Closed Trajectories in Planar Piecewise-Smooth Systems of Liénard-Type with a Line of Discontinuity

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Contents

1. Introduction

In this thesis we consider planar piecewise-smooth systems of Liénard-type with a line of discontinuity, i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} & \text{, if } x > 0 \\ \begin{pmatrix} y - F^-(x) \\ -g^-(x) \end{pmatrix} & \text{, if } x < 0, \end{cases} \tag{1.1}$$

where $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ are smooth functions for $x \geq 0$ and $x \leq 0$, respectively.

Our aim is to analyse the dynamical and bifurcation behaviour of system (1.1). First of all we identify the motivation for studying this kind of systems. For this we break system (1.1) down into its component pieces, which are "piecewise-smooth" and "piecewise-smooth of Liénard-type". In the first section we motivate the analysis of general piecewise-smooth systems and in the second section we identify the motivation for the analysis of piecewise-smooth systems of Liénard-type. After discussing the already considered bifurcation phenomena for non-smooth systems by other authors the objective of this thesis is given: the analysis of the piecewise-smooth system (1.1) in order to get *analytical and global results on dynamical and bifurcation behaviour.* We finally give an outline of the thesis.

1.1. Motivation for the analysis of piecewise-smooth systems

There are many applications in different parts of science, in particular in mechanics and engineering, which are modelled by piecewise-smooth systems of ordinary differential equations. These systems are given by

$$\dot{x} = f(x, \mu)$$

with $f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ being smooth on a finite number of domains G_i, $i = 1, ..., N$, and losing smoothness on the boundaries $M_{i,j}$ between adjacent domains G_i and G_j. To be more precise, we have the restriction $f(x) = f_i(x)$ if $x \in G_i$, $i = 1, ..., N$, with f_i smooth on $\overline{G_i}$. Thereby, $\mu \in \mathbb{R}^m$ is a parameter vector and $x \in \mathbb{R}^n$ the state vector, see [Filippov,1988]. Famous examples of applications, which are modelled by piecewise-systems, are mechanical systems with dry friction between two surfaces. This may be the noise of a squeaking chalk on a blackboard or the sound of a violin.

More relevant applications include noise generation in railway wheels, chattering of machine tools or squealing car brakes, see [Bishop et al.,1995], [Kunze & Kuepper,1997] or [Rudolph & Popp,2001]. Other examples are electric circuits, which include switching of a system component (diode, transistor, etc.), and are therefore modelled by piecewise-smooth systems, see [Stoker,1950] or [Andronov et al.,1966]. Moreover, piecewise-smooth systems have a long tradition in control theory, see [Flügge-Lotz,1947], [Lefschetz,1965], [Andronov et al.,1966], [Sontag,1990] or [Utkin,1992], where relay feedback is one of the most commonly used control techniques. These systems use switching components and are thus non-smooth. Examples are velocity control units of electric motors, course controllers for water torpedoes, electro mechanic temperature controllers or automatic course controllers for an aircraft, see [Popow,1958]. These systems tend to undesirable oscillations without external excitation. Although the detection of these self-oscillations have been of great interest for more than a century, there are still unsolved problems. It has been shown that piecewise-smooth systems exhibit a richness of different dynamical behaviour and bifurcations. Many of these phenomena are generated by the interaction between the system trajectories and the boundaries of the different domains of phase space, where the system is non-smooth. One particularly interesting type of solutions, which only exist for systems with discontinuous right-hand sides, are the so-called sliding motion solutions. They occur if trajectories do not cross the set of discontinuity, but continue their motion within it. For control theory the application of sliding motion is of great importance, in particular, for the control of electrical motors, see [Utkin,1992].

The last decades have witnessed an explosive development in the theory of smooth dynamical systems, see for instance [Guckenheimer & Holmes,1983], [Arrowsmith & Place,1990], [Troger & Steindl,1991], [Hale & Kocak,1991], [Perko,1991], [Seydel,1994] or [Kuznetsov,1998], but a number of real-world systems controlled by switching actions cannot be explained in terms of standard bifurcations of smooth systems. From a mathematical point of view, piecewise-smooth systems are not easy to handle, because their right-hand sides are not differentiable or even discontinuous. Since many concepts of classical dynamical systems and bifurcation theory do rely on smoothness, it is necessary to find a different concept to analyse piecewise-smooth systems.

1.2. Motivation for the analysis of piecewise-smooth systems of Liénard-type

For smooth systems Hilbert's 16th problem (third part) is a main source of motivation to study the number of periodic solutions of planar systems with polynomial nonlinearity. At the turn of the 19th century, the famous mathematician David Hilbert presented a list of 23 outstanding problems at the Second International Congress of Mathematicians. Hilbert's 16th problem asks for the maximum number of periodic solutions of a planar system with an polynomial nonlinearity of degree n, see [Hilbert,1902]. In the simplest case, where $n = 2$, it is still an open problem although Hilbert's 16th problem has generated much interesting mathematical research in recent years, see [Ilyashenko,2002]. In case of transcendental nonlinearities the problem is even more difficult to solve.

In 1928 the French physicist A. Liénard studied a special planar system, the so-called Liénard-system, of the form

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x) \end{aligned} \tag{1.2}$$

with $F, g : \mathbb{R} \longrightarrow \mathbb{R}$ smooth, $x, y \in \mathbb{R}$, which is equivalent to the second-order equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.3}$$

where $f(x) = F'(x)$. He determined the exact number of periodic solutions with certain conditions for the functions F and g, see [Liénard,1928] or [Ye,1986].

Equation (1.3) includes the famous *van der Pol equation (1926)*

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

with $\mu \in \mathbb{R}$, $g(x) = x$ and $F(x) = \frac{1}{3}\mu x^3 - \mu x$ of an electrical circuit with a triode valve, see for example [Guckenheimer & Holmes,1983]. Another special case of equation (1.3) is the *Duffing equation (1918)*

$$\ddot{x} + \delta\dot{x} - \beta x + x^3 = 0$$

with $\delta, \beta \in \mathbb{R}$, $g(x) = -\beta x + x^3$ and $F(x) = \delta x$. It describes a nonlinear oscillator with a cubic stiffness term to derive the hardening spring effect observed in many mechanical problems, see for example [Guckenheimer & Holmes,1983].

Control theory or other parts of engineering deal with many applications that can be written in the form of a piecewise-smooth system of Liénard-type, which means as system (1.2) with piecewise-smooth functions g and F. A control system is given by a system of the form

$$\begin{aligned} \dot{z} &= kz - \xi b \\ \dot{\xi} &= \varphi(\sigma) \\ \sigma &= cz - \rho\xi \end{aligned} \tag{1.4}$$

with $k, c, b, \rho, x, \xi, \sigma \in \mathbb{R}$, see [Lefschetz,1965]. The nonlinearity is caused by the characteristic function $\varphi(\sigma)$ of the control mechanism. σ is the so-called feedback signal. Characteristic functions may be smooth, continuous or even discontinuous. Typical examples of characteristic functions are given in Figure 1.1. Many examples use piecewise-constant characteristics like $\varphi(\sigma) = a \cdot \mathrm{sgn}(\sigma)$, $a \in \mathbb{R}$, see Figure 1.1(a). Transforming equation (1.4), we get a Liénard-system with $F(x)$ linear in x and $g(x) = C\varphi(x)$, $C \in \mathbb{R}$.

Mechanical or electronic nonlinear vibration systems are described by the second-order equation

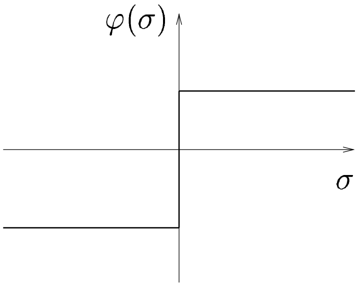
$$m\ddot{x} + \varphi(\dot{x}) + f(x) = 0 \tag{1.5}$$

where $m\ddot{x}$ is the inertia force, $-\varphi(\dot{x})$ the damping force and $-f(x)$ the restoring or spring force, see [Stoker,1950] or [Popow,1958]. Such equations arise, for example, in the case of a pendulum when damping forces are present, or in problems concerning unsteady motions of synchronous electrical machinery and in a variety of other physical problems. In the case of $f(x)$ linear in x, equation (1.5) can be transformed to a Liénard-system with $g(x)$ linear and $F(x) = C\varphi(x)$, $C \in \mathbb{R}$. As for control systems, the characteristic function φ can be smooth, continuous or even discontinuous. In [Giannakopoulos & Oster,1997], a mathematical model is considered for neural dynamics of a simple network consisting of two nerve cells of the form $\dot{u} = Au + b\varphi(u_1) + c$, with $u, b, c \in \mathbb{R}^2$ and A a 2×2 real matrix and a sigmoid function φ, see Figure 1.1(e). Another example with a smooth characteristic, corresponding to the type given in Figure 1.1(d), is an electrical circuit involving vacuum tubes, see [Stoker,1950, pp. 119-125]. A valve generator, assuming that the characteristic saturates, can be modelled by a continuous characteristic corresponding to the type given in Figure 1.1(b), see [Andronov et al.,1966, pp. 461-468]. Assuming a hard mode of excitation, the valve generator also can be modelled by a discontinuous characteristic corresponding to the type given in Figure 1.1(a), see [Andronov et al.,1966, pp. 468-480]. There are a number of well known cases of mechanical systems where self-excited oscillations, oscillations built up from an equilibrium in the absence of external forces, result from friction. For instance, modeling a block on a rough belt leads to a discontinuous characteristic as in Figure 1.1(c), see [Stoker,1950, pp. 126-127].

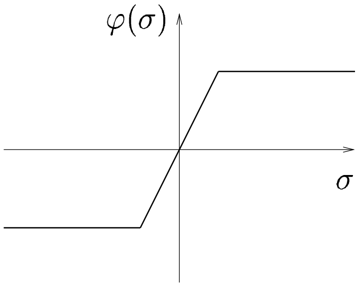
1.3. Literature survey

The fundamental work of Filippov, see [Filippov,1988], concerning non-smooth systems extends discontinuous differential equations to differential inclusions, i.e. differential equations whose right-hand sides are multi-valued. More results on differential inclusions can be found in [Aubin & Cellina,1984] or [Clarke,1998]. Results of more theoretical character concerning non-smooth dynamical systems, or multi-valued differential equations, are given in [Kunze,2000] and [Deimling,1992].

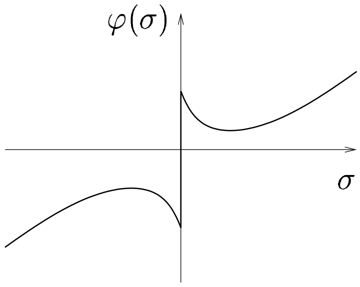
Because of the widespread interest in bifurcation theory for piecewise-smooth systems, researchers from different areas of science and engineering have independently developed different methods of studying these bifurcations. A general theory of bifurcations in piecewise-smooth systems is still lacking. Filippov has made a major contribution by characterising all types of local singular points in a planar system with a line of discontinuity and indicating their codimension one bifurcations, see [Filippov,1988]. In [Kuznetsov et al.,2002] this approach is continued by deriving a catalogue of global and local codimension one sliding bifurcations. Both use the classical approach of topological equivalence, see [Guckenheimer & Holmes,1983] or [Kuznetsov,1998], for defining bifurcations. Using this definition of bifurcation the appearance or disappearance of



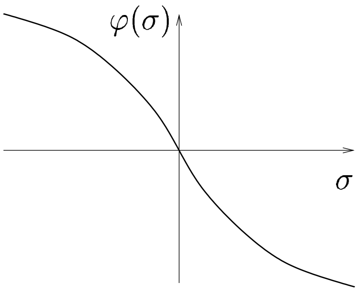
(a) Control system with on/off switch



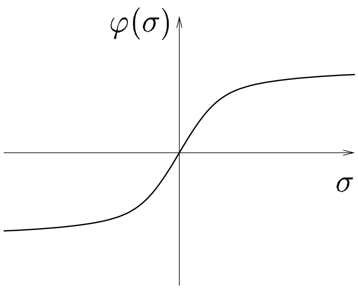
(b) Characteristic with saturation



(c) Typical characteristic for mechanical systems



(d) Electrical system involving vacuum tubes



(e) Sigmoid characteristic

Figure 1.1.: Typical characteristics φ

sliding motion at a particular parameter value is a bifurcation, even if it leaves the number of attractors of the system unchanged. In [Kuznetsov et al.,2002] only bifurcations from singular points on the line of discontinuity in consequence of sliding motion are considered. Normal forms for these bifurcations are given, but without proofs.
Leine et al. have defined the so-called *discontinuous bifurcations*, see [Leine et al.,2000] or [Leine,2000]. They have linearised the flow of both domains considering equilibria situated at the line of discontinuity. A discontinuous bifurcation occurs if there exists a convex combination of these linearisations containing an eigenvalue which crosses the imaginary axis. Using this approach the mathematical base and the reduction to the original piecewise-smooth system are not clear.
In Russian literature the so-called *C-bifurcations* are given, see [Feigin et al.,1999]. These bifurcations cannot be observed in smooth systems and include all bifurcations which can be explained in terms of interactions of equilibria or periodic solutions with the line of discontinuity. In [Dankowicz & Nordmark,1999] the so-called *grazing bifurcation* is considered using the concept of a discontinuity map. Another type of bifurcation, the *sliding bifurcation*, is considered in [di Bernardo et al.,2002]. This bifurcation occurs when a periodic solution without sliding motion contacts the line of discontinuity resulting in a periodic solution with sliding motion. With [di Bernardo et al.,2001] a first attempt to form a basis of a consistent theory of bifurcations of piecewise-smooth systems is made.

From an engineering point of view local solutions to control problems are often sufficient. Having in mind the linearisation principle of control theory "Designs based on linearisations work locally for the original system", see [Sontag,1990], piecewise-linear systems are of great importance in control theory. In [Freire et al.,1998] a piecewise-linear system is considered which is continuous but not smooth on the $y-$axis. They determine bifurcations and the existence of periodic, homoclinic or heteroclinic solutions. In [Llibre & Sotomayor,1996] the authors prove similar bifurcations for a piecewise-linear system $\dot{x} = Ax + \varphi(kx)b$ with A a 2×2 real matrix, $b, k, x \in \mathbb{R}^2$ and $\varphi(\xi) = \begin{cases} s\xi & , \text{if } |s\xi| \le 1 \\ 1 & , \text{if } |s\xi| > 1 \end{cases}$ with $s > 0$ a continuous piecewise-linear function.
In [Freire et al.,1999], the piecewise-linear system $\dot{x} = Ax + \varphi(x_1)b$ with A a 2×2 real matrix, $b, x \in \mathbb{R}^2$ and $\varphi(\xi) = \begin{cases} \mathrm{sgn}(\xi) & , \text{if } |\xi| > 1 \\ \xi & , \text{if } |\xi| \le 1 \end{cases}$ a continuous piecewise-linear function is considered. The authors prove a limit-cycle bifurcation, when the origin is a center. A. Teruel has presented a complete analysis of a continuous planar piecewise-linear system in his PhD thesis, see [Teruel,2000]. These three systems are continuous but not smooth on two straight lines and $\mathbb{Z}_2-$symmetric. However, sliding motion solutions cannot occur in these systems. Symmetric piecewise-linear systems $\dot{x} = Ax - \mathrm{sgn}(Cx)B$ with A a $n \times n$ real matrix and $B, C, x \in \mathbb{R}^n$ with $n > 2$ are considered in [di Bernardo, Johansson & Vasca,2001]. The main aim of this paper is to study the bifurcation phenomena leading to the formation of asymmetric periodic orbits and periodic orbits with sliding motion. They give analytical conditions for these bifurcations and determine numerically these bifurcations for a special system with $n = 3$.
There are more publications about piecewise-smooth planar systems with a line of

discontinuity. But their results are local and do not consider sliding motion solutions. The Liénard-system $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$ with $g(x) = x$ and $f(x, \dot{x})$ being discontinuous on a line is studied in [Coll et al.,1999] when the origin is a center. The authors have derived the general expression of the Lyapunov constants with either a vertical or horizontal straight line of discontinuity to study the stability of the origin. S. Moritz has proved a kind of Hopf-bifurcation for a planar piecewise-smooth system with a line of discontinuity in her diploma thesis. She assumes that the origin is a stable and an unstable focus for the systems in the first and the second half-plane, respectively, see [Moritz,2000] or [Küpper & Moritz,2001]. In [Küpper & Zou,2001(1)] a more general system is considered and in [Küpper & Zou,2001(2)] the system is discontinuous on the two axes. It is assumed that the linear parts of the piecewise-smooth system in all domains are given in canonical normal form. It is not for sure what will happen if they are not given in canonical normal form which is the case in general.

There exists so far no complete analysis of a planar piecewise-smooth system with a line of discontinuity including all bifurcations with and without sliding motion, not even in the case of a piecewise-linear system. In [Giannakopoulos & Pliete,2001] and [Giannakopoulos & Pliete,2002] we have derived a complete analysis including bifurcations of periodic, homoclinic and heteroclinic solutions, with and without sliding motion, for a planar piecewise-linear system with $\mathbb{Z}_2-$symmetry which is discontinuous on the $y-$axis. This system has the form $\dot{u} = Au + \text{sgn}(w^T u)v$ with A a 2×2 real matrix, $u, v, w \in \mathbb{R}^2$ and either $4\det(A) > (\text{tr}(A))^2$, see [Giannakopoulos & Pliete,2001], or $\det(A) < 0$, see [Giannakopoulos & Pliete,2002]. A first attempt for the analysis of all other cases $0 \leq 4\det(A) \leq (\text{tr}(A))^2$ can be found in [Pliete,1998].

1.4. Objective of the thesis

There are several reasons for considering piecewise-smooth planar systems of Liénard-type with a line of discontinuity. One main reason is that this type of systems arises from many applications of control theory, mechanics or engineering. Examples from control theory are velocity control units of electric motors and course or temperature controllers, see [Flügge-Lotz,1947], [Popow,1958], [Lefschetz,1965], [Andronov et al.,1966], [Sontag,1990] or [Utkin,1992]. Noise generation in railway wheels, chattering of machine tools or squealing of car brakes, see [Bishop et al.,1995], [Kunze & Kuepper,1997] or [Rudolph & Popp,2001], are typical examples for mechanical systems with dry friction between surfaces. Other applications are electric circuits including a switching component, see [Stoker,1950] or [Andronov et al.,1966].

It is a usual method to extend piecewise-smooth systems to differential inclusions, which means that the right-hand side of the non-smooth differential equation is replaced by a set-valued right-hand side. The existing theory on differential inclusions, see [Aubin & Cellina,1984], [Filippov,1988], [Deimling,1992], [Clarke,1998] or [Kunze,2000], provides some basic concepts. But methods as linearisation, Liénard's method, stability or bifurcation theory, which lead to analytical results in the case of smooth systems, are missing for piecewise-smooth systems due to the lack of smoothness.

In the smooth case, local approximation provides good properties. With polynomial nonlinearity equation (1.3) has been analysed very intensively. But even when f and g are polynomials of lower degree, a complete analysis without numerical methods cannot be presented, see for example [Khibnik et al.,1998] where f and g are polynomials of degree 2 and 3, respectively. Another approach is the approximation by transcendental functions. In [Giannakopoulos & Oster,1997] the authors approximate a sigmoid function by $\varphi(\sigma) = \frac{1}{1+\exp(-4\sigma)}$. They need local and numerical methods to determine all bifurcation sets. Not even in special cases a complete analysis can be given without numerical methods, see for example [Khibnik et al.,1998] or [Kooij & Giannakopoulos,2000].

When we approximate the smooth nonlinearity in equation (1.3) by piecewise-linear functions the corresponding phase portraits are similar. However, we lose local properties. But in return, we obtain global results without using numerical methods. The connection between piecewise-linear systems and their smooth counterparts is still an open mathematical problem. However, numerical calculations confirm the approach of using piecewise-linear systems.

1.5. Outline of the thesis

Before we start with the analysis of the dynamical and bifurcation behaviour of the piecewise-smooth system (1.1), we need to give a precise definition of solutions of piecewise-smooth systems. In chapter 2, we will first extend the piecewise-smooth system to a differential inclusion and define standard and sliding motion solutions. Afterwards we will define singular points, closed trajectories, with and without sliding motion, and the term bifurcation for piecewise-smooth systems.
Chapter 3 will deal with analytical results for the piecewise-smooth system. We will determine and characterise all singular points and verify the existence and uniqueness of the corresponding initial value problem in dependency on parameters. Afterwards we will present some lemmas on non-existence of closed trajectories assuming certain conditions for the functions $F^{\pm}$ and $g^{\pm}$. Assuming that the piecewise-smooth system is $\mathbb{Z}_2-$symmetric, we will be able to prove the existence of a unique closed trajectory without sliding motion. We will finally prove a local Hopf-like bifurcation of a periodic solution with sliding motion. All these results will be new to our knowledge.
One main tool for detecting closed trajectories in piecewise-smooth systems is the determination of appropriate discrete-time maps. We can define a Poincaré-map as the composition of these maps. The fixed points of the Poincaré-map correspond to the closed trajectories. We will obtain global results using this approach, which is our main goal. For piecewise-smooth systems it is in general not possible to analytically determine these discrete-time maps. But that is possible for piecewise-linear systems. For this reason we will consider piecewise-linear systems in chapter 4. After introducing the system and characterising all equilibria in $G_{\pm}$, we will derive the discrete-time maps with their domains and properties. With the aid of these maps we will obtain stronger results on non-existence of closed trajectories than in the case of piecewise-smooth systems. We will provide a general concept for determining closed trajectories. We will prove that the existence of a focus in G_+ or G_- is necessary for the existence of closed trajectories of type II and identify a global Hopf-like bifurcation of a periodic solution with sliding

motion. Furthermore, we will show that this periodic solution with sliding motion will become homoclinic and will disappear.
Only for determining closed trajectories of type I, we will have to consider 144 different cases depending on the eigenvalues of $A^{\pm}$. Assuming $\mathbb{Z}_2-$symmetry in the piecewise-linear system, we will be able to reduce the 144 to only 5 different cases. Thus, in chapter 5, we will derive a special normal form of the piecewise-linear system with $\mathbb{Z}_2-$symmetry. After summarizing all results on singular points and non-existence of closed trajectories, we will provide a general concept for the determination of closed trajectories of type I and their orbital stability. Then we will present complete analyses of dynamical and bifurcation behaviour in the 5 different cases including bifurcation diagrams. The analysis of two of these cases, the focus and the saddle case, have been published in a modified form in [Giannakopoulos & Pliete,2001] and [Giannakopoulos & Pliete,2002], respectively.

2. Preliminaries

We consider the planar piecewise-smooth (PWS) system of Liénard-type being discontinuous on the $y-$axis, i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \binom{y-F^+(x)}{-g^+(x)} & , \text{if } x > 0 \\ \binom{y-F^-(x)}{-g^-(x)} & , \text{if } x < 0, \end{cases} \tag{2.1}$$

where $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ are smooth functions for $x \geq 0$ and $x \leq 0$, respectively.

Our aim is to analyse system (2.1) in terms of equilibria, periodic, homoclinic or heteroclinic solutions and their bifurcations. For this we need to give a precise definition for solutions of PWS systems by introducing a differential inclusion corresponding to the PWS system, see [Filippov,1988] or [Aubin & Cellina,1984]. First we need some notation. We set

$$f^+(x,y) := \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix}, \text{ if } (x,y) \in \overline{G_+}$$
$$f^-(x,y) := \begin{pmatrix} y - F^-(x) \\ -g^-(x) \end{pmatrix}, \text{ if } (x,y) \in \overline{G_-}$$

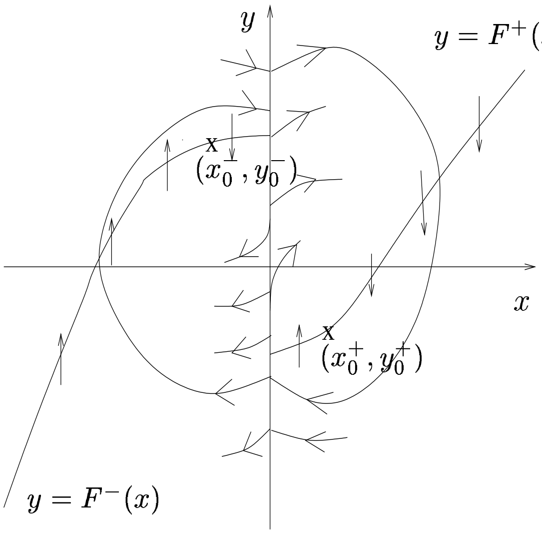
where

$$G_+ := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x > 0 \right\}$$
$$G_- := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x < 0 \right\}$$

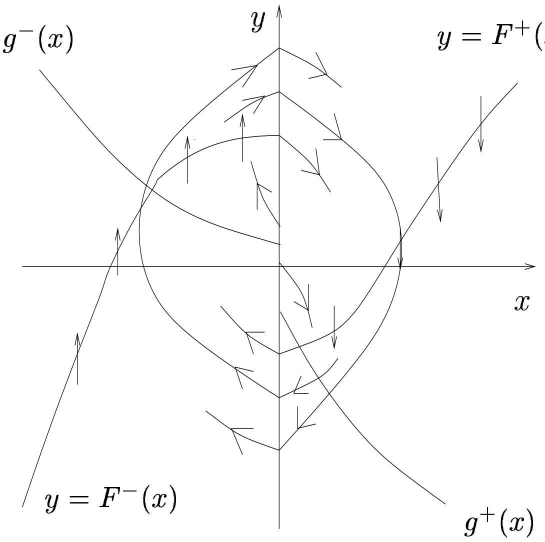
and $\overline{G_\pm}$ is the closure of $G_\pm$. Furthermore, we define the line of discontinuity

$$M := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x = 0 \right\}. \tag{2.2}$$

Example 2.1 (Phase portrait).
We will see in chapter 3 that we can analytically derive bifurcation phenomena for system (2.1) *if we assume, for example, that* F^+ *and* F^- *are strictly increasing functions for* $x > 0$ *and* $x < 0$*, respectively. The vector field is vertical (parallel to the* $y-$*axis) on the curves* $y = F^+(x)$ *and* $y = F^-(x)$ *for* $x > 0$ *and* $x < 0$*, respectively. The direction depends on the sign of* $g^\pm(x)$*. It changes if there exists a zero* x_0^+ *of* $g^+(x)$ *for* $x > 0$ *or* x_0^- *of* $g^-(x)$ *for* $x < 0$*, respectively. Then* $(x_0^\pm, y_0^\pm)$ *with* $y_0^\pm := F^\pm(x_0^\pm)$ *is an equilibrium of the PWS system* (2.1) *in* $G_\pm$*. Figure 2.1(a) shows a typical phase portrait if there exist such* $x_0^+ > 0$ *and* $x_0^- < 0$*, whereas in Figure 2.1(b) a typical phase portrait is shown if these numbers do not exist.*



(a) Two equilibria $(x_0^\pm, y_0^\pm)$ in $G_\pm$ (denoted by x)



(b) No equilibria in $G_\pm$

Figure 2.1.: Typical phase portrait with $F^\pm$ strictly increasing and $g^\pm$ strictly decreasing

On M system (2.1) is not defined. If for $(0, y) \in M$, the transversal components $f_1^+(0, y)$ and $f_1^-(0, y)$ (the subindex $_1$ denotes the first component of the corresponding vector) have the same sign, the orbit crosses M. But if $f_1^+(0, y)$ and $f_1^-(0, y)$ have opposite signs, the motion on M could be defined in different ways. The most natural definition is the *simplest convex definition* as in [Filippov,1988], which we use here. Another approach leading to the same result for equation (2.1) is Utkin's *equivalent control method*, see [Utkin,1992].
Replacing the right-hand side of equation (2.1) with the set-valued function

$$F(x, y) := \begin{cases} \{f^+(x, y)\} & \text{, if } (x, y) \in G_+ \\ \{\alpha f^+(0, y) + (1 - \alpha) f^-(0, y) : \alpha \in [0, 1]\} & \text{, if } (x, y) \in M \\ \{f^-(x, y)\} & \text{, if } (x, y) \in G_- \end{cases} \tag{2.3}$$

with $f^\pm(x, y)$ as above, we obtain the differential inclusion

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in F(x, y). \tag{2.4}$$

Note that $F(x, y)$ is the convex combination of $f^+(x, y)$ and $f^-(x, y)$ for $(x, y) \in M$.

Example 2.2 (Set-valued sgn−function).
The set-valued sgn−function is given by the convex combination of $\{-1\}$ *and* $\{1\}$ *for* $x = 0$*, i.e.*

$$Sgn(x) := \begin{cases} \{-1\} & \text{, if } x < 0 \\ [-1, 1] & \text{, if } x = 0 \\ \{1\} & \text{, if } x > 0, \end{cases}$$

see Figure 2.2.

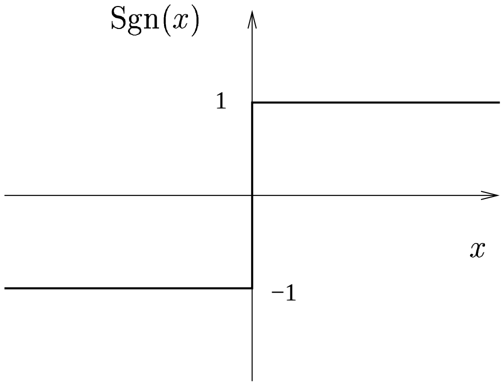


Figure 2.2.: Set-valued sgn−function

2.1. Definition of solutions

Following [Filippov,1988] we define a *solution of equation* (2.1) as an absolutely continuous function $\binom{x(t)}{y(t)}$ defined on an interval $J \subset \mathbb{R}$ which satisfies $\binom{\dot{x}(t)}{\dot{y}(t)} \in F(x,y)$ for almost all $t \in J$. We distinguish between two different types of solutions of equation (2.1). The first type consists of *standard solutions* which only have a finite number of intersection points with M, i.e. they cross M or they do not intersect M at all. The second type consists of *solutions with sliding motion* which intersect M and remain on M for a finite time or never leave M. We define the *sliding motion interval* I_s as the subset of M in which the vector field is not transversal, i.e.

$$I_s := \{0\} \times \{y \in \mathbb{R} : f_1^+(0,y) f_1^-(0,y) \leq 0\}.$$

I_s is called *attractive* (*repulsive*) if the vector fields of G_+ and G_- are both oriented towards (away from) I_s which is equivalent to $f_1^+(0,y) < 0$ and $f_1^-(0,y) > 0$ ($f_1^+(0,y) > 0$ and $f_1^-(0,y) < 0$) for all $(0,y) \in \overset{\circ}{I_s}$.
For each point $(0,y) \in I_s$ we define the following convex combination f^0 of the two vectors f^+ and f^-:

$$f^0(y) := \alpha f_2^+(0,y) + (1-\alpha) f_2^-(0,y) \text{ with } \alpha := \frac{f_1^-(0,y)}{f_1^-(0,y) - f_1^+(0,y)} \in [0,1]$$

Thereby, the subindex $_i$ denotes the ith coordinate of the corresponding vector $f^{\pm}(0,y)$. Note that $\left\{ \binom{0}{f^0(y)} \right\} = F(0,y) \cap M$, see equations (2.2) and (2.3). Thus, if $\phi(t)$ satisfies

$$\dot{y}(t) = f^0(y) \tag{2.5}$$

and $\binom{0}{y(t)} \in I_s$ for all $t \in J$, then $\binom{0}{\phi(t)}$ is a *sliding motion solution* of equation (2.1).

Lemma 2.3 (Sliding motion interval and vector field).
We set

$$\begin{aligned} b_1^{\pm} &:= -\, F^{\pm}(0) \\ b_2^{\pm} &:= -\, g^{\pm}(0). \end{aligned} \tag{2.6}$$

1. *a) If* $b_1^+ > b_1^-$ *then* $I_s = \{0\} \times [-b_1^+, -b_1^-]$ *and it is repulsive.*

   b) If $b_1^+ < b_1^-$ *then* $I_s = \{0\} \times [-b_1^-, -b_1^+]$ *and it is attractive.*

   c) If $b_1^+ = b_1^-$ *then* $I_s = \{(0, -b_1^+)\}$.

2. *If* $b_1^+ \neq b_1^-$ *then*

$$f^0(y) = \frac{b_2^+ - b_2^-}{b_1^- - b_1^+} y + \frac{b_1^- b_2^+ - b_1^+ b_2^-}{b_1^- - b_1^+}$$

   for all $(0, y) \in I_s$.

Proof. The assertions follow straight from the definitions above. ☐

2.2. Definition of singular points

In this section we give definitions for singular points of equation (2.1). In [Filippov,1988] all local singularities of two-dimensional autonomous systems with separated lines of discontinuity or a pointwise discontinuity are topologically classified. Because of the simple structure of equation (2.1) it is sufficient to consider two types of singular points, equilibria and tangential points.

Definition 2.4 (Equilibria, see [Filippov,1988]).
A point $(x_0, y_0) \in \mathbb{R}^2$ *is called* **equilibrium** *of equation* (2.1) *if*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in F(x_0, y_0)$$

with F *as in* (2.3).

Equilibria which belong to M are sometimes called *pseudo-equilibria*, see [Kuznetsov et al.,2002], or *quasi-equilibria*, see [Filippov,1988].

In addition to equilibria there is another type of singular points in PWS systems. These singular points form the boundary of the sliding motion interval where at least the vector field of one side is tangential. Of special interest are those tangential points at which a sliding motion solution can leave I_s or a solution from $G_\pm$ can reach it, see Figure 2.3.

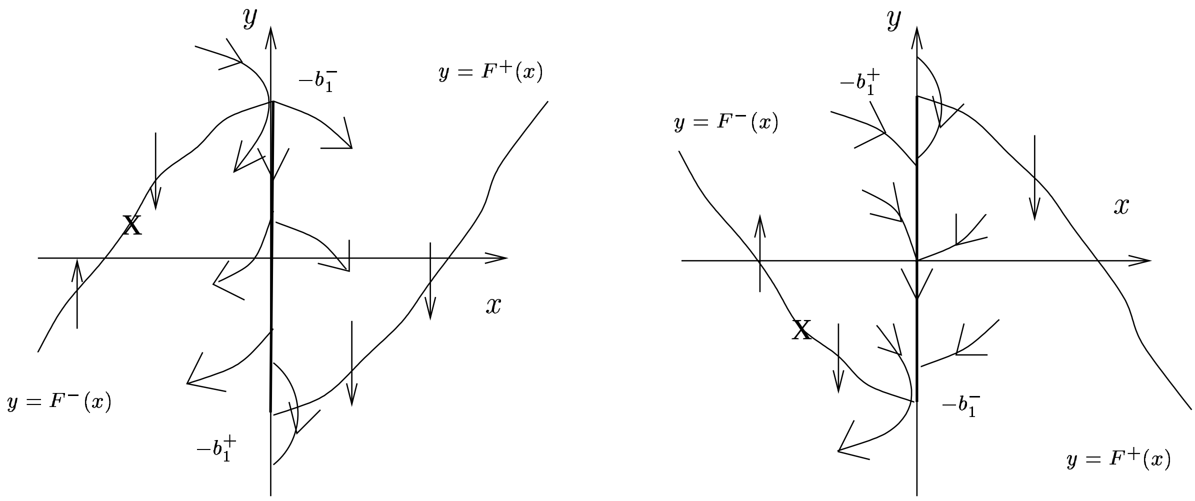
Definition 2.5 (Tangential, leaving, reaching points, see Figure 2.3).

1. *A point* $(0, y_t) \in M$ *is called* **tangential point of** $\mathbf{G_+}$ ($\mathbf{G_-}$) *of system* (2.1) *if* $f_1^+(0, y_t) = 0$ ($f_1^-(0, y_t) = 0$).

2. *A tangential point* $(0, y_t)$ *of* G_+ (G_-) *is called* **leaving point of** $\mathbf{G_+}$ ($\mathbf{G_-}$) *of system* (2.1) *if the following two conditions hold:*

   a) $f^0(y_t)(y - y_t) < 0$ *for a* $y \neq y_t$ *with* $(0, y) \in I_s$.

   b) There exists a $\delta > 0$ *such that for all* $(0, y) \in M \backslash I_s$ *with* $|y - y_t| < \delta$ *there holds* $f_1^+(0, y) > 0$ ($f_1^-(0, y) < 0$).



a) Reaching point of G_- b) Leaving point of G_-

Figure 2.3.: Examples of tangential points of $G_\pm$; x denotes an equilibrium

3. *A tangential point* $(0, y_t)$ *of* G_+ (G_-) *is called* **reaching point of** G_+ (G_-) *of system* (2.1) *if the following two conditions are fulfilled:*
   a) $f^0(y_t)(y - y_t) > 0$ *for a* $y \neq y_t$ *with* $(0, y) \in I_s$.
   b) *There exists a* $\delta > 0$ *such that for all* $(0, y) \in M \backslash I_s$ *with* $|y - y_t| < \delta$ *there holds* $f_1^+(0, y) < 0$ $(f_1^-(0, y) > 0)$.

In [Kuznetsov et al.,2002] tangential points are separated in another way, visible and invisible tangential points. A tangential point is a *visible* (*invisible*) point of $G_\pm$ if the trajectory starting at this point belongs to $G_\pm$ $(G_\mp)$ for all sufficiently small $|t| \neq 0$. For example, $(0, -b_1^+)$ is an invisible and $(0, -b_1^-)$ is a visible point of G_+ and G_-, respectively, in Figure 2.3 a). Note that leaving and reaching points are visible.

2.3. Definition of closed trajectories

In this section we give a definition for closed trajectories of system (2.1). These trajectories are closed curves in the plane. Let $(x(t), y(t))$ be a periodic solution of system (2.1). We denote the corresponding trajectory by γ. We distinguish between two different types of closed trajectories which can appear in PWS systems:

I) The closed trajectory γ lies in G_+ and G_- and crosses M twice, i.e. $\gamma \subset G_+ \cup M \cup G_-$, $\gamma \cap G_+ \neq \emptyset$, $\gamma \cap G_- \neq \emptyset$ and there exist $y_1 < y_2$, such that $\gamma \cap M = \{(0, y_1), (0, y_2)\}$.

II) The closed trajectory γ lies in $\overline{G_+}$ and/or $\overline{G_-}$ and the corresponding solution stays for a whole time-interval in M (it 'slides' along the $y-$axis). This means that $\gamma \subset \overline{G_\pm}$ or $\gamma \subset \overline{G_+} \cup \overline{G_-}$ and there exist at least $y_1 < y_2$ such that $[(0, y_1), (0, y_2)] \subset \gamma \cap M$.

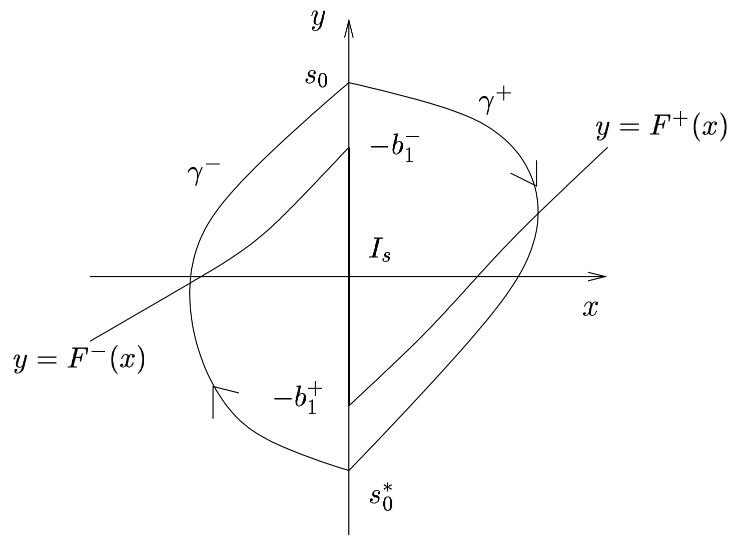


Figure 2.4.: Closed trajectory of type I

A closed trajectory of type I is characterised as follows. Starting from a point $(0, s_0)$ with $s_0 \in M^+$ the trajectory crosses M at a point $(0, s_0^*)$ with $s_0^* \in M^-$ after finite time t_0^+. Thereby, M^+ (M^-) is defined as the set of points $s_0 \in \mathbb{R}$ ($s_0^* \in \mathbb{R}$) with $s_0 > -b_1^+$ ($s_0^* < -b_1^-$) and every solution starting at $(0, s_0)$ ($(0, s_0^*)$) intersects M again after finite time. Then, the system changes the properties of flow, the trajectory continues on the other side of M until reaching it again at point $(0, s_1) \in M$ after finite time t_0^-. The trajectory is closed if we have $s_0 = s_1$, see Figure 2.4.
We can introduce appropriate discrete-time maps to characterise these solutions. Denote by $\gamma^\pm$ the part of γ in $G_\pm$. Then $\Pi^+ : M^+ \longrightarrow M^-$ is defined as the map which maps $(0, s_0)$ to $(0, s_0^*)$ along γ^+:

$$\begin{aligned} \Pi^+ : M^+ &\longrightarrow M^- \\ s_0 &\longmapsto s_0^* = y(t_0^+) \end{aligned}$$

Similarly, $\Pi^- : M^- \longrightarrow M^+$ is defined as the map which maps $(0, s_0^*)$ to $(0, s_1)$ along γ^-:

$$\begin{aligned} \Pi^- : M^- &\longrightarrow M^+ \\ s_0^* &\longmapsto s_1 = y(t_0^-) \end{aligned}$$

Finally, we define the 1-dimensional Poincaré-map as the composition of Π^+ and Π^-, $\Pi := \Pi^- \circ \Pi^+$. A fixed point s_0 of Π correspond to a closed trajectory of type I starting at $(0, s_0)$ with $s_0 \in M^+$. The Poincaré-map Π is smooth and therefore we can use it to consider existence and orbital stability of closed trajectories of type I.

