Some results in supergeometry

Harmonic maps from super Riemann surfaces

and

Automorphism supergroups of supermanifolds

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Kurzzusammenfassung

Die vorliegende Arbeit besteht aus zwei unabhängigen und eigenständigen Teilen.

Gegenstand des ersten Teils sind harmonische Abbildungen von super-Riemannschen Flächen nach komplex-projektiven Räumen und projektiven Räumen bezüglich des Superschiefkörpers $\mathbb{D}$. In beiden Fällen wird die Theorie der Gauß-Transformierten entwickelt und der Begriff der Isotropie studiert, insbesondere mit Hinblick auf den Zusammenhang zu holomorphen Differentialen auf der super-Riemannschen Fläche. Überdies geben wir eine Definition für harmonische Abbildungen endlichen Typs für eine spezielle Klasse von Abbildungen nach $\mathbb{C}P^{n|n+1}$ und erhalten so eine Klassifikation bestimmter harmonischer super-Tori. Ferner untersuchen wir die Gleichungen, die von den unterliegenden Objekten erfüllt werden und geben ein Beispiel eines harmonischen super-Torus in $\mathbb{D}P^2$ dessen unterliegende Abbildung nicht harmonisch ist.

Im zweiten Teil studieren wir einen klassischen Satz, der besagt, dass die Gruppe der Automorphismen einer Mannigfaltigkeit, die eine G-Struktur endlichen Typs erhalten, eine Lie-Gruppe bildet, im Kontext von Supermannigfaltigkeiten. Wir verallgemeinern dieses Theorem auf die Kategorie der *cs* Mannigfaltigkeiten und illustrieren es anhand einiger, sowohl klassische Objekte verallgemeinernder als auch genuin supergeometrischer, Beispiele. Insbesondere ist es nötig eine neue Klasse von Supermannigfaltigkeiten einzuführen - gemischte Supermannigfaltigkeiten.

Abstract

This thesis consists of two independent and self-contained parts.

The first part is concerned with harmonic maps form super Riemann surfaces in complex projective spaces and projective spaces associated with the super skew-field $\mathbb{D}$. In both cases, we develop the theory of Gauß transforms and study the notion of isotropy, in particular its relation to holomorphic differentials on the super Riemann surface. Moreover, we give a definition of finite type harmonic maps for a special class of maps into $\mathbb{C}P^{n|n+1}$ and thus obtain a classification for certain harmonic super tori. Furthermore, we investigate the equations satisfied by the underlying objects and give an example of a harmonic super torus in $\mathbb{D}P^2$ whose underlying map is not harmonic.

In the second part, we study a classical theorem stating that the group of automorphisms of a manifold M preserving a G-structure of finite type is a Lie group in the context of supermanifolds. We generalize this statement to the category of *cs* manifolds and give some examples, some of which being generalizations of classical notions, others being particular to the super case. Notably, we have to introduce a new class of supermanifolds which we call mixed supermanifolds.

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Harmonic maps from super Riemann surfaces

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1 Introduction

Harmonic maps and supergeometry

The purpose of this study is to prove some foundational results about harmonic maps in supergeometry. More precisely, we study harmonic maps from a super Riemann surface into complex projective spaces and in special cases into general complex Grassmannians.

Harmonic maps from Riemann surfaces into various target spaces are by now a classical topic in differential geometry. Such maps occur naturally in surface theory, for instance. The parametrization of a surface in $\mathbb{R}^3$ is minimal if and only if it is conformal and harmonic. It has constant mean curvature if and only if its Gauß map is harmonic. We refer to [33] for a treatment of these results. In the context of the anti-self-dual Yang-Mills equation, such maps appear as a symmetry reduction from four to two dimensions [53]. Consequently, a central problem is to develop techniques which allow for a classification and construction of such maps. For a review of this broad subject, we refer the reader to the survey articles [24, 25]. Closer to the specific subject of the present article are [11, 18, 56, 57].

Supergeometry is the extension of ordinary geometry which allows for commuting and anti-commuting coordinate functions. Many notions, constructions, and results from differential geometry carry over to the graded setting directly. In particular, there is a notion of Riemannian supermanifolds. However, the Riemannian structure might be even or odd. Another genuinely supergeometric notion is supersymmetry, the simplest instance of which is the concept of a super Riemann surface. The complex analytic properties have been studied in pioneering works in the 1980s, among others [2, 51], and more recently in [54, 55].

In this setting there exists a natural notion of harmonic maps from super Riemann surfaces to Riemannian supermanifolds [20, 34, 36, 46] which are the central objects of this article. In view of the plethora of results available in the non-graded setup, it is beyond the scope of this thesis to give a comprehensive treatment. Instead, we will concentrate on some selected aspects.

Gauß transform, isotropy and harmonic maps of finite type

In order to put our results into context, we first give a brief account on the relevant results in the ungraded setting. The energy of a map $f\colon \Sigma \to M$ between a compact Riemann surface and a Riemannian manifold is defined by

$$E(f) = \int_\Sigma \langle df_\mathbb{C}|_{T\Sigma^{(0,1)}}, df_\mathbb{C}|_{T\Sigma^{(1,0)}}\rangle_\mathbb{C},$$

where $\langle -, - \rangle_\mathbb{C}$ denotes the complex bilinear extension of the given Riemannian structure to $TM_\mathbb{C}$. Critical points are called harmonic maps and are characterized in a local complex coordinate z by

$$\nabla^{LC}_{\partial_{\bar z}}(df_{\mathbb{C}})(\partial_z) = 0,$$

where ∇^{LC} denotes the pullback of the Levi-Civita connection. Due to a result of Koszul and Malgrange [39], a complex vector bundle with connection (E, ∇) on a Riemann surface has a holomorphic structure such that the holomorphic sections are locally characterized by

$$\nabla_{\partial_{\bar z}} s = 0.$$

Using this result, harmonicity can be stated in a coordinate free manner. The map f is harmonic if and only if $(df_{\mathbb{C}})|_{T\Sigma^{(1,0)}}$ is a holomorphic section of $(T\Sigma^{(1,0)})^* \otimes TM_{\mathbb{C}}$, where the second factor is equipped with the Koszul-Malgrange structure. In particular, the differential either vanishes identically or its zeros are isolated.

In the case $M = \mathbb{C}P^n$, the harmonic map equation is equivalent to

$$\nabla^{LC}_{\partial_{\bar z}} df^{(1,0)}(\partial_z) = 0, \tag{1.1}$$

where $df_{\mathbb{C}} = df^{(1,0)} + df^{(0,1)}$ according to the type decomposition on $\mathbb{C}P^n$. In view of the isomorphism $(T\mathbb{C}P^n)^{(1,0)} \cong \mathrm{Hom}(\gamma, \gamma^{\perp})$, where γ is the tautological line bundle, if f is not antiholomorphic, $df^{(1,0)}(\partial_z)$ defines a line in $\mathbb{C}^{1+n}$ outside a discrete set of points. One can always extend this to a give a new map $f_1 \colon \Sigma \to \mathbb{C}P^n$, the Gauß transform. If f is not holomorphic, one can similarly produce a new map f_{-1} starting from $df^{(1,0)}(\partial_{\bar z})$. The central observation is that $f_{\pm 1}$ are harmonic again [22, 26]. This process can be iterated and gives the harmonic sequence

$$\ldots, f_{-2}, f_{-1}, f, f_1, f_2, \ldots$$

The harmonic map is called isotropic if this sequence is finite

$$f_{-l(f)}, \ldots, f_{-2}, f_{-1}, f, f_1, f_2, \ldots, f_{k(f)},$$

which forces the leftmost (resp. rightmost) map to be holomorphic (resp. antiholomorphic). Furthermore, the harmonic map is said to be full if $\oplus_i f_i = \underline{\mathbb{C}^{n+1}}_{\Sigma}$.

Theorem 1.2 ([22], [26, Thm. 6.9])**.** *For every $0 \leq r \leq n+1$ the assignment $f \mapsto f_r$ gives a bijective correspondence between full holomorphic maps $f \colon \Sigma \to \mathbb{C}P^n$ and full isotropic harmonic maps $g \colon \Sigma \to \mathbb{C}P^n$ with $l(g) = r$. The inverse is given by $g \mapsto g_{-l(g)}$.*

For a Riemann sphere, any harmonic map is isotropic, so that this theorem accounts for all full harmonic maps.

However, this is not necessarily the case for a torus $T^2 = \mathbb{C}/\Omega$. We shall especially be interested in the case where the map is $(n+1)$-orthogonal, meaning that any consecutive

$n+1$ lines in the harmonic sequence are mutually orthogonal, and non-isotropic. The harmonic sequence is in this situation infinite and in fact periodic, $f_k = f_{n+1+k}$.

Remark 1.3. In [6], harmonic maps of this type are called superconformal. In view of the next section and the following material, this terminology would be very unfavourable in the context of the present article.

The classification result for such maps is quite different in nature compared to the previous result and is based on the notion of harmonic maps of finite type. This approach has been developed and applied in a series of papers [5, 10, 28, 49]. The special situation we consider was dealt with in the ungraded case in [6]. The case of general harmonic tori in $\mathbb{C}P^n$ has been settled in [8].

In order to explain this notion and the results, we need to back up and introduce new objects. In the case at hand, the harmonic sequence determines a lift

$$\tilde{f}\colon \Sigma \to SU(n+1)/T,$$

where T is a maximal torus. The relevant structure of the co-domain is the structure of a $(n+1)$-symmetric space, i.e., it is equipped with an automorphism of order $n+1$, which leads after complexification to a decomposition

$$\mathfrak{psl}(n+1) = \bigoplus_{i=0}^{n} \mathcal{M}_i.$$

At this point, the only special property of these eigenspaces is the following. The pullback of $\tilde{f}$ along $p\colon \mathbb{C} \to T^2$ has a lift $F\colon \mathbb{C} \to SU(n+1)$ and it follows from the definition of the Gauß transform, that the pullback of the Maurer-Cartan form along F takes the form

$$F^*\alpha_z = A_{z,0} + A_{z,1}, \tag{1.4}$$

where $A_{z,i}$ takes values in $\mathcal{M}_i$ and $A_{z,1}$ satisifes a non-degeneracy condition given in terms of an invariant polynomial. This is actually a property of the map $\tilde{f}$ and such maps are called primitive. The concept of finite type harmonic maps is to construct solutions to (1.4) by solving two commuting ordinary differential equations. Then any of f, $\tilde{f}$, or F is called of finite type if it can be obtained from this construction. (This will be made more precise in our situation in Section 6.4.4.)

These commuting ordinary differential equations are constructed from the real and imaginary part of a complex vector field defined on

$$\Lambda_d = \{\sum_{i=-d}^{d} \xi_i \lambda^i \mid \xi \in \mathfrak{psl}(n+1),\ \bar{\xi}_i = \xi_{-i}\},\ \Lambda_{d,\tau} = \{\xi \in \Lambda_d \mid \xi_i \in \mathcal{M}_i\},$$

where $d \equiv 1 \pmod{n+1}$. This is given by

$$Z(\xi) = [\xi, \frac{1}{2}\xi_{d-1} + \lambda\xi_d]. \tag{1.5}$$

Theorem 1.6 ([6, Section 3])**.** *We have that:*

(a) This defines two commuting vector fields: $[Z, \bar{Z}] = 0$.

(b) Given any initial condition $\xi_0 \in \Lambda_{d,\tau}$*, there is a unique* $\xi \colon \mathbb{C} \to \Lambda_{d,\tau}$ *such that*

$$\xi(0) = \xi_0, \ \partial_z \xi = Z(\xi).$$

(c) For any such solution, the 1*-form defined by* $\beta_z = \xi_d + \frac{1}{2}\xi_{d-1}$ *is flat and integrates to a primitive map* $F \colon \mathbb{C} \to SU(n+1)$ *with* $F(0) = \mathrm{id}$.

The classification result is then:

Theorem 1.7 ([6, Cor. 4.7])**.** *Any* $(n+1)$*-orthogonal and non-isotropic harmonic torus* $T^2 \to \mathbb{C}P^n$ *is of finite type.*

Summary of results

On a supermanifold of dimension $(1|1)$ it makes sense to consider the square root of a conformal structure – a superconformal structure. The local model is $\mathbb{C}^{1|1}$ with coordinates z and ϑ together with

$$D = \partial_\vartheta - \vartheta\partial_z, \ D^2 = \frac{1}{2}[D, D] = -\partial_z.$$

Super Riemann surfaces are obtained by globalizing this notion. Notably, as usual in supergeometry, to make the theory sufficiently rich, it is necessary to work in families of super Riemann surfaces over a purely odd base $B = (*, \Lambda)$, where Λ is a complex Grassmann algebra. The space of even lines in $\mathbb{C}^{1+n|m}$ is $\mathbb{C}P^{n|m}$ and similarly as in the ungraded case, Equation (1.1), a map $f \colon \Sigma \to \mathbb{C}P^{n|m}$ is harmonic if

$$\nabla^{LC}_{\bar{D}} df^{(1,0)}(D) = 0. \tag{1.8}$$

The construction of the Gauß transform parallels the ungraded case. The isomorphism $T\mathbb{C}P^{n|m} \cong \underline{\mathrm{Hom}}(\gamma, \gamma^\perp)$ shows that $df^{(1,0)}(D)$ defines an odd line in $\mathbb{C}^{1+n|m}$ away from points where the differential degenerates. Our analysis shows that one cannot hope to define the Gauß transform in general. There are two caveats. Firstly, working over a purely odd base B leads to technical restrictions. Secondly, working in $(1|1)$ dimensions has the effect that certain ideals are no longer automatically invertible as is the case of a single complex dimension. However, in favorable cases one can define the Gauß transform on a blow up:

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{f}_1} & \mathbb{C}P^{m-1|n+1} \\ \downarrow & & \\ \Sigma & & \end{array} .$$

This blow up only modifies the odd directions of Σ, the underlying Riemann surface stays untouched. Although the resulting supermanifold is not longer a super Riemann surface, but only a parabolic super Riemann surface, the notion of harmonic maps is still defined. Similarly, one can discuss a Gauß transform associated with $df^{(1,0)}(\bar{D})$, possibly defined on a different blow up. Again, these Gauß transforms are harmonic. Under suitable assumptions, this process can be iterated to give the harmonic sequence

$$\ldots, \tilde{f}_{-2}, \tilde{f}_{-1}, \tilde{f}, \tilde{f}_1, \tilde{f}_2, \ldots,$$

which is defined on some blow up $p\colon \tilde{\Sigma} \to \Sigma$, and where $\tilde{f} = f \circ p$. The harmonic map is called isotropic if this sequence is finite

$$\tilde{f}_{-l(\tilde{f})}, \ldots, \tilde{f}_{-2}, \tilde{f}_{-1}, \tilde{f}, \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{k(\tilde{f})},$$

which forces the leftmost (resp. rightmost) map to be holomorphic (resp. antiholomorphic). Furthermore, if one defines a harmonic map to be full if $\oplus_i \tilde{f}_i = \underline{\mathbb{C}^{1+n|m}}_{\tilde{\Sigma}}$, then we obtain the following result. Here, the assumption that the ramification be invertible will be explained later and ultimately stems from the two aforementioned caveats.

Theorem A (see Theorem 6.16)**.** *For a full isotropic harmonic map $\tilde{\Sigma} \to \mathbb{C}P^{n|m}$, we have that $|n+1-m| \leq 1$. For every $0 \leq r \leq n+1+m$, the assignment $f \mapsto f_r$ gives a bijective correspondence between full holomorphic maps $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ with invertible ramification and full isotropic harmonic maps $g\colon \tilde{\Sigma} \to M_r$ with invertible ramification and $l(g) = r$. Here, $M_r = \mathbb{C}P^{n|m}$ if r is even and $M_r = \mathbb{C}P^{m-1|n+1}$ if r is odd. The inverse is given by $g \mapsto g_{-l(g)}$.*

Unlike the ungraded case, in supergeometry there is another instance, where a similar classification is available. This concerns harmonic maps into $\mathbb{D}P^n$ – the projective space associated with the super skew field $\mathbb{D}$.

Theorem B (see Theorem 7.12)**.** *For every $0 \leq r \leq n+1$ the assignment $f \mapsto f_r$ gives a bijective correspondence between full holomorphic maps $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ with invertible ramification and full isotropic harmonic maps with invertible ramification $g\colon \tilde{\Sigma} \to \mathbb{D}P^n$ such that $l(g) = r$. The inverse is given by $g \mapsto g_{-l(g)}$.*

Moreover, we also study periodic harmonic sequences in $\mathbb{C}P^{n|n+1}$ and show that there is a notion of finite type. This is particularly suited to study harmonic super tori. Later we will be more precise, but for now we use the loose notation $\Sigma = \mathbb{C}^{1|1}/\Omega$ and suppress B. In this situation, we have again a lift $\tilde{f}\colon \Sigma \to PSU(n+1|n+1)/T$ and the latter space is $2(n+1)$-symmetric with decomposition

$$\mathfrak{psl}(n+1|n+1) = \bigoplus_{i=0}^{2n+1} \mathcal{M}_i.$$

Pulling back along $\mathbb{C}^{1|1} \to \Sigma$, one can find a framing $F\colon \mathbb{C}^{1|1} \to PSU(n+1|n+1)$ of $\tilde{f}$ which will satisfy

$$F^*\alpha_D = A_{D,0} + A_{D,1},$$

where $A_{D,i}$ has values in $\mathcal{M}_i$. In order to define a vector field analogous to (1.5), there are several issues to overcome which will be settled in Section 6.4. Firstly, note that, D being odd, this vector field should be odd. Secondly, one crucial ingredient in the proof of Theorem 1.7 is that $A_{z,1}$ is semisimple which never holds for $A_{D,1}$:

$$A_{D,1}^2 \in \operatorname{im}(\operatorname{ad}(A_{D,1})) \cap \ker(\operatorname{ad}(A_{D,1})).$$

Moreover, we show that there are two invariants $P_1(f)$, $P_2(f)$ which are induced from two $\mathfrak{pst}_{\mathbb{C}}$-invariant polynomials on $\mathcal{M}_1$ as opposed to one $\mathfrak{psl}(n+1)$-invariant polynomial in the ungraded case [6]. This leads to the additional assumption in our theorem compared to Theorem 1.7. Also, an essential ingredient in Section 6.4.4 is the ellipticity of certain operators. We are not aware of any general result on elliptic operators on super Riemann surfaces, and the fact that we can apply analogous arguments as in the ungraded case relies on special properties of the situation at hand. Then we have the following result.

Theorem C (see Theorem 6.49)**.** *Any harmonic super torus $f\colon \Sigma \to \mathbb{C}P^{n|n+1}$ with invertible ramification and periodic harmonic sequence is of finite type if $P_1(f)/P_2(f)$ is constant.*

We also study the analogous situation for maps into $\mathbb{D}P^{2n}$. It turns out that one cannot expect a finite type classification as previously. We do not know how to overcome this problem, however, our analysis still leads to the following result.

Theorem D (see Theorem 7.31)**.** *There is a harmonic super torus $f\colon \Sigma \to \mathbb{D}P^2$ with invertible ramification and periodic harmonic sequence such that the underlying map $T^2 \to \mathbb{C}P^2$ is not harmonic.*

Relation to other work

Although the problem of developing the supersymmetric version of harmonic map theory has already been posed in [52, Problem 14], there are only a few results available in the literature. One of the first sources, where this problem has been taken up is [46].

The paper by Khemar [36] lays the foundation for all the results based on the zero curvature formulation (see Proposition 3.16). These are formulated therein for Lie groups as co-domain, but works equally well in the setting of Lie supergroups. More recently, there has been increasing interest in supersymmetric harmonic maps into $\mathbb{C}P^n$. Among others, see [21]. In view of the dimensional restrictions which we obtain for the co-domain, these results are largely independent of ours, though the methods are similar. Moreover, in work of Chen et al., for instance [15], the ordinary harmonic map equation is coupled to a nonlinear Dirac equation for a spinor. The underlying data of a supersymmetric harmonic map is similar (Section 4), however, in our setup, the spinor is an odd quantity.

In the present treatment we focus on working in the general setup of an arbitrary super Riemann surface whenever possible. From this point of view, the occurrence of parabolic super Riemann surfaces is quite natural. For instance, in [36] only $\mathbb{C}^{1|1}$ is considered and in [21] an additional boundary condition is imposed ([21, Equ. (11)]), which is however not appropriate to define maps on the super sphere.

Outline

This work is structured as follows. In Section 2, we introduce all relevant notions from supergeometry. Super Riemann surfaces are introduced in Section 3. Besides the basic definitions and examples, this section also contains the construction of blow ups of certain ideal sheaves which naturally appear in the context of the Gauß transform. Then we move on to discuss generalities about harmonic maps in this setting in Section 4. In particular, we derive the underlying equations. Section 5 contains a discussion of harmonic maps into Lie supergroups formulated for the special case of $U(n|m)$, i.e., the zero curvature formulation, framings, and a discussion of the underlying map from the point of view of elliptic integrable systems. Sections 6 and 7 on harmonic maps into $\mathbb{C}P^{n|m}$ and $\mathbb{D}P^n$ contain the main results. In both cases, we first prove basic results about the Gauß transform, study isotropy, and give basic examples. Then we go on to discuss harmonic maps with periodic harmonic sequences. This leads to a finite type classification in the case of $\mathbb{C}P^{n|n+1}$. This is accompanied by a detailed study of the case $\mathbb{C}P^{1|2}$ and certain special maps into $\mathbb{D}P^n$. The pay-off of which is a wealth of examples in the former case and an example of a supersymmetric harmonic map whose underlying map is non-harmonic in the latter case.

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2 Supergeometry

2.1 Recollections on supergeometry

We start with introducing the most important concepts from supergeometry and our conventions. As a general reference, we refer for instance to [13, 31] and for Lie supergroups in particular to [4].

2.1.1 Supermanifolds

A real (resp. complex) super vector space is a $\mathbb{Z}/2$-graded real (resp. complex) vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A morphism is a grading preserving linear (resp. complex linear) homomorphism. The parity reversed super vector space will be denoted by $\Pi V = V_{\bar{1}} \oplus V_{\bar{0}}$. The resulting category is closed symmetric monoidal with respect to the evident notion of tensor product and internal hom object denoted by $\underline{\mathrm{Hom}}(-,-)$.

We let $\mathbb{A}(V)$ denote the locally ringed superspace over $\mathbb{R}$ (resp. $\mathbb{C}$) given by the topological space $V_{\bar{0}}$ together with the sheaf of superalgebras $\mathcal{O}_V = \mathrm{C}^{\infty}_{V_{\bar{0}}} \otimes_{\mathbb{R}} \wedge^{\bullet} V_{\bar{1}}^*$ (resp. $\mathcal{H}_{V_{\bar{0}}} \otimes_{\mathbb{C}} \wedge^{\bullet} V_{\bar{1}}^*$). Here $C^{\infty}_{V_{\bar{0}}}(-)$ denotes the sheaf of real smooth functions and $\mathcal{H}_{V_{\bar{0}}}(-)$ denotes the sheaf of holomorphic functions. A smooth (resp. complex) supermanifold is a locally ringed superspace over $\mathbb{R}$ (resp. $\mathbb{C}$) with Hausdorff second countable base which is locally isomorphic to some $\mathbb{A}(V)$. A morphism of supermanifolds is a morphism of locally ringed superspaces. The respective category of supermanifolds will be denoted by SMan and $\mathrm{SMan}_{\mathbb{C}}$. The sheaf of ideals given by all nilpotent functions on a real or complex supermanifold M will be denoted by $\mathcal{J}_M$. This gives rise to the underlying manifold $i_M \colon M_0 \to M$ with sheaf of functions $\mathcal{O}_{M_0} = \mathcal{O}_M/\mathcal{J}_M$. A morphism of supermanifolds $M_1 \to M_2$ is then given by the data of a smooth map $f_0 \colon (M_1)_0 \to (M_2)_0$ and map of sheaves of superalgebras $f^\sharp \colon \mathcal{O}_{M_2} \to (f_0)_* \mathcal{O}_{M_1}$.

2.1.2 Functor of points approach

Often, it is convenient to use the Yoneda embedding to study a supermanifold M through its associated functor of points

$$\mathrm{SMan}^{\mathrm{op}} \longrightarrow \mathrm{Set}, \; T \mapsto \mathrm{SMan}(T, M).$$

Usually, elements of $\mathrm{SMan}(T, M)$ will be referred to as T-valued points of M. This works equally well in $\mathrm{SMan}_{\mathbb{C}}$.

2.1.3 Tangent bundles

The sections of the smooth (resp. holomorphic) tangent sheaf $\mathcal{T}_M$ over U_0 are given by the real (resp. complex) linear derivations of $\mathcal{O}_M|_{U_0}$. This forms a locally free sheaf of $\mathcal{O}_M$-modules and can be used to build the vector bundle $TM \to M$. Having this, for a smooth supermanifold, besides the notion of an even Riemannian metric, there is also the notion of an odd Riemannian metric. These have associated Levi-Civita connections. We refer to [29, 31] and the references therein.

2.1.4 Lie supergroups

A smooth (resp. complex) Lie supergroup is a group object in the respective category of supermanifolds. Given a subgroup $H \subset G$ such that $H_0 \subset G_0$ is closed, there exists an induced homogeneous manifold G/H. The projection is a principal H-bundle and has the universal property for quotients.

2.1.5 The forgetful functor

Any complex super vector space V has an underlying real super vector space $u(V)$. This assignment extends to a forgetful functor from complex supermanifolds to smooth supermanifolds which we will denote by $u(M)$. In the same way, a holomorphic vector bundle $E \to M$ has an underlying complex vector bundle $u(E) \to u(M)$. From now on, by abuse of notation, we will suppress $u(-)$ in the following and instead, in case of potential ambiguity, we will emphasize which structure we consider in the situation at hand.

2.2 The supergroup $U(n|m)$

Let T be a real supermanifold and $E = \underline{\mathbb{C}^{n|m}}_T$ the trivial complex vector bundle of rank $(n|m)$ over T. The standard basis of $\mathbb{C}^{n|m}$ is denoted by $\{e_1, \ldots, e_n, \epsilon_1, \ldots, \epsilon_m\}$. A homogeneous section f of $\underline{\mathrm{End}}(E) = \underline{\underline{\mathrm{End}}(\mathbb{C}^{n|m})}_T$ satisfies for homogeneous $a, b \in \Gamma(\mathcal{O}_T)$ and homogeneous sections v and w of E

$$f(av + bw) = (-1)^{|a||f|} af(v) + (-1)^{|b||f|} bf(w).$$

With respect to the standard basis, f is represented by a matrix

$$\Big(f(e_1), f(e_2), \ldots, f(\epsilon_m)\Big) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \; f(v) = \begin{pmatrix} A & (-1)^{|v|+1} B \\ (-1)^{|v|} C & D. \end{pmatrix} v.$$

We introduce a super hermitian form on E, by setting

$$\langle v, v' \rangle = \sum_{k=1}^{n} \overline{f_k} f'_k + i \sum_{l=1}^{m} (-1)^{|g'_l|} \overline{g_l} g'_l, \; v = \sum_{k=1}^{n} f_k e_k + \sum_{l=1}^{m} g_l \epsilon_l, \; v' = \sum_{k=1}^{n} f'_k e_k + \sum_{l=1}^{m} g'_l \epsilon_l.$$

Lemma 2.1. *This assignment is non-degenerate, supersymmetric and sesquilinear.*

Proof. We calculate

$$\begin{aligned}\overline{\langle v, v'\rangle} &= \sum_{k=1}^{n} f_k\overline{f_k'} - i\sum_{l=1}^{m}(-1)^{|g_l'|}g_l\overline{g_l'}\\ &= \sum_{k=1}^{n}(-1)^{|f_k||f_k'|}\overline{f_k'}f_k - i\sum_{l=1}^{m}(-1)^{|g_l'|}(-1)^{|g_l||g_l'|}\overline{g_l'}g_l\\ &= (-1)^{|v||v'|}(\sum_{k=1}^{n}\overline{f_k'}f_k - i\sum_{l=1}^{m}((-1)^{|g_l'|}(-1)^{|g_l||g_l'|}(-1)^{|v||v'|}(-1)^{|g_l|})(-1)^{|g_l|}\overline{g_l'}g_l)\\ &= (-1)^{|v||v'|}\langle v', v\rangle,\end{aligned}$$

where we used $|f_k||f_k'| = |v||v'|$ and $|g_l| + |g_l'| + |g_l g_l'| = 1 + |v||v'|$. And, moreover, for homogeneous x we have

$$\begin{aligned}\langle v, xv'\rangle &= \sum_{k=1}^{n}\overline{f_k}(xf_k') + i\sum_{l=1}^{m}(-1)^{|g_l'|+|x|}\overline{g_l}(xg_l')\\ &= (-1)^{|v||x|}x\sum_{k=1}^{n}\overline{f_k}f_k' + i(-1)^{|v||x|}\sum_{l=1}^{m}(-1)^{|g_l'|}\overline{g_l}g_l'\\ &= (-1)^{|v||x|}x\langle v, v'\rangle.\end{aligned}$$

Non-degeneracy is readily seen. □

Lemma 2.2. *The adjoint with respect to $\langle -, -\rangle$ of the homogeneous endomorphism given by the matrix*

$$M = \begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

is given by

$$M^* = \begin{pmatrix} A^* & iC^*\\ iB^* & D^*\end{pmatrix}.$$

Proof. For instance, for the upper right corner we find

$$\begin{aligned}i\bar{M}_{lk} &= \langle Me_k, \epsilon_l\rangle\\ &= \langle e_k, M^*\epsilon_l\rangle\\ &= (M^*)_{kl}.\end{aligned}$$

The other cases are similar. □

The general linear group $Gl(n|m)$ is the open subsupermanifold of $\underline{\mathrm{End}}(\mathbb{C}^{n|m})$ given by the invertible endomorphisms. In particular, this endows $Gl(n|m)$ with a natural complex structure. The preimage of $1_{n|m}$ under the submersion $A \mapsto AA^*$ onto the linear subspace of $\underline{\mathrm{End}}(\mathbb{C}^{n|m})$ given by endomorphisms satisfying $B^* = B$ yields the unitary group $U(n|m)$. The T-valued points are given by even endomorphisms f which are unitary in the sense that

$$\langle f(x), f(y)\rangle = \langle x, y\rangle.$$

Equivalently, $f^*f = 1_{n|m}$. If we represent such f as above by a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then the condition reads

$$A^*A - iC^*C = 1_n,\ A^*B + iC^*D = 0,\ B^*B + iD^*D = i\,1_m.$$

Similarly, for an odd T-valued point the condition is $f^*f = i\,1_{n|m}$. The Lie superalgebra $\mathfrak{u}(n|m)$ is by definition the Lie superalgebra of left-invariant vector fields on $U(n|m)$. If we denote by μ the multiplication on $U(n|m)$, then left-invariance means for a derivation

$$(I \otimes X) \circ \mu^\sharp = \mu^\sharp \circ X.$$

The bracket is given by the supercommutator of vector fields. Elements can be represented by anti-hermitian matrices $f^* = -f$:

$$\mathfrak{u}(n|m) = \{\begin{pmatrix} A & B \\ -iB^* & D \end{pmatrix} \mid A \in \mathfrak{u}(n),\ D \in \mathfrak{u}(m),\ B \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)\}$$

Then the bracket is given by the supercommutator of linear endomorphisms. This defines a real form of $\underline{\mathrm{End}}(\mathbb{C}^{n|m})$ and $U(n|m)$ is a real form of the complex Lie supergroup $Gl(n|m)$.

The $\mathrm{ad}_{\mathfrak{u}(n|m)}$-invariant metric given by the super trace $(X, Y) = -\mathrm{str}(XY)$, where

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D),$$

induces a pseudo-Riemannian metric on $U(n|m)$.

The Berezinian of a matrix defines a group homomorphism

$$\mathrm{Ber}\colon U(n|m) \longrightarrow U(1),\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \det(A - BD^{-1}C)\det(D)^{-1}.$$

This assignment is to be understood to be a definition on the level of functor of points. That is, a T-valued point of $U(n|m)$ is mapped to the T-valued point of $U(1)$ as indicated. The kernel of this group homomorphism is the special unitary supergroup $SU(n|m)$ with

Lie superalgebra

$$\mathfrak{su}(n|m) = \{A \in \mathfrak{u}(n|m) \mid \mathrm{str}(A) = 0\}.$$

This group has a nontrivial center given by multiples of the identity if $n = m$. The quotient by this subgroup is the projective special unitary supergroup $PSU(n|n)$ with Lie superalgebra

$$\mathfrak{psu}(n|n) = \mathfrak{su}(n|n)/\langle i \cdot \mathrm{id}\rangle.$$

These construction pass to the complexfication and in this way one obtains the Lie superalgebras $\mathfrak{sl}(n|m)$ and $\mathfrak{psl}(n|n)$.

Left translation gives a trivialization

$$TU(n|m) \longrightarrow \underline{\mathfrak{u}(n|m)}_{U(n|m)} \tag{2.3}$$

and hence the Maurer-Cartan form $\alpha \in \Omega^1(TU(n|m), \mathfrak{u}(n|m))$. This connection is flat:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Under the trivialization (2.3), the Levi-Civita connection takes the form $d + \frac{1}{2}\alpha$ (cf. [31, Cor. 1]).

Example 2.4. Consider a homogeneous section f of $\underline{\mathbb{C}^{n|m}}_T$ such that $i_T^*(f)$ is nowhere vanishing, where $i_T\colon T_0 \to T$ is the canonical inclusion. In particular, $\langle f, f\rangle$ is invertible. Equivalently, f spans a locally free $\mathcal{O}_T$-module. In this situation we have a projection onto the line l spanned by f, which is given by the formula

$$\pi_l(a) = (-1)^{(|f|+|a|)|f|}\frac{\langle f, a\rangle}{\langle f, f\rangle} f.$$

Then we have

$$\begin{aligned}\pi_l(\pi_l(a)) &= (-1)^{(|f|+|a|)|f|}\pi_l(\frac{\langle f, a\rangle}{\langle f, f\rangle} f)\\ &= \pi_l(a),\end{aligned}$$

$$\begin{aligned}\langle \pi_l(a), a'\rangle &= (-1)^{(|a|+|f|)|f|}\overline{\left(\frac{\langle f, a\rangle}{\langle f, f\rangle}\right)}\langle f, a'\rangle\\ &= \frac{\langle a, f\rangle}{\langle f, f\rangle}\langle f, a'\rangle\\ &= \langle a, (-1)^{(|f|+|a'|)|f|}\frac{\langle f, a'\rangle}{\langle f, f\rangle} f\rangle\\ &= \langle a, \pi_l(a')\rangle,\end{aligned}$$

and $\pi_l - \pi_{l^\perp} = 2\pi_l - \mathrm{id}$ is a T-valued point of $U(n|m)$. Similar considerations apply to construct the projection onto arbitrary subbundles F of $\underline{\mathbb{C}^{n|m}}_T$. Since $(\pi_l - \pi_{l^\perp})^* = (\pi_l - \pi_{l^\perp})^{-1}$, it actually takes values in $SU(n|m)$.

2.3 The supergroup $Q(n)$

A super division algebra is a superalgebra such that any nonzero homogeneous element is invertible (cf. [19]). The super divison algebra $\mathbb{D}$ over $\mathbb{C}$ is defined by

$$\mathbb{D} := \mathbb{C}[j]/(ji = ij, j^2 = 1),$$

where j is odd. A T-valued point of $\mathbb{D}$ will be written in the form $a + bj$. Throughout, $\mathbb{D}^n$ will be considered as a left $\mathbb{D}$-module. Left multiplication by j is denoted by J_n and we have

$$J_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

In particular, J_n is unitary. The subgroup of $U(n|n)$ given by all matrices which graded commute with J_n is denoted by $Q(n)$. On T-valued points, these are unitary endomorphisms whose representing matrix have the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

In particular,

$$\mathrm{Ber}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(1 + 1/2[A^{-1}B, A^{-1}B]) = 1,$$

so that $Q(n) \subset SU(n|n)$. On the infinitesimal level, we obtain

$$\mathfrak{q}(n) = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A \in \mathfrak{u}(n),\ B = iB^*\}.$$

There are odd analogues of the super trace and the Berezinian. The former gives rise to the subalgebra

$$\mathfrak{sq}(n) = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A \in \mathfrak{u}(n),\ B = iB^*,\ \mathrm{otr}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \mathrm{tr}(B) = 0\}.$$

The latter is defined by

$$\mathrm{odet}\colon Q(n) \longrightarrow \mathbb{C}^{0|1},\ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto \mathrm{tr}(\ln(1 + A^{-1}B)),$$

where we use the formal definition $\ln(1+Z) = \sum_n (-1)^{n+1} Z^n/n$. This is again understood to be a definition on the level of T-valued points. The sum converges since B is odd. The kernel is the subgroup $SQ(n)$, which has a non-trivial center spanned by the identity. The quotient by this subgroup gives the projective special queer Lie supergroup $PSQ(n)$.

In the case $n = 2$, we can write an even $X \in \mathfrak{q}(2)$ in the form $x \cdot \mathrm{id}_{1|1} + \xi \tilde{J}$ and we will often use the shorthand $x + \xi\tilde{J}$, where

$$\tilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For later reference we collect some facts concerning subbundles $V \subset \underline{\mathbb{D}^n}_T$ which are invariant under the left action of $\mathbb{D}$.

Lemma 2.5. *Consider a rank one $\mathbb{D}$-subbundle $V \subseteq \underline{\mathbb{D}^n}_T$. If the bundle is trivial then there is an even trivializing section v which satisfies $\langle v, J_n v\rangle = 0$. This generator is unique up to left multiplication by an even invertible element $(a + bj)$ such that $\bar{b} = -i b\bar{a}/a$.*

Proof. Starting with any even trivializing section w, one can take $v = (1 + \frac{\langle w, J_n w\rangle}{2\langle w,w\rangle} J_n)w$. Given such v, then

$$\langle (a + bJ_n)v, (a - bJ_n)J_n v\rangle = (-\bar{a}b + \bar{b}ai)\langle v, v\rangle$$

vanishes if and only if $\bar{b} = -ib\,\bar{a}/a$. □

Definition 2.6. An even trivializing section v which satisfies $(v, J_n v) = 0$ is called isotropic.

Example 2.7. We consider an even isotropic generator l of the $\mathbb{D}$-submodule $L \subseteq \underline{\mathbb{D}^n}_T$ We have that

$$\pi_L v = \frac{\langle l, v\rangle}{\langle l, l\rangle} l + (-1)^{1+|v|} \frac{\langle J_n l, v\rangle}{\langle J_n l, J_n l\rangle} J_n l.$$

Now we use $J_n^* = iJ_n$ to compute

$$\begin{aligned}
\pi_L J_n v &= \frac{\langle l, J_n v\rangle}{\langle l, l\rangle} l + (-1)^{|v|} \frac{\langle J_n l, J_n v\rangle}{\langle J_n l, J_n l\rangle} J_n l \\
&= \frac{\langle J_n l, v\rangle}{\langle J_n l, J_n l\rangle} l + (-1)^{|v|} \frac{\langle l, v\rangle}{\langle l, l\rangle} J_n l \\
&= J_n \pi_L v.
\end{aligned}$$

Moreover, $(\pi_L - \pi_{L^\perp})^* = (\pi_L - \pi_{L^\perp})^{-1}$ and so $\pi_L - \pi_{L^\perp}$ defines a T-valued point of $SQ(n)$. By similar methods the projection onto an arbitrary J_n-invariant subbundle F can be shown to commute with J_n.

2.4 The Grassmannians $Gr_{k|l}(\mathbb{C}^{n|m})$

The Grassmannian $Gr_{k|l}(\mathbb{C}^{n|m})$ is the supermanifold which classifies $(k|l)$-subbundles of the trivial bundle $\mathbb{C}^{n|m}$. For convenience, we give a detailed treatment of the homogeneous geometry. As a smooth supermanifold we define

$$Gr_{k|l}(\mathbb{C}^{n|m}) = U(n|m)/U(k|l) \times U(n-k|m-l).$$

As usual, we will use the notation $\mathbb{C}P^{n|m} = Gr_{1|0}(\mathbb{C}^{1+n|m})$. This comes with a tautological flag $\gamma = [\mathbb{C}^{k|l}] \subset \underline{\mathbb{C}^{\mathrm{n}|\mathrm{m}}}_{Gr_{k|l}(\mathbb{C}^{n|m})}$. Here the square brackets denote the bundle associated with the $U(k|l) \times U(n-k|m-l)$ representation where the first factor acts through the tautological representation and the second factor acts trivially. The last isomorphism is given by

$$[\mathbb{C}^{k|l}] \longrightarrow \underline{\mathbb{C}^{n|m}}_{Gr_{k|l}(\mathbb{C}^{n|m})},\ [g,v] \mapsto ([g], g(v)).$$

Pullback of this flag sets up the bijective correspondence between smooth maps into $Gr_{k|l}(\mathbb{C}^{n|m})$ and rank $(k|l)$ subbundles of the trivial rank $(n|m)$ bundle. The tautological bundle has a connection coming from the inclusion into the trivial bundle. That is, for a local complex derivation X and a local section of ρ, we set

$$\nabla^{\gamma}_X(\rho) = \pi_\gamma X(\rho).$$

Similarly, $\gamma^\perp$ is endowed with an analogous connection.

As concerns the tangent bundle we obtain as a $U(k|l) \times U(n-k|m-l)$ representation

$$T_{[\mathrm{id}]} Gr_{k|l}(\mathbb{C}^{n|m}) \cong \underline{\mathrm{Hom}}(\mathbb{C}^{k|l}, \mathbb{C}^{n-k|m-l}),$$

and using left translation thus

$$TGr_{k|l}(\mathbb{C}^{n|m}) \cong \underline{\mathrm{Hom}}(\gamma, \gamma^\perp).$$

From this we obtain a $U(n|m)$-invariant almost complex structure on $Gr_{k|l}(\mathbb{C}^{n|m})$. In view of the Newlander-Nirenberg theorem for supermanifolds [44], a proof of the integrability can be obtained along the lines of [38, Prop. X.6.5]. On local sections we have

$$TGr_{k|l}(\mathbb{C}^{n|m})^{(1,0)} \xrightarrow{\cong} \underline{\mathrm{Hom}}(\gamma, \gamma^\perp),\ Z \mapsto \pi_{\gamma^\perp} Z(\rho),$$

where ρ is a local section of γ. The connections on γ and $\gamma^\perp$ induce a connection on $\underline{\mathrm{Hom}}(\gamma, \gamma^\perp)$ given for local section F and ρ by

$$(\nabla^{LC}_X F)(\rho) = \pi_{\gamma^\perp} X(F(\rho)) - (-1)^{|X||F|} F(\pi_\gamma X(\rho)).$$

In particular, this connection preserves the type decomposition of $TGr_{k|l}(\mathbb{C}^{n|m})_{\mathbb{C}}$. This is the Levi-Civita connection of the underlying metric of the hermitian structure determined on homogeneous local sections F and G by

$$\langle F, G\rangle_{Gr_{k|l}(\mathbb{C}^{n|m})} = \mathrm{str}(F^*G).$$

We should point out that the underlying metric is only definite in the case $k = 0$ or $l = 0$.

Given a map $f\colon M \to Gr_{k|l}(\mathbb{C}^{n|m})$ from a complex manifold into $Gr_{k|l}(\mathbb{C}^{n|m})$, the complexified differential $df_{\mathbb{C}}$ decomposes into two summands according to the type composition of $Gr_{k|l}(\mathbb{C}^{n|m})$:

$$df_{\mathbb{C}} = df^{(1,0)} + df^{(0,1)}.$$

We have then the following.

Proposition 2.8. *The following are equivalent.*

(a) The map f is holomorphic.

(b) $df^{(1,0)}|_{TM^{(0,1)}} = 0$.

(c) For any local section ρ of γ and any section $\bar{Z}$ of $TM^{(0,1)}$, we have

$$\pi_{f^*(\gamma^{\perp})}\bar{Z}(\rho) = 0.$$

(d) The subbundle $f^*(\gamma) \subset \underline{\mathbb{C}^{n|m}}_M$ is holomorphic.

Proof. The map is holomorphic if and only if df is complex linear which is equivalent to (b) for the same reasons as in the ungraded setting. Parts (b) and (c) are equivalent in view of the isomorphism (2.4). Part (c) is equivalent to the statement that smooth sections ρ of $f^*(\gamma)$ are closed under applying sections $\bar{Z}$ of $TM^{(0,1)}$ which is equivalent to define a holomorphic subbundle of the trivial bundle. □

Finally, there is a totally geodesic embedding $Gr_{k|l}(\mathbb{C}^{n|m}) \to U(n|m)$ which is given on T-valued points by

$$V \mapsto (\pi_V - \pi_{V^{\perp}}) = 2\pi_V - 1.$$

Remark 2.9. As smooth supermanifolds, the Grassmannians are split

$$Gr_{k|l}(\mathbb{C}^{n|m}) \cong (Gr_k(\mathbb{C}^n) \times Gr_l(\mathbb{C}^m), \wedge^{\bullet}[\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^m) \oplus \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^n)]^*).$$

However, for instance the complex supermanifold $Gr_{1|1}(\mathbb{C}^{2|2})$ is non-split [42, Chapter 4 §3 Example 16].

