitute the *string representation* $M(w)$.

Example A.3.1. *Let $\mathfrak{B} = \mathfrak{B}^{(1)}$. We construct the representation $M(w)$ of the usual string*

$$w = \underset{\#1}{\varrho_1} \sim \underset{\#1}{\varrho_2} - \underset{\#1}{\zeta_2} \sim \underset{\#1}{\zeta_1} - \underset{\#2}{\varrho_1} \sim \underset{\#2}{\varrho_2} - \underset{\#2}{\zeta_2} \sim \underset{\#2}{\zeta_1} - \underset{\#3}{\varrho_1} \sim \underset{\#3}{\varrho_2}$$

We have written the occurrence number under the respective letter. The algorithm above yields the following matrices:

$$A = \begin{array}{c} \\ \varrho_2^{\#1} \\ \varrho_2^{\#2} \\ \varrho_2^{\#3} \end{array} \begin{array}{c} \begin{array}{cc} \zeta_2^{\#1} & \zeta_2^{\#2} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \\ \varrho_1^{\#1} \\ \varrho_1^{\#2} \\ \varrho_1^{\#3} \end{array} \begin{array}{c} \begin{array}{cc} \zeta_1^{\#1} & \zeta_1^{\#2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

These canonical forms correspond to the preprojective representation $\Bbbk^2 \overset{A}{\underset{B}{\rightrightarrows}} \Bbbk^3$ *of the Kronecker quiver.*

The next Remark collects some properties of string representations.

Remark A.3.2. *Let $w = \alpha_1\rho_1\alpha_2 \ldots \rho_k\alpha_k$ be some usual string of the bunch of chains $\mathfrak{B}$. Let $M(w) = (M_i)_{i \in I}$ be the string representation of w.*

(1) For every $i \in I$ any row or any column of the block matrix M_i has at most one *non-zero entry. In other words, the block matrices M_i are "sparse".*

(2) the block matrices M_i have full row rank *for all $i \in I$ if and only if the ends of w satisfy the following two conditions*:

(a) Either $\alpha_1 \in \mathfrak{C}$ is a column letter, or $\alpha_1 \in \mathfrak{R}$ is a row letter and $\rho_1 = -$.

(b) Either $\alpha_k \in \mathfrak{C}$ is a column letter, or $\alpha_k \in \mathfrak{R}$ is a row letter and $\rho_{k-1} = -$.

(3) The block matrices M_i are invertible *for all $i \in I$ if and only if $\rho_1 = -$ and $\rho_{k-1} = -$.*

Remark A.3.3. *Let us recall the notion of defect for the Kronecker quiver. Let V be a representation of the Kronecker quiver with dimension vector $\underline{\dim}(V) = (n_1, n_2)$. The* defect *of V is defined by $\delta(V) = 2\,(n_1 - n_2)$. The defect $\delta(V)$ is related to the position of V in the Auslander-Reiten quiver:*

$$\begin{array}{lll} \delta(V) < 0 & \textit{if and only if} & V \textit{ is preprojective} \\ \delta(V) = 0 & \textit{if and only if} & V \textit{ is regular} \\ \delta(V) > 0 & \textit{if and only if} & V \textit{ is preinjective} \end{array}$$

We refer to [**SS07**, Section XI.1] *for details on the defect for hereditary algebras.*

Example A.3.4. *Let $\mathfrak{B}^{(1)}$ be the bunch of chains from Example A.1.5 of the Kronecker quiver. The usual strings of $\mathfrak{B}^{(1)}$ correspond to the following representations of the Kronecker quiver:*

usual string w	$M(w)$	*type*
$\zeta_2 \sim \zeta_1 - (\varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 -)^n\ \varrho_1 \sim \varrho_2$	$\Bbbk^{n+1} \underset{J}{\overset{\mathrm{Id}}{\rightrightarrows}} \Bbbk^{n+1}$	*regular*
$(\varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_1 -)^n\ \varrho_1 \sim \varrho_2$	$\Bbbk^{n} \underset{B'}{\overset{A'}{\rightrightarrows}} \Bbbk^{n+1}$	*preprojective*
$\zeta_1 \sim \zeta_2 - (\varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2 -)^n\ \varrho_2 \sim \varrho_1$	$\Bbbk^{n+1} \underset{\mathrm{Id}}{\overset{J}{\rightrightarrows}} \Bbbk^{n+1}$	*regular*
$\zeta_1 \sim \zeta_2\ (-\varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_2)^n$	$\Bbbk^{n+1} \underset{B''}{\overset{A''}{\rightrightarrows}} \Bbbk^{n}$	*preinjective*
where $n \in \mathbb{N}_0$		

The matrices in the representations $M(w)$ are given as follows:

$$J = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \quad A' = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad B' = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad A'' = \begin{bmatrix} 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \quad B'' = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}$$

Remark A.3.5. *There are two series of* string *representations M such that $\delta(M) = 0$, or, equivalently, $\tau(M) \cong M$, where τ is the Auslander-Reiten translation of Kronecker quiver representations. This means, that the usual strings cannot be characterized in terms of the defect alone.*

A.3.1.2 Band representations

In this subsection we assume again that $\Bbbk$ is an algebraically closed field. Let Ω be some band of the bunch of chains $\mathfrak{B}$ in the sense of Definition A.2.18. In particular, Ω is given by (w, m, λ), where

$$w = \alpha_1 \sim \alpha_2 - \dots \alpha_{2k-1} \sim \alpha_{2k} -$$

is a cyclic and non-periodic word of even length, $m \in \mathbb{N}^+$ and $\lambda \in \Bbbk^*$.
The *band representation* $M(\Omega)$ is constructed as follows:

(1) *Representation of the finite subword*:
First, we view the subword $\dot{w} = \alpha_1 \sim \alpha_2 \dots \sim \alpha_{2k}$ as a finite word of $\mathfrak{A}_{\mathfrak{B}}$ and construct the representation $M(\dot{w})$ as described in Subsection A.3.1.1.

(2) *Blowing up with multiplicity m*:
Next, for every $i \in I$ we replace every zero entry in the block matrix M_i of $M(\dot{w})$ by a square zero matrix of size m and every identity in M_i by an identity matrix of size m.

(3) *Placing the Jordan block with eigenvalue λ*:
At last, we consider the subword $\alpha_{2k} - \alpha_1$ in w. There is an index $i \in I$ and there are $\zeta \in \mathfrak{C}_i$ and $\varrho \in \mathfrak{R}_i$ such that

either (a) $\alpha_{2k} - \alpha_1 = \zeta - \varrho$ or (b) $\alpha_{2k} - \alpha_1 = \varrho - \zeta$

In both cases we consider the block matrix M_i :

(a) In the first case, we replace the zero matrix at the last m columns of the vertical stripe ζ and the first m rows of the horizontal stripe ϱ in the block matrix M_i by the Jordan block $J_m(\lambda)$ with size m and eigenvalue λ.

(b) Similarly, in the second case, we place $J_m(\lambda)$ at the last m rows of the horizontal stripe ϱ and the first m columns of the vertical stripe ζ in the block matrix M_i.

The block matrices $(M_i)_{i \in I}$ yield the *band representation* $M(\Omega)$.

Remark A.3.6. *Let Ω be some band of the bunch of chains $\mathfrak{B}$ and let $M(\Omega) = (M_i)_{i \in I}$ be the band representation of Ω. Then for every $i \in I$ the following holds:*

(1) Any row (or any column) of the block matrix M_i has one or two *non-zero entries,*

(2) The block matrix M_i is invertible.

Example A.3.7. *The only band representation of the bunch of chains $\mathfrak{B}^{(1)}$ is given as follows*:

band word w	$M(w,m,\lambda)$ *where $m \in \mathbb{N}^+$ and $\lambda \in \mathbb{k}^*$*	*type*
$\underset{\#1}{\varrho_2} \sim \underset{\#1}{\varrho_1} - \underset{\#1}{\zeta_1} \sim \underset{\#1}{\zeta_2} \overset{\lambda}{-},$	$\mathbb{k}^m \underset{J_\lambda}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathbb{k}^m$ *where* $J_\lambda = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$	*regular*

At last, let us note that the present example together with example A.3.4 yield a complete description of indecomposable representations of the Kronecker quiver via bunches of chains.

A.3.1.3 Main Theorem

The construction algorithm of canonical forms described above yields *all* representations of any bunch of chains:

Theorem A.3.8. [**Bon88, Bon91**] *Let $\mathfrak{B}$ be any bunch of chains.*

(1) Let M be an indecomposable representation of $\mathfrak{B}$. Then there is some usual string $\Omega = w$ or some band $\Omega = (w, m, \lambda)$ of $\mathfrak{B}$ such that $M \cong M(\Omega)$, where $M(\Omega)$ denotes the canonical representation as constructed in Subsection A.3.1.1 respectively A.3.1.2.

(2) Two string or bands Ω and Υ of $\mathfrak{B}$ are equivalent if and only if there is an isomorphism $M(\Omega) \cong M(\Upsilon)$ of the corresponding string respectively band representations.

A.3.2 Canonical forms for bunches of semichains

Let $\mathfrak{B}$ be any bunch of semichains. Let Ω be some string or band of $\mathfrak{B}$ and let w denote the word of Ω.

Before we can describe the canonical form $M(\Omega)$ of $\mathfrak{B}$ we need to convert w into a new word $\overset{\leftrightarrow}{w}$ in two steps:

(1) Every special subword $\alpha \sim \alpha$ of w will be decorated an arrow $\overrightarrow{\alpha \sim \alpha}$ or $\overleftarrow{\alpha \sim \alpha}$.

(2) Every special letter α in w will be supplied by a sign α^+ or α^-.

The new word $\overset{\leftrightarrow}{w}$ will be the main input for the construction of the canonical form $M(\Omega)$.

A.3.2.1 Word with arrows and signs

As above, let w be a string or band word of $\mathfrak{A}_{\mathfrak{B}}$.

Arrows of special subwords

Let k denote the length of w. The arrows of special subwords in w are oriented as follows:

(1) First, we define the *ambient word* $\overline{w}$ of w as follows:

$$\overline{w} = \begin{cases} w & \text{if } w \text{ is usual or cyclic,} \\ w^{rev} \sim w & \text{if only the left end of } w \text{ is special,} \\ w \sim w^{rev} & \text{if only the right end of } w \text{ is special,} \\ w^{rev} \sim w \sim w^{rev} & \text{if } w \text{ is bispecial.} \end{cases}$$

(2) Let $1 \leq j < k$ be an index such that $\alpha_j \rho_j \alpha_{j+1} = \alpha \sim \alpha$ for some special letter $\alpha \in \mathfrak{A}_{\mathfrak{B}}$. We view w as a subword of its ambient word $\overline{w}$. Let Υ be the *maximal symmetric subword* of $\overline{w}$ such that ρ_j is the relation *in the middle* of Υ. In particular, Υ has the form $v^{rev}\rho_j v$ for some subword v of $\overline{w}$:

$$\overline{w} = \ldots \alpha_{j-m}\rho_{j-m} \underbrace{\alpha_{j-m+1}\rho_{j-m+1} \cdots \alpha_{j-1}\rho_{j-1}\alpha_j}_{v^{rev}} \underbrace{\rho_j}_{\sim} \underbrace{\alpha_{j+1}\rho_{j+1}\alpha_{j+2} \cdots \rho_{j+m-1}\alpha_{j+m}}_{v} \rho_{j+m}\alpha_{j+m+1} \cdots$$

In the notations above, the subword Υ has some even length $2m \geq 2$ and

$$\alpha_j = \alpha_{j+1}, \quad \alpha_{j-1} = \alpha_{j+2}, \quad \ldots \quad \alpha_{j-m+1} = \alpha_{j+m}.$$

We have to distinguish between two cases, where the first case is rather rare:

(a) Assume that $\Upsilon = \overline{w}$. In this case, it follows that $\overline{w} = w$ and w must be a cyclic symmetric word such that $\alpha_{j-i} = \alpha_{j+i+1}$ for all $i \in \mathbb{N}_0$. In other words, the relation ρ_j lies at the symmetry axis of the cyclic word w. In this case, we set

$$\overleftrightarrow{\alpha_j \sim \alpha_{j+1}} = \begin{cases} \overleftarrow{\alpha_j \sim \alpha_{j+1}} & \text{if } \alpha_j \in \mathfrak{C}, \\ \overrightarrow{\alpha_j \sim \alpha_{j+1}} & \text{if } \alpha_j \in \mathfrak{R}. \end{cases} \tag{A.3.1}$$

(b) From now on, we consider the case that $\Upsilon \neq \overline{w}$, that is, Υ is a proper subword of $\overline{w}$. In this setup, we are going to define two symbols γ_l and γ_r which will satisfy an order relation $\gamma_l < \gamma_r$ or $\gamma_l > \gamma_r$. There are the following three possibilities:

(i) If the left end α_{j-m+1} of Υ is also the left end of $\overline{w}$, we set $\gamma_l = \infty$ and $\gamma_r = \alpha_{j+m+1}$. In particular, this means that $\gamma_l > \gamma_r$.

(ii) Similarly, if the right end α_{j+m} of Υ is the same as the right end of $\overline{w}$, we set $\gamma_l = \alpha_{j-m}$ and $\gamma_r = \infty$. That is, we obtain that $\gamma_l < \gamma_r$.

(iii) At last, if no end of Υ is an end of $\overline{w}$, we set $\gamma_l = \alpha_{j-m}$ and $\gamma_r = \alpha_{j+m+1}$. In this case, it holds that $\gamma_l \neq \gamma_r$ and there is some $i \in I$ such that either γ_l and $\gamma_r \in \overline{\mathfrak{C}}_i$, or γ_l and $\gamma_r \in \overline{\mathfrak{R}}_i$. This means that γ_l and γ_r are comparable. It follows that either $\gamma_l < \gamma_r$ or $\gamma_l > \gamma_r$ in the total order of $\overline{\mathfrak{C}}_i$ respectively $\overline{\mathfrak{R}}_i$.

In all three cases above the orientation of $\alpha_j \sim \alpha_{j+1}$ is determined as follows:

- Assume that $\alpha_j \in \mathfrak{C}$, then we set

$$\overleftrightarrow{\alpha_j \sim \alpha_{j+1}} = \begin{cases} \overleftarrow{\alpha_j \sim \alpha_{j+1}} & \text{if either } \gamma_l < \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{R} \cup \{\infty\}, \\ & \quad \text{or } \gamma_l > \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{C} \cup \{\infty\}; \\ \overrightarrow{\alpha_j \sim \alpha_{j+1}} & \text{if either } \gamma_l > \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{R} \cup \{\infty\}, \\ & \quad \text{or } \gamma_l < \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{C} \cup \{\infty\}. \end{cases} \tag{A.3.2}$$

- If $\alpha_j \in \mathfrak{R}$, we reverse the orientations in the cases above:

$$\overleftrightarrow{\alpha_j \sim \alpha_{j+1}} = \begin{cases} \overrightarrow{\alpha_j \sim \alpha_{j+1}} & \text{if either } \gamma_l < \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{R} \cup \{\infty\}, \\ & \quad \text{or } \gamma_l > \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{C} \cup \{\infty\}; \\ \overleftarrow{\alpha_j \sim \alpha_{j+1}} & \text{if either } \gamma_l > \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{R} \cup \{\infty\}, \\ & \quad \text{or } \gamma_l < \gamma_r \text{ and } \gamma_l, \gamma_r \in \mathfrak{C} \cup \{\infty\}. \end{cases} \tag{A.3.3}$$

Step (2) is carried out for every index $1 \leq j < k$ such that $\alpha_j = \alpha_{j+1}$ in w.

Example A.3.9. *We consider some string and band words of $\mathfrak{B}^{(2)}$.*

(1) Let w be the only non-periodic word of $\mathfrak{A}^{(2)}_{\mathfrak{B}}$. Since w is symmetric we obtain that

$$\overset{\leftrightarrow}{w} = \varrho_2 \sim \varrho_1 - \overleftarrow{\zeta_1 \sim \zeta_1} - \varrho_1 - \varrho_2 - \overleftarrow{\zeta_2 \sim \zeta_2} -,$$

(2) Let w be the usual word from Example A.2.7. Then

$$\overset{\leftrightarrow}{w} = \underbrace{\varrho_1 \sim \varrho_2 - \overrightarrow{\zeta_2 \sim \zeta_2} - \varrho_2 - \varrho_1}_{\gamma_l = \infty > \gamma_r = \zeta_1} - \overleftarrow{\zeta_1 \sim \zeta_1} - \varrho_1 \sim \varrho_2,$$

(3) Let $w = \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \zeta_1 \sim \zeta_1$ *be the special word. In this case* $\overline{w} = w^{\mathrm{rev}} \sim w$*. The arrows are oriented as follows*:

$$\overline{w} = \zeta_1 \sim \zeta_1 - \dots \varrho_2 \sim \varrho_1 - \underbrace{\zeta_1 \sim \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2 \sim \zeta_2 - \varrho_2 \sim \varrho_1 - \overbrace{\zeta_1 \sim \zeta_1}^{\varrho_1 < \infty}}_{\varrho_1 < \infty}$$

$$\textit{It follows that} \qquad \overleftrightarrow{w} = \zeta_1 - \varrho_1 \sim \varrho_2 - \overleftarrow{\zeta_2 \sim \zeta_2} - \varrho_2 \sim \varrho_1 - \overleftarrow{\zeta_1 \sim \zeta_1}$$

(4) For the bispecial string $\Omega = (w, m, +, -)$ *with* $w = \zeta_1 - \varrho_1 \sim \varrho_2 - \zeta_2$ *and any* $m \in \mathbb{N}^+$ *it holds that* $\overleftrightarrow{w} = \zeta_1^+ - \varrho_1 \sim \varrho_2 - \zeta_2^-$*.*

Signs of special letters and elementary subwords

Special letters in $\overleftrightarrow{w}$ are decorated by signs as follows:

(1) Every special subword $\overleftrightarrow{\alpha \sim \alpha}$ in $\overleftrightarrow{w}$ is changed into $\overleftrightarrow{\alpha^+ \sim \alpha^-}$.

(2) If w has a special end, there are two cases to consider:

- Assume that $\Omega = (w, \varepsilon)$ is a special string, where w is some special word and $\varepsilon \in \{+, -\}$. In this case, the only special end α of w is changed into α^{ε}.
- Assume that $\Omega = (w, m, \varepsilon_1 \varepsilon_2)$ is bispecial string, where w is a bispecial word, $m \in \mathbb{N}^+$ and $\varepsilon_1, \varepsilon_2 \in \{+, -\}$. Then the left end α_1 in w is replaced by α^{ε_1} while the right end α_k becomes $\alpha_k^{\varepsilon_2}$.

In other words, the signs of special ends in $\overleftrightarrow{w}$ are determined by the sign data of the string Ω.

The outcome is again denoted by $\overleftrightarrow{w}$. Let us note that every special letter $\alpha \in \mathfrak{A}_{\mathfrak{B}}$ has been replaced by some special label α^+ or $\alpha^- \in \mathfrak{B}$.

At last, we will need the following notions for the construction algorithm of string and band representations.

Definition A.3.10. *Let* $\overleftrightarrow{w}$ *be the decorated word of some string or band* Ω*. A subword* v *of* w *is* elementary *if* v *is given by one of the following four subwords of* $\overleftrightarrow{w}$:

$$\alpha - \beta, \qquad \overrightarrow{\alpha^+ \sim \alpha^-} - \beta, \qquad \alpha - \overleftarrow{\beta^+ \sim \beta^-} \qquad \textit{or} \qquad \overrightarrow{\alpha^+ \sim \alpha^-} - \overleftarrow{\beta^+ \sim \beta^-},$$

where α, β *denote arbitrary labels of* $\mathfrak{B}$*, while* $\alpha^+, \alpha^-, \beta^+, \beta^-$ *denote special labels of* $\mathfrak{B}$*.*

A.3.2.2 Usual, special and bispecial strings with trivial multiplicity

Let $\Omega = w$ be a usual string, $\Omega = (w, \varepsilon)$ a special string or let $\Omega = (w, m, \varepsilon_1, \varepsilon_2)$ be a bispecial string of $\mathfrak{B}$ with *trivial* multiplicity $m = 1$. Let $\overleftrightarrow{w}$ be the decorated word as constructed in the previous subsection.

The construction of the string representation $M(\Omega) = (M_i)_{i\in I}$ is described below. We note that all steps *except the fourth step* are the exactly same as for string representations of bunches of chains.

(1) For every letter $\alpha \in \mathfrak{A}_{\mathfrak{B}}$ let n_α denote the total number of times α appears as a letter in the word w.

(2) For every $i \in I$ the horizontal and vertical stripes of the block matrix M_i of M have the following size:

- For each $\zeta \in \mathfrak{C}_i$ the vertical stripe labeled by ζ of the block matrix M_i has exactly n_ζ columns.
- Similarly, for each $\varrho \in \mathfrak{R}_i$, the horizontal stripe ϱ of the block matrix M_i has n_ϱ rows.

(3) For each letter α_j of w, where $1 \leq j \leq k$, we count its *occurrence number*, that is, the number of times α_j appears as a letter in the subword $\alpha_1 \dots \alpha_j$ of length j in w.

(4) Every *elementary* subword of w contributes a non-zero entry in some block matrix M_i of M in the following way:

(a) Let $\upsilon = \alpha_j \dots \alpha_{j+l}$ be any elementary subword of $\overset{\leftrightarrow}{w}$. In particular, the subword υ has one of the four types

$$\alpha_j - \alpha_{j+1}, \qquad \overrightarrow{\alpha_j \sim \alpha_{j+1}} - \alpha_{j+2},$$
$$\alpha_j - \overleftarrow{\alpha_{j+1} \sim \alpha_{j+2}} \quad \text{or} \quad \overrightarrow{\alpha_j \sim \alpha_{j+1}} - \overleftarrow{\alpha_{j+2} \sim \alpha_{j+3}}.$$

In any case, we have that $\alpha_j - \alpha_{j+l}$ in $\mathfrak{A}_{\mathfrak{B}}$. This implies that there is an index $i \in I$ and there are labels $\zeta \in \mathfrak{C}_i$, $\varrho \in \mathfrak{R}_i$ such that either $\alpha_j = \zeta$ and $\alpha_{j+l} = \varrho$, or $\alpha_j = \varrho$ and $\alpha_{j+l} = \zeta$.

Next, we consider the matrix $M_{\varrho\zeta}$ in the intersection of the horizontal stripe ζ and the vertical stripe ϱ of the block matrix M_i.

(b) Let p be the occurrence number of ϱ and q be the occurrence number of ζ. Then the (p, q)-th entry of the matrix $M_{\varrho\zeta}$ of the block matrix M_i is set to be 1.

The two steps above are carried out for every elementary subword in w.

(5) All remaining entries of the block matrices $(M_i)_{i\in I}$ are set to be zero.

This yields the *string representation* $M(\Omega)$ of $\mathfrak{B}$.

