Mathematische Annalen (2025) 391:6451–6465
https://doi.org/10.1007/s00208-024-03079-1

Mathematische Annalen

Volumes of components of Lelong upper level sets II

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Received: 3 May 2024 / Revised: 10 December 2024 / Accepted: 17 December 2024 /
Published online: 13 January 2025

Abstract

Let X be a compact Kähler manifold of dimension n, and let T be a closed positive (1, 1)-current in a nef cohomology class on X. We establish an optimal upper bound for the volume of components of Lelong upper level sets of T in terms of cohomology classes of non-pluripolar self-products of T.

Mathematics Subject Classification 32U15 · 32Q15

1 Introduction

The aim of this paper is to investigate the singularities of closed positive currents on compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n, and let T be a closed positive (1, 1)-current on X. We are interested in understanding the set of points where T has a strictly positive Lelong numbers. By the celebrated upper semi-continuity of Lelong numbers by Siu [13], we know that this set is a countable union of proper analytic subsets on X. Our goal is to estimate the size of this upper level set. The problem was first studied by Demailly in [5, 6]. In this paper, we provide in some sense a generalization of Demailly's estimate. To delve into details, let us first introduce some necessary notions.

Let ω be a fixed smooth Kähler form on X. We equip X with the Riemannian metric induced by ω. Let S be a closed positive (p, p)-current for some $0 \leq p \leq n$. We define the mass $\|S\|$ of S to be equal to $\int_X S \wedge \omega^{n-p}$. For an analytic set V of pure dimension l in X, we recall that

$$\mathrm{vol}(V) = \frac{1}{l!} \int_{\mathrm{Reg}\, V} \omega^l,$$

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where $\operatorname{Reg} V$ is the regular locus of V. Given a closed positive (p, p)-current R, we denote by $\{R\} \in H^{p,p}(X, \mathbb{R})$ the cohomology class of R. We say $\alpha \in H^{p,p}(X, \mathbb{R})$ is *pseudoeffective* if $\alpha = \{R\}$ for some closed positive (p, p)-current R. Let $\alpha, \beta \in H^{p,p}(X, \mathbb{R})$. We say $\alpha \geq \beta$ if $\alpha - \beta$ is a pseudoeffective class.

For every $x \in X$, let $\nu(T, x)$ denote the Lelong number of T at x. For every irreducible analytic subset V in X, we recall that the generic Lelong number $\nu(T, V)$ of T along V is defined as $\inf_{x\in V}\{\nu(T, x)\}$. The Lelong number is a notion measuring the singularities of T. We refer to [4] for the basics of Lelong numbers. For every constant $c > 0$, we denote $E_c(T) := \{x \in X|\nu(T, x) \geq c\}$ and $E_+(T) := \{x \in X|\nu(T, x) > 0\}$. By [13], $E_c(T)$ is a proper analytic subset in X, and $E_+(T) = \cup_{m\in\mathbb{N}^*} E_{1/m}(T)$ is a countable union of analytic sets.

Let W be an irreducible analytic subset of dimension m in X. We denote by $E_+^W(T) := \{x \in W|\nu(T, x) > \nu(T, W)\}$ the Lelong upper level set of T on W, which is also a countable union of proper analytic subsets in W. Let $V \subset E_+^W(T)$ be an irreducible analytic set. We say that V is *maximal* if there is no irreducible analytic subset V' of $E_+^W(T)$ such that V is a proper subset of V'. We call V *a component of the Lelong upper level set of T along W*, and let $\mathscr{V}_{T,W}$ be the set of such components V. Observe that $\mathscr{V}_{T,W}$ has at most countably many elements. For $0 \leq l \leq m$, we denote by $\mathscr{V}_{l,T,W}$ the set of $V \in \mathscr{V}_{T,W}$ such that $\dim V = l$.

Write $T = dd^c u$ locally, where u is a plurisubharmonic (psh in short) function. We define $T|_{\operatorname{Reg} W}$ to be $dd^c(u|_{\operatorname{Reg} W})$ if $u \not\equiv -\infty$ on $\operatorname{Reg} W$, and $T|_{\operatorname{Reg} W} := 0$ otherwise. One sees that this definition is independent of the choice of u. Thus, $T|_{\operatorname{Reg} W}$ is a current on $\operatorname{Reg} W$. Here is our main result in the paper.

Theorem 1.1 *Let α be a nef $(1, 1)$-class and let W be an irreducible analytic subset of dimension m in X. Let T be a closed positive current in α such that $\nu(T, W) = 0$. Let $1 \leq m' \leq m$ be an integer. Then, we have*

$$\sum_{V\in\mathscr{V}_{m-m',T,W}} \nu(T, V)^{m'} \operatorname{vol}(V) \leq \frac{1}{(m-m')!} \int_{\operatorname{Reg} W} \left(\alpha^{m'} - \langle (T|_{\operatorname{Reg} W})^{m'}\rangle\right) \wedge \omega^{m-m'}, \tag{1.1}$$

where in the integral, we identify α with a smooth closed form in α.

