The Equivariant Mayer–Vietoris Spectral Sequence and Degenerated Grassmannians

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Abstract

This thesis develops methods to compute equivariant cohomology of projective unions with torus actions via a detailed formulation of the Mayer–Vietoris spectral sequence, with particular focus on localization, change of tori, and connections to GKM theory. While the equivariant cohomology of each component in a projective union is simple, the combinatorial interaction of the cover is more subtle and is studied using simplicial complexes and poset cohomology. As an application, degenerated Grassmannians, arising as special fibers of semi-toric degenerations from Hodge-type Seshadri stratifications, are analyzed, and the torsion-free part of their equivariant cohomology under the Grassmannian torus action is described.

Zusammenfassung

Diese Dissertation entwickelt Methoden zur Berechnung der äquivarianten Kohomologie projektiver Vereinigungen mit Toruswirkungen anhand einer detaillierten Formulierung der Mayer–Vietoris-Spektralsequenz, mit besonderem Schwerpunkt auf Lokalisierung, Wechsel von Tori und Verbindungen zur GKM-Theorie. Während die äquivariante Kohomologie jeder einzelnen Komponente einer projektiven Vereinigung einfach ist, erweist sich die kombinatorische Wechselwirkung der Überdeckung als subtiler und wird mithilfe von simplizialen Komplexen und Poset-Kohomologie untersucht. Als Anwendung werden degenerierte Grassmannsche Mannigfaltigkeiten, die als spezielle Fasern semitorischer Degenerationen aus Seshadri-Stratifizierungen vom Hodge-Typ auftreten, analysiert, und der torsionsfreie Teil ihrer äquivarianten Kohomologie unter der Grassmannschen Toruswirkung wird beschrieben.

Contents

1 Introduction

Toric degenerations provide a powerful method in algebraic geometry by relating complicated algebraic varieties to toric varieties, whose structure is exceptionally well understood. This perspective facilitates the study of geometric, topological, and representation-theoretic aspects of the original varieties and has been developed in several areas of mathematics, among them geometric invariant theory, representation theory, and Newton–Okounkov theory (see, for example, [Hu08; FN24; HK15]). Semi-toric degenerations generalize this framework by degenerating to unions of toric varieties, appearing for example in the theory of Hodge algebras [DEP82] and in standard monomial theory [CFL23]. More specifically, in [CFL23] the authors introduced the method of a *Seshadri stratification*, which serves as an alternative to Newton–Okounkov theory and produces semi-toric degenerations governed by finite posets. For instance, in the case of Grassmannians, taking Schubert subvarieties as strata yields a degeneration whose special fiber is a *projective union*, that is, a projective variety whose irreducible components are projective spaces intersecting along linear subspaces. The resulting *degenerated Grassmannians* show that certain projective unions naturally inherit rich combinatorial structures closely tied to representation theory. Moreover, in this case the action of the diagonal torus of GL_n on the Grassmannian extends to the limit space. This illustrates a more general phenomenon: in favorable settings, for suitable degenerations the group action extends to the degenerated variety.

The fact that in natural settings projective unions arise equipped with substantial internal structure and group symmetries motivates the study of their *equivariant cohomology*, with degenerated Grassmannians providing a particularly significant example. To the best of our knowledge, no general computation of their equivariant cohomology is available, and addressing this gap is the aim of the present work.

To approach this problem, we require a method that computes the cohomology of a space from the cohomology of its parts. The classical tool for this purpose is the *Mayer–Vietoris sequence*, originally developed in algebraic topology as a long exact sequence for the union of two subspaces. Its generalization to finite covers yields the *Mayer–Vietoris spectral sequence*, which encodes the cohomology of a space in terms of the cohomology of covering pieces and their intersections. The first page of the spectral sequence naturally reflects the simplicial structure of the nerve of the cover and, under favorable conditions, collapses to simplicial cohomology. In this way, the spectral sequence provides a bridge between geometric decompositions and combinatorial data.

Although variants of the Mayer–Vietoris spectral sequence have appeared in several settings, a systematic construction in equivariant cohomology is not available in the literature. The purpose of this thesis is to provide such a construction and to apply it to compute the equivariant cohomology of projective unions, with particular attention to those arising from Seshadri stratifications of Grassmannians.

Equivariant cohomology forms a bridge between topology and representation theory. It was first introduced by Borel [Bor60] in the context of understanding how cohomological information reflects realizable group actions on a topological space X. When a Lie group G acts on X, the *Borel construction* encodes the symmetries of the action into the topology of X by considering the quotient

$$\mathbb{E}G \times^G X \coloneqq (\mathbb{E}G \times X)/\left((eg, x) \sim (e, gx)\right).$$

Here, $\mathbb{E}G$ denotes a contractible space with a free G-action, known as the *classifying space*. The equivariant cohomology of X is then defined as

$$H^i_G(X) \coloneqq H^i(\mathbb{E}G \times^G X, R),$$

with coefficients in a ring R, typically taken to be a field or the ring of integers. This construction produces a cohomological invariant reflecting both the topology of X and the symmetry imposed by the group action.

Wherever geometry and symmetry interact, equivariant cohomology plays a decisive role. In algebraic geometry, it provides a framework for studying varieties with group actions, including computations of intersection-theoretic invariants [AF24; Bri98]. In representation theory, it supports geometric constructions of representations, such as the Springer correspondence [CG97]. In symplectic topology, it plays a central role in the study of Hamiltonian group actions and moment maps [AB84]. Combinatorially, it provides explicit descriptions in the case of toric varieties, where the equivariant cohomology is determined by the fan [Ful93], and in Schubert calculus, where it encodes the combinatorics of Weyl groups, root systems, and the Bruhat order [BL00].

A central feature of equivariant cohomology is localization, which allows global invariants to be accessed through fixed-point data. For a torus action, the restriction map

$$H^*_T(X) \longrightarrow H^*_T(X^T),$$

often reduces complicated cohomological problems to explicit computations at the fixed points. When X^T is finite, $H^*_T(X)$ is frequently free over the symmetric algebra on the character lattice of T (see Definition 2.26), and under favorable conditions it can be recovered entirely from its restriction to X^T. This principle forms the theory of Goresky–Kottwitz–MacPherson [GKM98], which provides an explicit combinatorial description of $H^*_T(X)$ in terms of labeled graphs encoding the orbit structure.

An important tool in homology theory, the *Mayer–Vietoris sequence* was originally developed by Mayer to prove a conjecture posed to him by Vietoris concerning Betti numbers. Its first appearance in the modern exact sequence form is in [ES52, Theorem 15.3], where it is introduced in the context of generalized homology theory. Consequently, the Mayer–Vietoris sequence applies to any (co)homology theory satisfying the Eilenberg–Steenrod axioms, taking the form of a long exact sequence

$$\ldots \longleftarrow h_{n-1}(X \cap Y) \longleftarrow h_n(X \cup Y) \longleftarrow h_n(X) + h_n(Y) \longleftarrow h_n(X \cap Y) \longleftarrow \ldots$$

which relates the homologies in a proper triad $(X \cup Y; X, Y)$. Classical examples of its frequent use include the computation of the homology of a sphere or the Klein bottle and the standard proof of the excision theorem relies on the Mayer–Vietoris principle ([Hat02, Theorem 2.20]).

A natural generalization of the Mayer–Vietoris sequence arises when considering a space X covered by a countable collection of subspaces $(X_i)_{i \in I}$. In the cohomological setting, one seeks to express $H^*(X)$ in terms of the cohomologies $H^*(X_i)$ of the covering pieces and the behavior of their intersections. This leads to the *Mayer–Vietoris spectral sequence*, with first page

$$E_1^{p,q} = \bigoplus_{i_0 < \ldots < i_p} H^q(X_{i_0} \cap \ldots \cap X_{i_p}),$$

and converging to $H^{p+q}(X)$ under suitable conditions (see Remark 3.13). An important simplicial aspect of this construction is that the differential on the first page is induced by the coboundary operator of the cochain complex associated with the nerve $\mathcal{N}(I)$ of the cover (see Definition 2.5). In the particularly convenient case where every non-empty finite intersection $X_{i_0} \cap \ldots \cap X_{i_p}$ is contractible, the spectral sequence degenerates at the first page, which then identifies with the simplicial cochain complex of $\mathcal{N}(I)$. In this situation, one recovers the *nerve lemma*, asserting that $H^*(X)$ is isomorphic to the simplicial cohomology of $\mathcal{N}(I)$ (see Theorem 2.6).

To our knowledge, the earliest constructions of the Mayer–Vietoris spectral sequence appear in [BT82] and [God73], where it is developed as a tool to relate Čech cohomology $\check{H}^q(X, \mathcal{F})$ to sheaf cohomology $H^q(X, \mathcal{F})$. For a good cover (that is, one in which all finite non-empty intersections are acyclic in positive degrees), the spectral sequence collapses to the Čech complex, yielding an isomorphism $\check{H}^q(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$ ([BT82, Theorem 8.9]). By Serre's vanishing theorem, if $\mathcal{F}$ is a quasi-coherent sheaf, then any cover by affine varieties provides an example of a good cover ([CLS11, Theorem 9.0.3]).

In recent years, the spectral sequence has found applications in diverse areas, including persistent homology [Tor23; LSV11; JT24], bounded cohomology [FM19], local cohomology [CHN23; Pas24], and the study of spherical arrangements [JOS94], each requiring adaptation to the relevant framework.

The Mayer–Vietoris sequence is widely used as a computational tool in the equivariant versions of various cohomological theories, including singular cohomology, symplectic cohomology, and K-theory (see, e.g., [FP07], [Ahn24], [HW18]). In these cases, the sequence is typically applied to two subspaces or in situations where inductive arguments suffice. By contrast, a systematic treatment of the Mayer–Vietoris spectral sequence in equivariant cohomology has not yet been developed.

The goal of this thesis is twofold. The first aim is to develop a framework for the Mayer–Vietoris spectral sequence in the setting of equivariant cohomology, clarifying for which covers the spectral sequence computes the cohomology and how the characteristic features of equivariant cohomology, such as its algebra structure over the equivariant coefficient ring, localization techniques, functoriality, and the GKM method, interact with the Mayer–Vietoris principle. The second aim is to apply this framework to the computation of equivariant cohomology for projective unions. As discussed, these varieties arise naturally as limit spaces in semi-toric degenerations and, even beyond this context, provide compelling test cases: their individual components are cohomologically simple, while the essential complexity lies in the gluing process. In representation-theoretic settings such as the degeneration of Grassmannians, the combinatorics governing the original variety persist in the union and are reflected in its equivariant cohomology. The Mayer–Vietoris spectral sequence provides the natural tool for accessing these global invariants from the cohomology of the components and their intersections.

Structurally, the thesis reflects this twofold aim. The first part, consisting of Chapter 2 to Chapter 4, provides the general setup and develops results for the Mayer–Vietoris spectral sequence, specifically in the torus-equivariant setting. The second part, i.e., Chapter 5 and Chapter 6, computes the equivariant cohomology of projective unions, with degenerated Grassmannians forming the motivating class of examples that is treated in further detail in the final chapter.

Throughout the first part, we work with coefficients in R, a Noetherian unique factorization domain of characteristic zero.

Chapter 2 serves as the preliminaries chapter, collecting definitions, standard results from the literature, and fixing notation. After briefly introducing the necessary notation for simplicial cohomology, Chapter 2.1 provides a more detailed introduction to equivariant cohomology, while Chapter 2.2 places particular emphasis on the concept of localization. The general presentation of Chapter 2.1 follows [AF24], while the discussion of localization draws primarily from [AF24], but also includes results from [Fra24], [FY19], and [AB84] to give additional context.

The necessary background on spectral sequences is taken from [McC01] and [God73], and is summarized in Appendix A and Appendix B.

Chapter 3.1 is devoted to the construction of the double complex underlying the Mayer–Vietoris spectral sequence. While this setup is standard, our original contribution begins with endowing it with an algebra structure and adapting it to the equivariant setting by introducing a multiplication over the equivariant coefficient ring (Definition 3.4, Lemma 3.7).

In Chapter 3.2, we derive the spectral sequence associated to this double complex and clarify which types of covered spaces satisfy the necessary conditions for the Mayer–Vietoris spectral sequence to apply (Theorem 3.22, Remark 3.13). Particular attention is given to its first page, which we refer to as the *Mayer–Vietoris complex*. In the cases relevant to this thesis, the spectral sequence collapses at the second page, leading to the following result, which forms the basis for our further computations.

Theorem 3.28. *Let X be a topological space with a continuous action of a Lie group G. Assume the cover $\mathfrak{M}$ by G-invariant subspaces is a good cover of X. Then the Mayer–Vietoris complex has cohomology isomorphic to the associated graded algebra of $H^*_G(X)$ as bigraded algebra over the equivariant coefficient ring Λ_G of G.*

As usual for spectral sequences, the target is the associated graded algebra $G(H^*_G(X))$ of $H^*_G(X)$, arising from the filtration induced by the double complex. Within this graded algebra, we single out the *first-column component*, defined as

$$\nu(X) \coloneqq H^*_G(X)/(H^*_G(X))_1,$$

namely the degree-zero summand of $G(H^*_G(X))$, equivalently the degree-zero cohomology of the Mayer–Vietoris complex (Definition 3.25). In general, we do not solve the full extension problem from $G(H^*_G(X))$ to $H^*_G(X)$. Instead, we extract useful information about $H^*_G(X)$, for example by relating $\nu(X)$ to its torsion-free part, by studying morphisms into spaces with trivial filtration, or by comparing cohomologies of different spaces through their associated graded algebras (see, e.g, Lemma 4.2, Corollary 3.42, Theorem 5.73). In other cases we are content to work with the associated graded algebra itself.

By formalizing the framework of covered spaces with a group action and introducing suitable morphisms, we define the category **GCov** (Chapter 3.3) and establish the functorial properties of the Mayer–Vietoris spectral sequence (Corollary 3.40).

This primarily serves to prepare the ground for the study of the spectral sequence in the torus-equivariant setting in Chapter 4, where we examine the compatibility of classical properties of equivariant cohomology with the Mayer–Vietoris principle. Although the discussion could, in principle, be carried out for more general groups, we restrict to torus actions. This restriction is both natural in the context of localization and sufficient for the applications pursued in this thesis.

By considering the restriction of covered spaces to torus fixed points, Chapter 4.1 establishes results on torsion in equivariant cohomology. In particular, we describe the image and kernel of the localization map and relate them to the first-column component $\nu(X)$ (Lemma 4.2). We then investigate under which conditions GKM-type properties lift from the covering pieces to the entire space:

Theorem 4.10. *Let T be a torus, X a fixed-point closed T-space, and $\mathfrak{M} = (M_i)_{i \in I}$ a good cover of X. Suppose that the Mayer–Vietoris complex is torsion-free. If the set of T-fixed points in X is finite and, for all $i \in I$, the localization of M_i is described by its moment graph, then the localization of X is described by its moment graph.*

Group change is another example of a concept from equivariant cohomology that integrates well with the Mayer–Vietoris principle:

A morphism of tori $\varphi\colon T' \to T$ induces a morphism of equivariant coefficient rings $\hat{\varphi}\colon \Lambda_T \to \Lambda_{T'}$. If a topological space X admits compatible actions of T and T', the *restriction of tori* for X is the induced map

$$H_T^*(X) \longrightarrow H_{T'}^*(X).$$

For a covered space, this map arises from a morphism of Mayer–Vietoris complexes

$$\mathrm{MV} \longrightarrow \mathrm{MV}'$$

corresponding to the two torus actions. Under suitable conditions, including that the cover consists of *equivariantly formal spaces* (Definition 2.26), this restriction of tori coincides with the canonical map

$$H(\mathrm{MV}) \longrightarrow H\left(\Lambda_{T'} \otimes_{\Lambda_T} \mathrm{MV}\right),$$

obtained by first extending scalars along $\hat{\varphi}$ on MV and then passing to cohomology, as shown in Lemma 4.37. Building on this, we employ Künneth formulas as developed in [McC01] to describe torus change for covered spaces in Corollary 4.41 and Theorem 4.43, and employ Koszul resolutions (Corollary 4.49) as a computational tool, following [Eis95], for evaluating the Künneth formula in special cases.

We now turn to the second part of the thesis, where the Mayer–Vietoris spectral sequence is applied to projective unions and degenerated Grassmannians. For simplicity, and although many results remain valid under weaker assumptions, we specialize to the case where R is a field of characteristic zero in the final two chapters.

Chapter 5 introduces *projective unions* in an ambient projective space $\mathbb{P}$, defined by

$$P_{\mathfrak{C}} := \bigcup_{i \in I} P_{C_i}, \qquad \mathfrak{C} = (C_i)_{i \in I},$$

where the homogeneous coordinates of $\mathbb{P}$ are indexed by A, and each subset $C_i \subseteq A$ determines a coordinate projective subspace $P_{C_i} \subseteq \mathbb{P}$. The projective union $P_{\mathfrak{C}}$ inherits a torus action from the action of T on the homogeneous coordinates of $\mathbb{P}$.

After establishing the applicability of the Mayer–Vietoris spectral sequence by constructing a suitable retraction (Chapter 5.1, Lemma 5.12), we begin our study of the Mayer–Vietoris complex MV in Chapter 5.2. As emphasized in the first part of the thesis,

$$G(H_T^*(P_{\mathfrak{C}})) \cong H(\mathrm{MV}),$$

and the central task is therefore to describe its cohomology. Our first approach gives a direct description of the first-column component together with an interpretation in terms of syzygy modules

(Remark 5.23, Corollary 5.29). In the case of a generic (i.e., maximally independent) torus action, we obtain an alternative description by decomposing the Mayer–Vietoris complex into simplicial complexes, yielding a splitting into simplicial cohomologies.

Lemma 5.52. *Assume that T acts generically on the projective union $P_{\mathfrak{C}}$. Then*

$$G(H_T^*(P_{\mathfrak{C}})) \cong \Lambda_K[\zeta] \otimes_R \left(\bigoplus_{S \subseteq A} \Lambda_S \otimes_R H^*(\Delta_S) \right).$$

Here the simplicial complexes Δ_S are defined by recording which intersections in $\mathfrak{C}$ fail to contain a given subset $S \subseteq A$. The coefficients for each simplicial cohomology are taken in the corresponding subring $\Lambda_S \subseteq \Lambda_T$ of the equivariant coefficient ring of T, while the factor $\Lambda_K[\zeta]$ accounts for the subtorus of T that acts trivially on $P_{\mathfrak{C}}$.

A dual description is developed in Chapter 5.6, where we study *poset unions*, i.e., projective unions defined by the combinatorics of a poset: If the homogeneous coordinates of $\mathbb{P}$ are indexed by a poset A, then the collection $\mathfrak{C}$ can be chosen as the collection of maximal chains in A. For generic torus actions, the equivariant cohomology of the resulting projective union P_A is then described in terms of the poset cohomology of subposets $S \subseteq A$:

Lemma 5.82. *Assume that T acts generically on the poset union P_A. Then*

$$G(H_T^*(P_A)) \cong \Lambda_K[\zeta] \otimes \left(\bigoplus_{S \subseteq A} \Lambda_{A \setminus S} \otimes H^*(S) \right).$$

In Chapter 5.3 and Chapter 5.5, we adapt the methods of Chapter 4 to projective unions.

Chapter 5.3 applies the results of Chapter 4.1: we examine torsion, describe the first-column component, and verify the GKM-type localization for projective unions (Corollaries 5.37, 5.38, Lemma 5.44). In addition, Lemma 5.40 identifies when a restriction between projective unions coincides with restriction to fixed points.

Torus change for projective unions is studied in Chapter 5.5, motivated by the fact that for arbitrary torus actions the cohomology can be related to the generic case via restriction of tori. Lemma 5.62 describes this process using a method analogous to the Künneth formulas of Chapter 4.2, while Theorem 5.73 introduces a recursive approach based on *semi-regular sequences* (Definition 5.71, Lemma 5.75) to study surjectivity of torus restrictions.

In this setting, we assume that the map on equivariant coefficient rings $\hat{\varphi}\colon \Lambda_T \to \Lambda_{T'}$ describing the torus change *introduces relations on* Λ_T, i.e., it takes the form of a quotient $\Lambda_T/\mathcal{R} \to \Lambda_{T'}$ (Definition 4.18). This is precisely the case relevant in Chapter 6.2. The quotient map

$$\mathrm{MV} \longrightarrow \overline{\mathrm{MV}} \coloneqq \mathrm{MV}/\mathcal{R}\,\mathrm{MV}$$

then induces the restriction of tori in cohomology,

$$H(\mathrm{MV}) \longrightarrow H(\overline{\mathrm{MV}}).$$

Given a sequence $a_1, \ldots, a_n$ in A, one considers for each partition $U \sqcup V = \{a_1, \ldots, a_n\}$ the restriction of tori for simplified Mayer–Vietoris complexes MV_V^U. If the sequence satisfies certain torsion conditions (Lemma 5.75), this yields a criterion for the surjectivity of the original restriction.

Theorem 5.73. *Suppose $a_1, \dots, a_n$ is a semi-regular sequence in A and that*

$$H(\mathrm{MV}_V^U) \longrightarrow H(\overline{\mathrm{MV}}_V^U), \quad [\![x]\!] \longmapsto [\![x]\!],$$

is surjective for all partitions U, V of $\{a_1, \dots, a_n\}$. Then the restriction of tori

$$H(\mathrm{MV}) \longrightarrow H(\overline{\mathrm{MV}}),$$

is surjective.

The final chapter of the thesis turns to the motivating example for studying equivariant cohomology of projective unions: the degenerated Grassmannian. Choosing Schubert varieties as strata and Plücker coordinates as extremal functions, the resulting Seshadri stratification of $\mathrm{Gr}(d, n)$ yields a union of projective spaces governed by combinatorial data closely related to that of the Grassmannian itself. Concretely, the degenerated Grassmannian $P_{d,n}$ is the poset union associated with the poset of Schubert varieties of $\mathrm{Gr}(d, n)$ ordered by the Bruhat order. Here we identify the poset of Schubert varieties with the poset $I(d, n)$ of strictly increasing d-tuples in $1, \dots, n$.

The poset description developed in Chapter 5.6 provides a natural framework for analyzing equivariant cohomology of $P_{d,n}$ under a generic torus action. Within this framework, Theorem 6.8 gives a compact description of the image of the restriction map

$$r \colon G(H_T^*(P_{2,n+1})) \longrightarrow G(H_T^*(P_{2,n})),$$

between the equivariant cohomology of degenerated Grassmannians. After fixing suitable basis elements g_S^i for the poset cohomology of any subposet $S \subseteq I(2, n)$, an arbitrary element can be expanded as

$$\underline{f} = \sum_{S,i} \beta_{S,i} g_S^i \in G(H_T^*(P_{2,n})).$$

In Definition 6.12 we introduce integers $l_S(i)$ that measure how long a basis element g_S^i persists when S is extended as a subposet of $I(2, n+1)$. The monomial $\eta_{l_S(i)}$ compensates this defect, leading to the following divisibility criterion:

Theorem 6.8. *We can choose bases of $H^*(S)$, for $S \subseteq I$, such that an element*

$$\underline{f} = \sum_{S,i} \beta_{S,i} g_S^i \in G(H_T^*(P_{2,n}))$$

lies in the image of r if and only if

$$\beta_{S,i} \text{ is divisible by } \eta_{l_S(i)}, \quad \text{for all } S, i.$$

In Chapter 6.2, we finally consider the degenerated Grassmannian with the torus action inherited from the Grassmannian itself. For this *Grassmannian torus action* (Definition 6.14), we compute in Example 6.19 the equivariant cohomology of $P_{2,5}$, using semi-regular sequences and the method of Theorem 5.73.

With respect to the Grassmannian torus action, the degenerated Grassmannian is a union of GKM-varieties and in particular contains finitely many points fixed by this action. In consequence, the torsion-free part of the equivariant cohomology of $P_{d,n}$ is given by its localization image which can be computed either with the help of Corollary 6.33 or by considering the first-column component (Corollary 5.38).

To describe the latter, we use the identification of maximal chains $\mathfrak{C}(d,n)$ in $I(d,n)$ with the set $\mathcal{SYT}(d,n)$ of standard Young tableaux of rectangular shape $d \times (n-d)$, as explained in Remark 6.5. The symmetric group on $d(n-d)$ symbols acts on unordered tableaux of this shape, and although this action does not restrict to the subset $\mathcal{SYT}(d,n)$, it motivates considering when tableaux are related by simple transpositions of consecutive integers $i, i+1$ (Remark 6.20).

We then define the graph $G_{d,n}$ with vertex set $\mathcal{SYT}(d,n)$ (equivalently $\mathfrak{C}(d,n)$), and edges corresponding to such transpositions, each labeled by a polynomial determined by the intersection of the associated maximal chains. The first-column component of $P_{d,n}$ then consists of tuples $(f_C)_{C\in\mathfrak{C}(d,n)}$ of polynomials in the equivariant cohomology rings of the projective spaces P_C, where $C \in \mathfrak{C}(d,n)$, subject to the following divisibility condition:

Theorem 6.32. *A tuple $(f_C)_{C\in\mathfrak{C}(d,n)}$ lies in $\nu(P_{d,n})$ if and only if*

$$f_C - f_D \in (\eta_{C\cap D}),$$

for all C, D that share an edge in $G_{d,n}$.

Finally, the following Corollary compares the equivalent descriptions for the torsion-free part of the equivariant cohomology of the degenerated Grassmannian when considered with respect to the Grassmannian torus action.

Corollary 6.34. *Consider the Grassmannian torus action of the diagonal torus $T \subseteq \mathrm{GL}_n$ induced by its action via standard characters $e_1, \dots, e_n$ on $\mathbb{C}^n$. The torsion-free part of $H_T^*(P_{d,n})$ is isomorphic to the localization image of $P_{d,n}$. Further, we obtain as equivalent descriptions*

$$\iota(P_{d,n}) = \{(u_I)_{I\in I(d,n)} \in \bigoplus_{I(d,n)} \Lambda_T \mid u_I - u_{I'} \textit{ is divisible by } \chi_I - \chi_{I'} \textit{ for all } I \le I'\},$$

where $\chi_I = \sum_{j\in I} e_j$, and

$$\iota(P_{d,n}) \cong \nu(P_{d,n})$$

$$= \Big\{(f_C)_{C\in\mathfrak{C}(d,n)} \in \bigoplus_{C\in\mathfrak{C}(d,n)} H_T^*(P_C) \mid f_C - f_D \in (\eta_{C\cap D}) \textit{ for all } C, D \textit{ that share an edge in } G_{d,n}\Big\}.$$

2 Preliminaries

In the first part of the thesis (Chapters 2–4), we assume that R is a Noetherian unique factorization domain of characteristic zero.

We briefly fix notation and conventions for simplicial cohomology. For background material, see [Hat02, Chapter 2.1].

Definition 2.1. A *simplicial complex* Δ on a finite vertex set V is a non-empty collection of subsets of V such that if $A \in \Delta$, then every subset $B \subseteq A$ also lies in Δ. The $(p+1)$-element subsets in Δ are called the *p-simplices* of Δ. We denote the set of p-simplices by Δ_p, and a simplex in Δ_p is said to have *length p*.
After fixing a total ordering on V, every p-simplex of Δ can be expressed as a strictly increasing tuple $(v_0, \dots, v_p)$ of elements of V. For simplices $\underline{v}$ and $\underline{w}$ we write $\underline{v} \subseteq \underline{w}$ if $\underline{v}$ is a *face* of $\underline{w}$, that is, if $\underline{v}$ is a subtuple of, or contained as a set in, $\underline{w}$. The 0-simplices, called the *vertices* of Δ, are considered either as one-element subsets or as elements of V. In particular, we write $v \in \underline{v}$ if the vertex v is a face of $\underline{v}$.
Given a set $\mathfrak{D}$ of subsets of V, we define the simplicial complex $\langle \mathfrak{D} \rangle$ to consist of all subsets of sets in $\mathfrak{D}$. The simplicial complex consisting of all subsets of V is called the *full simplex* on V.

Convention 2.2. Throughout this thesis, we allow set-theoretic operations to be applied to strictly increasing tuples. That is, for an element $v \in V$ and $\underline{v}, \underline{w}$ strictly increasing tuples with entries in V, we regard $\underline{v}, \underline{w}$ as subsets of V, perform the operations

$$\underline{v} \cap \underline{w}, \quad \underline{v} \cup \underline{w} \quad \underline{v} \setminus \underline{w}, \quad \text{and} \quad \underline{v} \setminus v := \underline{v} \setminus \{v\},$$

and reinterpret the resulting sets as strictly increasing tuples. Similarly, the relations

$$v \in \underline{v}, \quad \text{and} \quad \underline{w} \subseteq \underline{v},$$

are understood via this identification.

Let Δ be a simplicial complex with vertex set V. Given $p \in \mathbb{Z}_{\geq 0}$, we write $C_p(\Delta, R)$, or simply $C_p(\Delta)$, for the module of *simplicial p-chains* with coefficients in R, the free R-module with basis given by the p-simplices of Δ.
The boundary map $\partial_p \colon C_p(\Delta) \to C_{p-1}(\Delta)$ is defined by

$$\partial_p(v_0, \dots, v_p) = \sum_{j=0}^{p} (-1)^j (v_0, \dots, \hat{v}_j, \dots, v_p),$$

where we set $\partial_0 = 0$. Since $\partial_{p-1}\partial_p = 0$, we obtain the *chain complex* $(C_*(\Delta), \partial)$ of R-modules with homology denoted by

$$H_p(\Delta) = \ker \partial_p / \operatorname{im} \partial_{p+1}, \quad p \geq 0.$$

The graded R-module $H_*(\Delta)$ is the *simplicial homology* of Δ.

Define $C^p(\Delta, R)$, or simply $C^p(\Delta)$, the module of *simplicial p-cochains* of Δ, as the free R-module dual to $C_p(\Delta)$. The coboundary map d is defined as the dual map to the boundary map ∂. In other words,

$$C^p(\Delta) = \hom(C_p(\Delta), R),$$

and

$$d^p \colon C^p(\Delta) \longrightarrow C^{p+1}(\Delta),$$
$$(d^p(\sigma))(v_0, \dots, v_p) = \sum_{j=0}^{p} (-1)^j \sigma(v_0, \dots, \hat{v}_j, \dots, v_p),$$

for $\sigma \in \hom(C_p(\Delta), R)$ and $p \in \mathbb{Z}_{\geq 0}$. The resulting complex $(C^*(\Delta), d)$ is called the *cochain complex* of Δ.

The product of two simplices $\underline{v} = (v_0, \dots, v_p)$ and $\underline{w} = (w_0, \dots, w_q)$ is defined to be the $(p+q)$-simplex

$$\underline{v} \cdot \underline{w} := (v_0, \dots, v_p, w_1, \dots, w_q),$$

if $v_p = w_0$ and $(v_0, \dots, v_p, w_1, \dots, w_q) \in \Delta_{p+q}$. Otherwise, the product is defined to be zero.

If we write $e_{(v_0,\dots,v_p)}$ for the basis element in $C^p(\Delta)$ dual to the p-simplex $(v_0, \dots, v_p)$ in Δ, then the cup product $\smile$ on $C^*(\Delta)$ is given by

$$e_{(v_0,\dots,v_p)} \smile e_{(w_0,\dots,w_q)} = e_{(v_0,\dots,v_p)\cdot(w_0,\dots,w_q)}.$$

Taking the cohomology of the cochain complex $(C^*(\Delta), d)$, we obtain

$$H^p(\Delta) := \ker d^p / \operatorname{im} d^{p-1}, \quad p \geq 0,$$

the *simplicial cohomology* of Δ. Since the differential d satisfies the Leibniz rule with respect to the cup product $\smile$ on $C^*(\Delta)$, the product descends to cohomology and induces a graded multiplication

$$\cdot \colon H^p(\Delta) \times H^q(\Delta) \longrightarrow H^{p+q}(\Delta),$$

which turns $H^*(\Delta)$ into a graded-commutative R-algebra with unit.

Example 2.3. Let Δ be the full simplex on V. It is a well-known fact that Δ is *acyclic*, meaning that the cochain complex $(C^*(\Delta), d)$ is exact in every positive degree. In particular,

$$H^0(\Delta) = R, \quad \text{and} \quad H^p(\Delta) = 0 \quad \text{for } p > 0.$$

Remark 2.4. Given two simplicial complexes Δ_1 and Δ_2 on a common vertex set V, both their union $\Delta := \Delta_1 \cup \Delta_2$ and their intersection $\Delta_1 \cap \Delta_2$ are simplicial complexes on V. The inclusion of $\Delta_1 \cap \Delta_2$ into Δ_1 and Δ_2 induces restriction morphisms

$$r_1 \colon H^*(\Delta_1) \longrightarrow H^*(\Delta_1 \cap \Delta_2), \quad r_2 \colon H^*(\Delta_2) \longrightarrow H^*(\Delta_1 \cap \Delta_2),$$

and the Mayer–Vietoris sequence computes the cohomology of Δ via the long exact sequence

$$\dots \longrightarrow H^p(\Delta) \longrightarrow H^p(\Delta_1) \oplus H^p(\Delta_2) \xrightarrow{r_1 - r_2} H^p(\Delta_1 \cap \Delta_2) \longrightarrow H^{p+1}(\Delta) \longrightarrow \dots$$

(See [Hat02, Chapter 2.2].)

If a simplicial complex Δ is covered by suitable subcomplexes, then its cohomology $H^*(\Delta)$ is isomorphic to the cohomology of the nerve of the cover.

Definition 2.5. Let Δ be a simplicial complex and $(\Delta_i)_{i \in I}$ a cover by subcomplexes, i.e., $\Delta = \bigcup_{i \in I} \Delta_i$. The *nerve of the cover* is the simplicial complex $\mathcal{N}((\Delta_i)_{i \in I})$ on I, whose p-simplices are the tuples $(v_0, \dots, v_p)$ such that

$$\Delta_{v_0} \cap \dots \cap \Delta_{v_p} \neq \varnothing.$$

Theorem 2.6. *Let Δ be a simplicial complex, and $(\Delta_i)_{i\in I}$ a cover by subcomplexes. If every non-empty intersection*

$$\Delta_{v_0} \cap \ldots \cap \Delta_{v_p}, \quad \text{where } v_0, \ldots, v_p \in I,$$

is acyclic, then Δ is homotopy equivalent to the nerve of the cover $\mathcal{N}((\Delta_i)_{i\in I})$.

Proof. [Bjö95, Theorem 10.6]. □

2.1 Equivariant Cohomology

The central algebraic invariant that we are concerned with in this thesis is equivariant cohomology with respect to a torus operation. First of all, we recall the definition of equivariant cohomology as introduced in [AF24] and subsequently discuss its fundamental properties. The focus remains on how this theory specializes to the case where the acting group is a torus, particularly in the context of localization, which will be treated in Chapter 2.2.
We start with the definition of equivariant cohomology using the *Borel construction* as proposed in [AF24, Chapter 2].
Let G be a Lie group. A fiber bundle

$$q\colon \mathbb{E} \longrightarrow \mathbb{B},$$

is called a *principal G-bundle* if G acts freely on $\mathbb{E}$ (on the right), the map q is isomorphic to the quotient map $\mathbb{E} \to \mathbb{E}/G$, and the base space $\mathbb{B}$ is both Hausdorff and paracompact. The last requirement ensures the existence of a *universal principal G-bundle* $\mathbb{E}G \to \mathbb{B}G$ with the property that any principal G-bundle $q\colon \mathbb{E} \to \mathbb{B}$ is the pullback of a map $\varphi\colon \mathbb{B} \to \mathbb{B}G$ in the following diagram.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & \mathbb{E}G \\ \downarrow q & & \downarrow \\ \mathbb{B} & \xrightarrow{\varphi} & \mathbb{B}G \end{array}$$

Here, the *classifying map* φ is defined uniquely up to homotopy. In fact, a principal bundle $\mathbb{E} \to \mathbb{B}$ is universal if and only if $\mathbb{E}$ is contractible.
Suppose X is a topological space equipped with a (continuous) left action by G. The *G-equivariant cohomology* of X with coefficients in R is a graded-commutative R-algebra $H^*_G(X)$. To define the graded components $H^p_G(X)$ in any range $p < N$, where N is a positive integer or ∞, consider a principal G-bundle $\mathbb{E} \to \mathbb{B}$ such that $\mathbb{E}$ is path-connected and satisfies $H^p(\mathbb{E}) = 0$ for $p < N$. The *G-balanced* product of $\mathbb{E}$ and X is defined as the quotient space

$$\mathbb{E} \times^G X := \mathbb{E} \times X / \sim,$$

where $(eg, x) \sim (e, gx)$ for all $e \in \mathbb{E}, x \in X$, and $g \in G$. Finally, the p-th equivariant cohomology group of X is defined by

$$H^p_G(X) := H^p(\mathbb{E} \times^G X, R), \quad p < N,$$

where $H^p(\cdot, R)$ denotes singular cohomology with coefficients in R. In [AF24, Proposition 2.2.2], it is shown that this definition is independent of the choice of the principal bundle $q\colon \mathbb{E} \to \mathbb{B}$. Consequently, we may fix q to be the universal principal bundle and since $\mathbb{E}G$ is contractible, we obtain the definition

$$H^p_G(X) := H^p(\mathbb{E}G \times^G X, R), \quad p \geq 0.$$

A first important distinction is that $H_G^*(X)$ carries a richer algebra structure than its counterpart $H^*(X)$ in ordinary singular cohomology. First of all, any G-equivariant morphism $f\colon X \to X'$ induces a morphism

$$\mathbb{E}G \times^G X \longrightarrow \mathbb{E}G \times^G X', \quad (e, x) \longmapsto (e, f(x)),$$

and hence a morphism of R-algebras

$$f^*\colon H_G^*(X') \longrightarrow H_G^*(X).$$

In particular, the trivial map to the one-point space $X \to \mathrm{pt}$, equips $H_G^*(X)$ with the structure of an algebra over the R-algebra

$$\Lambda_G \coloneqq H_G^*(\mathrm{pt}),$$

which we will refer to as the *equivariant coefficient ring* of G. Further, as seen from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ & \searrow \quad \swarrow & \\ & \mathrm{pt} & \end{array}$$

the induced map $f^*\colon H_G^*(X') \to H_G^*(X)$ is a morphism of Λ_G-algebras.