Example A.3.11. *Let $\mathfrak{B}^{(2)}$ be the bunch of semichains of the four subspace quiver. We consider the strings from Example A.2.10.*

(1) In the first case of a usual string we have

$$\overset{\leftrightarrow}{w} = \underset{\#1}{\varrho_1} \sim \underset{\#1}{\varrho_2} - \overrightarrow{\underset{\#1}{\zeta_2} \sim \underset{\#1}{\zeta_2}} - \underset{\#2}{\varrho_2} \sim \underset{\#2}{\varrho_1} - \overleftarrow{\underset{\#1}{\zeta_1} \sim \underset{\#1}{\zeta_1}} - \underset{\#3}{\varrho_1} \sim \underset{\#3}{\varrho_2}$$

Then $M(w)$ is given by

$$A = \begin{array}{cc} & \begin{array}{cc} \zeta_1^+ & \zeta_1^- \end{array} \\ \begin{array}{c} \varrho_1^{\#1} \\ \varrho_1^{\#2} \\ \varrho_1^{\#3} \end{array} & \left[\begin{array}{c|c} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{array}\right] \end{array} \qquad B = \begin{array}{cc} & \begin{array}{cc} \zeta_2^+ & \zeta_2^- \end{array} \\ \begin{array}{c} \varrho_2^{\#1} \\ \varrho_2^{\#2} \\ \varrho_2^{\#3} \end{array} & \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{array}\right] \end{array} \qquad V = \begin{tikzcd}[column sep=small] \Bbbk \arrow[drr] & \Bbbk \arrow[dr] & & \Bbbk \arrow[dl] & \Bbbk \arrow[dll] \\ & & \Bbbk^3 & & \end{tikzcd}$$

$M(w)$ corresponds to the preprojective representation $V = \tau^{-1}P_0$ of the four subspace quiver.

(2)

$$\overset{\leftrightarrow}{w} = \underset{\#1}{\zeta_1^+} - \underset{\#1}{\varrho_1} \sim \underset{\#1}{\varrho_2} - \overleftarrow{\underset{\#1}{\zeta_2^+} \sim \underset{\#1}{\zeta_2^-}} - \underset{\#2}{\varrho_2} \sim \underset{\#2}{\varrho_1} - \overleftarrow{\underset{\#2}{\zeta_1^+} \sim \underset{\#1}{\zeta_1^-}}$$

Then $M(\Omega)$ is given by

$$A = \begin{array}{cc} & \begin{array}{ccc} \zeta_1^+ & \zeta_1^+ & \zeta_1^- \end{array} \\ \begin{array}{c} \varrho_1^{\#1} \\ \varrho_1^{\#2} \end{array} & \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \end{array} \qquad B = \begin{array}{cc} & \begin{array}{cc} \zeta_2^+ & \zeta_2^- \end{array} \\ \begin{array}{c} \varrho_2^{\#1} \\ \varrho_2^{\#2} \end{array} & \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right] \end{array} \qquad V = \begin{tikzcd}[column sep=small] \Bbbk^2 \arrow[drr] & \Bbbk \arrow[dr] & & \Bbbk \arrow[dl] & \Bbbk \arrow[dll] \\ & & \Bbbk^2 & & \end{tikzcd}$$

(3) Let $\Omega = (w, 1, +, -)$ be the bispecial string with the word $\overset{\leftrightarrow}{w} = \zeta_1^+ - \varrho_1 \sim \varrho_2 - \zeta_2^-$. Then $M(\Omega)$ yields the following representation of the four subspace quiver:

$$A = \begin{array}{cc} & \zeta_1^+ \\ \varrho_1 & \left[\begin{array}{c} 1 \end{array}\right] \end{array} \qquad B = \begin{array}{cc} & \zeta_2^- \\ \varrho_2 & \left[\begin{array}{c} 1 \end{array}\right] \end{array} \qquad V = \begin{tikzcd}[column sep=small] \Bbbk \arrow[drr] & 0 \arrow[dr] & & 0 \arrow[dl] & \Bbbk \arrow[dll] \\ & & \Bbbk & & \end{tikzcd}$$

The construction described above will also serve as first step in the construction of band representations as well as bispecial string representations of higher multiplicity of $\mathfrak{B}$. We will describe some properties of string representations in Subsection A.3.3 below.

A.3.2.3 Bands

Let $\Bbbk$ be an algebraically closed field and let Ω be some band of the bunch of semichains $\mathfrak{B}$. Then Ω is given by (w, m, λ), where

$$w = \alpha_1 \sim \alpha_2 - \ldots \alpha_{2k-1} \sim \alpha_{2k} -$$

is a band word of even length, $m \in \mathbb{N}^+$ and $\lambda \in \Bbbk^*$ such that $\lambda \neq (-1)^{\varsigma(w)+1}$ if w is symmetric.

Let $\overset{\leftrightarrow}{w}$ be the decorated word of w. Next, we describe the construction of the band representation $M(\Omega) = (M_i)_{i \in I}$.

(1) *Representation of the cyclic word w*:
First, we construct the representation $M(w)$ of the cyclic word w as if w was a usual string by the rules of Subsection A.3.2.2. Note that we consider the indices in the word w modulo its length $2k$.

(2) *Blowing up with multiplicity m*:
Next, for every $i \in I$ we replace every zero entry in the block matrix M_i of $M(w)$ by a square zero matrix of size m and every identity in M_i by an identity matrix of size m.

(3) *Placement of the Jordan block with eigenvalue λ*:
We reconsider the *maximal* elementary subword v in $\overleftrightarrow{w}$ containing the last relation $\rho_{2k} = -$. In particular, v is given by one of the following four words:

$$\alpha_{2k} - \alpha_1, \quad \overrightarrow{\alpha_{2k-1} \sim \alpha_{2k}} - \alpha_1, \quad \alpha_{2k} - \overleftarrow{\alpha_1 \sim \alpha_2} \quad \text{or} \quad \overrightarrow{\alpha_{2k-1} \sim \alpha_{2k}} - \overleftarrow{\alpha_1 \sim \alpha_2},$$

Let α_j denote the left end, and α_{j+l} the right end of v, where $l = 1, 2$ or 3. There is an index $i \in I$ and there are $\zeta \in \mathfrak{C}_i$ and $\varrho \in \mathfrak{R}_i$ such that

$$\text{either} \qquad \text{(a)} \;\; \alpha_j - \alpha_{j+l} = \zeta - \varrho \qquad \text{or} \qquad \text{(b)} \;\; \alpha_j - \alpha_{j+l} = \varrho - \zeta.$$

In both cases we consider the block matrix M_i :

(a) In the first case, we replace the identity matrix at the last m columns of the vertical stripe ζ and the first m rows of the horizontal stripe ϱ in the block matrix M_i by the Jordan block $J_m(\lambda)$ with size m and eigenvalue λ.

(b) Similarly, in the second case, we place $J_m(\lambda)$ at the last m rows of the horizontal stripe ϱ and the first m columns of the vertical stripe ζ in the block matrix M_i.

The block matrices $(M_i)_{i\in I}$ yield the *band representation* $M(\Omega)$.

Example A.3.12. *Any band of $\mathfrak{B}^{(2)}$ is given up to equivalence by Let $\Omega = (w, m, \lambda)$, where*

$$\overleftrightarrow{w} = \overleftarrow{\underset{\#1}{\zeta_1^+} \sim \underset{\#1}{\zeta_1^-}} - \underset{\#1}{\varrho_1} - \underset{\#1}{\varrho_2} - \overleftarrow{\underset{\#1}{\zeta_2^+} \sim \underset{\#1}{\zeta_2^-}} - \underset{\#2}{\varrho_2} \sim \underset{\#2}{\varrho_1} -, \qquad m \in \mathbb{N}^+ \quad \textit{and} \quad \lambda \in \Bbbk^* \backslash \{1\}.$$

Then $M(\Omega)$ is given by two matrices

$$A = \begin{array}{c} \\ \varrho_1 \\ \varrho_1 \end{array} \begin{array}{c} \begin{array}{cc} \zeta_1^+ & \zeta_1^- \end{array} \\ \left[\begin{array}{c|c} 0 & \mathrm{Id}_m \\ \mathrm{Id}_m & J_m(\lambda) \end{array}\right] \end{array} \qquad B = \begin{array}{c} \\ \varrho_2 \\ \varrho_2 \end{array} \begin{array}{c} \begin{array}{cc} \zeta_2^+ & \zeta_2^- \end{array} \\ \left[\begin{array}{c|c} \mathrm{Id}_m & \mathrm{Id}_m \\ 0 & \mathrm{Id}_m \end{array}\right] \end{array}$$

which can be viewed as a representation of the four subspace quiver:

$$\begin{tikzcd} \Bbbk^m \arrow[drr, "A_+"'] & \Bbbk^m \arrow[dr, "A_-"] & & \Bbbk^m \arrow[dl, "B_+"'] & \Bbbk^m \arrow[dll, "B_-"] \\ & & \Bbbk^{2m} & & \end{tikzcd}$$

Remark A.3.13. *There is one exceptional case in the construction of* symmetric bands*. Let $\Omega = (w, m, \lambda)$ be some band of $\mathfrak{B}$ such that the band word w is symmetric. Let $\overleftrightarrow{w}$ be the word with oriented arrows of w. Then Ω is equivalent to $\Omega' = (w^{\mathrm{rev}}, m, \lambda)$. However, in the construction of the canonical form $M(\Omega')$ the word $\overleftrightarrow{w}^{\mathrm{rev}}$ with arrows has to be redefined. The arrows of the decorated word $\overleftrightarrow{w}^{\mathrm{rev}}$ are given by reversing* all *arrows of $\overleftrightarrow{w}$. This convention ensures that $M(w, m, \lambda) \cong M(w^{\mathrm{rev}}, m, \lambda)$ for symmetric bands.*

Remark A.3.14. *Let* $\Bbbk$ *be an* arbitrary *field and* $\Omega = (w, f^m)$ *be some band of* $\mathfrak{B}$.

(1) In this setup, the band representation $M(\Omega)$ *is constructed in the same way as describe above but instead of the Jordan block* $J_m(\lambda)$ *we use the Frobenius block with characteristic polynomial* f^m *in the last step.*

(2) Let $M(\Omega) = (M_i)_{i \in I}$ *be the corresponding band representation. Then for every* $i \in I$ *the following holds:*

(a) The block matrix M_i *of the band representation* $M(\Omega)$ *is invertible.*
In other terms, $M(\Omega) \in \operatorname{Rep}^*(\mathfrak{B})$.

(b) Any row (or any column) of the block matrix M_i *has from* one to five *non-zero entries.*

(3) Any band of $\mathfrak{B}$ *is equivalent to some band* $\Omega = (w, f^m)$ *such that the maximal elementary subword* v *in the last step of the algorithm is given by a subword of length* two. *In this case, the block matrices of the band representation* $M(\Omega)$ *have even* at most four *non-zero entries in any row or any column.*

A.3.2.4 Bispecial strings of higher multiplicity

Let $\Omega = (w, m, \varepsilon_1, \varepsilon_2)$ be a bispecial string, where

$$w = \alpha_1 - \alpha_2 \sim \alpha_3 - \alpha_4 \sim \ldots \alpha_{k-1} - \alpha_k$$

is a bispecial word of $\mathfrak{A}_{\mathfrak{B}}$, the multiplicity $m \in \mathbb{N}^+$ is *arbitrary* and $\varepsilon_1, \varepsilon_2 \in \{+, -\}$. As usual, let $\overset{\leftrightarrow}{w}$ denote the word with arrows and signs as defined in Subsection A.3.2.1.

To define bispecial string representations we need the following notation:

- Let $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$ denote the opposite signs of ε_1 respectively ε_2.
- Let δ_{ij} denote the square matrix of size m with 1 at entry (i, j) and zero elsewhere.
- Let A_m and B_m denote the following square matrices of size m :

$$\mathrm{A}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i,2i+1} = \begin{bmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \qquad \mathrm{B}_m = \mathrm{Id}_m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \delta_{2i-1,2i} = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \tag{A.3.4}$$

For bispecial words w of length $k = 2$, we refer to Remark A.3.16 for the construction of $M(\Omega)$. If the word w has length $k \geq 4$, the bispecial string representation $M(\Omega)$ is constructed as follows:

(1) *String representation with trivial multiplicity*:
We construct the string representation $M(\Omega)$ as for bispecial strings of multiplicity one following Subsection A.3.2.2.

(2) *Blowing up by multiplicity* m :
Similarly to the case of bands, we replace identity entries in all block matrices of $M(\Omega)$ by identity matrices of size m and all zero entries by square zero matrices of size m.

(3) *Twisting the beginning* :
We reconsider the first elementary subword $\alpha_1 - \alpha_2$ in $\overset{\leftrightarrow}{w}$, where $\alpha_1 \in \mathfrak{B}$ is a special end. Then there is some index $i \in I$, some labels $\zeta \in \mathfrak{C}_i$ and $\varrho \in \mathfrak{R}_i$ such that

(a) Either $\alpha_1 - \alpha_2 = \zeta - \varrho$

(b) Or $\alpha_1 - \alpha_2 = \varrho - \zeta$.

In both cases, there is an identity matrix Id_m at the first m rows of the horizontal stripe ϱ and the first m columns of the vertical stripe ζ in the block matrix M_i. This identity matrix is replaced by the square matrix A_m from (A.3.4).

(a) In the first case, the column label ζ is special and is actually decorated by the first sign: $\zeta = \zeta^{\varepsilon_1}$. In this case, every *even-numbered* column in the first m columns of the vertical stripe ζ^{ε_1} in the block matrix M_i is *relabeled* by $\zeta^{\bar{\varepsilon}_1}$.

(b) In the second case, the row label $\varrho = \varrho^{\varepsilon_1}$ is special and decorated by the first sign ε_1. Similarly to the case above, every *even-numbered* row in the first m rows of the horizontal stripe ϱ^{ε_1} in the block matrix M_i is *relabeled* by $\varrho^{\bar{\varepsilon}_1}$.

For the sake of completeness let us list the relabeled matrices A_m for both cases:

case (a):	case (b):
$\mathrm{A}_m = \begin{array}{cc} & \begin{array}{cccccc} \zeta^{\varepsilon_1} & \zeta^{\bar{\varepsilon}_1} & \zeta^{\varepsilon_1} & \zeta^{\bar{\varepsilon}_1} & \zeta^{\varepsilon_1} & \cdots \end{array} \\ \begin{array}{c} \varrho \\ \varrho \\ \varrho \\ \varrho \\ \varrho \\ \vdots \end{array} & \begin{bmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \end{array}$	$\mathrm{A}_m = \begin{array}{cc} & \begin{array}{cccccc} \zeta & \zeta & \zeta & \zeta & \zeta & \cdots \end{array} \\ \begin{array}{c} \varrho^{\varepsilon_1} \\ \varrho^{\bar{\varepsilon}_1} \\ \varrho^{\varepsilon_1} \\ \varrho^{\bar{\varepsilon}_1} \\ \varrho^{\varepsilon_1} \\ \vdots \end{array} & \begin{bmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \end{array}$

(4) *Twisting the end*:
We reconsider the last subword $\alpha_{k-1} - \alpha_k$ in w. Similarly to the previous step, there are some index $i \in I$ and labels $\zeta \in \mathfrak{R}_i$ and $\varrho \in \mathfrak{C}_i$ such that

(a) either $\alpha_{k-1} - \alpha_k = \varrho - \zeta$,

(b) or $\alpha_{k-1} - \alpha_k = \zeta - \varrho$.

In both cases, we replace the identity matrix at the last m rows of the horizontal stripe ϱ and the last m columns of the vertical stripe ζ in the block matrix M_i. by the second square matrix B_m from (A.3.4).

(a) In the first case, $\zeta = \zeta^{\varepsilon_2}$ is actually decorated by the second sign ε_2. In this case, every *even-numbered* column of the last m columns in the vertical stripe ζ^{ε_2} of the block matrix M_i is *relabeled* by $\zeta^{\bar{\varepsilon}_2}$.

(b) Similarly, if w has right end $\varrho = \varrho^{\varepsilon_2}$, we *relabel* every *even-numbered* row in the last m rows of horizontal stripe ϱ^{ε_2} of the block matrix M_i by $\varrho^{\overline{\varepsilon}_2}$.

The relabeled matrices B_m have the following form in the two cases:

case (a):	case (b):
$\mathrm{B}_m = \begin{array}{cc} & \begin{array}{cccccc} \zeta^{\varepsilon_2} & \zeta^{\overline{\varepsilon}_2} & \zeta^{\varepsilon_2} & \zeta^{\overline{\varepsilon}_2} & \zeta^{\varepsilon_2} & \cdots \end{array} \\ \begin{array}{c} \varrho \\ \varrho \\ \varrho \\ \varrho \\ \varrho \\ \vdots \end{array} & \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \end{array}$	$\mathrm{B}_m = \begin{array}{cc} & \begin{array}{cccccc} \zeta & \zeta & \zeta & \zeta & \zeta & \cdots \end{array} \\ \begin{array}{c} \varrho^{\varepsilon_2} \\ \varrho^{\overline{\varepsilon}_2} \\ \varrho^{\varepsilon_2} \\ \varrho^{\overline{\varepsilon}_2} \\ \varrho^{\varepsilon_2} \\ \vdots \end{array} & \begin{bmatrix} 1 & 1 & & & & \cdots \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \end{array}$

The outcome is the bispecial string representation $M(\Omega)$.

Example A.3.15. *Let* $\Omega = (w, 5, +, -)$ *be the bispecial string with the word* $\overset{\leftrightarrow}{w} = \zeta_1^+ - \varrho_1 \sim \varrho_2 - \zeta_2^-$. *Then* $M(\Omega)$ *is given by the following partitioned matrices* A *and* B *:*

$$A = \begin{array}{c} \begin{array}{ccc|cc} \zeta_1^+ & \zeta_1^+ & \zeta_1^+ & \zeta_1^- & \zeta_1^- \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \qquad B = \begin{array}{c} \begin{array}{cc|ccc} \zeta_2^+ & \zeta_2^+ & \zeta_2^- & \zeta_2^- & \zeta_2^- \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

These matrices give rise to the following quiver representation:

$$\begin{tikzcd} \Bbbk^3 \arrow[drr, "A_+"'] & \Bbbk^2 \arrow[dr, "A_-"] & & \Bbbk^2 \arrow[dl, "B_+"'] & \Bbbk^3 \arrow[dll, "B_-"] \\ & & \Bbbk^5 & & \end{tikzcd}$$

The representation $M(\Omega)$ *of* $\mathfrak{B}^{(2)}$ *corresponds to a representation* V *in a homogeneous tube of rank* two *of the four subspace quiver* Q_2.

Remark A.3.16. *Let* Ω *be a bispecial string* $(w, m, \varepsilon_1, \varepsilon_2)$, *where* $w = \alpha_1 - \alpha_2$ *is the shortest possible bispecial word,* $m \in \mathbb{N}^+$, *and* $\varepsilon_1, \varepsilon_2 \in \{+, -\}$. *Then either* $\alpha_1 - \alpha_2 = \varrho - \zeta$ *or* $\zeta - \varrho$ *for some index* $i \in I$, *labels* $\zeta \in \mathfrak{C}_i$ *and* $\varrho \in \mathfrak{R}_i$. *In both cases, the bispecial string representation* $M(\Omega)$ *has only one only non-empty block* *matrix M_i, which is given by the following square matrix:*

$$\mathrm{C}_m = \mathrm{Id}_m + \sum_{i=1}^{m-1} \delta_{i,i+1} = \begin{array}{c|cccccc} & \zeta^{\varepsilon_1} & \zeta^{\overline{\varepsilon}_1} & \zeta^{\varepsilon_1} & \zeta^{\overline{\varepsilon}_1} & \zeta^{\varepsilon_1} & \cdots \\ \hline \varrho^{\varepsilon_2} & 1 & 1 & & & & \cdots \\ \varrho^{\overline{\varepsilon}_2} & & 1 & 1 & & & \\ \varrho^{\varepsilon_2} & & & 1 & 1 & & \\ \varrho^{\overline{\varepsilon}_2} & & & & 1 & 1 & \\ \varrho^{\varepsilon_2} & & & & & 1 & \\ \vdots & & & & & & \ddots \end{array}$$

A.3.3 Main theorem and properties of canonical forms

As above, let $\mathfrak{B}$ be any bunch of semichains.

Theorem A.3.17. [**Bon88, Bon91**] *Let $\mathfrak{B}$ be any bunch of semichains.*

(1) Let M be an indecomposable representation of $\mathfrak{B}$. Then there is some usual string $\Omega = w$, some special string $\Omega = (w, \varepsilon)$, some bispecial string $\Omega = (w, m, \varepsilon_1, \varepsilon_2)$ or some band $\Omega = (w, m, \lambda)$ of $\mathfrak{B}$ such that $M \cong M(\Omega)$, where $M(\Omega)$ denotes the canonical representation as constructed in Subsection A.3.2.2, A.3.2.3 respectively A.3.2.4.

(2) Two string or bands Ω and Υ of $\mathfrak{B}$ are equivalent if and only if there is an isomorphism $M(\Omega) \cong M(\Upsilon)$ of the corresponding string respectively band representations.

Remark A.3.18. *Let V be a representation of the four subspace quiver $Q^{(2)}$ with dimension vector $\underline{\dim}(V) = (n_0, n_1, n_2, n_3, n_4)$ where n_0 is the dimension of the vector space at the only sink. The* defect *of V is defined by $\delta(V) = n_1 - n_2 - n_3 - n_4 - 2n_0$. The defect $\delta(V)$ tells us the position of V in the Auslander-Reiten quiver*:

$$\begin{aligned} \delta(V) < 0 &\quad \textit{if and only if} \quad V \textit{ is preprojective} \\ \delta(V) = 0 &\quad \textit{if and only if} \quad V \textit{ is regular} \\ \delta(V) > 0 &\quad \textit{if and only if} \quad V \textit{ is preinjective} \end{aligned}$$

More details on the defect in the representation of hereditary algebras can be found in [**SS07**, Section XI.1]

Example A.3.19. *Let $\mathfrak{B}^{(2)}$ be the bunch of semichains of the four subspace quiver Q_2. Strings and bands of $\mathfrak{B}^{(2)}$ correspond to the following indecomposable representations of Q_2:*

Ω	*type of quiver representation $V(\Omega)$*
usual string	*preprojective or preinjective or τ-periodic of rank two*
special string	*preinjective or preinjective*
bispecial string	*τ-periodic of rank two*
band	*τ-invariant*

There is no direct correspondence between the four classes and the homological classes of quiver representations.

The following remark summarizes the properties of string representations of $\mathfrak{B}$:

Remark A.3.20. *Let* $\Omega = w$ *be usual string,* $\Omega = (w, \varepsilon)$ *a special string, or* $\Omega = (w, m, \varepsilon_1, \varepsilon_2)$ *be a bispecial string of* $\mathfrak{B}$. *Assume that the word in* Ω *is given by* $w = \alpha_1\rho_1\alpha_2 \dots \rho_{k-1}\alpha_k$. *The corresponding string representation* $M(\Omega) = (M_i)_{i\in I}$ *has the following properties*:

(1) the matrix M_i *has full row rank for every* $i \in I$ *if and only if the ends of* w *satisfy the following two conditions*:
   - *Either* $\alpha_1 \in \mathfrak{C}$, *or* $\alpha_1 \in \mathfrak{R}$ *and* $\rho_1 = -$.
   - *Either* $\alpha_k \in \mathfrak{C}$, *or* $\alpha_k \in \mathfrak{R}$ *and* $\rho_{k-1} = -$.

(2) The matrix M_i *is invertible for every* $i \in I$ *if and only if* $\rho_1 = -$ *and* $\rho_{k-1} = -$.

(3) The block matrices of $M(\Omega)$ *are "sparse" in the sense that for any* $i \in I$ *any row or any column of the block matrix* M_i *has* at most four *non-zero entries.*

A.4 The involution and the defect of regular strings and bands

Let $\mathfrak{B}$ be any bunch of semichains. The bunch of semichains $\mathfrak{B}$ carries a natural involution σ. This involution can be viewed as a functor on its matrix representations:

Definition A.4.1. *Let* M *be a representation of* $\mathfrak{B}$. *The representation* $\sigma(M)$ *of* $\mathfrak{B}$ *is defined by interchanging the labels* α^+ *and* α^- *in all matrices of* M *for any pair of special elements of* $\mathfrak{B}$.