2.5 The projective spaces $\mathbb{D}P^n$

Now we shall discuss certain submanifolds of $Gr_{1|1}(\mathbb{C}^{k+1|k+1})$. The projective space $\mathbb{D}P^n$ is the supermanifold of $(1|1)$-planes in $\mathbb{D}^{n+1} = \mathbb{C}^{n+1|n+1}$ which are invariant under J_{n+1}. These supermanifolds have been introduced by Manin (cf. [40, 42]). This parallels very much the discussion of the previous section. As a smooth supermanifold we define

$$\mathbb{D}P^n = Q(1+n)/Q(1) \times Q(n).$$

We have a tautological flag $\gamma_{\mathbb{D}} = [\mathbb{D}^1] \subset \underline{\mathbb{D}^{1+\mathrm{n}}}_{\mathbb{D}P^n}$. Again, the square brackets mean the bundle associated with the indicated $Q(1) \times Q(n)$ representation. Pullback of this flag sets up the bijective correspondence between smooth maps into $\mathbb{D}P^n$ and rank $(1|1)$ subbundles of the trivial rank $(n+1|n+1)$ bundle which are invariant under J_{n+1}. In view of Example 2.7, the tautological bundle has a connection similarly defined as in Section 2.4. The tangent bundle is of the form

$$T\mathbb{D}P^n \cong \underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \gamma_{\mathbb{D}}^{\perp}).$$

From this we obtain a $Q(1+n)$-invariant almost complex structure, which is integrable since the inclusion into $Gr_{1|1}(\mathbb{C}^{1+n|1+n})$ respects the almost complex structures. Then on local sections we have

$$(T\mathbb{D}P^n)^{(1,0)} \cong \underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \gamma_{\mathbb{D}}^{\perp}),\ Z \mapsto \pi_{\gamma_{\mathbb{D}}^{\perp}} Z(\rho),$$

where ρ is a local section of $\gamma_{\mathbb{D}}$. Again, this connection preserves the type decomposition. There is an odd hermitian metric on $\underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \gamma_{\mathbb{D}}^{\perp})$, given for homogeneous local sections F and G by

$$\langle F, G\rangle_{\mathbb{D}P^n} = \mathrm{otr}(F^* G).$$

Notice that the super trace vanishes identically. The Levi-Civita connection is given by

$$(\nabla_X^{LC} F)(\rho) = \pi_{\gamma_{\mathbb{D}}^{\perp}} X(F(\rho)) - (-1)^{|X||F|} F(\pi_{\gamma_{\mathbb{D}}} X(\rho)),$$

where ρ is a local section of $\gamma_{\mathbb{D}}$. From Proposition 2.8 and the above discussion, we can conclude for a map $f\colon M \to \mathbb{D}P^n$ from a complex manifold M:

Proposition 2.10. *The following are equivalent.*

(a) The map f is holomorphic.

(b) $df^{(1,0)}|_{TM^{(0,1)}} = 0$.

(c) For any local section ρ of γ and any section of $\bar{Z}$, we have

$$\pi_{f^*(\gamma_{\mathbb{D}}^{\perp})} \bar{Z}(\rho) = 0.$$

(d) *The subbundle* $f^*(\gamma_{\mathbb{D}}) \subset \underline{\mathbb{D}^{n+1}}_M$ *is holomorphic.*

There is again a totally geodesic embedding $\mathbb{D}P^n \to Q(1+n)$ which is given on T-valued points by

$$V \mapsto (\pi_V - \pi_{V^\perp}) = 2\pi_V - 1.$$

Remark 2.11. The split model for the underlying smooth supermanifold is

$$\mathbb{D}P^n \cong (\mathbb{C}P^n, \wedge^\bullet[\underline{\mathrm{Hom}}(\mathbb{C}, \mathbb{C}^n]^*).$$

However, as a complex supermanifold $\mathbb{D}P^n$ is non-split for $n \geq 2$ (cf. [47]).

3 Super Riemann surfaces

Most of the objects we have introduced so far are a rather direct generalization of ungraded differential geometric notions. In contrast, super Riemann surfaces, which form the central objects of this article, are not of this sort but truly supergeometric in nature. As a general reference, especially for the material presented in Sections 3.1, 3.2, we refer to [43, 55]. For parabolic super Riemann surfaces (see Section 3.4) we also point out [54].

3.1 Basics

Let Λ be a complex Grassmann algebra and $B := \mathrm{Spec}(\Lambda) = (\mathrm{pt}, \Lambda)$ the associated complex supermanifold. A B-family of supermanifolds is a complex supermanifold M together with a holomorphic submersion $\pi\colon M \to B$. The relative tangent bundle TM/B is defined to be the kernel of $TM \to \pi^*TB$. Its dimension is the relative dimension of M over B. Given another complex Grassmann algebra Λ' with associated supermanifold B' and any morphism $f\colon B' \to B$, then $M \to B$ can be pulled back along f to give a family $M' \to B'$.

Definition 3.1. A super Riemann surface over B is a B-family $\pi\colon \Sigma \to B$ of complex supermanifolds of relative dimension $1|1$ together with a totally non-integrable holomorphic subbundle $\mathcal{D} \subseteq T\Sigma/B$ of rank $0|1$.

The condition means that the Lie bracket of vector fields induces an isomorphism of holomorphic vector bundles

$$\mathcal{D} \otimes \mathcal{D} \longrightarrow (T\Sigma/B)/\mathcal{D},\ X \otimes Y \mapsto [X, Y] \bmod \mathcal{D}.$$

Using that the Berezinian behaves well with respect to short exact sequences, one obtains for the holomorphic cotangent bundle

$$\mathrm{Ber}(\Sigma) = \mathrm{Ber}(T\Sigma^*)$$

$$
\begin{aligned}
&\cong \mathrm{Ber}(\mathcal{D}^{-1}) \otimes \mathrm{Ber}(\mathcal{D}^{-2}) \\
&= \mathcal{D}^{-1}.
\end{aligned}
$$

Consequently, the complexified Berezinian of the underlying smooth manifold is

$$
\begin{aligned}
\mathrm{Ber}_{\mathbb{R}}(\Sigma) \otimes \mathbb{C} &= \mathrm{Ber}(T\Sigma^*) \otimes_{\mathbb{C}} \mathrm{Ber}(\overline{T\Sigma^*}) \\
&= (\mathcal{D} \otimes \bar{\mathcal{D}})^{-1}.
\end{aligned}
$$

In order to make sense of component fields in subsequent sections, it is useful to introduce the notion of an underlying even manifold [34].

Definition 3.2. Let Σ be a super Riemann surface over B. An underlying even manifold is a complex supermanifold $|\Sigma|$ of dimension $(1|0)$ over B with $|\Sigma|_0 = \Sigma_0$ together with an embedding $\iota_B \colon |\Sigma| \to \Sigma$ of complex supermanifolds over B such that the pullback along $\mathrm{pt} \to B$ is the canonical inclusion $\Sigma_0 \to \Sigma$.

Remark 3.3. If $B = \mathrm{pt}$, then there is a unique underlying even manifold given by the standard embedding $i_{\mathrm{pt}} \colon \Sigma_0 \to \Sigma$. For general B, there always exists such an embedding which is however not unique [34].

3.2 Examples of super Riemann surfaces

We now discuss the most relevant examples of super Riemann surfaces.

3.2.1 The superconformal plane

We consider $\mathbb{C}^{1|1}_B = \mathbb{C}^{1|1} \times B$ together with the distribution generated by $D = \partial_\vartheta - \vartheta\partial_z$. A local superconformal coordinate system on a super Riemann surface is an isomorphism $(z, \vartheta) \colon U \to \mathbb{C}^{1|1}_B$ of super Riemann surfaces over B. Locally such always exist. Unless specified otherwise, in the following, when working in local coordinates, we will always tacitly assume that the coordinates are superconformal. A change of such superconformal coordinates takes the special form

$$
\tilde{z} = u(z) - \vartheta\eta(z)\sqrt{u'(z)}, \ \ \tilde{\vartheta} = \eta(z) + \vartheta\sqrt{u'(z) + \eta'(z)\eta(z)}.
$$

Under such a coordinate change, we have

$$
D = f\tilde{D}, \ \ f = D\tilde{\vartheta} = \sqrt{u'(z) + \eta'(z)\eta(z)} - \vartheta\eta'(z).
$$

For later use, we note at this point, that if we want $(D\tilde{\vartheta})^n$ to be a fixed invertible function, then any other superconformal coordinate system $(\tilde{z}', \tilde{\vartheta}')$ which achieves this is obtained from $(\tilde{z},\tilde{\vartheta})$ by a combination of superconformal translations and rotations in the following way. There is an nth root of unity $\sqrt{\omega}$ and a point $(z_0,\eta_0)\colon B\to\mathbb{C}^{1|1}$, such that

$$\tilde{z}' = (\omega u(z) - \sqrt{\omega}\eta(z)\eta_0 + z_0) - \vartheta(\sqrt{\omega}\eta(z)+\eta_0)\sqrt{\omega u'(z) - \sqrt{\omega}\eta'(z)\eta_0},$$

$$\tilde{\vartheta}' = (\sqrt{\omega}\eta(z)+\eta_0) + \vartheta\sqrt{\omega u'(z) + \omega\eta'(z)\eta(z)}.$$

Remark 3.4. Finally, we remark that locally up to superconformal change of coordinates in the co-domain all underlying even manifolds are equivalent to the standard embedding $\mathbb{C}^{1|0}\times B\to\mathbb{C}^{1|1}\times B$.

3.2.2 Split super Riemann surfaces

There is a super Riemann surface associated to any Riemann surface Σ_0 together with a choice of spin structure, i.e., a holomorphic line bundle L which satisfies $L^2\cong T\Sigma^*$. For the complex supermanifold $\Sigma=(\Sigma_0,\wedge^\bullet\mathcal{L})$, $\mathcal{L}$ the sheaf of holomorphic sections of L, we have

$$\mathcal{T}\Sigma \cong \mathcal{O}_\Sigma\otimes_{\mathcal{O}_{\Sigma_0}}(\mathcal{T}\Sigma_0\oplus\mathcal{L}^*).$$

As vector bundle, we define $\mathcal{D}=\mathcal{O}_\Sigma\otimes_{\mathcal{O}_{\Sigma_0}}\mathcal{L}^*$, and the inclusion $\mathcal{D}\to\mathcal{T}\Sigma$ is induced by

$$\mathcal{L}^* \xrightarrow{\;\triangle\;} \mathcal{L}^*\oplus\mathcal{L}^* \xrightarrow{\;\cong\;} (\mathcal{L}\otimes T\Sigma_0)\oplus\mathcal{L}^*\subset\mathcal{T}\Sigma.$$

Pullback along $B\to\mathrm{pt}$ yields the split family $\Sigma_B=\Sigma\times B\to B$. Split super Riemann surfaces come with a choice of an underlying even manifold

$$\iota_B=(i_\Sigma\times B)\colon |\Sigma_B|=\Sigma_0\times B\longrightarrow\Sigma\times B=\Sigma_B.$$

Morphisms from split super Riemann surfaces can be understood in terms of more elementary objects on $|\Sigma_B|$. For this, we consider a supermanifold M with connection ∇^M. The complexified structural morphism

$$f^\sharp\colon f_0^{-1}\mathcal{O}_{M,\mathbb{C}}\longrightarrow\mathcal{O}_{\Sigma,\mathbb{C}}\cong(\mathcal{O}_{\Sigma_0,\mathbb{C}}\oplus(\mathcal{L}\oplus\bar{\mathcal{L}})\oplus(\mathcal{L}\otimes\bar{\mathcal{L}}))\otimes_{\mathbb{C}}(\mathcal{O}_B\otimes\mathbb{C})$$

commutes with complex conjugation and thus is equivalently given by the components

$$\tilde{f}^\sharp\colon f_0{}^{-1}\mathcal{O}_{M,\mathbb{C}}\longrightarrow\mathcal{O}_{\Sigma_0,\mathbb{C}}\otimes_{\mathbb{C}}(\mathcal{O}_B\otimes\mathbb{C})=\mathcal{O}_{|\Sigma_B|,\mathbb{C}}\,,$$

$$X\colon f_0{}^{-1}\mathcal{O}_{M,\mathbb{C}}\longrightarrow\mathcal{L}\otimes_{\mathbb{C}}(\mathcal{O}_B\otimes\mathbb{C})=\iota_B^*(\mathcal{D}^*),$$

$$F\colon f_0{}^{-1}\mathcal{O}_{M,\mathbb{C}}\longrightarrow(\mathcal{L}\otimes\bar{\mathcal{L}})\otimes_{\mathbb{C}}(\mathcal{O}_B\otimes\mathbb{C})=\iota_B^*(\mathcal{D}\otimes\bar{\mathcal{D}})^*.$$

In the following, $df_{\mathbb{C}}$ refers to the complexified differential and we use the type decomposition $\mathcal{D} \otimes \mathbb{C} \cong \mathcal{D} \oplus \bar{\mathcal{D}}$.

Proposition 3.5. *In the situation above, we have in local superconformal coordinates.*

(a) $\tilde{f}^\sharp$ *is a real algebra morphism and* $\tilde{f}^\sharp = (f \circ \iota_B)^\sharp$.

(b) $X(D)$ *defines an odd complex derivation along* $\tilde{f}^\sharp$ *and* $X(D) = \iota_B^*(df_{\mathbb{C}}(D))$.

(c) $\tilde{F}(D, \bar{D}) = F(D, \bar{D}) - (\nabla^M d(-))(X(D), \bar{X}(\bar{D}))$ *defines an even complex derivation along* $\tilde{f}^\sharp$ *and* $\tilde{F}(D, \bar{D}) = \iota_B^*(\nabla_D^M df_{\mathbb{C}}(\bar{D}))$.

This sets up a bijection between the set of morphisms $\Sigma_B \to M$ *and triples* $(\tilde{f}, X, \tilde{F})$, *where*

$$\tilde{f} \colon |\Sigma_B| \to M, \;\; X \in \Gamma(\iota_B^*(\mathcal{D}^*) \otimes_{\mathbb{C}} \tilde{f}^* \Pi(TM_{\mathbb{C}}))_{\bar{0}}, \;\; \tilde{F} \in \Gamma(\iota_B^*(\mathcal{D} \otimes \bar{\mathcal{D}})^* \otimes_{\mathbb{C}} \tilde{f}^*(TM_{\mathbb{C}}))_{\bar{0}}.$$

Proof. This is proved in [36, Section 1]. □

3.2.3 Genus 0

In order to obtain a superization of the Riemann sphere, we consider two copies of $U_i = \mathbb{C}^{1|1} \times B$ which are glued along

$$(U_1 - 0)|_{U_1 \cap U_2 - 0} \longrightarrow (U_2 - 0)|_{U_1 \cap U_2 - 0}, \quad \psi^\sharp(z, \vartheta) = (1/z, \vartheta/z).$$

On U_2 we let $\mathcal{D}$ be generated by $\partial_\vartheta + \vartheta \partial_z$. Then we have $\tilde{z} = 1/z$, $\tilde{\vartheta} = \vartheta/z$ and compute

$$\begin{aligned}\partial_\vartheta &= \frac{\partial \tilde{\vartheta}}{\partial \vartheta} \partial_{\tilde{\vartheta}} + \frac{\partial \tilde{z}}{\partial \vartheta} \partial_{\tilde{z}} \\ &= 1/z \, \partial_{\tilde{\vartheta}},\end{aligned}$$

$$\begin{aligned}\partial_z &= \frac{\partial \tilde{\vartheta}}{\partial z} \partial_{\tilde{\vartheta}} + \frac{\partial \tilde{z}}{\partial z} \partial_{\tilde{z}} \\ &= -\vartheta/z^2 \partial_{\tilde{\vartheta}} - 1/z^2 \partial_{\tilde{z}} \\ &= -1/z(\tilde{\vartheta} \partial_{\tilde{\vartheta}} + \tilde{z} \partial_{\tilde{z}}).\end{aligned}$$

Hence

$$\begin{aligned}\partial_\vartheta + \vartheta \partial_z &= \tilde{z} \partial_{\tilde{\vartheta}} - \tilde{\vartheta}(\tilde{\vartheta} \partial_{\tilde{\vartheta}} + \tilde{z} \partial_{\tilde{z}}) \\ &= \tilde{z}(\partial_{\tilde{\vartheta}} - \tilde{\vartheta} \partial_{\tilde{z}}).\end{aligned}$$

So on U_1 we take $\mathcal{D}$ to be generated by $\partial_\vartheta - \vartheta \partial_z$. This way we obtain a split super Riemann $\Sigma \cong \mathbb{C}P^{1|1} \times B$ and $\mathcal{D} \cong \mathcal{O}_{\mathbb{C}P^{1|1}} \otimes_{\mathcal{O}_{\mathbb{C}P^1}} \mathcal{O}(1)$. From this we can conclude $\Gamma(\mathcal{D}^{-k}) = 0$ for all

$k \geq 1$. One can show that all compact genus 0 super Riemann surface are isomorphic to this one.

3.2.4 Genus 1

There are four different superizations of a torus corresponding to the four different spin structures. Recall that the parity of a spin structure is defined to be parity of the number of global holomorphic sections. There is one odd spin structure, in which cases the super tori are constructed as follows. We look at the group structure on $\mathbb{C}^{1|1}_B$ which is given on T-valued points by

$$(z, \vartheta) \cdot (z', \vartheta') = (z + z' - \vartheta\vartheta', \vartheta + \vartheta').$$

There is a right-invariant superconformal structure on $\mathbb{C}^{1|1}_B$ generated by $\partial_\vartheta - \vartheta\partial_z$. The right translations

$$S = R_{(1,0)},\ T = R_{(\tau,\delta)},$$

where $(\tau, \delta)\colon B \to \mathbb{H} \times \mathbb{C}^{0|1}$, generate a group $\mathbb{Z} \oplus \mathbb{Z}$ of superconformal automorphisms and the quotient

$$\Sigma_{\tau,\delta} = \mathbb{C}^{1|1}_B / \langle S, T \rangle$$

exists and inherits a superconformal structure. In addition to the even parameter τ, this family has an an odd parameter δ which causes these families to be non-split in general. In fact, we have

$$\Gamma(\mathcal{O}_{\Sigma_{\tau,\delta}}) = \{a + \alpha\vartheta \mid a,\ \alpha \in \Lambda,\ \delta\alpha = 0\}.$$

Moreover, D transforms trivially under $R_{(1,0)}$ and $R_{(\tau,\delta)}$ so that $\mathcal{D}$ is trivial and we have $\Gamma(\mathcal{D}^k) = \Gamma(\mathcal{O}_{\Sigma_{\tau,\delta}})$ for any $k \in \mathbb{Z}$. In particular, these families are non-isomorphic in general. Still, for any δ there is a smooth isomorphism

$$\begin{array}{ccc} \Sigma_{\tau,0} & \longrightarrow & \Sigma_{\tau,\delta}\ , \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

which is the identity when restricted along $\mathrm{pt} \to B$. One way to see this, though not very explicit, is the following. For any smooth supermanifold M, it is known that $M \cong \Pi i^*(TM)_{\bar{1}}$ as supermanifolds under M_0, where $i\colon M_0 \to M$. In the case at hand, since the tangent bundle is trivial, there exists an isomorphism $\psi\colon \Sigma_{\tau,\delta} \to (\underline{\mathbb{C}^{0|1}}_{(\Sigma_{\tau,\delta})_0}) \times B$. Composing with

$$(\pi_{(\underline{\mathbb{C}^{0|1}}_{(\Sigma_{\tau,\delta})_0}) \times B}, \pi_\Sigma \circ \psi^{-1})\colon (\underline{\mathbb{C}^{0|1}}_{(\Sigma_{\tau,\delta})_0}) \times B \longrightarrow (\underline{\mathbb{C}^{0|1}}_{(\Sigma_{\tau,\delta})_0}) \times B$$

gives an isomorphism over B. The co-domain of ψ stays unchanged when pulled back along $B \to \mathrm{pt} \to B$, but the left hand side becomes $\Sigma_{\tau,0}$, and the resulting isomorphism

$\Sigma_{\tau,\delta} \cong (\underline{\mathbb{C}^{0|1}}_{(\Sigma_{\tau,\delta})_0}) \times B \cong \Sigma_{\tau,0}$ has the desired properties.

There are three even spin structures. The resulting super tori are split and can be constructed as follows. They are quotients of $\mathbb{C}^{1|1}_B$ by the group of automorphisms generated by $S = c^{\epsilon_1} \circ R_{(1,0)}$ and $T = c^{\epsilon_2} \circ R_{(\tau,0)}$ where c is the group automorphism $c(z,\vartheta) = (z, -\vartheta)$ and $(\epsilon_1, \epsilon_2) \in \{(0,1), (1,1), (1,0)\}$. This makes use of the fact that c is also an automorphism of the superconformal structure on $\mathbb{C}^{1|1}_B$. The resulting super Riemann surface $\Sigma = (\Sigma_0, \wedge^\bullet L \otimes \Lambda)$ is split and L is an even spin structure on Σ_τ. The holomorphic line bundle L does not have non-trivial global sections, so that $\Gamma(\mathcal{O}_\Sigma) = \Gamma(\mathcal{O}_{T^2} \oplus L) \otimes \Lambda = \Lambda$. Also $\mathcal{D}$ is nontrivial and does not have global holomorphic sections, however $\mathcal{D}^{\otimes 2}$ is always trivial, and hence

$$\Gamma((\mathcal{D}^*)^{\otimes 2k}) = \Lambda.$$

If we choose a universal covering $p\colon \mathbb{C}^{1|1}_B \to \Sigma$ of a super torus then we will always identify sections of $\mathcal{D}^{\otimes 2k}$ with holomorphic functions by means of the trivialization induced by p.

For later use, we note the following property.

Proposition 3.6. *Let Σ be a super torus associated with (τ, δ) or $((\epsilon_1, \epsilon_2), \tau)$ as above with universal covering $p\colon \mathbb{C}^{1|1}_B \to \Sigma$. Given an even section of $\mathcal{D}^{\otimes 2l}$ of the form $s = (x + \vartheta\xi) D^{\otimes 2l}$, where x is invertible and $\xi\delta = 0$ or $\xi = 0$ in the respective cases, there is a superconformal isomorphism of $\mathbb{C}^{1|1}_B$ which descends to a torus Σ'*

$$\begin{array}{ccc} \mathbb{C}^{1|1}_B & \longrightarrow & \mathbb{C}^{1|1}_B \\ \downarrow & & \downarrow \\ \Sigma' & \longrightarrow & \Sigma \end{array}$$

such that s pulls back to $1 \cdot D^{\otimes 2l}$.

Proof. In the case of split tori the section is constant, so that one can scale the superconformal plane suitably. (The coordinate change for scalings is similar to that for rotations (cf. Section 3.2.1).)

The argument is similar in the non-split case, but less trivial since then the function $s = (x + \vartheta\xi) D^{\otimes 2l}$, where $\delta\xi = 0$, is in general non-constant. We look at the superconformal change of coordinates

$$f\colon \mathbb{C}^{1|1}_B \longrightarrow \mathbb{C}^{1|1}_B,\ (\tilde{z}, \tilde{\vartheta}) = f(z, \vartheta) = (az - \vartheta c z \sqrt{a}, cz + \vartheta\sqrt{a}),$$

where a is an even invertible constant and c is an odd constant such that $c\delta = 0$. This has the effect that $F\tilde{D} = D$, with $F = \sqrt{a} - \vartheta c$, and thus leads to

$$x + \vartheta\xi = F^{-2l} = \frac{1}{a^l} + \vartheta \frac{(2l)c}{a^{l+1/2}}.$$

Now we only need to see that such resulting f descends to $\Sigma' \to \Sigma$ for a suitable torus Σ'. I.e., for suitable (τ', δ') and (κ', η')

$$R_{(\tau,\delta)} \circ f = f \circ R_{(\tau',\delta')}, \;\; R_{(1,0)} \circ f = f \circ R_{(\kappa',\eta')}.$$

The ansatz $c\delta' = 0$ leads to the following. First equation, left hand side:

$$\begin{aligned} R_{\tau,\delta}(az - \vartheta cz\sqrt{a}, cz + \vartheta\sqrt{a}) &= (az - \vartheta cz\sqrt{a} + \tau - (cz + \vartheta\sqrt{a})\delta, cz + \vartheta\sqrt{a} + \delta) \\ &= (az - \vartheta cz\sqrt{a} + \tau - \vartheta\sqrt{a}\delta, cz + \vartheta\sqrt{a} + \delta). \end{aligned}$$

First equation, right hand side:

$$\begin{aligned} f(z + \tau' - \vartheta\delta', \vartheta + \delta') &= (a(z + \tau' - \vartheta\delta') - (\vartheta + \delta')c(z + \tau')\sqrt{a}, c(z + \tau') + (\vartheta + \delta')\sqrt{a}) \\ &= (az - \vartheta cz\sqrt{a} + (a\tau' - \vartheta(a\delta' + c\tau'\sqrt{a})), cz + \vartheta\sqrt{a} + (c\tau' + \delta'\sqrt{a})). \end{aligned}$$

This yields $\tau' = \tau/a$, $\delta' = (1/\sqrt{a})(\delta - c\tau')$. Similarly, the second equation yields

$$\kappa' = 1/a, \;\; \eta' = (1/\sqrt{a})(-c/a).$$

Thus setting $T' = R_{(\tau',\delta')}$, $S' = R_{(\kappa',\eta')}$, we can define $\Sigma' = \mathbb{C}_B^{1|1}/\langle S', T'\rangle$. □

3.3 Points, Divisors, Infinitesimal Neighbourhoods

A point of Σ/B is a section P of the projection $\pi\colon \Sigma \to B$. It determines an ideal $I_P \subset \mathcal{O}_\Sigma$ which in superconformal coordinates $(z, \vartheta)\colon U \to \mathbb{C}^{1|1}$ is generated by $(z - z_0 - \vartheta\vartheta_0, \vartheta - \vartheta_0)$, where $(z_0, \vartheta_0) = (z, \vartheta) \circ P$. Using the superconformal coordinate transformation $\tilde{z} = z - z_0 + \vartheta\vartheta_0$, $\tilde{\vartheta} = \vartheta - \vartheta_0$, one can always assume $(z_0, \vartheta_0) = (0, 0)$.

A generalization is the notion of an infinitesimal neighbourhood of a point given by I_P. A subsheaf of ideals $J_P \subset \mathcal{O}_\Sigma$ is called infinitesimal neighbourhood if it is isomorphic to I_P away from P and if there are superconformal coordinates on U containing P such that $(J_P)|_U$ is given by $(z^k, \vartheta z^l)$ for some weights (k, l), where k, $l \geq 0$. Conversely, this gives a construction of such ideals:

$$(J_P)(V_0) = \begin{cases} \mathcal{O}_\Sigma(V_0), \text{ if } P_0 \notin V_0, \\ \{f \in \mathcal{O}_\Sigma(V_0) \mid \mathrm{res}^V_{V\cap U} f \in (z^k, \vartheta z^l)\}, \text{ if } P_0 \in V_0. \end{cases}$$

Notice that in the case $l \geq k$ we have $(z^k, \vartheta z^l) = (z^k)$ and the integer k is well-defined. (Though l is certainly not.) In the other cases we have:

Lemma 3.7. *The property $k > l$ is well-defined and in this case the pair (k, l) is well-defined.*

Proof. Suppose $(z^a, \vartheta z^b) = (\tilde{z}^m, \tilde{\vartheta}\tilde{z}^n)$. We show that $m > n$ implies $a > b$. For, suppose

$a \leq b$. Then $(z^a, \vartheta z^b) = (z^a)$. Reducing modulo all nilpotents shows $a = m$. Moreover, $\tilde{\vartheta}\tilde{z}^n = gz^a$. Again, applying $\tilde{D} = uD$, u a unit, implies $n \geq a$. A contradiction. Now, suppose $(z^a, \vartheta z^b) = (\tilde{z}^m, \tilde{\vartheta}\tilde{z}^n)$, where $a > b$ and $m > n$. Setting all nilpotents equal to zero, we obtain $(z_0^a) = (\tilde{z}_0^m)$ which implies $a = m$. Then we have $\vartheta z^b = x\tilde{z}^a + y\tilde{\vartheta}\tilde{z}^n$. Applying $D = u\tilde{D}$, u a unit, and then reducing modulo all nilpotents shows that $b \geq n$ and the same argument shows $n \geq b$. □

A superconformal coordinate system (z, ϑ) such that $(J_P)|_U = (z^k, \vartheta z^l)$ is called compatible.

Lemma 3.8. *Two compatible coordinate systems determine the same* $0|1$*-dimensional submanifold: if* $(z^k, \vartheta z^l) = (\tilde{z}^k, \tilde{\vartheta}\tilde{z}^l)$*, then* $(z) = (\tilde{z})$ *and* $(z, \vartheta) = (\tilde{z}, \tilde{\vartheta})$*.*

Proof. The statement is clear in the case $k \leq l$. Now we assume $k > l$. We have

$$\tilde{z}^k = az^k + b\vartheta z^l, \ \tilde{\vartheta}\tilde{z}^l = cz^k + d\vartheta z^l,$$

and since $k > l$ we can assume in addition that a, b, c and d only depend on z. On the other hand

$$\tilde{z} = u - \vartheta\eta\sqrt{u'}, \ \tilde{\vartheta} = \eta + \vartheta\sqrt{u' + \eta\eta'}$$

and hence

$$\tilde{z}^k = u^k - \vartheta(ku^{k-1}\eta\sqrt{u'}), \ \tilde{\vartheta}\tilde{z}^l = \eta u^l + \vartheta\sqrt{u' + \eta\eta'}u^l.$$

Comparing coefficients yields $u^k = az^k$, $\eta u^l = cz^k$. That is

$$u = a^{1/k}z, \ \eta = a^{-l/k}cz^{k-l}.$$

Hence

$$\begin{aligned}\tilde{z} &= a^{1/k}z - \vartheta ca^{-l/k}z^{k-l}\sqrt{u'} \\ &= z(a^{1/k} - \vartheta ca^{-l/k}z^{k-l-1}\sqrt{u'}).\end{aligned}$$

This implies $(z) = (\tilde{z})$ and from this the claim follows. □

3.4 Blow ups of J_P and parabolic structures

We now construct the blow up of an ideal of the form J_P. This parallels the construction of the blow up of a coherent sheaf of ideals in ordinary algebraic geometry. A reference for blow ups of points is [54]. To the author's knowledge, blow ups of infinitesimal neighbourhoods as discussed here have so far not appeared in the literature, which is why we give the details here.

3.4.1 Construction

Let P be a point and J_P an infinitesimal neighbourhood of weight (k, l). Now assume that $k > l$. We set $n = k - l > 0$. The blow up of Σ along J_P is defined as follows. Locally on some open neighbourhood $U \subseteq \Sigma$ we choose compatible superconformal coordinates $(z, \vartheta)\colon U \to \mathbb{C}^{1|1}$ such that $I_P = (z, \vartheta)$, $J_P = (z^k, \vartheta z^l)$. We set $\tilde{\Sigma}_0 = \Sigma_0$. The supermanifold $\tilde{\Sigma}/B$ is covered by $\Sigma - P$ and U, i.e. $(\mathcal{O}_{\tilde{\Sigma}})|_{\Sigma_0 - P_0} = \mathcal{O}_{\Sigma - P}$ and $(\mathcal{O}_{\tilde{\Sigma}})|_{U_0} = \mathcal{O}_U$ and the transition function is

$$(\mathcal{O}_U)|_{U-P} \longrightarrow (\mathcal{O}_{\Sigma-P})|_{U-P}, \ (z, \vartheta) \mapsto (z, z^{-n}\vartheta).$$

This comes with a projection $p\colon \tilde{\Sigma} \to \Sigma$ which is defined to be the identity on $\tilde{\Sigma} - P = \Sigma - P$ and by $p^\sharp(z) = z$, $p^\sharp(\vartheta) = z^n\vartheta$ on U. We have that $p^*(I_P) = I_{\tilde{Z}}$ for a $0|1$-dimensional submanifold $\tilde{Z}$, and by construction

$$p^*(J_P) = I_{\tilde{Z}}^{\otimes k}$$

is locally free of rank $(1|0)$. The universal property of $\tilde{\Sigma} \to \Sigma$ is captured in the following.

Proposition 3.9. *Given a point P on Σ and an infinitesimal neighbourhood J_P, there is a supermanifold $p\colon \tilde{\Sigma} \to \Sigma$ such that $p^*(J_P)$ is locally free of rank $(1|0)$ and which has the following universal property: if $f\colon X \to \Sigma$ is any holomorphic map, then f factors through $p\colon \tilde{\Sigma} \to \Sigma$ if and only if $f^*(J_P)$ is locally free of rank $(1|0)$. In this case the lift is unique. In particular, $\tilde{\Sigma} \to \Sigma$ is unique up to unique isomorphism.*

Proof. We have already proved the first part. Now for the factorization property, assume that the map factors through p, i.e. $f = p \circ \tilde{f}$, then $f^*(J_P) = \tilde{f}^*(I_{\tilde{Z}})$ and thus is locally free of rank $(1|0)$. Conversely, we shrink the U we used to construct $\tilde{\Sigma}$ such that $f^*(J_P)$ is locally free on $f^{-1}(U)$. We write $s = f^\sharp(z)$, and $\eta = f^\sharp(\vartheta)$. A general generator of $f^*(J_P)$ is of the form $as^k + b\eta$, but then s^k is also a generator. Since it is locally free we have an a such that

$$s^l\eta = as^k.$$

Then it is clear that on $p^{-1}(U)$ we have to set

$$\tilde{f}^\sharp(\tilde{z}) = s, \ \tilde{f}^\sharp(\tilde{\vartheta}) = a,$$

and on the complementary part $\Sigma - P$ we are forced to use the map given by f. This shows existence and uniqueness. $\square$

3.4.2 Lifting the superconformal structure to a parabolic structure

Consider a blow up $p\colon \tilde{\Sigma} \to \Sigma$. Near P it is defined by $p^\sharp(z) = \tilde{z}$ and $p^\sharp(\vartheta) = \tilde{z}^n\tilde{\vartheta}$. We calculate that

$$\partial_\vartheta - \vartheta\partial_z = 1/z^n\partial_{\tilde{\vartheta}} - z^n\tilde{\vartheta}\partial_{\tilde{z}} = 1/\tilde{z}^n(\partial_{\tilde{\vartheta}} - \tilde{z}^{2n}\tilde{\vartheta}\partial_{\tilde{z}}).$$

So we see that $\mathcal{D}$ lifts to a distribution $\tilde{\mathcal{D}}$. However, this is not a superconformal structure, rather

$$\tilde{\mathcal{D}}^2 \cong (T\tilde{\Sigma}/B)/\mathcal{D} \otimes \mathcal{O}(-2n\tilde{Z}),$$

i.e., the square $\tilde{D}^2$ vanishes along $\tilde{Z}$ to order $2n$. (Here $\mathcal{O}(-2n\tilde{Z})$ denotes the line bundle determined by $I_{\tilde{Z}}^{\otimes 2n}$.) Following [54, Section 3.3], we call such a structure parabolic of order $2n$. For our purpose, a parabolic super Riemann surfaces will be the blow up of some ideal J_P on a super Riemann surface. This is not the most general case (cf. Loc. cit), but sufficient in our case. These blow ups can be viewed as special punctures on the super Riemann surface. As such they lead to additional moduli parameter. For further discussion we also refer to [30, 55].

3.5 Holomorphic sections of vector bundles and their regularity

On Σ/B we consider a holomorphic vector bundle E of rank $n|m$ and an even non-zero holomorphic section $s \in \Gamma(\mathcal{E})$, where $\mathcal{E}$ denotes the sheaf of holomorphic sections. Consider a point $p \in \Sigma_0$ such that $s^{\mathrm{red}}(p) = 0$. We choose a neighbourhood U of p on which the bundle is trivial and a local trivialization $\mathcal{E}|_U \cong \langle s_1, \dots, s_n | p_1, \dots p_m\rangle$ in which we have $s = \sum f_i s_i + \sum g_i p_i$. We say that the zero is regular if the ideal (f_i, g_i) is an infinitesimal neighbourhood of a point P: $(f_i, g_i) = J_P$. This is independent of the chosen local trivialization of the vector bundle. The goal is to find a line bundle $\mathcal{L}$ with a section $\mathcal{O}_\Sigma \to \mathcal{L}$ and a diagram

$$\begin{array}{ccc} \mathcal{O}_\Sigma & \xrightarrow{f \mapsto fs} & \mathcal{E}, \\ \big\downarrow & \nearrow_{s} & \\ \mathcal{L} & & \end{array}$$

where s is nowhere vanishing. If $k \leq l$, the vanishing ideal of s defines a line bundle and, using the notation from the previous sections, one can use $\mathcal{L}^{\otimes k}$, where $\mathcal{L} = I_Z$. The proof of this is similar to the case $k > l$ which we treat now. In this case the vanishing ideal defines a line bundle only after passing to a blow up $\tilde{\Sigma}$ of Σ. We now show that we can construct such an extension on $\tilde{\Sigma}$.

We set $n = k - l > 0$. Let $p\colon \tilde{\Sigma} \to \Sigma$ be the blow up of J_P. We set $Z = p^{-1}(P)$. We have seen that this defines a line bundle $\mathcal{L}$ (strictly speaking an isomorphism class). If Z is given by $\{z = 0\}$, then $\mathcal{L}$ is defined by the patching $(\mathcal{O}_U)|_{U-Z} \to (\mathcal{O}_{\tilde{\Sigma}})|_{U-Z}$, $f \mapsto z^{-1}f$. There is an arrow $\mathcal{O}_{\tilde{\Sigma}} \to \mathcal{L}^{\otimes k}$ given by $f \mapsto z^k f$ on U and the identity away from U. The section

s determines a section $p^*(s)$ of $\tilde{\mathcal{E}} = p^*\mathcal{E} = \mathcal{O}_{\tilde{\Sigma}} \otimes_{p^{-1}\mathcal{O}_\Sigma} p^{-1}\mathcal{E}$. We want to define $\tilde{s}$ in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\tilde{\Sigma}} & \xrightarrow{f \mapsto f p^*(s)} & \tilde{\mathcal{E}} \\ \big\downarrow & \nearrow_{\tilde{s}} & \\ \mathcal{L}^{\otimes k} & & \end{array} .$$

On $\tilde{\Sigma} - Z$ we need commutativity of

$$\begin{array}{ccc} \mathcal{O}_{\tilde{\Sigma}}|_{\Sigma - Z} & \longrightarrow & \tilde{\mathcal{E}}|_{\Sigma - Z} \\ \big\downarrow{=} & \nearrow_{\tilde{s}} & \\ \mathcal{L}^{\otimes k}|_{\Sigma - Z} = \mathcal{O}_{\tilde{\Sigma}}|_{\Sigma - Z} & & \end{array} .$$

To extend this to U, we notice that in view of the definition of the structure sheaf of $\tilde{\Sigma}$, we have $p^*(s)|_U = z^k \tilde{t}$ for some non-vanishing $\tilde{t} \in \Gamma(\tilde{\mathcal{E}}|_U)$. The extension is then given by $f \mapsto \tilde{t}$. By the definition of the transition function of $\mathcal{L}^{\otimes n}$, we see that on U we can use

$$\begin{array}{ccc} \mathcal{O}_U & \longrightarrow & \tilde{\mathcal{E}}|_U \\ \big\downarrow{z^k} & \nearrow_{\tilde{t}} & \\ \mathcal{L}^{\otimes k}|_U = \mathcal{O}_U & & \end{array} .$$

To summarize, we have proved the following.

Proposition 3.10. *Given a holomorphic section s of a vector bundle E over the super Riemann surface Σ/B. If the zeros of s are regular, then there exists a blow $p\colon \tilde{\Sigma} \to \Sigma$, a holomorphic line bundle L on $\tilde{\Sigma}$ with a holomorphic section f and an extension of $p^*(s)$ to a nowhere vanishing holomorphic section $\tilde{s}$ of $L^* \otimes p^*E$: $\tilde{s}(f) = p^*s$.*

Remark 3.11. Since we are working over a Grassmann algebra, not every zero is regular although over $B = \mathrm{pt}$ all zeros are regular.

3.6 Connections on super Riemann surfaces

3.6.1 Koszul-Malgrange holomorphic structures

Let $E \to \Sigma/B$ be a complex vector bundle, $\mathcal{E}$ its sheaf of sections. A connection on E is a complex linear map $\mathcal{E} \to \Omega^1_{\Sigma/B,\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{E}$ which satisfies the Leibniz rule. Here, $\Omega^1_{\Sigma/B,\mathbb{C}}$ denotes the sheaf of complex 1-forms $(\mathcal{T}\Sigma/B)^* \otimes \mathbb{C}$. The curvature of the connection is the endomorphism-valued two-form given by

$$R(X,Y) = \nabla_X \nabla_Y - (-1)^{|X||Y|} \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Definition 3.12. A connection is called partially flat if in local superconformal coordinates $R(D,D) = R(D^2,D) = R(\bar{D},\bar{D}) = R(\bar{D}^2,\bar{D}) = 0$.

Lemma 3.13. *(a) Let ∇ be a connection on $E \to \Sigma/B$. Then any (smooth) splitting of the inclusion $\mathcal{D} \to T\Sigma$ determines a unique partially flat connection $\tilde{\nabla}$ such that $\nabla|_{\mathcal{D}\oplus\bar{\mathcal{D}}} = \tilde{\nabla}|_{\mathcal{D}\oplus\bar{\mathcal{D}}}$.*

(b) Consider a complex vector bundle $E \to \Sigma/B$ with partially flat connection ∇. Then there is a unique holomorphic structure on E such that its holomorphic sections are locally characterized by $\nabla_{\bar{D}}s = 0$.

Proof. Part (a) is clear. For (b) we refer to [50]. □

Theorem 3.14 (Koszul-Malgrange structure for super Riemann surfaces)**.** *Let E be a complex vector bundle on the super Riemann surface Σ with connection ∇. Then E admits a unique holomorphic structure such that the holomorphic sections are locally characterized by $\nabla_{\bar{D}}s = 0$.*

Proof. We can choose a smooth splitting of $T\Sigma/B \to \mathcal{D}^{\otimes 2}$ and then apply Lemma 3.13. □

For later reference, we note at this point the following.

Proposition 3.15. *Let Σ/B be a super Riemann surface with underlying even supermanifold $\iota_B\colon |\Sigma| \to \Sigma$. Let s be an even section of a vector bundle $E \to \Sigma/B$ with connection ∇. Then $s = 0$ if and only if locally*

$$\iota_B^*(s) = 0,\ \iota_B^*(\nabla_D s) = 0,\ \iota_B^*(\nabla_{\bar{D}} s) = 0,\ \iota_B^*(\nabla_D \nabla_{\bar{D}} s) = 0.$$

Proof. The problem is local, so we may suppose that $\Sigma = \mathbb{C}^{1|1} \times B$ and $|\Sigma| = \mathbb{C}^{1|0} \times B$ with the standard embedding. The vector bundle is trivial which allows us to write $s = s_0 + \vartheta s_\vartheta + \bar{\vartheta} s_{\bar{\vartheta}} + \vartheta\bar{\vartheta} s_{\vartheta\bar{\vartheta}}$ where the components are sections of $i_B^*(E)$. The first equation says that $s_0 = 0$. Now we have $\nabla_D s = Ds + A_D(s)$ for some odd endomorphism A_D. Hence

$$\begin{aligned} 0 &= \iota_B^*(Ds + A_D(s)) \\ &= s_\vartheta + \iota_B^*(A_D)(\iota_B^* s) \\ &= s_\vartheta. \end{aligned}$$

Similarly, $s_{\bar{\vartheta}} = 0$. Finally,

$$\begin{aligned} 0 &= \iota_B^*((D + A_D)\nabla_{\bar{D}}(s)) \\ &= \iota_B^*(D\nabla_{\bar{D}}(s)) + (\iota_B^* A_D)(\iota_B^*(\nabla_{\bar{D}} s)) \\ &= \iota_B^*(D\bar{D}s) + \iota_B^*(D(A_{\bar{D}} s)) \end{aligned}$$

$$= -s_{\vartheta\bar{\vartheta}} + \iota_B^*(DA_{\bar{D}}(s) - A_{\bar{D}}(Ds))$$
$$= -s_{\vartheta\bar{\vartheta}}.$$

□

3.6.2 Flat $\mathfrak{g}$-valued 1-forms

For later reference we will also give a short review of $\mathfrak{g}$-valued 1-forms on $\mathbb{C}_B^{1|1}$, where $\mathfrak{g}$ is a Lie superalgebra. The curvature of such a form α is given by

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha].$$

The key point is that, in presence of a superconformal structure, flatness is encoded in a single equation as in the case of an ordinary Riemann surface if the connection is partially flat. We decompose the complexified form according to $\mathcal{D} \otimes \mathbb{C} \cong \mathcal{D} \oplus \bar{\mathcal{D}}$ and obtain the two maps α_D, $\alpha_{\bar{D}} = \overline{\alpha_D} \colon \mathbb{C}_B^{1|1} \to \Pi\mathfrak{g}_{\mathbb{C}}$. The most important facts are summarized in the following.