Ordinary cohomology can be seen as a special case of equivariant cohomology by considering the trivial action of the one-element group $G = 1$. In this situation, the universal principle bundle is the identity on the one-point space $q\colon \mathrm{pt} \to \mathrm{pt}$, the Borel construction for a topological space X yields,

$$\begin{array}{c} \mathrm{pt} \times^1 X = X \\ \downarrow \\ \mathrm{pt} \end{array}$$

and following the definition of equivariant cohomology, we obtain $H_1^*(X) = H^*(X)$.
Returning to the setting that X admits the action of a Lie group G, we can compare ordinary and G-equivariant cohomology of X by viewing $\mathbb{E}G \times^G X$ as a fiber bundle with base space $\mathbb{B}G$ and fiber X, via projection on the first factor in $\mathbb{E}G \times^G X \to \mathbb{B}G$.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{E}G \times^G X \\ \downarrow & & \downarrow \\ \mathrm{pt} & \hookrightarrow & \mathbb{B}G \end{array}$$

By examining pullbacks in the diagram above, we obtain a forgetful morphism of Λ_G-algebras,

$$H_G^*(X) \longrightarrow H^*(X),$$

where $H^*(X)$ is interpreted as Λ_G-algebra via the canonical map

$$\Lambda_G = H^*(\mathbb{B}G) \longrightarrow R = H^*(\mathrm{pt}).$$

This comparison is a specific instance of the property that equivariant cohomology is functorial in both the group and the space:

Suppose X and X' are topological spaces with actions by Lie groups G and G', respectively. Let $\varphi\colon G \to G'$ be a continuous group homomorphism. A morphism $f\colon X \to X'$ is *equivariant with respect to* φ if

$$f(gx) = \varphi(g)f(x), \quad \text{for all } x \in X \text{ and } g \in G.$$

Lemma 2.7. *Let $f\colon X \to X'$ be a morphism that is equivariant with respect to a group homomorphism φ. In this case, f induces a ring homomorphism*

$$f^*\colon H^*_{G'}(X') \longrightarrow H^*_G(X).$$

In particular, φ induces a ring homomorphism

$$\Lambda_{G'} \longrightarrow \Lambda_G,$$

and viewing $H^*_G(X)$ as a $\Lambda_{G'}$-algebra via restriction of scalars along this map, the morphism f^* becomes a morphism of $\Lambda_{G'}$-algebras.

Proof. [AF24, Chapter 3.2]. □

Lemma 2.8. *In the situation of Lemma 2.7, if both $\varphi\colon G \to G'$ and $f\colon X \to X'$ are homotopy equivalences, then f^* is an isomorphism of $\Lambda_{G'}$-algebras.*

Proof. [AF24, Corollary 3.3.4]. □

The two opposite extremes of a G-action on a space X are the free and the trivial action. In the case of a trivial action, equivariant cohomology is given by an extension of scalars from ordinary cohomology.

Lemma 2.9. *Suppose G acts trivially on X, and that Λ_G is a free R-module. Then*

$$H^*_G(X) \cong \Lambda_G \otimes_R H^*(X),$$

as Λ_G-algebras.

Proof. [AF24, Proposition 3.4.2]. □

We write $G \setminus X$ for the quotient space of X by the action of G. The following fact partly motivates the way equivariant cohomology is defined in [Bor60].

Lemma 2.10. *Suppose G acts freely on X. Then $H^p_G(X) = H^p(G \setminus X)$ for all $p \geq 0$.*

Proof. [AF24, Proposition 3.4.1]. □

Remark 2.11. Given two Lie groups G, G', we have that $\mathbb{E}G \times \mathbb{E}G' \to \mathbb{B}G \times \mathbb{B}G'$ is the universal principal bundle for $G \times G'$. It follows from Lemma 2.9, that if either Λ_G or $\Lambda_{G'}$ is free over R, then the map $\Lambda_G \times \Lambda_{G'} \to \Lambda_{G\times G'}$ induces an isomorphism of algebras

$$\Lambda_G \otimes \Lambda_{G'} \longrightarrow \Lambda_{G\times G'}.$$

(See [AF24, Exercise 3.2.4].)

The last concept we want to generalize from ordinary to equivariant cohomology is that of Chern classes. A vector bundle $E \to X$ is called a *G-equivariant vector bundle* on X if G acts linearly on the fibers in such a way that the projection is equivariant. In other words, there is a morphism of groups $G \to \mathrm{GL}(E)$, and for each $x \in X$ with fiber E_x, the action satisfies $g.E_x \subseteq E_{gx}$. A G-equivariant vector bundle determines an ordinary vector bundle

$$\mathbb{E}G \times^G E \longrightarrow \mathbb{E}G \times^G X,$$

and we can define the *equivariant Chern classes* of $E \to X$ as

$$c_k^G(E) \coloneqq c_k(\mathbb{E}G \times^G E) \in H_G^{2k}(X),$$

i.e., as the ordinary Chern classes of the bundle $\mathbb{E}G \times^G E \to \mathbb{E}G \times^G X$.

Remark 2.12. Suppose $\varphi\colon G \to G'$ is a continuous group homomorphism and $f\colon X \to Y$ is a morphism that is equivariant with respect to φ. A G'-equivariant vector bundle $E' \to X'$ can be regarded as G-equivariant via φ, and the pullback $E \coloneqq f^*E'$ defines a G-equivariant vector bundle on X. The induced morphism $f^*\colon H^*_{G'}(X') \to H^*_G(X)$ then satisfies

$$f^*(c_k^{G'}(E')) = c_k^G(E) = c_k^G(f^*E').$$

In the second half of this subsection, we take an initial look at the case where G is a torus. We begin by introducing the basic concepts, following [CLS11, Chapter 1.1].

An algebraic group T is called a torus of rank m if it is isomorphic to $(\mathbb{C}^\times)^m$. The character lattice of T, denoted by M, is the group of all algebraic group homomorphisms $T \to \mathbb{C}^\times$, written in additive notation. It is a free abelian group of rank m, and choosing an isomorphism $T \cong (\mathbb{C}^\times)^m$ corresponds to fixing a basis of standard characters $e_1, \dots, e_m$ of $M \cong \mathbb{Z}^m$.

A one-parameter subgroup of T is a morphism of algebraic groups $\mathbb{C}^\times \to T$. We write N for the group of all one-parameter subgroups of T, again using additive notation. The group N is a free abelian group of rank m, and choosing an isomorphism $T \cong (\mathbb{C}^\times)^m$ corresponds to fixing a basis of $N \cong \mathbb{Z}^m$.

Remark 2.13. There is a natural pairing

$$\langle \cdot, \cdot \rangle\colon M \times N \longrightarrow \mathbb{Z},$$

between M and N, where $\langle \chi, \sigma \rangle$ is equal to the exponent associated to the composition

$$\chi \circ \sigma\colon \mathbb{C}^\times \longrightarrow \mathbb{C}^\times.$$

This pairing identifies M with $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, i.e., M and N are dual to each other. The bases induced by an isomorphism $T \cong (\mathbb{C}^\times)^m$ are dual bases of N and M.

Remark 2.14. We have a canonical isomorphism

$$N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong T, \quad \sigma \otimes t \longmapsto \sigma(t),$$

which, after fixing isomorphisms $N \cong \mathbb{Z}^m$ and $T \cong (\mathbb{C}^\times)^m$, can be identified with

$$\mathbb{Z}^m \otimes \mathbb{C}^\times \cong (\mathbb{C}^\times)^m, \quad (b_1, \dots, b_m) \otimes t \longmapsto (t^{b_1}, \dots, t^{b_m}).$$

There is a natural correspondence between the sublattices of a character lattice M and the closed subgroups of its respective torus T. We discuss more details of this correspondence in Chapter 4.2.

Definition 2.15. Suppose $M' \subseteq M$ is a sublattice (i.e., a subgroup). The corresponding subgroup of the torus T is defined as

$$T_{M'} := \{t \in T \mid \chi(t) = 1 \text{ for all } \chi \in M'\}.$$

Conversely, for a subgroup $T' \subseteq T$, the corresponding sublattice of M is given by

$$M_{T'} := \{\chi \in M \mid \chi\,|_{T'} = 0\}.$$

Remark 2.16. The correspondence introduced above is one-to-one: for every sublattice $M' \subseteq M$, there exists a basis $\mathcal{B}$ of M such that a subset of $\mathcal{B}$ generates a sublattice containing M' with finite index. With respect to the corresponding identification $T \cong (\mathbb{C}^\times)^m$, one then checks that $M' = M_{T_{M'}}$. The equality $T' = T_{M_{T'}}$ follows, for example, from [Hum81, Proposition 16.1].

For the remainder of this thesis, whenever a torus T of rank m is considered, we implicitly fix an isomorphism $T \cong (\mathbb{C}^\times)^m$ together with a corresponding basis of standard characters $e_1, \ldots, e_m$ of its character lattice M.

To set up equivariant cohomology with respect to T, we begin by introducing the universal principal T-bundle.

The space $\mathbb{E} = (\mathbb{C}^\infty \setminus \{0\})^m$ is contractible by Kuiper's theorem ([Kui65]), and the coordinate-wise scaling action of T on $\mathbb{E}$ is free. The quotient

$$\mathbb{E}/T \cong (\mathbb{CP}^\infty)^m,$$

is a CW-complex and therefore both paracompact and Hausdorff ([Hat02, p.522] and [Hat03, p.35]), which lets us conclude that

$$q\colon \mathbb{E}T = (\mathbb{C}^\infty \setminus \{0\})^m \longrightarrow \mathbb{B}T = (\mathbb{CP}^\infty)^m,$$

is the universal principal bundle of T.

Let $\mathscr{O}_i(-1)$ denote the tautological line bundle on the ith factor of $(\mathbb{CP}^\infty)^m$, and set $x_i := c_1(\mathscr{O}_i(-1))$ to be its Chern class. The singular cohomology of $(\mathbb{CP}^\infty)^m$, which is equal to the equivariant coefficient ring of T, is given by

$$H^*((\mathbb{CP}^\infty)^m, R) = R[x_1, \ldots, x_m],$$

a graded polynomial ring with indeterminates $x_1, \ldots, x_m$ of degree two. If $m = 0$, then by definition $R[x_1, \ldots, x_m] = R$.

A more intrinsical description of Λ_T is given in terms of the character lattice of T:

Given a character $\chi \in M$ consider the equivariant complex line bundle $\mathbb{C}_\chi$ over a point, defined by the T-action $t \cdot v = \chi(t)v$ for $t \in T$ and $v \in \mathbb{C}_\chi$. Since

$$\mathbb{C}_{\chi_1+\chi_2} = \mathbb{C}_{\chi_1} \otimes \mathbb{C}_{\chi_2}, \quad \chi_1, \chi_2 \in M,$$

the map

$$\theta\colon M \longrightarrow \Lambda_T^{\oplus 2}, \quad \chi \longmapsto c_1^T(\mathbb{C}_\chi),$$

defines a group monomorphism. For the standard characters $e_1, \dots, e_m \in M$, we have

$$c_1^T(\mathbb{C}_{e_i}) = c_1(\mathscr{O}_i(-1)),$$

as can be seen from the fiber product

$$\begin{array}{ccc} (\mathbb{C}^\infty \setminus \{0\})^m \times^T \mathbb{C}_{e_i} & \xrightarrow{\sim} & \mathscr{O}_i(-1) \\ \downarrow & & \downarrow \\ (\mathbb{C}^\infty \setminus \{0\})^m \times^T \mathrm{pt} & \xrightarrow{\sim} & (\mathbb{CP}^\infty)^m \end{array}$$

Therefore, $\theta(e_i) = x_i$, and we obtain an isomorphism

$$\Lambda_T = \operatorname{Sym}_R^* M \coloneqq \operatorname{Sym}^*(M \otimes_{\mathbb{Z}} R).$$

Again note that, by definition of the symmetric algebra, $\operatorname{Sym}_R^*(0) = R$.
For most of this thesis, the topological space X will be a union of projective spaces, and by that argument, the equivariant cohomology of a complex projective space is a natural first example.
Suppose we have a complex vector bundle $E \to X$ of rank r over a complex smooth projective variety X. The projective bundle $\mathbb{P}(E) \to X$, obtained by projectivizing E, is the fiber bundle whose fiber over a point $x \in X$ is the projective space of lines in the fiber E_x. This construction equips $H^*(\mathbb{P}(E))$ with the structure of an $H^*(X)$-algebra. More precisely,

$$H^*\big(\mathbb{P}(E)\big) = \big(H^*(X)[\zeta]\big)/\big(\zeta^r + c_1(E)\zeta^{r-1} + \ldots + c_r(E)\big),$$

where ζ is the Chern class of $\mathscr{O}(1)$, the line bundle dual to the tautological line bundle $\mathscr{O}(-1)$ on $\mathbb{P}(E)$, and $c_1(E), \dots, c_r(E)$ are the Chern classes of $E \to X$ (see [AF24, Section A.5]).

Example 2.17. Suppose a torus T acts on a complex vector space $V = \mathbb{C}^{k+1}$ by

$$t.(v_0, \dots, v_k) = (\chi_0(t)v_0, \dots, \chi_k(t)v_k), \quad t \in T, (v_0, \dots, v_k) \in V,$$

for characters $\chi_0, \dots, \chi_k \in M$. Then the equivariant Chern classes of the T-equivariant vector bundle $V \to \mathrm{pt}$ are given by

$$c_i^T(V) = \mathbf{e}_i(\chi_0, \dots, \chi_k), \quad 0 \le i \le k,$$

where $\mathbf{e}_i$ is the elementary symmetric polynomial of degree i. Identifying the projective bundles

$$\mathbb{E}T \times^T \mathbb{P}(V) \cong \mathbb{P}(\mathbb{E}T \times^T V)$$

over $\mathbb{B}T$, we also identify their respective tautological line bundles. It follows that the equivariant cohomology ring is given by

$$H_T^*\big(\mathbb{P}(V)\big) = \Lambda_T[\zeta]/\big(\zeta^{k+1} + \mathbf{e}_1(\chi_0, \dots, \chi_k)\zeta^k + \ldots + \mathbf{e}_k(\chi_0, \dots, \chi_k)\big),$$

or equivalently,

$$H_T^*\big(\mathbb{P}(V)\big) = \Lambda_T[\zeta]/\left(\prod_{i=0}^{k}(\zeta + \chi_i)\right).$$

See [AF24, Chapter 2.6] for more details.

2.2 Localization

Restriction to fixed points is a method exclusive to equivariant settings and arguably one of the biggest advantages of equivariant cohomology over ordinary cohomology. The theory is especially well-structured when we consider cohomology with respect to a torus as in this case, the restriction to fixed points becomes an isomorphism in cohomology after inverting sufficiently many torus characters. Moreover, a large class of spaces can be described completely in terms of the zero and one-dimensional orbits of the torus action by a technique commonly referred to as the *GKM method*.

We start this subsection by introducing the *main localization theorem* before taking a closer look at the GKM method. Concerning the latter, there are two aspects to consider: under which conditions the method is applicable, and how explicit the resulting description is. Our discussion of these questions is adapted to the general setting of this thesis and for the most part is based on the respective chapters in [AF24]. Nonetheless, we provide sufficient context to relate the results we present to the broader theory (see in particular [Fra24]).

Let T be a torus of rank m with character lattice M, generated by its standard characters $e_1, \ldots, e_m$. As seen in the previous subsection, its equivariant coefficient ring is given by

$$\Lambda_T = \operatorname{Sym}_R^* M \cong R[e_1, \ldots, e_m].$$

Let X be a T-space, that is, a topological space with a (continuous) action of T. We denote the set of T-fixed points by X^T. The inclusion $\iota \colon X^T \hookrightarrow X$ induces a restriction map, called the *localization map*,

$$\iota^* \colon H_T^*(X) \longrightarrow H_T^*(X^T).$$

Definition 2.18. We denote the image of the localization map ι^* by $\iota_T(X)$ and its kernel by $\tau_T(X)$. If the context is clear, we will omit the index and simply write $\iota(X)$ and $\tau(X)$.

Example 2.19. Continuing Example 2.17, suppose that V is a complex vector space of dimension $k+1$, and that the torus T acts on $\mathbb{P}(V)$ via characters $\chi_0, \ldots, \chi_k$. The fixed point set $(\mathbb{P}(V))^T$ depends on the choice of characters: suppose $\bigsqcup_{j=1}^s F_j$ is the partition of $\{0, \ldots, k\}$ such that $\chi_u = \chi_v$ if and only if $u, v \in F_j$ for some $1 \leq j \leq s$. The connected components of $(\mathbb{P}(V))^T$ are the projective subspaces defined as the vanishing loci

$$V(y_i \mid i \notin F_j),$$

for $1 \leq j \leq s$, where $y_0, \ldots, y_k$ denote the homogeneous coordinates on $\mathbb{P}(V)$. The restriction map

$$H_T^*\big(\mathbb{P}(V)\big) \longrightarrow H_T^*\big(V(y_i \mid i \notin F_j)\big),$$

is given by the projection

$$\Lambda_T[\zeta]/\left(\prod_{i=0}^{k}(\zeta + \chi_i)\right) \longrightarrow \Lambda_T[\zeta]/\left(\prod_{i \in F_j}(\zeta + \chi_i)\right),$$

as can be seen from Remark 2.12, where ζ denotes the Chern class of the line bundle $\mathscr{O}(1)$ on both $\mathbb{P}(V)$ and the subspace $V(y_i \mid i \notin F_j)$. Therefore, we obtain the localization map

$$\iota^* \colon H_T^*\big(\mathbb{P}(V)\big) \longrightarrow \Lambda_T[\zeta]/\mathcal{Q}_1 \times \ldots \times \Lambda_T[\zeta]/\mathcal{Q}_s, \quad f \longmapsto (f_1, \ldots, f_s),$$

where $\mathcal{Q}_j$ is the ideal generated by $\prod_{i\in F_j}(\zeta+\chi_i)$. Here, the kernel $\tau(X)$ is equal to the intersection of the ideals $\mathcal{Q}_1,\dots,\mathcal{Q}_s$. Since $\Lambda_T[\zeta]$ is a unique factorization domain, we can conclude $\tau(X)=0$.

Remark 2.20. The most common setting for localization is when X^T is finite. In this case, every fixed point is *isolated*, meaning that every fixed point forms a connected component of X^T. The equivariant and the ordinary cohomology of X^T are given by

$$H_T^*(X^T) = \bigoplus_{p\in X^T} H_T^*(\mathrm{pt}) = \Lambda_T^{X^T}, \quad \text{and} \quad H^*(X^T) = \bigoplus_{p\in X^T} H^*(\mathrm{pt}) = R^{X^T}.$$

A direct justification to consider specifically the restriction to fixed points is provided by the following *main localization theorem*, which applies in greater generality, involving arbitrary subgroup $K\subseteq T$ and the set of K-fixed points X^K.

Theorem 2.21. *Let X be a complex algebraic variety with a torus action by T, and let $K\subseteq T$ be a subgroup. Let M_K be the corresponding sublattice of M (see Definition 2.15). If $S(K)$ denotes the multiplicative subset of Λ_T generated by the characters in $M\setminus M_K$, then the restriction map*

$$S(K)^{-1}(\iota_K^*)\colon S(K)^{-1}H_T^*(X) \longrightarrow S(K)^{-1}H_T^*(X^K),$$

is an isomorphism.

Proof. This follows from [AF24, Theorem 7.1.1], since there is a one-to-one correspondence between closed subgroups of T and the subgroups of M (see Remark 2.16). □

Remark 2.22. From the proof of [AF24, Theorem 7.1.1], it follows that it is sufficient to localize at a multiplicative subset $L\subseteq S(K)$ generated by finitely many characters in order to obtain the isomorphism in Theorem 2.21. For instance, in the situation of Example 2.19, it suffices to take all nonzero differences $\chi_u-\chi_v$, with $0\le u,v\le k$, as generators of L.

Remark 2.23. In its original form, the localization theorem was formulated in [AB84] for compact manifolds X with a smooth torus action. The general setup of localization and GKM-theory assumes X to be a "sufficiently nice" space together with a continuous torus action. Besides compact manifolds, admissible classes of spaces include finite CW-complexes ([FP07]), topological manifolds ([AFP14]), and complex algebraic varieties, which are the relevant class in this thesis.

The most important special case of Theorem 2.21 arises when $K=T$, in which case the localization map ι^* becomes an isomorphism to $H_T^*(X^T)$ after inverting every nonzero product of characters in Λ_T. In particular, the kernel $\tau(X)$ is a torsion module over Λ_T, i.e.,

$$\tau(X)\subseteq \operatorname{tor}(\Lambda_T, H_T^*(X)) := \{f\in H_T^*(X) \mid \exists\lambda\in\Lambda_T\setminus\{0\} : \lambda f = 0\},$$

and we obtain the following corollary.

Corollary 2.24. *We have $\tau(X)\subseteq \operatorname{tor}(\Lambda_T, H_T^*(X))$ with equality if and only if $H_T^*(X^T)$ is torsion-free over Λ_T.*

Remark 2.25. The equivariant cohomology $H_T^*(X^T)$ is torsion-free over Λ_T if X^T is finite (Remark 2.20), or, more generally, if $H^*(X^T)$ is torsion-free over R (Lemma 2.9).
On the other hand, let X be any variety such that $H^*(X)$ has torsion over R, and equip X with the trivial T-action. Then $\tau(X)=0$, while $H_T^*(X)$ has torsion over Λ_T.

Relating $H_T^*(X)$ to $H_T^*(X^T)$ via the localization map is the first step towards describing the cohomology of X in terms of its *equivariant skeleton*. The latter can be introduced as the filtration

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m = X,$$

of X, where X_i denotes the union of all T-orbits of dimension less than or equal to i. The most relevant parts of the filtration are the fixed points $X^T = X_0$, and the set of one-dimensional torus orbits X_1. By combining the localization map and the connecting morphism δ of the long exact sequence associated to the pair (X_1, X_0), we obtain the *Chang-Skjelbred sequence*

$$0 \longrightarrow H_T^*(X) \xrightarrow{\iota^*} H_T^*(X^T) \xrightarrow{\delta} H_T^{*+1}(X_1, X^T),$$

introduced in [CS74]. Here, following the Borel construction, the equivariant cohomology of the pair (X_1, X^T) is defined as the singular cohomology of the pair $(\mathbb{E}T \times^T X_1, \mathbb{E}T \times^T X^T)$. If the Chang–Skjelbred sequence is exact, then $H_T^*(X)$ embeds into $H_T^*(X^T)$ and can be computed as the kernel of the connecting morphism δ. In other words, the exact sequence allows one to describe $H_T^*(X)$ in terms of the zero- and one-dimensional orbits of T. This method was popularized in [GKM98], where an explicit description of $\ker \delta$ was given, and has since become known as the *GKM method*.
Alluding to the first of the two questions mentioned in the introduction of this subsection, the GKM method is applicable if the Chang-Skjelbred sequence is exact, and the T-spaces that can be expected to satisfy this property are called *equivariantly formal*.
The original definition of equivariant formality in [GKM98] assumes cohomology with coefficients in a field, and a major part of the discussion of GKM-theory in literature has taken place in this specific setting. We adopt a more general definition given in [Fra24], which allows for integral coefficients.

Definition 2.26. Let G be a Lie group acting on a topological space X. If $H_G^*(X)$ is a free module over Λ_G, then X is called *equivariantly formal* with respect to the action of G and the coefficient ring R.

Lemma 2.27. *Suppose X is equivariantly formal with respect to the action of G and the coefficient ring R. Then:*

1. *The restriction map $H_G^*(X) \to H^*(X)$ is surjective, with kernel generated by the kernel of the forgetful morphism $\Lambda_G \to R$.*
2. *For any G' acting on X through a morphism $G' \to G$, the corresponding homomorphism*

$$H_G^*(X) \otimes_{\Lambda_G} \Lambda_{G'} \longrightarrow H_{G'}^*(X),$$

is an isomorphism.

3. *If $G = T$ is a torus, then the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective.*

Proof. [AF24, Proposition 5.3.1 and Corollary 5.3.3]. The last point follows from Theorem 2.21. □

Remark 2.28. The initial version in [GKM98] considers coefficients in $\mathbb{R}$ and defines a space X to be equivariantly formal if the Serre spectral sequence associated with the fibration $\mathbb{E} \times^G X \to \mathbb{B}G$ collapses at the second page. When working over a field, this definition is equivalent to the one

given in Definition 2.26 and also to the first two properties listed in Lemma 2.27.
A compact reference for this equivalence can be found in [Fra24], where the author also notes that it does not extend to integral coefficients unless $H^*(X)$ is assumed to be free over $\mathbb{Z}$.
If G is a compact Lie group and $H^*(X)$ vanishes in odd degrees, or if X is a smooth complete variety and $G = T$ is a torus, then X is equivariantly formal when coefficients are taken in a field ([AF24, end of Chapter 5]). For integer coefficients, a smooth complete variety X with a T-action is equivariantly formal if X has finitely many T-fixed points ([AF24, Corollary 5.3.3]). More generally, for coefficients in an arbitrary ring, X is equivariantly formal if $H^*(X)$ is free over R and concentrated in even degrees ([FP11, Remark 2.3]).

As shown in [CS74, Lemma 2.3], the Chang-Skjelbred sequence is exact for equivariantly formal spaces when coefficients are taken in a field (see Remark 2.29 for the converse). For integer coefficients, connectedness of all isotropy groups in X in addition to equivariant formality implies exactness ([Fra24, Theorem 3.1], or [FP08, Theorem 2.4], for slightly weaker assumptions). Finally, if X is a complex algebraic variety, equivariant formality is a sufficient condition for exactness of the Chang-Skjelbred sequence, if $H^*(X)$ is free over an arbitrary unique factorization domain R (Lemma 2.27 (3) together with [AF24, Theorem 7.3.4]).

Remark 2.29. The *Atiyah-Bredon sequence*

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X)_0 \longrightarrow H_T^{*+1}(X_1, X_0) \longrightarrow H_T^{*+2}(X_2, X_1) \longrightarrow \cdots$$

$$\longrightarrow H_T^{*+m}(X_m, X_{m-1}) \longrightarrow 0$$

(see [Bre74]) is an extension of the Chang-Skjelbred sequence, which can be obtained from the equivariant skeleton by considering the inclusions of pairs $(X_{i+1}, X_{i-1}) \hookrightarrow (X_{i+1}, X_i)$, for $1 \leq i \leq m-1$. Sufficient and necessary conditions for exactness of this sequence are discussed in detail in [FP11], [FP07] and [AFP14]. Among other things, it is stated in [FP07, Theorem 1.1] and [AFP14, Proposition 5.12], that X is equivariantly formal if the Atiyah-Bredon sequence is exact.

Example 2.30. In the situation of Example 2.19, we can directly compute that $\{\zeta^j \mid j = 0, \dots, k-1\}$ is a Λ_T-module basis of $H_T^*\big(\mathbb{P}(V)\big)$, and hence $\mathbb{P}(V)$ is equivariantly formal.

The *Grassmann varieties* form the second class of varieties relevant to our thesis and provide a further example of equivariantly formal spaces.

Example 2.31. Let V be an n-dimensional complex vector space with basis $v_1, \dots, v_n$, and suppose T acts via distinct characters $\psi_1, \dots, \psi_n$ by

$$t.v_i = \psi_i(t)v_i, \quad t \in T, 1 \leq i \leq n.$$

The Grassmann variety $X = \mathrm{Gr}(d, n)$ of d-dimensional subspaces of V is a smooth projective variety with a T-action induced by the action on V. There are finitely many T-fixed points parametrized by coordinate subspaces of V: Let $I(d, n)$ denote the set of strictly increasing d-tuples with entries in $\{1, \dots, n\}$. Then,

$$X^T = \{q_{i_1,\dots,i_d} \coloneqq \langle v_{i_1}, \dots, v_{i_d} \rangle \mid (i_1, \dots, i_d) \in I(d, n)\}.$$

By Remark 2.28, it follows that $\mathrm{Gr}(d, n)$ is equivariantly formal with respect to the action of T and coefficients in a field or in $\mathbb{Z}$.

Given an equivariantly formal T-space X with exact Chang–Skjelbred sequence, we have

$$H_T^*(X) = \iota(X) = \ker \delta.$$

One of the main results in [GKM98] is the explicit description of $\ker \delta$ for the class of so-called *GKM-varieties*.
As before, since the original setting assumes coefficients in a field, we adopt the more general formulation from [AF24] and present the main result in Theorem 2.38.

Let X be a smooth algebraic variety and let $p \in X^T$ be a fixed point. The T-action on X induces a diagonalizable action on the tangent space T_pX via differentiation, and the associated weights are characters of T, which we call the characters of T at p. All weights on T_pX are nonzero if and only if p is an isolated fixed point ([AF24, Lemma 5.1.5]).

Definition 2.32. We call $\eta \in M$ a *primitive* character if the only expression $\eta = c' \cdot \eta'$, with $c' \in \mathbb{Z}_{>0}$ and $\eta' \in M$, is given by $c' = 1$ and $\eta' = \eta$. Any nonzero character $\chi \in M$ can be written uniquely as $c \cdot \eta$, where c is a positive integer and η is a primitive character. We call c the *coefficient* and η the *direction* of χ.
Two characters are *parallel* if their directions are the same or opposite. Two characters are called *relatively prime* if they are non-parallel and their coefficients are relatively prime in R.

Definition 2.33. Let X be a nonsingular variety that is equivariantly formal with respect to the action by T. If X has finitely many T-fixed points and for each $p \in X^T$, the weights of the T-action on the tangent space T_pX are pairwise relatively prime, then X is called a *GKM-variety* with respect to the torus T.

Let $C \subseteq X$ be a T-invariant curve. For any $x \in C \setminus C^T$, the action map

$$T \longrightarrow T \cdot x, \quad t \longmapsto t.x,$$

can be identified with a character χ of T. Up to the sign, this character is independent from any choices which justifies calling χ the character of the curve C. If the action of T on C is trivial, we set $\chi = 0$.
A *T-curve* $C \subseteq X$ is the closure of a one-dimensional T-orbit in X. As above, any T-curve has an associated nonzero character χ (for more details see [AF24, Chapter 7.2]).

Lemma 2.34. *Suppose X is a smooth k-dimensional variety with an isolated fixed point p and characters $\rho_1, \dots, \rho_k$ at p.*

1. *If no two characters at p are parallel, then there are exactly k T-curves in X through p. These curves are all nonsingular at p and have characters $\rho_1, \dots, \rho_k$.*
2. *If two characters have the same direction, then there are infinitely many T-curves in X through p.*
3. *If two characters have opposite directions, then there are infinitely many T-curves through every T-invariant neighborhood of p.*

Proof. [AF24, Chapter 7, Proposition 2.3]. □

Remark 2.35. The boundary of a one-dimensional T-orbit is a union of at most two fixed points (see [Hat02, Proposition 8.3]). This means that every T-curve is the closure of a uniquely determined T-invariant curve in X.
If X is a projective variety, then each T-curve is obtained by adding exactly two fixed points to the respective one-dimensional T-orbit ([GKM98, Chapter 7.1]).

Remark 2.36. When R is a field, a GKM-variety is often defined as an equivariantly formal projective variety with finitely many zero- and one-dimensional T-orbits ([GKM98, Chapter 7.1]). By Lemma 2.34 and Remark 2.35, this corresponds to a special case of Definition 2.33.
For integral coefficients, Remark 2.28 implies that a smooth projective variety with finitely many fixed points and relatively prime tangent weights at each fixed point is a GKM-variety.

The localization image of a GKM-variety is encoded by its *moment graph*, which can be defined more generally for varieties with finitely many fixed points.

Definition 2.37. Let X be a variety with a torus action by T, and suppose X^T is finite. The *moment graph* of X is an undirected graph Γ_X with vertex set X^T. There is an edge e, denoted $p \overset{e}{\text{---}} q$, between two distinct fixed points $p, q \in X^T$ for each T-curve in X connecting them. Each edge is labelled by the character of the corresponding T-curve.
The *graph equivariant cohomology* of Γ_X is defined as the graded subalgebra of $\Lambda_T^{X^T}$ given by

$$H^*(\Gamma_X) := \{(u_p)_{p\in X^T} \in \Lambda_T^{X^T} \mid \forall p, q \in X^T : u_p - u_q \text{ is divisible by } \chi_e, \text{ for all } e \text{ with } p \overset{e}{\text{---}} q\}.$$

If $\iota(X) = H^*(\Gamma_X)$, we say that the localization of X is described by its moment graph.

Theorem 2.38. *Suppose X is a GKM-variety. Then $\iota(X) \cong H_T^*(X)$ and the localization of X is described by its moment graph Γ_X, i.e.,*

$$H_T^*(X) \cong H^*(\Gamma_X).$$

Proof. This follows from Lemma 2.27 and [AF24, Theorem 7.1.1]. □

Example 2.39. We have seen in Example 2.30 that $\mathbb{P}(V)$ is equivariantly formal and we know from Example 2.19 that $(\mathbb{P}(V))^T$ is finite if and only if T acts on $\mathbb{P}(V)$ by pairwise distinct characters $\chi_0, \ldots, \chi_k$. In this case, the fixed points are given as the coordinate lines

$$p_i = (0 : \ldots : 0 : 1 : 0 : \ldots : 0),$$

and restriction to a fixed point p_i is the quotient map

$$\iota_i^* \colon H_T^*\big(\mathbb{P}(V)\big) \longrightarrow \Lambda_T[\zeta]/(\zeta + \chi_i).$$

In other words, for $f \in H_T^*\big(\mathbb{P}(V)\big)$, the map ι_i^* evaluates f at $\zeta = -\chi_i$, and the localization map becomes

$$\iota^* \colon H_T^*\big(\mathbb{P}(V)\big) \longrightarrow \Lambda_T[\zeta]/(\zeta + \chi_0) \times \ldots \times \Lambda_T[\zeta]/(\zeta + \chi_k) \cong \Lambda_T^{k+1},$$

with $\iota^*(f) = (f(\zeta = -\chi_i))_{i=0,\ldots,k}$.

For a fixed p_i, the weights in the tangent space $T_{p_i}\mathbb{P}(V)$ are the characters $\chi_j - \chi_i$, where $j \neq i$. Hence, $\mathbb{P}(V)$ is a GKM variety if and only if for any fixed p_i the k characters $\chi_j - \chi_i$ (for $j \neq i$) are pairwise relatively prime.

In that case, there are finitely many T-curves in $\mathbb{P}(V)$, and by Remark 2.35, there is exactly one T-curve C_{ij} connecting each pair of fixed points p_i and p_j. The character of C_{ij} is given by $\chi_i - \chi_j$, and Theorem 2.38 shows that ι^* is an isomorphism between

$$H_T^*\big(\mathbb{P}(V)\big) = \Lambda_T[\zeta]/\left(\prod_{i=0}^{k}(\zeta + \chi_i)\right),$$

and

$$H^*(\Gamma_X) = \{(u_i)_{i=0,\dots,k} \in \Lambda_T^{k+1} \mid \forall 0 \le i < j \le k : u_i - u_j \text{ is divisible by } \chi_i - \chi_j\}.$$

Example 2.40. For this example, assume that R is either $\mathbb{Z}$ or a field of characteristic zero. We have seen that the Grassmannian $\mathrm{Gr}(d,n)$ with respect to the natural action introduced in Example 2.31 is equivariantly formal and its fixed points are indexed by strictly increasing d-tuples with entries in $\{1,\dots,n\}$. For $J \in I(d,n)$, the weights on the tangent space $T_{p_J}\mathrm{Gr}(d,n)$ are given by the characters $\psi_i - \psi_j$, where $i \in J, j \notin J$. In order to apply Theorem 2.38, we assume that the weights on each tangent space are pairwise relatively prime, i.e., that $\mathrm{Gr}(d,n)$ is a GKM-variety.
Let $J_1, J_2 \in I(d,n)$. A pair of fixed points q_{J_1}, q_{J_2} is connected by a T-curve $C_{J_1 J_2}$ if and only if $J_1 \setminus \{j_1\} = J_2 \setminus \{j_2\}$ for some $1 \le j_1, j_2 \le n$, and the character of $C_{J_1 J_2}$ is given by $\psi_{j_1} - \psi_{j_2}$ (see [KT03, Chapter 2.3]). We can conclude, that

$$\begin{aligned} H_T^*(\mathrm{Gr}(d,n)) &\cong \iota(\mathrm{Gr}(d,n)) \\ &= \left\{ (u_J)_{J \in I(d,n)} \in \Lambda_T^{I(d,n)} \mid \begin{array}{l} \text{for all } J_1 \setminus \{j_1\} = J_2 \setminus \{j_2\} \text{ with } 1 \le j_1, j_2 \le n, \\ u_{J_1} - u_{J_2} \text{ is divisible by } \psi_{j_1} - \psi_{j_2} \end{array} \right\}. \end{aligned}$$

We conclude our discussion of the GKM method with the following remark concerning varieties whose cohomology is not torsion-free.

Remark 2.41. A necessary condition for applying the GKM method to a variety X is that $\tau(X) = 0$. If X has finitely many fixed points but its equivariant cohomology is not torsion-free over Λ_T, then Corollary 2.24 implies that X is neither a GKM-variety nor equivariantly formal. Nevertheless, it may still be possible to compute $\iota(X)$ using the Chang–Skjelbred sequence. If X is 'equivariantly formal modulo torsion', that is, if $\iota(X) = H_T^*(X)/\operatorname{tor}(\Lambda_T, H_T^*(X))$ is a free Λ_T-module, then [AF24, Theorem 7.3.4] can be used to show exactness of the Chang–Skjelbred sequence at $H_T^*(X^T)$.
Likewise, even though Theorem 2.38 does not apply directly, this does not prevent the possibility that $\iota(X)$ is still determined by the moment graph. In Chapter 5, we will study varieties whose equivariant cohomology is neither torsion-free nor free modulo torsion, yet whose localization image is nevertheless encoded by their moment graph (see Theorem 4.10).

