

Singularities of Closed Positive Currents

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Zusammenfassung

Diese Dissertation befasst sich mit der Analyse der Singularitäten geschlossener positiver Ströme auf kompakten komplexen Mannigfaltigkeiten und besteht aus sechs Kapiteln.

Im zweiten Kapitel werden Grundlagen zur Pluripotentialtheorie und zur komplexen Geometrie dargestellt. Im dritten Kapitel führen wir im Hermiteischen Kontext das Konzept der relativen nicht-pluripolaren Produkte ein, eine Methode zur Definition von Keilprodukten geschlossener positiver $(1, 1)$ -Ströme mit einem geschlossenen positiven (p, p) -Strom auf einer Hermiteschen Mannigfaltigkeit. Wir besprechen die Konstruktion dieses Produkts, die Monotonieeigenschaft sowie den Zusammenhang mit Dichteströmen, welche eine allgemeinere Methode zur Definition von Keilprodukten höherer Bi-Grad-Ströme darstellen.

Das vierte Kapitel konzentriert sich auf die Abschätzung der Größe der Singularitätsmenge eines gegebenen geschlossenen positiven Stroms T , wobei die Singularitätsmenge die Menge der Punkte ist, an denen T positive Lelong-Zahlen besitzt. Genauer gesagt, leiten wir obere Schranken für das Volumen beliebiger maximal irreduzibler analytischer Mengen innerhalb der oberen Lelong-Niveaumengen von geschlossenen positiven Strömen her. Darüber hinaus zeigen wir im Fall des Bi-Grads $(1, 1)$, dass diese Schranken optimal sind.

Im fünften Kapitel vergleichen wir die Lelong-Zahlen geschlossener positiver $(1, 1)$ -Ströme innerhalb derselben großen Klasse. Genauer gesagt: Seien T, T' zwei geschlossene positive $(1, 1)$ -Ströme in derselben Klasse, wobei T' weniger singulär als T ist. Wir geben eine obere Schranke für die Differenz der Lelong-Zahlen von T und T' an, ausgedrückt durch die Differenz der nicht-pluripolaren Massen von T' und T .

Abschließend stellen wir im sechsten Kapitel eine Methode zur Definition der Lelong-Zahl nicht-pluripolarer (m, m) -Kohomologieklassen auf einer kompakten Kähler-Mannigfaltigkeit vor. Wir vergleichen die Cup-Produkte und die relativen nicht-pluripolaren Produkte. Es stellt sich heraus, dass zwei Produkte stets verschieden sind, falls die Lelong-Zahl der nicht-pluripolaren Kohomologiekategorie von null verschieden ist. Als Anwendung präsentieren wir eine höherdimensionale Version der Zariski-Zerlegung.

Abstract

This thesis focus on analyzing the singularities of closed positive currents on compact complex manifolds, it consists of six chapters.

In the second chapter, some crucial background in pluripotential theory and complex geometry is reviewed. In the third chapter, we introduce the relative non-pluripolar products in Hermitian setting, which is a way defining the wedge products of closed positive $(1, 1)$ -currents and a closed positive (p, p) -current on Hermitian manifolds. We discuss the construction of this product, the monotonicity property, and its relation with density currents, which is a general way defining wedge products of higher bi-degree currents.

The fourth chapter focuses on estimating the size of the singular locus of a given closed positive current T , where the singular locus is the set of points at which T has positive Lelong numbers. Specifically, we established upper bounds for the volumes of arbitrary maximal irreducible analytic sets contained in the Lelong upper level sets of closed positive currents. Moreover, in the bi-degree $(1, 1)$ case, we proved that these bounds are optimal.

In the fifth chapter, we compare the Lelong numbers of closed positive $(1, 1)$ -currents in the same big class. More precisely, let T, T' be two closed positive $(1, 1)$ -currents in the same class such that T is less singular than T' . We provide an upper bound of the difference of the Lelong numbers of T and T' , in terms of the difference of the non-pluripolar masses of T and T' respectively.

Finally, in the sixth chapter, we provide a method for defining the Lelong number of non-pluripolar cohomology (m, m) -classes on compact Kähler manifolds. We compare the cup products and the relative non-pluripolar products. It turns out that the Lelong number of non-pluripolar cohomology classes obstructs the equality of these two classes. As an application, we present a higher-dimensional version of the Zariski decomposition.

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CHAPTER 1

Introduction

Currents, which generalize differential forms in the sense of distributions, play a fundamental role in complex geometry and pluripotential theory. Understanding the singularities of currents is crucial and has numerous applications in both complex geometry and complex dynamics. This thesis is devoted to the study of singularities of currents on complex manifolds. It is organized into six chapters and is based on my four articles [43, 42, 33, 38].

1.0.1. Relative non-pluripolar product on Hermitian manifolds. The first part of my thesis aims to study the wedge products of currents on Hermitian manifolds, and it is based on the article [33]. Currents, as a generalization of differential forms, are not necessarily smooth, and providing a suitable way to define their wedge products on manifolds has been a crucial problem in pluripotential theory.

This problem was first studied by Bedford and Taylor in [1], where they considered closed positive currents T and P on a complex manifold, with $T = dd^c u$ of bi-degree $(1, 1)$ and u a locally bounded potential. In this setting, they defined

$$T \wedge P := dd^c(uP).$$

The boundedness condition on u was slightly relaxed in [16], where Demailly imposes assumptions on the singularities of T .

The non-pluripolar product was introduced and studied in [2, 8], provided a way defining products of closed positive $(1, 1)$ -currents on complex manifolds. Later, in [51], this was generalized to the relative setting, where one could consider the product of closed positive $(1, 1)$ -currents $T_1, \dots, T_m$ and a closed positive (p, p) -current T . The first step of the construction is to reduce to the bounded potential cases. By cutting down the potential function u_j of T_j by $-k$, we set

$$R_k = dd^c \max\{u_1, -k\} \wedge \dots \wedge dd^c \max\{u_m, -k\} \wedge T.$$

When the ambient manifold is Kähler, one could obtain the uniformly boundedness condition on the sequence of currents $\{\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k\}$, which implies the existence of the limit

$$\langle T_1 \wedge \cdots \wedge T_m \frown T \rangle := \lim_{k \rightarrow \infty} \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k,$$

and it is called the non-pluripolar product of $T_1, \dots, T_m$ relative to T .

In [33], we show that the relative non-pluripolar product is also well-defined on compact Hermitian manifolds, with some restriction on the Hermitian form.

THEOREM 1.1. (Theorem 3.4) *Let X be a compact complex manifold of dimension n endowed with a Hermitian form ω satisfying*

$$dd^c \omega = dd^c(\omega^2) = 0.$$

Let T_j be a closed positive $(1,1)$ -current on X for $1 \leq j \leq m$, and let T be a closed positive (p,p) -current on X such that $p + m \leq n$. Then the relative non-pluripolar product $\langle T_1 \wedge \cdots \wedge T_m \frown T \rangle$ is well defined.

The monotonicity property [8, 55, 11, 51], which captures the mass loss in the non-pluripolar product, asserts that on a compact Kähler manifold, if closed positive $(1,1)$ -currents T_j and T'_j lie in the same Dolbeault cohomology class and T'_j is less singular than T_j , which means that the potential of T'_j is greater than or equal to that of T_j up to a constant, for all $1 \leq j \leq m$. Then, the non-pluripolar products of T_j and T'_j satisfy the inequality:

$$(1.0.1) \quad \{\langle T_1 \wedge \cdots \wedge T_m \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \rangle\},$$

where $\{R\}$ denotes the Dolbeault cohomology class of the current R . In [33], we prove that on a Hermitian manifold with the assumption on the Hermitian form given in Theorem 1.1, the monotonicity property in the relative setting holds.

THEOREM 1.2. (Theorem 3.13) *Let X be a compact Hermitian manifold of dimension n , endowed with a Hermitian form ω satisfying*

$$dd^c \omega = dd^c(\omega^2) = 0.$$

Let $\{\theta_1\}_{\partial\bar{\partial}}, \dots, \{\theta_m\}_{\partial\bar{\partial}}$ be Bott–Chern pseudoeffective classes on X . For each $1 \leq j \leq m$, let $T_j, T'_j \in \{\theta_j\}_{\partial\bar{\partial}}$ be closed positive $(1,1)$ -currents, and let T be a closed positive (p,p) -current such that $m + p \leq n$. Assume that T'_j is less singular than T_j for all $1 \leq j \leq m$.

Then, we have

$$\int_X \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle \wedge \omega^{n-m-p} \leq \int_X \langle T'_1 \wedge \cdots \wedge T'_m \wedge T \rangle \wedge \omega^{n-m-p}.$$

1.0.2. Volumes of Components of Lelong Upper Level Sets. The second part of my thesis is based on [43], and focuses on analyzing the singular loci (Lelong upper level set) of a closed positive current T on a compact Kähler manifold. That is, the set of points where T has positive Lelong numbers.

The motivation for this study is inspired by algebraic geometry. For example, let $\mathcal{C}$ be an algebraic curve in $\mathbb{P}^2$ of degree d . Estimating the singularities of $\mathcal{C}$ is a classical problem. By Bézout's theorem, $\mathcal{C}$ has at most $\frac{1}{2}(d-1)(d-2)$ singular points. Closed positive currents are far-reaching generalizations of analytic sets, and we aim to investigate the size of their singular loci. More precisely, the volumes of their Lelong upper level sets.

This problem was first studied by Demailly [18] in the case of closed positive $(1,1)$ -currents, where he provided volume estimates for certain maximal irreducible analytic subsets contained in the Lelong upper level sets, using the regularization theorem for closed positive $(1,1)$ -currents. This result was later extended to currents of higher bi-degree in [36, 46] via the Lelong–Skoda transformation. However, neither result could handle analytic sets of arbitrary dimension contained in the Lelong upper level set.

Let T be a closed positive (p,p) -current on a compact Kähler manifold X , and let W be an irreducible analytic subset of dimension m . In [22], the authors established upper bounds for the volumes of maximal irreducible analytic subsets $V \subseteq W$ that are contained in the Lelong upper level sets of closed positive currents on W . We denote the set of such analytic subsets of dimension ℓ by $\mathcal{V}_{\ell,T,W}$.

The upper bound in [22] depends on $\text{vol}(W)$ and $\|T\|$, where

$$\text{vol}(W) := \frac{1}{m!} \int_{\text{Reg}(W)} \omega^m, \quad \text{and} \quad \|T\| := \int_X T \wedge \omega^{n-p}.$$

Here, $\text{Reg}(W)$ denotes the regular locus of W .

In joint work with Duc-Viet Vu [43], we established a finer upper bound in terms of the cohomology classes of the non-pluripolar self-products of T .

Before stating our main theorem, we first recall the notion of Lelong number. Let T be a closed positive (p,p) -current on X . We denote by $\nu(T, x)$ the Lelong

number of T at a point $x \in X$ (see Section 2.1.4 for more details). Let $V \subseteq X$ be an irreducible analytic subset. We denote by $\nu(T, V)$ the *generic Lelong number* of T along V , defined by

$$\nu(T, V) = \inf_{x' \in V} \nu(T, x').$$

THEOREM 1.3. (*Theorem 4.1*) *Let α be a nef $(1, 1)$ -class on a compact Kähler manifold X , and let $W \subseteq X$ be an irreducible analytic subset of dimension m . Let $T \in \alpha$ be a closed positive current such that $\nu(T, W) = 0$. Let $1 \leq m' \leq m$ be an integer. Then, we have*

$$\sum_{V \in \mathcal{V}_{m-m', T, W}} \nu(T, V)^{m'} \text{vol}(V) \leq \frac{1}{(m-m')!} \int_{\text{Reg} W} (\alpha^{m'} - \langle (T|_{\text{Reg} W})^{m'} \rangle) \wedge \omega^{m-m'},$$

where in the integral, we identify α with a smooth closed form in α .

Our approach relies on the relative non-pluripolar product and its connection with density currents, a concept for defining the intersection of higher-degree currents.

1.0.3. Lelong numbers of currents with full mass intersection. The third part of my thesis, based on [42], investigates the difference in singularities between two closed positive $(1, 1)$ -currents T and T' in the same cohomology class. Assuming T is less singular than T' , we compare their Lelong numbers and derive bounds in terms of their non-pluripolar masses.

This problem is motivated by a fundamental question in pluripotential theory. Specifically, the phenomenon of mass loss in non-pluripolar products ([3, 8, 51]). In general, non-pluripolar products of cohomologous currents T and T' may fail to remain in the same cohomology class. Nevertheless, we have the monotonicity property (1.0.1), which captures the mass loss of non-pluripolar product. When the equality in (1.0.1) holds, $T_1, \dots, T_m$ is said to have *full mass intersection* with respect to $T'_1, \dots, T'_m$. Understanding obstructions to full mass intersection is an important problem, where the singularities of T and T' play a key role.

For the self-intersection case, a characterization of currents with full mass intersection was provided in [11] via the singularity types of associated envelopes. Vu [53] later studied this problem in the mixed setting, assuming $T = T_{\min}$ has minimal singularities. In my recent work [42], I extended Vu's results to the setting of prescribed singularities. We first recall the definition of an $\mathcal{I}$ -model

(see [14] for further details). Let $T = dd^c u + \theta$ be a closed positive $(1, 1)$ -current. We say that T is $\mathcal{I}$ -model if

$$u = P^\theta[u]_{\mathcal{I}},$$

where

$$P^\theta[u]_{\mathcal{I}} := (\sup \{w \in \text{PSH}(X, \theta) \mid w \leq 0, \mathcal{I}(tw) \subseteq \mathcal{I}(tu), \forall t \geq 0\})^*$$

is the $\mathcal{I}$ -envelope of u , and $\mathcal{I}(tu)$ denotes the multiplier ideal sheaf, locally generated by holomorphic functions f such that $|f|^2 e^{-tu}$ is integrable.

The following is our first main result.

THEOREM 1.4. (Theorem 5.1) *Let X be a compact Kähler manifold of dimension n . Let $\{\theta_1\}, \dots, \{\theta_m\}$ be big classes and let $T'_j, T_j \in \{\theta_j\}$ be closed positive $(1, 1)$ -currents such that for $1 \leq j \leq m$,*

- (1) $\int_X \langle T_j^n \rangle > 0$.
- (2) T_j is $\mathcal{I}$ -model.
- (3) T_j is less singular than T'_j .

Let V be a proper irreducible analytic subset such that $\dim(V) \geq n - m$. If $T'_1, \dots, T'_m$ are of full mass intersection with respect to $T_1, \dots, T_m$, that is,

$$\{\langle \wedge_{j=1}^m T_j \rangle\} = \{\langle \wedge_{j=1}^m T'_j \rangle\}.$$

Then there exists $1 \leq j \leq m$ such that $v(T'_j, V) = v(T_j, V)$.

For the case $m = n$, we have the following quantitative result:

THEOREM 1.5. (Theorem 5.2) *Let X be a compact Kähler manifold, and let $\mathcal{B}$ be a closed cone in the cone of big classes. There exists a constant $C > 0$ only dependent on the manifold X and the cone $\mathcal{B}$ such that for every $x_0 \in X$, $\{\theta_j\} \in \mathcal{B}$ and $T'_j, T_j \in \{\theta_j\}$, $1 \leq j \leq n$, satisfying the conditions in Theorem 5.1. The following inequality holds.*

$$(1.0.2) \quad \int_X \left(\langle \wedge_{j=1}^n T_j \rangle - \langle \wedge_{j=1}^n T'_j \rangle \right) \geq C \prod_{j=1}^n (v(T'_j, x_0) - v(T_j, x_0)).$$

My approach refines Vu's argument in [53] by using relative non-pluripolar products and introducing the notion of difference envelopes of currents.

1.0.4. Singularities of $(n - 1, n - 1)$ -classes. The fourth part of my thesis is based on the article [38] and focuses on understanding the singularities of

non-pluripolar cohomology (m, m) -classes on compact Kähler manifolds. More precisely, let α be a big $(1, 1)$ -class, and let $\langle \alpha^m \rangle := \{ \langle T_{\alpha, \min}^m \rangle \}$ denote the self non-pluripolar product of α , where $T_{\alpha, \min}$ is a current with minimal singularities in the class α . We define the Lelong number of $\langle \alpha^m \rangle$ at a point $x \in X$ to be the Lelong number of $\langle T_{\alpha, \min}^m \rangle$ at x . We first prove that this definition is independent of the choice of $T_{\alpha, \min}$ (Theorem 6.1).

The main result of this chapter is a comparison between the cup products and the relative non-pluripolar products.

THEOREM 1.6. (Theorem 6.2) *Let X be a compact Kähler manifold of dimension n . Let $T_1, \dots, T_{n-1}, T$ be closed positive $(1, 1)$ -current on X . Put*

$$\beta := \{ \langle \wedge_{j=1}^{n-1} T_j \rangle \}, \gamma := \{ \langle \wedge_{j=1}^{n-1} T_j \wedge T \rangle \}.$$

Then we have $\beta \wedge \{T\} \geq \gamma$, where $\beta \wedge \{T\}$ is the cup product of β and $\{T\}$. Moreover, if $\beta \wedge \{T\} = \gamma$, then

$$v(\langle \wedge_{j=1}^{n-1} T_j \rangle, x) \cdot v(T, x) = 0,$$

for every $x \in X$.

As an application of Theorem 1.6, we have the following vanishing theorem for Lelong numbers.

THEOREM 1.7. (Theorem 6.3) *Let X be a projective manifold, and let α be a big $(1, 1)$ -class on X . Then, we have*

$$v(\langle \alpha^{n-1} \rangle, x) = 0 \text{ for every } x \in X.$$

When X is a compact Kähler surface and α is a big class, the non-pluripolar part $\langle \alpha \rangle$ coincides with the nef part in the Zariski decomposition. In [6], it was shown that for a pseudoeffective class α on a compact Kähler manifold, the non-pluripolar part $\langle \alpha \rangle$ is nef if and only if its Lelong number vanishes everywhere. Theorem 1.6 can thus be viewed as a higher-dimensional analogue of the Zariski decomposition, in the sense that the non-pluripolar product $\langle \alpha^{n-1} \rangle$ has vanishing Lelong number at every point.

CHAPTER 2

Preliminary

2.1. Notations and definitions

Let $\mathbb{N}$ denote the set of natural numbers, and set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Let $n \in \mathbb{N}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we define $|\alpha| := \sum_{j=1}^n \alpha_j$.

Let $\Omega \subseteq \mathbb{C}^n$ be an open set, and let $(z_1, \dots, z_n)$ be the holomorphic coordinates on Ω , where

$$z_j = x_{2j-1} + ix_{2j}, \quad 1 \leq j \leq n.$$

Let $\alpha^1 = (\alpha_1^1, \dots, \alpha_n^1)$, $\alpha^2 = (\alpha_1^2, \dots, \alpha_n^2) \in \mathbb{N}_0^n$ be two multi-indices. We define the following:

$$\begin{aligned} \partial_{z_j} &:= \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), & \bar{\partial}_{z_j} &:= \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right), \\ \partial_z^{\alpha^1} &:= \partial_{z_1}^{\alpha_1^1} \cdots \partial_{z_n}^{\alpha_n^1}, & \bar{\partial}_z^{\alpha^2} &:= \bar{\partial}_{z_1}^{\alpha_1^2} \cdots \bar{\partial}_{z_n}^{\alpha_n^2}, \\ \partial &:= \sum_{j=1}^n \frac{\partial}{\partial z_j}, & \bar{\partial} &:= \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j}, \\ dz_j &:= dx_{2j-1} + i dx_{2j}, & d\bar{z}_j &:= dx_{2j-1} - i dx_{2j}. \end{aligned}$$

Let $I, J \subseteq \{1, \dots, n\}$, with $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_k\}$. We put

$$dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_s}, \quad d\bar{z}_J := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_k}.$$

We also define the differential operators

$$d := \partial + \bar{\partial}, \quad d^c := \frac{1}{2\pi i} (\bar{\partial} - \partial),$$

so that $dd^c = \frac{1}{\pi i} \partial \bar{\partial}$.

2.1.1. Differential forms and Currents. We recall some standard definitions and notations, following [16, 19].

Let $p, q \leq n$, and let

$$u = \sum_{|I|=p, |J|=q} u_{IJ} dz_I \wedge d\bar{z}_J$$

be a differential (p, q) -form on Ω . We denote by

$$p_K^s(u) := \sup_{x \in K} \max_{\substack{|I|=p, |J|=q \\ |\alpha^1| + |\alpha^2| \leq s}} \left| \partial_z^{\alpha^1} \partial_{\bar{z}}^{\alpha^2} u_{IJ}(x) \right|$$

the seminorm associated with the compact set $K \subseteq \Omega$ and $s \in \mathbb{N}$. We denote by $\mathcal{E}^{p,q}(\Omega)$ be the space of differential (p, q) -forms on Ω with the topology induced by the seminorms p_K^s , where $K \subseteq \Omega$ and $s \in \mathbb{N}$. Let $\mathcal{D}^{p,q}(\Omega)$ be the space of differential (p, q) -forms with compact support induced with the restriction topology.

A *current* T on Ω of bi-degree (p, q) (or bi-dimension $(n - p, n - q)$) is a continuous linear functional

$$T: \mathcal{D}^{n-p, n-q}(\Omega) \rightarrow \mathbb{C},$$

and we denote by $\langle T, \alpha \rangle \in \mathbb{C}$ the action of the current T on a test form $\alpha \in \mathcal{D}^{n-p, n-q}(\Omega)$. We denote by $\mathcal{D}'^{n-p, n-q}(\Omega)$ the space of currents of bi-degree (p, q) on Ω , which is the topological dual of $\mathcal{D}^{n-p, n-q}(\Omega)$. Currents can be viewed as a generalization of differential forms in the sense of distributions. Specifically, any current T can be written as

$$T = \sum_{|I|=p, |J|=q} T_{IJ} dz_I \wedge d\bar{z}_J,$$

where each coefficient T_{IJ} is a distribution on Ω .

Let $T \in \mathcal{D}'^{p,q}(\Omega)$. The *support* of T is denoted by $\text{supp}(T)$, which is the smallest closed set $B \subseteq \Omega$ such that the restriction of T to $\mathcal{D}^{n-p, n-q}(\Omega \setminus B)$ is zero.

Let $T \in \mathcal{D}'^{p,q}(\Omega)$, ∂T and $\bar{\partial} T$ are currents of bi-degree $(p+1, q)$ and $(p, q+1)$ respectively, defined by

$$\langle \partial T, \alpha \rangle := (-1)^{p+q+1} \langle T, \partial \alpha \rangle, \quad \langle \bar{\partial} T, \beta \rangle := (-1)^{p+q+1} \langle T, \bar{\partial} \beta \rangle,$$

for any $\alpha \in \mathcal{D}^{n-p-1, n-q}(\Omega)$, $\beta \in \mathcal{D}^{n-p, n-q-1}(\Omega)$. We say that T is *∂ -closed* (respectively, *$\bar{\partial}$ -closed*) if $\partial T = 0$ (respectively, $\bar{\partial} T = 0$). We say that T is *closed* if $dT := \partial T + \bar{\partial} T = 0$ on Ω .

Let $\{T_j\}_{j \in \mathbb{N}}$ be a sequence of (p, q) -currents on Ω . We say that T_j *converges weakly* to $T \in \mathcal{D}'^{p,q}(\Omega)$ if

$$\langle T_j, \alpha \rangle \rightarrow \langle T, \alpha \rangle,$$

for any $\alpha \in \mathcal{D}^{n-p, n-q}(\Omega)$.

A differential (p, p) -form α on Ω is said to be *positive* if, at each point, it can be written as a finite linear combination of forms of the type

$$i\gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i\gamma_p \wedge \bar{\gamma}_p,$$

where each γ_j is a $(1, 0)$ -form. A (p, p) -form β is said to be *weakly positive* if $\beta \wedge \alpha$ is positive for every positive $(n - p, n - p)$ -form α on Ω .

Let $T \in \mathcal{D}'^{p,p}(\Omega)$. We say that T is *positive* (respectively, *weakly positive*) if $\langle T, \alpha \rangle \geq 0$ for every weakly positive (respectively, positive) $(n - p, n - p)$ -form α . Any positive (p, p) -current

$$T = \sum_{|I|=p, |J|=p} T_{IJ} dz_I \wedge d\bar{z}_J$$

has real coefficients; that is, $T_{IJ} = \overline{T_{JI}}$. Let $T_1, T_2 \in \mathcal{D}'^{p,p}(\Omega)$. We write $T_1 \geq T_2$ if $T_1 - T_2$ is a positive (p, p) -current.

REMARK 2.1. Let X be a complex manifold of dimension n . By choosing holomorphic coordinate charts that cover X , one can extend the definitions of currents and the associated differential operators to the global setting.

Let X be a compact complex manifold of dimension n , and let ω be a Hermitian form on X . Let T be a closed positive (p, p) -current, $p \leq n$. We define the *mass* of T by

$$\|T\| := \int_X T \wedge \omega^{n-p}.$$

Here, we give some examples of positive currents.

EXAMPLE 2.2.

- (1) Let γ be a positive (p, p) -form on Ω . By the definition of weakly positive forms, we get

$$\gamma \wedge \beta \geq 0,$$

for any weakly positive $(n - 1, n - p)$ -form β . Hence α is also a positive (p, p) -current on Ω .