Proposition 3.16. *We have that:*

(a) A partially flat connection is flat if and only if $D\alpha_{\bar{D}} + \bar{D}\alpha_D + [\alpha_D, \alpha_{\bar{D}}] = 0$.

(b) If α *is flat, then there is a unique map* $F \colon \mathbb{C}_B^{1|1} \to G$, $F(0) = 1$, *such that* $F^{-1}dF = \alpha$.

(c) For a flat connection α, *we have that* $\alpha_z = -(D\alpha_D + \alpha_D^2)$.

Proof. This is proved in [36, Thm. 5]. □

Remark 3.17. Here and in the following we make use of the shorthand $\alpha_D^2 = \frac{1}{2}[\alpha_D, \alpha_D]$.

4 Generalities on harmonic maps

On a compact super Riemann surface the objects which can be integrated are sections of the Berezinian $\mathrm{Ber}_{\mathbb{C}}(\Sigma/B)$ (cf. [55]). Hence, given a map $f \colon \Sigma/B \to M$ into a supermanifold with even or odd Riemann metric, we can define the energy

$$E(f) = \int_{\Sigma/B} \langle df_{\mathbb{C}}|_{\mathcal{D}}, df_{\mathbb{C}}|_{\bar{\mathcal{D}}} \rangle_{\mathbb{C}} \in \Gamma(\mathcal{O}_B).$$

Here $\langle -, - \rangle_{\mathbb{C}}$ denotes the complex bilinear extension of the Riemannian structure on M. Critical points are called harmonic maps. Sometimes, we will add the adjective "supersymmetric" to distinguish such maps from ordinary harmonic maps from a Riemann surface into a

Riemannian manifold. The resulting Euler-Lagrange equation reads in local superconformal coordinates (cf. [36, Sect. 2])

$$\nabla^{LC}_{\bar{D}}(df_{\mathbb{C}}(D)) = 0.$$

(Here, the connection is understood to be the pullback of the Levi-Civita connection on M along f.) In other words, the differential $df_{\mathbb{C}}|_{\mathcal{D}}$ is a holomorphic section of $\mathcal{D}^* \otimes (f^*TM_{\mathbb{C}})$, where the second tensor factor is equipped with the Koszul-Malgrange holomorphic structure. In this respect, supersymmetric harmonic maps behave formally exactly the same way as in the ungraded setting (cf. [11]).

We now compute the underlying equation in local superconformal coordinates. Using that the Levi Civita connection is torsion-free, we find:

$$\nabla^{LC}_{\bar{D}}\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D) = (\frac{1}{2}R(\bar{D},\bar{D}) - \nabla^{LC}_{\bar{\partial}})(df_{\mathbb{C}}(D)),$$

and

$$\begin{aligned}
\nabla^{LC}_{\bar{D}}\nabla^{LC}_{D}\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D) &= (R(\bar{D},D) - \nabla^{LC}_{D}\nabla^{LC}_{\bar{D}})(\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D)) \\
&= R(\bar{D},D)(\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D)) - \nabla^{LC}_{D}(\frac{1}{2}R(\bar{D},\bar{D}) - \nabla^{LC}_{\bar{\partial}})(df_{\mathbb{C}}(D)) \\
&= R(\bar{D},D)(\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D)) - \frac{1}{2}(\nabla^{LC}_{D}R)(\bar{D},\bar{D})(df_{\mathbb{C}}(D)) \\
&\quad - \frac{1}{2}R(\nabla^{LC}_{D} df_{\mathbb{C}}(\bar{D}),\bar{D})(df_{\mathbb{C}}(D)) + \frac{1}{2}R(\bar{D},\nabla^{LC}_{D} df_{\mathbb{C}}(\bar{D}))(df_{\mathbb{C}}(D)) \\
&\quad - \frac{1}{2}R(\bar{D},\bar{D})(\nabla^{LC}_{D} df_{\mathbb{C}}(D)) + \nabla^{LC}_{D}\nabla^{LC}_{D} df_{\mathbb{C}}(\bar{\partial}) \\
&= R(\bar{D},D)(\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D)) - \frac{1}{2}(\nabla^{LC}_{D}R)(\bar{D},\bar{D})(df_{\mathbb{C}}(D)) \\
&\quad - R(\nabla^{LC}_{D} df_{\mathbb{C}}(\bar{D}),\bar{D})(df_{\mathbb{C}}(D)) + \frac{1}{2}(R(\bar{D},\bar{D})(df_{\mathbb{C}}(\partial)) + R(D,D)(df_{\mathbb{C}}(\bar{\partial}))) \\
&\quad - \nabla^{LC}_{\partial} df_{\mathbb{C}}(\bar{\partial}).
\end{aligned}$$

Given an underlying even supermanifold $\iota_B\colon |\Sigma| \to \Sigma$, then in view of Proposition 3.15, we find that the harmonic map equation is equivalent to:

$$\iota_B^*(\nabla^{LC}_{\bar{D}} df_{\mathbb{C}}(D)) = 0,$$

$$\iota_B^*((\frac{1}{2}R(\bar{D},\bar{D}) - \nabla^{LC}_{\bar{\partial}})df_{\mathbb{C}}(D)) = 0,$$

$$\iota_B^*(-\frac{1}{2}(\nabla^{LC}_{D}R)(\bar{D},\bar{D})(df_{\mathbb{C}}(D)) + \frac{1}{2}(R(\bar{D},\bar{D})(df_{\mathbb{C}}(\partial)) + R(D,D)(df_{\mathbb{C}}(\bar{\partial}))) - \nabla^{LC}_{\partial} df_{\mathbb{C}}(\bar{\partial})) = 0.$$

For a split super Riemann surface, we can conclude now:

Proposition 4.1. *Let $\Sigma_B = \Sigma \times B$ be a split super Riemann surface and M a supermanifold with Riemannian structure. Harmonic maps $f\colon \Sigma_B \to M$ are in one to one correspondence* *with pairs*

$$\tilde{f}\colon |\Sigma_B| \to M,\ X \in \Gamma(\iota_B^*(\mathcal{D}^*) \otimes_{\mathbb{C}} \tilde{f}^*(TM_{\mathbb{C}}))_{\bar{0}},$$

locally, setting $\psi = X(D)$*, subject to*

$$\nabla^{LC}_{\bar{\partial}}\psi = \frac{1}{2}R(\bar{\psi},\bar{\psi})\psi, \tag{4.2}$$

$$\nabla^{LC}_{\bar{\partial}} d\tilde{f}_{\mathbb{C}}(\partial) = \frac{1}{2}(R(\bar{\psi},\bar{\psi})(d\tilde{f}_{\mathbb{C}}(\partial)) + R(\psi,\psi)(d\tilde{f}_{\mathbb{C}}(\bar{\partial})) - (\nabla_{\psi}R)(\bar{\psi},\bar{\psi})\psi). \tag{4.3}$$

Proof. This follows from the above computation together with Proposition 3.5. □

The underlying map is harmonic if and only if $\nabla^{LC}_{\bar{\partial}} d\tilde{f}_{\mathbb{C}}(\partial) = 0$. If the co-domain is purely even, then the right hand side of (4.3) vanishes after restriction along $\mathrm{pt} \to B$. However, this is not necessarily the case if the co-domain is a supermanifold. In particular, it is then natural question, if coupled solutions to these equations, i.e., such that the underlying map is not harmonic, exist. We will construct an example in Section 7.4.4.

One should point out the structural similarity to the equations for Dirac harmonic maps [14] in which the tension field of a map is coupled to a spinor. The difference to the situation at hand is that in our situation ψ is an odd quantity. For instance, for the curvature R we do not necessarily have $R(\psi,\psi) = 0$. For a discussion of the analogous problem of finding truly coupled solutions in this context see [35].

Remark 4.4. These component equations have been derived in [36] as well in the case $\Sigma = \mathbb{C}^{1|1}_B$ and M an ordinary Riemannian manifold.

5 Harmonic maps into $U(n|m)$

5.1 Zero curvature representation

Since the Levi-Civita connection on $U(n|m)$ is given by $d + \frac{1}{2}\alpha$, where α is the Maurer-Cartan form, a map $f\colon \Sigma \to U(n|m)$ is harmonic if and only if

$$\bar{D}\alpha_D + \frac{1}{2}[\alpha_{\bar{D}}, \alpha_D] = 0.$$

Let $\alpha|_{\mathcal{D}\otimes\mathbb{C}} = \alpha' + \alpha''$ be the type decomposition. As in Lemma 3.13, in presence of a splitting of the inclusion $\mathcal{D} \to \mathcal{T}\Sigma/B$, any connection defined on $\mathcal{D}$ extends to a partially flat connection. The following result is to be understood by using this construction. We have the following characterization of harmonicity.

Lemma 5.1. *We have*

$$D\alpha_{\bar{D}} + \frac{1}{2}[\alpha_D, \alpha_{\bar{D}}] = 0$$

if and only if the loop of connections determined by

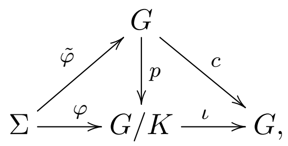
$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha''$$

is flat for all $\lambda \in S^1$.

Proof. See [36, Thm. 6]. $\square$

5.2 Harmonic maps and framings

Recall from Section 2.4 that there is the totally geodesic embedding $\iota\colon Gr_{k|l}(\mathbb{C}^{n|m}) \to U(n|m)$, $V \mapsto \pi_V - \pi_{V^\perp}$. It follows that the problem of harmonic maps into Grassmannians is reduced to the problem of harmonic maps into $U(n|m)$. However, it is also convenient to study harmonic maps $\Sigma \to Gr_{k|l}(\mathbb{C}^{n|m})$ via framings. We will write $G = U(n|m)$, $K = U(k|l) \times U(n-k|m-l)$. By a framing, we mean a lift in the commutative diagram



where p is the projection and $c = \iota \circ p$. On the level of Lie superalgebras ι induces an Ad_K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If we denote the projection onto the respective summand by a subscript $\mathfrak{k}$ or $\mathfrak{p}$, then for the Maurer-Cartan form α we obtain

$$\mathrm{Ad}(g^{-1})c^*\alpha = (-2)\alpha_{\mathfrak{p}}.$$

Setting $\beta = \varphi^*\iota^*\alpha$ and $\tilde{\beta} = \tilde{\varphi}^*\alpha$, we also have

$$\begin{aligned}\beta &= \tilde{\varphi}^* c^*\alpha \\ &= (-2)\mathrm{Ad}(\tilde{\varphi})\tilde{\beta}_{\mathfrak{p}}.\end{aligned}$$

We also extend the projections onto $\mathfrak{m}$ and $\mathfrak{p}$ linearly to the projections onto $\mathfrak{m}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ on the complexification $\mathfrak{g}_{\mathbb{C}}$. Starting from the harmonic map equation

$$\bar{D}\beta_D = -\frac{1}{2}[\beta_{\bar{D}}, \beta_D],$$

we can now compute both sides in terms of $\tilde{\beta}$ and obtain

$$\bar{D}\beta_D = (-2)\mathrm{Ad}(\tilde{\varphi})[\tilde{\beta}_{\bar{D}}, \tilde{\beta}_{\mathfrak{p},D}] + (-2)\mathrm{Ad}\tilde{\varphi}(\bar{D}\tilde{\beta}_{\mathfrak{p},D}),\ -\frac{1}{2}[\beta_{\bar{D}}, \beta_D] = (-2)\mathrm{Ad}(\tilde{\varphi})[\beta_{\mathfrak{p},\bar{D}}, \beta_{\mathfrak{p},D}].$$

So, finally, the harmonic map equation in terms of the framing $\tilde{\varphi}$ reads

$$\bar{D}\tilde{\beta}_{\mathfrak{p},D} = -[\tilde{\beta}_{\mathfrak{k},\bar{D}}, \tilde{\beta}_{\mathfrak{p},D}].$$

5.3 Supersymmetric harmonic maps and elliptic integrable systems

In view of the discussion in Section 4, it is natural to study how supersymmetric harmonic maps relate to harmonic maps. In the case of a Lie group as co-domain, as was observed by Khemar [36], the underlying map of a supersymmetric harmonic map is a solution to a special instance of a broad class of integrable equations, the elliptic integrable systems. For a comprehensive treatment of those, we refer to [36, 37]. We give a brief account on such and also compute the equations for the underlying map in this setup.

5.3.1 Basics on elliptic integrable systems

Let $\mathfrak{g}$ be a Lie (super)algebra with an automorphism τ of order k'. We write $k' = 2k$ or $k' = 2k + 1$. After complexification, there is a decomposition into the eigenspaces of τ

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{l \in \mathbb{Z}/k'} \mathfrak{g}_l,$$

where $\mathfrak{g}_l$ corresponds to the eigenvalue $e^{\frac{2\pi i l}{k'}}$. The mth elliptic equation is the flatness of the loop of $\mathfrak{g}$-valued 1-forms on $\mathbb{C}^{1|1}_B$ determined by

$$\alpha_{\lambda,D} = \sum_{i=0}^{m} \lambda^i \alpha_{D,i}, \ \lambda \in S^1,$$

where α_i is a section of $\mathcal{D}^* \otimes \mathfrak{g}_i$.

There are three cases. We put $m_1 = 0$ and for $k' > 1$ set $m_{k'} = k$, if $k' = 2k$, and $m_{k'} = k + 1$, if $k' = 2k + 1$. Then the cases are primitive if $m < m_{k'}$, determined if $m_{k'} \leq m \leq k' - 1$, and underdetermined if $m \geq k'$. Now let $\iota_B \colon |\mathbb{C}^{1|1}_B| = \mathbb{C}^{1|0}_B \to \mathbb{C}^{1|1}_B$ denote the standard underlying manifold. The mth elliptic equation on $|\mathbb{C}^{1|1}_B|$ is the flatness of

$$\sum_{i=0}^{m} \lambda^i \alpha_i, \ \lambda \in S^1,$$

where now α_i is a section of $(\mathcal{T}(|\mathbb{C}^{1|1}_B|/B)^{(1,0)})^* \otimes \mathfrak{g}_i$.

Remark 5.2. Originally, the notion of an elliptic integrable system was formulated by Khemar in the ungraded setting [37]. We have only made the straightforward necessary adaptions to the supersymmetric setting.

We have the following basic observation [36].

Lemma 5.3. *Consider a supersymmetric solution to the* m*th elliptic integrable system* $\alpha_{\lambda,D}$*. Then* $\beta_z := \iota_B^* \alpha_{\lambda,z}$ *is a solution to the* $(2m)$*th elliptic system on* $|\mathbb{C}_B^{1|1}|$.

Proof. This follows form the fact that flatness in the supersymmetric sense of $\alpha_{\lambda,D}$ implies flatness of $\alpha_{\lambda,z}$ in the non-super sense. The doubling $m \mapsto 2m$ is a result of $\alpha_{\lambda,z} = -(D\alpha_{\lambda,D} + \alpha_{\lambda,D}^2)$. □

Remark 5.4. For supersymmetric harmonic map into a Lie supergroup we have $k' = 1$, $m = 1$. Thus the underlying map has $k' = 1$, $m = 2$. As Khemar shows [37, p. 46], this underdetermined system is equivalent to the system with $k' = 3$ and $m = 2$ where one considers $\mathfrak{g}^3$ together with the cyclic permutation. This is in contrast with the ordinary harmonic map equation. There $k' = 1$, $m = 1$ and this underdetermined system is equivalent to the system with $k' = 2$ and $m = 1$ where on $\mathfrak{g}^2$ one considers the automorphism given by cyclic permutation.

Now, for a supersymmetric harmonic map into a symmetric space, we have for the pullback of the Maurer-Cartan form of a framing the equations

$$D\alpha_{\bar{D}} + \bar{D}\alpha_D + [\alpha_D, \alpha_{\bar{D}}] = 0,\ \bar{D}\alpha_{\mathfrak{p},D} = -[\alpha_{\mathfrak{k},\bar{D}}, \alpha_{\mathfrak{p},D}].$$

These are equivalent to the flatness of $\alpha_{\lambda,D} = \alpha_{\mathfrak{k},D} + \lambda\alpha_{\mathfrak{p},D}$, $\lambda \in S^1$. Consequently, the underlying map is related to the second system of a symmetric pair, $k' = 2$. This underdetermined system is in turn equivalent to the second system associated to the 4-symmetric space $\mathfrak{g}^2$, $(a, b) \mapsto (b, \tau(a))$.

5.3.2 The underlying map of a supersymmetric harmonic map in terms of a framing

We now compute the equation for the underlying map of a supersymmetric harmonic map in terms of a framing. This provides an alternative point of view on the equations derived in Section 4. We have

$$\alpha_{\mathfrak{p},z} = -(D\alpha_{\mathfrak{p},D} + [\alpha_{\mathfrak{k},D}, \alpha_{\mathfrak{p},D}]),\ \alpha_{\mathfrak{k},z} = -(\alpha_{\mathfrak{p},D}^2 + (D\alpha_{\mathfrak{k},D} + \alpha_{\mathfrak{k},D}^2)).$$

Then we compute

$$\begin{aligned}\bar{D}(-D\alpha_{\mathfrak{p},D}) &= D\bar{D}\alpha_{\mathfrak{p},D} \\ &= -D[\alpha_{\mathfrak{k},\bar{D}}, \alpha_{\mathfrak{p},D}] \\ &= -[D\alpha_{\mathfrak{k},\bar{D}}, \alpha_{\mathfrak{p},D}] + [\alpha_{\mathfrak{k},\bar{D}}, D\alpha_{\mathfrak{p},D}].\end{aligned}$$

Therefore,

$$\begin{aligned}
-\bar\partial(-D\alpha_{\mathfrak{p},D}) &= -[\bar D D\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}] - [D\alpha_{\mathfrak{k},\bar D},\bar D\alpha_{\mathfrak{p},D}] + [\bar D\alpha_{\mathfrak{k},\bar D}, D\alpha_{\mathfrak{p},D}] - [\alpha_{\mathfrak{k},\bar D},\bar D D\alpha_{\mathfrak{p},D}]\\
&= [D\bar D\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}] + [D\alpha_{\mathfrak{k},\bar D},[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]] + [\bar D\alpha_{\mathfrak{k},\bar D}, D\alpha_{\mathfrak{p},D}] - [\alpha_{\mathfrak{k},\bar D}, D[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]]\\
&= [D\bar D\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}] + D[\alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}] + [\bar D\alpha_{\mathfrak{k},\bar D}, D\alpha_{\mathfrak{p},D}]\\
&= D[\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}],
\end{aligned}$$

$$\bar D[\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] = [\bar D\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] + [\alpha_{\mathfrak{k},D},[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]],$$

$$\begin{aligned}
-\bar\partial[\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] &= [\bar D\bar D\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] + [\bar D\alpha_{\mathfrak{k},D},\bar D\alpha_{\mathfrak{p},D}] + [\bar D\alpha_{\mathfrak{k},D},[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]] - [\alpha_{\mathfrak{k},D},\bar D[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]]\\
&= [\bar D^2\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] - [\alpha_{\mathfrak{k},D},\bar D[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]]\\
&= [\bar D^2\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] - [\alpha_{\mathfrak{k},D},[\bar D\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]] - [\alpha_{\mathfrak{k},D},[\alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}]]\\
&= [\bar D^2\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] - [[\alpha_{\mathfrak{k},D},\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}],\alpha_{\mathfrak{p},D}] - [\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D},[\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}]].
\end{aligned}$$

Putting things together, we obtain

$$\begin{aligned}
\bar D\bar D\alpha_{\mathfrak{p},z} &= -[\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},z}] + [D(\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}),\alpha_{\mathfrak{p},D}] - [\bar D^2\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{p},D}] + [[\alpha_{\mathfrak{k},D},\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}],\alpha_{\mathfrak{p},D}]\\
&= -[\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},z}] + [D(\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}) - \bar D^2\alpha_{\mathfrak{k},D} + [\alpha_{\mathfrak{k},D},\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}],\alpha_{\mathfrak{p},D}].
\end{aligned}$$

We have

$$\bar D^2\alpha_{\mathfrak{k},D} = -(\bar D D\alpha_{\mathfrak{k},\bar D} + \bar D[\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{k},\bar D}] + \bar D[\alpha_{\mathfrak{p},D},\alpha_{\mathfrak{p},\bar D}]),$$

so we conclude

$$\begin{aligned}
D(\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}) - \bar D^2\alpha_{\mathfrak{k},D} + [\alpha_{\mathfrak{k},D},\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D}] &= D\alpha^2_{\mathfrak{k},\bar D} + [\bar D\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{k},\bar D}] + [\bar D\alpha_{\mathfrak{p},D},\alpha_{\mathfrak{p},\bar D}]\\
&\quad - [\alpha_{\mathfrak{p},D},\bar D\alpha_{\mathfrak{p},\bar D}] + [\alpha_{\mathfrak{k},D},\alpha^2_{\mathfrak{k},\bar D}]\\
&= [D\alpha_{\mathfrak{k},\bar D} + \bar D\alpha_{\mathfrak{k},D},\alpha_{\mathfrak{k},\bar D}] - [[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}],\alpha_{\mathfrak{p},\bar D}]\\
&\quad - [\alpha_{\mathfrak{p},D},\bar D\alpha_{\mathfrak{p},\bar D}] + [\alpha_{\mathfrak{k},D},\alpha^2_{\mathfrak{k},\bar D}]\\
&= -[[\alpha_{\mathfrak{p},D},\alpha_{\mathfrak{p},\bar D}],\alpha_{\mathfrak{k},\bar D}] - [[\alpha_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},D}],\alpha_{\mathfrak{p},\bar D}]\\
&\quad - [\alpha_{\mathfrak{p},D},\bar D\alpha_{\mathfrak{p},\bar D}]\\
&= [\alpha_{\mathfrak{p},D},\alpha_{\mathfrak{p},\bar z}].
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\bar D\bar D\alpha_{\mathfrak{p},z} &= -[\bar D\alpha_{\mathfrak{k},\bar D} + \alpha^2_{\mathfrak{k},\bar D},\alpha_{\mathfrak{p},z}] - [\alpha_{\mathfrak{p},\bar z},\alpha^2_{\mathfrak{p},D}]\\
&= [\alpha_{\mathfrak{k},\bar z},\alpha_{\mathfrak{p},z}] + [\alpha^2_{\mathfrak{p},\bar D},\alpha_{\mathfrak{p},z}] + [\alpha^2_{\mathfrak{p},D},\alpha_{\mathfrak{p},\bar z}].
\end{aligned}$$

Hence:

$$\bar{\partial}\alpha_{\mathfrak{p},z} = -[\alpha_{\mathfrak{k},\bar{z}}, \alpha_{\mathfrak{p},z}] - ([\alpha^2_{\mathfrak{p},\bar{D}}, \alpha_{\mathfrak{p},z}] + [\alpha^2_{\mathfrak{p},D}, \alpha_{\mathfrak{p},\bar{z}}]).$$

In particular, due to the second summand the underlying map needs not be harmonic.

6 Harmonic maps into $Gr_{k|l}(\mathbb{C}^{n|m})$ and the special case $\mathbb{C}P^{n|m}$

The aim of this section is to prove supersymmetric versions of by now classic results on harmonic maps into Grassmannians. There are in principle many ways to present the material. For our purposes, it is convenient to follow the exposition in [12]. For alternative approaches we refer at least to [16, 26]. Throughout, $\Sigma \to B$ denotes a connected super Riemann surface.

6.1 The Gauß transform

Let $f\colon \Sigma \to Gr_{k|l}(\mathbb{C}^{n|m})$ be harmonic. From now on we will tacitly identify f with the subbundle $f^*(\gamma) \subset \underline{\mathbb{C}^{n+1|m}}_\Sigma$ which it defines. The type decomposition of $TGr_{k|l}(\mathbb{C}^{n|m})_\mathbb{C}$ induces a decomposition of the complexified differential $df_\mathbb{C} = df^{(1,0)} + df^{(0,1)}$. In local superconformal coordinates and for a local section of the bundle f, we have that $df^{(1,0)}(D)$ is given by (cf. Section 2.4)

$$A_{f,f^\perp,D}\colon f \longrightarrow f^\perp,\ A_{f,f^\perp,D}(\rho) = \pi_{f^\perp} D\rho.$$

Similarly, $df^{(1,0)}(\bar{D})$ is given by

$$A_{f,f^\perp,\bar{D}}\colon f \longrightarrow f^\perp,\ A_{f,f^\perp,\bar{D}}(\rho) = \pi_{f^\perp} \bar{D}\rho.$$

More generally, following [12], a decomposition into orthogonal subbundles $\oplus_{i=1}^l \varphi_i = \underline{\mathbb{C}^{n|m}}_\Sigma$ leads in a local superconformal coordinate to the second fundamental forms

$$A_{\varphi_i,\varphi_j,D}\colon \varphi_i \longrightarrow \varphi_j,\ A_{\varphi_i,\varphi_j,D}(\rho) = \pi_{\varphi_j} D\rho,$$

$$A_{\varphi_i,\varphi_j,\bar{D}}\colon \varphi_i \longrightarrow \varphi_j,\ A_{\varphi_i,\varphi_j,\bar{D}}(\rho) = \pi_{\varphi_j} \bar{D}\rho.$$

The inclusion into the trivial bundle induces a hermitian metric $\langle -,- \rangle_{\varphi_i}$ and a compatible connection ∇^{φ_i} on each φ_i. Then we have

$$A_{\varphi_i,\varphi_j,D} = -A^*_{\varphi_j,\varphi_i\bar{D}}. \tag{6.1}$$

Remark 6.2. The second fundamental forms are actually sections of $\mathcal{D}^* \otimes \underline{\mathrm{Hom}}(\varphi_i, \varphi_j)$. For our purposes it is however always sufficient to work in local superconformal coordinates.

We have the following basic characterization of harmonic maps.

Lemma 6.3. *(a) The map f is holomorphic (resp. antiholomorphic) if and only if $A_{f,f^\perp,\bar{D}}$ (resp. $A_{f,f^\perp,D}$) vanishes.*

(b) The map f is harmonic if and only if

$$A_{f,f^\perp,D} \circ \nabla^f_{\bar{D}} = -\nabla^{f^\perp}_{\bar{D}} \circ A_{f,f^\perp,D},$$

i.e., $A_{f,f^\perp,D}$ is a holomorphic section of $\underline{\mathrm{Hom}}(f, f^\perp)$ equipped with its Koszul-Malgrange structure. Equivalently, $A_{f^\perp,f,\bar{D}}$ is an antiholomorphic section of $\underline{\mathrm{Hom}}(f^\perp, f)$. In particular, f is harmonic if and only if $f^\perp$ is harmonic.

Proof. The first part is a reformulation of Proposition 2.8. By definition, f is harmonic if and only if $\nabla^{LC}_{\bar{D}}(df_{\mathbb{C}}(\mathcal{D})) = 0$. Since $(T\mathbb{C}P^{n|m})^{(1,0)}$ and $(T\mathbb{C}P^{n|m})^{(0,1)}$ are parallel with respect to the Levi-Civita connection (cf. Section 2.4), this is equivalent to $\nabla^{LC}_{\bar{D}} df^{(1,0)}(D) = 0$ and $\nabla^{LC}_{\bar{D}} df^{(0,1)}(D) = 0$ and the latter is the complex conjugate of the former. The last claim follows from (6.1). □

The fundamental insight is that one can make use of the holomorphicity of $A_{f,f^\perp,D}$ to produce a new harmonic map from f. Since it is this case we are mainly interested in, we now specialize to $\mathbb{C}P^{n|m}$. (See also the remark below.) We assume that the zeros of $A_{f,f^\perp,D}$ are regular (cf. Section 3.5). In this case, due to holomorphicity, the zeros are isolated. Now we make use of Proposition 3.10 and obtain a blow up $\tilde{p}\colon \tilde{\Sigma} \to \Sigma$ and a line bundle $\mathcal{L}$ on $\tilde{\Sigma}$ such that $\tilde{p}^* A_{f,f^\perp,D}$ extends to a nowhere vanishing holomorphic section of

$$\mathcal{L} \otimes \tilde{p}^* \underline{\mathrm{Hom}}(f, f^\perp) = \mathcal{L} \otimes \underline{\mathrm{Hom}}(\tilde{f}, \tilde{f}^\perp),$$

where we set $\tilde{f} = f \circ \tilde{p}$. Hence, by means of the (non-holomorphic) inclusion

$$\tilde{f}^\perp \subset \underline{\mathbb{C}^{n+1|m}}_{\tilde{\Sigma}},$$

we obtain a nowhere vanishing odd inclusion of the line bundle $\mathcal{L}^* \otimes \tilde{f}$ into the trivial bundle. This defines a new map, the Gauß transform, $\tilde{f}_1 \colon \tilde{\Sigma} \to \mathbb{C}P^{m-1|n+1}$. Under similar assumptions we also obtain $\hat{f}_{-1} \colon \hat{\Sigma} \to \mathbb{C}P^{m-1|n+1}$ from $A_{f,f^\perp,\bar{D}}$, where $\hat{p}\colon \hat{\Sigma} \to \Sigma$ is a possibly different blow up and $\hat{f} = f \circ \hat{p}$.

Remark 6.4. The analogous construction, "filling out the zeros" of a holomorphic section of a bundle $\underline{\mathrm{Hom}}(E, F)$, is available in the ungraded setting for all Grassmannians [12, Prop. 2.2]. The proof makes use of the Plücker embedding to reduce the general case effectively to the above case. There is no Plücker embedding for the super Grassmannians (cf. [48]) and we do not know if in the graded setting such a generalization is feasible or not.

From this discussion, it is evident that we need to extend the notion of harmonic maps to parabolic super Riemann surfaces.

Definition 6.5. A map $g\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ is called harmonic if away from the degeneracy locus $\nabla^{LC}_{\bar{D}}(dg^{(1,0)}(D)) = 0$.

Remark 6.6. As concerns the theory of harmonic maps, the main difference between super Riemann surfaces and parabolic super Riemann surfaces is the non-existence of a Koszul-Malgrange structure on complex vector bundles with connection in the latter case. This is due to the fact that there is no associated partially flat connection along the degeneracy locus Z of the superconformal structure. For the same reason, a zero curvature representation is not available. However, these structures exist on the complement $\tilde{\Sigma} - Z$, which turns out to be enough for our purposes. The above definition is then equivalent to $g|_{\tilde{\Sigma}-Z}$ being harmonic.

We will now prove the central theorem, which is a supersymmetric generalization of the analogous statement in the non-graded case [22, 23, 26].

Theorem 6.7. *Let $f\colon \Sigma \to \mathbb{C}P^{n|m}$ be a harmonic map such that the zeros of $A_{f,f^\perp,D}$ and $A_{f,f^\perp,\bar{D}}$ are regular. Then the Gauß transforms $\tilde{f}_1$, $\hat{f}_{-1}$ exist on possibly different blow ups $\tilde{p}\colon \tilde{\Sigma} \to \Sigma$, and $\hat{p}\colon \hat{\Sigma} \to \Sigma$. They are harmonic and, moreover, $(\tilde{f}_1)_{-1}$ and $(\hat{f}_{-1})_1$ exist on $\tilde{\Sigma}$ resp. $\hat{\Sigma}$ and coincide with $\tilde{f} = f \circ \tilde{p}$ and $\hat{f} = f \circ \hat{p}$ respectively.*

Proof. The case $\hat{f}_{-1}$ being similar, we only prove that $\tilde{f}_1$ is harmonic. For this, we can work in a local superconformal coordinate away from the degeneracy locus. We can follow the reasoning in [12, Prop. 2.3]. Let R denote the orthogonal complement of $\tilde{f} \oplus \tilde{f}_1$. By the definition of $\tilde{f}_1$ and R, clearly $A_{\tilde{f},R,D} = 0$. This implies that $A_{\tilde{f},\tilde{f}^\perp,D}$ is holomorphic if and only if $A_{\tilde{f},\tilde{f}_1,D}$ is holomorphic. Now we show that $\tilde{f}$ being harmonic implies $A_{\tilde{f}_1,R,\bar{D}} = 0$. To see this, we pick a local trivializing section ρ of $\tilde{f}$ which is holomorphic in the Koszul-Malgrange structure. Then harmonicity implies $\nabla^{\tilde{f}^\perp}_{\bar{D}} A_{\tilde{f},\tilde{f}^\perp,D}(\rho) = 0$, which proves the desired equality since $A_{\tilde{f},\tilde{f}^\perp,D}(\rho)$ spans $\tilde{f}_1$ outside a discrete set. This implies that $A_{\tilde{f},\tilde{f}_1,D}$ is holomorphic if and only if $A_{\tilde{f}+R,\tilde{f}_1,D}$ is holomorphic, which is equivalent to $\tilde{f}_1$ being harmonic. The last statement follows from $A_{\tilde{f}_1,R,\bar{D}} = 0$ and the fact that for local trivializing sections $\tilde{\rho}_1$ of $\tilde{f}_1$ and $\tilde{\rho}$ of $\tilde{f}$:

$$\langle \tilde{\rho}, A_{\tilde{f}_1,\tilde{f},\bar{D}}(\tilde{\rho}_1)\rangle_{\tilde{f}} = -\langle A_{\tilde{f},\tilde{f}_1,D}(\tilde{\rho}), \tilde{\rho}_1\rangle_{\tilde{f}_1}.$$

□

6.2 Isotropic harmonic maps

We now define and study the class of full isotropic harmonic maps and arrive at an analogous result as in [26, Thm 6.9]. We show that, as in the non-graded setting, such are characterized by the vanishing of a series of holomorphic differentials. A genuine feature in the graded setting is that the two parameters $(n+1|m)$ for the co-domain $\mathbb{C}P^{n|m}$ are restricted by the property $|n+1-m| \leq 1$. Σ denotes a connected super Riemann surface and $\tilde{\Sigma}$ denotes a connected parabolic super Riemann surface with degeneracy locus Z. For the following it will be convenient to introduce a slightly different perspective on the harmonic map equation (cf. [26]). On $\mathbb{C}P^{n|m}$ we have the following exact sequence

$$0 \longrightarrow \gamma^* \otimes \gamma \longrightarrow \gamma^* \otimes \underline{\mathbb{C}^{n+1|m}} \xrightarrow{\pi} \gamma^* \otimes \gamma^\perp \longrightarrow 0.$$

The first bundle has a canonical section, the identity, which henceforth gives a canonical section Φ of $\gamma^* \otimes \underline{\mathbb{C}^{n+1|m}}$. For a map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$, we will freely identify $f^*\Phi$ and Φ.

We equip this bundle with the tensor product of the canonical and the flat connection, denoted by ∇^H. Then, if V is a section of $f^* \otimes \underline{\mathbb{C}^{n+1|m}}$ and ρ is local trivializing section of f, we have that

$$(\nabla^H_D V)(\rho) = DV(\rho) - (-1)^{|V|} V(\pi_\rho D\rho).$$

This connection is compatible with the hermitian metric defined for local sections F and G by

$$\langle F, G\rangle = \mathrm{str}(F^* G).$$

We start with a general observation:

Lemma 6.8. *Consider a smooth map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$. For any section V of $f^* \otimes \mathbb{C}^{n+1|m}$*

$$(\nabla^H_D \nabla^H_{\bar{D}} + \nabla^H_{\bar{D}} \nabla^H_D) V = \kappa_1 V.$$

Moreover, $\kappa_1 = -(|\nabla^H_D \Phi|^2 + |\nabla^H_{\bar{D}}\Phi|^2)$.

Proof. This follows from the fact that the curvature of the tensor product of connections is the difference of the curvatures of these connections and the flat connection has no curvature. The equality for κ_1 follows from a direct calculation. □

Lemma 6.9. *We have:*

(a) $\nabla^H_D \Phi$ is perpendicular to Φ and projects to $A_{f,f^\perp,D}$ under π.

(b) The map f is harmonic if and only if $\pi(\nabla^H_{\bar{D}} \nabla^H_D \Phi) = 0$. In fact, f is harmonic if and only if

$$\nabla^H_{\bar{D}} \nabla^H_D \Phi + \langle \nabla^H_D \Phi, \nabla^H_D \Phi\rangle \Phi = 0.$$

Proof. Part (a) follows from the local calculation $(\nabla^H_D f^*\Phi)(\rho) = \pi_{f^\perp} D\rho$. Part (b) is a reformulation of Lemma 6.3 (b). □

Definition 6.10. A smooth map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ is isotropic if in any local superconformal coordinate and for any local section ρ of f and all $k \geq 0$:

$$\langle \Phi(\rho), D^k \nabla_D^H \Phi(\rho) \rangle_{\mathbb{C}^{1+n|m}} = 0.$$

Equivalently, locally

$$\langle (\nabla_{\bar{D}}^H)^\alpha \Phi, (\nabla_D^H)^\beta \Phi \rangle = 0, \ \alpha, \beta \geq 1.$$

Definition 6.11. A map $\varphi\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ is full if, except for at a discrete set of points, we have

$$\operatorname{span}\{x^*(\nabla_{\bar{D}}^H)^k \Phi, x^*\Phi, x^*(\nabla_D^H)^l \Phi \mid k, \ l \geq 0\} = \mathbb{C}^{1+n|m},$$

where $x\colon \mathrm{pt} \to \tilde{\Sigma}$.

Remark 6.12. Our definition of fullness is strictly stronger than the convention used in [26]. Therein, a map is defined to be full if it does not factor through a strictly smaller complex projective space. Real analyticity of harmonic maps can be used to show that this notion of fullness implies ours. The converse is clear and holds in our situation as well. Thus, weakening our hypothesis would require to study the analytic properties of supersymmetric harmonic maps. This is not within the scope of this article.

We will now study the interplay between fullness, isotropy, and the harmonic map equation. We seek an analogue of [26, Thm. 6.9], however, in view of our slightly different setup, we cannot directly appeal to the results in [26].

Lemma 6.13. *Let $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ be a full isotropic and harmonic map such that $f_{\pm 1}$ exist on $\tilde{\Sigma}$. Then $f_{\pm 1}$ are full and isotropic.*

Proof. We indicate the line of argument in the case f_1. For details, we refer to [26, Prop. 5.9]. We will work in a local superconformal coordinate chart. In local coordinates we have $\Phi^{f_1}(\rho_1) = \nabla_D^H \Phi^f(\rho)$. Any of $(\nabla_D^{H_{f_1}})^k \Phi^{f_1}$ is a linear combination of $D^k \nabla_D^{H_f} \Phi^f(\rho)$, $k \geq 0$, while, in view of the second part of Theorem 6.7, any of $(\nabla_{\bar{D}}^{H_{f_1}})^k \Phi^{f_1}$ is a linear combination of $\bar{D}^k \Phi^f(\rho)$ and $\Phi^f(\rho)$. These are orthogonal in view of isotropy of f. This also implies that f_1 is full again. □

We call a full isotropic and harmonic map 1-regular if each of f_1 or f_{-1} either exists on $\tilde{\Sigma}$ or it is antiholomorphic respectively holomorphic. Notice that if $p\colon \hat{\Sigma} \to \tilde{\Sigma}$ is a blow up, disjoint from the degeneracy locus of the superconformal structure, and if $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ is a full isotropic map, then $f \circ p$ is full isotropic. We can now make the following definition:

Definition 6.14. (a) A full isotropic harmonic map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ has invertible ramification if all iterated Gauß transforms $f_{\pm r}$ are 1-regular.

(b) A full isotropic harmonic map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ has regular ramification if there exists a blow up $p\colon \hat{\Sigma} \to \tilde{\Sigma}$, disjoint from the degeneracy locus of the superconformal structure, such that $f \circ p$ has invertible ramification.

Remark 6.15. For $B = \mathrm{pt}$ all harmonic maps have regular ramification.

So, starting from a full isotropic harmonic map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ with invertible ramification, there are natural numbers $k = k(f)$ and $l = l(f)$, and the sequence of Gauß transforms takes the form

$$f_{-l}, \dots, f = f_0, f_1, \dots, f_k.$$

In view of Theorem 6.7 and Lemma 6.13 each constituent is full and isotropic. Moreover, each map has invertible ramification, which is also a consequence of the second part of Theorem 6.7 and f can be reconstructed from either f_{-l} or f_k. The maps f_{-l}, and f_k are holomorphic and anti-holomorphic respectively and by counting dimensions we see that $|n+1-m| \leq 1$. Thus we have proved the supersymmetric version of [26, Thm. 6.9]:

Theorem 6.16. *Consider $(n+1|m) = (n+1|n+1+\epsilon)$. For every $0 \leq r \leq n+1+m$, the assignment $f \mapsto f_r$ gives a bijective correspondence between full holomorphic maps $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ with invertible ramification and full isotropic harmonic maps $g\colon \Sigma \to M_r$ with invertible ramification such that $l(g) = r$. Here, $M_r = \mathbb{C}P^{n|m}$ if r is even and $M_r = \mathbb{C}P^{m-1|n+1}$ if r is odd. The inverse is given by $g \mapsto g_{-l(g)}$.*

Remark 6.17. In view of Corollary 6.20 below, this theorem applies to the genus 0 case. The theorem shows that any full harmonic map with regular ramification is related purely algebraically to a holomorphic map on a blow up. Some examples will be given in the next section.

We study now conditions under which these isotropy assumptions are satisfied. In local superconformal coordinates on some U, we set

$$(\eta_{\alpha,\beta})_U = \langle (\nabla^H_{\bar{D}})^\alpha \Phi, (\nabla^H_D)^\beta \Phi \rangle.$$

Lemma 6.18. *Let $f\colon \tilde{\Sigma} \to \mathbb{C}P^{n|m}$ be harmonic. We have that $(\eta_{0,1})_U = (\eta_{1,0})_U = 0$. Moreover, if $(\eta_{\alpha,\beta})_U = 0$ for all $1 \leq \alpha + \beta \leq r$ and all U, then $(\eta_{\alpha+1,\beta})_U$ and $(\eta_{\alpha,\beta+1})_U$ yield global holomorphic sections of $(\mathcal{D}^{\otimes(\alpha+\beta+1)})^{-1}$.*

Proof. The proof is the same as in the ungraded case [26, Lem. 7.2]. If the assumptions are satisfied, then $(\eta_{\alpha+1,\beta})_U$, $(\eta_{\alpha,\beta+1})_U$ have the correct transformation behaviour. Holomorphicity follows essentially from the impliciation of Lemma 6.8, that for $\alpha \geq 1$, we have

$$\nabla^H_{\bar{D}} (\nabla^H_D)^\alpha \Phi(\rho) \in \mathrm{span}\{ (\nabla^H_D)^l \Phi(\rho) \mid 0 \leq l \leq \alpha - 1 \},$$

and similarly for D and $\bar{D}$ interchanged. □

In view of this lemma, isotropy of a harmonic map is encoded in terms of holomorphic differentials. In particular, $\eta_{1,1}$, a quadratic holomorphic differential, is always defined.

Definition 6.19. A map is called weakly conformal if $\eta_{1,1} = 0$.

Corollary 6.20. *Any harmonic super sphere is isotropic.*

Proof. We have seen earlier that on a super sphere $\Gamma(\mathcal{D}^{-k}) = 0$ for all $k \geq 1$, hence Lemma 6.18 applies. □

We conclude with an interesting fact:

Lemma 6.21. *For any weakly conformal harmonic map $\mathbb{C}_B^{1|1} \to \mathbb{C}P^{n|m}$, the underlying map obtained by restriction along $\iota\colon \mathbb{C} \to \mathbb{C}_B^{1|1}$ is harmonic.*

Proof. For the proof we make use of the formula derived in Section 5.3.2. We choose a framing and then have

$$\alpha_{\mathfrak{p},D} = \begin{pmatrix} 0 & \tilde{v}^\dagger & \tilde{w}^\dagger \\ v & 0 & 0 \\ w & 0 & 0 \end{pmatrix},$$

where v and $\tilde{v}$ are odd and w and $\tilde{w}$ are even. Since the map is weakly conformal, we have

$$\alpha_{\mathfrak{p},D}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v\tilde{v}^\dagger & v\tilde{w}^\dagger \\ 0 & -w\tilde{v}^\dagger & w\tilde{w}^\dagger \end{pmatrix}.$$

From $\iota^* v = 0$ and $\iota^* \tilde{v} = 0$, we then obtain that the action of $\alpha_{\mathfrak{p},D}^2$ on $\alpha_{\mathfrak{p},\bar{z}}$ vanishes after restriction along ι. □

Remark 6.22. More generally, this argument shows that the underlying map of a conformal (i.e., such that the images of both second fundamental forms are orthogonal) supersymmetric harmonic map into a Grassmannian $Gr_{k|0}(\mathbb{C}^{n|m})$ is harmonic. However, this argument fails for Grassmannians of the form $Gr_{k|l}(\mathbb{C}^{n|m})$, k, $l \neq 0$. An explicit counterexample will be constructed in Section 7.4.4 below.

6.3 Examples for the Gauß transform

Holomorphic maps from a super sphere into $\mathbb{C}P^{n|m}$ can be written down explicitly in terms of polynomials, which makes it possible to give explicit examples of full isotropic harmonic maps. It is a more subtle question to determine the ramification type of such maps. A general discussion of this issue is not within the scope of this paper. Let $\Sigma = \mathbb{C}P^{1|1} \times B$ be the super sphere as constructed in Section 3.2.3. We will describe all maps in the chart U_1.