3 The Mayer–Vietoris Spectral Sequence

Suppose $M = M_1 \cup M_2$ is the union of two topological spaces and consider the short exact sequence of cochain groups

$$0 \longrightarrow C^n(M_1 + M_2) \longrightarrow C^n(M_1) \oplus C^n(M_2) \longrightarrow C^n(M_1 \cap M_2) \longrightarrow 0,$$

where $C^n(M_1 + M_2)$ denotes the group of cochains supported on chains contained in either M_1 or M_2. If the inclusion $C^n(M) \hookrightarrow C^n(M_1 + M_2)$ induces an isomorphism in cohomology (for example, if the interiors of M_1 and M_2 cover M), then we obtain the Mayer–Vietoris sequence for singular cohomology as the associated long exact sequence

$$\cdots \longrightarrow H^{n-1}(M_1 \cap M_2) \longrightarrow H^n(M_1 \cup M_2) \\ \longrightarrow H^n(M_1) \oplus H^n(M_2) \longrightarrow H^n(M_1 \cap M_2) \longrightarrow \cdots$$

(see [Hat02] for details.)
The Mayer–Vietoris spectral sequence generalizes this procedure to a countable cover $M = \bigcup_{i \in I} M_i$ via the spectral sequence of a suitable double complex (Appendix B). In this thesis we restrict to finite covers, construct the corresponding double complex, and compute the associated spectral sequence. This follows the standard generalization of the Mayer–Vietoris principle ([God73; BT82]), with the goal of rigorously introducing the concept in the equivariant setting. Beyond the usual construction, we equip the spectral sequence with an algebra structure over the equivariant coefficient ring, formalize the category of covered spaces to which it applies, and establish its functorial properties. The presentation is rounded out with special cases and examples relevant to later computations.

3.1 The Construction of the Differential Graded Algebra

The following construction of a double complex is the first step in the natural generalization of the Mayer–Vietoris sequence to a finite cover. For references see e.g. [God73] mainly Chapter 2.5, or [FM19, Chapter 1]. A multiplicative structure on the associated double complex is included below. While such structures are standard and appear in related contexts (e.g., de Rham cohomology; see [Nic11, Example 5.9]), the compatibility of the product with both differentials is verified here in detail, as this is often stated without full proof.

Let X be a topological space and $q \in \mathbb{N}$. Following the usual definition of singular cohomology (see [Hat02, Chapter 3]), we write $C_q(X)$ for the *group of singular q-chains*, the free R-module with basis the q-simplices in X, and $C^q(X) = \hom(C_q(X), R)$ for the *group of singular q-cochains* on X. If X is the empty set we define $C_q(X)$ and $C^q(X)$ to be zero.

Suppose now that we are given a finite covering $\mathfrak{M} = (M_i)_{i \in I}$ by subspaces of X with index set I. We write $\mathcal{I}$ for the full simplex on I, and $\mathcal{I}_p$ for the set of p-simplices in $\mathcal{I}$, where $p \in \mathbb{Z}_{\geq 0}$. As in Definition 2.1, we fix an ordering on I and identify $\mathcal{I}_p$ with the set of strictly increasing $(p+1)$-tuples with entries in I. Finally, given $\underline{i} = (i_0, \ldots, i_p) \in \mathcal{I}_p$, we define

$$M_{\underline{i}} \coloneqq M_{i_0} \cap \ldots \cap M_{i_p}.$$

The double complex of R-modules we want to investigate is given by

$$K^{p,q} = \bigoplus_{\underline{i} \in \mathcal{I}_p} C^q(M_{\underline{i}}).$$

Its two differentials are induced by the coboundary operator in singular cohomology and the coboundary operator in the simplicial cohomology of $\mathcal{I}$, respectively. More precisely,

$$d' \colon \bigoplus_{\underline{i} \in \mathcal{I}_p} C^q(M_{\underline{i}}) \longrightarrow \bigoplus_{\underline{i} \in \mathcal{I}_{p+1}} C^q(M_{\underline{i}}),$$

is defined as

$$(d'(\varphi))_{\underline{j}} = \sum_{k=0}^{p+1} (-1)^k \varphi_{j_0,\ldots,\hat{j}_k,\ldots,j_{p+1}}|_{M_{\underline{j}}},$$

where $\varphi \in \bigoplus_{\underline{i} \in \mathcal{I}_p} C^q(M_{\underline{i}})$ and $\underline{j} = (j_0, \ldots, j_{p+1}) \in \mathcal{I}_{p+1}$. In what follows, we will, by abuse of notation, omit explicitly indicating when we restrict a cochain to a subspace if it is clear from the context.

The second differential

$$d'' \colon \bigoplus_{\underline{i} \in \mathcal{I}_p} C^q(M_{\underline{i}}) \longrightarrow \bigoplus_{\underline{i} \in \mathcal{I}_p} C^{q+1}(M_{\underline{i}}),$$

reads as

$$d''(\varphi)_{\underline{i}} = (-1)^p \partial_{\mathbf{sg}}(\varphi_{\underline{i}}),$$

where $\underline{i} \in \mathcal{I}_p$, and where $\partial_{\mathbf{sg}} \colon C^q \to C^{q+1}$ is the coboundary operator in singular cohomology.

Lemma 3.1. *The differentials d' and d'' of the double complex K satisfy*

$$d'd'' + d''d' = 0.$$

Consequently, (K, d) is a differential graded module considered with the total grading and differential $d = d' + d''$ (see Appendix B).

Proof. [God73, Chapter 2 and start of Section 4.5]. □

The next step is to introduce a multiplicative structure so that we obtain a differential graded algebra. The correct definition combines the Whitney product in the singular cohomology of the spaces $M_{\underline{i}}$ with the product of simplices in the simplicial complex $\mathcal{I}$.

Remark 3.2. As for simplicial cohomology, the cup product in singular cohomology is induced by the Whitney product on the level of cochains, i.e., a bilinear map

$$\omega \colon C^s(X) \times C^r(X) \longrightarrow C^{s+r}(X), \quad (\varphi, \psi) \longmapsto \omega(\varphi, \psi),$$

which is associative and satisfies the Leibniz rule

$$\partial_{\mathbf{sg}}(\omega(\varphi, \psi)) = \omega(\partial_{\mathbf{sg}}(\varphi), \psi) + (-1)^s \omega(\varphi, \partial_{\mathbf{sg}}(\psi)).$$

Suppose we have an r-simplex $\underline{i} = (i_0, \dots, i_r)$ and an s-simplex $\underline{j} = (j_0, \dots, j_s)$ in $\mathcal{I}$, respectively. If we set $M_0 := \varnothing$, then the expression $M_{\underline{i}\cdot\underline{j}}$ is well-defined (see Chapter 1 for the simplicial cup product), and for $a_{\underline{i}} \in C^u(M_{\underline{i}})$ and $b_{\underline{j}} \in C^v(M_{\underline{j}})$ we can set

$$a_{\underline{i}} \times b_{\underline{j}} := \omega(a_{\underline{i}}, b_{\underline{j}}) \in C^{u+v}(M_{\underline{i}\cdot\underline{j}}),$$

where we implicitly restrict $a_{\underline{i}}$ and $b_{\underline{j}}$ to $M_{\underline{i}\cdot\underline{j}}$ in order to form the Whitney product.
The product $\times$ satisfies associativity and can be extended bilinearly to equip K with a product $K^{r,u} \times K^{s,v} \to K^{r+s,u+v}$.

Lemma 3.3. *For $a \in K^{r,u}$ and $b \in K^{s,v}$ we have that*

$$d'(a \times b) = d'(a) \times b + (-1)^r a \times d'(b),$$

as well as

$$d''(a \times b) = (-1)^s d''(a) \times b + (-1)^{u+r} a \times d''(b).$$

Proof. Suppose we have $a_{\underline{i}} \in C^u(M_{\underline{i}})$ and $b_{\underline{j}} \in C^v(M_{\underline{j}})$ so that $\underline{i}\cdot\underline{j} \neq 0$. To compute $d'(a_{\underline{i}} \times b_{\underline{j}})$ we fix a simplex $\underline{k}$ in $\mathcal{I}$ which contains one additional vertex ℓ compared to $\underline{i}\cdot\underline{j}$. We write

$$\ell_i = \max(\min(n \mid i_n > \ell), -1) + 1,$$

and similarly ℓ_j and $\ell_{i\cdot j}$. We have

$$d'(a_{\underline{i}} \times b_{\underline{j}})_{\underline{k}} = (-1)^{\ell_{i\cdot j}} \omega(a_{\underline{i}}, b_{\underline{j}}),$$

while

$$(a_{\underline{i}} \times d'(b_{\underline{j}}))_{\underline{k}} = \begin{cases} (-1)^{\ell_j} \omega(a_{\underline{i}}, b_{\underline{j}}), & \text{if } \ell > j_0, \\ 0, & \text{else,} \end{cases}$$

and

$$(d'(a_{\underline{i}}) \times b_{\underline{j}})_{\underline{k}} = \begin{cases} (-1)^{\ell_i} \omega(a_{\underline{i}}, b_{\underline{j}}), & \text{if } \ell < j_0, \\ 0, & \text{else.} \end{cases}$$

Consequently, this shows

$$d'(a_{\underline{i}} \times b_{\underline{j}}) = d'(a_{\underline{i}}) \times b_{\underline{j}} + (-1)^r a_{\underline{i}} \times d'(b_{\underline{j}}).$$

Now assume that $\underline{i}\cdot\underline{j} = 0$. The first case is that $i_r > j_0$ and consequently

$$a_{\underline{i}} \times b_{\underline{j}} = d'(a_{\underline{i}}) \times b_{\underline{j}} = a_{\underline{i}} \times d'(b_{\underline{j}}) = 0.$$

In the case that $i_r < j_0$ let us write $\underline{i}_j = (i_0, \dots, i_r, j_0)$ and $\underline{j}_i = (i_r, j_0, \dots, j_s)$. We still have $a_{\underline{i}} \times b_{\underline{j}} = 0$ but this time

$$(d'(a_{\underline{i}}) \times b_{\underline{j}})_{\underline{i}\cdot\underline{j}} = d'(a_{\underline{i}})_{\underline{i}_j} \times b_{\underline{j}} = (-1)^{r+1} \omega(a_{\underline{i}}, b_{\underline{j}}),$$

$$(a_{\underline{i}} \times d'(b_{\underline{j}}))_{\underline{i}\cdot\underline{j}} = a_{\underline{i}} \times d'(b_{\underline{j}})_{\underline{j}_i} = (-1)^r \omega(a_{\underline{i}}, b_{\underline{j}}).$$

As

$$(d'(a_{\underline{i}}) \times b_{\underline{j}})_{\underline{k}} = (a_{\underline{i}} \times d'(b_{\underline{j}}))_{\underline{k}} = 0,$$

for any other simplex $\underline{k}$, we can finally conclude that

$$d'(a \times b) = d'(a) \times b + (-1)^r a \times d'(b),$$

for all $a \in K^{r,u}, b \in K^{s,v}$.
Concerning the equation including d'' suppose that we have $a_{\underline{i}} \in C^u(M_{\underline{i}})$ and $b_{\underline{j}} \in C^v(M_{\underline{j}})$ so that $\underline{i} \cdot \underline{j} \neq 0$. It is a consequence of the Leibniz rule for the Whitney product that

$$\begin{aligned} d''(a_{\underline{i}} \times b_{\underline{j}}) &= (-1)^{r+s}\partial_{\mathbf{sg}}(\omega(a_{\underline{i}}, b_{\underline{j}})) = (-1)^{r+s}(\omega(\partial_{\mathbf{sg}}(a_{\underline{i}}), b_{\underline{j}}) + (-1)^u \omega(a_{\underline{i}}, \partial_{\mathbf{sg}}(b_{\underline{j}}))) \\ &= (-1)^s d''(a_{\underline{i}}) \times b_{\underline{j}} + (-1)^{u+r} a_{\underline{i}} \times d''(b_{\underline{j}}). \end{aligned}$$

If $\underline{i} \cdot \underline{j} = 0$, we obtain

$$a_{\underline{i}} \times b_{\underline{j}} = d''(a_{\underline{i}}) \times b_{\underline{j}} = a_{\underline{i}} \times d''(b_{\underline{j}}) = 0,$$

and end up with

$$d''(a \times b) = (-1)^s d''(a) \times b + (-1)^{u+r} a \times d''(b),$$

for all $a \in K^{r,u}, b \in K^{s,v}$. □

We now define the final product on K.

Definition 3.4. Let $a \in K^{r,u}$ and $b \in K^{s,v}$. We define

$$a \cdot b = (-1)^{us} a \times b \in K^{r+s,u+v}.$$

The two canonical filtrations on any double complex are the row-wise and column-wise filtration. These are given by

$${}'K_p = \bigoplus_{i \geq p} K^{i,j} \quad \text{and} \quad {}''K_p = \bigoplus_{j \geq p} K^{i,j},$$

respectively. Both are regular, since $K^{p,q} = 0$ if either $p < 0$ or $q < 0$ (see Example A.1).

Lemma 3.5. *The triple $(K, d, \cdot)$ is a differential graded algebra. Both canonical filtrations of K as a double complex are compatible with the product and the grading.*

Proof. The multiplication is bilinear and associative, as can be seen from the computation:
Let $a \in K^{r,u}, b \in K^{s,v}$ and $c \in K^{w,z}$. Then

$$a(bc) = (-1)^{u(s+w)+vw} a \times b \times c = (-1)^{us+(u+v)w} a \times b \times c = (ab)c.$$

Lemma 3.3 shows that both d' and d'' satisfy the Leibniz rule with respect to $\cdot$, hence so does $d = d' + d''$. Compatibility of the filtrations with the product and the grading follows from the definitions. □

We follow the Borel construction (see Chapter 2.1) to present the corresponding double complex for equivariant cohomology.
Suppose that X is a G-space, i.e., a topological space together with an action by a Lie group G, and fix the classifying space $\mathbb{B}G$ for G together with the universal bundle

$$\mathbb{E}G \longrightarrow \mathbb{B}G.$$

Next, assume that the cover $\mathfrak{M}$ is G-invariant, in other words, M_i is G-invariant for every $i \in I$. For any simplex $\underline{i}$ in $\mathcal{I}$, we write $C^q_G(M_{\underline{i}}) \coloneqq C^q(\mathbb{E}G \times^G M_{\underline{i}})$ for the group of equivariant q-cochains on $M_{\underline{i}}$. The equivariant version of the double complex K is defined by

$$K^{p,q}_G \coloneqq \bigoplus_{\underline{i} \in \mathcal{I}_p} C^q_G(M_{\underline{i}}),$$

and $(K_G, d, \cdot)$ is a filtered differential graded R-algebra by the same construction as above.
To complete the equivariant version we include the structure as an algebra over Λ_G. To this end, consider the morphisms of graded rings

$$\alpha_i \colon C^*_G(\mathrm{pt}) \longrightarrow C^*_G(M_i), \quad i \in I,$$

which endow $H^*_G(M_i)$ with the structure of a Λ_G-algebra. We can extend these morphisms by mapping into each component

$$\alpha \colon C^q_G(\mathrm{pt}) \longrightarrow K^{0,q}_G = \bigoplus_{i \in I} C^q_G(M_i), \quad \sigma \longmapsto (\alpha_i(\sigma))_{i \in I},$$

and restrict to the subring

$$C_G \coloneqq \{\text{cocycles in } C^*_G(\mathrm{pt})\}.$$

Remark 3.6. Every element in $\alpha(C^q_G(\mathrm{pt}))$ lies in the kernel of d', and the image of any cocycle is annihilated by d''. Consequently, $\alpha(C_G) \subseteq \ker d$.
If $\sigma = \partial_{\mathbf{sg}}(\tau)$ is a coboundary in C_G, then

$$d(\alpha(\tau)) = d''(\alpha(\tau)) = \alpha(\sigma),$$

which shows that coboundaries in C_G are mapped into $\operatorname{im} d \cap \operatorname{im} d''$.

Lemma 3.7. *The triple $(K_G, d, \cdot)$ is a differential graded algebra over the graded ring C_G (see Remark A.18).*

Proof. The morphism $C_G \to K$ turns K_G into a C_G-algebra. Since the image of C_G is contained in $\ker d$ by Remark 3.6, the differential d is a graded C_G-linear morphism. □

Example 3.8. Suppose we cover X by two G-invariant subspaces M_1 and M_2. In this case $K^{p,q}_G = 0$ for $p > 1$, and the full double complex is given by

$$\begin{array}{ccccc}
C^0_G(M_1) \oplus C^0_G(M_2) & \xrightarrow{d'} & C^0_G(M_{(1,2)}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle d''} & & \downarrow & & \\
C^1_G(M_1) \oplus C^1_G(M_2) & \longrightarrow & C^1_G(M_{(1,2)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
C^2_G(M_1) \oplus C^2_G(M_2) & \longrightarrow & C^2_G(M_{(1,2)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots
\end{array}$$

Remark 3.9. Let $\mathrm{PC}(X)$ denote the set of the path-connected components of X. Then G acts on $\mathrm{PC}(X)$ with orbit space $\mathrm{PC}(X)/G$, and we can define

$$\mathrm{PC}_G(X) \coloneqq \left\{ \bigcup_{Y \in \mathscr{G}} Y \mid \mathscr{G} \in \mathrm{PC}(X)/G \right\}.$$

Note that $\mathrm{PC}_G(X)$ is the set of minimal G-invariant unions of path-connected components of X (minimal with respect to inclusion).
By construction, the double complex K_G splits into a direct sum of double complexes

$$K_G = \bigoplus_{Z \in \mathrm{PC}_G(X)} K_Z,$$

where

$$K_Z^{p,q} \coloneqq \bigoplus_{\underline{i} \in \mathcal{I}_p} C_G^q(M_{\underline{i}} \cap Z)$$

for $Z \in \mathrm{PC}_G(X)$. Since both differentials and the product on K_G are compatible with this splitting, K_G splits into a direct sum of differential graded C_G-algebras.

Example 3.10. Suppose $X = \{p_1, \ldots, p_n\}$ is a union of finitely many G-fixed points. Given a G-invariant cover $\mathfrak{M}$ of X, Remark 3.9 yields a decomposition of the double complex K_G into subcomplexes $K_{G,i}$, for $1 \leq i \leq n$, where

$$K_{G,i}^{p,q} = \bigoplus_{\substack{\underline{i} \in \mathcal{I}_p \\ p_i \in M_{\underline{i}}}} C_G^q(p_i).$$

3.2 The Spectral Sequence

The canonical spectral sequences associated with the double complex K are used to study the G-equivariant cohomology of X. In particular, the row filtration gives rise to the *Mayer–Vietoris spectral sequence*, whose construction is described, for example, in [God73, Chapter 2.5]. We reproduce this in rigorous detail while simultaneously incorporating the algebra structure over the equivariant coefficient ring. Along the way, the *Mayer–Vietoris complex* and the *first-column component* (Definitions 3.18, 3.25) are introduced, both playing a central role in the following chapters. We discuss the associated graded algebra of $H_G^*(X)$, arising from the induced filtration and forming the target of the spectral sequence (Remarks 3.23, 3.24), and provide examples where the filtration is trivial and hence the graded algebra coincides with the equivariant cohomology of X. Finally, in Remark 3.34, the Mayer–Vietoris sequence is recovered as a special case of the developed framework.

Let X be a topological space with an action of a Lie group G, and let $\mathfrak{M} = (M_i)_{i \in I}$ be a finite cover by G-invariant subspaces. Since we will work in the equivariant setting throughout, we will, from now on, write K instead of K_G for the associated double complex.
Recall that the two natural filtrations on K, the row-wise and column-wise filtration, are given by

$$'K_p = \bigoplus_{\substack{i \geq p \\ j \geq 0}} K^{i,j} \quad \text{and} \quad ''K_p = \bigoplus_{\substack{j \geq p \\ i \geq 0}} K^{i,j},$$

each inducing its own spectral sequence (see Appendix B).

Definition 3.11. The spectral sequence associated with the row-wise filtration of the double complex K is called the *Mayer–Vietoris spectral sequence* associated to a cover $\mathfrak{M} = (M_i)_{i \in I}$ of X.

We first consider the spectral sequence $\{''E_r, '' d_r\}$ associated with the column-wise filtration on K, using the subalgebra

$$L \coloneqq \{x \in K^{0,*} \mid d'(x) = 0\}.$$

Although only the column-wise filtration on L is nontrivial, we may still regard L as a double complex, inheriting both filtrations and the total grading from K.

By the first part of Remark 3.6, the image of C_G lies in L, so both L and K are graded C_G-algebras. By the second part of Remark 3.6, coboundaries in C_G map to coboundaries in K and in L, respectively. Consequently, both $H^*(K)$ and $H^*(L)$ are naturally Λ_G-algebras.

Lemma 3.12. *The injection $L \hookrightarrow K$ induces an isomorphism of graded Λ_G-algebras in cohomology*

$$H^*(L) \longrightarrow H^*(K).$$

Proof. A known fact ([FM19, Lemma 1.2]) is that for every q, the complex

$$0 \longrightarrow K^{0,q} \longrightarrow K^{1,q} \longrightarrow K^{2,q} \longrightarrow \dots$$

is exact at every position apart from zero. As a consequence, $''E_2^{p,q} = 0$ for $p \neq 0$, and using Theorem B.1, we deduce that the injection $L \hookrightarrow K$ induces an isomorphism in cohomology, i.e., an isomorphism of graded R-algebras

$$H^*(L) \longrightarrow H^*(K).$$

This inclusion is part of the commutative diagram

$$\begin{array}{ccc} L & \hookrightarrow & K \\ & \nwarrow \quad \nearrow & \\ & C_G & \end{array}$$

and therefore we obtain an isomorphism of Λ_G-algebras. □

In order to compute the equivariant cohomology of X with help of the Mayer–Vietoris spectral sequence the cover $\mathfrak{M}$ needs to satisfy an additional requirement.
Similar as for the module of cocycles, we write

$$C_*^G(X) \coloneqq C_*(\mathbb{E}E \times^G X),$$

and let $C_n^{G,\mathfrak{M}}(X)$ denote the free R-module with basis all n-simplices that are contained in one of the sets $\mathbb{E}G \times^G M_i$ with $i \in I$. Its dual module $C^*_{G,\mathfrak{M}}(X)$ can be naturally identified with L, since an element in $\ker d'$ is precisely a cochain in K^{0*} that agrees on all q-simplices shared by multiple sets in the cover.
By the same argument, the morphism of R-complexes

$$C_G^*(X) \longrightarrow K^{0,*},$$

obtained by restriction in each component has image contained in $\ker d'$ and hence in L.

Remark 3.13. In all subsequent results of this chapter we will explicitly require that

$$C^*_G(X) \longrightarrow C^*_{G,\mathfrak{M}}(X) = L, \tag{1}$$

induced by the inclusion $C^{G,\mathfrak{M}}_*(X) \hookrightarrow C^G_*(X)$, is an isomorphism in cohomology.

It is therefore justified to later introduce this as the required condition on a cover $\mathfrak{M}$ of X in order to apply the Mayer–Vietoris spectral sequence with respect to $\mathfrak{M}$ to compute $H^*_G(X)$ (see Definition 3.36).

The two most important classes of covers satisfying the requirement in Remark 3.13 are presented in the following remarks.

Remark 3.14. Assume that the interiors of the M_i are G-invariant and cover X. Using similar notation as above, we write $C^{\mathfrak{M}}_n(X)$ for the free group with basis all n-simplices contained in one of the sets M_i of the cover.
In this case, the inclusion of cochain complexes

$$C^*(X) \longrightarrow C^*_{\mathfrak{M}}(X),$$

yields an isomorphism in cohomology ([Hat02, Proposition 2.21]).
As we have assumed that the interiors of the M_i are G-invariant, we may replace X by $\mathbb{E}G \times^G X$ and M_i by $\mathbb{E}G \times^G M_i$ to conclude that

$$C^*_G(X) \longrightarrow C^*_{G,\mathfrak{M}}(X),$$

also induces an isomorphism in cohomology.

Remark 3.15. A generalization of Remark 3.14 is the setting where each M_i is a deformation retract of some open neighborhood U_i, and where $M_{\underline{i}}$ is a deformation retract of

$$U_{\underline{i}} \coloneqq U_{i_0} \cap \ldots \cap U_{\mathcal{I}_p},$$

for all $\underline{i} = (i_0, \ldots, i_p) \in \mathcal{I}$, and X is a deformation retract of $\bigcup_{i \in I} U_i$. If $\mathfrak{U} = (U_i)_{i \in I}$ is a G-invariant cover of X, then the cover $\mathfrak{M}$ satisfies the required condition. For a detailed application, see Chapter 5.2.

Remark 3.16. By Lemma 3.12, the inclusion $L \hookrightarrow K$ induces an isomorphism in cohomology. Thus, the map (1) induces an isomorphism in cohomology if and only if the restriction to the first column

$$C^*_G(X) \longrightarrow K,$$

does so.

Corollary 3.17. *Suppose that the restriction of complexes*

$$C^*_G(X) \longrightarrow C^*_{G,\mathfrak{M}}(X),$$

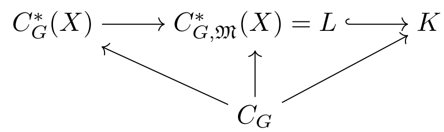
induces an isomorphism in cohomology. Then there is an isomorphism of graded Λ_G-algebras

$$H^*_G(X) \longrightarrow H^*(K).$$

Proof. By assumption we have an isomorphism of graded R-modules

$$\kappa\colon H_G^*(X) \longrightarrow H^*(L).$$

Now the product on K restricts to the usual cup product in cohomology on every component of K^{0*}, hence κ is an isomorphism of R-algebras. Since scalar multiplication by Λ_G on $H_G^*(X)$ is precisely induced by the morphism $X \to \mathrm{pt}$, we obtain the commutative diagram



and κ is an isomorphism of Λ_G-algebras. With Lemma 3.12 we derive the required isomorphism. □

We now turn to the spectral sequence $\{E_r, d_r\}$ of K induced by the row-wise filtration, i.e., the Mayer–Vietoris spectral sequence.
The row-wise filtration on K induces a filtration on $H^*(K)$, whose associated bigraded Λ_G-algebra we denote by $G(H^*(K))$ for the remainder of this section (see Appendix A). By Corollary 3.17, $H_G^*(X)$ inherits this filtration, which we denote by

$$(H_G^*(X))_p, \quad p \geq 0,$$

and the associated bigraded algebra $G(H_G^*(X))$ is isomorphic, as a bigraded Λ_G-algebra, to $G(H^*(K))$.

The first and the second page of $\{E_r, d_r\}$ admit the explicit descriptions

$$E_1^{p,q} = H^q(K^{p,*}, d''),$$

as well as

$$E_2^{p,q} = H^p(H^q(K, d''), d'),$$

see Appendix B. A direct computation gives

$$E_1^{p,*} = H\left(\bigoplus_{\underline{i}\in\mathcal{I}_p} C_G^*(M_{\underline{i}}), d''\right) = \bigoplus_{\underline{i}\in\mathcal{I}_p} H_G^*(M_{\underline{i}}),$$

and the differential $d_1\colon E_1^{p,*} \to E_1^{p+1,*}$ is induced by d', namely,

$$(d_1(\underline{f}))_{\underline{j}} = \sum_{k=0}^{p+1} (-1)^k \left(f_{j_0,\ldots,\hat{j}_k,\ldots,j_{p+1}}\right)|_{M_{\underline{j}}},$$

where $\underline{f} \in \bigoplus_{\underline{i}\in\mathcal{I}_p} H_G^*(M_{\underline{i}})$ and $\underline{j} = (j_0, \ldots, j_{p+1}) \in \mathcal{I}_{p+1}$.

Definition 3.18. The first page of the Mayer–Vietoris spectral sequence will be called the *Mayer–Vietoris complex* and denoted

$$(\mathrm{MV}(G, X, \mathfrak{M}), d) := (E_1, d_1),$$

or simply MV when G,X and $\mathfrak{M}$ are clear from the context.
As a complex with respect to the first grading, we set

$$\mathrm{MV}^i := E_1^{i,*}, \quad d^i := d|_{\mathrm{MV}^i} \colon \mathrm{MV}^i \longrightarrow \mathrm{MV}^{i+1},$$

and denote its cohomology by

$$H^p(\mathrm{MV}) := E_2^{p,*}, \quad \text{or simply} \quad H(\mathrm{MV}) = E_2.$$

When it is important to specify the bigraded components, we write

$$\mathrm{MV}^{p,q} := E_1^{p,q}, \quad H^{p,q}(\mathrm{MV}) := E_2^{p,q}.$$

In what follows, we shall regard the Mayer–Vietoris complex and its cohomology as monograded or bigraded objects, depending on the context.

Remark 3.19. Let $\underline{i} \in \mathcal{I}_r$ and $\underline{j} \in \mathcal{I}_s$, as well as $x \in H_G^p(M_{\underline{i}})$ and $y \in H_G^q(M_{\underline{j}})$. In MV, their product is the cup product of their restrictions to $H_G^*(M_{\underline{i}\cdot\underline{j}})$, adjusted by the factor $(-1)^{ps}$ (see Definition 3.4).

Suppose for the remainder of this subsection that the restriction

$$C_G^*(X) \longrightarrow C_{G,\mathfrak{M}}^*(X),$$

induces an isomorphism in cohomology.

Lemma 3.20. *The spectral sequence $\{E_r, d_r\}$ is a spectral sequence of C_G-algebras, where the C_G-action induces a Λ_G-algebra structure on E_r for all $r \geq 1$. Its infinity page E_∞ is isomorphic to $G(H_G^*(X))$ as a bigraded Λ_G-algebra.*

Proof. For the chosen filtration on K, the product satisfies,

$${}'K_s \cdot {}'K_t \subseteq {}'K_{s+t}, \text{ and } C_G \cdot {}'K_s \subseteq {}'K_s, \quad s,t \in \mathbb{N}.$$

Thus, the filtration is compatible with both the product and the action by C_G, and the associated spectral sequence is a spectral sequences of C_G-algebras (see Appendix A), with

$$E_\infty \cong G(H^*(K)),$$

as a bigraded C_G-algebra (see [McC01, Chapter 2.3]).
Since the coboundaries in C_G act trivially on E_1 (Remark 3.6, second part), the C_G-action on E_r factors through Λ_G for all $r \geq 1$. Moreover, the isomorphism $E_\infty \cong G(H^*(K))$ is an isomorphism of Λ_G-algebras. The claim then follows from Corollary 3.17. □

Converging to $G(H_G^*(X))$ as bigraded algebra means, expressed in graded components, that

$$E_\infty^{p,q} \cong (H_G^{p+q}(X))_p / (H_G^{p+q}(X))_{p+1},$$

see Appendix A.

Remark 3.21. We will henceforth consider the Mayer–Vietoris spectral sequence $\{E_r, d_r\}$ as a spectral sequence of Λ_G-algebras. This is justified by the fact that, for $r \geq 1$, each E_r is a differential bigraded Λ_G-algebra, and the canonical isomorphisms

$$H(E_r, d_r) \cong E_{r+1}, \quad r \geq 1, \quad \text{and} \quad E_\infty \cong H^*(K),$$

given in Theorem A.3 and Theorem A.5, are isomorphisms of (bigraded) Λ_G-algebras.

We summarize the construction developed in this chapter so far in the following theorem.

Theorem 3.22. *The Mayer–Vietoris spectral sequence $\{E_r, d_r\}$ is a spectral sequence of Λ_G-algebras whose first page is the Mayer–Vietoris complex and with*

$$E_\infty \cong G(H^*_G(X)),$$

as bigraded Λ_G-algebra.

Remark 3.23. For the remainder of this thesis, we compute and analyze equivariant cohomology via Theorem 3.22. A notable caveat is that the Mayer–Vietoris spectral sequence yields the associated graded algebra $G(H^*_G(X))$ rather than the cohomology ring itself. Determining $H^*_G(X)$ from $G(H^*_G(X))$ is an extension problem, and while in certain cases, such as some of those considered in the rest of this section, it is trivial, in general we do not resolve it here. Instead, we compute $G(H^*_G(X))$ via the spectral sequence and use the result to deduce properties of $H^*_G(X)$ (see, for example, Corollary 3.42, Lemma 4.2, and Theorem 5.73).

Remark 3.24. If E_∞ is projective as an R-module, then $\mathrm{Ext}^1_R(E_\infty, B)$ vanishes for any R-module B, and consequently

$$G(H^*_G(X)) \cong H^*_G(X),$$

as R-modules (see [McC01, p. 32]). Similarly, if E_∞ is projective as a Λ_G-module, then

$$G(H^*_G(X)) \cong H^*_G(X),$$

as Λ_G-modules.

Definition 3.25. We set the *first-column component* of X with respect to the cover $\mathfrak{M}$ to be the graded Λ_G-algebra $H^*_G(X)/(H^*_G(X))_1$. We denote it by $\nu(X, \mathfrak{M})$, or simply $\nu(X)$ if it is clear from the context which cover is considered.

Remark 3.26. We have

$$\nu(X) \cong \bigcap_{r \geq 1} \ker d^0_r,$$

which is a Λ_G-subalgebra of $\bigcap_{r\geq 1} E^{0,*}_r$. If $(H^*_G(X))_1 = 0$, then

$$\nu(X) = H^*_G(X) = G(H^*_G(X)).$$

In the most relevant cases, including the example classes considered later in this thesis, the Mayer–Vietoris spectral sequence collapses at the second page. In other words, the cohomology of the Mayer–Vietoris complex is the object of computation.

Definition 3.27. If the Mayer–Vietoris spectral sequence associated to a cover $\mathfrak{M} = (M_i)_{i \in I}$ of X collapses at the second page we call $\mathfrak{M}$ a *good cover* of X.

Theorem 3.28. *Suppose $\mathfrak{M}$ is a good cover of X. The Mayer–Vietoris complex has cohomology*

$$H^{*,*}(\mathrm{MV}) \cong G(H_G^*(X)),$$

as bigraded Λ_G-algebra. In graded components,

$$H^{p,q}(\mathrm{MV}) \cong (H_G^{p+q}(X))_p / (H_G^{p+q}(X))_{p+1},$$

and in particular, $\nu(X) \cong \ker d^0$.

Proof. This is a consequence of the definition of a good cover, together with Theorem 3.22 and Remark 3.26. □

Example 3.29. If for any p and any $\underline{i} \in \mathcal{I}_p$ there are no nonzero elements of odd degree in the cohomology ring $H_G^*(M_{\underline{i}})$, then the Mayer–Vietoris spectral sequence collapses at the second page due to Remark A.6.
Further examples of good covers will follow in Example 3.30, Lemma 3.31, and Remark 3.34.

Example 3.30. Let X be a space with a G-action and I an index set containing the symbol 0.
The first tautological cover is given by $M_0 = X$ and $M_i = \varnothing$ for every $i \in I \setminus \{0\}$. In this case,

$$\mathrm{MV}^* = \mathrm{MV}^0 = H_G^*(X),$$

so we are in the situation of Remark A.10. The Mayer–Vietoris spectral sequence collapses at the first page, $(H_G^*(X))_1 = 0$, and $G(H_G^*(X)) = H_G^*(X)$.
The second tautological cover is given by $M_i = X$ for all $i \in I$, i.e., $\mathfrak{M} = (X)_{i \in I}$. In this case, the Mayer–Vietoris complex is $C^*(\mathcal{I}, H_G^*(X))$, the cochain complex of the full simplex $\mathcal{I}$ with coefficients in $H_G^*(X)$. As stated in Example 2.3, MV^* is exact in every degree except 0, or equivalently, $E_2^{p,*} = 0$ unless $p = 0$. Again by Remark A.10, the spectral sequence collapses at the second page, $(H_G^*(X))_1 = 0$, and

$$G(H_G^*(X)) = H_G^*(X) = \ker d^0.$$

In both parts of the example, the cover is by definition a good cover of X.

The previous example highlights the simplicial nature of the Mayer–Vietoris complex. It can be viewed as a modification of the cochain complex of $\mathcal{I}$, obtained by restricting to the respective intersections of the subspaces in the cover. In the extreme case where every element of the cover equals X, these restrictions are identities, and the complex reduces to the simplicial skeleton.
More generally, if the covering subspaces are sufficiently large, applying the Mayer–Vietoris spectral sequence becomes redundant: the covering elements exhibit essentially the same cohomology as X, and the spectral sequence collapses on the first page to the first-column component of X, as shown in the following lemma.

Lemma 3.31. *With the notation from Remark 3.9, suppose that for each $Z \in \mathrm{PC}_G(X)$ there exists an $M \in \mathfrak{M}$ such that $Z \subseteq M$. Then*

$$\nu(X) = H_G^*(X),$$

and

$$H_G^*(X) \cong H^0\,\mathrm{MV},$$

as graded Λ_G-algebra. In particular, $\mathfrak{M}$ is a good cover of X.

Proof. By assumption,

$$H_G^*(X) = \bigoplus_{Z \in \mathrm{PC}_G(X)} H_G^* Z.$$

As pointed out in Remark 3.9 and since the condition from Corollary 3.17 is satisfied for every connected component of X, we may split the Mayer–Vietoris spectral sequence and compute the cohomology for each $Z \in \mathrm{PC}_G(X)$ separately. It therefore suffices to prove the claim in the case that $X = M_0$ where 0 is the unique minimal element in I.
For $x \in \mathrm{MV}^p$ with $p \geq 1$, define $y \in \mathrm{MV}^{p-1}$ by

$$y_{\underline{i}} := \begin{cases} x_{(0,i_0,\ldots,i_{p-1})}, & \text{if } i_0 > 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\underline{i} \in \mathcal{I}_{p-1}$.
If $d_1(x) = 0$, then for any $\underline{j} \in \mathcal{I}_p$ with $j_0 > 0$ we have

$$x_{\underline{j}} = \sum_{k=0}^{p} (-1)^k x_{(0,j_0,\ldots,\hat{j}_k,\ldots,j_p)},$$

which implies $d_1(y) = x$. Thus the Mayer–Vietoris complex is exact in every degree except 0. As in Example 3.30 it follows that $\mathfrak{M}$ is a good cover of X,

$$(H_G^*(X))_1 = 0,$$

and

$$G(H_G^*(X)) = H_G^*(X) \cong H^0(\mathrm{MV}).$$

□

Remark 3.32. The two opposite extremes for Lemma 3.31 are, on the one hand, Example 3.30, where the cover contains a subspace $M_0 = X$, and, on the other hand, Example 3.33, where X consists of isolated fixed points.