We consider now closed trajectories of type II. The solution starting at a point $(0, s_0)$ with $s_0 \in M^+$ may intersect with I_s after finite time t_0^+ at $(0, \sigma_0)$. This can happen at any point of I_s if it is attractive. Otherwise, it can only happen at reaching points of

G_+. The system changes the properties of flow, the trajectory continues along I_s and the solution solves now the differential equation (2.5). If I_s is repulsive, the trajectory can leave I_s at any point $(0, \sigma_0^*)$. Otherwise, it can leave I_s only at leaving points of G_+ or G_-. We can introduce another discrete-time map

$$\begin{aligned} \Pi^0 : I_s &\longrightarrow I_s \\ \sigma_0 &\longmapsto \sigma_0^* = y(\tau^0) \end{aligned}$$

for the sliding motion solution, which maps $(0, \sigma_0)$ to $(0, \sigma_0^*)$ along γ^0, where γ^0 is the sliding motion part of γ. Then, γ leaves I_s after finite time τ^0 or it stays for all times $(\tau^0 = \infty)$ on I_s. Π^0 is defined by the solution $y(t)$ of system (2.5). A solution $\phi^0(t)$ of system (2.5) with initial value $y(0) = \sigma_0$, $(0, \sigma_0) \in I_s$, is given by

$$y(t) = \sigma_0 e^{\beta^0 t} - y_0$$

where

$$\begin{aligned} \beta^0 &:= \frac{b_2^+ - b_2^-}{b_1^- - b_1^+}, \\ y_0 &:= \frac{b_1^+ b_2^- - b_1^- b_2^+}{b_2^+ - b_2^-}. \end{aligned} \tag{2.7}$$

If $(0, y(t))$ leaves I_s after finite time $\tau^0 > 0$ at $(0, \sigma_0^*)$, then there holds

$$\Pi^0(\sigma_0) = \phi^0(\tau^0) = \sigma_0 e^{\beta^0 \tau^0} - y_0 = \sigma_0^*.$$

A closed trajectory of type II can be described as composition of appropriate discrete-time maps Π^+, Π^- and Π^0. Note that this composed Poincaré-map is continuous, but not differentiable at the points where the trajectory reaches or leaves I_s.

2.4. Definition of bifurcation

Our aim is to apply bifurcation analysis to the PWS system (2.1). The bifurcation theory for smooth systems is not applicable to PWS systems because their right-hand sides are not differentiable. However, PWS systems can undergo bifurcations which have similarities, but also discrepancies, with bifurcations of smooth systems.
We say that (x_0, y_0, μ_0) is a *bifurcation point* regarding the parameter μ of a PWS system if the number of attractors or the topological structure of the phase portrait changes when μ passes through μ_0. These can be bifurcations of solutions without sliding motion which have similarities with those of smooth systems. But in addition, there can occur bifurcations in consequence of sliding motion solutions which do not exist in this form for smooth systems.

3. Results for the piecewise-smooth system

In this chapter, we analytically derive results on dynamical and bifurcation behaviour of the PWS system (2.1). These results, to our knowledge, are new in this general form. As mentioned in the introduction other publications deal with special cases of system (2.1), piecewise-linear systems or system (2.1) with continuous functions $F^{\pm}$ and $g^{\pm}$. First, we determine and characterise all singular points of the discontinuity M in dependency on the parameters $b_1^{\pm}$ and $b_2^{\pm}$. Afterwards, we provide a lemma on uniqueness and existence of the corresponding initial value problem, again in dependency on $b_1^{\pm}$ and $b_2^{\pm}$. Section 3.3 summarizes results on non-existence of closed trajectories. We give necessary conditions for the existence of closed trajectories of type II. Assuming additional conditions for the functions $F^{\pm}$ and $g^{\pm}$ we get results on non-existence of closed trajectories in G_+, G_- or of type I. Section 3.4 deals with a Hopf-like bifurcation of a closed trajectory of type I of system (2.1) with the additional assumption of $\mathbb{Z}_2-$symmetry. In section 3.5 we finally prove a local Hopf-like bifurcation of a periodic solution with sliding motion which occurs when an equilibrium coincides with a leaving or reaching point.

3.1. Singular points

In this section we specify and characterise the singular points of system (2.1) in dependency on the parameters $b_1^{\pm}$ and $b_2^{\pm}$. First, we remark that there are equilibria $(x_0^{\pm}, y_0^{\pm})$ in $G_{\pm}$ if and only if we have $g^{\pm}(x_0^{\pm}) = 0$ and $y_0^{\pm} = F^{\pm}(x_0^{\pm})$ with $x_0^{+} > 0$ and $x_0^{-} < 0$, respectively. These equilibria lie in a domain where system (2.1) is smooth. Therefore, they can be characterised by the eigenvalues of the linearisation of the right-hand sides $f^{\pm}$ at $(x_0^{\pm}, y_0^{\pm})$. In $M \backslash I_s$ are no singular points because at these points the vector field is transversal to M. Therefore, we only consider I_s.

Lemma 3.1 (Equilibria of $\mathbf{I_s}$, see Figure 3.1).
Consider the PWS system (2.1) *with* $b_1^{\pm} := -F^{\pm}(0)$ *and* $b_2^{\pm} := -g^{\pm}(0)$.

1. *If* $b_2^{+} = b_2^{-} = 0$, *all points of* $\mathring{I}_s$ *are equilibria. They are unstable if* $b_1^{+} > b_1^{-}$ *and stable if* $b_1^{+} < b_1^{-}$.

2. *If* $b_2^{+} b_2^{-} < 0$, *there exists exactly one equilibrium* $(0, y_0) \in I_s$ *with*

$$y_0 := \frac{b_1^{+} b_2^{-} - b_1^{-} b_2^{+}}{b_2^{+} - b_2^{-}}. \tag{3.1}$$

a) If $b_2^+ > 0 > b_2^-$, then $(0, y_0)$ is a saddle point.

b) If $b_2^+ < 0 < b_2^-$ and $b_1^+ > b_1^-$, then $(0, y_0)$ is an unstable node.

c) If $b_2^+ < 0 < b_2^-$ and $b_1^+ < b_1^-$, then $(0, y_0)$ is a stable node.

3. In all other cases there are no equilibria in I_s.

Proof. $(0, y_0)$ is an equilibrium of I_s if $f^0(y_0) = 0$ and $(0, y_0) \in I_s$.

1. In case of $b_2^+ = b_2^-$ we get $f^0(y) = b_2^+$. If $b_2^+ \neq 0$ there are no equilibria. If $b_2^+ = 0$ all points of $\mathring{I}_s$ are equilibria. Due to the repulsion (attraction) of I_s for $b_1^+ > b_1^-$ ($b_1^+ < b_1^-$) the points of $\mathring{I}_s$ are unstable (stable).

2. In case of $b_2^+ \neq b_2^-$ there exists exactly one zero of f^0:

$$f^0(y_0) = 0 \quad \Leftrightarrow \quad y_0 = \frac{b_1^+ b_2^- - b_1^- b_2^+}{b_2^+ - b_2^-}$$

First, we verify for which conditions we have $(0, y_0) \in \mathring{I}_s$ and in that case we consider the flow in a neighbourhood of $(0, y_0)$, in order to characterise this equilibrium. For this it is necessary to consider four different cases.

(i) $b_1^+ > b_1^-$, $b_2^+ > b_2^-$:

There holds $(0, y_0) \in \mathring{I}_s$ if and only if $y_0 \in]-b_1^+, -b_1^-[$.

$$\begin{cases} y_0 & > -b_1^+ \\ y_0 & < -b_1^- \end{cases} \quad \Leftrightarrow \quad \begin{cases} b_1^+ b_2^- - b_1^- b_2^+ & > -b_1^+(b_2^+ - b_2^-) \\ b_1^+ b_2^- - b_1^- b_2^+ & < -b_1^-(b_2^+ - b_2^-) \end{cases} \quad \Leftrightarrow \quad \begin{cases} b_2^+ & > 0 \\ b_2^- & < 0 \end{cases}$$

Therefore, $(0, y_0) \in \mathring{I}_s$ if and only if $b_2^+ > 0 > b_2^-$. I_s is repulsive because of $b_1^+ > b_1^-$. We have $f^0(y_0) = 0$, and for the first derivative of f^0 we get

$$f^{0\prime}(y) = \frac{b_2^+ - b_2^-}{b_1^- - b_1^+} < 0$$

which means that f^0 is strictly decreasing. From this we obtain

$$f^0(y) \begin{cases} < 0 & , \text{if } y > y_0 \\ > 0 & , \text{if } y < y_0 \end{cases}$$

and consequently, $(0, y_0)$ is a saddle point, see Figure 3.1(i).

(ii) $b_1^+ > b_1^-$, $b_2^+ < b_2^-$:

Analogous to (i), there holds $(0, y_0) \in \mathring{I}_s$ if and only if $b_2^+ < 0 < b_2^-$ and then it is an unstable node, see Figure 3.1(ii).

(iii) $b_1^+ < b_1^-$, $b_2^+ > b_2^-$:

There holds $(0, y_0) \in \mathring{I}_s$ if and only if $y \in]-b_1^-, -b_1^+[$. This is, analogous to (i), equivalent to $b_2^+ > 0 > b_2^-$ and then it is a saddle point, see Figure 3.1(iii).

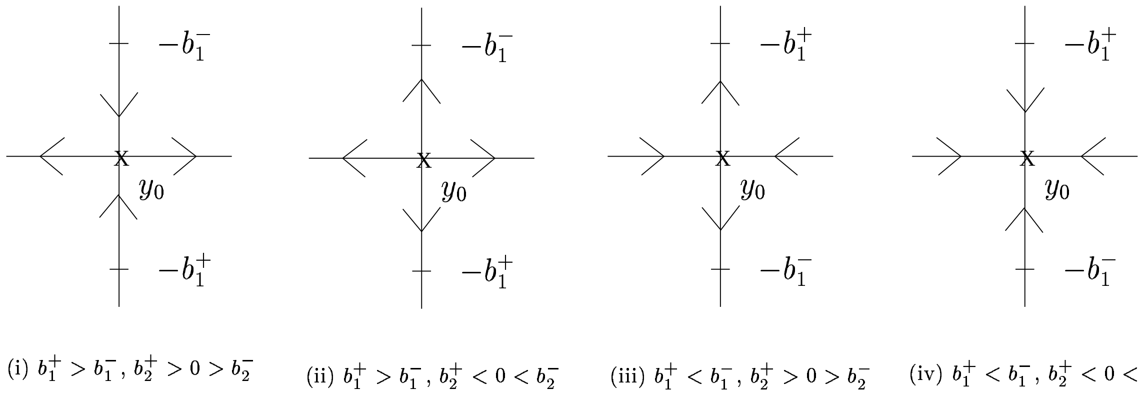


Figure 3.1.: Characterisation of the equilibrium $(0, y_0)$ in dependency on $b_1^\pm$ and $b_2^\pm$

(iv) $b_1^+ < b_1^-$, $b_2^+ < b_2^-$:

Analogous to (iii), there holds $(0, y_0) \in \mathring{I}_s$ if and only if $b_2^+ < 0 < b_2^-$ and then it is a stable node, see Figure 3.1(iv).

□

Now, we consider the tangential points of I_s and the conditions under which they are leaving or reaching points.

Lemma 3.2 (Tangential points, see Figure 2.3).
Consider the PWS system (2.1) with $b_1^\pm := -F^\pm(0)$.

1. *$(0, -b_1^+)$ is the unique tangential point of G_+.*
2. *$(0, -b_1^-)$ is the unique tangential point of G_-.*

Proof.

$$f_1^\pm(0, y_t) = 0 \quad \Leftrightarrow \quad y_t = F^\pm(0) = -b_1^\pm$$

□

Lemma 3.3 (Leaving and reaching points, see Figure 2.3).
Consider the PWS system (2.1) with $b_1^\pm := -F^\pm(0)$ and $b_2^\pm := -g^\pm(0)$.

1. *$(0, -b_1^+)$ is a leaving (reaching) point of G_+ if and only if $b_2^+ > 0$ and $b_1^+ < b_1^-$ $(b_1^+ > b_1^-)$.*
2. *$(0, -b_1^-)$ is a leaving (reaching) point of G_- if and only if $b_2^- < 0$ and $b_1^+ < b_1^-$ $(b_1^+ > b_1^-)$.*

Proof. Because of Lemma 3.2 we only need to consider the points $(0, -b_1^\pm)$:

1. We verify the conditions of Definition 2.5 for $(0, -b_1^+)$. For the conditions 2.a) and 3.a) we choose $y = -b_1^-$:

$$\begin{aligned} f^0(-b_1^+)(-b_1^- + b_1^+) < 0 \quad &\Leftrightarrow \quad b_2^+(b_1^+ - b_1^-) < 0 \\ f^0(-b_1^+)(-b_1^- + b_1^+) > 0 \quad &\Leftrightarrow \quad b_2^+(b_1^+ - b_1^-) > 0 \end{aligned}$$

For verifying the conditions 2.b) and 3.b) we choose a small $\delta > 0$ and $y = -b_1^+ + c$ with

$$c \in \begin{cases}]0, \delta[& , \text{if } b_1^+ < b_1^- \\ ]-\delta, 0[& , \text{if } b_1^+ > b_1^- . \end{cases}$$

Then there holds $(0, y) \in M \backslash I_s$ and $|-b_1^+ - y| = |c| < \delta$:

$$\begin{aligned} f_1^+(0,y) > 0 \;&\Leftrightarrow\; y + b_1^+ > 0 \;\Leftrightarrow\; c > 0 \;\Leftrightarrow\; b_1^+ < b_1^- \\ f_1^+(0,y) < 0 \;&\Leftrightarrow\; y + b_1^+ < 0 \;\Leftrightarrow\; c < 0 \;\Leftrightarrow\; b_1^+ > b_1^- \end{aligned}$$

Altogether, we get the assertion, see Figure 2.3.

2. We verify the conditions of Definition 2.5 for $(0, -b_1^-)$ analogous to 1., by choosing $y = -b_1^+$ for the conditions 2.a) and 3.a). For verifying condition 2.b) and 3.b) we choose a small $\delta > 0$ and $y = -b_1^- + c$ with

$$c \in \begin{cases}]0, \delta[& , \text{if } b_1^- < b_1^+ \\ ]-\delta, 0[& , \text{if } b_1^- > b_1^+ \end{cases} .$$

Altogether, we get the assertion.

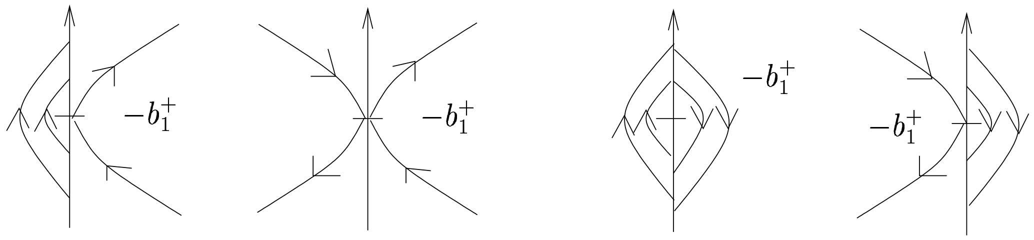
□

3.2. Existence and uniqueness of the initial value problem

In this section we consider the initial value problem corresponding to system (2.1) with initial values in $(x_0, y_0) \in \mathbb{R}^2$ concerning existence and uniqueness in dependency on the parameters $b_1^\pm$ and $b_2^\pm$.

Lemma 3.4 (Existence and Uniqueness).
Consider the PWS system (2.1) with $b_1^\pm := -F^\pm(0)$ and $b_2^\pm := -g^\pm(0)$.

1. *If $b_1^+ < b_1^-$, system (2.1) with initial values $(x_0, y_0) \in \mathbb{R}^2$ has a unique solution.*
2. *If $b_1^+ > b_1^-$, system (2.1) with initial values*
   a) *$(x_0, y_0) \in \mathbb{R}^2 \backslash I_s$ has a unique solution,*
   b) *$(x_0, y_0) \in \mathring{I}_s$ has at least three solutions,*
   c) *$(x_0, y_0) = (0, -b_1^+)$ $((x_0, y_0) = (0, -b_1^-))$ has at least two solutions, provided $b_2^+ \geq 0$ $(b_2^- \leq 0)$,*
   d) *$(x_0, y_0) = (0, -b_1^+)$ $((x_0, y_0) = (0, -b_1^-))$ has a unique solution, provided $b_2^+ < 0$ $(b_2^- > 0)$.*
3. *If $b_1^+ = b_1^-$, system (2.1) with initial value $(x_0, y_0) = (0, -b_1^+)$ has at least two solutions, provided $b_2^+ > 0$ and $b_2^- < 0$. In all other cases it has a unique solution, see Figure 3.2.*



$b_2^+ > 0$, $b_2^- > 0$ $\quad$ $b_2^+ > 0$, $b_2^- < 0$ $\quad$ $b_2^+ < 0$, $b_2^- > 0$ $\quad$ $b_2^+ < 0$, $b_2^- < 0$

Figure 3.2.: Vector field in a neighbourhood of $(0, -b_1^+)$ in dependency on the sign of $b_2^\pm$; $b_1^+ = b_1^-$

Proof. In all cases with initial values $(x_0, y_0) \in \mathbb{R}^2 \backslash I_s$ or in 3. with $(x_0, y_0) \neq (0, -b_1^+)$ the vector field is smooth if $(x_0, y_0) \in \mathbb{R}^2 \backslash M$ or transversal to M if $(x_0, y_0) \in M \backslash I_s$. It exists therefore a unique solution for these initial values and we only consider $(x_0, y_0) \in I_s$ in 1. and 2. and $(x_0, y_0) = (0, -b_1^+)$ in 3..

1. If $b_1^+ < b_1^-$ then I_s is attractive. Therefore, for initial values $(x_0, y_0) \in \overset{\circ}{I}_s$, there are no solutions which evolve in $G_\pm$. The only existing solution is the sliding motion solution.
   Consider now $(0, -b_1^+)$ $((0, -b_1^-))$ as initial value. In case of $b_2^+ > 0$ $(b_2^- < 0)$ this tangential point is a leaving point, see Lemma 3.3, and the only solution is the leaving solution in G_+ (G_-). In case of $b_2^+ < 0$ $(b_2^- > 0)$ only the sliding motion solution exists, and in case of $b_2^+ = 0$ $(b_2^- = 0)$ the equilibrium $(0, -b_1^+)$ $((0, -b_1^-))$ itself is the only solution.

2. If $b_1^+ > b_1^-$ then I_s is repulsive. Therefore, for initial values $(x_0, y_0) \in \overset{\circ}{I}_s$, in addition to the sliding motion solution exists a solution which evolves in G_+ and a different one which evolves in G_- exist.
   With initial value $(0, -b_1^+)$ $((0, -b_1^-))$ a solution evolves in G_- (G_+). $(0, -b_1^+)$ $((0, -b_1^-))$ is a reaching point if $b_2^+ > 0$ $(b_2^- < 0)$, see Lemma 3.3. This means that additionally the sliding motion solution exists, whereas in case of $b_2^+ < 0$ $(b_2^- > 0)$ the sliding motion solution does not exist. $(0, -b_1^+)$ $((0, -b_1^-))$ is an equilibrium and the only solution if $b_2^+ = 0$ $(b_2^- = 0)$.

3. For $b_1^+ = b_1^-$ we consider the vector field in a neighbourhood of the singular point $(0, -b_1^+)$ in four different cases in dependency on the sign of b_2^+ and b_2^-. There holds
   $$f_1^\pm(0, y) \begin{cases} > 0 & \text{, if } y > -b_1^\pm \\ < 0 & \text{, if } y < -b_1^\pm \end{cases} \quad \text{and} \quad f_2^\pm(0, y) = b_2^\pm.$$
   Considering the vector field in all different cases, see Figure 3.2, we can conclude that there are at least two solutions in case of $b_2^+ > 0$ and $b_2^- < 0$. In all other cases there exists a unique solution.

$\square$

3.3. Non-existence of closed trajectories

In this section we prove some lemmas on non-existence of closed trajectories. First, we give necessary conditions for the existence of closed trajectories of type II. Assuming additional conditions for the functions $F^{\pm}$ and $g^{\pm}$, we can prove lemmas on non-existence of closed trajectories in G_+, G_- or of type I.

Lemma 3.5 (Non-existence of closed trajectories of type II).
Consider the PWS system (2.1) *with* $b_1^{\pm} := -F^{\pm}(0)$ *and* $b_2^{\pm} := -g^{\pm}(0)$. *The conditions* $b_1^+ \neq b_1^-$ *and* $b_2^+ > 0$ *or* $b_2^- < 0$ *are necessary for the existence of closed trajectories of type II.*

Proof. The sliding motion interval I_s only exists for $b_1^+ \neq b_1^-$. It is repulsive (attractive) if $b_1^+ > b_1^-$ ($b_1^+ < b_1^-$), see Lemma 2.3. In case of $b_1^+ > b_1^-$ a solution of $G_{\pm}$ can only reach I_s at $(0, -b_1^{\pm})$. In case of $b_1^+ < b_1^-$ a solution can only leave I_s into $G_{\pm}$ at $(0, -b_1^{\pm})$. Therefore, $(0, -b_1^{\pm})$ must be a reaching or leaving point, which holds if and only if $b_2^+ > 0$ or $b_2^- < 0$, see Lemma 3.3. □

Lemma 3.6 (Non-existence of closed trajectories in $\mathbf{G_+}$).
Consider the PWS system (2.1). *Suppose that* F^+ *is strictly monotonous on* $]0, \alpha^+[$ *with* $\alpha^+ > 0$. *Then system* (2.1) *has no closed trajectory in* $]0, \alpha^+[\times\mathbb{R}$.

Proof. System (2.1) is smooth on G_+. Therefore, we can apply Bendixson's Criterion, see for instance [Guckenheimer & Holmes,1983], to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} = \begin{pmatrix} f_1^+(x,y) \\ f_2^+(x,y) \end{pmatrix}$$

for $(x,y) \in G_+$. Hence, because of

$$\frac{\partial f_1^+}{\partial x}(x,y) + \frac{\partial f_2^+}{\partial y}(x,y) = -F^{+\prime}(x) \neq 0$$

for all $x \in]0, \alpha^+[$, a closed trajectory cannot exist in $]0, \alpha^+[\times\mathbb{R}$. □

Analogous to Lemma 3.6 we can prove the non-existence of closed trajectories in G_-.

Lemma 3.7 (Non-existence of closed trajectories in $\mathbf{G_-}$).
Consider the PWS system (2.1). *Suppose that* F^- *is strictly monotonous on* $]\alpha^-, 0[$ *with* $\alpha^- < 0$. *Then system* (2.1) *has no closed trajectory in* $]\alpha^-, 0[\times\mathbb{R}$.

The following two lemmas deal with the non-existence of closed trajectories of type I. For the first one we need the monotony of the functions F^+ and F^-. We modify the proof of the Bendixson's Criterion for Liénard systems, see proof of Lemma 3.6, and obtain the following lemma.

Lemma 3.8 (Non-existence of closed trajectories of type I).
Consider the PWS system (2.1) *with* $b_1^{\pm} := -F^{\pm}(0)$. *Suppose that* F^+, g^+ *and* F^-, g^- *are continuously differentiable functions on the intervals* $[0, \alpha^+[$ *and* $]\alpha^-, 0]$, *respectively, where* $\alpha^- < 0 < \alpha^+$. *Assume that either*

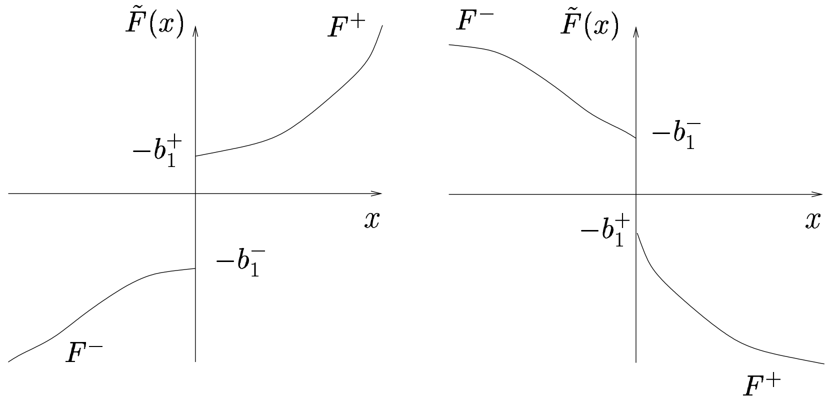


Figure 3.3.: Strictly increasing and decreasing discontinuous function $\tilde{F}$

1. *F^+ and F^- are strictly increasing on $[0, \alpha^+[$ and $]\alpha^-, 0]$, respectively, and $b_1^+ \leq b_1^-$ or*

2. *F^+ and F^- are strictly decreasing on $[0, \alpha^+[$ and $]\alpha^-, 0]$, respectively, and $b_1^+ \geq b_1^-$.*

Then there are no closed trajectories of type I in $]\alpha^-, \alpha^+[\times\mathbb{R}$.

The assertion still holds if F^+ or F^- is only increasing (decreasing) and $b_1^+ < b_1^-$ ($b_1^+ > b_1^-$).

Remark 3.9 (Geometrical relevance of the conditions of Lemma 3.8).
Defining the discontinuous function

$$\tilde{F}(x) := \begin{cases} F^+(x) & \text{, if } x > 0 \\ F^-(x) & \text{, if } x < 0 \end{cases}$$

the condition 1. (2.) means that $\tilde{F}$ is strictly increasing (decreasing) on $]\alpha^-, \alpha^+[$, see Figure 3.3.

Proof of Lemma 3.8.
The non-existence of closed trajectories of type I is a consequence of Green's Theorem, see [Apostol,1957]. Assume that system (2.1) has a closed trajectory γ of type I which intersects M twice at $(0, s_0)$ and $(0, s_0^*)$ with $s_0 > s_0^*$. Defining $\gamma^{\pm} := \gamma_{|G_{\pm}}$,

$$\begin{aligned} \gamma_0^+ : [0,1] &\longrightarrow M \\ t &\longmapsto \begin{pmatrix} 0 \\ s_0^* \end{pmatrix} + t \begin{pmatrix} 0 \\ s_0 - s_0^* \end{pmatrix} \text{ and} \\ \gamma_0^- : [0,1] &\longrightarrow M \\ t &\longmapsto \begin{pmatrix} 0 \\ s_0 \end{pmatrix} + t \begin{pmatrix} 0 \\ s_0^* - s_0 \end{pmatrix}, \end{aligned}$$

$\gamma^+ \cup \gamma_0^+$ and $\gamma^- \cup \gamma_0^-$ are closed trajectories in $\overline{G_+}$ and $\overline{G_-}$, respectively. Let D_+ and D_- be the domains enclosed by $\gamma^+ \cup \gamma_0^+$ and $\gamma^- \cup \gamma_0^-$, respectively, see Figure 3.4.

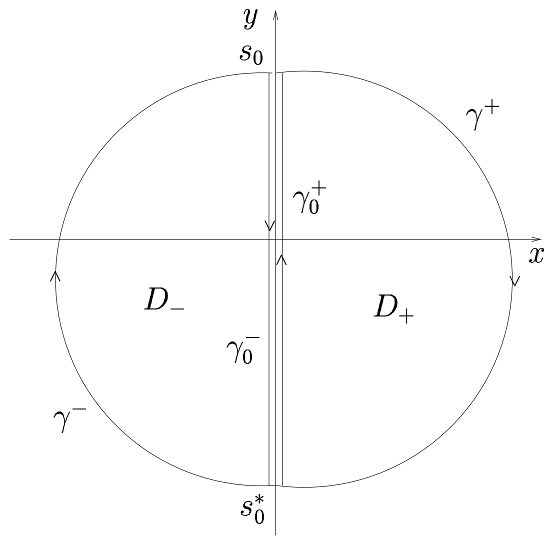


Figure 3.4.: Closed trajectory of type I

$f^{\pm}$ is a continuously differentiable vector field on $D_{\pm}$ and therefore, we can apply Green's Theorem. From that we get

$$\int_{\partial D_{\pm}} f^{\pm}(z)n^{\pm}(z)dz = \int_{D_{\pm}} div f^{\pm}(z)dz = -\int_{D_{\pm}} F^{\pm\prime}(x)dz$$

where $z := (x, y)$ and $n^{\pm}(z)$ is the normal vector corresponding to $\partial D_{\pm}$. On the other hand, $\gamma^{\pm} \cup \gamma_0^{\pm}$ is a negatively oriented curve and thus, there holds for its interior $D_{\pm}$:

$$\int_{\partial D_{\pm}} f^{\pm}(z)n^{\pm}(z)dz = -\int_{\gamma^{\pm}} f^{\pm}(z)n^{\pm}(z)dz - \int_{\gamma_0^{\pm}} f^{\pm}(z)n^{\pm}(z)dz$$

Since $\gamma^{\pm}$ is part of a trajectory of the system $\dot{z} = f^{\pm}(z)$, there holds $\int_{\gamma^{\pm}} f^{\pm}(z)dz = 0$ and consequently,

$$\begin{aligned}
\int_{D_+} F^{+\prime}(x)dz &= -\int_{\partial D_+} f^+(z)n^+(z)dz = \int_{\gamma_0^+} f^+(z)n^+(z)dz \\
&= \int_{\gamma_0^+} \left\langle f^+(z), \begin{pmatrix}1\\0\end{pmatrix}\right\rangle dz = \int_0^1 (s_0^* + t(s_0 - s_0^*) - F^+(0))(s_0 - s_0^*)dt \\
&= (s_0 - s_0^*)(\frac{1}{2}s_0 + \frac{1}{2}s_0^* + b_1^+), \\
\int_{D_-} F^{-\prime}(x)dz &= -\int_{\partial D_-} f^-(z)n^-(z)dz = \int_{\gamma_0^-} f^-(z)n^-(z)dz \\
&= \int_{\gamma_0^-} \left\langle f^-(z), \begin{pmatrix}1\\0\end{pmatrix}\right\rangle dz = \int_0^1 (s_0 + t(s_0^* - s_0) - F^-(0))(s_0^* - s_0)dt \\
&= (s_0^* - s_0)(\frac{1}{2}s_0 + \frac{1}{2}s_0^* + b_1^-).
\end{aligned}$$

Assume now that F^+ and F^- are strictly increasing and $b_1^+ \leq b_1^-$. Then we get

$$0 < \int_{D_+} F^{+\prime}(x)dz + \int_{D_-} F^{-\prime}(x)dz = \underbrace{(s_0 - s_0^*)}_{>0}\underbrace{(b_1^+ - b_1^-)}_{\leq 0}$$

which conflicts with $s_0 > s_0^*$. Analogous we get a contradiction if we assume that F^+ and F^- are strictly decreasing and $b_1^+ \geq b_1^-$. □

The next lemma provides the non-existence of closed trajectories of type I without F^+ and F^- necessarily being monotonous.

Lemma 3.10 (Non-existence of closed trajectories of type I, see Figure 3.5). *Consider the PWS system (2.1). Suppose that F^+, g^+ and F^-, g^- are continuously differentiable functions on the intervals $[0, \alpha^+[$ and $]\alpha^-, 0]$, respectively, where $\alpha^- < 0 < \alpha^+$. Assume that*

1. *there exists an $x_0^+ \in]0, \alpha^+[$ with*

   a)

$$g^+(x) \begin{cases} < 0 & , \text{if } x \in [0, x_0^+[\\ = 0 & , \text{if } x = x_0^+ \\ > 0 & , \text{if } x \in]x_0^+, \alpha^+[\end{cases}$$

   b)

$$F^+(x) \begin{cases} < y_0^+ & , \text{if } x \in [0, x_0^+[\\ = y_0^+ & , \text{if } x = x_0^+ \\ > y_0^+ & , \text{if } x \in]x_0^+, \alpha^+[\end{cases}$$

   where $y_0^+ := F^+(x_0^+)$;

2. *there exists an $x_0^- \in]\alpha^-, 0[$ with*

   a)

$$g^-(x) \begin{cases} < 0 & , \text{if } x \in]\alpha^-, x_0^-[\\ = 0 & , \text{if } x = x_0^- \\ > 0 & , \text{if } x \in]x_0^-, 0] \end{cases}$$

   b)

$$F^-(x) \begin{cases} < y_0^- & , \text{if } x \in]\alpha^-, x_0^-[\\ = y_0^- & , \text{if } x = x_0^- \\ > y_0^- & , \text{if } x \in]x_0^-, 0] \end{cases}$$

   where $y_0^- := F^-(x_0^-)$;

3.

$$y_0^+ \geq y_0^- .$$

Then system (2.1) has no closed trajectory of type I in $]\alpha^-, \alpha^+[\times \mathbb{R}$.

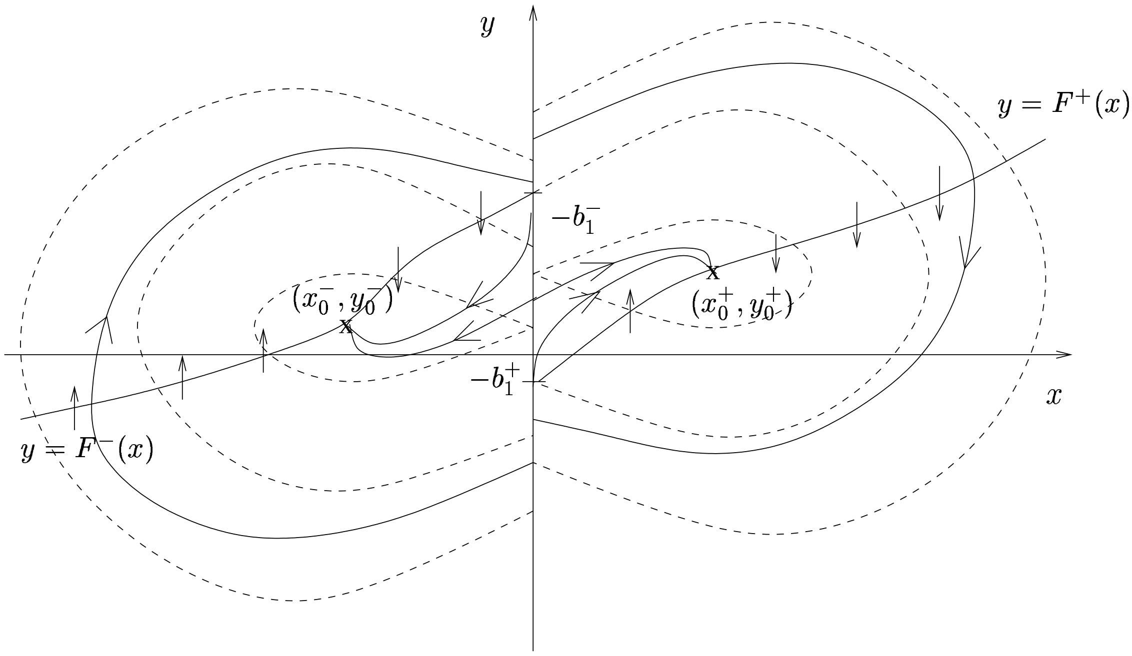


Figure 3.5.: Phase portrait of system (2.1) and level curves (dashed lines) of $\Lambda^{\pm}$

Proof. We prove this result by using appropriate functions

$$\Lambda^{\pm}(x,y) := \frac{1}{2}(y - y_0^{\pm})^2 + \int_0^x g^{\pm}(s)ds$$

and considering their level curves. This is an often used method in the theory of smooth Liénard-systems, see [Levinson & Smith, 1942]. For the rate of change of trajectories along a level curve of $\Lambda^{\pm}$ we find

$$\frac{d\Lambda^+}{dt}(x,y) = (y - y_0^+)\frac{dy}{dt} + g^+(x)\frac{dx}{dt} = -g^+(x)(F^+(x) - y_0^+) < 0$$

and

$$\frac{d\Lambda^-}{dt}(x,y) = (y - y_0^-)\frac{dy}{dt} + g^-(x)\frac{dx}{dt} = -g^-(x)(F^-(x) - y_0^-) < 0$$

for $x \geq 0$ and $x \leq 0$, respectively. Therefore, any trajectory of system (2.1) intersects level sets of $\Lambda^+(x,y)$, $x \geq 0$ and $\Lambda^-(x,y)$, $x \leq 0$ in the exterior-to-interior direction.

Next we consider a trajectory $\gamma^+ \in \overline{G^+}$ of system (2.1) starting at $(0, s_0)$. The condition $s_0 > F^+(0) = -b_1^+$ is necessary for the existence of γ^+. If $y_0^+ > -b_1^+$, every trajectory starting at $(0, s_0)$ with $s_0 \in]-b_1^+, y_0^+]$ reaches the equilibrium (x_0^+, y_0^+), because of $\frac{d\Lambda^+}{dt}(x,y) < 0$ for $x \geq 0$. Consequently, assume $s_0 > y_0^+$ and let $(0, s_0')$ be the intersection point of the level set of $\Lambda^+(x,y)$, $x \geq 0$ with M below y_0^+, which also contains $(0, s_0)$, see Figure 3.6. Because of

$$\Lambda^+(0, s_0') = \Lambda^+(0, s_0) = \text{const},$$
$$s_0' - y_0^+ < 0,$$
$$s_0 - y_0^+ > 0$$

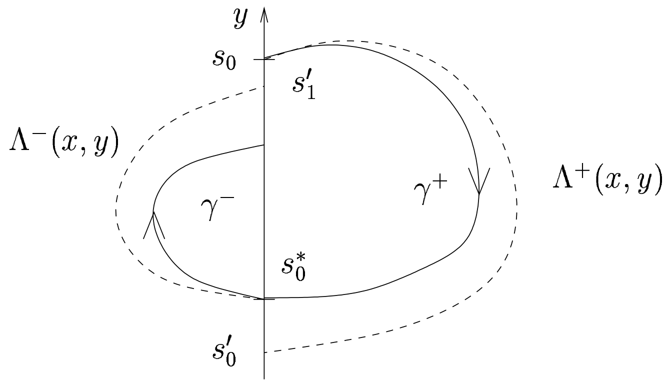


Figure 3.6.: Trajectories $\gamma^\pm$ and level curves $\Lambda^\pm$ (dashed lines)

we obtain

$$s_0' = -s_0 + 2y_0^+.$$

If γ^+ does not intersect M again, then there cannot exist a closed trajectory of type I. That is why we assume that there is a different intersection point $(0, s_0^*)$ of γ^+ with M. Because of $\frac{d\Lambda^+}{dt}(x,y) < 0$, $x \geq 0$ it follows

$$s_0^* > s_0' = -s_0 + 2y_0^+,$$

see Figure 3.6.
If $s_0^* \geq -b_1^- = F^-(0)$ then the trajectory cannot continue into G_- and thus there cannot exist a closed trajectory of type I. If $y_0^- < -b_1^-$ every trajectory γ^- starting at $(0, s_0^*)$ with $s_0^* \in \,]y_0^-, -b_1^-[$ reaches the equilibrium (x_0^-, y_0^-) because of $\frac{d\Lambda^-}{dt}(x,y) < 0$ for $x \leq 0$. Consequently, assume $s_0^* < y_0^-$ and consider a trajectory γ^- of system (2.1) starting at $(0, s_0^*)$. Let $(0, s_1')$ be the intersection point of the level set of $\Lambda^-(x,y)$, $x \leq 0$ with M above y_0^-, which also contains $(0, s_0^*)$, see Figure 3.6. On account of

$$\begin{aligned} \Lambda^-(0, s_1') = \Lambda^-(0, s_0^*) &= \text{const}, \\ s_1' - y_0^- &> 0, \\ s_0^* - y_0^- &< 0 \end{aligned}$$

we obtain

$$s_1' = -s_0^* + 2y_0^-.$$

If we assume that $\gamma^+ \cup \gamma^-$ is a closed trajectory of type I then γ^- intersects M again at $(0, s_0)$. Concerning $\frac{d\Lambda^-}{dt}(x,y) < 0$, $x \leq 0$ it follows

$$s_0 < s_1' = -s_0^* + 2y_0^- < s_0 + 2(y_0^- - y_0^+)$$

which conflicts with $y_0^+ \geq y_0^-$. □

Remark 3.11. *In [Kooij & Giannakopoulos,2000] the authors prove a similar result for smooth systems. We proved Lemma* (3.10) *by modifying the proof given in [Kooij & Giannakopoulos,2000]. As a consequence of losing smoothness we obtain a more global result.*

Considering the proof of Lemma 3.10 we can conclude that we have no closed trajectories in $G_\pm$ if the conditions of Lemma 3.10 are fulfilled. Note that the Lemmas 3.6 and 3.7 need the monotony of the functions $F^\pm$.