The action of the involution σ on strings and bands of $\mathfrak{B}$ can be described as follows:

Lemma A.4.2. *Let* Ω *be any string or band of* $\mathfrak{B}$ *and let* $M(\Omega)$ *be its canonical form in* $\mathrm{Rep}(\mathfrak{B})$. *In the following, for any sign* $\varepsilon = \pm$ *we denote by* $\overline{\varepsilon} = \mp$ *the opposite sign. Let* $\sigma(\Omega)$ *be defined by changing the sign data of* Ω :

	usual string	*special string*	*bispecial string*	*band*
Ω	w	(w, ε_1)	$(w, m, \varepsilon_1, \varepsilon_2)$	(w, m, λ)
$\sigma(\Omega)$	w	$(w, \overline{\varepsilon}_1)$	$(w, m, \overline{\varepsilon}_1, \overline{\varepsilon}_2)$	(w, m, λ)

(A.4.1)

(*where* $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, $m \in \mathbb{N}^+$ *and* $\lambda \in \Bbbk^* \backslash \Delta$)

Let $M(\sigma(\Omega))$ *be the canonical form of the string respectively band* $\sigma(\Omega)$. *Then* $\sigma(M(\Omega)) \cong M(\sigma(\Omega))$.

PROOF. Let w be the string respectively band word of w.

We note that the representation $\sigma(M(\Omega))$ is given by interchanging labels α^+ and α^- in all matrices of the canonical form $M(\Omega)$ for any pair of special labels of $\mathfrak{B}$.

The main observation is that $\sigma(M(\Omega))$ is *the same* as the canonical representation of Ω except that we interchange all signs in the string or band word $\overleftrightarrow{w}$.

Equivalently, we may construct the representation from the *reversed* word w^{rev} and take opposite signs of the ends in $\overleftrightarrow{w}^{\mathrm{rev}}$.

In particular, $\sigma(M(\Omega))$ is *equal* to the string or band representations in the second row :

	usual string	special string	bispecial string	band
Ω	w	(w,ε)	$(w,m,\varepsilon_1,\varepsilon_2)$	(w,m,λ)
$\sigma(M(\Omega))$	$M(w^{\mathrm{rev}})$	$(w^{\mathrm{rev}},\overline{\varepsilon})$	$M(w^{\mathrm{rev}},m,\overline{\varepsilon}_2,\overline{\varepsilon}_1)$	$M(w^{\mathrm{rev}},m,\lambda)$
$M(\sigma(\Omega))$	$M(w)$	$(w,\overline{\varepsilon})$	$M(w,m,\overline{\varepsilon}_1,\overline{\varepsilon}_2)$	$M(w,m,\lambda)$

By the definition of the equivalence relation of strings and non-symmetric bands the representations of the second row are isomorphic to the representations of the third row. This implies the statement. $\square$

By the Lemma above, the involution σ preserves any any usual string or band of $\mathfrak{B}$, but no special or bispecial string.

Beside the involution, there is the following natural invariant of representations of $\mathfrak{B}$:

Definition A.4.3. *(1) Let M be a representation of $\mathfrak{B}$. The* defect $\delta(M)$ *is given by the sum of the total number of columns of all vertical stripes indexed by* free *column labels and the total number of rows of all horizontal stripes indexed by* free *row labels.*

(2) Let Ω be a string or band of $\mathfrak{B}$. The defect $\delta(\Omega)$ *is given by the number of* free *ends of w.*

Lemma A.4.4. *Let Ω be a string or band of $\mathfrak{B}$. Let $M(\Omega)$ be its corresponding canonical form in* $\mathrm{Rep}^*(X)$. *Then $\delta(\Omega)=\delta(M(\Omega))$.*

PROOF. Let w denote the word of the string respectively band Ω. Any subword $\alpha\sim\beta$ of w is given either by *tied* letters (if $\alpha\neq\beta$) or *special* letters (if $\alpha=\beta$). In particular, any free letter of w must be a free end of w. This implies the claim. $\square$

At last, we consider the following type of matrix representations:

Definition A.4.5. *(1) A representation $M=(M_i)_{i\in I}$ of $\mathfrak{B}$ is* regular *if the matrix M_i is invertible for any $i\in I$.*

(2) A string or band Ω of $\mathfrak{B}$ will be called regular *if its canonical form $M(\Omega)$ is a regular representation.*

In the following let $\mathrm{Rep}^*(\mathfrak{B})$ *denote the full subcategory of* $\mathrm{Rep}(\mathfrak{B})$.

By Remark A.3.14 any band of $\mathfrak{B}$ is regular.

Lemma A.4.6. *Let* Ω *be a* regular *string or band of* $\mathfrak{B}$. *Then the following holds*:

$$\begin{array}{lcl} \delta(\Omega) = 2 & \Leftrightarrow & \Omega \textit{ is a usual string} \\ \delta(\Omega) = 1 & \Leftrightarrow & \Omega \textit{ is a special string} \\ \delta(\Omega) = 0 & \Leftrightarrow & \Omega \textit{ is a bispecial string or a band} \end{array}$$

PROOF. • If Ω is a band, then $\delta(\Omega) = 0$, since the word w has no ends.

• Otherwise, let $w = \alpha_1\rho_1 \dots \rho_{k-1}\alpha_k$ be the string word in Ω. Since $M(\Omega)$ is regular the relations ρ_1 and ρ_{k-1} are given by $-$ and it holds that $k \geq 2$ by Remark A.3.20. This means that any end of w is free or special. It follows that

$$2 - \delta(\Omega) = \text{ number of special ends of } w$$

These considerations imply the claim. □

Finally, we obtain the following intrinsic characterization of the four classes of string and band representations in $\operatorname{Rep}^*(\mathfrak{B})$:

Proposition A.4.7. *Let* $\mathfrak{B}$ *be any bunch of semichains,* Ω *be a regular string of band of* $\mathfrak{B}$ *and* $M = M(\Omega)$ *be the canonical form in* $\operatorname{Rep}^*(\mathfrak{B})$ *constructed from* Ω. *Then the following statements hold*:

$$\begin{array}{lll} (1)\ \Omega \textit{ is a usual string } w & \Leftrightarrow \delta(M) > 0 \textit{ and } \sigma(M) \cong M & \Leftrightarrow \delta(M) = 2 \\ (2)\ \Omega \textit{ is a special string } (w, \varepsilon_1) & \Leftrightarrow \delta(M) > 0 \textit{ and } \sigma(M) \ncong M & \Leftrightarrow \delta(M) = 1 \\ (3)\ \Omega \textit{ is a bispecial string } (w, m, \varepsilon_1, \varepsilon_2) & \Leftrightarrow \delta(M) = 0 \textit{ and } \sigma(M) \ncong M & \\ (4)\ \Omega \textit{ is a band } (w, m, \lambda) & \Leftrightarrow \delta(M) = 0 \textit{ and } \sigma(M) \cong M & \end{array}$$

$$\scriptstyle \textit{where } \varepsilon_1, \varepsilon_2 \in \{+,-\} \textit{ and } \lambda \in \Bbbk \setminus \Delta$$

PROOF. This follows from Lemmas A.4.6 and A.4.2. □

This characterization shows that the category $\operatorname{Rep}^*(\mathfrak{B})$ has better functorial properties than the category $\operatorname{Rep}(\mathfrak{B})$.

APPENDIX B

Semiperfect rings and derived categories

This appendix contains technical preliminaries for Part 2 of this thesis. Throughout this appendix all rings will be Noetherian and all modules finitely generated.

B.1 Semiperfect rings and orders

B.1.1 Projective covers and characterizations of semiperfect rings

Definition B.1.1. *Let M be any Λ-module and P be a projective module. An epimorphism $\pi : P \twoheadrightarrow M$ is a* projective cover *of M if* $\ker \pi \subseteq \operatorname{rad} P$.

In some literature projective covers are defined via superfluous submodules:

Definition B.1.2. *Let $K \subseteq P$ be Λ-modules. The submodule K is superfluous in P if and only if for any submodule U of P the equality $K + U = P$ implies that $U = P$.*

Lemma B.1.3. *Let $\pi : P \twoheadrightarrow M$, where $P \in \Lambda\text{-}\mathrm{proj}$ and $M \in \Lambda\text{-}\mathrm{mod}$. Then* $\ker \pi \subseteq \operatorname{rad} P$ *if and only if* $\ker \pi$ *is superfluous in P.*

Projective covers are "minimal" epimorphisms in the following sense:

Lemma B.1.4. *Consider the following diagram:*

$$\begin{tikzcd} & P \arrow[d, two heads, "\pi"] \\ P' \arrow[r, two heads, "\pi'"] & M \end{tikzcd}$$

where P' and P are projective Λ-modules, M is any Λ-module, π' is a projective cover and π is an epimorphism. Then $P \cong P' \oplus P''$ for some $P'' \in \Lambda\text{-}\mathrm{proj}$ such that $P'' \subseteq \ker \pi$.

Definition B.1.5. *A ring Λ is* semi–perfect *if any Λ-module has a projective cover.*

Theorem B.1.6. *A ring Λ is* semi–perfect *if and only if there are pairwise orthogonal idempotents $e_1, \ldots, e_n \in \Lambda$ such that*

$$\sum_{i=1}^{n} e_i = 1_\Lambda, \qquad \textit{and} \qquad e_i \Lambda e_i \textit{ is a } \text{local} \textit{ ring for any } i \in \{1, \ldots, n\}.$$

Let Λ be semi-perfect. Then the following statements hold:

(1) Any indecomposable projective Λ-module is isomorphic to $P_i = \Lambda e_i$ for some idempotent e_i in the decomposition of 1_Λ.

(2) There is a bijection between isomorphism classes of indecomposable projective Λ-modules and isomorphism classes of simple Λ-modules. This bijection is given by sending an indecomposable projective Λ-module P to $\operatorname{top} P$.

(3) The category Λ - mod *has the* Krull-Remak-Schmidt property*, that is, every Λ-module has a direct sum decomposition into indecomposable Λ-modules, which is unique up to permutation and isomorphism of summands.*

B.1.2 Completed path algebras of quivers

Let us introduce some notation for this subsection:

- Let (Q, I) be any finite quiver (possibly with oriented cycles).
- Let $A = \Bbbk Q/I$ denote its path algebra, and let $\mathfrak{a}$ be the arrow ideal in A.
- Let $\widehat{A}$ be the completion of the path algebra A with respect to the ideal $\mathfrak{a}$.

Definition B.1.7. *(1) A finite–dimensional representation V of (Q, I) is* nilpotent *if all cyclic paths in $\Bbbk Q/I$ act as nilpotent operators on V.*

(2) A finite-dimensional A-module N is nilpotent *if there is an $n \in \mathbb{N}^+$ such that $\mathfrak{a}^n N = 0$.*

Let us note that we assume any nilpotent representation to be finite-dimensional.

Let $\operatorname{nil.rep}(Q, I)$ denote the category of nilpotent representations of the quiver (Q, I). Let $\widehat{A}$ - fd. mod be the category of finite-dimensional $\widehat{A}$-modules.

Proposition B.1.8. *In the notations above, the following statements are true*:

(1) The completed path algebra $\widehat{A}$ is semi-perfect.

(2) There is an equivalences of categories:

$$\widehat{A}\text{ - fd. mod} \xrightarrow{\sim} \operatorname{nil.rep}(Q, I). \tag{B.1.1}$$

Remark B.1.9. *If (Q, I) has no oriented cycles, then $\widehat{A} = A$ and the above Proposition states the well-known equivalence of categories* $A\text{ - mod} \xrightarrow{\sim} \operatorname{Rep}(Q, I)$.

B.1.3 Standard duality for infinite-dimensional algebras

Let Λ be any (possibly infinite-dimensional) associative $\Bbbk$-algebra. In this subsection we consider the standard duality

$$\mathbb{D} = \operatorname{Hom}_{\Bbbk}(\_, \Bbbk) : \Lambda\text{ - fd. mod} \xrightarrow{\sim} \Lambda^{op}\text{ - fd. mod}, \qquad \mathbb{D}^2 \cong \operatorname{Id}.$$

The following Lemma is well-known in the context of finite-dimensional algebras:

Lemma B.1.10. *Let* $M \in \Lambda$ - fd. mod. *Then for any $i \in \mathbb{N}^+$ it holds that*

$$\widetilde{\mathbb{D}}(\operatorname{soc}^i M) \cong (\widetilde{\mathbb{D}} M)/(\operatorname{rad}^i \widetilde{\mathbb{D}} M) \quad \text{and} \quad \widetilde{\mathbb{D}}(M/\operatorname{rad}^i M) \cong \operatorname{soc}^i \widetilde{\mathbb{D}} M.$$

In particular, the radical and the socle series of M have equal length $\operatorname{ll}(M)$.

Remark B.1.11. *Let us note that for any module* $M \in \Lambda\text{-fd.mod}$ *and any* $n \in \mathbb{N}^+$:

$$\mathrm{ll}(M) \geq n \quad \Longleftrightarrow \quad \mathrm{rad}^n M = 0 \quad \Longleftrightarrow \quad M \in (\Lambda / \mathrm{rad}^n \Lambda)\text{-mod}$$

In particular, the Loewy length $\mathrm{ll}(M)$ *can be seen as "nilpotency degree" of* M.

B.1.4 Orders and Cohen-Macaulay modules

Let $\mathbf{R}$ be a regular Noetherian ring of Krull dimension d.

Definition B.1.12. *Let* Λ *be a ring and let* $C(\Lambda)$ *denote its center. The ring* Λ *is an* $\mathbf{R}$-order *if the following two conditions are satisfied*:

(1) There is a ring homomorphism $f : \mathbf{R} \longrightarrow \Lambda$ *such that* $f(\mathbf{R}) \subseteq C(\Lambda)$,

(2) Λ *is a finitely generated and projective as an* $\mathbf{R}$*-module.*

Proposition B.1.13. *Let* $\mathbf{R}$ *be a complete local ring and* Λ *be an order over* $\mathbf{R}$. *Then* Λ *is semiperfect.*

Proposition B.1.14. [**IR08**, Proposition 2.1] *Let* Λ *be any* $\mathbf{R}$*-order. Then the following holds*:

$$\mathrm{gldim}\,\Lambda = \mathrm{pd}(\mathrm{top}\,\Lambda) = \max\{\ \mathrm{pd}\,S \mid S \textit{ is a simple } \Lambda\textit{-module}\ \}.$$

The following statement is attributed by Goto and Nishida in [**GN02**] to Auslander:

Proposition B.1.15. [**GN02**, Corollary 3.5 (3)] *For any* $V \in \Lambda\text{-mod}$ *it holds that*

$$\mathrm{id}\,V = \sup_{j \in \mathbb{N}_0} \{\ \mathrm{Ext}^j_\Lambda(\mathrm{top}\,\Lambda, V) \neq 0\ \}$$

Remark B.1.16. *Any* $\mathbf{R}$*-order* Λ *has* no finitely generated injective Λ*-modules.*

B.1.5 Matlis duality

Let $\mathbf{R} = \Bbbk[\![x_1, \ldots, x_d]\!]$ be the ring of formal power series of Krull dimension d. Let Λ be an $\mathbf{R}$-order and $\omega = \mathrm{Hom}_{\mathbf{R}}(\Lambda, \mathbf{R})$ its canonical bimodule. Any Hom-functor on a derived category will be actually a *right-derived* Hom-functor in this subsection.

Proposition B.1.17. [**Ooi76**, Proposition 1.1] *Let* Λ *be an* $\mathbf{R}$*-order. Then there is an isomorphism of the standard duality* $\widetilde{\mathbb{D}}$ *to the* Matlis duality

$$\widetilde{\mathbb{D}} := \mathrm{Hom}_{\mathbf{R}}(\_, \mathrm{E}) \cong \mathrm{Hom}_{\Bbbk}(\_, \Bbbk) : \mathbf{R}\text{-fd.mod} \xrightarrow{\ \sim\ } \mathbf{R}\text{-fd.mod}$$

where $\mathrm{E} = \mathrm{E}(\Bbbk)$ *is the injective hull of the residue field* $\Bbbk = \mathbf{R}/\mathfrak{m}$.

Remark B.1.18. *The isomorphism above extends to an isomorphism of functors* $\widetilde{\mathbb{D}} = \mathrm{Hom}_{\mathbf{R}}(\_, \mathrm{E}) \cong \mathrm{Hom}_{\Bbbk}(\_, \Bbbk) : \Lambda\text{-fd.mod} \xrightarrow{\ \sim\ } \Lambda^{op}\text{-fd.mod}$.
Moreover, the Matlis duality is an exact functor satisfying $\widetilde{\mathbb{D}}^2 \cong \mathrm{Id}$.

Next, we consider the *derived* Matlis duality

$$\widetilde{\mathbb{D}} : \mathrm{D^b}(\Lambda\text{-fd.mod}) \xrightarrow{\ \sim\ } \mathrm{D^b}(\Lambda^{op}\text{-fd.mod})\,.$$

Using Lemma B.2.6 we may view Matlis duality as a functor on derived categories with finite-dimensional homology:

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\Lambda\text{ - fd. mod}) \arrow[r, "\widetilde{\mathbb{D}}", "\sim"'] \arrow[d, dashed, "\wr"] & \mathrm{D}^{\mathrm{b}}(\Lambda^{op}\text{ - fd. mod}) \arrow[d, dashed, "\wr"] \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \arrow[r, dashed, "\widetilde{\mathbb{D}}", "\sim"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda^{op}) \end{tikzcd}$$

On these categories the functor $\widetilde{\mathbb{D}}$ admits the following description.

Lemma B.1.19. **[IR08]** *Under the assumptions above, there are isomorphisms of functors*

$$[-d] \circ \widetilde{\mathbb{D}} \cong \mathbb{R}\operatorname{Hom}_\Lambda(\_\,, \omega) \tag{B.1.2}$$

$$\cong \operatorname{Hom}_\Lambda(\_\,, \Lambda) \circ \tau^{-1} : \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda^{op}) \tag{B.1.3}$$

PROOF. In the following all isomorphisms will be isomorphisms of functors.

(1) By [**IR08**, Lemma 3.6] it holds that

$$[-d] \circ \widetilde{\mathbb{D}} \cong \operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R}) : \mathrm{D}^{\mathrm{b}}(\mathbf{R}\text{ - fd. mod}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathbf{R}\text{ - fd. mod})$$

It is noted in the proof of [**IR08**, Theorem 3.8] that the last isomorphism commutes with the right action of Λ. That is we obtain

$$[-d] \circ \widetilde{\mathbb{D}} \cong \operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R}) : \mathrm{D}^{\mathrm{b}}(\Lambda\text{ - fd. mod}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda^{op}\text{ - fd. mod}).$$

By [**Fox79**] there is another way to express the functor $\operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R})$:

$$\operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R}) \cong \operatorname{Hom}_{\mathbf{R}}(\Lambda \underset{\Lambda}{\otimes} \_\,, \mathbf{R}) \cong \operatorname{Hom}_\Lambda(\_\,, \operatorname{Hom}_{\mathbf{R}}(\Lambda, \mathbf{R})) = \operatorname{Hom}_\Lambda(\_\,, \omega)$$

This yields the first isomorphism in (B.1.2).

(2) For the second isomorphism, by the proof of [**IR08**, Proposition 3.5 (2)] it holds that

$$\operatorname{Hom}_\Lambda(\_\,, \Lambda) \cong \operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R}) \circ \tau : \mathrm{D}^-(\Lambda) \xrightarrow{\sim} \mathrm{D}^-(\Lambda) \xrightarrow{\sim} \mathrm{D}^+(\Lambda^{op}).$$

This implies that

$$\begin{aligned} \operatorname{Hom}_\Lambda(\_\,, \omega) &\cong \operatorname{Hom}_{\mathbf{R}}(\_\,, \mathbf{R}) \\ &\cong \operatorname{Hom}_\Lambda(\_\,, \Lambda) \circ \tau^{-1} : \mathrm{D}^-(\Lambda) \xrightarrow{\sim} \mathrm{D}^-(\Lambda) \xrightarrow{\sim} \mathrm{D}^+(\Lambda^{op}) \end{aligned}$$

Restricting this isomorphism to $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ yields the second isomorphism in (B.1.2). □

If the global dimension of Λ is finite, the above Lemma describes how to compute the Matlis duality in terms of *projective* complexes with finite-dimensional homology.

Corollary B.1.20. *Let M be a finite-dimensional Λ-module. Then the standard dual of M is given by*

$$\widetilde{\mathbb{D}}(M) = \operatorname{Hom}_{\Bbbk}(M, \Bbbk) \cong \operatorname{Ext}^d_\Lambda(M, \omega)$$

Let us note that there is an analogous statement in the commutative setup.

Lemma B.1.21. *For any $M \in \Lambda$ - fd. mod it holds that* $\operatorname{pd} M = \operatorname{id} \mathbb{D}(M)$.

Assume that $\Lambda \cong \Lambda^{op}$. Then $\mathbb{D}$ can be viewed as a contravariant endofunctor

$$\mathbb{D} \cong \mathrm{Hom}_{\Bbbk}(\_, \Bbbk) : \Lambda\text{-fd.mod} \xrightarrow{\sim} \Lambda\text{-fd.mod}, \qquad \mathbb{D}^2 \cong \mathrm{Id}, \quad \mathbb{D}(S) \cong S.$$

Corollary B.1.22. *Let $\Lambda \cong \Lambda^{op}$. Then $\operatorname{pd} S = \operatorname{id} S$ for any simple Λ-module S.*

Definition B.1.23. *Let Λ be an $\mathbf{R}$-order and let $L \in \Lambda\text{-mod}$.*
The module L is a maximal Cohen-Macaulay module *if and only if L is finitely generated and projective as an $\mathbf{R}$-module.*

In this thesis we call "maximal Cohen-Macaulay modules" just "Cohen-Macaulay modules".

Lemma B.1.24. *Let $\mathbf{R} = \Bbbk[\![x]\!]$ and $Q(\mathbf{R}) = \Bbbk((x))$. Let Λ be an* isolated *one-dimensional $\mathbf{R}$-order, that is, let $Q(\mathbf{R}) \otimes_{\mathbf{R}} \Lambda$ be semisimple.*
Then a Λ-module L is Cohen-Macaulay if and only if L is isomorphic to some submodule of some projective Λ-module.

B.2 Derived categories of semiperfect rings

Throughout this section, let Λ be a *semiperfect* ring of *finite* global dimension.

B.2.1 Main definitions

By a projective *complex* $P_\bullet$ of Λ-modules we will mean a *chain complex* of projective Λ-modules

$$P_\bullet = \quad \ldots \longrightarrow P_{i+1} \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \ldots, \qquad d_i \cdot d_{i+1} = 0.$$

The shifted complex $P_\bullet[1]$ is given by shifting $P_\bullet$ to the left:

$$P_\bullet[1] = \quad \ldots \longrightarrow \underset{i+1}{P_i} \xrightarrow{-d_i} \underset{i}{P_{i-1}} \xrightarrow{-d_{i-1}} \underset{i-1}{P_{i-2}} \xrightarrow{-d_{i-2}} \ldots,$$

Let us introduce the following three categories:

- Let $\mathrm{D}^{\mathrm{b}}(\Lambda) := \mathrm{D}^{\mathrm{b}}(\Lambda\text{-mod})$ denote the *bounded derived category* of Λ-modules.
- Let $\mathrm{K}^{\mathrm{b}}(\Lambda\text{-proj})$ be the *bounded homotopy category* of projective complexes.
- Similarly, let $\mathrm{K}^{\mathrm{b}}(\Lambda\text{-mod})$ be the *bounded homotopy category* of complexes of Λ-modules.