- (2) Let X be a compact complex manifold of dimension n , and let $V \subseteq X$ be an analytic subset of dimension m . We denote by $[V]$ the $(n-m, n-m)$ -current of integration along V , defined by

$$\langle [V], \alpha \rangle := \int_{\text{Reg}(V)} \alpha, \quad \text{for all } \alpha \in \mathcal{D}^{m,m}(X).$$

Since the integral of a weakly positive form over V is non-negative, the current $[V]$ is positive. Moreover, by Stokes' theorem, $[V]$ is closed.

An upper semi-continuous function $u: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be *plurisubharmonic* (*psh* for short) if it satisfies

$$u(x) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + e^{i\theta} \zeta) d\theta,$$

for all $x \in \Omega$ and $\zeta \in \mathbb{C}^n$ with $|\zeta|$ sufficiently small.

Plurisubharmonic functions are central to pluripotential theory, which is closely linked to the theory of currents. Indeed, any closed positive $(1,1)$ -current T on Ω can be locally written as $T = dd^c u$ for some psh function u . Conversely, given any psh function u on Ω , the current $dd^c u$ defines a closed positive $(1,1)$ -current on Ω .

Let $T = dd^c u$ be a closed positive $(1,1)$ -current on Ω , where u is a psh function on Ω . The *singular locus* of T is denoted and defined as

$$I_T := \{x \in \Omega \mid u(x) = -\infty\}.$$

2.1.2. Cohomology classes of currents. Let X be a compact complex manifold of dimension n . Let $p, q \leq n$. The *Dolbeault* and *Bott–Chern cohomology groups* of currents on X are defined as follows:

$$H^{p,q}(X, \mathbb{C}) = \{T \in \mathcal{D}'^{p,q}(X) \mid \bar{\partial}T = 0\} / \{\bar{\partial}T' \mid T' \in \mathcal{D}'^{p,q-1}(X)\},$$

$$H_{\text{BC}}^{p,q}(X, \mathbb{C}) = \{T \in \mathcal{D}'^{p,q}(X) \mid dT = 0\} / \{\partial\bar{\partial}T' \mid T' \in \mathcal{D}'^{p-1,q-1}(X)\}.$$

Let T be a $\bar{\partial}$ -closed current on X . We denote by $\{T\}$ the *Dolbeault cohomology class* of T . Similarly, for a d -closed current T'' , we denote by $\{T''\}_{\partial\bar{\partial}}$ its *Bott–Chern cohomology class*.

Note that $\{\partial\bar{\partial}T' \mid T' \in \mathcal{D}'^{p-1,q-1}(X)\} \subseteq \{\bar{\partial}T' \mid T' \in \mathcal{D}'^{p,q-1}(X)\}$. Hence, we have the nature inclusion

$$i: H_{\text{BC}}^{p,q}(X, \mathbb{C}) \rightarrow H^{p,q}(X, \mathbb{C}).$$

If X is furthermore a compact Kähler manifold, then by the $\partial\bar{\partial}$ -lemma, the inclusion map i is an isomorphism.

We now assume that X is a compact Kähler manifold. Let $H^{1,1}(X, \mathbb{R})$ denote the space of real $(1,1)$ -Dolbeault cohomology classes on X , and let $\alpha \in H^{1,1}(X, \mathbb{R})$. In the following, we recall several positivity notions for $(1,1)$ -classes. We say

- (1) α is *Kähler* if there exists a Kähler form in α .
- (2) α is *nef* if for any $\epsilon > 0$, the class $\alpha + \{\epsilon\omega\}$ is Kähler.
- (3) We say that α is *big* if there exists a closed positive $(1,1)$ -current $T \in \alpha$ such that $T \geq \epsilon\omega$ for some $\epsilon > 0$. Such a current T is called a *Kähler current*.
- (4) α is *pseudoeffective* if there exists a closed positive $(1,1)$ -current $T \in \alpha$.

DEFINITION 2.3.

- (1) Let $\mathcal{K} \subseteq H^{1,1}(X, \mathbb{R})$ be the set of Kähler classes. $\mathcal{K}$ is an open convex cone, and is called the *Kähler cone*.
- (2) We denote by $\overline{\mathcal{K}} \subseteq H^{1,1}(X, \mathbb{R})$ the set of nef classes, which is the closure of $\mathcal{K}$.
- (3) Let $\mathcal{E} \subseteq H^{1,1}(X, \mathbb{R})$ be the set of cohomology classes of closed positive $(1,1)$ -currents. It is a closed convex cone, and is called the *pseudoeffective cone*.
- (4) $\mathcal{E}^0 \subseteq H^{1,1}(X, \mathbb{R})$ is the set of big classes, which is the interior of $\mathcal{E}$, and is called the *cone of big $(1,1)$ -classes*.

Let α, β be two pseudoeffective classes. We write $\alpha \geq \beta$ if $\alpha - \beta$ is a pseudoeffective class; that is, there exists a closed positive $(1,1)$ -current $T \in \alpha - \beta$.

Note that if $\alpha \geq \beta$ and $\beta \geq \alpha$, then there exist closed positive currents $T_{\alpha-\beta} \in \alpha - \beta$ and $T_{\beta-\alpha} \in \beta - \alpha$. Hence, the current $T_{\alpha-\beta} + T_{\beta-\alpha} \in \{0\}$ is closed and positive. Since the zero cohomology class $\{0\}$ contains only the zero closed positive current, it follows that $T_{\alpha-\beta} = T_{\beta-\alpha} = 0$. This shows that $\alpha = \beta$.

2.1.3. Classical intersection of $(1,1)$ -currents. Let X be a complex manifold of dimension n . Let P be a closed positive $(1,1)$ -current, and let T be a closed positive (p,p) -current with $p < n$. In general, defining the wedge product $P \wedge T$ on X is not easy when P and T are not smooth. However, under certain

regularity conditions on P , Bedford and Taylor introduced a natural way to define the product in [1]. This construction is known as the classical product. We review the definition here.

Since P is a closed positive $(1, 1)$ -current, it can be locally written as $P = dd^c u$ for some psh function u . If we further assume that u is locally bounded, then the product uT is a well-defined Borel measure, and we define

$$dd^c u \wedge T := dd^c(uT).$$

When P and T are smooth differential forms, this definition agrees with the classical wedge product, as ensured by Stokes' theorem. The current $dd^c u \wedge T$ is closed, and it can be shown to be positive by applying a smooth regularization to the psh function u .

The products defined above were generalized by Demailly in [16, Chapter III, Section 4], under the assumption that X can be covered by Stein open sets Ω satisfying

$$\partial\Omega \cap L(u) \cap \text{supp}(T) = \emptyset,$$

where $L(u)$ denotes the *unbounded locus* of u , that is, the set of points $x \in X$ such that u is unbounded in every neighborhood of x . Under this assumption, it can be ensured that uT is a well-defined measure, and consequently, the wedge product $dd^c u \wedge T = dd^c(uT)$ also makes sense.

Now, consider closed positive $(1, 1)$ -currents $T_1, \dots, T_m$ and a closed positive (p, p) -current T with $m + p \leq n$. Locally, we write $T_j = dd^c u_j$ for some psh function u_j . The product of $T_1, \dots, T_m$ and T is locally defined by

$$\begin{aligned} dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge T &= dd^c (u_1 dd^c u_2 \wedge \dots \wedge dd^c u_m \wedge T) \\ &= dd^c (dd^c (u_2 dd^c u_3 \wedge \dots \wedge dd^c u_m \wedge T)) \\ &\vdots \\ &= dd^c (\dots dd^c (u_m T) \dots). \end{aligned}$$

In [23], Fornæss and Sibony gave a concrete way to check the well-definedness of $dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge T$ in terms of the calculation of the Hausdorff dimensions of the bounded locus of potential functions u_j .

THEOREM 2.4. ([23, Corollary 3.6] or [16, Chapter III, Theorem 4.5]) *Let $u_1, \dots, u_m$ be psh functions on some local chart of X , and let T be a closed positive (p, p) -current*

on X such that $m + p \leq n$. Then, the current

$$dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

is well-defined if the Hausdorff measure

$$\mathcal{H}_{2(n-p)-2k+1}(L(u_{j_1}) \cap \cdots \cap L(u_{j_k}) \cap \text{supp } T) = 0,$$

for any indices $\{j_1, \dots, j_k\} \subseteq \{1, \dots, m\}$.

Currents can be regarded as generalizations of analytic sets (see Example 2.2, (2)). The following proposition illustrates how the classical product can be interpreted as an intersection of analytic sets.

PROPOSITION 2.5. ([16, Chapter III, Proposition 4.12]) *Let $Z_1, \dots, Z_q$ be hypersurfaces on X . Let $\{C_k\}_{k \in \mathbb{N}}$ be the irreducible components of $\cap_{j=1}^q Z_j$. If hypersurfaces $\{Z_j\}$ satisfy*

$$\dim(Z_{j_1} \cap \cdots \cap Z_{j_m}) = n - m,$$

for any index $\{j_1, \dots, j_m\} \subseteq \{1, \dots, q\}$. Then, there exist integers $m_k > 0$ such that

$$[Z_1] \cap \cdots \cap [Z_q] = \sum_k m_k [C_k].$$

Now, let's recall an important property of the classical product.

THEOREM 2.6. ([51, Theorem 2.9]) *Let $\Omega \subseteq \mathbb{C}^n$ be an open set. Let $m, l \leq n$. Let T be a closed positive current on X on Ω , and u_j, u'_j be locally bounded psh functions on Ω , $1 \leq j \leq m$. Let v_k, v'_k be locally bounded psh functions on Ω , $1 \leq k \leq l$. Assume that $u_j = u'_j$ on $W := \cap_{k=1}^l \{v_k > v'_k\}$ for $j = 1, \dots, m$. Then we have*

$$\mathbb{1}_W dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T = \mathbb{1}_W dd^c u'_1 \wedge \cdots \wedge dd^c u'_m \wedge T.$$

We note that the above theorem also works when we replace the locally bounded psh function by the locally bounded quasi-plurisubharmonic function, where we will recall the definition of quasi-plurisubharmonic function in Section 2.2.

2.1.4. Lelong numbers of currents. The Lelong number, introduced in [32], is a notion that measures the singularities of closed positive currents. Closed positive currents can be viewed as a generalization of analytic sets, and the Lelong number serves as the analytic analogue of the multiplicity of an analytic set. Let's review the definition now.

Let X be a complex manifold of dimension n , and let T be a closed positive (p, p) -current on X . Let $x \in X$, and let Ω be a coordinate chart centered at x . The *Lelong number* of T at the point x is denoted and defined by

$$v(T, x) := \lim_{r \rightarrow 0^+} v(T, x, r), \quad \text{where} \quad v(T, x, r) = \frac{1}{r^{2(n-p)}} \int_{B(x, r)} T(z) \wedge dd^c \|z\|^2.$$

In [32], Lelong showed that $v(T, x, r)$ is non-negative and monotonically increasing in r . Hence, the limit defining the Lelong number exists. Later, Siu proved in [41] that the definition is independent of the choice of the coordinate chart Ω .

When $T = [V]$ is the current of integration along an analytic set $V \subseteq X$, Thie showed in [44] that the Lelong number of T at x coincides with the multiplicity of V at x .

In the case where T is of bi-degree $(1, 1)$, it can be locally written as $T = dd^c u$ on Ω for some psh function u . In this setting, Lelong showed in [32] that

$$v(T, x) = \sup \left\{ \gamma \in \mathbb{R}^+ \mid u(z) \leq \gamma \log |z - x| + \mathcal{O}(1) \text{ at } x \right\}.$$

We review a key property concerning the Lelong number of intersections of closed positive $(1, 1)$ -currents. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents, and let T be a closed positive (p, p) -current on X such that $m + p \leq n$. Locally, we can write $T_j = dd^c u_j$ for some psh function u_j . We have the following proposition.

PROPOSITION 2.7. ([16, Chapter III, Corollary 7.9]) *Let Ω be a local coordinate chart on X . For each $1 \leq j \leq m$, we write $T_j = dd^c u_j$, where u_j is a psh function on Ω . If the product*

$$dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge T$$

is well-defined. Then, we have

$$v(dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge T, x) \geq v(dd^c u_1, x) \cdots v(dd^c u_m, x) v(T, x),$$

for every $x \in \Omega$.

Next, we review the upper level set and some of its properties.

COROLLARY 2.8. ([41]) *Let T be a closed positive (p, p) current in X , and let $c > 0$. Then, the upper level set*

$$E_c(T) := \{x \mid \nu(T, x) \geq c\}$$

is an analytic set of dimension less than or equal to $n - p$.

We now recall the generic Lelong number.

LEMMA 2.9. ([41]) *Let T be a closed positive (p, p) -current on X , and let $V \subseteq X$ be an irreducible analytic subset. Then, there exists a countable family of proper analytic subsets $\{V_j\}$ of V such that*

$$\nu(T, x) = \inf_{x' \in V} \{\nu(T, x')\},$$

for $x \in V \setminus \bigcup V_j$. The right-hand side of the above equation is called the generic Lelong number of T along V and is denoted by $\nu(T, V)$.

We conclude this section by recalling Siu's celebrated decomposition theorem.

THEOREM 2.10. ([41]) *Let T be a closed positive (p, p) -current. Then T can be written as the convergent series as follows:*

$$T = \sum_{k=1}^{\infty} \nu(T, V_k)[V_k] + R,$$

where V_k is an irreducible analytic set of dimension $n - p$, and R is a residual current such that

$$\dim E_c(R) < n - p$$

for every $c > 0$.

2.2. Quasi-plurisubharmonic functions, singularity types and envelopes

Let X be a compact Kähler manifold of dimension n . By the maximum principle, any psh function on X must be constant. Therefore, to develop the global pluripotential theory on X , we introduce the notion of quasi-plurisubharmonic functions.

We say that an upper semi-continuous function $u: X \rightarrow \mathbb{R} \cup \{-\infty\}$ is *quasi-plurisubharmonic* (*quasi-psh* for short) if it can be locally written as the sum of a plurisubharmonic (psh) function and a smooth function.

Let θ be a closed real $(1, 1)$ -form on X . A quasi-psh function u is said to be θ -plurisubharmonic (θ -psh for short) if

$$dd^c u + \theta \geq 0.$$

We denote by $\text{PSH}(X, \theta)$ the set of θ -psh functions on X .

Quasi-psh functions play a central role in the theory of currents on X . For instance, given any closed positive $(1, 1)$ -current T on X , and any closed real $(1, 1)$ -form θ cohomologous to T , the current T can be written as

$$T = dd^c u + \theta,$$

for some $u \in \text{PSH}(X, \theta)$. We also use θ_u to represent the current T .

Let $u, v \in \text{PSH}(X, \theta)$. We say that u is *less singular* (*more singular*) than v if $v \leq u + \mathcal{O}(1)$ ($u \leq v + \mathcal{O}(1)$), and we denote this by $v \preceq u$ ($v \succeq u$). The equivalence relation of the above ordering is denoted by $\simeq$, and we denote by $[u]$ the *singularity type* of u , that is, the equivalence class containing u .

For $u, v \in \text{PSH}(X, \theta)$, we say u is *less $\mathcal{I}$ -singular* (*more $\mathcal{I}$ -singular*) than v if $\mathcal{I}(\lambda v) \subseteq \mathcal{I}(\lambda u)$ ($\mathcal{I}(\lambda v) \supseteq \mathcal{I}(\lambda u)$) for all constant $\lambda > 0$. Here, $\mathcal{I}(\lambda u)$ denotes the *multiplier ideal sheaf* of λu , which is locally generated by holomorphic function f such that $|f|^2 e^{-\lambda u}$ is locally integrable. We denote this relation by $v \preceq_{\mathcal{I}} u$ ($v \succeq_{\mathcal{I}} u$), and denote the corresponding equivalence relation by $\simeq_{\mathcal{I}}$. The $\mathcal{I}$ *singularity type* of u is denoted by $[u]_{\mathcal{I}}$. Understanding the singularity type plays an important role in pluripotential theory; see [11, 13, 14].

DEFINITION 2.11. Let θ be a closed real $(1, 1)$ -form on X , and let u be a θ -psh function. We define the *upper envelope* and the $\mathcal{I}$ -*envelope* of u as

$$P^\theta[u] := (\sup\{w \in \text{PSH}(X, \theta) | w \leq 0, \quad w \preceq u\})^* \in \text{PSH}(X, \theta) \quad \text{and}$$

$$P^\theta[u]_{\mathcal{I}} := (\sup\{w \in \text{PSH}(X, \theta) | w \leq 0, \quad tw \preceq_{\mathcal{I}} tu, \quad t \geq 0\})^* \in \text{PSH}(X, \theta),$$

respectively. Here $(\cdot)^*$ means the upper semi-continuous envelope.

For $u \in \text{PSH}(X, \theta)$, note that

$$u \leq P^\theta[u] \leq P^\theta[u]_{\mathcal{I}}.$$

If u satisfies $u = P^\theta[u]$ (respectively, $u = P^\theta[u]_{\mathcal{I}}$), then we say that u is *model* (respectively, $\mathcal{I}$ -*model*).

EXAMPLE 2.12.

- (1) Set $V_\theta := \sup\{v \in \text{PSH}(X, \theta) | v \leq 0\}$, which is of minimal singularities in $\text{PSH}(X, \theta)$. Since V_θ is less singular than all θ -psh functions, it is model and $\mathcal{I}$ -model.
- (2) Let $u \in \text{PSH}(X, \theta)$. By [14, Proposition 2.18], we have

$$P^\theta[P^\theta[u]_{\mathcal{I}}]_{\mathcal{I}} = P^\theta[u]_{\mathcal{I}}.$$

Therefore, $P^\theta[u]_{\mathcal{I}}$ is $\mathcal{I}$ -model.

Let $T_1 = dd^c u_1 + \theta$ and $T_2 = dd^c u_2 + \theta$ be two closed positive $(1, 1)$ -currents in the same cohomology class. We say that T_1 is less singular than T_2 if $u_1 \succeq u_2$, and we denote this by $T_1 \succeq T_2$.

Let $T = \theta_u = dd^c u + \theta$ be a closed positive $(1, 1)$ -current. The *upper envelope* and $\mathcal{I}$ -*envelope* of T are defined as

$$\begin{aligned} P[T] &:= \theta_{P^\theta[u]} = dd^c P^\theta[u] + \theta \text{ and} \\ P[T]_{\mathcal{I}} &:= \theta_{P^\theta[u]_{\mathcal{I}}} = dd^c P^\theta[u]_{\mathcal{I}} + \theta, \end{aligned}$$

respectively.

LEMMA 2.13. *The envelopes $P[T]$ and $P[T]_{\mathcal{I}}$ are well-defined.*

PROOF. We first note that for $u, v \in \text{PSH}(X, \theta)$ having the same singularity type, the envelopes satisfy $P^\theta[u] = P^\theta[v]$ and $P^\theta[u]_{\mathcal{I}} = P^\theta[v]_{\mathcal{I}}$.

Let θ' be a smooth closed real $(1, 1)$ -form cohomologous to θ . Write $T = dd^c u' + \theta'$, where $u' \in \text{PSH}(X, \theta')$.

By the $\partial\bar{\partial}$ -lemma, there exists a smooth function $f \leq 0$ on X such that $dd^c f = \theta - \theta'$. Now, note that

$$\begin{aligned} T &= dd^c u + \theta \\ &= dd^c(u + f) + \theta', \end{aligned}$$

which shows that $[u + f] = [u']$, and we get

$$\begin{aligned} dd^c P^\theta[u] + \theta &= dd^c(P^\theta[u] + f) + \theta' \\ &= dd^c P^{\theta'}[u + f] + \theta' \\ &= dd^c P^{\theta'}[u'] + \theta'. \end{aligned}$$

Therefore, we get the well-definedness of $P[T]$. For the well-definedness of $P[T]_{\mathcal{I}}$, the proof is similar. $\square$

Let $f: X \rightarrow [-\infty, \infty]$ be a function on X . The *envelope with respect to f* is defined as

$$P^\theta(f) := (\sup\{v \in \text{PSH}(X, \theta) \mid v \leq f\})^*.$$

For convenience, we set $\sup \emptyset = -\infty$. Note that $P^\theta(f) \in \text{PSH}(X, \theta)$ if and only if there exists $v \in \text{PSH}(X, \theta)$ with $v \leq f$. This notation was introduced in [39] and has subsequently appeared in several articles (see [12, 13, 14]).

Now, we introduce the difference envelope of currents.

DEFINITION 2.14. Let $T_1 = dd^c u_1 + \theta_1, T_2 = dd^c u_2 + \theta_2$ be closed positive $(1, 1)$ -currents, where $u_1 \in \text{PSH}(X, \theta_1), u_2 \in \text{PSH}(X, \theta_2)$. If there exists $u_3 \in \text{PSH}(X, \theta_1 - \theta_2)$ such that $u_3 + u_2 \preceq u_1$. Then, we define the *difference envelope* of T_1 and T_2 as

$$P(T_1 - T_2) := dd^c P^{\theta_1 - \theta_2}(u_1 - u_2) + (\theta_1 - \theta_2).$$

Here, we note that if $u_1(x) = u_2(x) = -\infty$ for $x \in X$, then we define $(u_1 - u_2)(x) = \infty$.

Similar to Remark 2.13, one can check that $P(T_1 - T_2)$ is independent of the choices of θ_j and u_j for $j = 1, 2$.

Here, we review some important results regarding the Monge-Ampère mass of envelopes of currents, where the notation $\langle \cdot \rangle$ denotes the non-pluripolar product, which will be introduced in Chapter 3.

PROPOSITION 2.15. ([11, Proposition 3.1]) *Let $\{\theta_1\}, \dots, \{\theta_n\}$ be pseudoeffective classes. Let $u_j \in \text{PSH}(X, \theta_j), 1 \leq j \leq n$. Then*

$$\int_X \langle (\theta_1)_{u_1} \wedge \dots \wedge (\theta_n)_{u_n} \rangle = \int_X \langle (\theta_1)_{P^{\theta_1}[u_1]} \wedge \dots \wedge (\theta_n)_{P^{\theta_n}[u_n]} \rangle.$$

In the case of self-intersection, the following result characterizes the full mass intersection problem.

THEOREM 2.16. ([11, Theorem 3.14]) *Let $\{\theta\}$ be a pseudoeffective class, and let $u \in \text{PSH}(X, \theta)$ such that $\int_X \langle \theta_u^n \rangle > 0$. Let $v \in \text{PSH}(X, \theta)$ such that $v \preceq u$. Then, the following is equivalent.*

- (1) $\int_X \langle \theta_v^n \rangle = \int_X \langle \theta_u^n \rangle$.
- (2) $P^\theta[u] = P^\theta[v]$.

2.3. Density currents

In this section, we recall some basic properties of density currents introduced by Dinh-Sibony in [21].

Let X be a complex Kähler manifold of dimension n , and V a smooth complex submanifold of X of dimension l . Let T be a closed positive (p, p) -current on X , where $0 \leq p \leq n$. Denote by $\pi : E \rightarrow V$ the normal bundle of V in X and $\bar{E} := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of E . We recall that $E = TX|_V / TV$, where TX and TV are the holomorphic tangent bundles of X and V respectively (this shows E is a holomorphic vector bundle). By abuse of notation, we also use π to denote the canonical projection from $\bar{E}$ to V .

Let U be an open subset of X with $U \cap V \neq \emptyset$. Let τ be a smooth diffeomorphism from U to an open neighborhood of $U \cap V$ in E which is the identity on $U \cap V$ such that the induced map of the differential $d\tau$ to $E|_{V \cap U}$ is the identity (because for every $x \in U \cap V$, $d\tau$ at x is the identity map on TV_x , it induces a linear map from $TX_x / TV_x = E_x$ to $TE_x / TV_x = E_x$). Such a map is called *an admissible map*. Note that in [21], to define an admissible map, it is required furthermore that $d\tau$ is $\mathbb{C}$ -linear at every point of V . This difference doesn't affect what follows. When U is a small enough tubular neighborhood of V , there always exists an admissible map τ by [21, Lemma 4.2]. In general, τ is not holomorphic. When U is a small enough local chart, we can choose a holomorphic admissible map by using suitable holomorphic coordinates on U . For $\lambda \in \mathbb{C}^*$, let $A_\lambda : E \rightarrow E$ be the multiplication by λ on fibers of E , which can be extended to $A_\lambda : \bar{E} \rightarrow \bar{E}$. A (p, p) -current on $\bar{E}$ is said to be V -conic if it is invariant under the action of $\{A_\lambda\}$. Here, we state an important result of density currents.

THEOREM 2.17. ([21, Theorem 4.6]) *Let X be a compact Kähler manifold, and $V \subseteq X$ be a smooth complex submanifold. Let T be a closed positive (p, p) -current on X , and let τ be an admissible map defined on a tubular neighborhood of V . Then, the family $(A_\lambda)_* \tau_* T$ is of mass uniformly bounded in λ on compact subsets in E , and if S is a limit current of the last family as $\lambda \rightarrow \infty$, then S is a closed positive current on E which can be extended trivially through $\bar{E} \setminus E$ to be a V -conic closed positive current on $\bar{E}$ such that the cohomology class $\{S\}$ of S in $\bar{E}$ is independent of the choice of S , and $\{S\}|_V = \{T\}|_V$, and $\|S\| \leq C\|T\|$ for some constant C independent of S and T , where $\{S\}|_V$ denotes the pull back of $\{S\}$ under the canonical inclusion map from V to $\bar{E}$.*

The current S in the above theorem is called a *tangent current to T along V* . Its cohomology class is called *the total tangent class of T along V* and is denoted by $\kappa^V(T)$. Tangent currents are not unique in general. By [21, Theorem 4.6] again, if

$$S = \lim_{k \rightarrow \infty} (A_{\lambda_k})_* \tau_* T$$

for some sequence $(\lambda_k)_k$ converging to ∞ , then for every open subset U' of X and every admissible map $\tau' : U' \rightarrow E$, we also have

$$S = \lim_{k \rightarrow \infty} (A_{\lambda_k})_* \tau'_* T.$$

This is equivalent to saying that tangent currents are independent of the choice of the admissible map τ .