Corollary 3.12 (Non-existence of closed trajectories in G_+).
Consider the PWS system (2.1)*. Suppose that F^+ and g^+ are continuously differentiable functions on the interval $[0, \alpha^+[$ with $\alpha^+ > 0$. Assume that condition 1. of Lemma 3.10 is fulfilled. Then system 2.1 has no closed trajectory in $[0, \alpha^+[\, \times \mathbb{R}$.*

Corollary 3.13 (Non-existence of closed trajectories in G_-).
Consider the PWS system (2.1)*. Suppose that F^- and g^- are continuously differentiable functions on the interval $]\alpha^-, 0]$ with $\alpha^- < 0$. Assume that condition 2. of Lemma 3.10 is fulfilled. Then system 2.1 has no closed trajectory in $]\alpha^-, 0] \times \mathbb{R}$.*

3.4. Existence and bifurcation of a closed trajectory of type I

In this section we prove the unique existence of a closed trajectory of type I for PWS systems with $\mathbb{Z}_2-$symmetry. For a corresponding smooth system this result has been proved by [Levinson & Smith, 1942]. Using the properties of the $\mathbb{Z}_2-$symmetry, it is sufficient to consider the system in one half-plane. In the different half-planes G_+ and G_- the PWS system (2.1) is also smooth and therefore, we can adopt the prove of the smooth system performing some modifications. As a consequence of this result, we can define a Hopf-like bifurcation of a closed trajectory of type I. That is why we assume that system (2.1) is $\mathbb{Z}_2-$symmetric. This means, if $(x(t), y(t))$ is a solution of (2.1) then $(-x(t), -y(t))$ is also a solution. This symmetry leads to special properties of the functions $F^\pm$ and $g^\pm$. Assume that $(x(t), y(t))$ with $x(t) > 0$ is a solution of the PWS system (2.1). Because of the $\mathbb{Z}_2-$symmetry, $(-x(t), -y(t))$ is also a solution with $-x(t) < 0$. Then there holds

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} \\ -\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} -y - F^-(-x) \\ -g^-(-x) \end{pmatrix} \end{aligned}$$

and consequently,

$$\begin{aligned} F^+(x) &= -\, F^-(-x) \\ g^+(x) &= -\, g^-(-x). \end{aligned} \tag{3.2}$$

Lemma 3.14 (Existence of a unique asymptotically stable closed trajectory of type I).
Consider system (2.1). *Suppose that* $F^+(x), g^+(x)$ *and* $F^-(x), g^-(x)$ *are smooth functions for* $x \geq 0$ *and* $x \leq 0$, *respectively. Assume that*

1. $$F^+(x) = -F^-(-x)$$
   $$g^+(x) = -g^-(-x)$$
   for all $x \geq 0$;
2. *there exists a zero* $x_0 > 0$ *of* F^+ *with* $F^+(x) < 0$ *for* $0 < x < x_0$ *and* $F^+(x) > 0$ *and* F^+ *is monotonically increasing for* $x > x_0$;
3. $g^+(x) > 0$ *for all* $x \geq 0$;
4. $\int_0^\infty {F^+}'(x)dx = \int_0^\infty g^+(x)dx = \infty$.

Then system (2.1) *has a unique asymptotically stable closed trajectory of type I.*

Remark 3.15 (Geometrical relevance of the conditions).
From the symmetry properties in condition 1. together with condition 2. we get $F^-(x) > 0$ *for* $-x_0 < x < 0$ *and* $F^-(x) < 0$ *and monotonically decreasing as* $x \to -\infty$ *for* $x < -x_0$. *Additionally, we obtain* $b_1^+ := -F^+(0) > 0$ *and* $b_1^- := -F^-(0) = -b_1^+ < 0$. *Considering the conditions 1. and 3. we get* $g^-(x) = -g^+(-x) < 0$ *for all* $x \leq 0$. *This leads to* $b_2^+ := -g^+(0) > 0$ *and* $b_2^- := -g^-(0) = -b_2^+ < 0$. *From Lemma 2.3 we obtain that* $I_s = [-b_1^+, b_1^+]$ *is repulsive and from Lemma 3.1 we get that* $(0,0)$ *is the only equilibrium of* I_s *which is an unstable node. Condition 3. provides that there exist no equilibria in* G_+ *and thus, because of the symmetry properties, no equilibria in* G_-, *either. If we have a closed trajectory* γ *of type I, then* γ *is symmetric with respect to the origin. Especially,* γ *crosses the* $y-$*axis twice at* $(0, s_0)$ *and* $(0, -s_0)$ *with* $s_0 > b_1^+$, *see Figure 3.7.*

Proof of Lemma 3.14. The authors of [Levinson & Smith, 1942] use Liénard's method to prove an analogous result for a corresponding smooth system. They only need the smoothness of the system in one half-plane, especially for $x \geq 0$. Because of the symmetry properties, it is sufficient to consider the part of γ running through G_+. Here, system (2.1) is smooth, and thus, all statements of the proof for smooth systems still hold. The only difference is that γ must start at a point $(0, y_0)$ with $y_0 > b_1^+$ instead of $y_0 > 0$ as in the smooth case, because in the PWS case γ surrounds the sliding motion interval I_s. □

Analogous to Lemma 3.14 we can prove a lemma on the unique existence of an unstable closed trajectory of type I.

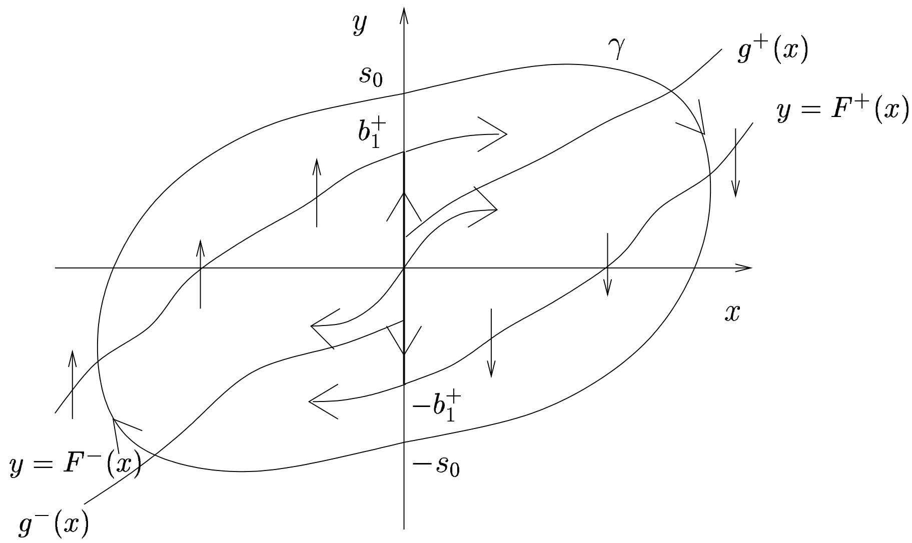


Figure 3.7.: Closed trajectory of type I in the presence of $\mathbb{Z}_2$−symmetry

Lemma 3.16 (Existence of a unique unstable closed trajectory of type I). *Consider system* (2.1)*. Suppose that $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ are smooth functions for $x \geq 0$ and $x \leq 0$, respectively. Assume that*

1.
$$F^+(x) = -F^-(-x)$$
$$g^+(x) = -g^-(-x)$$
for all $x \geq 0$;

2. *there exists a zero $x_0 > 0$ of F^+ with $F^+(x) > 0$ for $0 < x < x_0$ and $F^+(x) < 0$ and F^+ is monotonically decreasing for $x > x_0$;*

3. *$g^+(x) > 0$ for all $x \geq 0$;*

4. *$\int_0^\infty F^{+\prime}(x)dx = -\infty$ and $\int_0^\infty g^+(x)dx = \infty$.*

Then (2.1) *has a unique unstable closed trajectory of type I.*

As a consequence of the above lemmas, we can find a Hopf-like bifurcation for PWS systems if we assume in addition that the functions $F^\pm$ are monotonous. With Hopf-like bifurcation we mean the bifurcation of a closed trajectory of type I, surrounding I_s which contains an equilibrium, when a parameter passes through zero. At the same time, this equilibrium changes stability.

Corollary 3.17 (Bifurcation of a stable closed trajectory of type I). *Consider system* (2.1)*. Suppose that $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ are smooth functions for $x \geq 0$ and $x \leq 0$, respectively. Assume that the conditions of Lemma 3.14 are fulfilled but with F^+ monotonically increasing for all $x \geq 0$. Define $b_1 := b_1^+ = -F^+(0)$. Then the origin is a stable node and there exists no closed trajectory of type I, provided $b_1 < 0$. When b_1 passes through zero the origin becomes unstable and a stable closed trajectory of type I occurs surrounding I_s.*

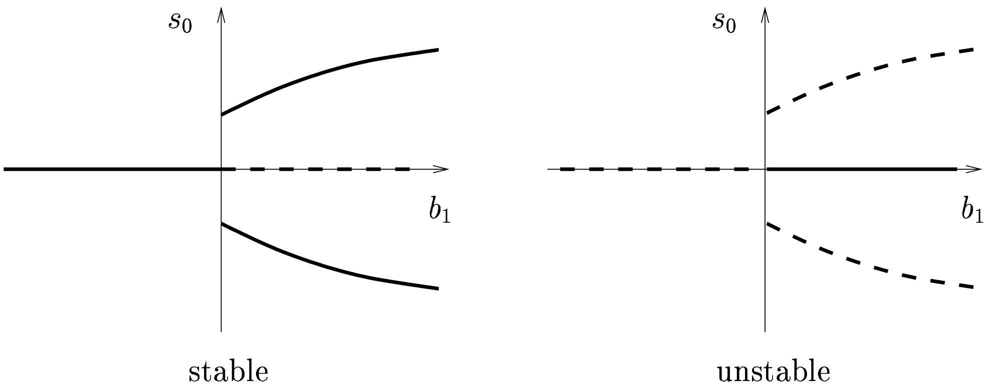


Figure 3.8.: Sketch of the Hopf-like bifurcation of a closed trajectory of type I

Proof. Because of the conditions 1. and 3. of Lemma 3.14, there holds $b_2^+ = -b_2^- < 0$. From Lemma 3.1 we get that $(0,0)$ is a unique equilibrium in I_s and it is a stable node, provided $b_1 < 0$, and an unstable node, provided $b_1 > 0$. On account of Lemma 3.8, there exists no closed trajectory of type I if $b_1 < 0$ and concerning Lemma 3.14 we obtain the existence of a stable closed trajectory of type I for $b_1 > 0$. This completes the proof. □

Corollary 3.18 (Bifurcation of an unstable closed trajectory of type I).
Consider system (2.1). Suppose that $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ are smooth functions for $x \geq 0$ and $x \leq 0$, respectively. Assume that the conditions of Lemma 3.16 are fulfilled but with F^- monotonically decreasing for all $x \geq 0$. Define $b_1 := b_1^+ = -F^+(0)$. Then the origin is an unstable node and there exists no closed trajectory of type I, provided $b_1 > 0$. When b_1 passes through zero the origin becomes stable and an unstable closed trajectory of type I occurs surrounding I_s.

Proof. Because of the conditions 1. and 3. of Lemma 3.16, there holds $b_2^+ = -b_2^- < 0$. From Lemma 3.1 we get that $(0,0)$ is a unique equilibrium in I_s and it is a stable node, provided $b_1 < 0$, and an unstable node, provided $b_1 > 0$. On account of Lemma 3.8, there exists no closed trajectory of type I if $b_1 > 0$ and concerning Lemma 3.16 we obtain the existence of an unstable closed trajectory of type I for $b_1 < 0$. This completes the proof. □

Remark 3.19 (Bifurcation diagram).
Drawing a bifurcation diagram, it looks similar to the bifurcation diagram of a Hopf bifurcation for smooth systems. But in case of the PWS system, the closed trajectory surrounds the sliding motion interval I_s, which means that its amplitude does not converge to zero as the bifurcation parameter b_1 converges to zero, see Figure 3.8. We cannot calculate the amplitude s_0 of the bifurcating closed trajectory in dependency on b_1. Therefore, Figure 3.8 should be seen as a sketch.

We present now an example in order to demonstrate that we can get the existence of a closed trajectory of type I even if the functions $F^{\pm}$, given by an application, are not monotonous.

Example 3.20.
Consider the second-order equation

$$\ddot{\xi} + \varphi(\dot{\xi}) + k\xi = 0 \tag{3.3}$$

with $k > 0$ *and the characteristic* φ *schematically indicated in Figure 3.9. We can transform equation* (3.3) *to a Liénard-system with* $F(x) = \varphi(x)$ *and* $g(x) = kx$. *All conditions of Lemma 3.14 are fulfilled and thus we have a stable closed trajectory of type I.*

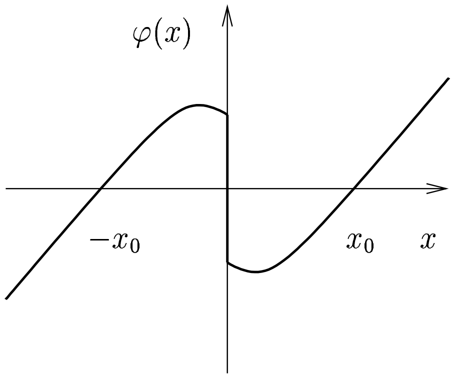


Figure 3.9.: The characteristic φ for equation (3.3)

3.5. Periodic solution with sliding motion in $\overline{G}_{\pm}$

In this section we prove a local Hopf-like bifurcation which arises in consequence of sliding motion in system (2.1). Hence, this kind of bifurcation cannot occur in case of smooth systems. It is comparable with the Hopf bifurcation of smooth systems but nevertheless different. In case of a smooth system $\binom{\dot{x}}{\dot{y}} = f(x, y, \mu)$, depending on a parameter $\mu \in \mathbb{R}$ with an equilibrium (x_0, y_0), a Hopf bifurcation occurs at a parameter value μ_0 when the Jacobian $Df(x_0, y_0, \mu_0)$ has a simple pair of pure imaginary eigenvalues. The local phase portrait changes when μ passes through the bifurcation value μ_0. A periodic solution is created when the stability of the equilibrium (x_0, y_0) changes, see for example [Guckenheimer & Holmes,1983, pp. 150-156]. In case of the PWS system (2.1) the Jacobian does not exist for $x_0 = 0$. Therefore, we cannot detect a Hopf-like bifurcation by considering the eigenvalues of the Jacobians of the right-hand sides $Df^{\pm}(x_0, y_0, \mu)$ of system (2.1) in dependency on μ. But nevertheless we can show that the local phase portrait changes when the parameter μ passes through μ_0.

3.5.1. Main results

Theorem 3.21 (Local existence of a periodic solution with sliding motion in $\overline{\mathbf{G}_+}$).
Consider the PWS system (2.1). *Suppose that* F^+ *and* g^+ *are continuously differentiable functions on the interval* $]\alpha^-, \alpha^+[$ *with* $\alpha^- < 0 < \alpha^+$. *Assume that*

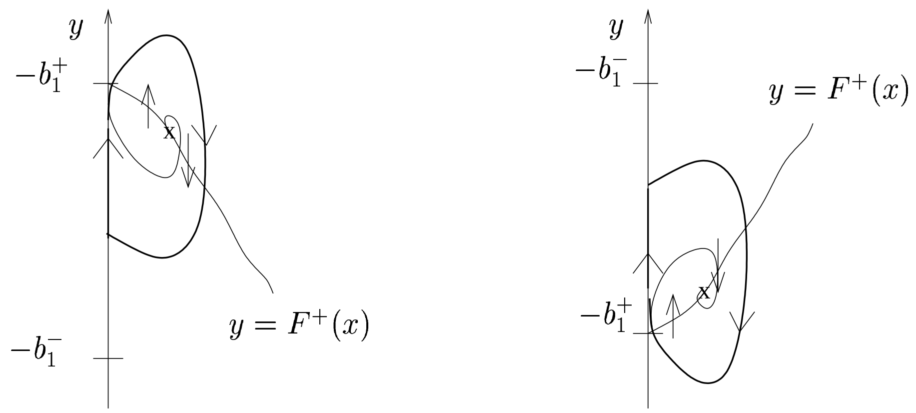
(A1+) there exists an $\varepsilon > 0$ *and an* x_0^+ *with* $0 \le x_0^+ < \varepsilon$ *such that*

$$g^+(x) \begin{cases} < 0 & , \text{ if } x \in \ [0, x_0^+[\\ = 0 & , \text{ if } x = x_0^+ \\ > 0 & , \text{ if } x \in \ ]x_0^+, \alpha^+[; \end{cases}$$

(A2+) ${a_{11}^+}^2 + 4a_{21}^+ < 0$ *where* $a_{11}^+ := -F^{+\prime}(0) \neq 0$ *and* $a_{21}^+ := -g^{+\prime}(0) < 0$;

(A3+) $a_{11}^+(b_1^+ - b_1^-) < 0$ *where* $b_1^\pm := -F^\pm(0)$.

Then there exists a periodic solution with sliding motion in $\overline{G_+}$ *surrounding a focus* (x_0^+, y_0^+). *The focus* (x_0^+, y_0^+) *is unstable (stable) and the periodic solution is stable (unstable), provided* $a_{11}^+ > 0$ $(a_{11}^+ < 0)$.



a) $b_1^+ < b_1^-$ and $F^{+\prime}(0) = -a_{11}^+ < 0$ b) $b_1^+ > b_1^-$ and $F^{+\prime}(0) = -a_{11}^+ > 0$

Figure 3.10.: Periodic solutions with sliding motion in $\overline{G_+}$; x denotes the focus (x_0^+, y_0^+).

Remark 3.22 (Geometric relevance of the conditions).
Condition (A1+) *signifies that* (x_0^+, y_0^+) *is an equilibrium in* G_+ *and that the vector field, which is vertical on the curve* $y = F^+(x)$, *changes its sign at* $x = x_0^+$. *If* $x_0^+ = 0$ *then* (x_0^+, y_0^+) *is a focus if and only if* ${a_{11}^+}^2 + 4a_{21}^+ < 0$ *and* $a_{11}^+ \neq 0$, *see condition (A2+). This still holds for* $x_0^+ > 0$ *sufficiently small. Condition* (A3+) *means that* (x_0^+, y_0^+) *is "on the same height" as* I_s, *that is* $(0, y_0^+) \in I_s$, *see Figure 3.10.*

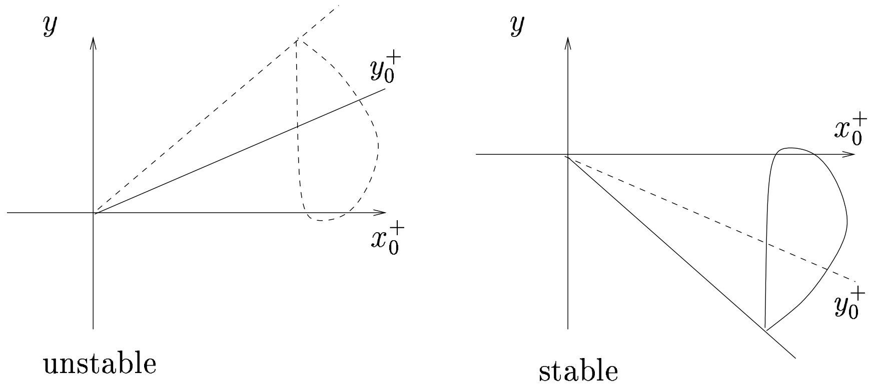


Figure 3.11.: Sketch of the Hopf-like bifurcation of a periodic solution with sliding motion, when x_0^+ passes through zero

Theorem 3.23 (Local existence of a periodic solution with sliding motion in $\overline{\mathbf{G_-}}$).
Consider the PWS system (2.1). Suppose that F^- and g^- are continuously differentiable functions on the interval $]\alpha^-, \alpha^+[$ with $\alpha^- < 0 < \alpha^+$. Assume that

(A1-) there exists an $\varepsilon > 0$ and an x_0^- with $-\varepsilon < x_0^- \leq 0$ such that

$$g^-(x) \begin{cases} < 0 \ , \ \text{if } x \in \,]\alpha^-, x_0^-[\\ = 0 \ , \ \text{if } x = x_0^- \\ > 0 \ , \ \text{if } x \in \,]x_0^-, 0]; \end{cases}$$

(A2-) ${a_{11}^-}^2 + 4a_{21}^- < 0$ where $a_{11}^- := -F^{-\prime}(0) \neq 0$ and $a_{21}^- := -g^{-\prime}(0) < 0$;

(A3-) $a_{11}^-(b_1^+ - b_1^-) < 0$ where $b_1^\pm = -F^\pm(0)$;

Then there exists a periodic solution with sliding motion in $\overline{G_-}$ surrounding a focus (x_0^-, y_0^-). The focus (x_0^-, y_0^-) is unstable (stable) and the periodic solution is stable (unstable), provided $a_{11}^- > 0$ ($a_{11}^- < 0$).

Remark 3.24 (Local Hopf-like bifurcation of a periodic solution with sliding motion).
We consider $x_0^+ \in \mathbb{R}$ as bifurcation parameter. If $x_0^+ < 0$ then there exists neither an equilibrium nor a periodic solution. When x_0^+ passes through zero the focus (x_0^+, y_0^+) occurs on M and at the same time a periodic solution with sliding motion bifurcates. The focus moves into G_+ as x_0^+ increases and is surrounded by the periodic solution. Considering a smooth system, the stability of an equilibrium changes, whereas in the PWS case a new equilibrium occurs on M, see Figure 3.11.

3.5.2. Proofs of the main results

In this subsection we prove Theorem 3.21. We omit the proof of Theorem 3.23, because it is analogous. The proof is divided into three parts: First, we show that the equilibrium

(x_0^+, y_0^+) is a focus for system (2.1) if $x_0^+ > 0$ sufficiently small. In the second part, we define an appropriate function in order to prove the local existence of a periodic solution with sliding motion in $\overline{G_+}$ if the focus is stable. Finally, we prove the existence of this periodic solution if the focus is unstable by reversing time in system (2.1).
(x_0^+, y_0^+) with $y_0^+ := F^+(x_0^+)$ is an equilibrium of system (2.1) if $x_0^+ > 0$. Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \tilde{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}, \tag{L}$$

where

$$\tilde{A} := \begin{pmatrix} \tilde{a}_{11} & 1 \\ \tilde{a}_{21} & 0 \end{pmatrix},$$

$\tilde{b}_1, \tilde{b}_2 \in \mathbb{R}$ and $\tilde{a}_{11}, \tilde{a}_{21} \neq 0$. Then, the linear system (L) has an equilibrium $(\tilde{x}_0, \tilde{y}_0) := (-\frac{\tilde{b}_2}{\tilde{a}_{21}}, \frac{\tilde{a}_{11}\tilde{b}_2}{\tilde{a}_{21}} - \tilde{b}_1)$. It coincides with the equilibrium (x_0^+, y_0^+) of the nonlinear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} \tag{NL}$$

when we define

$$\begin{aligned}
\tilde{b}_1 :=& -y_0^+ + F^{+'}(x_0^+)x_0^+, \\
\tilde{b}_2 :=& g^{+'}(x_0^+)x_0^+, \\
\tilde{a}_{11} :=& -F^{+'}(x_0^+) = a_{11}^+ + O(x_0^+), \\
\tilde{a}_{21} :=& -g^{+'}(x_0^+) = a_{21}^+ + O(x_0^+).
\end{aligned}$$

Here, with $O(x)$ we mean $\lim_{x\to 0}\left|\frac{O(x)}{x}\right| \leq C \in \mathbb{R}$. For x_0^+ sufficiently small there holds, because of $(A2+)$,

$$\tilde{a}_{11}^2 + 4\tilde{a}_{21} = {a_{11}^+}^2 + 4a_{21}^+ + O(x_0^+) < 0.$$

This means that $(\tilde{x}_0, \tilde{y}_0)$ is a focus of the linear system (L) which is unstable if $\tilde{a}_{11} > 0$ and stable if $\tilde{a}_{11} < 0$. The stability properties still hold for $a_{11}^+ > 0$ and $a_{11}^+ < 0$, respectively, for $x_0^+ > 0$ sufficiently small because of $\tilde{a}_{11} = a_{11}^+ + O(x_0^+)$.
Using the new notation and Taylor expansion around x_0^+, the nonlinear system (NL) has the form

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} y - y_0^+ - F^{+'}(x_0^+)(x - x_0^+) \\ -g^{+'}(x_0^+)(x - x_0^+) \end{pmatrix} + O((x - x_0^+)^2) \\
&= \tilde{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} + o(|x - x_0^+|).
\end{aligned}$$

Here, with $o(x)$ we mean $\lim_{x\to 0}\left|\frac{o(x)}{x}\right| = 0$. Setting $u := x - x_0^+$ and $v := y - y_0^+$, we can apply Theorem 2.2. in [Coddington & Levinson,1955, p. 376] and get that $(x_0^+, y_0^+) = (\tilde{x}_0, \tilde{y}_0)$ is also a focus with the same stability properties of the nonlinear system (NL).

We define now an appropriate function for proving the existence of the periodic solution in $\overline{G_+}$ with sliding motion if $x_0^+ > 0$ by

$$\Lambda(x,y) := \frac{1}{2}(y - y_0^+)^2 + \int_0^x g^+(s)ds$$

for $x \in [0, \alpha^+[\,, y \in \mathbb{R}$. For this we consider the level curves of Λ.

We consider now the two different cases corresponding to $(A3+)$.

Case 1: $b_1^+ > b_1^-$ and $a_{11}^+ < 0$:
For the rate of change of trajectories $(x(t), y(t))$ of system (2.1) in $\overline{G_+}$ along a level curve of Λ we find

$$\frac{d\Lambda}{dt}(x,y) = (y - y_0^+)\frac{dy}{dt} + g^+(x)\frac{dx}{dt} = -g^+(x)(F^+(x) - y_0^+).$$

For $x_0^+ > 0$ sufficiently small there holds $\tilde{a}_{11} = a_{11}^+ + O(x_0^+) < 0$ and consequently, for all $x \in [0, \alpha^+[$ and $|x - x_0^+| < \delta$ with $\delta > 0$ sufficiently small

$$F^+(x) - y_0^+ = F^+(x) - F^+(x_0^+) = -\tilde{a}_{11}(x - x_0^+) + o(|x - x_0^+|)$$
$$\begin{cases} < 0 & \text{, if } x < x_0^+ \\ > 0 & \text{, if } x > x_0^+. \end{cases}$$

Together with $(A1+)$ we get

$$\frac{d\Lambda}{dt}(x,y) < 0 \text{ for all } |x - x_0^+| < \delta, y \in \mathbb{R}.$$

Therefore, any trajectory of system (2.1) intersects a level set $\Lambda(x,y)$, $x \geq 0$ in the exterior-to-interior direction.

Next, we consider a trajectory γ of system (2.1) starting at $(0, s_0) \in I_s$ with $s_0 > y_0^+$. Due to the repulsion of $I_s = [-b_1^+, -b_1^-]$, there exists a periodic solution with sliding motion in $\overline{G_+}$ only if $(0, -b_1^+)$ is a reaching point, which is the case only if $b_2^+ > 0$, see Lemma 3.3. This yields

$$0 = g^+(x_0^+) = g^+(0) + g^{+\prime}(0)x_0^+ + o(x_0^+) = -b_2^+ - a_{21}^+ x_0^+ + o(x_0^+)$$
$$\Leftrightarrow \quad b_2^+ = -a_{21}^+ x_0^+ + o(x_0^+)$$

and consequently, $b_2^+ > 0$ if $x_0^+ > 0$ sufficiently small, where $b_2^+ := -g^+(0)$. Thus, $(0, -b_1^+)$ is a reaching point.
If γ reaches $(0, -b_1^+)$ and starts at a point $(0, \bar{s})$ with $\bar{s} \leq \min\{-b_1^-, y_0\}$, then γ together with the sliding motion solution forms a periodic solution. For the definition of y_0 see Lemma 3.1. Let s_0' be the y-coordinate of the intersection of the level set of $\Lambda(x,y)$ with the y-axis below y_0^+ which also contains $(0, s_0)$, see Figure 3.12. Because of

$$\Lambda(0, s_0') = \Lambda(0, s_0) = \text{const},$$
$$s_0' - y_0^+ < 0,$$
$$s_0 - y_0^+ > 0,$$

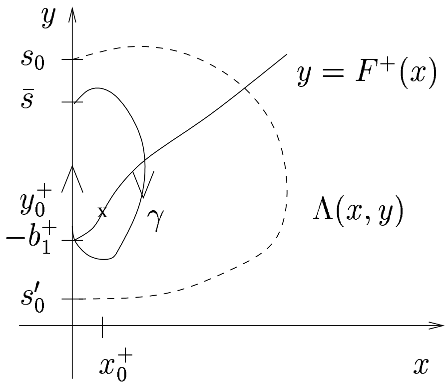


Figure 3.12.: Periodic solution γ and level curve Λ (dashed line); x denotes the focus (x_0^+, y_0^+)

we have

$$s_0' = -s_0 + 2y_0^+.$$

In case of $s_0' > -b_1^+$, γ reaches the stable focus (x_0^+, y_0^+), owing to the repulsion of I_s. If we assume $x_0^+ = 0$ there holds $y_0^+ = -b_1^+$, and setting $s_0 := \min\{y_0, -b_1^-\}$, we get

$$\begin{aligned} s_0' &= -s_0 + 2y_0^+ = -s_0 - 2b_1^+ \\ &= -b_1^+ + \begin{cases} b_1^- - b_1^+ & , \text{ if } b_2^- \geq 0 \\ \frac{(b_1^- - b_1^+)b_2^+}{b_2^+ - b_2^-} & , \text{ if } b_2^- < 0 \end{cases} < -b_1^+, \end{aligned}$$

where $b_2^- := -g^-(0)$. For $x_0^+ > 0$ sufficiently small it still holds

$$s_0' = -s_0 + 2y_0^+ = -s_0 + 2F^+(0) + O(x_0^+) = -s_0 - 2b_1^+ + O(x_0^+) < -b_1^+.$$

Consequently, the trajectory γ which reaches $(0, -b_1^+)$ for $x_0^+ > 0$ starts at a point $(0, \bar{s})$ with $(0, \bar{s}) \in]-b_1^+, \min\{-b_1^-, y_0\}[$ and creates a periodic solution in $\overline{G_+}$ with sliding motion surrounding the focus (x_0^+, y_0^+). Because of the repulsion of I_s and the stability of (x_0^+, y_0^+), it is unstable.

Case 2: $b_1^+ < b_1^-$ and $a_{11}^+ > 0$:
We consider now instead of system (2.1) the corresponding system with reversed time

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{cases} \begin{pmatrix} -v + F^+(u) \\ g^+(v) \end{pmatrix} & , \text{ if } u > 0 \\ \begin{pmatrix} -v + F^-(u) \\ g^-(v) \end{pmatrix} & , \text{ if } u < 0. \end{cases} \tag{3.4}$$

Note that $((x(t), y(t))$ is a solution of system (2.1) if and only if $(u(-t), v(-t))$ is a solution of system (3.4) and (x_0^+, y_0^+) is also a stable focus of system (3.4). As in case 1 the sliding motion interval $I_s = [-b_1^-, -b_1^+]$ is repulsive and $(0, -b_1^+)$ is a reaching point for $x_0^+ > 0$ sufficiently small.
For the rate of change of trajectories $(u(t), v(t))$ of system (3.4) in $\overline{G_+}$ along a level curve of Λ we find

$$\frac{d\Lambda}{dt}(u, v) = (u - y_0^+)\frac{dv}{dt} + g^+(u)\frac{du}{dt} = g^+(u)(F^+(u) - y_0^+).$$

For $x_0^+ > 0$ sufficiently small there holds $\tilde{a}_{11} = a_{11}^+ + O(x_0^+) > 0$ and consequently for all $u \in\ [0, \alpha^+[$ and $|u - x_0^+| < \delta$ with $\delta > 0$ sufficiently small

$$F^+(u) - y_0^+ = F^+(u) - F^+(x_0^+) = -\tilde{a}_{11}(u - x_0^+) + o(u - x_0^+)$$
$$\begin{cases} > 0 & \text{, if } u < x_0^+ \\ < 0 & \text{, if } u > x_0^+. \end{cases}$$

Together with $(A1+)$ we obtain

$$\frac{d\Lambda}{dt}(u, v) < 0 \text{ for all } |u - x_0^+| < \delta, v \in \mathbb{R}.$$

Analogous to case 1, we get the existence of a periodic solution $\tilde{\gamma}$ in $\overline{G_+}$ with sliding motion for $x_0^+ > 0$ sufficiently small which reaches I_s at $(0, -b_1^+)$, surrounds the stable focus (x_0^+, y_0^+) and is unstable. $\gamma := -\tilde{\gamma}$ is then a periodic solution of system (2.1) in $\overline{G_+}$ with sliding motion which leaves $(0, -b_1^+)$, surrounds the unstable focus (x_0^+, y_0^+) and is stable. This completes the proof.

□

In this chapter, we presented for the PWS system (2.1) a complete characterisation of singular points in the sliding motion interval I_s and the existence and uniqueness of the corresponding initial value problem in dependency on the parameters $b_1^\pm$ and $b_2^\pm$. We analytically proved some results on non-existence of closed trajectories with certain conditions for the functions $F^\pm$ and $g^\pm$. Furthermore, we proved the existence of a unique closed trajectory of type I assuming additionally $\mathbb{Z}_2-$symmetry. These results were all global. The last section provided a Hopf-like bifurcation of closed trajectories of type II. This result was local. In case of piecewise-linear systems we can prove an analogous global result. In addition, it is possible to prove stronger results on existence and non-existence of closed trajectories, because in the case of piecewise-linear systems we can analytically determine the discrete-time maps, which are one main tool for detecting closed trajectories. Therefore, we consider piecewise-linear systems in the following.

4. Results for the piecewise-linear system

As described in section 2.3 the main tool for detecting closed trajectories in PWS systems is the determination of discrete-time maps Π^+, Π^- and Π^0 in the different domains G_+, G_- and I_s. These maps can be used to define a 1-dimensional Poincaré-map which maps a point s_0 with $(0, s_0) \in M$ to a point $\Pi(s_0)$ with $(0, \Pi(s_0)) \in M$. In general, it is not possible to analytically determine Π^+ and Π^- for system (2.1). But in case of piecewise-linear (PWL) systems we can analytically determine these maps, make statements about their properties in dependence on the parameters and obtain global results.
In [Pliete,1998] we make a first step in this direction. But there, we only consider a PWL system with $A := A^+ = A^-$. We do not stress the bifurcation phenomena and highlight the results only in the symmetric case, that is if $b_1^+ = -b_1^-$ and $b_2^+ = -b_2^-$. For the special cases $\operatorname{tr}(A)^2 - 4\det(A) < 0$, $\operatorname{tr}(A) \neq 0$ and $\det(A) < 0$ we give complete analyses in a modified form in [Giannakopoulos & Pliete,2001] and [Giannakopoulos & Pliete,2002], respectively.

In this chapter we consider the PWL system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^+ \binom{x}{y} + \binom{b_1^+}{b_2^+} & \text{, if } x > 0 \\ A^- \binom{x}{y} + \binom{b_1^-}{b_2^-} & \text{, if } x < 0 \end{cases} \tag{4.1}$$

with

$$A^\pm := \begin{pmatrix} a_{11}^\pm & 1 \\ a_{21}^\pm & 0 \end{pmatrix}, \quad a_{11}^\pm, a_{21}^\pm, b_1^\pm, b_2^\pm \in \mathbb{R}.$$

Note that we get such a system if we linearise the right-hand sides of system (2.1) at $x = 0$:

$$f^\pm(0,0) = \begin{pmatrix} -F^\pm(0) \\ -g^\pm(0) \end{pmatrix} =: \begin{pmatrix} b_1^\pm \\ b_2^\pm \end{pmatrix}$$
$$Df^\pm(0,0) = \begin{pmatrix} -F^{\pm\prime}(0) & 1 \\ -g^{\pm\prime}(0) & 0 \end{pmatrix} =: \begin{pmatrix} a_{11}^\pm & 1 \\ a_{21}^\pm & 0 \end{pmatrix}$$

Before determining the discrete-time maps $\Pi^\pm$ we identify and characterise the equilibria of system (4.1) in $G_\pm$. We do this because the maps $\Pi^\pm$ depend on the eigenvalues of $A^\pm$ and consequently on the kind of equilibria.

Lemma 4.1 (Equilibria in $\mathbf{G}_{\pm}$).
There exists an equilibrium $(x_0^+, y_0^+) := \left(-\frac{b_2^+}{a_{21}^+}, \frac{a_{11}^+ b_2^+}{a_{21}^+} - b_1^+\right)$ *in* G_+ *and an equilibrium* $(x_0^-, y_0^-) := \left(-\frac{b_2^-}{a_{21}^-}, \frac{a_{11}^- b_2^-}{a_{21}^-} - b_1^-\right)$ *in* G_- *if* $a_{21}^+ \neq 0$, $\frac{b_2^+}{a_{21}^+} < 0$ *and* $a_{21}^- \neq 0$, $\frac{b_2^-}{a_{21}^-} > 0$, *respectively.*

1. *If* $a_{21}^{\pm} < 0$ *and* $a_{11}^{\pm} = 0$, *then* $(x_0^{\pm}, y_0^{\pm})$ *is a center.*
2. *If* ${a_{11}^{\pm}}^2 + 4a_{21}^{\pm} \geq 0$ *and* $a_{21}^{\pm} < 0$, *then* $(x_0^{\pm}, y_0^{\pm})$ *is a node, unstable if* $a_{11}^{\pm} > 0$ *and stable if* $a_{11}^{\pm} < 0$.
3. *If* ${a_{11}^{\pm}}^2 + 4a_{21}^{\pm} > 0$ *and* $a_{21}^{\pm} > 0$, *then* $(x_0^{\pm}, y_0^{\pm})$ *is a saddle point.*
4. *If* ${a_{11}^{\pm}}^2 + 4a_{21}^{\pm} < 0$ *and* $a_{11}^{\pm} \neq 0$, *then* $(x_0^{\pm}, y_0^{\pm})$ *is a focus, unstable if* $a_{11}^{\pm} > 0$ *and stable if* $a_{11}^{\pm} < 0$.

Considering the discrete-time maps Π^+ and Π^- we conclude results on existence and non-existence of closed trajectories of type I. Moreover, we give necessary conditions on the existence of closed trajectories of type II and we finally prove a theorem on the bifurcation of a periodic solution with sliding motion in $\overline{G_{\pm}}$ of the PWL system (4.1). In case of system (2.1) we proved a Hopf-like bifurcation of a periodic solution with sliding motion in $\overline{G_{\pm}}$. This result was local, whereas in case of the PWL system (4.1) we will obtain a global result.

4.1. Existence and properties of the discrete-time maps $\Pi^{\pm}$

In this section we determine for which conditions the discrete-time maps $\Pi^{\pm}$ exist. We consider $b_1^{\pm}$ and $b_2^{\pm}$ as bifurcation parameters. Nevertheless, we need to differentiate between several cases, depending on the eigenvalues of $A^{\pm}$. We will show that these maps are in each case strictly decreasing and convex, concave or a straight line.

We only determine the map Π^+ in detail. The determination of Π^- is analogous.

Assume $s_0 \in M^+$ is given. Then, the solution ϕ of equation (4.1) with initial value $\binom{\phi_1(0)}{\phi_2(0)} = \binom{0}{s_0}$ is given by

$$\begin{aligned}\phi(t) &= e^{A^+ t}\left(\begin{pmatrix}0\\ s_0\end{pmatrix} + (A^+)^{-1}\begin{pmatrix}b_1^+\\ b_2^+\end{pmatrix}\right) - (A^+)^{-1}\begin{pmatrix}b_1^+\\ b_2^+\end{pmatrix}\\ &= -\frac{1}{a_{21}^+}\begin{pmatrix} b_2^+(\alpha_{12}^+(t)a_{11}^+ + 1 - \alpha_{11}^+(t)) - \alpha_{12}^+(t)a_{21}^+(s_0 + b_1^+)\\ b_2^+(\alpha_{22}^+(t)a_{11}^+ - a_{11}^+ - \alpha_{21}^+(t)) - \alpha_{22}^+(t)(s_0 + b_1^+) + a_{21}^+ b_1^+\end{pmatrix},\end{aligned}$$

where $a_{21}^+ \neq 0$ and $\alpha^+(t) := e^{A^+ t}$ is the transition matrix, see Proposition C.1. We get $s_0 \in M^+$ if $s_0 > -b_1^+$ and there exists a minimal $t_0^+ > 0$ with $\phi_1(t_0^+) = 0$. Then, the discrete-time map Π^+ is defined with $\Pi^+(s_0) = \phi_2(t_0^+)$. If there exists a $t_0^+ > 0$ with $\alpha_{12}^+(t_0^+) = 0$, Π^+ exists for all $s_0 > -b_1^+$ in the case of $b_2^+ = 0$ and it is not defined in the case of $b_2^+ \neq 0$. For $a_{21}^+ = 0$ we need to compute ϕ separately, because in this case A^+ is not regular.