Proposition B.2.1. *Since Λ has finite global dimension and enough projectives, there are the following equivalences of categories*:

$$\mathrm{K}^{\mathrm{b}}(\Lambda\text{-proj}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\Lambda) \xleftarrow{\sim} \mathrm{K}^{\mathrm{b}}(\Lambda\text{-mod})$$

In fact, we will always work with the homotopy category $\mathrm{K}^{\mathrm{b}}(\Lambda\text{-proj})$ of projective complexes instead of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ whenever possible.

Remark B.2.2. *Since Λ is semiperfect, any complex in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is isomorphic to some* minimal complex*, that is, a projective complex $P_\bullet$ such that for any $i \in \mathbb{Z}$ it holds that $\operatorname{im} d_{i+1} \subseteq \operatorname{rad} P_i$.*

The following fact is one of our main motivations to study the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

Proposition B.2.3. *Since Λ has finite global dimension, there is a full and faithful functor*

$$\begin{array}{ccc} \Lambda\text{-mod} & \overset{\mathbf{R}}{\hookrightarrow} & \mathrm{D}^{\mathrm{b}}(\Lambda) \\ M & \longmapsto & P_\bullet^M := \textit{projective resolution of } M \end{array}$$

Moreover, for any Λ-modules M and N and any $j \in \mathbb{Z}$ it holds that

$$\operatorname{Ext}^j_\Lambda(M,N) \cong \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\Lambda)}(P_\bullet^M, P_\bullet^N[j]) \qquad \textit{for any } j \in \mathbb{Z}.$$

The structure of derived categories is quite explicit for rings of global dimension 0 and 1.

Proposition B.2.4. *Let Λ be a semisimple ring. Then there is an equivalence of categories*:

$$\mathrm{D}^{\mathrm{b}}(\Lambda) \xrightarrow{\sim} \prod_{i\in\mathbb{Z}} \Lambda\text{-proj}$$

Moreover, any object in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is isomorphic to a bounded projective complex with zero *differentials.*

Theorem B.2.5 (**[Dol60]**). *Let Γ be a hereditary ring. Then the indecomposable objects in $\mathrm{D}^{\mathrm{b}}(\Gamma)$ are given by shifts of projective resolutions of indecomposable Γ-modules*:

$$\operatorname{ind}[\mathrm{D}^{\mathrm{b}}(\Gamma)] = \coprod_{d\in\mathbb{Z}} \{\ P_\bullet^N[d] \mid N \in \operatorname{ind}[\Gamma\text{-mod}]\ \}$$

Next, let us assume that Λ is a semiperfect $\Bbbk$-*algebra* of finite global dimension. This assumption ensures that a Λ-module has finite length if and only if it is finite-dimensional.

- Let $\mathrm{D}^{\mathrm{b}}(\Lambda\text{-fd.mod})$ be the bounded derived category of finite-dimensional Λ-modules.
- Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ denote the full subcategory of projective complexes $P_\bullet$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$ such that the homology $\mathbf{H}_i(P_\bullet)$ is finite-dimensional for any degree $i \in \mathbb{Z}$.

Lemma B.2.6. **[IR08**, Lemma 2.5], **[BK12**, Lemma 2.4] *For a $\Bbbk$-algebra Λ as above, there is an equivalence of categories* $\mathrm{D}^{\mathrm{b}}(\Lambda\text{-fd.mod}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

In particular, we may view finite-dimensional Λ-modules by *projective* complexes with finite-dimensional homology:

$$\Lambda\text{-fd.mod} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\Lambda\text{-fd.mod}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda\text{-proj})$$

B.2.2 Projective presentations as a subcategory of the derived category

In this subsection, let Λ be any semiperfect ring. We will view the minimal projective presentations of Λ-modules as a subcategory of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$.

For any projective complex $P_\bullet$ of the derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ let us say that

- $P_\bullet$ is a *minimal complex* if $\operatorname{im} d_j \subseteq \operatorname{rad} P_{j-1}$ for any $j \in \mathbb{Z}$.
- $P_\bullet$ has *length* n for some $n \in \mathbb{N}^+$, if $P_\bullet$ is minimal and

$$P_\bullet = \ \ldots 0 \longrightarrow \underset{\neq 0}{P_n} \xrightarrow{d_n} \underset{\neq 0}{P_{n-1}} \xrightarrow{d_{n-1}} \ \ldots \ \longrightarrow \underset{\neq 0}{P_1} \xrightarrow{d_1} \underset{\neq 0}{P_0} \longrightarrow 0 \ldots$$

- $P_\bullet$ is a *minimal presentation* if $P_\bullet = P_1 \xrightarrow{d_1} \underset{\neq 0}{P_0}$ has length at most one,

  $\operatorname{im} d_1 \subseteq \operatorname{rad} P_0$ and $\ker d_1 \subseteq \operatorname{rad} P_1$.

In the following we will call indecomposable minimal presentations just *indecomposable presentations* for brevity. Next, we consider the following two subcategories in $\mathrm{D}^{\mathrm{b}}(\Lambda)$:

- Let $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ denote the full subcategory of minimal presentations in $\mathrm{D}^{\mathrm{b}}(\Lambda)$.
- Let $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ denote the full subcategory of minimal presentations $P_\bullet$ in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ such that the homology $\mathbf{H}_0(P_\bullet)$ has finite dimension.

The categories $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ and $\Lambda\text{-}\mathrm{mod}$ are related in the following way:

Lemma B.2.7. *The homology functor* $\mathbf{H}_0\colon \Lambda\text{-}\overrightarrow{\mathrm{proj}} \longrightarrow \Lambda\text{-}\mathrm{mod}$ *is full, dense and respects isomorphism classes of objects, that is*:

$$P'_\bullet \cong P''_\bullet \quad \textit{if and only if} \quad \mathbf{H}_0(P'_\bullet) \cong \mathbf{H}_0(P''_\bullet) \quad \textit{for any } P'_\bullet, P''_\bullet \in \Lambda\text{-}\overrightarrow{\mathrm{proj}}.$$

The homology functor restricts to the full subcategory $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ *of* $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ *and the category* $\Lambda\text{-}\mathrm{fd.\,mod}$ *of finite-dimensional* Λ*-modules.* □

Corollary B.2.8. *There is a bijection between isomorphism classes of indecomposable* Λ*-modules and isomorphism classes of indecomposable presentations in* $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$:

$$\operatorname{ind}[\Lambda\text{-}\mathrm{mod}] \overset{1:1}{\longleftrightarrow} \operatorname{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}]$$

This bijection restricts to isomorphism classes of indecomposable objects of the following subcategories:

$$\operatorname{ind}[\Lambda\text{-}\mathrm{fd.\,mod}] \overset{1:1}{\longleftrightarrow} \operatorname{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}]$$

Next, we are going to describe the indecomposable presentations in more detail.

Lemma B.2.9. *Let* $P_\bullet = \ (P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \ \ldots \ \longrightarrow P_0) \ \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ *be an* indecomposable *minimal complex of length* n *for some* $n \geq 1$*. Then* $\ker d_n \subseteq \operatorname{rad} P_n$.

Proof. We prove that $\ker d_n \subseteq \operatorname{rad} P_n$ by contradiction.
Assume that $\ker d_n \not\subseteq \operatorname{rad} P_n$. Let $\pi\colon P_n \longrightarrow M = P_n/\ker d_n$ be the canonical

epimorphism and let $\pi'\colon P'_n \longrightarrow M$ be a projective cover of M. By Lemma B.1.4 $P_n \cong P'_n \oplus P''_n$ for some module P''_n such that $P''_n \subseteq \ker\pi = \ker d_n$. Since $\ker d_n \not\subseteq \operatorname{rad} P_n$, it holds that $P''_n \neq 0$. It follows that there are the following isomorphisms of chain complexes:

$$\begin{aligned} P_\bullet &\cong (P'_n \oplus P''_n \xrightarrow{[d'_n,0]} P_{n-1} \longrightarrow \dots \longrightarrow P_0) \\ &\cong (P'_n \xrightarrow{d'_n} P_{n-1} \longrightarrow \dots \longrightarrow P_0) \oplus P''_n[n]. \end{aligned}$$

Since $P''_n \neq 0$ and $\operatorname{im} d'_n = \operatorname{im} d_n \subseteq \operatorname{rad} P_{n-1}$, this yields a decomposition of the complex $P_\bullet$ into non-zero objects in the category $\mathrm{D^b}(\Lambda)$. This is a contradiction to the indecomposability of $P_\bullet$. $\square$

Setting $n = 1$ in Lemma B.2.9 yields the following statement:

Corollary B.2.10. *The indecomposable presentations in* $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ *are given exactly by the "indecomposable minimal complexes from* $\mathrm{D^b}(\Lambda)$ *of length at most one"*:

$$\operatorname{ind}[\Lambda\text{-}\overrightarrow{\mathrm{proj}}] = \Big\{ P_\bullet \in \operatorname{ind}[\mathrm{D^b}(\Lambda)] \textit{ with } P_\bullet = P_1 \xrightarrow{d_1} \underset{\neq 0}{P_0} \ \textit{ and } \operatorname{im} d_1 \subseteq \operatorname{rad} P_0 \Big\}$$

In particular, the indecomposable presentations in $\Lambda\text{-}\overrightarrow{\mathrm{proj}}_{\mathrm{fd}}$ *coincide with "indecomposable minimal complexes from* $\mathrm{D^b}(\Lambda)$ *of length at most one with finite-dimensional homology at degree zero".*

Remark B.2.11. *The category* $\Lambda\text{-}\overrightarrow{\mathrm{proj}}$ *of minimal presentations can be seen as an "intermediate step" between* Λ*-modules and projective resolutions in the derived category.*
More precisely, there is the following diagram of categories and resolution or homology functors:

$$\begin{tikzcd} \Lambda\text{-}\mathrm{mod} \arrow[rr, shift left, "\mathbf{R}"] \arrow[dr, dashed, shift right, "\textit{taking min.pres.}"'] & & \mathrm{D^b}(\Lambda) \arrow[ll, shift left, "\mathbf{H}_0"] \arrow[dl, dashed, shift left, "\textit{truncation}"] \\ & \Lambda\text{-}\overrightarrow{\mathrm{proj}} \arrow[ul, shift right, "\mathbf{H}'_0"'] \arrow[ur, shift left, "\mathbf{R}'"] & \end{tikzcd}$$

where $\mathbf{R}' = \mathbf{R} \circ \mathbf{H}'_0$ *and* $\mathbf{H}'_0 = \mathbf{H}_0 \circ \mathbf{R}'$. *The dashed arrows define maps on objects, but are not functorial.*

B.2.3 Auslander-Reiten translation for one-dimensional orders

Throughout this section, let $\mathbf{R} = \Bbbk[\![x]\!]$ denote the local ring of formal power series, and let Λ be an $\mathbf{R}$-order of finite global dimension. In this subsection we recall the Auslander-Reiten formula for the order Λ.
Let $\omega = \operatorname{Hom}_{\mathbf{R}}(\Lambda, \mathbf{R})$ be the *canonical* or *dualizing bimodule* of Λ.
The bimodule ω induces the left-derived tensor product

$$\tau = \omega \underset{\Lambda}{\otimes} \_ \colon \mathrm{D^b}(\Lambda) \longrightarrow \mathrm{D^b}(\Lambda).$$

This functor is called the *Auslander-Reiten translation.*

Theorem B.2.12. [**vdB04**, Lemma 6.4.1], [**IR08**, Theorem 3.7] *Let Λ be an* **R**-*order of finite global dimension.*

(1) The functor τ is an auto-equivalence of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ and preserves the subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$.

(2) For any $X, Y \in \mathrm{D}^{\mathrm{b}}(\Lambda)$ such that X or $Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ there is a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\Lambda)}(X, Y) \cong \mathbb{D}\,\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}(\Lambda)}(Y, \tau(X)), \qquad \text{where } \mathbb{D} = \mathrm{Hom}_{\Bbbk}(\_\,, \Bbbk).$$

In different terms, the theorem states that the triangulated category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ has a *relative Serre functor* $\mathbb{S} = \tau \circ [1] = \omega \underset{\Lambda}{\otimes} \_[1]$.

Definition B.2.13. *Let $\mathcal{T}$ be a triangulated category and $X, Y, Z \in \mathcal{T}$. A distinguished triangle $X \longrightarrow Y \longrightarrow Z \xrightarrow{g} X[1]$ in $\mathcal{T}$ is an* Auslander-Reiten triangle *if the following conditions hold:*

(1) $g \neq 0$, and X and Z are indecomposable.

(2) for any indecomposable object $W \in \mathcal{T}$ and any non-isomorphism $f : W \longrightarrow Z$ it holds that $g \cdot f = 0$.

Theorem B.2.12 has the following important consequence:

Corollary B.2.14. *Let $X_\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fd}}(\Lambda)$ be any indecomposable complex with finite-dimensional homology. Then there is an Auslander-Reiten triangle*

$$\tau(X) \longrightarrow E \longrightarrow X \longrightarrow \tau(X)[1]\,.$$

B.2.4 An invariant of left-perpendicular categories

Let $\mathcal{T}$ be any triangulated category and $S \in \mathcal{T}$. In this subsection we consider an invariant which characterizes objects from the left-perpendicular category

$${}^{\perp}S := \{\ X \in \mathcal{T} \mid \mathrm{Ext}^i_{\mathcal{T}}(X, S) = 0 \ \text{ for any } i \in \mathbb{Z}\ \}.$$

Definition B.2.15. *Let $S \in \mathcal{T}$. For any $X \in \mathcal{T}$ and $k \in \mathbb{Z}$ we set*

$$\delta^{(k)}_S(X) = \dim_{\Bbbk} \mathrm{Ext}^k_{\mathcal{T}}(X_\bullet, S), \qquad \text{and} \qquad \delta_S(X) = \sum_{k \in \mathbb{Z}} \delta^{(k)}_S(X).$$

We recall that a full subcategory in $\mathcal{T}$ is *triangulated* if it is closed under cones and the shift functor. A triangulated subcategory of $\mathcal{T}$ is *thick* if it also closed under direct summands.

Let us note some basic properties of the invariant δ:

Lemma B.2.16. *Let $S \in \mathcal{T}$ and let $X, Y, Z \in \mathcal{T}$. Then the following statements hold:*

(1) $\delta_S(X) = 0$ if and only if $X \in {}^{\perp}S$.

(2) $\delta_S(X \oplus Y) = \delta_S(X) + \delta_S(Y)$.

(3) Let us assume that there is a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]\,.$$

Then $\delta_S(Y) \leq \delta_S(X) + \delta_S(Z)$.

In particular, ${}^{\perp}S$ is a thick subcategory of $\mathcal{T}$. □

(4) Let $\langle S\rangle$ be the smallest thick subcategory in $\mathcal{T}$ containing S.
Then there is an equality of categories:

$$\begin{aligned} {}^{\perp}\langle S\rangle :=& \{\ X \in \mathcal{T} \mid \operatorname{Ext}^i_{\mathcal{T}}(X, Y) = 0 \ \ \textit{for any } i \in \mathbb{Z} \textit{ and any } Y \in \langle S\rangle\ \} \\ =&{}^{\perp}S. \end{aligned}$$

For simple Λ-modules S the invariant δ_S is a Betti number in the following sense:

Lemma B.2.17. *Let P be an indecomposable projective Λ-module. Set $S = \operatorname{top} P$. For any minimal projective complex $P_\bullet \in \mathrm{D^b}(\Lambda)$ and any $k \in \mathbb{Z}$ the following holds:*

$$\delta_S^{(k)}(P_\bullet) = \textit{number of direct summands of type } P \textit{ in } P_k.$$

In particular, $\delta_S(P_\bullet)$ is the total number of direct summands of type P in $P_\bullet$. □

B.2.5 Representation theory of the two-cycle quiver

In this section we will recall the well-understood representation theory of the following quiver:

$$\widetilde{Q} \qquad \begin{tikzcd} \underset{\diamond}{\bullet} \arrow[r, bend left, "\mathsf{b}"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}"] \end{tikzcd}$$

B.2.5.1 Abelian category of the two-cycle quiver

Let Γ denote the completion of the path algebra $\Bbbk\widetilde{Q}$ with respect to the arrow ideal (b, a). Let $\mathbf{R}$ be the local ring of formal power series $\Bbbk[\![x]\!]$ and $\mathfrak{m} = (x)$ its maximal ideal. The algebra Γ is isomorphic to the following subalgebra of $\operatorname{Mat}_{2\times 2}(\mathbf{R})$:

$$\Gamma \xrightarrow{\ \sim\ } \begin{array}{c} \begin{array}{cc} \scriptstyle\widetilde{P}_\diamond & \scriptstyle\widetilde{P}_\star \end{array} \\ \begin{bmatrix} \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} \end{bmatrix} \end{array}$$

In the following we set $c := b\,a$ and $f := a\,b$. The indecomposable projective modules of Γ and their radicals are given as follows:

$$\widetilde{P}_\diamond = \Gamma e_\diamond = \begin{pmatrix} \mathbf{R} \\ \mathbf{R} \end{pmatrix} = \langle\ e_\diamond, c^{n-1}\,b, f^n \mid n \in \mathbb{N}^+\ \rangle \qquad \operatorname{rad} \widetilde{P}_\diamond = \begin{pmatrix} \mathfrak{m} \\ \mathbf{R} \end{pmatrix}$$

$$\widetilde{P}_\star = \Gamma e_\star = \begin{pmatrix} \mathfrak{m} \\ \mathbf{R} \end{pmatrix} = \langle\ e_\star, a\,c^{n-1}, c^n \mid n \in \mathbb{N}^+\ \rangle \qquad \operatorname{rad} \widetilde{P}_\star = \begin{pmatrix} \mathfrak{m} \\ \mathfrak{m} \end{pmatrix} \cong \widetilde{P}_\diamond$$

Both radicals are maximal submodules and projective. Equivalently, Γ is a *hereditary Nakayama* algebra. The modules of such algebras are well-understood:

Theorem B.2.18 ([**Dro81**]). *Let H be a hereditary Nakayama algebra (of possibly infinite dimension). Let M be an indecomposable non-projective H-module.*
Then there is some indecomposable projective H-module $\widetilde{P}$ and some $n \in \mathbb{N}^+$ such that

$$M \cong \widetilde{P} / \operatorname{rad}^n(\widetilde{P}).$$

In particular, hereditary Nakayama algebras have *discrete* or *countable* representation type.

Corollary B.2.19. *Let M be an indecomposable finite-dimensional module over the completed algebra Γ of the two-cycle quiver. By the theorem B.2.18 there is some $n \in \mathbb{N}^+$ such that M is isomorphic to one of the following four modules:*

module	$\widetilde{P}_\diamond/(c^{n-1}b)$	$\widetilde{P}_\diamond/(f^n)$	$\widetilde{P}_\star/(ac^{n-1})$	$\widetilde{P}_\star/(c^n)$
notation	$(2n-1)_\diamond$	$(2n)_\diamond$	$(2n-1)_\star$	$(2n)_\star$

Let us comment on the above notation.
The indecomposable Γ-module M denoted by $(n)_i$ for some $n \in \mathbb{N}^+$ and $i \in \{\diamond, \star\}$ is isomorphic to the quotient

$$(n)_i \cong \widetilde{P}_i / \operatorname{rad}^n(\widetilde{P}_i).$$

Here, the number n is equal to the following parameters:

- the length of M or the dimension of M as $\Bbbk$-vector space.
- The Loewy length of M, that is, the smallest number such that that $\operatorname{rad}^n(M) = 0$ and $\operatorname{rad}^{n-1}(M) \neq 0$.
- the nilpotency degree of M; that is, the smallest $n \in \mathbb{N}^+$ such that $c^n M = 0$ and $f^n M = 0$

Let $\operatorname{nil.rep}(\widetilde{Q})$ denote the finite-dimensional representations of the quiver $\widetilde{Q}$ on which c acts as a nilpotent operator. By Corollary B.1.8, the category of finite-dimensional Γ-modules is equivalent to the category of such representations:

$$\Gamma\text{-}\operatorname{fd.mod} \xrightarrow{\sim} \operatorname{nil.rep}_{\Bbbk}(\widetilde{Q}) \,. \tag{B.2.1}$$

To present the indecomposable nilpotent representations of the quiver $\widetilde{Q}$ we need some notation.
Let $n \in \mathbb{N}^+$. Let Id_n denote the identity matrix of size n and $J_n(0)$ denote the Jordan block of size n and eigenvalue 0. Furthermore, let $G_n : \Bbbk^{n-1} \longrightarrow \Bbbk^n$ and $H_n \in \Bbbk^n \longrightarrow \Bbbk^{n-1}$ be the following matrices:

$$G_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad H_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}$$

Corollary B.2.19 implies the following statement:

Corollary B.2.20. *Let V be an indecomposable nilpotent finite-dimensional representation of $\widetilde{Q}$. Then there is an $n \in \mathbb{N}^+$ such that V is isomorphic to one of the following four representations:*

$$
\begin{array}{cccc}
(2n-1)_\diamond & (2n)_\diamond & (2n-1)_\star & (2n)_\star \\
\begin{tikzcd} \Bbbk^n \arrow[r, bend left, "H_n"] & \Bbbk^{n-1} \arrow[l, bend left, "G_n"] \end{tikzcd} &
\begin{tikzcd} \Bbbk^n \arrow[r, bend left, "\mathrm{Id}_n"] & \Bbbk^{n} \arrow[l, bend left, "J_n"] \end{tikzcd} &
\begin{tikzcd} \Bbbk^{n-1} \arrow[r, bend left, "G_n"] & \Bbbk^{n} \arrow[l, bend left, "H_n"] \end{tikzcd} &
\begin{tikzcd} \Bbbk^n \arrow[r, bend left, "J_n"] & \Bbbk^{n} \arrow[l, bend left, "\mathrm{Id}_n"] \end{tikzcd}
\end{array}
$$

With the description above it is straightforward to compute the following invariants of each indecomposable module:

M	$\underline{\dim}(M)$	$\operatorname{top}(M)$	$\operatorname{soc}(M)$	$\operatorname{rad}(M)$	$M/\operatorname{soc}(M)$	$\operatorname{ll}(M)$
$(2n)_\diamond$	(n,n)	$S_\diamond$	$S_\star$	$(2n-1)_\star$	$(2n-1)_\diamond$	$2n$
$(2n)_\star$	(n,n)	$S_\star$	$S_\diamond$	$(2n-1)_\diamond$	$(2n-1)_\star$	$2n$
$(2n+1)_\diamond$	$(n+1,n)$	$S_\diamond$	$S_\diamond$	$(2n)_\star$	$(2n)_\diamond$	$2n+1$
$(2n+1)_\star$	$(n,n+1)$	$S_\star$	$S_\star$	$(2n)_\diamond$	$(2n)_\star$	$2n+1$

At last, we will describe the morphism structure in $\Gamma\text{-}\mathrm{mod}$. For each indecomposable finite-dimensional Γ-module M we have the natural morphisms

$$\operatorname{rad}(M) \hookrightarrow M \qquad \text{and} \qquad M \twoheadrightarrow M/\operatorname{soc}(M)$$

In fact, the radical embeddings and the socle projections are exactly all irreducible morphisms in $\Gamma\text{-}\mathrm{fd.mod}$. The Auslander-Reiten quiver of $\Gamma\text{-}\mathrm{fd.mod}$ is given by one homogeneous tube of rank two:

$$
\begin{tikzcd}
\vdots & & \vdots & & \vdots \\
(6)_\diamond \arrow[dr, two heads] \arrow[dd, dash, dashed] & & (6)_\star \arrow[dr, two heads] & & (6)_\diamond \arrow[dd, dash, dashed] \\
& (5)_\diamond \arrow[ur, hook] \arrow[dr, two heads] & & (5)_\star \arrow[ur, hook] \arrow[dr, two heads] & \\
(4)_\star \arrow[dr, two heads] \arrow[dd, dash, dashed] & & (4)_\diamond \arrow[ur, hook] \arrow[dr, two heads] & & (4)_\star \arrow[dd, dash, dashed] \\
& (3)_\star \arrow[ur, hook] \arrow[dr, two heads] & & (3)_\diamond \arrow[ur, hook] \arrow[dr, two heads] & \\
(2)_\diamond \arrow[dr, two heads] \arrow[uurr, phantom, ""] & & (2)_\star \arrow[ur, hook] \arrow[dr, two heads] & & (2)_\diamond \\
& (1)_\diamond \arrow[ur, hook] & & (1)_\star \arrow[ur, hook] &
\end{tikzcd}
$$

The derived category of the two-cycle quiver

Next, we will describe the structure of the bounded derived category $\mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$ of Γ-modules. The category $\Gamma\text{-}\mathrm{mod}$ can be viewed as a subcategory of $\mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$:

$$\Gamma\text{-}\mathrm{mod} \hookrightarrow \mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$$

The functor between these categories maps a Γ-module to its projective resolution.

