Algebro-Geometric Aspects of the Classical Yang-Baxter Equation

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Zusammenfassung

In dieser Arbeit untersuchen wir die Klassische Yang-Baxter Gleichung und die Verallgemeinerte Klassische Yang-Baxter Gleichung von einem algebro-geometrischen Standpunkt aus. Im ersten Kapitel stellen wir ein Verfahren vor, um ausgehend von bestimmten Garben von Lie Algebren auf algebraischen Kurven eine Lösung der Verallgemeinerten Klassischen Yang-Baxter Gleichung zu konstruieren. Weiterhin geben wir ein Kriterium an, um die Unitarität dieser Lösungen zu überprüfen. Im zweiten Kapitel behandeln wir eine spezielle Klasse von Lösungen, nämlich diejenigen, die aus Garben von Endomorphismen mit Spur null von einfachen Vektorbündeln auf der nodalen kubischen Kurve entstehen. Diese Lösungen sind quasi-trigonometrisch und wir erläutern, wie sie in das Klassifikationsschema solcher Lösungen passen. Desweiteren geben wir eine konkrete Formel für diese Lösungen an. Im abschließenden, dritten Kapitel zeigen wir, dass man alle unitären, rationalen Lösungen der Klassischen Yang-Baxter Gleichung durch Anwendung des Verfahrens aus Kapitel eins auf Garben von Lie-Algebren auf der kuspidalen kubischen Kurve erhalten kann.

Abstract

In this thesis we consider algebro-geometric aspects of the Classical Yang-Baxter Equation and the Generalised Classical Yang-Baxter Equation. In chapter one we present a method to construct solutions of the Generalised Classical Yang-Baxter Equation starting with certain sheaves of Lie algebras on algebraic curves. Furthermore we discuss a criterion to check unitarity of such solutions. In chapter two we consider the special class of solutions coming from sheaves of traceless endomorphisms of simple vector bundles on the nodal cubic curve. These solutions are quasi-trigonometric and we describe how they fit into the classification scheme of such solutions. Moreover, we describe a concrete formula for these solutions. In the third and final chapter we show that any unitary, rational solution of the Classical Yang-Baxter Equation can be obtained via the method of chapter one applied to a sheaf of Lie algebras on the cuspidal cubic curve.

Contents

Introduction

In the late seventies and early eighties the so-called Leningrad school of mathematical physics under Faddeev realised that one could study the solutions of a class of non-linear partial differential equations including the KdV-equations by viewing them as an infinite-dimensional integrable system. The Hamiltonian structure is defined in terms of so-called r-matrices (see for example [20] or [37]). Many of the important examples, e.g. instances of the Toda lattice, come from solutions of the Classical Yang-Baxter Equation (CYBE), that is germs of meromorphic functions $\mathsf{r} : \mathbb{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)] + [\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)] + [\mathsf{r}^{13}(x,z), \mathsf{r}^{23}(y,z)] = 0,$$

where $\mathfrak{g}$ is a finite-dimension simple Lie algebra over $\mathbb{C}$ and $(-)^{ij}$ denotes a certain element in the triple tensor product $\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. For example $(-)^{13}$ is given by the composite

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\iota\otimes\iota} \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \xrightarrow{\sigma_{13}} \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}),$$

where the first map is given by the canonical inclusion and the second one is induced by $x \otimes y \mapsto x \otimes 1 \otimes y$.
The discovery of the importance of r-matrices was the starting point of the mathematical exploration of solutions of the CYBE. The first and probably most important theorem in the theory was discovered by Belavin and Drinfeld ([3], [4], [5]) and reads as follows:

Theorem (Belavin-Drinfeld)**.** *Let $\mathsf{s}(x,y)$ be a non-degenerate solution of the CYBE. Then:*

1. *$\mathsf{s}(x,y)$ is gauge-equivalent to a solution $\mathsf{r}(z)$ which only depends on the difference $z = x - y$.*

2. *$\mathsf{r}(z)$ is automatically unitary, i.e.*

$$\mathsf{r}(z) = -\mathsf{r}^{21}(-z).$$

3. *$\mathsf{r}(z)$ extends to a meromorphic function on the whole complex plane $\mathbb{C}$ with only simple poles and Laurent expansion*

$$\mathsf{r}(z) = \frac{\Omega}{z} + \mathcal{O}(1)$$

around 0 *(up to multiplication by a scalar).*

4. *The poles of* $\mathsf{r}(z)$ *form a lattice* $\Gamma \subset \mathbb{C}$ *and* $\mathsf{r}(z)$ *is of one of three types:*
   (a) *rational if* $\mathsf{rank}(\Gamma) = 0$*. This means that* $\mathsf{r}(z)$ *is equivalent to* $P(z)$ *for some rational function* P*.*
   (b) *trigonometric if* $\mathsf{rank}(\Gamma) = 1$*. This means that* $\mathsf{r}(z)$ *is equivalent to* $P(\mathsf{exp}(z))$ *for some rational function* P*.*
   (c) *elliptic if* $\mathsf{rank}(\Gamma) = 2$*. This means that* $\mathsf{r}(z)$ *is equivalent to* $Q(z)$ *for some elliptic function* Q*.*

   The latter only exist for $\mathfrak{g} = \mathsf{sl}_n$*.*

Belavin and Drinfeld also classified trigonometric and elliptic solutions up to gauge equivalence and a few years later Stolin ([42], [43], [44]) found a different, natural Lie theoretic problem, whose answers are in one-to-one correspondence with unitary, rational solutions of the CYBE. For a textbook introduction to these results see [13] and [19].
Cherednik ([14]) realised that one could associate a solution of the CYBE or of some similar equation to sheaves of Lie algebras $\mathcal{A}$ on smooth curves C whose underlying coherent $\mathcal{O}_C$-module is locally free as long as the sheaf cohomology of $\mathcal{A}$ vanishes. At the beginning of the millennium Polishchuk ([36]) found another approach to this construction: He associates a unitary solution of the CYBE for $\mathfrak{sl}_n$ with any simple vector bundle on a reduced curve of arithmetic genus one using derived categories and A_∞ structures ([36]) as long as the A_∞-structure is cyclic with respect to the canonical Serre duality pairing. His work was clarified and expanded by Burban-Kreussler ([11]) and Burban-Henrich ([10]). The latter also worked out how the rational solutions one obtains from simple vector bundles on the cuspidal cubic curve fit into Stolin's classification scheme.

In this thesis we discuss three advances in the theory of the CYBE:

1. We describe a geometric set-up from which one can obtain solutions of the CYBE and of its closely related cousin, the Generalised Classical Yang-Baxter Equation. This approach is related to that of Cherednik, but also works for singular curves C and sheaves of Lie algebras whose underlying coherent $\mathcal{O}_C$-module is only torsionfree. Our approach has the advantage of not only producing more general theorems (for example, it is not restricted to $\mathfrak{sl}_n$), but also of having easier proofs: The

usage of A_∞ categories is avoided and instead the main results are deduced from residue considerations and the residue theorem. Since the cyclicity of A_∞-structures on singular cubic curves is a delicate and complicated matter this means that our theorem is also more widely applicable than the ones known before.

2. We work out how to concretely calculate the solutions coming from simple vector bundles on the nodal cubic and describe how these solutions fit into the classification framework of so-called quasi-trigonometric solutions given by Khoroshkin, Pop, Samsonov, Stolin and Tolstoy ([25]). Furthermore, a closed combinatorial formula for these solutions is worked out.

3. Finally, we show that all rational solutions in the sense of Stolin come from geometric data as considered in item one. In this way, we can reprove some of Stolin's original results and give a different and more geometric approach to his classification. These results rely crucially on the usage of sheaves of Lie algebras whose underlying coherent sheaves of modules are only torsionfree, since not all rational solutions come from ones whose underlying sheaves of modules are vector bundles.

Let us describe these results in more detail:
Let C be a projective, integral curve over $\mathbb{C}$ and let $\mathcal{A}$ be a sheaf of Lie algebras on C with vanishing sheaf cohomology. We fix a smooth, affine open subscheme $U \subseteq C$ such that the sheaf of Kähler differentials Ω_U is the trivial line bundle and such that there exists a simple Lie algebra $\mathfrak{g}$ and an isomorphism of Lie algebras $\mathcal{A}(U) \cong \mathfrak{g} \otimes_K \mathcal{O}_C(U)$. We choose a trivialising 1-form $\omega \in \mathcal{O}_C(U)$. In the geometric approach described previously ω is always a global, non-vanishing 1-form and since any of these only differ by a scalar on a curve of arithmetic genus one, there is essentially no dependence on it. In our approach the choice of ω matters, as we will soon see.
Let $D \subseteq C \times U$ denote the closed subscheme whose closed points are given by (u, u) for $u \in U$. Then there exists the so-called residue sequence

$$0 \to \mathcal{A} \boxtimes \mathcal{A}|_U \to \mathcal{A} \boxtimes \mathcal{A}|_U(D) \xrightarrow{\mathsf{res}_D^\omega} \mathcal{A}|_U \otimes \mathcal{A}|_U \to 0$$

which is such that res_D^ω is an isomorphism on global sections. Pulling back a Casimir element $\Omega \in \Gamma(U, \mathcal{A} \otimes \mathcal{A})$ to $\Gamma\big(C \times U, \mathcal{A} \otimes \mathcal{A}|_U(D)\big)$ gives a distinguished element $\mathsf{r}^\omega \in \Gamma\big(C \times U, \mathcal{A} \otimes \mathcal{A}|_U(D)\big)$ called the Szegö kernel (see chapter five of [6] or [14] for similar constructions). This is discussed in chapter one where we also prove the following theorem.

Theorem A. *The Szegö kernel* $\mathsf{r} = \mathsf{r}^\omega$ *satisfies the Generalised Classical Yang-Baxter Equation. That is for any* $x, y, z \in U$ *all distinct, we have*

$$\big[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)\big] + \big[\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)\big] + \big[\mathsf{r}^{32}(z,y), \mathsf{r}^{13}(x,z)\big] = 0.$$

Moreover, it is non-degenerate in the sense that there exists a non-empty open subset $V \subseteq U$ *such that the endomorphism of* $\mathfrak{g}$ *corresponding to* $\mathsf{r}(v_1, v_2)$ *under the isomorphism* $\sigma : \mathfrak{g} \otimes \mathfrak{g} \cong \mathsf{End}(\mathfrak{g})$ *induced by the Killing form is invertible for any* $v_1, v_2 \in V$.

Solutions $\mathsf{r}(x,y)$ coming from vector bundles are all unitary, i.e.

$$\mathsf{r}(x,y) = -\mathsf{r}^{21}(y,x),$$

where $(-)^{21}$ denotes the automorphism of $\mathfrak{g} \otimes \mathfrak{g}$ which exchanges the two tensor factors. Furthermore, any solution of the GCYBE which is unitary also satisfies the CYBE. Therefore it is of interest to understand which of the solutions we construct satisfy the unitarity condition. This is achieved by the following theorem, which is discussed at the end of the first chapter. Note that this shows that unitary of a solution may very well depend on the chosen 1-form ω.

Theorem B. 1. *The unitarity of* r^ω *is equivalent to the vanishing of*

$$\sum_{z \in C \setminus U} \mathsf{res}_z\big(\langle a, b\rangle \omega\big)$$

for any tuple $(x,y) \in U \times U \setminus D$, *any* $a \in \mathsf{H}^0\big(C, \mathcal{A}(x)\big)$ *and any* $b \in \mathsf{H}^0\big(C, \mathcal{A}(y)\big)$.

2. *If there exists a global, nowhere vanishing 1-form* ω *and a global bilinear form* $\langle -, - \rangle : \mathcal{A} \otimes \mathcal{A} \to \mathcal{O}_C$ *which restricts to the Killing form over* U, *the resulting Szegö kernel* r^ω *is unitary.*

Using these results one can show that a simple vector bundle $\mathcal{V}$ on the nodal cubic curve C produces a unitary, non-degenerate solution of the CYBE via the sheaf of Lie algebras $\mathsf{Ad}(\mathcal{V})$ given as the kernel of the trace map

$\mathcal{E}nd(\mathcal{V}) \to \mathcal{O}_C$. It turns out that these solutions are quasi-trigonometric, i.e. of the form

$$\mathsf{r}(x,y) = \frac{x\Omega}{y-x} + p(x,y),$$

where $p(x,y) \in \mathfrak{g} \otimes \mathfrak{g}[x,y]$ is a polynomial with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$. This situation is studied in chapter two, where we use results of Burban, Drozd and Greuel on the category of vector bundles on the nodal cubic curve ([7]) and end up with a unitary solution $\mathsf{r}_{n,d}$ of the CYBE for any pair of coprime positive integers n, d with $1 \leq d \leq n$. Quasi-trigonometric solutions have been classified by Khoroshkin, Pop, Samsonov, Stolin and Tolstoy (KPSST for short) in terms of certain Lie-theoretic data ([25]). This classification is recalled at the beginning of chapter two. Later on, we describe a geometric construction of the datum corresponding to a solution as in the previous theorem and prove that the two correspond to each other.

Theorem C. *Let $W(n,d) \subseteq \mathfrak{sl}_n\big((t)\big) \oplus \mathfrak{sl}_n$ be the Lie subalgebra*

$$t^2\mathfrak{sl}_n\Big(K\big[[t]\big]\Big) + t\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} + (\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, 0) + (\begin{pmatrix} A & tB \\ t^{-1}C & D \end{pmatrix}, \begin{pmatrix} D & C \\ B & A \end{pmatrix}),$$

for the $(n-d, d)$ block decomposition of all $n \times n$-matrices in the first entry of the tuples. Then $W(n,d)$ satisfies all properties required by the KPSST classification and the corresponding quasi-trigonometric solution is $\mathsf{r}_{n,d}$.

Furthermore one can also write down a concrete formula for the solution $\mathsf{r}_{n,d}$ using explicit but quite complicated combinatorial functions τ, ϵ and ψ which depend on the roots of $\mathfrak{sl}_n$ as well as e and d, and certain matrices J and K_i, which only depend on e and d. The result reads as follows:

Theorem D. *The quasi-trigonometric solution $\mathsf{r}_{n,d}$ is explicitly given by*

$$\Big(\sum_{\alpha\in\Delta^+} F_\alpha \otimes \Big(xE_\alpha + \sum_{k=1}^{\epsilon(\alpha)} \big(-yE_{\tau^k(\alpha)} + xE_{\tau^k(\alpha)}\big)$$

$$-y^2E_{\tau^{\epsilon(\alpha)+1}(\alpha)} + xyE_{\tau^{\epsilon(\alpha)+1}(\alpha)} - x^2E_{\psi(\alpha)} + x\sum_{k\geq 1}\big(-xE_{\psi^{k+1}(\alpha)} + yE_{\psi^k(\alpha)}\big)\Big)$$

$$+\sum_{\alpha\in\Delta^+} E_\alpha \otimes \Big(F_\alpha y + \sum_{k\geq 1}\big(-xF_{\psi^k(\alpha)} + F_{\psi^k(\alpha)}y\big)\Big)$$

$$+\sum_{i=1}^{n-1} G_i \otimes \Big(xK_i - JK_iJ^{-1}y\Big)\Big) \cdot \frac{1}{y-x}.$$

The previous two theorems hold true for all simple vector bundles on the nodal cubic curve. One could hope to obtain similar results for simple vector bundles on cycles of projective lines and we are able to prove their analogues for the next simplest case, namely for some simple vector bundles on a cycle of two projective lines. These solutions recover the so-called Cremmer-Gervais solutions ([15]) and are also discussed in chapter two.
As we noted before there exists classifications of trigonometric and quasi-trigonometric solutions of the CYBE, but these are very complicated. The geometric method discussed above helps us single out interesting solutions, e.g. the solutions $\mathsf{r}_{n,d}$ coming from simple vector bundles on the nodal cubic curve.

Finally, in chapter three, we focus on unitary, rational solutions of the CYBE, that is solutions of the form

$$\mathsf{r}(x,y) = \frac{\Omega}{y-x} + p(x,y),$$

where $p(x,y) \in \mathfrak{g} \otimes \mathfrak{g}[x,y]$ is a polynomial with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$. Burban-Henrich ([10]) described the solutions corresponding to simple vector bundles on the cuspidal cubic curve. These turned out to be rational. Using our generalised set-up to produce solutions of the CYBE and using the Krichever correspondence ([31]) as a method to obtain sheaves of Lie algebras from certain Lie subalgebras of $\mathfrak{g}\big((T)\big)$ we can show that much more is true:

Theorem E. *Given any unitary, rational solution $\mathsf{r}(x,y)$ of the CYBE there exists a sheaf of Lie algebras $\mathcal{A}$ on the cuspidal cubic curve $C = V(Y^2Z - X^3)$ satisfying all of the assumptions above such that the Szegö kernel associated with $\mathcal{A}$ and the global 1-form $d\frac{X}{Y}$ coincides with r.*

There is a classification of rational solutions due to Stolin in terms of certain Lie subalgebras of $\mathfrak{g}\big((T)\big)$ ([42], [43], [44]). We prove the following theorem in chapter three, which can be thought of as a generalisation of parts of the Krichever correspondence, to obtain a geometric approach to these Lie subalgebras. It is in spirit with Drinfeld's idea to study Manin triples geometrically (see section three of [17]).

Theorem F. *Let X be an integral, projective curve with Gorenstein point x. Assume that $X \setminus \{x\}$ is smooth and let $c_1, \dots, c_n$ be the preimages of x under the normalisation map $C \to X$. Let ω be a rational 1-form on X without poles along $X \setminus \{x\}$. Then to any torsion-free sheaf $\mathcal{F}$ on X of rank r with a symmetric, non-degenerate form $\langle -, - \rangle$ over $X \setminus \{x\}$ one can associate a decomposition*

$$\Big(K\big((t_1)\big) \times \dots \times K\big((t_n)\big)\Big)^{\oplus r} = \mathcal{O}_X\big(X \setminus \{x\}\big)^{\oplus r} \oplus \widehat{\mathcal{F}_x}.$$

Moreover, this decomposition is lagrangian with respect to the form

$$\{\alpha, \beta\} = \sum_{i=1}^{n} \mathsf{res}_{c_i}\big(\langle \alpha, \beta \rangle \omega\big)$$

if there exists a closed point y in $X \setminus \{x\}$ with $\mathsf{res}_y\big(\langle f, g \rangle \omega\big) = 0$ for any $f, g \in \mathsf{H}^0\big(X \setminus \{y\}, \mathcal{F}\big)$.

Finally, we use the previous two results and the faithfully-flat covering

$$U \mathbin{\dot\cup} \mathsf{spec}\big(\widehat{\mathcal{O}_{X,x}}\big) \to C$$

associated to a curve C and a closed point $x \in C$ with complement the open subscheme $U = C \setminus \{x\}$ to reprove Stolin's classification of rational, unitary solutions of the CYBE geometrically.

Theorem G. *Rational, unitary solutions of the CYBE are in bijection with Lie subalgebras $W \subseteq \mathfrak{g}\big((t^{-1})\big)$ satisfying*

1. $\mathfrak{g}\big((t^{-1})\big) = \mathfrak{g}[t] \oplus W$.
2. *W contains $t^{-N}\mathfrak{g}\big[[t^{-1}]\big]$ for some $N \geq 1$.*
3. *W is lagrangian with respect to the form $\mathsf{res}_{t^{-1}=0}\langle -, - \rangle$.*

Under this bijection, two solutions r_1 and r_2 are gauge equivalent (in the sense of Stolin) if and only if there exists an automorphism of Lie algebras $\sigma \in \mathsf{Aut}\big(\mathfrak{g}[t]\big)$ such that the corresponding subalgebras W_1 and W_2 are mapped to each other via σ.

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1 The Classical Yang-Baxter Equation via Szegö Kernels and Residues

1.1 The Generalised Classical Yang-Baxter Equation

Let K be a field (usually and classically taken to be the complex numbers $\mathbb{C}$) and $\mathfrak{g}$ a finite-dimensional Lie algebra over K. Recently mathematical physicists have become interested in non-skewsymmetric solutions of the Classical Yang-Baxter Equation ([40]) or solutions of the Generalised Classical Yang-Baxter Equation. Since the latter equation is not as well known as the Classical Yang-Baxter Equation, we define it in this section and discuss some of its properties. Note that some variant of it is already discussed by Cherednik in [14].

Definition 1.1. *For a meromorphic function $\mathsf{r}(x,y) : K \times K \to \mathfrak{g} \otimes \mathfrak{g}$ the Generalised Classical Yang-Baxter Equation (GCYBE for short) is given by*

$$\left[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)\right] + \left[\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)\right] + \left[\mathsf{r}^{32}(z,y), \mathsf{r}^{13}(x,z)\right] = 0,$$

where r^{ij} for $i \neq j$ and both in $\{1,2,3\}$ denotes the function

$$K^2 \to \mathfrak{g} \otimes_K \mathfrak{g} \to \mathcal{U}(\mathfrak{g}) \otimes_K \mathcal{U}(\mathfrak{g}) \xrightarrow{\iota^{ij}} \mathcal{U}(\mathfrak{g}) \otimes_K \mathcal{U}(\mathfrak{g}) \otimes_K \mathcal{U}(\mathfrak{g}),$$

where ι^{ij} is the morphism of rings which maps $a \otimes b$ to the elementary tensor with a in the i-th spot, b in the j-th spot and the identity element in the remaining tensor factor.
A solution r of the GCYBE is called unitary if

$$\mathsf{r}^{12}(x,y) = -\mathsf{r}^{21}(y,x)$$

for all x and y at which r is defined.

Remarks.

1. *A unitary solution of the GCYBE satisfies the well-known Classical Yang-Baxter Equation (CYBE for short)*

$$\left[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)\right] + \left[\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)\right] + \left[\mathsf{r}^{13}(x,z), \mathsf{r}^{23}(y,z)\right] = 0.$$

2. *If we talk about meromorphic functions on an abstract field K, we consider $K = \mathbb{A}^1_K$ equipped with its Zariski topology. In the case of $\mathbb{C}$ we*

may also consider the classical topology. Via analytification, solutions in the Zariski topology also lead to solutions in the classical topology. Therefore we will focus on the Zariski topology.

As the following lemma (and Example 3.2) shows, there are more ways to produce new solutions of the GCYBE from old ones than there are for the CYBE (for which only the first two items are true).

Lemma 1.1. *Let $\mathsf{r}(x, y)$ be a solution of the GCYBE. Then the following are also solutions of the GCYBE:*

1. *For any meromorphic function $\phi : K \to \mathsf{Aut}(\mathfrak{g})$ we consider the function $\phi(\mathsf{r})$ given by $(x, y) \mapsto \big(\phi(x) \otimes \phi(y)\big)\big(\mathsf{r}(x, y)\big)$.*

2. *For any (holomorphic) automorphism χ of K, we consider the function r^χ given by $(x, y) \mapsto \mathsf{r}\big(\chi(x), \chi(y)\big)$.*

3. *For any function $\psi : K \to K$, we consider the function $\psi \cdot \mathsf{r}$ given by $(x, y) \mapsto \psi(y) \cdot \mathsf{r}(x, y)$.*

Proof. The first two items are well-known. Nevertheless we shall sketch a proof for completeness.

1. If one applies $\phi \otimes \phi \otimes \phi$ to the GCYBE for r one arrives at the GCYBE for $\phi(\mathsf{r})$, hence the claim.

2. Obvious.

3. The GCYBE for $\chi \cdot \mathsf{r}$ reads

$$\big[\chi(y)\mathsf{r}^{12}(x, y), \chi(z)\mathsf{r}^{13}(x, z)\big] + \big[\chi(y)\mathsf{r}^{12}(x, y), \chi(z)\mathsf{r}^{23}(y, z)\big]$$
$$+\big[\chi(y)\mathsf{r}^{32}(z, y), \chi(z)\mathsf{r}^{13}(x, z)\big]$$

   and is therefore equal to $\chi(y)\chi(z)$ times the GCYBE for r.

□

From now on we will assume that $\mathfrak{g}$ is a simple, finite-dimensional Lie algebra over K with a fixed Killing form $\langle -, - \rangle$.

Definition 1.2. *A solution* $\mathsf{r}(x,y)$ *of the GCYBE is called non-degenerate if there exists a non-empty open subset* $U \subseteq K$ *such that the linear map associated to* $\mathsf{r}(x,y)$ *under the isomorphism* $\mathfrak{g}\otimes\mathfrak{g} \cong \mathsf{End}(\mathfrak{g})$ *induced by the Killing form is an isomorphism for any* $x, y \in U$ *for which* $\mathsf{r}(x,y)$ *is defined.*

Lemma 1.2. *Any non-degenerate solution of the GCYBE which also satisfies the CYBE is generically unitary, i.e. there exists a non-empty open subset* $U \subseteq K$ *such that*

$$\mathsf{r}(x,y) = -\mathsf{r}^{21}(y,x)$$

for all $x, y \in U$ *for which* $\mathsf{r}(x,y)$ *is defined.*

Proof. If we subtract the CYBE for r from the GCYBE for r we arrive at

$$\left[\mathsf{r}^{13}(x,z), \mathsf{r}^{32}(z,y) + \mathsf{r}^{23}(y,z)\right] = 0.$$

Since r is non-degenerate, this implies the vanishing of the second factor of the bracket for generic z which proves the claim. □

The following is the simplest example of a solution of the GCYBE which does not solve the CYBE. We will give a proof of this fact below and then another, more conceptual one later on.

Example 1.1. *Let* $\mathfrak{g}$ *be a simple Lie algebra with Casimir element* Ω*. Then*

$$\mathsf{r}(x,y) = \frac{y\Omega}{x-y}$$

solves the GCYBE, but not the CYBE.