DEFINITION 2.18. ([21, Definition 3.1]) Let F be a complex manifold and $\pi_F : F \rightarrow V$ a holomorphic submersion. Let S be a positive current of bi-degree (p, p) on F . The h -dimension of S with respect to π_F is the largest integer q such that $S \wedge \pi_F^* \theta^q \neq 0$ for some Hermitian metric θ on V .

By a bi-degree reason, the h -dimension of S is in $[\max\{l - p, 0\}, \min\{\dim F - p, l\}]$. We have the following description of currents with minimal h -dimension.

LEMMA 2.19. ([21, Lemma 3.4]) Let $\pi_F : F \rightarrow V$ be a holomorphic submersion. Let S be a closed positive current of bidegree (p, p) on F , with $p < l$, and of h -dimension $l - p$ with respect to π_F . Then there exists a closed positive current S' on V such that

$$S = \pi_F^* S'.$$

By [21, Lemma 3.8], the h -dimensions of tangent currents to T along V are the same and this number is called the *tangential h -dimension of T along V* .

Let $m \geq 2$ be an integer. Let T_j be a closed positive current of bi-degree (p_j, p_j) for $1 \leq j \leq m$ on X and let $T_1 \otimes \cdots \otimes T_m$ be the tensor current of $T_1, \dots, T_m$ which is a current on X^m . A *density current* associated to $T_1, \dots, T_m$ is a tangent current to $\otimes_{j=1}^m T_j$ along the diagonal Δ_m of X^m . Let $\pi_m : E_m \rightarrow \Delta_m$ be the normal bundle of Δ_m in X^m . Denote by $[V]$ the current of integration along V . When $m = 2$ and $T_2 = [V]$, the density currents of T_1 and T_2 are naturally identified with the tangent currents to T_1 along V (see [48, Lemma 2.3]).

The unique cohomology class of density currents associated to $T_1, \dots, T_m$ is called *the total density class of $T_1, \dots, T_m$* . We denote the last class by $\kappa(T_1, \dots, T_m)$.

The *tangential h -dimension* of $\otimes_{j=1}^m T_j$ along Δ_m is called the *density h -dimension* of $T_1, \dots, T_m$.

DEFINITION 2.20. ([21, Definition 5.9]) Let T_j be a closed positive current of bi-degree (p_j, p_j) on X , $1 \leq j \leq m$, with $\sum_{j=1}^m p_j \leq n$. Assume that *density h -dimension* of $T_1, \dots, T_m$ is equal to $n - \sum_{j=1}^m p_j$, which is minimal. We say that the *Dinh–Sibony product*

$$T_1 \wedge \cdots \wedge T_m$$

exists if there is a unique density current R associated to $T_1, \dots, T_m$ which is the pull back of a current S on the diagonal $\Delta_m \subset X^m$ by $\pi_m : X^m \rightarrow \Delta_m$. In this case, we define

$$T_1 \wedge \cdots \wedge T_m := S.$$

The following lemma describes the relationship between the total density class and the wedge products of the cohomology classes of the given currents.

LEMMA 2.21. ([21, Section 5]) Let T_j be a closed positive current of bi-degree (p_j, p_j) on X for $1 \leq j \leq m$ such that $\sum_{j=1}^m p_j \leq n$. Assume that the *density h -dimension* of $T_1, \dots, T_m$ is minimal, i.e, equals to $n - \sum_{j=1}^m p_j$. Then the total density class of $T_1, \dots, T_m$ is equal to $\pi_m^*(\wedge_{j=1}^m \{T_j\})$, where $\pi_m : \bar{E} \rightarrow \Delta_m$ is the normal bundle of Δ_m .

Let $h_{\bar{E}}$ be the Chern class of the dual of the tautological line bundle of $\bar{E}$. By [21, Page 535], we have

$$(2.3.1) \quad \kappa^V(T) = \sum_{j=\max\{0, l-p\}}^{\min\{l, n-p-1\}} \pi^*(\kappa_j^V(T)) \wedge h_{\bar{E}}^{p-(l-j)},$$

where $\pi : \bar{E} \rightarrow V$ is the canonical projection and $\kappa_j^V(T) \in H^{2l-2j}(V, \mathbb{R})$. The *tangential h -dimension* of T along V is exactly equal to the maximal j such that $\kappa_j^V(T) \neq 0$, and it was known that the class $\kappa_j^V(T)$ is pseudoeffective ([21, Lemma 3.15]).

THEOREM 2.22. ([21, Proposition 4.13]) Let V' be a submanifold of V and let T be a closed positive current on X . Let T_∞ be a tangent current to T along V . Let s be the *tangential h -dimension* of T_∞ along V' . Then, the *tangential h -dimension* of T along V' is at most s , and we have

$$\kappa_s^{V'}(T) \leq \kappa_s^{V'}(T_\infty).$$

As a consequence, we obtain the following result.

COROLLARY 2.23. ([43, Corollary 2.6]) *Let T_j be a closed positive current on X for $1 \leq j \leq m$. Then, for every $x \in X$ and for every density current S associated to $T_1, \dots, T_m$, we have*

$$(2.3.2) \quad \nu(S, x^m) \geq \nu(T_1, x) \cdots \nu(T_m, x),$$

$x^m = (x, \dots, x) \in \Delta_m \subset E$, where Δ_m is the diagonal of X^m and E is the normal bundle over Δ_m .

PROOF. Let $x \in X$. Let $\pi : E \rightarrow \Delta_m$ be the canonical projection from the normal bundle of the diagonal Δ_m of X^m in X^m . Put $T := \otimes_{j=1}^m T_j$ and $V' := \{x^m\}$. By [36, Lemma 2.4], we have $\nu(T, x^m) \geq \nu(T_1, x) \cdots \nu(T_m, x)$. By [21, Proposition 5.6], we have

$$\kappa_0^{V'}(S) = \nu(S, x^m) \delta_{x^m}, \quad \kappa_0^{V'}(T) = \nu(T, x^m) \delta_{x^m},$$

where δ_{x^m} is the Dirac measure on x^m (notice here $\dim V' = 0$). This combined with Theorem 2.22 applied to X^m , $T := \otimes_{j=1}^m T_j$, Δ_m the diagonal of X^m and $V' := \{x^m\}$ implies

$$\nu(S, x^m) \geq \nu(T, x^m) \geq \nu(T_1, x) \cdots \nu(T_m, x).$$

Hence, the desired inequality follows. The proof is finished. $\square$

The above corollary generalizes the well-known comparison of Lelong numbers of intersection of $(1, 1)$ -currents due to Demailly [16, Chapter III, Corollary 7.9] in the compact setting. It is probably the first result dealing with comparison of Lelong numbers for intersection of currents of arbitrary bi-degree.

CHAPTER 3

Relative non-pluripolar products on Hermitian manifold

3.1. Construction and some properties

The non-pluripolar product is a notion of wedge product for closed positive $(1, 1)$ -currents. It was introduced and studied in [3, 8, 24], and later generalized to the relative setting on Kähler manifolds in [51]. In [33], we consider a more general setting and show that, under certain assumptions on the Hermitian metric ω of a complex manifold X , the relative non-pluripolar product remains well-defined on X .

This section is based on the article [33]. We review the construction of the relative non-pluripolar product in the Hermitian setting and discuss some of its properties.

Let X be a compact complex manifold of dimension n . Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents and let T be a closed positive (p, p) -current such that $m + p \leq n$. Let Ω be a local coordinate chart of X , and for $1 \leq j \leq m$, we write $T_j = dd^c u_j$ for some psh function u_j on Ω . Let $k \in \mathbb{N}$, and set $u_{j,k} = \max\{u_j, -k\}$, which is a bounded psh function on Ω . Set

$$R_k := dd^c u_{1,k} \wedge \dots \wedge dd^c u_{m,k} \wedge T$$

to be the classical product of $dd^c u_1, \dots, dd^c u_m$, and T introduced in 2.1.3.

Note that by Theorem 2.6, we have the following.

$$\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k = \mathbb{1}_{\cap_{j=1}^m \{u_j > -k'\}} R_{k'}$$

for $k \leq k'$.

Under the assumption that the mass of $\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k$ is uniformly bounded, it follows that the limit

$$\lim_{k \rightarrow \infty} \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k$$

exists. This is established in the following lemma.

LEMMA 3.1. ([51, Lemma 3.1] or [33, Lemma 2.3]) If

$$(3.1.1) \quad \sup_{k \in \mathbb{N}} \left\{ \left\| \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \right\|_K \right\} < +\infty,$$

for every compact set $K \subseteq \Omega$, then the limit current

$$\lim_{k \rightarrow \infty} \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k =: R$$

exists. Moreover, for any Borel form Φ supported in an open subset $U \subseteq \Omega$ with bounded coefficients on U , we have

$$(3.1.2) \quad \lim_{k \rightarrow \infty} \left\langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \right\rangle = \langle R, \Phi \rangle.$$

PROOF. Let K be a compact set in X . For $\ell \in \mathbb{N}$, set

$$B_\ell = \cap_{j=1}^m \{u_j > -(\ell + 1)\} - \cap_{j=1}^m \{u_j > -\ell\}.$$

We have

$$\begin{aligned} \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k &= \mathbb{1}_{\cap_{j=1}^m \{u_j > 0\}} R_k + \sum_{\ell=0}^{k-1} \mathbb{1}_{B_\ell} R_k \\ &= \mathbb{1}_{\cap_{j=1}^m \{u_j > 0\}} R_0 + \sum_{\ell=0}^{k-1} \mathbb{1}_{B_\ell} R_{\ell+1}. \end{aligned}$$

By the assumption (3.1.1). One sees that

$$\sum_{\ell=\ell'}^{k-1} \mathbb{1}_{B_\ell} R_{\ell+1}$$

converges to 0 as $k - 1 > \ell'$ and $k, \ell' \rightarrow \infty$. This shows that the limit

$$\lim_{k \rightarrow \infty} \mathbb{1}_{\cap_{j=1}^m \{u_j > -\infty\}} R_k$$

exists.

We now prove (3.1.2). Let Φ be an $(n - p - m, n - p - m)$ -form supported in U , whose coefficients are bounded Borel functions. Let $\epsilon > 0$. Fix $k_0 \in \mathbb{N}$ such that

$$|\langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi' \rangle - \langle R, \Phi' \rangle| \leq \epsilon, \quad \text{for } k > k_0.$$

By Lusin's theorem, there exists a $(n - p - m, n - p - m)$ -form Φ' with continuous coefficients and compact support such that

$$\|R\|_{\{x \in \Omega | \Phi(x) \neq \Phi'(x)\}} \leq \epsilon.$$

Since $\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \leq R$ for $k \in \mathbb{N}$, we also get

$$\|\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k\|_{\{x \in \Omega \mid \Phi(x) \neq \Phi'(x)\}} \leq \epsilon, \quad \text{for } k \in \mathbb{N}.$$

For $k \geq k_0$, we have

$$\begin{aligned} & |\langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle - \langle R, \Phi \rangle| \\ & \leq |\langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle - \langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi' \rangle| + |\langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi' \rangle - \langle R, \Phi' \rangle| \\ & \quad + |\langle R, \Phi' \rangle - \langle R, \Phi \rangle| \\ & \leq 3\epsilon. \end{aligned}$$

This shows $\lim_{k \rightarrow \infty} \langle \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k, \Phi \rangle = \langle R, \Phi \rangle$, and the proof is completed. $\square$

By applying (3.1.2) from Lemma 3.1, we immediately obtain the following remark.

REMARK 3.2.

$$\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k = \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R, \quad \mathbb{1}_{\cup_{j=1}^m \{u_j = -\infty\}} R = 0.$$

Now, let us move on to the global setting. First, express each current T_j as

$$T_j = dd^c \tilde{u}_j + \theta_j,$$

where $\tilde{u}_j \in \text{PSH}(X, \theta_j)$ for $1 \leq j \leq m$. For each $k \in \mathbb{N}$, define

$$\tilde{u}_{j,k} := \max\{\tilde{u}_j, -k\},$$

and set

$$\tilde{R}_k := (dd^c \tilde{u}_{1,k} + \theta_1) \wedge \cdots \wedge (dd^c \tilde{u}_{m,k} + \theta_m) \wedge T.$$

By Theorem 2.6 and the fact that $\{\tilde{u}_j > -k\} = \{\tilde{u}_{j,k} > -k\}$, we have

$$(3.1.3) \quad \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k = \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_l$$

for every $l \geq k$.

Note that, in general, $\tilde{R}_k$ is not necessarily a positive current. However, the current

$$\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -\infty\}} \tilde{R}_k$$

is positive (see Lemma 3.3).

Analogously to Lemma 3.1, with the uniformly bounded mass condition on $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$, the sequence

$$\left\{ \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k \right\}$$

converges weakly to some closed positive current.

LEMMA 3.3. ([51, Lemma 3.2]) *The current $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ is positive on X . Furthermore, if we assume*

$$(3.1.4) \quad \sup_{k \in \mathbb{N}} \left\{ \left\| \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k \right\|_K \right\} < +\infty,$$

for every compact set $K \subseteq X$. Then $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ weakly converges to a positive current $\tilde{R}$ as $k \rightarrow \infty$.

PROOF. For every point x in X , let U be an open neighborhood of x such that $T_j = dd^c u_j$ for $j = 1, \dots, m$, where u_j is a psh function on U . By the local $\partial\bar{\partial}$ -lemma, we could shrink U and then find a smooth function τ_j on U such that $u_j = \tilde{u}_j + \tau_j$ on U with $dd^c \tau_j = \theta_j$. We could assume that τ_j is bounded on U by shrinking U again. Choose $c_0 \in \mathbb{N}$ with

$$c_0 > \sum_{j=1}^m \sup_U |\tau_j|.$$

On the subset $\{\tilde{u}_j > -k\} = \{u_j > -k + \tau_j\}$ of U , we have

$$\tilde{u}_{j,k+c_0} + \tau_j = \tilde{u}_j + \tau_j = u_j = u_{j,k+c_0}.$$

Combining this with (3.1.3), we have

$$\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k = \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_{k+c_0} = \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_{k+c_0}$$

on U . We deduce that $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ is positive on U , hence on X .

Now, we claim that with the assumption (3.1.4), the limit current $\tilde{R}$ exists. Since

$$\{u_j > -k\} \subseteq \{\tilde{u}_j > -(k+c_0)\} \subseteq \{u_j > -(k+2c_0)\}$$

on U for every k , we have

$$\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k = \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_{k+c_0} \leq \mathbb{1}_{\cap_{j=1}^m \{u_j > -(k+c_0)\}} R_{k+c_0}$$

and

$$\begin{aligned}\mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k &= \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_{k+2c_0} \\ &\leq \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -(k+c_0)\}} R_{k+2c_0} = \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -(k+c_0)\}} \tilde{R}_{k+c_0}\end{aligned}$$

on U for every k . In other words, we have

$$(3.1.5) \quad \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \leq \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -(k+c_0)\}} \tilde{R}_{k+c_0} \leq \mathbb{1}_{\cap_{j=1}^m \{u_j > -(k+2c_0)\}} R_{k+2c_0}$$

on U for every k . Combined with Lemma 3.1, we deduce that $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ weakly converges to the positive current R on U as $k \rightarrow \infty$. Also, by (3.1.5), we see that R is independent of the local potential u_j . Hence, $\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k$ weakly converges to a positive current $\tilde{R}$ on X as $k \rightarrow \infty$, and $\tilde{R} = R$ on U . $\square$

When the limit of $\{\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k\}$ exists, we denote it by

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle := \lim_{k \rightarrow \infty} \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k,$$

which is called the *non-pluripolar product* relative to T of $T_1, \dots, T_m$. One can see from Lemma 3.1 and Lemma 3.3, that the limit always exists if condition (3.1.4) holds, or condition (3.1.1) holds for any local coordinate chart Ω .

When T is the current of integration along X , we observe that the relative non-pluripolar product $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$ coincides with the non-pluripolar product, which is denoted by $\langle T_1 \wedge \cdots \wedge T_m \rangle$.

When X is a compact Kähler manifold, condition (3.1.4) always holds (see [51, Lemma 3.4] or [8, Proposition 1.4]), which implies the well-definedness of the relative non-pluripolar product. The following theorem considers a more general setting: a compact complex manifold X equipped with a Hermitian form ω satisfying

$$(3.1.6) \quad dd^c \omega = dd^c(\omega^2) = 0.$$

One can see that the above condition is equivalence to

$$(3.1.7) \quad dd^c(\omega)^k = 0, \quad 1 \leq k \leq n.$$

THEOREM 3.4. *Let X be a compact complex manifold of dimension n endowed with a Hermitian form ω satisfying (3.1.6). Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents and let T be a closed positive (p, p) -current on X such that $p + m \leq n$. Then the*

relative non-pluripolar product

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$$

is well-defined.

PROOF. First, recall that we write $T_j = dd^c \tilde{u}_j + \theta_j$, $\tilde{u}_j \in \text{PSH}(X, \theta_j)$, $1 \leq j \leq m$. For $k \in \mathbb{N}$, we set

$$\tilde{u}_{j,k} := \max\{\tilde{u}_j, -k\}, \quad \tilde{R}_k := (dd^c \tilde{u}_{1,k} + \theta_1) \wedge \cdots \wedge (dd^c \tilde{u}_{m,k} + \theta_m) \wedge T.$$

Let $C > 0$ be a constant such that $C\omega + \theta_j > 0$ for $1 \leq j \leq m$. Note that we have

$$\begin{aligned} & \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k \\ & \leq \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} (dd^c \tilde{u}_{1,k} + \theta_1 + C\omega) \wedge \cdots \wedge (dd^c \tilde{u}_{m,k} + \theta_m + C\omega) \wedge T. \end{aligned}$$

By the assumption on ω (3.1.7) and integrating by part, we obtain

$$\begin{aligned} & \|\mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} \tilde{R}_k\| \\ & \leq \|(dd^c \tilde{u}_{1,k} + \theta_1 + C\omega) \wedge \cdots \wedge (dd^c \tilde{u}_{m,k} + \theta_m + C\omega) \wedge T\| \\ & = \int_X (dd^c \tilde{u}_{1,k} + \theta_1 + C\omega) \wedge \cdots \wedge (dd^c \tilde{u}_{m,k} + \theta_m + C\omega) \wedge T \wedge \omega^{n-m-p} \\ & = \int_X (\theta_1 + C\omega) \wedge \cdots \wedge (\theta_m + C\omega) \wedge T \wedge \omega^{n-m-p}, \end{aligned}$$

which is independent of k . Hence, condition (3.1.4) holds, and the relative non-pluripolar product

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$$

is well-defined. □

The following example, inspired by [45], shows that there exist many n -dimensional non-Kähler compact complex manifolds which can be endowed with a Hermitian form ω satisfying condition (3.1.6).

EXAMPLE 3.5. A Gauduchon form ω on a n -dimensional complex manifold, is a Hermitian form satisfying $\partial\bar{\partial}\omega^{n-1} = 0$. Every compact complex manifold has a Gauduchon form. Let Y be a Hopf surface, which is a 2-dimensional non-Kähler compact complex manifold, there is a Gauduchon form ω_Y on Y satisfied $\partial\bar{\partial}\omega_Y = 0$.

Let Z be an n -dimensional compact Kähler manifold endowed with a Kähler form ω_Z . Let $X := Y \times Z$. Let $p_Y : X \rightarrow Y$ and $p_Z : X \rightarrow Z$ be the natural projections. Then X is an $(n+2)$ -dimensional non-Kähler compact complex manifold, with a Hermitian form $\omega_X := p_Y^* \omega_Y + p_Z^* \omega_Z$ satisfied $\partial\bar{\partial}\omega_X = \partial\omega_X \wedge \bar{\partial}\omega_X = 0$.

A set $A \subseteq X$ is a complete pluripolar set if $A = \{x | u(x) = -\infty\}$, for some quasi-psh function u on X . Here, we state some important properties of relative non-pluripolar product.

PROPOSITION 3.6. ([51, Proposition 3.5]) *Let X be a complex manifold of dimension n . Let $T_1, \dots, T_m$ be closed positive $(1,1)$ -current, and let T be a closed positive (p,p) -current on X such that $m+p \leq n$. Assume the non-pluripolar product*

$$\langle T_1 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$$

is well-defined on X . Then, the following hold.

- (1) $\langle T_1 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$ is symmetric with respect to $T_1, \dots, T_m$.
- (2) Let $\lambda > 0$, then $\langle (\lambda T_1) \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle = \lambda \langle T_1 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$.
- (3) Let A be a complete pluripolar set such that T has no mass on A , then

$$\langle T_1 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$$

also has no mass on A .

- (4) Let T'_1 be a closed positive $(1,1)$ -currents on X . Assume that $\langle T'_1 \wedge T_2 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$ is well-defined. Then $\langle (T_1 + T'_1) \wedge T_2 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle$ is well-defined and satisfies

$$\begin{aligned} & \langle (T_1 + T'_1) \wedge T_2 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle \\ & \leq \langle T_1 \wedge T_2 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle + \langle T'_1 \wedge T_2 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle. \end{aligned}$$

The equality occurs if T has non mass on $I_{T_1} \cup I_{T'_1}$.

- (5) Let $1 \leq l \leq m$ be an integer. Let T''_j be a closed positive $(1,1)$ -current on X for $j = 1, \dots, l$. Assume $T_j \leq T''_j$ for every j and T has no mass on $\cup_{j=1}^l I_{T''_j - T_j}$. Then we have

$$\langle T_1 \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle \leq \langle T''_1 \wedge \dots \wedge T''_l \wedge T_{l+1} \wedge \dots \wedge T_m \mathbin{\dot{\wedge}} T \rangle.$$

- (6) Let $1 \leq l \leq m - 1$ be an integer. Assume $R := \langle T_{l+1} \wedge \cdots \wedge T_m \wedge T \rangle$ and $\langle T_1 \wedge \cdots \wedge T_l \wedge T \rangle$ are well-defined. Then we have

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge \cdots \wedge T_l \wedge R \rangle.$$

- (7) Let A be a complete pluripolar set. Then we have

$$\mathbb{1}_{X \setminus A} \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge \cdots \wedge T_m \wedge (\mathbb{1}_{X \setminus A} T) \rangle.$$

In particular, the equality

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \langle T_1 \wedge \cdots \wedge T_m \wedge T' \rangle$$

holds, where $T' := \mathbb{1}_{X \setminus \cup_{j=1}^m I_{T_j}} T$.

REMARK 3.7. In the case of the non-pluripolar product ($T = [X]$), the no-mass condition in Proposition 3.6 (4) and (5) is automatically satisfied. Consequently, the inequality in (5) generalized the case of the non-pluripolar product, while the inequality in (4) becomes an equality, which reflects the multi-linearity of the non-pluripolar product.

The following proposition describes the relationship between relative non-pluripolar products and classical products introduced in Section 2.1.3.

PROPOSITION 3.8. ([51, Proposition 3.6]) Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -current, and let T be a closed positive (p, p) current on X such that $m + p \leq n$.

- (1) If T is of bi-degree $(1, 1)$, then

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \mathbb{1}_{X \setminus I_T} \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle.$$

- (2) If the product $T_1 \wedge \cdots \wedge T_m \wedge T$ is well-defined in the sense in section 2.1.3. Then,

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = \mathbb{1}_{X \setminus \cup_{j=1}^m I_{T_j}} T_1 \wedge \cdots \wedge T_m \wedge T.$$

When the current $T = [V]$, the current of integration along some analytic subset V , we have the following.

LEMMA 3.9. ([52, Lemma 2.3]) Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents on X . Then the following properties hold:

- (1) If V is contained in $\cup_{j=1}^m I_{T_j}$, then $\langle T_1 \wedge \cdots \wedge T_m \wedge [V] \rangle = 0$ and there is $1 \leq j_0 \leq m$ so that $V \subseteq I_{T_{j_0}}$.

(2) If V is not contained in $\cup_{j=1}^m I_{T_j}$, then

$$\langle \wedge_{j=1}^m T_j \wedge [V] \rangle = i_* \langle T_{1,V} \wedge \cdots \wedge T_{m,V} \rangle,$$

where $i: \text{Reg}(V) \rightarrow X$ is the natural inclusion, and $T_{j,V} := dd^c(u_j|_{\text{Reg}(V)})$ if $dd^c u_j = T_j$ locally.

THEOREM 3.10. ([51, Theorem 3.7]) Let T_j be closed positive $(1,1)$ -currents, $1 \leq j \leq m$. Let T be a closed positive (p,p) -current such that $m + p \leq n$. If

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle$$

is well-defined, then it is closed.