We have some comments on (1.1). To see why the term

$$I := \int_{\operatorname{Reg} W} \left(\alpha^{m'} - \langle (T|_{\operatorname{Reg} W})^{m'}\rangle\right) \wedge \omega^{m-m'}$$

is non-negative, one can consider the case where W is smooth. Then, by a monotonicity of non-pluripolar products (see Theorem 3.3 below), the cohomology class $(\alpha|_W)^{m'} - \{\langle (T|_W)^{m'}\rangle\}$ is pseudoeffective. Hence the integral in the right-hand side of (1.1) is non-negative. In the general case where W is singular, one can use either a desingularisation of W or interpret I as the mass of some non-pluripolar product relative to $[W]$ (the current of integration along W); see Lemma 3.2 below. We underline however that in order to prove Theorem 1.1, it is not possible to use desingularisation of W to reduce to the case where W is smooth. The reason is that in the process of desingularisation, one has to blow up submanifolds of W which in general could be some components of the Lelong upper level sets of T on W.

In [9], a less precise upper bound of volume of components of the Lelong upper level set was given in terms of the volume of W and the mass of T; see also Theorem 3.7 for a more general statement. If we consider $W = X$, then the generic Lelong number of T along W is zero. Thus, by Theorem 1.1, we have the following result.

Corollary 1.2 *Let α be a nef $(1,1)$-class, and T be a closed positive current in α. For $0 \le l \le n$, let $\mathscr{V}_{l,T}$ be the set of $V \in \mathscr{V}_{T,X}$ such that* $\dim V = l$.

Let $1 \le m' \le n$ be an integer. Then, we have

$$\sum_{V \in \mathscr{V}_{n-m',T}} \nu(T,V)^{m'} \operatorname{vol}(V) \le \frac{1}{(n-m')!} \int_X \left(\alpha^{m'} - \{\langle T^{m'} \rangle\}\right) \wedge \omega^{n-m'}. \tag{1.2}$$

Corollary 1.2 generalizes [6, Corollary 7.6] by Demailly in which it was assumed additionally that the components of the upper Lelong level set of T are only of dimension 0 (hence the cohomology class of T is necessarily nef, see [6, Lemma 6.3]). The feature of Corollary 1.2 is that it holds for any current in a nef class. The estimate (1.2) is optimal in the case where all of components of the Lelong upper level set of T have the same dimension. For example, we consider $X = \mathbb{P}^n$, $z \in \mathbb{C}^n \subset \mathbb{P}^n$ and $T = \frac{1}{2} dd^c \log \frac{\|z\|^2}{1+\|z\|^2} + \omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{P}^n$. In this case, we see that 0 is the only point at which the Lelong number of T is positive and $\nu(T,0) = 1$, and (1.2) (for $m = m' = n$) becomes an equality.

In general, if we consider the relative setting as in Theorem 1.1 (when W is not necessarily equal to X), then our main result (Theorem 1.1) is not satisfactory because it requires that $\nu(T,W) = 0$, hence, we can not apply it to the case where T is the current of integration along a curve $\mathcal{C}$ in a complex Kähler surface and $W = \mathcal{C}$). In Theorem 3.7 below, we are able to treat the case where $\nu(T,W) > 0$ but the estimate is not explicit due to the presence of a constant c in the right-hand side. In this regard, the estimate in [6, Theorem 1.7] is stronger than ours for dimension 2 (see the discussion after [6, Theorem 1.7] in [6]). On the other hand, as explained in [9], the feature of Theorem 1.1 is that it gives bounds for volumes of all of components of Lelong upper level sets whereas [6, Theorem 1.7] does not allow us to treat all of components in general.

This paper refines and substitutes [17]. The proof of Theorem 1.1 requires both the theory of density currents in [8] and relative non-pluripolar products in [18] (see also [1, 2]). One of the keys is Theorem 3.6 below following from a general comparison of Lelong numbers for density currents, as stated below.

Corollary 1.3 (Corollary 2.6 in Sect. 2) *Let T_j be a closed positive current on X for $1 \le j \le m$. Then, for every $x \in X$ and for every density current S associated to $T_1, \dots, T_m$, we have*

$$\nu(S, x^m) \ge \nu(T_1, x) \cdots \nu(T_m, x),$$

$x^m = (x, \dots, x) \in \Delta_m \subset E$, where Δ_m is the diagonal of X^m and E is the normal bundle over Δ_m.

The above corollary generalizes the well-known comparison of Lelong numbers of intersection of $(1, 1)$-currents due to Demailly [4, Chapter III, Corollary 7.9] in the compact setting. It is probably the first result dealing with comparison of Lelong numbers for intersection of currents of arbitrary bi-degree. As we will see, this corollary is more or less a direct consequence of [8, Proposition 4.13].

The organization of this paper is as follows. In Sect. 2, we recall basic properties of density currents from [8]. In Sect. 3, we discuss the connection between the non-pluripolar product and density currents, and prove Theorem 1.1.

2 Density currents

We first recall some basic properties of density currents introduced by Dinh–Sibony in [8].

Let X be a complex Kähler manifold of dimension n, and V a smooth complex submanifold of X of dimension l. Let T be a closed positive (p, p)-current on X, where $0 \leq p \leq n$. Denote by $\pi : E \to V$ the normal bundle of V in X and $\overline{E} := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of E. We recall that $E = TX|_V/TV$, where TX and TV are the holomorphic tangent bundles of X and V respectively (this shows E is a holomorphic vector bundle). By abuse of notation, we also use π to denote the canonical projection from $\overline{E}$ to V.