Proof. It is well-known (and we will also prove this later on) that $\frac{\Omega}{x-y}$ is a unitary solution of the CYBE. Therefore part three of the previous lemma implies that r is a solution of the GCYBE.
For the second part of the statement we start with the CYBE for r, which is

$$\frac{yz}{(x-y)(x-z)}[\Omega^{12},\Omega^{13}] + \frac{yz}{(x-y)(y-z)}[\Omega^{12},\Omega^{23}] + \frac{z^2}{(x-z)(y-z)}[\Omega^{13},\Omega^{23}].$$

Multiplying by $(x-y)(x-z)(y-z)$ and using Lemma 1.14 this turns out to be equal to

$$y^2z[\Omega^{12},\Omega^{13}] + yz^2[\Omega^{23},\Omega^{13}] + xyz[\Omega^{12},\Omega^{23}] + xz^2[\Omega^{13},\Omega^{23}].$$

Setting $y = 0$ and $x \neq z$ both non-zero produces something different from zero. Therefore r does not satisfy the CYBE. Of course, one could also argue using Lemma 1.2, since r is clearly not unitary on any open subset $U \subseteq K$, but is non-degenerate (by inspection or by using our geometric results). $\square$

1.2 The Relative Residue Sequence

Let $\pi : X \to B$ be a relative curve with reduced fibres over a smooth base B and assume that a relative dualising sheaf $\Omega_{X/B}$ exists and is locally free. Furthermore assume that there exists a section σ of π such that the image of σ is contained in the regular locus of X. It follows that $D = \mathsf{im}(\sigma)$ is a Cartier divisor of X and we consider the relative residue sequence

$$0 \to \Omega_{X/B} \to \Omega_{X/B}(D) \xrightarrow{\mathsf{res}_D} \mathcal{O}_D \to 0.$$

Note that $\Omega_{X/B}$ denotes the relative dualising sheaf of the map π and not necessarily the sheaf of relative Kähler differentials (although the two agree non-canonically on the smooth part of the map). This sequence has the following properties (see chapter five of [11]):

Remarks. 1. *The sequence is compatible with base change in the following sense: If $f : B' \to B$ is any morphism and*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

denotes a pullback square, then the pullback along g of the relative residue sequence for π produces the relative residue sequence for π'.

2. *In particular, the induced sequence after restricting to a fibre F of π over some point $b \in B$ is really just the ordinary residue sequence*

$$0 \to \Omega_F \to \Omega_F\big(\sigma(b)\big) \xrightarrow{\mathsf{res}_{\sigma(b)}} \kappa_{\sigma(b)} \to 0.$$

3. *Locally res_D is given as follows:*
Let $b \in B$ and let $U = \mathsf{spec}(R)$ be an affine open subset of B containing b and let $V = \sigma(U) \times \sigma(U)$. Denote the maps of rings associated to π

and σ by f and g. Assume that U is small enough such that D is given by the vanishing locus of $s \otimes 1 - 1 \otimes s$ for some $s \in R$. Then $\Omega_{V/U}$ is generated by ds and $\mathsf{res}_D(\frac{\alpha}{s}ds) = \mathsf{m}(\alpha)$, where $\mathsf{m} : R \otimes_K R \to R$ is the multiplication map.

4. *In the case we are interested (see the next paragraph for definitions) the relative residue sequence can be constructed by applying the functor $\mathcal{H}om(-, \Omega_{X/B})$ to the exact sequence*

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

defining the diagonal D and using a piece of the resulting long exact sequence of $\mathcal{E}xt$-sheafs.

Now we specialise the general construction to the following special case: Let C be a reduced, projective, Gorenstein curve over an algebraically closed field K of characteristic zero and let F be some irreducible, affine open subset of E which is contained in the set of smooth points such that the ideal sheaf of the diagonal $D \bigcap (F \times F)$ is generated by $s \otimes 1 - 1 \otimes s$ for some $s \in \Gamma(F, \mathcal{O}_C)$. Then we take $B = F$, $X = C \times F$, π the canonical projection and σ to be the restriction of the diagonal $\Delta : C \to C \times C$ to F.

Lemma 1.3. *A relative dualising sheaf for the morphism π exists and it is locally free of rank one.*

Proof. Since all fibres are Gorenstein and in particular Cohen-Macaulay, the base is smooth and in particular Cohen-Macaulay and π is faithfully flat, X is Cohen-Macaulay by standard results on faithfully flat maps (see for example chapter 23 in [30]) and therefore the relative dualising sheaf exists by chapter six of [28]. That it is locally free of rank one follows from the same argument if one replaces Cohen-Macaulay by Gorenstein, since the fibres of π are all of dimension one. $\square$

In the case discussed before the previous lemma, we do not really care about the original relative residue sequence, but about its tensor product with the line bundle $\Omega^\vee_{X/B}$, i.e. the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \xrightarrow{\mathsf{res}_D} \delta_*(\mathcal{H}om_F(\Omega_F, \mathcal{O}_F)) \to 0.$$

To identify the rightmost term as we have done above, we use the following lemma.

Lemma 1.4. *The canonical morphism of sheaves*

$$\delta_*(\mathcal{O}_B) \otimes \mathcal{H}om_X(\Omega_{X/B}, \mathcal{O}_X) \to \delta_*\big(\mathcal{H}om_F(\Omega_F, \mathcal{O}_F)\big)$$

given on an open subset $V \subseteq X$ *by* $1 \otimes f \mapsto f|_{D \bigcap V}$ *is an isomorphism.*

Proof. The canonical morphism is given as the composite of the canonical isomorphism

$$\mathcal{F} \otimes \mathcal{H}om_X(\mathscr{L}, \mathcal{O}_X) \to \mathcal{H}om_X(\mathscr{L}, \mathcal{F})$$

(which holds true for any line bundle $\mathscr{L}$ and coherent sheaf $\mathcal{F}$ on X) and the canonical isomorphism

$$\mathcal{H}om_X(\Omega_{X/B}, \delta_*\mathcal{O}_B) = \delta_*\mathcal{H}om_B(\Omega_{X/B}|_D, \delta_*\mathcal{O}_B)$$

coming essentially from the universal property of a quotient module. It remains to identify $\Omega_{X/B}|_D$, but since $\Omega_{X/B} = (\mathsf{pr}_1)^*\Omega_C$ by the base change property of the relative dualising sheaf (where $\mathsf{pr}_1 : C \times F \to C$ denotes the canonical map onto the first factor), this restriction turns out to be Ω_F. $\square$

Under this identification the map res_D is given by

$$\frac{1}{s} \mapsto (ds \mapsto 1)$$

Finally, Ω_F is the trivial line bundle and we choose a trivialisation $\omega = \frac{ds}{\phi} \in \Omega_F(F)$ where s is given as before and $\phi \in \mathcal{O}_F(F)$ is a unit. We end up with the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \xrightarrow{\mathsf{res}_D^\omega} \mathcal{O}_D \to 0.$$

Again, it is important to have a concrete description of the last non-trivial map, which is locally (under the same assumptions and using the same notations as before) given by

$$\frac{g}{s \otimes 1 - 1 \otimes s} \mapsto \frac{\mathsf{m}(g)}{\phi}.$$

The following is an important example of the situation discussed above. It is in this context that the CYBE has mainly been studied by Burban, Henrich, Kreussler and Polishchuk.

Example 1.2. *Let $C = E$ be a singular Weierstraß curve over an algebraically closed field K and let F be its regular part. Then F is isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$ (as schemes) and if T is a coordinate on F, then s can be chosen to be $1 \otimes T - T \otimes 1$, or rather suggestively if we identify $K[T] \otimes_K K[T]$ with $K[X,Y]$, we can choose $s = X - Y$.*

1.3 The Sheaf Of Universal Enveloping Algebras

Since we will be dealing with sheaves of Lie algebras, we include a short discussion of sheaves of universal enveloping algebras now: Let S be a K-scheme and $\mathcal{A}$ a (quasi-coherent) sheaf of Lie algebras on S. Then for each affine open subset $U \subseteq S$, consider the universal enveloping algebra A_U of $\Gamma(U, \mathcal{A})$. Then if $D(f) \subseteq U$ is a standard open subset, $A_{D(f)}$ is canonically isomorphic to $(A_U)_f$, since they share the same universal property with regard to $\Gamma\big(D(f), \mathcal{A}\big)$ and therefore these rings glue to a sheaf of rings $\mathcal{U}$ on S whose underlying sheaf of modules is a quasi-coherent sheaf of $\mathcal{O}_S$-modules. In total, we have shown the following lemma.

Lemma 1.5. *If $\mathcal{A}$ is a quasi-coherent sheaf of Lie algebras on a scheme S (in the sense that its underlying sheaf of $\mathcal{O}_S$-modules is quasi-coherent), then there exists a quasi-coherent sheaf of rings $\mathcal{U}$ and a morphism $\iota : \mathcal{A} \to \mathcal{U}$ of sheaves of $\mathcal{O}_S$-modules such that on each affine open subset $U \subseteq S$ the algebra of sections $\Gamma(U, \mathcal{U})$ is the universal enveloping algebra of the Lie algebra $\Gamma(U, \mathcal{A})$ and $\iota(U)$ is the canonical map from the Lie algebra into its enveloping algebra.*

We need to have some control over the morphism ι in the form of the following injectivity criterion.

Lemma 1.6. *In the notation of the previous lemma assume either that $\mathsf{char}(K) = 0$ or that S is irreducible of dimension one, $\mathcal{A}$ torsionfree and that the restriction of $\mathcal{A}$ to some non-empty open subset $U \subseteq S$ is given by the sheaf of Lie algebras associated to $\mathfrak{g} \otimes_K \mathcal{O}_S(U)$ for some simple Lie algebra $\mathfrak{g}$. Then ι is a monomorphism.*

Proof. If the characteristic of K is zero, then the restriction of ι to any affine open subset is injective by the PBW-theorem, see for instance the version by Deligne and Morgan in [16] or the one in [12].
In the second case described in the lemma $\iota|_U$ is injective and hence the kernel

of ι is supported at most at finitely many closed points. $\mathcal{A}$ being torsionfree, $\mathsf{ker}(\iota)$ must therefore be zero. $\square$

1.4 Construction of the Szegö Kernel

In this section, we want to construct potential solutions of the GCYBE starting with sheaves of Lie algebras satisfying certain conditions. We shall apply the short exact sequence constructed in the previous section to a sheaf of Lie algebras $\mathcal{A}$ on E which is supposed to satisfy the following assumptions:

1. $\mathcal{A}$ is a coherent $\mathcal{O}_C$-module (sometimes we will also say that $\mathcal{A}$ is a coherent sheaf of Lie algebras).
2. The sheaf cohomology of $\mathcal{A}$ vanishes (in particular, $\mathcal{A}$ is torsionfree).
3. $\mathcal{A}|_F \cong \mathfrak{g} \otimes_K \widetilde{\Gamma(F, \mathcal{O}_F)}$ as sheaves of Lie algebras for a semisimple Lie algebra $\mathfrak{g}$ over K.

Lemma 1.7. *Tensoring the short exact residue sequence (for concreteness in its third form) with $\mathcal{A} \boxtimes \mathcal{A}\big|_F$ produces the exact sequence*

$$0 \to \mathcal{A} \boxtimes \mathcal{A}|_F \to \mathcal{A} \boxtimes \mathcal{A}|_F(D) \xrightarrow{\mathsf{res}_D^\omega} \delta_*(\mathcal{A}|_F \otimes \mathcal{A}|_F) \to 0.$$

Proof. To prove that the sequence is exact, it is sufficient to show that the sheaf $\mathcal{T}or^1_{\mathcal{O}_X}(\mathcal{A} \boxtimes \mathcal{A}|_F, \mathcal{O}_D)$ is zero. This is achieved by noticing that it is at most supported on the "restricted diagonal" $D \subseteq F \times F$ and that the restriction of $\mathcal{A}$ to F is a vector bundle. $\square$

Now we apply the derived functor $\mathbb{R}\Gamma(C \times F, -)$ to the sequence obtained in the previous lemma.

Lemma 1.8. *Application of $\mathbb{R}\Gamma(C \times F, -)$ gives an isomorphism*

$$\mathsf{res}_D^\omega : \Gamma\big(C \times F, \mathcal{A} \boxtimes \mathcal{A}|_F(D)\big) \to \Gamma(F, \mathcal{A} \otimes \mathcal{A}).$$

Proof. By the Künneth theorem (see for instance tag 0BED in [41]) the sheaf cohomology group $\mathbb{R}^i\Gamma(X, \mathcal{A} \boxtimes \mathcal{A}|_F)$ is given by

$$\mathbb{R}^i\Gamma\big(X, \mathcal{A} \boxtimes \mathcal{A}|_F\big) = \bigoplus_{n+m=i} \mathsf{H}^n(C, \mathcal{A}) \otimes \mathsf{H}^m\big(F, \mathcal{A}|_F\big)$$

and hence vanishes for all i by the second assumption we made about $\mathcal{A}$. The result is now a direct consequence of the long exact sequence associated to sheaf cohomology and standard facts about the compatibility of $\mathbb{R}\Gamma$ with affine morphisms. □

Next we fix a Killing form $\langle -, - \rangle$ with Casimir element Ω for $\mathfrak{g}$ and denote its extension to $\mathfrak{g} \otimes_K \Gamma(F, \mathcal{O}_C)$ by Ω, too. Note that the induced Killing form on $\Gamma(F, \mathcal{A})$ is independent of the chosen isomorphism $\mathcal{A}|_F \cong \mathfrak{g} \otimes_K \widetilde{\Gamma(F, \mathcal{O}_F)}$ and therefore so is the element $\Omega \in \Gamma(F, \mathcal{A} \otimes \mathcal{A})$.

Lemma 1.9. *There is a unique element* $\mathsf{r} \in \Gamma\big(C \times F, \mathcal{A} \boxtimes \mathcal{A}|_F(D)\big)$ *such that* $\mathsf{res}_D^\omega(\mathsf{r}) = \Omega$. *Furthermore,*

$$\mathsf{r}|_{F \times F} = \frac{\phi}{1 \otimes s - s \otimes 1} \Omega + p$$

for some $p \in \Gamma(F, \mathcal{A}) \otimes \Gamma(F, \mathcal{A})$.

Proof. That there is exactly one such element follows from the bijectivity of the residue map on global sections, which was established in Lemma 1.8. The concrete description of $\mathsf{r}|_{F \times F}$ follows from the concrete description of the residue map and the fact that $\Gamma(F \times F, \mathcal{A} \boxtimes \mathcal{A}) = \Gamma(F, \mathcal{A}) \otimes_K \Gamma(F, \mathcal{A})$, since F is affine. □

Remark. *The notation* $\frac{\phi \Omega}{1 \otimes s - s \otimes 1}$ *is slightly ambiguous since* $\phi \in \mathcal{O}_C(F)$ *and not* $\phi \in \mathcal{O}_{C \times F}\big(F \times F\big)$. *But it is fine to read* $\phi \otimes 1$ *or* $1 \otimes \phi$ *instead, because the difference of the two is a multiple of* $1 \otimes s - s \otimes 1$, *by one of the assumptions we previously made about* F. *In view of Lemmas 1.1 and 1.10 we usually choose to view it as* $1 \otimes \phi$.
What is most important at the moment is not the concrete form of $\mathsf{r}|_{F \times F}$ *but that it comes from a section over* $C \times F$.

Definition 1.3. *The element* r *constructed above is called the Szegö kernel associated with* $\mathcal{A}$ *and* ω. *Sometimes the dependence on* ω *is suppressed. On the other hand, if we want to express the dependence, we will write* r^ω.

Remark. *If one is only interested in whether the Szegö kernel satisfies the GCYBE, the dependence on* ω *disappears, since any change of* ω *is given by application of a unit of* $\Gamma(F, \mathcal{O})$ *and the resulting Szegö kernels will also only differ by this unit (see Lemma 1.1).*

However, if one is interested in unitarity or solutions of the CYBE, the choice of ω matters as we have seen in Example 1.1.

Actually, we can say a bit more.

Lemma 1.10. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two sheaves of Lie algebras on C satisfying our running assumptions and let ω and ω' be two 1-forms generating Ω_F. Then the following are true:*

1. *r^ω and $\mathsf{r}^{\omega'}$ are related by an equivalence as in part three of Lemma 1.1.*
2. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as sheaves of Lie algebras, then the corresponding Szegö kernels are related by an equivalence of type one in Lemma 1.1.*

Proof. 1. Since $\Omega_F(F)$ is free of rank one, there exists a unit $\gamma \in \mathcal{O}_C(F)$ such that $\omega = \gamma \cdot \omega'$ and hence from the concrete description of $\mathsf{res}_D^?$ given in the previous section, one can see that the two residue maps differ by multiplication with γ. Therefore, so do the Szegö kernels.

2. Given an isomorphism of sheaves of Lie algebras $\psi : \mathcal{A}_1 \cong \mathcal{A}_2$, we have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_1 \boxtimes \mathcal{A}_1|_F \arrow[r] \arrow[d, "\psi\boxtimes\psi|_F"] & \mathcal{A}_1 \boxtimes \mathcal{A}_1|_F(D) \arrow[r, "\mathsf{res}_D^\omega"] \arrow[d, "\psi\boxtimes\psi|_F(D)"] & \mathcal{A}_1|_F \otimes \mathcal{A}_1|_F \arrow[r] \arrow[d, "\psi|_F\otimes\psi|_F"] & 0 \\ 0 \arrow[r] & \mathcal{A}_2 \boxtimes \mathcal{A}_2|_F \arrow[r] & \mathcal{A}_2 \boxtimes \mathcal{A}_2|_F(D) \arrow[r, "\mathsf{res}_D^\omega"] & \mathcal{A}_2|_F \otimes \mathcal{A}_2|_F \arrow[r] & 0 \end{tikzcd}$$

which is such that the Casimir element of the upper sequence is mapped to the Casimir element of the lower sequence and hence the same is true for the Szegö kernels.
Therefore, we may just take the polynomial map $K \to \mathsf{Aut}(\mathfrak{g})$ associated to $\psi|_F$ as the desired gauge equivalence.

$\square$

The following lemma relates our Szegö kernels to residues and evaluation maps, allowing for easier computations.

Lemma 1.11. *Let $(x, y) \in F \times F \setminus D$. Then the image of*

$$\mathsf{r}|_{(x,y)} \in \mathcal{A}\Big|_x \otimes \mathcal{A}\Big|_y$$

under the canonical isomorphism $\mathcal{A}\big|_x \otimes \mathcal{A}\big|_y \to \mathsf{Hom}\big(\mathcal{A}\big|_y, \mathcal{A}\big|_x\big)$ *(which sends* $a \otimes b$ *to* $\langle b, -\rangle a$*) is given by*

$$\mathsf{ev}_x \circ (\mathsf{res}^\omega_y)^{-1},$$

where $\mathsf{ev}_x : \mathsf{H}^0\big(C, \mathcal{A}(y)\big) \to \mathcal{A}(y)\big|_x \cong \mathcal{A}\big|_x$ *is the canonical evaluation map and* $\mathsf{res}^\omega_y : \mathsf{H}^0\big(C, \mathcal{A}(y)\big) \to \mathcal{A}\big|_y$ *denotes the map* $\mathsf{res}_y(- \cdot \omega)$*.*

Proof. Choose an orthonormal basis $\{G_i\}$ of $\mathfrak{g}$ with respect to $\langle -, -\rangle$ and let $\{F_i\}$ be the corresponding basis of $\mathsf{H}^0\big(C, \mathcal{A}(y)\big)$ given as preimages of the G_i under res^ω_y. Write $\mathsf{r}|_{C\times\{y\}} = \sum F_i \otimes g_i$ for some $g_i \in \mathcal{A}\big|_y$.
Write $g_i = \sum \lambda_{ij} G_j$. Because of $\Omega = \sum G_i \otimes G_i$ and $\mathsf{res}^\omega_y(F_i) = G_i$ we must have $\lambda_{ij} = \delta_{ij}$, that is $g_i = G_i$ and therefore the image of $\mathsf{r}|_{(x,y)}$ in $\mathsf{Hom}\big(\mathcal{A}\big|_x, \mathcal{A}\big|_y\big)$ is the map which sends G_i to $\mathsf{ev}_x(F_i)$. In other words, the claim is shown. $\square$

Remark. *Note that* $\mathsf{res}^\omega_y : \mathsf{H}^0\big(C, \mathcal{A}(y)\big) \to \mathcal{A}|_y$ *is really an isomorphism, since the base change property of the relative residue sequence gives a short exact sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{A}(y) \xrightarrow{\mathsf{res}^\omega_y} \mathcal{A}\big|_y \to 0$$

and the sheaf cohomology of $\mathcal{A}$ *(i.e. kernel and cokernel of the map under discussion) vanishes.*

Any unitary solution of the CYBE constructed via the geometric method mentioned previously is non-degenerate, i.e. the linear map $\mathsf{ev}_x \circ (\mathsf{res}^\omega_y)^{-1}$ is generically invertible. The same is true for the r-matrices we consider in the sense that the image of $\mathsf{r}|_{(x,y)}$ in $\mathsf{End}(\mathfrak{g})$ is an invertible linear map for (x, y) in a non-empty open subset of $F \times F \setminus D$ by the following proposition:

Proposition 1.1. *There is a non-empty open subset* $U \subseteq J(C)$ *of the Jacobian of* C *such that for all line bundles* $\mathcal{L}$ *parametrised by* V*, the cohomology group* $\mathsf{H}^0(C, \mathcal{A} \otimes \mathcal{L})$ *vanishes. In particular, the element* r *constructed above is (generically) non-degenerate.*

Proof. Denote the Jacobian of C by $J(C)$, its Poincaré bundle by $\mathcal{P}$ and consider the coherent sheaf $\mathcal{C} := \mathcal{P} \otimes \pi^*\mathcal{A}$ on $J(C) \times C$, where π is the projection onto the second factor. The first projection is a projective morphism with relatively ample line bundle which is a pullback of an ample line bundle $\mathcal{O}_C(1)$ on C along π. Therefore, the Hilbert polynomial of $\mathcal{C}$ on the fibre corresponding to $\mathscr{L} \in J(C)$ is given by

$$n \mapsto \chi\big(\mathcal{A} \otimes \mathscr{L} \otimes \mathcal{O}_C(n)\big) = \chi(\mathcal{A}) + \mathsf{deg}\big(\mathscr{L} \otimes \mathcal{O}_C(n)\big)\mathsf{rk}(\mathcal{A}).$$

Since the degree of $\mathscr{L}$ is zero, the Hilbert polynomial is independent of the fibre. Flattening stratifications (see for example chapter four of [34]) now imply that $\mathcal{C}$ is actually flat over $J(C)$. Therefore we may apply the semicontinuity theorem (see chapter five in [33]), which tells us that

$$\mathcal{L} \mapsto \mathsf{dim}_K \mathsf{H}^0(C, \mathcal{L} \otimes \mathcal{A})$$

is upper semicontinuous. Since $\mathsf{H}^0(C, \mathcal{A})$ vanishes by assumption, we conclude that there is a dense open subset $U \subseteq J(C)$ such that $\mathsf{H}^0(C, \mathcal{L} \otimes \mathcal{A}) = 0$ for all line bundles $\mathcal{L} \in U$.
To prove the second part of the proposition we use Lemma 1.11. The invertibility of the evaluation map

$$\mathsf{H}^0\big(C, \mathcal{A}(a)\big) \xrightarrow{\mathsf{ev}_b} \mathcal{A}\big|_b$$

is equivalent to the vanishing of the vector space $\mathsf{H}^0\big(C, \mathcal{A} \otimes \mathcal{O}_C(a-b)\big)$ for $a, b \in F$, since both spaces have the same (finite) K-dimension. Now consider the preimage of U under the map

$$F \times F \to j(C)$$

given by

$$(a, b) \mapsto \mathcal{O}_C(a-b)$$

and call it V. Since $\mathcal{O}_C \in U$, $V \neq \emptyset$ and is therefore a dense, open subset of the irreducible scheme $F \times F$. By construction the map ev_b considered above is invertible for any pair of distinct points $(a, b) \in V$ proving the second claim. □

1.5 The Szegö Kernel Satisfies the GCYBE

In this section, we prove that the Szegö kernel constructed in the previous section satisfies the Generalised Classical Yang-Baxter Equation (GCYBE for short), i.e. that for any $x, y, z \in F$, no two of which are equal, we have

$$\big[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)\big] + \big[\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)\big] + \big[\mathsf{r}^{32}(z,y), \mathsf{r}^{13}(x,z)\big] = 0,$$

where we consider $\mathsf{r} \in \mathsf{H}^0\big(F \times F, \mathcal{A} \boxtimes \mathcal{A}(D)\big)$ as a meromorphic map $F \times F \to \mathfrak{g} \otimes \mathfrak{g}$ and where r^{ij} denotes the map $F \times F \to \mathcal{U}(\mathfrak{g})^{\otimes 3}$ which inserts a 1 in the copy not labelled i or j.
Given any element $\mathsf{e} \in \mathsf{H}^0\big(C \times F, \mathcal{A} \boxtimes \mathcal{A}|_F(D)\big)$, we will first construct a new element

$$\mathsf{CYB}(e) \in \mathsf{H}^0\big(C \times F \times F, \mathcal{A} \boxtimes \mathcal{A}|_F \boxtimes \mathcal{A}|_F(E)\big),$$

where E denotes the divisor given by points of $C \times F \times F$ with (at least) two agreeing coordinates. This element $\mathsf{CYB}(e)$ will have the property that its evaluation at some point $(x, y, z) \in C \times F \times F \setminus E$ will be an instance of the GCYBE.

Lemma 1.12. *Let R be a ring and denote by $(a \otimes b)^{ij}$ for $i, j \in \{1, 2, 3\}$ not equal to each other, the elementary tensor of $R \otimes R \otimes R$ with a in i-th place, b in j-th place and a one in the other place. Then*

$$\begin{aligned}
\big[(a \otimes b)^{12}, (c \otimes d)^{13}\big] &= [a, c] \otimes b \otimes d \\
\big[(a \otimes b)^{12}, (c \otimes d)^{23}\big] &= a \otimes [b, c] \otimes d \\
\big[(a \otimes b)^{13}, (c \otimes d)^{23}\big] &= a \otimes c \otimes [b, d].
\end{aligned}$$

Proof. We will only prove the first equality, the others are shown in a similar fashion:

$$\begin{aligned}
\big[(a \otimes b)^{12}, (c \otimes d)^{13}\big] &= (a \otimes b \otimes 1)(c \otimes 1 \otimes d) - (c \otimes 1 \otimes d)(a \otimes b \otimes 1) \\
&= ac \otimes b \otimes d - ca \otimes b \otimes d \\
&= [a, c] \otimes b \otimes d
\end{aligned}$$

□

Denote the corresponding map of rings $R \otimes R \to R \otimes R \otimes R$ also by $(-)^{12}$, respectively $(-)^{13}$ (and similarly for the other cases).
We can now define $\mathsf{CYB}(e)$. If $e \in \mathsf{e} \in \mathsf{H}^0\big(C \times F, \mathcal{A} \boxtimes \mathcal{A}|_F(D)\big)$, consider its image in $\mathsf{H}^0\big(C \times F, \mathcal{U} \boxtimes \mathcal{U}(D)\big)$, where $\mathcal{U}$ denotes the sheaf of universal enveloping algebras of $\mathcal{A}$ constructed in Lemma 1.5. We call this section e, too and consider its image $e^{12} \in \mathsf{H}^0\big(C \times F \times F, \mathcal{U} \boxtimes \mathcal{U} \boxtimes \mathcal{U}(E_{12})\big)$, where $E_{12} \subseteq C \times F \times F$ denotes the divisor of points whose first two coordinates agree. Similarly, we can define elements e^{13} etc.