PROOF. For convenience, we set $\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle = R$. Let U be a local chart, and write $T_j = dd^c u_j$, where $u_j \leq 0$ is a plurisubharmonic function on U , for $1 \leq j \leq m$. Let ρ be a positive $(n - m - p - 1, n - m - p - 1)$ -form with constant coefficients on U . To prove that R is closed, it suffices to show that

$$d(\rho \wedge R) = 0 \quad \text{on } U.$$

For each $k \in \mathbb{N}$, set

$$\psi_k := \frac{1}{k} \max \left\{ \sum_{j=1}^m u_j, -k \right\} + 1,$$

which defines a plurisubharmonic function on U satisfying $0 \leq \psi_k \leq 1$. Moreover, ψ_k increases pointwise to $\mathbb{1}_{\cap_{j=1}^m \{u_j > -\infty\}}$.

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $g(0) = 0$, $g(1) = 1$, and $g'(0) = g'(1) = 0$.

Since R has no mass on the set $\cup_{j=1}^m \{u_j = -\infty\}$, it follows that $g(\psi_k)R \rightarrow R$ as $k \rightarrow \infty$. Thus, to prove the theorem, it suffices to show that

$$d(\rho \wedge R) = \lim_{k \rightarrow \infty} d(g(\psi_k) \rho \wedge R) = 0.$$

Since $g(\psi_k) = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$, we have

$$d(g(\psi_k) \rho \wedge R) = d(g(\psi_k) \rho \wedge R_k) = -g'(\psi_k) \rho \wedge d\psi_k \wedge R_k$$

for each $k \in \mathbb{N}$.

Let η be a smooth 1-form with compact support in U , and let $U_1 \Subset U_2 \Subset U$ be relatively compact open subsets. By the Cauchy–Schwarz inequality, we have

$$(3.1.8) \quad \left| \int_{U_1} g'(\psi_k) d\psi_k \wedge \eta \wedge \rho \wedge R_k \right|^2 \leq \left(\int_{U_1} \tau d\psi_k \wedge d^c \psi_k \wedge \rho \wedge R_k \right) \left(\int_{U_1} (g'(\psi_k))^2 \eta \wedge \bar{\eta} \wedge \rho \wedge R_k \right)$$

where τ is a smooth cutoff function with $\tau \equiv 1$ on the support of η .

Now, by the Chern–Levine–Nirenberg inequality and the identity

$$d\psi_k \wedge d^c \psi_k = dd^c(\psi_k^2) - \psi_k dd^c \psi_k,$$

we obtain

$$(3.1.9) \quad \|\tau d\psi_k \wedge d^c \psi_k \wedge \rho \wedge R_k\|_{U_1} \lesssim \|\psi_k \rho \wedge R_k\|_{U_2},$$

where the norm on the right-hand side is uniformly bounded in k .

On the other hand, since $g'(\psi_k) = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$, we have

$$(g'(\psi_k))^2 R_k = (g'(\psi_k))^2 R.$$

Using the fact that $g'(\psi_k) \rightarrow 0$ pointwise on $\cap_{j=1}^m \{u_j > -\infty\}$, and that R has no mass on $\cup_{j=1}^m \{u_j = -\infty\}$, it follows that

$$(g'(\psi_k))^2 R_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence,

$$\int_{U_1} (g'(\psi_k))^2 \eta \wedge \bar{\eta} \wedge \rho \wedge R_k \rightarrow 0.$$

This, combined with (3.1.8) and (3.1.9), completes the proof. $\square$

The following lemmas will be used in the Chapter 5. We assume X to be a compact Kähler manifold in the rest of this section.

LEMMA 3.11. ([42, Lemma 2.7]) *Let $\{\theta_1\}, \dots, \{\theta_m\}$ be pseudoeffective classes, $1 \leq m \leq n$. For $1 \leq j \leq m$, let $T'_j, T_j \in \{\theta_j\}$ be closed positive $(1,1)$ -currents such that $\text{supp}(T'_j) \subseteq W_j$ for some complete pluripolar set W_j . Then, the following equality holds.*

$$\langle \wedge_{j=1}^m (T_j + T'_j) \rangle = \langle \wedge_{j=1}^m T_j \rangle.$$

PROOF. By the multi-linearity of non-pluripolar product (Proposition 3.6 (4)), we have

$$\langle \wedge_{j=1}^m (T_j + T'_j) \rangle = \langle \wedge_{j=1}^m T_j \rangle + \sum_{s=1}^m R_s,$$

where $R_s = c_j \langle \wedge_{j=1}^s T'_j \wedge \wedge_{\ell=s+1}^m T_\ell \rangle$, $c_j \in \mathbb{N}^+$.

By the construction of the non-pluripolar product, we see that R_s is supported in $W = \cup_{j=1}^n W_j$, where W is a complete pluripolar set. On the other hand, the non-pluripolar product of currents does not charge mass on complete pluripolar set. Therefore, we get $R_s = 0$ for $1 \leq s \leq m$, which completes the proof. $\square$

LEMMA 3.12. ([42, Lemma 2.8]) *Let $\{\theta_1\}, \dots, \{\theta_m\}$ be pseudoeffective $(1,1)$ -classes, $1 \leq m \leq n$. Let $T'_j, T_j \in \{\theta_j\}$ be closed positive $(1,1)$ -currents such that T_j is less singular than T'_j , $1 \leq j \leq m$. Set $T_j^p := \mathbb{1}_{I_{T_j}} T_j$. If there exists a Kähler current $P_j \in \{\theta_j\}$ with analytic singularities that is more singular than T_j . Then, $\mathbb{1}_{I_{T'_j}} T'_j \geq T_j^p$ for $1 \leq j \leq m$, and the following holds.*

$$(3.1.10) \quad \langle \wedge_{j=1}^m T'_j \rangle = \langle \wedge_{j=1}^m (T'_j - T_j^p) \rangle.$$

PROOF. The proof strategy follows [53]. First, by applying Lemma 3.11, we immediately obtain (3.1.10). We now show that $\mathbb{1}_{I_{T'_j}} T'_j \geq T_j^p$ holds for $1 \leq j \leq m$. Since T_j is less singular than P_j , we get $\text{supp}(T_j^p) \subseteq I_{P_j} = V_j$, where V_j is an analytic subset. By the first support theorem ([16, Chapter III, Theorem 2.10]), the analytic subset V_j is of dimension $n - 1$. Furthermore, the second support theorem ([16, Chapter III, Theorem 2.13]) implies that

$$T_j^p = \sum_{V_{j,k}} \lambda_{j,k} [V_{j,k}],$$

where $\{V_{j,k}\}$ are the irreducible components of V_j . Since T_j is less singular than T'_j , we get

$$v(T'_j, V_{j,k}) \geq v(T_j, V_{j,k}) \geq v(T_j^p, V_{j,k}) = \lambda_{j,k}$$

for each irreducible component $V_{j,k}$. This shows that $T'_j \geq T_j^p$, and we get $\mathbb{1}_{I_{T'_j}} T'_j \geq T_j^p$. This completes the proof. $\square$

3.2. Monotonicity property

Let X be a compact Hermitian manifold of dimension n , and let θ be a closed $(1,1)$ -form on X . Recall that in Section (2.1.2), we denote by $\{\theta\}_{\partial\bar{\partial}}$ the Bott-Chern cohomology class of θ . We say $\{\theta\}_{\partial\bar{\partial}}$ is pseudoeffective if there exists a closed positive $(1,1)$ -current $T \in \{\theta\}_{\partial\bar{\partial}}$. Let $T_1, T_2 \in \{\theta\}_{\partial\bar{\partial}}$ be closed positive $(1,1)$ -currents. We write $T_1 = dd^c u_1 + \theta$ and $T_2 = dd^c u_2 + \theta$ respectively, where $u_1, u_2 \in \text{PSH}(X, \theta)$. Recall that T_1 is said to be less singular (more singular) than T_2 if $u_1 \geq u_2 + \mathcal{O}(1)$ ($u_1 \leq u_2 + \mathcal{O}(1)$).

We now aim to prove the monotonicity property of relative non-pluripolar products on certain compact Hermitian manifolds, stated below.

THEOREM 3.13. ([33, Theorem 3.4]) *Let X be a compact Hermitian manifold of dimension n , endowed with a Hermitian form ω satisfying condition (3.1.6). Let $\{\theta_1\}_{\partial\bar{\partial}}, \dots, \{\theta_m\}_{\partial\bar{\partial}}$ be pseudoeffective classes on X , where $m \leq n$. For $1 \leq j \leq m$, let $T_j, T'_j \in \{\theta_j\}_{\partial\bar{\partial}}$ be closed positive $(1,1)$ -currents, and let T be a closed positive (p,p) -current such that $m + p \leq n$. Assume that T'_j is less singular than T_j , for $1 \leq j \leq m$. Then, we have*

$$\|\langle T_1 \wedge \dots \wedge T_m \wedge T \rangle\| \leq \|\langle T'_1 \wedge \dots \wedge T'_m \wedge T \rangle\|.$$

Before we go into the proof of Theorem 3.13, we first introduce the concept of capacity. Let $U \subseteq \mathbb{C}^n$ be an open set, and let $K \subseteq U$ be a Borel set. Let T be a closed positive (p,p) -current on U . We define

$$\text{cap}_T(K, U) = \sup \left\{ \int_K (dd^c u)^{n-p} \wedge T \mid -1 \leq u \leq 0 \text{ and } u \text{ is psh on } U \right\}$$

Now, we review some important results in [51]. For the reader's convenience, we include the proofs.

LEMMA 3.14. ([51, Lemma 4.1]) *Let $U \subseteq \mathbb{C}^n$ be an open set. For $1 \leq j \leq m$, let u_j be a psh function, and let u_j^ℓ be a sequence of psh functions on U such that $u_j^\ell \geq u_j$ and $u_j^\ell \rightarrow u_j$ in L^1_{loc} , as $\ell \rightarrow \infty$. Let T be a closed positive (p,p) -current on U such that $m + p \leq n$. Assume that the non-pluripolar products $\langle \wedge_{j=1}^m dd^c u_j \wedge T \rangle$, $\langle \wedge_{j=1}^m dd^c u_j^\ell \wedge T \rangle$ are well-defined. Then,*

$$\liminf_{\ell \rightarrow \infty} \int_U \langle \wedge_{j=1}^m dd^c u_j^\ell \wedge T \rangle \wedge \Phi \geq \int_U \langle \wedge_{j=1}^m dd^c u_j \wedge T \rangle \wedge \Phi,$$

for every positive $(n - m - p, n - m - p)$ -form Φ with compact support in U .

PROOF. Let $k \in \mathbb{N}$, and let $u_{j,k} = \max\{u_j, -k\}$, $u_{j,k}^\ell = \max\{u_j^\ell, -k\}$. Set

$$\begin{aligned} R &= \langle \wedge_{j=1}^m dd^c u_j \wedge T \rangle, & R^\ell &= \langle \wedge_{j=1}^m dd^c u_j^\ell \wedge T \rangle, \\ R_k &= \wedge_{j=1}^m dd^c u_{j,k} \wedge T, & R_k^\ell &= \wedge_{j=1}^m dd^c u_{j,k}^\ell \wedge T. \end{aligned}$$

Let Φ be a positive $(n - m - p, n - m - p)$ -form with compact support in U . By Lemma 3.1 and the fact that $u_j^\ell > u_j$, we get

$$\begin{aligned} (3.2.1) \quad \int_U R^\ell \wedge \Phi &\geq \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j^\ell > -k\}} R^\ell \wedge \Phi \\ &= \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j^\ell > -k\}} R_k^\ell \wedge \Phi \\ &\geq \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k^\ell \wedge \Phi. \end{aligned}$$

Let $\epsilon > 0$, by [51, Theorem 2.4], there exists an open set $U' \subseteq U$ such that the restriction of u_j on $U \setminus U'$ is continuous and $\text{cap}_T(U, U') \leq \epsilon$. By [40, Theorem 20.4], the $u_j|_{U \setminus U'}$ can be continuously extended to $\tilde{u}_j$ on U .

Since $u_j^\ell \rightarrow u_j$ in L_{loc}^1 , we have $u_{j,k}^\ell \rightarrow u_{j,k}$ in L_{loc}^1 , as $\ell \rightarrow \infty$, for each $k \in \mathbb{N}$. Hence, by [51, Theorem 2.6], $R_k^\ell \rightarrow R_k$ weakly, as $\ell \rightarrow \infty$ for each $k \in \mathbb{N}$. Therefore, we have

$$(3.2.2) \quad \liminf_{\ell \rightarrow \infty} \int_U \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_k^\ell \wedge \Phi \geq \int_U \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_k \wedge \Phi.$$

By the facts that $u_{j,k}^\ell$ is uniformly bounded in ℓ and $\text{cap}_T(U, U') < \epsilon$, we get

$$\begin{aligned} (3.2.3) \quad &\left| \int_U \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_k^\ell \wedge \Phi - \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k^\ell \wedge \Phi \right| < c\epsilon, \\ &\left| \int_U \mathbb{1}_{\cap_{j=1}^m \{\tilde{u}_j > -k\}} R_k \wedge \Phi - \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \wedge \Phi \right| < c\epsilon \end{aligned}$$

for some $c > 0$ only depends on k . By (3.2.1), (3.2.2) and (3.2.3), we have

$$\liminf_{\ell \rightarrow \infty} \int_U R^\ell \wedge \Phi \geq \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R_k \wedge \Phi = \int_U \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} R \wedge \Phi.$$

Finally, letting $k \rightarrow \infty$, we get the desired inequality. $\square$

When T_j and T_j' have the same type of singularity, we have the following proposition.

PROPOSITION 3.15. ([51, Proposition 4.2]) Let X be a compact complex manifold. For $1 \leq j \leq m$, let T_j, T'_j be two closed positive $(1, 1)$ -currents in the same cohomology class and the same singularity type. Let T be a close positive (p, p) -current such that $m + p \leq n$. Assume that for $J, J' \subseteq \{1, \dots, m\}$ such that $J \cap J' = \emptyset$, the non-pluripolar product

$$\langle \wedge_{j \in J} T_j \wedge \wedge_{j' \in J'} T'_{j'} \wedge T \rangle$$

is well-defined. Then, for every dd^c -closed $(n - m - p, n - m - p)$ -form Φ on X , we have

$$\int_X \langle \wedge_{j=1}^m T_j \wedge T \rangle \wedge \Phi = \int_X \langle \wedge_{j=1}^m T'_j \wedge T \rangle \wedge \Phi.$$

PROOF. First, we write $T_j = dd^c u_j + \theta_j$ and $T'_j = dd^c u'_j + \theta_j$, where $u_j, u'_j \in \text{PSH}(X, \theta_j)$. Set $u_{j,k} = \max\{u_j, -k\}$ and $u'_{j,k} = \max\{u'_j, -k\}$, and let $T_{j,k} = dd^c u_{j,k} + \theta_j$, $T'_{j,k} = dd^c u'_{j,k} + \theta_j$. Set $w_{j,k} = u_{j,k} - u'_{j,k}$. Since T_j, T'_j are of the same singularity type. We may further assume that $|w_{j,k}| \leq 1$ for $1 \leq j \leq m$, and $k \in \mathbb{N}$. Set

$$\psi_k = k^{-1} \max\{(u_j + u'_j), -k\} + 1.$$

Note that ψ_k is a quasi-psh function which increases to $\mathbb{1}_{X \setminus \cup_{j=1}^m \{u_j = -\infty\}}$ as $k \rightarrow \infty$, and $\psi_k = 0$ on $\{u_j \leq -k\} \cup \{u'_j \leq -k\}$. Now, set $S = \langle \wedge_{j=1}^m T_j \wedge T \rangle - \langle \wedge_{j=1}^m T'_j \wedge T \rangle$. Since $\psi_k S \rightarrow S$ weakly as $k \rightarrow \infty$, to prove the desired equality, we need to claim the following.

$$(3.2.4) \quad \int_X \psi_k S \wedge \Phi \rightarrow 0, \quad k \rightarrow \infty,$$

for any dd^c -closed $(n - m - p, n - m - p)$ -form on X . By the construction of ψ_k , one sees that

$$\begin{aligned} \psi_k S &= \psi_k \wedge_{j=1}^m T_{j,k} \wedge T - \psi_k \wedge_{j=1}^m T'_{j,k} \wedge T \\ &= \sum_{s=1}^m \psi_k \wedge_{j'=1}^{s-1} T'_{j',k} \wedge dd^c w_{s,k} \wedge \wedge_{j=s+1}^m T_{j,k} \wedge T. \end{aligned}$$

Let $J, J' \subseteq \{1, \dots, m\}$ be such that $J \cap J' = \emptyset$. We set

$$R_{JJ'} = \langle \wedge_{j \in J} T_j \wedge \wedge_{j' \in J'} T'_{j'} \wedge T \rangle, \quad R_{JJ',k} = \wedge_{j \in J} T_{j,k} \wedge \wedge_{j' \in J'} T'_{j',k} \wedge T.$$

To prove (3.2.4), we show

$$\int_X \psi_k dd^c w_{s,k} \wedge R_{JJ',k} \wedge \Phi \rightarrow 0, \quad k \rightarrow \infty,$$

for $s \notin J \cup J'$. Note that the current $R_{JJ',k}$ is not necessarily positive in general. Hence, we could not estimate its mass directly, and we need to modify it slightly. Set $B_k = \cap_{j \in J} \{u_j > -k\} \cap \cap_{j' \in J'} \{u'_{j'} > -k\}$ and let $A = \cup_{j=1}^m \{u_j = -\infty\}$. Set $\tilde{R}_{JJ'} = \mathbb{1}_{X \setminus A} R_{JJ'}$. It is not difficult to see that $\tilde{R}_{JJ'}$ is closed and positive. Furthermore, $\tilde{R}_{JJ'}$ and $R_{JJ',k}$ satisfy the following relation:

$$\psi_k \tilde{R}_{JJ'} = \psi_k R_{JJ'} = \psi_k R_{JJ',k}.$$

By integration by parts, we have

$$\begin{aligned} \int_X \psi_k dd^c w_{s,k} \wedge R_{JJ',k} \wedge \Phi &= \int_X w_{s,k} dd^c (\psi_k R_{JJ',k} \wedge \Phi) \\ &= \int_X w_{s,k} dd^c (\psi_k \tilde{R}_{JJ'} \wedge \Phi). \end{aligned}$$

Since $dd^c(\psi_k \tilde{R}_{JJ'} \wedge \Phi)$ is a top-degree current and $\tilde{R}_{JJ'}$ is closed,

$$dd^c(\psi_k \tilde{R}_{JJ'} \wedge \Phi) = dd^c \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi + 2d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'}.$$

We can write $d^c \Phi$ as a linear combination of a $(1,0)$ -form wedged with a $(n-m-p, n-m-p)$ -form and a $(0,1)$ -form wedged with a $(n-m-p, n-m-p)$ -form. For convenience, we express $d^c \Phi = \tau_1 \wedge \Phi_1 + \tau_2 \wedge \Phi_2$, where τ_1 is a $(1,0)$ -form and τ_2 is a $(0,1)$ -form. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\int_X w_{s,k} d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'} \\ &\leq \left(\int_X |w_{s,k}|^2 \partial \psi_k \wedge \bar{\partial} \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi_2 \right)^{\frac{1}{2}} \left(\int_X \tau_2 \wedge \bar{\tau}_2 \wedge \tilde{R}_{JJ'} \wedge \Phi_2 \right)^{\frac{1}{2}} \\ &\quad + \left(\int_X |w_{s,k}|^2 \bar{\partial} \psi_k \wedge \partial \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi_1 \right)^{\frac{1}{2}} \left(\int_X \tau_1 \wedge \bar{\tau}_1 \wedge \tilde{R}_{JJ'} \wedge \Phi_1 \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_X d\psi_k \wedge d^c \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi_0 \right)^{\frac{1}{2}} \|R_{JJ'}\|^{\frac{1}{2}}, \end{aligned}$$

where $\Phi_0 = c\omega^{n-|J|-|J'|-1}$ is a positive form, and c is a constant such that $\Phi_1, \Phi_2 \leq \Phi_0$. Now, note that $\{\lim_{k \rightarrow \infty} \psi_k < 1\}$ equals the complete pluripolar set A . By [51, Remark 2.7] and the fact that $\tilde{R}_{JJ'}$ has no mass on A , we get

$$\lim_{k \rightarrow \infty} d\psi_k \wedge d^c \psi_k \wedge \tilde{R}_{JJ'} = \lim_{k \rightarrow \infty} (dd^c \psi_k^2 - \psi_k dd^c \psi_k) \wedge \tilde{R}_{JJ'} = 0.$$

Therefore, we have

$$\lim_{k \rightarrow \infty} \int_X w_{s,k} d\psi_k \wedge d^c \Phi \wedge \tilde{R}_{JJ'} = 0.$$

Using a similar argument, one also obtains

$$\lim_{k \rightarrow \infty} \int_X w_{s,k} dd^c \psi_k \wedge \tilde{R}_{JJ'} \wedge \Phi = 0.$$

This completes the proof. $\square$

We now begin the proof of Theorem 3.13, following the strategy introduced in [51].

PROOF OF THEOREM 3.13. First, we write $T_j = dd^c u_j + \theta_j$ and $T'_j = dd^c u'_j + \theta_j$, where $u_j, u'_j \in \text{PSH}(X, \theta_j)$. Set $u'_{j,\ell} = \max\{u'_j - \ell, u_j\} \in \text{PSH}(X, \theta_j)$, which decreases to u_j as $\ell \rightarrow \infty$. Note that $u'_{j,\ell}$ and u'_j are of the same singularity type. Set $T'_{j,\ell} = dd^c u'_{j,\ell} + \theta_j$. By Proposition 3.15, we get

$$(3.2.5) \quad \|\langle T'_{1,\ell} \wedge \cdots \wedge T'_{m,\ell} \wedge T \rangle\| = \|\langle T'_1 \wedge \cdots \wedge T'_m \wedge T \rangle\|,$$

for any $\ell \in \mathbb{N}$. Let S be a limit current of $\langle T'_{1,\ell} \wedge \cdots \wedge T'_{m,\ell} \wedge T \rangle$. By Lemma 3.14, we have

$$S \geq \langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle.$$

Combining this with (3.2.5), we obtain

$$\|\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\| \leq \|\langle T'_1 \wedge \cdots \wedge T'_m \wedge T \rangle\|.$$

$\square$

REMARK 3.16. In Theorem 3.13, when T is also a closed positive $(1, 1)$ -current, and $T' \in \{T\}$ is another closed positive $(1, 1)$ -current that is less singular than T . Then, we further obtain the following.

$$\|\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\| \leq \|\langle T'_1 \wedge \cdots \wedge T'_m \wedge T' \rangle\|.$$

REMARK 3.17. We note that in [51, Theorem 4.4], the monotonicity holds in the sense of cohomology classes. More precisely, if X is a compact Kähler manifold, then

$$\{\langle T_1 \wedge \cdots \wedge T_m \wedge T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \wedge T' \rangle\},$$

where T_j, T'_j are closed positive $(1, 1)$ -currents in the same pseudoeffective Dolbeault cohomology class $\{\theta_j\}$. This inequality holds because the Poincaré duality applies to Dolbeault cohomology groups on Kähler manifolds. However, the duality does not generally hold for the Bott–Chern cohomology on Hermitian manifolds in general. Therefore, the statement of monotonicity in Theorem 3.13 is formulated only in terms of masses.

3.3. Demailly's analytic approximation

In this section, we recall Demailly's analytic approximation theorem, introduced in [18, 19]. We also discuss the convergence of the Monge–Ampère masses associated with the currents in Demailly's approximation sequence.

Let θ be a closed real $(1, 1)$ -form, and let $u \in \text{PSH}(X, \theta)$. We say u is of *analytic singularities* if it can be expressed locally as

$$c \log \sum_{j=1}^N |f_j|^2 + g,$$

where $c > 0$, $f_1, \dots, f_N$ are some holomorphic functions, and g is a bounded function. Let T be a closed positive $(1, 1)$ -current. We say T is of *analytic singularities* if its global potential function is of analytic singularities.

THEOREM 3.18. ([19, Corollary 14.13]) *Let $\{\theta\}$ be a pseudoeffective class, and $u \in \text{PSH}(X, \theta)$. Then, there exists a sequence $u_k^D \in \text{PSH}(X, \theta + \epsilon_k \omega)$, where ϵ_k decreases to 0, such that*

- (1) *The sequence (u_k^D) satisfies $u_k^D \geq u$ and $u_k^D \rightarrow u$ in L^1 .*
- (2) *u_k^D has analytic singularities.*
- (3) *$\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly on X , where*

$$T := dd^c u + \theta, \quad T_k := dd^c u_k^D + (\theta + \epsilon_k \omega).$$

The analytic approximation theorem preserves the ordering of the singularity types, as stated below.

LEMMA 3.19. ([42, Lemma 2.11]) *Let $\{\theta\}$ be a pseudoeffective class, and let $u, v \in \text{PSH}(X, \theta)$ be such that $v \preceq u$. Let $u_k^D, v_k^D \in \text{PSH}(X, \theta + \epsilon_k \omega)$ be the approximation sequences of u and v given by Theorem 3.18. Then, we have $v_k^D \preceq u_k^D$ for each k .*

The Monge-Ampère masses of the sequence $\{u_k^D\}$ have the following convergence property.