Let U be an open subset of X with $U \cap V \neq \varnothing$. Let τ be a smooth diffeomorphism from U to an open neighborhood of $U \cap V$ in E which is the identity on $U \cap V$ such that the induced map of the differential $d\tau$ to $E|_{V \cap U}$ is the identity (because for every $x \in U \cap V$, $d\tau$ at x is the identity map on TV_x, it induces a linear map from $TX_x/TV_x = E_x$ to $TE_x/TV_x = E_x$). Such a map is called *an admissible map*. Note that in [8], to define an admissible map, it is required furthermore that $d\tau$ is $\mathbb{C}$-linear at every point of V. This difference doesn't affect what follows. When U is a small enough tubular neighborhood of V, there always exists an admissible map τ by [8, Lemma 4.2]. In general, τ is not holomorphic. When U is a small enough local chart, we can choose a holomorphic admissible map by using suitable holomorphic coordinates on U. For $\lambda \in \mathbb{C}^*$, let $(A_\lambda) : E \to E$ be the multiplication by λ on fibers of E, which can be extended to $(A_\lambda) : \overline{E} \to \overline{E}$. A (p, p)-current on $\overline{E}$ is said to be V-conic if it is invariant under the action of (A_λ). Here is the first fundamental result for density currents.

Theorem 2.1 ([8, Theorem 4.6]) *Let τ be an admissible map defined on a tubular neighborhood of V. Then, the family $(A_\lambda)_* \tau_* T$ is of mass uniformly bounded in λ on compact subsets in E, and if S is a limit current of the last family as $\lambda \to \infty$, then S is a closed positive current on E which can be extended trivially through $\overline{E} \backslash E$ to be a V-conic closed positive current on $\overline{E}$ such that the cohomology class $\{S\}$ of S in $\overline{E}$ is independent of the choice of S, and $\{S\}|_V = \{T\}|_V$, and $\|S\| \leq C\|T\|$ for some constant C independent of S and T, where $\{S\}|_V$ denotes the pull-back of $\{S\}$ under the canonical inclusion map from V to $\overline{E}$.*

The current S in the above theorem is called *a tangent current to T along V*. Its cohomology class is called *the total tangent class of T along V* and is denoted by $\kappa^V(T)$. Tangent currents are not unique in general. By [8, Theorem 4.6] again, if

$$S = \lim_{k\to\infty} (A_{\lambda_k})_* \tau_* T$$

for some sequence $(\lambda_k)_k$ converging to ∞, then for every open subset U' of X and every admissible map $\tau' : U' \to E$, we also have

$$S = \lim_{k\to\infty} (A_{\lambda_k})_* \tau'_* T.$$

This is equivalent to saying that tangent currents are independent of the choice of the admissible map τ.

Definition 2.2 ([8, Definition 3.1]) Let F be a complex manifold and $\pi_F : F \to V$ a holomorphic submersion. Let S be a positive current of bi-degree (p, p) on F. The h-dimension of S with respect to π_F is the biggest integer q such that $S \wedge \pi_F^* \theta^q \neq 0$ for some Hermitian metric θ on V.

By a bi-degree reason, the *h-dimension* of S is in $[\max\{l-p, 0\}, \min\{\dim F - p, l\}]$. We have the following description of currents with minimal *h-dimension*.

Lemma 2.3 ([8, Lemma 3.4]) *Let $\pi_F : F \to V$ be a holomorphic submersion. Let S be a closed positive current of bi-degree (p, p) on F of h-dimension $(l - p)$ with respect to π_F. Then $S = \pi^* S'$ for some closed positive current S' on V.*

By [8, Lemma 3.8], the *h-dimensions* of tangent currents to T along V are the same and this number is called the *tangential h-dimension of T along V*.

Let $m \geq 2$ be an integer. Let T_j be a closed positive current of bi-degree (p_j, p_j) for $1 \leq j \leq m$ on X and let $T_1 \otimes \cdots \otimes T_m$ be the tensor current of $T_1, \ldots, T_m$ which is a current on X^m. A *density current* associated to $T_1, \ldots, T_m$ is a tangent current to $\otimes_{j=1}^m T_j$ along the diagonal Δ_m of X^m. Let $\pi_m : E_m \to \Delta$ be the normal bundle of Δ_m in X^m. Denote by $[V]$ the current of integration along V. When $m = 2$ and $T_2 = [V]$, the density currents of T_1 and T_2 are naturally identified with the tangent currents to T_1 along V (see [15, Lemma 2.3]).

The unique cohomology class of density currents associated to $T_1, \ldots, T_m$ is called *the total density class of* $T_1, \ldots, T_m$. We denote the last class by $\kappa(T_1, \ldots, T_m)$. The *tangential h-dimension of* $T_1 \otimes \cdots \otimes T_m$ *along* Δ_m is called the *density h-dimension* of $T_1, \ldots T_m$.

Lemma 2.4 ([8, Section 5]) *Let T_j be a closed positive current of bi-degree (p_j, p_j) on X for $1 \leq j \leq m$ such that $\sum_{j=1}^m p_j \leq n$. Assume that the density h-dimension of $T_1, \ldots, T_m$ is minimal,* i.e*, equals to $n - \sum_{j=1}^m p_j$. Then the total density class of $T_1, \ldots, T_m$ is equal to $\pi_m^*(\wedge_{j=1}^m \{T_j\})$.*

Let $h_{\overline{E}}$ be the Chern class of the dual of the tautological line bundle of $\overline{E}$. By [8, Page 535], we have

$$\kappa^V(T) = \sum_{j=\max\{0,l-p\}}^{\min\{l,n-p-1\}} \pi^*\left(\kappa_j^V(T)\right) \wedge h_{\overline{E}}^{p-(l-j)}, \tag{2.1}$$

where $\pi : \overline{E} \to V$ is the canonical projection and $\kappa_j^V(T) \in H^{2l-2j}(V, \mathbb{R})$. The tangential h-dimension of T along V is exactly equal to the maximal j such that $\kappa_j^V(T) \neq 0$, and it was known that the class $\kappa_j^V(T)$ is pseudoeffective ([8, Lemma 3.15]).