By isotropy and for dimensional reasons, all full harmonic maps $\Sigma \to \mathbb{C}P^{1|1}$ are holomorphic or antiholomorphic. In this case, we can explicitly compute the whole harmonic sequence:

$$f = (p \quad r \mid q), \; f_1 = ((p,u) \quad (r,u) \mid (q,u)),$$

$$f_2 = c.c.(-Dqr + qDr \quad Dqp - qDp \mid \frac{-i}{Dq}((r,p)^{D^2} - D(p,r))q + i(p,r)),$$

where *c.c.* denotes complex conjugation. Here,

$$u = \langle f, f \rangle, \; (a,b)^X = (Xa)b - (-1)^{|a||b|}(Xb)a,$$

and $(a,b) = (a,b)^D$. More generally, if we start from a holomorphic map

$$f \colon \Sigma \longrightarrow \mathbb{C}P^{n|m}, \; f = (p_0 \quad \dots \quad p_n \mid p_{1+n} \quad \dots \quad p_{n+m}),$$

then

$$f_1 = ((p_0,u) \quad \dots \quad (p_n,u) \mid (p_{1+n},u) \quad \dots \quad (p_{n+m},u)).$$

For instance, we can consider the holomorphic map defined by

$$f = (1 \quad \sqrt{\binom{n}{1}}z \quad \sqrt{\binom{n}{2}}z^2 \quad \dots \quad z^n \mid \vartheta \quad \vartheta\sqrt{\binom{n-1}{1}}z \quad \dots \quad \vartheta z^{n-1}).$$

This is a supersymmetric generalization of the Veronese curve. In this case, the successive derivatives give an ascending sequence of vector bundles

$$\mathrm{span}\{f\} \subset \mathrm{span}\{f, Df\} \subset \ldots \subset \mathrm{span}\{f, Df, D^2f, \ldots, D^{2n}f\} = \underline{\mathbb{C}^{n+1|n}}_\Sigma,$$

defined on all of Σ. So that this is an example of a full isotropic holomorphic map with invertible ramification. For instance, we have, up to an invertible factor,

$$f_1 = ((f_1)_0 \quad \dots \quad (f_1)_n \mid (f_1)_{1+n} \quad \dots \quad (f_1)_{1+n+n}),$$

$$(f_1)_k = \sqrt{\binom{n}{k}}z^{k-1}(\vartheta(-k + n\frac{|z|^2}{1+|z|^2}) + \bar{\vartheta}((-i)\frac{z}{1+|z|^2})), \; 0 \le k \le n,$$

$$(f_1)_{1+n+l} = \sqrt{\binom{n-1}{l}}z^l, \; 0 \le l \le n.$$

An example for regular but non-invertible ramification can be constructed as follows. We consider the holomorphic map into $\mathbb{C}P^{2|1}$ given by

$$f = (\sqrt{2}z \quad 1 \quad z^2 \mid \vartheta(z - P)),$$

where P is a purely nilpotent. We have

$$|f|^2 = (1+|z|^2)(1+|z|^2 - i\bar{\vartheta}\vartheta h_P),\ Df = (\sqrt{2}\vartheta \quad 0 \quad 2\vartheta z \mid z - P),$$

where $h_P = |z-P|^2/(1+|z|^2)$. Furthermore,

$$(f, Df) = (1+|z|^2)(2\vartheta\bar{z} + i\bar{\vartheta}h_P),\ \frac{(f,Df)}{(f,f)} = \vartheta\frac{2\bar{z}}{1+|z|^2} + \bar{\vartheta}\frac{ih_P}{1+|z|^2},$$

$$\begin{aligned} f_1 = (\sqrt{2}\vartheta \quad 0 \quad 2\vartheta z \mid z-P) - \\ \frac{1}{1+|z|^2}(\sqrt{2}z(2\bar{z}\vartheta + ih_P\bar{\vartheta}) \quad (2\vartheta\bar{z} + \bar{\vartheta}ih_P) \quad z^2(2\vartheta\bar{z} + \bar{\vartheta}ih_P) \mid i\bar{\vartheta}\vartheta h_P(z-P)). \end{aligned}$$

It is clear that the underlying map degenerates at $z=0$. In order to find the full ramification divisor, we need to express f_1 in terms of holomorphic sections of $f^*\underline{\mathrm{Hom}}(\gamma, \gamma^\perp)$. For this, we put

$$X_1 = (\sqrt{2}\bar{\vartheta}\vartheta z \quad \bar{\vartheta}\vartheta \quad \bar{\vartheta}\vartheta z^2 \mid 1).$$

Then

$$\bar{D}X_1 = \vartheta f,\ (f, X_1) = \bar{\vartheta}\vartheta(1+|z|^2)^2 + i\bar{\vartheta}(\bar{z} - \bar{P})$$

and

$$\frac{(f,X_1)}{(f,f)} = \bar{\vartheta}\vartheta + i\bar{\vartheta}\frac{\bar{z}-\bar{P}}{1+|z|^2}.$$

Then

$$\begin{aligned} Z_1 &= X_1 - \frac{(f,X_1)}{(f,f)}f \\ &= (-i\bar{\vartheta}\frac{(\bar{z}-\bar{P})\sqrt{2}z}{(1+|z|^2)^2} \quad -i\bar{\vartheta}\frac{\bar{z}-\bar{P}}{(1+|z|^2)^2} \quad -i\bar{\vartheta}\frac{(\bar{z}-\bar{P})z^2}{(1+|z|^2)^2} | 1 - i\bar{\vartheta}\vartheta\frac{|z-P|^2}{(1+|z|^2)^2}) \end{aligned}$$

is holomorphic. (Recall that this means $V \perp f$ and $\bar{D}V \sim f$.) We compute

$$\begin{aligned} f_1 - (z-P)Z_1 &= \vartheta(\sqrt{2}(1-2h) \quad -2\frac{\bar{z}}{1+|z|^2} \quad 2z(1-h) \mid 0) \\ &= \vartheta Y. \end{aligned}$$

We note that $Y \perp f$ and

$$\bar{D}Y = -2\bar{\vartheta}\frac{1}{(1+|z|^2)^2}(\sqrt{2}z \quad 1 \quad z^2 \mid 0).$$

For a solution ψ of

$$\bar{D}\psi = -2\bar{\vartheta}\vartheta\frac{z-P}{(1+|z|^2)^2},\ \vartheta\psi = 0,$$

we define $X_2 = (Y \mid \psi)$ and set

$$Z_2 = X_2 - \frac{(f, X_2)}{(f, f)} f.$$

Then Z_2 is holomorphic and since $\vartheta(f, X_2) = \vartheta\psi = 0$ we have

$$f_1 = (z - P)Z_1 + \vartheta Z_2.$$

One easily checks that f_1 does not have a zero on U_2, so that in this example the Gauß transform is defined on the blow up of the super sphere along the infinitesimal neighbourhood $(z - P, \vartheta)$.

6.4 $2(n+1)$-orthogonal non-isotropic harmonic maps in $\mathbb{C}P^{n|n+1}$

In the previous section, we studied harmonic maps for which all $\eta_{k,l}$ vanished. Such maps were related to holomorphic maps via the Gauß transform. We will now describe a class of tori for which $\eta_{k,l} = 0$ for $k + l \leq 2n + 1$, but $\eta_{1,2n+1} \neq 0$. The essential ingredient is that on a torus, $\eta_{1,2n+1}$ is always a globally defined holomorphic function. The key assumption for the following discussion is that this function be invertible. The classification scheme differs substantially from the methods employed previously. In the ungraded case this goes back to [6, 10].

6.4.1 The $2(n+1)$-symmetric space $PSU(n+1|n+1)/PST$

We set $T = U(1)^{\times 2(n+1)} \subset U(n+1|n+1)$. Then PST is a torus in $PSU(n+1|n+1)$. An element of $\mathfrak{pst}_{\mathbb{C}}$ is of the form

$$\operatorname{diag}(\sigma_0, \ldots \sigma_n, \sigma_{n+1}, \ldots, \sigma_{2n+1}) + \langle \mathrm{id} \rangle,$$

and the roots are of the form $\sigma_i - \sigma_j$. We set $\alpha_l = \sigma_{l+n+1} - \sigma_l$, $0 \leq l \leq n$, $\alpha_{n+l'} = \sigma_{l'} - \sigma_{n+l'}$, $1 \leq l' \leq n$ and $\alpha_{2n+1} = -\sum_{k=0}^{2n} \alpha_k$. Notice that $\sum_{l=0}^{n} \alpha_l = \sum_{l=n+1}^{2n+1} \alpha_l = 0$. Let E_l be the root vector corresponding to α_l with non-zero entry equal to 1. Then we define $B_\tau = \sum_k E_k$. For example, for $n = 2$, we have

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now we fix a $2(n+1)$th simple root of unity ω and set

$$\tau = \mathrm{diag}(1, \omega^2, \omega^4, \ldots, \omega^{2n}, \omega, \omega^3, \ldots, \omega^{2n+1}).$$

The adjoint action of τ defines an automorphism of order $2(n+1)$ and after complexification we obtain a decomposition into eigenspaces

$$\mathfrak{psl}(n+1|n+1) = \bigoplus_{i=0}^{2n+1} \mathcal{M}_i.$$

Here $\mathcal{M}_i$ corresponds to the eigenvalue ω^i. For instance, $\mathcal{M}_1$ is the sum of the root spaces of α_l, $0 \leq l \leq 2n+1$. In the case $n = 2$, $\mathcal{M}_1$ consists of matrices of the type

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \end{pmatrix}.$$

There are two $PST_{\mathbb{C}}$-invariant supersymmetric polynomials of degree $n+1$:

$$P_i \in \mathrm{Sym}^*_{n+1}(\Pi\mathcal{M}_1), \ \ i = 1, 2.$$

Writing an element in the form $\xi_i = \sum_k a_i^k E_k$, they are given by the following sums over the symmetric group Σ_{n+1} :

$$P_1(\xi_0, \ldots, \xi_n) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \prod_{k=0}^{n} a_k^{\sigma(k)},$$

$$P_2(\xi_0, \ldots, \xi_n) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \prod_{k=n+1}^{2(n+1)} a_k^{\sigma(k)}.$$

Moreover, we set $P_i(\xi) = P_i(\xi, \ldots, \xi)$ and $P(\xi) = P_1(\xi)P_2(\xi)$.

Remark 6.23. There are also two invariant polynomials on $\mathcal{M}_2$. However, for a commutator $[X, X]$, where $X \in \mathcal{M}_1$, these two coincide and are equal to P. In view of this, P_1 and P_2 are more elementary.

Definition 6.24. A map $\xi \colon T \to \Pi\mathcal{M}_1$ is called cyclic if $P(\xi)$ is invertible.

Lemma 6.25. *Consider a map* $\xi\colon T \to \Pi\mathcal{M}_1$. *Then there exists* $t\colon T \to PT_{\mathbb{C}}$ *such that*

$$\mathrm{Ad}(t)(B_\tau) = \xi$$

if and only if $P_1(\xi)P_2(\xi) = 1$. *In this case* $P_2(\xi) = P_1(\xi)^{-1} = \mathrm{Ber}(t)$.

Proof. The ansatz $t = \mathrm{diag}(1, \lambda_1, \ldots, \lambda_{2n+1})$ yields a unique solution in $PT_{\mathbb{C}}$. $\square$

6.4.2 Primitive maps

Now let Σ be a connected super Riemann surface.

Definition 6.26. A map $f\colon \Sigma \to PSU(n+1|n+1)/PST$ is called primitive if $df_{\mathbb{C}}|_{\mathcal{D}} \in \Gamma((\mathcal{D})^* \otimes [\mathcal{M}_1])$ and it is cyclic at one point.

A framing of $f\colon \Sigma \to PSU(n+1|n+1)/PST$ is a map $\tilde{f}\colon \Sigma \to PSU(n+1|n+1)$ such that

$$\begin{array}{ccc} & & PSU(n+1|n+1) \\ & \nearrow & \downarrow \\ \Sigma & \longrightarrow & PSU(n+1|n+1)/PST \end{array}$$

commutes. In this situation, we set $A = \tilde{f}^*\alpha$, where α denotes the Maurer-Cartan form. Then the primitivity of f is locally characterized by

$$A_D = A_{D,0} + A_{D,1},$$

where $A_{D,i}$ has values in $\mathcal{M}_i$ and $A_{D,1}$ is cyclic at one point. The Maurer-Cartan equation for A takes the form

$$\bar{D}A_{D,0} + DA_{\bar{D},0} + [A_{D,0}, A_{\bar{D},0}] + [A_{D,1}, A_{\bar{D},1}] = 0,$$

$$\bar{D}A_{D,1} + [A_{\bar{D},0}, A_{D,1}] = 0.$$

Similarly as in the case of harmonic maps (Section 5.1), these equation have a zero curvature formulation which also provides the link to Section 5.3.

Lemma 6.27. *A is flat if and only if A_λ determined by $A_{\lambda,D} = A_{D,0} + \lambda A_{D,1}$ is flat for all $\lambda \in S^1$.*

Proof. This follows immediately from the Maurer-Cartan equation. $\square$

A primitive map has two invariants.

Lemma 6.28. *Given a primitive map $f\colon \Sigma \to PSU(n+1|n+1)/PST$. Then*

$$P_i(f) := P_i(\Pi df_{\mathbb{C}}|_{\mathcal{D}}) \in \Gamma(((\Pi\mathcal{D})^{\otimes(n+1)})^*)_{\bar{0}},\ i \in \{1,2\}$$

are holomorphic sections. In particular, $P_i(f)$ is invertible except for at a discrete set of points.

Proof. The invariant is constructed using functoriality and that $\mathrm{Sym}_{n+1}(\Pi\mathcal{D}) \cong (\Pi\mathcal{D})^{\otimes(n+1)}$. Holmorphicity can be checked locally. So we may suppose that f admits a framing $\tilde{f}$. We then find

$$\begin{aligned}\bar{D}P_i(f) &= \bar{D}P_i(A_{D,1})\\ &= (n+1)P_i(\bar{D}A_{D,1}, A_{D,1}, \ldots, A_{D,1})\\ &= (n+1)P_i(-[A_{\bar{D},0}, A_{D,1}], A_{D,1}, \ldots, A_{D,1})\\ &= 0.\end{aligned}$$

□

The last ingredient we need is the notion of a Toda frame [6]. For this, we will make use of a decomposition $\mathfrak{pt}_{\mathbb{C}} = \mathfrak{pst}_{\mathbb{C}} \oplus \langle M\rangle$, where $\mathrm{str}(M) = 1$. We assume that $P(f)$ is invertible at a point and hence in a coordinate neighbourhood U of that point. We notice that the invariant $P(f)$ is not invariant under change of superconformal coordinates in the domain, however the ratio $P_1(f)/P_2(f)$ is invariant. The latter is defined uniquely by requiring $P_1(f) = (P_1(f)/P_2(f))P_2(f)$ since both $P_i(f)$ are invertible by assumption. A framing $\tilde{f}$ into $PSU(n+1|n+1)$ defined on $U \to \Sigma$ is called a Toda frame if there exists a superconformal isomorphism $a\colon U \to U$ such that $P(a^*f) = 1$ and there exists a map $\Omega = \tilde{\Omega} + \chi M\colon U \to i\mathfrak{pst} \oplus \langle M\rangle$ such that

$$a^*A_D = D\tilde{\Omega} + \mathrm{Ad}(\exp(\Omega))(B_\tau). \tag{6.29}$$

It is worth mentioning that, using Lemma 6.25, $P_1(f)/P_2(f) = P_1(a^*f)/P_2(a^*f) = P_2(a^*f)^{-2} = \exp(\mathrm{str}(\Omega))^{-2}$ is holomorphic, so that χ is holomorphic.

The Maurer-Cartan equation for a^*A gives rise to the analogue of the affine Toda field equation [6, Equ. (2.12)]:

$$2\bar{D}D\Omega - i\left(\sum_{k=0}^{n} e^{2(\alpha_k(\tilde{\Omega})+\mathrm{Re}(\alpha_k(\chi M)))}(-\alpha_k^\sharp) + \sum_{k=n+1}^{2n+1} e^{2(\alpha_k(\tilde{\Omega})+\mathrm{Re}(\alpha_k(\chi M)))}\alpha_k^\sharp\right) = 0, \tag{6.30}$$

where $(-)^\sharp\colon \mathfrak{pst}_{\mathbb{C}}^* \to \mathfrak{pst}_{\mathbb{C}}$ is the isomorphism induced by the super trace and $\mathrm{Re}(-)$ denotes the real part. The main structural difference comes from the contributions of χ, which are not present in the ungraded case.

The theorem below is an adaption of [6, Thm. 2.5].

Theorem 6.31. *Let $f\colon \Sigma \to PSU(n{+}1|n{+}1)/PST$ be primitive and assume that f is cyclic at a point p_0. There exists a Toda frame in some neighbourhood of p_0. The superconformal isomorphism $a\colon U \to U$ is unique up to superconformal translation and rotation by a $2(n+1)$th root of unity and the Toda frame is unique for fixed such a.*

Proof. On a small coordinate neighbourhood U of p_0, f is cyclic and we can choose any framing $\tilde{f}\colon U \to PSU(n{+}1|n{+}1)$. Since $P(f)$ is holomorphic and invertible and U is simply connected we can change superconformal coordinates $a\colon U \to U$ such that $P(a^*f) = 1$. This coordinate transformation is unique up to superconformal rotation by a $2(n+1)$th root of unity and translation (Section 3.2.1). Since U is simply connected, by Lemma 6.25 we can moreover find $\eta\colon U \to \mathfrak{pt}_{\mathbb{C}}$ such that

$$a^*A_{D,1} = \mathrm{Ad}(\exp(\eta))(B_\tau).$$

If we write $\eta = \tilde{\eta} + \chi M$, we can decompose $\tilde{\eta} = \tilde{\Omega} + \tilde{\Lambda}$ such that $\overline{\tilde{\Lambda}} = \tilde{\Lambda}$, $\overline{\tilde{\Omega}} = -\tilde{\Omega}$. Then $\tilde{f}\exp((a^{-1})^*\tilde{\Lambda})$ is the desired Toda frame. We have

$$-[a^*A_{\bar{D},0}, a^*A_{D,1}] = \bar{D}a^*A_{D,1} = [\bar{D}\Omega, a^*A_{D,1}],$$

which implies $\bar{D}\tilde{\Omega} = \bar{D}\Omega = -a^*A_{\bar{D},0}$ and complex conjugation gives $D\tilde{\Omega} = a^*A_{D,0}$. If $\tilde{f}_1$, $\tilde{f}_2$ are two such framings, then, putting $A^i = \tilde{f}_i^*\alpha$, we have

$$a^*A^i_D = D\tilde{\Omega}_i + \mathrm{Ad}(\exp(\Omega_i))(B_\tau)$$

for suitable $\Omega_i = \tilde{\Omega}_i + \chi_i M$ as before. Moreover, we have $\tilde{f}_2 = \tilde{f}_1\exp(\Lambda)$ for some $\mathfrak{pst}$-valued function Λ. Hence, we necessarily have $\exp(\chi_1) = \exp(\chi_2)$ and

$$D\tilde{\Omega}_2 + \mathrm{Ad}(\exp(\Omega_2))(B_\tau) = D\tilde{\Omega}_1 + Da^*\Lambda + \mathrm{Ad}(\exp(\Omega_1 - a^*\Lambda))(B_\tau).$$

This implies $\Omega_1 = \Omega_2$ and $\exp(a^*\Lambda)$ is central. Since the center of $PSU(n+1|n+1)$ is trivial, we therefore have $\tilde{f}_1 = \tilde{f}_2$. □

Corollary 6.32. *Let Σ be a super torus and consider a primitive map $\Sigma \to PSU(n+1|n+1)/PST$. Then there exists a Toda frame on the universal covering $\mathbb{C}^{1|1}_B \to \Sigma$ and a superconformal isomorphism $a\colon \mathbb{C}^{1|1}_B \to \mathbb{C}^{1|1}_B$, unique up to superconformal rotation by a $2(n+1)$th root of unity and translation, such that (6.29) and (6.30) hold. Moreover, the Toda frame and Ω factor through some finite covering $\Sigma' \to \Sigma$.*

Proof. We let $\mathbb{C}^{1|1}_B \to \Sigma$ denote the universal covering and choose generators S and T of the group $\mathbb{Z} \oplus \mathbb{Z}$ defining the torus Σ (Section 3.2.4). Since $P(f)$ is a globally defined invertible holomorphic function on the torus, we can assume by Proposition 3.6 that $P(f) = 1$. On $\mathbb{C}_B^{1|1}$ we can proceed as in Theorem 6.31 to obtain the Toda frame. The pullback of the Maurer-Cartan form for any of $\tilde{f} \circ S^2$ and $\tilde{f} \circ T^2$ is again of the form (6.29). Hence as in the proof of the previous theorem, we have $\tilde{f} = \tilde{f} \circ S^2 = \tilde{f} \circ T^2$ and $\tilde{\Omega} = \tilde{\Omega} \circ S^2 = \tilde{\Omega} \circ T^2$ and similarly for $\exp(\chi)$ which implies the result. □

Remark 6.33. A similar ambiguity concerning the double periodicity of the Toda frame appears in the non-graded analogue [6, Cor. 2.7]. However, for a different reason, namely the nontrivial center of $SU(n)$. The center of $PSU(n+1|n+1)$ is trivial, and the ambiguity comes from the superconformal automorphisms defining the super torus. Moreover, in the non-graded case Ω always factors through the original torus, which we cannot conclude in our situation.

6.4.3 Primitive maps from pseudo-commuting flows

Now we relate the material from the previous section to a certain class of harmonic maps. We consider a full harmonic map $f\colon \Sigma \to \mathbb{C}P^{n|n+1}$ with invertible ramification, where Σ is a torus. Furthermore, we assume that the map is $2(n+1)$-orthogonal and is non-isotropic if $P(f)$ is invertible at one point (hence everywhere). In this situation, we have the harmonic sequence

$$f_{-l}, \ldots, f_0, \ldots, f_k,$$

which determines a map

$$f\colon \Sigma \longrightarrow PSU(n+1|n+1)/PST.$$

This map is primitive by the construction of the Gauß transform.

Proposition 6.34 ([6, Theorem 4.6])**.** *Conversely, any primitive map f into $PSU(n+1|n+1)/PST$ determines by projection onto $\mathbb{C}P^{n|n+1}$ a harmonic map which is $2(n+1)$-orthogonal and non-isotropic and whose Gauß transforms give back f.*

Proof. For a primitive map f into $PSU(n+1|n+1)/PST$ one can use the computation in Section 5.3.2 to show that any of the $2(n+1)$ projections onto $\mathbb{C}P^{n|n+1}$ is harmonic. The successive Gauß transforms give back f by construction. □

We will now show how the machinery of [6, 10] can be adapted to give a method to produce primitive maps such that the ratio $P_1(f)/P_2(f) = C$ is constant. Let $\Pi\mathcal{M}_1^1$ be the open submanifold of $\Pi\mathcal{M}_1$ given by such elements with real positive entries and $P_1(\xi) = P_2(\xi) = 1$. (Positivity here refers to positivity of the non-Grassmann-valued part.) Let $\mathcal{M}_2^1 \subset \mathcal{M}_2$ be the submanifold consisting of matrices with positive real entries such that both invariant polynomials equal unity. For a super vector space there is a map of supermanifolds $\Pi V \to V \otimes V$ which is given on T-valued points by $v \mapsto v \otimes v$. Composing with the Lie bracket gives the squaring map on $\Pi\mathcal{M}_1^1$.

Lemma 6.35. *Squaring defines an isomorphism* $\Pi\mathcal{M}_1^1 \to \mathcal{M}_2^1$.

Proof. If $\xi = \sum_{k=0}^{n} a_k E_k + \sum_{l=n+1}^{2n+1} b_l E_l$, then from ξ^2 one can reconstruct a_i/a_n, b_i/b_n. Since $P_1(\xi) = 1/P_2(\xi) = 1$ are fixed, this gives back a_i and b_i by positivity. The same argument shows injectivity. □

Remark 6.36. For the formulation of this lemma, the assumption that $P_1(f)/P_2(f)$ is constant is indispensable.

Lemma 6.37. *The following diagram commutes and all arrows are isomorphisms:*

$$\begin{array}{ccc} & & \Pi\mathcal{M}_1^1 \\ & \nearrow^{\mathrm{Ad}(\exp(-))(B_\tau)} & \\ i\mathfrak{pst}_{\mathbb{C}} & & \big\downarrow (-)^2 \\ & \searrow_{\mathrm{Ad}(\exp(-))(B_\tau^2)} & \\ & & \mathcal{M}_2^1 \end{array}$$

Proof. It is enough to show that the lower horizontal map is an isomorphism, this is similar to the proof of Lemma 6.25. □

Lemma 6.38. *The adjoint action of* B_τ *is injective on* $\mathcal{M}_0$.

Proof. It follows from the description of B_τ in terms of α_i that any element in the kernel is a multiple of the identity. □

Since the constant C is B-valued in general, we need to extend this in the following way. For a smooth supermanifold T we set $T(B) = \mathrm{Hom}(B, T)$, where $\mathrm{Hom}(-,-)$ denotes the internal hom object in supermanifolds. We take the splitting matrix $\mathfrak{pt}_{\mathbb{C}} = \mathfrak{pst}_{\mathbb{C}} \oplus \langle M \rangle$, $\mathrm{str}(M) = 1$, a suitable constant c such that $\mathrm{Ad}(\exp(cM))(B_\tau)$ has $P_1/P_2 = C$, and set

$$\Pi\mathcal{M}_1^C = \mathrm{Ad}(\exp(cM))(\Pi\mathcal{M}_1^1(B)).$$

We now take M to be of the form $M = \mathrm{diag}(1/(n+1)\, 1_{n+1}, 0_{n+1})$. Then $\mathrm{Ad}(\exp(cM))$ acts trivially on $\mathcal{M}_2^1$ and we have:

Lemma 6.39. *Squaring defines an isomorphism* $\Pi\mathcal{M}_1^C \to \mathcal{M}_2^1(B)$.

Proof. This follows from Lemma 6.35 by applying $\mathrm{Ad}(\exp(cM))$. □

Lemma 6.40. *The following diagram commutes and all arrows are isomorphisms:*

$$\begin{array}{ccc} & & \mathcal{M}_1^C \\ & \nearrow^{\mathrm{Ad}(\exp(-+cM))(B_\tau)} & \\ \mathfrak{ipst}_{\mathbb{C}}(B) & & \Big\downarrow (-)^2 \\ & \searrow_{\mathrm{Ad}(\exp(-))(B_\tau^2)} & \\ & & \mathcal{M}_2^1(B) \end{array}$$

Proof. This follows from Lemma 6.37. □

Let $d \in \mathbb{N}$, $d = 2 \pmod{2(n+1)}$. We denote by $\sqrt{-}\colon \mathcal{M}_2^1(B) \to \mathcal{M}_1^C$ the inverse map of squaring. We define

$$\Lambda_d = \{\sum_{i=-d}^{d} \lambda^i \xi_i \mid \bar{\xi_i} = \xi_{-i}\}.$$

On this space τ acts by $\tau \cdot \xi = \tau(\xi(\omega^{-1}-))$. Then we set

$$\Lambda_{d,\tau} = \{\xi \in \Lambda_d \mid \tau \cdot \xi = \xi\},\ \Lambda_d^+ = \{\xi \in \Lambda_d \mid \xi_d \in \mathcal{M}_2^1\},$$

$$\Lambda_d^*(B) = \{\xi \in \Lambda_d(B) \mid \xi_d \in \mathcal{M}_2^1(B),\ \xi_{d-1} \in \mathrm{im}(\mathrm{ad}(\sqrt{\xi_d})\},\ \Lambda_{d,\tau}^*(B) = \Lambda_d^*(B) \cap \Lambda_{d,\tau}(B).$$

Depending on the constant C, we now define a certain complex vector field. For this, we consider the assignment

$$Z\colon \Lambda_d^*(B) \longrightarrow \Lambda_d(B),\ Z\xi = [\xi, \tfrac{1}{2}\mathrm{ad}(\sqrt{\xi_d})^{-1}\xi_{d-1} + \lambda\sqrt{\xi_d}].$$

This is well-defined since $[\xi_d, \sqrt{\xi_d}] = 0$. Unlike in the non-graded case, this is not sufficient to show that this defines a vector field on $\Lambda_d^*(B)$. However, we can view Z as a vector field along the inclusion $\Lambda_d^*(B) \to \Lambda_d(B)$. In the following, we let $Z(-)$ denote the result of applying this vector field to a function. Recall that a vector field along the inclusion of a submanifold $M \to N$ can be applied to a function on N and returns a function on M. For instance, on $\Lambda_d^+(B)$ we have the function $\tilde{\Omega}$ defined by $\sqrt{\xi_d} = \mathrm{Ad}(\exp(\tilde{\Omega} + cM))(B_\tau)$. Then ξ_d and $\tilde{\Omega}$ will be considered as matrix-valued functions on $\Lambda_d(B)$ and $\Lambda_d^+(B)$ respectively.

Lemma 6.41. *We have that:*

(a) Z defines a vector field along the inclusion $\Lambda_d^*(B) \to \Lambda_d^+(B)$.

(b) As such it restricts to a vector field on $\Lambda_d^*(B)$ and $\Lambda_{d,\tau}^*(B)$.

Proof. We set $\tilde{\xi}_{d-1} := \mathrm{ad}(\sqrt{\xi_d})^{-1}(\xi_{d-1})$.

(a) : We need to check that the vector field acts tangentially on the top term. We have

$$(Z\xi)_d = 1/2[\tilde{\xi}_{d-1}, \xi_d],\ (\bar{Z}\xi)_d = -1/2[\bar{\tilde{\xi}}_{d-1}, \xi_d],$$

which means

$$Z(\xi_d) = 1/2[\tilde{\xi}_{d-1}, \xi_d], \bar{Z}(\xi_d) = -1/2[\bar{\tilde{\xi}}_{d-1}, \xi_d].$$

From this we conclude that this defines two real vector fields and both invariant polynomials of $\mathcal{M}_2(B)$ are preserved. Hence the result.

(b) : Using (a), the properties of a derivation, and the fact that the adjoint action of ξ_d is injective on $\mathcal{M}_0$, it follows now that

$$Z(\tilde{\Omega}) = 1/2\tilde{\xi}_{d-1},\ Z(\sqrt{\xi_d}) = 1/2[\tilde{\xi}_{d-1}, \sqrt{\xi_d}],\ \bar{Z}(\sqrt{\xi_d}) = -1/2[\bar{\tilde{\xi}}_{d-1}, \sqrt{\xi_d}].$$

Now we check that Z acts tangentially on the $(d-1)$st term. The condition on ξ_{d-1} for elements in $\Lambda_d^*(B)$ can be equivalently formulated as

$$P_i(\xi_{d-1}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) = 0.$$

We calculate

$$\begin{aligned} ZP_i(\xi_{d-1}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) &= P_i(Z\xi_{d-1}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) - nP_i(\xi_{d-1}, Z\sqrt{\xi_d}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) \\ &= P_i([\xi_{d-2}, \sqrt{\xi_d}], \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) \\ &\quad - nP_i(\xi_{d-1}, 1/2[\tilde{\xi}_{d-1}, \sqrt{\xi_d}], \sqrt{\xi_d}, \dots, \sqrt{\xi_d}]) \\ &= P_i([\xi_{d-2}, \sqrt{\xi_d}], \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) - n/2P_i(\xi_{d-1}, \xi_{d-1}, \sqrt{\xi_d}, \dots \sqrt{\xi_d}]. \end{aligned}$$

The first term vanishes by $\mathrm{ad}_{\mathfrak{pst}_{\mathbb{C}}}$-invariance. The second term vanishes since the P_i are supersymmetric. Moreover,

$$\begin{aligned} \bar{Z}P_i(\xi_{d-1}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) &= P_i(\bar{Z}\xi_{d-1}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) - nP_i(\xi_{d-1}, \bar{Z}\sqrt{\xi_d}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) \\ &= (-1/2)(P_i([\bar{\tilde{\xi}}_{d-1}, [\tilde{\xi}_{d-1}, \sqrt{\xi_d}]], \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) \\ &\qquad - nP_i(\xi_{d-1}, [\bar{\tilde{\xi}}_{d-1}, \sqrt{\xi_d}], \sqrt{\xi_d}, \dots \sqrt{\xi_d}])), \end{aligned}$$

where in the first line we dropped one of the first summands due to $\mathrm{ad}_{\mathfrak{pst}_{\mathbb{C}}}$-invariance. This sum vanishes as a result of $\mathrm{ad}_{\mathfrak{pst}_{\mathbb{C}}}$-invariance. In fact, it is a derivative of

$$P([\tilde{\xi}_{d-1}, \sqrt{\xi_d}], \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) = P(\xi_{d-1}, \sqrt{\xi_d}, \sqrt{\xi_d}, \dots, \sqrt{\xi_d}) = 0.$$

Finally, $\tau \cdot Z \cdot \tau^{-1} = Z$, hence the last statement. □

The following result is the supersymmetric version of [6, Section 3], [10, Thm. 2.1].

Theorem 6.42. *We have the following:*

(a) *We have* $[Z, \bar{Z}] = 0$ *and for every* $\xi_0 \in \Lambda_d^*(B)$ *there exists an open neighbourhood* $U \subset \mathbb{C}_B^{1|1}$ *of* 0 *and* $\xi \colon U \to \Lambda_d^*$ *uniquely specified by*

$$\xi(0) = \xi_0, \;\; D\xi^\sharp = \xi^\sharp \circ Z.$$

(b) *If* C *lies in* $\mathbb{C} \subset \Gamma(\mathcal{O}_B)$*, then any initial condition has a unique maximal flow defined on* $\mathbb{C}_B^{1|1}$.

(c) *Given a local flow* $\xi \colon U \to \Lambda_d^*$*, then*

$$A_{\lambda,D} = \frac{1}{2}\mathrm{ad}(\sqrt{\xi_d})^{-1}\xi_{d-1} + \lambda\sqrt{\xi_d}, \;\; \lambda \in S^1,$$

integrates to a unique loop of primitive maps

$$G_\lambda \colon U \longrightarrow PSU(n+1|n+1)/PST, \;\; G_\lambda(0) = \mathrm{id}.$$

Proof. We have seen in the previous proof that

$$\bar{Z}\sqrt{\xi_d} = -1/2[\bar{\tilde{\xi}}_{d-1}, \sqrt{\xi_d}].$$

We also have

$$\begin{aligned}[\bar{Z}\tilde{\xi}_{d-1}, \sqrt{\xi_d}] &= \bar{Z}\xi_{d-1} + [\tilde{\xi}_{d-1}, \bar{Z}\sqrt{\xi_d}] \\ &= -1/2[\bar{\tilde{\xi}}_{d-1}, [\tilde{\xi}_{d-1}, \sqrt{\xi_d}]] + [\xi_d, \overline{\sqrt{\xi_d}}] + [\tilde{\xi}_{d-1}, (-1/2)[\bar{\tilde{\xi}}_{d-1}, \sqrt{\xi_d}]],\end{aligned}$$

so that $\bar{Z}\tilde{\xi}_{d-1} = -[\overline{\sqrt{\xi_d}}, \sqrt{\xi_d}]$. Using this, we find

$$\begin{aligned}\bar{Z}Z(\xi) = \bar{Z}[\xi, 1/2\tilde{\xi}_{d-1} + \lambda\sqrt{\xi_d}] = [[\xi, 1/2\bar{\tilde{\xi}}_{d-1} + \lambda^{-1}\overline{\sqrt{\xi_d}}], 1/2\tilde{\xi}_{d-1} + \lambda\sqrt{\xi_d}] \\ + [\xi, -1/2[\overline{\sqrt{\xi_d}}, \sqrt{\xi_d}] + \lambda(-1/2)[\bar{\tilde{\xi}}_{d-1}, \sqrt{\xi_d}]]].\end{aligned}$$

The Jacobi identity implies now $(Z\bar{Z} + \bar{Z}Z)(\xi) = 0$. This is the integrability condition for a local $\mathbb{C}^{1|1}$ action, see [3, Thm. 1] and so (a) is proved. For (b), we note that the relevant vector field is already defined on Λ_d^* and it is sufficient to check that Z is complete there. We compute

$$Z^2\xi = [\xi, 1/2\xi_{d-2} + \lambda\xi_{d-1} + \lambda^2\xi_d],$$

which on the underlying purely even manifold takes the form

$$\tilde{Z^2}(\tilde{\xi}) = [\tilde{\xi}, 1/2\tilde{\xi}_{d-2} + \lambda^2\tilde{\xi}_d].$$

Now one uses the argument from [10, Proof of Thm. 2.1] to show that this vector field is complete: an ad-invariant inner product on the compact Lie algebra $\mathfrak{psu}(n+1|n+1)_{\bar{0}}$ induces on $(\Lambda_d)_{\bar{0}}$ the inner product

$$(\xi, \xi) = \sum_{i=0}^{d} (\xi_i, \xi_{-i}),$$

which is invariant under $\tilde{Z}^2$ and $\tilde{\tilde{Z}}^2$. Consequently, the flow is tangential to the spheres in $(\Lambda_d)_{\bar{0}}$ and thus is complete. This implies the assertion by [3, Thm. 2].

The flatness of the form determined by $A_{\lambda,D}$ follows from the above calculation so that (c) follows from Proposition 3.16. □

6.4.4 Finite type classification of $2(n+1)$-orthogonal non-isotropic harmonic tori

In the previous section we saw how primitive maps with constant $P_1(f)/P_2(f)$ can be obtained by integrating two pseudo-commuting vector fields. We will show now a partial converse for maps from a super torus. We consider a primitive map $f\colon \Sigma \to PSU(n+1|n+1)/PST$ from a super torus Σ with $P(f)$ invertible and constant $P_1(f)/P_2(f) = C$. Let $p\colon \mathbb{C}_B^{1|1} \to \Sigma$ denote the universal covering. In view of Corollary 6.32, we obtain from f a map g defined on a finite covering $\Sigma' \to \Sigma$ with universal cover $p'\colon \mathbb{C}_B^{1|1} \to \Sigma'$ and the following properties. We have $P(g) = 1$, there is a framing $\tilde{g}$ on $\mathbb{C}_B^{1|1}$ which factors through Σ' and such that (6.29) holds, and, moreover, such that Ω factors through Σ'. It follows from the construction in Corollary 6.32 that Σ' is always odd and we will use the trivilization of $\mathcal{D}$ induced by p' in the following.

Definition 6.43. The maps f and g are of finite type if there exists $d = 2 \pmod{2(n+1)}$ and a map $\xi\colon \Sigma' \to \Lambda_d^*$ such that $(p')^*\xi$ is a solution to the flow in Theorem 6.42 and

$$\tilde{g}^*\alpha_D = A_D = \frac{1}{2}\mathrm{ad}(\sqrt{(p')^*\xi_d})^{-1}(p^*\xi_{d-1}) + \sqrt{p^*\xi_d}.$$

The key for the next proposition is:

Lemma 6.44. *In this situation $A_{D,1}^2$ is semisimple in the sense that we have a bundle decomposition*

$$\underline{\mathfrak{sl}(n+1|n+1)}_{\mathbb{C}_B^{1|1}} = \ker(\mathrm{ad}(A_{D,1}^2)) \oplus \mathrm{im}(\mathrm{ad}(A_{D,1}^2)),$$

$$\underline{\mathfrak{psl}(n+1|n+1)}_{\mathbb{C}_B^{1|1}} = \ker(\mathrm{ad}(A_{D,1}^2)) \oplus \mathrm{im}(\mathrm{ad}(A_{D,1}^2)).$$

Proof. It is enough to check this for B_τ^2, which can be done directly. □

Proposition 6.45. *We put* $\mathcal{D} = d|_{\mathcal{D}\oplus\bar{\mathcal{D}}} + \mathrm{ad}(A_\lambda)$, *where* $A_{\lambda,D} = A_{D,0} + \lambda A_{D,1}$. *There is a formal series* $\xi = \sum_{i\le d} \lambda^i \xi_i$ *such that*

$$\xi_i\colon \mathbb{C}_B^{1|1} \to \mathfrak{psl}(n+1|n+1),\ \mathcal{D}\xi = 0,\ \xi_d = A_{D,1}^2,\ \xi_{d-1} = 2[A_{D,0}, A_{D,1}].$$

Moreover, the ξ_i *can be taken to factor through* Σ'.

Proof. We can follow the ideas of the proof in the non-graded case [10, Thm. 7.1]. Since one has to carefully distinguish between $A_{D,1}^2$ and $A_{D,1}$, which coincide in the non-graded setup, we provide the relevant details here. As a shorthand, we write $E = \underline{\mathfrak{sl}(n+1|n+1)}_{\mathbb{C}_B^{1|1}}$. To start with, we choose any lift of $\widetilde{A_{D,0}}$ to the maximal torus in $\mathfrak{sl}(n+1|n+1)$. This lift is unique up to a central element. Since the kernel of $\mathrm{ad}(A_{D,1}^2)$ restricted to this torus consists of central elements and due to the direct sum decomposition of Lemma 6.44, we may assume without loss of generality that $\widetilde{A_{D,0}}$ lies in $\mathrm{im}(A_{D,1}^2)$.

We notice that $d|_{\mathcal{D}\oplus\bar{\mathcal{D}}} A_{D,1}^2 + \mathrm{ad}(\widetilde{A_0})(A_{D,1}^2) = -\mathrm{ad}(Q)(A_{D,1}^2)$, where Q is given by $Q_D = -2\widetilde{A_{D,0}}$, and $Q_{\bar{D}} = 0$. Hence, if we set $D^\nabla = d|_{\mathcal{D}\oplus\bar{\mathcal{D}}} + \mathrm{ad}(\widetilde{A_0}) + \mathrm{ad}(Q)$, then $D^\nabla A_{D,1}^2 = 0$. This connection is independent of the choice of the lift of $A_{D,0}$ and $V := \ker(A_{D,1}^2)$, $V^\perp := \mathrm{im}(A_{D,1}^2)$ defines a direct sum decomposition of E into D^∇-parallel subbundles. Moreover, with respect to composition of matrices, we have

$$VV \subset V,\ V^\perp V \subset V^\perp,\ VV^\perp \subset V^\perp. \tag{6.46}$$

We make the following ansatz

$$\xi = (1+W)^{-1} A_{D,1}^2 (1+W),$$

where $W_i = \sum_{i\ge 1} \lambda^{-i} W_i$ and each W_i is a section of $V^\perp$. We need to solve

$$D_D^\nabla \xi = [\xi, -Q_D + \lambda A_{D,1}],\ D_{\bar{D}}^\nabla \xi = [\xi, -Q_{\bar{D}} + \lambda^{-1} A_{\bar{D},1}],$$

and, since

$$D^\nabla \xi = [\xi, (1+W)^{-1} D^\nabla W],$$

this is equivalent to

$$D_D^\nabla W (1+W)^{-1} - (1+W)(-Q_D + \lambda A_{D,1})(1+W)^{-1} = \omega_D \in \ker(A_{D,1}^2),$$

$$D_{\bar{D}}^\nabla W (1+W)^{-1} - (1+W)(-Q_{\bar{D}} + \lambda^{-1} A_{\bar{D},1})(1+W)^{-1} = \omega_{\bar{D}} \in \ker(A_{D,1}^2).$$

We solve this first for the D-direction. The equation is equivalent to

$$D_D^\nabla W - (1+W)(-Q_D + \lambda A_{D,1}) = \omega_D (1+W).$$

Splitting this equation according to $E = V \oplus V^\perp$ gives in view of (6.46)

$$(WQ_D)^V - \lambda A_{D,1} = \omega_D,$$

$$D_D^\nabla W + Q_D + (WQ_D)^\perp - \lambda W A_{D,1} = \omega_D W,$$

and hence we need to solve

$$\lambda[A_{D,1}, W] = (WQ_D)^V W - (WQ_D)^\perp - Q_D - D_D^\nabla W.$$

The first equation is

$$[A_{D,1}, W_1] = 2Q_D,$$

where W_1 is in $V^\perp$. By assumption, we have U, such that $[A_{D,1}^2, U] = 2Q_D$. Then we may take $W_1 = [A_{D,1}, U^\perp]$. The remaining W_i can be constructed inductively now, since $\mathrm{ad}(A_{D,1})$ is an isomorphism on $V^\perp$. Thus, we have found ξ such that $\mathcal{D}_D\xi = 0$. We claim that $\mathcal{D}_{\bar{D}}\xi = 0$. To see this, we set $\mathcal{D}_{\bar{D}}\xi = (1+W)^{-1}\sigma(1+W)$ and using $\mathcal{D}_D\mathcal{D}_{\bar{D}} = -\mathcal{D}_{\bar{D}}\mathcal{D}_D$, we find

$$\begin{aligned} D_D^\nabla \sigma &= (1+W)D_D^\nabla \mathcal{D}_{\bar{D}}\xi(1+W)^{-1} + [D_D^\nabla(1+W)^{-1}, \sigma] \\ &= (1+W)(\mathcal{D}_D + \mathrm{ad}(Q_D) - \mathrm{ad}(\lambda A_{D,1}))\mathcal{D}_{\bar{D}}\xi(1+W)^{-1} + [D_D^\nabla(1+W)^{-1}, \sigma] \\ &= [(1+W)(Q_D - \lambda A_{D,1})(1+W)^{-1} + D_D^\nabla(1+W)^{-1}, \sigma] \\ &= [\omega_D, \sigma]. \end{aligned} \tag{6.47}$$

Moreover, from $A_{D,1}^2 = (1+W)\xi(1+W)^{-1}$ one obtains that all components of σ are sections of $V^\perp$. Now, as in [10, Lem. 7.3], one can conclude $\sigma = 0$. Indeed, in view of (6.47), the summand $\lambda A_{D,1}$ in ω_D causes a potential first non-trivial coefficient of σ to lie in the kernel of $\mathrm{ad}(A_{D,1})$, which is a contradiction. The formal Killing field $\lambda^d\xi$ satisfies now all requirements. □

Any such formal series is called an adapted formal Killing field. We can average $1/2(n+1)(\sum_{k=0}^{2n+1}\tau^k)(\xi)$ and thus obtain a τ-invariant adapted formal Killing, which factors through a super torus. We now show that this implies that there is a τ-invariant adapted polynomial Killing field, i.e., a formal sequence which is bounded from below. We follow the ideas in [32, Section 25 II], but have to make some adjustments on the way. We consider a formal adapted Killing field which is τ-invariant $Y = \sum_{i\le 2} Y_i\lambda^i$. We can always assume this form, since $\lambda^{l(2(n+1))}Y$ is again a τ-invariant adapted formal Killing field for any $l \ge 0$. The top power of λ of a formal adapted Killing field is called the degree of Y. The equations satisfied by Y are

$$DY = [Y, A_{D,0} + \lambda A_{D,1}],\ \bar{D}Y = [Y, A_{\bar{D},0} + \lambda^{-1}A_{\bar{D},1}].$$

We set

$$W' = DY_{\geq 0} - [Y_{\geq 0}, A_{D,0} + \lambda A_{D,1}], \; W'' = \bar{D}Y_{\geq 0} - [Y_{\geq 0}, A_{\bar{D},0} + \lambda^{-1}A_{\bar{D},1}],$$

and notice that

$$DY_{\geq 0} - [Y_{\geq 0}, A_{D,0} + \lambda A_{D,1}] = -DY_{<0} + [Y_{<0}, A_{D,0} + \lambda A_{D,1}].$$

Hence W' can only have a constant term: $W' = DY_0 - [Y_0, A_{D,0}] = DY_0$. Likewise,

$$\bar{D}Y_{\geq 0} - [Y_{\geq 0}, A_{\bar{D},0} + \lambda^{-1}A_{\bar{D},1}] = -\bar{D}Y_{<0} + [Y_{<0}, A_{\bar{D},0} + \lambda^{-1}A_{\bar{D},1}], \; W'' = -\lambda^{-1}[Y_0, A_{\bar{D},1}].$$

The goal is now to construct a τ-invariant adapted Killing field such that W' and W'' vanish. Thus, this can be accomplished by constructing a τ-invariant adapted Killing field such that Y_0 vanishes. Evaluating the λ^0-coefficient of $\bar{D}DY$ shows that

$$\bar{D}DY_0 = [[Y_0, A_{\bar{D},1}], A_{D,1}].$$

As a shorthand we write this equation as $\bar{D}DY_0 = LY_0$. Shifting by $\lambda^{l(2(n+1))}$ shows that any $Y_{-l(2(n+1))}$, $l \geq 0$, is a solution of this equation.