PROPOSITION 3.20. ([15, Proposition 3.4]) *Let $\{\theta\}$ be a big class, and let $u \in \text{PSH}(X, \theta)$. Let $u_k^D \in \text{PSH}(X, \theta + \epsilon_k \omega)$ be the approximation sequences of u given by Theorem 3.18. Then*

$$\int_X \langle (\theta + \epsilon_k \omega)_{u_k^D}^n \rangle \searrow \int_X \langle (\theta)_{P^\theta[u]_{\mathcal{I}}}^n \rangle, \text{ as } k \rightarrow \infty.$$

The convergence property above can be generalized to the mixed setting as follows. The result is similar to [57, Theorem 4.2]. We present a proof for the reader's convenience.

LEMMA 3.21. ([42, Lemma 2.13]) *Let $\{\theta_1\}, \dots, \{\theta_n\}$ be big classes. Let $u_j \in \text{PSH}(X, \theta_j)$ be such that $\int_X \langle (\theta_j)_{u_j}^n \rangle > 0$ for $1 \leq j \leq n$. Then, we have*

$$\int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{u_{j,k}^D} \rangle \searrow \int_X \langle \wedge_{j=1}^n (\theta_j)_{P^{\theta_j}[u_j]_{\mathcal{I}}} \rangle, \text{ as } k \rightarrow \infty,$$

where $u_{j,k}^D$ is the approximation sequence of u_j given by Theorem 3.18.

PROOF. Since $u_{j,k}^D$ is of analytic singularities, we get $[u_{j,k}^D] = [P^{\theta_j + \epsilon_k \omega}[u_{j,k}^D]_{\mathcal{I}}]$ ([29, Theorem 4.3]). By [15, Proposition 3.3], we have $P^{\theta_j + \epsilon_k \omega}[u_{j,k}^D]_{\mathcal{I}} \searrow P^{\theta_j}[u_j]_{\mathcal{I}}$ as $k \rightarrow \infty$. Hence,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{u_{j,k}^D} \rangle \\ &= \lim_{k \rightarrow \infty} \int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{P^{\theta_j + \epsilon_k \omega}[u_{j,k}^D]_{\mathcal{I}}} \rangle \\ &\geq \int_X \langle \wedge_{j=1}^n (\theta_j)_{P^{\theta_j}[u_j]_{\mathcal{I}}} \rangle. \end{aligned}$$

We now prove

$$(3.3.1) \quad \lim_{k \rightarrow \infty} \int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{u_{j,k}^D} \rangle \leq \int_X \langle \wedge_{j=1}^n (\theta_j)_{P^{\theta_j}[u_j]_{\mathcal{I}}} \rangle.$$

By [13, Lemma 4.3] and Proposition 3.20, we can choose

$$b_{j,k} \in \left(1, \left(\frac{\int_X \langle (\theta_j + \epsilon_k \omega)_{u_{j,k}^D}^n \rangle}{\int_X \langle (\theta_j + \epsilon_k \omega)_{u_{j,k}^D}^n \rangle - \int_X \langle (\theta_j + \epsilon_k \omega)_{P^{\theta_j}[u_j]_{\mathcal{I}}}^n \rangle} \right)^{\frac{1}{n}} \right),$$

which increases to ∞ as $k \rightarrow \infty$, and $\varphi_{j,k} \in \text{PSH}(X, \theta_j + \epsilon_k \omega)$ such that

$$b_{j,k}^{-1} \varphi_{j,k} + (1 - b_{j,k}^{-1}) u_{j,k}^D \leq P^{\theta_j} [u_j]_{\mathcal{I}}.$$

By the monotonicity theorem of non-pluripolar product (see Theorem 3.13), we then have

$$\prod_{j=1}^n (1 - b_{j,k}^{-1}) \int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{u_{j,k}^D} \rangle \leq \int_X \langle \wedge_{j=1}^n (\theta_j + \epsilon_k \omega)_{P^{\theta_j} [u_j]_{\mathcal{I}}} \rangle.$$

Let $k \rightarrow \infty$, then we get (3.3.1). This completes the proof. $\square$

REMARK 3.22. The result of Proposition 3.20 extends to the case where θ is merely pseudoeffective. This follows from the continuity of the volume function on the cone of pseudoeffective $(1, 1)$ -classes in $H^{1,1}(X, \mathbb{R})$ (see [5]). However, it is not clear whether the same extension holds in the mixed setting of Lemma 3.21, where the proof relies on [13, Lemma 4.3], and the non-collapsing condition for the masses of $(1, 1)$ -currents plays an important role.

3.4. Relation between density currents and relative non-pluripolar products

In this section, we discuss the relationship between density currents and relative non-pluripolar products, which will have applications in Chapter 4. We assume X to be a compact Kähler manifold in this section. We begin by introducing the following lemma in the case where each T_j has a bounded potential.

LEMMA 3.23. ([49, Lemma 3.4]) *Let ψ be a locally bounded quasi-psh function on X^{m+1} . Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents with bounded potentials. Let $R := \otimes_{j=1}^m T_j \otimes T$. Then the unique tangent current to ψR along Δ_{m+1} is $\pi_{m+1}^* ((\psi|_{\Delta_{m+1}}) \wedge_{j=1}^m T_j \wedge T)$*

Now, let us turn to setting of the relative non-pluripolar product.

THEOREM 3.24. ([49, Theorem 3.5]) *Let R_∞ be a density current associated to $T_1, \dots, T_m, T$. Then we have*

$$(3.4.1) \quad \pi_{m+1}^* \langle \wedge_{j=1}^m T_j \wedge T \rangle \leq R_\infty,$$

where π_{m+1} is the canonical projection from the normal bundle of the diagonal Δ_{m+1} of X^{m+1} to Δ_{m+1} , and as usual we identify Δ_{m+1} with X .

PROOF. Since the problem is local, we could choose a local coordinate chart Ω , and write $T_j = dd^c u_j$, for some psh function $u_j < 0$ on Ω . For $k \in \mathbb{N}$, set

$$u_{j,k} := \max\{u_j, -k\} \text{ and } T_{j,k} := dd^c u_{j,k}$$

We set

$$R_k := \bigotimes_{j=1}^m T_{j,k} \otimes T \text{ and } R := \bigotimes_{j=1}^m T_j \otimes T.$$

For $1 \leq j \leq m$, let $p_j: X^{m+1} \rightarrow X$ be the projection from X^{m+1} to the j th-component. Set

$$\psi := \sum_{j=1}^m p_j^* u_j, \quad \text{and} \quad \psi_k := k^{-1} \max\{\psi, -k\}.$$

Note that $\psi_k + 1 = 0$ on $\bigcup_{j=1}^m p_j^* \{u_j \leq -k\}$.

Let $\epsilon > 0$, by [51, Theorem 2.4] and [40, Theorem 20.4], there exists a continuous function $v_{j,k}$ on Ω such that

$$\text{cap}_{T_j}(\{u_{j,k} \neq v_{j,k}\}, \Omega) \leq \epsilon, \quad \text{cap}_{T_{j,r}}(\{u_{j,k} \neq v_{j,k}\}, \Omega) \leq \epsilon,$$

for any $r \in \mathbb{N}$. Set

$$\tilde{\psi}_k := k^{-1} \max\left\{\sum_{j=1}^m p_j^* u_j, -k\right\},$$

which is continuous on Ω . Since

$$\{\tilde{\psi}_k \neq \psi_k\} \subseteq \bigcup_{j=1}^m p_j^* \{v_{j,k} \neq u_{j,k}\},$$

we get

$$\begin{aligned} \int |\psi_k - \tilde{\psi}_k| R_r \wedge \omega^{m(n-1)+n-p} &\lesssim \int_{\bigcup_{j=1}^m p_j^* \{v_{j,k} \neq u_{j,k}\}} R_r \wedge \omega^{m(n-1)+n-p} \\ (3.4.2) \quad &\lesssim \sum_{j=1}^m \|T_{j,r}\|_{\{u_{j,k} \neq v_{j,k}\}} \leq m\epsilon. \end{aligned}$$

On the other hand, one can show that

$$(3.4.3) \quad \int |\psi_k - \tilde{\psi}_k| R \wedge \omega^{m(n-1)+n-p} \lesssim m\epsilon,$$

by the same strategy. Since $\tilde{\psi}_k$ is continuous, we get

$$\tilde{\psi}_k R_r \rightarrow \tilde{\psi}_k R$$

weakly, when $r \rightarrow \infty$. This, combined with (3.4.2) and (3.4.3), then we get

$$\psi_k R_r \rightarrow \psi_k R$$

weakly, when $r \rightarrow \infty$. This implies

$$\begin{aligned} -\psi_k R &= -(\psi_k + 1)R + R \\ &= -\lim_{r \rightarrow \infty} (\psi_k + 1)R_r + R \\ &= -\lim_{r \rightarrow \infty} (\psi_k + 1)R_k + R \\ (3.4.4) \quad &= -(\psi_k + 1)R_k + R, \end{aligned}$$

Let $T_{k,\infty}$, R_∞ be tangent currents of $-\psi_k R$, R along $\Delta_{m+1} \subset X^{m+1}$ respectively. By Lemma 3.23, we get that the tangent current of $(\psi_k + 1)R_k$ equals $\pi^*((\rho_k + 1) \wedge_{j=1}^m T_{jk} \wedge T)$, where $\pi: E \rightarrow \Delta_{m+1}$ is the normal bundle of Δ_{m+1} in X^{m+1} , and ρ_k is the restriction of ψ_k on Δ_{m+1} . Now, by taking the tangent current of both sides of (3.4.4), we get

$$(3.4.5) \quad R_{k,\infty} = R_\infty - \pi^*((\rho_k + 1) \wedge_{j=1}^m T_{jk} \wedge T).$$

Since $\rho_k + 1 = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$, we then have

$$(\rho_k + 1) \wedge_{j=1}^m T_{jk} \wedge T = (\rho_k + 1) \langle \wedge_{j=1}^m T_j \wedge T \rangle,$$

which converges to $\langle \wedge_{j=1}^m T_j \wedge T \rangle$ as $k \rightarrow \infty$. This combines with (3.4.5) implies

$$\pi^*(\langle \wedge_{j=1}^m T_j \wedge T \rangle) \leq R_\infty.$$

This completes the proof. □

We will need the following to estimate the density h-dimension of currents, which is a special case of [49, Proposition 3.6].

PROPOSITION 3.25. ([43, Proposition 3.5]) *Let P and T be closed positive currents of bi-degree $(1, 1)$ and (p, p) respectively on X , $1 \leq p \leq n$. Assume that T has no mass on I_p . Then, for every density current S associated to P, T , the h -dimension of S is equal to $n - p - 1$.*

For every pseudoeffective (p, p) -class γ on X , we put $\|\gamma\| := \int_X \Theta \wedge \omega^{n-p}$, where Θ is any closed smooth form in γ . This definition is independent of the choice of Θ and is non-negative due to the pseudoeffectivity of γ . The following

theorem describes the relationship between the cup product of cohomology classes and the class of the relative non-pluripolar product.

THEOREM 3.26. ([43, Theorem 3.6]) *Let P and T be closed positive currents of bi-degree $(1, 1)$ and (p, p) respectively on X , where $1 \leq p \leq n - 1$. Assume that T has no mass on I_p . Then, the cohomology class*

$$\gamma := \{P\} \wedge \{T\} - \{P \wedge T\}$$

is pseudoeffective and we have

$$(3.4.6) \quad \|\gamma\| \geq \sum_V \nu(P, V) \nu(T, V) n_V! \operatorname{vol}(V),$$

where the sum is taken over every irreducible subset V of dimension at least $n - p - 1$ in X , and $n_V := \dim V$.

We note that by the proof below, we see that any irreducible subset V such that $\dim V \geq n - p - 1$ and $\nu(T, V) > 0$, $\nu(P, V) > 0$ must satisfy $\dim V = n - p - 1$.

PROOF. Let $\mathcal{V}$ be the set of irreducible analytic subsets V of dimension at least $n - p - 1$ in X such that $\nu(T, V) > 0$ and $\nu(P, V) > 0$. We note that in (3.4.6), it is enough to consider $V \in \mathcal{V}$. We will see below that $\mathcal{V}$ has at most countable elements.

Observe that if $\nu(P, x) > 0$, then $x \in I_p$. Hence, by hypothesis, the trace measure of T has no mass on the set $\{x \in X : \nu(P, x) > 0\}$. This allows us to apply Proposition 3.25 to P and T to obtain that the *density h -dimension* of P and T is minimal. Using this and Lemma 2.21 gives

$$(3.4.7) \quad \kappa(P, T) = \pi^*(\{P\} \wedge \{T\}),$$

where π is the canonical projection from the normal bundle of the diagonal Δ of X^2 to Δ .

Let S be a density current associated to P and T . Since the *h-dimension* of S is minimal, using Lemma 2.19, we get that there exists a current S' on X such that $S = \pi^* S'$ (recall Δ is identified with X). Since the relative non-pluripolar product is dominated by density currents (Theorem 3.24), the current $S' - \langle P \wedge T \rangle$ is closed and positive. Moreover, by (3.4.7), the cohomology class of the last current is equal to γ . It follows that γ is pseudoeffective.

It remains to prove (3.4.6). Let $V \in \mathcal{V}$. By definition, the generic Lelong number of T along V is positive. Since T is of bi-degree (p, p) , the dimension of V must be at most $n - p$. Hence, we have two possibilities: either $\dim V = n - p - 1$ or $\dim V = n - p$. Indeed, the latter case cannot occur. Suppose that such a V exists. Then, we consider two cases: whether T has mass on V or not. If T has no mass on V , then $\nu(T, V) = 0$, which leads to a contradiction. If T has mass on V , which is contained in I_P (for $\nu(P, V) > 0$), then this contradicts the hypothesis that T has no mass on I_P .

Let $V \in \mathcal{V}$. Since the Lelong numbers are preserved by submersion maps ([36, Proposition 2.3]), by applying Corollary 2.23 to P, T and generic $x \in V$, we obtain

$$\nu(S', V) = \nu(S, V) \geq \nu(P, V)\nu(T, V).$$

This combined with the fact that $\dim V = n - p - 1$ implies

$$S' \geq \nu(P, V)\nu(T, V) [V].$$

We deduce that

$$\begin{aligned} S' &\geq \langle P \wedge T \rangle + \mathbb{1}_{I_P} S' \\ &\geq \langle P \wedge T \rangle + \sum_{V \in \mathcal{V}} \nu(P, V)\nu(T, V) [V]. \end{aligned}$$

The second inequality comes from Siu's decomposition theorem (Theorem 2.10), and this also shows that $\mathcal{V}$ has at most countable elements. The desired assertion follows and the proof is finished. $\square$

CHAPTER 4

Volumes of components of Lelong upper level sets

4.1. Introduction

This chapter is based on the article [43] and aims to investigate the singularities of closed positive currents on compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n , and let T be a closed positive $(1, 1)$ -current on X . We are interested in understanding the set of points where T has strictly positive Lelong numbers. By the celebrated upper semi-continuity of Lelong numbers by Siu [41], we know that this set is a countable union of proper analytic subsets on X . Our goal is to estimate the size of this upper level set. The problem was first studied by Demailly in [17, 18]. In this chapter, we provide in some sense a generalization of Demailly's estimate. To delve into details, let us first introduce some necessary notions.

Let ω be a fixed smooth Kähler form on X . We equip X with the Riemannian metric induced by ω . For an analytic set V of dimension l in X , the volume of V is denoted and defined as follows.

$$\text{vol}(V) := \frac{1}{l!} \int_{\text{Reg} V} \omega^l,$$

where $\text{Reg} V$ is the regular locus of V .

Let T be a closed positive $(1, 1)$ -current, we recall that the Lelong number of T at $x \in X$ is denoted and defined by

$$\nu(T, x) = \sup \left\{ \gamma \in \mathbb{R}^+ \mid u(z) \leq \gamma \log |z - x| + \mathcal{O}(1) \text{ at } x \right\}.$$

We recall that the generic Lelong number $\nu(T, V)$ of T along V is defined as $\inf_{x \in V} \{\nu(T, x)\}$. For every constant $c > 0$, we recall that

$$E_c(T) := \{x \in X \mid \nu(T, x) \geq c\} \quad \text{and} \quad E_+(T) := \{x \in X \mid \nu(T, x) > 0\}.$$

By Corollary 2.8, $E_c(T)$ is a proper analytic subset in X , and

$$E_+(T) = \bigcup_{m \in \mathbb{N}^*} E_{1/m}(T)$$

is a countable union of analytic sets.

Let W be an irreducible analytic subset of dimension m in X . We denote by

$$E_+^W(T) := \{x \in W \mid \nu(T, x) > \nu(T, W)\}$$

the Lelong upper level set of T on W , which is also a countable union of proper analytic subsets in W . Let $V \subseteq E_+^W(T)$ be an irreducible analytic set. We say that V is *maximal* if there is no irreducible analytic subset V' of $E_+^W(T)$ such that V is a proper subset of V' . We call V a *component of the Lelong upper level set of T along W* , and let $\mathcal{V}_{T,W}$ be the set of such components V . Observe that $\mathcal{V}_{T,W}$ has at most countably many elements. For $0 \leq l \leq m$, we denote by $\mathcal{V}_{l,T,W}$ the set of $V \in \mathcal{V}_{T,W}$ such that $\dim V = l$.

Write $T = dd^c u$ locally, where u is a psh function. We define $T|_{\text{Reg}W}$ as $dd^c(u|_{\text{Reg}W})$ if $u \not\equiv -\infty$ on $\text{Reg}W$, and $T|_{\text{Reg}W} := 0$ otherwise. One sees that this definition is independent of the choice of u . Thus, $T|_{\text{Reg}W}$ is a current on $\text{Reg}W$. Here is our main result.

THEOREM 4.1. ([43, Theorem 1.1]) *Let α be a nef $(1, 1)$ -class and let W be an irreducible analytic subset of dimension m in X . Let T be a closed positive current in α such that $\nu(T, W) = 0$. Let $1 \leq m' \leq m$ be an integer. Then, we have*

$$(4.1.1) \quad \sum_{V \in \mathcal{V}_{m-m', T, W}} \nu(T, V)^{m'} \text{vol}(V) \leq \frac{1}{(m-m')!} \int_{\text{Reg}W} (\alpha^{m'} - \langle (T|_{\text{Reg}W})^{m'} \rangle) \wedge \omega^{m-m'},$$

where in the integral, we identify α with a smooth closed form in α .

We have some comments on (4.1.1). To see why the term

$$I := \int_{\text{Reg}W} (\alpha^{m'} - \langle (T|_{\text{Reg}W})^{m'} \rangle) \wedge \omega^{m-m'}$$

is non-negative, one can consider the case where W is smooth. Then, by the monotonicity of non-pluripolar products (see Remark 3.17), the cohomology class $(\alpha|_W)^{m'} - \{\langle (T|_W)^{m'} \rangle\}$ is pseudoeffective. Hence, the integral on the right-hand side of (4.1.1) is non-negative. In the general case where W is singular, one can use a desingularisation of W or interpret I as the mass of some non-pluripolar product relative to $[W]$ (the current of integration along W); see Lemma 3.9 below. We underline however that in order to prove Theorem 4.1, it is not possible to use desingularisation of W to reduce to the case where W is smooth. The reason

is that in the desingularisation process, one has to blow up submanifolds of W which in general could be some components of the Lelong upper level sets of T on W .

In [22], a less precise upper bound of volume of components of the Lelong upper level set was given in terms of the volume of W and the mass of T ; see also Theorem 4.3 for a more general statement. If we consider $W = X$, then the generic Lelong number of T along W is zero. Thus, by Theorem 4.1, we have the following result.

COROLLARY 4.2. ([43, Corollary 1.2]) *Let α be a nef $(1, 1)$ -class, and let T be a closed positive current in α . For $0 \leq l \leq n$, let $\mathcal{V}_{l,T}$ be the set of $V \in \mathcal{V}_{T,X}$ such that $\dim V = l$. Let $1 \leq m' \leq n$ be an integer. Then, we have*

$$(4.1.2) \quad \sum_{V \in \mathcal{V}_{n-m',T}} \nu(T, V)^{m'} \text{vol}(V) \leq \frac{1}{(n-m')!} \int_X (\alpha^{m'} - \{\langle T^{m'} \rangle\}) \wedge \omega^{n-m'}.$$

Corollary 4.2 generalizes [18, Corollary 7.6] by Demailly, in which it was assumed additionally that the components of the upper Lelong level set of T are only of dimension 0 (hence the cohomology class of T is necessarily nef, see [18, Lemma 6.3]). The feature of Corollary 4.2 is that it holds for any current in a nef class. The estimate (4.1.2) is optimal in the case where all of components of the Lelong upper level sets of T have the same dimension. For example, we consider $X = \mathbb{P}^n$, $z \in \mathbb{C}^n \subset \mathbb{P}^n$, and $T = \frac{1}{2} dd^c \log \frac{\|z\|^2}{1+\|z\|^2} + \omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{P}^n$. In this case, we see that 0 is the only point at which the Lelong number of T is positive and $\nu(T, 0) = 1$, and (4.1.2) (for $m = m' = n$) becomes an equality.

In general, if we consider the relative setting as in Theorem 4.1 (when W is not necessarily equal to X), then our main result (Theorem 4.1) is not satisfactory because it requires that $\nu(T, W) = 0$, hence, we can not apply it to the case where T is the current of integration along a curve $\mathcal{C}$ in a complex Kähler surface and $W = \mathcal{C}$. In Theorem 4.3 below, we are able to treat the case where $\nu(T, W) > 0$ but the estimate is not explicit due to the presence of a constant c in the right-hand side. In this regard, the estimates in [18, Theorem 1.7] are stronger than ours for dimension 2 (see the discussion after [18, Theorem 1.7] in [18]). On the other hand, as explained in [22], the feature of Theorem 4.1 is that it gives

bounds for volumes of all components of Lelong upper level sets whereas [18, Theorem 1.7] does not allow us to treat all components in general.

This chapter refines and substitutes [50]. The proof of Theorem 4.1 requires both the theory of density currents in [21] and relative non-pluripolar products in [51] (see also [3, 8]). One of the keys is Theorem 3.26 following from a general comparison of Lelong numbers for density currents.

4.2. Proof of Theorem 4.1

PROOF OF THEOREM 4.1. It suffices to consider the case where α is Kähler by using $\alpha + \epsilon\{\omega\}, T + \epsilon\omega$ instead of α, T and letting $\epsilon \rightarrow 0$. Hence, from now on we assume that α is Kähler. By abuse of notation, we also denote by α a smooth Kähler form in α . By Lemma 3.9, the right-hand side of (4.1.1) can be written as

$$\begin{aligned} & \frac{1}{(m-m')!} \int_{\text{Reg} W} (\alpha^{m'} - \langle (T|_{\text{Reg} W})^{m'} \rangle) \wedge \omega^{m-m'} \\ &= \frac{1}{(m-m')!} \|\langle \alpha^{m'} \wedge [W] \rangle - \langle T^{m'} \wedge [W] \rangle\|. \end{aligned}$$

Step 1. First, we focus on the case where T has analytic singularities. Set $S = \langle T^{m'-1} \wedge [W] \rangle$. Since α is Kähler, by the monotonicity of non-pluripolar product (Theorem 3.13) and Proposition 3.6 (6), we get

$$\begin{aligned} (4.2.1) \quad \|\langle \alpha^{m'} \wedge [W] \rangle - \langle T^{m'} \wedge [W] \rangle\| &\geq \|\langle \alpha \wedge T^{m'-1} \wedge [W] \rangle - \langle T^{m'} \wedge [W] \rangle\| \\ &= \|\alpha \wedge S - \langle T \wedge S \rangle\| \end{aligned}$$

We now show that S has no mass on I_T . For $m' > 1$, this directly follows from the definition of non-pluripolar product. For $m' = 1$, the current S is just $[W]$. Since we assume that T has analytic singularities, the polar locus I_T is an analytic subset and it does not contain W . Hence, $[W]$ also has no mass on I_T . Therefore, we can apply Theorem 3.26 to T, S , and get

$$(4.2.2) \quad \|\alpha \wedge \{S\} - \{\langle T \wedge S \rangle\}\| \geq (m-m')! \sum_{V \in \mathcal{V}_{m-m', T, W}} \nu(T, V) \nu(S, V) \text{vol}(V)$$

Let $V \in \mathcal{V}_{m-m', T, W}$ and let $\text{Sing}(I_T \cap W)$ be the singular locus of the analytic set $I_T \cap W$. Since T has analytic singularities, the Lelong number $\nu(T, x)$ is strictly positive if and only if x belongs to I_T . This coupled with the maximality of V

implies that V is contained in $I_T \cap W$, and is one of the irreducible components. Let $K_1, \dots, K_s$ be the irreducible components of $I_T \cap W$. Observe that the set $\text{Sing}(I_T \cap W)$ consists of singular points of irreducible components and their intersection points. By rearranging the index, we may assume $V = K_1$. Set

$$U := X \setminus \text{Sing}(K_1) \cup K_2 \cdots \cup K_s.$$

Now, we prove that the intersection $T^{m'-1} \wedge [W]$ is well-defined on U , in the sense in Section 2.1.3. Notice that $V \setminus \text{Sing}(I_T \cap W)$ is contained in $\text{Reg}(V)$, and is of dimension $m - m'$. Consequently, for $0 \leq j' \leq m' - 1$,

$$\begin{aligned} \mathcal{H}_{2m-2j'+1}(L(T)|_U \cap W) &= \mathcal{H}_{2m-2j'+1}(I_T \cap W \cap U) \\ &= \mathcal{H}_{2m-2j'+1}(V \setminus \text{Sing}(I_T \cap W)) \\ &= 0, \end{aligned}$$

where $L(T)$ is the set of $x \in X$ such that the local potential of T is unbounded on any neighborhood of x . This allows us to apply Theorem 2.4 and get the well-definedness of $T^{m'-1} \wedge [W]$ on U . By applying Proposition 3.8 to $\langle T^{m'-1} \wedge [W] \rangle$, we obtain

$$(4.2.3) \quad S = \langle T^{m'-1} \wedge [W] \rangle = \mathbb{1}_{U \setminus I_T} T^{m'-1} \wedge [W].$$

Actually, the equality also holds on $U \cap I_T$. To show this, we need to check that $T^{m'-1} \wedge [W]$ has no mass on $U \cap I_T$. Since $\dim(U \cap I_T \cap W) = m - m'$ and $T^{m'-1} \wedge [W]$ is of bi-dimension $(m - m' + 1, m - m' + 1)$, the current $T^{m'-1} \wedge [W]$ must have no mass on $U \cap I_T \cap W$. Also, by the fact $\text{supp}(T^{m'-1} \wedge [W]) \subseteq W$, the current $T^{m'-1} \wedge [W]$ also has no mass on $(U \cap I_T) \setminus W$. Therefore, the equality (4.2.3) extends to U . This implies that the Lelong number $\nu(S, V)$ equals $\nu(T^{m'-1} \wedge [W], V \setminus \text{Sing}(I_T \cap W))$ (remember that we consider the current $T^{m'-1} \wedge [W]$ on U , and $V \setminus \text{Sing}(I_T \cap W)$ is an analytic subset in U), and then we have

$$\begin{aligned} \nu(S, V) &= \nu(T^{m'-1} \wedge [W], V \setminus \text{Sing}(I_T \cap W)) \\ &\geq \nu(T, V \setminus \text{Sing}(I_T \cap W))^{m'-1} \cdot \nu([W], V \setminus \text{Sing}(I_T \cap W)) \\ (4.2.4) \quad &\geq \nu(T, V)^{m'-1}, \end{aligned}$$

where the first inequality comes from Proposition 2.7 (see also Corollary 2.23 for a more general version). By (4.2.1), (4.2.2) and (4.2.4), the desired inequality follows in the case where T has analytic singularities.