Theorem 2.5 ([8, Proposition 4.13]) *Let V' be a submanifold of V and let T be a closed positive current on X. Let T_∞ be a tangent current to T along V. Let s be the tangential h-dimension of T_∞ along V'. Then, the tangential h-dimension of T along V' is at most s, and we have*

$$\kappa_s^{V'}(T) \le \kappa_s^{V'}(T_\infty).$$

As a consequence, we obtain the following result.

Corollary 2.6 *Let T_j be a closed positive current on X for $1 \le j \le m$. Then, for every $x \in X$ and for every density current S associated to $T_1, \dots, T_m$, we have*

$$\nu(S, x^m) \ge \nu(T_1, x) \cdots \nu(T_m, x), \tag{2.2}$$

$x^m = (x, \dots, x) \in \Delta_m \subset E$, where Δ_m is the diagonal of X^m and E is the normal bundle over Δ_m.

Proof Let $x \in X$. Let $\pi : E \to \Delta_m$ be the canonical projection from the normal bundle of the diagonal Δ_m of X^m in X^m. Put $T := \otimes_{j=1}^m T_j$ and $V' := \{x^m\}$. By [12, Lemma 2.4], we have $\nu(T, x^m) \ge \nu(T_1, x) \cdots \nu(T_m, x)$. By [8, Proposition 5.6], we have

$$\kappa_0^{V'}(S) = \nu(S, x^m)\delta_{x^m}, \quad \kappa_0^{V'}(T) = \nu(T, x^m)\delta_{x^m},$$

where δ_{x^m} is the Dirac measure on x^m (notice here $\dim V' = 0$). This combined with Theorem 2.5 applied to X^m, $T := \otimes_{j=1}^m T_j$, Δ_m the diagonal of X^m and $V' := \{x^m\}$ implies

$$\nu(S, x^m) \ge \nu(T, x^m) \ge \nu(T_1, x) \cdots \nu(T_m, x).$$

Hence, the desired inequality follows. The proof is finished. □

3 Relative non-pluripolar products

We first recall basic facts about the relative non-pluripolar product of currents and discuss its connection with the density current.

Non-pluripolar product were introduced in [1, 2], and we follow [18], where the second author extended the construction to the case of higher bi-degree, known as the relative non-pluripolar product. For reader's convenience, we explain briefly how to do it.

Let X be a compact Kähler manifold of dimension n. Let $T_1, \dots, T_m$ be closed positive $(1,1)$-currents on X, and T be a closed positive (p,p)-current. Write $T_j = dd^c u_j + \theta_j$, where θ_j is a smooth form and u_j is a θ_j-psh function. Put

$$R_k := \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -k\} + \theta_j) \wedge T$$

for $k \in \mathbb{N}$. By the strong quasi-continuity of bounded psh functions ([18, Theorems 2.4 and 2.9]), we have

$$R_k = \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -l\} + \theta_j) \wedge T$$

for every $l \geq k \geq 1$. One can check that R_k is positive (see [18, Lemma 3.2]).

By [18, Lemma 3.4], the current R_k is of mass bounded uniformly in k and $(R_k)_k$ converges to a closed positive current as $k \to \infty$. This limit is denoted by $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle$, and is called the relative non-pluripolar product relative to T of $T_1, \dots, T_m$.

For every closed positive $(1,1)$-current P, we denote by I_P the set of $x \in X$ so that local potentials of P are equal to $-\infty$ at x. Note that I_P is a complete pluripolar set. The following is deduced from [18, Proposition 3.5].

Proposition 3.1 (i) *For $R := \langle \wedge_{j=l+1}^m T_j \dot{\wedge} T \rangle$, we have $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle = \langle \wedge_{j=1}^l T_j \dot{\wedge} R \rangle$.*
(ii) *For every complete pluripolar set A, we have*

$$\mathbf{1}_{X \setminus A} \langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle = \big\langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \dot{\wedge} (\mathbf{1}_{X \setminus A} T) \big\rangle.$$

In particular, the equality

$$\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle = \langle \wedge_{j=1}^m T_j \dot{\wedge} T' \rangle$$

holds, where $T' := \mathbf{1}_{X \setminus \cup_{j=1}^m I_{T_j}} T$.

By [18, Lemma 3.1], it follows that $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle$ has no mass on $\cup_{j=1}^m I_{T_j}$. Furthermore, by Proposition 3.1 (ii), we get that if T has no mass on A, then so does $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle$.

Let V be an irreducible analytic set in X. For the case $T = [V]$, we have the following lemma.