Example 3.33. If $X = \{p_1, \ldots, p_n\}$ is a finite union of G-fixed points, then each p_s lies in some $M \in \mathfrak{M}$. By Lemma 3.31 this implies that $\mathfrak{M}$ is a good cover of X, that $(H_G^*(X))_1 = 0$, and hence $H_G^*(X) \cong H^0(\mathrm{MV})$. As in Example 3.10, we split MV into subcomplexes $(\mathrm{MV}^*, d) = \bigoplus_{s=1}^n (\mathrm{MV}_s^*, d_s)$, where

$$\mathrm{MV}_s^p = \bigoplus_{\substack{\underline{i} \in \mathcal{I}_p \\ p_s \in M_{\underline{i}}}} \Lambda_G.$$

If $\mathcal{I}^s$ denotes the full simplex on the vertex set $\{i \in I \mid p_s \in M_i\}$, then $\mathrm{MV}_s^* = C^*(\mathcal{I}^s, \Lambda_G)$, which is exact except for $\ker d_s^0 = \Lambda_G$ (cf. Example 3.30). Consequently, $H_G^*(X) = \Lambda_G^n$.

Remark 3.34. As stated in the beginning of this chapter, the classical Mayer–Vietoris sequence computes the cohomology of a space X from a cover $\mathfrak{M} = (M_1, M_2)$ by two subspaces whose interiors cover X.
Assume the cover is G-invariant. Both the classical sequence and the spectral sequence require that the restriction $C_G^*(X) \to C_{G,\mathfrak{M}}^*(X)$ induces an isomorphism in cohomology, which in this case

follows from Remark 3.14.
The Mayer–Vietoris sequence arises from the short exact sequence of cochain complexes

$$0 \longrightarrow C^q_{G,\mathfrak{M}}(X) \xrightarrow{\iota} C^q_G(M_1) \oplus C^q_G(M_2) \xrightarrow{r_1 - r_2} C^q_G(M_1 \cap M_2) \longrightarrow 0$$

yielding the long exact sequence

$$\dots \longrightarrow H^q_G(X) \longrightarrow H^q_G(M_1) \oplus H^q_G(M_2) \xrightarrow{r_1^q - r_2^q} H^q_G(M_1 \cap M_2) \xrightarrow{\delta} H^{q+1}_G(X) \longrightarrow \dots$$

where δ denotes the connecting morphism.
From this, the equivariant cohomology of X can be expressed as

$$H^q_G(X)/\delta\left(\operatorname{coker}(r_1^{q-1} - r_2^{q-1})\right) \cong \ker(r_1^q - r_2^q), \quad q \geq 0.$$

On the other hand, the double complex for the Mayer–Vietoris spectral sequence in this case (Example 3.8) gives a Mayer–Vietoris complex

$$\mathrm{MV}^0 \xrightarrow{d^0} \mathrm{MV}^1 \longrightarrow 0,$$

and, as observed in Remark A.6, we have $E_2 = E_\infty$, i.e., (M_1, M_2) is a good cover of X. By Theorem 3.22,

$$H(\mathrm{MV}) \cong G(H^*_G(X)),$$

and since $d^0 = r_1 - r_2$ by construction, it follows that

$$\ker(r_1^q - r_2^q) = H^{0,q}(\mathrm{MV}) \cong H^q_G(X)/(H^q_G(X))_1,$$

and

$$\operatorname{coker}(r_1^{q-1} - r_2^{q-1}) = H^{1,q-1}(\mathrm{MV}) \cong (H^q_G(X))_1 \quad q \geq 0.$$

The two descriptions coincide, since the filtration on $H^*_G(X)$ satisfies

$$(H^*_G(X))_1 = \delta(H^*_G(X)).$$

Example 3.35. In the setting of Remark 3.34, suppose X admits a G-invariant cover by two subspaces M_1 and M_2 such that the restriction map

$$d^0 = r_1^* - r_2^* \colon H^*_G(M_1) \oplus H^*_G(M_2) \longrightarrow H^*_G(M_1 \cap M_2),$$

is surjective. Then

$$(H^{*+1}_G X)_1 = \operatorname{coker}(r_1^* - r_2^*) = 0,$$

and moreover

$$H^*_G(X) \cong \ker(r_1^* - r_2^*) = \{(x, y) \in H^*_G(M_1) \oplus H^*_G(M_2) \mid r_1^*(x) = r_2^*(y)\}.$$

3.3 Morphisms of Covered Spaces

To define the category of *covered spaces*, we both formalize the properties that a G-space equipped with a cover must satisfy for the Mayer–Vietoris spectral sequence to apply and introduce a natural notion of morphism. This allows us to regard the spectral sequence as a functor from the category of covered spaces to the category of bigraded algebras (Corollary 3.40). Morphisms of covered spaces follow the Mayer–Vietoris principle: a morphism between covered spaces is determined by its restrictions to the subspaces of the cover. In the remaining subsection, we explore their natural properties and obtain Corollary 3.42 and Corollary 3.47, which provide further instances where local analysis yields information about the global equivariant cohomology.

Definition 3.36. Let **BGradAlg** denote the category of bigraded algebras with bigraded algebra morphisms. We define **GCov** to be the category whose objects are triples $(G, X, \mathfrak{M})$, where G is a Lie group, X is a G-space and $\mathfrak{M}$ is a covering of X by G-invariant subspaces such that the restriction map

$$C^*_G(X) \longrightarrow C^*_{G,\mathfrak{M}}(X),$$

induces an isomorphism in cohomology. The objects of **GCov** are called *covered spaces*.
Morphisms in **GCov** exist only between objects whose coverings share the same index set I. A morphism

$$(\varphi, f)\colon (H, Y, (N_i)_{i\in I}) \longrightarrow (G, X, (M_i)_{i\in I}),$$

is given by a continuous group homomorphism $\varphi\colon H \to G$ and a φ-equivariant continuous map $f\colon Y \to X$ such that $f(N_i) \subseteq M_i$ for all $i \in I$.
If (φ, f) is a morphism in **GCov**, we also call f a morphism in **GCov** and say that it is compatible with the coverings.

Remark 3.37. Common examples of objects $(G, X, (M_i)_{i\in I})$ in **GCov** arise when either the interiors of M_i are G-invariant and cover X, or when each M_i is a deformation retract of some G-invariant neighborhood U_i such that $M_{\underline{i}}$ is a deformation retract of $U_{\underline{i}}$ for all $\underline{i} \in \mathcal{I}$ (see Remark 3.14 and Remark 3.15).
Both settings are stable under restriction to a G-invariant subspace. That is, if $Y \subseteq X$ is G-invariant and we are in either of the two situations above, then the triple $(G, Y, (M_i \cap Y)_{i\in I})$ is again an object of **GCov**.

Remark 3.38. Suppose $(G, X, (M_i)_{i\in I})$ and $(H, Y, (N_j)_{j\in J})$ are objects in **GCov**, let $\varphi\colon H \to G$ be a continuous group homomorphism, and let $f\colon Y \to X$ be a φ-equivariant continuous map.
If for every N_i there exists some M_j with $f(N_i) \subseteq M_j$, then, by adding additional copies of existing components or the empty set to the coverings $(N_j)_{j\in J}$ and $(M_i)_{i\in I}$, we may arrange that $I = J$ and $f(N_i) \subseteq M_i$ for all $i \in I$, in which case f is a morphism in **GCov**.

Lemma 3.39. *Let*

$$(\varphi, f)\colon (H, Y, (N_i)_{i\in I}) \longrightarrow (G, X, (M_i)_{i\in I}),$$

be a morphism in **GCov**. *Then*

$$f^*\colon H^*_G(X) \longrightarrow H^*_H(Y),$$

is compatible with filtrations, i.e., it induces a bigraded algebra morphism

$$\hat{f}\colon G(H^*_G(X)) \longrightarrow G(H^*_H(Y)).$$

Proof. By definition of **GCov**, to each object $(G, X, (M_i)_{i\in I})$ there is an associated Mayer–Vietoris spectral sequence constructed from the differential graded algebra

$$K_X^{p,q} := \bigoplus_{\underline{i}\in\mathcal{I}_p} C_G^q(M_{\underline{i}}).$$

The assumptions on $f\colon Y \to X$ give maps

$$f_{\underline{i}}\colon C_G^*(M_{\underline{i}}) \longrightarrow C_H^*(N_{\underline{i}}),$$

for all simplices $\underline{i}$ in $\mathcal{I}$. These assemble to a morphism of double complexes and differential graded algebras

$$f_K\colon K_X \longrightarrow K_Y.$$

The induced map

$$H(f_K)\colon H^*(K_X) \longrightarrow H^*(K_Y),$$

satisfies

$$H(f_K)\left((H^*(K_X))_p\right) \subseteq (H^*(K_Y))_p, \quad p \geq 0,$$

where the filtrations on $H^*(K_X)$ and $H^*(K_Y)$ come from the row-wise filtration of K_X and K_Y. Moreover, we have a commutative diagram

$$\begin{array}{ccc} H^*(K_X) & \xrightarrow{H(f_K)} & H^*(K_Y) \\ \downarrow & & \downarrow \\ H_G^*(X) & \xrightarrow{f^*} & H_H^*(Y), \end{array}$$

in which the vertical maps are the isomorphisms from Corollary 3.17. Since the filtrations on $H_G^*(X)$ and $H_H^*(Y)$ are defined via these vertical isomorphisms, the claim follows. □

Corollary 3.40. *The Mayer–Vietoris spectral sequence defines a functor*

$$\mathbf{GCov} \longrightarrow \mathbf{BGradAlg},$$

that sends a triple $(G, X, \mathfrak{M})$ *to the bigraded algebra* $G(H_G^*(X))$ *and a morphism* f *in* **GCov** *to the induced morphism* $\hat{f}$ *in* **BGradAlg**.

Proof. As shown in the proof of the previous Lemma, any morphism f in **GCov** induces a morphism of spectral sequences of algebras $\{f_r\}_{r\geq 0}$, with f_∞ equal to $\hat{f}$ (see Remarks A.13 and A.17). □

Remark 3.41. Given a morphism

$$(\varphi, f)\colon (H, Y, \mathfrak{N}) \longrightarrow (G, X, \mathfrak{M}),$$

in **GCov**, the proof of Lemma 3.39 shows that the induced morphism on the first page of the spectral sequences, i.e., on the corresponding Mayer–Vietoris complexes,

$$f_1 : \mathrm{MV}(G, X, \mathfrak{M}) \longrightarrow \mathrm{MV}(H, Y, \mathfrak{N}),$$

is given componentwise by

$$f_{\underline{i}}^* : H_G^*(M_{\underline{i}}) \longrightarrow H_H^*(N_{\underline{i}}), \quad \underline{i} \in \mathcal{I}.$$

Corollary 3.42. *Let*

$$(\varphi, f)\colon (H, Y, \mathfrak{N}) \longrightarrow (G, X, \mathfrak{M}),$$

be a morphism in **GCov**, *and let* $f_{\underline{i}}$ *be its restriction to* $M_{\underline{i}}$, *where* $\underline{i} \in \mathcal{I}$. *If*

$$f_{\underline{i}}^*\colon H_G^*(M_{\underline{i}}) \longrightarrow H_H^* N_{\underline{i}},$$

is an isomorphism of graded Λ_G*-algebras for all* $\underline{i} \in \mathcal{I}$, *then*

$$f^*\colon H_G^*(X) \longrightarrow H_H^*(Y),$$

is an isomorphism of graded Λ_G*-algebras.*

Proof. By Corollary 3.40, the morphism f induces a morphism of spectral sequences $\{f_r\}_{r\geq 0}$. The assumption implies f_1 is an isomorphism (see Remark 3.41), so the result follows from Theorem A.14 together with Remark A.17. □

Remark 3.43. In Theorem A.14 the isomorphism can occur on any page of the spectral sequence. Consequently, $\hat{f}$ is an isomorphism of bigraded algebras if and only if f^* is an isomorphism of graded algebras.

Example 3.44. Let $(H, Y, (N_i)_{i\in I})$ and $(G, X, (M_i)_{i\in I})$ be objects in **GCov** with associated Mayer–Vietoris spectral sequences $\{E(Y)_r\}$ and $\{E(X)_r\}$. Suppose

$$f\colon (H, Y, (N_i)_{i\in I}) \longrightarrow (G, X, (M_i)_{i\in I})$$

is a morphism in **GCov** and that $(H_G^*(X))_1 = 0$. This holds, for example, if $\mathfrak{M} = (X)_{i\in I}$ (see Example 3.30). In this case $G(H_G^*(X)) = H_G^*(X)$ and

$$\hat{f}\colon H_G^*(X) \longrightarrow G(H_H^*(Y)),$$

maps into $\nu(Y)$. Let

$$x \in \bigcap_{r\geq 1} \ker d_r^0 \subseteq \bigcap_{r\geq 1} E(X)_r^{0,*},$$

see Remark 5.23. If, for some $r \geq 1$, the restriction of f_r to the first column

$$f_r|_{E(X)_r^{0,*}}\colon E(X)_r^{0,*} \longrightarrow E(Y)_r^{0,*}$$

maps x to zero, then $\hat{f}(x) = 0$.

Now suppose that $(H_H^*(Y))_1 = 0$ (see Example 3.30). Then f^* factors through $\hat{f}$:

$$H_G^*(X) \twoheadrightarrow \nu(X) \xrightarrow{\hat{f}} H_H^*(Y), \qquad f^* = \hat{f}\circ(H_G^*(X) \twoheadrightarrow \nu(X)).$$

In particular, $(H_G^*(X))_1 \subseteq \ker f^*$.

Example 3.45. Suppose $(M_i)_{i\in I}$ is a G-invariant cover of X that satisfies the condition of either Remark 3.14 or Remark 3.15. If $Y \subseteq X$ is a G-invariant subspace, then also $(G, Y, (M_i \cap Y)_{i\in I})$ is an object in **GCov** and the inclusion $r\colon Y \hookrightarrow X$ is compatible with the coverings. The induced morphism on the first page of the Mayer–Vietoris spectral sequence consists of the restrictions

$$r_{\underline{i}}^*\colon H_G^*(M_{\underline{i}}) \longrightarrow H_G^*(M_{\underline{i}} \cap Y), \quad \underline{i} \in \mathcal{I}.$$

Example 3.46. If $(G, X, \mathfrak{N} = (N_i)_{i\in I})$ is an object in **GCov**, then so is $(G, X, \mathfrak{M} = (X)_{i\in I})$. The identity

$$\mathrm{id}\colon (G, X, (N_i)_{i\in I}) \longrightarrow (G, X, (X)_{i\in I}),$$

is a morphism in **GCov** with

$$\hat{\mathrm{id}} : H^*_G(X) \longrightarrow \nu(X, \mathfrak{N}) \subseteq G(H^*_G(X)),$$

by Example 3.44.
Similarly to Example 3.45, let

$$r_i\colon N_i \hookrightarrow X, \quad i \in I,$$

denote the inclusions. On the first page of the spectral sequences, the morphism id_1 inducing $\hat{\mathrm{id}}$ is given by the restrictions

$$r^*_{\underline{i}}\colon H^*_G(X) \longrightarrow H^*_G(N_{\underline{i}}), \quad i \in \mathcal{I}.$$

Corollary 3.47. *Suppose $(G, X, (M_i)_{i\in I})$ is an object in* **GCov** *with $(H^*_G(X))_1 = 0$. If two elements $x, y \in H^*_G(X)$ restrict identically to $H^*_G(M_i)$ for all $i \in I$, then $x = y$.*

Proof. Consider the two covers of X introduced in Example 3.46. The identity map id is then a morphism in **GCov**, and since $(H^*_G(X))_1 = 0$ for the filtrations induced by both covers, we have $\hat{\mathrm{id}} = \mathrm{id}$. The assumption $r^*_i(x) = r^*_i(y)$ for all $i \in I$ means that

$$(x - y)_{i\in I} \in \mathrm{MV}^0(G, X, (X)_{i\in I}),$$

lies in the kernel of the restriction of id_1 to the first column

$$\mathrm{id}_1|_{\mathrm{MV}^0(G,X,(X)_{i\in I})}\colon \mathrm{MV}^0(G, X, (X)_{i\in I}) \longrightarrow \mathrm{MV}^0(G, X, (M_i)_{i\in I}).$$

By Example 3.44, this implies $\hat{\mathrm{id}}(x - y) = 0$, hence $x = \mathrm{id}(x) = \mathrm{id}(y) = y$. □

Remark 3.48. For a morphism $f\colon (H, Y, \mathfrak{N}) \to (G, X, \mathfrak{M})$ in **GCov**, if both $\mathfrak{M}$ and $\mathfrak{N}$ are good covers for X and Y, respectively, then the morphism induced by f_1 in cohomology is already equal to $\hat{f}$.

Example 3.49. Consider a morphism $f\colon (H, Y, \mathfrak{N}) \to (G, X, \mathfrak{M})$ in **GCov**, where $\mathfrak{N} = (N_1, N_2)$ and $\mathfrak{M} = (M_1, M_2)$ are covers by two subsets. Denote the differentials of the respective Mayer–Vietoris complexes by d and c. As noted in Remark 3.34, both $\mathfrak{N}$ and $\mathfrak{M}$ are good covers, and the induced morphism $\hat{f}$ arises from the morphism f_1 between the corresponding Mayer–Vietoris complexes:

$$\begin{array}{ccc} H^*_H(N_1) \oplus H^*_H(N_2) & \xrightarrow{d} & H^*_H(N_1 \cap N_2) \\ \downarrow{\scriptstyle (f_1, f_2)} & & \downarrow{\scriptstyle f_{(1,2)}} \\ H^*_G(M_1) \oplus H^*_G(M_2) & \xrightarrow{c} & H^*_G(M_1 \cap M_2) \end{array}$$

If both d and c are surjective, then we are in the situation of Example 3.35, and the induced morphism in cohomology takes the form

$$\hat{f} = f^*\colon H^*_H(Y) = \ker(d) \xrightarrow{(f_1, f_2)} H^*_G(X) = \ker(c).$$

Finally, suppose in addition that (f_1, f_2) is surjective and that

$$d^{-1}(\ker f_{(1,2)}) \subseteq \ker(f_1, f_2).$$

Then f^* is surjective as well. Indeed, given $x \in \ker(c)$ with $x = (f_1, f_2)(y)$, we have

$$y - z \in \ker(d), \qquad x = (f_1, f_2)(y - z),$$

for any $z \in d^{-1}(d(y))$.

4 The Mayer–Vietoris Spectral Sequence for Torus Action

We have constructed the Mayer–Vietoris spectral sequence and, in Chapter 3.3, taken a first look at how the Mayer–Vietoris principle, namely, deriving information about $H_G^*(X)$ from the cohomologies of the covering spaces, applies in the context of morphisms. In what follows, we further develop this idea of obtaining global information from local data, with emphasis on the case most relevant to our purposes, where $G = T$ is a torus.
The first subsection addresses the compatibility of restriction to fixed points with the spectral sequence. Subsequently, we examine the behavior under changes of the torus action, with particular attention to the case where the elements of the cover are equivariantly formal.

In this subsection, let $T \cong (\mathbb{C}^\times)^m$ denote a complex algebraic torus of rank m. We keep the notation for covered spaces and the Mayer–Vietoris spectral sequence from Chapter 3.

4.1 Localization and the Moment Graph

The concept of localization was introduced in Chapter 2.2, where we discussed under which conditions, and to what extent, the equivariant cohomology of a variety is determined by its restriction to the fixed points. Here, we revisit this concept in the setting of covered spaces. Under suitable assumptions, we describe the kernel and image of the localization map and relate them to the first-column component (Lemma 4.2, Corollary 4.3). In Theorem 4.10, we express the Mayer–Vietoris complex in terms of fixed points, assuming all elements of the cover are GKM-varieties.

In this subsection, unless stated otherwise, the term *torsion-free* refers to torsion-freeness over the ring Λ_T.

Let X be a variety with a T-action and a cover by T-invariant subvarieties $\mathfrak{M} = (M_i)_{i\in I}$ such that $(T, X, \mathfrak{M})$ is an object of **GCov**. Denote by X^T the set of T-fixed points in X. It is natural to apply the localization idea to covered spaces by considering the cover

$$\mathfrak{M}^T \coloneqq (M_i^T)_{i\in I},$$

and examining the Mayer–Vietoris spectral sequence in this setting.
When X^T is finite, which is the classical situation in the literature, $H_T^*(X^T)$ is a (torsion-)free Λ_T-module with trivial filtration $(H_T^*(X^T))_1 = 0$ (see Example 3.33), and $(T, X^T, \mathfrak{M}^T)$ is automatically an object of **GCov**. All assumptions made in this chapter are satisfied in this case, which will therefore serve as the motivating example for the discussion.
Throughout this chapter we assume that $(T, X^T, \mathfrak{M}^T)$ is an object in **GCov**. In addition to the case where X^T is finite, this holds in the most relevant situations, as noted in Remark 3.37.

Let

$$\{E_r, d_r\} \quad \text{and} \quad \{E_r^T, d_r^T\},$$

denote the spectral sequences associated to the covered spaces X and X^T, respectively. The localization map

$$\iota^*\colon H_T^*(X) \longrightarrow H_T^*(X^T),$$

is induced by the inclusion $\iota\colon X^T \hookrightarrow X$. We denote by $\iota(X)$ and $\tau(X)$ its image and kernel, respectively, as in Definition 2.18. By assumption,

$$\iota\colon (T, X^T, \mathfrak{M}^T) \longrightarrow (T, X, \mathfrak{M})$$

is a morphism in **GCov** and therefore induces algebra morphisms on the pages of the spectral sequences

$$\iota_r\colon E_r \longrightarrow E_r^T, \quad r \geq 0,$$

as well as a morphism of bigraded algebras

$$\hat{\iota}\colon G(H_T^*(X)) \longrightarrow G(H_T^*(X^T)).$$

On the first page we obtain the bigraded Λ_T-algebra morphism

$$\iota_1\colon \operatorname{MV}(T, X, \mathfrak{M}) \longrightarrow \operatorname{MV}(T, X^T, \mathfrak{M}^T)$$

given by the sum of the respective localization maps

$$\iota_{\underline{i}}^*\colon H_T^*(M_{\underline{i}}) \longrightarrow H_T^*(M_{\underline{i}}^T), \quad \underline{i} \in \mathcal{I}.$$

We have already seen that $\mathfrak{M}^T$ is a good cover of X^T when X^T is finite. The following lemma slightly generalizes this.

Lemma 4.1. *If $\mathfrak{M}$ is a good cover of X and E_2^T is torsion-free, then $\mathfrak{M}^T$ is a good cover of X^T.*

Proof. By Theorem 2.21, there exists a sufficiently large multiplicative subset $S \subseteq M$ such that

$$S^{-1}\iota_r\colon S^{-1}E_r \longrightarrow S^{-1}E_r^T$$

is an isomorphism for $r = 1$, and hence for all $r \geq 1$ (see Theorem A.14 applied to the spectral sequences associated to the double complexes localized at S). In the commutative diagram

$$\begin{array}{ccc} S^{-1}E_2 & \xrightarrow{S^{-1}d_2} & S^{-1}E_2 \\ \big\downarrow{\scriptstyle S^{-1}\iota_2} & & {\scriptstyle S^{-1}\iota_2}\big\downarrow \\ S^{-1}E_2^T & \xrightarrow{S^{-1}d_2^T} & S^{-1}E_2^T \end{array}$$

the morphism $S^{-1}d_2$ vanishes since $\mathfrak{M}$ is a good cover of X. As the vertical maps are isomorphisms, $S^{-1}d_2^T$ vanishes as well, and torsion-freeness of E_2^T implies $d_2^T = 0$. Hence $\mathfrak{M}^T$ is a good cover of X^T. □

Lemma 4.2. *If $H_T^*(M_i)$ is torsion-free for all $i \in I$ and $(H_T^*(X^T))_1 = 0$, then*

$$\tau(X) = (H_T^*(X))_1 = \operatorname{tor}(\Lambda_T, H_T^*(X)) \quad \textit{and} \quad \nu(X) \cong \iota(X).$$

Proof. If $(H_T^*(X^T))_1 = 0$, the second part of Example 3.44 yields

$$(H_T^*(X))_1 \subseteq \tau(X),$$

and there is a surjective morphism

$$\nu(X) = H_T^*(X)/(H_T^*(X))_1 \xrightarrow{\iota^*} \iota(X).$$

On the other hand, Corollary 2.24 gives

$$\tau(X) \subseteq \operatorname{tor}(\Lambda_T, H_T^*(X)).$$

Since $\nu(X) \subseteq \bigoplus_{i\in I} H_T^*(M_i)$ is torsion-free by assumption, it follows that

$$\operatorname{tor}(\Lambda_T, H_T^*(X)) \subseteq (H_T^*(X))_1 \subseteq \tau(X) \subseteq \operatorname{tor}(\Lambda_T, H_T^*(X)).$$

In particular, the surjective map above is also injective, hence an isomorphism. □

Corollary 4.3. *If $H_T^*(M_i)$ is torsion-free for all $i \in I$, $(H_T^*(X^T))_1 = 0$, and $\mathfrak{M}$ is a good cover of X, then*

$$\iota(X) \cong \ker d^0 = H^0(\mathrm{MV}(T, X, \mathfrak{M})).$$

Proof. This follows from Lemma 4.2 and Theorem 3.28. □

Remark 4.4. If X^T is finite and all M_i are equivariantly formal, then all assumptions of Lemma 4.2 are satisfied, giving an explicit description of $\tau(X)$.
In Chapter 5 we will present examples of spaces covered by equivariantly formal spaces for which $(H_T^*(X))_1 \neq 0$, and thus X is not equivariantly formal (see Lemma 2.27).

The second half of this subsection is devoted to characterizing the localization image of X, that is, to determining conditions under which $\iota(X)$ can be described via the moment graph of X. For $i \in I$, consider the inclusion

$$r_i \colon M_i \longrightarrow X,$$

as a morphism in **GCov** by equipping M_i with the trivial cover $(N_j)_{j\in I}$ defined by $N_i = M_i$ and $N_j = \varnothing$ for all $j \in I \setminus \{i\}$ (see Remark 3.38). The morphism $(r_i)_1$ induced on the first page of the Mayer–Vietoris spectral sequence is the projection

$$\mathrm{MV}(T, X, \mathfrak{M}) \longrightarrow H_T^*(M_i),$$

and, as seen in Example 3.44, the map r_i^* factors through $\hat{r}_i$. Similarly, the inclusion

$$t_i \colon M_i^T \longrightarrow X^T,$$

is a morphism in **GCov**, with $(t_i)_1$ given by the projection

$$\mathrm{MV}(T, X^T, \mathfrak{M}^T) \longrightarrow H_T^*(M_i^T),$$

and t_i^* factors through $\hat{t}_i$.

Lemma 4.5. *The localization image of X satisfies*

$$\iota(X) \subseteq \bigcap_{i\in I} (t_i^*)^{-1}(\iota(M_i)).$$

Proof. For all $i \in I$, the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{r_i} & M_i \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota_i} \\ X^T & \xleftarrow[t_i]{} & M_i^T \end{array}$$

induces the commutative diagram in cohomology

$$\begin{array}{ccc} H_T^*(X) & \xrightarrow{r_i^*} & H_T^*(M_i) \\ {\scriptstyle\iota^*}\downarrow & & \downarrow{\scriptstyle\iota_i^*} \\ H_T^*(X^T) & \xrightarrow[t_i^*]{} & H_T^*(M_i^T) \end{array}$$

and therefore

$$\iota^*(H_T^*(X)) \subseteq (t_i^*)^{-1}(\iota^*(H_T^*(M_i))).$$

□

Lemma 4.6. *If E_1 is torsion-free, $\mathfrak{M}$ is a good cover of X, and $(H_T^*(X^T))_1 = 0$, then*

$$\iota(X) = \bigcap_{i \in I} (t_i^*)^{-1}(\iota(M_i)).$$

Proof. For $i, j \in I$, consider the restriction maps

$$r_{ij}^* \colon H_T^*(M_i) \longrightarrow H_T^*(M_{(i,j)}), \qquad t_{ij}^* \colon H_T^*(M_i^T) \longrightarrow H_T^*(M_{(i,j)}^T).$$

Let $x \in \bigcap_{i \in I} (t_i^*)^{-1}(\iota(M_i))$ with $t_i^*(x) = \iota_i^*(m_i)$ for some $m_i \in H_T^*(M_i)$. Then

$$0 = (t_{ij}^* t_i^* - t_{ji}^* t_j^*)(x) = t_{ij}^* \iota_i^*(m_i) - t_{ji}^* \iota_j^*(m_j) = \iota_{(i,j)}^*\big(r_{ij}^*(m_i) - r_{ji}^*(m_j)\big).$$

Since $\mathrm{MV}(T, X, \mathfrak{M})$ is torsion-free and $\ker \iota_{(i,j)}^* = \tau(M_{(i,j)}) \subseteq \mathrm{tor}(\Lambda_T, H_T^*(M_{(i,j)}))$ by Corollary 2.24, it follows that

$$r_{ij}^*(m_i) - r_{ji}^*(m_j) = 0.$$

Hence

$$y := (m_i)_{i \in I} \in \bigoplus_{i \in I} H_T^*(M_i)$$

lies in $\ker d^0$, which equals $\nu(X)$ since $\mathfrak{M}$ is a good cover of X. Therefore $y \in G(H_T^*(X))$, and any representative $\tilde{y}$ of y in $H_T^*(X)$ satisfies $r_i^*(\tilde{y}) = r_i^*(y)$, as r_i^* factors through $\hat{r}_i$. It follows that

$$t_i^*(\iota^*(\tilde{y})) = \iota_i^*(r_i^*(\tilde{y})) = \iota_i^*(m_i) = t_i^*(x)$$

for all $i \in I$. By Corollary 3.47, this implies $\iota^*(\tilde{y}) = x$. □

We now turn to the moment graph of X.

Definition 4.7. A cover $(T, X, (M_i)_{i\in I})$ is called *fixed-point closed* if

$$M_i^T = \overline{M_i}^T,$$

for all $i \in I$, i.e., the closure of M_i has the same set of T-fixed points as M_i.

Lemma 4.8. *Let X be a variety with a T-action such that X^T is finite. If $(T, X, (M_i)_{i\in I})$ is a fixed-point closed cover by subvarieties in* **GCov***, then the moment graph of X is the union of the moment graphs of the M_i.*

Proof. Let C be a T-curve in X, i.e., the closure of an orbit $T \cdot x$ for some $x \in X$. By Remark 2.35 and the fixed-point closed property, any M_i containing x must also contain C. If C connects two distinct fixed points of X, the edge corresponding to C in the moment graph Γ_X already appears as an edge in the moment graph Γ_{M_i} of each M_i containing x. Since

$$X^T = \bigcup_{i\in I} M_i^T,$$

the vertex set of Γ_X is the union of the vertex sets of the Γ_{M_i}, and the same holds for the edge sets. Thus Γ_X is the union of the moment graphs Γ_{M_i}, for $i \in I$. □

Example 4.9. Let $T = (\mathbb{C}^\times)^2$ act on $\mathbb{P}^1$ via relatively prime characters χ_0 and χ_1 as in Example 2.19. Consider the cover by the affine patches

$$V_0 = \{(z_0 : z_1) \in \mathbb{P}^1 \mid z_0 \neq 0\}, \quad V_1 = \{(z_0 : z_1) \in \mathbb{P}^1 \mid z_1 \neq 0\}.$$

Then $(T, \mathbb{P}^1, (V_0, V_1))$ is an object in **GCov** that is not fixed-point closed. The moment graphs of V_0 and V_1 both consist of a single vertex, so their union does not recover the moment graph of $\mathbb{P}^1$.

Theorem 4.10. *Suppose* MV *is torsion-free, X^T is finite, $\mathfrak{M}$ is a good cover of X, and $(T, X, (M_i)_{i\in I})$ is fixed-point closed.*
If the localization of each M_i is described by its moment graph Γ_{M_i}, then the localization of X is described by Γ_X.

Proof. Since X^T is finite, all assumptions of Lemma 4.6 are satisfied. For each $i \in I$, the restriction map

$$t_i^* \colon \Lambda_T^{X^T} \longrightarrow \Lambda_T^{M_i^T},$$

is simply the projection onto the factors indexed by M_i^T. Thus

$$\iota(X) = \bigcap_{i\in I} (t_i^*)^{-1}(\iota(M_i)),$$

is the subalgebra of $\Lambda_T^{X^T}$ consisting of all tuples $(u_p)_{p\in X^T}$ such that:
Let $p, q \in X^T$ and assume that $j \in I$ is such that p, q are contained in M_j. Then for each T-curve C in M_j connecting p and q the difference $u_p - u_q$ has to be divisible by the character of the T-curve C.

By Lemma 4.8, this exactly describes the subalgebra $H^*(\Gamma_X)$. Hence the localization of X is described by Γ_X. □

4.2 Change of Tori

An important further property of equivariant cohomology is its functoriality, not only in the space but also in the acting group. Let X be a topological space and $\varphi\colon T_2 \to T_1$ a morphism of tori. If T_1 and T_2 act on X compatibly with φ, the *change of tori*, or *torus change*, is given by the induced morphism of algebras

$$H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X).$$

Relating the cohomology with respect to different torus actions is often useful, for example, when approximating $H^*_{T_2}(X)$ via a more accessible action of a torus T_1. This approach will be employed in Chapter 6.2, and in the present subsection, we prepare for it by examining torus change in the setting of covered spaces.

We begin by recalling the functorial behavior of the character lattice, with emphasis on subgroups and quotients (see [Hum81, end of Sections 16.1 and 16.2]). We introduce the notion of *introduced relations* (Definition 4.18), recall torus change for an ordinary G-space, and work out the details of splitting off a subtorus with trivial action (Example 4.30). Torus change for covered spaces is then described explicitly in Lemma 4.37 for the case where the cover consists of equivariantly formal spaces. In this situation, understanding torus change amounts to understanding the map

$$H(\mathrm{MV}(T_1, X, \mathfrak{M})) \longrightarrow H(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \mathrm{MV}(T_1, X, \mathfrak{M})),$$

for which we introduce Künneth formulas (Theorem 4.40, Theorem 4.43). We conclude the chapter with a brief introduction of the Koszul complex following [Eis95], and present as an application the evaluation of the Künneth formula in special cases (Corollary 4.49).

We recall the notation introduced in Chapter 2.1, beginning with Remark 2.13 and the subsequent definitions and remarks. In the following, if x is an element of a module, group, or algebra, its residue class in any quotient thereof will be denoted by $[\![x]\!]$.

For a torus T, taking the character lattice M defines a contravariant functor

$$F\colon \{\text{algebraic tori}\} \longrightarrow \{\text{finitely generated free abelian groups}\}.$$

Under F, a morphism of tori $\varphi\colon T_2 \to T_1$ is sent to the morphism of character lattices

$$F(\varphi)\colon M_1 \longrightarrow M_2, \quad \sigma \longmapsto \sigma \circ \varphi.$$

Let N denote the group of one-parameter subgroups of T. A quasi-inverse

$$Q\colon \{\text{finitely generated free abelian groups}\} \longrightarrow \{\text{algebraic tori}\},$$

to F is obtained by sending a finitely generated free abelian group M to the torus

$$\mathrm{Hom}(M, \mathbb{Z}) \otimes \mathbb{C}^\times = N \otimes \mathbb{C}^\times,$$

and a morphism $\gamma\colon M_1 \to M_2$ to the morphism of algebraic tori

$$\gamma^* \otimes \mathrm{id}_{\mathbb{C}^\times} : T_2 \cong N_2 \otimes \mathbb{C}^\times \longrightarrow T_1 \cong N_1 \otimes \mathbb{C}^\times,$$

given by

$$\delta \otimes t \longmapsto \gamma^*(\delta) \otimes t = (\delta \circ \gamma) \otimes t,$$

where $\gamma^* : N_2 \to N_1$ is the dual map induced by γ (see Remark 2.13 and Remark 2.14). In fact, the category of algebraic tori and the category of finitely generated free abelian groups are canonically equivalent under F.

By the functoriality of equivariant cohomology, any morphism $\varphi\colon T_2 \to T_1$ induces a morphism of equivariant coefficient rings $\Lambda_{T_1} \to \Lambda_{T_2}$, which coincides with the morphism

$$\operatorname{Sym}^*_R M_1 \to \operatorname{Sym}^*_R M_2,$$

induced by $F(\varphi)\colon M_1 \to M_2$. Summarizing, we have the following canonical one-to-one correspondences between morphisms:

$$\{T_2 \to T_1\} \xleftrightarrow{1:1} \{M_1 \to M_2\} \xleftrightarrow{1:1} \{N_2 \to N_1\} \xleftrightarrow{1:1} \{\Lambda_{T_1} \to \Lambda_{T_2}\}.$$

In Definition 2.15, we introduced the one-to-one correspondence between subgroups of T and sublattices of its character lattice M. Recall, that for a sublattice $M' \leq M$ we set

$$T_{M'} = \{t \in T \mid \chi(t) = 1 \text{ for all } \chi \in M'\},$$

and for a subgroup $T' \leq T$ we set

$$M_{T'} = \{\chi \in M \mid \chi\,|_{T'} = 0\}.$$

Note that this is not the correspondence given by the functors F and Q.