Lemma 4.2 (Properties of the discrete-time map Π^+).
Consider the PWL system (4.1). Then there holds for the half discrete-time map Π^+:

1. ${a_{11}^+}^2 + 4a_{21}^+ > 0$, $a_{21}^+ \neq 0$ *(saddle/node-case)*
   Π^+ *exists if and only if* $b_2^+ < 0$.
   Let $\eta^+ + \omega^+$ *and* $\eta^+ - \omega^+$ *with* $\eta^+ := \frac{a_{11}^+}{2}$ *and* $\omega^+ := \frac{1}{2}\sqrt{{a_{11}^+}^2 + 4a_{21}^+}$ *be the eigenvalues of* A^+.

   a) *If* $\eta^+ - \omega^+ < 0$ *and* $\eta^+ + \omega^+ < 0$ *(stable node),*
   $$\Pi^+ : \quad]-b_1^+, \infty[\quad \longrightarrow \quad]\frac{b_2^+}{a_{21}^+}(\eta^+ + \omega^+) - b_1^+, -b_1^+[$$
   is strictly decreasing and convex.

   b) *If* $\eta^+ - \omega^+ > 0$ *and* $\eta^+ + \omega^+ > 0$ *(unstable node),*
   $$\Pi^+ : \quad]-b_1^+, \frac{b_2^+}{a_{21}^+}(\eta^+ - \omega^+) - b_1^+[\quad \longrightarrow \quad]-\infty, -b_1^+[$$
   is strictly decreasing and concave.

   c) *If* $\eta^+ - \omega^+ < 0$ *and* $\eta^+ + \omega^+ > 0$ *(saddle point),*
   $$\Pi^+ : \quad]-b_1^+, \frac{b_2^+}{a_{21}^+}(\eta^+ - \omega^+) - b_1^+[\quad \longrightarrow \quad]\frac{b_2^+}{a_{21}^+}(\eta^+ + \omega^+) - b_1^+, -b_1^+[$$
   is strictly decreasing, concave if $\eta^+ > 0$ *and convex if* $\eta^+ < 0$.

   d) *If* $\eta^+ = 0$ *(saddle point),*
   $$\Pi^+ : \quad]-b_1^+, -\frac{b_2^+\omega^+}{a_{21}^+} - b_1^+[\quad \longrightarrow \quad]\frac{b_2^+\omega^+}{a_{21}^+} - b_1^+, -b_1^+[$$
   is a straight line with slope -1.

   $T^+ :=]0, \infty[$ *is the domain of the switching time* t_0^+.

2. ${a_{11}^+}^2 + 4a_{21}^+ < 0$, $a_{21}^+ \neq 0$ *(focus-case)*
   Π^+ *exists for all* $b_2^+ \in \mathbb{R}$.
   Let $\eta^+ + i\omega^+$ *and* $\eta^+ - i\omega^+$ *with* $\eta^+ := \frac{a_{11}^+}{2}$ *and* $\omega^+ := \frac{1}{2}\sqrt{-{a_{11}^+}^2 - 4a_{21}^+}$ *be the eigenvalues of* A^+.
   $$\Pi^+ : M^+ \longrightarrow \hat{M}^+$$
   is strictly decreasing, concave if $\eta^+ > 0$, *convex if* $\eta^+ < 0$ *and a straight line with slope* -1 *if* $\eta^+ = 0$. *The domains* M^+ *and* $\hat{M}^+$ *are given by*
   $$M^+ := \begin{cases}]-b_1^+, \infty[& \text{, if } b_2^+ \leq 0 \text{ or } \eta^+ \geq 0 \\ ]\overline{s}^+, \infty[& \text{, if } b_2^+ > 0 \text{ and } \eta^+ < 0 \end{cases}$$

where $\overline{s}^+ := -\frac{b_2^+}{\omega^+}e^{-\eta^+ t_1^0}\sin(\omega t_1^0) - b_1^+$ *and* t_1^0 *is the unique zero of* $\Xi_1(t;\eta^+,\omega^+)$ *in* $]\frac{\pi}{\omega^+},\frac{2\pi}{\omega^+}[$, *see Proposition B.1.*

$$\hat{M}^+ := \begin{cases}]-\infty, -b_1^+[& \text{, if } b_2^+ \leq 0 \text{ or } \eta^+ \leq 0 \\ ]\hat{s}^+, -b_1^+[& \text{, if } b_2^+ > 0 \text{ and } \eta^+ > 0 \end{cases}$$

where $\hat{s}^+ := \frac{b_2^+}{a_{21}^+}\frac{\eta^+\sin(\omega^+ t_1^0) - 2\omega^+\sinh(\eta^+ t_1^0)}{\sin(\omega t_1^0)} - b_1^+$.

$$T^+ := \begin{cases}]0, \frac{\pi}{\omega^+}[& \text{, if } b_2^+ < 0 \\ ]\frac{\pi}{\omega^+}, t_1^0[& \text{, if } b_2^+ > 0, \eta^+ \neq 0 \\ ]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[& \text{, if } b_2^+ > 0, \eta^+ = 0 \\ \{\frac{\pi}{\omega^+}\} & \text{, if } b_2^+ = 0 \end{cases}$$

is the domain of the switching time t_0^+.

3. ${a_{11}^+}^2 + 4a_{21}^+ = 0$, $a_{11}^+ \neq 0 \neq a_{21}^+$ *(node-case)*
Π^+ *exists if and only if* $b_2^+ < 0$.
Let $\eta^+ := \frac{a_{11}^+}{2}$ *be the eigenvalue of* A^+.

$$\Pi^+ : M^+ \longrightarrow \hat{M}^+$$

is strictly decreasing and concave if $\eta^+ > 0$ *and convex if* $\eta^+ < 0$. *The domains* M^+ *and* $\hat{M}^+$ *are given by*

$$M^+ := \begin{cases}]-b_1^+, -\frac{b_2^+}{\eta^+} - b_1^+[& \text{, if } \eta^+ > 0 \\ ]-b_1^+, \infty[& \text{, if } \eta^+ < 0, \end{cases}$$

$$\hat{M}^+ := \begin{cases}]-\infty, -b_1^+[& \text{, if } \eta^+ > 0 \\ ]-\frac{b_2^+}{\eta^+} - b_1^+, -b_1^+[& \text{, if } \eta^+ < 0. \end{cases}$$

$T^+ :=]0,\infty[$ *is the domain of the switching time* t_0^+.

4. $a_{21}^+ = 0, a_{11}^+ \neq 0$
Π^+ *exists if and only if* $b_2^+ < 0$.

$$\Pi^+ : M^+ \longrightarrow \hat{M}^+$$

is strictly decreasing and concave if $a_{11}^+ > 0$ *and convex if* $a_{11}^+ < 0$. *The domains* M^+ *and* $\hat{M}^+$ *are given by*

$$M^+ := \begin{cases}]-b_1^+, -\frac{b_2^+}{a_{11}^+} - b_1^+[& \text{, if } a_{11}^+ > 0 \\ ]-b_1^+, \infty[& \text{, if } a_{11}^+ < 0, \end{cases}$$

$$\hat{M}^+ := \begin{cases}]-\infty, -b_1^+[& \text{, if } a_{11}^+ > 0 \\ ]-\frac{b_2^+}{a_{11}^+} - b_1^+, -b_1^+[& \text{, if } a_{11}^+ < 0. \end{cases}$$

$T^+ :=]0,\infty[$ *is the domain of the switching time* t_0^+.

Proof. The determination of the discrete-time map Π^+ is very technical and therefore located in Appendix D. □

Lemma 4.3 (Properties of the discrete-time map Π^-).
Consider the PWL system (4.1). Then there holds for the half discrete-time map Π^-:

1. $a_{11}^{-\,2} + 4a_{21}^- > 0$, $a_{21}^- \neq 0$ *(saddle/node-case)*
   Π^- *exists if and only if* $b_2^- > 0$.
   Let $\eta^- + \omega^-$ *and* $\eta^- - \omega^-$ *with* $\eta^- := \frac{a_{11}^-}{2}$ *and* $\omega^- := \frac{1}{2}\sqrt{a_{11}^{-\,2} + 4a_{21}^-}$ *be the eigenvalues of* A^-.

   a) *If* $\eta^- - \omega^- < 0$ *and* $\eta^- + \omega^- < 0$ *(stable node),*
   $$\Pi^- : \quad]-\infty, -b_1^-[\quad \longrightarrow \quad]-b_1^-, \frac{b_2^-}{a_{21}^-}(\eta^- + \omega^-) - b_1^-[$$
   is strictly decreasing and concave.

   b) *If* $\eta^- - \omega^- > 0$ *and* $\eta^- + \omega^- > 0$ *(unstable node),*
   $$\Pi^- : \quad]\frac{b_2^-}{a_{21}^-}(\eta^- - \omega^-) - b_1^-, -b_1^-[\quad \longrightarrow \quad]-b_1^-, \infty[$$
   is strictly decreasing and convex.

   c) *If* $\eta^- - \omega^- < 0$ *and* $\eta^- + \omega^- > 0$ *(saddle point),*
   $$\Pi^- : \quad]\frac{b_2^-}{a_{21}^-}(\eta^- - \omega^-) - b_1^-, -b_1^-[\quad \longrightarrow \quad]-b_1^-, \frac{b_2^-}{a_{21}^-}(\eta^- + \omega^-) - b_1^-[$$
   is strictly decreasing, convex if $\eta^- > 0$ *and concave if* $\eta^- < 0$.

   d) *If* $\eta^- = 0$ *(saddle point),*
   $$\Pi^- : \quad]-\frac{b_2^-\omega^-}{a_{21}^-} - b_1^-, -b_1^-[\quad \longrightarrow \quad]-b_1^-, \frac{b_2^-\omega^-}{a_{21}^-} - b_1^-[$$
   is a straight line with slope -1.

   $T^- :=]0, \infty[$ *is the domain of the switching time* t_0^-.

2. $a_{11}^{-\,2} + 4a_{21}^- < 0$, $a_{21}^- \neq 0$ *(focus-case)*
   Π^- *exists for all* $b_2^- \in \mathbb{R}$.
   Let $\eta^- + i\omega^-$ *and* $\eta^- - i\omega^-$ *with* $\eta^- := \frac{a_{11}^-}{2}$ *and* $\omega^- := \frac{1}{2}\sqrt{-a_{11}^{-\,2} - 4a_{21}^-}$ *be the eigenvalues of* A^-.
   $$\Pi^- : M^- \longrightarrow \hat{M}^-$$
   is strictly decreasing, convex if $\eta^- > 0$, *concave if* $\eta^- < 0$ *and a straight line with slope* -1 *if* $\eta^- = 0$. *The domains* M^- *and* $\hat{M}^-$ *are given by*
   $$M^- := \begin{cases}]-\infty, -b_1^-[& , \text{ if } b_2^- \geq 0 \text{ or } \eta^- \geq 0 \\ ]-\infty, \overline{s}^-[& , \text{ if } b_2^- < 0 \text{ and } \eta^- < 0 \end{cases}$$

where $\overline{s}^- := -\frac{b_2^-}{\omega^-}e^{-\eta^- t_1^0}\sin(\omega t_1^0) - b_1^-$ *and* t_1^0 *is the unique zero of* $\Xi_1(t;\eta^-,\omega^-)$ *in* $]\frac{\pi}{\omega^-}, \frac{2\pi}{\omega^-}[$, *see Proposition B.1.*

$$\hat{M}^- := \begin{cases}]-b_1^-, \infty[& , \textit{ if } b_2^- \geq 0 \textit{ or } \eta^- \leq 0 \\ ]-b_1^-, \hat{s}^-[& , \textit{ if } b_2^- < 0 \textit{ and } \eta^- > 0 \end{cases}$$

where $\hat{s}^- := \frac{b_2^-}{a_{21}^-}\frac{\eta^- \sin(\omega^- t_1^0) - 2\omega^- \sinh(\eta^- t_1^0)}{\sin(\omega t_1^0)} - b_1^-$.

$$T^- := \begin{cases}]\frac{\pi}{\omega^-}, t_1^0[& , \textit{ if } b_2^- < 0, \eta^- \neq 0 \\ ]\frac{\pi}{\omega^-}, \frac{2\pi}{\omega^-}[& , \textit{ if } b_2^- < 0, \eta^- = 0 \\ ]0, \frac{\pi}{\omega^-}[& , \textit{ if } b_2^- > 0 \\ \{\frac{\pi}{\omega^-}\} & , \textit{ if } b_2^- = 0 \end{cases}$$

is the domain of the switching time t_0^-.

3. ${a_{11}^-}^2 + 4a_{21}^- = 0$, $a_{11}^- \neq 0 \neq a_{21}^-$ *(node-case)*
   Π^- *exists if and only if* $b_2^- > 0$.
   Let $\eta^- := \frac{a_{11}^-}{2}$ *be the eigenvalue of* A^-.

$$\Pi^- : M^- \longrightarrow \hat{M}^-$$

is strictly decreasing and convex if $\eta^- > 0$ *and concave if* $\eta^- < 0$. *The domains* M^- *and* $\hat{M}^-$ *are given by*

$$M^- := \begin{cases}]-\frac{b_2^-}{\eta^-} - b_1^-, -b_1^-[& , \textit{ if } \eta^- > 0 \\ ]-\infty, -b_1^-[& , \textit{ if } \eta^- < 0, \end{cases}$$

$$\hat{M}^- := \begin{cases}]-b_1^-, \infty[& , \textit{ if } \eta^- > 0 \\ ]-b_1^-, -\frac{b_2^-}{\eta^-} - b_1^-[& , \textit{ if } \eta^- < 0. \end{cases}$$

$T^- :=]0, \infty[$ *is the domain of the switching time* t_0^-.

4. $a_{21}^- = 0, a_{11}^- \neq 0$
   Π^- *exists if and only if* $b_2^- > 0$.

$$\Pi^- : M^- \longrightarrow \hat{M}^-$$

is strictly decreasing and convex if $a_{11}^- > 0$ *and concave if* $a_{11}^- < 0$. *The domains* M^- *and* $\hat{M}^-$ *are given by*

$$M^- := \begin{cases}]-\frac{b_2^-}{a_{11}^-} - b_1^-, -b_1^-[& , \textit{ if } a_{11}^- > 0 \\ ]-\infty, -b_1^-[& , \textit{ if } a_{11}^- < 0, \end{cases}$$

$$\hat{M}^- := \begin{cases}]-b_1^-, \infty[& , \textit{ if } a_{11}^- > 0 \\ ]-b_1^-, -\frac{b_2^-}{a_{11}^-} - b_1^-[& , \textit{ if } a_{11}^- < 0. \end{cases}$$

$T^- :=]0, \infty[$ *is the domain of the switching time* t_0^-.

Remark 4.4 (The cases $\mathbf{a_{11}^+ = a_{21}^+ = 0}$ and $\mathbf{a_{11}^- = a_{21}^- = 0}$).
For a solution in G_+ starting at a point $\binom{0}{s_0}$ with $s_0 > -b_1^+$ there holds

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}b_2^+t^2 + s_0t + b_1^+t \\ b_2^+t + s_0 \end{pmatrix}.$$

$x(t)$ is equal to zero if and only if $t = 0$. This means that $M^+ = \emptyset$ and Π^+ is not defined. Analogous, we have $M^- = \emptyset$ and Π^- is not defined.

4.2. Closed trajectories of type I

First, we summarize the results on non-existence of closed trajectories in G_+, G_- and of type I for the PWL system (4.1) as a consequence of Lemmas 3.6, 3.7, 3.8 and 3.10. From the properties of the discrete-time maps Π^+ and Π^- we can conclude statements on the maximal number of closed trajectories of type I. Afterwards we introduce a general concept for determining them.

Corollary 4.5 (Non-existence of closed trajectories in $\mathbf{G_+}$).
Consider the PWL system (4.1). If we assume that $a_{11}^+ \neq 0$ or ${a_{11}^+}^2 + 4a_{21}^+ \geq 0$ then there exists no closed trajectory in G_+.

Corollary 4.6 (Non-existence of closed trajectories in $\mathbf{G_-}$).
Consider the PWL system (4.1). If we assume that $a_{11}^- \neq 0$ or ${a_{11}^-}^2 + 4a_{21}^- \geq 0$ then there exists no closed trajectory in G_-.

Corollary 4.7 (Non-existence of closed trajectories of type I).
Consider the PWL system (4.1). If one of the conditions

1. $a_{11}^+(b_1^+ - b_1^-) \geq 0$ *and* $a_{11}^-(b_1^+ - b_1^-) \geq 0$
2. $a_{21}^+ < 0$, $a_{11}^+ < 0$, $a_{21}^- < 0$, $a_{11}^- < 0$, $b_2^+ > 0$, $b_2^- < 0$ *and* $\frac{a_{11}^+b_2^+}{a_{21}^+} - b_1^+ \geq \frac{a_{11}^-b_2^-}{a_{21}^-} - b_1^-$

holds, then system (4.1) has no closed trajectory of type I.

From the properties of the discrete-time maps Π^+ and Π^- we can draw consequences on the maximal number of closed trajectories of type I.

Corollary 4.8 (Existence of closed trajectories of type I).
Consider the PWL system (4.1). There exist at most two closed trajectories of type I for all $b_1^\pm, b_2^\pm, a_{21}^\pm \in \mathbb{R}$, provided $a_{11}^+ \neq 0$ or $a_{11}^- \neq 0$. In case of $a_{11}^+ = a_{11}^- = 0$ there either exist no or infinitely many closed trajectories of type I.

Proof. The existence of a closed trajectory of type I containing the point $\binom{0}{s_0}$ with $s_0 \in M^+$ is equivalent to the existence of the non-trivial fixed point s_0 of the Poincaré-map $\Pi := \Pi^- \circ \Pi^+$. As a consequence of Lemma 4.3 we have that Π^- is strictly decreasing and thus invertible:

$$s_0 = \Pi(s_0) = \Pi^-(\Pi^+(s_0)) \quad \Leftrightarrow \quad (\Pi^-)^{-1}(s_0) = \Pi^+(s_0)$$

This means that there exists a closed trajectory of type I if and only if Π^+ and $(\Pi^-)^{-1}$ have an intersection point in their domains. Due to Lemmas 4.2 and 4.3 it follows that Π^+ and Π^- are strictly decreasing and convex, concave or a straight line. Thus $(\Pi^-)^{-1}$ has the same properties as Π^-. Two functions which are strictly decreasing and concave, convex or a straight line can have at most two intersection points, provided that not both functions are straight lines. If both functions are straight lines, they have no intersection points if they are parallel or they have infinitely many if they coincide. This is the case if and only if $a_{11}^+ = a_{11}^- = 0$. $\square$

We introduce now a general concept for determining closed trajectories of type I. A closed trajectory of type I of the PWS system (2.1) is characterised as follows, see section 2.3. Starting from a point $\binom{0}{s_0}$ with $s_0 \in M^+$ the trajectory crosses M at a point $\binom{0}{s_0^*}$ with $s_0^* \in M^-$. Here, the system changes the properties of flow, evolves on G_- until reaching M again at a point $\binom{0}{s_1}$. The trajectory is closed if we have $s_0 = s_1$ which is equivalent to $s_0 = \Pi^-(s_0^*)$. We get the discrete-time maps $\Pi^+(s_0(t_0^+))$ and $\Pi^-(s_0^*(t_0^-))$ from Lemmas 4.2 and 4.3. The times t_0^+ and t_0^- are defined as the minimal numbers greater zero with $\phi_1^\pm(t_0^\pm) = 0$, where $\phi^\pm$ is the solution of system (4.1) in $\overline{G_\pm}$ with initial values $\phi^+(0) = \binom{0}{s_0}$ and $\phi^-(0) = \binom{0}{s_0^*}$, respectively.
In the case of $a_{21}^\pm \neq 0$, using the transition matrix, see Proposition C.1, we obtain

$$\begin{aligned}\phi^+(t) &= \left[e^{A^+t}\left(\begin{pmatrix}0\\ s_0\end{pmatrix} + (A^+)^{-1}\begin{pmatrix}b_1^+\\ b_2^+\end{pmatrix}\right) - (A^+)^{-1}\begin{pmatrix}b_1^+\\ b_2^+\end{pmatrix}\right]\\ &= -\frac{1}{a_{21}^+}\begin{pmatrix} b_2^+(\alpha_{12}^+(t)a_{11}^+ + 1 - \alpha_{11}^+(t)) - \alpha_{12}^+(t)a_{21}^+(s_0 + b_1^+)\\ b_2^+(\alpha_{22}^+(t)a_{11}^+ - a_{11}^+ - \alpha_{21}^+(t)) - \alpha_{22}^+(t)(s_0 + b_1^+) + a_{21}^+ b_1^+\end{pmatrix}.\end{aligned}$$

$\phi_1^+(t_0^+) = 0$ is equivalent to $\alpha_{12}^+(t_0^+) = 0$, $b_2^+ = 0$ and $s_0 > -b_1^+$ arbitrary or

$$s_0 = \frac{b_2^+(\alpha_{12}^+(t_0^+)a_{11}^+ + 1 - \alpha_{11}^+)}{\alpha_{12}^+(t_0^+)a_{21}^+} - b_1^+. \tag{s0}$$

The domains of s_0 and t_0^+ are given in Lemma 4.2. Using equation (s0) we get

$$s_0^* = \phi_2^+(t_0^+) = \frac{b_2^+(\alpha_{22}^+(t_0^+) + \alpha_{12}^+(t_0^+)a_{11}^+ - det(\alpha^+(t_0^+)))}{a_{21}^+\alpha_{12}^+(t_0^+)} - b_1^+. \tag{s0*}$$

Note that s_0 and s_0^* are functions of t_0^+ depending on b_1^+ and b_2^+.

In case of $\alpha_{12}^-(t_0^-) \neq 0$, s_1 is given by $\phi_2^-(t_0^-)$, where ϕ^- is the solution of system (4.1) with initial value $\phi^-(0) = \binom{0}{s_0^*}$ and s_0^* is given by equation (s0*):

$$s_1 = -\frac{1}{a_{21}^-}\left(b_2^-(\alpha_{22}^-(t_0^-)a_{11}^- - a_{11}^- - a_{21}^-(t_0^-)) - \alpha_{22}^-(t_0^-)(s_0^* + b_1^-) - b_1^-\right)$$

The domain of t_0^- is given in Lemma 4.3. Note that s_1 is a function of t_0^- depending on t_0^+, $b_1^\pm$ and $b_2^\pm$.
Assuming that there exists a closed trajectory of type I, there holds $s_1 = s_0$ which is equivalent to

$$s_0(t_0^+; b_1^+, b_2^+) = -\frac{1}{a_{21}^-}\left(b_2^-(\alpha_{22}^-(t_0^-)a_{11}^- - a_{11}^- - a_{21}^-(t_0^-)) - \alpha_{22}^-(t_0^-)(s_0^*(t_0^+; b_1^+, b_2^+) + b_1^-) - b_1^-\right). \tag{4.2}$$

This equation depends on the transition matrices $e^{A^{\pm}t_0^{\pm}}$ which depend for their part on the eigenvalues of $A^{\pm}$. In case of $a_{21}^{\pm} \neq 0$ we have to consider $10 \times 10 = 100$ different cases. In case of either $a_{21}^{+} = 0$ or $a_{21}^{-} = 0$ we get at each time 2×10 more cases and $2 \times 2 = 4$ cases for $a_{21}^{+} = a_{21}^{-} = 0$, see Lemmas 4.2 and 4.3. This means that we have all together 144 different cases. However, in the next chapter we will show that the PWL system (4.1) with $\mathbb{Z}_2-$symmetry has only two parameters b_1, b_2 and that a complete analysis using the above mentioned approach is accessible, see also [Pliete,1998], [Giannakopoulos & Pliete,2001] and [Giannakopoulos & Pliete,2002].

4.3. Non-existence of closed trajectories of type II

In consequence of Lemma 3.5 we get necessary conditions for the existence of closed trajectories of type II for the PWS system (2.1). In the case of the PWL system (4.1) we can prove stronger result.

Lemma 4.9 (Non-existence of closed trajectories of type II).
Consider the PWL system (4.1). *One of the following conditions is necessary for the existence of a closed trajectory of type II:*

1. ${a_{11}^{+}}^2 + 4a_{21}^{+} < 0$, $a_{11}^{+} \neq 0$, $b_1^{+} \neq b_1^{-}$ *and* $b_2^{+} > 0$.
2. ${a_{11}^{-}}^2 + 4a_{21}^{-} < 0$, $a_{11}^{-} \neq 0$, $b_1^{+} \neq b_1^{-}$ *and* $b_2^{-} < 0$.

This means that the existence of a focus in G_+ or G_- is necessary for the existence of a closed trajectory of type II.

Proof. In consequence of Lemma 3.5 we get that the conditions $b_1^{+} \neq b_1^{-}$ and $b_2^{+} > 0$ or $b_2^{-} < 0$ are necessary for the existence of a closed trajectory of type II. So, assume that $b_2^{+} > 0$. We only prove this case. The case $b_2^{-} < 0$ is analogous.

In case of $b_1^{+} < b_1^{-}$, all sliding motion solutions $(0, y(t))$ with $y(0) \geq \min\{y_0, -b_1^{-}\}$ leave $I_s = \{0\} \times [-b_1^{-}, -b_1^{+}]$ at $(0, -b_1^{+})$ after finite time because of the attraction of I_s and $(0, -b_1^{+})$ is a leaving point. Without loss of generality, assume that the solution ϕ leaves I_s at $(0, -b_1^{+})$ at time $t = 0$. Then ϕ evolves in G_+ for $t > 0$ and there holds

$$\begin{aligned} \phi(t) &= e^{A^+t}\left[\begin{pmatrix} 0 \\ -b_1^{+} \end{pmatrix} + (A^+)^{-1}\begin{pmatrix} b_1^{+} \\ b_2^{+} \end{pmatrix}\right] - (A^+)^{-1}\begin{pmatrix} b_1^{+} \\ b_2^{+} \end{pmatrix} \\ &= -\frac{b_2^{+}}{a_{21}^{+}}\begin{pmatrix} \alpha_{12}^{+}(t)a_{11}^{+} + 1 - \alpha_{11}^{+}(t) \\ \alpha_{22}^{+}(t)a_{11}^{+} - a_{11}^{+} - \alpha_{21}^{+}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^{+} \end{pmatrix} \end{aligned}$$

if $a_{21}^{+} \neq 0$ and

$$\phi(t) = \frac{b_2^{+}}{{a_{11}^{+}}^2}\begin{pmatrix} e^{a_{11}^{+}t} - 1 - a_{11}^{+}t \\ {a_{11}^{+}}^2 t \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^{+} \end{pmatrix}$$

if $a_{21}^{+} = 0$. Assume now ${a_{11}^{+}}^2 + 4a_{21}^{+} \geq 0$ or $a_{11}^{+} = 0$. A necessary condition for the existence of a closed trajectory of type II is that ϕ intersects with M again after finite time $t > 0$. But if we consider the first coordinate $\phi_1(t)$ of the solution with $t > 0$ we get that

$$\phi_1(t) = \begin{cases} -\frac{b_2^+}{a_{21}^+}(\alpha_{12}^+(t)a_{11}^+ + 1 - \alpha_{11}^+(t)) & \text{, if } a_{21}^+ \neq 0 \\ \frac{b_2^+}{{a_{11}^+}^2}(e^{a_{11}^+ t} - 1 - a_{11}^+ t) & \text{, if } a_{21}^+ = 0, a_{11}^+ \neq 0 \end{cases}$$

$$= \begin{cases} -\frac{b_2^+}{a_{21}^+\omega^+}\Xi_1(t;0,\omega^+) & \text{, if } a_{21}^+ < 0, a_{11}^+ = 0 \\ -\frac{b_2^+}{\omega^+}\frac{\Xi_2(t;\eta^+,\omega^+)}{a_{21}^+} & \text{, if } {a_{11}}^2 + 4a_{21}^+ > 0, a_{21}^+ \neq 0 \\ -\frac{b_2^+ e^{\eta^+ t}}{a_{21}^+}\Xi_5(t;\eta^+) & \text{, if } {a_{11}^+}^2 + 4a_{21}^+ = 0, a_{11}^+ \neq 0 \neq a_{21}^+ \\ -\frac{b_2^+}{{a_{11}^+}^2}\Xi_7(t;a_{11}^+) & \text{, if } a_{21}^+ = 0, a_{11}^+ \neq 0 \end{cases}$$

is greater zero for all $t > 0$ or in case of $a_{11}^+ = 0$ and $a_{21}^+ < 0$ for all $t \in]0, \frac{2\pi}{\omega^+}[$. Thereby, we have $\eta^+ := \frac{a_{11}^+}{2}$, $\omega^+ := \frac{1}{2}\sqrt{|{a_{11}^+}^2 + 4a_{21}^+|}$ and Ξ_1, Ξ_2, Ξ_5 and Ξ_7 as in Propositions B.1, B.2, B.5 and B.7.

In case of $b_1^+ > b_1^-$, a solution in G_+ can reach $I_s = \{0\} \times [-b_1^+, -b_1^-]$ only at the reaching point $(0, -b_1^+)$. Without loss of generality, assume that the solution ϕ reaches I_s at $(0, -b_1^+)$ at time $t = 0$. Then ϕ evolves in G_+ for $-t < 0$ and there holds

$$\begin{aligned} \phi(-t) &= e^{-A^+ t}\left[\begin{pmatrix} 0 \\ -b_1^+ \end{pmatrix} + (A^+)^{-1}\begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix}\right] - (A^+)^{-1}\begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \\ &= -\frac{b_2^+}{a_{21}^+}\begin{pmatrix} \alpha_{12}^+(-t)a_{11}^+ + 1 - \alpha_{11}^+(-t) \\ \alpha_{22}^+(-t)a_{11}^+ - a_{11}^+ - \alpha_{21}^+(-t) \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^+ \end{pmatrix} \end{aligned}$$

if $a_{21} \neq 0$ and

$$\phi(-t) = \frac{b_2^+}{{a_{11}^+}^2}\begin{pmatrix} e^{-a_{11}^+ t} - 1 + a_{11}^- t \\ -{a_{11}^+}^2 t \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^+ \end{pmatrix}$$

if $a_{21}^+ = 0$. Assume now ${a_{11}^+}^2 + 4a_{21}^+ \geq 0$ or $a_{11}^+ = 0$. A necessary condition for the existence of a closed trajectory of type II is that ϕ intersects with M again after finite time $-t < 0$. But if we consider the first coordinate $\phi_1(-t)$ of the solution with $-t < 0$ we get that

$$\phi_1(-t) = \begin{cases} -\frac{b_2^+}{a_{21}^+\omega^+}\Xi_1(t;0,\omega^+) & \text{, if } a_{21}^+ < 0, a_{11}^+ = 0 \\ -\frac{b_2^+}{\omega^+}\frac{\Xi_2(t;-\eta^+,\omega^+)}{a_{21}^+} & \text{, if } {a_{11}^+}^2 + 4a_{21}^+ > 0, a_{21}^+ \neq 0 \\ -\frac{b_2^+ e^{\eta^+ t}}{a_{21}^+}\Xi_5(t;-\eta^+) & \text{, if } {a_{11}^+}^2 + 4a_{21}^+ = 0, a_{11}^+ \neq 0 \neq a_{21}^+ \\ -\frac{b_2^+}{{a_{11}^+}^2}\Xi_7(t;-a_{11}^+) & \text{, if } a_{21}^+ = 0, a_{11}^+ \neq 0 \end{cases}$$

is greater zero for all $-t < 0$ or $-t \in]0, \frac{2\pi}{\omega^+}[$ in case of $a_{21}^+ < 0$ and $a_{11}^+ = 0$. From this we obtain the assertion. □

Example 4.10 (Geometrical proof for the saddle case).
In [Giannakopoulos & Pliete,2002] we use a more geometrical approach to prove the non-existence of closed trajectories of type II in G_+ for the saddle-case, i.e. for the case $a_{11}^{+\,2} + 4a_{21}^+ > 0$, $a_{21}^+ > 0$.

Lemma 4.11.

1. *System* (4.1) *has a sliding motion solution if and only if* $b_1^+ \neq b_1^-$ *and* $b_2^+ \neq b_2^-$.
2. *If* $b_1^+ < b_1^-$ *and* $b_2^+ < 0$*, there are no sliding motion solutions which can leave* I_s *into* G_+*, see Figure 4.1(a).*
3. *If* $b_1^+ < b_1^-$ *and* $b_2^+ > 0$*, all sliding motion solutions leaving* I_s *at* $(0, -b_1^+)$ *after a finite time, evolve in* G_+ *and become unbounded as* $t \to \infty$*, see Figure 4.1(b).*
4. *If* $b_1^+ > b_1^-$ *and* $b_2^+ < 0$*, there are no solutions of* G_+ *which enter* $\mathring{I}_s$*, see Figure 4.1(c).*
5. *If* $b_1^+ > b_1^-$ *and* $b_2^+ > 0$*, any solution with sliding motion reaching* I_s *at* $(0, -b_1^+)$ *from* G_+ *satisfies either* $(x(t), y(t)) \to (0, y_0)$ *if* $b_2^- < 0$, $|x(t)| + |y(t)| \to \infty$ *as* $t \to \infty$ *or* $(x(t), y(t))$ *evolves in* G_-*, see Figure 4.1(d).*

Note that if $b_1^+ \neq b_1^-$ *and* $b_2^+ > 0$ *the domain* $D^+ := \left\{ \binom{x}{y} \in \mathbb{R}^2 : x > 0, y > -a_{11}^+ x - b_1^+ \right\}$ *is positive invariant under the flow on* G_+.

4.4. Bifurcation of a periodic solution with sliding motion in $\overline{G_\pm}$

In section 3.5 we proved the bifurcation of a periodic solution in $\overline{G_\pm}$ with sliding motion for the PWS system (2.1). We could only prove a local result, i.e. for sufficiently small values of the parameters $x_0^\pm$. In case of the PWL system (4.1) we can prove a global analogue. We can determine the whole parameter interval for which the periodic solution exists and can show that this periodic solution becomes homoclinic and disappears.

4.4.1. Main results

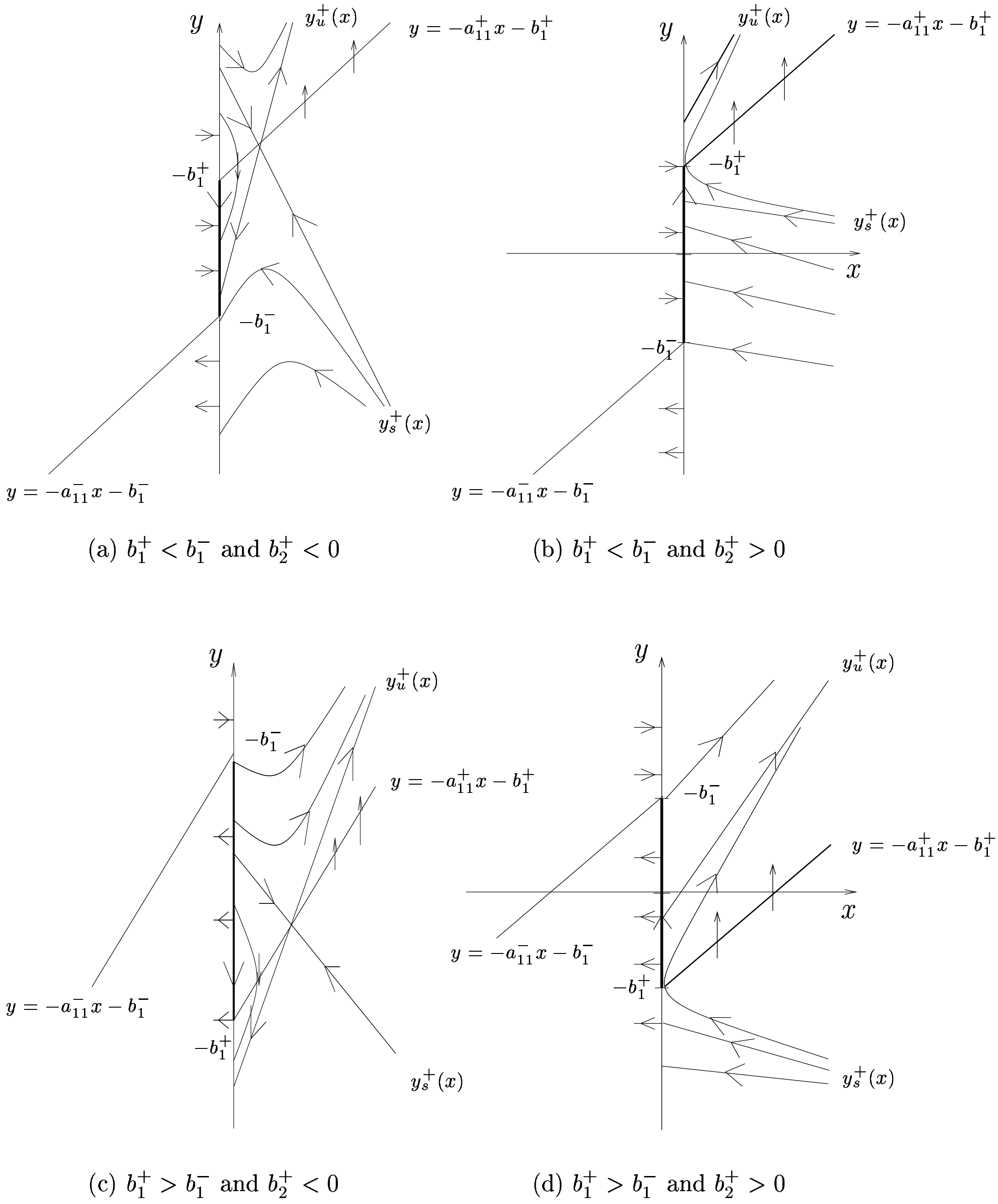
Theorem 4.12 (Hopf-like bifurcation of a periodic solution with sliding motion in $\overline{\mathbf{G}_+}$).
Consider the PWL system (4.1)*. Assume that*

(L1+) $a_{11}^{+\,2} + 4a_{21}^+ < 0$, $a_{11}^+ \neq 0$;

(L2+) $a_{11}^+(b_1^+ - b_1^-) < 0$.

Then

1. *there exist no equilibria and no closed trajectories in* $\overline{G_+}$*, provided* $b_2^+ < 0$;



(a) $b_1^+ < b_1^-$ and $b_2^+ < 0$

(b) $b_1^+ < b_1^-$ and $b_2^+ > 0$

(c) $b_1^+ > b_1^-$ and $b_2^+ < 0$

(d) $b_1^+ > b_1^-$ and $b_2^+ > 0$

Figure 4.1.: Phase portrait in a neighbourhood of I_s for the saddle-case, i.e. ${a_{11}^+}^2 + 4a_{21}^+ > 0$, $a_{21}^+ > 0$ and $a_{11}^\pm < 0$. Thereby, $y_u^+(x)$ and $y_s^+(x)$ are the unstable and stable manifolds of the saddle point (x_0^+, y_0^+), respectively.

2. *there exists a focus* $(x_0^+, y_0^+) := \left(-\frac{b_2^+}{a_{21}^+}, \frac{a_{11}^+ b_2^+}{a_{21}^+} - b_1^+\right)$ *for all* $b_2^+ > 0$. *It is unstable (stable) if* $a_{11}^+ > 0$ $(a_{11}^+ < 0)$;

3. *there exists a number* $b_2^{p+} > 0$ *such that there exists a periodic solution with sliding motion in* $\overline{G_+}$ *for all* $b_2^+ \in]0, b_2^{p+}[$. *It surrounds* (x_0^+, y_0^+) *and is stable (unstable) if* $a_{11}^+ > 0$ $(a_{11}^+ < 0)$;

4. *there exists a homoclinic solution with sliding motion in* $\overline{G_+}$ *to the equilibrium* $(0, y_0) \in I_s$ *for* $b_2^+ = b_2^{p+}$, *provided* $b_2^- < 0$. *It is unstable (stable) if* $a_{11}^+ > 0$ $(a_{11}^+ < 0)$;

5. *there exists no homoclinic or periodic solution with sliding motion in* $\overline{G_+}$ *for* $b_2^+ > b_2^{p+}$.

Theorem 4.13 (Hopf-like bifurcation of a periodic solution with sliding motion $\overline{\mathbf{G}_-}$).
Consider the PWL system (4.1). *Assume that*

(L1-) $a_{11}^{-\,2} + 4a_{21}^- < 0$, $a_{11}^- \neq 0$;

(L2-) $a_{11}^-(b_1^+ - b_1^-) < 0$.

Then

1. *there exist no equilibria and no closed trajectories in* $\overline{G_-}$, *provided* $b_2^- > 0$;

2. *there exists a focus* $(x_0^-, y_0^-) := \left(-\frac{b_2^-}{a_{21}^-}, \frac{a_{11}^- b_2^-}{a_{21}^-} - b_1^-\right)$ *for all* $b_2^- < 0$. *It is unstable (stable) if* $a_{11}^- > 0$ $(a_{11}^- < 0)$;

3. *there exists a number* $b_2^{p-} < 0$ *such that there exists a periodic solution with sliding motion in* $\overline{G_-}$ *for all* $b_2^- \in]b_2^{p-}, 0[$. *It surrounds* (x_0^-, y_0^-) *and is stable (unstable) if* $a_{11}^- > 0$ $(a_{11}^- < 0)$;

4. *there exists a homoclinic solution with sliding motion in* $\overline{G_-}$ *to the equilibrium* $(0, y_0) \in I_s$ *for* $b_2^- = b_2^{p-}$, *provided* $b_2^+ > 0$. *It is unstable (stable) if* $a_{11}^- > 0$ $(a_{11}^- < 0)$;

5. *there exists no homoclinic or periodic solution with sliding motion in* $\overline{G_-}$ *for* $b_2^- < b_2^{p-}$.