By Theorem B.2.5 of Dold any indecomposable projective complex $\widetilde{P}_\bullet \in \mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$ is isomorphic to some shift of a projective resolution of some indecomposable Γ-module. Using the description of indecomposable Γ-modules in B.2.19 we obtain the following statement:

Corollary B.2.21. *Let $\widetilde{P}_\bullet$ be an indecomposable projective complex in $\mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$. Then there is some shift $d \in \mathbb{Z}$ such that $\widetilde{P}_\bullet$ is isomorphic to some projective resolution of the following list:*

$$\begin{array}{rccc@{\qquad}rccc}
\textit{notation} & & \widetilde{P}_\bullet & & \textit{notation} & & \widetilde{P}_\bullet & \\
\widetilde{P}_\diamond[d] & 0 & \longrightarrow & \widetilde{P}_\diamond & \widetilde{P}_\star[d] & 0 & \longrightarrow & \widetilde{P}_\star \\
(2n-1)_\diamond[d] & \widetilde{P}_\star & \xrightarrow{\cdot c^{n-1}b} & \widetilde{P}_\diamond & (2n-1)_\star[d] & \widetilde{P}_\diamond & \xrightarrow{\cdot ac^{n-1}} & \widetilde{P}_\star \\
(2n)_\diamond[d] & \widetilde{P}_\diamond & \xrightarrow{\cdot f^{n}} & \widetilde{P}_\diamond & (2n)_\star[d] & \widetilde{P}_\star & \xrightarrow{\cdot c^{n}} & \widetilde{P}_\star \\
\scriptstyle\textit{degree:} & \scriptstyle d+1 & & \scriptstyle d & \scriptstyle\textit{degree:} & \scriptstyle d+1 & & \scriptstyle d
\end{array}$$

Let us describe the morphisms in $\mathrm{D^b}(\Gamma\text{-}\mathrm{mod})$.

First, there are the morphisms coming from irreducible homomorphisms of modules over Γ at each degree $d \in \mathbb{Z}$:

$$\begin{tikzcd}
\vdots \arrow[d, two heads] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
(4)_\diamond[d] \arrow[d, two heads] & \widetilde{P}_\diamond \arrow[r, "\cdot f^2"] \arrow[d, "\cdot a"'] & \widetilde{P}_\diamond \arrow[d, equal] \\
(3)_\diamond[d] \arrow[d, two heads] & \widetilde{P}_\star \arrow[r, "\cdot bf"] \arrow[d, "\cdot b"'] & \widetilde{P}_\diamond \arrow[d, equal] \\
(2)_\diamond[d] \arrow[d, two heads] & \widetilde{P}_\diamond \arrow[r, "\cdot f"] \arrow[d, "\cdot a"'] & \widetilde{P}_\diamond \arrow[d, equal] \\
(1)_\diamond[d] & \widetilde{P}_\star \arrow[r, "\cdot b"] & \widetilde{P}_\diamond
\end{tikzcd}
\qquad
\begin{tikzcd}
(1)_\star[d] \arrow[d, hook] & \widetilde{P}_\diamond \arrow[r, "\cdot a"] \arrow[d, equal] & \widetilde{P}_\star \arrow[d, "\cdot b"] \\
(2)_\diamond[d] \arrow[d, hook] & \widetilde{P}_\diamond \arrow[r, "\cdot f"] \arrow[d, equal] & \widetilde{P}_\diamond \arrow[d, "\cdot a"] \\
(3)_\star[d] \arrow[d, hook] & \widetilde{P}_\diamond \arrow[r, "\cdot ac"] \arrow[d, equal] & \widetilde{P}_\star \arrow[d, "\cdot b"] \\
(4)_\diamond[d] \arrow[d, hook] & \widetilde{P}_\diamond \arrow[r, "\cdot f^2"] \arrow[d, equal] & \widetilde{P}_\diamond \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

There is the dual picture:

$$
\begin{tikzcd}
\vdots \arrow[d, two heads] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
(4)_\star[d] \arrow[d, two heads] & \widetilde{P}_\star \arrow[r, "\cdot c^2"] \arrow[d, "\cdot b"'] & \widetilde{P}_\star \arrow[d, equal] \\
(3)_\star[d] \arrow[d, two heads] & \widetilde{P}_\diamond \arrow[r, "\cdot ac"] \arrow[d, "\cdot a"'] & \widetilde{P}_\star \arrow[d, equal] \\
(2)_\star[d] \arrow[d, two heads] & \widetilde{P}_\star \arrow[r, "\cdot c"] \arrow[d, "\cdot b"'] & \widetilde{P}_\star \arrow[d, equal] \\
(1)_\star[d] & \widetilde{P}_\diamond \arrow[r, "\cdot a"] & \widetilde{P}_\star
\end{tikzcd}
\qquad
\begin{tikzcd}
(1)_\diamond[d] \arrow[d, hook] & \widetilde{P}_\star \arrow[r, "\cdot b"] \arrow[d, equal] & \widetilde{P}_\diamond \arrow[d, "\cdot a"] \\
(2)_\star[d] \arrow[d, hook] & \widetilde{P}_\star \arrow[r, "\cdot c"] \arrow[d, equal] & \widetilde{P}_\star \arrow[d, "\cdot b"] \\
(3)_\diamond[d] \arrow[d, hook] & \widetilde{P}_\star \arrow[r, "\cdot cb"] \arrow[d, equal] & \widetilde{P}_\diamond \arrow[d, "\cdot a"] \\
(4)_\star[d] \arrow[d, hook] & \widetilde{P}_\star \arrow[r, "\cdot c^2"] \arrow[d, equal] & \widetilde{P}_\star \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Second, there are morphisms corresponding to extensions of Γ-modules.

$$
\begin{tikzcd}
(2n-1)_\star[d] \arrow[d] & 0 \arrow[r] \arrow[d] & \widetilde{P}_\diamond \arrow[r, "\cdot ac^{n-1}"] \arrow[d, equal] & \widetilde{P}_\star \arrow[d, "\cdot b"] \\
(2n)_\diamond[d] \arrow[d] & 0 \arrow[r] \arrow[d] & \widetilde{P}_\diamond \arrow[r, "\cdot f^n"] \arrow[d, equal] & \widetilde{P}_\diamond \arrow[d] \\
\widetilde{P}_\diamond[d+1] \arrow[d] & 0 \arrow[r] \arrow[d] & \widetilde{P}_\diamond \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
(2n)_\diamond[d+1] \arrow[d] & \widetilde{P}_\diamond \arrow[r, "\cdot f^n"] \arrow[d, "\cdot a"'] & \widetilde{P}_\diamond \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
(2n-1)_\diamond[d+1] & \widetilde{P}_\star \arrow[r, "\cdot c^{n-1}b"] & \widetilde{P}_\diamond \arrow[r] & 0
\end{tikzcd}
$$

Of course, there are the same morphisms with $\widetilde{P}_\star$ in the middle degree of the diagram, indicated below:

$$(2n-1)_\diamond[d] \longrightarrow (2n)_\star[d] \longrightarrow \widetilde{P}_\star[d+1]$$

$$\widetilde{P}_\star[d+1] \longrightarrow (2n-1)_\star[d+1] \longrightarrow (2n)_\star[d+1]$$

At last there are non-bijective endomorphisms of shifted projectives:

$$\begin{array}{cccc}
\widetilde{P}_{\diamond}[d] & \widetilde{P}_{\diamond}[d] & \widetilde{P}_{\star}[d] & \widetilde{P}_{\star}[d] \\
\big\downarrow{\scriptstyle \cdot f^n} & \big\downarrow{\scriptstyle \cdot ac^{n-1}} & \big\downarrow{\scriptstyle \cdot c^n} & \big\downarrow{\scriptstyle \cdot c^{n-1}b} \\
\widetilde{P}_{\diamond}[d] & \widetilde{P}_{\star}[d] & \widetilde{P}_{\diamond}[d] & \widetilde{P}_{\star}[d]
\end{array}$$

Let us remark that any morphism which starts or ends in a shifted projective module $\widetilde{P}_{\diamond}[d]$ or $\widetilde{P}_{\star}[d]$ is not an irreducible morphism in $\mathrm{D}^{\mathrm{b}}(\Gamma\text{-}\mathrm{mod})$.

Lemma B.2.22. *Let ψ be a morphism of indecomposable projective complexes in* $\mathrm{D}^{\mathrm{b}}(\Gamma\text{-}\mathrm{mod})$. *Then ψ is a composition of some morphisms from the diagrams above.*

Remark B.2.23. *Let $\psi : \Lambda \xrightarrow{\sim} \Gamma$ be an isomorphism of rings. Then ψ gives rise to an equivalence of categories $\psi^* : \Gamma\text{-}\mathrm{mod} \longrightarrow \Lambda\text{-}\mathrm{mod}$ as follows. For any Γ-module M the module $\psi^*(M)$ has M as underlying set. The ring Λ acts on the module $\psi^*(M)$ by $(a, m) \longmapsto \psi(a) \cdot m$.*

B.3 Spherical twist functors

Throughout this section let Λ be some ring of finite global dimension and $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\Lambda)$.

B.3.1 Spherical twists

Let $S \in \mathcal{T}$. We will recall some properties of the *dual twist functor* $\mathbb{T}_S^{\vee}$ associated to S.

We need to introduce the following notation:

- Let $X, Y \in \mathcal{T}$. We denote by $\mathrm{Hom}^{\bullet}_{\mathcal{T}}(X, Y)$ the complex in $\mathcal{T}$ given by
$$\mathrm{Hom}^{\bullet}_{\mathcal{T}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_{\mathcal{T}}(X, Y)[-i]$$
and zero differentials.

Definition B.3.1. *The* dual twist functor *associated to S is given by*

$$\mathbb{T}_S^{\vee}(X) := \mathrm{cone}\Big(X \xrightarrow{\mathrm{ev}} \mathrm{Hom}^{\bullet}_{\mathrm{E}^{op}}(\mathrm{Hom}^{\bullet}_{\mathcal{T}}(X, S), S) \Big)[-1] \qquad \text{for any } X \in \mathcal{T}, \tag{B.3.1}$$

where $\mathrm{E} := \mathrm{End}_{\mathcal{T}}(S)$ *and* ev *is the natural evaluation map.*

Definition B.3.2. *Let $S \in \mathcal{T}$ and $n \in \mathbb{N}_0$. Then S is n-*spherical *if and only if the following two conditions hold*:

(1) S is n-Calabi-Yau: $\mathrm{Hom}_{\mathcal{T}}(S, Y) \cong \mathbb{D}\,\mathrm{Hom}_{\mathcal{T}}(Y, S[n])$ *for any $Y \in \mathcal{T}$.*

(2) S is n-spherelike: $\operatorname{Hom}_{\mathcal{T}}(S, S[i]) \cong \begin{cases} \Bbbk & i \in \{0,n\}, \\ 0 & i \notin \{0,n\}. \end{cases}$

The notion of spherical objects is motivated by the following result:

Proposition B.3.3 ([**ST01**, Proposition 2.10])**.** *Let $S \in \mathcal{T}$ be an n-spherical object for some $n \in \mathbb{N}_0$. Then the dual twist functor $\mathbb{T}_S^{\vee}$ is an* autoequivalence *of $\mathcal{T}$.*

B.3.2 The subcategory generated by a spherical object

In this subsection we will consider the following setup:

- Let $\mathcal{T} = \mathrm{D^b}(\Lambda\text{-}\mathrm{mod})$ be the bounded derived category of some order Λ of finite global dimension.
- Let S be an n-spherical object in $\mathcal{T}$ with $n \in \mathbb{N}^{\geq 2}$.
- Let $\langle S \rangle$ denote thick category generated by S, that is, the smallest full subcategory of $\mathcal{T}$ containing S and closed under cones, shifts and direct summands.

The results cited in this subsection describe the thick subcategory $\langle S \rangle$. These statements are formulated in terms of *differential graded* algebras. However, the formalism of dg-algebras can be treated as a black box and will not be needed in the sequel.

- We will consider the polynomial ring $\Bbbk[t]$ as a dg-algebra with trivial differential and $\deg t = -n+1$.
- Let $\mathrm{D}(\Bbbk[t])$ denote the derived category of dg-modules over $\Bbbk[t]$.
- let $\mathrm{D_{fd}}(\Bbbk[t])$ be the full subcategory of $\mathrm{D}(\Bbbk[t])$ of dg-modules with homology of finite total dimension.

Theorem B.3.4. [**KYZ09**, Section 2] *There is an equivalence of categories:*

$$\langle S \rangle \xrightarrow{\sim} \mathrm{D_{fd}}(\Bbbk[t])$$

There is an explicit description of the category on the right:

Theorem B.3.5. [**Jor04**, Proposition 8.10, Theorem 8.13]*,*

(1) The indecomposable objects of $\mathrm{D_{fd}}(\Bbbk[t])$ are given by $\Bbbk[t]/(t^m)[d]$ for some shift $d \in \mathbb{N}$ and $m \in \mathbb{N}^+$.

(2) The Auslander-Reiten quiver of $\mathrm{D_{fd}}(\Bbbk[t])$ is given by $n-1$ components of type $\mathbb{Z}\mathbb{A}_\infty$.

Remark B.3.6. *Jorgensen's result on the Auslander-Reiten components $\mathrm{D_{fd}}(\Bbbk[t])$ was actually proven under the assumption that the field $\Bbbk$ has characteristic zero. His result was extended to arbitrary fields by Schmidt* [**Sch10**]*.*

B.3.3 Generalized twist functors

There is another type of autoequivalences on triangulated categories beside spherical twists. These autoequivalences are twist functors associated to some sequence of certain special objects. They have been studied under the names *tubular mutations* [**Mel97**], *generalized spherical collections* [**vR12**] and *exceptional cycles* [**BPP13**].

We review the definition and main properties of generalized twists following [**vR12**].

Definition B.3.7. *Let $\mathcal{S} = \{ S_1, S_2, \dots S_d \}$ be a collection of some objects from $\mathcal{T}$.*

(1) The twist functor $\mathbb{T}_{\mathcal{S}} \colon \mathcal{T} \longrightarrow \mathcal{T}$ associated to $\mathcal{S}$ *is defined as*

$$\mathbb{T}_{\mathcal{S}}(X) = \operatorname{cone}\Big(\bigoplus_{j=1}^{n} \operatorname{Hom}^{\bullet}_{\mathcal{T}}(S_j, X) \otimes_{\Bbbk} S_j \xrightarrow{[\mathrm{ev}_1, \dots, \mathrm{ev}_n]} X \Big) \qquad \text{for any } X \in \mathcal{T}.$$

(2) The dual twist functor $\mathbb{T}^{\vee}_{\mathcal{S}} \colon \mathcal{T} \longrightarrow \mathcal{T}$ associated to $\mathcal{S}$ *is defined as*

$$\mathbb{T}^{\vee}_{\mathcal{S}}(X) = \operatorname{cone}\Big(X \xrightarrow{\begin{bmatrix} \mathrm{coev}_1 \\ \vdots \\ \mathrm{coev}_n \end{bmatrix}} \bigoplus_{j=1}^{n} \operatorname{Hom}^{\bullet}_{\Bbbk}(\operatorname{Hom}^{\bullet}_{\mathcal{T}}(X, S_j), S_j)[-1] \Big) \quad \text{for any } X \in \mathcal{T}.$$

Proposition B.3.8. *For any collection $\mathcal{S}$ of objects in $\mathcal{T}$ the functor $\mathbb{T}^{\vee}_{\mathcal{S}}$ is left adjoint to $\mathbb{T}_{\mathcal{S}}$.*

Generalized twist functors are interesting for the following type of collections:

Definition B.3.9. *Let $\mathcal{S} = (S_1, S_2, \dots, S_d)$ be a sequence of objects from $\mathcal{T}$ for some $d \in \mathbb{N}^{+}$ and let $n \in \mathbb{N}_0$. The sequence $\mathcal{S}$ is a* generalized n-spherical collection *if it satisfies the following two conditions*:

(1) $\mathbb{S}(S_j) \cong S_{j+1}[n]$ for any $j \in \mathbb{Z}_d$,

(2) for any $i, j \in \mathbb{Z}_d$ it holds that

$$\operatorname{Ext}^{q}_{\mathcal{T}}(S_i, S_j) \cong \begin{cases} \Bbbk & \text{if } q = 0 \text{ and } j = i, \text{ or } q = n \text{ and } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

In the following let us call “generalized spherical collections” just “spherical collections” for brevity.

Theorem B.3.10. [**vR12**, Theorem 3.11] *Let $\mathcal{S}$ be an n-spherical sequence of objects in $\mathcal{T}$ for some $n \in \mathbb{N}_0$. Then the twist functor $\mathbb{T}_{\mathcal{S}}$ is an auto-equivalence of $\mathcal{T}$ with quasi-inverse $\mathbb{T}^{\vee}_{\mathcal{S}}$.*

Remark B.3.11. *(1) A single object forms a n-spherical collection if and only if it is n-spherical in the sense of Definition B.3.2.*

(2) Let $\mathcal{S} = (S_1, S_2, \dots, S_d)$ be a n-spherical collection with $d \geq 2$. Then for any $i \in \mathbb{Z}_d$ the object S_i is exceptional *and* fractionally $(d\,n, d)$-Calabi-Yau:

$$\operatorname{Ext}^{q}_{\mathcal{T}}(S_i, S_i) = \begin{cases} \Bbbk & q = 0 \\ 0 & q \neq 0 \end{cases} \qquad \text{and} \qquad \mathbb{S}^{d}(S_i) \cong S_i[d\,n].$$

(3) Vice versa, let $S \in \mathcal{T}$ be exceptional and $(d\,n, d)$-Calabi-Yau for some $n \in \mathbb{N}_0$ and $d \geq 2$. Then the sequence $(\, S, \mathbb{S}(S), \mathbb{S}^2(S), \ldots \mathbb{S}^{d-1}(S)\,)$ is generalized n-spherical.

We recall that $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\Lambda)$ was the bounded derived category of a some ring Λ with finite global dimension. In this context, it is possible to express twist functors as "standard functors", that is, as tensor products with bimodules:

Theorem B.3.12. [**IR08**, Theorem 6.14] *Let Λ be an order of finite global dimension and let $\mathcal{S}$ be a generalized spherical collection $\mathcal{S}$ in $\mathrm{D}^{\mathrm{b}}(\Lambda)$. Then there is an isomorphism of functors*

$$\mathbb{T}^{\vee}_{\mathcal{S}}(\_) \cong \mathbb{T}^{\vee}_{\mathcal{S}}(\Lambda) \underset{\Lambda}{\otimes} \_ \qquad \textit{on } \mathrm{D}^{\mathrm{b}}(\Lambda).$$

This theorem was proven by Iyama and Reiten for spherical twists. Their proof generalizes directly to the setup above.

We will apply the theorem above frequently in the following special situation:

Corollary B.3.13. *For Λ as above, let $\mathcal{S} = (S_1, S_2, \ldots, S_d)$ be a generalized n-spherical sequence of* simple *Λ-modules. Let $\pi_{\mathcal{S}}$ denote the canonical projection*

$$\pi_{\mathcal{S}} : \Lambda \twoheadrightarrow \bigoplus_{j=1}^{d} S_j$$

considered as a morphism of Λ-bimodules. Then there is an isomorphism of functors

$$\mathbb{T}^{\vee}_{\mathcal{S}}(\_) \cong \ker(\pi_{\mathcal{S}}) \underset{\Lambda}{\otimes} \_ \qquad \textit{on } \mathrm{D}^{\mathrm{b}}(\Lambda).$$

PROOF. Under the assumptions above, we need to show that

$$\mathbb{T}^{\vee}_{\mathcal{S}}(\Lambda) \cong \ker(\pi_{\mathcal{S}})$$

We have to compute

$$\mathbb{T}^{\vee}_{\mathcal{S}}(\Lambda) = \mathrm{cone}\Big(\Lambda \xrightarrow{\begin{bmatrix} \mathrm{coev}_1 \\ \vdots \\ \mathrm{coev}_d \end{bmatrix}} \bigoplus_{j=1}^{d} \mathrm{Hom}^{\bullet}_{\mathrm{E}^{op}}(\mathrm{Hom}^{\bullet}_{\Lambda}(\Lambda, S_j), S_j) \Big)[-1],$$

where Λ is considered as a stalk complex in degree zero and $\mathrm{E} = \mathrm{End}_{\Lambda}(S) \cong \mathbb{k}^d$. Since Λ is projective it follows that

$$\mathrm{Hom}^{\bullet}_{\Lambda}(\Lambda, S_j) = \mathrm{Hom}_{\Lambda}(\Lambda, S_j) \cong S_j \quad \text{for any} \quad 1 \leq j \leq d.$$

It is straightforward to check that there is an isomorphism of Λ-bimodules

$$\mathrm{Hom}^{\bullet}_{\mathbb{k}}(S_j, S_j) = \mathrm{Hom}_{\mathbb{k}}(S_j, S_j) \cong S_j$$

It follows that

$$\mathbb{T}^{\vee}_{\mathcal{S}}(\Lambda) \cong \mathrm{cone}\Big(\Lambda \xrightarrow{\begin{bmatrix} \mathrm{coev}_1 \\ \vdots \\ \mathrm{coev}_d \end{bmatrix}} \bigoplus_{j=1}^{d} S_j \Big)[-1],$$

where both complexes are stalk complexes of Λ-bimodules at degree zero. It can be checked that the map coev_j is the natural projection $\Lambda \twoheadrightarrow S_j$ for any $1 \leq j \leq d$, which implies the claim. $\square$

B.3.4 A Lemma on generalized spherical twists

There are many examples of triangulated categories $\mathcal{T}$ with Auslander-Reiten translation τ and objects $X \in \mathcal{T}$ of higher τ-period:

$$\tau^m(X) \cong X \text{ for some } m \in \mathbb{N}^+ \quad \text{and } \tau(X) \ncong X$$

Next, we show that this cannot happen for generalized n-spherical twists with $n \neq 1$:

$$\text{if } (\mathbb{T}_{\mathcal{S}}^{\vee})^m(X) \cong X \text{ for some } m \in \mathbb{N}^+, \quad \text{then } \mathbb{T}_{\mathcal{S}}^{\vee}(X) \cong X.$$

Let $\mathcal{S}$ be a collection of objects from $\mathcal{T}$. Let us recall our notation for the left-perpendicular category associated to $\mathcal{S}$:

$${}^{\perp}\mathcal{S} := \{ \ X \in \mathcal{T} \mid \ \operatorname{Ext}^i_{\mathcal{T}}(X, S) = 0 \ \text{ for any } i \in \mathbb{Z} \ \text{ and for all } \ S \in \mathcal{S} \ \}.$$