Remark 6.48. In the non-graded setup one makes now use of the fact that the analogous equation for the coefficient of λ^0 is an elliptic equation on a torus and hence a finite linear combination of $\lambda^{l(2(n+1))}Y$ satisfies $W' = W'' = 0$. It is true that a family of doubly periodic solutions Y_0^k, $k \in \mathbb{N}$, of $\bar{D}DY_0^k = LY_0^k$ necessarily satisfies a non-trivial relation. This follows from similar principles as we will encounter shortly. However, since we are working over the basis B, when applied to the family $\lambda^{l(2(n+1))}Y$ this would not necessarily give an adapted Killing field. Still, an extension of this idea applies in the present situation.

We first assume that Σ' is split. Filtering the Grassmann algebra by choosing a basis indexed by multi-indices I, we have:

$$\bar{D}DY_{0,<|I|} = (L_{<|I|}Y_{0,<|I|})_{<|I|},$$

$$\bar{D}DY_I = (L_{<|I|}Y_{0,<|I|})_I + L_I Y_{0,\varnothing} + L_\varnothing Y_{0,I}.$$

Assuming for a moment that we know that $\bar{D}D - L_\varnothing$ has finite-dimensional kernel, we can do an induction on $|I|$. The hypothesis reads:

For every $l \geq 0$ there are infinitely many τ-invariant adapted formal Killing fields $X^{l,k}$ of different degrees such that $X^{l,k}_{(-l)(2(n+1)),<|I|} = 0$.

For $|I| = 0$ this holds, since we can take the family $\lambda^{l(2(n+1))}Y$. For $|I|$ large enough, $X^{0,k}$ is a τ-invariant adapted polynomial Killing field. For the inductive step, we choose any $l \geq 0$. We consider an index I_1 with $|I_1| = |I|$. Then by the above consideration, the $X^{l,k}_{(-l)(2(n+1)),I_1}$ satisfy a relation and we can arrange to obtain a new τ-invariant polynomial Killing field $\tilde{X}^l$, such that $\tilde{X}^l_{(-l)(2(n+1)),<|I|} = \tilde{X}^l_{(-l)(2(n+1)),I_1} = 0$. Since l was arbitrary and we can shift τ-invariant adapted Killing fields, we see that we obtain from this that for each $l \geq 0$ there are infinitely many τ-invariant adapted formal Killing fields X^k of different degrees such that $X^k_{(-l)(2(n+1)),<|I|} = X^k_{(-l)(2(n+1)),I_1} = 0$. Repeating this for the finitely many other multi-indices such that $|I_j| = |I|$ finishes the induction.

We now argue that $\bar{D}D - L_\varnothing$ has finite-dimensional kernel. If A^i denote the components of the $\mathcal{M}_0$-valued function A, then the above equation is of the form

$$\bar{D}D(A^i) = \sum_k L_{i,k}(D, \bar{D})A^k.$$

Here, due to the special form of L, in our situation each $L_{i,k}$ has the form $(L_{i,k})_0 + \vartheta\bar{\vartheta}(L_{i,k})_{\vartheta\bar{\vartheta}}$ and A takes the form $A^i = A^i_0 + \vartheta\bar{\vartheta}A^i_{\vartheta\bar{\vartheta}}$, if even, and $A^i = \vartheta A^i_{\vartheta} + \bar{\vartheta}A^i_{\bar{\vartheta}}$, if odd. Writing out the components shows that in each case we have an elliptic operator on some trivial vector bundle. Hence double periodicity implies that the kernel is finite-dimensional.

In the case of a non-split super torus $\delta \neq 0$ we cannot directly argue like this, however, we can circumvent this problem with the following manoeuvre. We know that there is a smooth isomorphism (cf. Section 3.2.4) $\Sigma_{(\tau,0)} \to \Sigma_{(\tau,\delta)}$ over B such that D takes the form

$$D \mapsto X = D + N$$

on the left hand side, where N vanishes after setting $B = \mathrm{pt}$. Now we study the same problem as before, with D replaced by X. In the expansion in auxiliary variables all entries are now doubly periodic and the nilpotent part of X does not change the underlying homogeneous equation. In particular, we can then apply the ellipticity argument as before.

Hence, we obtain a τ-invariant complex polynomial solution

$$\tilde{\xi} = \sum_{k=0}^{d} \lambda^k \xi_k, \ \tilde{\xi}_d = A_{D,1}, \ \tilde{\xi}_{d-1} = 2[A_{D,0}, A_{D,1}],$$

where $d = 2 \pmod{2(n+1)}$ and then

$$\eta = (1/(2n+2)\textstyle\sum_k \tau^k)(\xi + \bar{\xi}) \colon \mathbb{C}^{1|1}_B \longrightarrow \Lambda^*_{d,\tau}$$

is a solution of the flow from Theorem 6.42. Moreover, it factors through Σ'. We thus have proved the main theorem of this section.

Theorem 6.49. *Every primitive map $\Sigma \to PSU(n+1|n+1)/PST$ from a super torus such that $P_1(f)/P_2(f)$ is constant is of finite type.*

Remark 6.50. The condition that $P_1(f)/P_2(f)$ be constant is vacuous for even spin structure tori.

6.4.5 Example: The case $n = 1$

In the following we shall calculate the zero curvature equation in the case of primitive maps into $PSU(2|2)/PST$ with $P = 1$ and $P_1 = 1/P_2 = i$ and interpret the result as a supersymmetric generalization of the sinh-Gordon equation.

Remark 6.51. At this point one should emphasize that a naive supersymmetric generalization of the form $\bar{D}Du = 2\lambda\cosh(u)$ or $\bar{D}Du = 2\lambda\sinh(u))$ does not reduce to the ordinary sinh-Gordon equation, but a sinh-Gordon equation with "wrong sign".

The construction of the Toda frame as discussed in Section 6.4.2 can be carried out explicitly in terms of the Gauß transforms. This works as in the ungraded case and also for general n (cf. [6, Section 4]). Locally we can choose holomorphic sections of f_i (endowed with the Koszul-Malgrange structure) and by abuse of notation we denote these by f_i as well. Together with the harmonic map equation this leads to

$$f_1 = Df_0 - D\log|f_0|^2 f_0, \quad \bar{D}f_1 = -\frac{|f_1|^2}{|f_0|^2} f_0$$

and in general

$$f_{p+1} = Df_p - D\log|f_p|^2 f_p, \quad \bar{D}f_p = (-1)^{|f_p|}\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}.$$

The compatibility equation for this system reads

$$\bar{D}D\log|f_p|^2 + (-1)^{|f_{p+1}|}\frac{|f_{p+1}|^2}{|f_p|^2} = -(-1)^{|f_p|}\frac{|f_p|^2}{|f_{p-1}|^2}.$$

We set $\omega_p = \log|f_p|$ and can rewrite this equation in the form

$$2\bar{D}D\omega_p = -(-1)^{|f_p|}(e^{2(\omega_p - \omega_{p-1})} - e^{2(\omega_{p+1}-\omega_p)}).$$

(Here, we choose, once and for all, a fixed $(1/2)\log(i)$.) In other words we have the equations:

$$\omega_0 + \omega_2 = \omega_1 + \omega_3 - \log(i),$$

$$2\bar{D}D\omega_0 = -(e^{2(\omega_0-\omega_3)} - e^{2(\omega_1-\omega_0)}),$$

$$2\bar{D}D\omega_1 = (e^{2(\omega_1-\omega_0)} - e^{2(\omega_2-\omega_1)}),$$

$$2\bar{D}D\omega_2 = -(e^{2(\omega_2-\omega_1)} - e^{2(\omega_3-\omega_2)}),$$

$$2\bar{D}D\omega_3 = (e^{2(\omega_3-\omega_2)} - e^{2(\omega_0-\omega_3)}).$$

Example 6.52. A particular solution is given by

$$(\omega_0, \omega_1, \omega_2, \omega_3) = (\log(1), (1/2)\log(i), \log(1), (1/2)\log(i)).$$

We note that we have

$$\begin{aligned}
2\bar{D}D(\omega_0 - \omega_2) &= -e^{2(\omega_0-\omega_3)} + e^{2(\omega_1-\omega_0)} + e^{2(\omega_2-\omega_1)} - e^{2(\omega_3-\omega_2)} \\
&= e^{2(\omega_1-\omega_2)} + e^{2(\omega_1-\omega_0)} + e^{2(\omega_2-\omega_1)} + e^{2(\omega_0-\omega_1)} \\
&= 2\text{cosh}(2(\omega_2 - \omega_1)) + 2\text{cosh}(2(\omega_1 - \omega_0)) \\
&= 4\text{cosh}(\omega_2 - \omega_0)\text{cosh}(\omega_2 + \omega_0 - \omega_1 - \omega_1) \\
&= -4i\text{cosh}(\omega_2 - \omega_0)\text{sinh}(\omega_3 - \omega_1) \\
&= (4i)\text{cosh}(\omega_0 - \omega_2)\text{sinh}(\omega_1 - \omega_3),
\end{aligned}$$

and similarly

$$\begin{aligned}
2\bar{D}D(\omega_1 - \omega_3) &= e^{2(\omega_1-\omega_0)} - e^{2(\omega_2-\omega_1)} - e^{2(\omega_3-\omega_2)} + e^{2(\omega_0-\omega_3)} \\
&= e^{2(\omega_1-\omega_0)} - e^{2(\omega_2-\omega_1)} + e^{2(\omega_0-\omega_1)} - e^{2(\omega_1-\omega_2)} \\
&= 2\text{cosh}(2(\omega_1 - \omega_0)) - 2\text{cosh}(2(\omega_2 - \omega_1)) \\
&= 4\text{sinh}(\omega_2 - \omega_0)\text{sinh}(\omega_1 - \omega_0 + \omega_1 - \omega_2) \\
&= (-4i)\text{sinh}(\omega_0 - \omega_2)\text{cosh}(\omega_1 - \omega_3).
\end{aligned}$$

Finally,

$$\begin{aligned}
\bar{D}D(\omega_0 + \omega_2) &= \bar{D}D(\omega_1 + \omega_3) \\
&= -e^{2(\omega_0-\omega_3)} + e^{2(\omega_1-\omega_0)} - e^{2(\omega_2-\omega_1)} + e^{2(\omega_3-\omega_2)} \\
&= e^{2(\omega_1-\omega_2)} - e^{2(\omega_2-\omega_1)} + e^{2(\omega_1-\omega_0)} - e^{2(\omega_0-\omega_1)} \\
&= 2\text{sinh}(2(\omega_1 - \omega_2)) + 2\text{sinh}(2(\omega_1 - \omega_0)) \\
&= 4\text{sinh}(\omega_1 + \omega_1 - \omega_2 - \omega_0)\text{cosh}(\omega_0 - \omega_2) \\
&= (4i)\text{cosh}(\omega_1 - \omega_3)\text{cosh}(\omega_0 - \omega_2).
\end{aligned}$$

So putting $f = \omega_0 - \omega_2$, $g = \omega_1 - \omega_3$, and $h = \omega_0 + \omega_2$, we obtain the equivalent system

$$\begin{aligned}
\bar{D}Df &= (2i)\text{cosh}(f)\text{sinh}(g), \\
\bar{D}Dg &= (-2i)\text{sinh}(f)\text{cosh}(g), \\
\bar{D}Dh &= (4i)\text{cosh}(g)\text{cosh}(f),
\end{aligned} \tag{6.53}$$

where f, g, and h are real functions. The most simple ansatz is $f = f_0 + i\bar{\vartheta}\vartheta F$, $g = g_0 + i\bar{\vartheta}\vartheta G$. Using that

$$\cosh(f) = \cosh(f_0) + i\bar{\vartheta}\vartheta F \sinh(f_0), \ \sinh(f) = \sinh(f_0) + i\bar{\vartheta}\vartheta F \cosh(f_0),$$

we obtain the equations

$$\begin{aligned} -iF + \bar{\vartheta}\vartheta\partial\bar{\partial} f_0 &= (2i)(\cosh(f_0) + i\bar{\vartheta}\vartheta \sinh(f_0)F)(\sinh(g_0) + i\bar{\vartheta}\vartheta \cosh(g_0)G), \\ -iG + \bar{\vartheta}\vartheta\partial\bar{\partial} g_0 &= (-2i)(\sinh(f_0) + i\bar{\vartheta}\vartheta \cosh(f_0)F)(\cosh(g_0) + i\bar{\vartheta}\vartheta \sinh(g_0)G). \end{aligned}$$

This system reduces to

$$F = -2\cosh(f_0)\sinh(g_0), \ G = 2\sinh(f_0)\cosh(g_0),$$

$$\bar{\partial}\partial f_0 = -2\sinh(2f_0), \ \bar{\partial}\partial g_0 = -2\sinh(2g_0).$$

In particular, we can choose $f_0 = g_0$, hence $-G = F$. With this choice, we have

$$\begin{aligned} \cosh(\omega_1 - \omega_3)\cosh(\omega_0 - \omega_2) &= (\cosh(g_0) + i\bar{\vartheta}\vartheta \sinh(g_0)G)(\cosh(f_0) + i\bar{\vartheta}\vartheta \sinh(f_0)F) \\ &= \cosh(g_0)\cosh(f_0), \end{aligned}$$

so that we can choose $h = h_0 + \vartheta\bar{\vartheta}(4i)\cosh(g_0)\cosh(f_0)$, for a harmonic function h_0. This analysis shows that there is a large class of examples coming from solutions to the ordinary sinh-Gordon equation:

Theorem 6.54. *Any doubly periodic solution to the* sinh-*Gordon equation superizes and gives a doubly periodic solution to (6.30) for* $n = 1$. *Given a doubly periodic solution to the* sinh-*Gordon equation, then if the associated non-conformal harmonic map is doubly periodic* $\Sigma^1_\tau \to \mathbb{C}P^1$ *with modular parameter* τ, *then the* 4-*orthogonal, in particular weakly conformal, non-isotropic harmonic map is doubly periodic* $\Sigma^{1|1}_{\tau,0} \to \mathbb{C}P^{1|2}$. *Here* $\Sigma_{(\tau,0)}$ *is the split odd super torus with modular parameters* $(\tau, 0)$.

Proof. We have already proved the first part. Now for the second part, let the solution to the superized sinh-Gordon equation (6.53) be given by the connection $A_D = A_{D,0} + A_{D,1}$. The integrated map which is defined on $\mathbb{C}^{1|1}$ will be doubly periodic if and only if the integrated map of the flat connection on $\mathbb{C}$ determined by $-(DA_{D,0} + A^2_{D,1})$ is doubly periodic (cf. [36, Proof of Thm. 5]), which is true by assumption. □

Remark 6.55. The non-conformal harmonic tori in $\mathbb{C}P^1$ have an explicit description. We refer to [45] and the references therein.

7 Harmonic maps into $\mathbb{D}P^n$

We will now study harmonic maps into $\mathbb{D}P^n \subset Gr_{1|1}(\mathbb{C}^{n+1|n+1})$. In spite of the superficial similarity of $\mathbb{D}$ and the quaternions $\mathbb{H}$, such maps turn out to behave very similar to (non-supersymmetric) harmonic maps into $\mathbb{C}P^n$. The key points are that the orthogonal complement of a J_{n+1}-invariant subbundle of the trivial bundle $\underline{\mathbb{D}^{n+1}}$ is J_{n+1}-invariant and the second fundamental form of J_{n+1}-invariant subbundles of $\underline{\mathbb{D}^{n+1}}$ commutes with J_{n+1}. These two facts imply that all Gauß transforms, a priori maps into $Gr_{1|1}(\mathbb{C}^{n+1|n+1})$, are in fact maps into $\mathbb{D}P^n$ again. This contrasts with ordinary harmonic maps into $\mathbb{H}P^n$, where the rank possibly drops under the Gauß transform. The map can happen to be "∂-reducible" in the terminology of [1].

7.1 The Gauß transform

We presented the material in Section 6.1 such that adaptation to the case $\mathbb{D}P^n$ is possible with ease. Let $f\colon \Sigma \to \mathbb{D}P^n$ be harmonic. Using the type decomposition of $T\mathbb{D}P^n_{\mathbb{C}}$, the complexified differential of f decomposes into two parts $df_{\mathbb{C}} = df^{(1,0)} + df^{(0,1)}$. In local superconformal coordinates and picking a local section of the bundle determined by f, we have that $df^{(1,0)}(D)$ is given by

$$A_{f,f^\perp,D}\colon f \longrightarrow f^\perp,\ A_{f,f^\perp,D}(\rho) = \pi_{f^\perp} D\rho.$$

Similarly, $df^{(1,0)}(\bar{D})$ is given by

$$A_{f,f^\perp,\bar{D}}\colon f \longrightarrow f^\perp,\ A_{f,f^\perp,\bar{D}}(\rho) = \pi_{f^\perp} \bar{D}\rho.$$

As before, a decomposition into orthogonal J_{n+1}-invariant subbundles $\oplus_{i=1}^{l}\varphi_i = \underline{\mathbb{D}^{n+1}}_\Sigma$ leads in a local superconformal coordinate to the second fundamental forms

$$A_{\varphi_i,\varphi_j,D}\colon \varphi_i \longrightarrow \varphi_j,\ A_{\varphi_i,\varphi_j,D}(\rho) = \pi_{\varphi_j} D\rho,$$

$$A_{\varphi_i,\varphi_j,\bar{D}}\colon \varphi_i \longrightarrow \varphi_j,\ A_{\varphi_i,\varphi_j,\bar{D}}(\rho) = \pi_{\varphi_j} \bar{D}\rho.$$

In view of the J_{n+1}-linearity of $\pi_f^\perp$, Example 2.7, they commute with J_{n+1}. With respect to the standard hermitian structure on $\mathbb{C}^{1+n|1+n}$ they satisfy as before

$$A_{\varphi_i,\varphi_j,D} = -A^*_{\varphi_j,\varphi_i\bar{D}}. \tag{7.1}$$

From Proposition 2.10 and Lemma 6.3, we obtain:

Lemma 7.2. *We have the following:*

(a) The map f is holomorphic (resp. antiholomorphic) if and only if $A_{f,f^\perp,\bar{D}}$ (resp. $A_{f,f^\perp,D}$) vanishes.

(b) The map f is harmonic if and only if

$$A_{f,f^\perp,D} \circ \nabla^f_{\bar{D}} = -\nabla^{f^\perp}_{\bar{D}} \circ A_{f,f^\perp,D},$$

i.e., $A_{f,f^\perp,D}$ is a holomorphic section of $\underline{\mathrm{Hom}}_{\mathbb{D}}(f, f^\perp)$ equipped with its Koszul-Malgrange structure. Equivalently, $A_{f^\perp,f,\bar{D}}$ is antiholomorphic. In particular, f is harmonic if and only if $f^\perp$ is harmonic.

We can make again use of the holomorphicity of $A_{f,f^\perp,D}$ to produce a new harmonic map from f. We assume that the zeros of $A_{f,f^\perp,D}$ are regular. In particular, due to holomorphicity, the zeros are isolated. From Proposition 3.10 we obtain a blow up $\tilde{p}\colon \tilde{\Sigma} \to \Sigma$ and a line bundle $\mathcal{L}$ on $\tilde{\Sigma}$ such that $\tilde{p}^* A_{f,f^\perp,D}$ extends to a nowhere vanishing holomorphic section of

$$\mathcal{L} \otimes \tilde{p}^* \underline{\mathrm{Hom}}_{\mathbb{D}}(f, f^\perp) = \mathcal{L} \otimes \underline{\mathrm{Hom}}_{\mathbb{D}}(\tilde{f}, \tilde{f}^\perp),$$

where we set $\tilde{f} = f \circ \tilde{p}$. In view of the inclusion

$$\tilde{f}^\perp \subset \underline{\mathbb{D}^{n+1}}_{\tilde{\Sigma}},$$

we obtain an inclusion of $\mathcal{L}^* \otimes \tilde{f}$ into the trivial bundle. This inclusion commutes with J_{n+1}, if we consider on the former the action on the second tensor factor. Hence this defines a new map, the Gauß transform, $\tilde{f}_1\colon \tilde{\Sigma} \to \mathbb{D}P^n$. Similarly, under suitable assumptions on $A_{\varphi,\varphi^\perp,\bar{D}}$, we obtain $\hat{f}_{-1}\colon \hat{\Sigma} \to \mathbb{D}P^n$, where $\hat{p}\colon \hat{\Sigma} \to \Sigma$ is a possibly different blow up, $\hat{f} = f \circ \hat{p}$.

Theorem 7.3. *Let $f\colon \Sigma \to \mathbb{D}P^n$ be a harmonic map such that the zeros of $A_{f,f^\perp,D}$ and $A_{f,f^\perp,\bar{D}}$ are regular. Then the Gauß transforms $\tilde{f}_1$, $\hat{f}_{-1}$ exist on possibly different blow ups $\tilde{p}\colon \tilde{\Sigma} \to \Sigma$, and $\hat{p}\colon \hat{\Sigma} \to \Sigma$. They are harmonic and, moreover, $(\tilde{f}_1)_{-1}$ and $(\hat{f}_{-1})_1$ exist on $\tilde{\Sigma}$ resp. $\hat{\Sigma}$ and coincide with $\tilde{f} = f \circ \tilde{p}$ and $\hat{f} = f \circ \hat{p}$ respectively.*

Proof. The proof is formally the same as in Theorem 6.7. □

Remark 7.4. The notion of harmonic maps is extended to parabolic super Riemann surfaces as before in Remark 6.6.

7.2 Isotropic harmonic maps

We now study isotropy properties of maps into $\mathbb{D}P^n$. Σ denotes a connected super Riemann surface and $\tilde{\Sigma}$ denotes a connected parabolic super Riemann surface with degeneracy locus Z.

It is again convenient to introduce a slightly different perspective on this. On $\mathbb{D}P^n$ we have the following exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \gamma_{\mathbb{D}}) \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \mathbb{D}^{n+1}) \xrightarrow{\pi} \underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \gamma_{\mathbb{D}}^{\perp}) \longrightarrow 0.$$

We note that these are merely complex vector bundles. The first bundle has a canonical section, the identity, which therefore gives a canonical section Φ of $\underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \mathbb{D}^{n+1})$. Again, on $\underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \mathbb{D}^{1+n})$ we have the connection ∇^H induced by the canonical and the flat connection. For a map $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$, we will freely identify $f^*\Phi$ and Φ.

Lemma 7.5. *We have:*

(a) $\nabla^H_D \Phi$ *is perpendicular to the* $\mathbb{D}$*-module spanned by* Φ *and projects to* $A_{f,f^{\perp},D}$ *under* π.

(b) The map f *is harmonic if and only if*

$$\pi(\nabla^H_{\bar{D}} \nabla^H_D \Phi) = 0.$$

Proof. This is a reformulation of the previous characterization as before. □

We have a again the general fact:

Lemma 7.6. *Consider a smooth map* $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$. *For any section* V *of* $\underline{\mathrm{Hom}}_{\mathbb{D}}(f, \mathbb{D}^{n+1})$

$$(\nabla^H_D \nabla^H_{\bar{D}} + \nabla^H_{\bar{D}} \nabla^H_D) V = V \circ \varphi,$$

where φ *is a section of* $\underline{\mathrm{Hom}}_{\mathbb{D}}(f, f)$.

Proof. This follows again from the fact that the curvature of the tensor product of connections is the difference of the curvatures of these connections and that the flat connection has no curvature. □

Definition 7.7. A smooth map $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ is isotropic if in any local superconformal coordinate and for any two local sections ρ_i of f :

$$\langle \Phi(\rho_1), D^k \nabla^H_D \Phi(\rho_2) \rangle_{\mathbb{C}^{1+n|1+n}} = 0, \ k \geq 0.$$

We note that the standard hermitian structure on $\underline{\mathrm{Hom}}_{\mathbb{D}}(\gamma_{\mathbb{D}}, \mathbb{D}^{1+n})$, given by

$$\langle \nabla^H_D \Phi, \nabla^H_{\bar{D}} \Phi \rangle_{f^* \otimes \mathbb{C}^{1+n|1+n}} ,$$

which is effectively a super trace, is always zero. However, we can reformulate the isotropy condition in the following way.

Remark 7.8. Equivalently, for any two local sections ρ_i of f :

$$\langle (\nabla^H_{\bar{D}})^\alpha \Phi(\rho_1), (\nabla^H_D)^\beta \Phi(\rho_2)\rangle_{\mathbb{C}^{1+n|1+n}} = 0,\ \alpha, \beta \geq 1.$$

Definition 7.9. A map $\varphi\colon \tilde{\Sigma} \to \mathbb{D}P^n$ is full if, except for at a discrete set of points, we have

$$\mathrm{span}_{\mathbb{D}}\{x^*(\nabla^H_{\bar{D}})^k\Phi, x^*\Phi, x^*(\nabla^H_D)^l\Phi \mid k,\ l \geq 0\} = \mathbb{D}^{1+n},$$

where $x\colon \mathrm{pt} \to \tilde{\Sigma}$.

Lemma 7.10. *Let $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ be a full isotropic and harmonic map such that $f_{\pm 1}$ exist on $\tilde{\Sigma}$. Then $f_{\pm 1}$ are full and isotropic.*

Proof. The same as in Lemma 6.13. □

We call a full isotropic and harmonic map 1-regular if each of f_1 and f_{-1} either exists on $\tilde{\Sigma}$ or it is antiholomorphic respectively holomorphic. If $p\colon \hat{\Sigma} \to \tilde{\Sigma}$ is a blow up, disjoint from the degeneracy locus of the superconformal structure, and if $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ is a full isotropic map, then $f \circ p$ is full isotropic.

Definition 7.11. (a) A full isotropic harmonic map $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ has invertible ramification if all iterated Gauß transforms $f_{\pm r}$ are 1-regular.

(b) A full isotropic harmonic map $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ has regular ramification if there exists a blow up $p\colon \hat{\Sigma} \to \tilde{\Sigma}$, disjoint from the degeneracy locus of the superconformal structure, such that $f \circ p$ has invertible ramification.

Starting from a full isotropic harmonic map $\tilde{f}\colon \tilde{\Sigma} \to \mathbb{D}P^n$ with invertible ramification, there are natural numbers $k = k(f)$, $l = l(f)$ such that the sequence of Gauß transforms takes the form

$$f_{-l}, \ldots, f = f_0, f_1, \ldots, f_k.$$

The maps f_{-l}, and f_k are holomorphic and anti-holomorphic respectively and by counting dimensions we see that $k + l = n$. In view of Theorem 7.3 and Lemma 7.10 each constituent is full and isotropic. Moreover, each map has invertible ramification, which is also a consequence of the second part of Theorem 7.3 and f can be reconstructed from either f_{-l} or f_k. Thus we have proved:

Theorem 7.12. *For every $0 \leq r \leq n+1$, the assignment $f \mapsto f_r$ gives a bijective correspondence between full holomorphic maps $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ with invertible ramification and full isotropic harmonic maps $g\colon \Sigma \to \mathbb{D}P^n$ with invertible ramification such that $l(g) = r$. The inverse is given by $g \mapsto g_{-l(g)}$.*

We now define holomorphic invariants which characterize isotropy as in Lemma 6.18. First in local superconformal coordinates on some U, we set

$$(\eta^1_{\alpha,\beta})_U = \langle (\nabla^H_{\bar{D}})^\alpha \Phi, (\nabla^H_D)^\beta \Phi \rangle_{\underline{\mathrm{Hom}}_{\mathbb{D}}(f,\mathbb{D}^{1+n})},$$

where $\langle -, - \rangle_{\underline{\mathrm{Hom}}_{\mathbb{D}}(f,\mathbb{D}^{1+n})}$ denotes the odd hermitian metric

$$\langle F, G \rangle_{\underline{\mathrm{Hom}}_{\mathbb{D}}(f,\mathbb{D}^{1+n})} = \mathrm{otr}(F^* G).$$

Then ∇^H is compatible with this metric. (Notice that $\mathrm{str}(F^*G)$ vanishes necessarily.) Moreover, we set

$$(\eta^2_{\alpha,\beta})_U = \langle (\nabla^H_{\bar{D}})^\alpha \Phi(\rho), (\nabla^H_D)^\beta \Phi(\rho) \rangle_{\mathbb{C}^{n+1|n+1}} / \langle \rho, \rho \rangle_{\mathbb{C}^{1+n|1+n}},$$

where ρ is a local isotropic trivializing section of f. In view of Lemma 2.5, this does not depend on the chosen isotropic vector section, provided $(\eta^1_{\alpha,\beta})_U = 0$.

Lemma 7.13. *Let $f\colon \tilde{\Sigma} \to \mathbb{D}P^n$ be harmonic. We have that $(\eta^1_{0,1})_U = (\eta^1_{1,0})_U = 0$ and $\eta^2_{1,0} = \eta^2_{0,1} = 0$. Moreover, if $(\eta^i_{\alpha,\beta})_U = 0$ for all $1 \leq \alpha + \beta \leq r$, all U, and $i \in \{1, 2\}$, then $(\eta^i_{\alpha+1,\beta})_U$ and $(\eta^i_{\alpha,\beta+1})_U$ yield global holomorphic sections of $\Pi^i(\mathcal{D}^{\otimes(\alpha+\beta+1)})^{-1}$.*

Proof. The proof follows along the lines of Lemma 6.18. □

The map f is isotropic if and only if all these invariants vanish.

Definition 7.14. A map is called weakly conformal if $\eta^1_{1,1} = 0$, $\eta^2_{1,1} = 0$.

Corollary 7.15. *Any harmonic super sphere is isotropic.*

Proof. This follows from Lemmas 7.13, since $\Gamma(\mathcal{D}^{-k}) = 0$ for all $k \geq 1$. □

Remark 7.16. (a) Again, for $B = \mathrm{pt}$ all maps have always regular ramification.

(b) In view of Corollary 7.15, Theorem 7.12 applies for instance in the genus 0 case.

Remark 7.17. In contrast to Lemma 6.21, the underlying map of a weakly conformal (supersymmetric) harmonic map into $\mathbb{D}P^n$ is in general not harmonic. We will construct an example of such a map in Section 7.4.4.

7.3 Examples for the Gauß transform

Examples of full isotropic harmonic maps from a super sphere into $\mathbb{D}P^n$ can be easily constructed as in Section 6.3. For dimensional reasons, any full harmonic map from a genus 0 super Riemann surface into $\mathbb{D}P^1$ is holomorphic or antiholomorphic. For harmonic spheres into $\mathbb{D}P^2$, we can use the construction of harmonic super spheres in $\mathbb{C}P^{1|1}$ from Section 6.3

to obtain full isotropic harmonic maps. This simply works by considering the holomorphic map defined by

$$f = (p \quad J_3 q \quad r).$$

The computation of the harmonic sequence can be carried out similarly.

7.4 Periodic harmonic sequences in $\mathbb{D}P^{2n}$

Analogous to Section 6.4, we now analyze harmonic maps $\Sigma \to \mathbb{D}P^{2n}$ with periodic harmonic sequence.

7.4.1 The $(2n+1)$-symmetric space $PSQ(2n+1)/PSQ(1)^{2n+1}$

We fix the torus $T = PS(U(1)^{2n+1}) \subset PSQ(2n+1)$ and set $\beta_l = \sigma_{l+1} - \sigma_l$, $l = 0, \ldots, 2n-1$, and $\beta_{2n} = -\sum_l \beta_l$. Each root space is 1|1-dimensional over $\mathbb{C}$. We consider the adjoint action of $\tau = \mathrm{diag}(1, \omega, \ldots, \omega^{2n} \mid 1, \omega, \ldots, \omega^{2n})$, where ω is a simple $(2n+1)$st root of unity. Then after complexification we have a decomposition into eigenspaces

$$\mathfrak{psq}(2n+1)_{\mathbb{C}} = \bigoplus_{i=0}^{2n} \mathcal{M}_i.$$

For instance, $\mathcal{M}_1$ is the sum of the root spaces of β_l, $l = 0, \ldots 2n$. In the following, we will consider matrices as $\mathbb{D}$-valued. Let E_l be the root vector for β_l with only one non-zero entry equal to 1. We set $B_\tau = \sum_k \tilde{J}^{-1} E_k$. In the case $n = 1$ we have

$$\mathcal{M}_1 = \{ \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \mid a, b, c \in \mathbb{D}\},\ E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ B_\tau = \begin{pmatrix} 0 & 0 & \tilde{J}^{-1} \\ \tilde{J}^{-1} & 0 & 0 \\ 0 & \tilde{J}^{-1} & 0 \end{pmatrix}.$$

We notice that $\mathcal{M}_0 = \mathfrak{ps}(\mathfrak{q}(1)^{2n+1})_{\mathbb{C}}$ is not abelian. The same τ gives also rise to a description of $PQ(2n+1)/P(Q(1)^{2n+1})$ as a $(2n+1)$-symmetric space. The only difference being that then $\mathcal{M}_0 = \mathfrak{p}(\mathfrak{q}(1)^{2n+1})_{\mathbb{C}}$. All $\mathcal{M}_i$, $i \neq 0$, are left unchanged. In order to obtain invariants for primitive maps, we need to understand the $\mathfrak{p}(\mathfrak{q}(1)^{2n+1})_{\mathbb{C}}$-invariant polynomials of $\mathcal{M}_1$. Writing $\xi_i = \sum_k a_i^k E_k$, we have

$$P_1^Q(\xi_0, \ldots, \xi_{2n}) = \frac{1}{(2n+1)!} \sum_{\sigma \in \Sigma_{2n+1}} \mathrm{otr}(a^0_{\sigma(0)} a^{2n}_{\sigma(1)} a^{2n-1}_{\sigma(2)} \ldots a^1_{\sigma(2n)}).$$

Moreover, concerning $\mathfrak{ps}(\mathfrak{q}(1)^{2n+1})_{\mathbb{C}}$, there is one more invariant polynomial given by

$$P_2^Q(\xi_0, \ldots, \xi_{2n}) = \frac{1}{(2n+1)!} \sum_{\sigma \in \Sigma_{2n+1}} \mathrm{odet}(a^0_{\sigma(0)} \tilde{J} a^{2n}_{\sigma(1)} \tilde{J} a^{2n-1}_{\sigma(2)} \tilde{J} \ldots a^1_{\sigma(2n)} \tilde{J}).$$

They define elements in $\mathrm{Sym}^*_{2n+1}(\Pi\mathcal{M}_1)$ and $\Pi\mathrm{Sym}^*_{2n+1}(\Pi\mathcal{M}_1)$, respectively. Again, we set $P_i^Q(\xi) = P_i^Q(\xi, \dots, \xi)$.

Definition 7.18. A map $\xi\colon T \to \Pi\mathcal{M}_1$ is called cyclic if $P_1^Q(\xi)$ is invertible.

Lemma 7.19. *Consider a map $\xi = \sum A_i E_i\colon T \to \Pi\mathcal{M}_1$. Then there exists $t \in PQ(1)^{2n+1}_{\mathbb{C}}$ such that*

$$\mathrm{Ad}(t)(B_\tau) = \xi$$

if and only if $P_1^Q(\xi) = 1$. In this case we have $P_2^Q(\xi) = 2\mathrm{odet}(t)$.

Proof. In view of $P_1^Q(\xi) = 1$, we can find $X = 1 + \xi\tilde{J}$ such that

$$t = \mathrm{diag}(A_0^{-1}\tilde{J}^3X^{-1}, \tilde{J}^4X, A_1\tilde{J}X^{-1}, A_2A_1\tilde{J}^2X, A_3A_2A_1\tilde{J}^3X^{-1}, \dots, A_{2n-1}\cdots A_1\tilde{J}^{2n-1}X^{-1}),$$

does the job. The second claim follows by inspection. □

7.4.2 Primitive maps

Definition 7.20. A map $f\colon \Sigma \to PSQ(2n+1)/T$ is called primitive if $df_{\mathbb{C}}|_{\mathcal{D}}$ has values in $[\mathcal{M}_1]$ and it is cyclic at one point.

The definition of a framing $\tilde{f}\colon \Sigma \to PSQ(2n+1)$ is as before (Section 6.4.2) and, again, primitivity of f is characterized by

$$A_D = A_{D,0} + A_{D,1},$$

where $A = \tilde{f}^*\alpha$ and $A_{D,i}$ has values in $\mathcal{M}_i$. Analogously as in Lemma 6.28, we have:

Lemma 7.21. *Given a primitive map, then $P_i^Q(f) := P_i^Q(\Pi df_{\mathbb{C}}|_{\mathcal{D}})$ are holomorphic sections of $\Pi^{i+1}((\Pi\mathcal{D})^*)^{\otimes(2n+1)}$.*

There is also a notion of Toda frame similar to Section 6.4.2. For this, we choose the complement of $\mathfrak{psq}(2n+1) \subset \mathfrak{pq}(2n+1)$ spanned by

$$M^Q = \mathrm{diag}(\tilde{J}, -\tilde{J}, \tilde{J}, \dots, \tilde{J}).$$

We assume that $P_1^Q(f)$ is invertible at a point and hence in a coordinate neighbourhood U. A framing $\tilde{f}\colon U \to PSQ(2n+1)$ is a Toda frame if there exists a superconformal isomorphism $a\colon U \to U$ such that $P_1^Q(a^*f) = 1$ and a map $\Omega = \tilde{\Omega} + \chi M^Q\colon U \to i\mathfrak{psq}(1)^{2n+1} \oplus \langle M^Q\rangle$ such that

$$a^*A_D = (D\tilde{\Omega} - \frac{1}{2}[\tilde{\Omega} - \bar{\chi}\bar{M}^Q, D\tilde{\Omega}]) + \mathrm{Ad}(\exp(\Omega))(B_\tau).$$

The Maurer-Cartan equations of such a framing reduce to

$$2\bar{D}D\tilde{\Omega} - 2[\bar{D}\tilde{\Omega}, D\tilde{\Omega}] + 1/2\bar{D}D[(\chi M^Q + \bar{\chi}\bar{M}^Q), \tilde{\Omega}] + [A_{D,1}, A_{\bar{D},1}] = 0. \tag{7.22}$$

Then we have that $P_2^Q(a^*f) = P_2^Q(f)/P_1^Q(f) = 2\mathrm{odet}(\exp(\Omega)) = 2\mathrm{otr}(\Omega)$ is holomorphic, so that χ is holomorphic. The additional summand in $A_{D,0}$ as compared with the Toda frames in Theorem 6.31 in the above formula results from non-commutativity of $Q(1)$:

Lemma 7.23. *For any derivation X, we have*

$$(X\exp(\Omega))\exp(\Omega)^{-1} = X\Omega + \frac{1}{2}[\Omega, X\Omega].$$

Proof. This is a direct calculation. □

Then we have:

Theorem 7.24. *Let $f\colon \Sigma \to PSQ(2n+1)/T$ be primitive and assume that f is cyclic at the point p_0. Then there exists a Toda frame in some neighbourhood of p_0 such that (7.22) holds. The superconformal isomorphism $a\colon U \to U$ is unique up to superconformal rotation by a $(2n+1)$st root of unity and translations and the Toda frame is unique for such a.*

Proof. The proof is similar to Theorem 6.31. On a coordinate neighbourhood U of p_0 we can find a framing $\tilde{f}$ with values in $PSQ(2n+1)$ and since $P_1^Q(f)$ is holomorphic and invertible and U is simply connected we can change superconformal coordinates such that $P_1^Q(a^*f) = 1$. This coordinate transformation is unique up to superconformal translation and rotation by a $(2n+1)$st root of unity. Since U is simply connected, we can find by Lemma 7.19 an $\eta\colon U \to \mathfrak{pq}(1)_{\mathbb{C}}^{2n+1}$ such that

$$a^*A_{D,1} = \mathrm{Ad}(\exp(\eta))(B_\tau).$$

Although $PQ(1)_{\mathbb{C}}^{2n+1}$ is non-abelian and the exponential map is not a group homomorphism, we can still find a decomposition $\exp(\eta) = \exp(\tilde{\Lambda})\exp(\tilde{\Omega} + \chi M^Q)$, where $\bar{\tilde{\Lambda}} = \tilde{\Lambda}$, $\bar{\tilde{\Omega}} = -\tilde{\Omega}$ have vanishing odd trace. We define $\Omega = \tilde{\Omega} + \chi M^Q$. We can gauge away $\tilde{\Lambda}$ and obtain the desired Toda frame $\tilde{f}$. Using Lemma 7.23, we find

$$-[a^*A_{\bar{D},0}, a^*A_{D,1}] = \bar{D}a^*A_{D,1} = [\bar{D}\Omega + 1/2[\Omega, \bar{D}\Omega], a^*A_{D,1}] = [\bar{D}\tilde{\Omega} + 1/2[\Omega, \bar{D}\tilde{\Omega}], a^*A_{D,1}]$$

and hence $a^*A_{\bar{D},0} = -\bar{D}\tilde{\Omega} - 1/2[\Omega, \bar{D}\tilde{\Omega}]$ and $a^*A_{D,0} = D\tilde{\Omega} - 1/2[\tilde{\Omega} - \bar{\chi}\bar{M}^Q, D\tilde{\Omega}]$. (Here we used that the stabilizer B_τ acting on odd T-valued points of $\mathfrak{pq}(1)_{\mathbb{C}}^{2n+1}$ is trivial due to the fact that $2n+1$ is odd.) The $\mathcal{M}_0$ component of the Maurer-Cartan equation is precisely (7.22). The uniqueness follows as in Theorem 6.31. □

Corollary 7.25. *Let Σ be a super torus and consider a primitive map $\Sigma \to PSQ(2n+1)/T$. Then there exists a Toda frame on the universal covering $\mathbb{C}_B^{1|1} \to \Sigma$ and a superconformal isomorphism $a\colon \mathbb{C}_B^{1|1} \to \mathbb{C}_B^{1|1}$, unique up to superconformal rotation by a $(2n+1)$st root of unity and translation, such that (7.22) holds. Moreover, the Toda frame and Ω factor through some finite covering $\Sigma' \to \Sigma$.*

Proof. This follows as before in Corollary 6.32. □

7.4.3 A class of $2n+1$-orthogonal non-isotropic harmonic maps

Starting from a full harmonic map from a super torus with invertible ramification which is $(2n+1)$-orthogonal and non-isotropic in the sense that $P_1^Q(f)$ is invertible at one point (and hence everywhere), the harmonic sequence

$$f_{-l}, \ldots, f_0, \ldots, f_k$$

determines a primitive map

$$\tilde{f} \colon \Sigma \longrightarrow PSQ(2n+1)/T.$$

(Since $2n+1$ is odd, this forces the super torus to be of the odd type.)