Remark 4.14.
In the case of $b_2^+ = b_2^{p+}$ $(b_2^- = b_2^{p-})$ *and* $b_2^- \geq 0$ $(b_2^+ \leq 0)$ *there is no equilibrium in* I_s. *Then the periodic solution still exists and no homoclinic solution exists.*

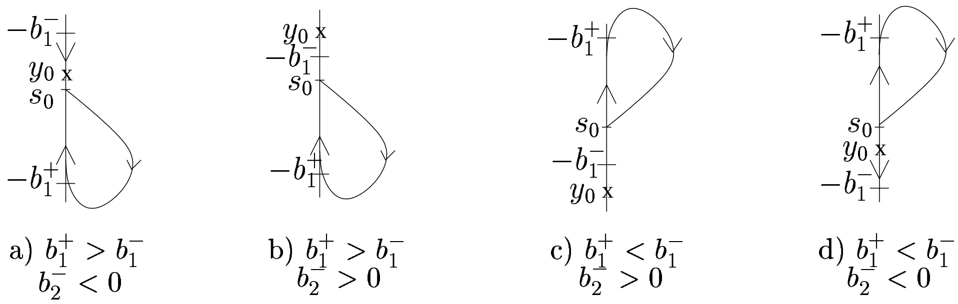


Figure 4.2.: Periodic solutions with sliding motion in $\overline{G_+}$ for $b_2^+ > 0$

4.4.2. Proof of Theorem 4.12

There exists an equilibrium $(x_0^+, y_0^+) := \left(-\frac{b_2^+}{a_{21}^+}, \frac{a_{11}^+ b_2^+}{a_{21}^+} - b_1^+\right)$ in G_+ of system (4.1) if and only if $b_2^+ > 0$ and then it is a focus, see $(L1+)$. As a consequence of Corollary 4.5 we get that there exist no closed trajectories in G_+.

In case of $b_1^+ > b_1^-$, the sliding motion interval $I_s = \{0\} \times [-b_1^+, -b_1^-]$ is repulsive. For the existence of a closed trajectory with sliding motion in $\overline{G_+}$ it is necessary that $(0, -b_1^+)$ is a reaching point. So, we get from Lemma 3.3 the necessary condition $b_2^+ > 0$. We consider the reaching solution ϕ of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A^+ \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \tag{4.3}$$

with initial value $(\phi_1(0), \phi_2(0)) = (0, -b_1^+)$. ϕ must reach I_s after finite time $-t_0 < 0$ at a point $(0, s_0)$ with $s_0 \leq \min\{y_0, -b_1^-\}$, where

$$y_0 := \frac{b_1^+ b_2^- - b_1^- b_2^+}{b_2^+ - b_2^-}.$$

This means that $(0, s_0) \in I_s$ and lies on the $y-$axis below the equilibrium $(0, y_0)$ if it exists, see Figures 4.2 a) and b).

In case $b_1^+ < b_1^-$, the sliding motion interval $I_s = \{0\} \times [-b_1^-, -b_1^+]$ is attractive. For the existence of a closed trajectory with sliding motion in $\overline{G_+}$ it is necessary that $(0, -b_1^+)$ is a leaving point. Again, we get from Lemma 3.3 the necessary condition $b_2^+ > 0$. We consider now the leaving solution ϕ of system (4.3) with initial value $(\phi_1(0), \phi_2(0)) = (0, -b_1^+)$. ϕ must reach I_s again after finite time $t_0 > 0$ at a point $(0, s_0)$ with $s_0 \geq \max\{y_0, -b_1^-\}$. This means that $(0, s_0) \in I_s$ and lies on the $y-$axis above the equilibrium $(0, y_0)$ if it exists, see Figure 4.2 c) and d).
Our next step is to determine the solution of the initial value problem. Defining

$$\eta^+ := \frac{a_{11}^+}{2} \text{ and}$$
$$\omega^+ := \frac{1}{2}\sqrt{-{a_{11}^+}^2 - 4a_{21}^+}$$

we get for the solution of the initial value problem

$$\phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{A^+t}\left(\begin{pmatrix} 0 \\ -b_1^+ \end{pmatrix} + (A^+)^{-1}\begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix}\right) - (A^+)^{-1}\begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix}.$$

Based on the definition of the transition matrix $\alpha^+(t) := e^{A^+t}$, see Proposition C.1, we get

$$\begin{aligned}
\phi(t) &= -\frac{b_2^+}{a_{21}^+}\begin{pmatrix} \alpha_{12}^+(t)a_{11}^+ + 1 - \alpha_{11}^+(t) \\ \alpha_{22}^+(t)a_{11}^+ - a_{11}^+ - \alpha_{21}^+(t) \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^+ \end{pmatrix} \\
&= -\frac{b_2^+}{\omega^+ a_{21}^+}\begin{pmatrix} e^{\eta^+t}(\eta^+\sin(\omega^+t) - \omega^+\cos(\omega^+t)) + \omega^+ \\ e^{\eta^+t}(\sin(\omega^+t)(\omega^{+2} - \eta^{+2}) + 2\omega^+\eta^+\cos(\omega^+t)) - 2\eta^+\omega^+ \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^+ \end{pmatrix}.
\end{aligned}$$

Assuming $b_2^+ > 0$ we verify for which conditions closed trajectories exit.

Case 1: $b_1^+ > b_1^-$: Assume that ϕ reaches M again after finite time $-t_0 < 0$ at $(0, s_0)$. Then there holds

$$\phi_1(-t_0) = 0 \Leftrightarrow \Xi_1(t_0; \eta^+, \omega^+) = 0$$

with

$$\Xi_1(t; \eta, \omega) := -e^{-\eta t}(\eta\sin(\omega t) + \omega\cos(\omega t)) + \omega. \tag{4.4}$$

From Proposition B.1 we get that there exists no solution of equation (4.4) , provided $\eta^+ \geq 0$. For $\eta < 0$ we get that $t_0 = t_1^0 \in]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$ solves equation (4.4). According to these assumptions we obtain for the intersection point $(0, s_0)$

$$\begin{aligned}
s_0 &= \phi_2(-t_1^0) \\
&= -\frac{b_2^+}{\omega^+ a_{21}^+}(e^{-\eta^+t_1^0}(-\sin(\omega^+t_1^0)(\omega^{+2} - \eta^{+2}) + 2\eta^+\omega^+\cos(\omega^+t_1^0)) - 2\eta^+\omega^+) - b_1^+ \\
&= \frac{-b_2^+e^{-\eta^+t_1^0}\sin(\omega^+t_1^0)}{\omega^+} - b_1^+.
\end{aligned}$$

Because of Lemma 3.1 we get $(0, y_0) \in I_s$ if and only if $b_2^+b_2^- < 0$. Therefore, the periodic solution with sliding motion exits for all $b_2^+ \in]0, b_2^{p+}[$ with

$$b_2^{p+} := \begin{cases} \frac{(b_1^- - b_1^+)\omega^+e^{\eta^+t_1^0}}{\sin(\omega^+t_1^0)} + b_2^- & \text{, if } b_2^- \leq 0 \\ \frac{(b_1^- - b_1^+)\omega^+e^{\eta^+t_1^0}}{\sin(\omega^+t_1^0)} & \text{, if } b_2^- > 0. \end{cases}$$

Case 2: $b_1^+ < b_1^-$: Assume that ϕ reaches M again after finite time $t_0 > 0$ at $(0, s_0)$. Then there holds

$$\phi_1(t_0) = 0 \Leftrightarrow \Xi_1(t_0; -\eta^+, \omega^+) = 0. \tag{4.5}$$

From Proposition B.1 we get that there exists no solution of equation (4.5), provided $\eta^+ \leq 0$. For $\eta^+ > 0$ we get that $t_0 = t_1^0 \in]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$ solves equation (4.5). According to these assumptions we obtain for the intersection point $(0, s_0)$

$$s_0 = \phi_2(t_1^0) = \frac{b_2^+e^{\eta^+t_1^0}\sin(\omega^+t_1^0)}{\omega^+} - b_1^+.$$

In this case, the periodic solution exists for all $b_2^+ \in]0, b_2^{p+}[$ with

$$b_2^{p+} := \begin{cases} \frac{(b_1^+ - b_1^-)\omega^+ e^{-\eta^+ t_1^0}}{\sin(\omega^+ t_1^0)} + b_2^- & , \text{if } b_2^- \leq 0 \\ \frac{(b_1^+ - b_1^-)\omega^+ e^{-\eta^+ t_1^0}}{\sin(\omega^+ t_1^0)} & , \text{if } b_2^- > 0. \end{cases}$$

In case of $b_2^- < 0$ the equilibrium $(0, y_0)$ exists and lies in I_s such that the periodic solution becomes a homoclinic solution to $(0, y_0)$, provided $b_2^+ = b_2^{p+}$.

□

4.4.3. Proof of Theorem 4.13

We can prove this theorem analogous to the proof of Theorem 4.12. There exists an equilibrium $(x_0^-, y_0^-) := \left(-\frac{b_2^-}{a_{21}^-}, \frac{a_{11}^- b_2^-}{a_{21}^-} - b_1^-\right)$ in G_- of system (4.1) if and only if $b_2^- < 0$ and then it is a focus, see $(L1-)$. As a consequence of Corallary 4.6 we know that there exist no closed trajectories in G_-. For the existence of a closed trajectory with sliding motion in $\overline{G_-}$ it is necessary that $(0, -b_1^-)$ is a reaching or leaving point. So we get from Lemma 3.3 the necessary condition $b_2^- < 0$.
Defining

$$\eta^- := \frac{a_{11}^-}{2} \text{ and}$$
$$\omega^- := \frac{1}{2}\sqrt{-{a_{11}^-}^2 - 4a_{21}^-}$$

we obtain for the solution of the initial value problem

$$\begin{aligned} \phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{A^- t}\left(\begin{pmatrix} 0 \\ -b_1^- \end{pmatrix} + (A^-)^{-1}\begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix}\right) - (A^-)^{-1}\begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \\ &= -\frac{b_2^-}{\omega^- a_{21}^-}\begin{pmatrix} e^{\eta^- t}(\eta^- \sin(\omega^- t) - \omega^- \cos(\omega^- t)) + \omega^- \\ e^{\eta^- t}(\sin(\omega^- t)({\omega^-}^2 - {\eta^-}^2) + 2\omega^- \eta^- \cos(\omega^- t)) - 2\eta^- \omega^- \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^- \end{pmatrix}. \end{aligned}$$

Assuming $b_2^- < 0$ we verify for which conditions closed trajectories exist.

<u>Case 1:</u> $b_1^+ > b_1^-$: We get for $\eta^+ < 0$

$$s_0 = \phi_2(-t_1^0) = \frac{-b_2^- e^{-\eta^- t_1^0} \sin(\omega^- t_1^0)}{\omega^-} - b_1^-,$$

where t_1^0 is the unique solution of $\Xi_1(t; \eta^-, \omega^-)$ in $]\frac{\pi}{\omega^-}, \frac{2\pi}{\omega^-}[$, see Proposition B.1. The periodic solution with sliding motion exits for all $b_2^- \in]b_2^{p-}, 0[$ with

$$b_2^{p-} := \begin{cases} \frac{(b_1^+ - b_1^-)\omega^- e^{\eta^- t_1^0}}{\sin(\omega^- t_1^0)} + b_2^+ & , \text{if } b_2^+ \geq 0 \\ \frac{(b_1^+ - b_1^-)\omega^- e^{\eta^- t_1^0}}{\sin(\omega^- t_1^0)} & , \text{if } b_2^+ < 0. \end{cases}$$

Case 2: $b_1^+ < b_1^-$: We get for $\eta^+ > 0$

$$s_0 = \phi_2(t_1^0) = \frac{b_2^- e^{\eta^- t_1^0} \sin(\omega^- t_1^0)}{\omega^-} - b_1^-,$$

where t_1^0 is the unique solution of $\Xi_1(t; -\eta^-, \omega^-)$ in $]\frac{\pi}{\omega^-}, \frac{2\pi}{\omega^-}[$, see Proposition B.1. In this case, the periodic solution exists for all $b_2^- \in]b_2^{p-}, 0[$ with

$$b_2^{p-} := \begin{cases} \frac{(b_1^- - b_1^+)\omega^- e^{-\eta^- t_1^0}}{\sin(\omega^- t_1^0)} + b_2^+ & \text{, if } b_2^+ \geq 0 \\ \frac{(b_1^- - b_1^+)\omega^- e^{-\eta^- t_1^0}}{\sin(\omega^- t_1^0)} & \text{, if } b_2^+ < 0. \end{cases}$$

In case of $b_2^+ > 0$ the equilibrium $(0, y_0)$ exists and lies in I_s such that the periodic solution becomes a homoclinic solution to $(0, y_0)$, provided $b_2^- = b_2^{p-}$.

□

Remark 4.15 (Bifurcation diagram).
We present a bifurcation diagram of the Hopf-like bifurcation of a periodic solution with sliding motion in $\overline{G_+}$ in case of $b_2^+ > 0$ and $b_2^- < 0$. In that case the equilibrium $(0, y_0)$ exists and it is unstable (stable) if $b_1^+ > b_1^-$ ($b_1^+ < b_1^-$). The focus (x_0^+, y_0^+) also exists and it is stable (unstable) if $a_{11}^+ < 0$ ($a_{11}^+ > 0$) which is because of (L2+) equivalent to $b_1^+ > b_1^-$ ($b_1^+ < b_1^-$). We draw the length of the sliding motion part of the periodic solution against b_2^+. Note that $y_0^+ = \frac{a_{11}^+ b_2^+}{a_{21}^+} - b_1^+$ and s_0 are linear in b_2^+, see Figure 4.3.

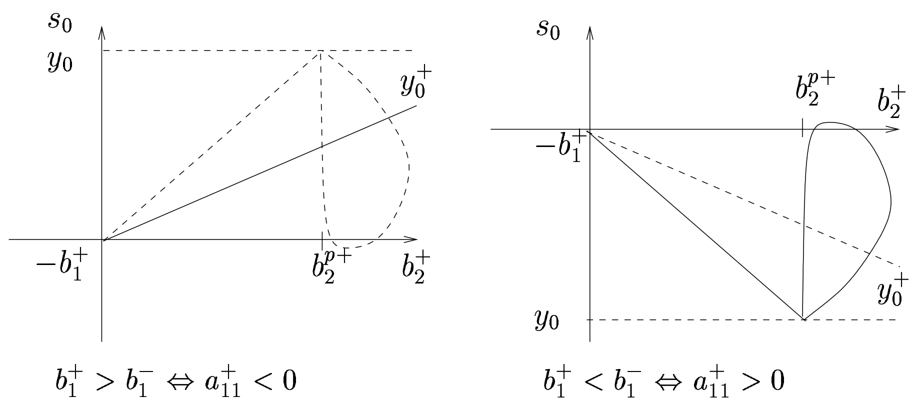


Figure 4.3.: Homoclinic bifurcation from a periodic solution in $\overline{G_+}$ at $b_2 = b_2^{p+}$

As mentioned at the end of section 4.2, only for determining closed trajectories of type I we have to consider 144 different cases, depending on the eigenvalues of $A^\pm$. Assuming $\mathbb{Z}_2$−symmetry for the PWL system (4.1), the 144 different cases can be reduced to only 5 different cases. For these 5 cases we can provide a complete analysis with global results and bifurcation diagrams depending on two parameters. Therefore, in the next chapter we consider system (4.1) with $\mathbb{Z}_2$−symmetry.

5. Results for the piecewise-linear system with $\mathbb{Z}_2-$symmetry

As mentioned before we consider now PWL systems with $\mathbb{Z}_2-$symmetry, because in that case we can provide a complete analysis including global results and bifurcation diagrams. Therefore, throughout this chapter we assume that system (4.1) is $\mathbb{Z}_2-$symmetric. This means if $(x(t), y(t))$ is also a solution of (4.1) then $(-x(t), -y(t))$ is a solution. After deriving a special normal form for that system, we investigate a complete analysis and present bifurcation diagrams for five different cases depending on the kind of equilibria in $G_{\pm}$. A first step is done in [Pliete,1998] and the focus- and saddle-cases are investigated in a modified form in [Giannakopoulos & Pliete,2001] and [Giannakopoulos & Pliete,2002], respectively.

5.1. Preliminaries

In this section we first derive a normal form of system (4.1) with $\mathbb{Z}_2-$symmetry. Afterwards we summerize all results on singular points, existence and non-uniqueness of the initial value problem and non-existence of closed trajectories. Finally, we derive a concept for determining existence and stability of closed trajectories of type I.

5.1.1. Derivation of the PWL system with $\mathbb{Z}_2-$symmetry

In this subsection we derive a normal form for the PWL system (4.1) with $\mathbb{Z}_2-$symmetry which is investigated in the following.
As shown in section 3.4, the symmetry properties lead to

$$A^+ \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} = A^- \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix}.$$

We thus get the conditions

$$\begin{aligned} b_1^+ &= -b_1^- =: \beta_1 \\ b_2^+ &= -b_2^- =: \beta_2 \\ a_{11}^+ &= a_{11}^- =: a_{11} \\ a_{21}^+ &= a_{21}^- =: a_{21} \end{aligned}$$

in system (4.1) and thus the PWL system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} a_{11} & 1 \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \operatorname{sgn}(u) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \tag{5.1}$$

Using time rescaling, we transform system (5.1) to an equivalent system in a special normal form.

Lemma 5.1. *System* (5.1) *is equivalent to system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_{p\delta} \begin{pmatrix} x \\ y \end{pmatrix} + sgn(x) b, \tag{5.2}$$

where

$$\begin{aligned} A_{p\delta} &= \begin{pmatrix} -\delta & 1 \\ -p & 0 \end{pmatrix}, \\ p &= \begin{cases} -\frac{a_{21}}{a_{11}} & , \text{ if } a_{11} \neq 0 \\ -a_{21} & , \text{ if } a_{11} = 0 \end{cases}, \\ \delta &= \begin{cases} 1 & , \text{ if } a_{11} \neq 0 \\ 0 & , \text{ if } a_{11} = 0 \end{cases} \end{aligned} \tag{5.3}$$

and $b = (b_1, b_2)^T$ *with*

$$\begin{aligned} b_1 &= \begin{cases} -\frac{\beta_1}{a_{11}} & , \text{ if } a_{11} \neq 0 \\ \beta_1 & , \text{ if } a_{11} = 0 \end{cases}, \\ b_2 &= \begin{cases} \frac{\beta_2}{a_{11}^2} & , \text{ if } a_{11} \neq 0 \\ \beta_2 & , \text{ if } a_{11} = 0 \end{cases}. \end{aligned}$$

Proof. Setting

$$\begin{aligned} x(t) &:= u(-\frac{t}{a_{11}}) \\ y(t) &:= -\frac{1}{a_{11}} v(-\frac{t}{a_{11}}) \end{aligned}$$

in case of $a_{11} \neq 0$ proves the lemma. $\square$

Note that if $\operatorname{tr}(A) = a_{11} > 0$, the transformation of system (5.1) to system (5.2) causes a time reversing. This implicates that stable equilibria and periodic solutions become unstable, and unstable equilibria (except saddle points) and periodic solutions become stable.

Equation (5.2) is the system we investigate in this chapter. We assume $\delta \in \{0, 1\}$ and $p, b_1, b_2 \in \mathbb{R}$.

5.1.2. Singular points, sliding motion, existence and non-uniqueness of the initial value problem

In this subsection we summarize the results on solutions of system (5.2) with initial values $(x(0), y(0)) = (x_0, y_0)$, see section 2.1 and Lemma 3.4. Afterwards we determine and characterise all singular points of system (5.2).

Lemma 5.2 (Sliding motion interval and vector field).
Consider system (5.2). *There holds*

1. *for the sliding motion interval* I_s*:*
   a) *if* $b_1 > 0$*, then* $I_s = \{0\} \times [-b_1, b_1]$ *is repulsive;*
   b) *if* $b_1 < 0$*, then* $I_s = \{0\} \times [b_1, -b_1]$ *is attractive;*
   c) *if* $b_1 = 0$*, then* $I_s = \{(0,0)\}$*;*
2. *for the vector field on* I_s *in case of* $b_1 \neq 0$*:*

$$f^0(y) = -\frac{b_2}{b_1} y$$

All results on existence and non-uniqueness of the initial value problem and singular points in I_s, given in Lemmas 3.4, 3.1 and 3.3, still hold if we set $b_1^+ = -b_1^- = b_1$, $b_2^+ = -b_2^- = b_2$ and $y_0 = 0$.

The next lemma provides a characterisation of the equilibria in $G_\pm$, see Lemma 4.1.

Lemma 5.3 (Equilibria in $\mathbf{G}_\pm$).
There are equilibria
$(x_0^\pm, y_0^\pm) := (\pm\frac{b_2}{p}, \pm\frac{\delta b_2}{p} \mp b_1)$ *in* $G_\pm$ *if and only if* $p \neq 0$ *and* $\frac{b_2}{p} > 0$.

1. *If* $p > 0$*,* $\delta = 0$*, then* $(x_0^\pm, y_0^\pm)$ *are centers.*
2. *If* $p < 0$*, then* $(x_0^\pm, y_0^\pm)$ *are saddle points.*
3. *If* $0 < p \leq \frac{1}{4}$*,* $\delta = 1$*, then* $(x_0^\pm, y_0^\pm)$ *are stable nodes.*
4. *If* $p > \frac{1}{4}$*,* $\delta = 1$*, then* $(x_0^\pm, y_0^\pm)$ *are stable foci.*

5.2. Existence and non-existence of closed trajectories

In this section we summarize the results on existence and non-existence of closed trajectories for system (5.2), see Lemmas 3.8, 3.10, 3.5, Theorem 3.14 and Corollary 4.8. As a consequence of Corollaries 4.5 and 4.6 we obtain the non-existence of closed trajectories in $G_\pm$ if $p \leq \frac{1}{4}$ or $\delta = 1$. In the case of $p > 0$, $\delta = 0$ and $b_2^+ > 0$ the equilibria $(x_0^\pm, y_0^\pm)$ exist and are centers.

Corollary 5.4 (Non-existence of closed trajectories of type I).
Consider system (5.2).

1. *If $\delta = 1$, the condition $b_1 > 0$ is necessary for the existence of closed trajectories of type I and then there exist at most two.*
2. *If $\delta = 0$, the condition $b_1 \geq 0$ is necessary for the existence of closed trajectories of type I and then there exist none or infinitely many.*
3. *If $\delta = 1$, $p > 0$, the condition $b_2 < b_1 p$ is necessary for the existence of closed trajectories of type I.*

Corollary 5.5 (Non-existence of closed trajectories of type II).
Consider system (5.2). *The conditions $b_1 > 0$, $b_2 > 0$, $\delta = 1$ and $p > \frac{1}{4}$ are necessary for the existence of closed trajectories of type II.*

Corollary 5.6 (Existence of closed trajectories of type I).
Consider system (5.2). *Assume that $b_1 > 0$, $b_2 < 0$, $\delta = 1$ and $p \geq 0$. Then there exists a unique stable closed trajectory of type I surrounding I_s.*

5.2.1. Determination of existence and stability of closed trajectories of type I

In the next sections we will detect closed trajectories of type I in different cases for system (5.2) depending on δ and p. Hence, we develop a general concept for determining closed trajectories of type I and their orbital stability.

Existence

For detecting closed trajectories of type I, we consider solutions of system (5.2) which start at a point $\binom{0}{s_0}$ with $s_0 \in M^+$ at $t = 0$, enter G_+ and cross M at a point $\binom{0}{s_0^*}$ with $s_0^* \in M^-$ after finite time $t_0^+ \in T^+$. Then they evolve in G_- until reaching M again at a point $\binom{0}{s_1}$ after finite time $t_0^- \in T^-$. For the definitions of $M^\pm$ and $T^\pm$ see Lemmas 4.2 and 4.3. The corresponding trajectories are closed if we have $s_0 = s_1$. From the symmetry properties of system (2.1) we obtain $t^* := t_0^+ = t_0^-$ and $s_0^* = -s_0$ in case of a closed trajectory.
If $\alpha_{12}(t^*) \neq 0$, from equation (s0) we get

$$s_0 = \frac{b_2}{p} \frac{\alpha_{11}(t^*) + \delta\alpha_{12}(t^*) - 1}{\alpha_{12}(t^*)} - b_1, \tag{5.4}$$

where

$$\alpha(t) := \begin{pmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{pmatrix}$$

is the transition matrix, see Proposition C.1, and equation (4.2) is equivalent to

$$\frac{b_2}{p}(\alpha_{11}(t^*) - \alpha_{22}(t^*) + 2\delta\alpha_{12}(t^*) + det(\alpha(t^*)) - 1) = 2b_1\alpha_{12}(t^*). \tag{5.5}$$

Equations (5.4) and (5.5) provide all parameters, initial values and intersection times for which system (5.2) has a periodic solution.

Stability

Assume that $\binom{0}{s_0}$ with $s_0 \in M^+$ is a fixed point of $\Pi := \Pi^- \circ \Pi^+$. This means that the solution starting at $\binom{0}{s_0}$ is periodic. From Lemmas 4.2 and 4.3 we get that Π^+ and Π^- are both strictly decreasing. Furthermore, Π^+ is convex and Π^- is concave if $\delta = 1$ and they are both straight lines if $\delta = 0$.

Lemma 5.7 (Properties of the Poincaré-map Π).
Consider system (5.2).

1. *If $\delta = 0$, the Poincaré-map Π is a straight line.*
2. *If $\delta = 1$, the Poincaré-map Π is strictly increasing and concave.*

Proof. From the definition of Π as a composition of Π^+ and Π^- it follows

$$\Pi'(s_0) = \Pi^{-\prime}(\Pi^+(s_0)) \cdot \Pi^{+\prime}(s_0) > 0,$$

where $\Pi^{-\prime}$ and $\Pi^{+\prime}$ are the first derivatives of Π^- and Π^+.

1. If $\delta = 0$, Π^+ and Π^- are straight lines, see Lemmas 4.2 and 4.3. Consequently, $\Pi := \Pi^- \circ \Pi^+$ is also a straight line.
2. If $\delta = 1$, there holds for the second derivative of Π

$$\begin{aligned} \Pi''(s_0) &= \frac{d}{ds}(\Pi'(s_0)) = \frac{d}{ds}(\Pi^{-\prime}(\Pi^+(s_0)) \cdot \Pi^{+\prime}(s_0)) \\ &= \underbrace{\Pi^{-\prime\prime}(\Pi^+(s_0))}_{<0}\underbrace{(\Pi^{+\prime}(s_0))^2}_{>0} + \underbrace{\Pi^{-\prime}(\Pi^+(s_0))}_{<0}\underbrace{\Pi^{+\prime\prime}(s_0)}_{>0} < 0. \end{aligned}$$

□

In the following sections we investigate the existence of closed trajectories of system (5.2) and their stability in five different cases. For this, let $(x(t), y(t))$ be a closed trajectory of system (5.2). We denote the corresponding trajectory by γ. Closed trajectories of system (5.2) are closed curves in the plane. From Lemma 5.4 we know that there exist no closed trajectories if $b_1 \leq 0$ and $\delta = 1$ or if $b_1 < 0$ and $\delta = 0$. As a consequence of Lemma 5.5 we obtain that closed trajectories of type II can only exist in the focus-case $\delta = 1$ and $p > \frac{1}{4}$. Consequently, throughout the following sections, we assume $b_1 \geq 0$ and consider only closed trajectories of type I, except for the focus-case. We present in all different cases depending on δ and p a complete analysis and bifurcation diagram with respect to the parameters b_1 and b_2 of system (5.2).

5.3. The center-case: $\delta = 0$ and $p > 0$

In this section we consider the center-case of system (5.2) which means we assume $\delta = 0$ and $p > 0$. We call it center-case, because the equilibria $(x_0^{\pm}, y_0^{\pm})$ are centers if they exist. We state our main results on existence, number and stability of closed trajectories in dependency on the parameters b_1 and b_2.

Theorem 5.8.
If $b_1 = 0$, for any s_0 with $s_0 > 0$ the solution starting at $(x = 0, y = s_0)$ is periodic and surrounds $(0,0)$. In all other cases there are no closed, heteroclinic or homoclinic trajectories.

Proof. From Lemma 5.5 we know that in this case there exist no closed trajectories of type II. Therefore, we only consider trajectories of type I.
Equation (5.4), which is equivalent to

$$s_0 = \frac{b_2}{\sqrt{p}} \frac{\cos(\sqrt{p}t^*) - 1}{\sin(\sqrt{p}t^*)} - b_1,$$

has a solution $s_0 \in M^+$ for all $b_2 \neq 0$ and under this assumption for all $t^* \in]0, \frac{\pi}{\sqrt{p}}[\cup]\frac{\pi}{\sqrt{p}}, \frac{2\pi}{\sqrt{p}}[$, see Lemma 4.2. In case of $b_2 < 0$, s_0 is a strictly increasing function of $t^* \in]0, \frac{\pi}{\sqrt{p}}[$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to \frac{\pi}{\sqrt{p}}-} s_0(t^*) = \infty$. In case of $b_2 > 0$, s_0 is a strictly decreasing function of $t^* \in]\frac{\pi}{\sqrt{p}}, \frac{2\pi}{\sqrt{p}}[$ with $\lim_{t^* \to \frac{\pi}{\sqrt{p}}+} s_0(t^*) = \infty$ and $\lim_{t^* \to \frac{2\pi}{\sqrt{p}}-} s_0(t^*) = -b_1$. Since

$$\alpha_{12} = \frac{\sin(\sqrt{p}t^*)}{\sqrt{p}} \neq 0,$$

equation (5.5) provides

$$0 = \frac{2b_1 \sin(\sqrt{p}t^*)}{\sqrt{p}}$$

for $t^* \in]0, \frac{\pi}{\sqrt{p}}[\cup]\frac{\pi}{\sqrt{p}}, \frac{2\pi}{\sqrt{p}}[$. This means that for any s_0 with $s_0 > 0$ the solution starting at $(x = 0, y = s_0)$ is periodic, provided $b_1 = 0$ and $b_2 \neq 0$. In case of $b_2 = 0$ system (5.2) is linear if $b_1 = 0$ and $(0,0)$ is a center. Otherwise there are no closed, homoclinic or heteroclinic trajectories. □

5.4. The saddle-case: $p < 0$

In this section we consider the saddle-case of system (5.2) which means we assume $p < 0$. We call it saddle-case, because the equilibria $(x_0^\pm, y_0^\pm)$ are saddle points if they exist. This case is in a modified form published in [Giannakopoulos & Pliete,2002].

5.4.1. Main results

We first state our main results on existence, number and stability of closed trajectories depending on the parameters δ, b_1 and b_2, see Figure 5.1 and [Giannakopoulos & Pliete,2002].

Theorem 5.9.

1. *If $\delta = 0$, $b_1 = 0$ and $b_2 < 0$, for any s_0 with $|s_0| < \frac{b_2\sqrt{-p}}{p}$, the solution starting at $(x = 0, y = s_0)$ is periodic and surrounds $(0,0)$. The two trajectories starting at $(x = 0, y = +\frac{b_2\sqrt{-p}}{p})$ and $(x = 0, y = -\frac{b_2\sqrt{-p}}{p})$ are heteroclinic. They connect the saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) forming a heteroclinic cycle.*

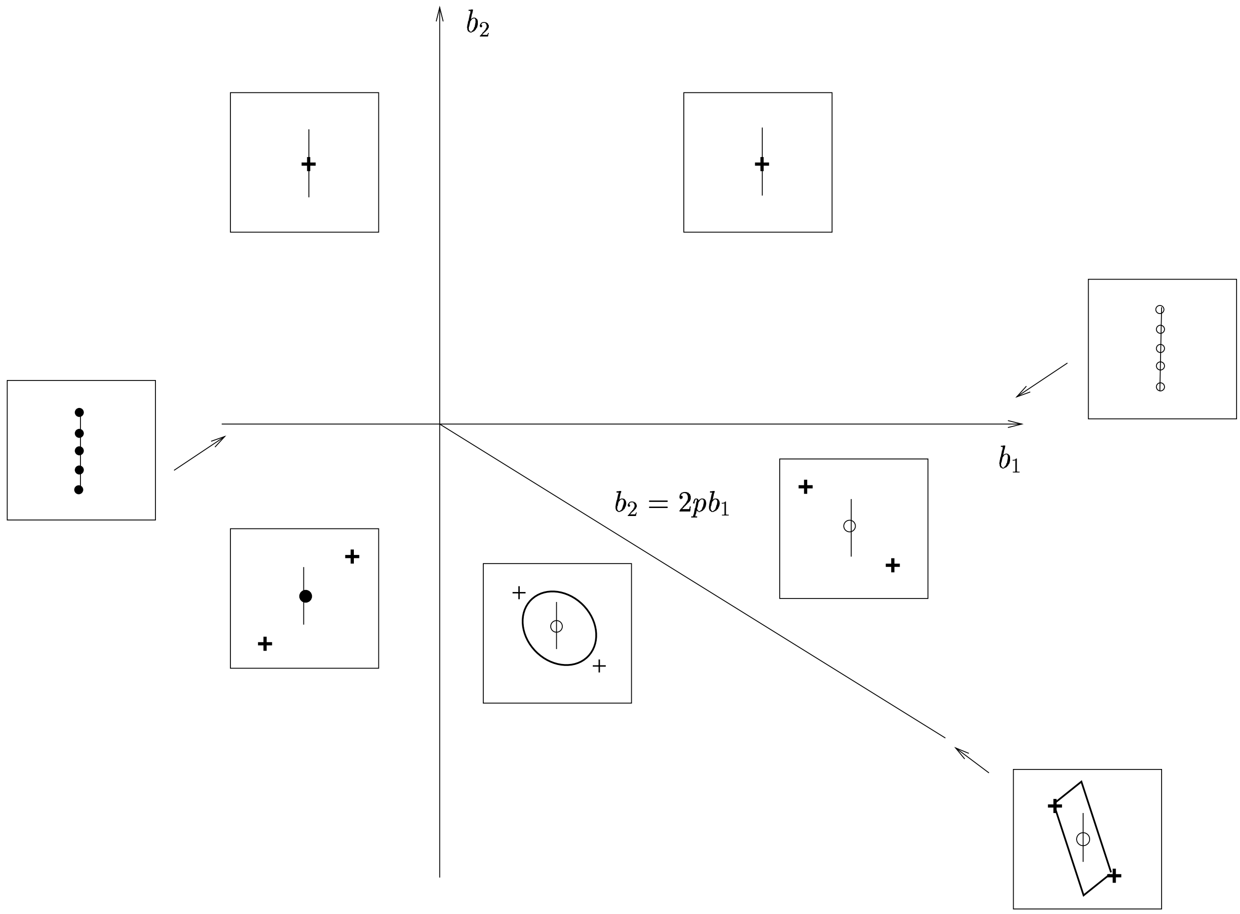


Figure 5.1.: The dependence of equilibria and closed trajectories on the parameters b_1 and b_2 with $\delta = 1$ and $p < 0$. For notion see Figure A.1.

2. *If $\delta = 1$, $b_1 > 0$ and $b_2 < 2pb_1$, there exists exactly one closed trajectory which is asymptotically stable and surrounds the sliding motion interval $I_s = \{0\} \times [-b_1, b_1]$.*

3. *If $\delta = 1$, $b_1 > 0$ and $b_2 = 2pb_1$, there exist exactly two heteroclinic trajectories connecting the saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) forming a heteroclinic cycle. The sliding motion interval I_s lies inside the interior of the heteroclinic cycle.*

In all other cases there are no closed, heteroclinic or homoclinic trajectories.

5.4.2. Proof of Theorem 5.9

Existence of closed trajectories

From Lemma 5.5 we know that in this case no closed trajectories of type II exist. Therefore, we only consider closed trajectories of type I.

1. In case $\delta = 0$ equation (5.4), which is equivalent to

$$s_0 = \frac{b_2}{p}\sqrt{-p}\frac{\cosh(\sqrt{-p}t^*) - 1}{\sinh(\sqrt{-p}t^*)} - b_1,$$

has a solution $s_0 \in M^+$ if and only if $b_2 < 0$ and under this assumption for all $t^* > 0$, see Lemma 4.2. Further s_0 is a strictly increasing function of $t^* > 0$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to \infty} s_0(t^*) = \frac{b_2\sqrt{-p}}{p} - b_1$. Since

$$\alpha_{12} = \frac{\sinh(\sqrt{-p}t^*)}{\sqrt{-p}} \neq 0,$$

equation (5.5) provides

$$0 = \frac{2b_1 \sinh(\sqrt{-p}t^*)}{\sqrt{-p}}$$

for $t^* > 0$. This means that for any s_0 with $|s_0| < \frac{b_2\sqrt{-p}}{p}$ the solution starting at $(x = 0, y = s_0)$ is periodic, provided $b_1 = 0$ and $b_2 < 0$. The solutions starting at $(x = 0, y = \frac{b_2\sqrt{-p}}{p})$ and $(x = 0, y = -\frac{b_2\sqrt{-p}}{p})$ are heteroclinic. They connect the saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) forming a heteroclinic cycle. Otherwise, there are no closed, homoclinic or heteroclinic trajectories.

2. In case $\delta = 1$ equation (5.4), which is equivalent to

$$s_0 = \frac{b_2}{2p} \frac{e^{-\frac{t^*}{2}} (\sinh(\omega t^*) + 2\omega \cosh(\omega t^*)) - 2\omega}{e^{-\frac{t^*}{2}} \sinh(\omega t^*)} - b_1,$$

with $\omega := \frac{1}{2}\sqrt{1-4p} > \frac{1}{2}$, has a solution $s_0 \in M^+$ if and only if $b_2 < 0$ and under this assumption for all $t^* > 0$, see Lemma 4.2. If $b_2 < 0$, s_0 is a strictly increasing function of $t^* > 0$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to +\infty} s_0(t^*) = \frac{b_2}{p}(\frac{1}{2} + \omega) - b_1$. Since

$$\alpha_{12}(t^*) = \frac{\sinh(\omega t^*) e^{-\frac{t^*}{2}}}{\omega} \neq 0,$$

equation (5.5) provides

$$\frac{b_2}{p}\left(\sinh(\omega t^*) - 2\omega \sinh(\frac{t^*}{2})\right) = 2b_1 \sinh(\omega t^*)$$

with $t^* > 0$ and $b_2 < 0$. This implies

$$b_2 = \beta(t^*) := \frac{2b_1 p \sinh(\omega t^*)}{\sinh(\omega t^*) - 2\omega \sinh(\frac{t^*}{2})} \tag{5.6}$$

with $t^* > 0$. β is a well defined function of t^* and satisfies

$$\lim_{t^* \to 0+} \beta(t^*) = -\infty$$
$$\lim_{t^* \to \infty} \beta(t^*) = 2pb_1.$$

For the first derivative of β there holds

$$\beta'(t^*) = \frac{2b_1 p\omega(\sinh(\omega t^*) \cosh(\frac{t^*}{2}) - 2\omega \cosh(\omega t^*) \sinh(\frac{t^*}{2}))}{(\sinh(\omega t^*) - 2\omega \sinh(\frac{t^*}{2}))^2}.$$

β' is positive for all $t^* > 0$. Consequently, β is strictly increasing.

We have proved the existence of exactly one closed trajectory, provided $b_1 > 0$ and $b_2 < 2pb_1$.

3. As shown in 2., there holds $b_2 = \lim_{t^* \to \infty} \beta(t^*) = 2pb_1$. From equation (5.4) we get
$$\begin{aligned} s_0 &= \lim_{t^* \to \infty} \frac{b_2}{p} \frac{\alpha_{11}(t^*) + \delta\alpha_{12}(t^*) - 1}{\alpha_{12}(t^*)} - b_1 \\ &= \lim_{t^* \to \infty} b_1 \frac{e^{-\frac{t^*}{2}}(\sinh(\omega t^*) + 2\omega\cosh(\omega t^*)) - 2\omega}{e^{\frac{-t^*}{2}} \sinh(\omega t^*)} - b_1 \\ &= 2b_1\omega. \end{aligned}$$
We need the following proposition which deals with the seperatrices of the saddle points.

Proposition 5.10. *The unstable manifold of the saddle point $(x_0^{\pm}, y_0^{\pm})$ is given by*
$$y_u^{\pm}(x) := (\frac{1}{2} + \omega)x \pm \frac{b_2}{p}(\frac{1}{2} - \omega) \mp b_1.$$
The stable manifold of the saddle point $(x_0^{\pm}, y_0^{\pm})$ is given by
$$y_s^{\pm}(x) := (\frac{1}{2} - \omega)x \pm \frac{b_2}{p}(\frac{1}{2} + \omega) \mp b_1.$$

Note that y_i^+ and y_i^-, $i \in \{u, s\}$, are defined for $x > 0$ and $x < 0$, respectively. For the intersection points of the seperatrices of the saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) with the $y-$axis there holds in case of $b_2 = 2b_1 p$:
$$\begin{aligned} y_u^{\pm}(0) &= \pm\frac{b_2}{p}(\frac{1}{2} - \omega) \mp b_1 = \mp 2\omega b_1 \\ y_s^{\pm}(0) &= \pm\frac{b_2}{p}(\frac{1}{2} + \omega) \mp b_1 = \pm 2\omega b_1 \end{aligned}$$
This means that if $b_1 > 0$ and $b_2 = 2pb_1$, the trajectories which start at $(x = 0, y = s_0)$ with $|s_0| = 2\omega b_1$ connect the saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) forming a heteroclinic cycle.