Lemma 1.13. *We have* $[e^{12}, e^{13}] \in \mathsf{H}^0\big(C\times F\times F, \mathcal{A}\boxtimes\mathcal{A}|_F\boxtimes\mathcal{A}|_F(E_{12}\cup E_{13})\big)$.

Proof. By the previous lemma we have to show that the canonical map

$$\mathsf{H}^0\big(C\times F\times F, \mathcal{A}\boxtimes\mathcal{A}|_F\boxtimes\mathcal{A}|_F(E_{12}\cup E_{13})\big) \to \mathsf{H}^0\big(C\times F\times F, \mathcal{U}\boxtimes\mathcal{U}\boxtimes\mathcal{U}(E_{12}\cup E_{13})\big)$$

is injective. Since $\mathcal{O}_{C\times F\times F}\big(E_{ij}\big)$ is a line bundle, it is sufficient to show that the map $\iota\boxtimes\iota\boxtimes\iota$ is injective. Since $\mathcal{A}|_F$ and $\mathcal{U}_F$ are free $\mathcal{O}_F$-modules and $\iota|_F$ is injective, this reduces to checking that $\iota : \mathcal{A}\to\mathcal{U}$ is injective, which is true by Lemma 1.6. □

In this way we have defined an element

$$[e^{12}, e^{13}] \in \mathsf{H}^0\big(C\times F\times F, \mathcal{A}\boxtimes\mathcal{A}|_F\boxtimes\mathcal{A}|_F(E_{12}\cup E_{13})\big).$$

Then finally we define $\mathsf{CYB}(e)$ as

$$\mathsf{CYB}(e) = \big[\mathsf{e}^{12}, \mathsf{e}^{13}\big] + \big[\mathsf{e}^{12}, \mathsf{e}^{23}\big] + \big[\mathsf{e}^{32}, \mathsf{e}^{13}\big].$$

Proposition 1.2. *Given an element* $\mathsf{e}\in\mathsf{H}^0\big(C\times F, \mathcal{A}\boxtimes\mathcal{A}|_F(D)\big)$ *there exists an element*

$$\mathsf{CYB}(\mathsf{e}) \in \mathsf{H}^0\big(C\times F\times F, \mathcal{A}\boxtimes\mathcal{A}|_F\boxtimes\mathcal{A}|_F(E)\big),$$

where E denotes the divisor given by points of $C\times F\times F$ with (at least) two agreeing coordinates. This element $\mathsf{CYB}(e)$ *has the property that its evaluation at some point $(x,y,z)\in C\times F\times F\setminus E$ gives an instance of the GCYBE.*

Of course we do not want to stop at constructing such an element, but want to show that it is actually zero, i.e. that the GCYBE is satisfied. This is achieved by the next theorem.

Theorem 1.1. *The element* $\mathsf{CYB}(\mathsf{r})$ *is zero. In other words, for any pairwise different $x,y,z\in F$, we have*

$$\big[\mathsf{r}^{12}(x,y), \mathsf{r}^{13}(x,z)\big] + \big[\mathsf{r}^{12}(x,y), \mathsf{r}^{23}(y,z)\big] + \big[\mathsf{r}^{32}(z,y), \mathsf{r}^{13}(x,z)\big] = 0.$$

Proof. We will need to apply residues several times. Let us start by using the residue sequence for $C\times F\times F\to F\times F$ and the section $(f_1,f_2)\mapsto(f_1,f_1,f_2)$.

As we have seen in the first part of this chapter, this gives a residue sequence

$$0 \to \mathcal{O}_{C\times F\times F} \to \mathcal{O}_{C\times F\times F}(E_{12}) \to \mathcal{O}_{E_{12}} \to 0.$$

Again, one can tensor this sequence with $\mathcal{A} \boxtimes \mathcal{A}|_F \boxtimes \mathcal{A}|_F =: \mathcal{A} \boxtimes (\mathcal{A}|_F)^{\boxtimes 2}$ without sacrificing exactness. Note again that the E_{ij} are Cartier divisors since they are contained in the smooth part of $C \times F \times F$. Therefore we can also tensor the sequence with the line bundle $\mathcal{O}_{C\times F\times F}(E_{23} \cup E_{13})$ to obtain the short exact sequence

$$0 \to \mathcal{A}\boxtimes(\mathcal{A}|_F)^{\boxtimes 2}(E_{23}\cup E_{13}) \to \mathcal{A}\boxtimes(\mathcal{A}|_F)^{\boxtimes 2}(E) \to \mathcal{A}\boxtimes\mathcal{A}|_F\boxtimes\mathcal{A}|_F(E)|_{E_{12}} \to 0.$$

Now if we apply the residue map coming from this short exact sequence to $\mathsf{CYB}(\mathsf{r})$, then the term $[\mathsf{r}^{32}, \mathsf{r}^{13}]$ vanishes, since it is a section of the sheaf $\mathcal{A} \boxtimes (\mathcal{A}|_F)^{\boxtimes 2}(E_{23} \cup E_{13})$. Furthermore, to check the vanishing of the whole term, we may restrict to $F \times F \times F$, since the sections of $\mathcal{O}_{E_{12}}$ over $C \times F \times F$ and $F \times F \times F$ agree. But over $F \times F$, the Szegö kernel r may be written as $\mathsf{r} = \frac{\phi}{s\otimes 1 - 1\otimes s}\Omega + p$ for some regular section $p \in \mathsf{H}^0(F \times F, \mathcal{A} \boxtimes \mathcal{A})$ as we noticed before.
Now if we apply the concrete description of the residue map to $\mathsf{CYB}(\mathsf{r})$ and use this concrete description of r, we arrive at

$$[\Omega^{12}, \frac{\phi}{s}\Omega^{13} + p^{13}] + [\Omega^{12}, \frac{\phi}{s}\Omega^{23} + p^{23}].$$

That this element vanishes follows from the lemma below, hence we conclude that $\mathsf{CYB}(\mathsf{r})$ is actually an element of $\mathsf{H}^0\big(C \times F \times F, \mathcal{A}\boxtimes(\mathcal{A}|_F)^{\boxtimes 2}(E_{23}\cup E_{13})\big)$. Using similar residue calculations, one shows

$$\mathsf{CYB}(\mathsf{r}) \in \mathsf{H}^0\big(C \times F \times F, \mathcal{A} \boxtimes (\mathcal{A}|_F)^{\boxtimes 2}\big),$$

but since this group is zero by the Künneth formula we have succeeded in proving the theorem. □

Lemma 1.14. *For any element $A \in \mathfrak{g} \otimes \mathfrak{g}$, we have*

$$\big[\Omega^{12}, A^{13} + A^{23}\big] = 0.$$

Proof. We might as well assume that $A = a \otimes b$ for some $a, b \in \mathfrak{g}$. Then

$$\begin{aligned}\left[\Omega^{12}, A^{13} + A^{23}\right] &= \left[\Omega^{12}, a \otimes 1 \otimes b + 1 \otimes a \otimes b\right] \\ &= \left[\Omega, a \otimes 1 + 1 \otimes a\right]^{12} \cdot 1 \otimes 1 \otimes b\end{aligned}$$

and the term inside the Lie brackets is zero, since Ω is an element of $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$, the $\mathfrak{g}$-invariants of $\mathfrak{g} \otimes \mathfrak{g}$. $\square$

1.6 Unitarity

In this section we establish a useful criterion for when the solution of the GCYBE constructed above satisfies the unitarity condition (which in turn implies that it is actually a solution of the CBYE as was already remarked earlier).

Lemma 1.15. *Let $\tau : F \times F \to F \times F$ be the morphism which flips the two terms of the product. Then the following are equivalent:*

1. $\mathsf{r} + \tau^*(\mathsf{r}) = 0$ *in* $\mathsf{H}^0\big(F \times F, \mathcal{A} \boxtimes \mathcal{A}(D)\big)$.
2. $\big\langle \mathsf{res}^{\omega}_y(b), \mathsf{ev}_y(a)\big\rangle = -\big\langle \mathsf{res}^{\omega}_x(a), \mathsf{ev}_x(b)\big\rangle$ *for all* $(x, y) \in F \times F \setminus D$, *all* $a \in \mathsf{H}^0\big(C, \mathcal{A}(x)\big)$ *and all* $b \in \mathsf{H}^0\big(C, \mathcal{A}(y)\big)$.

Proof. Since K is algebraically closed and C is reduced, it is certainly true that the first condition is equivalent to

$$\mathsf{r}|_{(x,y)} = -\big(\mathsf{r}|_{(y,x)}\big)^{21}$$

for any $(x, y) \in F \times F \setminus D$.
Denote by

$$t : \mathsf{Hom}_K\big(\mathcal{A}\big|_x, \mathcal{A}\big|_y\big) \to \mathsf{Hom}_K\big(\mathcal{A}\big|_y, \mathcal{A}\big|_x\big)$$

the isomorphism

$$f \mapsto \mathsf{d}^{-1} \circ f \circ \mathsf{d},$$

where d denotes the isomorphisms induced by the Killing form $\langle -, - \rangle$. Then the diagram

$$\begin{tikzcd} \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y \arrow[r, "\tau"] \arrow[d, "\mathsf{can}"] & \mathcal{A}\big|_y \otimes \mathcal{A}\big|_x \arrow[d, "\mathsf{can}"] \\ \mathsf{Hom}_K\big(\mathcal{A}\big|_x, \mathcal{A}\big|_y\big) \arrow[r, "t"] & \mathsf{Hom}_K\big(\mathcal{A}\big|_y, \mathcal{A}\big|_x\big), \end{tikzcd}$$

commutes, where

$$\mathsf{can} : \mathfrak{g} \otimes \mathfrak{g} \to \mathsf{Hom}_K(\mathfrak{g}, \mathfrak{g})$$

is given by

$$v \otimes w \mapsto \langle v, - \rangle w.$$

Using the commutativity of the diagram above and Lemma 1.11, we conclude that condition 1. above is equivalent to

$$t\big(\mathsf{ev}_y \circ (\mathsf{res}_x^\omega)^{-1}\big) = -\mathsf{ev}_x \circ (\mathsf{res}_y^\omega)^{-1}$$

for any $(x, y) \in F \times F \setminus D$. The claim now follows from the definition of the map t. □

Lemma 1.16. *Let $(x, y) \in F \times F \setminus D$, let $U = F \setminus \{x, y\}$ and denote the embedding $U \hookrightarrow C$ by j. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{A}(x) \otimes \mathcal{A}(y) \arrow[r, "\mathsf{unit}^{\otimes 2}"] \arrow[d, "\mathsf{res}_x^\omega \otimes \mathsf{ev}_x"'] & j_*(\mathcal{A}|_U) \otimes j_*(\mathcal{A}|_U) \arrow[d, "j_*\langle -,- \rangle"] \\
\mathcal{A}\big|_x \otimes \mathcal{A}\big|_x \arrow[d, "\langle -,- \rangle"'] & j_*(\mathcal{O}_C|_U) \arrow[d, "1 \mapsto \omega"] \\
\kappa_x & j_*(\Omega_C|_U) \arrow[l, "j_*(\mathsf{res}_x)"']
\end{tikzcd}$$

Proof. Since both ways around the diagram are morphisms of sheaves which end in a sheaf, it is sufficient to consider the presheaf tensor product as a starting point for both morphisms. In other words, it is sufficient to show commutativity for all $V \subseteq C$ open, $a, b, \in \mathcal{A}(V)$, $\lambda \in \mathcal{O}_C(x)$ and $\mu \in \mathcal{O}_C(y)$. Then an easy calculation shows that both ways round the diagram give the same element (namely $\big\langle \mathsf{ev}_x(a), \mathsf{ev}_x(b) \big\rangle \cdot \mathsf{ev}_x(\mu) \cdot \mathsf{res}_x^\omega(\lambda)$) in κ_x. □

Using these preliminaries we can now prove the main result on unitarity of solutions.

Theorem 1.2. *1. The unitarity of r is equivalent to the vanishing of*

$$\sum_{z \in C \setminus F} \mathsf{res}_z\big(\langle a, b \rangle \omega\big)$$

for any closed point $(x, y) \in F \times F \setminus D$ and any pair of global sections $a \in \mathsf{H}^0\big(C, \mathcal{A}(x)\big)$ and $b \in \mathsf{H}^0\big(C, \mathcal{A}(y)\big)$.

2. *If there exists a global, nowhere vanishing 1-form $\omega \in \Omega_C$ and a global bilinear form $\langle -, - \rangle : \mathcal{A} \otimes \mathcal{A} \to \mathcal{O}_C$ which restricts to the Killing form over F, the resulting Szegö kernel (using this 1-form) is unitary.*

Proof. 1. By Lemmas 1.15 and 1.16, the unitarity of r is equivalent to the vanishing of

$$\mathsf{res}_x\big(\langle a, b\rangle\omega\big) + \mathsf{res}_y\big(\langle a, b\rangle\omega\big)$$

for any closed point $(x, y) \in F \times F \setminus D$ and any pair of global sections $a \in \mathsf{H}^0\big(C, \mathcal{A}(x)\big)$ and $b \in \mathsf{H}^0\big(C, \mathcal{A}(y)\big)$. By the residue theorem (see for example the last chapter of [1], chapter two of [39] or [45]) the sum in the statement vanishes if we sum over all closed points of C, but over any point of F which is not equal to x or y the residue obviously vanishes, hence the result.

2. Since ω is a global, nowhere vanishing 1-form and $\langle -, - \rangle$ takes values in $\mathcal{O}_C$, the element $\mathsf{res}_p\big(\langle a, b\rangle\omega\big)$ is zero for any closed point p of C which is not equal to x or y. The proof then proceeds just as in part one.

□

Remarks. *1. The second part of the previous theorem can be used to prove the unitarity of the solutions of the CYBE considered by Polishchuk ([36]), Burban-Kreussler ([11]) and Burban-Henrich ([10]), since any sheaf of Lie algebras which is given as* Ad *of a simple vector bundle comes with a global symmetric form and there exist global, nowhere vanishing 1-forms on Weierstraß curves.*

2. Notice that the above proof works pointwise, i.e. $\mathsf{r}|_{(x,y)} = -\mathsf{r}^{21}|_{(y,x)}$ is equivalent to the vanishing of $\mathsf{res}_x\big(\langle a, b\rangle\omega\big) + \mathsf{res}_y\big(\langle a, b\rangle\omega\big)$ for any $a \in \mathsf{H}^0\big(C, \mathcal{A}(x)\big)$ and any $b \in \mathsf{H}^0\big(C, \mathcal{A}(y)\big)$.

Example 1.3. *We want to reconsider Example 1.1 from a geometric perspective. To do so, let us consider the curve $C = \mathbb{P}^1$ and fix some point $p \in C$ with complement U. Consider $\mathcal{I} = \mathcal{O}_C(-1)$ as the ideal sheaf of the reduced subscheme structure on $\{p\}$ and let $\mathcal{E}$ be the vector bundle of $n \times n$-matrices with values in $\mathcal{I}$. Let $\mathcal{A}$ be the kernel of the trace map $\mathcal{E} \to \mathcal{I}$.*

Then $\mathcal{A}$ is a sheaf of Lie algebras with vanishing cohomology, which agrees with the sheaf associated to $\mathfrak{sl}_n\big(\mathcal{O}_C(U)\big)$ over U. We take the standard form $(A, B) \mapsto \mathsf{tr}(AB)$ as its Killing form. Furthermore, U satisfies the assumptions needed to construct a Szegö kernel, since $U \cong \mathbb{A}^1$.
Let us fix a coordinate X on U and let V be $\mathsf{spec}\big(K[X, X^{-1}]\big) \subseteq U$. Then there are two natural choices of a trivialising 1-form which we may take on V, namely

1. *$\omega_1 = dX$ and*
2. *$\omega_2 = \frac{dX}{X}$.*

Let us now calculate the corresponding evaluation and residue maps:
It is easy to see that once we fix an identification

$$\mathsf{H}^0(V, \mathcal{A}) = \mathfrak{sl}_n\big(K[X, X^{-1}]\big),$$

we have

$$\mathsf{H}^0\big(V, \mathcal{A}(a)\big) = \frac{1}{X-a}\mathfrak{sl}_n\big(K[X, X^{-1}]\big)$$

canonically for any point $a \in V$. For any point $b \in V \setminus \{a\}$, the evaluation at b map is given by inserting b instead of X, whereas the residue at a is given by multiplication by $X - a$ on $\frac{1}{X-a}\mathfrak{sl}_n(K)$ and zero on terms of higher degree (in $X - a$).
In this way, we see that the Szegö kernels associated to the two 1-forms are given as follows:

1. *$\mathsf{r}_1(X, Y) = \frac{\Omega}{X-Y}$ and*
2. *$\mathsf{r}_2(X, Y) = \frac{Y\Omega}{X-Y}$,*

where Ω denotes a Casimir element for the Killing form.
In particular, we recover the result that both satisfy the GCYBE.
In order to understand (potential) unitarity of these solutions from the geometric point of view, we first note that under the identification of $\mathsf{H}^0\big(V, \mathcal{A}(a)\big)$ above, $\mathsf{H}^0\big(C, \mathcal{A}(a)\big)$ corresponds to $\frac{\mathfrak{g}}{X-a}$. Therefore

$$\Big\langle \mathsf{H}^0\big(C, \mathcal{A}(a)\big), \mathsf{H}^0\big(C, \mathcal{A}(b)\big) \Big\rangle = \frac{K}{(X-a)(X-b)}.$$

But, since $\frac{dX}{(X-a)(X-b)}$ *has a non-zero residue at precisely the two closed points* a *and* b*, whereas* $\frac{dX}{X(X-a)(X-b)}$ *has a non-zero residue at precisely three points, the residue theorem together with Theorem 1.2 tells us that solution* r_1 *is unitary and thus satisfies the CYBE, whereas solution* r_2 *is not unitary at any point. By Lemma 1.2 it cannot satisfy the CYBE since it is non-degenerate by Proposition 1.1, but does not satisfy the unitarity condition.*

2 Calculation of some Quasi-Trigonometric Solutions of the Classical Yang-Baxter Equation Associated with Simple Vector Bundles on the Nodal Cubic Curve

2.1 The KPSST-Theory of Quasi-Trigonometric Solutions

We begin with a short reminder about the first steps in the classification of quasi-trigonometric solutions of the CYBE by Khoroshkin, Pop, Samsonov, Stolin and Tolstoy ([25], [35]) which is about a one-to-one correspondence between such solutions and certain Lie subalgebras of a certain infinite-dimensional Lie algebra called $\mathcal{D}(\mathfrak{g})$, because we will later identify the subalgebras corresponding to certain quasi-trigonometric solutions coming from stable vector bundles on the nodal cubic curve.
Let $\mathfrak{g}$ be a semisimple Lie algebra over $K = \mathbb{C}$ with Casimir element Ω and fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Fix an orthonormal basis (for the Killing form on $\mathfrak{g}$) for $\mathfrak{h}$ to obtain a basis $\{E_\alpha, F_\alpha, H_i\}_{\alpha,i} = \{G_1, \ldots, G_m\}$ of $\mathfrak{g}$. Define $\mathcal{D}(\mathfrak{g})$ to be the Lie algebra

$$\mathcal{D}(\mathfrak{g}) = \mathfrak{g}\big((t^{-1})\big) \times \mathfrak{g}$$

and endow it with the symmetric, non-degenerate form

$$\{-,-\} : \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g}) \to K$$

given by

$$\{(A,a),(B,b)\} = \mathsf{res}_t \langle A, B\rangle \omega - \langle a, b\rangle,$$

where $\langle -,-\rangle$ denotes the Killing form on $\mathfrak{g}$ (and its canonical extension to $\mathfrak{g}\big((t^{-1})\big)$), $\omega = \frac{dt^{-1}}{t}$, A and B are elements of $\mathfrak{g}\big((t^{-1})\big)$ and a and b are elements of $\mathfrak{g}$.
Note that the morphism

$$\begin{aligned} \mathfrak{g}[t] &\to \mathcal{D}(\mathfrak{g}) \\ P(t) &\mapsto \big(P(t), P(0)\big) \end{aligned}$$

is injective and its image (also denoted by $\mathfrak{g}[t]$) is a lagrangian subalgebra of $\mathcal{D}(\mathfrak{g})$ with respect to the form $\{-,-\}$.
There is a standard decomposition

$$\mathcal{D}(\mathfrak{g}) = \mathfrak{g}[t] \oplus \left(t^{-1}\mathfrak{g}\big[[t^{-1}]\big] \oplus (\mathfrak{n}_+, 0) \oplus \big\{(h,-h)\big|h \in \mathfrak{h}\big\} \oplus (0, \mathfrak{n}_-)\right).$$

We consider the associated projection $\pi : \mathcal{D}(\mathfrak{g}) \to \mathfrak{g}[t]$ and the associated standard quasi-trigonometric solution

$$\mathsf{r}_{st}(x,y) = \frac{x\Omega}{y-x} + \frac{1}{2}\left(\Omega + \sum_{\alpha \in \Delta_+} F_\alpha \wedge E_\alpha\right),$$

where Δ_+ denotes the set of positive roots.

Definition 2.1. *A solution r of the CYBE for $\mathfrak{g}$ is called quasi-trigonometric if*

$$\mathsf{r}(x,y) = \frac{x\Omega}{y-x} + p(x,y)$$

for some polynomial $p(x,y) \in \mathfrak{g} \otimes_K \mathfrak{g}[x,y]$.

We can now state the main result of KPSST. We have chosen to change the order of the direct sum as this is more in line with our geometric results, but since all solutions are unitary, this does not change the content of the theorem.

Theorem. *There is a bijection between unitary quasi-trigonometric solutions of the CYBE and lagrangian Lie subalgebras $W \subseteq \mathcal{D}(\mathfrak{g})$ satisfying*

1. *$\mathcal{D}(\mathfrak{g}) = \mathfrak{g}[t] \oplus W$.*
2. *There exists a natural number $n \in \mathbb{N}$ such that $t^{-n}\mathfrak{g}\big[[t^{-1}]\big]$ is contained in W.*

Remark. *Given W the associated solution is constructed as follows:*

1. *Calculate the unique set of elements $F_{n,i} \in W$ such that*

$$\{F_{n,i}, G_j t^m\} = \delta_{ij}\delta_{mn}.$$

2. *Calculate $X_{i,n} = (\pi \otimes \pi)(G_i t^n \otimes F_{i,n})(x,y)$.*

3. Then

$$\mathsf{r}_W(x,y) = \mathsf{r}_{st}(x,y) + \sum_{i,n} X_{i,n}$$

is the associated quasi-trigonometric solution of the CYBE.

2.2 Derivation of the Computational Version of the Approach via Residues and Evaluations

In this section we associate a Szegö kernel with every pair of relatively prime, positive integers n and d. Let $K = \mathbb{C}$ be the field of complex numbers and fix the nodal cubic curve $E = V(Y^2Z - X^2(X+Z))$ with normalisation $\mathbb{P}^1$ (and coordinates $[z_0, z_1]$ and normalisation map ν such that $0 = [0,1]$ and $\infty = [1,0]$ are mapped to the singular point of E). Let U denote the regular part of E. We fix isomorphisms

$$\mathcal{O}_{\mathbb{P}^1}(c)\big|_{\mathbb{P}^1\setminus\{0\}} \cong \mathcal{O}_{\mathbb{P}^1\setminus\{0\}} \qquad\qquad p \mapsto \frac{p}{z_0^c}$$

and

$$\mathcal{O}_{\mathbb{P}^1}(c)\big|_{\mathbb{P}^1\setminus\{\infty\}} \cong \mathcal{O}_{\mathbb{P}^1\setminus\{\infty\}} \qquad\qquad p \mapsto \frac{p}{z_1^c}.$$

Definition 2.2. *Let e and d denote two non-negative integers. Let $\mathsf{BM}(E)$ denote the category whose objects are $(e+d)\times(e+d)$ matrices with entries in $K\times K$ with block structure (e,d), i.e. any matrix $M \in \mathsf{BM}(E)$ is given as*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M_1 is a $d\times d$ matrix and M_4 is a $e\times e$ matrix.
A morphism between two matrices M and N of sizes $m = m_0 + m_1$ and $n = n_0 + n_1$ is given by a pair (F, f), where $f \in \mathsf{Mat}_{n\times m}(K)$ and $F = \begin{pmatrix} F_1 & 0 \\ F_3 & F_4 \end{pmatrix}$ is given by $F_1 \in \mathsf{Mat}_{n_0\times m_0}(K)$, $F_3 \in \mathsf{Mat}_{n_1\times m_0}(K\times K)$ and $F_4 \in \mathsf{Mat}_{n_1\times m_1}(K)$ subject to the condition $FM = Nf$, where K is considered as a subring of $K\times K$ via the diagonal embedding. Composition of morphisms in $\mathsf{BM}(E)$ is given by the usual matrix product (in each entry of the tuple (F,f) separately).

Recall the following special case of a theorem of Drozd and Greuel (see for instance [7] or [9]).

Theorem. *Let* $\mathsf{VB}^{(0,1)}(E)$ *denote the full subcategory of the category of vector bundles on* E *of objects whose pull-back to the normalisation* $\mathbb{P}^1$ *of* E *only contains direct summands of the form* $\mathcal{O}_{\mathbb{P}^\mathbb{1}}$ *and* $\mathcal{O}_{\mathbb{P}^\mathbb{1}}(1)$*. Then* $\mathsf{VB}^{(0,1)}(E)$ *and* $\mathsf{BM}(E)$ *are equivalent.*

Definition 2.3. *Let* e *and* d *be coprime positive integers. Define a matrix* $J = J(e,d)$ *inductively as follows:*

- $J(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- *If* $J(e,d) = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$ *for* $J_1 \in \mathsf{Mat}_{e\times e}(K)$ *and* $J_4 \in \mathsf{Mat}_{d\times d}(K)$*, then*
$$J(e+d,d) = \begin{pmatrix} J_1 & J_2 & 0 \\ 0 & 0 & I \\ J_3 & J_4 & 0 \end{pmatrix}.$$
- *If* $J(e,d) = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$ *for* $J_1 \in \mathsf{Mat}_{e\times e}(K)$ *and* $J_4 \in \mathsf{Mat}_{d\times d}(K)$*, then*
$$J(e,d+e) = \begin{pmatrix} 0 & 0 & I \\ J_3 & J_4 & 0 \\ J_1 & J_2 & 0 \end{pmatrix}.$$

Remark. *If one follows the proof of Theorem 9.19 in [11], one ends up with a slightly different definition of the normal form* J*, but one can easily see that the two are isomorphic as*

$$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ J_1 & 0 & J_2 \\ J_3 & 0 & J_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \\ J_3 & J_4 & 0 \\ J_1 & J_2 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

In fact, our choice of J *is actually the inverse of the one defined in that article.*

Our choice of J turns out to be well-suited to calculations with certain block matrices which is the content of the following lemma and the reason for our choice of J.