As follows from the proof of Theorem 7.24, for such primitive maps, we neither have $DA_{D,1} \neq [A_{D,0}, A_{D,1}]$ nor $[A_{D,0}, A_{\bar{D},0}] = 0$ in general. However, the machinery used in Section 6.4.3 produces maps satisfying these constraints. In view of this, one might not expect and we cannot give a general finite type classification along the lines of Section 6.4.4. However, it turns out that quite drastic assumptions still lead to sufficiently interesting examples.

It is useful to use a slightly different setup. In the situation of Theorem 7.24, in a suitable coordinate with $P_1^Q(f) = 1$, we can also write

$$A_{D,1} = \mathrm{Ad}(\exp(\eta))(B_\tau), \ \exp(\eta) = \exp(\Lambda')(\exp(\Omega'),$$

where Λ', and Ω' are $\mathfrak{pq}(1)_{\mathbb{C}}^{2n+1}$-valued and $\bar{\Lambda}' = \Lambda'$, $\bar{\Omega}' = -\Omega'$. At the cost of obtaining a $\mathfrak{pq}(2n+1)_{\mathbb{C}}$-valued form we can gauge away Λ'. Thus such a form will integrate to a framing of a primitive map into $PQ(2n+1)/PQ(1)^{2n+1}$. In this new gauge

$$A'_{D,1} = \mathrm{Ad}(\exp(\Omega'))(B_\tau), \ A'_{D,0} = D\Omega' - 1/2[\Omega', D\Omega']$$

and the Maurer Cartan equation reads:

$$2\bar{D}D\Omega' - 2[D\Omega', \bar{D}\Omega'] + [A'_{D,1}, A'_{\bar{D},1}] = 0. \tag{7.26}$$

We write $A'_{D,1} = \sum_k AiE_i$ and $A_i\tilde{J} = a_i(1 + \alpha_i\tilde{J})$. In this situation all $A_i\tilde{J}$ are real.

Proposition 7.27. *If* $[A'_{D,0}, A'_{\bar{D},0}] = 0$ *and* $A_i\tilde{J}$ *commute pairwise, then the Maurer Cartan equation is equivalent to the system:*

$$\bar{D}D\log(a_i) = \frac{i}{2}(a_{i+1}^2 - a_{i-1}^2), \ i = 0, \ldots, 2n, \tag{7.28}$$

$$\bar{D}D\alpha_i = i(a_{i+1}^2\alpha_{i+1} - a_{i-1}^2\alpha_{i-1}),\ i = 0, \ldots, 2n, \tag{7.29}$$

subject to $\alpha_i\alpha_j = 0$, *and* $[D\Omega', \bar{D}\Omega'] = 0$. *Here, we set* $\alpha_{2n+1} = \alpha_0$ *and* $a_{2n+1} = a_0$.

In particular, if we denote by $p_i\colon \mathbb{D} \to \Pi^i\mathbb{C}$ *the projection onto the ith summand, then the form determined by* $p_0A_{D,0} + p_1A_{D,1}$ *satisfies the Maurer-Cartan equation (7.22).*

Proof. We know that

$$\begin{aligned}\exp(\Omega') &= \operatorname{diag}(A_0^{-1}\tilde{J}^3X^{-1}, \tilde{J}^4X, A_1\tilde{J}X^{-1}, (A_2A_1)\tilde{J}^2X, (A_3A_2A_1)\tilde{J}^3X^{-1}, \ldots, \\ &\qquad (A_{2n-1}\cdots A_1)\tilde{J}^{2n-1}X^{-1}) \\ &= \operatorname{diag}(\tilde{J}(A_0\tilde{J})^{-1}\tilde{J}^{-1}X^{-1}, X, (A_1\tilde{J})X^{-1}, (A_2\tilde{J})(\tilde{J}A_1)X, (A_3\tilde{J})(\tilde{J}A_2)(A_1\tilde{J})X^{-1}, \\ &\qquad \ldots, (A_{2n-1}\tilde{J})(\tilde{J}A_{2n-2})\ldots(A_1\tilde{J})X^{-1}),\end{aligned}$$

where $X = 1 + \chi\tilde{J}$ is determined in terms of $A_i\tilde{J}$ and is real. Since X and $A_i\tilde{J}$ commute, we have

$$\begin{aligned}\Omega' &= \log(\operatorname{diag}(\tilde{J}(A_0\tilde{J})^{-1}\tilde{J}^{-1}X^{-1}, X, (A_1\tilde{J})X^{-1}, (A_2\tilde{J})(\tilde{J}A_1)X, (A_3\tilde{J})(\tilde{J}A_2)(A_1\tilde{J})X^{-1}, \ldots, \\ &\qquad (A_{2n-1}\tilde{J})(\tilde{J}A_{2n-2})\cdots(A_1\tilde{J})X^{-1})) \\ &= \operatorname{diag}(-\log(a_0) + (\alpha_0)\tilde{J}, 0 + 0\cdot\tilde{J}, \log(a_1) + (\alpha_1)\tilde{J}, \log(a_2a_1) + (\alpha_2 - \alpha_1)\tilde{J}, \ldots, \\ &\qquad \log(a_{2n-1}\cdots a_1) + (\alpha_{2n-1} - \alpha_{2n-2} + \alpha_{2n-3}\ldots + \alpha_1)\tilde{J}) \\ &\quad + \operatorname{diag}(-\chi\tilde{J}, \chi\tilde{J}, \ldots, -\chi\tilde{J}).\end{aligned}$$

Moreover, $A_i^*A_i = ia_i^2(1 + 2\alpha_i\tilde{J})$ and $A_iA_i^* = ia_i^2(1 - 2\alpha_i\tilde{J})$, so that

$$\begin{aligned}[A_{D,1}, A_{\bar{D},1}] &= (-1)\operatorname{diag}(A_0^*A_0 + A_{2n}A_{2n}^*, A_1^*A_1 + A_0A_0^*, \ldots, A_{2n}^*A_{2n} + A_{2n-1}A_{2n-1}^*) \\ &= (-i)\operatorname{diag}(a_0^2 + a_{2n}^2 + (2a_0^2\alpha_0 - 2a_{2n}^2\alpha_{2n})\tilde{J}, \ldots, \\ &\qquad a_{2n}^2 + a_{2n-1}^2 + (2a_{2n}^2\alpha_{2n} - 2a_{2n-}^2\alpha_{2n-1})\tilde{J}).\end{aligned}$$

Comparing with (7.26) then gives the result. ☐

Remark 7.30. (a) On a torus the solution space to (7.29) is finite-dimensional as follows from ellipticity considerations similar to those in Section 6.4.4. A solution given by (α_i) to (7.29) can always be arranged to satisfy the constraints by adding an additional Grassmann variable η to the base B and then considering $(i\eta\bar{\eta})\alpha_i$.

(b) Up to a change of basis, the form $p_0A_{D,0} + p_1A_{D,1}$ fits into the framework of Section 6.4. In particular, local solutions can be constructed from Theorem 6.42.

7.4.4 Example: The case $n = 1$

We study the case $n = 1$. Similar as in Section 6.4.5, we have a trivial solution $a_1 = a_2 = a_3 = 1$. We consider $\omega = e^{2\pi i/3}$ and set

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ A = \begin{pmatrix} \tilde{J}^{-1} & 0 & 0 \\ 0 & \omega \tilde{J}^{-1} & 0 \\ 0 & 0 & \omega^2 \tilde{J}^{-1} \end{pmatrix}.$$

Then $[\bar{A}, A] = 0$ in $\mathfrak{psq}(3)_{\mathbb{C}}$ and the framing is given by

$$\exp(-zA^2 - \bar{z}\bar{A}^2)(1 + \vartheta A + \bar{\vartheta}\bar{A} + \bar{\vartheta}\vartheta \bar{A}A)(v \mid (A^{-1}\tilde{J}^{-1})v \mid (A^{-1}\tilde{J}^{-1})^2 v).$$

The matrix on the left hand side is

$$\begin{pmatrix} e^{z-\bar{z}} & 0 & 0 \\ 0 & e^{\omega^2 z - \omega \bar{z}} & 0 \\ 0 & 0 & e^{\omega z - \omega^2 \bar{z}} \end{pmatrix}$$

and the Maurer-Cartan form is given by

$$A_D = A_{D,1} = \begin{pmatrix} 0 & 0 & \tilde{J}^{-1} \\ \tilde{J}^{-1} & 0 & 0 \\ 0 & \tilde{J}^{-1} & 0 \end{pmatrix}.$$

The associated map to $PSQ(3)$ is periodic with respect to the lattice spanned by $2\pi/\sqrt{3}$ and $2\pi i$.

Now we try to extend this non-trivially according to Proposition 7.27. The simplest ansatz is to take α_i to be harmonic. From (7.29) we thus obtain $\alpha_0 = \alpha_1 = \alpha_2$. For instance, we can take $\alpha_i = \alpha_C = C(\vartheta - i\bar{\vartheta})$, where C is real.

We thus obtain

$$A'_{D,1} = \begin{pmatrix} 0 & 0 & \tilde{J}^{-1} + \alpha_C \\ \tilde{J}^{-1} + \alpha_C & 0 & 0 \\ 0 & \tilde{J}^{-1} + \alpha_C & 0 \end{pmatrix},$$

and

$$\exp(\Omega') = \begin{pmatrix} 1 + \alpha_C/2\tilde{J} & 0 & 0 \\ 0 & 1 + \alpha_C/2\tilde{J} & 0 \\ 0 & 0 & 1 + \alpha_C/2\tilde{J} \end{pmatrix}.$$

Hence

$$A'_{\bar{D},0} = -(-i)(C/2)\mathrm{diag}(\tilde{J}, \tilde{J}, \tilde{J}), \ A'_{D,0} = (C/2)\mathrm{diag}(\tilde{J}, \tilde{J}, \tilde{J}),$$

which shows that the constraints in Proposition 7.27 are satisfied.

Theorem 7.31. *There is a* 1*-parameter family of* 3*-orthogonal non-isotropic harmonic maps* $f_C\colon \mathbb{C}^{1|1} \to \mathbb{D}P^2$ *such that the underlying maps* $\tilde{f}_C\colon \mathbb{C} \to \mathbb{C}P^2$ *are non-harmonic except for* $C = 0$*. The map* f_C *factors through a split super torus* $\Sigma_{\tau,0}$ *with*

$$\tau = 2\pi/(\sqrt{3}(1+C)) + i2\pi/(1-C),\ C \neq 1.$$

If $C = 1$*, the map is constant in* y*.*

Proof. We define f_C to be the map obtained by integrating the above form. To show that the underlying map is not harmonic and to compute the periods, we need to study the underlying map. From the above, using that for a flat connection $\alpha_z = -(D\alpha_D + \alpha_D^2)$, we see that the relevant connection is given by

$$\alpha_z = \begin{pmatrix} 0 & 1 & -C \\ -C & 0 & 1 \\ 1 & -C & 0 \end{pmatrix},\ \alpha_{\bar{z}} = -\begin{pmatrix} 0 & -C & 1 \\ 1 & 0 & -C \\ -C & 1 & 0 \end{pmatrix}.$$

Setting $G = U(3)$, $K = U(1) \times U(2)$, we thus see that

$$[\alpha_{\mathfrak{k},\bar{z}}, \alpha_{\mathfrak{p},z}] = -[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -C \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -C \\ -C & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}],$$

which vanishes if and only if $C = 0$, so that the underlying map is harmonic if and only if $C = 0$. Moreover,

$$\alpha_x = (1+C)\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},\ \alpha_y = 1/(-i)(1-C)\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

have eigenvalues given by $\{0, \pm i\sqrt{3}\}$ and $\{-1, 2\}$ respectively. Hence the associated 1-parameter groups have periods $2\pi/(\sqrt{3}(1+C))$ and $2\pi/(1-C)$, $C \neq 1$ respectively. If $C = 1$, $\alpha_y = 0$. □

Remark 7.32. In view of the computation of the component fields in Section 4, we have thus constructed doubly periodic solutions to the equations in Proposition 4.1 which are coupled in the sense that the underlying map is not harmonic.

There is yet another extension to this system making use of Remark 7.30. Namely, the ansatz $\alpha_0 = \varphi_0\vartheta - i\bar{\varphi}_0\bar{\vartheta}$, $\alpha_1 = \varphi_1\vartheta - i\bar{\varphi}_1\bar{\vartheta}$, and $\sum_i \alpha_i = 0$ leads to the equations

$$\bar{\partial}\varphi_0 = -(2\bar{\varphi}_1 + \bar{\varphi}_0),\ \bar{\partial}\varphi_1 = (2\bar{\varphi}_0 + \bar{\varphi}_1).$$

This system is equivalent to $\partial\bar{\partial}\varphi_0 = -3\varphi_0$ and automatically $\varphi_1 = -1/2(\partial\bar{\varphi}_0 + \varphi_0)$. The condition $\alpha_0\alpha_1 = 0$ is then equivalent to

$$\varphi_0\bar{\partial}\varphi_0 - \bar{\varphi}_0\partial\bar{\varphi}_0 = 0.$$

Considering the ansatz $\varphi_0 = Ae^{i\sqrt{3}x} + Be^{-i\sqrt{3}x}$, this is satisfied if $\varphi_0 = A\cos(\sqrt{3}x)$ for real A. Then $\varphi_1 = -A/2(-\sqrt{3}\sin(\sqrt{3}x) + \cos(\sqrt{3}x))$. After adding an additional parameter η, we thus obtain a doubly periodic solution to the Maurer-Cartan equation. The resulting map will still be periodic with period $\pi/\sqrt{3} + 2\pi i$ since the underlying map, setting $\eta = 0$, has this property.

8 Outlook

There is a wealth of problems which we did not address. We shall highlight only a few.

Non-conformal harmonic tori

Originally, the notion of finite type harmonic maps led to a classification of all non-conformal tori in compact rank one Riemannian symmetric spaces [10]. Moreover, Burstall gave a finite type description of all non-isotropic harmonic tori in $\mathbb{C}P^n$ [8]. In view of our results in Section 6.4.4, it is a natural question whether such other finite type classification results have analogous supersymmetric versions. This is particularly interesting since in this case Lemma 6.21 does no longer hold in general. However, in view of the special properties of the situation employed in Section 6.4.4, e.g. the existence of suitable invariant polynomials and the special orbit structure which allowed to define the complex vector field Z, such an extension seems to be non-obvious.

Harmonic maps of finite uniton number

Theorem 1.2 has been vastly generalized by Uhlenbeck [52] and Burstall and Rawnsley gave a comprehensive treatment for harmonic spheres in symmetric spaces [11]. It is not clear how these results generalize to the graded setting. For instance, the statement in [46, p.8 l.-5 – l.-3] is not comprehensible since the arguments in [52] rely crucially on the existence of kernel and image bundles of holomorphic endomorphisms similar as used in the Gauß transform. In view of our results, this seems to be a subtle issue.

Spaces of harmonic maps

In [17, 41], the authors showed that in the case of harmonic spheres in $\mathbb{C}P^2$, one can build in certain situations smooth manifolds of harmonic maps. In view of the structural similarity, it would be interesting to study a similar problem for full isotropic super spheres in $\mathbb{D}P^2$.

Harmonic maps into $\overline{\mathbb{D}}P^n$

Finally, there is also an interesting super division algebra over $\mathbb{R}$:

$$\overline{\mathbb{D}} := \mathbb{C}[j]/(ji = -ij, j^2 = 1).$$

As a vector space it isomorphic to $\mathbb{C}^{1|1}$ and we will represent an element in the form $a + bj$. We let $J_{\overline{\mathbb{D}}}$ denote the operation of left multiplication by j, hence this takes (for even elements) the form

$$J_{\overline{\mathbb{D}}}(a, b) = (-\bar{b}, \bar{a}).$$

Any map $T \to \overline{\mathbb{D}} - 0$ can be written as $f + gj$ and then

$$\frac{1}{f + gj} = \frac{1}{f}\left(1 - \frac{\bar{g}g}{\bar{f}f}\right) + \frac{g}{\bar{f}f}j.$$

We define $\overline{\mathbb{D}}P^n$ to be covered by $n + 1$ copies of $\overline{\mathbb{D}}^n$ with the usual transition functions. This comes with the tautological flag

$$\begin{array}{ccc} \gamma_{\overline{\mathbb{D}}} = \overline{\mathbb{D}} \times_{\overline{\mathbb{D}}^\times} (\overline{\mathbb{D}}^{1+n} - 0) & \longrightarrow & \underline{\overline{\mathbb{D}}^{1+n}}_{\overline{\mathbb{D}}P^n} \\ & \searrow & \downarrow \\ & & \overline{\mathbb{D}}P^n, \end{array}$$

where the top map is just $(\lambda, a) \mapsto ([a], \lambda a)$.

Lemma 8.1. *Smooth maps $T \to \overline{\mathbb{D}}P^n$ are in bijective correspondence with complex subbundles $L \subset \underline{\overline{\mathbb{D}}^n}_T$ which are invariant under $J_{\overline{\mathbb{D}}}$.*

Proof. This bijective correspondence is given again by pulling back the tautological flag. □

Notice that $J_{\overline{\mathbb{D}}}$ is not unitary. Rather, for even vectors f and g we have

$$\langle J_{\overline{\mathbb{D}}} f, J_{\overline{\mathbb{D}}} g\rangle = i\langle \bar{f}, \bar{g}\rangle.$$

As a consequence, there is no natural connection on $\iota\colon \overline{\mathbb{D}}P^n \hookrightarrow Gr_{1|1}(\mathbb{C}^{1+n|1+n})$ as was the case for $\mathbb{D}P^n$. We define $f\colon \Sigma \to \overline{\mathbb{D}}P^n$ to be harmonic if the composite map $\iota \circ f\colon \Sigma \to Gr_{1|1}(\mathbb{C}^{1+n|1+n})$ is harmonic. In view of the discussion in Section 6.1, there is a notion of Gauß transform for such maps. Moreover, there is a fibration sequence

$$\mathbb{C}P^{0|1} \longrightarrow \mathbb{C}P^{n|n+1} \longrightarrow \overline{\mathbb{D}}P^n.$$

Recall that a holomorphic map into a complex projective space is harmonic. On general grounds, if a 1-regular isotropic holomorphic map $f\colon \tilde{\Sigma} \to \mathbb{C}P^{1|2}$ is $\mathcal{D}$-horizontal, i.e.,

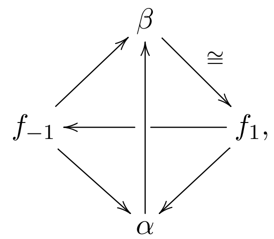
$\langle J_{\overline{\mathbb{D}}} f, Df\rangle = 0$, then its projection to $\overline{\mathbb{D}}P^1$ is harmonic. (This also follows from similar considerations as in the proof we give below.) The following partial converse illustrates that our definition of harmonicity is not arbitrary.

Lemma 8.2. *Consider a parabolic super Riemann surface $\tilde{\Sigma}$ and a harmonic map $f\colon \tilde{\Sigma} \to \overline{\mathbb{D}}P^1$. Assume that $f_{\pm 1}$ exist and the map is weakly conformal $f_{-1} \perp f_1$. Then:*

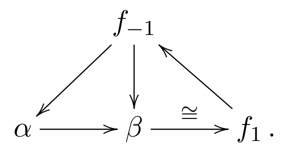
(a) There is a harmonic lift $\beta\colon \tilde{\Sigma} \to \mathbb{C}P^{1|2}$ which satisfies $\beta_1 \perp J_{\overline{\mathbb{D}}}\beta$.

(b) The differential of $d\beta^{(1,0)})|_{\bar{\mathcal{D}}}$ is vertical, i.e., contained in $\underline{\mathrm{Hom}}(\beta, f/\beta)$ if and only if $\alpha = \beta^{\perp} \subset f$ is isotropic: $\langle J_{\overline{\mathbb{D}}}\alpha, \alpha\rangle = 0$. In this case, β is isotropic and $\mathcal{D}$-horizontal.

Proof. On $\tilde{\Sigma}$ the relevant diagram for the second fundamental forms is (cf. [12, Thm. 3.7], Section 6.2)



where $\alpha \oplus \beta = f$, $\alpha \perp \beta$, and $\alpha = \ker(A_{f,f^{\perp},D})$. The arrow $f_1 \to \alpha$ vanishes by $f_1 \perp f_{-1}$. Thus, the diagram is actually of the form



Using similar arguments as in [12, Prop. 1.5], one can show that the map determined by β is harmonic. It clearly satisfies the stated horizontality conditions.

Let v be an even local trivializing section of α. We compute

$$\begin{aligned} \bar{D}\beta &= \bar{D}\left(J_{\overline{\mathbb{D}}}v - \frac{\langle v, J_{\overline{\mathbb{D}}}v\rangle}{\langle v, v\rangle} v\right) \\ &= -J_{\overline{\mathbb{D}}}(Dv) - \bar{D}\left(\frac{\langle v, J_{\overline{\mathbb{D}}}v\rangle}{\langle v, v\rangle}\right) v + \frac{\langle v, J_{\overline{\mathbb{D}}}v\rangle}{\langle v, v\rangle}\bar{D}v. \end{aligned}$$

By assumption, the vector $\bar{D}v$ spans a locally free module except for at isolated points. So the map is vertical if and only if $\langle v, J_{\overline{\mathbb{D}}}v\rangle = 0$.

□

Remark 8.3. This result suggests a similarity between the theory of supersymmetric harmonic maps into $\overline{\mathbb{D}}P^n$ and harmonic maps into quaternionic projective spaces. Harmonic spheres in $\mathbb{H}P^n$ spaces have been classified [1] and the above is analogous to the classification of harmonic spheres in $\mathbb{H}P^1$ (cf. [1, Equ. 6.2]). Moreover, this result is reminiscent of [27, Lem. 2.7]. In the terminology introduced therein, (b) says that f is "quaternionic holomorphic".

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Automorphism supergroups of supermanifolds

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Contents

1 Introduction

In this article, we study geometric structures on *cs* manifolds and their automorphisms. Super-Riemannian structures on *cs* manifolds play a prominent role in the work of Zirnbauer [16]. In particular, the so-called Riemannian symmetric superspaces are worth mentioning. Other instances of geometric structures on supermanifolds appeared in the context of supergravity theories [14].

By a geometric structure on a manifold M we mean a reduction of the structure group of the frame bundle $L(M)$ to some closed subgroup $G \leqslant GL(V)$. Depending on the context, there might be additional conditions like 1-flatness. A classical theorem states that the group of automorphisms of such a G-structure is a Lie group provided it is of finite type. (See [12] and the references therein.) This includes for instance the isometry group of a Riemannian manifold.

In this work, we study the analogous structures in the category of *cs* manifolds (cf. [7]). First, we lay the necessary foundations for the definition of a G-structure. This leads naturally to the notion of mixed supermanifolds as follows. The frame bundle of an ordinary manifold locally modelled on the vector space V is obtained from a cocycle $U_{ij} \to GL(V)$ by glueing. Suppose M is a *cs* manifold (called supermanifolds in this article) which is locally modelled on the super vector space $V_{\bar{0}} \oplus V_{\bar{1}}$. Here, $V_{\bar{0}}$ is a real and $V_{\bar{1}}$ is a complex vector space. Then the analogous cocycle takes values in the mixed Lie supergroup $GL(V)$ which has as body the mixed manifold $GL(V_{\bar{0}}) \times GL(V_{\bar{1}})$. It is crucial to keep the complex analytic structure on the second factor. After having developed the basic theory of mixed supermanifolds, one can define G-structures, prolongations and G-structures of finite type along the lines of the classical definitions. Our main result concerns the functor of automorphisms of a G-structure of finite type that is in addition admissible. In this situation, if restricted to purely even supermanifolds, the functor is representable by a mixed Lie group and it is finite-dimensional in the sense that the higher points are determined by the Lie superalgebra of infinitesimal automorphisms of the G-structure, which we prove to be finite-dimensional (Theorem 4.11). Representability can fail for two reasons here, due to the fact that the higher points of the functor of automorphisms contain all infinitesimal automorphisms of the G-structure. For a representable functor these are necessarily all complete and decomposable, which means that they admit a decomposition of the form

$X + iY$ for two real complete vector fields. The theory of G-structures can be developed for real supermanifolds without the need for enlarging the category. Moreover, there is no need for imposing an additional property on a G-structure of finite type. The only obstruction for the representability of the functor of automorphisms of finite type is the completeness of the infinitesimal automorphisms.

The paper is organized as follows. In Section 2 we introduce mixed supermanifolds. After giving the basic definitions, we give a short account on mixed Lie supergroups and principal bundles. We then show that mixed supermanifolds are the natural home for constructions such as tangent bundles and frame bundles as well as their mixed forms, the real tangent bundles and real frame bundles. In contrast to what the name suggests, mixed supermanifolds are not supermanifolds with extra structure as we show in the appendix (Proposition 7.1). Moreover, therein we prove that, for our purposes, mixed supermanifolds cannot be avoided (Proposition 7.2).

In Section 3 we define a geometric structure to be a reduction of the real frame bundle of a mixed supermanifold and construct its prolongation. In the super context it is advisable to make the constructions in such a way that functoriality is evident. A subtlety is that the standard prolongation has to be refined to a real prolongation, which is again a geometric structure in the sense of our definition. The existence is ensured if the G-structure is admissible.

In Section 4 we define the functor of automorphisms of a G-structure. Due to functoriality, prolongation gives rise to inclusions of functors of automorphisms. Then we treat the case of a $\{1\}$-structure. We show that the underlying functor is representable and the Lie superalgebra of infinitesimal automorphisms is finite-dimensional. An important ingredient is that even real vector fields possess a flow, as we show in the appendix. Similar results on the functor of automorphisms of an admissible G-structure of finite type can then be deduced by embedding it into the functor of automorphisms of a $\{1\}$-structure.

Everything we have said has a direct analogue in the category of real supermanifolds, except that there are no complications caused by mixed structures and admissibility. The completeness issues remain. The analogous theorems are stated in Section 5.

Finally, in Section 6 we discuss some examples. We treat even and odd metric structures on supermanifolds and construct a canonical admissible geometric structure of finite type associated to the superization of a Riemannian spin manifold as studied in [1, 14].

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2 Recollections on supergeometry

1 Mixed supermanifolds

A *complex super vector space* is a $\mathbb{Z}/2$-graded complex vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A morphism is a grading preserving complex linear homomorphism. The resulting category is closed symmetric monoidal with respect to the evident notion of tensor product and inner hom objects.

A *general mixed super vector space* consists of the data $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$ where V is a complex super vector space, $V_{\mathbb{R}} \subseteq V$ is a real sub super vector space, $V_{\mathbb{C}} \subseteq V$ is a complex sub super vector space such that $V_{\mathbb{C}} \subseteq V_{\mathbb{R}}$ and the canonical map $\mathbb{C} \otimes V_{\mathbb{R}}/V_{\mathbb{C}} \to V/V_{\mathbb{C}}$ is an isomorphism. A *mixed super vector space* is a general mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$ such that $(V_{\mathbb{R}})_{\bar{1}} = (V_{\mathbb{C}})_{\bar{1}} = V_{\bar{1}}$. The class of these contains the classes of *super vector spaces* and *complex super vector spaces* as the extreme cases where $V_{\mathbb{C}} = V_{\bar{1}}$ and $V_{\mathbb{C}} = V$, respectively. A *real super vector space* is a general mixed super vector space of the form $(V, V_{\mathbb{R}}, 0)$. For our purposes it is not necessary to discuss the various notions of morphisms of general mixed super vector spaces at this point.

Example 2.1. One way to produce (general) mixed super vector spaces is the following. Suppose we are given a real sub super vector space $V_{\mathbb{R}}$ of a complex super vector space W. The kernel of the induced map $f\colon \mathbb{C} \otimes V_{\mathbb{R}} \to W$ is of the form $\{i \otimes v - 1 \otimes iv \mid v \in V_{\mathbb{C}}\} \cong \overline{V_{\mathbb{C}}}$ for a complex subspace $V_{\mathbb{C}} \subseteq W$ contained in $V_{\mathbb{R}}$. Then $(V = \mathrm{im}(f), V_{\mathbb{R}}, V_{\mathbb{C}})$ is a general mixed super vector space. Of course, conversely, given a general mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$, $V_{\mathbb{C}}$ can be recovered from this by applying this procedure to $V_{\mathbb{R}} \to V$. In particular, the pair $(V, V_{\mathbb{R}})$ determines $V_{\mathbb{C}}$ and the pair $(V_{\mathbb{R}}, V_{\mathbb{C}})$ determines V up to isomorphism.

This leads to various notions of supermanifolds. We will first introduce the relevant notions at the level of manifolds (without grading). Consider a purely even mixed vector space $V_{\mathbb{C}} \subseteq V_{\mathbb{R}} \subseteq V$. We denote by $\mathbb{A}(V_{\mathbb{R}})$ the locally ringed space over $\mathbb{C}$ given by the topological space $V_{\mathbb{R}}$ together with the sheaf $\mathcal{O}_{V_{\mathbb{R}}}$ of partially holomorphic functions, i.e., complex valued smooth functions whose differential is complex linear in the fibre $\mathbb{A}(V_{\mathbb{R}}) \times V_{\mathbb{C}} \subseteq \mathbb{A}(V_{\mathbb{R}}) \times V_{\mathbb{R}} = T\mathbb{A}(V_{\mathbb{R}})$.

Remark 2.2. More concretely, if we choose an isomorphism $V \cong \mathbb{C}^n \times \mathbb{C}^m$ such that $V_{\mathbb{R}} \cong \mathbb{R}^n \times \mathbb{C}^m$ and $V_{\mathbb{C}} \cong \mathbb{C}^m$, then these are complex smooth functions $\psi(x, z)$ on open subspaces of $\mathbb{R}^n \times \mathbb{C}^m$ which are holomorphic in z.

Definition 2.3. A *mixed manifold* consists of a locally ringed space $(M_0, \mathcal{O}_{M_0})$ over $\mathbb{C}$ with a second countable Hausdorff base which is locally isomorphic to $\mathbb{A}(V_{\mathbb{R}})$ for some mixed vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$. The subsheaf of real-valued functions is denoted by $\mathcal{O}_{M_0,\mathbb{R}}$. The full subcategory of locally ringed spaces over $\mathbb{C}$ with objects mixed manifolds is denoted by M^{μ}.

Remark 2.4. These are precisely the smooth manifolds locally of the form $\mathbb{R}^n \times \mathbb{C}^m$ with transition functions $(x,z) \mapsto (\varphi(x), \psi(x,z))$, where $\psi(x,z)$ is holomorphic in z. Put differently, these are manifolds endowed with a Levi flat CR-structure (cf. [2]).

Consider now a mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$. We denote by $\mathbb{A}(V_{\mathbb{R}})$ the locally ringed superspace over $\mathbb{C}$ given by the topological space $V_{\mathbb{R}\bar{0}}$ together with the structure sheaf $\mathcal{O}_{\mathbb{A}(V_{\mathbb{R}\bar{0}})} \otimes_{\mathbb{C}} \bigwedge V_{\bar{1}}^*$. Given a mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$, we can forget the mixed structure and consider the mixed super vector space (V, V, V). The associated locally ringed space will be denoted by $\mathbb{A}(V)$.

Definition 2.5. A *mixed supermanifold* is a locally ringed superspace $M = (M_0, \mathcal{O}_M)$ over $\mathbb{C}$ with a second countable Hausdorff base which is locally isomorphic to $\mathbb{A}(V_{\mathbb{R}})$ for some mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$. The full subcategory of locally ringed superspaces over $\mathbb{C}$ with objects mixed supermanifolds is denoted by SM^μ. The category SM^μ contains the full subcategories SM and $\mathsf{SM}^{\mathbb{C}}$ of *supermanifolds* and *complex supermanifolds* as the extreme cases where $V_{\mathbb{C}} = V_{\bar{1}}$ and $V_{\mathbb{C}} = V$, respectively.

The sheaf of nilpotent functions on a mixed (real) supermanifold M will be denoted by $\mathcal{J}_M$. The mixed (real) supermanifold structure on M induces the structure of a mixed (real) manifold on the locally ringed space $(M_0, \mathcal{O}_M/\mathcal{J}_M)$ which we abbreviate by abuse of notation by M_0. Moreover, we set $\mathcal{O}_{M_0} := \mathcal{O}_M/\mathcal{J}_M$. Then the inclusion $i\colon \mathsf{M}^\mu \to \mathsf{SM}^\mu$ has the right adjoint $r\colon \mathsf{SM}^\mu \to \mathsf{M}^\mu$, $M \mapsto M_0$.

Given a mixed supermanifold, we define the sheaf of *real functions* to be the pullback in the square of (real) supercommutative superalgebras

$$\begin{array}{ccc} \mathcal{O}_{M,\mathbb{R}} & \longrightarrow & \mathcal{O}_{M_0,\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathcal{O}_M & \longrightarrow & \mathcal{O}_{M_0}. \end{array}$$

We will often consider a mixed supermanifold as a set-valued functor on SM^μ by the assignment $T \mapsto \mathsf{SM}^\mu(T, M)$. Then there is a natural transformation of functors $M \to r^* i^* M = r^* M_0$ which is given by sending a map $T \to M$ to its underlying map $T_0 \to M_0$. The second part of the next lemma is only the first encounter of the typical reality condition enforced by a mixed structure.

Lemma 2.6. *Consider a mixed super vector space* $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$.

(a) There is a natural isomorphism $\mathsf{SM}^\mu(M, \mathbb{A}(V)) \cong \Gamma(\mathcal{O}_M \otimes_{\mathbb{C}} V)_{\bar{0}}$.

(b) *The following diagram is a pullback of functors on* SM^μ*:*

$$\begin{array}{ccc} \mathbb{A}(V_\mathbb{R}) & \longrightarrow & r^*\mathbb{A}(V_{\mathbb{R}\bar{0}}) \\ \downarrow & & \downarrow \\ \mathbb{A}(V) & \longrightarrow & r^*\mathbb{A}(V_{\bar{0}}). \end{array}$$

In other words, we have

$$\mathsf{SM}^\mu(M, \mathbb{A}(V_\mathbb{R})) \cong \Gamma(\mathcal{O}_{M,\mathbb{R},\bar{0}} \otimes_\mathbb{R} (V_\mathbb{R}/V_\mathbb{C})_{\bar{0}}) \oplus \Gamma(\mathcal{O}_{M,\bar{0}} \otimes_\mathbb{C} (V_\mathbb{C})_{\bar{0}}) \oplus \Gamma(\mathcal{O}_{M,\bar{1}} \otimes_\mathbb{C} V_{\bar{1}}).$$

Proof. The proof is similar to that in [6, Theorem 4.1.11]. $\square$

Corollary 2.7. *The category* SM^μ *admits all finite products and the full subcategory* M^μ *is closed under finite products in* SM^μ*.*

Let M_0 be a mixed manifold. Consider the sheaf $\mathcal{T}_{M_0}$ whose sections over U_0 are complex linear derivations of $\mathcal{O}_{M_0}|_{U_0}$ and the subsheaf $\mathcal{T}_{M_0,\mathbb{R}}$ of those derivations which restrict to derivations of $\mathcal{O}_{M_0,\mathbb{R}}|_{U_0}$. Then $\mathcal{T}_{M_0,\mathbb{R}}$ contains a complex ideal $\mathcal{T}_{M_0,\mathbb{C}}$ of derivations which annihilate $\mathcal{O}_{M_0,\mathbb{R}}|_{U_0}$ and the quotient by this sheaf is isomorphic to the sheaf of derivations of $\mathcal{O}_{M_0,\mathbb{R}}$.

Now, if M is a mixed supermanifold, the complex tangent sheaf is the sheaf $\mathcal{T}_M$ whose sections over U_0 are the complex linear superderivations of $\mathcal{O}_M|_{U_0}$. By analogy with the definition of the real functions, one defines the *real tangent sheaf* by the pullback

$$\begin{array}{ccc} \mathcal{T}_{M,\mathbb{R}} & \longrightarrow & \mathcal{T}_{M_0,\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathcal{T}_M & \longrightarrow & \mathcal{T}_{M_0}, \end{array}$$

where the lower arrow takes a vector field to its underlying vector field.

An important point is that, although $\mathcal{T}_{M,\mathbb{R}}$ is not closed under brackets, its even part is and consists of those derivations which restrict to derivations of $\mathcal{O}_{M,\mathbb{R}}$. In analogy, one defines $\mathcal{T}_{M,\mathbb{C}} \subseteq \mathcal{T}_{M,\mathbb{R}}$ in terms of $\mathcal{T}_M$, $\mathcal{T}_{M_0}$ and $\mathcal{T}_{M_0,\mathbb{C}}$. Then $(\mathcal{T}_{M,\mathbb{C}})_{\bar{0}} \subseteq (\mathcal{T}_{M,\mathbb{R}})_{\bar{0}}$ is an ideal.

The tangent space T_mM at $m \in M_0$ is the complex super vector space of complex derivations $\mathcal{O}_{M,m} \to \mathbb{C}$. This comes with a mixed structure by considering the real subspace $(T_mM)_\mathbb{R}$ consisting of those derivations which induce a derivation $\mathcal{O}_{M_0,\mathbb{R},m} \to \mathbb{R}$ together with its complex subspace $(T_mM)_\mathbb{C}$ of those derivations in $(T_mM)_\mathbb{R}$ which vanish on $\mathcal{O}_{M_0,\mathbb{R},m}$.

2 Mixed Lie supergroups

In this section we give a brief review of basic results concerning mixed Lie supergroups.

Definition 2.8. A *mixed Lie supergroup* is a group object in SM^μ.

2.1 Equivalence of mixed Lie supergroups and mixed super pairs

First we characterize mixed Lie groups, i.e., mixed Lie supergroups with trivial odd direction. For a real (resp. mixed) Lie group G we will use the notation $\mathrm{Lie}_\mathbb{R}(G)$ (resp. $\mathrm{Lie}_\mathbb{C}(G)$) for the Lie algebra of left-invariant derivations of the sheaf of real valued smooth functions (resp. sheaf of complex functions).

We define a *mixed pair* to be a pair $(\mathfrak{g}_\mathbb{C}, G^{sm})$ consisting of a real Lie group G^{sm} and an $\mathrm{Ad}_{G^{sm}}$-invariant ideal $\mathfrak{g}_\mathbb{C} \subseteq \mathrm{Lie}_\mathbb{R}(G^{sm})$ endowed with a complex structure which is respected by the adjoint action of G^{sm}.

A morphism of such pairs is a morphism of Lie groups such that the differential at the identity respects the complex ideals.

Lemma 2.9. *The categories of mixed Lie groups and mixed pairs are equivalent.*

Proof. This follows from the Baker–Campbell–Hausdorff formula as in the case of complex analytic structures on Lie groups. □

As usual, the adjoint representation of a mixed Lie group G is the differential at the identity of the conjugation action of G on itself. It can be seen as a mixed morphism $G \times \mathbb{A}(\mathfrak{g}_\mathbb{R}) \to \mathbb{A}(\mathfrak{g}_\mathbb{R})$.

Now, we turn our attention to mixed Lie supergroups. A *mixed super pair* consists of a pair $(\mathfrak{g}, G_0)$ where G_0 is a mixed Lie group and $\mathfrak{g}$ is a complex Lie superalgebra together with

(a) an isomorphism $\mathrm{Lie}_\mathbb{C}(G_0) \cong \mathfrak{g}_{\bar{0}}$, and

(b) an action $\sigma\colon G_0 \times \mathbb{A}(\mathfrak{g}) \to \mathbb{A}(\mathfrak{g})$ such that $\sigma(g)|_{\mathbb{A}(\mathfrak{g}_{\bar{0},\mathbb{R}})} = \mathrm{Ad}_{G_0}$ and the differential of σ acts as the adjoint representation

$$d\sigma(X)(Y) = [X, Y].$$

There is an evident notion of a morphism of mixed super pairs, and the following result follows along the same lines as the corresponding for real and complex Lie supergroups.

Proposition 2.10. *The categories of mixed super pairs and mixed Lie supergroups are equivalent.*

Proof. See [6, 7.4]. □

An important notion is the following.

Definition 2.11. A *mixed real form* of a complex Lie supergroup G is a mixed Lie supergroup $G_{\mathbb{R}}$ together with a group morphism $i\colon G_{\mathbb{R}} \to G$ such that $i_0\colon (G_{\mathbb{R}})_0 \to G_0$ is the inclusion of a closed subgroup and $di_e\colon T_e(G_{\mathbb{R}}) \to T_e(G)$ is an isomorphism.

Remark 2.12. Any mixed real form $G_{\mathbb{R}} \leqslant G$ yields a mixed real form $(G_{\mathbb{R}})_0 \leqslant G_0$. Conversely, given a mixed real form $(G_0)_{\mathbb{R}} \leqslant G_0$, the pullback

$$\begin{array}{ccc} G_{\mathbb{R}} & \longrightarrow & r^*(G_0)_{\mathbb{R}} \\ \downarrow & & \downarrow \\ G & \longrightarrow & r^*G_0 \end{array}$$

is representable and defines a mixed real form of G. For that reason, we will adopt the notation $(G_{\mathbb{R}})_0 = (G_0)_{\mathbb{R}} = G_{0,\mathbb{R}}$.

Example 2.13. Finally, we come to discuss the example of linear supergroups. Let $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$ be a mixed super vector space. Then we have the complex Lie supergroup $GL(V)$ given by the complex group $GL(V_{\bar{0}}) \times GL(V_{\bar{1}})$ and the Lie superalgebra $\mathfrak{gl}(V)$. An element of $GL(V)(T)$ is given by an automorphism over T of the trivial vector bundle $\underline{V}_T = T \times \mathbb{A}(V) \to T$.

Consider the subgroups of those even invertible isomorphisms of V respecting $V_{\mathbb{C}}$ or the pair $V_{\mathbb{C}} \subseteq V_{\mathbb{R}}$. We will denote them by

$$GL^{\mu}(V)_{0,\mathbb{R}} \leqslant GL^{\mu}(V)_0 \leqslant GL(V)_0.$$

We then define the two group-valued functors $GL^{\mu}(V)$ and $GL^{\mu}(V)_{\mathbb{R}}$ on SM^{μ} by the pullback

$$\begin{array}{ccc} GL^{\mu}(V)_{(\mathbb{R})} & \longrightarrow & r^*GL^{\mu}(V)_{0,(\mathbb{R})} \\ \downarrow & & \downarrow \\ GL(V) & \longrightarrow & r^*GL(V)_0, \end{array}$$

where it is understood that the quantities in parentheses are only present in the latter case.

The inclusion $\mathrm{Lie}_{\mathbb{R}}(GL^{\mu}(V)_{0,\mathbb{R}}) \subseteq \mathfrak{gl}(V)_{\bar{0}}$ only defines a mixed structure in the cases $V_{\mathbb{C}} = V_{\bar{1}}$ and $V_{\mathbb{C}} = V$. In this case $GL^{\mu}(V)_{\mathbb{R}}$ is representable and is a mixed real form of $GL(V)$. In general, $GL^{\mu}(V)_{(\mathbb{R})}$ is not representable.

2.2 Actions of mixed Lie supergroups and their point functors

A *left action* of the mixed Lie supergroup G on the mixed supermanifold M is given by a unital and associative map $a\colon G \times M \to M$. The map $a^{\sharp}$ can be made explicit in terms of two more basic objects. First, let $\underline{a}$ denote the action $G_0 \times M \to G \times M \to M$. Then any

$g \in G_0$ (considered as a map $g\colon \mathbb{A}(\{0\}) \to G$) gives a map

$$a_g\colon M \cong \mathbb{A}(\{0\}) \times M \xrightarrow{g\times M} G_0 \times M \xrightarrow{a} M.$$

Secondly, the action gives rise to a Lie superalgebra antimorphism

$$\rho\colon \mathfrak{g} \longrightarrow \Gamma(\mathcal{T}_M),\ X \mapsto (e \times M)^\sharp \circ (X \otimes 1) \circ a^\sharp \tag{2.14}$$

and we have

(a) $\rho|_{\mathfrak{g}_{\bar{0}}}(X) = (X \otimes 1) \circ \underline{a}^\sharp$, and

(b) $\rho(g \cdot Y) = (a_g^{-1})^\sharp \cdot \rho(Y) \cdot a_g^\sharp$.

Conversely, given an action $\underline{a}\colon G_0 \times M \to M$ and ρ satisfying (a) and (b), then one can construct an action $G \times M \to M$ (cf. [6, Prop. 8.3.2, 8.3.3]).

Now let G be a mixed Lie group and M a mixed supermanifold and consider an action $a^{sm}\colon G^{sm} \times M \to M$. This gives rise to a Lie algebra morphism $\mathfrak{g}_\mathbb{R} \to \Gamma(\mathcal{T}_{M,\mathbb{R}})_{\bar{0}}$. The connection between such an action and an action of G is made precise in the next lemma.