Stability of closed trajectories

Assume $\delta = 1$, $b_1 > 0$, $b_2 < 2b_1 p$ and $s_0 \in M^+$ is a fixed point of $\Pi := \Pi^- \circ \Pi^+$. This means that the solution starting at $\binom{0}{s_0}$ is periodic. From Lemmas 4.2 and 4.3 we get for the half Poincaré-maps
$$\begin{aligned} \Pi^+ : \quad]-b_1, \frac{b_2}{p}(\frac{1}{2} + \omega) - b_1[\quad &\longrightarrow \quad]\frac{b_2}{p}(\frac{1}{2} - \omega) - b_1, -b_1[\,, \\ \Pi^- : \quad]-\frac{b_2}{p}(\frac{1}{2} + \omega) + b_1, b_1[\quad &\longrightarrow \quad]b_1, -\frac{b_2}{p}(\frac{1}{2} - \omega) + b_1[\,. \end{aligned}$$
The periodic solution is asymptotically stable if the first derivative of Π at the fixed point s_0 satisfies $|\Pi'(s_0)| < 1$ and unstable if $|\Pi'(s_0)| > 1$.
From Lemma 5.7 we know that Π is strictly increasing and concave. Furthermore, we know that Π has exactly one nontrivial fixed point $s_0 \in]-b_1, \frac{b_2}{p}(\frac{1}{2} + \omega) - b_1[$. Since $\lim_{s \to \frac{b_2}{p}(\frac{1}{2}+\omega)-b_1} \Pi(s) < \frac{b_2}{p}(\frac{1}{2} + \omega) - b_1$, the monotony and concavity of Π provide $0 < \Pi'(s_0) < 1$ and thus, the closed trajectory corresponding to the fixed point s_0 is asymptotically stable.

5.4.3. Description of the occurring bifurcation phenomena

Figure 5.1 can also be seen as a 2-dimensional bifurcation diagram. We distinguish between three different types of bifurcations.

1. *Pitchfork-like bifurcation of equilibria on* $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_2 = 0\}$. For $b_2 > 0$, the trivial solution $(0,0)$ is a unique equilibrium of system (5.2) and it is a saddle point. Passing from $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_2 > 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_2 < 0\}$ two saddle points branch from $(0, -b_1)$ and $(0, b_1)$, and $(0,0)$ itself becomes a stable node if $b_1 < 0$ and an unstable node if $b_1 > 0$. For $b_2 = 0$ the sliding motion interval I_s consists of equilibria.

2. *Hopf-like bifurcation of periodic solutions without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 = 0, \ b_2 < 0\}$. When we cross the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 = 0, b_2 < 0\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 < 0, b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, b_2 < 0\}$ the asymptotically stable equilibrium $(0,0)$ becomes unstable and a stable periodic solution bifurcates from $(0,0)$ surrounding the sliding motion interval I_s.

3. *Heteroclinic cycle bifurcation from a periodic solution without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, b_2 = 2pb_1\}$. For $b_1 > 0$ and $b_2 < 2pb_1$ there is a unique asymptotically stable periodic solution without sliding motion surrounding the equilibrium $(0,0)$ and the sliding motion interval I_s. Crossing the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, b_2 = 2pb_1\}$ the periodic solution becomes a heteroclinic cycle consisting of two heteroclinic trajectories connecting the two saddle points (x_0^+, y_0^+) and (x_0^-, y_0^-) and disappears.

Remark 5.11 (Comparison with piecewise-continuous and smooth cases). *We compare now the bifurcation diagram of the PWL saddle-case with a continuously PWL and a smooth saddle-case. A similar bifurcation diagram to Figure 5.1 for equation* (5.2)*, where the nonlinearity is a continuous and piecewise-linear function $\varphi(\sigma)$ of the form $\varphi(\sigma) = s\sigma$ if $|s\sigma| \le 1$ and $\varphi(\sigma) = 1$ if $|s\sigma| > 1$ with $s > 0$, can be found in [Llibre & Sotomayor,1996]. The phase portraits of $\dot{u} = Au + \varphi(w^T u)v$ with* $\det(A) < 0$, *$tr(A + svw^T) < 0$ and* $\det(A + svw^T) > 0$ *presented in [Llibre & Sotomayor,1996] relate to those of* (5.2) *with $b_1 > 0$ and $b_2 < 0$ except the time direction.*

A smooth example of the saddle-case can be found in [Guckenheimer & Holmes,1983, pp. 371-373]. The nonlinearity is a polynomial of degree 3. The smooth Liénard-system (1.2) *is given by $F(x) = \frac{1}{3}x^3 - \mu_2 x$ and $g(x) = -x^3 - \mu_1 x$ with parameters $\mu_1, \mu_2 \in \mathbb{R}$. Note that the system is $\mathbb{Z}_2$−symmetric. This system provides locally around $x = 0$ and $\mu_1 = \mu_2 = 0$ all bifurcation phenomena in dependency on μ_1 and μ_2 as the PWL system* (5.2) *in dependency on b_1 and b_2. The bifurcation sets in the smooth case are locally separated by straight lines as in the PWL case.*

5.5. The node-case: $\delta = 1$, $0 < p \le \frac{1}{4}$

In this section we consider the node-case of system (5.2) which means we assume $\delta = 1$ and $0 < p \le \frac{1}{4}$. We call it node-case, because the equilibria $(x_0^\pm, y_0^\pm)$ are nodes if they

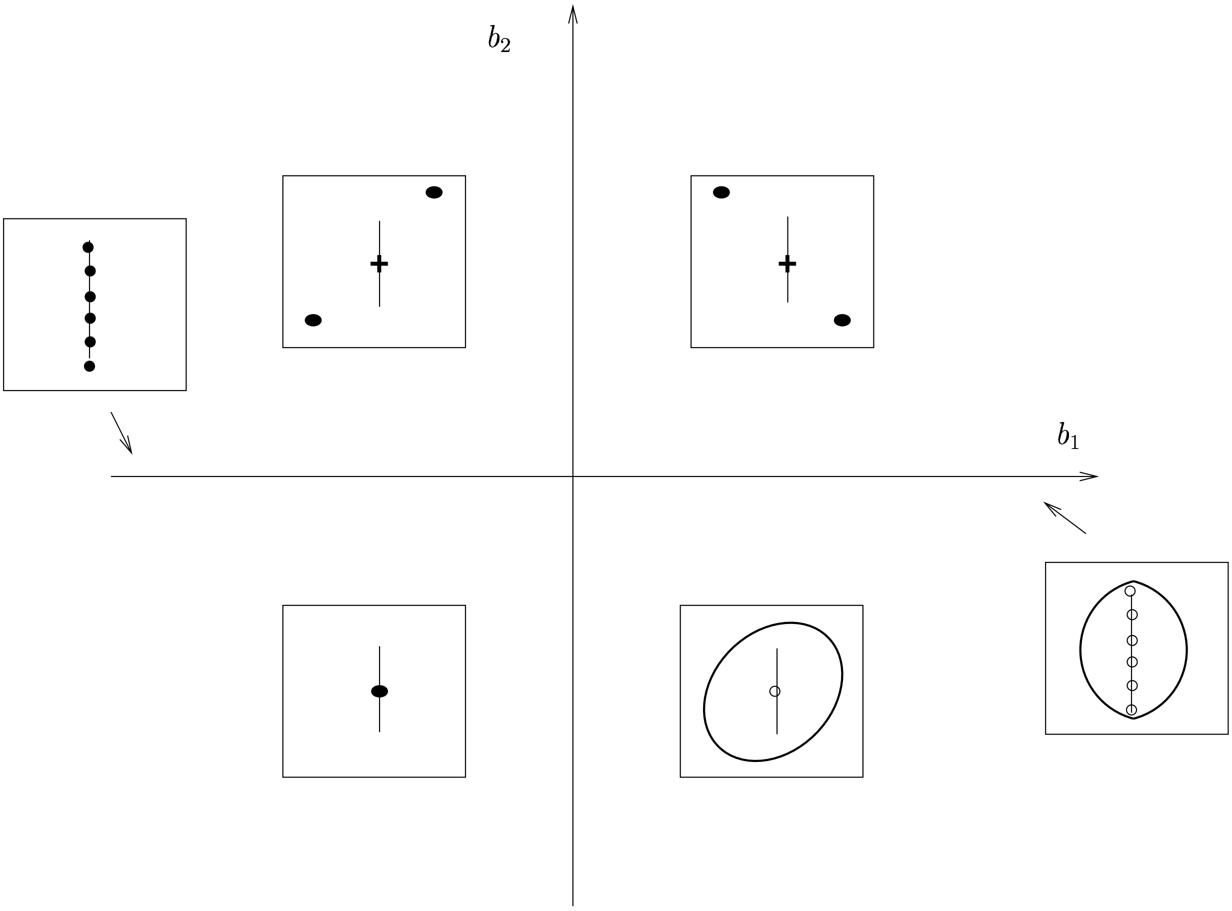


Figure 5.2.: The dependence of equilibria and closed trajectories on the parameters b_1 and b_2 with $\delta = 1$ and $0 < p \leq \frac{1}{4}$. For notation see Figure A.1.

exist.

5.5.1. Main results

We first state our main results on existence, number and stability of closed trajectories depending on the parameters b_1 and b_2, see Figure 5.2.

Theorem 5.12.

1. *If $b_1 > 0$ and $b_2 < 0$, there exists exactly one closed trajectory which is asymptotically stable and surrounds the sliding motion interval $I_s = \{0\} \times [-b_1, b_1]$.*

2. *If $b_1 > 0$ and $b_2 = 0$, there exist exactly two heteroclinic trajectories connecting the equilibria $(0, -b_1)$ and $(0, b_1)$ forming a heteroclinic cycle. The sliding motion interval I_s lies inside the interior of the heteroclinic cycle.*

In all other cases there are no closed, heteroclinic or homoclinic trajectories.

5.5.2. Proof of Theorem 5.12

Note that we get 1. as a consequence of Corollary 3.17. Nevertheless, we determine this periodic solution to verify its disappearance in a heteroclinic cycle.

Existence of closed trajectories

From Lemma 5.5 we know that in this case there exist no closed trajectories of type II. Therefore, we only consider trajectories of type I.

1. In case of $0 < p < \frac{1}{4}$ we can adopt the proof of Theorem 5.9 for the case $\delta = 1$ and $p < 0$ because the transition matrices coincide in these cases. Thus, there exists a closed trajectory if and only if $b_2 = \beta(t^*)$ with $t^* > 0$ and β as in (5.6). Now, it holds

$$\lim_{t^* \to 0+} \beta(t^*) = -\infty,$$
$$\lim_{t^* \to \infty} \beta(t^*) = 0$$

and β is still a strictly increasing function of $t^* > 0$. We have proved the existence of exactly one closed trajectory, provided $b_1 > 0$ and $b_2 < 0$. Note that if $b_2 < 0$, s_0 is a strictly increasing function of $t^* > 0$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to \infty} s_0(t^*) = \infty$.

In case of $p = \frac{1}{4}$ equation (5.4), which is equivalent to

$$s_0 = \frac{b_2}{pt^*}(1 + \frac{1}{2}t^* - e^{\frac{t^*}{2}}) - b_1,$$

has a solution $s_0 \in M^+$ if and only if $b_2 < 0$ and under this assumption for all $t^* > 0$, see Lemma 4.2. If $b_2 < 0$, s_0 is a strictly increasing function of $t^* > 0$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to +\infty} s_0(t^*) = \infty$. Since

$$\alpha_{12}(t^*) = e^{-\frac{t^*}{2}} t^* \neq 0,$$

equation (5.5) provides

$$\frac{b_2}{p}(t^* - e^{\frac{t^*}{2}} + e^{-\frac{t^*}{2}}) = 2b_1 t^*$$

with $t^* > 0$ and $b_2 < 0$. This implies

$$b_2 = \beta(t^*) := \frac{b_1 t^*}{2(t^* + e^{-\frac{t^*}{2}} - e^{\frac{t^*}{2}})}$$

with $t^* > 0$ and $p = \frac{1}{4}$. β is a well defined function of t^* and satisfies

$$\lim_{t^* \to 0+} \beta(t^*) = -\infty$$
$$\lim_{t^* \to \infty} \beta(t^*) = 0.$$

For the first derivative of β there holds

$$\beta'(t^*) = \frac{b_1}{2} \frac{t^* \cosh(\frac{t^*}{2}) - 2\sinh(\frac{t^*}{2})}{(t^* + e^{-\frac{t^*}{2}} - e^{\frac{t^*}{2}})^2}.$$

β' is positive for all $t^* > 0$. Consequently, β is strictly increasing.

We have proved the existence of exactly one closed trajectory, provided $b_1 > 0$ and $b_2 < 0$.

2. As shown in 1. there holds $b_2 = \lim_{t^* \to \infty} \beta_2(t^*) = 0$. From equation (5.4) we get

$$\lim_{t^* \to \infty} s_0(t^*) = -b_1.$$

This means that if $b_1 > 0$ and $b_2 = 0$, the trajectories which start at the equilibria $(0, -b_1)$ and $(0, b_1)$ form a heteroclinic cycle.

Stability of closed trajectories

Assume $b_1 > 0$, $b_2 < 0$ and $s_0 \in M^+$ is a fixed point of $\Pi := \Pi^- \circ \Pi^+$. This means, that the solution starting at $\binom{0}{s_0}$ is periodic. From Lemmas 4.2 and 4.3 we get for the half Poincaré-maps

$$\begin{aligned} \Pi^+ : \quad]-b_1, \infty[\quad &\longrightarrow \quad]\frac{b_2}{p}(\frac{1}{2} - \omega) - b_1, -b_1[\,, \\ \Pi^- : \quad]-\infty, b_1[\quad &\longrightarrow \quad]b_1, -\frac{b_2}{p}(\frac{1}{2} - \omega) + b_1[\,. \end{aligned}$$

The periodic solution is asymptotically stable if the first derivative of Π at the fixed point s_0 satisfies $|\Pi'(s_0)| < 1$ and unstable if $|\Pi'(s_0)| > 1$. From Lemma 5.7 we know that Π is strictly increasing and concave. Furthermore, we know that Π has exactly one nontrivial fixed point $s_0 \in]-b_1, \infty[$. Since $\Pi(s)$ is defined for $s \to \infty$, the monotony and concavity of Π provide $0 < \Pi'(s_0) < 1$ and thus, the closed trajectory corresponding to the fixed point s_0 is asymptotically stable.

5.5.3. Description of the occurring bifurcation phenomena

Figure 5.2 can also be seen as a 2-dimensional bifurcation diagram. We distinguish between three different types of bifurcations.

1. *Pitchfork-like bifurcation of equilibria on* $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 = 0\}$. For $b_2 > 0$, the trivial solution $(0, 0)$ is a unique equilibrium of system (5.2) and it is a saddle point. Passing from $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 > 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 < 0\}$ two nodes branch from $(0, -b_1)$ and $(0, b_1)$, and $(0, 0)$ itself becomes a stable node if $b_1 < 0$ and an unstable node if $b_1 > 0$. For $b_2 = 0$ the sliding motion interval I_s consists of equilibria.

2. *Hopf-like bifurcation of periodic solutions without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 :$ $b_1 = 0,\, b_2 < 0\}$. When we cross the half-line

$\{(b_1, b_2) \in \mathbb{R}^2 : b_1 = 0, b_2 < 0\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 < 0, b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 > 0, b_2 < 0\}$ the asymptotically stable equilibrium $(0, 0)$ becomes unstable and a stable periodic solution bifurcates from $(0, 0)$ surrounding the sliding motion interval I_s.

3. *Heteroclinic cycle bifurcation from a periodic solution without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 > 0, b_2 = 0\}$. For $b_1 > 0$ and $b_2 < 0$ there is a unique asymptotically stable periodic solution without sliding motion surrounding the equilibrium $(0, 0)$ and the sliding motion interval I_s. Crossing the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 > 0, b_2 = 0\}$ the periodic solution becomes a heteroclinic cycle consisting of two heteroclinic trajectories connecting the two equilibria $(0, -b_1)$ and $(0, b_1)$ and disappears.

5.6. The focus-case: $\delta = 1$ and $p > \frac{1}{4}$

In this section we consider the focus-case of system (5.2) which means we assume $\delta = 1$ and $p > \frac{1}{4}$. We call it focus-case, because the equilibria $(x_0^{\pm}, y_0^{\pm})$ are foci if they exist. This case is in a modified form published in [Giannakopoulos & Pliete,2001].

5.6.1. Main results

We first state our main results on existence, number and stability of closed trajectories of type I and II in dependence on the parameters b_1 and b_2, see Figure 5.3 and [Giannakopoulos & Pliete,2001].

Theorem 5.13 (Closed trajectories of type I).
There exist positive real numbers $0 < b_2^0 < b_2^t$ such that for $b_1 > 0$ the following hold:

1. *If $b_2 < b_2^0$, there exists a unique closed trajectory of type I which is asymptotically stable and surrounds the sliding motion interval $I_s := \{0\} \times [-b_1, b_1]$.*
2. *If $b_2 = b_2^0$, there exist exactly two closed trajectories γ_1 and γ_2 of type I.*
   γ_1 is unstable and intersects the line of discontinuity M at $(0, -b_1)$ and $(0, b_1)$.
   The interior $\mathring{I}_s$ of the sliding motion interval I_s lies inside the interior of γ_1.
   γ_2 is asymptotically stable and surrounds γ_1.
3. *If $b_2 \in]b_2^0, b_2^t[$, there exist exactly two closed trajectories γ_1 and γ_2 of type I.*
   γ_1 is unstable and surrounds the sliding motion interval I_s.
   γ_2 is asymptotically stable and encircles γ_1.
4. *If $b_2 = b_2^t$, there exists a unique closed trajectory of type I surrounding the sliding motion interval I_s. It is semi-stable, more precisely, it is unstable from inside and asymptotically stable from outside.*
5. *For $b_2 > b_2^t$ there are no closed trajectories of type I.*

For $b_2 > 0$ all existing closed trajectories of type I surround the equilibria $(x_0^{\pm}, y_0^{\pm})$.
b_2^0 and b_2^t are given in Proposition 5.16, used in the proof of this theorem, see next subsection.

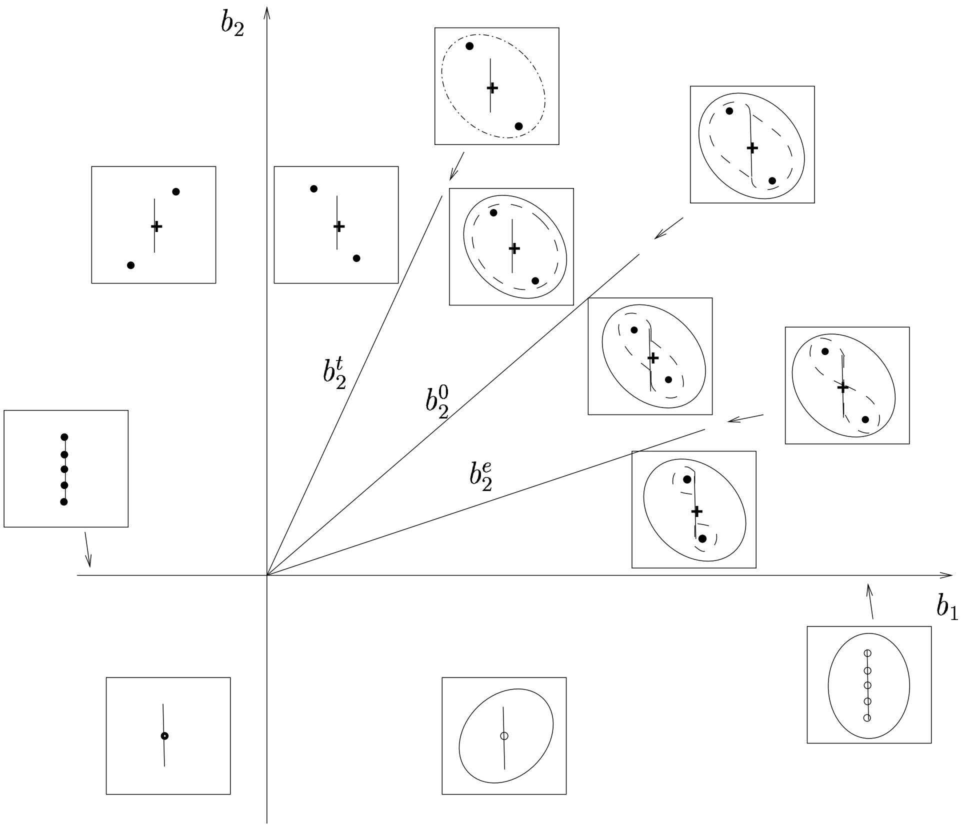


Figure 5.3.: The dependence of equilibria and closed trajectories on the parameters b_1 and b_2 with $\delta = 1$ and $p > \frac{1}{4}$. For notation see Figure A.1.

Theorem 5.14 (Closed and homoclinic trajectories of type II).
For given $b_1 > 0$ there is a positive real number b_2^e with $0 < b_2^e < b_2^0 < b_2^t$, where b_2^0 and b_2^t are given in Proposition 5.16, such that:

1. *If $0 < b_2 < b_2^e$, there exist two closed trajectories γ^- and γ^+ of type II. Both trajectories are unstable. $\gamma^\pm$ lies in $\overline{G_\pm}$ and surrounds the equilibrium $(x_0^\pm, y_0^\pm)$.*

2. *If $b_2 = b_2^e$, there exist exactly two homoclinic trajectories γ^- and γ^+ to the equilibrium $(0,0)$. $\gamma^\pm$ lies in $\overline{G_\pm}$ and surrounds the equilibrium $(x_0^\pm, y_0^\pm)$.*

3. *If $b_2^e < b_2 < b_2^0$, there exists a unique closed trajectory of type II. It is unstable, lies in $\overline{G_+} \cup \overline{G_-}$ and surrounds the three equilibria (x_0^-, y_0^-), $(0,0)$ and (x_0^+, y_0^+).*

4. *If $b_2 \leq 0$ or $b_2 \geq b_2^0$, there are neither homoclinic nor closed trajectories of type II.*

Remark 5.15.
By homoclinic trajectory we mean a solution $(x(t), y(t))$ of system (5.2) with $\lim_{t\to\pm\infty}(x(t), y(t)) = (0,0)$.

5.6.2. Proof of Theorem 5.13

Note that we get the existence of a unique asymptotically stable closed trajectory of type I from Corollary 3.17, provided $b_1 > 0$ and $b_2 < 0$. Nevertheless, we determine this closed trajectory to verify what happens to it if b_2 becomes greater than 0.

Existence of closed trajectories

Equation (5.4), which is equivalent to

$$s_0 = \frac{b_2}{2p}\frac{\sin(\omega t^*) + 2\omega\cos(\omega t^*) - 2\omega e^{\frac{t^*}{2}}}{\sin(\omega t^*)} - b_1,$$

with $\omega := \frac{1}{2}\sqrt{4p-1}$, has a solution $s_0 \in M^+$ for all $b_2 \in \mathbb{R}$ and $t^* \in T^+$, see Lemma 4.2. Assume now $b_2 \neq 0$. Then there holds

$$\alpha_{12}(t^*) = \frac{\sin(\omega t^*)e^{-\frac{t^*}{2}}}{\omega} \neq 0$$

and equation (5.5) provides

$$\frac{b_2}{p}\left(\sin(\omega t^*) - 2\omega\sinh(\frac{t^*}{2})\right) = 2b_1\sin(\omega t^*)$$

with $t^* \in T^+$ and $b_2 \neq 0$. This implies

$$b_2 = \beta(t^*) := \frac{2b_1 p\sin(\omega t^*)}{\sin(\omega t^*) - 2\omega\sinh(\frac{t^*}{2})} \tag{5.7}$$

with $t^* \in T^+$. In case of $b_2 = 0$ there holds $t^* = \frac{\pi}{\omega}$, see Lemma 4.2.

Proposition 5.16.
The function β defined in (5.7) is continuously differentiable for all $t \in]0, t_1^0[$, where t_1^0 is the unique zero of

$$\Xi_1(t; -\frac{1}{2}, \omega) = \frac{e^{\frac{t}{2}}}{2}(\sin(\omega t) - 2\omega\cos(\omega t)) + \omega$$

in $]\frac{\pi}{\omega}, \frac{2\pi}{\omega}[$, see Proposition B.1. If $b_1 > 0$, there exists a unique number $t^t \in]\frac{\pi}{\omega}, t_1^0[$ such that there holds, see Figure 5.4:

1. *β is strictly increasing in $]0, t^t[$;*
2. *β is strictly decreasing in $]t^t, t_1^0[$;*
3. *$\lim_{t\to 0+}\beta(t) = -\infty$, $\beta(\frac{\pi}{\omega}) = 0$ and $0 < b_2^0 < b_2^t$, where $b_2^0 := \beta(t_1^0)$ and $b_2^t := \beta(t^t)$.*

Proof. It is easy to see that β is continuously differentiable for $t \in]0, t_1^0[$ and for the first derivative of β we obtain

$$\beta'(t^*) = \frac{2b_1 p\omega(\sin(\omega t^*)\cosh(\frac{t^*}{2}) - 2\omega\cos(\omega t^*)\sinh(\frac{t^*}{2}))}{(\sin(\omega t^*) - 2\omega\sinh(\frac{t^*}{2}))^2}.$$

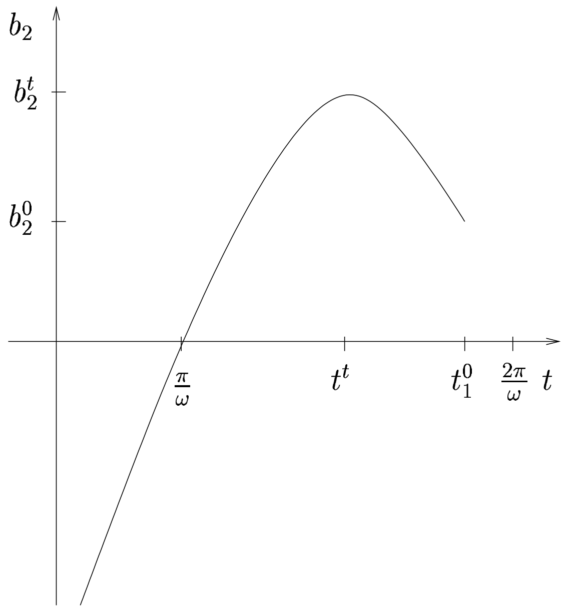


Figure 5.4.: The function β

There exists a unique number $t^t \in \,]\frac{\pi}{\omega}, t_1^0[$, given by

$$\beta'(t^t) = 0 \Leftrightarrow \sin(\omega t^t)\cosh(\frac{t^t}{2}) - 2\omega\cos(\omega t^t)\sinh(\frac{t^t}{2}) = 0,$$

such that β is strictly increasing in $]0, t^t[$ and strictly decreasing in $]t^t, t_1^0[$. Furthermore we have that $\lim_{t\to 0+}\beta(t) = -\infty$, $\beta(\frac{\pi}{\omega}) = 0$ and $0 < b_2^0 < b_2^t$, where $b_2^0 := \beta(t_1^0)$ and $b_2^t := \beta(t^t)$. This completes the proof. □

We have proved the existence of exactly one closed trajectory of type I, provided $b_1 > 0$ and $b_2 < b_2^0$ or $b_2 = b_2^t$, respectively. If $b_1 > 0$ and $b_2^0 \le b_2 < b_2^t$ we have exactly two and in all other cases no closed trajectories of type I.

Stability of closed trajectories

We get the stability properties of the closed trajectories of type I from the following lemma:

Lemma 5.17.
Suppose $b_1 > 0$. With b_2^0 and b_2^t as in Proposition 5.16 there holds:

(a) If $b_2 < b_2^0$, Π has a unique fixed point $s^* > \bar{s}^+$ which is attracting.

(b) If $b_2 = b_2^0$, Π has exactly two fixed points $s_1^* = \bar{s}^+$ and $s_2^* > \bar{s}^+$. s_1^* is repelling from the right and s_2^* is attracting.

(c) If $b_2^0 < b_2 < b_2^t$, Π has exactly two fixed points s_1^* and s_2^* with $\bar{s}^+ < s_1^* < s_2^*$. s_1^* is repelling and s_2^* is attracting.

(d) If $b_2 = b_2^t$, Π has a unique fixed point $s^* > \bar{s}^+$ which is attracting from the left and repelling from the right.

Thereby, there holds $\bar{s}^+ := -\frac{b_2 e^{\frac{t_1^0}{2}} \sin(\omega t_1^0)}{\omega} - b_1$*, see Lemma 4.2.*

Remark 5.18.
In case $b_2 = b_2^0$, if we say $\bar{s}^+$ is a fixed point of Π, we mean $\lim_{s\to\bar{s}^+ +} \Pi(s) = \bar{s}^+$.

Proof of the lemma. From Lemma 4.2 we get the half Poincaré-maps $\Pi^+ : M^+ \longrightarrow \hat{M}^+$ and $\Pi^- : M^- \longrightarrow \hat{M}^-$. The periodic solution is asymptotically stable if the first derivative of $\Pi := \Pi^- \circ \Pi^+$ at the fixed point s_0 satisfies $|\Pi'(s_0)| < 1$ and unstable if $|\Pi'(s_0)| > 1$. From Lemma 5.7 we know that Π is strictly increasing and concave.
We distinguish between the following cases:

1. If $b_2 < b_2^0$, there exists exactly one nontrivial fixed point
$$s^* \in \begin{cases}]-b_1, \infty[& , \text{ if } b_2 \le 0 \\ ]\bar{s}^+, \infty[& , \text{ if } 0 < b_2 < b_2^0. \end{cases}$$
Since $\Pi(s)$ is defined for $s \to \infty$, the monotony and concavity of Π provide $0 < \Pi'(s^*) < 1$.

2. If $b_2 = b_2^0$, there exist two nontrivial fixed points $s_1^* = \bar{s}^+$ and $s_2^* > \bar{s}^+$ of Π. There holds
$$\lim_{s\to s_1^*+} \Pi'(s) = \lim_{s\to \bar{s}^+ +} \Pi^{-\prime}(\Pi^+(s))\Pi^{+\prime}(s) = \lim_{t^*\to t_1^0 +} \frac{\Xi_1^2(t^*; \frac{1}{2}, \omega)}{\Xi_1^2(t^*; -\frac{1}{2}, \omega)} = \infty.$$
Since $\Pi(s)$ is defined for $s \to \infty$, the monotony and concavity of Π provide $0 < \Pi'(s_2^*) < 1$.

3. If $b_2^0 < b_2 < b_2^t$, the monotony and concavity of Π provide $0 \le \Pi'(s_2^*) < 1 < \Pi'(s_1^*)$ for the fixed points $s_1^* < s_2^*$ of Π.

4. If $b_2 = b_2^t$, the unique fixed point s^* of Π fulfills
$$\Pi'(s^*) = \Pi^{-\prime}(\Pi^+(s))\Pi^{+\prime}(s) = \frac{\Xi_1^2(t^t; \frac{1}{2}, \omega)}{\Xi_1^2(t^t; -\frac{1}{2}, \omega)}.$$
t^t is defined as the solution of $\beta'(t^t) = 0$ which is equivalent to $\Xi_1(t^t; \frac{1}{2}, \omega) = \Xi_1(t^t; -\frac{1}{2}, \omega)$. Thus, we have $\Pi'(s^*) = 1$. Since Π is concave, it follows, s^* is attracting from the left and repelling from the right.

Hence, the lemma is proved. □

5.6.3. Proof of Theorem 5.14

For $b_2 > 0$ there exists $\bar{s}^+ > -b_1$ such that a trajectory starting at $(0, \bar{s}^+)$ at time $t = 0$ evolves in G_+ and reaches I_s at $(0, -b_1)$ tangentially after time t_1^0. Because of the symmetry properties there exists another trajectory starting at $(0, -\bar{s}^+)$ at time $t = 0$ which evolves in G_- and reaches I_s at $(0, b_1)$ tangentially after time t_1^0. If $\bar{s}^+ \in I_s$ we can get closed trajectories with sliding motion. Lemma 4.2 provides

$$\bar{s}^+ = \frac{-b_2 e^{\frac{t_1^0}{2}} \sin(\omega t_1^0)}{\omega} - b_1$$

such that $\bar{s}^+$ can be interpreted as a strictly increasing function of $b_2 > 0$. Note that t_1^0 does not depend on b_2. There holds

$$\bar{s}^+ = 0 \quad \Leftrightarrow \quad b_2 = \frac{-b_1 \omega e^{-\frac{t_1^0}{2}}}{\sin(\omega t_1^0)} =: b_2^e,$$

$$\bar{s}^+ = b_1 \quad \Leftrightarrow \quad b_2 = \frac{-2b_1 \omega e^{-\frac{t_1^0}{2}}}{\sin(\omega t_1^0)} = b_2^0$$

with b_2^0 as in Proposition 5.16.

1.,2. The conditions $(L1\pm)$ and $(L2\pm)$ of Theorems 4.12 and 4.13 are fulfilled. These proofs provide

$$b_2^{p+} = -b_2^{p-} = b_2^e.$$

Thus, we get the existence of a closed trajectory of type II for $0 < b_2 < b_2^e$ and the existence of a homolicinic trajectory with sliding motion in $\overline{G_+}$ and $\overline{G_-}$ to $(0,0)$ for $b_2 = b_2^e$, respectively. We determine the Poincaré-map of the closed trajectory in $\overline{G_+}$, the determination of the Poincaré-map of the closed trajectory in $\overline{G_-}$ is analogous, as composition of discrete-time maps Π^+ and Π^0 in G_+ and I_s, respectively. The condition $0 < b_2 \leq b_2^e$ is equivalent to the condition $-b_1 < \bar{s}^+ \leq 0$. Assume that there exists a closed trajectory γ with sliding motion which starts at $(0, \bar{s}^+) \in I_s$, evolves in G_+, reaches $(0, -b_1)$ tangentially after finite time t_1^0 and reaches, moving along I_s for finite time τ^0, again $(0, \bar{s}^+)$, see Figure 5.5(a). Let $(x^+(t), y^+(t))$ be the solution corresponding to γ in G_+ for $t \in [0, t_1^0]$ and $(0, y^0(t))$ be the solution in I_s for $t \in [0, \tau^0]$, see Figure 5.5(a). Using this, we get the Poincaré-map $\Pi := \Pi^0 \circ \Pi^+$ with

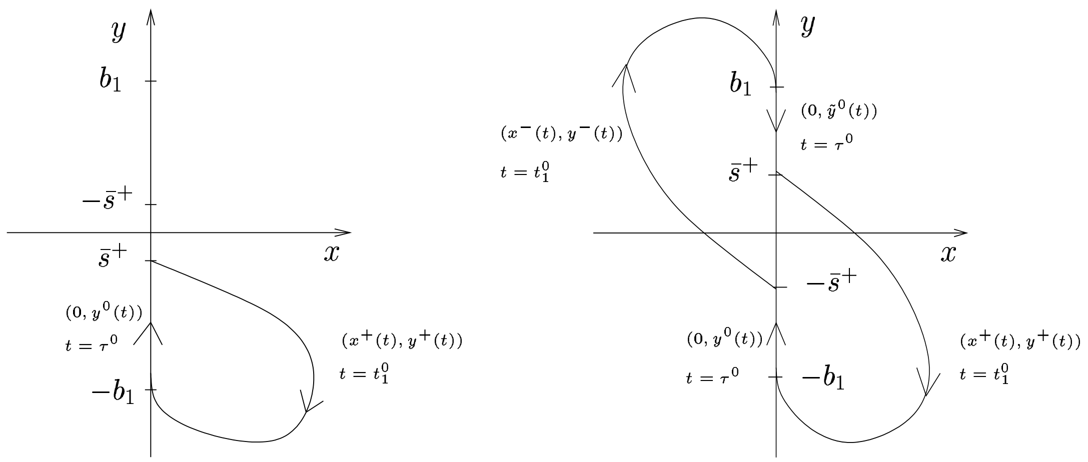
$$\Pi^+(\bar{s}^+) = y^+(t_1^0) = -b_1,$$
$$\Pi^0(-b_1) = y^0(\tau^0) = \bar{s}^+.$$

For the first derivative of Π there holds

$$\Pi'(s) = \Pi^{0\prime}(\Pi^+(s)) \cdot \Pi^{+\prime}(s)$$

and therefore, at the fixed point $\bar{s}^+$

$$\Pi'(\bar{s}^+) = e^{\beta_0 \tau^0} \lim_{t \to t_1^0} \frac{\Xi_1(t; \frac{1}{2}, \omega)}{\Xi_1(t; -\frac{1}{2}, \omega)} = \infty$$



(a) $b_1 > 0$ and $0 < b_2 < b_2^e$ (b) $b_1 > 0$ and $b_2^e < b_2 < b_2^0$

Figure 5.5.: Closed trajectories with sliding motion

with Ξ_1 as in Proposition B.1 and β_0 as in (2.7). Hence, the closed trajectory of type II is unstable.

3. If $b_2^e < b_2 < b_2^0$, which is equivalent to $0 < \bar{s}^+ < b_1$, we determine the Poincaré-map Π as a composition of discrete-time maps Π^+, Π^-, Π^0 and $\tilde{\Pi}^0$ in G_+, G_- and I_s, respectively, see section 2.3. Assume that there exists a closed trajectory γ with sliding motion which starts at $(0, \bar{s}^+) \in I_s$, evolve in G_+, reaches $(0, -b_1)$ tangentially after finite time t_1^0 and moves along I_s for finite time τ^0. Then γ leaves, because of the symmetry properties, I_s at $(0, -\bar{s}^+)$, evolves for time t_1^0 in G_-, reaches $(0, b_1)$ tangentially and reaches again $(0, \bar{s}^+)$ moving along I_s for time τ^0, see Figure 5.5(b). Let $(x^+(t), y^+(t))$ be the solution corresponding to γ in G_+ for $t \in [0, t_1^0]$ and $(0, y^0(t))$ be the first part of the solution in I_s for $t \in [0, \tau^0]$, see Figure 5.5(b). Analogously, let $(x^-(t), y^-(t))$ be the solution corresponding to γ in G_- for $t \in [0, t_1^0]$ and $(0, \tilde{y}^0(t))$ be the second part of the solution in I_s for $t \in [0, \tau^0]$. Using this, we get the Poincaré-map $\Pi := \tilde{\Pi}^0 \circ \Pi^- \circ \Pi^0 \circ \Pi^+$ with

$$\begin{aligned} \Pi^+(\bar{s}^+) &= y^+(t_1^0) = -b_1, \\ \Pi^0(-b_1) &= y^0(\tau^0) = -\bar{s}^+, \\ \Pi^-(-\bar{s}^+) &= y^-(t_1^0) = b_1, \\ \tilde{\Pi}^0(b_1) &= \tilde{y}^0(\tau^0) = \bar{s}^+. \end{aligned}$$

Note that, because of the symmetry properties, there hold

$$\begin{aligned} \Pi^- &= -id_{\mathbb{R}} \circ \Pi^+ \circ -id_{\mathbb{R}}, \\ \tilde{\Pi}^0 &= -id_{\mathbb{R}} \circ \Pi^0 \circ -id_{\mathbb{R}} \end{aligned}$$

and therefore, we get for the Poincaré-map $\Pi = (-id_{\mathbb{R}} \circ \Pi^0 \circ \Pi^+)^2$. For the first derivative of Π there holds

$$\Pi'(s) = \Pi^{0\prime}(\Pi^+(-\Pi^0(\Pi^+(s)))) \cdot \Pi^{+\prime}(-\Pi^0(\Pi^+(s))) \cdot \Pi^{0\prime}(\Pi^+(s)) \cdot \Pi^{+\prime}(s)$$

and therefore, at the fixed point $\bar{s}^+$

$$\Pi'(\bar{s}^+) = e^{2\beta_0 \tau^0} \lim_{t \to t_1^0} \frac{\Xi_1^2(t; \frac{1}{2}, \omega)}{\Xi_1^2(t; -\frac{1}{2}, \omega)} = \infty$$

with Ξ_1 as in Proposition B.1 and β_0 as in (2.7). Hence, the closed trajectory of type II is unstable.

4. If $b_2 \geq b_2^0$, we have $\bar{s}^+ \geq b_1$ and thus, $(0, \bar{s}^+)$ is no longer in $\overset{\circ}{I_s}$. On the other hand, the only trajectory processing in G_+ which can reach I_s starts at $(0, \bar{s}^+)$. This provides the non-existence of closed trajectories with sliding motion.

   If $b_2 \leq 0$ Lemma 3.5 provides the non-existence of closed trajectories with sliding motion.

5.6.4. Description of the occurring bifurcation phenomena

Figure 5.3 can also be seen as a 2-dimensional bifurcation diagram. All the lines in Figure 5.3, except b_2^0, consist of bifurcation points. We distinguish between five different types of bifurcations.

1. *Pitchfork-like bifurcation of equilibria on* $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 = 0\}$. For $b_2 < 0$, the trivial solution $(0,0)$ is a unique equilibrium of system (5.2) and it is a stable node if $b_1 < 0$ and an unstable node if $b_1 > 0$. Passing from $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_2 > 0\}$ two foci branch from $(0, -b_1)$ and $(0, b_1)$, and $(0,0)$ itself becomes a saddle point. For $b_2 = 0$, the sliding motion interval I_s consists of equilibria.