Remark 4.11. Let T' be a subgroup of T. The character group of T' is naturally isomorphic to the quotient $M/M_{T'}$, which is not necessarily free. Additionally, the inclusion $\iota\colon T' \hookrightarrow T$ induces the projection

$$F(\iota)\colon M \to M/M_{T'}.$$

If $\overline{M}$ is a quotient of M, then the associated torus $T' \coloneqq Q(\overline{M})$ is naturally embedded in T and the projection $\pi\colon M \to \overline{M}$ induces the inclusion

$$Q(\pi)\colon T' \hookrightarrow T.$$

Conversely, let M' be a sublattice of M and $\overline{T} \coloneqq Q(M')$. The inclusion $\iota : M' \hookrightarrow M$ induces a morphism of tori

$$Q(\iota) : T \to \overline{T},$$

whose kernel is precisely $T_{M'}$.

If $\overline{T}$ is a quotient of T, then the associated lattice $M' \coloneqq F(\overline{T})$ is naturally embedded in M and the projection $\pi : T \to \overline{T}$ induces the inclusion

$$F(\pi) : M' \hookrightarrow M.$$

We give a list of selected facts related to saturation of a sublattice. For the proofs we refer to [Hum81, end of Chapters 16.1 and 16.2].

Definition 4.12. A sublattice $M' \leq M$ is called *saturated* in M if

$$p \cdot \chi \in M' \Rightarrow \chi \in M'$$

for all $\chi \in M$ and $p \in \mathbb{Z} \setminus \{0\}$.
The *saturation* $S(M')$ of M' is the intersection of all saturated sublattices of M containing M'. Equivalently, $S(M')$ is the smallest saturated sublattice of M containing M'.

Lemma 4.13. *Let T be a torus with character lattice M, and let $M' \leq M$ be a sublattice with corresponding subgroup $T' \coloneqq T_{M'}$ of T. Then:*

1. *T' is a subtorus of T if and only if T' is connected.*
2. *T' is connected if and only if M' is saturated in M.*
3. *T/T' is a torus if and only if M' is saturated in M.*
4. *There exists a lattice complement to M', i.e., a saturated sublattice $M'' \leq M$ such that $M = M' \oplus M''$ if and only if M' is saturated.*
5. *There exists a torus complement to T', i.e., a subtorus $T'' \subseteq T$ such that $T = T' \times T''$ if and only if T' is a subtorus of T.*

Lemma 4.14. *Let T be a torus with character lattice M, and let $M' \leq M$ be a saturated sublattice with lattice complement M''. Denote by $T' \coloneqq T_{M'}$ and $T'' \coloneqq T_{M''}$ the corresponding subtori of T. Then:*

1. *The character lattice of T' is naturally isomorphic to M'', and the character lattice of T'' is naturally isomorphic to M'.*
2. *The inclusion $M' \hookrightarrow M$ corresponds to the quotient map of the tori $T \to T/T' \cong T''$.*
3. *The quotient map $M \to M/M''$ of lattices corresponds to the inclusion $T' \hookrightarrow T$.*

Before turning to equivariant cohomology, we discuss the behavior of equivariant coefficient rings under torus change.

Definition 4.15. Let $\varphi\colon T_2 \to T_1$ be a morphism of tori with corresponding morphisms

$$\varphi^* = Q(\varphi)\colon M_1 \to M_2, \quad \hat{\varphi}\colon \Lambda_{T_1} \to \Lambda_{T_2},$$

of character lattices and equivariant coefficient rings.
We write

$$\mathcal{R}_\varphi \coloneqq \ker \hat{\varphi} = \ker \varphi^* \cdot \Lambda_{T_1},$$

or simply $\mathcal{R}$, for the kernel of $\hat{\varphi}$. The subtorus of T_2 corresponding to the saturation $S(\operatorname{im} \varphi^*)$ of $\operatorname{im} \varphi^*$ in M_2 is denoted by

$$T_2^S \coloneqq T_{S(\operatorname{im} \varphi^*)}.$$

Corollary 4.16. *Assume that* $\operatorname{im}\varphi^*$ *is saturated in* M_2 *with corresponding subtorus* $T_2' \coloneqq T_{\operatorname{im}\varphi^*}$ *of* T_2*. Then*

$$\Lambda_{T_2} \cong \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2'},$$

and under this identification,

$$\hat{\varphi}\colon \Lambda_{T_1} \longrightarrow \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2'}, \quad x \longmapsto [\![x]\!] \otimes 1.$$

Proof. By Lemma 4.13, we can choose a lattice complement M_2'' of $M_2' \coloneqq \operatorname{im}\varphi^*$ in M_2, with corresponding subtori T_2'' and T_2' such that $T_2 = T_2' \times T_2''$. By Lemma 4.14, the character lattice of T_2' is naturally isomorphic to M_2'', and the character lattice of T_2'' is naturally isomorphic to M_2'. Therefore

$$\Lambda_{T_2} = \operatorname{Sym}^*_R(M_2) = \operatorname{Sym}^*_R(M_2') \otimes \operatorname{Sym}^*_R(M_2'') = \Lambda_{T_2'} \otimes \Lambda_{T_2''}.$$

(compare Remark 2.11).
We obtain an isomorphism of abelian groups

$$\varphi^*\colon M_1/\ker\varphi \cong M_2',$$

and may regard φ^* as the morphism

$$M_1 \longrightarrow M_2 = M_2' \oplus M_2'' \cong M_1/\ker\varphi \oplus M_2'', \quad x \longmapsto ([\![x]\!], 0).$$

Passing to symmetric algebras gives

$$\operatorname{Sym}^*_R(M_1/\ker\varphi) = \Lambda_{T_1}/\mathcal{R}_\varphi \cong \operatorname{Sym}^*_R(M_2') = \Lambda_{T_2''},$$

and $\hat{\varphi}$ is identified with

$$\hat{\varphi}\colon \Lambda_{T_1} \longrightarrow \Lambda_{T_2} = \Lambda_{T_2'} \otimes \Lambda_{T_2''} \cong \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2'}, \quad x \longmapsto [\![x]\!] \otimes 1.$$

□

Lemma 4.17. *Assume every nonzero integer is invertible in* R*. Then*

$$\Lambda_{T_2} \cong \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2'},$$

and under this identification,

$$\hat{\varphi}\colon \Lambda_{T_1} \longrightarrow \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2'}, \quad x \longmapsto [\![x]\!] \otimes 1.$$

Proof. As in the proof of Corollary 4.16, choose a lattice complement M_2'' of $M_2' \coloneqq S(\operatorname{im}\varphi^*)$ in M_2, with corresponding subtori T_2'' and T_2^S, respectively. Since every nonzero integer is invertible in R, the induced map

$$\varphi^* \otimes 1\colon M_1/\ker\varphi \otimes_{\mathbb{Z}} R \cong M_2' \otimes_{\mathbb{Z}} R,$$

is an isomorphism of R-modules.
Passing to symmetric algebras yields an isomorphism of R-algebras

$$\hat{\varphi}\colon \Lambda_{T_1}/\mathcal{R}_\varphi \cong \Lambda_{T_2''}.$$

The claimed decomposition of Λ_{T_2} and the description of $\hat{\varphi}$ follow exactly as in Corollary 4.16. □

Definition 4.18. If we are in the situation of Corollary 4.16 or Lemma 4.17, i.e., if

$$\Lambda_{T_2} \cong \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2^S},$$

we say that φ *introduces the relations* $\mathcal{R}_\varphi$ *on* Λ_{T_1}.

Remark 4.19. The kernel of φ is precisely the subgroup corresponding to the sublattice $\operatorname{im}\varphi^*$. Indeed, every character of T_2 that factors through $\ker\varphi$ is the restriction of some character of T_1. If we replace $\operatorname{im}\varphi^*$ by its saturation, the corresponding subtorus is generally contained in, but not equal to $\ker\varphi$.
In Cor 4.16 and Lemma 4.17, this means that T_2' coincides with $\ker\varphi$, while T_2^S is contained in $\ker\varphi$.
Conversely, $\operatorname{im}\varphi$ is the subtorus corresponding to $\ker\varphi^* \leq M_1$. Note that $\operatorname{im}\varphi$ is always a subtorus of T_1, and $\mathcal{R}_\varphi = \ker\varphi^*$ is always a saturated sublattice of M_1.

Definition 4.20. Let X be a topological space with actions of both T_1 and T_2. A morphism $\varphi\colon T_2 \to T_1$ is *compatible* (with the given actions) if the identity map on X is φ-equivariant.

Remark 4.21. In this situation, the T_2-action on X is completely determined by φ. It is therefore equivalent to start with a T_1-action on X together with a morphism $\varphi\colon T_2 \to T_1$, and then define the T_2-action via φ.

For the remainder of this subsection, suppose that X has actions by both T_1 and T_2, and that $\varphi\colon T_2 \to T_1$ is compatible.

The morphism $\hat\varphi\colon \Lambda_{T_1} \to \Lambda_{T_2}$ allows us to change coefficients as follows. First, restriction of scalars along $\hat\varphi$ yields a graded Λ_{T_1}-module morphism

$$\theta_\varphi\colon H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X),$$

induced by the identity map on X. Second, extension of scalars along $\hat\varphi$ is given by the graded Λ_{T_2}-module morphism

$$\Theta_\varphi\colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X), \quad \lambda \otimes x \longmapsto \lambda \cdot \theta_\varphi(x).$$

When the morphism φ is clear from the context, we simply write θ and Θ.

Definition 4.22. We call θ_φ the *restriction of tori* and Θ_φ the *extension of tori* associated to φ.

In later chapters, we will use extension of tori to describe $H^*_{T_2}(X)$ in terms of the often simpler equivariant cohomology $H^*_{T_1}(X)$. This method applies particularly well to the examples presented below.

Example 4.23. If $T_2 = 1$ is trivial, then the restriction of tori coincides with the forgetful morphism

$$\theta\colon H^*_{T_1}(X) \longrightarrow H^*(X),$$

introduced in Chapter 2.1.
If instead $T_1 = 1$, then compatibility of $\varphi\colon T_2 \to 1$ forces T_2 to act trivially on X. In this case, the restriction of tori is given by

$$\theta\colon H^*(X) \longrightarrow \Lambda_{T_2} \otimes H^*(X), \quad x \longmapsto 1 \otimes x.$$

In both situations, the extension of tori yields an isomorphism.

Example 4.24. If X is equivariantly formal with respect to the T_1-action, then by Lemma 2.27 the extension of tori

$$\Theta_\varphi \colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X),$$

is isomorphism.

Example 4.25. If X is a finite union of T_1-fixed (and hence also T_2-fixed) points, then restriction and extension of tori are given by

$$\theta \colon \Lambda^X_{T_1} \longrightarrow \Lambda^X_{T_2}, \quad \text{and} \quad \Theta \colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} \Lambda^X_{T_1} \cong \Lambda^X_{T_2},$$

where Λ_{T_1} acts diagonally on $\Lambda^X_{T_2}$ via φ.

We will use GKM varities as a recurring example in the context of torus change.

Remark 4.26. Suppose X is a GKM-variety of dimension k with respect to both the T_1-action and the T_2-action. If $p \in X^{T_2} \setminus X^{T_1}$ were fixed by T_2 but not by T_1, then the T_1-orbit of p would be contained in X^{T_2} and would have positive dimension. This contradicts the fact that both X^{T_1} and X^{T_2} are finite, so we must have $X^{T_1} = X^{T_2}$.

For $p \in X^{T_1}$, the T_1-action on T_pX is determined by weights $\rho_1, \dots, \rho_k$ that are pairwise relatively prime and correspond to the T_1-curves through p (see Definition 2.33 and Lemma 2.34). The T_2-action on T_pX is given by characters $\hat\varphi(\rho_1), \dots, \hat\varphi(\rho_k)$ which are pairwise relatively prime by assumption. Moreover, each $\hat\varphi(\rho_i)$ corresponds to a T_2-curve that coincides with the T_1-curve associated to ρ_i.

Thus, both actions have the same fixed points and fixed curves, and the characters associated to curves are related via $\hat\varphi$. Equivalently, the moment graphs Γ_1 and Γ_2 for the respective T_1- and T_2-actions have the same vertex and edge set, and their edge labels are obtained from one another via $\hat\varphi$.

Example 4.27. Let X be a GKM-variety with respect to both the T_1- and T_2-actions. The extension of tori Θ is summarized in the following commutative diagram.

$$\begin{array}{ccccc}
\Lambda_{T_2} \otimes H^*_{T_1}(X) & \xrightarrow{\sim} & \Lambda_{T_2} \otimes H^*(\Gamma_1) & \hookrightarrow & \Lambda_{T_2} \otimes \Lambda^{X^{T_1}}_{T_1} \\
\Theta \downarrow \wr & & \wr \downarrow & & \wr \downarrow \\
H^*_{T_2}(X) & \xrightarrow{\sim} & H^*(\Gamma_2) & \hookrightarrow & \Lambda^{X^{T_2}}_{T_2}
\end{array}$$

It is possible to split off the action of a subtorus that acts trivially.

Lemma 4.28. *Let T be a torus acting on X. If T splits into subtori $T = T' \times T''$ such that the action restricted to T' is trivial, then*

$$H^*_T(X) = \Lambda_{T'} \otimes_R H^*_{T''}(X).$$

Proof. [Fra17, Lemma 3.1]. □

Remark 4.29. If $\varphi' \colon T'_2 \to T'_1$ and $\varphi'' \colon T''_2 \to T''_1$ are morphism of tori, then

$$\varphi' \times \varphi'' \colon T'_2 \times T''_2 \longrightarrow T'_1 \times T''_1,$$

is a morphism of tori, and the induced morphism

$$\Lambda_{T'_2 \times T''_2} = \Lambda_{T'_2} \otimes \Lambda_{T''_2} \longrightarrow \Lambda_{T'_1 \times T''_1} = \Lambda_{T'_1} \otimes \Lambda_{T''_1},$$

is $\hat\varphi' \otimes \hat\varphi''$.

The next example generalizes Example 4.23 and shows that splitting off tori with trivial action also works in the context of a change of tori.

Example 4.30. Assume that both T_1 and T_2 split into subtori

$$T_1 = T_1' \times T_1'', \quad \text{and} \quad T_2 = T_2' \times T_2'',$$

such that the action of T_1' and T_2' on X is trivial, and suppose that the morphism $\psi\colon T_2 \to T_1$ is compatible. In this case, the image $\psi(T_2')$ acts trivially on X, and we may assume that T_1' has been chosen large enough to guarantee $\psi(T_2') \subseteq T_1'$. Define $\psi'\colon T_2'' \to T_1'$ and $\psi''\colon T_2'' \to T_1''$ such that, for all $t'' \in T_2''$, we have

$$\psi(t'') = (\psi'(t''), \psi''(t'')).$$

Then the morphism φ

$$\varphi\colon T_2' \times T_2'' \longrightarrow T_1' \times T_1'', \quad (t', t'') \longmapsto (\psi(t', 1), \psi''(t'')),$$

is the product of two morphisms:

$$\varphi' := \psi(\cdot, 1)\colon T_2' \longrightarrow T_1', \quad \varphi'' := \psi''(1, \cdot)\colon T_2'' \longrightarrow T_1''.$$

Moreover, φ is compatible: Let $x \in X$ and $t = (t', t'') \in T_2' \times T_2''$. We have

$$t.x = t''.x = \psi(t'').x = \psi''(t'').x = (\psi(t', 1), \psi''(t'')).x = \varphi(t).x.$$

We now demonstrate the change of tori with respect to φ:
According to Lemma 4.28,

$$H^*_{T_1}(X) = \Lambda_{T_1'} \otimes H^*_{T_1''}X, \quad \text{as well as} \quad H^*_{T_2}(X) = \Lambda_{T_2'} \otimes H^*_{T_2''}X,$$

and the induced morphism $\hat\varphi$ splits as

$$\hat\varphi' \otimes \hat\varphi''\colon \Lambda_{T_1'} \otimes \Lambda_{T_1''} \longrightarrow \Lambda_{T_2'} \otimes \Lambda_{T_2''},$$

see Remark 4.29. Now θ_φ can be expressed as

$$\theta_\varphi\colon \Lambda_{T_1'} \otimes H^*_{T_1''}X \longrightarrow \Lambda_{T_2'} \otimes H^*_{T_2''}X, \quad \lambda \otimes x \longmapsto \hat\varphi'(\lambda) \otimes \theta_{\varphi''}(x).$$

Since

$$\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \left(\Lambda_{T_1'} \otimes_R H^*_{T_1''}X\right) = \left(\Lambda_{T_2'} \otimes_R \Lambda_{T_2''}\right) \otimes_{\Lambda_{T_1'} \otimes \Lambda_{T_1''}} \left(\Lambda_{T_1'} \otimes_R H^*_{T_1''}X\right),$$

we can simplify the right-hand side step by step:

$$\left(\Lambda_{T_2'} \otimes_{\Lambda_{T_1'}} \Lambda_{T_1'}\right) \otimes_R \left(\Lambda_{T_2''} \otimes_R H^*_{T_1''}X\right) = \Lambda_{T_2'} \otimes_R \left(\Lambda_{T_2''} \otimes_{\Lambda_{T_1''}} H^*_{T_1''}X\right).$$

Therefore, we may write

$$\Theta_\varphi\colon \Lambda_{T_2'} \otimes_R \left(\Lambda_{T_2''} \otimes_{\Lambda_{T_1''}} H^*_{T_1''}X\right) \longrightarrow \Lambda_{T_2'} \otimes_R H^*_{T_2''}X,$$

$$\lambda \otimes (\mu \otimes x) \longmapsto \lambda \otimes \Theta_{\varphi''}(\mu \otimes x).$$

Example 4.31. Suppose that T_2' is a subtorus of the kernel of a compatible morphism $\psi\colon T_2 \to T_1$. In this case, we can pick a torus complement T_2'' of T_2' and apply Example 4.30, by considering the splittings

$$T_2 = T_2' \times T_2'', \quad T_1 = T_1' \times T_1'',$$

where we set $T_1' = 1$ and $T_1'' = T_1$. Note that $\psi(T_2') \subseteq T_1'$ and that ψ splits into morphisms

$$\psi'\colon T_2' \longrightarrow T_1' = 1, \quad \text{and} \quad \psi'' = \varphi\,|_{T_2''}\colon T_2'' \longrightarrow T_1'' = T_1.$$

The morphism φ constructed in Example 4.30 is, in this case, equal to ψ, and we obtain the descriptions

$$\theta_\psi\colon H^*_{T_1}(X) \longrightarrow \Lambda_{T_2'} \otimes H^*_{T_2''}X, \quad x \longmapsto 1 \otimes \theta_{\psi''}(x),$$

and

$$\Theta_\psi\colon \Lambda_{T_2'} \otimes_R \left(\Lambda_{T_2''} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)\right) \longrightarrow \Lambda_{T_2'} \otimes_R H^*_{T_2''}X,$$
$$\lambda \otimes (\mu \otimes x) \longmapsto \lambda \otimes \Theta_{\psi''}(\mu \otimes x).$$

Let us briefly consider the case where φ introduces relations on Λ_1, and adapt Example 4.27 as well as Example 4.31 to this setting.

Corollary 4.32. *Suppose that φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1}. Then $\hat{\varphi}\colon \Lambda_{T_1} \to \Lambda_{T_2}$ is flat if and only if φ is surjective.*

Proof. By Remark 4.19, φ is surjective if and only if $\ker\varphi = 0$. If $\hat{\varphi}$ is flat, then, since Λ_{T_1} is a domain, the Λ_{T_1}-module $\Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2^S}$ is torsion-free, which implies $\mathcal{R}_\varphi = 0$. Conversely, if $\mathcal{R}_\varphi = 0$, then $\Lambda_{T_2} = \Lambda_{T_1} \otimes \Lambda_{T_2^S}$ is free over Λ_{T_1}, hence flat. □

Remark 4.33. Suppose that φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1}. By Remark 4.19, T_2^S is contained in $\ker\varphi$. After choosing a torus complement T_2'' to T_2^S in T_2 and fixing the morphism $\varphi'' \coloneqq \varphi\,|_{T_2''}$, we can apply Example 4.31 to obtain

$$\theta_\varphi\colon H^*_{T_1}(X) \longrightarrow \Lambda_{T_2^S} \otimes H^*_{T_2''}X, \quad x \longmapsto 1 \otimes \theta_{\varphi''}(x).$$

Moreover, both θ and $\theta_{\varphi''}$ factor through $\mathcal{R}_\varphi H^*_{T_1}(X)$. In particular, we can define

$$\overline{\theta}_\varphi\colon H^*_{T_1}(X)/\mathcal{R}_\varphi H^*_{T_1}(X) \longrightarrow \Lambda_{T_2^S} \otimes H^*_{T_2''}X, \quad [\![x]\!] \longmapsto 1 \otimes \theta_{\varphi''}(x).$$

To describe Θ, observe that

$$\Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X) = \left(\Lambda_{T_2^S} \otimes_R \Lambda_{T_2''}\right) \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)$$
$$\cong \Lambda_{T_2^S} \otimes_R \left(\Lambda_{T_1}/\mathcal{R}_\varphi\Lambda_{T_1} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)\right) \cong \Lambda_{T_2^S} \otimes_R \left(H^*_{T_1}(X)/\mathcal{R}_\varphi H^*_{T_1}(X)\right),$$

and hence

$$\Theta\colon \Lambda_{T_2^S} \otimes_R \left(H^*_{T_1}(X)/\mathcal{R}_\varphi H^*_{T_1}(X)\right) \longrightarrow \Lambda_{T_2^S} \otimes_R H^*_{T_2''}X,$$
$$\lambda \otimes [\![x]\!] \longmapsto \lambda \otimes \overline{\theta}_\varphi([\![x]\!]) = \lambda \otimes \theta_{\varphi''}(x).$$

Example 4.34. If φ introduces relations on Λ_{T_1}, then by Remark 4.33 equivariantly formal varieties admit surjective restrictions of tori. In particular, this applies to projective space $\mathbb{P}$ with the torus action of Example 2.30.

Example 4.35. In the situation of Example 4.27, suppose φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1} and, for the sake of simplicity assume that T_2^S is trivial:

$$\Lambda_{T_1}/\mathcal{R}_\varphi \cong \Lambda_{T_2}.$$

We can rewrite the diagram from the original example as follows:

$$\begin{array}{ccccc} H^*_{T_1}(X)/\mathcal{R}_\varphi H^*_{T_1}(X) & \xrightarrow{\sim} & S_1/\left(\mathcal{R}_\varphi S_1\right) & \hookrightarrow & \left(\Lambda_{T_1}/\mathcal{R}_\varphi\right)^{X^{T_1}} \\ \Theta \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ H^*_{T_2}(X) & \xrightarrow{\sim} & S_2 & \hookrightarrow & \Lambda_{T_2}^{X^{T_2}} \end{array}$$

We now turn to covered spaces and consider change of tori in the context of the Mayer–Vietoris spectral sequence.

Let X be a space with cover $\mathfrak{M}$, equipped with actions by tori T_1 and T_2, and let $\varphi\colon T_2 \to T_1$ be a compatible morphism such that both $(T_1, X, \mathfrak{M})$ and $(T_2, X, \mathfrak{M})$ are objects of **GCov**. As defined earlier, the restriction of tori θ is induced by the identity on X, and by the above conditions, this defines a morphism in **GCov**.

The filtration of $H^*_{T_1}(X)$ is compatible with θ and induces a filtration of $\Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)$, which is itself compatible with Θ. We therefore obtain bigraded algebra morphisms of associated graded algebras

$$\hat{\theta}\colon G(H^*_{T_1}(X)) \longrightarrow G(H^*_{T_2}(X)),$$

and

$$\hat{\Theta}\colon G(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)) \longrightarrow G(H^*_{T_2}(X)).$$

By construction,

$$G(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X)) = \Lambda_{T_2} \otimes_{\Lambda_{T_1}} G(H^*_{T_1}(X)),$$

and as before, $\hat{\Theta}$ is determined by $\hat{\theta}$ via

$$\hat{\Theta}(\lambda \otimes x) = \lambda \otimes \hat{\theta}(x), \quad \text{for } \lambda \in \Lambda_{T_2}, x \in G(H^*_{T_1}(X)).$$

Remark 4.36. We have discussed before that θ is an isomorphism if and only $\hat{\theta}$ is an isomorphism (see Remark 3.43). As a consequence of Remark A.15, the same logical equivalence holds between the isomorphism property of Θ and that of $\hat{\Theta}$.

As a morphism in **GCov**, the restriction of tori θ is given by the respective restrictions of tori

$$\theta_{\underline{i}}\colon H^*_{T_1} M_{\underline{i}} \longrightarrow H^*_{T_2} M_{\underline{i}},$$

for $\underline{i} \in \mathcal{I}$, on the Mayer–Vietoris complexes. Under suitable assumptions, this approach can be used to compute the change of tori for X. Let us abbreviate $(\mathrm{MV}, d) = (\mathrm{MV}(T_1, X, \mathfrak{M}), d)$.

Lemma 4.37. *Let X be a space with good cover $\mathfrak{M}$, equipped with actions by tori T_1 and T_2, and let $\varphi\colon T_2 \to T_1$ be a compatible morphism, such that both $(T_1, X, \mathfrak{M})$ and $(T_2, X, \mathfrak{M})$ are objects of* **GCov**. *If $M_{\underline{i}}$ is equivariantly formal with respect to T_1 for all $\underline{i} \in \mathcal{I}$, then $\hat{\theta}$ and $\hat{\Theta}$ are given by*

$$\hat{\theta}\colon H(\mathrm{MV}) \longrightarrow H\left(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \mathrm{MV}\right), \quad [\![x]\!] \longmapsto [\![1 \otimes x]\!],$$

and

$$\hat{\Theta}\colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H(\mathrm{MV}) \longrightarrow H\left(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \mathrm{MV}\right), \quad \lambda \otimes [\![x]\!] \longmapsto [\![\lambda \otimes x]\!].$$

Proof. For $\underline{i} \in \mathcal{I}$, we have

$$H^*_{T_2} M_{\underline{i}} \cong \Lambda_{T_2} \otimes H^*_{T_1} M_{\underline{i}},$$

by equivariant formality of $M_{\underline{i}}$. Consequently, the restriction of tori can be written as

$$\theta_{\underline{i}} \colon H^*_{T_1} M_{\underline{i}} \to H^*_{T_2} M_{\underline{i}} \cong \Lambda_{T_2} \otimes H^*_{T_1} M_{\underline{i}},$$

and on the Mayer–Vietoris complexes, θ takes the form

$$\theta_1 \colon \operatorname{MV} \longrightarrow \Lambda_{T_2} \otimes_{\Lambda_{T_1}} \operatorname{MV}, \quad x \longmapsto 1 \otimes x.$$

Since $\mathfrak{M}$ is good cover, the induced morphism $\hat{\theta}$ is

$$H(\operatorname{MV}) \longrightarrow H\left(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \operatorname{MV}\right), \quad [\![x]\!] \longmapsto [\![1 \otimes x]\!],$$

and, as $\hat{\Theta}$ is determined by $\hat{\theta}$, we arrive at the statement. □

We now adapt the result to the case that φ introduces relations.

Corollary 4.38. *Let X be a space with good cover $\mathfrak{M}$, equipped with actions by tori T_1 and T_2, and let $\varphi \colon T_2 \to T_1$ a compatible morphism, such that both $(T_1, X, \mathfrak{M})$ and $(T_2, X, \mathfrak{M})$ are objects in* **GCov**. *Suppose that φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1} and that $M_{\underline{i}}$ is equivariantly formal with respect to T_1 for all $\underline{i} \in \mathcal{I}$, then $\hat{\theta}$ and $\hat{\Theta}$ are given by*

$$\hat{\theta} \colon H(\operatorname{MV}) \longrightarrow \Lambda_{T_2^S} \otimes_R H\left(\operatorname{MV}/\mathcal{R}_\varphi \operatorname{MV}\right), \quad [\![x]\!] \longmapsto 1 \otimes [\![x]\!],$$

and

$$\hat{\Theta} \colon \Lambda_{T_2^S} \otimes_R \left(H(\operatorname{MV})/\mathcal{R}_\varphi H(\operatorname{MV})\right) \longrightarrow \Lambda_{T_2^S} \otimes_R H\left(\operatorname{MV}/\mathcal{R}_\varphi \operatorname{MV}\right), \quad \lambda \otimes [\![x]\!] \longmapsto \lambda \otimes [\![x]\!].$$

Proof. If φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1}, then

$$\Lambda_{T_2} \cong \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2^S},$$

and $\hat{\varphi}$ is identified with

$$\hat{\varphi} \colon \Lambda_{T_1} \longrightarrow \Lambda_{T_1}/\mathcal{R}_\varphi \otimes \Lambda_{T_2^S}.$$

Since

$$H\left(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \operatorname{MV}\right) = H\left(\Lambda_{T_2^S} \otimes_R \Lambda_{T_1}/\mathcal{R}_\varphi \Lambda_{T_1} \otimes_{\Lambda_{T_1}} \otimes \operatorname{MV}\right),$$

and $\Lambda_{T_2^S}$ is a flat R-module, the claim follows from Lemma 4.37. □

Remark 4.39. Suppose we are in the situation of Corollary 4.38. Then

$$H(\operatorname{MV}/\mathcal{R}_\varphi \operatorname{MV}) = \frac{d^{-1}(\mathcal{R}_\varphi \operatorname{MV})}{\mathcal{R}_\varphi \operatorname{MV} + \operatorname{im} d}, \qquad H(\operatorname{MV})/\mathcal{R}_\varphi H(\operatorname{MV}) = \ker d \big/ \big(\mathcal{R}_\varphi \ker d + \operatorname{im} d\big).$$

In particular, the kernels of the morphisms associated with the change of tori are

$$\ker \hat{\theta} = \frac{\ker d \cap \mathcal{R}_\varphi \operatorname{MV} + \operatorname{im} d}{\operatorname{im} d}, \qquad \ker \hat{\Theta} = \Lambda_{T_2^S} \otimes \frac{\big(\mathcal{R}_\varphi \operatorname{MV} \cap \ker d\big) + \operatorname{im} d}{\mathcal{R}_\varphi \ker d + \operatorname{im} d}.$$

If the assumptions of Lemma 4.37 are satisfied, then describing $H^*_{T_2}(X)$ in terms of $H^*_{T_1}(X)$ reduces to comparing $H(\mathrm{MV})$ with $H(\Lambda_{T_2} \otimes \mathrm{MV})$. To carry out this procedure, we use the structure of MV as a complex and apply Künneth formulas, the standard tool for computing the cohomology of a tensor product of complexes. We state the *Künneth theorem* in the form given in [McC01, Chapter 2.3, Theorem 2.12]. Here, $\mathrm{Tor}_i^R(-,-)$ denotes the i-th left derived functor of the tensor product over R.

Theorem 4.40. *Let (J, d_J) and (L, d_L) be differential graded R-modules, and assume that for each n the kernel*

$$Z^n(L) = \ker\left(d_L \colon L^n \to L^{n-1}\right),$$

and the image

$$B^n(L) = \mathrm{im}\left(d_L \colon L^{n-1} \to L^n\right),$$

are flat R-modules. Then, for all n there are short exact sequences

$$0 \longrightarrow \bigoplus_{r+s=n} H^r(L) \otimes H^s(J) \xrightarrow{\kappa} H^n(L \otimes J) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}_1^R(H^r(L), H^s(J)) \longrightarrow 0,$$

where $\kappa([\![a]\!] \otimes [\![c]\!]) = [\![a \otimes c]\!]$.

If Λ_{T_2} is flat, we obtain the expected isomorphism.

Corollary 4.41. *Suppose, in the setting of Lemma 4.37, that $\hat{\varphi}$ is a flat ring homomorphism. Then*

$$\Theta \colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X),$$

is an isomorphism.

Proof. Apply Theorem 4.40 with $J = \mathrm{MV}$ (with its complex structure), $L = \Lambda_{T_2}$, and $d_L = 0$, and combine the result with Lemma 4.37 to conclude that $\kappa = \hat{\Theta}$ is an isomorphism. By Remark 4.36, it follows Θ is also an isomorphism. □

Remark 4.42. In the typical case where the module Λ_{T_2} is not flat over Λ_{T_1}, the Mayer–Vietoris complex must satisfy the flatness conditions required to apply the Künneth Theorem. More precisely, if $\ker d^n$ and $\mathrm{im}\, d^n$ are flat Λ_{T_1}-modules for all n, then there are short exact sequences as in Theorem 4.40, which simplify to

$$0 \longrightarrow \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^n(\mathrm{MV}) \longrightarrow H^n(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \mathrm{MV}) \longrightarrow \mathrm{Tor}_1^{\Lambda_{T_1}}(\Lambda_{T_2}, H^{n-1}(\mathrm{MV})) \longrightarrow 0.$$

The *Künneth spectral sequence* (see Theorem B.3) generalizes the Künneth theorem by relaxing the flatness assumption:

Theorem 4.43. *If* MV *is flat over Λ_{T_1}, then there is a (second-quadrant) spectral sequence with second page*

$$E_2^{p,q} = \mathrm{Tor}_{-p}^{\Lambda_{T_1}}(\Lambda_{T_2}, H^q(\mathrm{MV})),$$

converging to $H^*(\Lambda_{T_2} \otimes_{\Lambda_{T_1}} \mathrm{MV})$, provided it converges.

While the Künneth spectral sequence offers a broad framework for computing the cohomology of tensor products of complexes, its practical usefulness depends strongly on the specific situation. In general, convergence is not guaranteed, and even when it converges, it may fail to collapse at the second page (see Remark B.4).

To conclude this chapter, we investigate the Künneth spectral sequence in the situation where φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1}.
When the ideal $\mathcal{R}_\varphi \subseteq \Lambda_{T_1}$ is generated by a regular sequence, a natural tool for computing the Tor-groups is the *Koszul complex*. The following definitions and results are taken from [Eis95, Chapter 17].

Definition 4.44. Let S be a Noetherian commutative ring and M an S-module. A sequence of elements $y_1, \dots, y_k$ in S is called an *M-regular sequence* if

$$(y_1, \dots, y_k)M \neq M,$$

and

$$y_i \text{ is not a zero-divisor in } M/(y_1, \dots, y_{i-1})M,$$

for all $i = 1, \dots, k$. Here, $(y_1, \dots, y_k)$ denotes the ideal in S generated by $y_1, \dots, y_k$.

Recall that the exterior algebra of an S-module N is defined as the quotient of the tensor algebra

$$T(N) \coloneqq S \oplus N \oplus (N \otimes N) \oplus \dots,$$

by the relations $a \otimes b = -(b \otimes a)$ and $a \otimes a = 0$ for all $a, b \in N$. We write $\bigwedge N$ for the exterior algebra of N and $a \wedge b$ for the product of two elements. The exterior algebra $\bigwedge N$ is a graded S-algebra whose homogeneous part $\bigwedge^n N$ is generated, as an S-module, by products of exactly n elements in N.

Definition 4.45. For an element $y \in N$, the *Koszul complex* $K(y)$ is the complex

$$\left(\bigwedge N, d_y\right) : 0 \longrightarrow S \longrightarrow N \longrightarrow \bigwedge^2 N \xrightarrow{d_y} \bigwedge^3 N \longrightarrow \dots,$$

with differential $d_y(a) = y \wedge a$. In particular, $d_y(1) = y$.

In practice, the module N will usually be a free S-module. In the case that $N = S^k$ and $y = (y_1, \dots, y_k)$, we write $K(y) = K(y_1, \dots, y_k)$. The following result relates regular sequences to Koszul complexes.

Theorem 4.46. *Let M be a finitely generated S-module and $y_1, \dots, y_k \in S$. If*

$$H^i(M \otimes K(y_1, \dots, y_k)) = 0,$$

for $i < r$, and

$$H^r(M \otimes K(y_1, \dots, y_k)) \neq 0,$$

then every maximal M-regular sequence in $I = (y_1, \dots, y_k)$ has length r.

Proof. [Eis95, Theorem 17.4]. □

Let I be an ideal in R that satisfies $IM \neq M$. The above theorem justifies calling the length of any maximal M-regular sequence in I the *M-depth* of I.

Corollary 4.47. *Let M be a finitely generated S-module. If $y_1, \dots, y_k$ is an M-regular sequence, then*

$$H^j(M \otimes K(y_1, \dots, y_k)) = 0,$$

for $j < k$, and

$$H^k(M \otimes K(y_1, \dots, y_k)) = M/(y_1, \dots, y_k)M.$$

Proof. [Eis95, Corollary 17.5]. □

Remark 4.48. The converse of Corollary 4.47 holds true in the case that S is a local ring ([Eis95, Theorem 17.6]). Similarly, if $y_1, \dots, y_k$ is not an M-regular sequence and the M-depth of the ideal $I = (y_1, \dots, y_k)$ is less than k, then

$$H^i(M \otimes K(y_1, \dots, y_k)) \neq 0,$$

for some $i < k$, by [Eis95, Corollary 17.12], together with [Eis95, Corollary 17.10].

If an ideal I is generated by an S-regular sequence $y_1, \dots, y_k$, then, by Corollary 4.47, the complex

$$S \otimes K(y_1, \dots, y_k) \longrightarrow S/(y_1, \dots, y_k),$$

is a free resolution of $S/(y_1, \dots, y_k)$ and we can use it to compute Tor groups.

Corollary 4.49. *Suppose that φ introduces the relations $\mathcal{R}_\varphi$ on Λ_{T_1}, that the Mayer–Vietoris complex* MV *is flat over R, and that $H(\mathrm{MV})$ is finitely generated over Λ_{T_1}. If $\mathcal{R}_\varphi$ is generated by a sequence that is both $H(\mathrm{MV})$-regular and Λ_{T_1}-regular, then*

$$\Theta\colon \Lambda_{T_2} \otimes_{\Lambda_{T_1}} H^*_{T_1}(X) \longrightarrow H^*_{T_2}(X),$$

is an isomorphism.