Lemma 2.15. *The action a^{sm} extends to an action $a\colon G \times M \to M$ if and only if $\mathfrak{g}$ fits into the following square*

$$\begin{array}{ccc} \mathfrak{g}_\mathbb{R} & \longrightarrow & \Gamma(\mathcal{T}_{M,\mathbb{R}})_{\bar{0}} \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \Gamma(\mathcal{T}_M)_{\bar{0}} \end{array}$$

the lower horizontal arrow being an antimorphism of complex Lie algebras. The extension is unique if it exists. Equivalently, the restriction of the upper horizontal arrow to $\mathfrak{g}_\mathbb{C}$ factors as a complex linear map through $\Gamma(\mathcal{T}_{M,\mathbb{C}})_{\bar{0}}$.

Proof. Uniqueness is clear since any element $X \in \mathfrak{g}$ can be written in the form $X_1 + iX_2$ for some $X_j \in \mathfrak{g}_\mathbb{R}$. If the extension in the diagram exists, then the differential $TG_\mathbb{R}^{sm} \times TM \to TM$ is complex linear on $TG_\mathbb{C}^{sm} \times TM \to TM$, which proves that the action extends to $G \times M$. $\square$

Let T be an arbitrary mixed supermanifold. Consider a morphism $\varphi_0\colon T \to G$ and a homogeneous derivation $X\colon \mathcal{O}_G \to (e_{T0})_*\mathcal{O}_T$ along $e_T\colon T \to * \to G$.

Given this, we construct a homogeneous derivation along φ_0 as follows:

$$\varphi_0 \cdot X\colon \mathcal{O}_G \xrightarrow{(\varphi_0\times T)^\sharp\circ(1\otimes X)\circ\mu^\sharp} (\mu_0)_*(\varphi_0 \times e_T)_{0_*}\mathcal{O}_{T\times T} \xrightarrow{\quad\Delta^\sharp\quad} (\varphi_0)_*\mathcal{O}_T.$$

Similarly, for two homogeneous derivations X and Y we set

$$X \cdot Y := \Delta^\sharp \circ (\mu_0)_*((X \otimes 1) \circ (1 \otimes Y)) \circ \mu^\sharp.$$

Now, suppose G acts on M and let X and Y be as above. We set

$$\rho(X)\colon \mathcal{O}_{T\times M} \xrightarrow{(1\otimes X\otimes 1)\circ(T\times a)^\sharp} \mathcal{O}_{T\times T\times M} \xrightarrow{(\Delta\times M)^\sharp} \mathcal{O}_{T\times M}.$$

Then $\rho(X)$ is the $\mathcal{O}_T$-linearization of $\rho(X)\circ p_T^\sharp$, where $p_T\colon T\times M\to M$ is the projection. From the associativity of the action it follows that

$$\rho(X\cdot Y) = (-1)^{|X||Y|}\rho(Y)\circ\rho(X).$$

Let $n\geq 0$, then $\Gamma(\mathcal{O}_{\mathbb{A}(\mathbb{C}^{0|n})})$ is the exterior algebra on generators η_i. As usual, given a non-empty subset $I\subset\{1,\dots,n\}$, we set $\eta^I = \prod_{i\in I}\eta_i$, where we implicitly use the ordering on I induced from the standard ordering on $\{1,\dots,n\}$.

Lemma 2.16. *Suppose G is mixed and acts on the mixed supermanifold M.*

(a) Any $\varphi\in G(\mathbb{A}(\mathbb{C}^{0|n})\times T)$ is uniquely determined by $\varphi_0\in G(T)$ and homogeneous derivations X_I along e_T of degree $|I|$ and

$$\varphi^\sharp = \varphi_0^\sharp\cdot\prod_{k=1}^{n}\Big(1+\sum_{k\in I\subseteq\{1,\dots,k\}}\eta^I X_I\Big).$$

(b) Moreover, under this identification, the morphism a_φ, defined as the composition

$$(\mathbb{A}(\mathbb{C}^{0|n})\times T\times a)\circ((\mathrm{pr}_{\mathbb{A}(\mathbb{C}^{0|n})\times T},\varphi)\times M)\colon \mathbb{A}(\mathbb{C}^{0|n})\times T\times M\longrightarrow \mathbb{A}(\mathbb{C}^{0|n})\times T\times M,$$

takes the form

$$a_\varphi^\sharp = \prod_{k=n}^{1}\Big(1+\sum_{k\in I\subseteq\{1,\dots,k\}}\eta^I\rho(X_I)\Big)\cdot a_{\varphi_0}^\sharp.$$

Proof. The first part is proved by induction on n and the second part then boils down to $(\mu\times M)^\sharp\circ a^\sharp = (G\times a)^\sharp\circ a^\sharp$. □

3 Mixed real forms of principal G-bundles

Suppose we are given a mixed supermanifold M and a group-valued functor G on SM^μ. A principal G-bundle is a functor P on SM^μ together with a right G-action and a map $\pi\colon P\to M$ equivariant with respect to the trivial action on M such that for each $m\in M_0$ there exist an open neighbourhood U and equivariant isomorphisms $U\times G\to P|_U$ over U. This reduces to the usual definition if G is representable.

Later we will need to build real forms of certain principal bundles. This will be done so with the help of the following lemma.

Lemma 2.17. *Let G be a complex Lie supergroup with mixed real form $G_{\mathbb{R}}$. Let $P \to M$ be a principal G-bundle over a mixed supermanifold M and $P_{0,\mathbb{R}} \to M_0$ a reduction of P_0 to $G_{0,\mathbb{R}}$. Then the pullback*

$$\begin{array}{ccc} P_{\mathbb{R}} & \longrightarrow & r^*(P_{0,\mathbb{R}}) \\ \downarrow & & \downarrow \\ P & \longrightarrow & r^*(P_0) \end{array}$$

is a principal $G_{\mathbb{R}}$-bundle.

Proof. We observe that $G_{\mathbb{R}}$ acts on $P_{\mathbb{R}}$ by the universal property of the pullback and the map $P_{\mathbb{R}} \to P \to M$ is equivariant with respect to this action. So we only need to show local triviality. We choose trivializations $\psi_i \colon U_i \times G \to P|_{U_i}$ on coordinate charts $U_i = \mathbb{A}(V_{\mathbb{R}})$ on M. They come with retractions $r_i \colon U_i \to (U_i)_0$. Without loss of generality, we may assume that $P_{0,\mathbb{R}}|_{(U_i)_0}$ is trivial, too, say by maps $\varphi_i \colon (U_i)_0 \times G_{0\mathbb{R}} \to P_{0,\mathbb{R}}|_{(U_i)_0}$. The φ_i induce trivializations $\tilde{\varphi}_i \colon (U_i)_0 \times G_0 \to P_0|_{(U_i)_0}$ which differ from $(\psi_i)_0$ by maps $g_i \colon (U_i)_0 \to G_0$ in the sense that

$$\tilde{\varphi}_i = (\psi_i)_0 \circ ((U_i)_0 \times a_0) \circ ((\mathrm{id}_{(U_i)_0}, g_i) \times G_0) \colon (U_i)_0 \times G_0 \longrightarrow P_0|_{(U_i)_0}.$$

Denoting by $\tilde{a}$ the composition $G_0 \times G \to G \times G \to G$, we now set

$$\tilde{\psi}_i = \psi_i \circ (U_i \times \tilde{a}) \circ (U_i \times g_i \times G) \circ ((\mathrm{id}_{U_i}, r_i) \times G) \colon U_i \times G \longrightarrow P|_{U_i},$$

which is still a trivialization. Then $(\tilde{\psi}_i)_0 = \tilde{\varphi}_i$, and the universal property of the pullback now shows that $\tilde{\psi}_i$, restricted to $U_i \times G_{\mathbb{R}}$, gives a trivialization of $P_{\mathbb{R}}|_{U_i}$. □

4 Tangent bundles and frame bundles of mixed supermanifolds

Suppose M is a mixed supermanifold locally modelled on the mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$. The sheaf $\mathcal{T}_M$ is locally free on V and glueing leads to the mixed total space $TM \to M$. If $i \colon M_0 \to M$ is the canonical inclusion, then

$$i^*TM = TM_{\bar{0}} \oplus TM_{\bar{1}}$$

for certain complex bundles $(TM)_{\bar{j}} \to M_0$ (in the category of mixed manifolds). Actually, we have $(TM)_{\bar{0}} = TM_0$.

Define $\underline{V}_T = T \times \mathbb{A}(V) \to T$ to be the trivial vector bundle over T with fibre $\mathbb{A}(V)$. There is a vector bundle of homomorphisms $\mathrm{Hom}(\underline{V}_M, TM) \to M$ and the T-points of the total space are given by squares of vector bundles

$$\begin{array}{ccc} \underline{V}_T & \xrightarrow{\varphi'} & TM \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & M. \end{array} \tag{2.18}$$

Equivalently, a T-point consists of a tuple (f,φ) consisting of a map $f\colon T \to M$ and a map $\varphi\colon \underline{V}_T \to f^*(TM)$ of vector bundles over T.

The frame bundle of M is the open subsupermanifold of $\mathrm{Hom}(\underline{V}_M, TM)$ characterized by

$$L(M)(T) = \{(f,\varphi) \in \mathrm{Hom}(\underline{V}_M, TM)(T) \mid \varphi \text{ isomorphism}\}.$$

In terms of squares: $(f,\varphi) \in L(M)(T)$ if and only if the associated square (2.18) is a pullback. This is a principal $GL(V)$-bundle over M.

We have $L(M)_0 = L(TM_{\bar{0}}) \times_M L(TM_{\bar{1}})$, and thus the mixed structure of M yields subbundles

$$L^\mu(M)_{0,\mathbb{R}} \longrightarrow L^\mu(M)_0 \longrightarrow L(M)_0,$$

where $L^\mu(M)_0$ (resp. $L^\mu(M)_{0,\mathbb{R}}$) is the subbundle of those frames which map $V_\mathbb{C}$ to $TM_\mathbb{C}$ (resp. $(V_\mathbb{R}, V_\mathbb{R})$ to $(TM_\mathbb{R}, TM_\mathbb{C})$). By pulling back, we obtain the bundles

$$\begin{array}{ccc} L^\mu(M)_{(\mathbb{R})} & \longrightarrow & r^*L^\mu(M)_{0,(\mathbb{R})} \\ \downarrow & & \downarrow \\ L(M) & \longrightarrow & r^*L(M)_0. \end{array}$$

The structure group of $L^\mu(M)_{(\mathbb{R})}$ is precisely $GL^\mu(V)_{(\mathbb{R})}$, and this functor of frames is representable precisely for supermanifolds and complex supermanifolds, that is, in terms of local models $V_\mathbb{C} \in \{V_{\bar{1}}, V\}$.

All these principal bundles have associated bundles that fit in a square

$$\begin{array}{ccc} TM_\mathbb{R} & \longrightarrow & r^*T(M_0)_\mathbb{R} \\ \downarrow & & \downarrow \\ TM & \longrightarrow & r^*T(M_0), \end{array}$$

which is a pullback in view of the pullback square defining $\mathcal{T}_{M,\mathbb{R}}$ in terms of $\mathcal{T}_M$, $\mathcal{T}_{M_0}$ and $\mathcal{T}_{M_0,\mathbb{R}}$.

3 Geometric structures on mixed supermanifolds

We can now define the notion of a geometric structure on a mixed supermanifold. Let $G \leqslant GL(V)$ be a closed mixed Lie subgroup, i.e., $G_0^{sm} \leqslant GL(V)_0^{sm}$ is closed and $G_0 \leqslant GL(V)_0$ is a mixed embedding.

1 Basic definitions

Definition 3.1. A *G-structure* on M is a reduction P of $L^\mu(M)_\mathbb{R}$ to G. Equivalently, it is a reduction P of $L(M)$ such that $P_0 \to L(M)_0$ factors through $L^\mu(M)_{0,\mathbb{R}}$.

Any G-structure P comes with a canonical 1-form $\vartheta\colon TP \to \underline{V}_P$. It sends a pair $(f, X) \in TP(T)$, considered as the data of a map $f = (\pi \circ f, \varphi)\colon T \to P$ and a section X of $f^*(TP)$, to the composite

$$T \xrightarrow{X} f^*(TP) \xrightarrow{f^*(d\pi)} (\pi \circ f)^*(TM) \xrightarrow{\varphi^{-1}} \underline{V}_T \xrightarrow{f \times \mathrm{id}_V} \underline{V}_P.$$

The differential of the canonical 1-form $\vartheta\colon TP \to \underline{V}_P$ is a 2-form $d\vartheta\colon \Lambda^2 TP \to \underline{V}_P$.

Lemma 3.2. *Let $\mathcal{V}\colon P \times \mathfrak{g} \to TP$ be the restriction of the differential of the action $P \times G \to P$. For all $A\colon S \to \underline{\mathfrak{g}}_P$ and $x\colon S \to TP$ with same underlying map $S \to P$ we have*

$$d\vartheta \circ (\mathcal{V}(A) \wedge x) = -A(\vartheta(x))\colon S \longrightarrow \underline{V}_P.$$

Proof. This is Proposition 4 in [9]. □

2 Prolongation

2.1 Unrestricted prolongation

Adapting the classical construction [12], we will in this subsection associate with a G-structure P on M a tower of prolongations

$$\cdots \longrightarrow P^{(k)} \longrightarrow P^{(k-1)} \longrightarrow \cdots \longrightarrow P^{(1)} \longrightarrow P^{(0)} = P \longrightarrow M,$$

where $P^{(i+1)} \to P^{(i)}$ is a reduction of $L(P^{(i)})$ to $G^{(i+1)}$. Here $G^{(0)} = G$ and $G^{(i)}$ is a vector group for all $i \geq 1$.

Remark 3.3. Given a super vector space, the associated supergroup structure on $\mathbb{A}(V)$ will be denoted by V. More generally, if a Lie supergroup G acts linearly on a complex super vector space V, then the associated semi-direct product will be denoted by $G \ltimes V$ instead of $G \ltimes \mathbb{A}(V)$.

It will be convenient to introduce a name for the representation of G on V: $\alpha\colon G \to GL(V)$. Applying $\mathbb{J}^1_V(-)$ to $G \to P \to M$ yields a principal $\mathbb{J}^1_V G$-bundle $\mathbb{J}^1_V G \to \mathbb{J}^1_V P \to \mathbb{J}^1_V M$ and the usual identification $TG \cong G \ltimes_{\mathrm{ad}} \mathfrak{g}$ gives an isomorphism of groups $\mathbb{J}^1_V(G) \cong G \ltimes_{\mathrm{ad}} \underline{\mathrm{Hom}}(V, \mathfrak{g})$, where G acts via its adjoint representation on $\mathfrak{g}$. The bundle of horizontal frames is defined by the pullback

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathbb{J}^1_V P \\ {\scriptstyle d\pi_*}\downarrow & & \downarrow \\ P & \longrightarrow & \mathbb{J}^1_V M. \end{array}$$

Its S-points are the squares

$$\begin{array}{ccc} \underline{V}_S & \xrightarrow{h} & TP \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & P \end{array}$$

such that the composite square

$$\begin{array}{ccccc} \underline{V}_S & \longrightarrow & TP & \longrightarrow & TM \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & P & \longrightarrow & M \end{array}$$

lies in $P(S)$. Moreover, $\mathcal{H}$ is the total space of a principal $G \ltimes_{\mathrm{ad}} \underline{\mathrm{Hom}}(V, \mathfrak{g})$-bundle with respect to the map $d\pi_*$. We need to construct an action of $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$. The group G acts via α on $\mathbb{J}^1_V(P)$ by precomposition. Together with the action of $\underline{\mathrm{Hom}}(V, \mathfrak{g}) \leqslant \mathbb{J}^1_V(G)$, this yields an action of $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$ on $\mathbb{J}^1_V(P)$, which restricts to an action on $\mathcal{H}$. The composition $\iota_P\colon \mathcal{H} \to \mathbb{J}^1_V P \to P$ is equivariant if we let $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$ act trivially on P. Moreover, $d\pi_*$ is equivariant with respect to this action if we let $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$ act on P via the projection to G.

The canonical vertical distribution $\mathcal{V}\colon \underline{\mathfrak{g}}_P \to TP$ gives rise to a map $\mathbb{J}^1_V P \to \mathbb{J}^1_{V \oplus \mathfrak{g}} P$ and the composition $\mathcal{H} \to \mathbb{J}^1_{V \oplus \mathfrak{g}} P$ factors through $L(P)$. Moreover, the $GL(V \oplus \mathfrak{g})$-action on $L(P)$ is seen to restrict to the action of $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g}) \leqslant GL(V \oplus \mathfrak{g})$. This identifies $\iota_P\colon \mathcal{H} \to P$ as a reduction of $L(P)$ to $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$.

As usual, $\mathfrak{g}^{(1)}$ is defined to be the kernel of the super-antisymmetrizer

$$\partial\colon \underline{\mathrm{Hom}}(V, \mathfrak{g}) \longrightarrow \underline{\mathrm{Hom}}(\Lambda^2 V, V), \; (\partial S)(v, w) = \tfrac{1}{2}(S(v)(w) - (-1)^{|v||w|} S(w)(v)),$$

and the first prolongation $P^{(1)} \to P$ is obtained from $\mathcal{H} \to P$ by two successive reductions of the structure group to $\mathfrak{g}^{(1)}$ using the following lemma.

Lemma 3.4. *Consider a short exact sequence of mixed Lie supergroups*

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1.$$

Let $\pi\colon P \to B$ be a G-principal bundle and assume that there is a G-equivariant map $f\colon P \to K$. Then $P/H \to B$ is a principal K-bundle and as such isomorphic to the trivial bundle. Moreover, the map $(\pi, f)\colon P \to B \times K$ is a principal H-bundle.

Proof. Since any map of principal bundles is an isomorphism, it suffices to construct a K-equivariant map $P/H \to B \times K$ over B. But such a map can be constructed from the G-equivariant map $(\pi, f)\colon P \to B \times K$ since H acts trivially on the target. $\square$

The first step is a reduction to $\underline{\mathrm{Hom}}(V, \mathfrak{g}) \leqslant G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$. We have two maps $d\pi_*$, $\iota_P\colon \mathcal{H} \to P$ over the same map to the base M. Fibrewise comparison yields a map $d\colon \mathcal{H} \to G$. It follows now from the equivariance properties of $d\pi_*$ and ι_P that d is $G \ltimes_\alpha \underline{\mathrm{Hom}}(V, \mathfrak{g})$-equivariant if we let this group act from the right on G by $g \cdot (g', \varphi) = (g')^{-1} g$. Now we can apply Lemma 3.4 and see that $(\iota_P, d)\colon \mathcal{H} \to P \times G$ is a principal $\underline{\mathrm{Hom}}(V, \mathfrak{g})$-bundle. Pulling back along the inclusion $P \times \{1\} \hookrightarrow P \times G$ yields the bundle of *compatible horizontal frames* $\mathcal{CH} \to P$, a reduction of $L(P)$ to the group $\underline{\mathrm{Hom}}(V, \mathfrak{g})$. Its S-points consist of those squares (f, h) such that $T(\pi) \circ h = f \in P(S)$.

The second reduction is a little bit more elaborate. For a section $v\colon T \to \underline{V}_T$ and a map $f\colon T \to P$, we will use the shorthand $v_f := (f \times \mathbb{A}(V)) \circ v\colon T \to \underline{V}_P$.

Lemma 3.5. *For all compatible horizontal frames $(f, h) \in \mathcal{CH}(T)$ and all sections $x\colon T \to \underline{V}_T$, we have $\vartheta(h(x)) = x_f$:*

$$\begin{array}{ccccc} T & \xrightarrow{x} & \underline{V}_T & \xrightarrow{h} & TP \\ & \searrow^{x_f} & & & \downarrow \vartheta \\ & & & & \underline{V}_P. \end{array}$$

Proof. This follows immediately from the definition. $\square$

Consider $(f, h) \in \mathcal{CH}(S)$. The *torsion* is defined to be the composition

$$c(f, h)\colon \Lambda^2 \underline{V}_S \xrightarrow{\Lambda^2 h} \Lambda^2 TP \xrightarrow{d\vartheta} \underline{V}_P.$$

Equivalently, it is given by a map

$$c'(f, h)\colon S \longrightarrow \underline{\mathrm{Hom}}(\Lambda^2 V, V).$$

By naturality, we obtain a map $c\colon \mathcal{CH} \to \underline{\mathrm{Hom}}(\Lambda^2 V, V)$. Now consider two distinguished squares over f with horizontal parts h and h'. As $\mathcal{CH} \to P$ is a principal $\underline{\mathrm{Hom}}(V, \mathfrak{g})$-bundle there is a unique map $S_{(f,h'),(f,h)}\colon \underline{V}_S \to \underline{\mathfrak{g}}_P$ over f such that $h' = h + \mathcal{V} \circ S_{(f,h'),(f,h)}$. By adjointness this can be viewed as a map $S'_{(f,h'),(f,h)}\colon S \to \underline{\mathrm{Hom}}(V,\mathfrak{g})$. Then, by Lemmas 3.2 and 3.5, we have that for any two sections $v, w\colon S \to \underline{V}_S$

$$\begin{aligned}
c(f,h')(v\wedge w) - c(f,h)(v\wedge w) &= d\vartheta\circ h'(v)\wedge h'(w) - d\vartheta\circ h(v)\wedge h(w)\\
&= d\vartheta\circ(h'(v)-h(v))\wedge h'(w) + d\vartheta\circ h(v)\wedge(h'(w)-h(w))\\
&= d\vartheta\circ(\mathcal{V}\circ S_{(f,h'),(f,h)}(v))\wedge h'(w) + d\vartheta\circ h(v)\wedge(\mathcal{V}\circ S_{(f,h'),(f,h)}(w))\\
&= -S_{(f,h'),(f,h)}(v)(\vartheta(h'(w))) - d\vartheta\circ(\mathcal{V}\circ S_{(f,h'),(f,h)}(w))\wedge h(v)\\
&= -S_{(f,h'),(f,h)}(v)(\vartheta(h'(w))) + S_{(f,h'),(f,h)}(w)(\vartheta(h(v)))\\
&= -S_{(f,h'),(f,h)}(v)(w_f) + S_{(f,h'),(f,h)}(w)(v_f).
\end{aligned}$$

In other words,

$$c'(f,h') - c'(f,h) = -2\partial S'_{(f,h),(f,h')}$$

and if we let $\underline{\mathrm{Hom}}(V,\mathfrak{g})$ act on $\underline{\mathrm{Hom}}(\Lambda^2 V, V)$ via $(-2)\partial$, then $c\colon \mathcal{CH} \to \underline{\mathrm{Hom}}(\Lambda^2 V, V)$ is $\underline{\mathrm{Hom}}(V,\mathfrak{g})$-equivariant. Now, we have the exact sequence

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \underline{\mathrm{Hom}}(V,\mathfrak{g}) \xrightarrow{\partial} \underline{\mathrm{Hom}}(\Lambda^2 V, V) \longrightarrow H^{0,2}(V,\mathfrak{g}) \longrightarrow 0.$$

(Here, $H^{0,2}(V,\mathfrak{g})$ denotes the $(0,2)$th Spencer cohomology [13].) Consequently, any splitting s of $\mathrm{im}(\partial) \to \underline{\mathrm{Hom}}(\Lambda^2 V, V)$ gives rise to an equivariant map $\mathcal{CH} \to \mathrm{im}(\partial)$ and Lemma 3.4 applied to the short exact sequence

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \underline{\mathrm{Hom}}(V,\mathfrak{g}) \xrightarrow{\partial} \mathrm{im}(\partial) \longrightarrow 0$$

shows that $\mathcal{CH} \to P \times \mathrm{im}(\partial)$ is a principal $\mathfrak{g}^{(1)}$-bundle. Finally, by pulling back along $P\times\{0\} \to P\times \mathrm{im}(\partial)$ one obtains the *first prolongation* $P^{(1)} \to P$, a reduction of $L(P)$ to $\mathfrak{g}^{(1)}$ which consists of those compatible horizontal frames with torsion contained in $C := \ker(s)$.

The higher prolongations are now defined inductively: $P^{(i+1)} := (P^{(i)})^{(1)}$. Setting $\mathfrak{g}^{(-1)} := V$ and $\mathfrak{g}^{(0)} := \mathfrak{g}$, we arrive at the following inductive description of $\mathfrak{g}^{(k)}$ for $k \geq 1$:

$$\mathfrak{g}^{(k)} = \{X \in \underline{\mathrm{Hom}}(\mathfrak{g}^{(-1)}, \mathfrak{g}^{(k-1)}) \mid X(v)(w) = (-1)^{|v||w|}X(w)(v) \text{ for all homog. } v,\ w\}.$$

By inspection, we have

$$(\mathfrak{g}^{(1)})_{\bar{0}} \subseteq (\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^{\mu} \subseteq \underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}},$$

i.e., any $f \in (\mathfrak{g}^{(1)})_{\bar{0}}$ satisfies $f(V_{\mathbb{C}}) \subseteq \mathfrak{g}_{\mathbb{C}}$. This implies that $P^{(k)} \subseteq L^{\mu}(P^{(k-1)})$.

2.2 The real prolongation

The prolongations $P^{(k+1)} \to P^{(k)}$ defined so far only provide reductions of $L^\mu(P^{(k-1)})$. To prove representability for the functor of automorphisms of a G-structure of finite type, we need to single out the *real prolongation* which provides a reduction of $L^\mu(P^{(k-1)})_{\mathbb{R}}$. For this to be possible, we need to impose a condition on the G-structure.

To that end, consider the subspaces

$$(\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^\mu_{\mathbb{R}} \subseteq (\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^\mu \subseteq \underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}}$$

consisting of even linear maps f satisfying $f(V_{\mathbb{C}}) \subseteq \mathfrak{g}_{\mathbb{C}}$ or $f(V_{\mathbb{R}}, V_{\mathbb{C}}) \subseteq (\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{C}})$, respectively.

Recall the bundle of compatible horizontal frames with the map $\mathcal{CH} \to P \times \mathrm{im}(\partial)$. One readily constructs $(\mathcal{CH}_0)^\mu_{\mathbb{R}} \subseteq (\mathcal{CH}_0)^\mu \subseteq \mathcal{CH}_0$ with structure groups $(\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^\mu_{(\mathbb{R})}$. Pullback along the inclusion $P_0 \times \{0\} \to P_0 \times \mathrm{im}(\partial)_{\bar{0}}$ yields $(P_0^{(1)})^\mu_{\mathbb{R}} \subseteq (P_0^{(1)})^\mu \subseteq P_0^{(1)}$ with structure groups given by the pullback

$$\begin{array}{ccc} (\mathfrak{g}^{(1)})_{\bar{0},\mathbb{R}} & \longrightarrow & (\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^\mu_{(\mathbb{R})} \\ \downarrow & & \downarrow \\ (\mathfrak{g}^{(1)})_{\bar{0}} & \longrightarrow & \underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}}, \end{array}$$

where once again, it is understood that the quantities in parentheses are only present for the case of $(P_0^{(1)})^\mu_{\mathbb{R}}$. Inductively, we obtain $(P_0^{(k)})^\mu_{\mathbb{R}} \subseteq (P_0^{(k)})^\mu \subseteq P_0^{(k)}$ with structure groups given by the pullback

$$\begin{array}{ccc} (\mathfrak{g}^{(k)})_{\bar{0},\mathbb{R}} & \longrightarrow & (\underline{\mathrm{Hom}}(V,\mathfrak{g}^{(k-1)})_{\bar{0}})^\mu_{(\mathbb{R})} \\ \downarrow & & \downarrow \\ (\mathfrak{g}^{(k)})_{\bar{0}} & \longrightarrow & \underline{\mathrm{Hom}}(V,\mathfrak{g}^{(k-1)})_{\bar{0}}. \end{array}$$

Definition 3.6. A G-structure is called *admissible* if, for all $k \geq 0$, $(\mathfrak{g}^{(k)})_{\bar{0},\mathbb{R}}$ defines a mixed structure on $(\mathfrak{g}^{(k)})_{\bar{0}}$.

Assume now that the G-structure is admissible. Since $(\mathfrak{g}^{(1)})_{\bar{0}} \subseteq (\underline{\mathrm{Hom}}(V,\mathfrak{g})_{\bar{0}})^\mu$, we have that $(P_0^{(1)})^\mu = P_0^{(1)}$ and the structure group of $(P_0^{(1)})^\mu_{\mathbb{R}} = P_{0,\mathbb{R}}^{(1)}$ is by definition $(\mathfrak{g}^{(1)})_{\bar{0},\mathbb{R}}$. Pulling back $r^*P_{0,\mathbb{R}}^{(1)} \to r^*P_0^{(1)}$ along $P^{(1)} \to r^*P_0^{(1)}$ gives the functor $P_{\mathbb{R}}^{(1)}$, which is representable in view of Lemma 2.17 and the assumption on the G-structure. All in all, this yields the real prolongation:

$$\cdots \longrightarrow P_{\mathbb{R}}^{(k)} \longrightarrow P_{\mathbb{R}}^{(k-1)} \longrightarrow \cdots \longrightarrow P_{\mathbb{R}}^{(1)} \longrightarrow P_{\mathbb{R}}^{(0)} = P \longrightarrow M.$$

The structure group of the kth real prolongation will be denoted by $G_{\mathbb{R}}^{(k)}$.

4 Automorphisms of G-structures

The main object of study in this paper is the functor of automorphisms of a G-structure, which we presently define.

1 The functor of automorphisms of a G-structure

Let M be a mixed supermanifold. An automorphism $f\colon S \times M \to S \times M$ over S is called an *S-family of automorphisms of M*. Such morphisms assemble to a functor $\underline{\mathrm{Diff}}(M)$ given by

$$\underline{\mathrm{Diff}}(M)(S) = \{f\colon S \times M \to S \times M \mid f \text{ an } S\text{-family of automorphisms of } M\}.$$

Moreover, for any Lie supergroup G and any principal G-bundle $P \to M$, we let $\underline{\mathrm{Diff}}(P)^G \subseteq \underline{\mathrm{Diff}}(P)$ be the subfunctor of equivariant automorphisms, i.e.

$$\underline{\mathrm{Diff}}(P)^G(S) = \{f \in \underline{\mathrm{Diff}}(P)(S) \mid f \text{ is } G\text{-equivariant}\}.$$

Note that if P is a G-structure, then inducing up from G to $GL(V)$ gives a map $\underline{\mathrm{Diff}}(P)^G \to \underline{\mathrm{Diff}}(L(M))^{GL(V)}$ and, moreover, the differential induces an inclusion of functors $L(-)\colon \underline{\mathrm{Diff}}(M) \to \underline{\mathrm{Diff}}(L(M))^{GL(V)}$.

Definition 4.1. The functor of automorphisms of a G-structure P on M is defined to be the pullback

$$\begin{array}{ccc} \underline{\mathrm{Aut}}(P) & \longrightarrow & \underline{\mathrm{Diff}}(M) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Diff}}(P)^G & \longrightarrow & \underline{\mathrm{Diff}}(L(M))^{GL(V)}. \end{array}$$

An S-point of $\underline{\mathrm{Aut}}(P)$ is called an *S-family of automorphisms of P*.

Definition 4.2. A homogeneous vector field $\mathcal{O}_M \to (p_{S_0})_*\mathcal{O}_{S\times M}$ along $p_S\colon S \times M \to M$ is called an *S-family of infinitesimal automorphisms of P* if the induced vector field $\mathcal{O}_{L(M)} \to (p_{S_0})_*\mathcal{O}_{S\times L(M)}$ extends to $\mathcal{O}_P \to (p_{S_0})_*\mathcal{O}_{S\times P}$. For $S = *$ this yields the Lie superalgebra $\mathfrak{aut}(P) \subseteq \Gamma(\mathcal{T}_M)$ of infinitesimal automorphisms of P. The even part has a real subalgebra defined by $\mathfrak{aut}(P)_{\bar{0},\mathbb{R}} := \mathfrak{aut}(P)_{\bar{0}} \cap \Gamma(\mathcal{T}_{M,\mathbb{R}})$.

Remark 4.3. There is no reason for $\mathfrak{aut}(P)_{\bar{0},\mathbb{R}}$ to be a mixed real form or even a real form of $\mathfrak{aut}(P)_{\bar{0}}$. For instance, on a purely odd supermanifold all vector fields are real. The latter would be a necessary condition for the automorphism group to be representable by a Lie supergroup. For this reason automorphism groups of G-structures are generically mixed supermanifolds.

In analogy with Lemma 2.16, one sees that any $\varphi \in \underline{\mathrm{Diff}}(M)(\mathbb{A}(\mathbb{C}^{0|n}) \times T)$, where T is a mixed supermanifold, can be uniquely written as

$$\varphi^\sharp = \prod_{k=n}^{1} \Big(1 + \sum_{k \in I \subseteq \{1,\dots,k\}} \eta^I X_I\Big) \cdot \varphi_0^\sharp$$

where X_I are vector fields along p_T of degree $|I|$ and $\varphi_0 \in \underline{\mathrm{Diff}}(M)(T)$.

Lemma 4.4. *Consider $\varphi \in \underline{\mathrm{Diff}}(M)(\mathbb{A}(\mathbb{C}^{0|n}) \times T)$. Then $\varphi \in \underline{\mathrm{Aut}}(P)(\mathbb{A}(\mathbb{C}^{0|n}) \times T)$ if and only if $\varphi_0 \in \underline{\mathrm{Aut}}(P)(T)$ and all X_I are T-families of infinitesimal automorphisms of P.*

Proof. The condition is clearly sufficient. So, assume that φ is an $\mathbb{A}(\mathbb{C}^{0|n}) \times T$-family of automorphisms of P. Then φ_0 is a T-family of such automorphisms since it is obtained by restricting along the inclusion $T \to \mathbb{A}(\mathbb{C}^{0|n}) \times T$. Now one proceeds by induction on n to show that all X_I are infinitesimal automorphisms of P. □

2 The automorphisms of a $\{1\}$-structure

We now come to the issue of representability of $\underline{\mathrm{Aut}}(P)$. Before proceeding to higher order G-structures we need to treat the simplest case $G = \{1\}$. We assume that M has finitely many connected components. A G-structure is simply a parallelization $\Phi\colon \underline{V_{\mathbb{R}}}_M \to TM_{\mathbb{R}}$. Such a Φ induces an even real vector field on $M \times \mathbb{A}(V_{\mathbb{R}})$:

$$Z\colon M \times \mathbb{A}(V_{\mathbb{R}}) \longrightarrow TM_{\mathbb{R}} \times \mathbb{A}(V_{\mathbb{R}}) \longrightarrow T(M \times \mathbb{A}(V_{\mathbb{R}}))_{\mathbb{R}}$$

and $\underline{\mathrm{Aut}}(\Phi)(S)$ consists of those automorphisms making the diagram

$$\begin{array}{ccc} S \times M \times \mathbb{A}(V_{\mathbb{R}}) & \xrightarrow{S\times\Phi} & S \times TM_{\mathbb{R}} \\ \Big\downarrow{\scriptstyle f\times V_{\mathbb{R}}} & & \Big\downarrow{\scriptstyle df} \\ S \times M \times \mathbb{A}(V_{\mathbb{R}}) & \xrightarrow{S\times\Phi} & S \times TM_{\mathbb{R}} \end{array}$$

commutative.

We first show that $i^*\underline{\mathrm{Aut}}(\Phi)$ is representable. To that end, we endow $\mathrm{Aut}(\Phi)_0 := \underline{\mathrm{Aut}}(\Phi)(*)$ with the structure of a Lie group acting on M.

Recall that there is a forgetful functor sending a mixed manifold to its underlying smooth manifold. (We prove in Proposition 7.1 below that such a functor does not exist for mixed supermanifolds.) Consider the underlying parallelization $\Phi_0\colon M_0 \times \mathbb{A}((V_{\mathbb{R}})_{\bar 0}) \to T(M_0)_{\mathbb{R}}$ and its underlying smooth morphism $\Phi_0^{sm}\colon M_0^{sm} \times (V_{\mathbb{R}})_{\bar 0} \to TM_0^{sm}$. In order to define a topology on $\mathrm{Aut}(\Phi)_0$, we need the following fact.

Lemma 4.5. *The forgetful map* $\mathrm{Aut}(\Phi)_0 \to \mathrm{Aut}(\Phi_0)$*, $s \mapsto s_0$, is injective.*

Proof. Deferred to Section 4. ☐

Moreover, we have that $\mathrm{Aut}(\Phi_0) \subseteq \mathrm{Aut}(\Phi_0^{sm})$ are precisely the elements which preserve the mixed structure on M_0^{sm}.

By a result of Kobayashi [12], any choice of representatives $x_i \in M_0$ $(i \in \{1, \dots l\})$ of $\pi_0(M_0)$ gives rise to a closed injection

$$\mathrm{Aut}(\Phi_0^{sm}) \longrightarrow \prod_{i=1}^{l} M_0^{sm},\ s \mapsto (s(x_i))$$

and with this topology, $\mathrm{Aut}(\Phi_0^{sm})$ is a Lie group such that the evaluation map $a_0^{sm} \colon \mathrm{Aut}(\Phi_0^{sm}) \times M_0^{sm} \to M_0^{sm}$ is smooth [3, Thm. 1.7]. This topology is the coarsest such that for all $f \in \Gamma(\mathcal{O}_{M_0^{sm}})$, the map $\mathrm{Aut}(\Phi_0^{sm}) \to \Gamma(\mathcal{O}_{M_0^{sm}}),\ s \mapsto s^\sharp(f)$, is continuous, where $\Gamma(\mathcal{O}_{M_0^{sm}})$ is considered as a Fréchet space with respect to the family of seminorms $|f|_{K,\partial} = \sup_K |\partial f|$, $K \subseteq M_0$ compact, ∂ differential operator. In this topology, $s_n \to s$ if and only if $s_n^\sharp(f) \to s^\sharp(f)$ in $\Gamma(\mathcal{O}_{M_0^{sm}})$ for all $f \in \Gamma(\mathcal{O}_{M_0^{sm}})$.

Being mixed is a closed condition (locally equations of the form $\partial_{\bar{z}} s^\sharp(f) = 0$ for all $f \in \mathcal{O}_M$), hence $\mathrm{Aut}(\Phi_0) \subseteq \mathrm{Aut}(\Phi_0^{sm})$ is closed. Then we get a Lie group $\mathrm{Aut}(\Phi)_0^{sm} \subseteq \mathrm{Aut}(\Phi_0)$, in view of the following lemma.

Lemma 4.6. *The subspace* $\mathrm{Aut}(\Phi)_0 \subseteq \mathrm{Aut}(\Phi_0)$ *is closed. The topology on* $\mathrm{Aut}(\Phi)_0$ *is such that $s_n \to s$ implies that for all pairs of coordinate charts U, V such that $s_n(U) \subseteq V$ for all n large enough, all the coefficients in the Taylor expansion of $s_n^\sharp(f)$, $f \in \Gamma(\mathcal{O}_M|_V)$, with respect to the odd coordinates, converge in $\mathcal{O}_{M_0^{sm}}(U_0)$.*

Proof. Deferred to Section 4. ☐

In particular, we have an action $a_0' \colon \mathrm{Aut}(\Phi)_0^{sm} \times M_0 \to M_0$ and this is a mixed map since it is so pointwise.

Lemma 4.7. *The map a_0' extends to the action*

$$a'^\sharp \colon \mathcal{O}_M \longrightarrow \mathcal{O}_{\mathrm{Aut}(\Phi)_0^{sm} \times M},\ f \mapsto (s \mapsto s^\sharp(f)).$$

Proof. Deferred to Section 4. ☐

As explained in Section 7, even real vector fields have unique maximal flows. Using this, the action above and the description of the topology on $\mathrm{Aut}(\Phi)_0^{sm}$, one obtains an isomorphism

$$\mathrm{Lie}_{\mathbb{R}}(\mathrm{Aut}(\Phi)_0^{sm}) \cong \mathfrak{aut}(\Phi)_{\bar{0},\mathbb{R}}^{c} := \{X \in \Gamma(\mathcal{T}_{M,\mathbb{R}})_{\bar{0}} \mid [X, Z] = 0, X \text{ is complete}\} \subseteq \mathfrak{aut}(\Phi).$$

Then $\mathbb{C}$-linearization yields a Lie algebra morphism

$$\mathbb{C} \otimes \mathfrak{aut}(\Phi)^c_{\bar{0},\mathbb{R}} \longrightarrow \Gamma(\mathcal{T}_M)_{\bar{0}}$$

and the kernel is of the form

$$\overline{\mathfrak{aut}}(\Phi)^c_{\bar{0},\mathbb{C}} := \{1 \otimes iv - i \otimes v \mid v \in \mathfrak{aut}(\Phi)^c_{\bar{0},\mathbb{C}}\}$$

for a complex invariant ideal $\mathfrak{aut}(\Phi)^c_{\bar{0},\mathbb{C}} \subseteq \mathfrak{aut}(\Phi)^c_{\bar{0},\mathbb{R}}$. This yields the mixed structure $\mathrm{Aut}(\Phi)_0$ on $\mathrm{Aut}(\Phi)^{sm}_0$ and, on general grounds, the quotient

$$(\mathbb{C} \otimes \mathfrak{aut}(\Phi)^c_{\bar{0},\mathbb{R}})/\overline{\mathfrak{aut}}(\Phi)^c_{\bar{0},\mathbb{C}} =: \mathfrak{aut}(\Phi)^{c,d}_{\bar{0}} \subseteq \mathfrak{aut}(\Phi)_{\bar{0}}$$

is the Lie algebra of left-invariant derivations of $\mathcal{O}_{\mathrm{Aut}(\Phi)_0}$. It is the algebra of *complete decomposable infinitesimal automorphisms* in the sense that any of its elements can be written as the sum $v + iw$ of complete real vector fields v and w. Moreover, with this structure $a\colon \mathrm{Aut}(\Phi)_0 \times M \to M$ is a mixed morphism, by Lemma 2.15.

Finite-dimensionality of the full algebra of infinitesimal automorphisms is ensured by the following lemma.

Lemma 4.8. *Assume that M_0 is connected. For every $p \in M_0$, evaluation $\mathfrak{aut}(\Phi) \to T_pM$, $X \mapsto X(p)$, is injective. If M_0 is not connected, the analogous statement holds true if one chooses one point for each connected component.*

Proof. Deferred to Section 4. □

Moreover, the conjugation action of $\mathrm{Aut}(\Phi)_0$ on $\Gamma(\mathcal{T}_M)$ restricts to an action on $\mathfrak{aut}(\Phi)$ and the differential of this representation is simply the restriction of the adjoint representation

$$\mathfrak{aut}(\Phi)^{c,d}_{\bar{0}} \times \mathfrak{aut}(\Phi) \longrightarrow \mathfrak{aut}(\Phi).$$

The following result shows that $\mathrm{Aut}(\Phi)_0$ has the correct topology and mixed structure.

Proposition 4.9. *The functors $i^*\underline{\mathrm{Aut}}(\Phi)$ and $\mathsf{M}^\mu(-, \mathrm{Aut}(\Phi)_0)$ are naturally isomorphic.*

Proof. Given a map $T_0 \to \mathrm{Aut}(\Phi)_0$, the action of the group yields a map $T_0 \times M \to T_0 \times M$. Conversely, take an element $f\colon T_0 \times M \to T_0 \times M$ in $\underline{\mathrm{Aut}}(\Phi)(T_0)$. The obvious candidate $\tilde{f}\colon T_0 \to \mathrm{Aut}(\Phi)_0$ is a smooth mixed map, since the composition

$$T_0 \longrightarrow \mathrm{Aut}(\Phi)_0 \longrightarrow M_0$$

with evaluation at some $m \in M_0$ equals $f_0(-, m_0)$, which is smooth and mixed. □

3 The automorphisms of a G-structure of finite type

Definition 4.10. An admissible G-structure is of *finite type* if there exists a $k \geq 0$ such that $G_{\mathbb{R}}^{(k+l)} = \{1\}$ for all $l \geq 0$.

The main theorem is as follows:

Theorem 4.11. *Suppose M has finitely many connected components and $P \to M$ is an admissible G-structure of finite type. Then $i^*\underline{\mathrm{Aut}}(P)$ is representable and its (real) Lie algebra consists of the complete real infinitesimal automorphisms of P, denoted by $\mathfrak{aut}(P)^c_{\bar{0},\mathbb{R}}$. Moreover, $\mathfrak{aut}(P)$ is finite-dimensional and the functor $\underline{\mathrm{Aut}}(P)$ is representable if and only if $\mathfrak{aut}(P)^{c,d}_{\bar{0}} = \mathfrak{aut}(P)_{\bar{0}}$.*

Proof. By applying the universal property of the pullback in the construction of $P_{\mathbb{R}}^{(1)}$, we see that there is a natural inclusion of group-valued functors $\underline{\mathrm{Aut}}(P) \to \underline{\mathrm{Aut}}(P_{\mathbb{R}}^{(1)})$. More generally, we have an embedding $\underline{\mathrm{Aut}}(P) \to \underline{\mathrm{Aut}}(P_{\mathbb{R}}^{(k)})$ for any $k \geq 0$. We choose $k \geq 0$ such that $G_{\mathbb{R}}^{(k+l)} = \{1\}$ for all $l \geq 0$. Then $\mathfrak{aut}(P)$ is finite-dimensional in view of Lemma 4.8. Let Φ be the given parallelization of the real tangent bundle of $P_{\mathbb{R}}^{(k-1)}$.