Lemma 2.1. *1. If we partition the rows of $J(e,d)$ as $d{+}e$ and the columns as $e+d$, then*

$$J(e,d) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

2. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be partitioned into blocks such that $A \in \mathsf{Mat}_{e\times e}(K)$ and $D \in \mathsf{Mat}_{d\times d}(K)$. Then

$$J(e,d)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} J(e,d) = \begin{pmatrix} D & C \\ B & A \end{pmatrix}.$$

Proof. 1. By induction on $n = e + d$.
For $J(1,1)$ it is true by definition. If $(0\ I) = (J_1 J_2)$ and $(I\ 0) = (J_3 J_4)$, then

$$J(e+d,d) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}$$

and

$$J(e,d+e) = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

Therefore the claim is true.

2. The claim follows from part one:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

□

Let n and d be coprime integers and let $\mathcal{V}$ be the stable vector bundle on the nodal cubic curve E of rank n and degree d given by the matrix $\big(I, J(e,d)\big) \in \mathsf{BM}(E)$, where $e + d = n$. Associated with this is a sheaf of Lie algebras $\mathcal{A} = \mathsf{Ad}(\mathcal{V})$ constructed as the kernel of the canonical trace map $\mathcal{E}nd(\mathcal{V}) \to \mathcal{O}_E$, some of whose properties are collected in the following lemmas.

Lemma 2.2. *Given any stable vector bundle $\mathcal{W}$ on E, the sheaf of Lie algebras one obtains from the above procedure is isomorphic to one of the ones* *constructed above.*

Proof. We know that the action of $\mathsf{Pic}^0(E)$ on stable vector bundles of a fixed rank n and degree d (via tensor product) is transitive and we also know that stable bundles only exist for n and d coprime (see for example [7], [9] or [11]), hence the result follows, because there is an isomorphism

$$\mathcal{E}nd(\mathcal{V}) \cong \mathcal{E}nd(\mathcal{V} \otimes \mathcal{L})$$

for $\mathcal{V}$ a vector bundle and $\mathcal{L}$ a line bundle, which respects the trace map. $\square$

Lemma 2.3. *The sheaf cohomology of $\mathcal{A}$ vanishes.*

Proof. By definition, $\mathcal{A}$ comes with the following short exact sequence:

$$0 \to \mathcal{A} \to \mathcal{E}nd(\mathcal{V}) \xrightarrow{\mathsf{tr}} \mathcal{O}_E \to 0$$

Since $\mathcal{V}$ is simple and E connected and since the characteristic of K is zero, the trace map induces an isomorphism

$$\mathsf{H}^0\big(E, \mathcal{E}nd(\mathcal{V})\big) = \mathsf{End}(\mathcal{V}) \cong K = \mathsf{H}^0(E, \mathcal{O}_E).$$

Hence the zero-th cohomology group of $\mathcal{A}$ vanishes.
Furthermore we have a non-degenerate, bilinear form $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_E$, which is given by $(a, b) \mapsto \mathsf{tr}(a \cdot b)$ and which identifies $\mathcal{A}$ with its dual sheaf $\mathcal{A}^\vee$. Since E is a Calabi-Yau curve, this allows us to conclude that $h^1(\mathcal{A}) = 0$ by Serre duality.
Finally, all higher sheaf cohomology groups vanish because E is a curve. $\square$

Remark. *One could also argue $h^1(\mathcal{A}) = 0$ via the Riemann-Roch theorem, since the fact that both $\mathcal{O}_E$ and $\mathcal{E}nd(\mathcal{V}) \cong \mathcal{V} \otimes \mathcal{V}^\vee$ are of degree zero implies that*

$$\chi(\mathcal{A}) = 0.$$

Lemma 2.4. *The restriction of $\mathcal{A}$ to U is given by $\mathfrak{sl}_n(K) \otimes_K \mathcal{O}_E(U)$ (or more precisely the quasi-coherent sheaf associated to this module).*

Proof. Since $R = \mathcal{O}_E(U)$ is a PID, the module $\Gamma(U, \mathcal{V})$ is a free module of rank n and since the global trace map descends to the usual trace map of finitely generated free R-modules, the claim follows at once. $\square$

Lemma 2.5. *Let $x \in U = K^*$. With the identifications made above the space of global sections $\mathsf{H}^0\big(E, \mathcal{A}(x)\big)$ corresponds to*

$$\left\{ \begin{pmatrix} A_0 + A_1 t & B \\ C_0 + C_1 t + C_2 t^2 & D_0 + D_1 t \end{pmatrix} \in \mathfrak{g}[t] \middle| \begin{array}{l} A_i \in \mathsf{Mat}_{e\times e}(K), \quad B \in \mathsf{Mat}_{e\times d}(K), \\ C_i \in \mathsf{Mat}_{d\times e}(K), \quad D_i \in \mathsf{Mat}_{d\times d}(K), \\ -JM_0 = xM_\infty J \end{array} \right\}$$

in the (matrix version of the) category of triples, where $\mathfrak{g} = \mathfrak{sl}_n(K)$,

$$M_0 = \begin{pmatrix} A_0 & B \\ C_0 & D_0 \end{pmatrix}$$

and

$$M_\infty = \begin{pmatrix} A_1 & B \\ C_2 & D_1 \end{pmatrix}.$$

Proof. We have $\nu^*(\mathcal{V}) = \mathcal{O}_{\mathbb{P}^1}^e \oplus \mathcal{O}_{\mathbb{P}^1}(1)^d$. Therefore $\nu^*(\mathcal{A}(x))$ is equal to all elements of trace zero in the matrix vector bundle

$$\begin{pmatrix} \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus e\cdot e} & \mathcal{O}_{\mathbb{P}^1}^{\oplus e\cdot d} \\ \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus d\cdot e} & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d\cdot d} \end{pmatrix}.$$

We thus need to understand the transition map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. But since the transition map for $\mathcal{V}$ is given by (id, J) and the transition map for $\mathcal{O}_E(x)$ is given by $(\mathsf{id}, -\frac{1}{x})$ the transition map for $\mathcal{A}(x)$ is given by $(\mathsf{id}, -\frac{1}{x}\mathsf{Ad}(J))$ and therefore the statement of the lemma follows from the usual identification of $\mathsf{H}^0(\mathbb{P}^1, \mathcal{O}(n))$ with polynomials (in $t = \frac{z_0}{z_1}$) of degree less or equal n and the definition of morphisms in the category of Burban-Drozd-Greuel triples. $\square$

Definition 2.4. *The vector space appearing in the above lemma will be denoted by $\mathsf{Sol}(e, d, x)$.*

Remark. *As the notation suggests, $\mathsf{Sol}(e, d, x)$ is independent of the concrete choice of $\mathcal{V}$, because already $\mathcal{A}$ does not depend on this choice (up to isomorphism), see Lemma 2.2.*

Lemma 2.6. *The residue and evaluation maps for $M \in \mathsf{Sol}(e, d, x)$ are given by*

$$\mathsf{res}_x(M) = \frac{M(x)}{x}$$

and

$$\mathsf{ev}_y(M) = \frac{M(y)}{y-x},$$

where we use $\omega = \frac{dt}{t}$ *as a non-vanishing one-form on* E *(*$t = \frac{z_0}{z_1}$ *as before) and* $x, y \in K^* = U$ *are two distinct points.*

Proof. This follows from Lemma 4.5 in [11]. □

Remark. *As was already done in the previous lemma, we will drop the form* ω *from the notation and only write* res_x *instead of* res_x^ω*.*

Putting these results together we arrive at the following theorem:

Theorem 2.1. *Given a stable vector bundle* $\mathcal{V}$ *of rank* n *and degree* d *on the nodal cubic curve* E *there exist unique elements* $e_\alpha^x, f_\alpha^x, g_i^x \in \mathsf{Sol}(e, d, x)$ *(for* α *a positive root of* $\mathfrak{g}$*,* $1 \leq i$ $\mathsf{rk}(\mathfrak{g})$*,* $x \in K^*$*) such that* $\frac{e_\alpha^x(x)}{x} = E_\alpha \in \mathfrak{g} = \mathfrak{sl}_n(K)$ *and similarly for* f_α^x *and* g_i^x*. Furthermore the meromorphic function*

$$\mathsf{r}(x, y) := \sum_\alpha \frac{F_\alpha \otimes e_\alpha^x(y)}{y-x} + \sum_\alpha \frac{E_\alpha \otimes f_\alpha^x(y)}{y-x} + \sum_i \frac{G_i \otimes g_i^x(y)}{y-x}$$

is a unitary solution of the Classical Yang-Baxter Equation.

Proof. As before, we take the global nowhere vanishing 1-form $\omega = \frac{dt}{t}$. By Theorem 1.1 and the assumptions described in that section we have to check that $h^0(\mathcal{A}) = h^1(\mathcal{A}) = 0$ and that $\mathcal{A}|_U$ is isomorphic to $\mathfrak{g} \otimes_K \mathcal{O}_E(U)$ for the term defined by residues and evaluations to be a solution of the GCYBE since the assumptions on $U = \mathbb{G}_m$ are automatically satisfied. The (remaining) assumptions on $\mathcal{A}$ are satisfied by Lemmas 2.4 and 2.3. Therefore the result follows from Lemmas 2.5 and 2.6.
Unitarity of the solution follows from the second part of Theorem 1.2 and the remark following it. □

Remark. *Of course one has a bit more freedom in the choice of bases. It is only important that the basis appearing on the left hand side of the tensor products is dual (for the Killing form) to the one whose inverse images under the residue map appear on the right hand side.*
We will use this more general version in Sections 2.6 and 2.7.

2.3 The Geometric Construction

As in the previous section, let $\mathcal{V}$ be the simple vector bundle on the nodal curve E of rank n and degree d with gluing map J and consider the associated sheaf of Lie algebras $\mathcal{A} = \mathsf{Ad}(\mathcal{V})$ constructed as the kernel of the trace map $\mathcal{E}nd(\mathcal{V}) \to \mathcal{O}_E$. This $\mathcal{A}$ is a vector bundle and has the property $\mathsf{H}^0(E, \mathcal{A}) = \mathsf{H}^1(E, \mathcal{A}) = 0$. Thus it can be used to construct a solution of the CYBE as described in the previous section.
In this section we want to construct a decomposition

$$\mathfrak{g}\big((z)\big) \times \mathfrak{g} = W(n, d) \oplus \mathfrak{g}[z^{-1}]$$

determined by $\mathcal{A}$ satisfying the assumptions of the theorem of Khoroshkin, Pop, Samsonov, Stolin and Tolstoy. For this, let $\nu : \mathbb{P}^1 \to E$ be the normalisation map and consider the sheaf of Lie algebras $\mathcal{B} := \nu^*(\mathcal{A})$. Let $X = \mathbb{P}^1$ and x one of the two points mapping to the singular point of E. Denote the complement $X \setminus \{x\}$ by U. $\mathcal{B}$ is then a direct sum of copies of $\mathcal{O}(-1)$, $\mathcal{O}$ and $\mathcal{O}(1)$ and therefore we have a short exact sequence of the form

$$0 \to \mathsf{H}^0(X, \mathcal{B}) \to \mathsf{H}^0(U, \mathcal{B}) \oplus \widehat{\mathcal{B}_x} \to Q(\mathcal{O}_X) \otimes \widehat{\mathcal{B}_x} \to 0$$

by Theorem 3.1 (note that its proof does not use anything we have shown up to now, hence the argument is not circular). Denote the other point mapping to the singular point of E by $-x$ and let us consider the induced morphism (where all new maps are the canonical ones and otherwise zero)

$$\beta : \mathsf{H}^0(U, \mathcal{B}) \oplus \widehat{\mathcal{B}_x} \to Q(\mathcal{O}_X) \otimes \widehat{\mathcal{B}_x} \times \mathcal{B}\big|_{-x} \times \mathcal{B}\big|_x.$$

Using J we define a new morphism α as the following composition:

$$\mathsf{H}^0(U, \mathcal{B}) \oplus \widehat{\mathcal{B}_x} \xrightarrow{\beta} Q(\mathcal{O}_X) \otimes \widehat{\mathcal{B}_x} \times \mathcal{B}\big|_{-x} \times \mathcal{B}\big|_x \xrightarrow{\begin{pmatrix} \mathsf{id} & 0 & 0 \\ 0 & \mathsf{id} & \mathsf{Ad}(J) \end{pmatrix}} Q(\mathcal{O}_X) \otimes \widehat{\mathcal{B}_x} \times \mathfrak{g}.$$

Theorem 2.2. *The morphism α is an isomorphism of K-vector spaces whose restriction to each summand is a morphism of Lie algebras.*

Proof. The second statement is clearly true.
For showing that α is an isomorphism, we describe the images of $\mathsf{H}^0(U, \mathcal{B})$ and $\widehat{\mathcal{B}_x}$ under α and then show that their sum is direct inside $\Big(Q(\mathcal{O}_X) \otimes \widehat{\mathcal{B}_x}\Big) \times \mathfrak{g}$

and spans the whole space. Note that this is sufficient as the first component of α restricted to either summand is injective, because $\mathcal{A}$ is a vector bundle. Since pull-back commutes with taking endomorphism sheaves of vector bundles, $\mathcal{B}$ is given as the kernel of the trace map $\mathcal{E}nd(\nu^*\mathcal{V}) \to \mathcal{O}_X$ and therefore our chosen trivialisations allow us to identify the image of $\widehat{\mathcal{B}_x}$ in $Q(\mathcal{O}_X)\otimes\widehat{\mathcal{B}_x} = \mathfrak{sl}_n\Big(K\big((t)\big)\Big)$ as $\mathsf{Ad}(T)\Big(\mathfrak{sl}_n(K)\big[[t]\big]\Big)$, where T is the diagonal matrix with e 1s on the diagonal followed by $n-e$ entries of t^{-1}. The morphism with target $\mathcal{B}|_{-x}$ is the evaluation at t^{-1}. Therefore the image of $\widehat{\mathcal{B}_x}$ under α is given by the Lie algebra

$$t^2\mathfrak{sl}_n\Big(K\big[[t]\big]\Big) + t\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} + \left(\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, 0\right) + \left(\begin{pmatrix} A & tB \\ t^{-1}C & D \end{pmatrix}, \begin{pmatrix} D & C \\ B & A \end{pmatrix}\right),$$

where all the matrices in the first entry are of the form $(e, n-e)\times(e, n-e)$ and have trace equal to zero.
By the choice of trivialisation the image of $\mathsf{H}^0(U,\mathcal{B})$ in $\mathfrak{sl}_n\Big(K\big((t)\big)\Big)\times\mathfrak{sl}_n$ is given by $\mathfrak{sl}_n\big(K[t^{-1}]\big)$, with diagonal embedding in degree zero.
It remains to check that these two Lie subalgebras have only $\{0\}$ as intersection and span all of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big)\times\mathfrak{sl}_n$: The only possible intersection happens in degree zero and is (potentially) given by elements of the form $\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} D & 0 \\ 0 & A \end{pmatrix}\right)$. The element $\begin{pmatrix} A - At & 0 \\ 0 & D - Dt \end{pmatrix}$ belongs to $\mathsf{Sol}(e, d, 1)$ if our original element happens to belong to the intersection and also to the kernel of res_1. Since the residue map is injective, the intersection is given by $\{0\}$. The spanning property is clearly true in all degrees except possibly for zero and one. But in degree zero it follows for dimension reasons and is hence true in degree one as well by the shape of the image of $\widehat{\mathcal{B}}_x$ under α. □

It turns out that these decompositions satisfy all properties demanded by KPSST for these to come from a quasi-trigonometric r-matrix. For easier notation, let's call the Lie subalgebra coming from $\widehat{\mathcal{B}_x}$ constructed above $W(n, d)$.

Proposition 2.1. *The vector space $W(n,d)$ is a lagrangian Lie subalgebra of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big)\times\mathfrak{sl}_n$ which contains $t^N\mathfrak{sl}_n\Big(K\big[[t]\big]\Big)$ for some $N > 0$.*

Proof. Of course, $W(n,d)$ is a Lie subalgebra containing $t^2\mathfrak{sl}_n\Big(K\left[[t]\right]\Big)$, hence we only need to show that it is lagrangian and since it has a lagrangian complement (namely $\mathfrak{sl}_n\big(K[t^{-1}]\big)$, we only need to show that the inner product of any two elements of $W(n,d)$ vanishes. But since the form is ad-invariant and $W(n,d)$ differs from the lagrangian subalgebra $\mathfrak{sl}_n\Big(K\left[[t]\right]\Big)$ only by some automorphism of ad-type, the result is clear. □

2.4 Comparison with Manin Triples

In this section, we prove that the r-matrices produced by the two methods agree. We do not calculate them completely, but rather try to compare the two methods of producing the r-matrices as much as possible. Nonetheless, later in this chapter a closed formula is also worked out.

Lemma 2.7. *Let $n, d \geq 1$ be natural numbers such that d and $n-d$ are positive and coprime. The following algorithm terminates: It starts with a pair of natural numbers (i,j) with $i \neq j$ and both between one and n. If both i and j are bigger than d, it subtracts d as many times as possible from both i and j making sure that both stay positive. If both are smaller or equal to d it adds $n-d$ as many times as possible to both making sure that both stay smaller or equal to n. In all other cases, it stops.*

Proof. Let us assume that the algorithm does not terminate. Then, since there are only a finite number of allowed pairs (i,j) with a fixed difference $i-j$ (which is left invariant by the algorithm), there exist such pairs (i,j) which are reached by the algorithm several times. In other words, there exist positive numbers a and b such that $ad = b(n-d)$ and since d and $n-d$ are coprime, this means that $a = x(n-d)$ and $b = xd$ for some positive number $x \in \mathbb{N}$. If we now choose a and b to be as small as possible, this provides a contradiction, since there are less than n pairs (i,j) with that fixed difference, while the algorithm needs $xd + x(n-d) = xn > n-1$ steps. □

In the following $\{H_i\}$ denotes an orthonormal basis of the diagonal matrices $\mathfrak{h}$ with vanishing trace and E_{ij} denotes the matrix with a one in row i and column j and zeroes everywhere else. We denote the union of these two sets by $\{G_i\}$. Note that the dual basis $\{G_i^\vee\}$ is actually equal to $\{G_i\}$ (as sets, but not point-wise). We will start with some general remarks and then proceed to prove more technical statements. At the end we will put these together to show the main theorem.

Lemma 2.8. *Assume that $w \in W(n,d)$ is given by*

$$W_n t^n + \ldots + W_2 t^2 + \begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix} t +$$

$$\left(\begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}, 0 \right) + \left(\begin{pmatrix} A & Bt \\ Ct^{-1} & D \end{pmatrix}, \begin{pmatrix} D & C \\ B & A \end{pmatrix} \right).$$

Then

$$\{gt^{-m}, w\} = \begin{cases} \langle g, W_m \rangle & m \geq 2 \\ \langle g, \begin{pmatrix} A_1 & B \\ C_1 & D_1 \end{pmatrix} \rangle & m = 1 \\ \langle g, \begin{pmatrix} A & 0 \\ C_2 & D \end{pmatrix} \rangle - \langle g, \begin{pmatrix} D & C \\ B & A \end{pmatrix} \rangle & m = 0 \end{cases}$$

for any $g \in \mathfrak{g}$.

Proof. This is true by the definition of the form $\{-,-\}$ and the definition of the map $\mathfrak{g}[t^{-1}] \to \mathcal{D}(\mathfrak{g})$. $\square$

Lemma 2.9. *1. The (unique) element $w_i \in W(n,d)$ with the property $\{w_i, G_j t^{-n}\} = \delta_{ij}\delta_{0n}$ is of the form*

$$\left(\begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix}, \begin{pmatrix} D_i & C_i \\ 0 & A_i \end{pmatrix} \right) + \begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} t^{-1}.$$

The entries satisfy

$$G_i^\vee = \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix} - \begin{pmatrix} D_i & C_i \\ 0 & A_i \end{pmatrix}.$$

2. The element $v_i \in W(n,d)$ with the property $\{v_i, G_j t^{-n}\} = \delta_{ij}\delta_{1n}$ is of the form

$$\begin{pmatrix} U_i' & B_i' \\ V_i' & W_i' \end{pmatrix} t + \left(\begin{pmatrix} A_i' & 0 \\ P_i' & D_i' \end{pmatrix}, \begin{pmatrix} D_i' & 0 \\ B_i' & A_i' \end{pmatrix} \right).$$

The entries satisfy

$$G_i^\vee = \begin{pmatrix} U_i' & B_i' \\ V_i' & W_i' \end{pmatrix}$$

and
$$\begin{pmatrix} A_i' & 0 \\ P_i' & D_i' \end{pmatrix} = \begin{pmatrix} D_i' & 0 \\ B_i' & A_i' \end{pmatrix}.$$

3. *The element $G_i^\vee t^m$ which is in $W(n,d)$ for $m \geq 2$ has the property*
$$\{G_j t^{-n}, G_i t^m\} = \delta_{ij}\delta_{mn}$$
for any j and n.

4. *If $G_i \in \mathfrak{h}$, then $w_i, v_i \in \mathfrak{h} \oplus \mathfrak{h}t$.*

5. *If $G_i = E_\alpha$, then $C_i = 0$ and A_i and D_i are lower triangular.*

6. *If $G_i = F_\alpha$, then $P_i = 0$.*

Proof. 1.-3.: Most of this is a direct consequence of Lemma 2.8, the shape of $W(n,d)$, the fact that the Killing form $\langle -, - \rangle$ is non-degenerate and that $\{G_i\}$ is an orthonormal basis for it.
The only thing that needs an explanation is why there is no C_i' matrix in the description of v_i, which follows from the fact that $\begin{pmatrix} 0 & 0 \\ B_i' & 0 \end{pmatrix}$ is a lower triangular matrix and Lemma 2.7.
4.-6.: This follows from Lemma 2.7. □

Proposition 2.2. *In the notation of the previous lemma define $\Phi_{i,x}$ by*
$$-x\begin{pmatrix} D_i + xD_i' & C_i \\ xB_i' & A_i + xA_i' \end{pmatrix} + t\begin{pmatrix} A_i + xA_i' & 0 \\ P_i - xC_i + xP_i' & D_i + xD_i' \end{pmatrix} + t^2\begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix}.$$

Then:

1. $\Phi_{i,x} \in \mathsf{Sol}(e,d,x)$
2. $\mathsf{res}_x(\Phi_{i,x}) = G_i^\vee$

Proof. 1. Let M be the element of $\mathfrak{g} \oplus \mathfrak{g}t \oplus \mathfrak{g}t^2$ defined in the statement. In the notation of Lemma 2.5, we first calculate M_0 and M_∞.
$$M_0 = -x\begin{pmatrix} D_i + xD_i' & C_i \\ xB_i' & A_i + xA_i' \end{pmatrix}$$

For the calculation of M_∞ one has to keep in mind that there are two different partitions of the matrix being used, but an application of Lemma 2.9 shows that

$$M_\infty = \begin{pmatrix} A_i + xA'_i & xB'_i \\ C_i & D_i + xD'_i \end{pmatrix}.$$

It remains to check that $-xJ^{-1}M_\infty J = M_0$. But

$$J^{-1}M_\infty J = \begin{pmatrix} D_i + xD'_i & C_i \\ xB'_i & A_i + xA'_i \end{pmatrix}$$

and therefore $\Phi_{i,x} = M \in \mathsf{Sol}(e, d, x)$.

2. By Lemma 2.6 we have

$$\mathsf{res}_x(\Phi_{i,x}) = \frac{1}{x}\Bigg(-x\begin{pmatrix} D_i + xD'_i & C_i \\ xB'_i & A_i + xA'_i \end{pmatrix} +$$

$$+x\begin{pmatrix} A_i + xA'_i & 0 \\ P_i - xC_i + xP'_i & D_i + xD'_i \end{pmatrix} + x^2\begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix}\Bigg).$$

By part two of Lemma 2.9

$$\begin{pmatrix} A'_i & 0 \\ P'_i & D'_i \end{pmatrix} = \begin{pmatrix} D'_i & 0 \\ B'_i & A'_i \end{pmatrix}.$$

Therefore the expression above reduces to

$$\mathsf{res}_x(\Phi_{i,x}) = \frac{1}{x}\Bigg(-x\begin{pmatrix} D_i & C_i \\ 0 & A_i \end{pmatrix} + x\begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix}\Bigg)$$

which is equal to $G_i^\vee$ by part one of Lemma 2.9.

□

If i is such that $G_i = F_\alpha$ for some positive root α we shall denote the corresponding matrices $A_i, B_i, \ldots$ by $A_\alpha, B_\alpha, \ldots$.
If i is such that $G_i = E_\alpha$ for some positive root α we shall denote the corresponding matrices $A_i, B_i, \ldots$ by $\overline{A_\alpha}, \overline{B_\alpha}, \ldots$.