2. *Hopf-like bifurcation of periodic solutions without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 = 0,\, b_2 < 0\}$. When we cross the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 = 0, b_2 < 0\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 < 0, b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 < 0\}$, the asymptotically stable equilibrium $(0,0)$ becomes unstable and a stable periodic solution without sliding motion bifurcates from $(0,0)$ surrounding the sliding motion interval I_s.

3. *Double Hopf-like bifurcation of periodic solutions with sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 = 0\}$. When we cross the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 = 0\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 > 0\}$ an unstable periodic solution with sliding motion, which lies in $\overline{G_-}$, branches from $(0, b_1)$. Because of the symmetry properties of system (5.2) a second unstable periodic solution with sliding motion, which lies in $\overline{G_+}$, branches simultaneously from $(0, -b_1)$.

4. *Double homoclinic bifurcation from periodic solutions with sliding motion on* b_2^e. For $b_1 > 0$ and $0 < b_2 < b_2^e(b_1)$ there are two small unstable periodic solutions with sliding motion in $\overline{G_+}$ and $\overline{G_-}$ surrounding the equilibria (x_0^+, y_0^+) and (x_0^-, y_0^-), respectively. Crossing the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2 = b_2^e(b_1)\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, 0 < b_2 < b_2^e(b_1)\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \; b_1 > 0, b_2^e(b_1) < b_2 < b_2^0(b_1)\}$ both small periodic solutions become simultaneously homoclinic to $(0,0)$

and disappear. At the same time an unstable big periodic solution with sliding motion surrounding the three equilibria (x_0^+, y_0^+), $(0,0)$ and (x_0^-, y_0^-) bifurcates from the double homoclinic trajectory. This periodic solution exists for $b_2^e(b_1) < b_2 < b_2^0(b_1)$ and $b_1 > 0$.

5. *Fold bifurcation of periodic solutions without sliding motion on b_2^t.* For $b_1 > 0$ and $b_2^0(b_1) < b_2 < b_2^t(b_1)$ there are two big periodic solutions without sliding motion surrounding the three equilibria (x_0^+, y_0^+), $(0,0)$, (x_0^-, y_0^-) and the sliding motion interval I_s. The inner one is unstable and the outer one is stable. Crossing the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, \, b_2 = b_2^t(b_1)\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, \, b_2^0(b_1) < b_2 < b_2^t(b_1)\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, \, b_2^t(b_1) < b_2\}$ these two big periodic orbits collide and disappear. Note that the map Π, which provides the existence of periodic solutions without sliding motion, see proof of Theorem 5.13, undergoes a fold bifurcation of fixed points at $b_2 = b_2^t(b_1)$ for any $b_1 > 0$, since $\Pi'(s^*) = 1$ and $\Pi''(s^*) \neq 0$, where s^* is the unique fixed point of Π, provided $b_2 = b_2^t(b_1)$ and $b_1 > 0$.

The half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0, \, b_2 = b_2^0(b_1)\}$ does not contain any bifurcation point. If $b_2 = b_2^0(b_1)$ and $b_1 > 0$, for the first derivative of Π there holds $\lim_{s \to b_2+} \Pi'(s) = +\infty$, see proof of Theorem 5.14.

Remark 5.19 (Comparison with the smooth case).
We compare now the bifurcation diagram of the PWL focus-case with three different smooth focus-cases. First, we consider an example which can be found in [Guckenheimer & Holmes,1983, pp. 373-376]. The nonlinearity is approximated by a polynomial of degree 3. The smooth Liénard-system (1.2) *is given by* $F(x) = \frac{1}{3}x^3 - \mu_2 x$ *and* $g(x) = x^3 - \mu_1 x$ *with parameters* $\mu_1, \mu_2 \in \mathbb{R}$. *Note that the system is* $\mathbb{Z}_2$−*symmetric. This system provides locally all bifurcation phenomena in dependency on* μ_1 *and* μ_2 *as the PWL system* (5.2) *is dependent on* b_1 *and* b_2. *The bifurcation sets in the smooth case are locally separated by straight lines as in the PWL case. In the smooth case, there is one more bifurcation line, in notation of the PWL case, between the lines* $b_2 = 0$ *and* $b_2 = b_2^e$ *for* $b_1 > 0$. *Considering the smooth case, in notation of the PWL case for a fixed* $b_1 > 0$, *a stable node becomes a saddle point and two stable foci appear at* $b_2 = 0$. *When* b_2 *increases the two small periodic solutions surrounding the foci are formed on a different bifurcation line. These two bifurcation lines coincide in the PWL case. A bifurcation analysis of equation*

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \mu_1 + \mu_2 x + \mu_3 y - x^3 - x^2 y
\end{aligned}
$$

with three parameters $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ *is given in [Khibnik et al.,1998].*

Second, we consider an example, given in [Giannakopoulos & Oster,1997] or [Kooij & Giannakopoulos,2000], where the nonlinearity is a transcendental function. They consider a planar system modelling neural activity of the form

$$
\begin{aligned}
\dot{u}_1 &= -u_1 + q_{11} f(u_1) - Q u_2 + E \\
\dot{u}_2 &= -u_2 + q_{21} f(u_1) + e_2
\end{aligned}
$$

with $q_{11}, q_{21}, e_2, Q, E \in \mathbb{R}$ and $f(u_1) := \frac{1}{1+\exp(-4u_1)}$ a sigmoid function. The authors provide bifurcation phenomena with regard to the parameters E and Q. The system can be transformed to a Liénard-system with $F(x) = 2x - a(f(x + \bar{u}_1) - f(\bar{u}_1))$ and $g(x) = x - (a - b)(f(x + \bar{u}_1) - f(\bar{u}_1))$, where $(\bar{u}_1, \bar{u}_2)$ is an equilibrium of the original system, $a := q_{11}$ and $b := Qq_{21}$. For this system, the authors locally find with the aid of numerical computations the same bifurcation phenomena as in the PWL case.

5.7. The case: $\delta = 1$, $p = 0$

In this section we consider the case of system (5.2) in which the determinant of A is zero, which means we assume $\delta = 1$ and $p = 0$. In this case there are no equilibria in $G_\pm$.

5.7.1. Main results

We first state our main results on existence, number and stability of closed trajectories in dependence on the parameters b_1 and b_2, see Figure 5.6.

Theorem 5.20.

1. *If $b_1 > 0$ and $b_2 < 0$, there exists exactly one closed trajectory which is asymptotically stable and surrounds the sliding motion interval $I_s = \{0\} \times [-b_1, b_1]$.*
2. *If $b_1 > 0$ and $b_2 = 0$, there exist exactly two heteroclinic trajectories connecting the equilibria $(0, -b_1)$ and $(0, b_1)$ forming a heteroclinic cycle. The sliding motion interval I_s lies inside the interior of the heteroclinic cycle.*

In all other cases there are no closed, heteroclinic or homoclinic trajectories.

5.7.2. Proof of Theorem 5.20

Note that we get 1. from Corollary 3.17. Nevertheless, we determine this periodic solution to verify its disappearance in a heteroclinic cycle.

Existence of closed trajectories

1. From Lemma 5.5 we know that in this case there exist no closed trajectories of type II. Therefore, we only consider closed trajectories of type I. In this case we cannot use equations (5.4) and (5.5), because A is singular and therefore, the solution ϕ of system (5.2) has another representation. From appendix D.4 we get that

$$s_0 = \frac{b_2(t^* - 1 + e^{-t^*})}{e^{-t^*} - 1} - b_1 \qquad \text{(s0)}$$

has a solution $s_0 \in M^+$ if and only if $b_2 < 0$ and under this assumption for all $t^* > 0$. If $b_2 < 0$, then s_0 is a strictly increasing function of $t^* > 0$ with $\lim_{t^* \to 0+} s_0(t^*) = -b_1$ and $\lim_{t^* \to +\infty} s_0(t^*) = \infty$. There holds

$$s_0^* = \Pi^+(s_0) = b_2 t^* + s_0 = \frac{b_2(t^* e^{-t^*} - 1 + e^{-t^*})}{e^{-t^*} - 1} - b_1.$$

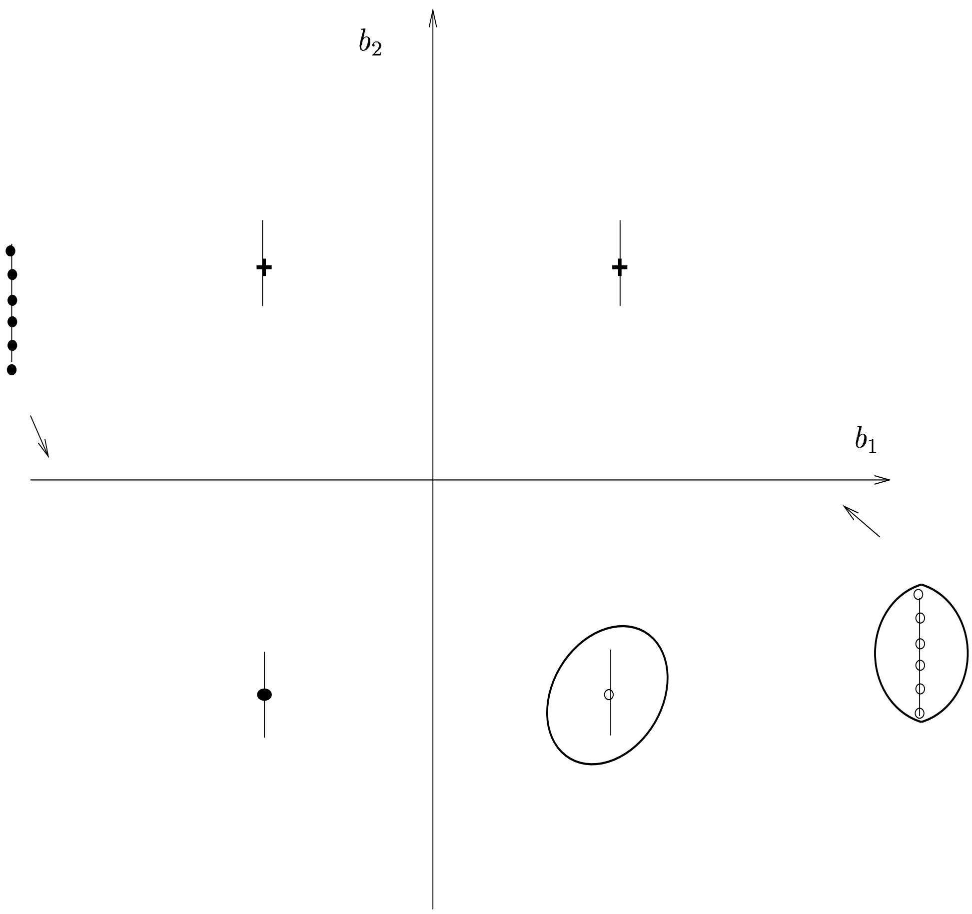


Figure 5.6.: The dependence of equilibria and closed trajectories on the parameters b_1 and b_2 with $\delta = 1$ and $p = 0$. For notation see Figure A.1.

Due to the symmetry properties we have $s_0^* = -s_0$ in the case of an existing closed trajectory, which is equivalent to

$$\frac{b_2(t^* e^{-t^*} + t^* - 2 + 2e^{-t^*})}{e^{-t^*} - 1} = 2b_1$$

with $t^* > 0$ and $b_2 < 0$. This implies

$$b_2 = \beta(t^*) := \frac{2b_1(e^{-t^*} - 1)}{t^* e^{-t^*} + t^* - 2 + 2e^{-t^*}}$$

with $t^* > 0$. β is a continuously differentiable function of $t^* > 0$ and satisfies

$$\lim_{t^* \to 0+} \beta(t^*) = -\infty$$
$$\lim_{t^* \to \infty} \beta(t^*) = 0.$$

For the first derivative of β there holds

$$\beta'(t^*) = \frac{2b_1(-2t^* e^{-t^*} - e^{-2t^*} + 1)}{(t^* e^{-t^*} + t^* - 2 + 2e^{-t^*})^2}.$$

β' is positive for all $t > 0$. Consequently, β is strictly increasing.

We have proved the existence of exactly one closed trajectory, provided $b_1 > 0$ and $b_2 < 0$.

2. As shown in 1. there holds $b_2 = \lim_{t^* \to \infty} \beta_2(t^*) = 0$. From equation (s0) we get

$$\lim_{t^* \to \infty} s_0(t^*) = -b_1.$$

This means that if $b_1 > 0$ and $b_2 = 0$, the trajectories, which start at the equilibria $(0, -b_1)$ and $(0, b_1)$, form a heteroclinic cycle.

Stability of closed trajectories

Assume $b_1 > 0$, $b_2 < 0$ and $s_0 \in M^+$ is a fixed point of $\Pi := \Pi^- \circ \Pi^+$. This means, that the solution starting at $\binom{0}{s_0}$ is periodic. From Lemmas 4.2 and 4.3 we get for the half Poincaré-maps

$$\begin{aligned} \Pi^+ : \quad]-b_1, \infty[\quad &\longrightarrow \quad]b_2 - b_1, -b_1[\,, \\ \Pi^- : \quad]-\infty, b_1[\quad &\longrightarrow \quad]b_1, -b_2 + b_1[\,. \end{aligned}$$

The periodic solution is asymptotically stable if the first derivative of Π at the fixed point s_0 satisfies $|\Pi'(s_0)| < 1$ and unstable if $|\Pi'(s_0)| > 1$. From Lemma 5.7 we know that Π is strictly increasing and concave. Furthermore, we know that Π has exactly one nontrivial fixed point $s_0 \in]-b_1, \infty[$. Since $\Pi(s)$ is defined for $s \to \infty$, the monotony and concavity of Π provide $0 < \Pi'(s_0) < 1$ and thus, the closed trajectory corresponding to the fixed point s_0 is asymptotically stable.

5.7.3. Description of the occurring bifurcation phenomena

Figure 5.2 can also be seen as a 2-dimensional bifurcation diagram. We distinguish between two different types of bifurcations.

1. *Hopf-like bifurcation of periodic solutions without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : b_1 = 0,\, b_2 < 0\}$. When we cross the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 = 0,\, b_2 < 0\}$ from $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 < 0,\, b_2 < 0\}$ to $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0,\, b_2 < 0\}$ the asymptotically stable equilibrium $(0, 0)$ becomes unstable and a stable periodic solution bifurcates from $(0, 0)$ surrounding the sliding motion interval I_s.

2. *Heteroclinic cycle bifurcation from a periodic solution without sliding motion on* $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0,\, b_2 = 0\}$. For $b_1 > 0$ and $b_2 < 0$ there is a unique asymptotically stable periodic solution surounding the equilibrium $(0, 0)$ and the sliding motion interval I_s. Crossing the half-line $\{(b_1, b_2) \in \mathbb{R}^2 : \ b_1 > 0,\, b_2 = 0\}$ the periodic solution becomes a heteroclinic cycle consisting of two heteroclinic trajectories connecting the two equilibria $(0, -b_1)$ and $(0, b_1)$ and disappears.

6. Conclusion and prospect

In this thesis, we have considered a planar piecewise-smooth system of Liénard-type with a line of discontinuity, i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} & , \text{if } x > 0 \\ \begin{pmatrix} y - F^-(x) \\ -g^-(x) \end{pmatrix} & , \text{if } x < 0, \end{cases} \tag{6.1}$$

where $F^+(x), g^+(x)$ and $F^-(x), g^-(x)$ have been smooth functions for $x \geq 0$ and $x \leq 0$, respectively. In chapter 2, we defined the terms solution, sliding motion, singular points, closed trajectories and bifurcation of piecewise-smooth systems.
Chapter 3 dealt with the piecewise-smooth system (6.1). We determined the sliding motion interval, characterised all singular points in it and analysed the corresponding initial value problem in terms of existence and uniqueness. Afterwards, we considered the piecewise-smooth system concerning the existence of closed trajectories. We could find necessary conditions for the existence of closed trajectories without sliding motion assuming additional conditions for the functions $F^\pm$ and $g^\pm$. If we also assumed $\mathbb{Z}_2-$symmetry for system (6.2), we could prove the unique existence of a closed trajectory without sliding motion. This led to a Hopf-like bifurcation of a periodic solution without sliding motion. All these results were global. We could finally prove a Hopf-like bifurcation of a periodic solution with sliding motion. However, this result was only local.

One main tool for detecting closed trajectories in piecewise-smooth systems was the determination of appropriate discrete-time maps. We could define a Poincaré-map as the composition of these maps. The fixed points of the Poincaré-map corresponded to closed trajectories. For piecewise-smooth systems it was in general not possible to analytically determine these discrete-time maps. But that was possible for piecewise-linear systems. Therefore, in chapter 4 we considered the piecewise-linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^+ \binom{x}{y} + \binom{b_1^+}{b_2^+} & , \text{if } x > 0 \\ A^- \binom{x}{y} + \binom{b_1^-}{b_2^-} & , \text{if } x < 0 \end{cases} \tag{6.2}$$

with

$$A^\pm := \begin{pmatrix} a_{11}^\pm & 1 \\ a_{21}^\pm & 0 \end{pmatrix}, \quad a_{11}^\pm, a_{21}^\pm, b_1^\pm, b_2^\pm \in \mathbb{R}.$$

For this system we could analytically determine the discrete-time maps. We got stronger necessary conditions for the existence of closed trajectories than for the piecewise-smooth system (6.1). Moreover, we proved the existence of at most two closed trajectories without sliding motion. We finally presented a global result on a Hopf-like bifurcation of a periodic solution with sliding motion and showed that this periodic solution disappeared in a homoclinic solution to an equilibrium of the sliding motion interval. However, the solutions of the piecewise-linear system (6.2) depended on the eigenvalues of $A^{\pm}$ and therefore, we had to differentiate between 144 different cases. Assuming $\mathbb{Z}_2-$symmetry in system (6.2) we could reduce the 144 different cases to only 5 using a special normal form. In chapter 5 we transformed system (6.2) with $\mathbb{Z}_2-$symmetry to an equivalent system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_{p\delta} \begin{pmatrix} x \\ y \end{pmatrix} + \operatorname{sgn}(x) b, \tag{6.3}$$

where

$$A_{p\delta} = \begin{pmatrix} -\delta & 1 \\ -p & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad p, b_1, b_2 \in \mathbb{R}, \ \delta \in \{0, 1\}.$$

For the 5 different cases, dependent on p and δ, we provided complete dynamical analyses including bifurcation diagrams. All these results were global and analytical.

Altogether we found seven different bifurcation phenomena in piecewise-smooth systems, which are listed in the following:

Bifurcations in piecewise-smooth systems

Consider the planar piecewise-smooth system $\binom{\dot{x}}{\dot{y}} = f(x, y, \mu)$ with $f : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$ smooth on the domains $G_+ := \{(x, y)^T \in \mathbb{R}^2 : x > 0\}$ and $G_- := \{(x, y)^T \in \mathbb{R}^2 : x < 0\}$ and discontinuous on the line $M := \{(x, y)^T \in \mathbb{R}^2 : x = 0\}$. $\mu \in \mathbb{R}$ is a parameter and I_s denotes the sliding motion interval.

1. **Pitchfork-like bifurcation of equilibria**
   For $\mu < \mu_0$ an equilibrium $(0, y_0) \in I_s$ is the unique equilibrium and it is unstable (stable). Passing through $\mu = \mu_0$ two unstable (stable) equilibria branch from the boundary points of the sliding motion interval and $(0, y_0)$ itself becomes stable (unstable). The two new equilibria lie in G_+ and G_-, respectively.

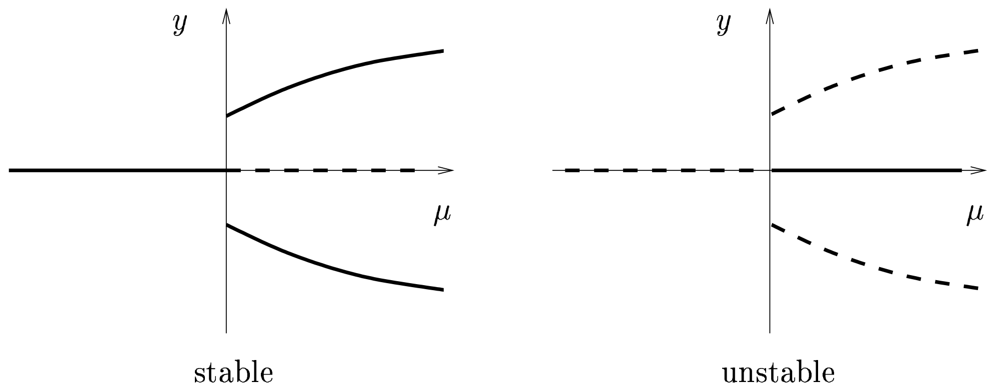


Figure 6.1.: Sketch of the Pitchfork-like bifurcation of equilibria, where $\mu_0 = 0$ and $y_0 = 0$

For the definition of pitchfork bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 145-150].

2. **Fold bifurcation of periodic solutions without sliding motion**
   For $\mu < \mu_0$ there exist two periodic solutions without sliding motion, one of them is unstable and the other one is stable. When μ passes through $\mu = \mu_0$ these two periodic solutions collide and disappear for $\mu > \mu_0$. The bifurcation diagram coincides with the one for smooth systems. For the definition of fold bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 156-157].

3. **Hopf-like bifurcation of periodic solutions without sliding motion**
   For $\mu < \mu_0$ there exists an equilibrium $(0, y_0) \in I_s$ and it is unstable (stable). When μ passes through $\mu = \mu_0$ the equilibrium $(0, y_0)$ becomes stable (unstable) and an unstable (stable) periodic solution without sliding motion bifurcates from $(0, y_0)$.

   The bifurcation diagram looks similar to the bifurcation diagram of a Hopf bifurcation for smooth systems, see for example [Guckenheimer & Holmes,1983, pp. 150-156]. The difference is that in this case the closed trajectory surrounds the sliding motion interval. Consequently, its amplitude does not converge to zero as $\mu \to \mu_0$.

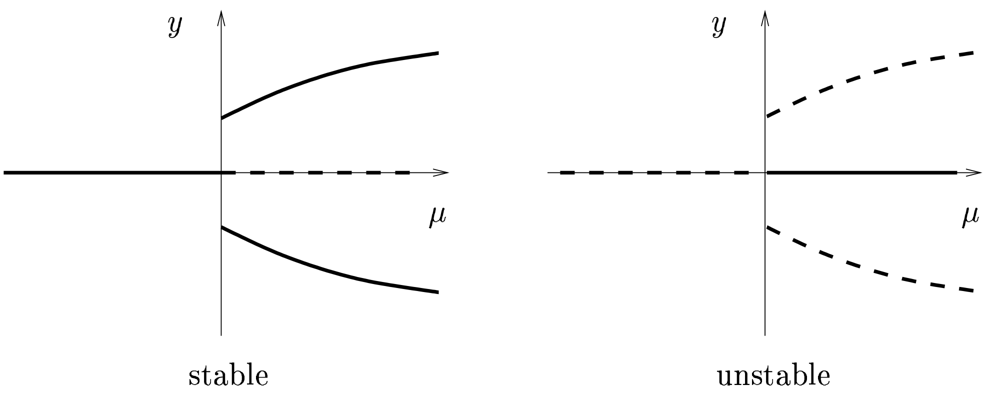


Figure 6.2.: Sketch of the Hopf-like bifurcation of a closed trajectory without sliding motion, where $\mu_0 = 0$ and $y_0 = 0$

4. **Heteroclinic cycle bifurcation from a periodic solution without sliding motion**
   For $\mu < \mu_0$ there is a unique asymptotically stable periodic solution without sliding motion surrounding an unstable equilibrium $(0, y_0) \in I_s$ and the sliding motion interval I_s. Passing through $\mu = \mu_0$ the periodic solution becomes a heteroclinic cycle consisting of two heteroclinic trajectories connecting two equilibria $(0, y_1)$ and $(0, y_2)$, especially the boundary points of I_s, and it disappears for $\mu > \mu_0$.

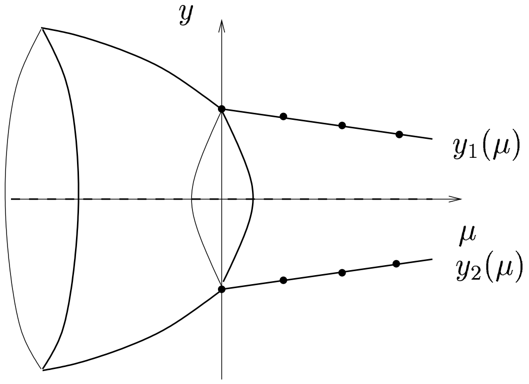


Figure 6.3.: Sketch of the heteroclinic cycle bifurcation of a closed trajectory without sliding motion, where $\mu_0 = 0$ and $y_0 = 0$

For the definition of heteroclinic cycle bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 290-295].

5. **Hopf-like bifurcation of periodic solutions with sliding motion**
   When μ passes through $\mu = \mu_0$ an unstable (stable) periodic solution with sliding motion in $\overline{G_\pm}$ bifurcates when a stable (unstable) focus $(x_0^\pm, y_0^\pm)$ occurs on M. As μ increases, the focus moves into $G_\pm$ and it is surrounded by the periodic solution with sliding motion.

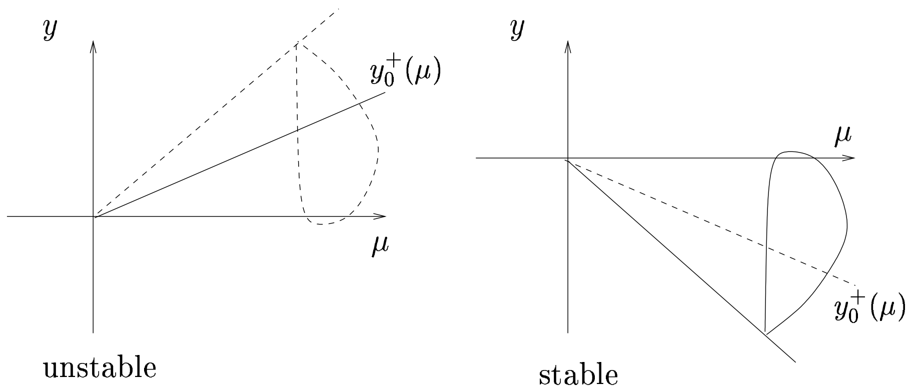


Figure 6.4.: Sketch of the Hopf-like bifurcation of a periodic solution with sliding motion, where $\mu_0 = 0$

For the definition of Hopf bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 150-156].

6. **Homoclinic bifurcation from a periodic solution with sliding motion**
   For $\mu < \mu_0$ there exists an unstable (stable) periodic solution with sliding motion in $\overline{G_\pm}$ surrounding a stable (unstable) focus $(x_0^\pm, y_0^\pm)$ in $G_\pm$. When μ passes through $\mu = \mu_0$ the periodic solution becomes homoclinic to a saddle point $(0, y_0) \in I_s$ and disappears for $\mu > \mu_0$.

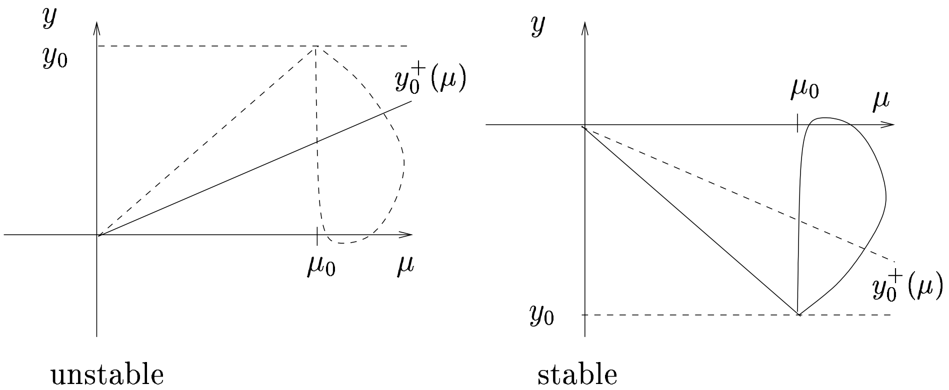


Figure 6.5.: Sketch of the homoclinic bifurcation from a periodic solution with sliding motion at $\mu = \mu_0$

For the definition of homoclinic bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 290-295].

7. **Double homoclinic bifurcation from periodic solutions with sliding motion**
   For $\mu < \mu_0$ there exist two unstable (stable) periodic solutions with sliding motion surrounding a stable (unstable) focus $(x_0^{\pm}, y_0^{\pm})$ in G_+ and G_-, respectively. When μ passes through $\mu = \mu_0$ both periodic solutions become simultaneously homoclinic to a saddle point $(0, y_0) \in I_s$, i.e. we obtain a eight-figure-configuration. For $\mu > \mu_0$ an unstable (stable) periodic solution in $\overline{G_+} \cup \overline{G_-}$ with sliding motion bifurcates from the double homoclinic trajectory surrounding the three equilibria.

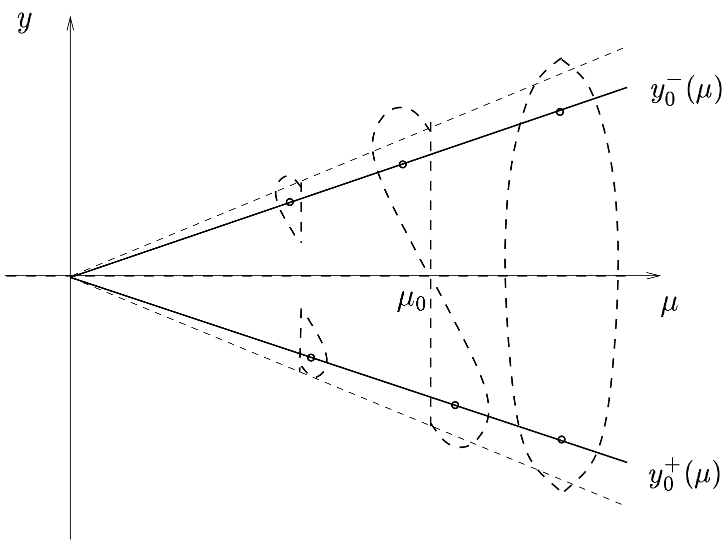


Figure 6.6.: Sketch of the double homoclinic bifurcation from a periodic solution with sliding motion, where $y_0 = 0$

For the definition of double homoclinic bifurcation for smooth systems see for example [Guckenheimer & Holmes,1983, pp. 290-295].

Although these bifurcation behaviour patterns are similar to their smooth counterparts, there is no general theory connecting the dynamical behaviour of piecewise-smooth systems with that of the corresponding smooth systems. This is still an open mathematical problem.

The piecewise-linear-case with $\mathbb{Z}_2-$symmetry has been completely analysed in this thesis. Further research could be done for the non-symmetric piecewise-linear systems. In a first step, one could try to reduce the 144 different cases to a manageable number of different cases. Moreover, one may determine a general concept for analysing the remaining cases as for the symmetric case.
In chapter 3 we proved the unique existence of a closed trajectory without sliding motion assuming $\mathbb{Z}_2-$symmetry. The proof was a modification of the proof for smooth Liénard-systems. This was possible because the proof used the smoothness of the system only in one half-plane. There has been done a lot of research into smooth Liénard-systems during the last decades. There are also proofs for non-symmetric systems, which use the smoothness only in one half-plane. These proofs are very technical, but nevertheless transferable to piecewise-smooth systems with a line of discontinuity. Further research will be done on this field.

A. Notation for the simplified bifurcation diagrams

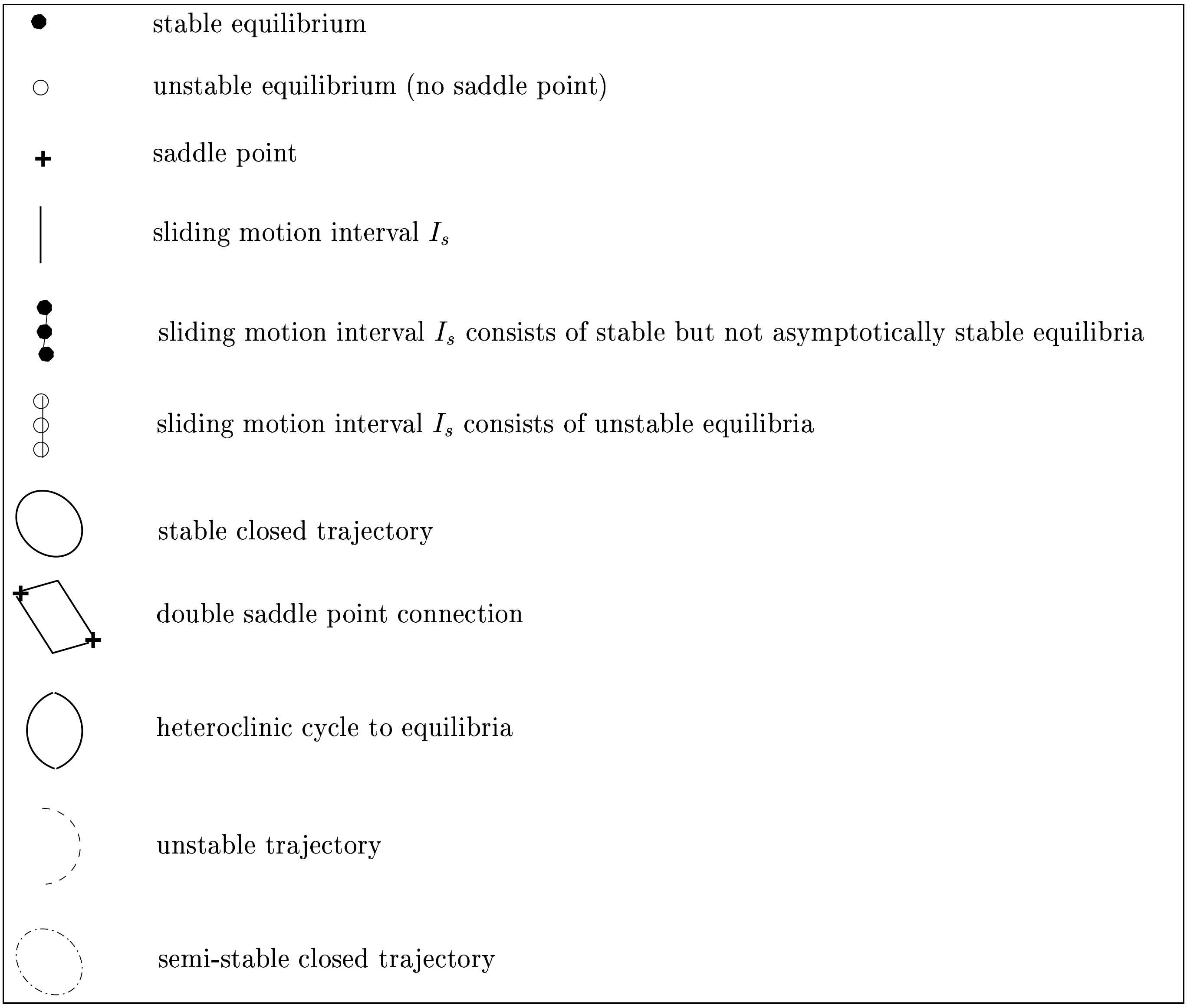


Figure A.1.: Notation for the simplified bifurcation diagrams

B. The auxiliary functions Ξ_i, $i = 1, \dots, 8$

Proposition B.1 (Properties of Ξ_1).
Defining the function $\Xi_1 :]0, \frac{2\pi}{\omega}[\longrightarrow \mathbb{R}$ is dependent on the parameters $\eta \in \mathbb{R}$ and $\omega > 0$ by

$$\Xi_1(t; \eta, \omega) := -e^{-\eta t}(\eta \sin(\omega t) + \omega \cos(\omega t)) + \omega$$

there holds, see Figure B.1:

1. *If $\eta \geq 0$, $\Xi_1(t; \eta, \omega) > 0$ for all $t \in]0, \frac{2\pi}{\omega}[$.*
2. *If $\eta < 0$, Ξ_1 has a unique zero $t_1^0 \in]\frac{\pi}{\omega}, \frac{2\pi}{\omega}[$ for all $\omega > 0$.*

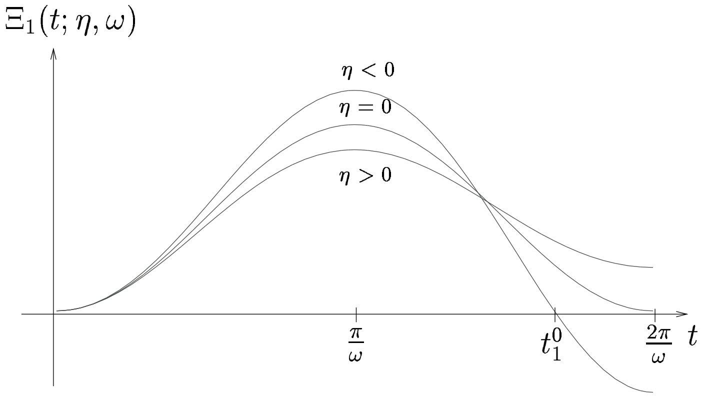


Figure B.1.: The function Ξ_1 for $\eta > 0$, $\eta = 0$ and $\eta < 0$

Proof. There holds

$$\begin{aligned}
\Xi_1(0; \eta, \omega) &= 0, \\
\Xi_1(\frac{\pi}{\omega}; \eta, \omega) &= (e^{-\frac{\eta\pi}{\omega}} + 1)\omega > 0, \\
\Xi_1(\frac{2\pi}{\omega}; \eta, \omega) &= (1 - e^{-\frac{2\pi\eta}{\omega}})\omega \begin{cases} > 0 & \text{, if } \eta > 0 \\ = 0 & \text{, if } \eta = 0 \\ < 0 & \text{, if } \eta < 0 \end{cases}
\end{aligned}$$

and for the derivative with respect to t we get

$$\frac{\partial \Xi_1}{\partial t}(t; \eta, \omega) = e^{-\eta t}(\eta^2 + \omega^2)\sin(\omega t) \begin{cases} > 0 & \text{, if } t \in]0, \frac{\pi}{\omega}[\\ < 0 & \text{, if } t \in]\frac{\pi}{\omega}, \frac{2\pi}{\omega}[\, . \end{cases}$$

From this we get the assertions. $\square$

Proposition B.2 (Properties of Ξ_2).
Defining the function $\Xi_2 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent on the parameters* $\eta \in \mathbb{R}$ *and* $\omega > 0$ *by*

$$\Xi_2(t; \eta, \omega) := e^{\eta t}(\eta \sinh(\omega t) - \omega \cosh(\omega t)) + \omega$$

there holds

$$\Xi_2(t; \eta, \omega) \begin{cases} > 0 & , \textit{ if } \eta^2 - \omega^2 > 0 \\ = 0 & , \textit{ if } \eta^2 - \omega^2 = 0 \\ < 0 & , \textit{ if } \eta^2 - \omega^2 < 0. \end{cases}$$

Proposition B.3 (Properties of Ξ_3).
Defining the function $\Xi_3 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent on the parameters* $\eta \in \mathbb{R}$ *and* $\omega > 0$ *with* $\omega^2 - \eta^2 \neq 0$ *by*

$$\Xi_3(t; \eta, \omega) := \frac{\omega \sinh(\eta t) - \eta \sinh(\omega t)}{\omega^2 - \eta^2}$$

there holds

$$\Xi_3(t; \eta, \omega) \begin{cases} < 0 & , \textit{ if } \eta > 0 \\ = 0 & , \textit{ if } \eta = 0 \\ > 0 & , \textit{ if } \eta < 0. \end{cases}$$

Proposition B.4 (Properties of Ξ_4).
Defining the function $\Xi_4 :]0, \frac{2\pi}{\omega}[\longrightarrow \mathbb{R}$ *is dependent on the parameters* $\eta \in \mathbb{R}$ *and* $\omega > 0$ *by*

$$\Xi_4(t; \eta, \omega) := \omega \sinh(\eta t) - \eta \sin(\omega t)$$

there holds

$$\Xi_4(t; \eta, \omega) \begin{cases} > 0 & , \textit{ if } \eta > 0 \\ = 0 & , \textit{ if } \eta = 0 \\ < 0 & , \textit{ if } \eta < 0. \end{cases}$$

Proposition B.5 (Properties of Ξ_5).
Defining the function $\Xi_5 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent on the parameter* $\eta \neq 0$ *by*

$$\Xi_5(t; \eta) := \eta t - 1 + e^{-\eta t}$$

there holds

$$\Xi_5(t; \eta) > 0.$$

Proposition B.6 (Properties of Ξ_6).
Defining the function $\Xi_6 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent of the parameter* $\eta \in \mathbb{R}$ *by*

$$\Xi_6(t; \eta) := \eta t - \sinh(\eta t)$$

there holds

$$\Xi_6(t; \eta) \begin{cases} < 0 & , \textit{ if } \eta > 0 \\ = 0 & , \textit{ if } \eta = 0 \\ > 0 & , \textit{ if } \eta < 0. \end{cases}$$

Proposition B.7 (Properties of Ξ_7).
Defining the function $\Xi_7 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent on the parameter* $a \neq 0$ *by*

$$\Xi_7(t; a) := at + 1 - e^{at}$$

it holds

$$\Xi_7(t; a) < 0.$$

Proposition B.8 (Properties of Ξ_8).
Defining the function $\Xi_8 :]0, \infty[\longrightarrow \mathbb{R}$ *is dependent on the parameter* $a \in \mathbb{R}$ *by*

$$\Xi_8(t; a) := at(1 + e^{-at}) - 2(1 - e^{-at})$$

there holds

$$\Xi_8(t; a) \begin{cases} > 0 & , \textit{if } a > 0 \\ = 0 & , \textit{if } a = 0 \\ < 0 & , \textit{if } a < 0. \end{cases}$$

C. The transition matrix e^{At}

Proposition C.1.
Let

$$A := \begin{pmatrix} a_{11} & 1 \\ a_{12} & 0 \end{pmatrix}$$

be a real $2 \times 2-$*matrix with* $a_{12} \neq 0$. *Then the transition matrix* $\alpha(t) := e^{At}$ *for* $t \in \mathbb{R}$ *depends on the eigenvalues of* A. *Defining* $\eta := \frac{a_{11}}{2}$ *and* $\omega := \frac{1}{2}\sqrt{|a_{11}^2 + 4a_{21}|}$ *there holds for the transition matrix:*

1. $a_{11}^2 + 4a_{21} > 0$:
$$\alpha(t) = \frac{e^{\eta t}}{2\omega}\begin{pmatrix} a_{11}\sinh(\omega t) + 2\omega\cosh(\omega t) & 2\sinh(\omega t) \\ 2a_{21}\sinh(\omega t) & -a_{11}\sinh(\omega t) + 2\omega\cosh(\omega t) \end{pmatrix}$$
$$\det(\alpha(t)) = e^{2\eta t}$$

2. $a_{11}^2 + 4a_{21} < 0$:
$$\alpha(t) = \frac{e^{\eta t}}{2\omega}\begin{pmatrix} a_{11}\sin(\omega t) + 2\omega\cos(\omega t) & 2\sin(\omega t) \\ 2a_{21}\sin(\omega t) & -a_{11}\sin(\omega t) + 2\omega\cos(\omega t) \end{pmatrix}$$
$$\det(\alpha(t)) = e^{2\eta t}$$

3. $a_{11}^2 + 4a_{21} = 0$:
$$\alpha(t) = \frac{e^{\eta t}}{2}\begin{pmatrix} a_{11}t + 2 & 2t \\ 2a_{21}t & -a_{11}t + 2 \end{pmatrix}$$
$$\det(\alpha(t)) = e^{2\eta t}$$

Proof.