Step 2. Now, we remove the assumption that T has analytic singularities. The argument we use is standard and is based on the work of Demailly in [18], see Section 3.3 for more detail. First, we write $T = dd^c u + \theta$, where θ is a closed smooth $(1,1)$ -form, and $u \in \text{PSH}(X, \theta)$. Demailly's analytic approximation theorem (Theorem 3.18) allows us to construct a sequence $u_k^D \in \text{PSH}(X, \theta + \epsilon_k \omega)$, where ϵ_k decreases to 0, such that

- (1) $u_k^D \geq u$ and u_k^D converges to u in L^1 .
- (2) u_k^D has analytic singularities.
- (3) $\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly on X , where

$$T_k = dd^c u_k^D + (\theta + \epsilon_k \omega).$$

By the monotonicity property of non-pluripolar product (Theorem 3.13), we have

$$\begin{aligned} & \|\langle \alpha^{m'} \wedge [W] \rangle - \langle T^{m'} \wedge [W] \rangle\| \\ (4.2.5) \quad &= \lim_{k \rightarrow \infty} \|\langle (\alpha + \epsilon_k \omega)^{m'} \wedge [W] \rangle - \langle (T + \epsilon_k \omega)^{m'} \wedge [W] \rangle\| \\ &\geq \limsup_{k \rightarrow \infty} \|\langle (\alpha + \epsilon_k \omega)^{m'} \wedge [W] \rangle - \langle T_k^{m'} \wedge [W] \rangle\| \end{aligned}$$

For every constant $r > 0$, set $A_r := \{V \in \mathcal{V}_{m-m', T, W} \mid \nu(T, V) \geq r\}$. Observe that A_r increases to $\mathcal{V}_{m-m', T, W}$ as $r \rightarrow 0$. Since $\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly and T_k is less singular than T , for every fixed $r > 0$ we have

$$A_r \subseteq \mathcal{V}_{m-m', T_k, W}$$

when k is large enough. By **Step 1**, we therefore have

$$\begin{aligned} \|\langle (\alpha + \epsilon_k \omega)^{m'} \wedge [W] \rangle - \langle T_k^{m'} \wedge [W] \rangle\| &\geq (m - m')! \sum_{V \in \mathcal{V}_{m-m', T_k, W}} \nu(T_k, V)^{m'} \text{vol}(V) \\ &\geq (m - m')! \sum_{V \in A_r} \nu(T_k, V)^{m'} \text{vol}(V). \end{aligned}$$

Letting $k \rightarrow \infty$ and using (4.2.5) give

$$\begin{aligned} \|\langle \alpha^{m'} \wedge [W] \rangle - \langle T^{m'} \wedge [W] \rangle\| &\geq (m - m')! \limsup_{k \rightarrow \infty} \sum_{V \in A_r} \nu(T_k, V)^{m'} \text{vol}(V) \\ &= (m - m')! \sum_{V \in A_r} \nu(T, V)^{m'} \text{vol}(V), \end{aligned}$$

for every constant $r > 0$. Letting $r \rightarrow 0$, we obtain the desired estimate. $\square$

For the general case where $\nu(T, W) > 0$. We could not directly compare the volume of Lelong upper level sets of T on W and the mass of $\{\langle \alpha^{m'} \wedge [W] \rangle\} - \{\langle T^{m'} \wedge [W] \rangle\}$. In this case, we have the following modified inequality which is stronger than [22, Theorem 1.1].

THEOREM 4.3. ([43, Theorem 3.7]) *Let α be a nef $(1, 1)$ -class. Let W be an irreducible analytic subset in X . Let T be a closed positive current in α . Let $1 \leq m' \leq m$ be an integer. Then we have*

$$(4.2.6) \quad (m - m')! \sum_{V \in \mathcal{V}_{m-m', T, W}} (\nu(T, V) - \nu(T, W))^{m'} \text{vol}(V) \leq \|\langle \alpha + c\{\omega\} \rangle^{m'} \wedge \{[W]\} - \{\langle (T + c\omega)^{m'} \wedge [W] \rangle\}\|,$$

where $c = c_1 \cdot \nu(T, W)$ and $c_1 > 0$ is a constant independent of α, T, W . In particular, there is a constant $c_2 > 0$ independent of α, T, W such that

$$(4.2.7) \quad \sum_{V \in \mathcal{V}_{m-m', T, W}} (\nu(T, V) - \nu(T, W))^{m'} \text{vol}(V) \leq c_2 \text{vol}(W) \|T\|^{m'}.$$

We will use the regularization theorem introduced in [18], which we state below.

THEOREM 4.4. ([18, Theorem 1.1]) *Let X be a compact complex manifold. Let $T = dd^c u + \theta$ be a closed almost positive $(1, 1)$ -current, $u \in \text{PSH}(X, \theta)$, such that $T \geq \gamma$ for some continuous real $(1, 1)$ -form γ . Let $\pi_X: \mathbb{P}(T^*X) \rightarrow X$ be the projection. Suppose that $\mathcal{O}_{TX}(1)$ is equipped with a smooth Hermitian metric such that*

$$c(\mathcal{O}_{TX}(1)) + \pi_X^* u \geq 0$$

for some non-negative smooth $(1, 1)$ -form u on X . Let ω be a Hermitian metric on X . Then, for $c > 0$, there exists a sequence of closed almost positive $(1, 1)$ -currents $T_{c,k}$ in

the class $\{\theta\}$, which converges weakly to T , and

$$T_{c,k} \geq \gamma - \min\{\lambda_k, c\}u - \epsilon_k \omega,$$

where

(1) λ_k is a decreasing sequence of continuous functions such that

$$\lim_{k \rightarrow \infty} \lambda_k(x) = v(T, x), \quad x \in X.$$

(2) $\epsilon_k \searrow 0$, as $k \rightarrow \infty$.

(3) $v(T_{c,k}, x) = \max\{v(T, x) - c, 0\}$, for $x \in X$.

REMARK 4.5. When (X, ω) is a compact Kähler manifold, one can choose $u = c_1 \omega$ for some sufficiently large constant $c_1 > 0$ in the above theorem, so that the required condition is always satisfied.

PROOF OF THEOREM 4.3. The inequality (4.2.7) follows directly from (4.2.6). The proof of (4.2.6) is similar to **Step 2** of Theorem 4.1, which is based on Demailly's regularization theorem (Theorem 4.4). For convenience, set $c_3 := v(T, W) > 0$. Theorem 4.4 allows us to cut down the Lelong upper level set $\{x \in X | v(T, V) \geq c_3\}$ from T . More precisely, there exists a sequence of almost positive closed $(1, 1)$ -currents $T_{c_3,k}$ in α such that

- (1) $T_{c_3,k} \geq -(c_1 \cdot c_3 + \epsilon_k) \omega$, where $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and $c_1 > 0$ is a constant independent of α, T and W .
- (2) The sequence of global potentials of $T_{c_3,k}$ decreases to the global potential of T .
- (3) $v(T_{c_3,k}, x) = \max\{v(T, x) - c_3, 0\}$.

Set $\tilde{T}_{c_3,k} = T_{c_3,k} + (c_1 \cdot c_3 + \epsilon_k) \omega$, which is a closed positive $(1, 1)$ -current. By Theorem 3.13, we have

$$\begin{aligned} & \|(\alpha + c_1 \cdot c_3 \{\omega\})^{m'} \wedge \{[W]\} - \langle (T + c_1 \cdot c_3 \omega)^{m'} \wedge [W] \rangle \| \\ & \geq \limsup_{k \rightarrow \infty} \|(\alpha + (c_1 \cdot c_3 + \epsilon_k) \{\omega\})^{m'} \wedge \{[W]\} - \langle \tilde{T}_{c_3,k}^{m'} \wedge [W] \rangle \|. \end{aligned}$$

Since $\nu(\tilde{T}_{c_3,k}, W) = 0$, we can apply Theorem 4.1 to the right-hand side of the above inequality and get

$$\begin{aligned}
 & \left\| \left(\alpha + (c_1 \cdot c_3 + \epsilon_k) \{ \omega \} \right)^{m'} \wedge \{ [W] \} - \langle \tilde{T}_{c_3,k}^{m'} \wedge [W] \rangle \right\| \\
 (4.2.8) \quad & \geq (m - m')! \sum_{V \in \mathcal{V}_{m-m', \tilde{T}_{c_3,k}, W}} \nu(\tilde{T}_{c_3,k}, V)^{m'} \text{vol}(V),
 \end{aligned}$$

By the above properties of $T_{c_3,k}$, we have

$$\mathcal{V}_{m-m', \tilde{T}_{c_3,k}, W} = \mathcal{V}_{m-m', T, W}.$$

Therefore, the right-hand side of (4.2.8) is equal to

$$\begin{aligned}
 & (m - m')! \sum_{V \in \mathcal{V}_{m-m', T, W}} \nu(\tilde{T}_{c_2,k}, V)^{m'} \text{vol}(V) \\
 & = (m - m')! \sum_{V \in \mathcal{V}_{m-m', T, W}} \left(\nu(T, V) - \nu(T, W) \right)^{m'} \text{vol}(V).
 \end{aligned}$$

This completes the proof. $\square$

REMARK 4.6. We note that by [46], for every closed positive (p, p) -current R on X , there always exists a closed positive $(1, 1)$ -current T whose Lelong numbers coincide with those of R . However, if we apply directly our result to current T , we will get an estimate of the Lelong upper level set for the current R . But there will be a constant appear in the right-hand side of (4.1.1) in Theorem 4.1, since the mass of T is bounded by a universal constant times the mass of R .

CHAPTER 5

Singularities of currents of full mass intersection**5.1. Introduction**

This chapter is based on the article [42]. Let X be a compact Kähler manifold of dimension n . Let $\{\theta_1\}, \dots, \{\theta_m\}$ be pseudoeffective classes on X , $1 \leq m \leq n$. Let $T_j, T'_j \in \{\theta_j\}$ be closed positive $(1, 1)$ -current, $1 \leq j \leq m$, such that T_j is less singular than T'_j , meaning that the potential of T_j is greater than the potential of T'_j , modulo an additive constant. The monotonicity property of the non-pluripolar product (Remark 3.17) plays a crucial role in pluripotential theory:

$$\{\langle \wedge_{j=1}^m T'_j \rangle\} \leq \{\langle \wedge_{j=1}^m T_j \rangle\},$$

When the equality holds, we say $T'_1, \dots, T'_m$ have full mass intersection with respect to $T_1, \dots, T_m$. The aim of this chapter is to investigate the singularities of T_j and T'_j when $T'_1, \dots, T'_m$ have full mass intersection with respect to $T_1, \dots, T_m$.

Let $\{\theta\}$ be a big class, we say that a closed positive current $T \in \{\theta\}$ is *big* if $\int_X \langle T^n \rangle > 0$. A characterization of the bigness of T is provided in [15, Proposition 3.6], which states that there exists a Kähler current $P \in \{\theta\}$ that is more singular than T . Let $u \in \text{PSH}(X, \theta)$. Recall that the $\mathcal{I}$ -model envelope $P^\theta[u]_{\mathcal{I}} \in \text{PSH}(X, \theta)$ is defined as

$$P^\theta[u]_{\mathcal{I}} := (\sup\{w \in \text{PSH}(X, \theta) \mid w \leq 0, \mathcal{I}(tw) \subseteq \mathcal{I}(tu), t \geq 0\})^*.$$

A closed positive $(1, 1)$ -current $T = dd^c u + \theta$ is $\mathcal{I}$ -model if $u = P^\theta[u]_{\mathcal{I}}$. The following is our main result. We consider the case where $T_1, \dots, T_m$ are big and $\mathcal{I}$ -model (this condition can be slightly relaxed, see Remark 5.8), and we compare the Lelong numbers of T_j and T'_j under the full mass intersection assumption. Here is the main result.

THEOREM 5.1. *Let $\{\theta_1\}, \dots, \{\theta_m\}$ be big classes, and let $T'_j, T_j \in \{\theta_j\}$ be closed positive $(1, 1)$ -currents such that*

- (1) *T_j is big and is $\mathcal{I}$ -model.*

(2) T_j is less singular than T'_j .

Let V be a proper irreducible analytic subset such that $\dim(V) \geq n - m$. If $T'_1, \dots, T'_m$ are of full mass intersection with respect to $T_1, \dots, T_m$. Then, there exists $1 \leq j \leq m$ such that $\nu(T'_j, V) = \nu(T_j, V)$.

Theorem 5.1 was proved in [24] for the case where $\theta_1 = \dots = \theta_n = \omega$ is a Kähler form, $T'_1 = \dots = T'_n$ and $T_1 = \dots = T_n \in \{\omega\}$ is of minimal singularities. This result was later generalized in [53] to the setting where $\{\theta_1\}, \dots, \{\theta_m\}$ are big classes, and each $T_j \in \theta_j$ is of minimal singularities. For the self-intersection case ($T'_1 = \dots = T'_n$, $T_1 = \dots = T_n$), a characterization of currents having full mass intersection was given in [11] in terms of the singularity types of the envelopes of the given currents.

When $m = n$, we have the following more general quantitative result.

THEOREM 5.2. *Let $\mathcal{B}$ be a closed cone in the cone of big classes. There exists a constant $C > 0$, depending only on the manifold X and the cone $\mathcal{B}$ such that for every $x_0 \in X$, $\{\theta_j\} \in \mathcal{B}$ and $T'_j, T_j \in \{\theta_j\}$, $1 \leq j \leq n$, satisfying the conditions in Theorem 5.1. The following inequality holds.*

$$(5.1.1) \quad \int_X \left(\langle \wedge_{j=1}^n T_j \rangle - \langle \wedge_{j=1}^n T'_j \rangle \right) \geq C \prod_{j=1}^n (\nu(T'_j, x_0) - \nu(T_j, x_0)).$$

We note that our proof strategy is based on [53], where we generalize Theorems 1.1 and 1.2 from that paper.

5.2. Proof of Theorem 5.1

Step 1. First, we focus on the case where T_j has analytic singularities and there exists a Kähler current $P_j \in \{\theta_j\}$ with analytic singularities that is more singular than T_j for each j . We also assume that V is a smooth submanifold of dimension greater than $n - m$.

Let $\epsilon > 0$ be such that $P_j \geq \epsilon\omega$ for $1 \leq j \leq m$. We recall that $T_j^p := \mathbb{1}_{I_{T_j}} T_j$. Our first step is to reduce the problem to the case where $T_j^p = 0$ for $1 \leq j \leq m$. To achieve this, we replace T_j , T'_j and P_j with $T_j - T_j^p$, $T'_j - T_j^p$ and $P_j - T_j^p$,

respectively. By Lemma 3.12, we obtain

$$\begin{aligned}
 (5.2.1) \quad & \|\langle \wedge_{j=1}^m T_j \rangle\| = \|\langle \wedge_{j=1}^m (T_j - T_j^p) \rangle\|, \\
 & \|\langle \wedge_{j=1}^m T'_j \rangle\| = \|\langle \wedge_{j=1}^m (T'_j - T_j^p) \rangle\|, \\
 & \|\langle \wedge_{j=1}^m P_j \rangle\| = \|\langle \wedge_{j=1}^m (P_j - T_j^p) \rangle\|.
 \end{aligned}$$

Note that the new currents $T_j - T_j^p$, $T'_j - T_j^p$, and $P_j - T_j^p$ satisfy the following properties.

PROPOSITION 5.3. *For $1 \leq j \leq m$, the following holds.*

- (1) $T_j - T_j^p$ is less singular than $T'_j - T_j^p$ and $P_j - T_j^p$.
- (2) $\mathbb{1}_{I_{T_j - T_j^p}}(T_j - T_j^p) = 0$.

PROOF. First, observe from the proof of Lemma 3.12 that

$$T_j^p = \sum_{V_{j,k}} \lambda_{j,k} [V_{j,k}],$$

where $\{V_{j,k}\}$ denote the irreducible components of $V_j = I_{P_j}$. Since T_j is less singular than both T'_j and P_j , the Lelong number of T_j along each hypersurface $V_{j,k}$ must be less than the corresponding Lelong numbers of T'_j and P_j , we therefore obtain that $T_j - T_j^p$ is less singular than both $T'_j - T_j^p$ and $P_j - T_j^p$. This completes the first part of the proposition.

The second part is relatively straightforward and follows from the inequality

$$0 \leq \mathbb{1}_{I_{T_j - T_j^p}}(T_j - T_j^p) \leq \mathbb{1}_{I_{T_j}}(T_j - T_j^p) = 0.$$

□

The equalities (5.2.1) and Proposition 5.3 explain why it suffices to prove the theorem with the currents $T_j - T_j^p$, $T'_j - T_j^p$ and $P_j - T_j^p$. For convenience, from now on we will continue to use T_j , T'_j and P_j to represent the currents $T_j - T_j^p$, $T'_j - T_j^p$ and $P_j - T_j^p$, respectively.

Let $\sigma: \widehat{X} \rightarrow X$ be the blow-up of X along V . We denote by $\widehat{V} = \sigma^{-1}(V)$ the exceptional divisor. In general, the pull back of the Kähler form ω by σ is not necessarily a Kähler form. However, we can construct a Kähler form on $\widehat{X}$ as follows. By [47, Lemma 3.25], there exists a closed smooth $(1, 1)$ -form ω_h that is

cohomologous to $-\widehat{V}$, and a constant $c_V > 0$ depends on V such that

$$(5.2.2) \quad \widehat{\omega} := c_V \sigma^* \omega + \omega_h > 0,$$

which is a Kähler form on $\widehat{X}$.

For $\delta \in (0, 1)$, set $P_j^\delta := (1 - \delta)T_j + \delta P_j$. Note that $P_j^\delta \geq \delta \epsilon \omega$ is a Kähler current on X . Now, we decompose the pull back of T_j , T'_j and P_j^δ by σ as follows.

$$\begin{aligned} \sigma^* T_j &= \lambda_j[\widehat{V}] + \eta_j, \\ \sigma^* T'_j &= \lambda'_j[\widehat{V}] + \eta'_j, \\ \sigma^* P_j^\delta &= \lambda_j^\delta[\widehat{V}] + \eta_j^\delta. \end{aligned}$$

Since the Lelong numbers are preserved by the blow-up map ([4, Corollary 1.1.8]), we get

$$\begin{aligned} \lambda_j &= \nu(\sigma^* T_j, \widehat{V}) = \nu(T_j, V), \\ \lambda'_j &= \nu(\sigma^* T'_j, \widehat{V}) = \nu(T'_j, V), \\ \lambda_j^\delta &= \nu(\sigma^* P_j^\delta, \widehat{V}) = \nu(P_j^\delta, V). \end{aligned}$$

We note that for any closed $(n - m, n - m)$ -form Φ , we have

$$\int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \Phi = \int_{\widehat{X}} \langle \wedge_{j=1}^m \eta_j \rangle \wedge \sigma^* \Phi \text{ and } \int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \Phi = \int_{\widehat{X}} \langle \wedge_{j=1}^m \eta'_j \rangle \wedge \sigma^* \Phi$$

LEMMA 5.4. *Let η_m be the $(1, 1)$ -current defined as above. Let $\tau_1, \dots, \tau_{m-1}$ be closed positive $(1, 1)$ -currents on $\widehat{X}$. Then, we have*

$$\langle \wedge_{j=1}^{m-1} \tau_j \wedge \eta_m \rangle = \langle \wedge_{j=1}^{m-1} \tau_j \wedge \eta_m \rangle.$$

PROOF. By Proposition 3.8, it suffices to show that $\mathbb{1}_{I_{\eta_m}} \eta_m = 0$. Note that

$$\begin{aligned} \mathbb{1}_{I_{\eta_m}} \eta_m &= \mathbb{1}_{I_{\eta_m} \setminus \widehat{V}} \eta_m \\ &\leq \sigma^*(\mathbb{1}_{\sigma(I_{\eta_m})} T_m) \\ &\leq \sigma^*(\mathbb{1}_{I_{T_m}} T_m). \end{aligned}$$

Since we assume $T_m^p = \mathbb{1}_{I_{T_m}} T_m = 0$, it follows that $\mathbb{1}_{I_{\eta_m}} \eta_m = 0$, which completes the proof. $\square$

Note that T_j is less singular than P_j^δ , hence

$$\sigma^* P_j^\delta = \eta_j^\delta + \lambda_j^\delta[\widehat{V}] \preceq \sigma^* T_j.$$

This implies

$$(5.2.3) \quad \eta_j^\delta \preceq P((\sigma^* T_j) - \lambda_j^\delta [\widehat{V}]).$$

Set

$$Q_j^\delta := \eta_j^\delta + \frac{\delta\epsilon}{2c_V} \omega_h \geq \frac{\delta\epsilon}{2c_V} \widehat{\omega}, \quad \widetilde{\eta}_j^\delta := P(P(\sigma^* T_j - \lambda_j^\delta [\widehat{V}]) - \frac{\delta\epsilon}{2c_V} [\widehat{V}]).$$

LEMMA 5.5. *The currents Q_j^δ is more singular than $\widetilde{\eta}_j^\delta$.*

PROOF. First, we write

$$\begin{aligned} \eta_j &= dd^c u_{\eta_j} + \sigma^* \theta_j + \lambda_j \omega_h, & u_{\eta_j} &\in \text{PSH}(\widehat{X}, \sigma^* \theta_j + \lambda_j \omega_h), \\ \eta_j^\delta &= dd^c u_j^\delta + \sigma^* \theta_j + \lambda_j^\delta \omega_h, & u_j^\delta &\in \text{PSH}(\widehat{X}, \sigma^* \theta_j + \lambda_j^\delta \omega_h), \\ P(\sigma^* T_j - \lambda_j^\delta [\widehat{V}]) &= dd^c u_j^p + \sigma^* \theta_j + \lambda_j^\delta \omega_h, & u_j^p &\in \text{PSH}(\widehat{X}, \sigma^* \theta_j + \lambda_j^\delta \omega_h), \\ [\widehat{V}] &= dd^c u_{\widehat{V}} - \omega_h, & u_{\widehat{V}} &\in \text{PSH}(\widehat{X}, -\omega_h). \end{aligned}$$

We normalize the potential function $u_{\widehat{V}}$ so that $u_{\widehat{V}} \leq 0$. Note that u_j^δ can also represent the potential of Q_j^δ . In other words, $u_j^\delta \in \text{PSH}(X, \sigma^* \theta_j + (\lambda_j^\delta + \frac{\delta\epsilon}{2c_V}) \omega_h)$ such that

$$Q_j^\delta = dd^c u_j^\delta + \sigma^* \theta_j + (\lambda_j^\delta + \frac{\delta\epsilon}{2c_V}) \omega_h.$$

By (5.2.3) and the fact that $u_{\widehat{V}} \leq 0$, we obtain

$$[u_j^\delta + \frac{\delta\epsilon}{2c_V} u_{\widehat{V}}] \preceq [u_j^\delta] \preceq [u_j^p],$$

which implies $Q_j^\delta \preceq P(P(\sigma^* T_j - \lambda_j^\delta [\widehat{V}]) - \frac{\delta\epsilon}{2c_V} [\widehat{V}]) = \widetilde{\eta}_j^\delta$. $\square$

We prove Theorem 5.1 by contradiction. Suppose $\lambda'_j = \nu(T'_j, V) > \nu(T_j, V) = \lambda_j$ for $1 \leq j \leq m$. For each j , take $\delta_j > 0$ small enough so that $\lambda'_j - \lambda_j^{\delta_j} - \frac{\delta_j \epsilon}{2c_V} > 0$. Here, we provide a precise method for choosing δ_j . First, note that

$$\lambda'_j - \lambda_j^{\delta_j} - \frac{\delta_j \epsilon}{2c_V} = (\lambda'_j - \lambda_j) - \delta_j ((\lambda^{P_j} - \lambda_j) - \frac{\epsilon}{2c_V}),$$

where $\lambda^{P_j} := \nu(\sigma^* P_j, \widehat{V}) = \nu(P_j, V)$. Since P_j is a current in $\{\theta_j\}$, we get $\lambda^{P_j} \leq c \|\{\theta_j\}\|$ for some constant c independent of the class $\{\theta_j\}$. Combining this with the equality above, we can choose

$$(5.2.4) \quad \delta_j = (c \|\{\theta_j\}\| + \frac{\epsilon}{2c_V})^{-1} \frac{\lambda'_j - \lambda_j}{2}.$$

LEMMA 5.6. $\sigma^*T'_j - (\lambda_j^{\delta_j} + \frac{\delta_j\epsilon}{2c_V})[\widehat{V}]$ is a closed positive current and is more singular than $\tilde{\eta}_j^{\delta_j}$.