We recall the following basic property of the dual twist functor $\mathbb{T}_{\mathcal{S}}^{\vee}$. Let $X \in \mathcal{T}$.

$$\text{if } X \in {}^{\perp}\mathcal{S}, \ \text{ then } \mathbb{T}_{\mathcal{S}}^{\vee}(X) \cong X. \tag{B.3.2}$$

The converse is true in the following setup:

Lemma B.3.14. *Let $X \in \mathcal{T}$ and let $\mathcal{S}$ be a generalized d-spherical sequence of objects from $\mathcal{T}$ for some $n \in \mathbb{N}_0$ with $n \neq 1$. Then the following conditions are equivalent*:

(1) $X \in {}^{\perp}\mathcal{S}$,

(2) $\mathbb{T}_{\mathcal{S}}^{\vee}(X) \cong X$,

(3) there is some $m \in \mathbb{N}^+$ such that $(\mathbb{T}_{\mathcal{S}}^{\vee})^m(X) \cong X$,

PROOF. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It remains to show that (3) implies (1):
Let $X \in \mathcal{T}$ and let $\mathcal{S} = \{ \ S_1, S_2, \dots, S_d \ \}$ be a generalized n-spherical sequence of objects from $\mathcal{T}$ with $n \neq 1$. Let $i \in \mathbb{Z}$ and $m \in \mathbb{N}^+$ such that $(\mathbb{T}_{\mathcal{S}}^{\vee})^m(X) \cong X$. Let S be the direct sum of all objects in $\mathcal{S}$. We claim that $\operatorname{Ext}^i_{\mathcal{T}}(X, S) = 0$.
Since $\mathcal{T}$ is bounded from the right, there is some $l_0 \in \mathbb{Z}$ such that $\operatorname{Ext}^l_{\mathcal{T}}(X, S) = 0$ for all $l \leq l_0$. In the following we choose $k \in \mathbb{Z}$ to be "small enough", more precisely, let $k \in \mathbb{Z}$ be such that $i + k\, m\, (n-1) < l_0$.
Since $\mathbb{T}_{\mathcal{S}}^{\vee}$ is fully faithful and $\mathbb{T}_{\mathcal{S}}^{\vee}(S) = S[n-1]$, it follows that

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{T}}(X, S) &\cong \operatorname{Ext}^i_{\mathcal{T}}((\mathbb{T}_{\mathcal{S}}^{\vee})^{k\,m}(X), (\mathbb{T}_{\mathcal{S}}^{\vee})^{k\,m}(S)) \cong \operatorname{Ext}^i_{\mathcal{T}}(X, S[k\,m\,(n-1)]) \\ &= \operatorname{Ext}^{i+k\,m\,(n-1)}_{\mathcal{T}}(X, S) = 0.\end{aligned}$$

It follows that $X \in {}^{\perp}\mathcal{S}$. $\square$

Remark B.3.15. *For* 1*-spherical twist functors the converse of* (B.3.2) *is* not *true in general. Consider the following setup:*

- *Let $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\mathbf{R})$, which is* 1*-Calabi-Yau, that is,* $\mathbb{S} \cong [1]$,
- *Let $S = \Bbbk$. Then S is a* 1*-spherical object in $\mathcal{T}$ such that* $\mathbb{T}_S^{\vee} \cong \tau \cong \mathrm{Id}$.
- *Now, let $X \in \mathrm{D}^{\mathrm{b}}(\mathbf{R})$ be the stalk complex of any non-zero $\mathbf{R}$-module.*

In this case, $\mathbb{T}_S(X) \cong \tau(X) \cong X$ but $\operatorname{Hom}_{\mathcal{T}}(X, S) = \operatorname{Hom}_{\mathbf{R}}(X, \Bbbk) \neq 0$, *so $X \notin {}^{\perp}S$.*

APPENDIX C

General facts in Lie theory

In the present appendix we collect some facts from the representation theory of semisimple Lie groups and Lie algebras.

C.1 Harish-Chandra's correspondence

In this section we recall the relationship between admissible representations of Lie groups and their Harish-Chandra modules.

C.1.1 Generalized Lorentz groups of rank one

In this Appendix we will be concerned with the following setup:

- Let G be a *linear semisimple group* in the sense of **[Kna86]**, that is, G is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, which is stable under complex conjugate transpose and has finite center.
- Let K be a *maximal compact subgroup* of G. Such a subgroup may be given as the fixed points of the Cartan involution θ :
$$K = G^\theta = \{ \ g \in G \mid \theta(g) := (\overline{g}^t)^{-1} = g \ \}.$$
  The choice of the maximal compact subgroup of G is not essential, since any two maximal compact subgroups of G are conjugate.
- Let $\mathfrak{g}_0$ denote the Lie algebra of G considered as a *real* Lie algebra. Let $\mathfrak{g}$ denote the *complexified Lie algebra* of $\mathfrak{g}_0$.

Our first goal is to describe *Harish-Chandra's correspondence*: a relationship between certain analytic representations of the Lie group G and certain algebraic modules associated to the Lie algebra $\mathfrak{g}$ and the compact group K.

In fact, we shall be concerned mainly with the following classical Lie groups:

Example C.1.1. *(1) Our main example will be the simplest non-compact Lie group*

$$\mathrm{SL}(2, \mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} : \ a\, d - b\, c = 1 \ \}$$

(2) We will view the special linear group $\mathrm{SL}(2,\mathbb{C})$ *as a* real *Lie group.*

(3) any $n \in \mathbb{N}^+$ *let* $\mathrm{SO}(n,1)$ *denote the generalized Lorentz group of rank one*:

$$\mathrm{SO}(n,1) = \{\ A \in \mathrm{SL}(n+1,\mathbb{R}) \mid A\,D\,A^T = D\ \} \quad \textit{where } D = \begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \tag{C.1.1}$$

The Lie group $\mathrm{SO}(n,1)$ *has two connected components. Its identity component* $\mathrm{SO}_e(n,1)$ *is given as follows*:

$$\mathrm{SO}_e(n,1) = \{\ A \in \mathrm{SO}(n,1) \mid a_{11} \geq 1\ \}$$

In these cases, the Lie algebra $\mathfrak{g}$ *and a maximal compact subgroup* K *are given as follows*:

G	$\mathfrak{g}$	K
$\mathrm{SL}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathrm{SO}(2)$
$\mathrm{SL}(2,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathrm{SU}(2)$
$\mathrm{SO}_e(n,1)$	$\mathfrak{so}(n+1)$	$\mathrm{SO}(n)$
$\mathrm{SO}(n,1)$	$\mathfrak{so}(n+1)$	$\mathrm{O}(n)$

Let us note that we do not assume that the Lie group G or K is connected. Moreover, Harish-Chandra's correspondence holds in the more general framework of *real reductive groups*, which have a more technical definition.

C.1.2 Admissible Representations of Lie groups

In this section we define the main class of analytic representations of Lie groups we will be interested in. Then we will consider some of these representations for the Lie group $\mathrm{SL}(2,\mathbb{R})$.

First, we need to define representations of Lie groups in general. Roughly speaking, a representation of a Lie group G on a topological space E is given by some continuous action of G on E. The technical details are given as follows:

Definition C.1.2. *As above, let* G *be a real linear semisimple Lie group.*

(1) Let E *be a complex topological vector space which is locally convex, complete and Hausdorff. Let* $\mathrm{Aut}(E)$ *denote the linear homeomorphisms of* E*. A* representation (π, E) *of* G *is a group homomorphism* $\pi\colon G \longrightarrow \mathrm{Aut}(E)$ *such that the action* $G \times E \longrightarrow E$ *given by* $(g,v) \longmapsto \pi(g)\,v$ *is continuous.*

(2) A representation (π, E) *of* G *is a Hilbert, Banach or Fréchet representation of* G *if* E *is a Hilbert, Banach respectively a Fréchet space.*

(3) A Hilbert representation (π, E) *is* unitary *if* $\pi(g)\cdot\pi(g)^* = \pi(g)^*\cdot\pi(g) = \mathrm{Id}$ *for any* $g \in G$*, where* $\pi(g)^*$ *denotes the adjoint operator of* $\pi(g)$*.*

Morphisms of representations of G *are given by continuous linear* G*-equivariant maps.*

The topological space for a representation of G will not be essential in the following. More importantly, we will distinguish between *irreducible* and *indecomposable* representations:

Definition C.1.3. *Let (π, E) be a representation of G.*

(1) A subrepresentation *of (π, E) is given by any* closed *subspace F of E such that $\pi(g)F \subseteq F$ for any $g \in G$. Note that such a subspace F yields a representation $(\pi|_F, F)$ of G.*

(2) (π, E) is irreducible *if its only subrepresentations are given by the trivial subrepresentations 0 and E.*

(3) (π, E) is completely reducible*, if E has a direct sum decomposition into irreducible subrepresentations*:

$$E = \widehat{\bigoplus_{i\in I}} E_i$$

If the index set I is infinite, the above direct sum is topological*, that is, the algebraic direct sum $\bigoplus_{i\in I} E_i$ is meant to be* dense *in E.*

(4) (π, E) is indecomposable *if any direct sum decomposition of E is given by the trivial subrepresentations 0 and E.*

Obviously, any irreducible representation is indecomposable.

Remark C.1.4. *Any* unitary *representation is completely reducible. In particular, any unitary representation is indecomposable if and only if it is irreducible.*

Over non-compact Lie groups there can be indecomposable representations which are neither irreducible nor completely reducible:

Example C.1.5. *Let $G = \mathrm{SL}(2,\mathbb{R})$. There is a natural action of G on $\mathbb{R}^2\backslash\{0\}$ by matrix multiplication*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,x + b\,y \\ c\,x + d\,y \end{pmatrix}$$

Let $\varepsilon \in \{0,1\}$ and $\nu \in \mathbb{C}$, and let $F = F_\varepsilon(\nu)$ be the following function space:

$$F = \left\{ \varphi : \mathbb{R}^2\backslash\{0\} \longrightarrow \mathbb{C} \text{ smooth} \;\middle|\; \begin{array}{l} \varphi(-x,-y) = (-1)^\varepsilon\,\varphi(x,y), \text{ and} \\ \varphi(\,t\,x, t\,y\,) = t^{\nu-1}\varphi(x,y) \text{ for any } t \in \mathbb{R}^+ \end{array} \right\}$$

There is a representation $\pi : G \longrightarrow \mathrm{Aut}(F)$ defined by

$$\pi(g)\cdot\varphi(x,y) = \varphi(\ g^{-1}\cdot(x,y)^T\ )$$

The space F of smooth functions carries a natural Fréchet topology. It can be checked that π defines an indecomposable Fréchet representation of $\mathrm{SL}(2,\mathbb{R})$.

Before we define admissible representations, let us first recall the representation theory of *compact* Lie groups.

For a compact Lie group K let us introduce the following notation:

- Let $\widehat{K}$ denote the set of isomorphism classes of irreducible representations of K.

- Let (π, E) be any representation of K and $\tau \in \widehat{K}$ be an irreducible representation of K on some space E_τ. Then we define the K-multiplicity of τ in π by $m_\tau = \dim_{\mathbb{C}} \operatorname{Hom}_K(E_\tau, E)$. We note that the multiplicity m_τ may be infinite.

The representation theory of compact Lie groups is well-understood:

Theorem C.1.6 (Peter-Weyl). *Let K be a compact Lie group.*

(1) Any irreducible representation of K is finite-dimensional and unitarizable.

(2) Any representation (π, E) of K can be decomposed into a (possibly infinite) direct sum of irreducible subrepresentations:

$$E = \widehat{\bigoplus_{\tau \in \widehat{K}}} E_\tau^{m_\tau}$$

In other words, any representation of K is completely reducible.

Let us note that for a representation E of K the multiplicity m_τ describes "how often the irreducible representation E_τ appears as a subrepresentation of E".

Example C.1.7. *Let $K = \mathrm{SO}(2)$, which is isomorphic to the circle group*:

$$\mathrm{SO}(2) = \{A_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mid 0 \le \vartheta < 2\,\pi\ \} \cong \mathbb{S}^1 = \{\ e^{i\vartheta} \mid 0 \le \vartheta < 2\pi\ \}$$

For any $k \in \mathbb{Z}$ let (π_k, E_k) denote the one-dimensional representation of K given by

$$\pi_k(A_\vartheta) \cdot v = e^{i\,k\,\vartheta}\, v \qquad \text{for any } v \in E_k = \mathbb{C}.$$

Then the following statements hold:

(1) Any irreducible representation of K is given by (π_k, E_k) for some $k \in \mathbb{Z}$.

(2) By the Peter-Weyl Theorem any representation E of K has a decomposition

$$E = \widehat{\bigoplus_{k \in \mathbb{Z}}} E_k^{m_k} \quad \text{for some } m_k \in \mathbb{N}_0 \cup \{\infty\}$$

From now, on let G again be any linear reductive group.

Definition C.1.8. *(1) A representation (π, E) of G is* weakly admissible *if the multiplicity $m_\tau = \dim_{\mathbb{C}} \operatorname{Hom}_K(E_\tau, E)$ is finite for any irreducible representation $\tau \in \widehat{K}$.*

(2) A weakly admissible representation (π, E) of G is admissible *if it is Noetherian, that is, every ascending chain of subrepresentations of E becomes stationary.*

We will note in the next section that any admissible representation of G satisfies also the decreasing chain condition on its subrepresentations. In other words, admissible representations have *finite length.*

The following theorem ensures that there are enough admissible representations of interest:

Theorem C.1.9 (Harish-Chandra). *Every irreducible unitary representation of G is admissible.*

Remark C.1.10. *There is an example of an irreducible but* not admissible *Banach representation of* $\mathrm{SL}(2,\mathbb{R})$ *by Soergel. On the other hand, according to W. Schmid "*all irreducible representations which have come up naturally in geometry, differential equations, physics, and number theory are admissible.*"* **[Sch05]**

Let $\mathrm{admrep}(G)$ denote the category of admissible representations over G.

Definition C.1.11. *(1) A morphism* $\phi : E \longrightarrow F$ *of admissible representations of* G *has* closed range *if* $\mathrm{coim}\,\phi = \phi(E)$ *is homeomorphic to* $\mathrm{im}\,\phi = \overline{\phi(E)}$*, the closure of* $\phi(E)$ *in* F*.*

(2) A sequence of representations $0 \longrightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} H \longrightarrow 0$ *is topologically exact, or an exact sequence in* $\mathrm{admrep}(G)$*, if* ϕ *has closed range.*

The following remark summarizes the properties of the category $\mathrm{admrep}(G)$

Remark C.1.12. *(1) The category* $\mathrm{admrep}(G)$ *is an additive category which admits kernels and cokernels.*

(2) Not every morphism in $\mathrm{admrep}(G)$ *has closed range. In particular,* $\mathrm{admrep}(G)$ *is not abelian. The category* $\mathrm{admrep}(G)$ *is* quasi-abelian *in the sense of P. Schneiders.*

(3) It follows by the Harish-Chandra correspondence that the category $\mathrm{admrep}(G)$ *is Hom-finite and every object in* $\mathrm{admrep}(G)$ *has finite length.*

C.1.3 Harish-Chandra modules

To define Harish-Chandra modules we need to define $(\mathfrak{g}, K)$-modules first.

Definition C.1.13. *Let* (π, E) *be a representation of the compact subgroup* K*. Then the subspace of* K*-finite vectors of* E *is given by*

$$E^{\mathrm{K\text{-}fin}} = \{\ v \in E \mid \dim_{\mathbb{C}} \langle\ \pi(K)\, v\ \rangle < \infty\ \}.$$

Definition C.1.14. *A* $(\mathfrak{g}, K)$*-module* V *is given by a Lie algebra representation* $\mathfrak{g}$ *on* V *and a Lie group representation of* K *on* V *such that*

(1) any vector is K*-finite, that is,* $V = V^{\mathrm{K\text{-}fin}}$*.*

(2) the differential of the group representation of K *coincides with the action of* $\mathfrak{g}$ *restricted to the real Lie algebra* $\mathfrak{k}_0$ *of* K*.*

(3) the action of $\mathfrak{g}$ *is* K*-equivariant, that is,*

$$\pi(k)\,(x\,v) = \pi(Ad(k)x)\pi(k)\,v \text{ for any } k \in K, x \in \mathfrak{g} \text{ and } v \in V. \qquad \text{(C.1.2)}$$

Remark C.1.15. *Assume that* G *is connected. Then also* K *is connected, and the third condition* (C.1.2) *is automatically satisfied.*

To define Harish-Chandra modules we need to introduce some finiteness conditions. In the following $Z(\mathfrak{g})$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$.

Definition C.1.16. *Let* V *be a* $(\mathfrak{g}, K)$*-module.*

- V *is* finitely generated *if* V *is finitely generated as a* $U(\mathfrak{g})$*-module.*

- *V is* weakly admissible *if the K-multiplicities m_τ are finite for each $\tau \in \widehat{K}$.*
- *V is* $Z(\mathfrak{g})$-finite *if there is an ideal $I \subset Z(\mathfrak{g})$ of finite codimension such that $IV = 0$.*
- *V has* finite length *if every descending and every ascending chain of $(\mathfrak{g}, K)$-submodules becomes stationary.*

Remark C.1.17. *Any finite-dimensional representation of the Lie algebra $\mathfrak{g}$ is a Harish-Chandra module. However, most Harish-Chandra modules are infinite-dimensional.*

These finiteness conditions are related in the following way:

Proposition C.1.18. *For any $(\mathfrak{g}, K)$-module V the following conditions are equivalent:*

(1) V is finitely generated and admissible,

(2) V is finitely generated and $Z(\mathfrak{g})$-finite,

(3) V is weakly admissible and $Z(\mathfrak{g})$-finite,

(4) V has finite length.

Definition C.1.19. *A $(\mathfrak{g}, K)$-module V is a* Harish-Chandra module *if V satisfies any of the equivalent conditions in Proposition C.1.18.*

Definition C.1.20. *Let V be a Harish-Chandra module. Then the* contragredient *or* K-finite dual $\mathbb{L}(V)$ *of V is defined as*

$$\mathbb{L}(V) := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^{\text{K -fin}} = \{\ \phi \in V^* \mid \dim_{\mathbb{C}} \langle K \cdot \phi \rangle_{\mathbb{C}} < \infty\ \}$$

That is, the module $\mathbb{L}(V)$ is given by the K-finite vectors of the algebraic dual of V.

In the following let $\mathcal{H}(\mathfrak{g}, K)$ denote the category of Harish-Chandra modules. This category has nice algebraic properties:

Theorem C.1.21. *(1) The category $\mathcal{H} = \mathcal{H}(\mathfrak{g}, K)$ is an abelian category.*

(2) Let V and W be Harish-Chandra modules.

- *It holds that the space $\mathrm{Hom}_{\mathcal{H}}(V, W)$ as well as $\mathrm{Ext}^j_{\mathcal{H}}(V, W)$ is* finite-dimensional *for any $j \in \mathbb{N}^+$.*
- *Moreover, it holds that $\mathrm{Ext}^j_{\mathcal{H}}(V, W) = 0$ for any $j \geq m = \dim(\mathfrak{g}/\mathfrak{k})$.*

In other words, $\mathcal{H}(\mathfrak{g}, K)$ is a Hom-finite category of finite global dimension.

(3) The contragredient duality $\mathbb{L} : \mathcal{H} \longrightarrow \mathcal{H}$ satisfies $\mathbb{L}^2 \cong \mathrm{Id}$.

In particular, the category $\mathcal{H}(\mathfrak{g}, K)$ has the Krull-Remak-Schmidt property.

Remark C.1.22. *The category $\mathcal{H}$ does not have projective objects.*

We will describe further properties of Harish-Chandra modules in Section C.2 below.

Harish-Chandra modules of $\mathrm{SL}(2,\mathbb{R})$

Let $G = \mathrm{SL}(2,\mathbb{R})$. Its complexified Lie algebra is given by $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and a maximal compact subgroup by $\mathrm{SO}(2)$. Let e, f, h be the standard basis of $\mathfrak{g}$:

$$[\,e,\,h\,] = 2\,e, \qquad [\,f,\,h\,] = 2\,f, \qquad [\,e,\,f\,] = h.$$

In particular, the subspace $\langle h\rangle$ is a Cartan subalgebra of $\mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Remark C.1.23. *A finitely generated $U(\mathfrak{g})$-module V is a* Harish-Chandra module *of G, or the pair $(\mathfrak{g}, K)$, if and only if V has a vector space decomposition*

$$V = \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda \quad \textit{such that} \;\; h|_{V_\lambda} = \lambda\,\mathrm{Id} \;\; \textit{and} \;\; \dim_{\mathbb{C}} V_\lambda < \infty \;\; \textit{for any } \lambda \in \mathbb{Z}.$$

Let $\mathcal{H} = \mathcal{H}(\mathfrak{g}, K)$ denote the category of Harish-Chandra modules of G. Let V be an object in $\mathcal{H}$. For a weight $\lambda \in \mathbb{Z}$ it holds that $e(V_\lambda) \subseteq V_{\lambda+2}$ and $f(V_\lambda) \subseteq V_{\lambda-2}$. For each $\lambda \in \mathbb{Z}$ we define the restricted operators of e and f as $e_\lambda = e\big|_{V_\lambda}$ and $f_\lambda = f\big|_{V_\lambda}$. In particular, a Harish-Chandra module V can be viewed by the following diagram:

$$\begin{tikzcd}
\cdots & V_{\lambda+4} \arrow[r, bend left, "f_{\lambda+4}"] & V_{\lambda+2} \arrow[l, bend left, "e_{\lambda+2}"] \arrow[r, bend left, "f_{\lambda+2}"] & V_\lambda \arrow[l, bend left, "e_\lambda"] \arrow[r, bend left, "f_\lambda"] & V_{\lambda-2} \arrow[l, bend left, "e_{\lambda-2}"] \arrow[r, bend left, "f_{\lambda-2}"] & V_{\lambda-4} \arrow[l, bend left, "e_{\lambda-4}"] & \cdots
\end{tikzcd}$$

A morphism of Harish-Chandra modules $\phi\colon V \longrightarrow W$ is given by morphisms $\phi_\lambda\colon V_\lambda \longrightarrow W_\lambda$, where $\lambda \in \mathbb{Z}$, which commute with the restricted operators.

Example C.1.24. *The following five series are well-known Harish-Chandra modules V in $\mathcal{H}$ with the property that* $\dim V_\lambda \leq 1$ *for all* $\lambda \in \mathbb{Z}$.

notation	*basis*	*relations*
$M(k)$, *where* $k \in \mathbb{Z}$	$\{v_i \mid i \in \mathbb{N}_0\}$	$e(v_0) = 0$ $e(v_{i+1}) = (i+1)(k-i)\, v_i$ $f(v_i) = v_{i+1}$ $h(v_i) = (k-2i)\, v_i$
$M^*(k)$, *where* $k \in \mathbb{Z}$	$\{v_i \mid i \in \mathbb{N}_0\}$	$e(v_i) = v_{i+1}$ $f(v_0) = 0$ $f(v_{i+1}) = -(i+1)(k+i)\, v_i$ $h(v_i) = (k+2i)\, v_i$
$W_\varepsilon(\tau)$, *where* $\varepsilon = 0$ *or* 1, $\tau \in \mathbb{C}$.	$\{v_i \mid i \in \mathbb{Z}\}$	$e(v_i) = \frac{1}{4}(\tau - (\varepsilon + 2i + 1)^2)\, v_{i+1}$ $f(v_i) = v_{i-1}$ $h(v_i) = (\varepsilon + 2i)\, v_i$
$W^*_\varepsilon(\tau)$, *where* $\varepsilon = 0$ *or* 1, $\tau \in \mathbb{C}$.	$\{v_i \mid i \in \mathbb{Z}\}$	$e(v_i) = v_{i+1}$ $f(v_i) = \frac{1}{4}(\tau - (\varepsilon + 2i - 1)^2)\, v_{i-1}$ $h(v_i) = (\varepsilon + 2i)\, v_i$
$X_k(\nu)$, *where* $\nu \in \mathbb{C}, k \in \mathbb{Z}$.	$\{v_i \mid i \in \mathbb{Z}\}$	$e(v_i) = \frac{1}{2}(\nu + (k + 2i + 1))\, v_{i+1}$ $f(v_i) = \frac{1}{2}(\nu - (k + 2i - 1)) v_{i-1}$ $h(v_i) = (k+2i)\, v_i$

(C.1.3)

Modules of type $M(k)$ are the Verma modules with highest weight k and $M^*(k)$ are the modules with lowest weight k.