Lemma 3.2 ([19, Lemma 2.3]) *Let $T_1, \dots, T_m$ be closed positive $(1,1)$-currents on X. Then the following properties hold:*

(i) *If V is contained in $\cup_{j=1}^m I_{T_j}$, then $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} [V] \rangle = 0$ and there is $1 \leq j_0 \leq m$ so that $V \subset I_{T_{j_0}}$.*

(ii) *If V is not contained in $\cup_{j=1}^m I_{T_j}$, then*

$$\langle \wedge_{j=1}^m T_j \dot{\wedge} [V] \rangle = i_* \langle T_{1,V} \wedge \cdots \wedge T_{m,V} \rangle,$$

where $i\colon\ Reg(V) \to X$ is the natural inclusion, and $T_{j,V} := dd^c(u_j|_{Reg(V)})$ if $dd^c u_j = T_j$ locally.

Let T, T' be closed positive $(1,1)$-currents in the same cohomology class on X. T' is said to be less singular than T if for every local potentials u of T and u' of T', $u \leq u' + O(1)$. Here is a crucial property of relative non-pluripolar products.

Theorem 3.3 ([18, Theorem 1.1]) *Let T'_j be closed positive $(1,1)$-current in the cohomology class of T_j on X such that T'_j is less singular than T_j for $1 \leq j \leq m$. Then we have*

$$\{\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \dot{\wedge} T \rangle\}.$$

Weaker versions of the above result were proved in [2, 3, 20]. Let $\alpha_1, \ldots, \alpha_m$ be big $(1,1)$-classed of X. A current $T_{j,\min} \in \alpha_j$ is said to have minimal singularities if it is less singular than any closed positive current in α_j.

By Theorem 3.3, the class $\{\langle T_{1,\min} \wedge \cdots \wedge T_{m,\min} \dot{\wedge} T \rangle\}$ is a well-defined pseudoeffective class which is independent of the choice of $T_{j,\min}$. We denote the last class by $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \dot{\wedge} T \rangle\}$. When $T \equiv 1$, we simply write $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ for $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \dot{\wedge} T \rangle\}$. In this case, the product $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ was introduced in [2].

Regarding the relation between relative non-pluripolar products and density currents, the following fact was proved in [16, Theorem 3.5], see also [10, 11].

Theorem 3.4 *Let R_∞ be a density current associated to $T_1, \ldots, T_m, T$. Then we have*

$$\pi_{m+1}^* \langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle \leq R_\infty, \tag{3.1}$$

where π_{m+1} is the canonical projection from the normal bundle of the diagonal Δ of X^{m+1} to Δ, and as usual we identified Δ with X.

We will need the following to estimate the *density h-dimension* of currents, which is a special case of [16, Proposition 3.6].

Proposition 3.5 *Let P and T be closed positive currents of bi-degree $(1,1)$ and (p,p) respectively on X, $1 \leq p \leq n$. Assume that T has no mass on I_P. Then, for every density current S associated to P, T, the h-dimension of S is equal to $n-p-1$.*

For every pseudoeffective (p,p)-class γ on X, we put $\|\gamma\| := \int_X \Theta \wedge \omega^{n-p}$, where Θ is any closed smooth form in γ. This definition is independent of the choice of Θ and is nonnegative because of the pseudoeffectivity of γ.

Theorem 3.6 *Let P and T be closed positive currents of bi-degree $(1, 1)$ and (p, p) respectively on X, where $1 \leq p \leq n-1$. Assume that T has no mass on I_P. Then, the cohomology class*

$$\gamma := \{P\} \wedge \{T\} - \{\langle P \dot{\wedge} T \rangle\}$$

is pseudoeffective and we have

$$\|\gamma\| \geq \sum_V \nu(P, V)\nu(T, V) n_V! \operatorname{vol}(V), \tag{3.2}$$

where the sum is taken over every irreducible subset V of dimension at least $n-p-1$ in X, and $n_V := \dim V$.

We note that by the proof below, we see that any irreducible subset V such that $\dim V \geq n-p-1$ and $\nu(T, V) > 0$, $\nu(P, V) > 0$ must satisfy $\dim V = n-p-1$.

Proof Let $\mathscr{V}$ be the set of irreducible analytic subsets V of dimension at least $n-p-1$ in X such that $\nu(T, V) > 0$ and $\nu(P, V) > 0$. We note that in (3.2), it is enough to consider $V \in \mathscr{V}$. We will see below that $\mathscr{V}$ has at most countable elements.

Observe that if $\nu(P, x) > 0$, then $x \in I_P$. Hence, by hypothesis, the trace measure of T has no mass on the set $\{x \in X : \nu(P, x) > 0\}$. This allows us to apply Proposition 3.5 to P and T to obtain that the *density h-dimension* of P and T is minimal. Using this and Lemma 2.4 gives

$$\kappa(P, T) = \pi^*(\{P\} \wedge \{T\}), \tag{3.3}$$

where π is the canonical projection from the normal bundle of the diagonal Δ of X^2 to Δ.

Let S be a density current associated to P and T. Since the *h-dimension* of S is minimal, using Lemma 2.3, we get that there exists a current S' on X such that $S = \pi^* S'$ (recall Δ is identified with X). Since the relative non-pluripolar product is dominated by density currents (Theorem 3.4), the current $S' - \langle P \dot{\wedge} T \rangle$ is closed and positive. Moreover, by (3.3), the cohomology class of the last current is equal to γ. It follows that γ is pseudoeffective.