PROOF. Since T_j is less singular than T'_j , we get

$$\sigma^*T'_j - \lambda_j^{\delta_j}[\widehat{V}] \preceq P(\sigma^*T_j - \lambda_j^{\delta_j}[\widehat{V}]).$$

Note that $\sigma^*T'_j - \lambda_j^{\delta_j}[\widehat{V}] = \eta'_j + (\lambda'_j - \lambda_j^{\delta_j})[\widehat{V}]$ is a closed positive current. This is because $\lambda'_j - \lambda_j^{\delta_j} > \lambda'_j - \lambda_j^{\delta_j} - \frac{\delta_j\epsilon}{2c_V} > 0$. By a similar process as above, we see that $\sigma^*T'_j - (\lambda_j^{\delta_j} + \frac{\delta_j\epsilon}{2c_V})[\widehat{V}] = \eta'_j + (\lambda'_j - \lambda_j^{\delta_j} - \frac{\delta_j\epsilon}{2c_V})[\widehat{V}]$ is a closed positive current, and satisfied

$$\sigma^*T'_j - (\lambda_j^{\delta_j} + \frac{\delta_j\epsilon}{2c_V})[\widehat{V}] \preceq P(P(\sigma^*T_j - \lambda_j^{\delta_j}[\widehat{V}]) - \frac{\delta_j\epsilon}{2c_V}[\widehat{V}]) = \tilde{\eta}_j^{\delta_j}.$$

□

LEMMA 5.7. We have the following inequality.

$$\{\langle \wedge_{j=1}^m \eta'_j \rangle\} + \{(\lambda'_m - \lambda_m) \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle\} \leq \{\langle \wedge_{j=1}^m P[\sigma^*T_j] \rangle\}.$$

PROOF. By the multi-linearity (taking T to be the current of integration along X in Proposition 3.6 (4)) and the monotonicity of non-pluripolar product (Theorem 3.13), we obtain

$$\begin{aligned} & \{\langle \wedge_{j=1}^m \eta'_j \rangle\} + \{(\lambda'_m - \lambda_m) \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle\} \\ & \leq \{\langle \wedge_{j=1}^{m-1} (\eta'_j + (\lambda'_j - \lambda_j^{\delta_j} - \frac{\delta_j\epsilon}{2c_V})[\widehat{V}]) \wedge \eta'_m \rangle\} + \{(\lambda'_m - \lambda_m) \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle\} \\ & \leq \{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge \eta'_m \rangle\} + \{(\lambda'_m - \lambda_m) \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle\} \text{ (by Lemma 5.6)} \\ & = \{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge (\eta'_m + (\lambda'_m - \lambda_m)[\widehat{V}]) \rangle\} \\ & \leq \{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge \eta_m \rangle\} \\ & = \{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge \eta_m \rangle\} \text{ (by Lemma 5.4)} \end{aligned}$$

Now, we show that $\{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge \eta_m \rangle\} \leq \{\langle \wedge_{j=1}^m P[\sigma^*T_j] \rangle\}$. Let

$$\tilde{\theta}_j^{\delta_j} := \sigma^*\theta_j + (\lambda_j^{\delta_j} + \frac{\delta_j\epsilon}{2c_V})\omega_h.$$

We write $\tilde{\eta}_j^{\delta_j} = dd^c \tilde{u}_j^{\delta_j} + \tilde{\theta}_j^{\delta_j}$, $\tilde{u}_j^{\delta_j} \in \text{PSH}(\widehat{X}, \tilde{\theta}_j^{\delta_j})$, and recall that

$$P(\sigma^* T_j - \lambda_j^{\delta_j} [\widehat{V}]) = dd^c u_j^p + (\tilde{\theta}_j^{\delta_j} - \frac{\delta_j \epsilon}{2c_V} \omega_h),$$

where

$$\begin{aligned} \tilde{u}_j^{\delta_j} &= (\sup\{v \in \text{PSH}(\widehat{X}, \tilde{\theta}_j^{\delta_j}) \mid v + \frac{\delta_j \epsilon}{2c_V} u_{\widehat{V}} \leq u_j^p\})^*, \\ u_j^p &= (\sup\{v \in \text{PSH}(\widehat{X}, \tilde{\theta}_j^{\delta_j} - \frac{\delta_j \epsilon}{2c_V} \omega_h) \mid v + \lambda_j^{\delta_j} u_{\widehat{V}} \leq \sigma^* u_j\})^*. \end{aligned}$$

We observe that $[\tilde{u}_j^{\delta_j} + \frac{\delta_j \epsilon}{2c_V} u_{\widehat{V}}] \preceq [P[u_j^p]]$ and $[u_j^p + \lambda_j^{\delta_j} u_{\widehat{V}}] \preceq [P[\sigma^* u_j]]$. This implies

$$\begin{aligned} (5.2.5) \quad [\tilde{u}_j^{\delta_j} + (\frac{\delta_j \epsilon}{2c_V} + \lambda_j^{\delta_j}) u_{\widehat{V}}] &\preceq [P[u_j^p] + \lambda_j^{\delta_j} u_{\widehat{V}}] \preceq [P[u_j^p + \lambda_j^{\delta_j} u_{\widehat{V}}]] \\ &\preceq [P[P[\sigma^* u_j]]] = [P[\sigma^* u_j]]. \end{aligned}$$

In other words, $\tilde{\eta}_j^{\delta_j} + (\frac{\delta_j \epsilon}{2c_V} + \lambda_j^{\delta_j}) [\widehat{V}] \preceq P[\sigma^* T_j]$. By the multi-linearity and the monotonicity of non-pluripolar product again, we get

$$\begin{aligned} \{\langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge \eta_m \rangle\} &\leq \{\langle \wedge_{j=1}^{m-1} (\tilde{\eta}_j^{\delta_j} + (\frac{\delta_j \epsilon}{2c_V} + \lambda_j^{\delta_j}) [\widehat{V}]) \wedge \eta_m \rangle\} \\ &\leq \{\langle \wedge_{j=1}^m P[\sigma^* T_j] \rangle\} \end{aligned}$$

□

By Lemma 5.7, Lemma 5.5 and Proposition 2.15, we now have

$$\begin{aligned} &\int_X \langle \wedge_{j=1}^m T_j' \rangle \wedge \omega^{n-m} \\ &= \int_{\widehat{X}} \langle \wedge_{j=1}^m \eta_j' \rangle \wedge (\sigma^* \omega)^{n-m} \\ &\leq \int_{\widehat{X}} \langle \wedge_{j=1}^m P[\sigma^* T_j] \rangle \wedge (\sigma^* \omega)^{n-m} - (\lambda_m' - \lambda_m) \int_{\widehat{X}} \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle \wedge (\sigma^* \omega)^{n-m} \\ &= \int_{\widehat{X}} \langle \wedge_{j=1}^m \sigma^* T_j \rangle \wedge (\sigma^* \omega)^{n-m} - (\lambda_m' - \lambda_m) \int_{\widehat{X}} \langle \wedge_{j=1}^{m-1} \tilde{\eta}_j^{\delta_j} \wedge [\widehat{V}] \rangle \wedge (\sigma^* \omega)^{n-m} \\ (5.2.6) \quad &\leq \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m} - (\lambda_m' - \lambda_m) \int_{\widehat{X}} \langle \wedge_{j=1}^{m-1} Q_j^{\delta_j} \wedge [\widehat{V}] \rangle \wedge (\sigma^* \omega)^{n-m} \end{aligned}$$

Recall that $Q_j^{\delta_j} = \eta_j^{\delta_j} + \frac{\delta_j \epsilon}{2c_V} \omega_h \geq \frac{\delta_j \epsilon}{2c_V} \hat{\omega}$ and $\sigma^* P_j^{\delta_j} = \lambda_j^{\delta_j} [\hat{V}] + \eta_j^{\delta_j}$. Since $P_j^{\delta_j}$ is of analytic singularities, so do $\eta_j^{\delta_j}$ and $Q_j^{\delta_j}$. This combines with the fact that $[\hat{V}] \not\subseteq I_{\eta_j^{\delta_j}}$, induces that $[\hat{V}]$ has no mass on $I_{Q_j^{\delta_j}}$. Therefore, we can apply Proposition 3.6 and get

$$\begin{aligned} \int_{\hat{X}} \langle \wedge_{j=1}^{m-1} Q_j^{\delta_j} \wedge [\hat{V}] \rangle \wedge (\sigma^* \omega)^{n-m} &\geq \prod_{j=1}^{m-1} \left(\frac{\delta_j \epsilon}{2c_V} \right) \int_{\hat{X}} \langle \hat{\omega}^{m-1} \wedge [\hat{V}] \rangle \wedge (\sigma^* \omega)^{n-m} \\ (5.2.7) \quad &= \prod_{j=1}^{m-1} \left(\frac{\delta_j \epsilon}{2c_V} \right) \int_{\hat{X}} [\hat{V}] \wedge (\sigma^* \omega)^{n-m} \wedge \hat{\omega}^{m-1} \neq 0 \end{aligned}$$

Combining (5.2.6) and (5.2.7), we get

$$\begin{aligned} (5.2.8) \quad &\int_X \langle \wedge_{j=1}^m T_j' \rangle \wedge \omega^{n-m} \\ &\leq \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m} - (\lambda'_m - \lambda_m) \prod_{j=1}^{m-1} \left(\frac{\delta_j \epsilon}{2c_V} \right) \int_{\hat{X}} [\hat{V}] \wedge (\sigma^* \omega)^{n-m} \wedge \hat{\omega}^{m-1}. \end{aligned}$$

Step 2. Now, we remove the analytic singularities assumption of T_j and P_j . Here, we note that the existence of the Kähler current $P_j \preceq T_j$ follows from [15, Proposition 3.6]. We apply Demailly's approximation theorem (Theorem 3.18) on the potentials of T_j , T_j' and P_j (u_j , u_j' and p_j), and get sequences $u_{j,k}^D, u_{j,k}'^D, p_{j,k}^D \in \text{PSH}(X, \theta_j + \epsilon_k \omega)$, where ϵ_k decreases to 0 such that

- (1) $u_{j,k}^D \searrow u_j, u_{j,k}'^D \searrow u_j'$, and $p_{j,k}^D \searrow p_j$.
- (2) $u_{j,k}^D, u_{j,k}'^D$, and $p_{j,k}^D$ have analytic singularities.
- (3) $\nu(T_{j,k}^D, x) \rightarrow \nu(T_j, x)$, $\nu(T_{j,k}'^D, x) \rightarrow \nu(T_j', x)$ and $\nu(p_{j,k}^D, x) \rightarrow \nu(p_j, x)$ uniformly on X , where

$$\begin{aligned} T_{j,k}^D &= dd^c u_{j,k}^D + (\theta_j + \epsilon_k \omega) \\ T_{j,k}'^D &= dd^c u_{j,k}'^D + (\theta_j + \epsilon_k \omega) \\ p_{j,k}^D &= dd^c p_{j,k}^D + (\theta_j + \epsilon_k \omega) \end{aligned}$$

By Lemma 3.19, the ordering of the singularity types is preserved after applying Demailly's approximation theorem. To be more precise, for each $k \in \mathbb{N}$, we have $[u_{j,k}^D] \succeq [u_{j,k}'^D], [p_{j,k}^D]$. Since P_j is a Kähler current for each j , there exists

$\epsilon > 0$ such that $P_j \geq \epsilon\omega$ for all j . By the construction of $P_{j,k}^D$, one sees that $P_{j,k}^D$ is also a Kähler current and satisfied $P_{j,k}^D \geq (\epsilon - \epsilon_k)\omega$.

For $1 \leq j \leq m$, set $\lambda_{j,k} := \nu(T_{j,k}^D, V)$, $\lambda'_{j,k} := \nu(T'_{j,k}, V)$, and let

$$\delta_{j,k} = (c\|\{\theta_j + \epsilon_k\omega\}\| + \frac{\epsilon}{2c_V})^{-1} \frac{\lambda'_{j,k} - \lambda_{j,k}}{2},$$

where c_V, c are constants in (5.2.4). Now, we apply (5.2.8) in **step 1**, then we obtain the following.

$$\begin{aligned} (5.2.9) \quad & \int_X \langle \wedge_{j=1}^m T'_{j,k} \rangle \wedge \omega^{n-m} \\ & \leq \int_X \langle \wedge_{j=1}^m T_{j,k}^D \rangle \wedge \omega^{n-m} \\ & \quad - (\lambda'_{m,k} - \lambda_{m,k}) \prod_{j=1}^{m-1} \left(\frac{\delta_{j,k}(\epsilon - \epsilon_k)}{2c_V} \right) \int_{\widehat{X}} [\widehat{V}] \wedge (\sigma^*\omega)^{n-m} \wedge \widehat{\omega}^{m-1}. \end{aligned}$$

By the monotonicity property of non-pluripolar product (Theorem 3.13), we have

$$(5.2.10) \quad \int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \omega^{n-m} \leq \int_X \langle \wedge_{j=1}^m T'_{j,k} \rangle \wedge \omega^{n-m}.$$

Since we assume that T_j is $\mathcal{I}$ -model for $1 \leq j \leq m$, Lemma 3.21 induces

$$\begin{aligned} (5.2.11) \quad & \int_X \langle \wedge_{j=1}^m T_{j,k}^D \rangle \wedge \omega^{n-m} = \int_X \langle \wedge_{j=1}^m (\theta_j + \epsilon_k\omega)_{u_{j,k}^D} \rangle \wedge \omega^{n-m} \\ & \searrow \int_X \langle \wedge_{j=1}^m (\theta_j)_{P[u_j]_{\mathcal{I}}} \rangle \wedge \omega^{n-m} \\ & = \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m}, \quad k \rightarrow \infty. \end{aligned}$$

Combining (5.2.9), (5.2.10), (5.2.11) and let $k \rightarrow \infty$. Then, we get (5.2.8)

$$\begin{aligned} & \int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \omega^{n-m} \\ & \leq \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m} - (\lambda'_m - \lambda_m) \prod_{j=1}^{m-1} \left(\frac{\delta_j \epsilon}{2c} \right)^{m-1} \int_{\widehat{X}} [\widehat{V}] \wedge (\sigma^*\omega)^{n-m} \wedge \widehat{\omega}^{m-1} \end{aligned}$$

for the general case. Finally, since we assume that $T'_1, \dots, T'_m$ is of relative full mass intersection with respect to $T_1, \dots, T_m$, the inequality above does not hold, and this makes a contradiction. Therefore, $\lambda'_j - \lambda_j = \nu(T'_j, V) - \nu(T_j, V) = 0$ for some $j = 1, \dots, m$.

Step 3. We get rid of the assumption that V is smooth. By using Hironaka's desingularization method (see [25] or [34, Theorem 2.1.13]), we get $\sigma': X' \rightarrow X$, which is a composition of finite many blow-ups along smooth centers, such that $V' = \sigma'^{-1}(V)$ is smooth.

Let $R_j = (\sigma')^* T_j$ and $R'_j = (\sigma')^* T'_j$. In general, $R'_1, \dots, R'_m$ do not necessarily have full mass intersection with respect to $R_1, \dots, R_m$. However, we have

$$\begin{aligned} \int_{X'} \langle \wedge_{j=1}^m R'_j \rangle \wedge ((\sigma')^* \omega)^{n-m} &= \int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \omega^{n-m} \\ &= \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m} \\ &= \int_{X'} \langle \wedge_{j=1}^m R_j \rangle \wedge ((\sigma')^* \omega)^{n-m}. \end{aligned}$$

Since $[V] \wedge \omega^{n-m} \neq 0$, it follows that $[V'] \wedge (\sigma'^*(\omega))^{n-m} \neq 0$. Hence, we can apply **step 1.** and **step 2.** in this setting, and get

$$\nu(T'_j, V) = \nu(R'_j, V') = \nu(R_j, V') = \nu(T_j, V)$$

for some j . This completes the proof of Theorem 5.1.

5.3. Proof of Theorem 5.2

Let $\mathcal{B}$ be a closed cone in the cone of big classes, and let $V = \{x_0\}$ be a point in X . First, note that to prove inequality (5.1.1), it suffices to consider $\{\theta_j\} \in \mathcal{S} \cap \mathcal{B}$ for $j = 1, \dots, m$, where $\mathcal{S}$ is the unit sphere in $H^{1,1}(X, \mathbb{R})$. Recall that in (5.2.8), we choose

$$\delta_j = (c\|\{\theta_j\}\| + \frac{\epsilon}{2c_V})^{-1} \frac{\lambda'_j - \lambda_j}{2},$$

for $j = 1, \dots, m$. Here, ϵ is derived from the Kähler currents $P_j \in \{\theta_j\}$ such that $P_j \geq \epsilon\omega$. Since $\mathcal{S} \cap \mathcal{B}$ is compact, we can choose ϵ to be independent of $\{\theta_j\} \in \mathcal{S} \cap \mathcal{B}$. Also, by the construction of $\widehat{X}$, the constant c_V in (5.2.2) is independent of $V = \{x_0\}$. Therefore, (5.2.8) become

$$\int_X \langle \wedge_{j=1}^n T_j \rangle - \langle \wedge_{j=1}^n T'_j \rangle \geq C \prod_{j=1}^n (\nu(T_j, x_0) - \nu(T, x_0)),$$

where $C = \prod_{j=1}^{n-1} (2c\|\theta_j\| + \frac{\epsilon}{c_V})^{-1} \text{vol}(\widehat{V})$ is a constant that depends only on the cone $\mathcal{B}$ and X . This completes the proof.

REMARK 5.8. Note that in Theorem 5.1 and Theorem 5.2, the assumption that T_j is $\mathcal{I}$ -model for $j = 1, \dots, m$ is too strong. Instead, we only need to assume that

$$\int_X \langle \wedge_{j=1}^m (\theta_j)_{P^{\theta_j}[u_j]_{\mathcal{I}}} \rangle \wedge \omega^{n-m} = \int_X \langle \wedge_{j=1}^m (\theta_j)_{u_j} \rangle \wedge \omega^{n-m} = \int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \omega^{n-m}.$$

5.4. Further discussion

Let T and T' be closed positive $(1, 1)$ -currents in the same cohomology class, with T' being less singular than T . In this section, we discuss some results concerning the case where T has full mass intersection with respect to T' .

THEOREM 5.9. *Let X be a compact Kähler manifold of dimension n . Let $\{\theta_1\}, \dots, \{\theta_m\}$ be big classes, and let $T_j, T'_j \in \{\theta_j\}$ be closed positive $(1, 1)$ -currents such that T'_j is less singular than T_j , and T_j is big, that is $\int_X \langle T_j^n \rangle > 0$, for $1 \leq j \leq m$. Let $V \subseteq X$ be an analytic subset of dimension p with $m \leq p$, and assume that V is not contained in $\bigcup_{j=1}^m I_{T_j}$. Assume that T_j has full mass intersection with respect to T'_j for each j , that is,*

$$\|\langle T_j^n \rangle\| = \|\langle T'_j{}^n \rangle\|, \quad \text{for } 1 \leq j \leq m.$$

Then, we have

$$\int_X \langle T_1 \wedge \dots \wedge T_m \wedge [V] \rangle \wedge \omega^{p-m} = \int_X \langle T'_1 \wedge \dots \wedge T'_m \wedge [V] \rangle \wedge \omega^{p-m}.$$

PROOF. By Theorem 3.13, we have

$$\int_X \langle T_1 \wedge \dots \wedge T_m \wedge [V] \rangle \wedge \omega^{p-m} \leq \int_X \langle T'_1 \wedge \dots \wedge T'_m \wedge [V] \rangle \wedge \omega^{p-m}.$$

We now show that the above inequality is actually an equality. First, we write

$$T_j = dd^c u_j + \theta_j, \quad T'_j = dd^c u'_j + \theta_j,$$

for $1 \leq j \leq m$, where $u_j, u'_j \in \text{PSH}(X, \theta_j)$. By Lemma 3.9, we have

$$\begin{aligned} \langle T_1 \wedge \dots \wedge T_m \wedge [V] \rangle &= (i)_* \langle T_1|_{\text{Reg}(V)} \wedge \dots \wedge T_m|_{\text{Reg}(V)} \rangle, \\ \langle T'_1 \wedge \dots \wedge T'_m \wedge [V] \rangle &= (i)_* \langle T'_1|_{\text{Reg}(V)} \wedge \dots \wedge T'_m|_{\text{Reg}(V)} \rangle, \end{aligned}$$

here $i: V \rightarrow X$ is the inclusion map, and

$$T_j|_{\text{Reg}(V)} = dd^c u_j|_{\text{Reg}(V)} + \theta_j|_{\text{Reg}(V)}, \quad T'_j|_{\text{Reg}(V)} = dd^c u'_j|_{\text{Reg}(V)} + \theta_j|_{\text{Reg}(V)},$$

$1 \leq j \leq m$. By [13, Lemma 4.3], for any $b_j \in (1, \infty)$, there exists $v_j \in \text{PSH}(X, \theta_j)$ such that

$$b_j^{-1}v_j + (1 - b_j^{-1})u'_j \leq u_j.$$

By the multi-linearity and monotonicity of the non-pluripolar product (Proposition 3.6 (4) and Theorem 3.13), we get

$$\begin{aligned} & \left\| \prod_{j=1}^m (1 - b_j^{-1}) \left\langle \wedge_{j=1}^m (dd^c u'_j|_{\text{Reg}(V)} + \theta_j|_{\text{Reg}(V)}) \right\rangle \right\| \\ & \leq \left\| \left\langle \wedge_{j=1}^m (dd^c u_j|_{\text{Reg}(V)} + \theta_j|_{\text{Reg}(V)}) \right\rangle \right\|. \end{aligned}$$

Let $b_j \rightarrow \infty$. Then we obtain the desired result. $\square$

CHAPTER 6

Singularities of $(n - 1, n - 1)$ -classes

6.1. Introduction

This chapter is based on the article [38], using a method that is slightly different from the one presented there. Let X be a compact Kähler manifold of dimension n . In this chapter, we study the singularities of non-pluripolar (m, m) -class, $m \leq n$, where we say that a (m, m) -class β is *non-pluripolar* if

$$\beta = \{\langle T_1 \wedge \cdots \wedge T_m \rangle\},$$

for some closed positive $(1, 1)$ -currents $T_1, \dots, T_m$. In particular, we denote and define

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle := \langle T_{\min, 1} \wedge \cdots \wedge T_{\min, m} \rangle,$$

where $T_{\min, j}$ is a current with minimal singularities in the pseudoeffective class α_j . Note that the definition does not depend on the choice of current with minimal singularities, thanks to Remark 3.17. We refer the reader to [8, 51] for more details.

The following is our first result.

THEOREM 6.1. ([38, Theorem 3.2]) *Let X be a compact Kähler manifold of dimension n . Let $m \leq n$ be a positive integer. For $1 \leq j \leq m$, let T_j, T'_j be closed positive $(1, 1)$ -currents on X such that: T_j, T'_j are in the same cohomology class and in the same singularity type. Then*

$$v(\langle T_1 \wedge \cdots \wedge T_m \rangle, x) = v(\langle T'_1 \wedge \cdots \wedge T'_m \rangle, x)$$

for every $x \in X$.

The above Theorem allows us to define the Lelong number of non-pluripolar cohomology class. Let $T_1, \dots, T_m$ be closed positive $(1, 1)$ -currents on X , and set $\beta := \langle T_1 \wedge \cdots \wedge T_m \rangle$. We define

$$v(\beta, x) := v(\langle T_1 \wedge \cdots \wedge T_m \rangle, x).$$

Let $\alpha_1, \dots, \alpha_m$ be pseudoeffective $(1, 1)$ -classes. We define

$$\nu(\langle \wedge_{j=1}^m \alpha_j \rangle, x) := \nu(\langle \wedge_{j=1}^m T_{\min, j}^m \rangle, x).$$

The second main result of this chapter is a comparison between the cup products and the relative non-pluripolar products. For a related result, we refer the reader to Theorem 3.26. We show that the Lelong number of a non-pluripolar cohomology class serves as the obstruction to the equality of these two products.