C.1.4 Harish-Chandra functor

Let G be any semisimple Lie group as at the beginning of this Appendix. Next, we define the Harish-Chandra functor $\mathrm{H} : \mathrm{admrep}(G) \longrightarrow \mathcal{H}(\mathfrak{g}, K)$ between the "analytic" category of admissible representations and the "algebraic" category of Harish-Chandra modules.

Definition C.1.25. *Let (π, E) be an admissible representation of G. Let* $\mathrm{H}(E)$ *be the K-finite vectors of E :*

$$\mathrm{H}(E) = \{\ v \in E \mid \dim_{\mathbb{C}} \langle\ \pi(K)\, v\ \rangle < \infty\}.$$

The definition of the functor H *on morphisms of admissible representations is given by restriction to their K-finite vectors.*

Remark C.1.26. *Let (π, E) be an admissible representation of G. By the Peter-Weyl Theorem the restriction $(\pi|_K, E)$ of the representation to the maximal compact subgroup K is given by a topological direct sum of finite-dimensional irreducible* *representations of K :*

$$E \cong \widehat{\bigoplus_{\tau\in\widehat{K}}} E_\tau^{m_\tau} \qquad \Rightarrow \qquad \mathrm{H}(E) = \bigoplus_{\tau\in\widehat{K}} E_\tau^{m_\tau}$$

The K-finite vectors $\mathrm{H}(E)$ of E are given exactly by the algebraic *direct sum of the same representations of K.*

The following Theorem is one of the main motivations to study Harish-Chandra modules:

Theorem C.1.27 (Harish-Chandra)**.** *The functor* $\mathrm{H} : \mathrm{admrep}(G) \longrightarrow \mathcal{H}(\mathfrak{g}, K)$ *has the following properties*:

(1) H *is well-defined, that is,* $\mathrm{H}(E)$ *is indeed a Harish-Chandra module in the sense of Definition C.1.19. Moreover, the functor* H *is* faithful *and* exact*.*

(2) More precisely, the following statements hold for any admissible representation E of G :

  (a) $\mathrm{H}(E)$ *is a* dense *subspace of E.*

  (b) The lattice of closed G-invariant subspaces of E *is the same as the* lattice of $(\mathfrak{g}, K)$-submodules of $\mathrm{H}(E)$*.*

(3) For any irreducible Harish-Chandra module V there exists an irreducible admissible representation E such that $\mathrm{H}(E) \cong V$ *in* $\mathcal{H}(\mathfrak{g}, K)$*.*

(4) two unitary irreducible representations U_1, U_2 are isomorphic in $\mathrm{admrep}(G)$ *if and only if there is an isomorphism* $\mathrm{H}(U_1) \cong \mathrm{H}(U_2)$ *in* $\mathcal{H}(\mathfrak{g}, K)$*.*

Corollary C.1.28. *Let $E \in \mathrm{admrep}(G)$. Then E has the same Jordan-Hölder-multiplicities as* $\mathrm{H}(E)$*.*

Remark C.1.29. *The Harish-Chandra functor extends to the categories*

$$\mathrm{H} : \mathrm{adm.\,Rep}(G) \longrightarrow (\mathfrak{g}, K)\text{ - Mod}$$

of weakly admissible representations of G and arbitrary $(\mathfrak{g}, K)$-modules. In this setup, Theorem C.1.27 remains valid.

The above Remark and Proposition C.1.18 yield the following characterization of admissible representations:

Corollary C.1.30. *For any weakly admissible representation E of G the following three conditions are equivalent*:

(1) E is admissible, that is, E is Noetherian,

(2) $\mathrm{H}(E)$ *is a Harish-Chandra module.*

(3) E has finite length.

Remark C.1.31. *There is the notion of contragredient duality* $\mathbb{L}_G :$ $\mathrm{adm.\,Banach.\,rep}(G) \longrightarrow \mathrm{adm.}$ *on admissible Banach representations. Let* $\mathbb{L} : \mathcal{H}(\mathfrak{g}, K) \longrightarrow \mathcal{H}(\mathfrak{g}, K)$ *be the contragredient duality of Harish-Chandra modules.. Then these two dualities commute* *with the Harish-Chandra functor*:

$$\begin{array}{ccc} \text{adm. Banach. rep}(G) & \xrightarrow{\mathrm{H}} & \mathcal{H}(\mathfrak{g}, K) \\ \big\downarrow{\scriptstyle \mathbb{L}_G} & & \big\downarrow{\scriptstyle \mathbb{L}} \\ \text{adm. Banach. rep}(G) & \xrightarrow{\mathrm{H}} & \mathcal{H}(\mathfrak{g}, K) \end{array} \qquad \mathrm{H} \circ \mathbb{L} \cong \mathbb{L}_G \circ \mathrm{H}$$

We refer to [**BK14**] *for the definition of* $\mathbb{L}_G$ *and a proof of this statement.*

C.1.5 Globalizations of Harish-Chandra modules

Definition C.1.32. *Let V be a Harish-Chandra module in $\mathcal{H}(\mathfrak{g}, K)$. An admissible representation (π, E) of G is a* globalization *of V if* $\mathrm{H}(E) \cong V$ *in $\mathcal{H}(\mathfrak{g}, K)$.*

By Theorem C.1.27 any *irreducible* Harish-Chandra module has a globalization. This is also true in general:

Proposition C.1.33 (Casselman)**.** *Let V be any Harish-Chandra module in $\mathcal{H}(\mathfrak{g}, K)$. Then V has some globalization (π, H) in* $\mathrm{admrep}(G)$*, where H is a Hilbert space.*

In other words, the Harish-Chandra functor $\mathrm{H} : \mathrm{admrep}(G) \longrightarrow \mathcal{H}(\mathfrak{g}, K)$ is also *dense.*

Remark C.1.34. *Let V be any Harish-Chandra module in $\mathcal{H}(\mathfrak{g}, K)$.*

- *As seen before, all globalizations of V must have the* same *lattice of closed G-invariant subspaces.*
- *In general, there can be* infinitely *many globalizations of V. In particular, the Harish-Chandra functor* H *is not full.*

However, if we restrict to certain admissible representations we get an equivalence of categories. To introduce these representations we need the following notions

Definition C.1.35. *Let (π, E) be some representation of G and $v \in E$.*

- *The vector v is C^∞ if the orbit map $\gamma_v \colon G \longrightarrow E \ , \ g \longmapsto \pi(g)\, v$ is a C^∞ map.*
- *Let (π, E) be a Banach representation. The vector v is* analytic *if the orbit map γ_v is real analytic;*

In the following let E^∞ denote the subspace of C^∞-vectors, and $E^{\text{K -fin}}$ the subspace of K-finite vectors of E. Similarly, E^ω will denote be the space of analytic vectors, in this case we will assume without comment that (π, E) is a Banach representation.

We recall that we assumed that $G \subseteq \mathrm{GL}(n, \mathbb{R})$ is some linear semisimple group.

Definition C.1.36. *(1) For any $g \in G$ the* norm of g *is given by*

$$\|g\| = \mathrm{trace}\left(g \cdot g^t + (g^{-1} \cdot (g^{-1})^t\right).$$

(2) Let (π, E) be a Fréchet representation of G. Then (π, E) has moderate growth *if for any semi-norm p on E there is a semi-norm q on E and some $n \in \mathbb{N}^+$*

such that

$$p(\pi(g)\,v) \leq \|g\|^n \cdot q(v) \text{ for all } g \in G.$$

(3) A Fréchet representation (π, E) *of moderate growth is* smooth, *if* $E = E^\infty$.

Theorem C.1.37 (Casselman-Wallach)**.** *There is an equivalence of the following categories*:

$$\begin{aligned} \mathrm{SAF}(G) &\xrightarrow{\mathrm{H}} \mathcal{H}(\mathfrak{g}, K), \\ E &\longmapsto E^{\mathrm{K\text{-}fin}} \end{aligned}$$

where $\mathrm{SAF}(G)$ *denotes the category of smooth admissible Fréchet globalizations of moderate growth.*

Let us note that the category $\mathrm{SAF}(G)$ is a category of certain topological representations of G which is equivalent to a category of certain algebraic representations of $\mathfrak{g}$.

Kashiwara and Schmid introduced minimal and maximal globalizations of Harish-Chandra modules. They have the following properties:

Theorem C.1.38 (Kashiwara-Schmid)**.** *There are two functors* mg *and* MG

$$\begin{tikzcd} \mathrm{admrep}(G) \arrow[r, "\mathrm{H}"] & \mathcal{H}(\mathfrak{g}, K) \arrow[l, bend right, "\mathrm{mg}"'] \arrow[l, bend left, "\mathrm{MG}"] \end{tikzcd}$$

with the following properties:

(1) $\mathrm{MG}(V)$ *is a nuclear Fréchet representation and* $\mathrm{mg}(V)$ *is a dual nuclear Fréchet representation.*

(2) Let $V \in \mathcal{H}(\mathfrak{g}, K)$ *and let* $E \in \mathrm{admrep}(G)$ *be a globalization of* V. *Then there are inclusions*

$$V \subseteq \mathrm{mg}(V) \subseteq E \subseteq \mathrm{MG}(V).$$

(3) the functor mg *is left adjoint to* H *and the functor* MG *is right adjoint to* H.

(4) the functors mg *and* MG *are both exact and fully faithful.*

C.2 Blocks of Harish-Chandra modules

We return to the algebraic setup of the pair $(\mathfrak{g}, K)$ and the category of Harish-Chandra modules $\mathcal{H}(\mathfrak{g}, K)$.

C.2.1 General block decomposition

Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$.

Definition C.2.1. *A character of $Z(\mathfrak{g})$ is given by some homomorphism*

$$\chi\colon Z(\mathfrak{g}) \longrightarrow \mathbb{C}$$

Let us denote the set of characters by $Z(\mathfrak{g})^*$.

Definition C.2.2. *Let $\chi \in Z(\mathfrak{g})^*$ be some character and let V be a $(\mathfrak{g}, K)$-module.*

(1) V has infinitesimal character *χ if*

$$(\chi(z) - z)\, v = 0 \qquad \text{for all } v \in V, z \in Z(\mathfrak{g}).$$

(2) V has generalized infinitesimal character *χ if there is some $n \in \mathbb{N}^+$ such that*

$$(z - \chi(z))^n v = 0 \qquad \text{for all } v \in V, z \in Z(\mathfrak{g}).$$

(3) V is quasi-simple *if V has some infinitesimal character.*

(4) The χ-primary component of V is given by

$V_\chi = \{\ v \in V \mid$ *there is some $n \in \mathbb{N}^+$ such that $(z - \chi(z))^n v = 0$ for all $z \in \mathbb{Z}(\mathfrak{g})$* $\}$

For every $\chi \in Z(\mathfrak{g})^*$ we denote by $\mathcal{H}_\chi(\mathfrak{g}, K)$ the full subcategory of Harish-Chandra modules with generalized infinitesimal character χ.

Theorem C.2.3. *The category of Harish-Chandra modules $\mathcal{H}(\mathfrak{g}, K)$ admits a decomposition into blocks*:

$$\mathcal{H}(\mathfrak{g}, K) = \bigoplus_{\chi \in Z(\mathfrak{g})^*} \mathcal{H}_\chi(\mathfrak{g}, K) \tag{C.2.1}$$

In other words, the following statements hold:

- *Any Harish-Chandra module $V \in \mathcal{H}(\mathfrak{g}, K)$ has a χ-primary decomposition*

$$V \cong \bigoplus_{\chi \in Z(\mathfrak{g})^*} V_\chi,$$

  where every subspace V_χ is a Harish-Chandra submodule of V.

- *For any two characters $\chi' \neq \chi''$, it holds that* $\mathrm{Ext}^j_{\mathcal{H}}(V_{\chi'}, V_{\chi''}) = 0$ *for all $j \in \mathbb{Z}$.*

Remark C.2.4. *Any block $\mathcal{H}_\chi(\mathfrak{g}, K)$ may decompose further into blocks.*

Let $\mathcal{H}_0$ denote the *principal block* of $\mathcal{H}$, that is, the smallest block containing the one-dimensional representation $\mathbb{C}$ of the Lie algebra $\mathfrak{g}$.

Remark C.2.5. *By a result of Zuckerman there are many* equivalent *blocks in the block decomposition* (C.2.1). *In particular, any block $\mathcal{H}_\chi$ containing a finite-dimensional representation of the Lie algebra $\mathfrak{g}$ is equivalent to the principal block $\mathcal{H}_0$.*

Theorem C.2.6 (Harish-Chandra)**.** *Let χ be some infinitesimal character of $Z(\mathfrak{g})^*$. Then there are finitely many irreducible Harish-Chandra modules with infinitesimal character χ.*

Theorem C.2.7 ([**BBG97**]). *Let $\chi \in Z(\mathfrak{g})^*$ and let d be the real rank of the Lie group G. Then there is some* order $\Lambda_\chi(G)$ *over the regular ring* $\mathbb{C}[\![x_1, \ldots, x_d]\!]$ *of Krull dimension d such that there is an equivalence of categories*

$$\mathcal{H}_\chi(\mathfrak{g}, K) \xrightarrow{\sim} \Lambda_\chi \text{ -fd. mod},$$

where Λ_χ -fd. mod *is the category of finite-dimensional modules over the order Λ_χ.*

Summarized, any block $\mathcal{H}_\chi$ of Harish-Chandra modules has only finitely many simple objects and admits a ring-theoretic description.

C.2.2 Blocks of Harish-Chandra modules of $\mathrm{SL}(2,\mathbb{R})$ via quivers

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$. In fact, the center $Z(\mathfrak{g})$ is given by the polynomial ring $\mathbb{C}[\Omega]$, where Ω is the Casimir operator of $\mathfrak{sl}(2,\mathbb{C})$:

- $\Omega = (h+1)^2 + 4\,f\,e = h^2 + 1 + 2\,f\,e + 2\,e\,f.$

Let us note that Ω acts as the identity on the trivial representation $\mathbb{C}$ of $\mathfrak{g}$.

In this setup, any infinitesimal character $\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ can be identified with a complex number.

Definition C.2.8. *Let $V = \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda$ be a Harish-Chandra module of G.*

(1) For $\xi = 0$ or 1 we set $V_\xi = \bigoplus_{\lambda\in 2\mathbb{Z}+\xi} V_\lambda$.

*(2) For any $\chi \in \mathbb{C}$ the χ-*primary component *V_χ of V is given by*

$$V_\chi = \{\ v \in V \mid (\Omega - \chi\ \mathrm{id})^n\, v = 0 \textit{ for some } n \in \mathbb{N}^+\ \}.$$

(3) For any $\xi \in \{0,1\}$ and $\xi \in \mathbb{C}$ we set $V_\xi^\chi = V_\xi \cap V^\chi$.

Remark C.2.9. *Let $V \in \mathcal{H}$, $\chi \in \mathbb{C}$ and $\xi \in \{0,1\}$. Then the subspace V_ξ^χ of V is a Harish-Chandra module.*

For any $\chi \in \mathbb{C}$ and $\xi \in \{0,1\}$ let $\mathcal{H}_\xi^\chi$ denote the full subcategory of objects V in $\mathcal{H}$ such that $V = V_\xi^\chi$.

Proposition C.2.10. *The category $\mathcal{H}$ admits a* block decomposition

$$\mathcal{H} = \bigoplus_{\chi\in\mathbb{C}} \mathcal{H}_0^\chi \oplus \mathcal{H}_1^\chi\,.$$

Theorem C.2.11 (Gelfand). *Let $\mathbf{R} = \mathbb{C}[\![x]\!]$ and $\mathfrak{m} = (x)$. For any $\chi \in \mathbb{C}$ and $\xi \in \{0,1\}$ there is a quiver $(Q,I)_\xi^\chi$ and an $\mathbf{R}$-order Λ_ξ^χ such that there are equivalences of categories*:

$$\mathcal{H}_\xi^\chi \xrightarrow{\sim} \mathrm{nil.\,rep}(Q,I)_\xi^\chi \xrightarrow{\sim} \Lambda_\xi^\chi \text{ -fd. mod}$$

For a more precise description let us set

$$\Delta_0 = \{\,(2n-1)^2 \mid n \in \mathbb{N}^+\,\} = \{\,1, 9, 25, \ldots\} \textit{ and}$$
$$\Delta_1 = \{\,(2n)^2 \mid n \in \mathbb{N}^+\,\} = \{\,4, 16, 36\ldots\}.$$

Then the the quivers $(Q,I)^{\chi}_{\xi}$ and their orders are given as follows:

conditions on ξ and χ	$(Q,I)^{\chi}_{\xi}$	Λ^{χ}_{ξ}	$\operatorname{gldim}\Lambda^{\chi}_{\xi}$
if $\xi = 0$ *and* $\chi \in \mathbb{C}\backslash\Delta_0$ *or* $\xi = 1$ *and* $\chi \in \mathbb{C}^*\backslash\Delta_1$	$\begin{tikzcd} \bullet \arrow[loop left] \end{tikzcd}$	$\mathbf{R}$	1
if $\xi = 1$ *and* $\chi = 0$	$\begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left] & \underset{-}{\bullet} \arrow[l, bend left] \end{tikzcd}$	$\begin{bmatrix} \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} \end{bmatrix}$	1
if $\xi = 0$ *and* $\chi \in \Delta_0$, *or* $\xi = 1$ *and* $\chi \in \Delta_1$	$\begin{tikzcd} \underset{+}{\bullet} \arrow[r, bend left, "\mathsf{b}_+"] & \underset{\star}{\bullet} \arrow[l, bend left, "\mathsf{a}_+"] \arrow[r, bend right, "\mathsf{a}_-"'] & \underset{-}{\bullet} \arrow[l, bend right, "\mathsf{b}_-"'] \end{tikzcd}$ $\mathsf{b}_+\,\mathsf{a}_+ = \mathsf{b}_-\,\mathsf{a}_-$	$\begin{bmatrix} \mathbf{R} & \mathfrak{m} & \mathfrak{m} \\ \mathbf{R} & \mathbf{R} & \mathfrak{m} \\ \mathbf{R} & \mathfrak{m} & \mathbf{R} \end{bmatrix}$	2

Let us denote by $\mathcal{H}_0 := \mathcal{H}_0^{\chi=1}$ the principal block containing the trivial representation $\mathbb{C}$ of $\mathfrak{g}$.

Assume that $\chi \in \Delta_\xi$. Then there is some $n \in \mathbb{N}_0$ such that $\chi = (n+1)^2$ and $\xi \sim n$ mod 2.

C.2.3 Classical admissible representations of $\mathrm{SL}(2,\mathbb{R})$

In the following we introduce some notation for classical admissible Hilbert representations of $G = \mathrm{SL}(2,\mathbb{R})$:

- for $n \in \mathbb{N}_0$ let Φ_n denote the finite-dimensional representation with highest weight n. In this notation $\dim_{\mathbb{C}} \Phi_n = n+1$.
- For $m \in \mathbb{N}_{\geq 2}$ let D_m^+ and D_m^- be the holomorphic and anti-holomorphic discrete series.
- Let D_1^+ and D_1^- denote the limits of discrete series.
- For $\nu \in \mathbb{C}$ let $\mathrm{P}_0(\nu)$ and $\mathrm{P}_1(\nu)$ denote the principal series of G.
  More precisely, the principal series is the induced representation
  $$\mathrm{P}_\varepsilon(\nu) = \mathrm{Ind}_{MAN}^{G}(\varepsilon \otimes \exp(\nu) \otimes \mathrm{Id})$$
  with sign representation ε on $M = \{\pm 1\}$ and representation $\exp(\nu)$ on the abelian subgroup $A \cong \mathbb{R}^+$.

Here are some well-known facts on principal series:

(1) A principal series $\mathrm{P}_1(\nu)$ is irreducible if and only if $\nu \notin 2\mathbb{Z}$.

(2) There is the special case $\mathrm{P}_1(0) \cong \mathrm{D}_1^+ \oplus \mathrm{D}_1^-$ in $\mathrm{admrep}(G)$,

(3) A principal series $\mathrm{P}_0(\nu)$ is irreducible if and only if $\nu \notin 2\mathbb{Z}+1$.

(4) If a principal series $\mathrm{P}_\varepsilon(\nu)$ is irreducible, then there is an isomorphism
$$\mathrm{P}_\varepsilon(\nu) \cong \mathrm{P}_\varepsilon(-\nu) \qquad \text{in } \mathrm{admrep}(G).$$

Let $n \in \mathbb{N}_0$. Let ε be the parity of n. Then the principal series $\mathrm{P}_\varepsilon(\pm(n+1))$ are not irreducible, but indecomposable. In particular, these principal series appear in the following short exact sequences in $\mathrm{admrep}(G)$:

$$\begin{gathered}0 \longrightarrow \mathrm{D}^-_{n+2} \oplus \mathrm{D}^+_{n+2} \longrightarrow \mathrm{P}_\varepsilon(n+1) \longrightarrow \Phi_n \longrightarrow 0\\ 0 \longrightarrow \Phi_n \longrightarrow \mathrm{P}_\varepsilon(-n-1) \longrightarrow \mathrm{D}^-_{n+2} \oplus \mathrm{D}^+_{n+2} \longrightarrow 0\end{gathered} \tag{C.2.2}$$

The two sequences can be related to each other via the duality $\mathbb{L}$ on $\mathrm{admrep}(G)$. There are two more analytic representations, which may be defined as quotients of principal series:

$$0 \longrightarrow \mathrm{D}^\pm_{n+2} \longrightarrow \mathrm{P}_\varepsilon(n+1) \longrightarrow R^\pm_{n+2} \longrightarrow 0 \tag{C.2.3}$$

Using Lie duality we may define $T^\mp := \mathbb{L}(R^\pm)$. In particular there are the two short exact sequences:

$$0 \longrightarrow T^\mp_{n+2} \longrightarrow \mathrm{P}_\varepsilon(-n-1) \longrightarrow \mathrm{D}^\mp_{n+2} \longrightarrow 0 \tag{C.2.4}$$

So far we have three closely related categories:

$$\mathrm{admrep}(G) \longrightarrow \mathcal{H}(\mathfrak{g}, K) \longrightarrow \mathrm{nil.\,rep}(Q, I)$$

In this section, we recall the description of Harish-Chandra modules and quiver representations of some well-known analytic representations of $\mathrm{SL}(2, \mathbb{R})$.

The following tables list all irreducible admissible representations of G, their Harish-Chandra modules and the corresponding quiver representations.