It remains to prove (3.2). Let $V \in \mathscr{V}$. By definition, the generic Lelong number of T along V is positive. Since T is of bi-degree (p, p), the dimension of V must be at most $n-p$. Hence, we have two possibilities: either $\dim V = n-p-1$ or $\dim V = n-p$. Indeed, the latter case cannot occur. Suppose that such a V exists. Then, we consider two cases: whether T has mass on V or not. If T has no mass on V, then $\nu(T, V) = 0$, which leads to a contradiction. If T has mass on V, which is contained in I_P (for $\nu(P, V) > 0$), then this contradicts the hypothesis that T has no mass on I_P.

Let $V \in \mathscr{V}$. Since the Lelong numbers are preserved by submersion maps ([12, Proposition 2.3]), by applying Corollary 2.6 to P, T and generic $x \in V$, we obtain

$$\nu(S', V) = \nu(S, V) \geq \nu(P, V)\nu(T, V).$$

This combined with the fact that $\dim V = n-p-1$ implies $S' \geq \nu(P, V)\nu(T, V)\,[V]$. We deduce that

$$\begin{aligned} S' &\geq \langle P \dot{\wedge} T\rangle + \mathbf{1}_{I_P} S' \\ &\geq \langle P \dot{\wedge} T\rangle + \sum_{V \in \mathscr{V}} \nu(P, V)\nu(T, V)\,[V]. \end{aligned}$$

The second inequality comes from Siu's decomposition theorem ([7, 2.18]), and this also shows that $\mathscr{V}$ has at most countable elements. The desired assertion follows and the proof is finished. □

We now prove Theorem 1.1.

Proof of Theorem 1.1 It suffices to consider the case where α is Kähler by using $\alpha + \epsilon\{\omega\}$, $T + \epsilon\omega$ instead of α, T and letting $\epsilon \to 0$. Hence we assume from now on that α is Kähler. By abuse of notation, we also denote by α a smooth Kähler form in α. By Lemma 3.2, the right hand-hand side of (1.1) can be written as

$$\frac{1}{(m-m')!}\int_{\mathrm{Reg}\, W}\left(\alpha^{m'} - \langle (T|_{\mathrm{Reg}\, W})^{m'}\rangle\right) \wedge \omega^{m-m'} = \frac{1}{(m-m')!}\left\|\langle \alpha^{m'} \dot{\wedge} [W]\rangle - \langle T^{m'} \dot{\wedge} [W]\rangle\right\|.$$

Step 1. We first focus on the case that T has analytic singularities. Set $S = \langle T^{m'-1} \dot{\wedge} [W]\rangle$. Since α is Kähler, by the monotonicity of non-pluripolar product (Theorem 3.3) and Proposition 3.1 (i), we get

$$\begin{aligned} \left\|\langle \alpha^{m'} \dot{\wedge} [W]\rangle - \langle T^{m'} \dot{\wedge} [W]\rangle\right\| &\geq \left\|\langle \alpha \wedge T^{m'-1} \dot{\wedge} [W]\rangle - \langle T^{m'} \dot{\wedge} [W]\rangle\right\| \\ &= \|\alpha \wedge S - \langle T \dot{\wedge} S\rangle\| \end{aligned} \tag{3.4}$$

We now show that S has no mass on I_T. For $m' > 1$, this directly follows from the definition of non-pluripolar product. For $m' = 1$, the current S is just $[W]$. Since we assume that T has analytic singularities, the polar locus I_T is an analytic subset and it doesn't contain W. Hence, $[W]$ also has no mass on I_T. Therefore, we can apply Theorem 3.6 to T, S, and get

$$\|\alpha \wedge \{S\} - \{\langle T \dot{\wedge} S\rangle\}\| \geq (m-m')! \sum_{V \in \mathscr{V}_{m-m',T,W}} \nu(T, V)\nu(S, V)\,\mathrm{vol}(V) \tag{3.5}$$

Let $V \in \mathscr{V}_{m-m',T,W}$ and let $\mathrm{Sing}(I_T \cap W)$ be the singular locus of the analytic set $I_T \cap W$. Since T has analytic singularities, the Lelong number $\nu(T, x)$ is strictly positive if and only if x belongs to I_T. This coupled with the maximality of V implies that V is contained in $I_T \cap W$, and is one of the irreducible components. Let $K_1, \ldots, K_s$ be the irreducible components of $I_T \cap W$. Observe that the set $\mathrm{Sing}(I_T \cap W)$ consists of singular points of irreducible components and their intersection points. By rearranging the index, we may assume $V = K_1$. Set

$$U := X \setminus \mathrm{Sing}(K_1) \cup K_2 \cdots \cup K_s.$$

Now, we prove that the intersection $T^{m'-1} \wedge [W]$ is well-defined on U, in the sense in [4, Chapter III, Theorem 4.5]. Notice that $V \setminus \mathrm{Sing}(I_T \cap W)$is contained in $\mathrm{Reg}(V)$, and is of dimension $m - m'$. Consequently, for $0 \le j' \le m' - 1$,

$$\begin{aligned}\mathscr{H}_{2m-2j'+1}(L(T)|_U \cap W) &= \mathscr{H}_{2m-2j'+1}(I_T \cap W \cap U)\\ &= \mathscr{H}_{2m-2j'+1}(V \setminus \mathrm{Sing}(I_T \cap W))\\ &= 0,\end{aligned}$$

where $L(T)$ is the set of $x \in X$ such that the local potential of T is unbounded on any neighborhood of x. This allows us to apply [4, Chapter III, Theorem 4.5] and get the well-definedness of $T^{m'-1} \wedge [W]$ on U.