Proposition 2.3. *The quasi-trigonometric solution* $\mathsf{r}_{n,d}$ *associated with the Lie algebra* $W(n,d)$ *is given by*

$$\mathsf{r}_{n,d}(x,y) = \frac{x\Omega}{x-y} + \sum_{G_i \in \mathfrak{h}} G_i \otimes \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} + \sum_{\alpha \in \Delta^+} E_\alpha \otimes \begin{pmatrix} \overline{A_\alpha} & 0 \\ \overline{P_\alpha} & D_\alpha \end{pmatrix}$$

$$+ \sum_\alpha F_\alpha \otimes \left(\begin{pmatrix} A_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ C_\alpha & 0 \end{pmatrix} \right) + \sum_\alpha x F_\alpha \otimes \begin{pmatrix} A'_\alpha & 0 \\ P'_\alpha & D'_\alpha \end{pmatrix}.$$

Proof. We follow the prescription given in Remark 2.1 and also use the notation established there.
The dual basis to $G_i t^{-n}$ is calculated in Lemma 2.9 and we immediately conclude that $X_{i,n} = 0$ for $n \geq 2$.
We first calculate $X_{i,0}$ and $X_{i,1}$ for those i such that $G_i \in \mathfrak{h}$:

$$G_i^\vee = G_i = \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} - \begin{pmatrix} D_i & 0 \\ 0 & A_i \end{pmatrix}$$

and π applied to the corresponding element of the dual basis is given by $\frac{1}{2}\begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} D_i & 0 \\ 0 & A_i \end{pmatrix}$. Therefore

$$X_{i,0} = \frac{1}{2}\left(\begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} + \begin{pmatrix} D_i & 0 \\ 0 & A_i \end{pmatrix} \right) \otimes G_i.$$

Since $V_i \in \mathfrak{h}$ we conclude $X_{i,1} = 0$.
Next, let i be such that $G_i = E_\alpha$ for some positive root α. Then $X_{i,1} = 0$ and Lemma 2.7 implies that $C_i = 0$ and that A_i and D_i are lower triangular matrices. Therefore $X_{i,0}$ is equal to

$$E_\alpha \otimes \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix}.$$

Finally, consider those i such that $G_i = F_\alpha$ for some positive root α. Then $P_i = 0$ and A_i and D_i are upper triangular by Lemma 2.7 and therefore $X_{i,0}$ is given by

$$F_\alpha \otimes \left(\begin{pmatrix} D_i & C_i \\ 0 & A_i \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} \right).$$

A similar argument shows

$$X_{i,1} = xF_\alpha \otimes \begin{pmatrix} A_i' & 0 \\ P_i' & D_i' \end{pmatrix}.$$

Using the relations in Lemma 2.9 once more we see that the $\mathsf{r}_{n,d}$ is given by the claimed expression. $\square$

Next, we want to calculate the geometric r-matrix in terms of the findings of Proposition 2.2.

Proposition 2.4. *The solution* r *coming from Theorem 2.1 is given by*

$$\mathsf{r}(x,y) = \frac{x\Omega}{x-y} + \sum_{G_i \in \mathfrak{h}} G_i \otimes \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} + \sum_{\alpha \in \Delta^+} E_\alpha \otimes \begin{pmatrix} \overline{A_\alpha} & \overline{0} \\ P_\alpha & D_\alpha \end{pmatrix}$$

$$+ \sum_\alpha F_\alpha \otimes \left(\begin{pmatrix} A_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ C_\alpha & 0 \end{pmatrix} \right) + \sum_\alpha xF_\alpha \otimes \begin{pmatrix} A_\alpha' & 0 \\ P_\alpha' & D_\alpha' \end{pmatrix}.$$

Proof. If i is such that $G_i \in \mathfrak{h}$, then $v_i = G_i t$ and therefore

$$\mathsf{ev}_y(\Phi_{i,x}) = \frac{1}{y-x}\left(-x \begin{pmatrix} D_i & 0 \\ 0 & A_i \end{pmatrix} + y \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} \right).$$

Inserting $x \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} - x \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix}$ into the bracket, things can be regrouped to show

$$\mathsf{ev}_y(\Phi_{i,x}) = \frac{xG_i}{y-x} + \begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix}$$

by another application of the relations in Lemma 2.9.
If i is such that $G_i = E_\alpha$ then Lemma 2.7 implies

$$\mathsf{ev}_y(\Phi_{i,x}) = \frac{1}{y-x}\left(-x \begin{pmatrix} D_i & 0 \\ 0 & A_i \end{pmatrix} + y \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix} \right).$$

Inserting $x \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix} - x \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix}$ into the bracket, things can be regrouped to show

$$\mathsf{ev}_y(\Phi_{i,x}) = \frac{xF_\alpha}{y-x} + \begin{pmatrix} A_i & 0 \\ P_i & D_i \end{pmatrix}$$

by another application of Lemma 2.9.
If i is such that $G_i = F_\alpha$ then

$$\mathsf{ev}_y(\Phi_{i,x}) = \frac{1}{y-x}\Bigg(- x \begin{pmatrix} D_i + xD_i' & C_i \\ xB_i' & A_i + xA_i' \end{pmatrix} +$$

$$+ y \begin{pmatrix} A_i + xA_i' & 0 \\ -xC_i + xP_i' & D_i + xD_i' \end{pmatrix} + y^2 \begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} \Bigg).$$

Using Lemma 2.9 several times and adding zero summands as before, we note the following equalities:

$$\begin{aligned} y\begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} - x\begin{pmatrix} D_i & C_i \\ 0 & A_i \end{pmatrix} &= xE_\alpha + (y-x)\begin{pmatrix} A_i & 0 \\ 0 & D_i \end{pmatrix} \\ y^2\begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} - yx\begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} &= y(y-x)\begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} \\ yx\begin{pmatrix} A_i' & 0 \\ P_i' & D_i' \end{pmatrix} - x^2\begin{pmatrix} D_i' & 0 \\ B_i' & A_i' \end{pmatrix} &= x(y-x)\begin{pmatrix} A_i' & 0 \\ P_i' & D_i' \end{pmatrix} \end{aligned}$$

This finishes the calculation of $\mathsf{ev}_y(\Phi_{i,x})$ and the proof. □

Collecting everything we have done so far, we end up with the following theorem.

Theorem 2.3. *The solution $\mathsf{r}_{n,d}$ of the CYBE associated to the stable vector bundle $\mathcal{V}$ of rank n and degree d is quasi-trigonometric and the Lie subalgebra of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n$ associated to this solution is $W(n,d)$.*

Proof. Both statements can be shown together by comparing the results of Proposition 2.3 and Proposition 2.4 and noting that they agree. □

2.5 Cremmer-Gervais solutions

One could hope to apply the method of Section 2.2 to construct the Manin triples corresponding to all simple vector bundles on cycles of projective lines.

Unfortunately this is not the case, but nonetheless we can describe the vector bundles giving the so-called Cremmer-Gervais solutions ([15]).
Therefore we consider a simple vector bundle $\mathcal{V}^c$ on a cycle of projective lines E, which is given by the trivial bundle of rank n on all components but one and by a stable bundle of rank n and degree d on the last, with just one non-trivial gluing map, which is given by the matrix J we have used before. In case we take the non-trivial component as supplying the spectral parameters, we end up with the same set-up as in Section 2.4. But taking another component, we arrive at different results. Let us denote the resulting sheaf of Lie algebras by $\mathcal{A}^c$. Note that we may and shall always assume that E is a cycle of two projective lines.
We begin by a concrete description of the global sections, just as before.

Lemma 2.10. *Let $x \in U = K^*$. With the identifications made in Section 2.2 the vector space $\mathsf{H}^0\big(E, \mathcal{A}^c(x)\big)$ corresponds to*

$$\left\{ x\begin{pmatrix} A & 0 \\ C_0 & D \end{pmatrix} - t\begin{pmatrix} D & C_1 \\ 0 & A \end{pmatrix} \in \mathfrak{g}[t] \,\middle|\, \begin{matrix} A \in \mathsf{Mat}_{e\times e}(K), C_i \in \mathsf{Mat}_{d\times e}(K), \\ D \in \mathsf{Mat}_{d\times d}(K) \end{matrix} \right\}$$

in the (matrix version of the) category of triples.

Proof. On one copy of $\mathbb{P}^1$, $\mathcal{A}^c$ is given by $\mathfrak{sl}_n(\mathcal{O}_{\mathbb{P}^1})$ and on the other copy, it is given by the pull-back of $\mathcal{A}$ to $\mathbb{P}^1$. Therefore if we take x in the first component $\mathcal{A}^c(x)$, is given by $\mathfrak{sl}_n(\mathcal{O}_{\mathbb{P}^1}(1))$ and the pull-back of $\mathcal{A}$ with only one non-trivial gluing map, which is given by $-\frac{1}{x}\mathsf{Ad}(J)$.
Therefore $\mathsf{H}^0\big(E, \mathcal{A}^c(x)\big)$ is identified with the subspace of

$$\left\{\Phi_0 z_0 + \Phi_1 z_1 \middle| \Phi_0, \Phi_\infty \in \mathfrak{sl}_n\right\} \times \left\{ \begin{pmatrix} A & 0 \\ C_0 z_0 + C_1 Z_1 & B \end{pmatrix} \right\}$$

given by those tuples such that

$$\begin{pmatrix} A & 0 \\ C_1 & B \end{pmatrix} = -\frac{1}{x} J\Phi_1 J^{-1}$$

and

$$\Phi_0 = \begin{pmatrix} A & 0 \\ C_2 & B \end{pmatrix}.$$

Here $A \in \mathsf{Mat}_{e\times e}(K)$ $B \in \mathsf{Mat}_{d\times d}(K)$ and $\mathsf{tr}(A) + \mathsf{tr}(B) = 0$.

The result now follows from Theorem 2.2 and Lemma 2.1. $\square$

Definition 2.5. *The vector space appearing in the above lemma will be denoted by* $\mathsf{Sol}^c(e, d, x)$.

Remark. *As the notation suggests,* $\mathsf{Sol}^c(e, d, x)$ *is independent of the concrete choice of* $\mathcal{V}$*, because already* $\mathcal{A}^c$ *does not depend on this choice (up to isomorphism).*

The same reasoning as in Section 2.2 implies the following two results:

Lemma 2.11. *The residue and evaluation maps in this context are given by*

$$\mathsf{res}_x(M) = \frac{M(x)}{x}$$

and

$$\mathsf{ev}_x(M) = \frac{M(y)}{y - x},$$

where we use $\omega = \frac{dt}{t}$ *as a non-vanishing 1-form on* E *(where* $t = \frac{z_0}{z_1}$ *as before).*

Theorem 2.4. *Given a simple vector bundle* $\mathcal{V}^c$ *as above there exist unique elements* $e_\alpha^x, f_\alpha^x, g_i^x \in \mathsf{Sol}^c(e, d, x)$ *(for* α *a positive root of* $\mathfrak{g}$*,* $1 \leq i$ $\mathsf{rk}(\mathfrak{g})$*,* $x \in K^*$*) such that* $\frac{e_\alpha^x(x)}{x} = E_\alpha \in \mathfrak{g}$ *and similarly for* f_α^x *and* g_i^x*. Furthermore the meromorphic function*

$$\mathsf{r}(x, y) := \sum_\alpha \frac{F_\alpha \otimes e_\alpha^x(y)}{y - x} + \sum_\alpha \frac{E_\alpha \otimes f_\alpha^x(y)}{y - x} + \sum_i \frac{G_i \otimes g_i^x(y)}{y - x}$$

is a unitary solution of the Classical Yang-Baxter Equation.

On the other hand, define the following Lie algebra.

$$W^c(n, d) := t\mathfrak{sl}_n\Big(K\big[[t]\big]\Big) + \left(\begin{pmatrix} D & C \\ 0 & A \end{pmatrix}, \begin{pmatrix} A & 0 \\ B & D \end{pmatrix}\right),$$

where $A \in \mathsf{Mat}_{e\times e}(K)$, $D \in \mathsf{Mat}_{d\times d}(K)$, $B \in \mathsf{Mat}_{d\times e}(K)$, $C \in \mathsf{Mat}_{d\times e}(K)$ and $\mathsf{tr}(A) + \mathsf{tr}(D) = 0$.

Proposition 2.5. *$W^c(n,d)$ is a lagrangian subalgebra of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n$ containing $t\mathfrak{sl}_n\Big(K\big[[t]\big]\Big)$ and is such that*

$$\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n = W^c(n,d) \oplus \mathfrak{sl}_n\big(K[t^{-1}]\big).$$

In other words, $W^c(n,d)$ corresponds to a quasi-constant quasi-trigonometric solution of the CYBE.

Proof. Let us first show that $\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n = W^c(n,d) \oplus \mathfrak{sl}_n\big(K[t^{-1}]\big)$. For this, we need only to consider degree zero. An element of the intersection of the two Lie algebras can be interpreted as an element of $\mathsf{Sol}^c(e,d,x)$ (see also the next lemma) via $(M,N) \mapsto -xM + Nt$. This map is obviously injective and so is res_x. But any element of the diagonal is mapped to zero under the composition and hence it turns out that the intersection has to be zero. For dimension reasons, this proves the decomposition.
Next we check that $W^c(n,d)$ is a lagrangian subalgebra of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n$ and since it has a Lagrangian complement, we need only check that the pairing of any two elements of $W^c(n,d)$ is zero. Visibly this only has to be checked for two elements of degree zero:

$$\begin{aligned}
\mathsf{tr}\left(\begin{pmatrix} D & C \\ 0 & A \end{pmatrix} - \begin{pmatrix} A & 0 \\ B & D \end{pmatrix}\right) &= \mathsf{tr}\left(\begin{pmatrix} D & C \\ 0 & A \end{pmatrix} - J\begin{pmatrix} D & C \\ 0 & A \end{pmatrix} J^{-1}\right) \\
&= \mathsf{tr}\left(\begin{pmatrix} D & C \\ 0 & A \end{pmatrix} - \begin{pmatrix} D & C \\ 0 & A \end{pmatrix}\right) \\
&= 0
\end{aligned}$$

□

The remaining claims follow from $t\mathfrak{sl}_n(K)\big[[t]\big] \subseteq W^c(n,d)$.

Remark. *If one applies the construction of section three, one arrives at the Lie algebra*

$$t\mathfrak{sl}_n\Big(K\big[[t]\big]\Big) + \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} D & C \\ B & A \end{pmatrix}\right),$$

which has a non-trivial intersection with $\mathfrak{sl}_n(K)[t^{-1}]$. The Lie subalgebra $W^c(n,d)$ can be thought of as a "deformation" of this other Lie algebra which has trivial intersection with $\mathfrak{sl}_n\big(K[t^{-1}]\big)$.

Again, it turns out that these two methods are related, which we will prove after some preliminary lemmas.

Lemma 2.12. *The map*

$$\Phi_x : \mathsf{Sol}^c(e,d,x) \to W^c(n,d)$$

given by

$$A + Bt \mapsto \big(B, -\frac{A}{x}\big)$$

is an isomorphism of K-vector spaces onto the degree zero part of $W^c(n,d)$. It has the property that

$$\Phi_x \circ \mathsf{res}_x^{-1}\big(\{F_\alpha, E_\alpha, G_i\}_{\alpha,i}\big)$$

is the dual basis to

$$\{E_\alpha, F_\alpha, G_i\}_{\alpha,i},$$

where $\{G_i\}$ denotes an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form and the E_α and F_α have their usual meaning.

Proof. By definition of the two vector spaces $\mathsf{Sol}(e,d,x)$ and $W^c(n,d)$ the image of Φ_x is contained in the degree zero part of $W^c(n,d)$. Furthermore, Φ_x is clearly injective and hence an isomorphism of K-vector spaces since its domain and target have the same (finite) K-dimension.
For the claim about the dual basis let $a, b \in \mathfrak{g}$ and assume $\mathsf{res}_x^{-1}(b) = A + Bt$ (i.e. $\frac{A}{x} + B = b$). We calculate:

$$\begin{aligned}\big\{(a,a), \Phi_x(\mathsf{res}_x^{-1}(b))\big\} &= \big\{(a,a), (B, -\frac{A}{x})\big\} \\ &= \big\langle a, B\big\rangle - \big\langle a, -\frac{A}{x}\big\rangle \\ &= \langle a, b\rangle\end{aligned}$$

Therefore the claim follows. □

Lemma 2.13. *If $\mathsf{res}_x^{-1}(E_\alpha) = xA + Bt$, then both A and B are strictly upper triangular. Similarly, if $\mathsf{res}_x^{-1}(F_\alpha) = xC + Dt$, then both C and D are strictly lower triangular.*

Proof. One can apply the algorithm of Lemma 2.7 to see that the obvious idea for finding the inverse image works and has the desired properties. □

Remark. *Lemma 2.16 is a more precise version of the previous lemma.*

Theorem 2.5. *The solution $\mathsf{r}^c_{n,d}$ of the CYBE associated to the stable vector bundle $\mathcal{V}^c$ of rank n and degree d is quasi-trigonometric and the Lie subalgebra of $\mathfrak{sl}_n\Big(K\big((t)\big)\Big) \times \mathfrak{sl}_n$ associated to this solution is $W^c(n,d)$.*

Proof. We shall prove both statements at once by showing that the solutions associated to $\mathcal{V}^c$ and $W^c(n,d)$ agree. This is done case-by-case:

1. If
$$\mathsf{res}_x^{-1}(E_\alpha) = xA + Bt,$$
then the corresponding term in the geometric solution is
$$\frac{x}{y-x} F_\alpha \otimes (A+B) + F_\alpha \otimes B = \frac{x}{y-x} F_\alpha \otimes E_\alpha + F_\alpha \otimes B.$$
By Lemma 2.12 we have to compare the last summand with
$$\pi \otimes \pi\big((F_\alpha, F_\alpha) \otimes (B, -A)\big) + F_\alpha \otimes E_\alpha.$$
The first term of the sum is equal to $F_\alpha \otimes -A$ by Lemma 2.13 and the definition of π. Therefore the two r-matrices agree on all terms of the form $F_\alpha \otimes -$, because $B = E_\alpha - A$.

2. If
$$\mathsf{res}_x^{-1}(F_\alpha) = xC + Dt$$
then the corresponding term in the geometric r-matrix is given by
$$\frac{x}{y-x} E_\alpha \otimes (C+D) + E_\alpha \otimes D = \frac{x}{y-x} E_\alpha \otimes F_\alpha + E_\alpha \otimes D.$$
The corresponding terms of the algebraic r-matrix are given by
$$\frac{x}{y-x} E_\alpha \otimes F_\alpha + \pi \otimes \pi\big((E_\alpha, E_\alpha), (D, -C)\big).$$
That the two terms are equal follows from the definition of π and Lemma 2.13.

3. Finally, we consider the terms of the form $G_i \otimes -$. If
$$\mathsf{res}_x^{-1}(G_i) = xE + Ft,$$

then the contribution to the geometric r-matrix is

$$\frac{x}{y-x} G_i \otimes G_i + G_i \otimes F.$$

Since $(F, -E)$ is dual to (G_i, G_i) (slight abuse of terminology) by Lemma 2.12, the contribution to the algebraic r-matrix is

$$\frac{x}{y-x} G_i \otimes G_i + G_i \otimes \frac{G_i}{2} + G_i \otimes \frac{-F+E}{2}.$$

Since $E+F = G_i$, we have $\frac{1}{2}(G_i + F - E) = F$ giving the desired result.

$\square$

Remark. *Actually, the solution $\mathsf{r}^c_{n,d}$ is quasi-constant, i.e. it is of the form*

$$\frac{x\Omega}{y-x} + c$$

for some constant $c \in \mathfrak{g} \otimes \mathfrak{g}$, as can be seen easily from the shape of $W^c(n,d)$.

2.6 A Closed Formula for $\mathsf{r}_{n,d}$

In this section, we develop the idea of Lemma 2.7 further to establish a closed formula for the quasi-trigonometric solutions $\mathsf{r}_{n,d}$ we have constructed in Section 2.4.

Let us start with the case of a stable vector bundle of rank n and degree d (with $1 \le d < n$) on the nodal cubic curve and the associated sheaf of Lie algebras $\mathcal{A}$. Recall that we write e for the positive integer $n-d$.

Definition 2.6. *Let Δ denote the set of roots of $\mathfrak{g} = \mathfrak{sl}_n$ thought of as tuples (i,j) with $i \neq j$ natural numbers between 1 and n. Define a function $\tau : \Delta \cup \{0\} \to \Delta \cup \{0\}$ by*

$$\tau\big((i,j)\big) = \begin{cases} (i+e, j+e) & i,j \le d \\ (i-d, j-d) & i,j > d \\ (i+e, j-d) & i \le d, j > d \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\tau(0) = 0.$$

Define another map $\psi : \Delta \cup \{0\} \to \Delta \cup \{0\}$ *by*

$$\psi\big((i,j)\big) = \begin{cases} (i+d, j+d) & i-j > 0, i, j \leq e \\ (i-e, j-e) & i-j > 0, i, j > e \\ (i+d, j-e) & i \leq e, j > e \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\psi(0) = 0.$$

Remarks. *1. Note that* τ *is a nilpotent map in the sense that there is some natural number d such that* τ^d *is the constant zero function. This property is a consequence of Lemma 2.7.*

2. τ *and* ψ *mimic the relations on the matrices* $M \in \mathsf{Sol}(e, d, x)$*.*

We can use these maps to give an explicit description of res_x^{-1} of upper and lower triangular matrices. For definiteness and to make computations easier we fix the following basis of the diagonal matrices $\mathfrak{h}$:

Definition 2.7. *Let* $1 \leq i \leq n-1$ *and let* H_i *denote the diagonal matrix which has exactly two non-zero entries, namely a* 1 *at position* (i,i) *and a* -1 *at position* $(i+e, i+e)$ *if* $i \leq d$ *or at position* $(i-d, i-d)$ *if* $i > d$*.*

Lemma 2.14. *1. Let* $\sigma : \{1, \ldots, n\} \to \{0, \ldots, n\}$ *be the function*

$$i \mapsto \begin{cases} i+e & i \leq d \\ i-d & n > i > d \\ 0 & n. \end{cases}$$

Denote the smallest natural number k *such that*

$$\sigma^k(i) = n$$

by $\phi(i)$*. Then the diagonal matrix* G_j *which has entry*

$$\frac{\phi(j) - n}{n}$$

at all entries $\left(\sigma^k(j), \sigma^k(j)\right)$ *and*

$$\frac{\phi(j)}{n}$$

at all other diagonal entries has trace zero.

2. *The set* $\{G_j\}_{1\leq j\leq n-1}$ *is the dual basis to* $\{H_i\}$.

Proof. 1. First of all, note that there is no positive integer k such that $\sigma^k(j) = j$, since for any integer j' $\sigma^k(j') = j'$ implies that there exists $l, m \in \mathbb{N}$ such that $j' = j' + le - md$ and therefore it takes at least $n = e + d$ iterations of σ before $\sigma^k(j')$ can be equal to j'. But therefore $\left\{j, \sigma(j), \sigma^2(j), \ldots, \sigma^{n-1}(j)\right\} = \{1, \ldots, n\}$ if it were true that $\sigma^n(j) = j$, which produces a contradiction, since $\sigma(n) = 0$.
Now we just add the entries of the diagonal:

$$\sum_{i=1}^{\phi(j)} \frac{\phi(j) - n}{n} + \sum_{\phi(j)+1}^{n} \frac{\phi(j)}{n} = \frac{n\phi(j)}{n} - \frac{\phi(j)n}{n} = 0$$

2. By definition of H_i, σ and G_i it is true that H_i and G_i pair to one. The case for $j \neq i$ follows from the observation that if $i < n$ has the property $\sigma(i) = \sigma^k(j)$ for some positive natural number k, then $i = \sigma^{k-1}(j)$, which is true by a case by case analysis and because $e + d = n > i, \sigma^{k-1}(j)$.

□

We need one more combinatorial function before we can describe the concrete formulae:

Definition 2.8. 1. *For* $\alpha \in \Delta \cup \{0\}$ *define* $|\alpha|$ *to be* 0 *if* $\alpha = 0$ *and* $i - j$ *if* $\alpha = (i, j)$.

2. *Consider the function* $\epsilon : \Delta \to \mathbb{N}$ *given by the smallest natural number* k *such that* $|\tau^{k+1}(\alpha)| < 0$ *if it exists and* $n + 1$ *otherwise.*

Lemma 2.15. *If* $\alpha \in \Delta$ *is a negative root, use the convention that* E_α *stands for* $F_{-\alpha}$. *Furthermore, we set* $E_0 = 0 = F_0$. *Then the following statements are true:*

1. $\mathsf{res}_x^{-1}(E_\alpha) = xE_\alpha + \sum_{k=1}^{\epsilon(\alpha)}\big(-tE_{\tau^k(\alpha)} + xE_{\tau^k(\alpha)}\big) - t^2E_{\tau^{\epsilon(\alpha)+1}(\alpha)}$
   $+ xtE_{\tau^{\epsilon(\alpha)+1}(\alpha)} - x^2E_{\psi(\alpha)} + x\sum_{k\geq 1}\big(-xE_{\psi^{k+1}(\alpha)} + tE_{\psi^k(\alpha)}\big)$

2. $\mathsf{res}_x^{-1}(F_\alpha) = F_\alpha t + \sum_{k\geq 1}\big(-xF_{\psi^k(\alpha)} + F_{\psi^k(\alpha)}t\big)$

3. *Let* $K_i = \mathsf{diag}\big(-\frac{1}{n},\ldots,-\frac{1}{n},\frac{n-1}{n},-\frac{1}{n},\ldots,-\frac{1}{n}\big)$*, where the entry* $\frac{n-1}{n}$ *is at the* $i-$*th spot. Then*

$$\mathsf{res}_x^{-1}(H_i) = xK_i - JK_iJ^{-1}t.$$

Proof. 1. Denote the element on the right hand side of the equal sign by M. Then first of all, note that the trace of M is zero, i.e. $M \in \mathfrak{g}\oplus\mathfrak{g}t\oplus\mathfrak{g}t^2$. Furthermore, note that no positive power of τ maps α to a root (i,j) with $i \leq e$ and $j > e$ and that no power greater or equal to two of ψ maps α into a root (i,j) with $i > d$ and $j \leq d$. Therefore M is actually an element of $\mathsf{Sol}(e,d,x)$ and since it is defined precisely so that $\mathsf{res}_x(M) = E_\alpha$, we are done.

2. Essentially the same arguments (but even easier) as in the previous part work.

3. Note that by definition the right hand side is an element of $\mathsf{Sol}(e,d,x)$. Furthermore Lemmas 2.1 and 2.6 imply that the residue map applied to the right hand side is equal to H_i.