We show that the inclusion

$$\underline{\mathrm{Aut}}(P)(*) \subseteq \underline{\mathrm{Aut}}(\Phi)(*) = \mathrm{Aut}(\Phi)_0^{sm}$$

is closed. Recall that the topology on $\mathrm{Aut}(\Phi)_0^{sm} \subseteq (P_{\mathbb{R}}^{(k-1)})_0^{sm}$ is such that $s_n \to s$ implies that locally all $s_n^\sharp(f)$, $f \in \Gamma(\mathcal{O}_{P_{\mathbb{R}}^{(k-1)}}|_V)$, converge in the closed subspace

$$\mathcal{O}_{P_{\mathbb{R}}^{(k-1)}}(U_0) \cong \bigoplus_{i=1}^{2^d} \mathcal{O}_{(P_{\mathbb{R}}^{(k-1)})_0}(U_0) \subseteq \bigoplus_{i=1}^{2^d} \mathcal{O}_{(P_{\mathbb{R}}^{(k-1)})_0^{sm}}(U_0),$$

where d denotes the odd dimension of $P_{\mathbb{R}}^{(k-1)}$.

Now assume $s_n \in \underline{\mathrm{Aut}}(P)(*)$ and $s_n^{(k)} \to \tilde{s}$. From the construction of the prolongation, it is clear that one obtains a diffeomorphism $s\colon M \to M$ with kth prolongation $s^{(k)}$ equal to $\tilde{s}$. From equivariance it now follows that s is actually in $\underline{\mathrm{Aut}}(P)(*)$.

Next, assume that the action $\mathrm{Aut}(P_{\mathbb{R}}^{(i+1)})_0^{sm} \times P_{\mathbb{R}}^{(i)} \to P_{\mathbb{R}}^{(i)}$ is smooth. Restricted to $\mathrm{Aut}(P_{\mathbb{R}}^{(i)})_0^{sm}$, it is pointwise equivariant, hence it is itself equivariant and thus descends to an action on $P_{\mathbb{R}}^{(i-1)}$. This action gives the identification of the Lie algebra of $\mathrm{Aut}(P)_0^{sm}$ with $\mathfrak{aut}(P)^c_{\bar{0},\mathbb{R}}$, and the mixed structure is now defined as in the case of $\mathrm{Aut}(\Phi)_0$. Then the action just defined refines to an action $\mathrm{Aut}(P)_0 \times M \to M$ by Lemma 2.15 and, using this, as in the similar situation of the automorphisms of a parallelization, one deduces that $i^*\underline{\mathrm{Aut}}(P) \cong \mathrm{Aut}(P)_0$.

Clearly, if $\underline{\mathrm{Aut}}(P)$ is representable, then $\mathfrak{aut}(P)$ can only consist of complete and decomposable vector fields. Conversely, if $\mathfrak{aut}(P)^{c,d}_{\bar{0}} = \mathfrak{aut}(P)_{\bar{0}}$, then

$$\mathrm{Aut}(P) = (\mathfrak{aut}(P), \mathrm{Aut}(P)_0)$$

forms a mixed super pair. The action defines a map $\mathsf{SM}^\mu(-, \mathrm{Aut}(P)) \to \underline{\mathrm{Diff}}(M)$, and in view of Lemma 2.16, it factors locally through an isomorphism to $\underline{\mathrm{Aut}}(P)$. Hence, it factors globally as an isomorphism $\mathsf{SM}^\mu(-, \mathrm{Aut}(P)) \cong \underline{\mathrm{Aut}}(P)$. □

4 Proofs of Lemmas 4.5, 4.6, 4.7, and 4.8

Proof of Lemma 4.5. Let $s \in \mathrm{Aut}(\Phi)_0$ be such that $s_0 = \mathrm{id}$. In order to see that this implies $s = \mathrm{id}$, we consider, for $k \geq 1$, the restriction of s to the $(k-1)$th infinitesimal neighbourhood

$$(s^{(k-1)})^\sharp \colon \mathcal{O}_M/\mathcal{J}^k \longrightarrow (s_0)_*\mathcal{O}_M/\mathcal{J}^k.$$

We have $(s^{(0)})^\sharp = s_0^\sharp = \mathrm{id}$.

Now, we choose a homogeneous basis $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}\}$ of $V_\mathbb{R}$ and local coordinates $\{q_1, \ldots, q_n, q_{n+1}, \ldots, q_{n+m}\}$ on an open subset U_0 containing $m \in M_0$. Here, the first n (resp. last m) entries are assumed to be even (resp. odd). In the given basis

$$Z_{v_k} = \sum_l A_{kl} \partial_{q_l}$$

for some even invertible matrix $A = (A_{kl}) \in GL_{\mathcal{O}_M}(\mathcal{O}_M(U_0)^{n|m})$. The requirement for f to lie in $\mathrm{Aut}(\Phi)_0$ reads

$$Jf = A^{-1} \circ f^\sharp(A)$$

where $Jf = (\partial_{q_i} f^\sharp(q_j))$ and we denote the natural extension of $f^\sharp$ to matrices by the same symbol.

So assume $(f^{(k-1)})^\sharp = \mathrm{id}$. We have

$$\begin{aligned} Jf + \mathcal{J}^k(U_0)^{(n|m)\times(n|m)} &= A^{-1} \circ f^\sharp(A) + \mathcal{J}^k(U_0)^{(n|m)\times(n|m)} \\ &= A^{-1} \circ (f^{(k-1)})^\sharp(A) + \mathcal{J}^k(U_0)^{(n|m)\times(n|m)} \\ &= \mathrm{id}_{n|m} + \mathcal{J}^k(U_0)^{(n|m)\times(n|m)}, \end{aligned}$$

and this implies $(f^{(k)})^\sharp = \mathrm{id}$. □

Proof of Lemma 4.6. Let $\{s_n\}$ be a sequence in $\mathrm{Aut}(\Phi)_0$ such that $\{(s_n)_0\}$ converges to some $\tilde{s}$. We have to show that $\tilde{s} = s_0$ for some suitable $s \in \mathrm{Aut}(\Phi)_0$ and that s_n converges to s. Without loss of generality all $(s_n)_0$ lie in one coordinate chart (in $\mathrm{Aut}(\Phi_0)$) and since a_0^{sm} is smooth we may choose open subspaces U and V with coordinates $\{p_i\}$ and $\{q_i\}$

respectively such that every s_n restricts to a map $U \to V$. Let us organise the coordinates into even and odd functions $\{p_i\} = \{x_i, \eta_j\}$, $\{q_i\} = \{y_i, \xi_j\}$.

In these coordinate charts the condition for s_n to lie in $\mathrm{Aut}(\Phi)_0$ reads

$$J(s_n) = A \cdot s_n^\sharp(B)$$

for certain invertible matrices A and B where $J(s_n) = (\partial_{p_i} s_n^\sharp(q_i))$. Starting from $s^{(0)^\sharp} := \tilde{s}^\sharp$, we inductively define $(s^{(k)})^\sharp \colon \mathcal{O}_M/\mathcal{J}^{k+1}(V_0) \to \mathcal{O}_M/\mathcal{J}^{k+1}(U_0)$ with reductions $\tilde{s}$. The construction will be such that the following holds: We have $(s_n^{(k)})^\sharp(f) \to (s^{(k)})^\sharp(f)$ for all $f \in (\mathcal{O}_M/\mathcal{J}^{k+1})(V_0)$. Here, $(\mathcal{O}_M/\mathcal{J}^{k+1})(U_0)$ is considered as a subspace of $\bigoplus \mathcal{O}_{M_0^{sm}}(U_0)$, where the number of summands is 2^m.

The respective lifts will be determined by the Jacobian $J(s^{(k)})$ which naturally has values in matrices of the form

$$\begin{pmatrix} \mathcal{O}_M/\mathcal{J}^{k+1} & \mathcal{O}_M/\mathcal{J}^{k} \\ \mathcal{O}_M/\mathcal{J}^{k+1} & \mathcal{O}_M/\mathcal{J}^{k} \end{pmatrix}.$$

There is a projection from $\mathcal{O}_M/\mathcal{J}^{k+1}$-valued matrices to such matrices. The image of a matrix A will be denoted by $A^\sim$.

Assume that k is even and $(s^{(k)})^\sharp$ has been constructed such that

$$J(s^{(k)}) = (A^{(k)}(s^{(k)})^\sharp B^{(k)})^\sim.$$

First, we have to set $(s^{(k+1)})^\sharp(q_i) = (s^{(k)})^\sharp(q_i)$ for q_i even. The odd-odd sector of the Jacobian determines $(s^{(k+1)})^\sharp(q_i)$ for q_i odd: In fact, it follows that

$$\begin{aligned} \partial_{\eta_i}(s^{(k+1)})^\sharp(\xi_j) &\overset{!}{=} (A^{(k+1)}(s^{(k+1)})^\sharp B^{(k+1)})_{ij}^\sim \\ &= (A^{(k)}(s^{(k)})^\sharp B^{(k)})_{ij} \\ &= \lim_n (A^{(k)}(s_n^{(k)})^\sharp B^{(k)})_{ij} \\ &= \lim_n \partial_{\eta_i}(s_n^{(k+1)})^\sharp(\xi_j). \end{aligned}$$

These derivatives fit together to give a well-defined $(s^{(k+1)})^\sharp(\xi_j)$ since the different partial derivatives fit together; that is, for any multiindex I, $|I| = k+1$, with η_i, $\eta_{i'} \in I$, we have

$$\partial_{I-\{\eta_i\}}\partial_{\eta_i}((s^{(k+1)})^\sharp(\xi_j)) = \epsilon_{i,i'}\partial_{I-\{\eta_{i'}\}}\partial_{\eta_{i'}}((s^{(k+1)})^\sharp(\xi_j))$$

since this equality holds for all s_n. With this definition we have $(s^{(k+1)})^\sharp = \lim_n (s_n^{(k+1)})^\sharp$, which ensures $J(s^{(k+1)}) = (A^{(k+1)}(s^{(k+1)})^\sharp B^{(k+1)})^\sim$ by continuity.

If k is odd and $(s^{(k)})^\sharp$ has been constructed in such a way that

$$J(s^{(k)}) = (A^{(k)} \cdot (s^{(k)})^\sharp B^{(k)})^\sim,$$

then one can proceed similarly. There are no changes in the pullbacks of odd coordinates and the pullbacks of the even coordinates are forced by the respective equation for the odd-even sector of the Jacobian. Again, $(s^{(k)})^\sharp = \lim(s_n^{(k)})^\sharp$. This yields the construction of $s|_U\colon \mathcal{O}_V \to (s_0)_*\mathcal{O}_U$. By uniqueness (Lemma 4.5), these $s|_U$ coincide where two coordinates patches overlap, and so we obtain the desired $s\colon M \to M$.

The statement concerning the topology is clear from the above considerations. □

Proof of Lemma 4.7. Similary as in the preceding lemma, starting from $((a')^{(0)})^\sharp := (a'_0)^\sharp$, we inductively construct $((a')^{(k)})^\sharp\colon \mathcal{O}_M/\mathcal{J}^{k+1} \to (a'_0)_*\mathcal{O}_{\mathrm{Aut}(\Phi)_0^{sm}\times M}/\mathcal{J}^{k+1}$. First we choose some neighbourhoods $W \subseteq \mathrm{Aut}(\Phi)_0^{sm}$ and $U, V \subseteq M$ given by coordinates $\{p_i\} = \{x_i, \eta_j\}$ and $\{q_i\} = \{y_i, \xi_j\}$ such that a'_0 restricts to

$$W \times U_0 \longrightarrow V_0.$$

Then, if A and B are as in the proof above, the map $(a')^\sharp$ to be constructed will be characterized by

$$J^{\mathrm{res}}(a') = A(a')^\sharp(B).$$

where $J^{\mathrm{res}}(a')$ denotes the submatrix $(\partial_{p_i}(a')^\sharp(q_j))$ of the Jacobian. So, assume $((a')^{(k)})^\sharp$ is constructed such that

$$J((a')^{(k)}) = (A^{(k)}((a')^{(k)})^\sharp B^{(k)})^\sim.$$

Suppose first that k is even. Looking at the odd-odd sector of the Jacobian gives

$$\partial_{\eta_i}((a')^{(k+1)})^\sharp(\xi_j) = (A^{(k)}((a')^{(k)})^\sharp B^{(k)})_{ij}.$$

These fit together since they do so pointwise, i.e. after specializing to any element $s \in \mathrm{Aut}(\Phi)_0^{sm}$. Moreover, the identity for the Jacobian holds true, since it holds true pointwise. □

Proof of Lemma 4.8. We follow [3, Lem. 2.4]. If $X \in \mathfrak{aut}(\Phi)$, then $X_{V_\mathbb{R}} := X \otimes \mathrm{id}_{V_\mathbb{R}}$ is a vector field on $M \times \mathbb{A}(V_\mathbb{R})$ which commutes with Z (as is seen in local coordinates).

Let Θ^Z be the maximal flow of the even real vector field Z (see Theorem 7.8), defined on $\mathcal{V} \subseteq \mathbb{R} \times M \times \mathbb{A}(V_\mathbb{R})$, and consider the composite $\Theta^{Z'} = \mathrm{pr}_1 \circ \Theta^Z\colon \mathcal{V} \to M$. Note that $\{1\} \times M \times \{0\} \subseteq \mathcal{V}$, so $\Theta^{Z'}(1, -)$ is defined on an open neighbourhood of $M \times \{0\}$.

We have the following: For all $p \in M_0$ there exists an open neighbourhood $p \in U_0 \subseteq M_0$ and an open subspace $V' \subseteq \mathbb{A}(V_\mathbb{R})$ such that for all $q \in U_0$ the map $\Theta^{Z'}(1, q, -)\colon V' \to M$ is a diffeomorphism onto an open subspace.

Indeed, the map $(\mathrm{pr}_1, \Theta^{Z'}(1, -))$ is defined on an open neighbourhood of $M \times \{0\}$ and its differential at $(p,0)$ is of the form

$$\begin{pmatrix} 1 & 0 \\ * & Z \end{pmatrix},$$

which is invertible.

Now, assume $\mathrm{inj}_p^\sharp \circ X = X(p) = 0$. Choose open subspaces $U \subseteq M$ and $V' \subseteq \mathbb{A}(V_\mathbb{R})$ such that $p \in U_0$ and $0 \in V'$ such that $\varphi := \Theta^Z(1,p,-)\colon V' \to U$ is an isomorphism. Then

$$\begin{aligned} \varphi^\sharp \circ X &= \mathrm{inj}_p^\sharp \circ \Theta^Z(1,-,-)^\sharp \circ \mathrm{pr}_1^\sharp \circ X \\ &= \mathrm{inj}_p^\sharp \circ \Theta^Z(1,-,-)^\sharp \circ X_V \circ \mathrm{pr}_1^\sharp \\ &= \mathrm{inj}_p^\sharp \circ X_V \circ \Theta^Z(1,-,-)^\sharp \circ \mathrm{pr}_1^\sharp \\ &= 0, \end{aligned}$$

where we have used Proposition 7.10 in the third line. Since $\varphi^\sharp$ is invertible, it follows that $X = 0$ on U.

This shows that the non-empty closed set $\{p \in M_0 | X(p) = 0\}$ is contained in the open subset $\{p \in M_0 | X_p = 0\}$. The converse inclusion holds always, so that both subsets agree and are open and closed, hence they are all of M_0 if M_0 is connected. More generally, the argument shows that $X(p) = 0$ implies $X = 0$ on the connected component containing p. □

5 G-structures of finite type on real supermanifolds

Results analogous to those obtained in the mixed case hold for real supermanifolds. Their proofs are simplifications of our previous arguments, so we only briefly comment on them to provide precise statements for future reference.

A real super vector space is $\mathbb{Z}/2$-graded real vector space $V = V_{\bar 0} \oplus V_{\bar 1}$. The model spaces for real supermanifolds are the affine spaces $\mathbb{A}(V) = (V_{\bar 0}, C^\infty_{V_{\bar 0}}(-) \otimes \bigwedge V_{\bar 1}^*)$.

Definition 5.1. A *real supermanifold* is a locally ringed superspace $M = (M_0, \mathcal{O}_M)$ over $\mathbb{R}$ with a second countable Hausdorff base that is locally isomorphic to $\mathbb{A}(V)$ for some real super vector space V. The full subcategory of locally ringed superspaces over $\mathbb{R}$ with objects real supermanifolds is denoted by $\mathsf{SM}_\mathbb{R}$.

Similarly as in the case of supermanifolds, a real supermanifold has a frame bundle $L(M)$, which is a principal $GL(V)$-bundle. In the real category, $GL(V)$ is a real Lie supergroup and so $L(M)$ is an object in the category of real manifolds. Furthermore, given a G-structure, i.e. a closed subgroup $G \leqslant GL(V)$ and a reduction P of $L(M)$ to G, one can define the prolongation without leaving the real category.

One has a functor $i\colon \mathsf{M} \to \mathsf{SM}_{\mathbb{R}}$ and similarly as in the case of mixed supermanifolds, one obtains the following result.

Theorem 5.2. *Suppose $P \to M$ is a G-structure of finite type and M has finitely many connected components. Then $i^*\underline{\mathrm{Aut}}(P)$ is representable and its Lie algebra consists of the complete infinitesimal automorphisms of P, denoted by $\mathfrak{aut}(P)^c_{\bar{0}}$. Moreover, $\mathfrak{aut}(P)$ is finite-dimensional. The functor $\underline{\mathrm{Aut}}(P)$ is representable if and only if $\mathfrak{aut}(P)^c_{\bar{0}} = \mathfrak{aut}(P)_{\bar{0}}$.*

6 Examples of G-structures of finite type

1 Riemannian structures on supermanifolds

In this section, we treat Riemannian structures on a supermanifold M locally modelled on the super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$.

1.1 Even Riemannian structures

Consider an even non-degenerate supersymmetric bilinear form $J\colon V \otimes V \to \mathbb{C}^{1|0}$ with components $J_i\colon V_{\bar{i}} \otimes V_{\bar{i}} \to \mathbb{C}$ ($i \in \{0, 1\}$). There is a Lie supergroup $OSp(V, J)$ which represents automorphisms of the trivial vector bundle endowed with J:

$$OSp(V, J)(S) = \{f \in GL(V)(S) \mid (S \times J) \circ (f \otimes f) = (S \times J)\}.$$

Proposition 6.1.

(a) Reductions of $L(M)$ to $OSp(V, J)$ are in bijective correspondence with even non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \underline{\mathbb{C}^{1|0}}_M$.

(b) $OSp(V, J) \leqslant GL(V)$ is of finite type, more precisely $\mathfrak{osp}(V, J)^{(1)} = 0$.

Proof. Suppose given an $OSp(V, J)$-structure on P. A local trivialization $P|_U \cong U \times OSp(V, J)$ induces a trivialization $TM|_U \cong \underline{V}_U$. In virtue of this isomorphism we use the constant metric on $\underline{V}_U$ given by J to define the form on $TM|_U$. This definition is independent of the choice of local trivialization and thus gives the required tensor. Conversely, if g is any metric then locally $(TM|_U, g|_U)$ is isomorphic to $(\underline{V}_U, J)$ (cf. [8, Sect. 2.8]). We then use the constant $OSp(V, J)$-structure on the latter to get an $OSp(V, J)$-structure on U and these fit together to give an $OSp(V, J)$-structure on M.

In order to show the second part, we observe that $\mathfrak{osp}(V, J)$ consists of those endomorphisms $A\colon V \to V$, whose homogeneous components A_i satisfy $J(A_i v, w) = -(-1)^{|A_i||v|} J(v, A_i w)$. Using a homogeneous basis $\{v_i\}$, the conditions for T to lie in $\mathfrak{osp}(V, J)^{(1)}$ read $T^i_{jk} = (-1)^{|v_i||v_j|} T^j_{ik}$ and $T^i_{jk} = -(-1)^{|v_j||v_k|} T^i_{kj}$, where we set $T^i_{jk} = J(T(v_i)v_j, v_k)$. Both together imply $T^i_{jk} = 0$. □

The underlying complex group of $OSp(V, J)$ is the product of the complex groups $O(V_{\bar{0}}, J_0) \times Sp(V_{\bar{1}}, J_1)$. Assume that J_0 restricts to a non-degenerate bilinear form $J_{0,\mathbb{R}}\colon (V_{\bar{0}})_{\mathbb{R}} \otimes (V_{\bar{0}})_{\mathbb{R}} \to \mathbb{R}$. Such a J gives rise to the mixed real form $OSp(V, J)_{\mathbb{R}} \to OSp(V, J)$ with underlying group $O((V_{\bar{0}})_{\mathbb{R}}, J_{0,\mathbb{R}}) \times Sp(V_{\bar{1}}, J_1)$. Moreover, $OSp(V, J)_{\mathbb{R}} \leqslant GL(V)_{\mathbb{R}}$.

Lemma 6.2. *The $OSp(V, J)_{\mathbb{R}}$-structures on M are in bijective correspondence with even non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \underline{\mathbb{C}^{1|0}}_M$ whose restriction to $(TM)_{\bar{0}} \otimes (TM)_{\bar{0}} \subseteq i^*(TM \otimes TM)$ induce a metric $T(M_0)_{\mathbb{R}} \otimes T(M_0)_{\mathbb{R}} \to \underline{\mathbb{R}^1}_{M_0}$ of the same signature as $J_{0,\mathbb{R}}$ on the underlying real manifold M_0.*

Proof. This follows readily from the definition of $OSp(V, J)_{\mathbb{R}}$. □

From Theorem 4.11, we obtain the following result.

Theorem 6.3. *Let $P \to M$ be a $OSp(V, J)_{\mathbb{R}}$-structure on a supermanifold with finitely many path components. If M_0 is complete and every Killing vector field is decomposable, then the isometry group functor $\underline{\mathrm{Aut}}(P)$ is representable.*

Remark 6.4. In the real category the only obstruction for representability is completeness of the Killing vector fields. In this setting, an isometry group was constructed by Goertsches [11]. (The completeness condition seems to be assumed implicitly.) Our results in the real case give a rederivation of this result.

Example 6.5. The isometry group of V with the $OSp(V, J)_{\mathbb{R}}$-structure as above is $OSp(V, J)_{\mathbb{R}} \ltimes V_{\mathbb{R}}$.

1.2 Odd Riemannian structures

In the super setting, there is an odd analogue of a Riemannian structure, given by an odd non-degenerate supersymmetric bilinear form $J\colon V \otimes V \to \mathbb{C}^{1|0}$. The Lie supergroup $P(V, J)$ is defined by the functor

$$P(V, J)(S) = \{f \in GL(V)(S) \mid (S \times J) \circ (f \otimes f) = (S \times J)\}.$$

As with the even case, one can show the following.

Proposition 6.6.

(a) *The $P(V, J)$-structures on $L(M)$ and the odd non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \underline{\mathbb{C}^{1|0}}_M$ are in one-to-one correspondence.*

(b) *$P(V, J) \leqslant GL(V)$ is of finite type, more precisely, $\mathfrak{p}(V, J)^{(1)} = 0$.*

We have $P(V,J)_0 \cong GL(V_{\bar{0}})$, which comes with the mixed real form given by $GL((V_{\bar{0}})_{\mathbb{R}})$ and thus gives rise to $P(V,J)_{\mathbb{R}} \leqslant GL(V)_{\mathbb{R}}$.

For any $P(V,J)$-structure P on M, we have that $P_0 \cong L(M_0)$ and hence, it admits the real form $P_{0,\mathbb{R}} \cong L(M_0)_{\mathbb{R}}$. Now, one easily concludes the following.

Proposition 6.7. *$P(V,J)_{\mathbb{R}}$-structures are in one-to-one correspondence with $P(V,J)$-structures.*

From Theorem 4.11, we obtain the following result.

Theorem 6.8. *Let $P \to M$ be a $P(V,J)_{\mathbb{R}}$-structure on a supermanifold with finitely many path components. If M_0 is complete and all infinitesimal automorphisms are decomposable, then the isometry group functor $\underline{\mathrm{Aut}}(P)$ is representable.*

2 Superization of Riemannian spin manifolds

Let (M_0, g_0) be a connected pseudo-Riemannian spin manifold endowed with a $Spin(V_{\bar{0}})$-structure

$$\rho(M_0)\colon Spin(M_0) \longrightarrow SO(M_0),$$

where we set $(V_{\bar{0}}, \alpha) = (T_m M_0, g_m)$ for some $m \in M_0$. Choose a real or complex $Cl(V_{\bar{0}}, \alpha)$- or $Cl(V_{\bar{0}}, \alpha) \otimes \mathbb{C}$-module $V_{\bar{1}}$. The spinor bundle is the associated bundle $\mathcal{S} = Spin(M_0) \times^{Spin(V_{\bar{0}})} V_{\bar{1}} \to M_0$, which we endow with the lift of the Levi-Civita connection. Then $TM_0 \oplus \mathcal{S} \to M_0$ admits a reduction to $Spin(V_{\bar{0}}) \leqslant GL(V_{\bar{0}}) \times GL(V_{\bar{1}})$ by means of

$$(\rho(M_0), \mathrm{id})\colon Spin(M_0) \longrightarrow SO(M_0) \times Spin(M_0).$$

The supermanifold M associated to this data is obtained by taking the exterior algebra of the dual $\mathcal{S}^*$:

$$M = (M_0, \Gamma(-, \textstyle\bigwedge \mathcal{S}^*)).$$

It is a real supermanifold or a supermanifold depending on whether $V_{\bar{1}}$ is chosen to be real or complex. Any vector field on M_0 can be extended to M by means of the dual connection on $\mathcal{S}^*$, $X \mapsto \nabla_X$, and, furthermore, dual spinors can be contracted with spinors. This yields an inclusion $\iota\colon TM_0 \oplus \Pi\mathcal{S} \to TM$ and hence a $Spin(V_{\bar{0}})$-structure $P_{Spin(V_{\bar{0}})} \subseteq L(M)_{\mathbb{R}}$.

Any $Spin(V_{\bar{0}})$-submodule $\mathcal{W} \subseteq \underline{\mathrm{Hom}}(V_{\bar{0}}, V_{\bar{1}})$ gives rise to a mixed Lie supergroup $Spin(V_{\bar{0}}) \ltimes \mathcal{W} \leqslant GL(V)_{\mathbb{R}}$. Consequently, by inducing up, any such $\mathcal{W}$ gives rise to a $Spin(V_{\bar{0}}) \ltimes \mathcal{W}$-structure on M:

$$P_{Spin(V_{\bar{0}}) \ltimes \mathcal{W}} := P_{Spin(V_{\bar{0}})} \times^{Spin(V_{\bar{0}})} (Spin(V_{\bar{0}}) \ltimes \mathcal{W}).$$

A particular choice is

$$\mathcal{W} = \{f_s \colon V_{\bar{0}} \to V_{\bar{1}} \mid s \in V_{\bar{1}},\ f_s(v_0) = v_0 s\}.$$

Proposition 6.9. *For this choice of $\mathcal{W}$, $Spin(V_{\bar{0}}) \ltimes \mathcal{W} \leqslant GL(V)$ is of finite type, provided that* $\dim M \geq 3$.

Proof. We show that any $f \in (\mathfrak{spin}(V_{\bar{0}}) \oplus \mathcal{W})^{(1)} \subset \underline{\mathrm{Hom}}(V, \mathfrak{spin}(V_{\bar{0}}) \oplus \mathcal{W})$ vanishes. Suppose f is even. The homomorphism $f|_{V_{\bar{0}}}$ has image in the image of $(\rho_*, \mathrm{id})\colon \mathfrak{spin}(V_{\bar{0}}) \to \mathfrak{o}(V_{\bar{0}}) \oplus \mathfrak{spin}(V_{\bar{0}})$. Since ρ_* is an isomorphism and $\mathfrak{o}(V_{\bar{0}})^{(1)} = 0$, we have $f|_{V_{\bar{0}}} = 0$. Then $f|_{V_{\bar{1}}} \in \mathrm{Hom}(V_{\bar{1}}, \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}))$ vanishes as well by supersymmetry. If f is odd, then $f|_{V_{\bar{1}}}$ has image in $\mathfrak{spin}(V_{\bar{0}}) \hookrightarrow \mathfrak{o}(V_{\bar{0}}) \oplus \mathfrak{spin}(V_{\bar{0}})$. Using that $(\mathfrak{spin}(V_{\bar{0}}) \oplus \mathcal{W}) \cap \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}}) = 0$ and that ρ_* is an isomorphism, we see that $f|_{V_{\bar{1}}} = 0$. Finally, we show that $f|_{V_{\bar{0}}} = 0$. If we choose an orthogonal basis $\{e_i\}$ of $V_{\bar{0}}$, normalized such that $(e_i, e_i)^2 = 1$, we have $f(e_i)(e_j) = e_j s_i$ for certain $s_i \in V_{\bar{1}}$. The condition on f then reads

$$e_i s_j = e_j s_i$$

for all i and j. This implies $s_j = 0$ if $\dim M \geq 3$: Using $e_i e_j + e_j e_i = -2(e_i, e_j)$, we have $s_j = -(e_i, e_i) e_i e_j s_i$. On one hand, if k, l and j are such that $l \neq j$ and $l \neq k$ we have

$$\begin{aligned} s_k &= -(e_l, e_l) e_l e_k s_l \\ &= -(e_l, e_l) e_l e_k (-(e_j, e_j) e_j e_l s_j) \\ &= -(e_j, e_j) e_k e_j s_j \end{aligned}$$

On the other hand,

$$s_k = -(e_j, e_j) e_j e_k s_j.$$

So, if in addition $k \neq j$ (hence all three are different), then

$$\begin{aligned} s_k &= -(e_j, e_j) \frac{1}{2}(e_k e_j + e_j e_k) s_j \\ &= (e_j, e_j)(e_k, e_j) s_j \\ &= 0. \end{aligned}$$

□

Remark 6.10. By a theorem of Cortés et al. [1], the vector field $\iota(s)$ associated with a spinor gives rise to an infinitesimal automorphism of $P_{Spin(V_0) \ltimes \mathcal{W}}$ if and only if s is a twistor spinor, i.e., there exists a spinor $\tilde{s}$ such that for all X we have $\nabla_X s = X \cdot \tilde{s}$.

7 Appendix

1 Non-existence of a forgetful functor $\mathsf{SM}^\mu \to \mathsf{SM}$

A mixed manifold M has an underlying manifold M^{sm} which comes with a functorial map $M^{sm} \to M$. For an affine space $M = \mathbb{A}(V, V_\mathbb{R}, V_\mathbb{C})$, the assignment is simply given by setting $M^{sm} = \mathbb{A}(\mathbb{C} \otimes V_\mathbb{R}, V_\mathbb{R}, 0)$, and the map $M^{sm} \to M$ is induced by the map $\mathbb{C} \otimes V_\mathbb{R} \to V$. We show that the analogous statement fails in the category of mixed supermanifolds. This is not surprising, insofar as there does not even exist a forgetful functor from complex supermanifolds to supermanifolds [15]. A by-product of the argument is a proof that there is no functorial way to split even complex functions on supermanifolds into two even real functions (Proposition 7.2).

Let $(V, V_\mathbb{R}, V_\mathbb{C})$ be a mixed super vector space. The natural choice for the underlying supermanifold is given by the affine space associated with the super vector space $u(V, V_\mathbb{R}, V_\mathbb{C}) = (\mathbb{C} \otimes (V_\mathbb{R})_{\bar{0}} \oplus V_{\bar{1}}, V_\mathbb{R}, V_{\bar{1}})$. The natural choice for the map

$$\epsilon_{(V,V_\mathbb{R},V_\mathbb{C})} \colon \mathbb{A}(u(V, V_\mathbb{R}, V_\mathbb{C})) \longrightarrow \mathbb{A}(V, V_\mathbb{R}, V_\mathbb{C})$$

is induced by the $\mathbb{C}$-linearization of the inclusion $(V_\mathbb{R})_{\bar{0}} \to V_{\bar{0}}$ and the identity on $V_{\bar{1}}$. Note that $u^2 = u$. However, these natural choices do not assemble to a forgetful functor from mixed supermanifolds to supermanifolds:

Proposition 7.1. *There is no functor $F\colon \mathsf{SM}^\mu \to \mathsf{SM}$ such that the following two conditions hold:*

(a) $F(\mathbb{A}(V, V_\mathbb{R}, V_\mathbb{C})) = \mathbb{A}(u(V, V_\mathbb{R}, V_\mathbb{C}))$ *and* $F(\mathbb{A}(\epsilon_{(V,V_\mathbb{R},V_\mathbb{C})})) = \mathrm{id}_{\mathbb{A}(u(V,V_\mathbb{R},V_\mathbb{C}))}$.

(b) $F|_{\mathsf{SM}} = \mathrm{id}_{\mathsf{SM}}$.

Proof. Assume that such a functor F existed. Consider $\mathbb{A}(\mathbb{C})$ and $\mathbb{A}(\mathbb{R}^2)$ with their standard monoid structure. Then we would have a commutative square

$$\begin{array}{ccc} \mathbb{A}(\mathbb{R}^2) \times \mathbb{A}(\mathbb{R}^2) & \xrightarrow{\mu_{\mathbb{R}^2}} & \mathbb{A}(\mathbb{R}^2) \\ \downarrow{\scriptstyle \epsilon_{\mathbb{C}\times\mathbb{C}}} & & \downarrow{\scriptstyle \epsilon_\mathbb{C}} \\ \mathbb{A}(\mathbb{C}) \times \mathbb{A}(\mathbb{C}) & \xrightarrow{\mu_\mathbb{C}} & \mathbb{A}(\mathbb{C}) \end{array}$$

and it would follow from the second assumption that F would take the monoid $\mathbb{A}(\mathbb{C})$ to the monoid $\mathbb{A}(\mathbb{R}^2)$.

Consider the supermanifold $M = \mathbb{A}(\mathbb{R}^2 \times \mathbb{C}^{0|2})$ with coordinates $(x, y, \vartheta_1, \vartheta_2)$ and consider the two maps $\varphi_z, \varphi_{\vartheta_1\vartheta_2} \colon M \to \mathbb{A}(\mathbb{C})$ given by $\varphi_z^\sharp(z) = x + iy$ and $\varphi_{\vartheta_1\vartheta_2}^\sharp(z) = \vartheta_1\vartheta_2$, respectively. Then we have $\varphi_z = \epsilon_\mathbb{C} \circ (x, y)$, so that we would obtain $F(\varphi_z) = F((x, y)) = (x, y)$.

For an arbitrary smooth function $\alpha\colon \mathbb{R}^2 \to \mathbb{C}$ we now define $f_\alpha\colon M \to M$ by

$$\begin{aligned} f_\alpha^\sharp(x) &= x + \alpha\vartheta_1\vartheta_2, \\ f_\alpha^\sharp(y) &= y + (-i)(1-\alpha)\vartheta_1\vartheta_2, \\ f_\alpha^\sharp(\vartheta_i) &= \vartheta_i. \end{aligned}$$

Then $\varphi_z \circ f_\alpha = \varphi_z + \varphi_{\vartheta_1\vartheta_2}$. However, on one hand

$$\begin{aligned} F(\varphi_z \circ f_\alpha)^\sharp &= F(f_\alpha)^\sharp \circ F(\varphi_z)^\sharp \\ &= f_\alpha^\sharp \circ F(\varphi_z)^\sharp \\ &= f_\alpha^\sharp \circ (x, y) \\ &= (x, y) + (\alpha\varphi_{\vartheta_1\vartheta_2}, (-i)(1-\alpha)\varphi_{\vartheta_1\vartheta_2}) \end{aligned}$$

and on the other hand,

$$\begin{aligned} F(\varphi_z + \varphi_{\vartheta_1\vartheta_2}) &= F(\varphi_z) + F(\varphi_{\vartheta_1\vartheta_2}) \\ &= (x, y) + F(\varphi_{\vartheta_1\vartheta_2}). \end{aligned}$$

This would imply $F(\varphi_{\vartheta_1\vartheta_2}) = (\alpha\varphi_{\vartheta_1\vartheta_2}, (-i)(1-\alpha)\varphi_{\vartheta_1\vartheta_2})$ for arbitrary $\alpha\colon \mathbb{R}^2 \to \mathbb{C}$, which is absurd. □

Similarly, one proves the following related proposition.

Proposition 7.2. *The natural transformation $\epsilon_\mathbb{C}\colon \mathbb{A}(\mathbb{R}^2) \to \mathbb{A}(\mathbb{C})$ between functors on* SM *admits no section.*

Proof. Assume that such a natural transformation F existed. We use the notation from the previous proof. We consider again $M = \mathbb{A}(\mathbb{R}^2 \times \mathbb{C}^{0|2})$ and the two maps φ_z and $\varphi_{\vartheta_1\vartheta_2}$. Then $F(\varphi_z) = (x + n, y + in)$ for a nilpotent function n on M. Defining f_α as previously, we have $\varphi_z \circ f_\alpha = \varphi_z + \varphi_{\vartheta_1\vartheta_2}$, and so $F(\varphi_z \circ f_\alpha)$ would be independent of α. However, we would have

$$\begin{aligned} F(\varphi_z \circ f_\alpha) &= f_\alpha^\sharp(x + n, y + in) \\ &= (x + \alpha\vartheta_1\vartheta_2 + n, y + (-i)(1-\alpha)\vartheta_1\vartheta_2 + in), \end{aligned}$$

a contradiction. □

2 Flows of even real vector fields on mixed supermanifolds

We outline the construction of flows of vector fields on mixed supermanifolds. In this setting, only even real vector fields can be integrated. We show that they have a unique maximal flow.

Let M be a mixed supermanifold and let X be an even real vector field. Let $\mathcal{V} \subseteq \mathbb{R}^1 \times M$ be open such that $\{0\} \times M \subseteq \mathcal{V}$. A morphism

$$\Theta^X : \mathbb{R}^1 \times M \supseteq \mathcal{V} \longrightarrow M$$

is called a *flow of X* if

(a) $\partial_t \circ \Theta^{X\sharp} = \Theta^{X\sharp} \circ X$, and

(b) $\Theta^X|_{\{0\}\times M} = \mathrm{id}_M$.

Following [10], an open subspace $\{0\} \times M \subseteq \mathcal{V} \subseteq \mathbb{R}^1 \times M$ such that, for all $m \in M_0$, $\mathcal{V} \cap (\mathbb{R}^1 \times \{m\})$ is an interval and a flow exists on $\mathcal{V}$ is called a *flow domain.*

First we show that a real vector field on a mixed manifold has a unique maximal flow. Let M be a mixed manifold and M^{sm} its underlying smooth manifold which comes with a map $i \colon M^{sm} \to M$. Then $(i^*\mathcal{T}_M), \overline{(i^*\mathcal{T}_M)} \subseteq \mathbb{C} \otimes \mathcal{T}_{M^{sm}}$ and we have the following exact sequence:

$$0 \longrightarrow i^*(\mathcal{T}_{M,\mathbb{C}} \oplus \bar{\mathcal{T}}_{M,\mathbb{C}}) \longrightarrow \mathbb{C} \otimes \mathcal{T}_{M^{sm}} \longrightarrow i^*\mathcal{T}_M/\mathcal{T}_{M,\mathbb{C}} \longrightarrow 0. \tag{7.3}$$

In fact, locally in a neighborhood of the form $(\mathbb{C}^{n_1+n_2}, \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}, \mathbb{C}^{n_2})$, $i^*\mathcal{T}_{M,\mathbb{C}}$ and $\overline{(i^*\mathcal{T}_{M,\mathbb{C}})}$ are spanned as $\mathcal{O}_{M^{sm}}$-modules by ∂_{z_i} and $\bar{\partial}_{z_i}$ ($i \in \{n_1+1, \ldots, n_1+n_2\}$), respectively.

Then we have the following observation.

Lemma 7.4. *For any real vector field X on M, there is a unique real vector field Y on M^{sm} such that $(\mathbb{C} \otimes Y)|_{\mathcal{O}_M} = X$.*

Proof. Consider two such real vector fields Y_1 and Y_2 on M^{sm}. Locally on the model space defined by $(\mathbb{C}^{n_1+n_2}, \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}, \mathbb{C}^{n_2})$, with coordinates $\{x = (x_1, \ldots x_{n_1}), z = (z_1, \ldots, z_{n_2})\}$, we have

$$X = \sum_i f_i(x)\partial_{x_i} + \sum_j g_j(x,z)\partial_{z_j}$$

for smooth real functions $f_i(x)$ and partially holomorphic functions $g_j(x,z)$. Hence we have

$$Y_l = \sum_i f_i(x)\partial_{x_i} + \sum_j g_j(x,z)\partial_{z_j} + \sum_j \bar{g}_j(x,z)\bar{\partial}_{z_j} \quad (l \in \{1,2\}),$$

which proves uniqueness. In order to prove existence, we choose a splitting of (7.3) in order to write $i^*X = X_\mathbb{R} + X_\mathbb{C}$, where $X_\mathbb{C} \in i^*\mathcal{T}_{M,\mathbb{C}}$. Then $Y = X_\mathbb{R} + X_\mathbb{C} + \bar{X}_\mathbb{C}$ is the desired vector field. ☐

Lemma 7.5. *Let $(V, V_\mathbb{R}, V_\mathbb{C})$ be a mixed vector space and let X be a real vector field on $U \subseteq \mathbb{A}(V_\mathbb{R})$ and Y the unique real vector field such that $(\mathbb{C} \otimes Y)|_{\mathcal{O}_U} = X$. The maximal* *flow $\Theta^Y : \mathcal{V}^{sm} \to U^{sm}$ of Y defines a morphism of mixed manifolds Θ^X which is the unique maximal flow of X.*

Proof. The proof of [4, Thm. 12.4.2] applies to show that for every $p \in U$ there is an open neighbourhood U' of p and an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times U' \subseteq \mathcal{V}^{sm}$ and $\Theta^Y|_{(-\epsilon,\epsilon)\times U'}$ is a mixed morphism. Since $\mathcal{V}^{sm}$ is a flow domain and since the flow is additive, we conclude that Θ^Y defines a mixed morphism. This is a flow morphism since Θ^Y is a flow for Y and $(\mathbb{C} \otimes Y)|_{\mathcal{O}_U} = X$. Uniqueness follows from uniqueness of the flow of Y and maximality is ensured by maximality of $\mathcal{V}^{sm}$. □

Lemma 7.6. *Let $(V, V_\mathbb{R}, V_\mathbb{C})$ be a mixed super vector space and let X be a real even vector field on the open subspace $U \subseteq \mathbb{A}(V_\mathbb{R})$. Furthermore, let $\tilde{X}$ be the underlying real vector field on $\mathbb{A}((V_{\bar{0}})_\mathbb{R})$ with maximal flow $\Theta^{\tilde{X}} : \mathcal{V}_0 \to U_0$. There is a unique flow morphism $\Theta^X : \mathcal{V} \to U$ where $\mathcal{V}_0$ is the maximal flow domain and $(\Theta^X)_0$ is the maximal flow of $\tilde{X}$.*

Proof. Following the proof given in [10, Lem. 2.1], the higher order terms of the flow Θ^X are constructed by solving linear ordinary differential equations. The unique solutions will automatically be partially holomorphic, since the initial condition, the identity, is partially holomorphic. So we get a flow $\Theta^X : \mathcal{V} \to U$ for X with $(\Theta^X)_0 = \Theta^{\tilde{X}}$ and $\mathcal{V} \subseteq \mathbb{R} \times U$ is the open sub supermanifold with base $\mathcal{V}_0$. □

By the same reasoning as in [10, Lem. 2.2] one can prove the existence of flow domains:

Lemma 7.7. *Let X be an even real vector field on the mixed supermanifold M. Then there exists a flow domain $\mathcal{V}$ for X. Furthermore, if $\mathcal{V}_i$, $i \in \{1, 2\}$, are flow domains with flows Θ_i^X, then $\Theta_1^X|_{\mathcal{V}_1\cap\mathcal{V}_2} = \Theta_2^X|_{\mathcal{V}_1\cap\mathcal{V}_2}$.*

Putting everything together we obtain the final result.

Theorem 7.8. *Let X be an even real vector field on the mixed supermanifold M with underlying real vector field $\tilde{X}$ on M_0. Then there exists a unique flow map $\Theta^X : \mathcal{V} \to M$ where $\mathcal{V}$ is the maximal flow domain for X. Moreover, $(\Theta^X)_0$ is the maximal flow of $\tilde{X}$.*

Proof. This follows from the above considerations by taking the union of all flow domains. □

Definition 7.9. An even real vector field is called complete if its maximal flow domain $\mathcal{V}$ equals $\mathbb{R} \times M$.

The following basic properties can be proved as in the classical case.

Proposition 7.10. *Suppose X is an even real vector field and Y is an arbitrary vector field on M.*

(a) $\mathcal{L}_X Y := \partial_t|_{t=0} (\Theta_t^X)^\sharp \circ Y \circ (\Theta_{-t}^X)^\sharp = [X, Y]$.

(b) If $[X, Y] = 0$*, then* $\Theta^{X^\sharp}$ *and* Y *commute.*

Proof. See for instance [5, Lem. 3.7, Cor. 3.8]. □

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