Proof. Let $y_1, \dots, y_k$ be a sequence generating $\mathcal{R}_\varphi$ that is both $H(\mathrm{MV})$-regular and Λ_{T_1}-regular Since Λ_{T_1} is Noetherian, we may set $S = \Lambda_{T_1}$ and consider the Koszul complex

$$K(y_1, \dots, y_k) \longrightarrow \Lambda_{T_1}/\mathcal{R}_\varphi,$$

as a free resolution of $\Lambda_{T_1}/\mathcal{R}_\varphi$. Using this resolution to compute Tor-groups, we obtain

$$\mathrm{Tor}_{-p}^{\Lambda_{T_1}}(\Lambda_{T_1}/\mathcal{R}_\varphi, H^*(\mathrm{MV})) = 0, \quad (p < 0), \quad \mathrm{Tor}_0^{\Lambda_{T_1}}(\Lambda_{T_1}/\mathcal{R}_\varphi, H^*(\mathrm{MV})) = H(\mathrm{MV})/\mathcal{R}_\varphi H(\mathrm{MV}),$$

because $H^*(\mathrm{MV}) \otimes K(y_1, \dots, y_k)$ is exact everywhere except in degree 0 (by the regularity assumption and Corollary 4.47). Since MV is flat, the Künneth spectral sequence (Theorem 4.43) applies, and we have just shown that its second page is concentrated in the first column, with

$$E_2^{0,*} \cong H(\mathrm{MV})/\mathcal{R}_\varphi H(\mathrm{MV}).$$

By Remark A.10 the sequence collapses at E_2, the filtration is trivial and we obtain a canonical isomorphism

$$E_2^{0,*} \longrightarrow H(\mathrm{MV}/\mathcal{R}_\varphi \mathrm{MV}), \quad [\![x]\!] \mapsto [\![x]\!].$$

Tensoring by $\Lambda_{T_2^S}$ shows that $\hat{\Theta}$ is an isomorphism, and hence, by Remark 4.36, so is Θ. □

Remark 4.50. By Remark 4.19, the sublattice $\ker \varphi^* \leq M_1$ is saturated, so without loss of generality $\mathcal{R}_\varphi$ can be taken to be generated by some of the variables in the polynomial ring $\Lambda_{T_1} = R[e_1, \ldots, e_n]$. In particular, $\mathcal{R}_\varphi$ is always generated by a Λ_{T_1}-regular sequence which means that the corresponding Koszul complex can at least be used to compute $\mathrm{Tor}_{-p}^{\Lambda_{T_1}}(\Lambda_{T_1}/\mathcal{R}_\varphi, H^q(\mathrm{MV}))$.

5 Projective Unions

This chapter concerns the main class of objects in this thesis. Their appeal is twofold. First, we aim to understand the equivariant cohomology of projective unions in order to study those arising in representation-theoretic or combinatorial contexts. The original motivation remains their role as special fibres of semi-toric degenerations for Hodge-type Seshadri stratifications in [CFL23]. Second, this class of varieties provides an excellent testing ground for the theory developed in the first part of the thesis. The canonical choice of cover comes from their very definition as a union, the torus action is defined globally, and the building blocks are simple yet non-trivial, since the torus-equivariant cohomology of a projective space has a transparent but sufficiently rich structure (see Example 2.17). Consequently, the difficulty in computing their cohomology lies in understanding the combinatorial interaction within the cover. This is precisely the situation suited for applying the Mayer–Vietoris principle, which is concerned with the gluing of multiple pieces rather than the complexity of the pieces themselves.

While the complements of subspace arrangements have been extensively studied in algebraic topology, combinatorial topology, and algebraic geometry (see, e.g., [FZ00; OT92; DP95]), projective unions have received far less direct attention, since as CW complexes their structure is, in principle, well understood. To our knowledge, there has been no systematic treatment of their equivariant cohomology.

We begin the chapter by clarifying the notion of a *projective union*, fixing notation, and verifying in Lemma 5.12 that projective unions are covered spaces in the sense of Definition 3.36. The first direct application of the Mayer–Vietoris spectral sequence will appear in Chapter 5.2. Chapter 5.4 establishes structural results in the less general setting of trivial and generic torus actions, whereas Chapter 5.6 turns to projective unions defined by the combinatorial data of a poset. The analysis of localization and torus change in the context of the Mayer–Vietoris spectral sequence, developed in Chapter 4, is then specialized to projective unions in Chapters 5.3 and 5.5.

While this chapter is devoted entirely to projective unions and develops a range of results that describe them in their own right, we recall that the original motivation lies in degenerated Grassmannians, and much of the material is formulated in a setting adapted to the requirements of the final chapter.

In the last two chapters we assume that R is a field of characteristic zero, and the notation introduced here and at the beginning of Chapter 5.2 will remain in use throughout. Following Chapter 4.2, we write $[\![x]\!]$ to denote the residue class of an element x in a quotient.

Let T be a complex algebraic torus of rank m. We fix an isomorphism $T \cong (\mathbb{C}^\times)^m$ and a basis of standard characters $x_1, \dots, x_m$ for the character lattice M of T. The equivariant coefficient ring of T is the graded R-algebra

$$\Lambda_T = R[x_1, \dots, x_m],$$

where each standard character has degree two. For a homogeneous element χ in Λ_T, we write $|\chi|$ for its degree, and we write Λ_T^q for the graded component of degree q.

Let A be a finite index set and $\{e_a \mid a \in A\}$ the canonical basis of the complex vector space $\mathbb{C}^A$. As in Example 2.17, suppose that T acts on $\mathbb{C}^A$ by

$$t.e_a = \chi_a(t)e_a, \quad t \in T, a \in A,$$

with characters $\{\chi_a \mid a \in A\} \subseteq M$. The induced action of T on the projectivization $\mathbb{P} \coloneqq \mathbb{P}(\mathbb{C}^A)$ is then given by

$$t.(z_a)_{a\in A} = (\chi_a(t)z_a)_{a\in A}, \quad t \in T, \quad (z_a)_{a\in A} \in \mathbb{P}.$$

Let y_a be the homogeneous coordinates on $\mathbb{P}$. For any subset $J \subseteq A$, define

$$V(y_a \mid a \notin J),$$

the vanishing set of all coordinates y_a with $a \notin J$. Equivalently,

$$P_J = \{(z_a)_{a\in A} \in \mathbb{P} \mid z_b = 0 \text{ for all } b \notin J\}.$$

Each P_J is a T-invariant subspace of $\mathbb{P}$, and throughout we will consider these subspaces with the T-action induced from $\mathbb{P}$.

Definition 5.1. A *projective union* in a projective space $\mathbb{P}$ is a subvariety of the form

$$P_{\mathfrak{C}} \coloneqq \bigcup_{C\in\mathfrak{C}} P_C$$

where $\mathfrak{C}$ is a finite collection of subsets of A. We consider $P_{\mathfrak{C}}$ with the T-action induced from $\mathbb{P}$.

The remainder of this thesis will work within the following setup and notation. Let $\mathfrak{C}$ be a finite collection of subsets of A, indexed by a set I,

$$\mathfrak{C} = (C_i \mid i \in I).$$

After fixing a total ordering on I, simplices are expressed as strictly increasing tuples, following the notation from Definition 2.1. Define

$$\overline{\mathfrak{C}} \coloneqq \bigcup_{i\in I} C_i \subseteq A, \quad \text{and} \quad \underline{\mathfrak{C}} \coloneqq \bigcap_{i\in I} C_i \subseteq A.$$

For each $i \in I$, set $P_i \coloneqq P_{C_i}$, the subspace of $\mathbb{P}$ associated to C_i. Let $\mathcal{I}$ denote the full simplex on I, and let $\mathcal{I}_p$ be the set of its p-simplices. For $p \in \mathbb{Z}_{\geq 0}$ and $\underline{i} = (i_0, \dots, i_p) \in \mathcal{I}_p$, define

$$C_{\underline{i}} \coloneqq C_{i_0} \cap \ldots \cap C_{i_p},$$

and, in line with the notation of Chapter 3.1,

$$P_{\underline{i}} \coloneqq P_{C_{\underline{i}}} = P_{i_0} \cap \ldots \cap P_{i_p}.$$

The central object of this chapter is the T-equivariant cohomology of the projective union

$$P_{\mathfrak{C}} = \bigcup_{i\in I} P_i \subseteq \mathbb{P}.$$

Since the cover is fully described by the collection $\mathfrak{C}$, we use the notation

$$\mathfrak{M}_{\mathfrak{C}} \coloneqq (P_i)_{i\in I}.$$

Example 5.2. As an illustration, consider the union of the three coordinate axes in $\mathbb{P}^2$. This is described by the collection

$$\mathfrak{C} = (C_1, C_2, C_3),$$

with index set $I = \{1, 2, 3\}$ and the subsets of $A = \{a, b, c\}$ given by

$$C_1 = \{b, c\}, \quad C_2 = \{a, c\}, \quad C_3 = \{a, b\}.$$

5.1 Retractions onto $P_{\mathfrak{C}}$

Let $P_{\mathfrak{C}}$ be a projective union with a T-action as introduced in Definition 5.1. The aim of this subsection is to show that $P_{\mathfrak{C}}$ is a covered space, i.e.,

$$(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}) \in \mathbf{GCov}\,.$$

Since each P_i is T-invariant, it suffices to prove that the morphism

$$\iota\colon C^*_T(P_{\mathfrak{C}}) \longrightarrow K,$$

induces an isomorphism in cohomology (see Definition 3.36 and Remark 3.16). We establish this by constructing suitable deformation retractions (see [Hat02, Chapter 0.1] for a reference).

Remark 5.3. Let Y be a subspace of a topological space X. A *retraction* is a continuous map

$$r\colon X \to Y,$$

whose restriction to Y is the identity. In this case Y is called a *retract* of X.
A *deformation retraction* of X onto Y is a continuous map

$$F\colon [0,1] \times X \longrightarrow X,$$

such that

$$F(0,x) = x, \quad F(1,x) \in Y, \quad \text{and} \quad F(t,y) = y,$$

for all $t \in [0,1]$ and $x \in X, y \in Y$. In this case Y is called a *deformation retract* of X.

Remark 5.4. Suppose $r\colon X \to Y$ is a retraction such that for each $z \in X$ the map

$$[0,1] \to X, \quad t \longmapsto (1-t)z + t \cdot r(z),$$

is continuous. Then the map

$$F\colon [0,1] \times X \longrightarrow X, \quad (t,z) \longmapsto (1-t)z + t \cdot r(z)$$

defines a deformation retraction of X onto Y.
If Y is a deformation retract of X with deformation retraction F, then X and Y are homotopy equivalent via the retraction

$$X \to Y, \quad x \longmapsto F(1,x),$$

and the inclusion $Y \hookrightarrow X$.

The set $\mathfrak{C}$ determines a simplicial complex $\langle\mathfrak{C}\rangle$ on $\overline{\mathfrak{C}}$, covered by $(C_i)_{i\in I}$ (see Definition 2.5 and Definition 2.1). If there exists $i \in I$ such that $\overline{\mathfrak{C}} = C_i$, then the cover $\mathfrak{M}_{\mathfrak{C}} = (P_i)_{i\in I}$ is trivial and the restriction map

$$C^*_T(P_{\mathfrak{C}}) \to C^*_{T,\mathfrak{M}_{\mathfrak{C}}}(P_{\mathfrak{C}}),$$

necessarily induces an isomorphism in cohomology.
Henceforth we assume

$$C_i \subsetneq \overline{\mathfrak{C}} \quad \text{for all } i \in I.$$

We equip $\mathbb{P}$ with the quotient topology induced by the standard Euclidean topology of $\mathbb{C}^A$ and begin by constructing the corresponding affinizated retraction.
For $i \in I$ define the continuous functions

$$c_i \colon \mathbb{C}^A \longrightarrow \mathbb{R}_{\geq 0}, \quad z = (z_a)_{a \in A} \longmapsto \frac{1}{|\overline{\mathfrak{C}} \setminus C_i|} \sum_{a \in \overline{\mathfrak{C}} \setminus C_i} |z_a|,$$

and

$$c \colon \mathbb{C}^A \longrightarrow \mathbb{R}_{\geq 0}, \quad z \longmapsto \min(c_i(z) \mid i \in I).$$

For $z \in \mathbb{C}^A$ set

$$I(z) := \{i \in I \mid c_i(z) = c(z)\},$$

and let $\underline{i}(z)$ denote the simplex in $\mathcal{I}$ with vertices $I(z)$.

Remark 5.5. By continuity of c and each c_i, for every $z \in \mathbb{C}^A$ there exists a neighborhood U of z such that

$$I(z') \subseteq I(z) \quad \text{for all } z' \in U.$$

Define $\hat{r} \colon \mathbb{C}^A \to \mathbb{C}^A$ by

$$\hat{r}(z)_a = \begin{cases} 0, & \text{if } a \notin C_{\underline{i}(z)}, \\ z_a, & \text{if } a \in \underline{\mathfrak{C}}, \\ \left(1 - \frac{c(z)}{\min(c_j(z) \mid a \notin C_j)}\right) z_a, & \text{if } a \in C_{\underline{i}(z)} \setminus \underline{\mathfrak{C}}. \end{cases}$$

Lemma 5.6. *Let*

$$A_{\mathfrak{C}} := \bigcup_{i \in I} V(e_a \mid a \notin C_i) \subseteq \mathbb{C}^A,$$

be the affinization of $P_{\mathfrak{C}}$, where e_a are the coordinate functions of $\mathbb{C}^A$. Then:

1. *$\hat{r}$ is continuous,*
2. *$\operatorname{im} \hat{r} \subseteq A_{\mathfrak{C}}$,*
3. *$\hat{r}(z) = z$ for all $z \in A_{\mathfrak{C}}$, and*
4. *if $z \in \mathbb{C}^A$ satisfies $\hat{r}(z) \neq 0$, then $\hat{r}(z') \neq 0$ for every*

$$z' \in L_z := \{tz + (1-t)\hat{r}(z) \mid t \in [0,1]\}.$$

Proof. For the second point, note that $C_{\underline{i}(z)} \subseteq C_i$ for every $z \in \mathbb{C}^A$ and $i \in I(z)$, hence $\hat{r}(z) \in A_{\mathfrak{C}}$.

We now prove (1), the continuity of $\hat{r}$. Let $z \in \mathbb{C}^A$ and choose a neighborhood U of z as in Remark 5.5, so that $I(z') \subseteq I(z)$ for all $z' \in U$. We verify continuity in each coordinate $a \in A$.
Case 1: If $a \in \underline{\mathfrak{C}}$, then $\hat{r}_a(z) = e_a(z) = z_a$, which is continuous.
Case 2: If $a \in C_{\underline{i}(z)} \setminus \underline{\mathfrak{C}}$, then $a \in C_{\underline{i}(z')} \setminus \underline{\mathfrak{C}}$ for all $z' \in U$. The quotient

$$z' \longmapsto \frac{c(z')}{\min(c_j(z') \mid a \notin C_j)},$$

is well-defined and continuous on U, so $\hat{r}_a$ is continuous at z.

Case 3: Suppose $a \notin C_{\underline{i}(z)}$ and let $(z_k)_{k\in\mathbb{N}}$ be a sequence in U converging to z. We have $\hat{r}(z_k)_a = 0$ for all k with $a \notin C_{\underline{i}(z_k)}$, and hence we may restrict our attention to the subsequence with $a \in C_{\underline{i}(z_k)}$ for all k. This implies that $I(z_k) \subsetneq I(z)$, and, by splitting into finitely many subsequences, we may assume that there exists some $j_0 \in I$ with $j_0 \in I(z) \setminus I(z_k)$ for all k. The quotient $c(z_k)/c_{j_0}(z_k)$ converges to one and therefore $(\hat{r}(z_k)_a)_k$ converges to zero, which is equal to $\hat{r}(z_k)_a$.

We now prove (3). Let $z \in A_{\mathfrak{C}}$, so that $z \in V(e_a \mid a \notin C_i)$ for some fixed $i \in I$. It follows that $c(z) = c_i(z) = 0$. If $j \in I(z)$, then $c_j(z) = 0$ as well, and hence $z \in V(e_a \mid a \notin C_j)$. Thus $z_a = 0$ for all $a \notin C_{\underline{i}(z)}$, and by the definition of $\hat{r}$ we have

$$\hat{r}(z)_a = \begin{cases} z_a, & \text{if } a \in C_{\underline{i}(z)}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we prove (4). Let $z' = tz + (1-t)\hat{r}(z)$ be a point on the line segment L_z with $t \in [0,1]$. For any $i \in I(z)$ and $j \in I \setminus I(z)$, we have $A \setminus C_i \subseteq A \setminus C_{\underline{i}(z)}$, and hence

$$c_i(z') = \frac{1}{|\overline{\mathfrak{C}} \setminus C_i|} \sum_{a\in\overline{\mathfrak{C}}\setminus C_i} t|z_a| = tc_i(z) < tc_j(z) = \frac{1}{|\overline{\mathfrak{C}} \setminus C_j|} \sum_{a\in\overline{\mathfrak{C}}\setminus C_j} t|z_a| \leq c_j(z').$$

Since $c_i(z') = tc_i(z)$ for all $i \in I(z)$, it follows that $I(z') = I(z)$. Moreover, $z_a \neq 0$ if and only if $z'_a \neq 0$ for all $a \in C_{\underline{i}(z)} = C_{\underline{i}(z')}$, and $\hat{r}(z) \neq 0$ if and only if

$$\sum_{a\in C_{\underline{i}(z)}} |z_a| \neq 0$$

which is equivalent to

$$\sum_{a\in C_{\underline{i}(z')}} |z'_a| \neq 0.$$

Thus $\hat{r}(z') \neq 0$. □

Remark 5.7. Let $z \in \mathbb{C}^A \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. By definition, $c_i(\lambda z) = |\lambda| c_i(z)$ for all $i \in I$, and hence $I(z) = I(\lambda z)$.

First, this implies that for all $a \in C_{\underline{i}(z)} \setminus \underline{\mathfrak{C}}$ the quotient

$$c(z)/\min(c_j(z) \mid a \notin C_j),$$

is invariant under scalar multiplication.

Second, z gets mapped to zero by $\hat{r}$ if and only if $z_a = 0$ for all $a \in C_{\underline{i}(z)}$. Therefore $\hat{r}(z) = 0$ if and only if $\hat{r}(\lambda z) = 0$. In other words, the closed subset

$$Z := \hat{r}^{-1}(0) \subseteq \mathbb{C}^A,$$

is invariant under multiplication by nonzero scalars.

By the second part of Remark 5.7 and Lemma 5.6, the set

$$U := \left(\mathbb{C}^A \setminus Z\right) / \sim,$$

is an open subset of $\mathbb{P}$ containing

$$P_{\mathfrak{C}} = (A_{\mathfrak{C}} \setminus \{0\}) / \sim .$$

Corollary 5.8. *The subspace $P_{\mathfrak{C}}$ is a retract of U.*

Proof. By the first part of Remark 5.7 we have the scaling property

$$\hat{r}(\lambda z) = \lambda \hat{r}(z), \tag{2}$$

for all $z \in \mathbb{C}^A \setminus \{0\}$ and $\lambda \neq 0$. Using the construction of U, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^A \setminus Z & \xrightarrow{\hat{r}} & A_{\mathfrak{C}} \\ \downarrow & & \downarrow \\ U & \xrightarrow[r]{} & P_{\mathfrak{C}} \end{array}$$

where the vertical arrows are projective quotients and r the projectivization of $\hat{r}$. We can use Lemma 5.6 together with the property (2) to characterize r: Since $\hat{r}$ is continuous, the quotient maps preserve continuity, and thus r is also continuous. Moreover, since $\hat{r}$ restricts to the identity on $A_{\mathfrak{C}}$, its projectivization r restricts to the identity on $P_{\mathfrak{C}}$. We conclude that r is a retraction, which proves the claim. □

Remark 5.9. By Lemma 5.6(4), for any $z \in \mathbb{C}^A \setminus Z$ the entire line segment L_z lies in $\mathbb{C}^A \setminus Z$. Consequently, for each $z \in U$ the map

$$[0,1] \to U, \quad t \longmapsto (1-t)z + t \cdot r(z),$$

is continuous. By Remark 5.4, this shows that $P_{\mathfrak{C}}$ is a deformation retract of U.

Finally, define for each $i \in I$ the open subset

$$U_i := \{z \in U \mid \sum_{a \in C_i} |z_a| \neq 0\} \subseteq \mathbb{P},$$

and for a simplex $\underline{i} = (i_0, \ldots, i_p) \in \mathcal{I}$ set

$$U_{\underline{i}} = U_{i_0} \cap \ldots \cap U_{i_p}.$$

Lemma 5.10. *For each $\underline{i} \in \mathcal{I}$, the projective space $P_{\underline{i}}$ is a deformation retract of $U_{\underline{i}}$.*

Proof. Fix $\underline{i} \in \mathcal{I}$. Define $r_i \colon [0,1] \times U_{\underline{i}} \to P_{\underline{i}}$ by

$$r_i(t,z)_a = \begin{cases} (1-t)z_a, & \text{if } a \notin C_{\underline{i}}, \\ z_a, & \text{if } a \in C_{\underline{i}}, \end{cases}$$

for $z = (z_a)_{a \in A} \in U_{\underline{i}}$. This is a deformation retraction of $U_{\underline{i}}$ onto $P_{\underline{i}}$. □

Corollary 5.11. *The inclusions induce isomorphisms of graded Λ_T-algebras*

$$H_T^*(U) \cong H_T^*(P_{\mathfrak{C}}), \quad H_T^*(U_{\underline{i}}) \cong H_T^*(P_{\underline{i}}),$$

for all $\underline{i} \in \mathcal{I}$.

Proof. Both U and the sets $U_{\underline{i}}$ are T-invariant. The inclusions

$$P_{\mathfrak{C}} \hookrightarrow U, \quad P_{\underline{i}} \hookrightarrow U_{\underline{i}},$$

are equivariant and homotopy equivalences by Corollary 5.8, Remark 5.9 and Lemma 5.10. Applying Lemma 2.8 yields the claimed isomorphisms. □

We are now in the situation described in Remark 3.15 and ready to prove the main result of this subsection.

Lemma 5.12. *The triple* $(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ *is an object in* **GCov**.

Proof. Let K_P and K_U be the double complexes associated to the covers $(P_i)_{i\in I}$ and $(U_i)_{i\in I}$, with the row-wise filtration. Since $(U_i)_{i\in I}$ is an open cover of U by T-invariant sets, Remark 3.14 shows that $(T, U, (U_i)_{i\in I}) \in$ **GCov**, i.e., $C_T^*(U) \hookrightarrow K_U$ induces an isomorphism in cohomology.
The inclusions $P_i \hookrightarrow U_i$ give a morphism of filtered complexes

$$\phi\colon K_P \to K_U,$$

and the induced morphism of spectral sequences $(\phi_n)_n$ consists of isomorphisms of bigraded modules for $n \geq 1$ (see Corollary 5.11 and Remark A.13). Theorem A.14 then yields that

$$H(\phi)\colon H^*(K_P) \longrightarrow H^*(K_U),$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} C_T^*(P_{\mathfrak{C}}) & \longrightarrow & K_P \\ \uparrow & & \uparrow{\scriptstyle \phi} \\ C_T^*(U) & \longrightarrow & K_U \end{array}$$

where vertical arrows come from inclusions and horizontal arrows from Remark 3.16. In cohomology, the vertical arrows are isomorphisms by the first part of the proof and Corollary 5.11. The lower horizontal arrow is an isomorphism since $(T, U, (U_i)_{i\in I}) \in$ **GCov**. Hence also $C_T^*(P_{\mathfrak{C}}) \to K_P$ induces an isomorphism, proving that $(T, P_{\mathfrak{C}}, (P_i)_{i\in I}) \in$ **GCov**. □

5.2 Applying the Mayer–Vietoris Spectral Sequence

After verifying that $(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ is a covered space, we are now ready to apply the theory developed in the first part of this thesis. Many of the assumptions made in Chapter 3 and Chapter 4 were motivated by the example class of projective unions, and in this chapter we carry out an initial and direct computation of the equivariant cohomology of $P_{\mathfrak{C}}$ using the Mayer–Vietoris spectral sequence.

In Lemma 5.13 we formulate the Mayer–Vietoris complex for projective unions, collect the relevant properties of $P_{\mathfrak{C}}$ in Remark 5.14, and establish the key result that the cohomology of $P_{\mathfrak{C}}$ is computed by this complex in Lemma 5.15. We then consider the example of a union of two projective spaces and encounter our first instance of a projective union with torsion over Λ_T (Examples 5.19 and 5.22). Furthermore, in Remark 5.21 we recover the Mayer–Vietoris complex as a special case of a complex of quotient rings equipped with a simplicial cochain differential, and in Corollary 5.29 we use this perspective to describe its cohomology in terms of syzygies. This description is then reapplied to the projective union from Example 5.22 in Example 5.30.

As indicated at the beginning of Chapter 5, the notation introduced below will be retained for the rest of this thesis.

Following the setup in the start of this chapter, we have seen that the equivariant cohomology of $\mathbb{P}$ is the graded Λ_T-algebra

$$H_T^*\mathbb{P} = \Lambda_T[\zeta]/\left(\prod_{a\in A}(\zeta+\chi_a)\right),$$

where ζ is the Chern class of $\mathscr{O}_{\mathbb{P}}(1)$, the line bundle dual to the tautological line bundle on $\mathbb{P}$. Each standard character in Λ_T and the class ζ have degree two.
For any subspace P_J determined by $J \subseteq A$, we consider the induced T-action and again denote by ζ the equivariant Chern class of $\mathscr{O}_{P_J}(1)$, the dual tautological line bundle. Define

$$\eta_a \coloneqq \zeta + \chi_a, \quad a \in A,$$

and for $J \subseteq A$ set

$$\eta_J \coloneqq \prod_{a\in J}\eta_a \in \Lambda_T[\zeta], \quad \eta_\varnothing = 1.$$

The equivariant cohomology ring is then

$$H_T^*(P_J) = \Lambda_T[\zeta]/\left(\eta_J\right),$$

which we abbreviate as $\Omega_J \coloneqq H_T^*(P_J)$.
Let $\mathfrak{C} = \{C_i \mid i \in I\}$ be a collection of subsets of A. For $i \in I$ set

$$\Omega_i \coloneqq \Omega_{C_i} \quad \eta_i \coloneqq \eta_{C_i},$$

and for $\underline{i} = (i_0 \dots, i_p) \in \mathcal{I}_p$ write

$$\Omega_{\underline{i}} \coloneqq \Omega_{C_{\underline{i}}} \quad \eta_{\underline{i}} \coloneqq \eta_{i_0\dots i_p} \coloneqq \eta_{C_{\underline{i}}}.$$

If $J_1 \subseteq J_2 \subseteq A$, the restriction map for the corresponding subspaces is given by the projection

$$\Omega_{J_2} = \Lambda_T[\zeta]/\left(\eta_{J_2}\right) \longrightarrow \Omega_{J_1} = \Lambda_T[\zeta]/\left(\eta_{J_1}\right), \quad \zeta \longmapsto \zeta,$$

where we identify the Chern class of $\mathscr{O}_{P_{J_2}}(1)$ with that of $\mathscr{O}_{P_{J_1}}(1)$. Thus Ω_{J_1} can be regarded as quotient of Ω_{J_2}, and, when it is clear from the context, we do not explicitly indicate when an element from Ω_{J_2} is being restricted to Ω_{J_1}.
Summarizing the above discussion, the Mayer–Vietoris complex of a projective union, as defined in Definition 3.18, has the following form.

Lemma 5.13. *The Mayer–Vietoris complex* $\mathrm{MV} = \mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ *has monograded components*

$$\mathrm{MV}^p = \bigoplus_{\underline{i}\in\mathcal{I}_p}\Omega_{\underline{i}},$$

and differential

$$d^p \colon \mathrm{MV}^p \longrightarrow \mathrm{MV}^{p+1}, \quad (d^p(\underline{f}))_{\underline{j}} = \sum_{k=0}^{p+1} (-1)^k f_{j_0,\dots,\hat{j}_k,\dots,j_{p+1}},$$

where $\underline{f} \in \bigoplus_{\underline{i}\in\mathcal{I}_p} \Omega_{\underline{i}}$ *and* $\underline{j} = (j_0,\dots,j_{p+1}) \in \mathcal{I}_{p+1}$. *Here, each term* $f_{j_0,\dots,\hat{j}_k,\dots,j_{p+1}}$ *is regarded as an element of the quotient* $\Omega_{\underline{j}}$ *of* $\Omega_{(j_0,\dots,\hat{j}_k,\dots,j_{p+1})}$.

Remark 5.14. For any $J \subseteq A$, the subspace P_J is a smooth projective variety with vanishing odd singular cohomology. Under the induced T-action, P_J is equivariantly formal and $H_T^*(P_J)$ free over Λ_T and concentrated in even degrees.
By Lemma 5.12, $(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ is an object in **GCov**, and $\mathfrak{M}_{\mathfrak{C}}$ is a good cover of $P_{\mathfrak{C}}$ (see Example 3.29).

Since the covering $\mathfrak{M}_{\mathfrak{C}}$ satisfies the hypotheses of Theorem 3.28, we may apply it to compute the equivariant cohomology of $P_{\mathfrak{C}}$ via its Mayer–Vietoris complex.

Lemma 5.15. *Let* $(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ *be a projective union. Then*

$$G(H_T^*(P_{\mathfrak{C}})) \cong H(\mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})),$$

as bigraded Λ_T*-algebra. In particular, the first-column component* $\nu(P_{\mathfrak{C}})$ *is equal to* $\ker d^0$.

Definition 5.16. We define the *standard generators* of MV as the elements

$$e_{\underline{i}} \in \mathrm{MV}, \quad \underline{i} \in \mathcal{I},$$

given by

$$(e_{\underline{i}})_{\underline{j}} = \begin{cases} 1, & \text{if } \underline{i} = \underline{j}, \\ 0, & \text{else.} \end{cases}$$

(The component 1 here is the unit in $\Omega_{\underline{i}}$.) The constant vector in the first-column component is

$$e_{\mathbf{1}} := \sum_{i\in I} e_i \in \mathrm{MV}^0,$$

and we set $e_0 := 0$.

Remark 5.17. Let $\underline{i} \in \mathcal{I}_r, \underline{j} \in \mathcal{I}_s$ and $f \in \Omega^p_{\underline{i}}, g \in \Omega^q_{\underline{j}}$. By Remark 3.19, the product in MV is

$$(f \cdot e_{\underline{i}})(g \cdot e_{\underline{j}}) = (-1)^{ps} fg e_{\underline{i}\cdot\underline{j}}.$$

Since f can be nonzero only if p is even, the sign factor $(-1)^{ps}$ is always 1 so

$$(f \cdot e_{\underline{i}})(g \cdot e_{\underline{j}}) = fg e_{\underline{i}\cdot\underline{j}}.$$

As before, the term fg is considered as the product of the respective restrictions of f and g to $\Omega_{\underline{i}\cdot\underline{j}}$.
In the following, we will mostly view MV as a module. Its multiplication, and hence also the multiplication on $H(\mathrm{MV})$, is entirely determined by the above rule.

Remark 5.18. We regard MV as differential $\Lambda_T[\zeta]$-algebra (hence also a $\Lambda_T[\zeta]$-module) via

$$\Lambda_T[\zeta] \longrightarrow \mathrm{MV}, \quad x \longmapsto x \cdot e_{\mathbf{1}} \in \mathrm{MV}^0 .$$

A generating set of MV as $\Lambda_T[\zeta]$-module is given by $\{e_{\underline{i}} \mid \underline{i} \in \mathcal{N}(\mathfrak{C})\}$.

Example 5.19. Let $\mathfrak{C} = (C_1, C_2)$ be a collection of two subsets of A, defining the union of two projective spaces in $\mathbb{P}$. We are in the situation of Remark 3.34, and the corresponding Mayer–Vietoris complex is

$$\Lambda_T[\zeta]/(\eta_1) \oplus \Lambda_T[\zeta]/(\eta_2) \xrightarrow{r_1^* - r_2^*} \Lambda_T[\zeta]/(\eta_{(1,2)}) \longrightarrow 0$$

where r_1^*, r_2^* denote the projection maps. Note that both r_1^*, r_2^* as well as $r_1^* - r_2^*$ are surjective, and hence Example 3.35 yields

$$H_T^*(P_{\mathfrak{C}}) \cong \{(f, g) \in \Lambda_T[\zeta]/(\eta_1) \oplus \Lambda_T[\zeta]/(\eta_2) \mid f - g \in (\eta_{(1,2)})\}.$$

Remark 5.20. In principle, any projective union can be obtained by successively considering the union of a projective union $P_{\mathfrak{C}}$ with a subspace $P_B \subseteq \mathbb{P}$, where $B \subseteq A$. Again, we obtain the Mayer–Vietoris spectral sequence on two terms (Remark 3.34)

$$H_T^*(P_{\mathfrak{C}}) \oplus H_T^*(P_B) \xrightarrow{d^0} H_T^*(P_{\mathfrak{C} \cap B}) \longrightarrow 0,$$

where $\mathfrak{C} \cap B := (C \cap B \mid C \in \mathfrak{C})$ and d^0 can be constructed as a morphism of covered spaces. Although the above complex is not exact in general, if d^0 is surjective, then

$$H_T^*(P_{\mathfrak{C}} \cup P_B) \cong \ker d^0 \subseteq H_T^*(P_{\mathfrak{C}}) \oplus H_T^*(P_B).$$

However, if d^0 is not surjective, the cokernel contributes to the E_2-page of the spectral sequence, and this cokernel may already involve the cohomology of the potentially complicated projective union $P_{\mathfrak{C} \cap B}$. In this case the filtration on $H_T^*(P_{\mathfrak{C}} \cup P_B)$ is no longer trivial, so the computation yields only the associated graded module at this stage. Since the Mayer–Vietoris spectral sequence is formulated in terms of the actual cohomology, not its associated graded object, the iterative procedure cannot be continued directly and therefore becomes impractical.

Remark 5.21. The Mayer–Vietoris complex of a projective union appears as a special case of the following general construction.

Let S be a Noetherian integral domain with unit, and fix elements $\mu_a \in S$ for all $a \in A$. As in the beginning of Chapter 5, for $B \subseteq A$ and $\underline{i} \in \mathcal{I}$ we set

$$\mu_B := \prod_{a \in B} \mu_a, \qquad \mu_{\underline{i}} := \mu_{C_{\underline{i}}}, \qquad \mu_{\varnothing} := 1.$$

With the earlier convention that $i \in I$ also denotes the 0-simplex in $\mathcal{I}$, this implies in particular that $\mu_i = \mu_{C_i}$.

Define the differential graded module $D(\mathfrak{C}, S)$ by

$$D^p(\mathfrak{C}, S) := \bigoplus_{\underline{i} \in \mathcal{I}_p} S/(\mu_{\underline{i}}),$$

with differential

$$d^p\colon D^p(\mathfrak{C},S)\longrightarrow D^{p+1}(\mathfrak{C},S),\quad (d^p(\underline{s}))_{\underline{j}}=\sum_{k=0}^{p+1}(-1)^k s_{j_0,\dots,\hat{j}_k,\dots,j_{p+1}},$$

where $s_{j_0,\dots,\hat{j}_k,\dots,j_{p+1}}$ is regarded as an element of the quotient $S/(\mu_{\underline{j}})$. The diagonal action of S on each summand turns $D(\mathfrak{C},S)$ into a differential graded S-module, and we write

$$H^*(\mathfrak{C},S),$$

for its cohomology.
If it is necessary to emphasize the chosen collection of elements $(\mu_a)_{a\in A}$, we write

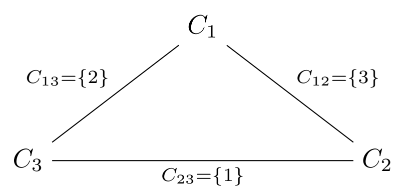
$$D(\mathfrak{C},S,(\mu_a)_{a\in A}),\qquad H^*(\mathfrak{C},S,(\mu_a)_{a\in A}).$$

If we take $S=\Lambda_T[\zeta]$ and $\mu_a=\eta_a$ for all $a\in A$, then $D(S,\mathfrak{C})$ is precisely the Mayer–Vietoris complex $\mathrm{MV}(T,P_{\mathfrak{C}},\mathfrak{M}_{\mathfrak{C}})$.

Example 5.22. Consider the union of the coordinate axes in $\mathbb{P}^2$. Let $A=\{1,2,3\}$ and

$$\mathfrak{C}_\vartriangle=\{C_1,C_2,C_3\},\quad C_1=\{2,3\},\quad C_2=\{1,3\},\quad C_3=\{1,2\},$$

and suppose T acts on $\mathbb{P}$ via the characters χ_1,χ_2,χ_3. In this example we abbreviate $i_0i_1\dots i_p$ for the p-simplex $(i_0,\dots,i_p)$ in $\mathcal{I}$. The nerve of the cover (C_1,C_2,C_3) is the hollow triangle.



For $P_\vartriangle\coloneqq P_{\mathfrak{C}_\vartriangle}$, the Mayer–Vietoris complex is

$$\Omega_1\oplus\Omega_2\oplus\Omega_3\xrightarrow{d^0}\Omega_{12}\oplus\Omega_{13}\oplus\Omega_{23}\longrightarrow 0.$$

More explicitly, we have

$$\begin{aligned}d^0\colon \Lambda_T[\zeta]/\left((\zeta+\chi_2)(\zeta+\chi_3)\right)\oplus\Lambda_T[\zeta]/\left((\zeta+\chi_1)(\zeta+\chi_3)\right)\oplus\Lambda_T[\zeta]/\left((\zeta+\chi_1)(\zeta+\chi_2)\right)\\ \longrightarrow\Lambda_T[\zeta]/\left(\zeta+\chi_3\right)\oplus\Lambda_T[\zeta]/\left(\zeta+\chi_2\right)\oplus\Lambda_T[\zeta]/\left(\zeta+\chi_1\right),\end{aligned}$$

with

$$(f_1,f_2,f_3)\longmapsto(f_1-f_2,f_1-f_3,f_2-f_3).$$

The kernel is equal to

$$\ker d^0=\{(f_1,f_2,f_3)\in\mathrm{MV}^0\mid f_1-f_2\in(\zeta+\chi_3),f_1-f_3\in(\zeta+\chi_2),f_2-f_3\in(\zeta+\chi_1)\},$$

and the image of d^0 is generated as $\Lambda_T[\zeta]$-module by

$$d^0(1,0,0)=(1,1,0),\quad d^0(0,1,0)=(1,0,1),\quad\text{and}\quad d^0(0,0,1)=(0,1,1).$$

The $\Lambda_T[\zeta]$-module morphism

$$\Lambda_T[\zeta] \longrightarrow \operatorname{coker} d^0, \quad f \longmapsto (f,0,0),$$

is surjective with kernel

$$\langle \zeta+\chi_1, \zeta+\chi_2, \zeta+\chi_3\rangle_{\Lambda_T[\zeta]}.$$

Thus

$$\operatorname{coker} d^0 \cong \Lambda_T / \left(\chi_1-\chi_2, \chi_1-\chi_3, \chi_2-\chi_3\right),$$

as $\Lambda_T[\zeta]$-modules, with ζ acting by $-\chi_1 = -\chi_2 = -\chi_3$. We obtain equalities of graded $\Lambda_T[\zeta]$-modules

$$\nu(P_\vartriangle) = \ker d^0, \quad (H_T^{*+1} P_\vartriangle)_1 = \Lambda_T / \left(\chi_1-\chi_2, \chi_1-\chi_3, \chi_2-\chi_3\right), \quad \text{and} \quad (H_T^{*+2} P_\vartriangle)_2 = 0.$$

Note that

$$\operatorname{tor}(\Lambda_T, H_T^*(P_\vartriangle)) = (H_T^*(P_\vartriangle))_1,$$

except for the case where $\chi_1 = \chi_2 = \chi_3$, i.e., when the T-action is trivial.