□

As a consequence of the previous results, we can give an "explicit" description of the quasi-trigonometric solution associated to $\mathcal{A}$:

Theorem 2.6. *Let* $\mathcal{A}$ *be the sheaf of Lie algebras associated to coprime natural numbers* $1 \leq d \leq n$ *and let* $e = n-d$*. Consider the functions* τ, ψ *and* ϵ *associated to* e *and* d *as given in Definitions 2.6 and 2.8. Then the quasi-trigonometric solution* $\mathsf{r}_{n,d}$ *associated with* $\mathcal{A}$ *is given by*

$$\Bigg(\sum_{\alpha\in\Delta^+} F_\alpha \otimes \Bigg(xE_\alpha + \sum_{k=1}^{\epsilon(\alpha)}\big(-yE_{\tau^k(\alpha)} + xE_{\tau^k(\alpha)}\big)$$

$$-y^2E_{\tau^{\epsilon(\alpha)+1}(\alpha)} + xyE_{\tau^{\epsilon(\alpha)+1}(\alpha)} - x^2E_{\psi(\alpha)} + x\sum_{k\geq 1}\big(-xE_{\psi^{k+1}(\alpha)} + yE_{\psi^k(\alpha)}\big)\Bigg)$$

$$+\sum_{\alpha\in\Delta^+} E_\alpha \otimes \left(F_\alpha y + \sum_{k\geq 1}\big(-xF_{\psi^k(\alpha)} + F_{\psi^k(\alpha)}y\big)\right)$$

$$+\sum_{i=1}^{n-1} G_i \otimes \left(xK_i - JK_iJ^{-1}y\right)\Bigg) \cdot \frac{1}{y-x}.$$

Proof. This is a direct consequence of Theorem 2.1 and Lemmas 2.14 and 2.15. □

Example 2.1. *We shall calculate the solutions corresponding to $n \in \{2,3\}$ and all possible d with the help of the previous theorem. Note that it seems unlikely that one may find these solutions without theory.*

- *If $n = 2$ and $d = 1$, then one easily checks $\tau\big((1,2)\big) = (2,1) = \psi\big((1,2)\big)$ and $\tau\big((2,1)\big) = 0 = \psi\big((2,1)\big)$. Furthermore,*

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2G_1 = 2K_1$$

and therefore the associated solution $\mathsf{r}_{2,1}(x,y)$ is given by

$$\frac{x\Omega}{y-x} + E \otimes F + (x-y)F \otimes F + \frac{1}{4}H \otimes H.$$

- *If $n = 3$ and $d = 1$, then the following table encodes the actions of τ and ψ:*

	(1,2)	(1,3)	(2,1)	(2,3)	(3,1)	(3,2)
τ	(3,1)	(3,2)	0	(1,2)	0	(2,1)
ψ	0	(2,1)	(3,2)	(3,1)	0	0

Furthermore:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad H_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \quad G_2 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$K_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad K_2 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

Therefore the associated solution $\mathsf{r}_{3,1}(x,y)$ is given by $\frac{1}{y-x}$ times the following expression

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ -y^2+xy & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & x \\ -x^2+xy & 0 & 0 \\ 0 & -y^2+xy-x^2+xy & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x-y & 0 \\ 0 & 0 & x \\ xy-x^2-y^2+yx & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & y-x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{2}{3}x+\frac{1}{3}y & 0 & 0 \\ 0 & -\frac{1}{3}x+\frac{1}{3}y & 0 \\ 0 & 0 & -\frac{1}{3}x-\frac{2}{3}y \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{3}x-\frac{2}{3}y & 0 & 0 \\ 0 & \frac{2}{3}x+\frac{1}{3}y & 0 \\ 0 & 0 & -\frac{1}{3}x+\frac{1}{3}y \end{pmatrix}.$$

- *If $n = 3$ and $d = 2$, then the following table encodes the actions of τ and ψ:*

	(1,2)	*(1,3)*	*(2,1)*	*(2,3)*	*(3,1)*	*(3,2)*
τ	*(2,3)*	*(2,1)*	*(3,2)*	*(3,1)*	*0*	*0*
ψ	*(3,1)*	*(3,2)*	*0*	*0*	*0*	*(2,1)*

Furthermore:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad G_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$K_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad K_2 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

Therefore the solution $\mathsf{r}_{3,2}(x,y)$ *is given by* $\frac{1}{y-x}$ *times the following expression*

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x-y \\ 2xy - x^2 - y^2 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & x \\ -y^2 + 2xy - x^2 & 0 & 0 \\ 0 & -x^2 + xy & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ xy - y^2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ y-x & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \\
&+ \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{2}{3}x + \frac{1}{3}y & 0 & 0 \\ 0 & -\frac{1}{3}x - \frac{2}{3}y & 0 \\ 0 & 0 & -\frac{1}{3}x + \frac{1}{3}y \end{pmatrix} \\
&+ \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{3}x + \frac{1}{3}y & 0 & 0 \\ 0 & \frac{2}{3}x + \frac{1}{3}y & 0 \\ 0 & 0 & -\frac{1}{3}x - \frac{2}{3}y \end{pmatrix}.
\end{aligned}$$

2.7 A Closed Formula for $\mathsf{r}^c_{n,d}$

In this section we will establish a "concrete" description of the quasi-constant Cremmer-Gervais solution $\mathsf{r}^c_{n,d}$ we constructed in Section 2.5. We will again need some combinatorially defined functions as in the previous section, but the overall answer will be easier as we are talking about quasi-constant solutions.
Note first of all, that we may reuse the results of the previous section for the diagonal matrices of trace zero. For the others, we need a different sort of τ and ψ which we define as follows (as before, we fix positive, coprime integers e, d with $1 \le d < n = e + d$.

Definition 2.9. *Let Δ denote the set of roots of $\mathfrak{g} = \mathfrak{sl}_n$ thought of as tuples (i, j) with $i \neq j$ natural numbers between 1 and n. Define a function*

$$\tau : \Delta \cup \{0\} \to \Delta \cup \{0\}$$

by

$$\tau\big((i,j)\big) = \begin{cases} (i+e, j+e) & i, j \le d \\ (i-d, j-d) & i, j > d \\ 0 & \textit{otherwise.} \end{cases}$$

Define a function

$$\psi : \Delta \cup \{0\} \to \Delta \cup \{0\}$$

by

$$\psi\big((i,j)\big) = \begin{cases} (i+d, j+d) & i, j \le e \\ (i-e, j-e) & i, j > e \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2.16. *Set $E_0 = F_0 = 0$. Then the following statements are true:*

1. $\mathsf{res}_x^{-1}(E_\alpha) = tE_\alpha + \sum_{k\ge 1}\big(- xE_{\tau^k(\alpha)} + tE_{\tau^k(\alpha)}\big)$
2. $\mathsf{res}_x^{-1}(F_\alpha) = xF_\alpha + \sum_{k\ge 1}\big(- tF_{\psi^k(\alpha)} + xF_{\psi^k(\alpha)}\big)$

Proof. The functions τ and ψ were defined precisely such that this result is true. □

Theorem 2.7. *Let $\mathcal{A}^c$ be the sheaf of Lie algebras associated to coprime natural numbers $1 \le d \le n$ and let $e = n - d$. Consider the functions τ*

and ψ associated to e and d given in Definition 2.9. Then the quasi-constant quasi-trigonometric solution $\mathsf{r}^c_{n,d}$ associated with $\mathcal{A}^c$ is given by

$$\Bigg(\sum_{\alpha\in\Delta^+} F_\alpha \otimes yE_\alpha + \sum_{k\geq 1} \big(-xE_{\tau^k(\alpha)} + yE_{\tau^k(\alpha)}\big)$$

$$+ \sum_{\alpha\in\Delta^+} E_\alpha \otimes xF_\alpha + \sum_{k\geq 1} \big(-yF_{\psi^k(\alpha)} + xF_{\psi^k(\alpha)}\big)$$

$$+ \sum_{i=1}^{n-1} G_i \otimes \big(xK_i - JK_iJ^{-1}y\big)\Bigg) \cdot \frac{1}{y-x}.$$

Proof. This follows from Lemmas 2.15 and 2.16 and Theorem 2.4. □

Remark. *At first sight, the above formula might not look like it describes a quasi-constant solution, but note that many of the linear factors cancel with the denominator as they appear in pairs $X \cdot A$ and $-y \cdot A$. The others can be rearranged with the help of $\frac{y}{y-x} = 1 + \frac{x}{y-x}$.*

Example 2.2. *We shall calculate the solution $\mathsf{r}^c_{2,1}$ via the previous theorem. As H_1, G_1 and K_1 are already known from calculations in the previous section and both τ and ϵ are the zero map, it is given by*

$$\frac{\begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}0&y\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\x&0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{pmatrix} \otimes \begin{pmatrix}\frac{x+y}{2}&0\\0&\frac{-x-y}{2}\end{pmatrix}}{y-x}.$$

Using the cancellation rules described in the previous remark, this expression is equal to

$$\frac{x\Omega}{y-x} + \begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{pmatrix} \otimes \begin{pmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{pmatrix},$$

which is clearly quasi-constant.

3 The Geometry of Rational Solutions of the Classical Yang-Baxter Equation

3.1 Mulase's Krichever Correspondence

In this subsection we will describe the main results of the Krichever correspondence à la Mulase ([31]) in the formulation we will need and use later on. Older accounts of the theory are due to Krichever and Mumford and can be found in [26] and [32].
Fix a field K and a finite-dimensional K-vector space V. Consider the $K\big((T)\big)$-module $V\big((T)\big)$. Assume $W \subseteq V\big((T)\big)$ is a K-subvectorspace such that $\mathsf{dim}_K W \bigcap V\big[[T]\big]$ and $\mathsf{dim}_K V\big((T)\big)/\Big(W + V\big[[T]\big]\Big)$ are finite and such that the ring $A = \Big\{a \in K\big((T)\big)\Big| a \cdot W \subseteq W\Big\}$ is strictly bigger than K.

Lemma. *A is noetherian and of Krull dimension one.*

We define a filtration on A via

$$A^{(n)} = T^{-n} K\big[[T]\big] \bigcap A$$

and consider the associated graded ring $\mathsf{gr}(A) = \bigoplus_{n \geq 0} A_n$.

Lemma. *The scheme $C = \mathsf{Proj}(\mathsf{gr}(A))$ is integral and one-dimensional. In a natural way $\mathsf{spec}(A)$ can be viewed as an affine open subset of C whose complement is given by a single smooth K-rational point.*

In the same way (or by using Grothendieck's fpqc-descent [21], [2] or [46]), one can define a torsionfree sheaf $\mathcal{F}$ on C using W:

Theorem (Mulase)**.** *There is a torsionfree sheaf $\mathcal{F}$ on C of rank $\mathsf{dim}_K V$ such that*

$$h^0(C, \mathcal{F}) = \mathsf{dim}_K\, W \bigcap V\big[[T]\big]$$
$$h^1(C, \mathcal{F}) = \mathsf{dim}_K\, V\big((T)\big)/\Big(W + V\big[[T]\big]\Big).$$

Remarks. *1. As usual, $h^i(C, \mathcal{F})$ denotes $\mathsf{dim}_K\, \mathsf{H}^i(C, \mathcal{F})$.*

2. *Essentially, this defines a bijection between W with a fixed A and torsionfree sheaves on the associated curve C: After fixing a formal parameter T of C at p and identifying the completion $\widehat{\mathcal{F}_p}$ with $V\big[[T]\big]$, W is given by the image of the canonical map $\mathsf{H}^p\big(C \setminus \{p\}, \mathcal{F}\big) \to Q\big(\widehat{\mathcal{F}_p}\big)$.*

3. *In the next section, we will see a functorial description of the description of the h^i which will not only work for smooth points, but more generally for Gorenstein points.*

4. *Mulase's construction is compatible with tensor products. In particular, if $V = \mathfrak{g}$ is a Lie algebra over K and $W \subseteq \mathfrak{g}((T))$ is a Lie subalgebra, $\mathcal{F}$ will be a sheaf of Lie algebras.*

3.2 Exact Krichever Sequence - Algebraic Preliminaries

In this section we state and prove some easy results on formal fibres of one-dimensional noetherian local rings and torsionfree modules over them. These will be used in the next section to construct a certain exact sequence related to the Krichever correspondence.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a commutative, one-dimensional, noetherian, local ring without zerodivisors. Let $S = R\backslash\{0\}$ and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then the localisation $S^{-1}\widehat{R}$ is equal to the total ring of quotients $Q(\widehat{R})$ of $\widehat{R}$.*

Proof. Since the inclusion $R \to \widehat{R}$ is flat, any element in S acts as a non-zerodivisor on $\widehat{R}$. Hence there is an inclusion $\widehat{R} \subseteq S^{-1}\widehat{R} \subseteq Q(\widehat{R})$. On the other hand, the maximal ideal $\mathfrak{m}_{\widehat{R}}$ is equal to $\widehat{R}\cdot\mathfrak{m}$ and hence the localisation $S^{-1}\widehat{R}$ is zero-dimensional. Being reduced, it is a product of fields and must therefore be equal to $Q(\widehat{R})$ (because any non-zerodivisor in $\widehat{R}$ will also act injectively on $S^{-1}\widehat{R}$ and in a finite product of fields non-zerodivisors and units agree). □

Remark. *By a theorem of Matsumura ([29]) which states that*

$$\mathsf{dim}(S^{-1}\widehat{R}) = \mathsf{dim}(R) - 1,$$

the previous lemma is not true if $\mathsf{dim}(R) \geq 2$.

Lemma 3.2. *In the situation of the previous lemma the canonical map*

$$Q(R)/R \to Q(\widehat{R})/\widehat{R}$$

is an isomorphism (of R-modules).

Proof. We consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d] & Q(R) \arrow[r] \arrow[d] & Q(R)/R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{R} \arrow[r] & Q(R) \otimes_R \widehat{R} \arrow[r] & (Q(R)/R) \otimes_R \widehat{R} \arrow[r] & 0.
\end{tikzcd}$$

The left and middle vertical maps are injective. By the snake lemma it is therefore sufficient to show that the canonical localisation map $\widehat{R}/R \to Q(R) \otimes_R (\widehat{R}/R)$ is bijective. By the previous lemma this amounts to showing that any element of $S = R\backslash\{0\}$ acts bijectively on $\widehat{R}/R$. In order to show this we fix $s \in S$ and use the snake lemma once again, this time applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "s"] & \widehat{R} \arrow[r] \arrow[d, "s"] & \widehat{R}/R \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & R \arrow[r] & \widehat{R} \arrow[r] & \widehat{R}/R \arrow[r] & 0.
\end{tikzcd}$$

The left and middle maps are injective, hence what we really have to show is that the morphism $R/(s) \to \widehat{R}/(s)$ is an isomorphism of R-modules. Since R is one-dimensional, the support of $R/(s)$ is zero-dimensional which implies the claim as $\widehat{R}/(s) = R/(s) \otimes_R \widehat{R}$ and $M \otimes_R \widehat{R} = M$ for any module M of finite length. $\square$

Remark. *If we assume in addition that R is a Gorenstein ring we can also give a different proof of this result: In this case, the injective hull $E_R(R/\mathfrak{m})$ of $R/\mathfrak{m}$ is given by $Q(R)/R$ and similarly $E_{\widehat{R}}(\widehat{R}/\mathfrak{m}_{\widehat{R}})$ is given by $Q(\widehat{R})/\widehat{R}$. But, by 10.2.10 of [8] we have a canonical isomorphism $E_R(R/\mathfrak{m}) \cong E_{\widehat{R}}(\widehat{R}\mathfrak{m}_{\widehat{R}})$.*

Corollary 3.1. *In the situation considered above, let M be a finitely generated torsion free R-module. Then the canonical map*

$$Q(M)/M \to Q(\widehat{M})/\widehat{M}$$

is an isomorphism.

Proof. By Lemma 3.1

$$Q(\widehat{M}) = Q(\widehat{R}) \otimes_{\widehat{R}} \widehat{M} = \widehat{S^{-1}M} = \widehat{Q(M)}.$$

Tensoring the isomorphism provided by Lemma 3.2 with M over R hence provides the claimed result. $\square$

Lemma 3.3. *Let R be a commutative ring, M an R-module and $\mathfrak{m} \subseteq R$ a maximal ideal such that for each $m \in M$ there exists a natural number $n \in \mathbb{N}$ such that $\mathfrak{m}^n \cdot m = 0$. Then every element of $S = R \backslash \mathfrak{m}$ acts as a unit on M.*

Proof. Take some element $s \in S$. We have to show that the homothety $s\cdot : M \to M$ is bijective. By assumption on M, there exists a filtration

$$M = \bigcup_{n \in \mathbb{N}} M_n,$$

where M_n denotes the submodule of all elements annihilated by $\mathfrak{m}^n$. As multiplication by s respects this filtration it is sufficient to consider $M = M_n$. Since $s \in S$, we can find $a \in R$ such that $as = 1$ in $R/(\mathfrak{m}^n)$, which implies the claim. $\square$

3.3 Exact Krichever Sequence

In this section we describe a certain exact sequence of abelian groups related to the Krichever correspondence. We begin with a few sheaf-theoretic preliminaries.

Lemma 3.4. *Let X be a noetherian scheme, $x \in X$ a closed point, $\mathcal{F}$ a quasi-coherent sheaf on X. Then there is a natural isomorphism of δ-functors*

$$\mathsf{H}^i_x\big(X, \mathcal{F}\big) \to \mathsf{H}^i_{\mathfrak{m}_x}\big(\mathcal{F}_x\big)$$

for all $i \in \mathbb{Z}$.

Proof. Let us fix some affine open subscheme $U \subseteq X$ containing x. Then (for example by Lemma 1.12 in [27]) we may restrict to $X = U = \mathsf{spec}(R)$

affine. Let $\mathcal{F} = \widetilde{M}$ for some R-module M and denote the maximal ideal corresponding to x by $\mathfrak{m}$. Then

$$\mathsf{H}^i_{\mathfrak{m}_x}\big(\mathcal{F}_x\big) \cong \mathsf{H}^i_{\mathfrak{m}}\big(M\big)_{\mathfrak{m}}$$

(for example by Corollary 4.3.3 in [8]). But, by Lemma 3.3,

$$\mathsf{H}^i_{\mathfrak{m}}\big(M\big)_{\mathfrak{m}} \cong \mathsf{H}^i_{\mathfrak{m}}\big(M\big)$$

and the latter is isomorphic to $\mathsf{H}^i_x\big(X, \mathcal{F}\big)$ by definition. □

Lemma 3.5. *Let $X = \mathsf{spec}(R)$ be a noetherian scheme, $\mathcal{F} = \widetilde{M}$ a quasi-coherent sheaf on X, $x \in X$ a closed point corresponding to the maximal ideal $\mathfrak{m} \subseteq R$ and $U = X \backslash \{x\}$. Then there exists an isomorphism*

$$\mathsf{H}^0(U, \mathcal{F}) \cong \mathsf{colim}_i \mathsf{Hom}(\mathfrak{m}^i, M).$$

Proof. Let $\mathfrak{m} = (f_1, \ldots, f_n)$ and therefore $U = D(f_1) \cup \ldots \cup D(f_n)$. An element of $\mathsf{H}^0(U, \mathcal{F})$ is therefore the same thing as a tuple

$$(m_1, \ldots, m_n) \in M_{f_1} \oplus \ldots \oplus M_{f_n}$$

with the property that m_i and m_j agree in $M_{f_i f_j}$ for all $1 \leq i, j \leq n$. To any morphism $f : \mathfrak{m}^l \to M$ we can associate such a tuple as follows:
We define m_i to be the image of 1 under the morphism

$$R_{f_i} = \mathfrak{m}^k_{f_i} \xrightarrow{f} M_{f_i}.$$

The fact that these elements come from the same morphism implies the compatibility condition since

$$\frac{f(f_j^k)}{f_j^k} = \frac{f_i^k f(f_j^k)}{f_i^k f_j^k} = \frac{f_j^k f(f_i^k)}{f_j^k f_i^k} = \frac{f(f_i^k)}{f_i^k}.$$

This defines a natural transformation of functors $\mathsf{Hom}(\mathfrak{m}^k, -) \to \mathsf{H}^0\big(U, \widetilde{(-)}\big)$ which is compatible with the colimit system of varying $k \in \mathbb{N}$.
Now consider the following diagram with exact rows, where all vertical morphisms are the canonical ones, respectively the one just defined and where

$\overline{R_k}$ denotes $R/(\mathfrak{m}^k)$:

$$\begin{tikzcd}
\mathsf{H}^0_{\mathfrak{m}}(M) \arrow[r, hook] & M \arrow[r] & \mathsf{H}^0(U,M) \arrow[r, two heads] & \mathsf{H}^1_{\mathfrak{m}}(M) \\
\mathsf{Hom}(\overline{R_k},M) \arrow[r, hook] \arrow[u] & \mathsf{Hom}(R,M) \arrow[r] \arrow[u] & \mathsf{Hom}(\mathfrak{m}^k,M) \arrow[r, two heads] \arrow[u] & \mathsf{Ext}^1(\overline{R_k},M) \arrow[u]
\end{tikzcd}$$

The commutativity of the first two squares is obvious and the commutativity of the third square is the naturality of the morphism we have defined together with the naturality of the snake lemma. Taking the colimit of the lower row and using the five lemma, we arrive at the natural isomorphism

$$\mathsf{H}^0(U,\mathcal{F}) \cong \mathsf{colim}_i \mathsf{Hom}(\mathfrak{m}^i, M)$$

since the outer two maps are isomorphisms by standard results about local cohomology. $\square$

Let us now fix the following data: X is a one-dimensional, integral, Gorenstein scheme with (possibly singular) closed point x such that $U = X\backslash\{x\}$ is affine, $\mathcal{F}$ is a torsionfree, coherent sheaf on X, $(R,\mathfrak{m})$ denotes $\big(\mathcal{O}_{X,x},\mathfrak{m}_x\big)$, $Q(R)$ its field of fractions, $\widehat{R}$ its $\mathfrak{m}$-adic completion and $Q(\widehat{R})$ the total ring of quotients of $\widehat{R}$. Fix an affine open subset $V = \mathsf{spec}(S)$ of X containing x and an S-module M such that $\widetilde{M} \cong \mathcal{F}|_V$.
We define the morphism $\alpha : \widehat{\mathcal{F}_x} \otimes_{\widehat{R}} Q(\widehat{R}) \to \mathsf{H}^1_x\big(X,\mathcal{F}\big)$ to be the composite

$$\begin{tikzcd}
\widehat{\mathcal{F}_x} \otimes_{\widehat{R}} Q(\widehat{R}) \arrow[r, "proj"] & \widehat{\mathcal{F}_x} \otimes_{\widehat{R}} Q(\widehat{R})/\widehat{R} & \mathcal{F}_x \otimes_R Q(R)/R \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
& \mathsf{H}^1_x\big(X,\mathcal{F}\big) & \mathcal{F}_x \otimes_R \mathsf{H}^1_x\big(X,\mathcal{O}_X\big). \arrow[l, "\cong"']
\end{tikzcd}$$

Following the compositional order, these morphisms are defined as follows:

- The first map is given by the projection map $Q(\widehat{R}) \to Q(\widehat{R})/\widehat{R}$ tensored with $\widehat{\mathcal{F}_x}$.
- The second map is given by the inverse of the morphism described in Lemma 3.1.

- Since R is a Gorenstein ring of dimension one

$$0 \to R \to Q(R) \to Q(R)/R \to 0$$

is an injective resolution of R and since $Q(R)/R$ is $\mathfrak{m}$-torsion we have

$$\mathsf{H}^1_x\big(X, \mathcal{O}_x\big) \cong Q(R)/R.$$

The vertical map is given by tensoring this isomorphism with $\mathcal{F}_x$.

- The last isomorphism comes from the description of local cohomology via the algebraic Cech complex and Lemma 3.1.

In terms of $N = \mathcal{F}_x$ the map $\alpha : N \otimes_R Q(\widehat{R}) = \widehat{N} \otimes_{\widehat{R}} Q(\widehat{R}) \to \mathsf{H}^1_{\mathfrak{m}}\big(N\big)$ can be described explicitly. Since N is torsionfree,

$$\mathsf{H}^1_{\mathfrak{m}}\big(N\big) = \mathsf{colim}_i \mathsf{Hom}\big(R/(\mathfrak{m}^i), Q(N)/N\big) = \mathsf{colim}_i \mathsf{Hom}\big(\widehat{R}/(\mathfrak{m}^i), Q(\widehat{N})/\widehat{N}\big),$$

where $Q(N)$ denotes the localisation of N at $R\backslash\{0\}$. An element $n \otimes \frac{1}{r}$ is mapped to the class of the morphism $1 \mapsto \frac{n}{r} + N$ (this also makes sense if everything is taken to be completed).

Lemma 3.6. *The map* $\mathsf{colim}\ \mathsf{Hom}(\mathfrak{m}^k, M) \to M_{\mathfrak{m}} \otimes_R Q(\widehat{R})$ *induced by the canonical morphism of schemes* $U \to \mathsf{spec}\big(K(X)\big) \to \mathsf{spec}\big(Q(\widehat{R})\big)$ *and the identification of Lemma 3.5 is given by*

$$(f : \mathfrak{m}^k \to M) \mapsto Q(f)(1).$$

Proof. If we go back to the proof of Lemma 3.5 we see that the map

$$\mathsf{colim}\ \mathsf{Hom}(\mathfrak{m}^k, M) \to Q(M)$$

corresponding to the restriction $\mathsf{H}^0(U, \widetilde{M}) \to Q(M) = \widetilde{M}_{(0)}$ is given by

$$(f : \mathfrak{m}^k \to M) \mapsto Q(f)(1).$$

Since the second morphism is just an inclusion, this finishes the proof. □

Construction. *Let*

$$f : \mathfrak{m}^k \to M$$

be given and define

$$g : R/(\mathfrak{m}^k) \to Q(M)/M$$

by the map on cokernels in the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^k \arrow[r] \arrow[dd, "f"] & R \arrow[r] \arrow[d] & R/(\mathfrak{m}^k) \arrow[r] \arrow[dd, dotted, "g"] & 0 \\
& & Q(R) = Q(\mathfrak{m}^k) \arrow[d, "Q(f)"] & & \\
0 \arrow[r] & M \arrow[r] & Q(M) \arrow[r] & Q(M)/M \arrow[r] & 0
\end{tikzcd}$$

Denote the induced map

$$\mathsf{Hom}(\mathfrak{m}^k, M) \to \mathsf{Hom}\big(R/(\mathfrak{m}^k), Q(M)/M\big)$$

by β.

Lemma 3.7. *The diagram*

$$\begin{tikzcd}
\mathsf{Hom}(\mathfrak{m}^k, M) \arrow[r] \arrow[dr, "\beta"'] & \mathsf{Ext}^1\big(R/(\mathfrak{m}^k), M\big) \\
& \mathsf{Hom}\big(R/(\mathfrak{m}^k), Q(M)/M\big) \arrow[u]
\end{tikzcd}$$

is commutative, where the unlabelled arrows denote the connecting morphisms of the two long exact sequences involved.