THEOREM 6.2. ([38, Theorem 1.1]) *Let X be a compact Kähler manifold of dimension n . Let $T_1, \dots, T_{n-1}$, and T be closed positive $(1, 1)$ -currents on X . Put*

$$\beta := \{\langle \wedge_{j=1}^{n-1} T_j \rangle\}, \gamma := \{\langle \wedge_{j=1}^{n-1} T_j \wedge T \rangle\}.$$

Then we have $\beta \wedge \{T\} \geq \gamma$, where $\wedge$ denotes the cup product. Moreover, if

$$\beta \wedge \{T\} = \gamma,$$

then $\nu(\beta, x) \cdot \nu(T, x) = 0$ for every $x \in X$.

Consider a compact Kähler surface Y and β a big cohomology class on Y . Let $T_{\min, \beta}$ be a current of minimal singularities in β . It is well-known that we can decompose $T_{\min, \beta} = T_1 + T_2$, where T_1 is an effective divisor and T_2 is a closed positive current in a nef class. Correspondingly we obtain $\beta = \{T_1\} + \{T_2\}$. Such a formula is called Zariski's decomposition; see [59]. Observe that $\{T_2\}$ is equal to $\langle \beta \rangle$. In the higher dimensional setting, an analogue of this decomposition also holds but the class $\{T_2\}$ is only nef in codimension 1, see [6] and also [58] for more information. An exact higher dimensional generalization of Zariski's decomposition is not true; *i.e.*, $\{T_2\}$ is not nef in general; see [6, A.2] or [37]. We refer to [35] for an equivalence between the existence of Zariski's decomposition and properties of restricted volumes, and to [31] for a version of Zariski's decomposition for $(n-1, n-1)$ -classes.

Let α be a pseudoeffective $(1, 1)$ -class. It is well-known that $\langle \alpha \rangle$ is nef if and only if $\nu(\langle \alpha \rangle, x) = 0$ for every $x \in X$; see [6]. As an application of the comparison theorem (Theorem 6.2), we get the following generalized Zariski decomposition for the higher-dimensional case.

THEOREM 6.3. ([38, Corollary 1.2]) *Let X be a projective manifold. Let α be a big cohomology. Then we have the following.*

$$(6.1.1) \quad \nu(\langle \alpha^{n-1} \rangle, x) = 0 \text{ for every } x \in X.$$

6.2. Lelong number of (m, m) -class

In this section, our aim is to prove Theorem 6.1. First, we recall the definition of pull back of closed positive $(1, 1)$ -currents. Let T be a closed positive $(1, 1)$ -current on a complex manifold Y . Let $\pi : X \rightarrow Y$ be a holomorphic map. Suppose that $\pi(X)$ is not contained in the singular set of T . Let φ be a local potential of T . Then we can define the pull back π^*T locally by $dd^c(\pi \circ \varphi)$. This defines a closed positive $(1, 1)$ -current on X and we have $\{\pi^*T\} = \pi^*\{T\}$.

Let $x \in X$, and let $\sigma : \hat{X} \rightarrow X$ be the blow-up of X at x . Let T be a closed positive (p, p) -current on X . Let T_1 be the pull back of T to $X \setminus \sigma^{-1}(x)$, which can be extended by 0 through the hypersurface $\sigma^{-1}(x) \simeq \mathbb{P}^{n-1}$. This is called the strict transform of T , and is denoted by $\sigma^\diamond(T)$, which is a closed positive (p, p) -current on $\hat{X}$. In general, $\{\sigma^\diamond(T)\}$ is not equal to $\sigma^*\{T\}$. By using Siu's description of Lelong numbers ([41]), we get that

$$(6.2.1) \quad \sigma^*\{T\} - \{\sigma^\diamond(T)\} = \nu(T, x)\{[H]\},$$

where H is a $(n - p)$ -dimensional linear subspace of the exceptional divisor $\sigma^{-1}(x) \simeq \mathbb{P}^{n-1}$.

PROOF OF THEOREM 6.1. Let $x \in X$ and $\sigma : \hat{X} \rightarrow X$ be the blow-up of X at x . Let $T = \langle T_1 \wedge \cdots \wedge T_m \rangle$ and $T' = \langle T'_1 \wedge \cdots \wedge T'_m \rangle$.

By (6.2.1), one remains to show that $\{T\} = \{T'\}$ and $\{\pi^\diamond(T)\} = \{\pi^\diamond(T')\}$. By Remark 3.17, it is clear that $\{T\} = \{T'\}$. We consider the non-pluripolar product $\langle \sigma^*T_1 \wedge \cdots \wedge \sigma^*T_m \rangle$. Since $\sigma^{-1}(x)$ is an analytic set, $\langle \sigma^*T_1 \wedge \cdots \wedge \sigma^*T_m \rangle$ puts no mass on $\pi^{-1}(x)$. Moreover, since σ is a bi-holomorphic map outside $\sigma^{-1}(x)$, $\langle \sigma^*T_1 \wedge \cdots \wedge \sigma^*T_m \rangle = \sigma^*T$ on $\hat{X} \setminus \sigma^{-1}(x)$. Thus, we get $\langle \sigma^*T_1 \wedge \cdots \wedge \sigma^*T_m \rangle = \pi^\diamond(T)$. Similarly, $\langle \sigma^*T'_1 \wedge \cdots \wedge \sigma^*T'_m \rangle = \sigma^\diamond(T')$. For $j = 1, \dots, m$, we have $\{\sigma^*T_j\} = \sigma^*\{T_j\} = \pi^*\{T'_j\} = \{\sigma^*T'_j\}$. By construction, we also have σ^*T_j and $\sigma^*T'_j$ are in the same singularity type. Thus, by Remark 3.17, $\{\sigma^\diamond(T)\} = \{\sigma^\diamond(T')\}$. We complete the proof. $\square$

In the self-intersection case, the following more general result holds, which shows that only the full mass intersection condition is needed.

THEOREM 6.4. *Let α be a big class. Let $T, T' \in \alpha$ be closed positive $(1, 1)$ -current such that T' is less singular than T . If $\int_X \langle T^n \rangle = \int_X \langle T'^n \rangle > 0$. Then, we get $\nu(\langle T^m \rangle, x) = \nu(\langle T'^m \rangle, x)$ for any $m \leq n$, and $x \in X$.*

PROOF. Let $P[T]$ be the envelope of T . We first claim that

$$\nu(\langle T^m \rangle, x) = \nu(\langle P[T]^m \rangle, x).$$

Let $\sigma: \widehat{X} \rightarrow X$ be the blow-up of X at x . Similarly to the proof of Theorem 6.1, we only need to show that $\{\langle T^m \rangle\} = \{\langle P[T]^m \rangle\}$ and $\{\langle \sigma^* T^m \rangle\} = \{\langle \sigma^* P[T]^m \rangle\}$. By Proposition 2.15, we have $\{\langle T^m \rangle\} = \{\langle P[T]^m \rangle\}$. On the other hand, by the facts that $\sigma^* P[T]$ is less singular than $\sigma^* T$, $P[\sigma^* T]$ is less singular than $\sigma^* P[T]$, and $\{\langle \sigma^* T^m \rangle\} = \{\langle P[\sigma^* T]^m \rangle\}$, we get $\{\langle \sigma^* T^m \rangle\} = \{\langle \sigma^* P[T]^m \rangle\}$. Hence, we now have $\nu(\langle T^m \rangle, x) = \nu(\langle P[T]^m \rangle, x)$. Since $P[T]$ and $P[T']$ are in the same singularity type (Theorem 2.16), by Theorem 6.1, the following holds.

$$\nu(\langle T^m \rangle, x) = \nu(\langle P[T]^m \rangle, x) = \nu(\langle P[T']^m \rangle, x) = \nu(\langle T'^m \rangle, x).$$

□

6.3. Volume of cohomology classes

In this section, we recall some regularity properties of the volume function of big cohomology classes.

Let X be a compact Kähler manifold of dimension n . Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class. We define the *volume of α* to be $\int_X \langle \alpha^n \rangle$, and denoted by $\text{vol}(\alpha)$. If α is the first Chern class of a line bundle $L \rightarrow X$, then

$$\text{vol}(\alpha) = \limsup_{k \rightarrow \infty} \frac{n!}{k^n} \dim H^0(X, L^k) = \lim_{k \rightarrow \infty} \frac{n!}{k^n} \dim H^0(X, L^k).$$

See [5, 20]. The study of the regularity of volume function plays an important role in complex geometry reference, see [9, 10, 54]. Here, we recall the following important result due to Witt Nyström [54].

THEOREM 6.5. ([54]) *Let X be a projective manifold of dimension n . Let $\alpha, \gamma \in H^{1,1}(X, \mathbb{R})$ such that α is big. Then we have*

$$\frac{d}{dt} \Big|_{t=0} \text{vol}(\alpha + t\gamma) = n \langle \alpha^{n-1} \rangle \wedge \gamma.$$

We refer to similar statements [9, 30] for the case where α is integral (and X is projective), and also to [7] for the case where X is Hyperkähler. It was conjectured in [7] that Theorem 6.5 is true for every compact Kähler manifold X . In this general setting, only partial derivatives along divisors of the volume functions were known. Let us recall this result in what follows.

Let V be a smooth submanifold of dimension k in X such that V is not contained in the non-Kähler locus of α . The *restricted volume* $\text{vol}_{X|V}(\alpha)$ of a big class α to V is defined as $\int_V \langle (T_{\min, \alpha}|_V)^k \rangle$, where $T_{\min, \alpha}$ is a current with minimal singularities in α . Since V is not contained in the non-Kähler locus of α , one sees that $\text{vol}_{X|V}(\alpha)$ is equal to the supremum of the integrals $\int_V \langle (T|_V)^k \rangle$ for all Kähler currents $T \in \alpha$ with analytic singularities; see [56, 35, 26]. We have the following formula for the partial derivative of the volume function along divisors also due to Witt Nyström.

THEOREM 6.6. ([56, Theorem C]) *Let X be a compact Kähler manifold of dimension n . For every big class $\alpha \in H^{1,1}(X, \mathbb{R})$, for every smooth hypersurface D which is not contained in the non-Kähler locus of α , and $\gamma := \{D\}$ (where $\{D\}$ denotes the cohomology class of $[D]$), there holds*

$$(6.3.1) \quad \frac{d}{dt} \Big|_{t=0} \text{vol}(\alpha + t\gamma) = n \text{vol}_{X|D}(\alpha).$$

We refer to ([52]) for a more general statement where a formula similar to (6.3.1) is proved for any hypersurface D on X . However, for what follows, Theorem 6.6 is sufficient. By [10, (8.5)], we have

$$\text{vol}_{X|D}(\alpha) \leq \langle \alpha^{n-1} \rangle \wedge \gamma,$$

for $\gamma = \{D\}$. It is still not known whether we have the equality in the general case (where X is only compact Kähler manifold). On the other hand, when X is projective, by Theorems 6.5 and 6.6, it is necessarily true that

$$(6.3.2) \quad \text{vol}_{X|D}(\alpha) = \langle \alpha^{n-1} \rangle \wedge \gamma,$$

for $\gamma = \{D\}$. Note that by Lemma 3.9, we have

$$\text{vol}_{X|D}(\alpha) = \int_X \langle T_{\min, \alpha}^{n-1} \wedge [D] \rangle.$$

By this and (6.3.2), we obtain the following.

$$(6.3.3) \quad \int_X \langle T_{\min, \alpha}^{n-1} \wedge [D] \rangle = \langle \alpha^{n-1} \rangle \wedge \gamma,$$

This is exactly one of the main points that we exploit in the proof of Theorem 6.3

6.4. Proof of Theorem 6.2 and 6.3

For the proof of Theorem 6.2, we first need the following comparison theorem between the density currents and the relative non-pluripolar products.

THEOREM 6.7. ([38, Theorem 4.5]) *Let X be a compact Kähler manifold of dimension n . Let $T_1, \dots, T_m$, and T be closed positive $(1, 1)$ -currents with $m + 1 \leq n$. Then every density current of $\langle \wedge_{j=1}^m T_j \rangle$, T has minimal h -dimension. Moreover, let R_∞ be a density current of T , $\langle \wedge_{j=1}^m T_j \rangle$, we have $R_\infty = \pi^*(R'_\infty)$ for some closed positive current R'_∞ on X , with*

$$\langle \wedge_{j=1}^m T_j \wedge T \rangle \leq R'_\infty,$$

where $\pi: E \rightarrow \Delta_2$ is the projection of the normal bundle of the diagonal $\Delta_2 \subseteq X^2$.

Before we start the proof of Theorem 6.7, we first review the following result, which is a special case of [27, Theorem 3.1].

THEOREM 6.8. ([27, Theorem 3.1]) *Let P, T be closed positive currents on some open domain $\Omega \subseteq \mathbb{C}^n$ of bi-degree $(1, 1)$ and (m, m) , respectively, where $m + 1 \leq n$. We write $P = dd^c u$, where u is a psh function on Ω . Suppose that there exists a closed positive $(m + 1, m + 1)$ -current R on Ω such that, for any sequence of smooth psh functions u_ℓ decreasing to u , we have*

$$dd^c u_\ell \wedge T \rightarrow R.$$

Then, we define $dd^c u \wedge T := R$. In this case, the Dinh-Sibony product of P and T is well-defined, and we have the following.

$$P \frown T = P \wedge T.$$

Now, let us start by considering the case of bounded potential.

Let $T_1, \dots, T_m, T$ be closed positive $(1, 1)$ -currents on X , where each T_j is of bounded potential, and $m + 1 \leq n$. Let $\Omega \subseteq X$ be a holomorphic coordinate, we write

$$T_j = dd^c u_j, \quad T = dd^c u,$$

where u_j is bounded on Ω .

Since u_j is bounded for $1 \leq j \leq m$, we see that u is locally integrable with respect to $\bigwedge_{j=1}^m T_j$. Hence, the product $T \wedge \left(\bigwedge_{j=1}^m T_j \right)$ is well-defined in the classical sense. Moreover, the currents T and $\bigwedge_{j=1}^m T_j$ satisfy the assumptions of Theorem 6.8.

Let ψ be a bounded positive psh function on Ω . Set

$$\mathbf{T} = T \otimes \left(\psi \bigwedge_{j=1}^m T_j \right).$$

Consider the local admissible map

$$(6.4.1) \quad \begin{aligned} \tau: \Omega &\rightarrow \mathbb{C}^n \times \Omega, \\ (x^1, x^2) &\longmapsto (x^1 - x^2, x^2). \end{aligned}$$

We set $(y^1, y^2) = (x^1 - x^2, x^2)$. Let $\pi: \mathbb{C}^n \times \Omega \rightarrow \Omega$ be the projective map. For $1 \leq j \leq m$, set $\tilde{u}_j(y^1, y^2) = \pi^* u_j = u_j(y^2)$, and set $\tilde{u}(y^1, y^2) = u(y^1 + y^2)$. Let $\tilde{\psi}(y^1, y^2) = \pi^* \psi$. Note that we have

$$\tau_*(\mathbf{T}) = \tau_* \left(T \otimes \left(\psi \bigwedge_{j=1}^m T_j \right) \right) = \tilde{\psi} dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right).$$

Let $\lambda \in \mathbb{C}^*$. By using the regularizations of ψ and u , one sees that

$$(A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \bigwedge_{j=1}^m dd^c \tilde{u}_j \right) = \psi(y^2) dd^c u(\lambda^{-1} y^1 + y^2) \wedge \left(\bigwedge_{j=1}^m dd^c u_j(y^2) \right).$$

The following lemma is similar to [28, Lemma 2.2]. We include a proof here for the reader's convenience.

LEMMA 6.9. *Let R be a limit current of the sequence*

$$(A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right).$$

Then we have

$$0 \leq R \leq \pi^* \left(\psi dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right) \right).$$

PROOF. Since $\tilde{\psi}$ is positive, the limit current R is also positive. Let ψ_ℓ be a sequence of smooth psh functions decreasing to ψ , and define $\tilde{\psi}_\ell(y^1, y^2) = \psi_\ell(y^2)$. We fix $\ell \in \mathbb{N}$. By Theorem 6.8, we have

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} (A_\lambda)_* \left(\tilde{\psi}_\ell dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right) \\ &= \lim_{\lambda \rightarrow \infty} \psi_\ell(y^2) (A_\lambda)_* \left(dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right) \\ &= \pi^* \left(\psi_\ell dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right) \right). \end{aligned}$$

Since $\psi_\ell \geq \psi$, it follows that

$$(A_\lambda)_* \left(\tilde{\psi}_\ell dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right) \geq (A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right),$$

for $\lambda \in \mathbb{C}^*$. Letting $\lambda \rightarrow \infty$, we obtain

$$\pi^* \left(\psi_\ell dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right) \right) \geq R, \quad \text{for any } \ell \in \mathbb{N}.$$

Finally, letting $\ell \rightarrow \infty$, we conclude that

$$\pi^* \left(\psi dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right) \right) \geq R.$$

This completes the proof. □

Now, we show that the inequality in Lemma 6.9 is actually equality.

LEMMA 6.10. *Under the assumptions above, the limit of*

$$(A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \bigwedge_{j=1}^m dd^c \tilde{u}_j \right)$$

is unique, and

$$(6.4.2) \quad \lim_{\lambda \rightarrow \infty} (A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \bigwedge_{j=1}^m dd^c \tilde{u}_j \right) = \pi^* \left(\psi dd^c u \wedge \bigwedge_{j=1}^m dd^c u_j \right).$$

PROOF. Consider the test form with compact support of the type $\Phi(y^1, y^2) = \Phi_1(y^1) \wedge \Phi_2(y^2)$. Since these forms generate the space of test forms, it suffices to prove (6.4.2) against $\Phi_1(y^1) \wedge \Phi_2(y^2)$. By Lemma 6.9, we only need to consider the case where $\Phi_1(y^1)$ is of full bi-degree. We can further ask $\Phi_1(y^1)$ to be a radial form with unit volume, that is, $\Phi_1(y^1) = \chi(\|y^1\|^2) i^n dy^1 \wedge d\bar{y}^1$ such that $\int \Phi_1(y^1) = 1$.

Set

$$u^\lambda(y^2) := \int_{y^1} \tilde{u}(\lambda^{-1}y^1, y^2) \wedge \Phi_1(y^1) = \int_{y^1} u(\lambda^{-1}y^1 + y^2) \wedge \Phi_1(y^1),$$

which is the convolution of $u(y^2)$, and it decreases to $u(y^2)$, as $\lambda \rightarrow \infty$. Set

$$R^\lambda := \psi dd^c u^\lambda \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right).$$

Using the regularizations of ψ and u , and applying Fubini's theorem, we obtain

$$\left\langle (A_\lambda)_* \left(\tilde{\psi} dd^c \tilde{u} \wedge \left(\bigwedge_{j=1}^m dd^c \tilde{u}_j \right) \right), \Phi_1(y^1) \wedge \Phi_2(y^2) \right\rangle = \left\langle R^\lambda, \Phi_2(y^2) \right\rangle.$$

Since u^λ decreases to u as $\lambda \rightarrow \infty$, we get

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \left\langle R^\lambda, \Phi_2(y^2) \right\rangle &= \left\langle \psi dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right), \Phi_2(y^2) \right\rangle \\ &= \left\langle \psi dd^c u \wedge \left(\bigwedge_{j=1}^m dd^c u_j \right), \pi_*(\Phi) \right\rangle \\ &= \left\langle \pi^* \left(\psi \bigwedge_{j=1}^m dd^c u_j \wedge dd^c u \right), \Phi \right\rangle. \end{aligned}$$

This completes the proof. □

We note that Lemma 6.10 also applies when T is a closed positive (p, p) -current with $p + m \leq n$ (see [38, Theorem 4.1]). However, the method used in the proof above cannot be applied in this case. Instead, we follow the approach from [49].

PROOF OF THEOREM 6.7. Since $\langle \wedge_{j=1}^m T_j \rangle$ is a non-pluripolar current, it puts no mass on $\{x : \nu(T, x) > 0\}$ (which is a pluripolar set). Thus, by Proposition 3.26, any density current of $\langle \wedge_{j=1}^m T_j \rangle$, T must have minimal h-dimension.

We now prove the second statement. Since this is a local problem, we only need to work on a holomorphic coordinate chart Ω . First, we write $T_j = dd^c u_j$, where $u_j \leq 0$ is psh function on Ω . For $k \in \mathbb{N}$,

set

$$u_{j,k} := \max\{u_j, -k\} \text{ and } T_{j,k} := dd^c u_{j,k}.$$

Since $T_{j,k}$ is of bounded potential, the classical product $\wedge_{j=1}^m T_{j,k}$ is well-defined. We set

$$R_k := T \otimes \wedge_{j=1}^m T_{j,k} \quad \text{and} \quad R := T \otimes \langle \wedge_{j=1}^m T_j \rangle.$$

For $1 \leq j \leq 2$, let $p_j: \Omega^2 \rightarrow \Omega$ be the projection from Ω^2 to the j th-component. Set

$$\begin{aligned} \rho &:= \sum_{j=1}^m u_j \quad \text{and} \quad \rho_k := k^{-1} \max\{\rho, -k\}, \\ \psi &:= p_2^* \rho \quad \text{and} \quad \psi_k := p_2^* \rho_k. \end{aligned}$$

Note that $\psi_k + 1 = 0$ on $\cup_{j=1}^m p_2^{-1}\{u_j \leq -k\}$. Hence, we get

$$\begin{aligned} -\psi_k R &= -(\psi_k + 1)R + R \\ (6.4.3) \quad &= -(\psi_k + 1)R_k + R, \end{aligned}$$

We choose the admissible map τ as in (6.4.1). By Lemma 6.10, we get that the tangent current of

$$(\psi_k + 1)R_k = T \otimes \left((\rho_k + 1) \mathbb{1}_{\cap_{j=1}^m \{u_j > -k\}} \langle \wedge_{j=1}^m T_j \rangle \right)$$

equals $\pi^*((\rho_k + 1) \wedge_{j=1}^m T_{j,k} \wedge T)$, where $\pi: E \rightarrow \Delta_2$ is the normal bundle of Δ_2 . Now, by taking the tangent current of both sides of (6.4.3), we get

$$(6.4.4) \quad R_{k,\infty} = R_\infty - \pi^* \left((\rho_k + 1) T \wedge \left(\bigwedge_{j=1}^m T_{j,k} \right) \right),$$

where $T_{k,\infty}$ and R_∞ are tangent currents of $-\psi_k R$ and R along Δ_2 , respectively. Since $\rho_k + 1 = 0$ on $\cup_{j=1}^m \{u_j \leq -k\}$, we then have

$$(\rho_k + 1) T \wedge \left(\bigwedge_{j=1}^m T_{j,k} \right) = (\rho_k + 1) \left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle,$$

which converges to $\left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle$ as $k \rightarrow \infty$. This combines with (6.4.4) implies

$$\pi^* \left(\left\langle \bigwedge_{j=1}^m T_j \wedge T \right\rangle \right) \leq R_\infty.$$

This completes the proof. $\square$

PROOF OF THEOREM 6.2. Let $x \in X$. Let S be the density current associated to $\left\langle \bigwedge_{j=1}^{n-1} T_j \right\rangle$ and T . By Theorem 6.7, we have

$$S \geq \left\langle \bigwedge_{j=1}^{n-1} T_j \wedge T \right\rangle.$$

By comparison of Lelong numbers (Corollary 2.23),

$$\nu(S, x) \geq \nu \left(\left\langle \bigwedge_{j=1}^{n-1} T_j \right\rangle, x \right) \nu(T, x).$$

It follows that

$$S \geq \left\langle \bigwedge_{j=1}^{n-1} T_j \wedge T \right\rangle + \nu \left(\left\langle \bigwedge_{j=1}^{n-1} T_j \right\rangle, x \right) \nu(T, x) \delta_x.$$

Since $\{S\} = \left\{ \left\langle \bigwedge_{j=1}^{n-1} T_j \right\rangle \right\} \wedge \{T\}$, the desired conclusion follows. $\square$

PROOF OF THEOREM 6.3. Assume that X is projective. Let $x \in X$. Choose a smooth hypersurface D passing through x . By (6.3.3), we see that

$$\langle \alpha^{n-1} \rangle \wedge \{D\} = \langle \alpha^{n-1} \wedge [D] \rangle.$$

Since $\nu([D], x) = 1$, by Theorem 6.2, we then get $\nu(\langle \alpha^{n-1} \rangle, x) = 0$. This completes the proof. $\square$

In the end of this section, we show that if T is not of minimal singularities in α , then $\nu(\langle T^{n-1} \rangle, x)$ could be strictly positive.

EXAMPLE 6.11. Let $X = \mathbb{P}^n$ and $[x_0 : \cdots : x_n]$ be the homogeneous coordinate. Let ω be the Fubini-Study form on $\mathbb{P}^n$ and $x = [0 : \cdots : 0 : 1]$. Set

$$T := dd^c \log \left(\frac{|x_0|^2 + \cdots + |x_{n-1}|^2}{|x_0|^2 + \cdots + |x_n|^2} \right) + \omega,$$

which is a closed positive $(1, 1)$ -current in the Kähler class $\{\omega\}$. By [16, Corollary 4.11], $S = T^{n-1}$ is classically well-defined. Since S has no mass on the point x , we get $S = \langle T^{n-1} \rangle$. Hence,

$$\nu(\langle T^{n-1} \rangle, x) = \nu(S, x) \geq \nu(T, x)^{n-1}.$$

By the construction of T , we can see that $\nu(T, x) > 0$. Therefore,

$$\nu(\langle T^{n-1} \rangle, x) > 0.$$

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