Remark 5.23. We have seen that the differential on MV is obtained by combining the coboundary operator from the simplicial cohomology of $\mathcal{I}$ with projections to smaller quotients of $\Lambda_T[\zeta]$. All data defining the Mayer–Vietoris complex, and hence the equivariant cohomology of $P_{\mathfrak{C}}$, are encoded by the sets in $\mathfrak{C}$ together with their intersections.
As in the previous example, it is therefore natural to visualize a projective union $P_{\mathfrak{C}}$ by the *weighted simplicial complex* whose underlying simplicial complex is $\mathcal{N}(\mathfrak{C})$ and whose weights are the subsets $C_{\underline{i}} \subseteq A$. For the first-column component of $P_{\mathfrak{C}}$ we obtain a compact, albeit not very insightful, description

$$\nu(P_{\mathfrak{C}}) = \{(f_i)_{i\in I} \in \mathrm{MV}^0 \mid f_i - f_j \in (\eta_{ij})\}.$$

In general, the cohomology of MV can be interpreted as *weighted simplicial cohomology*. This reformulation does not significantly aid the actual computation of $H(\mathrm{MV})$, but it highlights the simplicial nature of the Mayer–Vietoris complex. We will not pursue this viewpoint further and refer the reader to [LR25].

Remark 5.24. Given collections of subsets $\mathfrak{C} = (C_i)_{i\in I}$ and $\mathfrak{D} = (D_j)_{j\in J}$ of A, then the corresponding projective unions satisfy $P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}$ if and only if there exists a map

$$\omega \colon J \longrightarrow I,$$

such that

$$D_j \subseteq C_{\omega(j)}, \quad j \in J.$$

As explained in Remark 3.38, we may, without loss of generality, assume, that ω is a bijection, or equivalently, that $I = J$ and

$$D_i \subseteq C_i, \quad i \in I.$$

After this adjustment, the inclusion

$$\iota_{\mathfrak{D}}^{\mathfrak{C}} \colon (T, P_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}) \longrightarrow (T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}),$$

is a morphism in **GCov**. The induced restriction

$$\hat{\iota}_{\mathfrak{D}}^{\mathfrak{C}} \colon G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_T^*(P_{\mathfrak{D}})),$$

between the associated graded algebras is equal to the map in cohomology induced by the morphism of Mayer-–Vietoris complexes

$$(\iota_{\mathfrak{D}}^{\mathfrak{C}})_1\colon \mathrm{MV}(T, P_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}) \longrightarrow \mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}),$$

whose components are the projections

$$H_T^*(P_{C_{\underline{i}}}) = \Lambda_T[\zeta]/(\eta_{C_{\underline{i}}}) \longrightarrow H_T^*(P_{D_{\underline{i}}}) = \Lambda_T[\zeta]/(\eta_{D_{\underline{i}}}), \quad \underline{i} \in \mathcal{I}.$$

We conclude this subsection with a brief discussion on how $H(\mathrm{MV})$ can be computed by considering syzygies instead of divisibility conditions. For a detailed introduction to syzygies we refer to Chapter 15 of [Eis95] or [Eis05].

Definition 5.25. Let S be an integral domain with unit and let $u_1, \dots, u_r$ be elements of an S-module L. The *syzygies of* $u_1, \dots, u_r$ *over* S are the elements of the S-module

$$\mathrm{Syz}_S(u_1, \dots, u_r) \coloneqq \{(a_1, \dots, a_r) \in S^r \mid a_1 u_1 + \ldots + a_r u_r = 0\} \subseteq S^r.$$

If a submodule $L' \subseteq L$ is generated by $u_1, \dots, u_r$, we write

$$\mathrm{Syz}_S(L') \coloneqq \mathrm{Syz}_S(u_1, \dots, u_r).$$

Note that $\mathrm{Syz}_S(L')$ depends on the chosen generating set of L'.

We begin with the more general framework introduced in Remark 5.21. Fix nonzero elements $\mu_a \in S$ for all $a \in A$. For both the complex $(D(\mathfrak{C}, S), d)$ and the complex $(C^*(\mathcal{I}, S), c)$, that is, the cochain complex of the full simplex $\mathcal{I}$ with coefficients in S, we define the standard generators

$$e_{\underline{i}}, \quad \underline{i} \in \mathcal{I},$$

as well as $e_{\mathbf{1}}$, in the same way as for the Mayer–Vietoris complex in Definition 5.16. By modifying the differential c, we obtain a new complex

$$\ldots \longrightarrow C^{p-1}(\mathcal{I}, S) \xrightarrow{\hat{c}^{p-1}} C^p(\mathcal{I}, S) \xrightarrow{\hat{c}^p} C^{p+1}(\mathcal{I}, S) \longrightarrow \ldots$$

with

$$\hat{c}(e_{\underline{j}}) = \sum_{\underline{i} \in \mathcal{I}_p} \frac{\mu_{\underline{j}}}{\mu_{\underline{i}}} c(e_{\underline{j}})_{\underline{i}}, \quad \underline{j} \in \mathcal{I}_{p-1}.$$

where the quotient $\mu_{\underline{j}}/\mu_{\underline{i}}$ is taken in S and is well-defined because $\mu_{\underline{i}}$ is nonzero and divides $\mu_{\underline{j}}$ whenever $c(e_{\underline{j}})_{\underline{i}} \neq 0$. We define the graded S-modules

$$\mathcal{S}^p(\mathfrak{C}, S) \coloneqq \ker \hat{c}^p, \quad \mathcal{U}^p(\mathfrak{C}, S) \coloneqq \operatorname{im} \hat{c}^{p-1},$$

and set

$$\mathcal{S}^*(\mathfrak{C}, S) = \bigoplus_{p \geq 0} \mathcal{S}^p(\mathfrak{C}, S), \quad \mathcal{U}^*(\mathfrak{C}, S) = \bigoplus_{p \geq 0} \mathcal{U}^p(\mathfrak{C}, S).$$

Furthermore, we write

$$H^*(\mathcal{I}, S), \quad \widehat{H}^*(\mathcal{I}, S) = \mathcal{S}^*(\mathfrak{C}, S)/\mathcal{U}^*(\mathfrak{C}, S),$$

for the cohomology of $(C^*(\mathcal{I}, S), c)$ and $(C^*(\mathcal{I}, S), \hat{c})$, respectively.

Remark 5.26. Suppose that $(a_{\underline{j}})_{\underline{j}\in\mathcal{I}_p} \in \mathcal{S}^p(\mathfrak{C}, S)$. Equivalently,

$$\hat{c}((a_{\underline{j}})_{\underline{j}\in\mathcal{I}_p})_{\underline{i}} = 0, \quad \underline{i} \in \mathcal{I}_{p+1},$$

which is the same as

$$\sum_{\underline{j}\in\mathcal{I}_p} a_{\underline{j}}\mu_{\underline{j}}c(e_{\underline{j}})_{\underline{i}} = \sum_{\underline{j}\in\mathcal{I}_p} a_{\underline{j}}c(\mu_{\underline{j}}e_{\underline{j}})_{\underline{i}} = 0, \quad \underline{i} \in \mathcal{I}_{p+1},$$

since S is a domain. Thus,

$$\mathcal{S}^p(\mathfrak{C}, S) = \operatorname{Syz}_S(c(\mu_{\underline{j}}e_{\underline{j}}) \mid \underline{j} \in \mathcal{I}_p),$$

and we call $\mathcal{S}^*(\mathfrak{C}, S)$ the *module of syzygies*.
The elements of $\mathcal{U}^*(\mathfrak{C}, S)$ are those universal relations obtained by eliminating the coefficient part via least common multiples of the $\mu_{\underline{j}}$, after which only the simplicial coboundary part remains. We will refer to them as *elementary syzygies*, and they are present for any choice of μ_a and, in particular, also hold when the μ_a are algebraically independent. In that case they are closely related to the familiar monomial syzygies (see, e.g., [Eis95, Chapter 15.2]).

Example 5.27. If $(a_i)_{i\in I} \in \ker \hat{c}^0$, then

$$\mu_i a_i = \mu_j a_j, \quad i, j \in I.$$

Define

$$L_\mu := \bigcap_{a\in\overline{\mathfrak{C}}} (\mu_a) \subseteq S.$$

If μ_a is nonzero for all $a \in A$, then

$$\widehat{H}^0(\mathcal{I}, S) = \mathcal{S}^0(\mathfrak{C}, S) = L_\mu \cdot \left(\frac{1}{\mu_i} e_i\right)_{i\in I}.$$

In particular, if S is a UFD, then $L_\mu = (\overline{\mu})$ is principal, where the least common multiple

$$\overline{\mu} := \operatorname{lcm}(\mu_a \mid a \in \overline{\mathfrak{C}}),$$

is well defined up to a unit. Hence

$$\widehat{H}^0(\mathcal{I}, S) = S \cdot \left(\frac{\overline{\mu}}{\mu_i}\right)_{i\in I}.$$

For the following Lemma we keep the notation as in Example 5.27. Note that the shift in degrees is a consequence of the long exact sequence we consider in order to obtain the formula.

Lemma 5.28. *Let S be a Noetherian integral domain, and let $\mu_a \in S$ be nonzero for all $a \in A$. Then there are isomorphisms of S-modules*

$$H^p(\mathfrak{C}, S) \cong \mathcal{S}^{p+1}(\mathfrak{C}, S)/\mathcal{U}^{p+1}(\mathfrak{C}, S), \quad p \geq 1,$$

as well as

$$H^0(\mathfrak{C}, S) \cong S/L_\mu \oplus \mathcal{S}^1(\mathfrak{C}, S)/\mathcal{U}^1(\mathfrak{C}, S).$$

Proof. The projections

$$\pi^p \colon C^p(\mathcal{I}, S) = \bigoplus_{\underline{i} \in \mathcal{I}_p} S \longrightarrow D^p(\mathfrak{C}, S) = \bigoplus_{\underline{i} \in \mathcal{I}_p} S/(\mu_{\underline{i}}), \quad p \geq 0,$$

assemble into a surjective morphism of complexes

$$\pi \colon C^*(\mathcal{I}, S) \longrightarrow D(\mathfrak{C}, S).$$

Moreover, the map

$$\xi : (C^*(\mathcal{I}, S), \hat{c}) \longrightarrow (\ker(\pi), c), \quad e_{\underline{i}} \longmapsto \mu_{\underline{i}} e_{\underline{i}},$$

is an isomorphism, giving rise to a short exact sequence

$$0 \longrightarrow (C^*(\mathcal{I}, S), \hat{c}) \xrightarrow{\ \xi\ } (C^*(\mathcal{I}), c) \xrightarrow{\ \pi\ } (D(\mathfrak{C}, S), d) \longrightarrow 0.$$

The complex $(C^*(\mathcal{I}), c)$ is exact in positive degree, so the associated long exact sequence in cohomology yields graded S-isomorphisms

$$H^p(\mathfrak{C}, S) \cong \widehat{H}^{p+1}(\mathcal{I}, S), \quad p \geq 1.$$

The beginning of the long exact sequence is

$$0 \longrightarrow \widehat{H}^0(\mathcal{I}, S) \longrightarrow H^0(\mathcal{I}, S) \longrightarrow H^0(\mathfrak{C}, S) \xrightarrow{\ \delta\ } \widehat{H}^1(\mathcal{I}, S) \longrightarrow 0,$$

with connecting morphism

$$\delta \colon H^0(\mathfrak{C}, S) \longrightarrow \widehat{H}^1(\mathcal{I}, S), \quad (a_j e_j)_{j \in J} \longmapsto \left(\frac{a_i - a_j}{\mu_{(i,j)}} e_{(i,j)} \right)_{(i,j)} \in \mathcal{I}_2.$$

Fixing the minimal element $i_0 \in I$, we obtain a right inverse to δ by

$$\widehat{H}^1(\mathcal{I}, S) \longrightarrow H^0(\mathfrak{C}, S), \quad (a_{(i,j)} e_{(i,j)})_{(i,j) \in \mathcal{I}_2} \longmapsto (a_{(i_0,j)} \mu_{(i_0,j)} e_j)_{j \in I},$$

where we set $a_{i_0 i_0} \mu_{i_0 i_0} = 0$. Furthermore,

$$H^0(\mathcal{I}, S) = S \cdot e_{\mathbf{1}}, \quad \xi\left(\widehat{H}^0(\mathcal{I}, S)\right) = L_\mu \cdot e_{\mathbf{1}},$$

as shown in Example 5.27. Hence we obtain a split exact sequence

$$0 \longrightarrow S/L_\mu \cdot e_I \longrightarrow H^0(\mathfrak{C}, S) \longrightarrow \widehat{H}^1(\mathcal{I}, S) \longrightarrow 0.$$

which proves the claim. □

In the following we write

$$\mathcal{S}^* := \mathcal{S}^*(\mathfrak{C}, \Lambda_T[\zeta]), \quad \mathcal{U}^* := \mathcal{U}^*(\mathfrak{C}, \Lambda_T[\zeta]).$$

Corollary 5.29. *We have isomorphisms of $\Lambda_T[\zeta]$-modules*

$$H^p(\mathrm{MV}) \cong \mathcal{S}^{p+1}/\mathcal{U}^{p+1}, \quad p \geq 1,$$

as well as

$$H^0(\mathrm{MV}) \cong \Lambda_T[\zeta]/(\overline{\eta}) \oplus \mathcal{S}^1/\mathcal{U}^1,$$

where

$$\overline{\eta} := \mathrm{lcm}(\eta_a \mid a \in \overline{\mathfrak{C}}),$$

denotes the least common multiple (well defined up to a unit).

Proof. Since $\Lambda_T[\zeta]$ is a UFD, the claim follows from Lemma 5.28 with $S = \Lambda_T[\zeta]$ and $\mu_a = \eta_a$ for all $a \in A$, together with Example 5.27. □

Example 5.30. We revisit Example 5.22. One computes

$$\mathcal{S}^2 = C^2(\mathcal{I}, \Lambda_T[\zeta]) = \Lambda_T[\zeta], \quad \mathcal{U}^2 = \Big\langle \zeta + \chi_1, \zeta + \chi_2, \zeta + \chi_3 \Big\rangle,$$

in agreement with the earlier result. Moreover,

$$\mathcal{S}^1 = \Big\{ (a_{12}, a_{13}, a_{23}) \in C^1(\mathcal{I}, \Lambda_T[\zeta]) = \Lambda_T[\zeta]^{\oplus 3} \mid a_{12}(\zeta + \chi_3) - a_{13}(\zeta + \chi_2) + a_{23}(\zeta + \chi_1) = 0 \Big\},$$

and

$$\mathcal{U}^1 = \Big\langle (\zeta + \chi_2, \zeta + \chi_3, 0), (\zeta + \chi_1, 0, -(\zeta + \chi_3)), (0, \zeta + \chi_1, \zeta + \chi_2) \Big\rangle.$$

According to Corollary 5.29, the module $H^0(\mathrm{MV})$ consists of the constant triples, i.e., elements (f_1, f_2, f_3) such that there exists some $f \in \Lambda_T[\zeta]/(\overline{\eta})$ whose restrictions equal the entries f_i, and of the non-constant triples $(f_1, f_2, f_3) \in \mathrm{MV}^0$ characterized by the coefficients of their differences

$$f_1 - f_2 = a_{12}(\zeta + \chi_3), \quad f_1 - f_3 = a_{13}(\zeta + \chi_2), \quad f_2 - f_3 = a_{23}(\zeta + \chi_1).$$

The coefficients of compatible differences form the module $\mathcal{S}^1$, and those lying in $\mathcal{U}^1$ correspond either to triples in the kernel of π or to constant triples whose difference coefficients still satisfy the conditions of $\mathcal{S}^1$. For example, the element $(\zeta + \chi_2, \zeta + \chi_3, 0) \in \mathcal{U}^1$ corresponds to $((\zeta + \chi_2)(\zeta + \chi_3), 0, 0) \in C^0(\mathcal{I}, \Lambda_T[\zeta])$, which maps to zero in MV^0.
To describe the non-constant triples in $H^0(\mathrm{MV})$, consider the morphism

$$\kappa : \mathcal{S}^1 \longrightarrow \mathcal{T} := \{(b_{12}, b_{13}) \in \Lambda_T^2/\big((\chi_2 - \chi_1, \chi_3 - \chi_1)\big) \mid b_{12}(\chi_3 - \chi_1) = b_{13}(\chi_2 - \chi_1)\},$$

$$(a_{12}, a_{13}, a_{23}) \longmapsto \Big(a_{12}(\zeta = -\chi_1), a_{13}(\zeta = -\chi_1) \Big),$$

that evaluates $a_{12}, a_{13} \in \Lambda_T[\zeta]$ at $\zeta = -\chi_1$. It can be verified directly that κ is well-defined, surjective and has kernel $\mathcal{U}^1$. In conclusion,

$$\mathcal{S}^1/\mathcal{U}^1 \cong \mathcal{T}.$$

As another example, the triple

$$\Big(0, (\chi_2 - \chi_1)(\zeta + \chi_3), (\chi_3 - \chi_1)(\zeta + \chi_2) \Big) \in \mathrm{MV}^0$$

is an element of $H^0(\mathrm{MV})$, and corresponds to

$$\kappa\Big((\chi_2-\chi_1,\chi_3-\chi_1,\chi_2-\chi_3)\Big) = \big(\chi_2-\chi_1,\chi_3-\chi_1\big) = 0$$

in $\mathcal{T}$. Note that

$$\big(\chi_2-\chi_1,\chi_3-\chi_1,\chi_2-\chi_3\big) \in \mathcal{U}^1,$$

and that

$$\Big(0,(\chi_2-\chi_1)(\zeta+\chi_3),(\chi_3-\chi_1)(\zeta+\chi_2)\Big) = \Big((\zeta+\chi_2)(\zeta+\chi_3),(\zeta+\chi_2)(\zeta+\chi_3),(\zeta+\chi_2)(\zeta+\chi_3)\Big)$$

is a constant triple.

An advantage of $\mathcal{S}$ as opposed to MV is that it is a domain, and therefore the description of torsion over $\Lambda_T[\zeta]$ is more straightforward in $\widehat{H}^*(\mathcal{I},\Lambda_T[\zeta])$ than in $H(\mathrm{MV})$.

Definition 5.31. Let L be an S-module and $\eta \in S$. The *annihilator* of η in L is defined as the S-submodule

$$\mathrm{Ann}(\eta, L) \coloneqq \{m \in L \mid \eta \cdot m = 0\} \subseteq L.$$

Define

$$M^p(\eta,\mathcal{U}) \coloneqq \{(a_{\underline{i}})_{\underline{i}\in\mathcal{I}_p} \in \mathcal{U}^p(\mathfrak{C},S) \mid a_{\underline{i}} \text{ is divisible by } \eta \text{ for all } \underline{i}\in\mathcal{I}_p\}, \quad p \geq 0.$$

The following Lemma relates the torsion in $\widehat{H}^*(\mathcal{I},S)$ to the torsion in $H^*(\mathfrak{C},S)$.

Lemma 5.32. *Let S be a Noetherian integral domain, and let $\mu_a \in S$ be nonzero for all $a \in A$. For any nonzero element $\eta \in S$, there are S-module isomorphisms*

$$\mathrm{Ann}(\eta, H^p(\mathfrak{C},S)) \cong M^{p+1}(\eta,\mathcal{U})/\hat{c}\,(\eta\cdot C^p(\mathcal{I},S))\,, \quad p\geq 1,$$

and

$$\mathrm{Ann}(\eta, H^0(\mathfrak{C},S)) \cong \mathrm{Ann}(\eta, S/L_\mu) \oplus M^1(\eta,\mathcal{U})/\hat{c}\left(\eta\cdot C^0(\mathcal{I},S)\right).$$

Proof. From Lemma 5.28 we obtain

$$\mathrm{Ann}(\eta, H^p(\mathfrak{C},S)) \cong \mathrm{Ann}(\eta,\widehat{H}^p(\mathcal{I},S)), \quad p\geq 1,$$

and

$$\mathrm{Ann}(\eta, H^0(\mathfrak{C},S)) \cong \mathrm{Ann}(\eta, S/L_\mu) \oplus \mathrm{Ann}(\eta,\widehat{H}^0(\mathcal{I},S)),$$

as S-modules. Since $C^*(\mathcal{I},S)$ is torsion-free over S, we have a surjective morphism

$$M^p(\eta,\mathcal{U}) \longrightarrow \mathrm{Ann}(\eta,\widehat{H}^p(\mathcal{I},S)), \quad (a_{\underline{i}})_{\underline{i}\in\mathcal{I}_p} \longmapsto \left[\!\!\left[\left(\frac{a_{\underline{i}}}{\eta}\right)_{\underline{i}\in\mathcal{I}_p}\right]\!\!\right],$$

where, for each $\underline{i}$, the notation $\frac{a_{\underline{i}}}{\eta}$ denotes the unique element $s\in S$ such that $s\eta = a_{\underline{i}}$.The kernel of this morphism is

$$\eta\cdot\mathcal{U}^p(\mathfrak{C},S) = \hat{c}\left(\eta\cdot C^{p-1}(\mathcal{I},S)\right),$$

and the claim follows. □

5.3 Localization for Projective Unions

As anticipated, unions of projective spaces provide a favorable setting for the method of restricting to T-fixed points. One of the main results of this theory, Theorem 2.21, is formulated for fixed points of arbitrary subgroups $K \subseteq T$. In the case of projective unions, many restrictions to smaller unions can be realized as restrictions to the fixed points of such subgroups, and these maps consequently become isomorphisms after inverting an appropriate set of characters.

This phenomenon is examined in Lemma 5.40 and Remark 5.42. Furthermore, the results from Chapter 4.1 are adapted to the present situation in Corollary 5.37, Corollary 5.38, and Lemma 5.44.

Remark 5.33. In Example 2.19, we considered the restriction of a projective space $\mathbb{P}$ to its set of T-fixed points: Let $F_1, \dots, F_s$ be the partition of A such that $\chi_a = \chi_b$ if and only if $a, b \in A$ lie in the same part of the partition. The fixed-point set $\mathbb{P}^T$ can then be expressed as the projective union

$$P_{\mathfrak{F}}, \qquad \mathfrak{F} = \{F_j \mid j = 1, \dots, s\}.$$

Its equivariant cohomology is

$$H_T^*(P_{\mathfrak{F}}) = \bigoplus_{j=1}^{s} \Lambda_T[\zeta]/(\eta_{F_j}),$$

since the sets F_j are disjoint. In particular, $H_T^*(P_{\mathfrak{F}})$ is free over Λ_T.
The restriction is given by the projections π_j onto each component $\Lambda_T[\zeta]/(\eta_{F_j})$, i.e.,

$$\iota^* \colon H_T^*\mathbb{P} \longrightarrow H_T^*(P_{\mathfrak{F}}), \quad f \longmapsto (\pi_1(f), \dots, \pi_s(f)).$$

Example 5.34. In the situation where $\mathbb{P}$ has finitely many T-fixed points (see Example 2.39), the partition $\mathfrak{F}$ consists of all singletons $F_a = \{a\}$, $a \in A$. In this case, each projection π_a is precisely the evaluation of $f(\zeta) \in \Lambda_T[\zeta]/(\eta_A)$ at $\zeta = -\chi_a$. Thus, a tuple $(p_a)_{a \in A} \in \Lambda_T^A$ lies in the image of ι^* if and only if there exists a polynomial $f(\zeta)$ in $\Lambda_T[\zeta]/(\eta_A)$ interpolating the values p_a.

Definition 5.35. Let $\mathfrak{C} = (C_i)_{i \in I}$ and $\mathfrak{D} = (D_j)_{j \in J}$ be collections of subsets of A. We write $\mathfrak{D} \subseteq \mathfrak{C}$ and say that $\mathfrak{D}$ is *finer* than $\mathfrak{C}$ if, for all $j \in J$, there exists an $i \in I$ such that $D_j \subseteq C_i$. The *intersection of collections* is defined as

$$\mathfrak{C} \cap \mathfrak{D} \coloneqq (C_i \cap D_j)_{(i,j) \in I \times J}.$$

Remark 5.36. The set of T-fixed points in a projective union $P_{\mathfrak{C}}$ is itself a projective union

$$P_{\mathfrak{C}}^T = P_{\mathfrak{C} \cap \mathfrak{F}} = P_{\mathfrak{C}} \cap P_{\mathfrak{F}}.$$

In particular, $P_{\mathfrak{C}}^T$ is finite precisely when $\mathfrak{C} \cap \mathfrak{F}$ consists of singletons. Equivalently, this holds exactly when, within each subset $C \in \mathfrak{C}$, all corresponding characters are distinct.

Remark 5.36 has the following consequence for the torsion in equivariant cohomology.

Corollary 5.37. *If R is a field, then the Λ_T-torsion submodule of $H_T^*(P_{\mathfrak{C}})$ coincides with the kernel of the localization map:*

$$\tau(P_{\mathfrak{C}}) = \operatorname{tor}(\Lambda_T, H_T^*(P_{\mathfrak{C}})).$$

Proof. This follows from Corollary 2.24, since

$$H_T^*(P_{\mathfrak{C}})^T = \Lambda_T \otimes_R H^*(P_{\mathfrak{C}\cap\mathfrak{F}}),$$

which is torsion-free over Λ_T by Corollary 5.48, a result that is independent of the present chapter. □

Corollary 5.38. *If $P_{\mathfrak{C}}$ has finitely many T-fixed points, then*

$$\tau(P_{\mathfrak{C}}) = \operatorname{tor}(\Lambda_T, H_T^*(P_{\mathfrak{C}})) = (H_T^*(P_{\mathfrak{C}}))_1, \qquad \iota(P_{\mathfrak{C}}) \cong \nu(P_{\mathfrak{C}}) = H^0(\mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})).$$

Proof. If $P_{\mathfrak{C}}$ has finitely many T-fixed points, then $(H_T^*(P_{\mathfrak{C}})^T)_1 = 0$, and Lemma 4.2 implies the claim. □

In Remark 5.24, we introduced inclusions of projective unions and saw that $P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}$ holds precisely when $\mathfrak{D} \subseteq \mathfrak{C}$. Moreover, we observed there that in this situation one may always assume $I = J$ and $D_i \subseteq C_i$ for all $i \in I$. With this adjustment, the inclusion

$$\iota_{\mathfrak{D}}^{\mathfrak{C}} \colon (T, P_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}) \longrightarrow (T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}),$$

is a morphism in **GCov**. The induced restriction on cohomology,

$$\hat{\iota}_{\mathfrak{D}}^{\mathfrak{C}} \colon G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_T^*(P_{\mathfrak{D}})),$$

between the associated graded algebras coincides with the cohomology map arising from the morphism of Mayer-–Vietoris complexes

$$(\iota_{\mathfrak{D}}^{\mathfrak{C}})_1 \colon \mathrm{MV}(T, P_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}) \longrightarrow \mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}),$$

whose components are the projections

$$H_T^*(P_{C_{\underline{i}}}) = \Lambda_T[\zeta]/(\eta_{C_{\underline{i}}}) \longrightarrow H_T^*(P_{D_{\underline{i}}}) = \Lambda_T[\zeta]/(\eta_{D_{\underline{i}}}), \quad \underline{i} \in \mathcal{I}.$$

Definition 5.39. For a subset $B \subseteq A$ and collection of subsets $\mathfrak{D} = (D_j)_{j\in J}$ set

$$L_B := \{\chi_a - \chi_b \mid a, b \in B\}, \quad L_{\mathfrak{D}} := \bigcup_{j\in J} L_{D_j},$$

and define the subgroup

$$T_{\mathfrak{D}} := T_{\langle L_{\mathfrak{D}}\rangle} = \bigcap_{\chi\in L_{\mathfrak{D}}} \ker\chi \subseteq T,$$

(see Definition 2.15).

Lemma 5.40. *For $\mathfrak{D} \subseteq \mathfrak{C}$, the inclusion $\iota_{\mathfrak{D}}^{\mathfrak{C}}$ is the restriction to the fixed points of a subgroup of T if and only if*

$$P_{\mathfrak{C}}^{T_{\mathfrak{D}}} = P_{\mathfrak{D}}.$$

Proof. Let $K \subseteq T$ be a subgroup with corresponding sublattice $M_K \leq M$. For any $i \in I$, the action of K on P_{C_i} is given by the characters

$$[\![\chi_a]\!], \quad a \in C_i,$$

viewed as elements of the character lattice M/M_K of K (Remark 4.11). By Example 2.19,

$$P_{D_i} \subseteq P_{C_i}^K$$

if and only if

$$[\![\chi_a]\!] = [\![\chi_{a'}]\!], \quad \forall a, a' \in D_i,$$

that is, if $L_{D_i} \subseteq M_K$. Since

$$P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}^K \iff P_{D_i} \subseteq P_{C_i}^K, \quad \forall i \in I,$$

we deduce

$$P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}^K \iff L_{\mathfrak{D}} = \bigcup_{i \in I} L_{D_i} \subseteq M_K.$$

Thus, $P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}^K$ if and only if M/M_K is a quotient of $M/M_{T_{\mathfrak{D}}} = M/\langle L_{\mathfrak{D}} \rangle$, equivalently if and only if $K \subseteq T_{\mathfrak{D}}$ (Remark 4.11).
Finally, if $P_{\mathfrak{D}} = P_{\mathfrak{C}}^K$ for some subgroup $K \subseteq T$, then $K \subseteq T_{\mathfrak{D}}$, and hence

$$P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}^{T_{\mathfrak{D}}} \subseteq P_{\mathfrak{C}}^K = P_{\mathfrak{D}},$$

which proves the claim. □

Remark 5.41. If $P_{\mathfrak{C}}^{T_{\mathfrak{D}}} = P_{\mathfrak{D}}$, then in particular,

$$P_{\mathfrak{C} \cap \mathfrak{F}} = P_{\mathfrak{C}} \cap P_{\mathfrak{F}} = P_{\mathfrak{C}} \cap \mathbb{P}^T = P_{\mathfrak{C}}^T \subseteq P_{\mathfrak{C}}^{T_{\mathfrak{D}}} = P_{\mathfrak{D}}.$$

This implies $\mathfrak{F} \cap \mathfrak{C} \subseteq \mathfrak{D}$ as well as $\overline{\mathfrak{C}} = \overline{\mathfrak{D}}$, since $\mathfrak{F}$ is a partition of A. Hence, these two properties provide necessary conditions for $\mathfrak{D}$ in Lemma 5.40.

Moreover, as seen in the proof of Lemma 5.40, if $P_{\mathfrak{D}}$ is the fixed-point set of a subgroup of T, then $T_{\mathfrak{D}}$ is the maximal subgroup with this property.

Remark 5.42. By the Localization Theorem 2.21, restrictions to fixed points become isomorphisms after localization. For the fact that it suffices to invert only finitely many characters, we refer to the proof of [AF24, Theorem 7.1.1]. We simply state the result here:

For $\mathfrak{D} \subseteq \mathfrak{C}$, define $S_{\mathfrak{D}}^{\mathfrak{C}}$ to be the multiplicative subset of Λ_T generated by the finite set $L_{\mathfrak{C}} \setminus L_{\mathfrak{D}}$. If $\iota_{\mathfrak{D}}^{\mathfrak{C}}$ is a restriction to fixed points, then Lemma 5.40 gives $P_{\mathfrak{D}} = P_{\mathfrak{C}}^{T_{\mathfrak{D}}}$, and $\iota_{\mathfrak{D}}^{\mathfrak{C}}$ becomes an isomorphism after localizing at $S_{\mathfrak{D}}^{\mathfrak{C}}$.

Another way to see the same fact is to regard $\iota_{\mathfrak{D}}^{\mathfrak{C}}$ as a morphism in **GCov**. If it is a restriction to fixed points, then for each $\underline{i} \in \mathcal{I}$ the component

$$H_T^*(P_{C_{\underline{i}}}) \longrightarrow H_T^*(P_{D_{\underline{i}}}),$$

is itself a restriction to fixed points, becoming an isomorphism after localization at $S_{\mathfrak{D}}^{C_{\underline{i}}}$, the multiplicative subset generated by $L_{C_{\underline{i}}} \setminus L_{\mathfrak{D}}$.

It follows that $(\iota_{\mathfrak{D}}^{\mathfrak{C}})_1$ becomes an isomorphism after localization at $S_{\mathfrak{D}}^{\mathfrak{C}}$, since

$$L_{\mathfrak{C}} \setminus L_{\mathfrak{D}} = \bigcup_{\underline{i} \in \mathcal{I}} L_{C_{\underline{i}}} \setminus L_{\mathfrak{D}},$$

and hence $(S_{\mathfrak{D}}^{\mathfrak{C}})^{-1}\iota_{\mathfrak{D}}^{\mathfrak{C}}$ is an isomorphism by Corollary 3.42.

Example 5.43. The localization map ι^* of $P_{\mathfrak{C}}$ is given by $\iota_{\mathfrak{C}\cap\mathfrak{F}}^{\mathfrak{C}}$. By definition, we have that $L_{\mathfrak{F}} = \{0\}$ as well as $T_{\mathfrak{F}} = T$, and ι^* becomes an isomorphism after inverting the characters in $L_{\mathfrak{C}} \setminus \{0\}$.

We conclude this subsection with a discussion of the moment graph of a projective union. In principle, nothing new needs to be proved here, we simply recollect results from previous chapters.

Lemma 5.44. *Assume that for each $i \in I$ the characters in L_{C_i} are relatively prime. Then the moment graph $\Gamma_{\mathfrak{C}}$ of $P_{\mathfrak{C}}$ has vertex set $\overline{\mathfrak{C}}$ and an edge between $a, b \in \overline{\mathfrak{C}}$ whenever there exists an $i \in I$ with $a, b \in C_i$. Moreover,*

$$\iota(P_{\mathfrak{C}}) \cong H^*(\Gamma_{\mathfrak{C}}).$$

Proof. In Example 2.39, we saw that each P_{C_i} has as moment graph the complete graph on the vertex set C_i and is a GKM-variety, provided the characters in L_{C_i} are relatively prime. Hence $P_{\mathfrak{C}}$ itself has finitely many T-fixed points and is fixed-point closed. Furthermore, by Remark 5.14, the Mayer-–Vietoris complex $\mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}})$ is torsion-free over Λ_T, and $\mathfrak{M}_{\mathfrak{C}}$ is a good cover. Therefore Lemma 4.8 and Theorem 4.10 apply, and the claim follows. □

5.4 Generic and Trivial Torus Action

In this subsection we examine two extremal cases of the torus action on a projective union $P_{\mathfrak{C}}$: the trivial action and the generic action, in which the characters are maximally independent. In both situations we obtain structural results expressing $G(H_T^*(P_{\mathfrak{C}}))$ in terms of simplicial cohomology (Lemma 5.47 and Lemma 5.52). As R-modules, the cohomology of $P_{\mathfrak{C}}$ is thus described entirely in simplicial terms. As a Λ_T-module, however, even the torsion-free part of the equivariant cohomology with respect to a generic torus action need not be free (Example 5.59).

The characters relevant for the equivariant cohomology of $P_{\mathfrak{C}}$ lie in the sublattice

$$M_{\mathfrak{C}} \coloneqq \langle \chi_a \mid a \in \overline{\mathfrak{C}} \rangle,$$

and we start with the case that $M_{\mathfrak{C}} = 0$, i.e., T acts trivially on $P_{\mathfrak{C}}$. Since Λ_T is free over R, Lemma 2.9 gives

$$H_T^*(P_{\mathfrak{C}}) = \Lambda_T \otimes_{\mathbb{Z}} H^*(P_{\mathfrak{C}}).$$

One option would be to compute the ordinary singular cohomology of $P_{\mathfrak{C}}$, e.g. by the Mayer–Vietoris spectral sequence. Instead, we remain in the equivariant setting and reproduce the same procedure under the assumption $M_{\mathfrak{C}} = 0$. In this case, the equivariant cohomology of P_i reduces to

$$\Omega_i = \Lambda_T[\zeta]/(\zeta^{|C_i|}),$$

and $H_T^*(P_{\mathfrak{C}})$ can be expressed in terms of simplicial cohomology of certain simplicial complexes.

Definition 5.45. For $t \in \mathbb{Z}_{\geq 0}$, let $\Delta(t)$ be the simplicial complex on I with p-simplices all $\underline{i} \in \mathcal{I}_p$ with $|C_{\underline{i}}| \geq t$. Its cochain differential is denoted by d_t.