By the continuity theorem of classical intersection ([4, Chapter III, Corollary 4.3]), we can apply [18, Proposition 3.6] to $\langle T^{m'-1} \dot{\wedge} [W] \rangle$, and obtain

$$S = \langle T^{m'-1} \dot{\wedge} [W] \rangle = \mathbf{1}_{U \setminus I_T} T^{m'-1} \wedge [W]. \tag{3.6}$$

Actually, the equality also holds on $U \cap I_T$. To show this, we need to check that $T^{m'-1} \wedge [W]$ has no mass on $U \cap I_T$. Since $\dim(U \cap I_T \cap W) = m - m'$ and $T^{m'-1} \wedge [W]$ is of bi-dimension $(m-m'+1, m-m'+1)$, the current $T^{m'-1} \wedge [W]$ must have no mass on $U \cap I_T \cap W$. Also, by the fact $\mathrm{Supp}(T^{m'-1} \wedge [W]) \subset W$, the current $T^{m'-1} \wedge [W]$ also has no mass on $(U \cap I_T) \setminus W$. Therefore, the equality (3.6) extends to U. This implies that the Lelong number $\nu(S, V)$ equals $\nu(T^{m'-1} \wedge [W], V \setminus \mathrm{Sing}(I_T \cap W))$ (remember that we consider the current $T^{m'-1} \wedge [W]$ on U, and $V \setminus \mathrm{Sing}(I_T \cap W)$ is an analytic subset in U), and we then have

$$\begin{aligned}\nu(S, V) &= \nu(T^{m'-1} \wedge [W], V \setminus \mathrm{Sing}(I_T \cap W))\\ &\ge \nu(T, V \setminus \mathrm{Sing}(I_T \cap W))^{m'-1} \nu([W], V \setminus \mathrm{Sing}(I_T \cap W))\\ &\ge \nu(T, V)^{m'-1},\end{aligned} \tag{3.7}$$

where the first inequality comes from [4, Chapter III, Corollary 7.9] (see also Corollary 2.6 for a more general version). By (3.4), (3.5) and (3.7), the desired inequality follows in the case where T has analytic singularities.

Step 2. Now, we remove the assumption that T has analytic singularities. The argument we use is standard and is based on the work of Demailly in [6]. First, we write $T = dd^c u + \theta$, where θ is a closed smooth $(1, 1)$-form and $u \in \mathrm{PSH}(X, \theta)$. Demailly's analytic approximation theorem (see [7, Corollary 14.13]) allows us to construct a sequence $u_k^D \in \mathrm{PSH}(X, \theta + \epsilon_k \omega)$, where ϵ_k decreases to 0, such that

(1) $u_k^D \ge u$ and u_k^D converges to u in L^1.

(2) u_k^D has analytic singularities.

(3) $\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly on X, where $T_k = dd^c u_k^D + (\theta + \epsilon_k \omega)$.

By the monotonicity property of non-pluripolar product (Theorem 3.3), we have

$$\begin{aligned}\left\|\langle\alpha^{m'}\dot{\wedge}[W]\rangle-\langle T^{m'}\dot{\wedge}[W]\rangle\right\| &= \lim_{k\to\infty}\left\|\langle(\alpha+\epsilon_k\omega)^{m'}\dot{\wedge}[W]\rangle-\langle(T+\epsilon_k\omega)^{m'}\dot{\wedge}[W]\rangle\right\| \\ &\geq \limsup_{k\to\infty}\left\|\langle(\alpha+\epsilon_k\omega)^{m'}\dot{\wedge}[W]\rangle-\langle T_k^{m'}\dot{\wedge}[W]\rangle\right\|\end{aligned} \tag{3.8}$$

For every constant $r>0$, set $A_r := \{V\in\mathscr{V}_{m-m',T,W}\,|\,\nu(T,V)\geq r\}$. Observe that A_r increases to $\mathscr{V}_{m-m',T,W}$ as $r\to 0$. Since $\nu(T_k,x)$ converges to $\nu(T,x)$ uniformly and T_k is less singular than T, for every fixed $r>0$ we have

$$A_r\subset\mathscr{V}_{m-m',T_k,W}$$

when k is large enough. By Step 1, we therefore have

$$\begin{aligned}\left\|\langle(\alpha+\epsilon_k\omega)^{m'}\dot{\wedge}[W]\rangle-\langle T_k^{m'}\dot{\wedge}[W]\rangle\right\| &\geq (m-m')!\sum_{V\in\mathscr{V}_{m-m',T_k,W}}\nu(T_k,V)^{m'}\operatorname{vol}(V)\\ &\geq (m-m')!\sum_{V\in A_r}\nu(T_k,V)^{m'}\operatorname{vol}(V).\end{aligned}$$

Letting $k\to\infty$ and using (3.8) give

$$\begin{aligned}\left\|\langle\alpha^{m'}\dot{\wedge}[W]\rangle-\langle T^{m'}\dot{\wedge}[W]\rangle\right\| &\geq (m-m')!\limsup_{k\to\infty}\sum_{V\in A_r}\nu(T_k,V)^{m'}\operatorname{vol}(V)\\ &= (m-m')!\sum_{V\in A_r}\nu(T,V)^{m'}\operatorname{vol}(V),\end{aligned}$$

for every constant $r>0$. Letting $r\to 0$, we obtain the desired estimate. □

For the general case where $\nu(T,W)>0$. We couldn't directly compare the volume of Lelong upper level sets of T on W and the mass of $\{\langle\alpha^{m'}\dot{\wedge}[W]\rangle\}-\{\langle T^{m'}\dot{\wedge}[W]\rangle\}\}$. In this case, we have the following modified inequality which is stronger than [9, Theorem 1.1].