TABLE C.2.1. **Irreducible admissible representations of G and their algebraic counterparts**

Blocks with one simple object		
admissible rep.	Harish-Chandra module	quiver representation
$\mathrm{P}_1(\nu) \quad \nu \in \mathbb{C}\backslash\{0, 2, 4, \ldots\}$	$X_1(\nu^2)$	$\circlearrowright \Bbbk$
$\mathrm{P}_0(\nu) \quad \nu \in \mathbb{C}\backslash\{1, 3, \ldots\}$	$X_0(\nu^2)$	
Blocks with two simple objects		
D^-_1	$M(-1)$	$0 \rightleftarrows \Bbbk$
D^+_1	$M^*(1)$	$\Bbbk \rightleftarrows 0$
Blocks with three simple objects		
D^-_{n+2}	$M(-n-2)$	$S_- = 0 \rightleftarrows 0 \rightleftarrows \Bbbk$

admissible rep.	Harish-Chandra module	quiver representation
Φ_n	$\mathbb{C}^{n+1}$	$S_\star = 0 \rightleftarrows \Bbbk \leftrightarrows 0$
D^+_{n+2}	$M^*(n+2)$	$S_+ = \Bbbk \rightleftarrows 0 \leftrightarrows 0$

Let us note that the principal block $\mathcal{H}_0 = \mathcal{H}_0^{\xi=1}$ contains the K-finite vectors of the following admissible representations :

$$E^{\text{K -fin}} \in \mathcal{H}_0 \quad \text{where} \quad E \cong \mathrm{D}_2^\pm,\ \Phi_0,\ \mathrm{P}_0(1),\ \mathrm{P}_0(-1),\ R_2^\pm \ \text{ or } \ T_2^\pm.$$

C.2.4 Standard representations of $\mathrm{SL}(2, \mathbb{R})$ and the Gelfand quiver

In this subsection we list some low-dimensional modules of the Gelfand quiver.

The K-finite vectors of some principal series $\mathrm{P}_\varepsilon(\nu)$ have infinitesimal character ν^2 and support type ε :

$$(\mathrm{P}_1(\nu))^{\text{K -fin}} \in \mathcal{H}_{odd}^{\chi=\nu^2} \qquad \text{and} \qquad (\mathrm{P}_0(\nu))^{\text{K -fin}} \in \mathcal{H}_{even}^{\chi=\nu^2}\ .$$

The main series of examples are the standard representations in Tables C.2.2 and C.2.3 below. They correspond to K-finite vectors of well-known analytic representations of $\mathrm{SL}(2, \mathbb{R})$.

Remark C.2.12. *Standard representations of the Gelfand quiver can be characterized as follows.*

Let M be some finite-dimensional module over the Gelfand order Λ. Then the following conditions are equivalent:

(1) M is isomorphic to some standard representation from Table C.2.2 or Table C.2.3,

(2) M is quasi-simple, that is, $\mathrm{End}_\Lambda(M) \cong \Bbbk$,

(3) M is an indecomposable Λ/I-module, where $I = (a_+b_+, c, a_-b_-)$.

Table C.2.2. **Standard representations of the Gelfand quiver**

V	Gelfand quiver representation	analytic representation of G	representation of $\mathfrak{sl}(2,\mathbb{C})$	projective resolution	$\sigma(V)$	$\tau(V)$	$\delta(V)$	class
$S_\star$	$0 \rightleftarrows \Bbbk \leftrightarrows 0$	one-dimensional representation Φ_1	one-dimensional representation $\mathbb{C}$	$P_\star \xrightarrow{\begin{bmatrix} \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}} \bigoplus_{i\in\{+,-\}} P_i \xrightarrow{[\cdot\mathsf{a}_+ \ \cdot\mathsf{a}_-]} P_\star$	$S_\star$	$S_\star[1]$	2	usual string
S_+	$\Bbbk \rightleftarrows 0 \leftrightarrows 0$	holomorphic discrete series D_2^+	lowest weight module M_2^*	$P_\star \xrightarrow{\cdot\mathsf{b}_+} P_+$	S_-	T_+	1	special string
S_-	$0 \rightleftarrows 0 \leftrightarrows \Bbbk$	anti-holomorphic discrete series D_2^-	highest weight module M_{-2}	$P_\star \xrightarrow{\cdot\mathsf{b}_-} P_-$	S_+	T_-		
R_-	$\Bbbk \rightleftarrows \Bbbk \leftrightarrows 0$	[...]	lowest weight module M_0^*	$P_- \xrightarrow{\cdot\mathsf{a}_-} P_\star$	R_+	$*$		
R_+	$0 \rightleftarrows \Bbbk \leftrightarrows \Bbbk$	$\mathcal{O}_{hol}(\overline{\mathbb{D}})$	highest weight module M_0	$P_+ \xrightarrow{\cdot\mathsf{a}_+} P_\star$	R_-			
T_-	$\Bbbk \rightleftarrows \Bbbk \leftrightarrows 0$	[...]	$W_0(1)/\operatorname{soc} W_0(1)$ or $X_0(1)/M_2$	$P_\star \xrightarrow{\begin{bmatrix} \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}} \bigoplus_{i\in\{+,-\}} P_i \xrightarrow{[\cdot\mathsf{a}_+\mathsf{b}_+ \ \cdot\mathsf{a}_-\mathsf{b}_+]} P_+$	T_+	S_-		
T_+	$0 \rightleftarrows \Bbbk \leftrightarrows \Bbbk$	[...]	$W_0^*(1)/\operatorname{soc} W_0^*(1)$ or $X_0^*(1)/M_2^*$	$P_\star \xrightarrow{\begin{bmatrix} \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}} \bigoplus_{i\in\{+,-\}} P_i \xrightarrow{[\cdot\mathsf{a}_+\mathsf{b}_- \ \cdot\mathsf{a}_-\mathsf{b}_-]} P_-$	T_-	S_+		

$*$ means that the complex $\tau(V)$ has homology in some non-zero degree.

Table C.2.3. **Standard representations of the Gelfand quiver II**

V	Gelfand quiver representation	analytic representation of G	representation of $\mathfrak{sl}(2,\mathbb{C})$	projective resolution	$\sigma(V)$	$\tau(V)$	$\delta(V)$	class
$X_\star$	$\mathbb{k}\ \substack{\dashrightarrow \\ \longleftarrow}\ \mathbb{k}\ \substack{\dashleftarrow \\ \longrightarrow}\ \mathbb{k}$	principal series $\mathrm{P}_0(1)$	$X_0(1)$.	$P_\star \xrightarrow{\cdot \mathsf{b}_\pm \mathsf{a}_\pm} P_\star$	$X_\star$	$Y_\star$	2	usual string
$Y_\star$	$\mathbb{k}\ \substack{\longrightarrow \\ \dashleftarrow}\ \mathbb{k}\ \substack{\longleftarrow \\ \dashrightarrow}\ \mathbb{k}$	principal series $\mathrm{P}_0(-1)$	$X_0(-1)$	$P_\star \xrightarrow{\begin{bmatrix} 0 \\ \cdot\mathsf{b}_+ \\ \cdot(-\mathsf{b}_-) \end{bmatrix}} \bigoplus_{i\in\{\star,+,-\}} P_i \xrightarrow{\begin{bmatrix} \cdot\mathsf{b}_+ & \cdot\mathsf{a}_+\mathsf{b}_+ & \cdot\mathsf{a}_-\mathsf{b}_+ \\ \cdot\mathsf{b}_- & \cdot\mathsf{a}_+\mathsf{b}_- & \cdot\mathsf{a}_-\mathsf{b}_- \end{bmatrix}} \bigoplus_{i\in\{+,-\}} P_i$	$Y_\star$	$*$		
B_{++}	$\mathbb{k}\ \substack{\longrightarrow \\ \dashleftarrow}\ \mathbb{k}\ \substack{\dashleftarrow \\ \longrightarrow}\ \mathbb{k}$	Whittaker model	$W_0(1)$	$P_+ \xrightarrow{\cdot\mathsf{a}_+\mathsf{b}_+} P_+$	B_{--}		0	bispecial string
B_{--}	$\mathbb{k}\ \substack{\dashrightarrow \\ \longleftarrow}\ \mathbb{k}\ \substack{\longleftarrow \\ \dashrightarrow}\ \mathbb{k}$	Whittaker model	$W_0^*(1)$	$P_- \xrightarrow{\cdot\mathsf{a}_-\mathsf{b}_-} P_-$	B_{++}			
dashed arrows denote zero maps and solid arrows denote identity maps.				$*$ means that the complex $\tau(V)$ has homology in some non-zero degree.				

C.3 Principal blocks of Lorentz groups of real rank one

Next, we consider the following setup:

We need to fix some notation. Let $n \in \mathbb{N}^+$.

- Let G be the Lie group $G_n^\# = \mathrm{SO}(n,1)$ or its identity component $G_n = \mathrm{SO}_e(n,1)$.
- Let $\mathfrak{g}$ be the complexified Lie algebra of G, and let K be the maximal compact subgroup of G.
- Let $\mathcal{H}_0(G)$ be the principal block of Harish-Chandra modules of the Lie group G.
- Let $(Q,I) = (Q,I)_n$ respectively $(Q,I)_n^\#$ be the corresponding Khoroshkin quiver as defined in Subsection 2.1.2 respectively Subsection 2.1.3.
- Let $\Lambda = \Lambda_n$ respectively $\Lambda_n^\#$ (if $G = G_n$ respectively $G_n^\#$) be the Khoroshkin order over $\mathbf{R} = \mathbb{C}[\![t]\!]$, that is the arrow ideal completion of the path algebra of the quiver $(Q,I)_n$ respectively $(Q,I)_n^\#$.

The following result was obtained by Khoroshkin:

Theorem C.3.1 ([**Kho81**]). *There are the following equivalences of categories:*

$$\mathcal{H}_0(G) \xrightarrow{\sim} \Lambda\text{ -fd. mod} \xrightarrow{\sim} \text{nil. rep}_{\mathbb{C}}(Q,I)$$

Under this equivalence, the trivial Harish-Chandra module corresponds to the simple quiver representation at the vertex marked by $\star$.

C.3.1 The defect and the relative Lie algebra cohomology for $\mathrm{SO}(n,1)$

Let us fix some connected Lorentz group $G_n = \mathrm{SO}_e(n,1)$ for some $n \in \mathbb{N}^+$. Let $\Lambda = \Lambda_n$ be the Khoroshkin order of the principal block $\mathcal{H}_0 = \mathcal{H}_0(G_n)$

In this subsection we relate the defect of a Harish-Chandra module to its relative Lie algebra cohomology.

On the one hand there is the following Lie-theoretic notion (see [**KV95**, Chapter II, Proposition 2.117])

Definition C.3.2. *Let G be a semisimple Lie group and V be a $(\mathfrak{g},K)$-module. For any $j \in \mathbb{Z}$ the j-th* relative Lie algebra cohomology *of V is defined as*

$$\mathbf{H}^j_{\mathfrak{g},K}(V) = \mathrm{Ext}^j_{\mathfrak{g},K}(\mathbb{C},V)$$

where $\mathbb{C}$ is the trivial $(\mathfrak{g},K)$-module.

Let us recall the notion of defect from Chapter 2:

Definition C.3.3. *Let V be a finitely generated Λ-module, where $\Lambda = \Lambda_n$.*

- *For any $j \in \mathbb{Z}$ the j-th defect number of V is given by*

$$\delta^{(j)}(V) = \dim \mathrm{Ext}^j_\Lambda(V, S_\star),$$

where $S_\star$ is the simple Λ_n-module corresponding to the Harish-Chandra module $\mathbb{C}$.

- *The total defect of V is given by*

$$\delta(V) = \sum_{j \in \mathbb{Z}} \dim \delta^{(j)}(V) = \sum_{j=0}^{n} \dim \delta^{(j)}(V)$$

There is the following diagram of categories and functors:

$$\Lambda\text{-mod} \longleftarrow \Lambda\text{-fd. mod} \xrightarrow{\sim} \mathcal{H}_0(\mathfrak{g}, K) \hookrightarrow (\mathfrak{g}, K)\text{-Mod}$$

Proposition C.3.4 (Casselman [**BW80**, Proposition I.5.5]). *Let G be any semisimple Lie group and $V, W \in \mathcal{H}(\mathfrak{g}, K)$. For any $j \in \mathbb{Z}$ it holds that $\mathrm{Ext}^j_{\mathcal{H}}(V, W) \cong \mathrm{Ext}^j_{\mathfrak{g},K}(V, W)$.*

The proof of Casselman's Proposition can be adapted to the following setup:

Proposition C.3.5. *Let A be a semiperfect $\Bbbk$-algebra. Let V, W be finite dimensional A-modules. For any $j \in \mathbb{Z}$ it holds that $\mathrm{Ext}^j_{A\text{-fd. mod}}(V, W) \cong \mathrm{Ext}^j_{A\text{-mod}}(V, W)$.*

Now we can relate the defect to the relative Lie algebra cohomology:

Corollary C.3.6. *As above let $\Lambda = \Lambda_n$. Let V be some finite-dimensional Λ-module Then the defect of V is the total dimension of the relative Lie algebra cohomology of V :*

$$\delta(V) = \sum_{j \in \mathbb{Z}} \dim \mathbf{H}^j_{\mathfrak{g},K}(V),$$

where on the right hand side the module V is viewed as a Harish-Chandra module.

Proof. Let $j \in \mathbb{Z}$. By Lemma 2.1.7 the simple module $S_\star$ is n-Calabi-Yau. Using Propositions C.3.5 and C.3.4 it follows that

$$\begin{aligned} \mathbb{D}\,\mathrm{Ext}^j_{\Lambda\text{-mod}}(V, S_\star) \cong \mathrm{Ext}^{n-j}_{\Lambda\text{-mod}}(S_\star, V) &\cong \mathrm{Ext}^{n-j}_{\Lambda\text{-fd. mod}}(S_\star, V) \\ &\cong \mathrm{Ext}^{n-j}_{\mathcal{H}}(\mathbb{C}, V) \cong \mathbf{H}^{n-j}_{\mathfrak{g},K}(V). \end{aligned}$$

This shows that $\delta^{(j)}(V) = \dim \mathbf{H}^{n-j}_{\mathfrak{g},K}(V)$. This implies the statement. □

C.3.2 Involutions on Harish-Chandra modules over $\mathrm{SO}(n,1)$ and $\mathrm{SO}_e(n,1)$

Next, we consider the involution on Harish-Chandra modules for Lorentz groups of real rank one.

Remark C.3.7. *To prevent some confusion about the various Lie groups encountered in this Appendix, let us note there are the following morphisms of Lie groups*:

$$\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1) \twoheadrightarrow \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(2,1) \hookrightarrow \mathrm{PGL}(2,\mathbb{R}) \cong \mathrm{SO}(2,1)$$

$$\mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}_0(3,1) \hookrightarrow \mathrm{SO}(3,1)$$

Moreover, the following non-isomorphic Lie groups have equivalent *principal blocks of Harish-Chandra modules*:

$$\mathcal{H}_0(\mathrm{SL}(2,\mathbb{R})) \xrightarrow{\sim} \mathcal{H}_0(\mathrm{SO}_0(2,1)) \qquad \textit{and} \qquad \mathcal{H}_0(\mathrm{SL}(2,\mathbb{C})) \xrightarrow{\sim} \mathcal{H}_0(\mathrm{SO}_0(3,1))\,.$$

In particular, this explains why the Khoroshkin quiver for $\mathrm{SO}_e(2,1)$ *is the same as the Gelfand quiver.*

Let $G = \mathrm{SO}_e(n,1)$ denote the identity component of the Lie group $G^{\#} = \mathrm{SO}(n,1)$.

In this subsection we recall Khoroshkin's results on the relationship between the categories of Harish-Chandra modules over the Lie groups G and $G^{\#}$.

Let $K = \mathrm{SO}(n)$, respectively $K^{\#} = \mathrm{O}(n)$, denote a maximal compact subgroup of G_n, respectively $G_n^{\#}$. The complexified Lie algebra of *both* Lie groups G_n and $G_n^{\#}$ is given by $\mathfrak{g} \cong \mathfrak{so}(n+1)$.

In this subsection let us denote the category of Harish-Chandra modules over G by $\mathcal{H}(\mathfrak{g}, K)$, and similarly we will write $\mathcal{H}(\mathfrak{g}, K^{\#})$ instead of $\mathcal{H}(G^{\#})$.

The compact group K is a normal subgroup of $K^{\#}$ of index two. In particular, the Harish-Chandra modules over the Lie groups G and $G^{\#}$ are related by certain induction and restriction functors:

$$\mathcal{H}(\mathfrak{g}, K) \underset{\mathrm{res}}{\overset{\mathrm{ind}}{\rightleftarrows}} \mathcal{H}(\mathfrak{g}, K^{\#})$$

There is a natural action of the component group $K^{\#}/K$ on both categories. Since $K^{\#}/K \cong \mathbb{Z}_2$, this action is actually an involution.

More precisely, this $\mathbb{Z}_2$-action can be described as follows.

In the notation (C.1.1) for the Lie group $G^{\#}$ a non-trivial representative of $K^{\#}/K$ is given by the matrix

$$k = \begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}$$

Since $k^2 = \mathrm{Id}$, the matrix k gives rise to an involution of the Lie group $G^{\#}$:

$$Ad_k : G^{\#} \xrightarrow{\sim} G^{\#} \qquad \text{via} \qquad g \longmapsto k \cdot g \cdot k^{-1}$$

The automorphism Ad_k preserves the compact subgroups K and $K^{\#}$ and induces an involution of the Lie algebra $\mathfrak{g}$.

Furthermore, the involution of the pair $(\mathfrak{g}, K)$ gives rise to an autoequivalence κ of the principal block $\mathcal{H}_0(\mathfrak{g}, K)$ of order two. Similarly, the involution of the pair $(\mathfrak{g}, K^{\#})$ gives rise to an involution functor $\kappa^{\#}$ on the category $\mathcal{H}_0(\mathfrak{g}, K^{\#})$.

Since both principal blocks can be described by Khoroshkin quivers or their orders, we obtain the following diagram of categories and functors:

$$\begin{tikzcd}
G^{\#} = \mathrm{SO}(n,1): & \mathcal{H}_0(\mathfrak{g},K^{\#}) \arrow[loop above, "\kappa^{\#}"] \arrow[r, "\sim"] \arrow[d, shift right, "\mathrm{res}"'] & \mathrm{nil.\,rep}(Q,I)_n^{\#} \arrow[r, "\sim"] \arrow[d, shift right, "\mathrm{res}"'] & \Lambda_n^{\#}\text{ - fd. mod} \arrow[loop above, "\kappa^{\#}"] \arrow[loop right, dotted, "\sigma=\mathrm{id}"] \arrow[d, shift right, "\mathrm{res}"'] \\
G = \mathrm{SO}_e(n,1): & \mathcal{H}_0(\mathfrak{g},K) \arrow[loop below, "\kappa"] \arrow[r, "\sim"] \arrow[u, shift right, "\mathrm{ind}"'] & \mathrm{nil.\,rep}(Q,I)_n \arrow[r, "\sim"] \arrow[u, shift right, "\mathrm{ind}"'] & \Lambda_n\text{ - fd. mod} \arrow[loop below, "\kappa\cong\sigma"] \arrow[u, shift right, "\mathrm{ind}"']
\end{tikzcd}$$

Remark C.3.8. *For any $n \in \mathbb{N}^+$, there is a functorial isomorphism $\kappa \cong \sigma : \Lambda_n\text{ - fd. mod} \longrightarrow \Lambda_n\text{ - fd. mod}$ where σ is the trivial involution if n is odd, and the involution of the special symmetry of the Khoroshkin quiver, if n is even.*

Remark C.3.9. *The involution $\kappa^{\#}$ is given as follows*:

- *if n is odd, the quiver $(Q,I)_n^{\#}$ is the direct product $(Q,I)_n^{\times 2}$. The involution $\kappa^{\#}$ corresponds to the symmetry interchanging both copies of $(Q,I)_n^{\#}$.*
- *if n is even, $\kappa^{\#}$ corresponds to the symmetry interchanging $+$ and $-$ in all but one vertex of $(Q,I)_n^{\#}$ (see* (2.1.7)*).*

In both cases, we have that $\kappa \ncong \sigma = \mathrm{Id}$.

Example C.3.10. *Let $G = \mathrm{SL}(2,\mathbb{R}) \subset G^{\#} = \mathrm{SL}^{\pm}(2,\mathbb{R})$ In this case, $K = SO(2) \subset K^{\#} = O(2)$. A non-trivial element of $K^{\#}/K$ is given by $k = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Conjugation with k induces an involution of $\mathrm{SL}(2,\mathbb{R})$. Under the isomorphism $G \cong \mathrm{SU}(1,1)$ the involution Ad_k corresponds to* complex conjugation*. The involution Ad_k induces the following involution on the $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$:*

$$Ad_k\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{sl}(2,\mathbb{C}) \qquad \begin{array}{l} e \longmapsto f \\ f \longmapsto e \\ h \longmapsto -h \end{array} \;,$$

where e, f and h denote the standard basis of $\mathfrak{sl}(2,\mathbb{C})$.

Theorem C.3.11. **[Kho81]** *Consider the following categories and functors*:

$$\begin{tikzcd}
\Lambda\text{ - fd. mod} \arrow[loop left, "\kappa"] \arrow[r, shift left, "\mathrm{ind}"] & \Lambda^{\#}\text{ - fd. mod} \arrow[l, shift left, "\mathrm{res}"] \arrow[loop right, "\kappa^{\#}"]
\end{tikzcd}$$

The functors (ind, res) *form an adjoint pair.*

(1) let N be some finite-dimensional $\Lambda^{\#}$-module.

- *If $N \ncong \kappa^{\#}(N)$, then $\mathrm{res}(N) \cong \mathrm{res}(\kappa^{\#}(N))$ is an indecomposable finite dimensional $\Lambda^{\#}$-module.*
- *If $N \cong \kappa^{\#}(N)$, then there is a direct sum decomposition $\mathrm{res}(N) \cong M \oplus \kappa(M)$ for some* indecomposable *finite-dimensional Λ-module M such that $M \ncong \kappa(M)$.*

(2) the analogous statements hold for the induction functor ind.

The next two remarks are valid for Khoroshkin orders Λ and $\Lambda^{\#}$ defined over any base field $\Bbbk$ with characteristic different from two.

In this case, the Khoroshkin orders Λ and $\Lambda^{\#}$ are related as follows:

Remark C.3.12. *The involution σ of the Khoroshkin order Λ, which interchanges interchanging $+$ and $-$, gives rise to a* skew group ring $\mathbb{Z}_2\#\Lambda$. *It can be shown that this ring is* Morita equivalent *to the ring $\Lambda^{\#}$:*

$$\mathbb{Z}_2\#\Lambda\text{ - mod} \xrightarrow{\ \sim\ } \Lambda^{\#}\text{ - mod}$$

Remark C.3.13. *Let us recall that the order $\Lambda^{\#}$ had "gentle" type, while Λ was "skew-gentle" by Remarks 2.1.13 and 2.1.4. By* [**Kho81**] *the natural restriction functor*

$$\mathrm{res}:\ \Lambda^{\#}\text{ - mod} \xrightarrow{\ \sim\ } \Lambda\text{ - mod}\,.$$

has the following property:

- *Every indecomposable Λ-module N is isomorphic to some* direct summand *of the restriction* $\mathrm{res}(M)$ *for some indecomposable $\Lambda^{\#}$-module M.*

In particular, Khoroshkin's method proves tameness of Λ and yields a parametrization of the indecomposable Λ-modules. However, it is not straightforward to derive all indecomposable Λ-modules by this approach. This situation is very similar to Knörrer's correspondence for matrix factorizations described in Remark 1.3.28. For further results on skew group algebras we refer to the work of Reiten and Riedtmann [**RR85**].

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