1. $a_{11}^2 + 4a_{21} > 0$:
A has two real eigenvalues $\lambda_{1,2} := \frac{a_{11}}{2} \pm \frac{1}{2}\sqrt{a_{11}^2 + 4a_{21}}$. Determining the corresponding eigenvectors we get the transition matrix.

2. $a_{11}^2 + 4a_{21} < 0$:
A has two conjugate complex eigenvalues $\lambda_{1,2} := \frac{a_{11}}{2} \pm \frac{i}{2}\sqrt{-a_{11}^2 - 4a_{21}}$. Determining the corresponding eigenvectors we get the transition matrix.

3. $a_{11}^2 + 4a_{21} = 0$:
A has one eigenvalue $\lambda_1 := \frac{a_{11}}{2}$ of multiplicity 2. Determining the corresponding eigenvector and a generalised eigenvector we get the transition matrix.

□

D. Determination of the discrete-time map $\Pi^{\pm}$

In this part of the appendix we determine the discrete-time map Π^+ and consider their properties as function of s_0, see Lemmas 4.2 and 4.3. We omit the determination of the discrete-time map Π^- because it is analogous to the one of Π^+. We only have to exchange $+$ and $-$ and replace $s_0 > -b_1^+$ by $s_0^* < -b_1^-$. The solution of the initial value problem corresponding to system (4.1) depends on the eigenvalues of $A^{\pm}$. Therefore, we differentiate between four different cases.

D.1. The case ${a_{11}^+}^2 + 4a_{21}^+ > 0$, $a_{21}^+ \neq 0$ (saddle point or node)

Using η^+ and ω^+ as defined in Lemma 4.2 and the definition of the transition matrix $\alpha^+(t_0^+)$ of Proposition C.1 we get

$$\alpha_{12}^+(t_0^+) = \frac{e^{\eta^+ t_0^+}\sinh(\omega^+ t_0^+)}{\omega^+} \neq 0$$

for all $t_0^+ > 0$.

$$\begin{aligned} s_0 &= \frac{b_2^+(\eta^+ \sinh(\omega^+ t_0^+) - \omega^+ \cosh(\omega^+ t_0^+) + \omega^+ e^{-\eta^+ t_0^+})}{a_{21}^+ \sinh(\omega^+ t_0^+)} \\ &= \frac{b_2^+ e^{-\eta^+ t_0^+}}{\sinh(\omega^+ t_0^+)} \frac{\Xi_2(t_0^+; \eta^+, \omega^+)}{a_{21}^+} - b_1^+ \end{aligned} \tag{D.1}$$

with

$$\Xi_2(t; \eta, \omega) := e^{\eta t}(\eta \sinh(\omega t) - \omega \cosh(\omega t)) + \omega$$

as in Proposition B.2. Note that $a_{21}^+ = {\omega^+}^2 - {\eta^+}^2$. Because of the necessary condition $s_0 > -b_1^+$, equation (D.1) can only have a solution $t_0^+ > 0$ if $b_2^+ < 0$. So that is why we assume $b_2^+ < 0$ in the following. For the derivative of s_0 with respect to t_0^+ we get

$$\begin{aligned} \frac{ds_0}{dt_0^+}(t_0^+) &= \frac{b_2^+ \omega^+}{a_{21}^+ \sinh^2(\omega^+ t_0^+)}(-e^{-\eta t_0^+}(\eta^+ \sinh(\omega^+ t_0^+) + \omega^+ \cosh(\omega^+ t_0^+)) + \omega^+) \\ &= \frac{b_2^+ \omega^+}{\sinh^2(\omega^+ t_0^+)} \frac{\Xi_2(t_0^+; -\eta^+, \omega^+)}{a_{21}^+} > 0. \end{aligned}$$

This means that s_0 as function of t_0^+ is strictly increasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\lim_{t\to 0+} s_0 = -b_1^+$$
$$\lim_{t\to\infty} s_0 = \begin{cases} \frac{b_2^+}{a_{21}^+}(\eta^+ - \omega^+) - b_1^+ & \text{, if } \eta^+ + \omega^+ > 0 \\ \infty & \text{, if } \eta^+ + \omega^+ < 0. \end{cases}$$

Consequently, there holds

$$M^+ = \begin{cases}]-b_1^+, \frac{b_2^+}{a_{21}^+}(\eta^+ - \omega^+) - b_1^+[& \text{, if } \eta^+ + \omega^+ > 0 \\ ]-b_1^+, \infty[& \text{, if } \eta^+ + \omega^+ < 0 \end{cases}$$

and we get

$$\Pi^+(s_0) = \phi_2(t_0^+) = \frac{-b_2^+ e^{\eta^+ t_0^+}}{\sinh(\omega^+ t_0^+)} \frac{\Xi_2(t_0^+; -\eta^+, \omega^+)}{a_{21}^+} - b_1^+,$$

which is defined for all $t_0^+ > 0$, $s_0 \in M^+$ and $b_2^+ < 0$. Π^+ is as function of s_0 smooth and therefore we can determine its derivatives.

$$\Pi^{+\prime}(s_0) = \frac{d}{ds_0}\Pi^+(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^+}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} = -\frac{\Xi_2(t_0^+; \eta^+, \omega^+)}{\Xi_2(t_0^+; -\eta^+, \omega^+)} < 0,$$

$$\Pi^{+\prime\prime}(s_0) = \frac{d}{ds_0}\Pi^{+\prime}(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^{+\prime}}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)}$$
$$= \underbrace{\frac{2a_{21}^{+\,3}\sinh^3(\omega^+ t_0^+)}{b_2^+\omega^+\Xi_2^3(t_0^+; -\eta^+, \omega^+)}}_{>0} \Xi_3(t_0^+; \eta^+, \omega^+) \begin{cases} < 0 & \text{, if } \eta^+ > 0 \\ = 0 & \text{, if } \eta^+ = 0 \\ > 0 & \text{, if } \eta + < 0 \end{cases}$$

with

$$\Xi_3(t; \eta, \omega) := \frac{\omega\sinh(\eta t) - \eta\sinh(\omega t)}{\omega^2 - \eta^2}$$

as in Proposition B.3. Consequently, Π^+ is strictly decreasing, concave if $\eta^+ > 0$, convex if $\eta^+ < 0$ and a straight line with slope -1 if $\eta^+ = 0$. For the limits of Π^+ we get

$$\lim_{s_0\to -b_1^+ +} \Pi^+(s_0) = \lim_{t_0^+\to 0+} \Pi^+(\Sigma^{-1}(t_0^+)) = -b_1^+$$
$$\lim_{s_0\to\infty} \Pi^+(s_0) = \lim_{t_0^+\to\infty} \Pi^+(\Sigma^{-1}(t_0^+)) = \begin{cases} -\infty & \text{, if } \eta^+ - \omega^+ > 0 \\ \frac{b_2^+}{a_{21}^+}(\eta^+ + \omega^+) - b_1^+ & \text{, if } \eta^+ - \omega^+ < 0. \end{cases}$$

D.2. The case ${a_{11}^+}^2 + 4a_{21}^+ < 0$, $a_{21}^+ \neq 0$ (focus)

Using η^+ and ω^+ as defined in Lemma 4.2 and the definition of the transition matrix $\alpha^+(t_0^+)$ of Proposition C.1 we get

$$\alpha_{12}^+(t_0^+) = \frac{e^{\eta^+ t_0^+} \sin(\omega^+ t_0^+)}{\omega^+} = 0$$

if $t_0^+ = \frac{\pi}{\omega^+}$. Note that the solution $\phi(t)$ is $\frac{2\pi}{\omega^+}-$periodic with respect to t_0^+. Because we are looking for the minimal t_0^+, we only consider the interval $]0, \frac{2\pi}{\omega^+}[$. If $t_0^+ = \frac{\pi}{\omega}$, equation $\phi_1(t) = 0$ has a solution for all $s_0 > -b_1^+$ if and only if $b_2^+ = 0$. Therefore, in this case there holds $M^+ =]-b_1^+, \infty[$. Assume now $b_2^+ \neq 0$ and $t_0^+ \in]0, \frac{\pi}{\omega^+}[\cup]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$. Then there holds

$$\begin{aligned} s_0 &= \frac{b_2^+}{a_{21}^+} \frac{\eta^+ \sin(\omega^+ t_0^+) - \omega^+ \cos(\omega^+ t_0^+) + \omega^+ e^{-\eta^+ t_0^+}}{\sin(\omega^+ t_0^+)} - b_1^+ \\ &= \frac{b_2^+ e^{-\eta^+ t_0^+}}{\sin(\omega^+ t_0^+)} \frac{\Xi_1(t_0^+; -\eta^+, \omega^+)}{a_{21}^+} - b_1^+ \end{aligned} \tag{D.2}$$

with

$$\Xi_1(t; \eta, \omega) := -e^{-\eta t}(\eta \sin(\omega t) + \omega \cos(\omega t)) + \omega$$

as in Proposition B.1. For the derivative of s_0 with respect to t_0^+ we get

$$\begin{aligned} \frac{ds_0}{dt_0^+}(t_0^+) &= \frac{b_2^+ \omega^+}{a_{21}^+ \sin^2(\omega^+ t_0^+)} (-e^{-\eta^+ t_0^+}(\eta^+ \sin(\omega^+ t_0^+) + \omega^+ \cos(\omega^+ t_0^+)) + \omega^+) \\ &= \frac{b_2^+ \omega^+}{a_{21}^+ \sin^2(\omega^+ t_0^+)} \Xi_1(t_0^+; \eta^+, \omega^+). \end{aligned}$$

In case of $b_2^+ < 0$ and $t_0^+ \in]0, \frac{\pi}{\omega}[$, we have that s_0 as function of t_0^+ is strictly increasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\begin{aligned} &\lim_{t \to 0+} s_0 = -b_1^+ \\ &\lim_{t \to \frac{\pi}{\omega^+}-} s_0 = \frac{b_2^+}{a_{21}^+} \lim_{t \to \frac{\pi}{\omega^+}-} \frac{\eta^+ \sin(\omega^+ t) - \omega^+ \cos(\omega^+) + \omega^+ e^{\eta^+ t_0^+}}{\sin(\omega^+ t)} - b_1^+ = \infty. \end{aligned}$$

Consequently, there holds $M^+ =]-b_1^+, \infty[$ in case of $b_2^+ < 0$.
In case of $b_2^+ > 0$ we have to differentiate between the cases $\eta^+ > 0$, $\eta^+ = 0$ and $\eta^+ < 0$. Then the equilibrium (x_0^+, y_0^+) exists and is an unstable focus (center, stable focus) if $\eta^+ > 0$ ($\eta^+ = 0$, $\eta^+ < 0$).

1. $\eta^+ > 0$: As shown in case 2 of the proof of Theorem 4.12, there exists a trajectory which starts at $(0, -b_1^+)$ and reaches M again after finite time t_1^0, where t_1^0 is the unique zero of $\Xi_1(t; -\eta^+, \omega^+)$ in $]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$, see Proposition B.1. There

holds $\frac{ds_0}{dt_0^+}(t_0^+) < 0$ which means that s_0 as function of t_0^+ is strictly decreasing and therefore invertible,

$$s_0 := \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\begin{aligned}
&\lim_{t\to t_1^0-} s_0 = -b_1^+ \\
&\lim_{t\to\frac{\pi}{\omega^+}+} s_0 = \frac{b_2^+}{a_{21}^+} \lim_{t\to\frac{\pi}{\omega^+}+} \frac{\eta^+\sin(\omega^+t) - \omega^+\cos(\omega^+t) + \omega^+e^{-\eta^+t}}{\sin(\omega^+t)} - b_1^+ = \infty.
\end{aligned}$$

Consequently, there holds $M^+ =]-b_1^+, \infty[$.

2. $\eta^+ = 0$: In this case we have

$$\frac{ds_0}{dt_0^+}(t_0^+) = \frac{b_2^+\omega^{+2}}{a_{21}^2\sin^2(\omega^+t_0^+)}(1-\cos(\omega^+t_0^+)) < 0$$

for all $t_0^+ \in]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$. This means that s_0 as function of t_0^+ is striclty decreasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\begin{aligned}
\lim_{t\to\frac{\pi}{\omega^+}+} s_0 &= \frac{b_2^+\omega^+}{a_{21}^+} \lim_{t\to\frac{\pi}{\omega^+}+} \frac{1-\cos(\omega^+t)}{\sin(\omega^+t)} - b_1^+ = \infty \\
\lim_{t\to\frac{2\pi}{\omega^+}-} s_0 &= \frac{b_2^+\omega^+}{a_{21}^+} \lim_{t\to\frac{2\pi}{\omega^+}-} \frac{1-\cos(\omega^+t)}{\sin(\omega^+t)} - b_1^+ = -b_1^+.
\end{aligned}$$

Consequently, there holds $M^+ =]-b_1^+, \infty[$.

3. $\eta^+ < 0$: As shown in case 1 of the proof of Theorem 4.12, there exists a trajectory which reaches $(0, -b_1^+)$ tangentially. This trajectory starts at $(0, \overline{s}^+)$ with

$$\overline{s}^+ := -\frac{b_2^+e^{-\eta^+t_1^0}\sin(\omega^+t_1^0)}{\omega^+} - b_1^+,$$

where t_1^0 is the unique zero of $\Xi_1(t;\eta^+,\omega^+)$ in $]\frac{\pi}{\omega^+}, \frac{2\pi}{\omega^+}[$, see Proposition B.1, and moreover the time the trajectory needs from $(0,\overline{s}^+)$ to $(0,-b_1^+)$. All trajectories starting at a point $(0, s_0)$ with $s_0 \in]-b_1^+, \overline{s}^+]$ reach the stable focus (x_0^+, y_0^+) without intersecting M again. Therefore, $s_0 > \overline{s}^+$ is a necessary condition for the existence of Π^+. For $t_0^+ \in]\frac{\pi}{\omega^+}, t_1^0[$ there holds $\frac{ds_0}{dt_0^+}(t_0^+) < 0$. This means that s_0 as function of t_0^+ is strictly decreasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\lim_{t \to t_1^0 -} s_0 = \overline{s}^+$$

$$\lim_{t \to \frac{\pi}{\omega^+}+} s_0 = \frac{b_2^+}{a_{21}^+} \lim_{t \to \frac{\pi}{\omega^+}+} \frac{\eta^+ \sin(\omega^+ t) - \omega^+ \cos(\omega^+) + \omega^+ e^{-\eta^+ t}}{\sin(\omega^+ t)} - b_1^+ = \infty.$$

Consequently, there holds $M^+ =]\overline{s}^+, \infty[$.

We get for

$$\Pi^+(s_0) = \phi_2(t_0^+) = \begin{cases} \frac{-b_2^+ e^{\eta^+ t_0^+}}{a_{21}^+ \sin(\omega^+ t_0^+)} \Xi_1(t_0^+; \eta^+, \omega^+) - b_1^+ & , \text{if } b_2^+ \neq 0 \\ -e^{\frac{\eta^+ \pi}{\omega^+}} s_0 - (1 + e^{\frac{\eta^+ \pi}{\omega^+}}) b_1^+ & , \text{if } b_2^+ = 0. \end{cases}$$

Π^+ is as function of s_0 smooth and therefore we can determine its derivatives.

$$\begin{aligned}
\Pi^{+\prime}(s_0) &= \frac{d}{ds_0} \Pi^+(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^+}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= \begin{cases} -\frac{\Xi_1(t_0^+; -\eta^+, \omega^+)}{\Xi_1(t_0^+; \eta^+, \omega^+)} & , \text{if } b_2^+ \neq 0 \\ -e^{\frac{\eta^+ \pi}{\omega^+}} & , \text{if } b_2^+ = 0 \end{cases} < 0 \\
\Pi^{+\prime\prime}(s_0) &= \frac{d}{ds_0} \Pi^{+\prime}(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^{+\prime}}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= \begin{cases} \frac{2{a_{21}^+}^2 \sin^3(\omega^+ t_0^+)}{b_2^+ \omega^+ \Xi_1^3(t_0^+; \eta^+, \omega^+)} \Xi_4(t_0^+; \eta^+, \omega^+) & , \text{if } b_2^+ \neq 0 \\ 0 & , \text{if } b_2^+ = 0 \end{cases} \\
& \begin{cases} < 0 & , \text{if } \eta^+ > 0 \text{ and } b_2^+ \neq 0 \\ = 0 & , \text{if } \eta^+ = 0 \text{ or } b_2^+ = 0 \\ > 0 & , \text{if } \eta^+ < 0 \text{ and } b_2^+ \neq 0 \end{cases}
\end{aligned}$$

with

$$\Xi_4(t; \eta, \omega) := \omega \sinh(\eta t) - \eta \sin(\omega t)$$

as in Proposition B.4. Consequently, Π^+ is strictly decreasing, concave if $\eta^+ > 0$, convex if $\eta^+ < 0$ and a straight line with slope -1 if $\eta^+ = 0$, provided $b_2^+ \neq 0$. In case of $b_2^+ = 0$, Π^+ is a straight line with slope $-e^{\frac{\eta^+ \pi}{\omega^+}}$.
We consider now the limits of Π^+ in the different cases.
The cases $b_2^+ \leq 0$ and $b_2^+ > 0$, $\eta^+ = 0$:

$$\lim_{s_0 \to -b_1^+ +} \Pi^+(s_0) = -b_1^+$$

$$\lim_{s_0 \to \infty} \Pi^+(s_0) = -\infty$$

The case $b_2^+ > 0$ and $\eta^+ \neq 0$:

$$\begin{aligned}
\lim_{s_0 \to -b_1^+ +} \Pi^+(s_0) &= \lim_{t_0^+ \to t_1^0 -} \Pi^+(\Sigma^{-1}(t_0^+)) \\
&= \begin{cases} \frac{b_2^+}{a_{21}^+} \frac{\eta^+ \sin(\omega^+ t_1^0) - 2\omega^+ \sinh(\eta^+ t_1^0)}{\sin(\omega^+ t_1^0)} - b_1^+ =: \hat{s}^+ & \text{, if } \eta^+ > 0 \\ -b_1^+ & \text{, if } \eta^+ < 0 \end{cases} \\
\lim_{s_0 \to \infty} \Pi^+(s_0) &= \lim_{t_0^+ \to \frac{\pi}{\omega^+} -} \Pi^+(\Sigma^{-1}(t_0^+)) = -\infty
\end{aligned}$$

D.3. The case ${a_{11}^+}^2 + 4a_{21}^+ = 0$, $a_{11}^+ \neq 0 \neq a_{21}^+$ (node)

Using η^+ as defined in Lemma 4.2 and the definition of the transition matrix $\alpha^+(t_0^+)$ of Proposition C.1 we get

$$\alpha_{12}^+(t_0^+) = e^{\eta^+ t_0^+} t_0^+ \neq 0$$

for all $t_0^+ > 0$.

$$\begin{aligned}
s_0 &= \frac{b_2^+}{a_{21}^+ t_0^+}(\eta^+ t_0^+ - 1 + e^{-\eta^+ t_0^+}) - b_1^+ \\
&= \frac{b_2^+}{a_{21}^+ t_0^+} \Xi_5(t_0^+; \eta^+) - b_1^+
\end{aligned} \tag{D.3}$$

with

$$\Xi_5(t;\eta) := \eta t - 1 + e^{-\eta t}$$

as in Proposition B.5. Because of the necessary condition $s_0 > -b_1^+$, equation (D.3) can only have a solution $t_0^+ > 0$ if $b_2^+ < 0$. So that is why we assume $b_2^+ < 0$ in the following. For the derivative of s_0 with respect to t_0^+ we get

$$\begin{aligned}
\frac{ds_0}{dt_0^+}(t_0^+) &= \frac{b_2^+ e^{-\eta^+ t_0^+}}{a_{21}^+ {t_0^+}^2}(-\eta^+ t_0^+ - 1 + e^{\eta^+ t_0^+}) \\
&= \frac{b_2^+ e^{-\eta^+ t_0^+}}{a_{21}^+ {t_0^+}^2} \Xi_5(t_0^+; -\eta^+) > 0.
\end{aligned}$$

This means that s_0 as function of t_0^+ is strictly increasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\begin{aligned}
\lim_{t \to 0+} s_0 &= -b_1^+ \\
\lim_{t \to \infty} s_0 &= \begin{cases} -\frac{b_2^+}{\eta^+} - b_1^+ & \text{, if } \eta^+ > 0 \\ \infty & \text{, if } \eta^+ < 0. \end{cases}
\end{aligned}$$

Consequently, there holds

$$M^+ = \begin{cases}]-b_1^+, -\frac{b_2^+}{\eta^+} - b_1^+[& \text{, if } \eta^+ > 0 \\ ]-b_1^+, \infty[& \text{, if } \eta^+ < 0 \end{cases}$$

and we get

$$\Pi^+(s_0) = \phi_2(t_0^+) = -\frac{b_2^+}{a_{21}^+ t_0^+}\Xi_5(t_0^+; -\eta^+) - b_1^+.$$

Π^+ is as function of s_0 smooth and therefore we can determine its derivatives.

$$\begin{aligned}
{\Pi^+}'(s_0) &= \frac{d}{ds_0}\Pi^+(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^+}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= -\frac{e^{\eta^+ t_0^+}\Xi_5(t_0^+; \eta^+)}{e^{-\eta^+ t_0^+}\Xi_5(t_0^+; -\eta^+)} < 0, \\
{\Pi^+}''(s_0) &= \frac{d}{ds_0}{\Pi^+}'(\Sigma^{-1}(t_0^+)) = \frac{\frac{d{\Pi^+}'}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= \underbrace{\frac{-2{t_0^+}^3}{b_2^+ e^{-3\eta^+ t_0^+}\Xi_5^3(t_0^+; -\eta^+)}}_{>0}\Xi_6(t_0^+; \eta^+)\begin{cases} < 0 & \text{, if } \eta^+ > 0 \\ > 0 & \text{, if } \eta^+ < 0 \end{cases}
\end{aligned}$$

with

$$\Xi_6(t; \eta) := \eta t - \sinh(\eta t)$$

as in Proposition B.6. Consequently, Π^+ is strictly decreasing, concave if $\eta^+ > 0$ and convex if $\eta^+ < 0$. For the limits of Π^+ we get

$$\begin{aligned}
\lim_{s_0 \to -b_1^+ +} \Pi^+(s_0) &= \lim_{t_0^+ \to 0+} \Pi^+(\Sigma^{-1}(t_0^+)) = -b_1^+ \\
\lim_{s_0 \to \infty} \Pi^+(s_0) &= \lim_{t_0^+ \to \infty} \Pi^+(\Sigma^{-1}(t_0^+)) = \begin{cases} -\infty & \text{, if } \eta^+ > 0 \\ -\frac{b_2^+}{\eta^+} - b_1^+ & \text{, if } \eta^+ < 0. \end{cases}
\end{aligned}$$

D.4. The case $a_{21}^+ = 0, a_{11}^+ \neq 0$

The solution of $\binom{\dot{x}}{\dot{y}} = A^+\binom{x}{y} + \binom{b_1^+}{b_2^+}$ with initial value $\binom{x(0)}{y(0)} = \binom{0}{s_0}$ with $s_0 > -b_1^+$ is given by

$$\phi(t) = \begin{pmatrix} (1 - e^{a_{11}^+ t})(-\frac{b_2^+}{{a_{11}^+}^2} - \frac{b_1^+ + s_0}{a_{11}^+}) - \frac{b_2^+ t}{a_{11}^+} \\ b_2^+ t + s_0 \end{pmatrix}$$

and we have

$$\begin{aligned}
s_0 &= \frac{b_2^+}{a_{11}^+}\frac{a_{11}^+ t_0^+ + 1 - e^{a_{11}^+ t_0^+}}{e^{a_{11}^+ t_0^+} - 1} - b_1^+ \\
&= \frac{b_2^+}{a_{11}^+(e^{a_{11}^+ t_0^+} - 1)}\Xi_7(t_0^+; a_{11}^+) - b_1^+ && \text{(D.4)}
\end{aligned}$$

with

$$\Xi_7(t; a) := at + 1 - e^{at}$$

as in Proposition B.7. Because of the necessary condition $s_0 > -b_1^+$, equation (D.4) can only have a solution $t_0^+ > 0$ if $b_2^+ < 0$. So that is why we assume $b_2^+ < 0$ in the following. For the derivative of s_0 with respect to t_0^+ we get

$$\begin{aligned}
\frac{ds_0}{dt_0^+}(t_0^+) &= \frac{b_2^+ e^{a_{11}^+ t_0^+}}{(e^{a_{11}^+ t_0^+} - 1)^2}(1 - a_{11}^+ t_0^+ - e^{-a_{11}^+ t_0^+}) \\
&= \frac{b_2^+ e^{a_{11}^+ t_0^+}}{(e^{a_{11}^+ t_0^+} - 1)^2}\Xi_7(t_0^+; -a_{11}^+) > 0.
\end{aligned}$$

This means that s_0 as function of t_0^+ is strictly increasing and therefore invertible,

$$s_0 =: \Sigma^{-1}(t_0^+) \quad \Leftrightarrow \quad t_0^+ = \Sigma(s_0),$$

as well as t_0^+ as function of s_0. By considering the limits of s_0 we get the domain M^+

$$\begin{aligned}
&\lim_{t\to 0+} s_0 = -b_1^+ \\
&\lim_{t\to\infty} s_0 = \begin{cases} -\frac{b_2^+}{a_{11}^+} - b_1^+ & \text{, if } a_{11}^+ > 0 \\ \infty & \text{, if } a_{11}^+ < 0. \end{cases}
\end{aligned}$$

Consequently, there holds

$$M^+ = \begin{cases}]-b_1^+, -\frac{b_2^+}{a_{11}^+} - b_1^+[& \text{, if } a_{11}^+ > 0 \\ ]-b_1^+, \infty[& \text{, if } a_{11}^+ < 0 \end{cases}$$

and we get

$$\Pi^+(s_0) = \phi_2(t_0^+) = -\frac{b_2^+ e^{a_{11}^+ t_0^+}}{a_{11}^+(e^{a_{11}^+ t_0^+} - 1)}\Xi_7(t_0^+; -a_{11}^+) - b_1^+.$$

Π^+ is as function of s_0 smooth and therefore we can determine its derivatives.

$$\begin{aligned}
\Pi^{+\prime}(s_0) &= \frac{d}{ds_0}\Pi^+(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^+}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= -\frac{\Xi_7(t_0^+; a_{11}^+)}{\Xi_7(t_0^+; -a_{11}^+)} < 0, \\
\Pi^{+\prime\prime}(s_0) &= \frac{d}{ds_0}\Pi^{+\prime}(\Sigma^{-1}(t_0^+)) = \frac{\frac{d\Pi^{+\prime}}{dt_0^+}(s_0)}{\frac{ds_0}{dt_0^+}(t_0^+)} \\
&= \underbrace{\frac{a_{11}^+(1 - e^{a_{11}^+ t_0^+})^3}{b_2^+ e^{a_{11}^+ t_0^+}\Xi_7^3(t_0^+; -a_{11}^+)}}_{<0}\Xi_8(t_0^+; a_{11}^+)\begin{cases} < 0 & \text{, if } a_{11}^+ > 0 \\ > 0 & \text{, if } a_{11}^+ < 0 \end{cases}
\end{aligned}$$

with

$$\Xi_8(t; a) := at(1 + e^{-at}) - 2(1 - e^{-at})$$

as in Proposition B.8. Consequently, Π^{+} is strictly decreasing, concave if $a_{11}^{+} > 0$ and convex if $a_{11}^{+} < 0$. For the limits of Π^{+} we get

$$\lim_{s_0 \to -b_1^{+}+} \Pi^{+}(s_0) = \lim_{t_0^{+} \to 0+} \Pi^{+}(\Sigma^{-1}(t_0^{+})) = -b_1^{+}$$

$$\lim_{s_0 \to \infty} \Pi^{+}(s_0) = \lim_{t_0^{+} \to \infty} \Pi^{+}(\Sigma^{-1}(t_0^{+})) = \begin{cases} -\infty & \text{, if } a_{11}^{+} > 0 \\ -\frac{b_2^{+}}{a_{11}^{+}} - b_1^{+} & \text{, if } a_{11}^{+} < 0. \end{cases}$$

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt habe, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegeben Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Prof. Dr. T. Küpper betreut worden.

Köln, Oktober 2002

(Teilpublikationen siehe Literaturverzeichnis [Giannakopoulos & Pliete,2001, Giannakopoulos & Pliete,2002])

Zusammenfassung

Wir betrachten planare stückweise-glatte Systeme in Liénard-Form, die auf der $y-$Achse unstetig sind. Solche Systeme sind gegeben durch

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y - F^+(x) \\ -g^+(x) \end{pmatrix} & \text{, falls } x > 0 \\ \begin{pmatrix} y - F^-(x) \\ -g^-(x) \end{pmatrix} & \text{, falls } x < 0, \end{cases} \tag{1}$$

wobei $F^+(x), g^+(x)$ und $F^-(x), g^-(x)$ jeweils glatte Funktionen für $x \geq 0$ und $x \leq 0$ sind. Unser Hauptziel ist die analytische Untersuchung von System (1) in Hinblick auf globales dynamisches Verhalten und Verzweigungsphänomene. System (1) beinhaltet zwei verschiedene interessante Komponenten, "stückweise Glattheit" und "Liénard-Form", die die Untersuchung motivieren.

Stückweise-glatte Systeme spielen eine wichtige Rolle bei der Modellierung von vielen Anwendungen aus den Bereichen der Kontrolltheorie, der Mechanik oder der Ingenieurwissenschaften. Beispiele aus der Mechanik sind Phänomene, die durch trockene Reibung hervorgerufen werden, wie Geräuschentwicklung bei Rädern von Zügen, quietschende Autobremsen oder Klappern von Maschinenteilen, siehe [Bishop et al.,1995], [Kunze & Kuepper,1997] oder [Rudolph & Popp,2001]. Anwendungen aus dem Bereich der Elektrotechnik sind elektronische Schaltkreise, siehe [Stoker,1950] oder [Andronov et al.,1966]. Des weiteren haben stückweise-glatte Systeme eine lange Tradition in der Kontrolltheorie, in der Rückkopplung eine der meist verwendeten Kontrolltechniken ist, siehe [Flügge-Lotz,1947], [Lefschetz,1965], [Andronov et al.,1966], [Sontag,1990] oder [Utkin,1992]. Beispiele hierfür sind Temperatur-Regler oder automatische Kurs-Regler von Flugzeugen oder Wassertorpedos, siehe [Popow,1958]. Unstetige Systeme sind insbesondere deshalb von großer Bedeutung, da sogenanntes Sliding Motion auftreten kann, d.h. Lösungen, die innerhalb der Unstetigkeit verlaufen. Sliding Motion wird beispielsweise zur Kontrolle von elektrischen Motoren benutzt wird, siehe [Utkin,1992]. Es hat sich gezeigt, dass stückweise-glatte Systeme vielfältige dynamische Verhaltensweisen und Verzweigungsphänomene hervorrufen.

Eine Motivation für die Untersuchung planarer Systeme in Hinblick periodischer Lösungen ist Hilbert's 16. Problem, also die Frage nach der maximalen Anzahl periodischer Lösungen in einem planaren System mit polynomialer Nichtlinearität in Abhängigkeit des Grades n der Polynome, siehe [Hilbert,1902]. Dieses Problem ist bis heute nicht einmal für den einfachsten Fall $n = 2$ gelöst, siehe [Ilyashenko,2002]. Glatte Systeme in Liénard-Form wurden, seitdem Liénard sie 1928 erstmalig untersucht hat, weiterhin sehr intensiv untersucht, siehe [Liénard,1928] oder [Levinson & Smith, 1942]. Die Resultate sind jedoch meist lokal, und numerische Methoden sind notwendig um die Analyse zu vervollständigen, siehe [Giannakopoulos & Oster,1997] oder [Khibnik et al.,1998]. Aus diesem Grund verwenden wir einen anderen Ansatz mit Hilfe stückweise-glatter Nichtlinearitäten, um analytisch globales Verhalten zu bekommen.

Während der letzten Jahrzehnte ist die Verzweigungstheorie und die Theorie der Dynamischen Systeme für glatte Systeme intensiv weiterentwickelt worden, siehe z.B. [Guckenheimer & Holmes,1983], [Arrowsmith & Place,1990] oder [Kuznetsov,1998]. Diese Theorie benötigt jedoch meistens die Glattheit des Systems, so dass sie auf stückweise-glatte Systeme nicht anwendbar ist. Üblicherweise werden stückweise-glatte Systeme zu Differentialinklusionen erweitert, d.h. die rechte Seite wird durch eine mengenwertige Funktion ersetzt, siehe [Aubin & Cellina,1984] oder [Filippov,1988]. Die existierende Theorie zu Differentialinklusionen liefert bisher nur einige Grundlagen, siehe [Aubin & Cellina,1984], [Filippov,1988], [Deimling,1992], [Clarke,1998] oder [Kunze,2000]. Deshalb sind neue Konzepte notwendig.

Zur Untersuchung von System (1) definieren wir zunächst die Begriffe Lösung, Sliding Motion, singuläre Punkte, geschlossene Trajektorie und Verzweigung für stückweise-glatte Systeme. Wir analysieren System (1) soweit das analytisch möglich ist. Wir bestimmen das Sliding Motion Interval, charakterisieren alle singuären Punkte, die darin enthalten sind, und untersuchen das entsprechende Anfangswertproblem in Hinblick auf Existenz und Eindeutigkeit in Abhängigkeit von Parametern. Ferner betrachten wir System (1) bezüglich der Existenz von geschlossenen Trajektorien. Unter gewissen Voraussetzungen an die Funktionen $F^{\pm}$ und $g^{\pm}$ beweisen wir die Nicht-Existenz von geschlossenen Trajektorien sowohl mit als auch ohne Sliding Motion. Wenn wir zusätzlich $\mathbb{Z}_2-$Symmetrie in System (1) voraussetzen, können wir die eindeutige Existenz einer asymptotisch stabilen periodischen Lösung ohne Sliding Motion nachweisen. Dies führt zu einer Hopf-ähnlichen Verzweigung einer periodischen Lösung ohne Sliding Motion. Alle diese Ergebnisse sind global. Schließlich finden wir eine Hopf-ähnliche Verzweigung einer periodischen Lösung mit Sliding Motion. Dies ist jedoch nur ein lokales Ergebnis.

Ein wichtiges Hilfsmittel für die Berechnung geschlossener Trajektorien sind Punkttransformationen. Eine geschlossene Trajektorie entspricht einem Fixpunkt der Poincaré-Abbildung. Die Poincaré-Abbildung kann als Komposition der Punkttransformationen in den einzelnen Gebieten der Glattheit bestimmt werden. Im Falle von System (1) können wir diese Punkttransformationen im Allgemeinen nicht analytisch berechnen. Wenn wir aber statt des stückweise-glatten Systems (1) ein stückweise-lineares System betrachten, so können wir diese Punkttransformationen analytisch berechnen. Dieses stückweise-lineare System ist gegeben durch

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases} A^{+}\binom{x}{y} + \binom{b_1^{+}}{b_2^{+}} & \text{, falls } x > 0 \\ A^{-}\binom{x}{y} + \binom{b_1^{-}}{b_2^{-}} & \text{, falls } x < 0, \end{cases} \tag{2}$$

wobei

$$A^{\pm} := \begin{pmatrix} a_{11}^{\pm} & 1 \\ a_{21}^{\pm} & 0 \end{pmatrix}, \quad a_{11}^{\pm}, a_{21}^{\pm}, b_1^{\pm}, b_2^{\pm} \in \mathbb{R}.$$

Wir berechnen die Punkttransformationen für System (2) und erhalten mit deren Hilfe stärkere Aussagen über Existenz und Nicht-Existenz von geschlossenen Trajektorien als für System (1). Außerdem können wir zeigen, dass die Punkttransformationen in diesem Fall streng monoton und entweder konkav, konvex oder eine Gerade sind. Als Konsequenz daraus erhalten wir die Existenz von maximal zwei geschlossenen Trajektorien ohne Sliding Motion. Schließlich können wir die Hopf-ähnliche Verzweigung von periodischen Lösungen mit Sliding Motion global nachweisen und zeigen, dass diese periodischen Lösungen zu homoklinen Lösungen mit Sliding Motion werden und dann verschwinden. Allerdings sind die Punkttransformationen abhängig von den Eigenwerten der Matrizen $A^{\pm}$ und aus diesem Grunde müssen wir 144 verschiedene Fälle betrachten. Wenn wir aber annehmen, dass System (2) $\mathbb{Z}_2-$symmetrisch ist, können wir das entsprechende System zu einem äquivalenten System in einer speziellen Normalform transformieren und dadurch die Anzahl der Fälle drastisch reduzieren. Dieses System in Normalform ist gegeben durch

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_{p\delta} \begin{pmatrix} x \\ y \end{pmatrix} + \operatorname{sgn}(x) b, \tag{3}$$

wobei

$$A_{p\delta} = \begin{pmatrix} -\delta & 1 \\ -p & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad p, b_1, b_2 \in \mathbb{R}, \ \delta \in \{0, 1\}.$$

In System (3) brauchen wir nur zwischen fünf verschiedenen Fällen in Abhängigkeit von p und δ zu unterscheiden. Für diese fünf verschiedenen Fälle liefern wir vollständige dynamische Analysen, inklusive Verzweigungsdiagrammen. Diese Ergebnisse sind alle analytisch und global und wir finden sieben unterschiedliche Verzweigungsphänomene, die in dieser Form bei glatten Systemen nicht auftreten.

Abstract

We consider planar piecewise-smooth systems of Liénard-type with a line of discontinuity. These systems arise from many applications in control theory, mechanics or engineering. We analyse such a system in terms of dynamical and bifurcation behaviour. For this we determine all equilibria, periodic, heteroclinic and homoclinic solutions in dependency on parameters. Our main goal is the analytical determination of global behaviour. This is in particular possible if the system is piecewise-linear. When we additionally assume that the system is $\mathbb{Z}_2-$symmetric we obtain a complete characterisation. As one result we detect bifurcation phenomena which do not exist in this form for smooth systems.

Kurzzusammenfassung

Wir betrachten planare stückweise-glatte Systeme in Liénard-Form, die unstetig auf einer Geraden sind. Diese Systeme finden sich in vielen Anwendungen im Bereich der Kontrolltheorie, Mechanik oder Ingenieurwissenschaften wieder. Wir analysieren solch ein System in Hinblick auf dynamisches Verhalten und Verzweigungsphänomene. Dazu bestimmen wir alle Gleichgewichtszustände, periodische, heterokline und homokline Lösungen in Abhängigkeit von Parametern. Unser Hauptziel ist die analytische Bestimmung von globalem Verhalten. Dies ist insbesondere dann möglich, wenn das System stückweise-linear ist. Wenn wir zusätzlich $\mathbb{Z}_2-$Symmetrie voraussetzen, können wir das System vollständig charakterisieren. Dabei finden wir Verzweigungsphänomene, die bei glatten Systemen in dieser Form nicht existieren.

Lebenslauf

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