Theorem 3.7 *Let α be a nef $(1,1)$-class. Let W be an irreducible analytic subset in X. Let T be a closed positive current in α. Let $1\leq m'\leq m$ be an integer. Then we have*

$$\begin{aligned}&(m-m')!\sum_{V\in\mathscr{V}_{m-m',T,W}}\big(\nu(T,V)-\nu(T,W)\big)^{m'}\operatorname{vol}(V)\\ &\quad\leq\left\|(\alpha+c\{\omega\})^{m'}\wedge\{[W]\}-\{\langle(T+c\omega)^{m'}\dot{\wedge}[W]\rangle\}\right\|\end{aligned}\tag{3.9}$$

where $c = c_1 \cdot \nu(T, W)$ *and* $c_1 > 0$ *is a constant independent of* α, T, W*. In particular, there is a constant* $c_2 > 0$ *independent of* α, T, W *such that*

$$\sum_{V \in \mathscr{V}_{m-m',T,W}} \big(\nu(T, V) - \nu(T, W)\big)^{m'} \operatorname{vol}(V) \leq c_2 \operatorname{vol}(W) \|T\|^{m'}. \tag{3.10}$$

Proof The desired inequality (3.10) follows directly from (3.9). The proof of (3.9) is similar to Step 2. of Theorem 1.1, which is again based on Demailly's regularization theorem ([6]). For convenience, set $c_3 := \nu(T, W) > 0$. The regularization theorem of currents introduced in [6] allows us to cut down the Lelong upper level set $\{x \in X | \nu(T, V) \geq c_3\}$ from T. More precisely, by [6, Theorem 1.1], there exists a sequence of almost positive closed $(1, 1)$-currents $T_{c_3,k}$ in α such that

(1) $T_{c_3,k} \geq -(c_1 \cdot c_3 + \epsilon_k)\omega$, where $\lim_{k\to\infty} \epsilon_k = 0$ and $c_1 > 0$ is a constant independent of α, T and W.
(2) The global potentials of $T_{c_3,k}$ decreases to the global potential of T.
(3) $\nu(T_{c_3,k}, x) = \max\{\nu(T, x) - c_3, 0\}$.

Set $\widetilde{T}_{c_3,k} = T_{c_3,k} + (c_1 \cdot c_3 + \epsilon_k)\omega$, which is a closed positive $(1, 1)$-current. By Theorem 3.3, we have

$$\begin{aligned}&\left\|(\alpha + c_1 \cdot c_3\{\omega\})^{m'} \wedge \{[W]\} - \langle (T + c_1 \cdot c_3\omega)^{m'} \dot{\wedge} [W]\rangle\right\| \\ &\quad \geq \limsup_{k\to\infty} \left\|\big(\alpha + (c_1 \cdot c_3 + \epsilon_k)\{\omega\}\big)^{m'} \wedge \{[W]\} - \langle \widetilde{T}^{m'}_{c_3,k} \dot{\wedge} [W]\rangle\right\|.\end{aligned}$$

Since $\nu(\widetilde{T}_{c_3,k}, W) = 0$, we can apply Theorem 1.1 to the right-hand side of the above inequality and get

$$\begin{aligned}&\left\|\big(\alpha + (c_1 \cdot c_3 + \epsilon_k)\{\omega\}\big)^{m'} \wedge \{[W]\} - \langle \widetilde{T}^{m'}_{c_3,k} \dot{\wedge} [W]\rangle\right\| \\ &\quad \geq (m - m')! \sum_{V \in \mathscr{V}_{m-m',\tilde{T}_{c_3,k},W}} \nu(\widetilde{T}_{c_3,k}, V)^{m'} \operatorname{vol}(V),\end{aligned} \tag{3.11}$$

By the above properties of $T_{c_3,k}$, we have

$$\mathscr{V}_{m-m',\tilde{T}_{c_3,k},W} = \mathscr{V}_{m-m',T,W}.$$

Therefore, the right-hand side of (3.11) is equal to

$$\begin{aligned}&(m - m')! \sum_{V \in \mathscr{V}_{m-m',T,W}} \nu(\widetilde{T}_{c_2,k}, V)^{m'} \operatorname{vol}(V) \\ &\quad = (m - m')! \sum_{V \in \mathscr{V}_{m-m',T,W}} \big(\nu(T, V) - \nu(T, W)\big)^{m'} \operatorname{vol}(V).\end{aligned}$$

This completes the proof. $\square$

Remark 3.8 We note that by [14], for every closed positive (p, p)-current R on X, there always exists a closed positive $(1, 1)$-current T whose Lelong numbers coincide with those of R. However, if we apply directly our result to current T, we will get an estimate of the Lelong upper level set for the current R. But there will be a constant appear in the right hand side of (1.1) in Theorem 1.1, since the mass of T is bounded by a universal constant times the mass of R.

Acknowledgements This research is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)-Projektnummer 500055552 and by the ANR-DFG grant QuaSiDy, grant no ANR-21-CE40-0016.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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