Proof. Fix some $f \in \mathsf{Hom}(\mathfrak{m}^k, M)$. Thinking in terms of Yoneda-Ext, the statement boils down to proving that the modules $A := (M \oplus R)/(f(\mathfrak{m}^k) \sim \mathfrak{m}^k)$ and $B := \big\{(x, y) \in Q(M) \oplus R/(\mathfrak{m}^k) \;\big|\; x + M = y \cdot Q(f)(1)\big\}$ are isomorphic.
Define a morphism $\Gamma : A \to Q(M) \oplus R/(\mathfrak{m}^k)$ by $(x, r) \mapsto \big(x + Q(f)(r), r\big)$. By construction the image of Γ is contained in B and we denote the corresponding morphism by Γ, too. If (x, r) is mapped to zero under Γ we have $r \in \mathfrak{m}^k$ and $-f(r) = x \in M$. On the other hand, given a tuple $(x, y) \in B$, pick a preimage r of y in R. Then $x - Q(f)(r) \in M$ and hence $\big(x - Q(f)(r), r\big)$ is mapped to (x, y) under Γ. $\square$

Lemma 3.8. *The diagram*

$$\begin{tikzcd} \mathsf{H}^0(U,\mathcal{F}) \arrow[r] \arrow[d, "can"'] & \mathsf{H}^1_x\big(X,\mathcal{F}\big) \\ N \otimes_R Q(\widehat{R}) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

is commutative.

Proof. By Lemma 3.3 and because the canonical map $\mathsf{H}^0(U,\mathcal{F}) \to N \otimes_R Q(\widehat{R})$ factors over $Q(R)$, we may assume that $X = V$ is affine and that $\mathcal{F} = \widetilde{M}$. Furthermore Lemmas 3.5, 3.6 and 3.7 allow us to prove commutativity of the diagram

$$\begin{tikzcd} \operatorname{colim} \mathsf{Hom}(\mathfrak{m}^k, M) \arrow[r, "\beta"] \arrow[d] & \operatorname{colim} \mathsf{Hom}\big(R/(\mathfrak{m}^k), Q(M)/M\big) \\ M \otimes_S R \otimes_R Q(\widehat{R}) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

instead. The stated commutativity is now clear, because both ways around the diagram give the class of the morphism $R \to Q(M)/M$ which sends 1 to $Q(f)(1) + M$. $\square$

Theorem 3.1. *There is a natural exact sequence*

$$0 \to \mathsf{H}^0(X,\mathcal{F}) \to \widehat{\mathcal{F}_x} \oplus \mathsf{H}^0(U,\mathcal{F}) \to Q(\widehat{R}) \otimes_{\widehat{R}} \widehat{\mathcal{F}_x} \to \mathsf{H}^1(X,\mathcal{F}) \to 0.$$

Proof. Consider the following commutative diagram in which the first row is exact and given by (a part of) the long exact sequence relating local and global cohomology:

$$\begin{tikzcd} \mathsf{H}^0(X,\mathcal{F}) \arrow[r] \arrow[d] & \mathsf{H}^0(U,\mathcal{F}) \arrow[r] \arrow[d] & \mathsf{H}^1_x\big(X,\mathcal{F}\big) \arrow[r] & \mathsf{H}^1(X,\mathcal{F}) \\ \widehat{\mathcal{F}_x} \arrow[r] & Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x} \arrow[ur, "\alpha"'] & & \end{tikzcd}$$

Note that the first morphism in the upper row is injective, since $\mathcal{F}$ is torsionfree and that the last morphism in the same row is surjective, since U is affine. Putting these two rows together we obtain a sequence of morphisms

$$0 \to \mathsf{H}^0(X, \mathcal{F}) \to \widehat{\mathcal{F}_x} \oplus \mathsf{H}^0(U, \mathcal{F}) \to Q(\widehat{R}) \otimes_{\widehat{R}} \widehat{\mathcal{F}_x} \to \mathsf{H}^1(X, \mathcal{F}) \to 0$$

as in the statement, where we change the map $\mathsf{H}^0(X, \mathcal{F}) \to \widehat{\mathcal{F}_x}$ by a minus sign in comparison to the commutative diagram above. Then the above sequence of maps is a complex, because $\widehat{\mathcal{F}_x}$ is contained in the kernel of α (actually it is equal to it!). Now, let us consider exactness of this complex step by step:

1. Given $s \in \mathsf{H}^0(X, \mathcal{F})$ which is mapped to zero in $\mathsf{H}^0(U, \mathcal{F})$ the section s presents itself as an element of the space $\mathsf{H}^0_x\big(X, \mathcal{F}\big)$. This space is zero since $\mathcal{F}$ is torsionfree.

2. Given two elements $a \in \widehat{\mathcal{F}_x}$ and $b \in \mathsf{H}^0(U, \mathcal{F})$ which are mapped to the same element in $Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x}$, they are also mapped to the same element in $\mathsf{H}^1_x(X, \mathcal{F})$, which means that b is mapped to zero in $\mathsf{H}^1_x\big(X, \mathcal{F}\big)$. Thus we may find $c \in \mathsf{H}^0(X, \mathcal{F})$ such that $b = c$ in $\mathsf{H}^0(U, \mathcal{F})$. Since the image of c in $\widehat{\mathcal{F}_x}$ and a agree in $Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x}$, they already agree in the submodule $\widehat{\mathcal{F}_x}$.

3. Given $d \in Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x}$ which maps to zero in $\mathsf{H}^1(X, \mathcal{F})$, we take a preimage e of $\alpha(d)$ in $\mathsf{H}^0(U, \mathcal{F})$. Then d minus the image of e in the space $Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x}$ is mapped to zero by α, hence comes from an element of $\mathsf{ker}(\alpha) = \widehat{\mathcal{F}_x}$.

4. Finally, it is clear that the morphism $Q(\widehat{R}) \otimes \widehat{\mathcal{F}_x} \to \mathsf{H}^1(X, \mathcal{F})$ is surjective, being the composition of two surjective morphisms. $\square$

3.4 Application to Manin Triples

Let us assume X and $\mathcal{F}$ as before and assume furthermore that the sheaf cohomology of $\mathcal{F}$ vanishes. Assume that we are given a non-zero rational 1-form $\omega \in \Omega_{K(X)/K}$ of X and a symmetric, non-degenerate bilinear form $\langle -, - \rangle : Q(\mathcal{F}) \otimes_{K(X)} Q(\mathcal{F}) \to K(X)$, too. We can extend this form in the obvious manner to a form $Q(\widehat{\mathcal{F}}) \otimes_{Q(\widehat{\mathcal{O}_{X,x}})} Q(\widehat{\mathcal{F}}) \to Q\big(\widehat{\mathcal{O}_{X,x}}\big)$ and then we can make it into a bilinear form $\{-, -\} : Q(\widehat{\mathcal{F}}) \otimes_K Q(\widehat{\mathcal{F}}) \to K$ via composition with the linear map $\mathsf{res}_x(- \cdot \omega) : Q\big(\widehat{\mathcal{O}_{X,x}}\big) \to K$.

Lemma 3.9. *The form $\{-,-\}$ is K-bilinear, symmetric and non-degenerate.*

Proof. The first two statements are clear. Let us fix $f \in Q(\widehat{\mathcal{F}}) \setminus \{0\}$ and assume that $\{f,-\}$ is zero. Since ω is non-zero and $< f, - >$ is surjective, it is sufficient to prove the following lemma by the definition of the residue in a singular point (see for example Theorem 11.11 and the discussion before that theorem in Kunz).

Lemma 3.10. *Let C be a smooth curve over K, $c_1, \ldots, c_n$ finitely many distinct closed points of C and $\omega \in \Omega_{K(C)/K}$ a non-zero, rational 1-form of C. Then there exists $k \in K(C)$ such that $\sum_{i=1}^n \mathsf{res}_{c_i}(k\omega) \neq 0$.*

Proof. Assume ω has a pole of order r_i at c_i and let π_i be a local parameter for c_i (normalised such that the value of ω's pole at c_i is one). Then the element $k = \pi_1^{r_1-1} \cdot \Pi_{i\neq 1}\pi_i^{r_i}$ does the job since

$$\mathsf{res}_{c_i}(k\omega) = \delta_{1i}. \qquad \square$$

As remarked above, this finishes the proof. $\square$

In his ICM talk Drinfeld ([17]) discussed ways to study Manin triples from an algebro-geometric point of view. The following theorem describes an approach to creating Manin triples from our geometric data.

Theorem 3.2. *Assume that X is moreover projective, that for any open subset $U' \subseteq U$, any $f, g \in \mathsf{H}^0(U', \mathcal{F})$ the sum $\sum_{u \in U'} \mathsf{res}_u\big(\langle f, g\rangle\omega\big)$ is equal to zero and that there exists some $y \in U$ such that for $V = X \setminus \{y\}$ and for any $f, g \in \mathsf{H}^0(V, \mathcal{F})$ the residue $\mathsf{res}_y\big(\langle f, g\rangle\omega\big)$ is zero. Then the decomposition*

$$Q(\widehat{\mathcal{F}_x}) = \widehat{\mathcal{F}_x} \oplus \mathsf{H}^0(U, \mathcal{F})$$

is lagrangian with respect to the form $\{-,-\}$, i.e.

$$\widehat{\mathcal{F}_x}^{\perp} = \widehat{\mathcal{F}_x} \text{ and}$$
$$\mathsf{H}^0(U, \mathcal{F})^{\perp} = \mathsf{H}^0(U, \mathcal{F}).$$

Proof. By Theorem 3.1 and because we assumed that both $\mathsf{H}^0(X, \mathcal{F})$ and $\mathsf{H}^1(X, \mathcal{F})$ vanish, there really is a decomposition

$$Q(\widehat{\mathcal{F}_x}) = \widehat{\mathcal{F}_x} \oplus \mathsf{H}^0(U, \mathcal{F}).$$

By Lemma 3.9 the form is non-degenerate. Therefore to show that the decomposition is lagrangian it is sufficient to show the inclusions $\widehat{\mathcal{F}_x} \subseteq \widehat{\mathcal{F}_x}^{\perp}$ and $\mathsf{H}^0(U, \mathcal{F}) \subseteq \mathsf{H}^0(U, \mathcal{F})^{\perp}$ since any element $a + b$ with $a \in \widehat{\mathcal{F}_x}$ and $b \in \mathsf{H}^0(U, \mathcal{F})$ which is in $\widehat{\mathcal{F}_x}$ automatically implies $b \in \widehat{\mathcal{F}_x}^{\perp}$ and hence $b = 0$. Similarly one can show the other equality.
These inclusions follow from the residue theorem (which we have at our disposal since X is projective) and the assumptions as follows:

1. If $f, g \in \mathsf{H}^0(U, \mathcal{F})$, then the sum over residues of $\langle f, g\rangle\omega$ at all closed points of X is zero, but since the same sum over all closed points of U is already zero and $X = U \cup \{x\}$, the residue at x must be zero, too.

2. By the Hartshorne-Lichtenbaum Vanishing Theorem ([22]) the open subscheme V is affine. Hence if $\alpha, \beta \in \widehat{\mathcal{F}_x}$ then there are f and g in $\mathsf{H}^0(V, \mathcal{F})$ such that $f - \alpha$ and $g - \beta$ are zero in $\mathcal{F} \otimes \kappa_x$ (here we used that V is affine). Therefore $\{f, g\}$ is equal to $\{\alpha, \beta\}$. The latter element is zero, because our assumptions imply

$$\begin{aligned} \sum_{z \in U} \mathsf{res}_z\big(\langle f, g\rangle\omega\big) &= \sum_{z \in V \bigcap U} \mathsf{res}_z\big(\langle f, g\rangle\omega\big) + \mathsf{res}_y\big(\langle f, g\rangle\omega\big) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

whereas the residue theorem implies

$$\sum_{z \in C} \mathsf{res}_z\big(\langle f, g\rangle\omega\big) = 0.$$

Therefore we have

$$\{f, g\} = \mathsf{res}_x\big(\langle f, g\rangle\omega\big) = 0$$

as claimed.

□

Remark. *As one can see from the proof, one does not need to assume the vanishing for all open subsets $U' \subseteq U$, but only for U and $V \bigcap U$. But in the applications we have in mind these more general assumptions are not needed.*

Corollary 3.2. *Let X be an integral, projective curve with Gorenstein point x. Assume that $X \setminus \{x\}$ is smooth and let $c_1, \ldots, c_n$ be the preimages of x under the normalisation map $C \to X$. Let ω be a rational 1-form on X without poles along $X \setminus \{x\}$. Then to any torsion-free sheaf $\mathcal{F}$ on X of rank r with a symmetric, non-degenerate form $\langle -, - \rangle$ over $X \setminus \{x\}$ one can associate a decomposition*

$$\Big(K\big((t_1)\big) \times \ldots \times K\big((t_n)\big)\Big)^{\oplus r} = \mathcal{O}_X\big(X \setminus \{x\}\big)^{\oplus r} \oplus \widehat{\mathcal{F}_x}$$

which is lagrangian with respect to the form

$$\{\alpha, \beta\} = \sum_{i=1}^{n} \mathsf{res}_{c_i}\big(\langle \alpha, \beta \rangle \omega\big)$$

if there exists a closed point p in $X \setminus \{x\}$ with $\mathsf{res}_p\big(\langle f, g \rangle \omega\big) = 0$ for any $f, g \in \mathsf{H}^0\big(X \setminus \{p\}, \mathcal{F}\big)$.

Proof. This follows from Theorem 3.2 if we take Hironaka-Matsumura ([24]) into account for the identification of $Q(\widehat{\mathcal{F}_x})$. $\square$

Example 3.1. *If the form $\langle -, - \rangle$ has the property that*

$$\big\langle \mathcal{F}(W), \mathcal{F}(W) \big\rangle \subseteq \mathcal{O}_X(W)$$

for any open subset $W \subseteq X$, then any point $y \in X \setminus \{x\}$ satisfies the last condition, i.e. the decomposition will always be lagrangian.
For example, this property is satisfied if there exists an isomorphism of sheaves of $\mathcal{O}_X$-modules $f : \mathcal{A} \cong \mathcal{A}^\vee$ extending the one coming from $\langle -, - \rangle$ over $X \setminus \{x\}$, since then $\langle -, - \rangle$ can be extended to all of $\mathcal{A}$ via the composite $\mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathsf{id} \otimes f} \mathcal{A} \otimes \mathcal{A}^\vee \xrightarrow{\mathsf{ev}} \mathcal{O}_X$.

3.5 Construction of Curve and Sheaf of Lie Algebras from a Rational Solution

The main results of this section are summarised in the following theorem:

Theorem 3.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field K with Casimir element Ω and let $\mathsf{r}(x, y) = \frac{\Omega}{y-x} + p(x, y)$ denote a*

unitary rational solution (in the sense of Stolin) of the Classical Yang-Baxter Equation.
Then there exists a (coherent) sheaf of Lie algebras $\mathcal{A}$ on the cuspidal cubic curve $C = V(Y^2Z - X^3) \subseteq \mathbb{P}^2_K$ with vanishing sheaf cohomology which is isomorphic to $\mathfrak{g} \otimes_K \Gamma\big(C^{\mathsf{smooth}}, \mathcal{O}_C\big)$ over the smooth part of C such that r is given by the Szegö kernel associated to $\mathcal{A}$ and the 1-form $d\frac{X}{Y}$.
Two rational solutions $\mathsf{r}_1(x, y)$ and $\mathsf{r}_2(x, y)$ are gauge equivalent (i.e. there exists $\sigma \in \mathsf{Aut}(\mathfrak{g}[T])$ such that $\mathsf{r}_2 = \sigma \otimes \sigma(\mathsf{r}_1)$) if and only if the two sheaves of Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to these solutions are isomorphic (as sheaves of Lie algebras).

Remark. *Here $d\frac{X}{Y}$ denotes the unique nowhere vanishing* 1*-form $\omega \in \Omega_C$ such that its restriction to the smooth part $\mathsf{spec}\big(K[Z]\big)$ is given by dZ for the coordinate $Z = \frac{X}{Y}$.*

The proof of the previous theorem will be given in several steps and begins with the following lemma whose one-variable version goes back to Semenov-Tian-Shansky ([38]). In contrast with the one-variable version, the version for two variables does not work for every solution of the CYBE (e.g. it is wrong for quasi-trigonometric solutions).

Lemma 3.11. *Let V be a finite-dimensional K-vector space, $f \in V \otimes V[X, Y]$ and $\langle -, - \rangle : V \otimes V \to K$ a symmetric, non-degenerate bilinear form. Let $\{G_i\}$ be an orthonormal basis for $\langle -, - \rangle$ and let $\Omega = \sum_i G_i \otimes G_i$. Consider*

$$F = \frac{\Omega}{Y - X} + f = \sum_n X^n \sum_i G_i \otimes f_{ni}$$

for $f_{ni} \in V[Y, Y^{-1}]$. Let $V(F) \subseteq V\big((Y)\big)$ denote the K-span of all the f_{ni}. Then the following holds true:

1. *$V(F) + V\big[[Y]\big] = V\big((Y)\big)$ and $V(F) \bigcap V\big[[Y]\big] = \{0\}$.*
2. *Consider the non-degenerate, symmetric bilinear form*

$$\{-, -\} : V\big((Y)\big) \otimes V\big((Y)\big) \to K$$

given by

$$f \otimes g \mapsto \mathsf{res}_Y\big(\langle f, g \rangle dY\big)$$

(where $\langle f, g\rangle$ is the obvious $K\big((Y)\big)$-bilinear extension of $\langle -, -\rangle$ to $V\big((Y)\big) \otimes V\big((Y)\big)$). Then

$$V(F)^{\perp} = V(F) \iff F(X, Y) = -\tau F(Y, X),$$

where $^{\perp}$ is taken with respect to $\{-, -\}$ and $\tau : V \otimes V \to V \otimes V$ is the automorphism exchanging the two tensor factors.

3. *The elements f_{ni} satisfy*

$$\{G_i X^n, f_{mj}\} = \delta_{ij}\delta_{nm}.$$

4. *If $V = \mathfrak{g}$ is a Lie algebra and F satisfies the CYBE, then $V(F)$ is a Lie subalgebra of $V\big((Y)\big)$.*

5. *Let $n > \mathsf{deg}(f)$ (where deg denotes the total degree). Then*

$$Y^{-n}K[Y^{-1}]V \subseteq V(F) \subseteq K[Y^{-1}]V \oplus YV \oplus \ldots \oplus Y^{n-1}V.$$

6. *Let $A = \Big\{a \in K\big((Y)\big) \Big| a \cdot V(F) \subseteq V(F)\Big\}$ be the stabiliser ring of $V(F)$. Then there exists $n \in \mathbb{N}$ such that $Y^{-n}K[Y^{-1}] \subseteq A \subseteq Y^n K[Y]$.*

Proof. 1. Since $f_{ni} = G_i Y^{-n-1} + \text{something} \in V[Y]$ both claims follow at once.

2. First of all, the form $\{-, -\}$ is non-degenerate, since $\{g, G_i Y^m\} = g_{mi}$ if we write

$$g = \sum_n \sum_{i=1}^{\mathsf{dim}_K V} g_{ni} G_i Y^n.$$

Next, we note that part 1. implies

$$V(F)^{\perp} = V(F) \iff V(F) \subseteq V(F)^{\perp}$$

and furthermore

$$V(F) \subseteq V(F)^{\perp} \iff (f_{ni}, f_{mj}) = 0 \;\; \forall i, j, n, m.$$

If we compute $\{f_{ni}, f_{mj}\}$ this turns out to be equal to

$$\langle G_j, \text{degree m term of } f_{ni}\rangle + \langle G_i, \text{degree n term of } f_{mj}\rangle$$

and is zero for $i \neq j$. In the case that i and j are equal, the first summand provides us with an entry $\lambda G_i \otimes G_i X^n Y^m$ of F and the second with one of the form $\mu G_i \otimes G_i X^m Y^n$ and hence the claim follows.

3. This follows from the form of the f_{ni} as $G_i Y^{-n} + p$, where $p \in V[Y]$.

4. The CYBE for F can be written as (see for instance a lemma in the part about Szegö kernels)

$$\begin{aligned}
0 = &\sum_{i,j,n,m} X^{n+m}[G_i, G_j] \otimes f_{ni}(Y) \otimes f_{mj}(Z) \\
&+ \sum_{i,j,n,m} +X^n G_i \otimes Y^m \big[f_{ni}(Y), G_j\big] \otimes f_{mj}(Z) \\
&+ \sum_{i,j,n,m} X^n G_i \otimes Y^m G_j \otimes \big[f_{ni}(Z), f_{mj}(Z)\big].
\end{aligned}$$

Comparing coefficients for $X^n G_i \otimes Y^m G_j$ thus gives the desired statement.

5. Any f_{mi} is of the form $G_i Y^{-m-1}+$ something $\in V \oplus YV \oplus \dots Y^{m-1}V$ and that something is zero for $m \geq \mathsf{deg}(f)$.

6. The claim follows directly from the previous item. □

Let us consider the easiest possible example. A slightly more complicated is discussed at the end of this section to show that not all rational solutions come from vector bundles.

Example 3.2. *Let* $\mathsf{r}(x,y) = \frac{\Omega}{y-x}$. *Then the associated Lie algebra* $\mathfrak{g}(\mathsf{r})$ *is obviously given by* $Y^{-1}\mathfrak{g}[Y^{-1}]$ *and its associated sheaf of Lie algebras is no other than the one considered in Example 1.3, i.e. the sheaf of matrices of trace zero with entries in* $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Next we apply the Krichever correspondence à la Mulase to the vector space F and are given an integral curve C with one smooth point x and a (coherent) sheaf of Lie algebras $\mathcal{A}$ on C with vanishing sheaf cohomology, such that $\widehat{\mathcal{A}_x} = \mathfrak{g}\big[[Y]\big]$ and such that $\mathsf{H}^0\big(C \setminus \{x\}, \mathcal{A}\big) = \mathfrak{g}(\mathsf{r})$.

Lemma 3.12. *The smooth locus of C is either $\mathbb{A}^1$ or $\mathbb{P}^1$. It is $\mathbb{P}^1$ if and only if $p = 0$ (recall that $\mathsf{r}(x, y) = \frac{\Omega}{y-x} + p(x, y)$). The only singular point of C (if it exists) is given by the maximal ideal of A which contains Y^{-n}.*

Proof. By the last part of the previous lemma and general facts about the Krichever correspondence, we know that the smooth locus of C consists of $\mathsf{spec}\big(K[Y, Y^{-1}]\big)$ and one or two additional closed points. Considering the smooth locus of C as an open subscheme of the normalisation of C (which is $\mathbb{P}^1$) allows to conclude its structure is either $\mathbb{A}^1 = \mathsf{spec}\big(K[Y]\big)$ or all of $\mathbb{P}^1$.
To show that the smooth locus is given by $\mathbb{P}^1$ if and only if $\mathsf{r} = \frac{\Omega}{y-x}$, it is sufficient to show that for a given Lie algebra $\mathfrak{g}$ there is at most one sheaf of Lie algebras $\mathcal{A}$ on $\mathbb{P}^1$ such that $\mathsf{H}^0(\mathbb{P}^1, \mathcal{A}) = \mathsf{H}^1(\mathbb{P}^1, \mathcal{A}) = 0$ and such that $\Gamma(\mathbb{G}_m, \mathcal{A}) = \mathfrak{g} \otimes \Gamma(\mathbb{G}_m, \mathcal{O}_{\mathbb{P}^1})$ (see also Example 3.2).
The vanishing of its sheaf cohomology implies that $\mathcal{A}$ is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and the restriction condition implies among other things that it is a direct sum of $\mathsf{dim}_K(\mathfrak{g})$ copies of that line bundle. Therefore as a coherent sheaf $\mathcal{A}$ is completely determined by the above data and since $\mathcal{H}om(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ is a vector bundle, too, (and hence in particular has injective restriction maps) the Lie algebra structure is also determined by the restriction condition. $\square$

We denote the singular point of C by y if it exists. Fix

$$F = \mathsf{spec}\big(K[Y]\big) = \mathbb{A}^1$$

inside the smooth part such that $x \in F$. To apply the construction of Szegö kernels to $\mathcal{A}$ we need to check that the restriction of $\mathcal{A}$ to F is given by $\mathfrak{g}[Y]$.

Lemma 3.13. $\mathsf{H}^0(F, \mathcal{A}) = \mathfrak{g}[Y]$.

Proof. Let M denote the A-module $\mathsf{H}^0(F, \mathcal{A})$. We note that $M_Y = \mathfrak{g}[Y, Y^{-1}]$ and that $\widehat{M_{(Y)}} = \mathfrak{g}\big[[Y]\big]$. Therefore the statement follows from

Lemma 3.14. *Let R be a principal ideal domain, M a finitely generated free R-module and let $\pi \in R$ be a prime element. Then $M = M_\pi \bigcap \widehat{M_{(\pi)}}$ as subsets of $Q\big(\widehat{M_{(\pi)}}\big)$.*

whose proof is easy. $\square$

To finish the proof of Theorem 3.3 we want to apply our results about the construction of Szegö kernels and therefore need to know that C is a Gorenstein curve. Fortunately, this can be achieved by shrinking A slightly.

Lemma 3.15. *Let $K[T^d, T^{d+1}, \ldots] \subseteq A \subseteq K[T]$ be a ring. Then there exists a subring $B \subseteq A$ with the same field of fractions, such that B is a Gorenstein ring.*

Proof. By shrinking A, we may assume that A is a monoid subalgebra of $K[T]$, say $A = K[\Gamma]$ for some monoid $\Gamma \subseteq \mathbb{N}$, which contains all natural numbers bigger than a certain $d \in \mathbb{N}$. Then consider the conductor c, which is the smallest possible such d. By an old result of Herzog and Kunz (see [23] for more general results and exercise 21.11 in [18] for this particular statement), the completion of A at the maximal ideal containing T^c is Gorenstein if and only if $\left|\{a \in \Gamma, a < c\}\right| = \left|a \notin \Gamma\right|$. Let $\Delta \subseteq \Gamma$ be the submonoid given by the next lemma, where this condition is satisfied and let B be its monoid algebra. Then since $B_{T^c} = A_{T^c} = K[T, T^{-1}]$ is Gorenstein, B is a Gorenstein ring with the same field of fractions as A. □

Lemma 3.16. *In the notation above, there exists a submonoid $\Delta \subseteq \Gamma$ with conductor d such that $\left|\{a \in \Delta, a < d\}\right| = \left|a \notin \Delta\right|$.*

Proof. Let c be the conductor of Γ and let $d = 2c$. Remove $d - 1$ and any positive number smaller than c from Γ and call the result Δ. Obviously the conductor of Δ is then d and for each decomposition $d - 1 = a + b$ in natural numbers a and b, Δ contains exactly one of a or b, because exactly one of them is bigger or equal to c. This proves $\left|\{a \in \Delta, a < d\}\right| = \left|a \notin \Delta\right|$. □

Next we want to show that the Szegö kernel associated to $\mathcal{A}$ and the global 1-form $d\frac{Y}{X}$ agrees with the element $\mathsf{r}(x, y)$ we started with (at least suitably interpreted).

Lemma 3.17. *If we denote the canonical morphism of schemes*

$$F \setminus \{x\} \times \mathsf{spec}\Big(K\big[[X]\big]\Big) \to C \times F$$

by j, where the coordinate on the second copy is called X, then

$$\Gamma\bigg(F \setminus \{x\} \times \mathsf{spec}\Big(K\big[[X]\big]\Big), j^*\big(\mathcal{A} \boxtimes \mathcal{A}|_F(D)\big)\bigg) = \Gamma(F, \mathcal{A}) \otimes_K \widehat{\mathcal{A}_{(X)}}.$$

Proof. This is an instance of the Künneth formula, see for instance tag 0BEC in [41]. □

Fixing coordinates also on the first copy, the equality above translates to

$$\Gamma\Big(\mathsf{spec}\big(K[Y,Y^{-1}]\big)\times\mathsf{spec}\Big(K\big[[X]\big]\Big), j^*\big(\mathcal{A}\boxtimes\mathcal{A}|_F(D)\big)\Big) = \mathfrak{g}[Y,Y^{-1}]\otimes_K\mathfrak{g}\big[[X]\big]$$

which is the version we will use in the following.

Proposition 3.1. *Let* r' *denote the Szegö kernel. Then we have*

$$j^*(\mathsf{r}') = \mathsf{r}.$$

Proof. Since j factors through $F\times F$, we may reduce to considering this scheme instead of $C\times F$ and, as we have seen before, over the scheme $F\times F$ the Szegö kernel r' is given by

$$\frac{\Omega}{Y-X}+p(X,Y)$$

for some polynomial $p(X,Y)\in\mathfrak{g}\otimes\mathfrak{g}[X,Y]$. Furthermore, we do not consider $j^*(\mathsf{r}')$ directly, but rather all of its components one at a time, by considering its images in the rings $K[Y,Y^{-1}][X]/(X^n)$ for all n.
Note that the composite

$$j_n : \mathsf{spec}\big(K[Y,Y^{-1}][X]/(X^n)\big)\to\mathsf{spec}\Big(K\big[[X]\big][Y,Y^{-1}]\Big)\to F\setminus\{p\}\times F$$

is actually equal to the map of schemes induced by the canonical quotient map of rings $K[X,Y,Y^{-1}]\to K[Y,Y^{-1}][X]/(X^n)$ which is the identity on Y and Y^{-1} and maps X to its coset. Therefore the Y-part of $j_n^*(\mathsf{r}')$ is actually given by elements of $\mathfrak{g}(\mathsf{r})=\mathsf{H}^0(C\setminus\{x\},\mathcal{A})$.
We need to calculate $j_m^*\big(\frac{1}{Y-X}\big)$. Since $Y-X\in\Gamma\big(F\times F,\mathcal{O}_{C\times F}\big)$ is mapped to $Y-X\in K\big[[X]\big][Y,Y^{-1}]$ and the latter has inverse $\sum_{n\geq 0}Y^{-n-1}X^n$ in the ring $K[Y,Y^{-1}]\big[[X]\big]$ which has the same reductions modulo powers of X as $K\big[[X]\big][Y,Y^{-1}]$, we conclude

$$j_m^*\Big(\frac{1}{Y-X}\Big)=\sum_{0\leq n\leq m-1}Y^{-n-1}X^n.$$

Now fix an orthonormal basis (with respect to $\langle -, - \rangle$) $\{G_i\}$ of $\mathfrak{g}$ and write

$$j_0^*(\mathsf{r}') = \sum_{i=1}^{\mathsf{dim}(\mathfrak{g})} R_{i0} \otimes G_i.$$

Since $\mathsf{r} = \frac{\Omega}{Y-X} + p(X, Y)$ for some $p(X, Y) \in \mathfrak{g} \otimes_K \mathfrak{g}[X, Y]$ we have

$$\{R_{i0}, G_j Y^m\} = \delta_{ij}\delta_{m0}.$$

Similarly (and inductively), we define elements $R_{in} \in \mathfrak{g}(\mathsf{r})$ via

$$j_n^*(\mathsf{r}') = \sum_{k \leq n} \sum_{i=1}^{\mathsf{dim}(\mathfrak{g})} R_{ik} \otimes G_i X^k.$$

Again, by the shape of r' we have

$$\{R_{in}, G_j Y^m\} = \delta_{ij}\delta_{mn}.$$

Since r is unitary, there is only one set of elements inside $\mathfrak{g}(\mathsf{r})$ with this property, namely $\{\mathsf{r}_{ni}\}$ and since

$$\mathsf{r} = \sum_n X^n \sum_i G_i \otimes f_{ni},$$

we must have $\mathsf{r} = j^*(\mathsf{r}')$ as claimed. $\square$

As was already mentioned in the introduction, some of the sheaves of Lie algebras $\mathcal{A}$ on C corresponding to rational solutions are not locally free. The following gives an explicit example of such a sheaf.

Example 3.3. *Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $\mathsf{r}(x, y) = \frac{\Omega}{y-x} + \frac{H \otimes E - E \otimes H}{2}$, where we use the standard basis*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2$. According to Example 3.2.16 of [13] the term r is a unitary, rational solution of the CYBE (by the final proof presented in this thesis, one can also argue instead that the space $\mathfrak{g}(\mathsf{r})$ is a Lie algebra).
Let us calculate the corresponding vector space $\mathfrak{g}(\mathsf{r})$: As the polynomial part is constant, $Y^{-2}\mathfrak{g}[Y^{-1}] \subseteq \mathfrak{g}(\mathsf{r})$ and as one can see directly the remaining basis vectors are given by $Y^{-1}H + E$, $Y^{-1}F - \frac{H}{2}$ and $Y^{-1}E$. One can check by hand that it is a Lie algebra.
Since $Y^{-1}H \notin \mathfrak{g}(\mathsf{r})$, we see that the ring of stabilisers A of $\mathfrak{g}(\mathsf{r})$ is given by $K[Y^{-2}, Y^{-3}]$.
To show that $\mathfrak{g}(\mathsf{r})$ is not a projective A-module, it is sufficient to show that its completion at the ideal (Y^{-2}, Y^{-3}) is not a free $K\big[[Y^{-2}, Y^{-3}]\big]$-module of rank three. To do so, we need only show that the number of minimal generators of the completion of $\mathfrak{g}(\mathsf{r})$ is bigger than three. By Nakayama's lemma it is therefore sufficient to consider the quotient

$$\mathfrak{g}(\mathsf{r})/\big((Y^{-2}, Y^{-3})\mathfrak{g}(\mathsf{r})\big).$$

This is a five-dimensional K-vector space, which can be seen as follows:

- $Y^{-4}\mathfrak{g}[Y^{-1}] = Y^{-2} \cdot Y^{-2}\mathfrak{g}[Y^{-1}]$ *is set to zero.*
- $Y^{-3}H$ *is zero, too, since it differs from $Y^{-3} \cdot (2Y^{-1} - H)$ only by an element which becomes zero itself.*
- *Therefore $Y^{-2}E$ and $Y^{-3}E$ are also set to zero.*
- *There is no way $Y^{-2}F$ can be in the space we set to zero.*
- $Y^{-3}F$ *equals $Y^{-2}\frac{H}{2}$ in the quotient space.*

3.6 Comparison with Stolin's Work

We will now compare our approach with the classification of rational solutions of the CYBE by Stolin and give a geometric way to obtain the Stolin decomposition associated to r. Let us first recall Stolin's result ([43]):

Theorem (Stolin)**.** *Rational, unitary solutions of the CYBE are in 1-1-correspondence with Lie subalgebras $W \subseteq \mathfrak{g}\big((t^{-1})\big)$ satisfying*

1. $\mathfrak{g}\big((t^{-1})\big) = \mathfrak{g}[t] \oplus W$.

2. *W contains $t^{-N}\mathfrak{g}\big[[t^{-1}]\big]$ for some $N \geq 1$.*

3. *W is lagrangian with respect to the form which picks out the t^{-1}-coefficient of $\langle -, - \rangle$.*

Definition 3.1. *If r is a rational solution of the CYBE we will call the associated W (or more precisely, the decomposition $\mathfrak{g}\big((t^{-1})\big) = \mathfrak{g}[t] \oplus W$) the Stolin decomposition of r.*

We continue to denote the singular point of C by y.

Theorem 3.4. *Choosing coordinates on F as above, the decomposition*

$$Q\big(\widehat{\mathcal{A}_y}\big) = \mathsf{H}^0(F, \mathcal{A}) \oplus \widehat{\mathcal{A}_y}$$

is the Stolin decomposition of r.

Proof. By Theorem 3.1 there is a decomposition $Q\big(\widehat{\mathcal{A}_y}\big) = \mathsf{H}^0(F, \mathcal{A}) \oplus \widehat{\mathcal{F}_y}$ associated to $\mathcal{A}$ and y and the two vector spaces appearing on the right hand side are actually Lie sub-algebras of $Q\big(\widehat{\mathcal{A}_y}\big)$. Fixing coordinates $F = \mathsf{spec}\big(K[Y]\big)$ as before and using $\Gamma(F, \mathcal{A}) = \mathfrak{g}[Y]$ this becomes

$$\mathfrak{g}\big((Y^{-1})\big) = \mathfrak{g}[Y] \oplus W_{\mathcal{A}}$$

for some suitable Lie subalgebra $W_{\mathcal{A}} \subseteq \mathfrak{g}\big((Y^{-1})\big)$. For this to be a decomposition in the sense of Stolin, we need to check the following:

1. $W_{\mathcal{A}}$ contains $Y^{-N}\mathfrak{g}\big[[Y^{-1}]\big]$ for some $N \geq 1$.

2. $W_{\mathcal{A}}$ is lagrangian with respect to the form $\mathsf{res}_{Y^{-1}}\big(\langle -, - \rangle dY\big)$.

The first part is achieved by Lemma 3.18 and an easy argument for finitely generated modules with the same localisation. The second part follows from Corollary 3.2 once we can show that the assumptions of that theorem are satisfied:
We take the Killing form on $Q(\mathcal{A})$ as the symmetric non-degenerate bilinear form in that theorem. Furthermore we take the point p in the statement of that result to be the neutral element of $\mathbb{G}_a = F$, i.e. the point x. Then the scheme $V := C \setminus \{p\} = \mathsf{spec}(A)$ is affine and $\mathsf{res}_p\big(\langle f, g \rangle dY\big) = 0$ for any $f, g \in \mathsf{H}^0(V, \mathcal{A})$ by Lemma 3.11.
We note that the Leibniz formula implies

$$0 = d(1) = d(Y \cdot Y^{-1}) = Y dY^{-1} + Y^{-1} dY$$

and therefore $\mathsf{res}_y(q \cdot dY)$ is given by the coefficient of $-Y^{-1}$ if we write q as a Laurent series in Y^{-1}. This implies that $\mathsf{res}_y\big(\langle -, - \rangle dY\big)$ is the correct bilinear form, at least up to a sign, but this sign doesn't matter if one only cares about lagrangianness.
Finally, we still have to show that the rational solution associated to the decomposition above is given by r, i.e. the rational solution we started with. Note that there are canonical morphisms

$$\alpha : \mathsf{H}^0(V, \mathcal{A}) \to \widehat{\mathcal{A}_y}$$

and

$$\beta : \mathsf{H}^0(F, \mathcal{A}) \to \widehat{\mathcal{A}_x},$$

where the second one turns out to be the canonical map $\mathfrak{g}[Y] \to \mathfrak{g}\big[[Y]\big]$ after our usual identifications, and hence a chosen basis of $\mathfrak{g}[Y]$ maps to a topological basis of $\mathfrak{g}\big[[Y]\big]$.
For any $a \in \mathsf{H}^0(V, \mathcal{A})$ and any $b \in \mathsf{H}^0(F, \mathcal{A})$ we have

$$\big\langle \alpha(a), b \big\rangle = \big\langle a, \beta(b) \big\rangle$$

since they may be computed on the level of the pullback of $\mathcal{A}$ to the generic point of C. But since the restriction of that element to $F \bigcap V$ is regular and dY is a global 1-form, the residue theorem implies that

$$\mathsf{res}_y\Big(\big\langle \alpha(a), b \big\rangle dY\Big) = -\mathsf{res}_x\Big(\big\langle a, \beta(b) \big\rangle dY\Big).$$

Given a dual basis in $\mathsf{H}^0(V, \mathcal{A})$ of the chosen topological basis, the corresponding dual basis in $\widehat{\mathcal{A}_y}$ to the chosen basis of $\mathsf{H}^0(F, \mathcal{A})$ is thus given by an application of α (note that the sign cancels with the other sign introduced above). But since the composite of α with $\widehat{\mathcal{A}_y} \subseteq Q\big(\widehat{\mathcal{A}_y}\big)$ is just given by the canonical map $\mathfrak{g}(\mathsf{r}) \to \mathfrak{g}\big((Y^{-1})\big)$ and both Stolin's r-matrix and the rational solution r we started with are given by considering the sum of the tensor product of the basis with its dual basis, the two solutions coincide. $\square$

Corollary 3.3. *We have $K[Y^{-2}, Y^{-3}] \subseteq A$. In other words, C can always be taken to be the cuspidal cubic $V(Y^2Z - X^3)$.*

Proof. We have to show that $Y^{-2}, Y^{-3} \in A$, which means showing that multiplication by these elements maps $\mathsf{H}^0(V, \mathcal{A}) = \mathfrak{g}(\mathsf{r})$ into itself by Mulase's

Krichever correspondence. Let $a, a' \in \mathsf{H}^0(V, \mathcal{A})$ and let $i \in \mathbb{N} \setminus \{0, 1\}$. Then since

$$\mathsf{res}_y\Big(\langle \alpha(a), \alpha(a')\rangle dY\Big) = -\mathsf{res}_x\Big(\langle a, a'\rangle dY\Big)$$

and because $\mathsf{H}^0(V, \mathcal{A})$ is lagrangian, it is sufficient to show that $\widehat{\mathcal{A}_x}$ is closed under multiplication by Y^{-i}.

But this is true by a result of Stolin (see section one of [43]), which states that up to some automorphism induced by the simply-connected Lie group of $\mathfrak{g}$, $\widehat{\mathcal{A}_y}$ contains $Y^{-1}\mathfrak{g}\big[[Y^{-1}]\big]$ since it is lagrangian by the previous theorem. $\square$

Here is the lemma, which was used in the proof of the previous theorem.

Lemma 3.18. *The local ring $R = \mathcal{O}_{C,y}$ contains $Y^{-d}K[Y^{-1}]$ for some natural number $d >> 0$.*

Proof. Let $S = \mathsf{H}^0(V, \mathcal{O}_C)$. Since the normalisation of C is $\mathbb{P}^1$, we can write $Y^{-1} = \frac{r}{s}$ for some elements $r, s \in S \setminus \{0\}$. Since $K[Y^{-1}]$ is factorial, we may write $r = P_1 \dots P_l Y^{n-1}$ and $s = P_1 \dots P_l Y^n$ for some prime elements $P_i \in K[Y^{-1}]$ and some $n \leq 0$. Now once we localise at the singular maximal ideal of S, all the P_i become units of R, since C and $\mathbb{P}^1$ are isomorphic away from y. This proves that R contains Y^{n-1} and Y^n and since n and $n-1$ are coprime, the claim follows. $\square$

We can also describe the Stolin decomposition in terms of the decomposition we started with.

Lemma 3.19. *Given a decomposition $\mathfrak{g}\big((Y)\big) = \mathfrak{g}\big[[Y]\big] \oplus \mathsf{H}^0(C \setminus \{x\}, \mathcal{A})$, the associated Stolin decomposition $\mathfrak{g}\big((Y^{-1})\big) = \widehat{\mathcal{A}_y} \oplus \mathfrak{g}[Y]$ is given by*

$$\widehat{\mathcal{A}_y} = \textit{power series in elements of } \mathsf{H}^0(C \setminus \{x\}, \mathcal{A}) \textit{ of decreasing degree.}$$

It is meant here, that since $\mathsf{H}^0(C \setminus \{x\}, \mathcal{A}) \subseteq \mathfrak{g}[Y, Y^{-1}]$ it makes sense to consider power series (in Y^{-1}) of its elements as long as only finitely many of these elements are non-zero in any given degree.

Proof. Note that the right hand side of the claimed equality is a subspace of the left hand side, since there is a canonical map $\mathsf{H}^0(C \setminus \{x\}, \mathcal{A}) \to \widehat{\mathcal{A}_y}$. Given any element $f \in \mathfrak{g}\big((Y^{-1})\big)$, we will write it as a sum of an element in

$\mathfrak{g}[Y]$ and an element of the right hand side, from which the claimed equality will follow. To achieve this, we need only consider elements of the form

$$f = \sum_{i=-n}^{-1} f_i Y^i,$$

since $Y^{-N}\mathfrak{g}[Y^{-1}] \subseteq \mathsf{H}^0\big(C \setminus \{x\}, \mathcal{A}\big)$ for some $N \geq 1$. But an element f of this sort can also be thought of as being an element of $\mathfrak{g}\big((Y)\big)$, where it can be written as $f = f_1 + f_2$ for some $f_1 \in \mathfrak{g}\big[[Y]\big]$ and $f_2 \in \mathsf{H}^0\big(C \setminus \{x\}, \mathcal{A}\big)$. Since f and f_2 only admit finitely many non-zero terms with positive powers of Y, the same must be true for f_1, which is therefore an element of $\mathfrak{g}[Y]$ finishing the proof. □

Example 3.4. *Either by hand or by using the previous lemma, one can show that the Stolin triple associated to the sheaf of Lie algebras constructed in Example 3.2 is just*

$$W = Y^{-1}\mathfrak{g}\big[[Y^{-1}]\big].$$

Next we deal with a geometrisation of gauge equivalence.

Proposition 3.2. *Let $\mathsf{r}_1(x, y)$ and $\mathsf{r}_2(x, y)$ be rational solutions of the CYBE and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the associated sheaves of Lie algebras on the cuspidal cubic curve C. Then the following are equivalent:*

- *The solutions r_1 and r_2 are gauge equivalent, i.e. there exists $\phi \in \mathsf{Aut}(\mathfrak{g}[T])$ such that*
$$\mathsf{r}_2 = \phi \otimes \phi(\mathsf{r}_1(x, y)).$$
- *There is an isomorphism of sheaves of Lie algebras $\mathcal{A}_1 \cong \mathcal{A}_2$.*

Proof. If the solutions are gauge equivalent and we fix an associated polynomial map ϕ, then a theorem of Stolin (see [43]) tells us that the associated Stolin decompositions are mapped to each other via ϕ.
Hence Grothendieck's fpqc-descent applied to morphisms of (quasi-)coherent sheaves implies that there exists an isomorphism of Lie algebras $\mathcal{A}_1 \cong \mathcal{A}_2$ which restricts to $\phi : \mathfrak{g}(\mathsf{r}_1) \cong \mathfrak{g}(\mathsf{r}_2)$ over $C \setminus \{x\}$.
On the other hand the second condition implies the first by Lemma 1.10. □

Remarks. *1. In the first part of the proof, one can circumvent using Stolin's result by showing by hand that a morphism ϕ as in the theorem takes $\mathfrak{g}(\mathsf{r}_1)$ bijectively to $\mathfrak{g}(\mathsf{r}_2)$.*

2. *The theorem could be stated more precisely by noticing that the same morphism (suitably interpreted) can be used for both statements. This more precise version follows from the proof.*

Finally, we can recover Stolin's theorem on Stolin decompositions geometrically. We have already described the maps involved and as one might suspect their bijectivity is essentially given by fpqc-descent.

Geometric proof of Stolin's theorem. We consider the assignment of Lemma 3.11 which associates to any rational, unitary solution of the CYBE a lagrangian Lie subalgebra $A \subseteq \mathfrak{g}\big((T)\big)$ such that $A \oplus \mathfrak{g}\big[[T]\big] = \mathfrak{g}\big((T)\big)$ and such that there exists a natural number N with $T^{-N}\mathfrak{g}[T^{-1}] \subseteq A \subseteq \mathfrak{g}(T)$. Fixing an orthonormal basis $\{G_i\}$ of $\mathfrak{g}$, we can recover the rational solution from the Lie algebra by considering the formal sum $\sum_{i,n} G_i T^n \otimes W_{i,n}$, where $\{W_{i,n}\}$ denotes the dual basis of $\{G_i T^n\}$.
Furthermore, starting with any such A, we can produce such a formal sum, since A is lagrangian and a vector space complement to $\mathfrak{g}\big[[T]\big]$ and the resulting formal sum will be rational (since $T^{-N}\mathfrak{g}[T^{-1}] \subseteq A \subseteq \mathfrak{g}(T)$). Applying the assignment of Lemma 3.11 to the formal sum, will return A, since it will clearly produce a subvectorspace of A which is a complement to $\mathfrak{g}\big[[T]\big]$ and hence the formal sum will also be unitary. That it satisfies the CYBE follows from the proof of Theorem 3.3.
Summing up, we have shown that the map sending r to $\mathfrak{g}(\mathsf{r})$ is a bijection between rational, unitary solution of the CYBE and lagrangian Lie subalgebras $A \subseteq \mathfrak{g}\big((T)\big)$ such that $A \oplus \mathfrak{g}\big[[T]\big] = \mathfrak{g}\big((T)\big)$ and such that there exists a natural number N with $T^{-N}\mathfrak{g}[T^{-1}] \subseteq A \subseteq \mathfrak{g}(T)$.
Next, notice that if X denotes a curve and $x \in X$ is a closed point, then the canonical map

$$\big(X \setminus \{x\}\big) \,\dot{\cup}\, \mathsf{spec}\big(\widehat{\mathcal{O}_{X,x}}\big) \to X$$

is a covering in the fpqc-topology. Moreover the fibre product of $X \setminus \{x\}$ and $\mathsf{spec}\big(\widehat{\mathcal{O}_{X,x}}\big)$ over X is given by $\mathsf{spec}\big(Q(\widehat{\mathcal{O}_{X,x}})\big)$ (the proof of this is essentially the same as that of Lemma 3.1).
Therefore Grothendieck's fpqc-descent provides us with an equivalence of tensor categories between the category of coherent sheaves on X and the category of triples of coherent sheaves on $X \setminus \{x\}$ and $\mathsf{spec}\big(\widehat{\mathcal{O}_{X,x}}\big)$ (coming from $\mathsf{spec}\big(\mathcal{O}_{X,x}\big)$) with a fixed isomorphism over $\mathsf{spec}\big(Q(\widehat{\mathcal{O}_{X,x}})\big)$, which restricts to an equivalence of the category of Lie algebra objects on both sides. If we take C to be the cuspidal cubic and x the smooth point we have

used before, the categories of objects of Lie algebras $\mathcal{A}$ with a fixed isomorphism $\mathcal{A}\big(\mathsf{spec}(\widehat{\mathcal{O}_{X,x}})\big) \cong \mathfrak{g}\big[[T]\big]$ are also equivalent. Under this equivalence the subcategory of those sheaves with vanishing cohomology corresponds to the subcategory of those tuples such that their sum inside the pullback to $\mathsf{spec}\big(Q(\widehat{\mathcal{O}_{X,x}})\big)$ is direct by Theorem 3.1. The same is true of the subcategories of those tuples which furthermore satisfy $T^{-n}\mathfrak{g}[T^{-1}] \subseteq \mathcal{A}\big(X \setminus \{x\}\big)$ and those such that furthermore $\mathcal{A}(F) = \mathfrak{g}[T]$ by Lemma 3.13. If we now also demand that $\mathcal{A}\big(X \setminus \{x\}\big) \subseteq \mathfrak{g}\big((T)\big)$ is lagrangian (for both categories) and pass to isomorphism classes, we end up with a bijection of sets between lagrangian Lie subalgebras $A \subseteq \mathfrak{g}\big((T)\big)$ such that $A \oplus \mathfrak{g}\big[[T]\big] = \mathfrak{g}\big((T)\big)$ and such that there exists a natural number N with $T^{-N}\mathfrak{g}[T^{-1}] \subseteq A \subseteq \mathfrak{g}(T)$ (note that A is always a $K[T^{-2}, T^{-3}]$-module by Corollary 3.3 and that its extension of scalars to $K\big((T)\big)$ is canonically isomorphic to $\mathfrak{g}\big((T)\big)$ because its extension to $K(T)$ is already canonically isomorphic to $\mathfrak{g}(T)$ by the assumption $T^{-N}\mathfrak{g}[T^{-1}] \subseteq A \subseteq \mathfrak{g}(T)$) and isomorphism classes of sheaves of Lie algebras $\mathcal{A}$ on the cuspidal cubic C with a fixed isomorphism $\mathcal{A}(F) = \mathfrak{g}[T]$, vanishing sheaf cohomology and such that $\widehat{\mathcal{A}_x} \subseteq Q\big(\widehat{\mathcal{A}_x}\big) = \mathfrak{g}\big((T)\big)$ is lagrangian.
The same kind of reasoning can be applied if we take y as the closed point instead. We will not provide full details again, but just sketch which properties correspond to each other: Theorem 3.4 provides a way to turn any such sheaf of Lie algebras $\mathcal{A}$ into a Stolin decomposition. Again, descent provides a map in the other direction, which is inverse to the one just described and we only need to check whether its image is the set of isomorphism classes of sheaves of coherent Lie algebras $\mathcal{A}$ on the cuspidal cubic C with a fixed isomorphism $\mathcal{A}(F) = \mathfrak{g}[T]$, vanishing sheaf cohomology and such that $\widehat{\mathcal{A}_x} \subseteq Q(\widehat{\mathcal{A}_x}) = \mathfrak{g}\big((T)\big)$ is lagrangian.
Clearly the result will be a sheaf of Lie algebras and $\mathcal{A}(F) = \mathfrak{g}[T]$ is true by one of the assumption on Stolin decompositions. The vanishing of the sheaf cohomology of $\mathcal{A}$ follows from Theorem 3.1 and another of the assumptions on Stolin decompositions, while the property of being lagrangian follows from the proof of Corollary 3.3 and the remaining assumption on being a Stolin decomposition. Finally, the condition $\widehat{\mathcal{A}_y}$ contains $Y^{-N}\mathfrak{g}[[Y^{-1}]]$ for some N is implied by Theorem 3.4 while it implies the coherence of $\mathcal{A}$ by Lemma 3.18.
Putting the steps discussed above together proves Stolin's theorem. $\square$

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Veröffentlichung

Orlov's Equivalence and Maximal Cohen-Macaulay Modules over the Cone of an Elliptic Curve, *Mathematische Nachrichten* Vol.287, Issue 13/2014 , page 1438-1455.