We form the differential bigraded R-module

$$C := \bigoplus_{t\geq 0} C^*(\Delta(t), R), \quad d := \sum_{t\geq 0} d_t, \quad C^{p,t} = C^p(\Delta(t), R).$$

For a p-simplex $\underline{i}$ in $\Delta(t)$ and a q-simplex $\underline{j}$ in $\Delta(s)$, denote the dual basis elements by $e^t_{\underline{i}}$ and $e^s_{\underline{j}}$, respectively. Define

$$e^t_{\underline{i}} \cdot e^s_{\underline{j}} := \begin{cases} (-1)^{tq} e^{t+s}_{\underline{i}} \smile e^{t+s}_{\underline{j}} = (-1)^{tq} e^{t+s}_{\underline{i}\cdot\underline{j}}, & \underline{i}, \underline{j} \in \Delta(t+s), \\ 0, & \text{otherwise.} \end{cases}$$

By bilinear extension, we obtain a product

$$\cdot \colon C^p(\Delta(t)) \times C^q(\Delta(s)) \longrightarrow C^{p+q}(\Delta(t+s)).$$

Lemma 5.46. *The differential bigraded module (C, d) is a differential bigraded algebra with respect to the above product.*

Proof. Let $\underline{i}$ be a p-simplex in $\Delta(t)$ and $\underline{j}$ a q-simplex in $\Delta(s)$, and assume both $\underline{i}$ and $\underline{j}$ are simplices in $\Delta(t+s)$.

First observe that

$$d_t(e^t_{\underline{i}}) \cdot e^s_{\underline{j}} = (-1)^{tq} d_{t+s}(e^{t+s}_{\underline{i}}) \smile e^{t+s}_{\underline{j}},$$

since every summand occurring in $d_t(e^t_{\underline{i}})$ but not in $d_{t+s}(e^{t+s}_{\underline{i}})$ is annihilated by the product. Similarly,

$$e^t_{\underline{i}} \cdot d_s(e^s_{\underline{j}}) = (-1)^{t(q+1)} e^{t+s}_{\underline{i}} \cdot d_{t+s}(e^{t+s}_{\underline{j}}).$$

It follows that

$$d(e^t_{\underline{i}} \cdot e^s_{\underline{j}}) = (-1)^{tq} d_{t+s}(e^{t+s}_{\underline{i}} \smile e^{t+s}_{\underline{j}})$$
$$= (-1)^{tq} \left(d_{t+s}(e^{t+s}_{\underline{i}}) \smile e^{t+s}_{\underline{j}} + (-1)^p e^{t+s}_{\underline{i}} \smile d_{t+s}(e^{t+s}_{\underline{j}}) \right),$$

by the Leibniz rule for the differential d_{t+s}. On the other hand,

$$d(e^t_{\underline{i}}) \cdot e^s_{\underline{j}} + (-1)^{t+p} e^t_{\underline{i}} \cdot d(e^s_{\underline{j}}) = d_t(e^t_{\underline{i}}) \cdot e^s_{\underline{j}} + (-1)^{t+p} e^t_{\underline{i}} \cdot d_s(e^s_{\underline{j}})$$
$$= (-1)^{tq} d_{t+s}(e^{t+s}_{\underline{i}}) \smile e^{t+s}_{\underline{j}} + (-1)^{t(q+1)+t+p} e^{t+s}_{\underline{i}} \cdot d_{t+s}(e^{t+s}_{\underline{j}}),$$

and we conclude

$$d(e^t_{\underline{i}} \cdot e^s_{\underline{j}}) = d(e^t_{\underline{i}}) \cdot e^s_{\underline{j}} + (-1)^{t+p} e^t_{\underline{i}} \cdot d(e^s_{\underline{j}}).$$

If either $\underline{i}$ or $\underline{j}$ is not a simplex in $\Delta(t+s)$, then

$$e^t_{\underline{i}} \cdot e^s_{\underline{j}} = d(e^t_{\underline{i}}) \cdot e^s_{\underline{j}} = e^t_{\underline{i}} \cdot d(e^s_{\underline{j}}) = 0,$$

and d satisfies the Leibniz rule in both cases.

Finally, for associativity, take a p-simplex $\underline{i}$ in $\Delta(t)$, a q-simplex $\underline{j}$ in $\Delta(s)$, and an v-simplex $\underline{\ell}$ in $\Delta(r)$, all lying in $\Delta(t+s+r)$. Then

$$e^t_{\underline{i}}(e^s_{\underline{j}} e^r_{\underline{\ell}}) = (-1)^{sv+t(q+v)} e^{t+s+r}_{\underline{i}\cdot\underline{j}\cdot\underline{\ell}} = (-1)^{tq+(t+s)v} e^{t+s+r}_{\underline{i}\cdot\underline{j}\cdot\underline{\ell}} = (e^t_{\underline{i}} e^s_{\underline{j}}) e^r_{\underline{\ell}}.$$

□

The cohomology of (C, d) is

$$H(C, d) = \bigoplus_{t \geq 0} H^*(\Delta(t)),$$

which inherits the bigrading and multiplication from (C, d).

Lemma 5.47. *If T acts trivially on $P_{\mathfrak{C}}$, then*

$$G(H_T^*(P_{\mathfrak{C}})) \cong \Lambda_T \otimes_R \left(\bigoplus_{t \geq 0} H^*(\Delta(t)) \right),$$

as bigraded Λ_T-algebra.

Proof. Notice that the Mayer–Vietoris complex (MV, d) associated to $P_{\mathfrak{C}}$ splits into subcomplexes

$$\mathrm{MV} = \bigoplus_{t \geq 0} \mathrm{MV}_t,$$

with each component given by

$$\mathrm{MV}_t \coloneqq \bigoplus_{p \geq 0} \bigoplus_{\underline{i} \in \Delta(t)_p} \Lambda_T \cdot \zeta^t = \Lambda_T \otimes \bigoplus_{p \geq 0} \bigoplus_{\underline{i} \in \Delta(t)_p} R \cdot \zeta^t.$$

The map

$$\Lambda_T \otimes C^*(\Delta(t)) \longrightarrow \mathrm{MV}_t, \quad e_{\underline{i}}^t \longmapsto \zeta^t \in \Omega_{\underline{i}},$$

is an isomorphism of differential bigraded Λ_T-modules, with grading

$$(\Lambda_T \otimes C^*(\Delta(t)))^{p,q} = \Lambda_T^{q-t} \otimes C^p(\Delta(t)), \quad \mathrm{MV}_t^{p,q} = \Lambda_T^{q-t} \otimes \bigoplus_{\underline{i} \in \Delta(t)_p} R \cdot \zeta^t.$$

The product on C was defined to ensure that

$$\Lambda_T \otimes C \longrightarrow \mathrm{MV},$$

is an isomorphism of differential bigraded algebras. Hence

$$H(\mathrm{MV}) \cong \Lambda_T \otimes_R \bigoplus_{t \geq 0} H^*(\Delta(t))$$

as bigraded Λ_T-algebras, and the claim follows from Theorem 3.28. □

Corollary 5.48. *If T acts trivially on $P_{\mathfrak{C}}$ and R is a field, then $H_T^*(P_{\mathfrak{C}})$ is torsion-free over Λ_T.*

Proof. Since R is a field, each cohomology group $H^*(\Delta(t))$ is free as an R-module. Lemma 5.47 then implies that the associated graded module $G(H_T^*(P_{\mathfrak{C}}))$ is torsion-free over Λ_T. But if $H_T^*(P_{\mathfrak{C}})$ contained torsion, the same would hold for its associated graded module, contradicting the previous conclusion. Therefore $H_T^*(P_{\mathfrak{C}})$ is torsion-free over Λ_T. □

Next, we turn to the case of a *generic* torus action on $P_{\mathfrak{C}}$, characterized by maximal independence of the characters. In this setting, the equivariant cohomology can again be compared to simplicial cohomology of suitable complexes.

Definition 5.49. We call the action of T on $P_{\mathfrak{C}}$ *generic* if $M_{\mathfrak{C}}$ is a saturated sublattice of M with basis $\{\chi_a \mid a \in \overline{\mathfrak{C}}\}$.

Assume for the rest of this subsection that T acts generically on $P_{\mathfrak{C}}$. Set $T'' := T_{M_{\mathfrak{C}}}$ and $M' := M_{\mathfrak{C}}$. Note that T'' acts trivially on $P_{\mathfrak{C}}$, since $\chi_a \mid_{T''} = 0$ for all $a \in \overline{\mathfrak{C}}$. Choose complements $M'' \subseteq M$ and $T' \leq T$ to M' and T'', respectively. By Lemma 4.14, the character lattices of T' and T'' are naturally identified with M' and M'', and we obtain the splitting

$$\Lambda_T = \Lambda_{T'} \otimes \Lambda_{T''} = \mathrm{Sym}^* M' \otimes \mathrm{Sym}^* M''.$$

Fix an index set K and characters $\chi_a \in M$, $a \in K$, forming a basis of M''.

A key feature of the generic action is that the elements

$$\eta_a = \zeta + \chi_a, \quad a \in O := \overline{\mathfrak{C}} \sqcup K,$$

together with ζ, form a set of algebraically independent generators of $\Lambda_T[\zeta]$. From now on we regard $\Lambda_T[\zeta]$ as a polynomial ring in these variables, so that η_S is a monomial, and every use of the term monomial refers to this interpretation.

For subsets $S \subseteq B \subseteq O$, define

$$\Lambda_B^S := R[\eta_a \mid a \in B \setminus S], \quad \Lambda^S := \Lambda_O^S, \quad \Lambda_B := \Lambda_B^{\varnothing}.$$

Definition 5.50. For subsets $S \subseteq B \subseteq \overline{\mathfrak{C}}$, let Δ_S^B be the simplicial complex on I whose p-simplices are all $\underline{i} \in \mathcal{I}_p$ with $C_{\underline{i}} \cap B \nsubseteq S$. The corresponding cochain differential is denoted by d_S^B. We also write $(\Delta_S, d_S) := (\Delta_S^{\overline{\mathfrak{C}}}, d_S^{\overline{\mathfrak{C}}})$.

We now construct the analogue of the complex from the trivial action case. Define the differential bigraded R-module

$$\mathrm{GE} := \bigoplus_{S \subseteq \overline{\mathfrak{C}}} \Lambda_S \otimes_R C^*(\Delta_S),$$

with differential $d := \sum_{S \subseteq \overline{\mathfrak{C}}} d_S$ and bigrading

$$\mathrm{GE}^{p,k} = \bigoplus_{\substack{S \subseteq \overline{\mathfrak{C}} \\ |S| \leq k}} \left(\Lambda_S^{k-|S|} \otimes C^p(\Delta_S) \right).$$

For $S, U \subseteq \overline{\mathfrak{C}}$, let $\chi_S \in \Lambda_S$ and $\chi_U \in \Lambda_U$ be monomials. If $\underline{i}$ is a p-simplex in Δ_S and $\underline{j}$ a q-simplex in Δ_U, we denote their dual basis elements in $C^p(\Delta_S)$ and $C^q(\Delta_U)$ by $e_{\underline{i}}^S$ and $e_{\underline{j}}^U$, respectively, and put

$$e_{\mathbf{1}}^S := \sum_{\substack{i \in I \\ C_i \nsubseteq S}} e_i^S.$$

Multiplication is defined by

$$(\chi_S \otimes e_{\underline{i}}^S) \cdot (\chi_U \otimes e_{\underline{j}}^U) = (-1)^{(|\chi_S| + |S|)q} \left(\chi_S \chi_U \eta_{S \cap U} \otimes e_{\underline{i} \cdot \underline{j}}^{S \cup U} \right),$$

whenever $\underline{i}$ and $\underline{j}$ are simplices in $\Delta_{S \cup U}$, and zero otherwise. Bilinear extension yields a product

$$\cdot \colon \Lambda_S \otimes C^p(\Delta_S) \times \Lambda_U \otimes C^q(\Delta_U) \longrightarrow \Lambda_{S \cup U} \otimes C^{p+q}(\Delta_{S \cup U}),$$

restricting to

$$\mathrm{GE}^{p,k} \times \mathrm{GE}^{q,v} \longrightarrow \mathrm{GE}^{p+q,k+v},$$

on the graded components. By arguments parallel to Lemma 5.46, one verifies:

Lemma 5.51. *The differential bigraded module* (GE, d) *is a differential bigraded algebra with respect to the product defined above.*

By mapping

$$\eta_a \longmapsto 1 \otimes e_{\mathbf{1}}^{\{a\}}, \quad a \in \overline{\mathfrak{C}},$$

we obtain a morphism $\Lambda_{\overline{\mathfrak{C}}} \to \mathrm{GE}$ which turns (GE, d) into a differential bigraded $\Lambda_{\overline{\mathfrak{C}}}$-algebra. The cohomology

$$H(\mathrm{GE}) = \bigoplus_{S \subseteq \overline{\mathfrak{C}}} \Lambda_S \otimes_R H^*(\Delta_S),$$

inherits a natural structure of a bigraded $\Lambda_T[\zeta]$-algebra. As in the case of the Mayer–Vietoris complex, we regard both GE and $H(\mathrm{GE})$ simultaneously as bigraded and as monograded objects, the latter with respect to the first grading.

Lemma 5.52. *Assume that T acts generically on $P_{\mathfrak{C}}$. With the notation above,*

$$G(H_T^*(P_{\mathfrak{C}})) \cong \Lambda_K[\zeta] \otimes_R H(\mathrm{GE}),$$

both as bigraded and monograded $\Lambda_T[\zeta]$-algebra.

Proof. First treat the case $M_{\mathfrak{C}} = M$. Define

$$\Xi\colon R[\zeta] \otimes_R \mathrm{GE} \longrightarrow \mathrm{MV}, \quad \zeta^t \otimes (\chi_S \otimes e_{\underline{i}}^S) \longmapsto \zeta^t \chi_S \eta_S e_{\underline{i}},$$

for $t \geq 0$ and $\chi_S \otimes e_{\underline{i}}^S \in \Lambda_S \otimes_R C^*(\Delta_S)$.
Because the action is generic, Ξ is an isomorphism of *differential* bigraded R-modules, where the left-hand side is graded by

$$(R[\zeta] \otimes_R \mathrm{GE})^{p,k} = \bigoplus_{\substack{S \subseteq \overline{\mathfrak{C}} \\ |S| \leq k}} \left(\bigoplus_{r+t+|S|=k} R\zeta^t \otimes \Lambda_S^r \otimes C^p(\Delta_S) \right).$$

By construction, the Λ_T-action and the product on GE are chosen to be compatible with Ξ, hence Ξ is in fact an isomorphism of differential bigraded Λ_T-algebras. Since $R[\zeta]$ is free over R, Theorem 3.28 yields

$$R[\zeta] \otimes_R H(\mathrm{GE}) \cong H(R[\zeta] \otimes \mathrm{GE}) \cong H(\mathrm{MV}) \cong G(H_T^*(P_{\mathfrak{C}})),$$

since $R[\zeta]$ is free over R.
In the general case, using the previously fixed decomposition $T \cong T' \times T''$ with T'' acting trivially on $P_{\mathfrak{C}}$, Lemma 4.28 gives

$$G(H_T^*(P_{\mathfrak{C}})) = \Lambda_{T''} \otimes G(H_{T'}^*(P_{\mathfrak{C}})) = \Lambda_K \otimes G(H_{T'}^*(P_{\mathfrak{C}})).$$

and the first part applied to T' yields the stated isomorphism. □

Remark 5.53. In the case of a generic action, Lemma 5.52 provides a simplicial description of $H(\mathrm{MV})$. As an R-module, the cohomology $G(H_T^*(P_{\mathfrak{C}}))$ decomposes completely into a direct sum of the cohomology rings of the simplicial complexes Δ_S. However, this splitting does not extend to the Λ_T-module structure, since the individual cohomology rings $\Lambda_S \otimes H^*(\Delta_S)$ are not preserved as Λ_T-submodules. This phenomenon is illustrated in Example 5.59, where we show that $\Lambda_K[\zeta] \otimes_R H^0(\mathrm{GE})$ does not need to be free over Λ_T.

Remark 5.54. Suppose we have an inclusion of projective unions $P_{\mathfrak{D}} \subseteq P_{\mathfrak{C}}$ as in Remark 5.24, with $\mathfrak{D} = (D_i)_{i\in I}$ and $\mathfrak{C} = (C_i)_{i\in I}$ such that $D_i \subseteq C_i$ for all $i \in I$. Let GE' denote the corresponding complex and Ξ' the corresponding isomorphism for $P_{\mathfrak{D}}$ (see Lemma 5.52).

Since $M_{\mathfrak{D}}$ is saturated in $M_{\mathfrak{C}}$, a basis $\{\chi_a \mid a \in K\}$ complementing $\{\chi_a \mid a \in \overline{\mathfrak{C}}\}$ can be extended to a basis complementing $\{\chi_a \mid a \in \overline{\mathfrak{D}}\}$ by setting

$$K' \coloneqq K \sqcup \overline{\mathfrak{C}} \setminus \overline{\mathfrak{D}}.$$

Thus, the collections of characters

$$\{\chi_a \mid a \in \overline{\mathfrak{D}} \cup K'\}, \quad \{\chi_a \mid a \in \overline{\mathfrak{C}} \cup K\},$$

coincide.

For $S \subseteq \overline{\mathfrak{C}}$ we have

$$D_{\underline{i}} \nsubseteq S \Rightarrow C_{\underline{i}} \nsubseteq S, \quad \underline{i} \in \mathcal{I},$$

so every simplex of $\Delta_S^{\overline{\mathfrak{D}}}$ is also a simplex of $\Delta_S^{\overline{\mathfrak{C}}}$, yielding an inclusion $\Delta_S^{\overline{\mathfrak{D}}} \hookrightarrow \Delta_S^{\overline{\mathfrak{C}}}$. This induces

$$\hat{r}_S \colon C^*(\Delta_S^{\overline{\mathfrak{C}}}) \longrightarrow C^*(\Delta_S^{\overline{\mathfrak{D}}}),$$

and we define

$$r_S \colon \Lambda_S \otimes C^*(\Delta_S^{\overline{\mathfrak{C}}}) \longrightarrow \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes \Lambda_{S\cap\overline{\mathfrak{D}}} \otimes C^*(\Delta_S^{\overline{\mathfrak{D}}}), \quad r_S(1 \otimes e_{\underline{i}}^S) = \eta_{S\setminus\overline{\mathfrak{D}}} \otimes 1 \otimes \hat{r}_S(e_{\underline{i}}^S),$$

using $\Lambda_S \cong \Lambda_{S\setminus\overline{\mathfrak{D}}} \otimes \Lambda_{S\cap\overline{\mathfrak{D}}} \hookrightarrow \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes \Lambda_{S\cap\overline{\mathfrak{D}}}$.

Collecting these maps gives

$$r \colon \mathrm{GE} = \bigoplus_{S\subseteq\overline{\mathfrak{C}}} \Lambda_S \otimes_R C^*(\Delta_S^{\overline{\mathfrak{C}}}) \longrightarrow \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes \mathrm{GE}' = \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes \bigoplus_{S\subseteq\overline{\mathfrak{D}}} \Lambda_S \otimes_R C^*(\Delta_S^{\overline{\mathfrak{D}}}).$$

The diagram

$$\begin{array}{ccc} \mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}) & \xrightarrow{\ \Xi\ } & \Lambda_K[\zeta] \otimes \mathrm{GE} \\ \big\downarrow & & \big\downarrow r \\ \mathrm{MV}(T, P_{\mathfrak{D}}, \mathfrak{D}) & \xrightarrow{\ \Xi'\ } & \Lambda_K[\zeta] \otimes \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes \mathrm{GE}' \end{array}$$

commutes. Hence the restriction

$$G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_T^*(P_{\mathfrak{D}})),$$

can be expressed, via Lemma 5.52, as

$$\Lambda_K[\zeta] \otimes H(\mathrm{GE}) \longrightarrow \Lambda_K[\zeta] \otimes \Lambda_{\overline{\mathfrak{C}}\setminus\overline{\mathfrak{D}}} \otimes H(\mathrm{GE}'),$$

and this map is exactly the one induced by r in cohomology.

As an application of Lemma 5.52, we can express $\iota(P_{\mathfrak{C}}) = \nu(P_{\mathfrak{C}})$ as $\Lambda_K[\zeta] \otimes_R H^0(\mathrm{GE})$ which is determined by the connected components of the 1-skeletons of the simplicial complexes Δ_S.

Definition 5.55. For $S \subseteq A$ let G_S be the 1-skeleton of Δ_S, that is, the graph with vertices $i \in I$ such that $C_i \nsubseteq S$ and an edge between distinct $i \neq j$ whenever $C_{(i,j)} \nsubseteq S$.

Remark 5.56. An element

$$\sum_{S \subseteq A} \sum_{\substack{i \in I \\ C_i \nsubseteq S}} \chi_{S,i} \otimes e_i^S,$$

of GE^0 lies in $H^0(\mathrm{GE})$ if and only if for every connected component of G_S, the coefficients $\chi_{S,i}$ are constant across all vertices in that component.

Even for a generic action, determining a minimal generating set over Λ_T for $G(H_T^*(P_{\mathfrak{C}}))$, or even for $\nu(P_{\mathfrak{C}})$, is difficult, despite the criterion of Remark 5.56. We will return to the description via GE in Chapters 5.6 and 6.1. For now, we employ the Mayer–Vietoris complex to analyze $\nu(P_{\mathfrak{C}})$ in the generic case. Specifically, we describe a test for freeness of $\nu(P_{\mathfrak{C}})$ over Λ_T.

Remark 5.57. The localization theorem ensures that, for a sufficiently large multiplicative subset $L \subseteq \Lambda_T$, the localized module $L^{-1}\iota(P_{\mathfrak{C}})$ is free of rank $|\overline{\mathfrak{C}}|$ over $L^{-1}\Lambda_T$. Consequently, if $\iota(P_{\mathfrak{C}})$ is free as a Λ_T-module, its rank must be $|\overline{\mathfrak{C}}|$. Thus, non-freeness can be shown by exhibiting $|\overline{\mathfrak{C}}| + 1$ elements in $H^0(\mathrm{GE}) = \iota(P_{\mathfrak{C}})$ that are independent over Λ_T.

Lemma 5.58. *Let $z \coloneqq |\overline{\mathfrak{C}}|$ and assume that the characters $\{\chi_a \mid a \in \overline{\mathfrak{C}}\}$ are pairwise distinct. Then the elements $\zeta^0 e_{\mathbf{1}}, \ldots, \zeta^{z-1} e_{\mathbf{1}}$ in $\nu(P_{\mathfrak{C}})$ are linearly independent over Λ_T.*

Proof. By construction, or by Corollary 5.29, the elements $\zeta^0 e_{\mathbf{1}}, \ldots, \zeta^{z-1} e_{\mathbf{1}}$ lie in $H^0(\mathrm{MV}) = \nu(P_{\mathfrak{C}}$. If they were linearly dependent, there would exist an $f \in \Lambda_T[\zeta]$ of degree less than z such that

$$f e_{\mathbf{1}} = 0 \text{ in } \mathrm{MV}^0 .$$

This holds if and only if f is divisible by $\overline{\eta}$, which is impossible since $\Lambda_T[\zeta]$ is a UFD and the characters χ_a are distinct. ∎

Example 5.59. We give an example of a projective union with generic torus action for which the first-column component is not free.
Suppose $A = \{1, \ldots, 8\}$ and $\mathfrak{C}_A = \{C_1, C_2, C_3, C_4\}$ with

$$C_1 = \{1,3,5\}, \quad C_2 = \{2,3,6\}, \quad C_3 = \{2,4,7\}, \quad C_4 = \{1,4,8\}.$$

For this example, we write

$$\eta_{i_0 \ldots i_t} \coloneqq \eta_{\{i_0, \ldots, i_t\}}, \quad i_0, \ldots i_t \in A.$$

$$\begin{array}{ccc} 135 & \overset{3}{\text{———}} & 236 \\ {\scriptstyle 1}\big| & & \big|{\scriptstyle 2} \\ 148 & \overset{4}{\text{———}} & 247 \end{array}$$

We claim that $\nu(P_{\mathfrak{C}_A})$ is not free over Λ_T by constructing a linearly independent set of size 9. Both $(\eta_{12}, \eta_{12}, 0, 0)$ and $(\eta_{34}, 0, 0, \eta_{34})$ lie in $H^0(\mathrm{GE})$. Suppose the 9 elements

$$e_0, \zeta e_0, \ldots, \zeta^6 e_0, (\eta_{12}, \eta_{12}, 0, 0), (\eta_{34}, 0, 0, \eta_{34}),$$

are linearly dependent. Then there exist $f \in \Lambda_T[\zeta]$ of degree at most 6 and $\alpha, \beta \in \Lambda_T$ such that

$$\eta_{247} \mid f, \quad \eta_{148} \mid f + \beta\eta_{34}, \quad \eta_{236} \mid f + \alpha\eta_{12},$$

and

$$\eta_{135} \mid f + \alpha\eta_{12} + \beta\eta_{34}.$$

Equivalently, there exist $a_0, a_1, a_2, a_3 \in \Lambda_T[\zeta]$ of degree at most 3 with

$$a_0\eta_{247} = f, \quad a_0\eta_{247} + a_1\eta_{148} = \beta\eta_{34}, \quad a_0\eta_{247} + a_2\eta_{236} = \alpha\eta_{12},$$

and

$$a_0\eta_{247} + a_3\eta_{135} = \alpha\eta_{12} + \beta\eta_{34}.$$

This yields

$$a_0\eta_{247} + a_1\eta_{148} + a_2\eta_{236} - a_3\eta_{135} = 0,$$

which forces one of η_{18}, η_{36} or η_{135} to divide a_0, and hence f. In any case this leads to a contradiction: for example, if $\eta_{135} \mid f$, then

$$\eta_{135} \mid \alpha\eta_{12} + \beta\eta_{34},$$

which is impossible since $\alpha, \beta \in \Lambda_T$.

5.5 Torus Change for Projective Unions

In the previous subsection we described the equivariant cohomology of a projective union $P_{\mathfrak{C}}$ under a generic torus action. If a torus $\overline{T}$ acts non-generically on $P_{\mathfrak{C}}$, that is, if the characters χ_a through which it acts are linearly dependent, the computation of equivariant cohomology becomes more involved. Nevertheless, one can always relate $\overline{T}$ to a sufficiently large torus T acting generically, such that the $\overline{T}$-equivariant cohomology of $P_{\mathfrak{C}}$ can be recovered from the T-equivariant cohomology via a change of tori.

We now make this approach precise, which motivates a closer analysis of torus change from Chapter 4.2 in the context of projective unions. We establish a criterion for when short exact sequences analogous to those of the Künneth theorem arise (Lemma 5.62) and we present an alternative method to decide whether the restriction of tori is surjective (Theorem 5.73).

For the remainder of this chapter we adopt the notation of Chapter 4.2, and in particular write $[\![x]\!]$ for the class of an element x whenever the quotient under consideration is clear from the context.

Let $\overline{T}$ act on $P_{\mathfrak{C}}$ via characters χ_a, and let T act via characters μ_a. Denote the corresponding character lattices by $\overline{M}$ and M, and write

$$(\mathrm{MV}, d) = (\mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}), d), \quad (\overline{\mathrm{MV}}, \overline{d}) = (\mathrm{MV}(\overline{T}, P_{\mathfrak{C}}, \mathfrak{C}), d),$$

for the associated Mayer–Vietoris complexes.

Remark 5.60. By construction of the torus action on a projective union, a morphism of tori

$$\varphi : \overline{T} \to T$$

is compatible with the actions on $P_{\mathfrak{C}}$ if and only if the induced map on character lattices

$$\varphi^* \colon M \longrightarrow \overline{M},$$

satisfies $\varphi^*(\mu_a) = \chi_a$ for all $a \in A$.

Remark 5.61. If $T \cong (\mathbb{C}^\times)^A$ is chosen with canonical characters μ_a, then its action on $P_{\mathfrak{C}}$ is generic by definition. In this case, the assignment

$$\varphi^*\colon M \longrightarrow \overline{M}, \quad \mu_a \longmapsto \chi_a,$$

defines a morphism of character lattices, with corresponding morphism of tori $\varphi\colon \overline{T} \to T$ compatible with the respective actions. The induced restriction and extension of tori

$$\theta\colon H_T^*(P_{\mathfrak{C}}) \longrightarrow H_{\overline{T}}^*(P_{\mathfrak{C}}), \quad \Theta\colon \Lambda_{\overline{T}} \otimes_{\Lambda_T} H_T^*(P_{\mathfrak{C}}) \longrightarrow H_{\overline{T}}^*(P_{\mathfrak{C}}),$$

relate the non-generic to the generic case.

For a general compatible morphism of tori $\varphi\colon \overline{T} \to T$, Remark 5.14 ensures that the hypotheses of Lemma 4.37 are satisfied. Consequently, the induced maps on associated graded algebras

$$\hat{\theta}\colon G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_{\overline{T}}^*(P_{\mathfrak{C}})), \quad \hat{\Theta}\colon \Lambda_{\overline{T}} \otimes_{\Lambda_T} G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_{\overline{T}}^*(P_{\mathfrak{C}})),$$

are explicitly given by

$$\hat{\theta}\colon H(\mathrm{MV}) \longrightarrow H\left(\Lambda_{\overline{T}} \otimes_{\Lambda_T} \mathrm{MV}\right) = H(\overline{\mathrm{MV}}), \quad [\![x]\!] \longmapsto [\![x]\!],$$

and

$$\hat{\Theta}\colon \Lambda_{\overline{T}} \otimes_{\Lambda_T} H(\mathrm{MV}) \longrightarrow H\left(\Lambda_{\overline{T}} \otimes_{\Lambda_T} \mathrm{MV}\right) = H(\overline{\mathrm{MV}}), \quad \lambda \otimes [\![x]\!] \longmapsto [\![\lambda \otimes x]\!].$$

Since R is a field, the morphism φ necessarily introduces relations $\mathcal{R}$ on Λ_T (see Lemma 4.17). It is therefore natural to formulate the results of this chapter in terms of introduced relations, which is also the form required in the next chapter.

For simplicity, we further assume that $\overline{T}^S = 1$, or equivalently, that $S(\operatorname{im}\varphi^*) = \overline{M}$ (see Definition 4.15). By Remark 4.11 this is the same as saying that $\overline{M}$ is a quotient of M, or that the morphism

$$\varphi\colon \overline{T} \hookrightarrow T,$$

is an embedding. Through this assumption, we may omit the factor $\Lambda_{\overline{T}^S}$ in the formulas, which will again be the relevant situation in the final chapter.

With these conventions we obtain

$$\Lambda_{\overline{T}} = \Lambda_T/\mathcal{R}, \quad \overline{\mathrm{MV}} = \mathrm{MV}/\mathcal{R}\,\mathrm{MV},$$

and hence

$$\hat{\theta}: H(\mathrm{MV}) \longrightarrow H(\mathrm{MV}/\mathcal{R}\,\mathrm{MV}),$$

as well as

$$\hat{\Theta}: H(\mathrm{MV})/\left(\mathcal{R}H(\mathrm{MV})\right) \longrightarrow H(\mathrm{MV}/\mathcal{R}\,\mathrm{MV}),$$

see Corollary 4.38.

Since MV is free over Λ_T, the map $\hat{\Theta}$ can in principle be studied via the Künneth spectral sequence. However, this approach is typically complicated. By contrast, the Künneth theorem yields a more accessible description, though it is not always applicable.

The following lemma gives a coarser description that holds under weaker assumptions than those required for the Künneth theorem.

Lemma 5.62. *For $p \geq 0$, set*

$$Z^p := \ker d^p, \qquad B^p := \operatorname{im} d^{p-1}.$$

Then there are short exact sequences

$$0 \longrightarrow (H^p(\mathrm{MV})/\mathcal{R}H^p(\mathrm{MV})) \xrightarrow{\hat{\Theta}} H^p(\mathrm{MV}/\mathcal{R}\,\mathrm{MV}) \longrightarrow \mathrm{Syz}_{H^{p+1}(\mathrm{MV})}(\mathcal{R})/\,\mathrm{Syz}_{Z^{p+1}}(\mathcal{R}) \longrightarrow 0,$$

for all $p \geq 0$, provided that

$$\mathcal{R}\,\mathrm{MV} \cap Z = \mathcal{R}Z.$$

Proof. Throughout, $[\![x]\!]$ denotes the residue class of $x \in \mathrm{MV}$ in the relevant quotient. Consider the short exact sequence of graded Λ_T-modules

$$0 \longrightarrow Z^p \longrightarrow \mathrm{MV}^p \longrightarrow B^{p+1} \longrightarrow 0,$$

viewed as a short exact sequence of differential graded modules with zero differentials on Z and B. If $\mathcal{R}\,\mathrm{MV} \cap Z = \mathcal{R}Z$, then the map

$$Z/\mathcal{R}Z \to \mathrm{MV}/\mathcal{R}\,\mathrm{MV},$$

is injective. By right exactness of the tensor product, we obtain a short exact sequence of differential graded modules

$$0 \longrightarrow Z^p/\mathcal{R}Z^p \longrightarrow \mathrm{MV}^p/\mathcal{R}\,\mathrm{MV}^p \longrightarrow B^{p+1}/\mathcal{R}B^{p+1} \longrightarrow 0.$$

The corresponding long exact sequence in cohomology is

$$\ldots \longrightarrow B^p/\mathcal{R}B^p \xrightarrow{\delta^p} Z^p/\mathcal{R}Z^p \xrightarrow{\phi} H^p(\mathrm{MV}/\mathcal{R}\,\mathrm{MV}) \longrightarrow B^{p+1}/\mathcal{R}B^{p+1} \xrightarrow{\delta^{p+1}} \ldots$$

where δ^p is the connecting homomorphism induced by the inclusion $B^p \hookrightarrow Z^p$, and ϕ is induced by the inclusion $Z^p \hookrightarrow \mathrm{MV}^p$. From this, we obtain short exact sequences

$$0 \longrightarrow H^p(\mathrm{MV})/\mathcal{R}H^p(\mathrm{MV}) \xrightarrow{\hat{\Theta}} H^p(\mathrm{MV}/\mathcal{R}\,\mathrm{MV}) \longrightarrow \ker \delta^{p+1} \longrightarrow 0, \qquad p \geq 0,$$

and after fixing generators $r_1, \ldots, r_t$ of $\mathcal{R}$, we can define a well-defined morphism

$$\kappa^p : \mathrm{Syz}_{H^p(\mathrm{MV})}(\mathcal{R}) \longrightarrow \ker \delta^p, \quad ([\![u_1]\!], \ldots, [\![u_t]\!]) \mapsto [\![u_1 r_1 + \ldots u_t r_t]\!] \in B^p/\mathcal{R}B^p.$$

If $[\![x]\!] \in \ker \delta^p$, then $x = z_1 r_1 + \ldots z_t r_t$ for some $z_i \in Z^p$ hence $\kappa^p([\![z_1]\!], \ldots, [\![z_t]\!]) = [\![x]\!]$, and κ^p is surjective. The kernel of κ^p consists precisely of tuples $([\![u_1]\!], \ldots, [\![u_t]\!])$ with

$$u_1 r_1 + \cdots + u_t r_t \in \mathcal{R}B^p,$$

i.e., there exist $b_i \in B^p$ such that $(u_1 - b_1)r_1 + \ldots + (u_t - b_t)r_t = 0$ and $([\![u_1]\!], \ldots, [\![u_t]\!]) = ([\![u_1 - b_1]\!], \ldots, [\![u_t - b_t]\!])$ in $\mathrm{Syz}_{H^p(\mathrm{MV})}(\mathcal{R})$. In other words,

$$(u_1 - b_1, \ldots, u_t - b_t) \in \mathrm{Syz}_{Z^p}(\mathcal{R}),$$

and we have shown that

$$\mathrm{Syz}_{H^p(\mathrm{MV})}(\mathcal{R})/\,\mathrm{Syz}_{Z^p}(\mathcal{R}) \longrightarrow \ker \delta^p,$$

is an isomorphism. □

Remark 5.63. The result of the previous lemma is not specific to projective unions. More generally, it requires only a covered space X, torus actions by T and $\overline{T}$, and a compatible morphism

$$\varphi\colon \overline{T} \hookrightarrow T,$$

which is an embedding, i.e., such that $\overline{T}^S = 1$.

By Remark 4.39, the assumption of Lemma 5.62, that is,

$$\mathcal{R}\,\mathrm{MV} \cap Z = \mathcal{R}Z,$$

is sufficient to ensure that $\hat{\Theta}$ is injective. In the case of equivariantly formal spaces, Example 4.24 shows that Θ is even an isomorphism.

The subtle point is surjectivity of $\hat{\theta}$ (or equivalently of $\hat{\Theta}$): namely, whether

$$\ker \overline{d} = d^{-1}(\mathcal{R}\,\mathrm{MV})/\mathcal{R}\,\mathrm{MV},$$

contains additional cocycles compared to $\ker d$. For projective unions, one can establish a recursive criterion for this by considering *semi-regular sequences.* Recall that the Mayer–Vietoris complex corresponding to $P_{\mathfrak{C}}$ together with its T-action is given by

$$\mathrm{MV} = \bigoplus_{p\geq 0} \bigoplus_{\underline{i}\in\mathcal{I}_p} \Lambda_T[\zeta]/(\eta_{\underline{i}}), \quad \eta_{\underline{i}} = \prod_{a\in C_{\underline{i}}} \eta_a, \quad \eta_a = \zeta + \mu_a.$$

Similarly, writing

$$\overline{\eta}_a := \zeta + \chi_a = \hat{\varphi}(\eta_a),$$

see Remark 5.60, we obtain

$$\overline{\mathrm{MV}} = \bigoplus_{p\geq 0} \bigoplus_{\underline{i}\in\mathcal{I}_p} \Lambda_{\overline{T}}[\zeta]/(\overline{\eta}_{\underline{i}}), \quad \overline{\eta}_{\underline{i}} = \prod_{a\in C_{\underline{i}}} \overline{\eta}_a.$$

In Remark 5.21 these appeared as a special case of the more general differential graded S-module $D(\mathfrak{C}, S)$. Concretely,

$$\mathrm{MV}(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}) = D(\mathfrak{C}, \Lambda_T[\zeta], (\eta_a)_{a\in A}),$$

and

$$\mathrm{MV}(\overline{T}, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}) = D(\mathfrak{C}, \Lambda_{\overline{T}}[\zeta], (\overline{\eta}_a)_{a\in A}).$$

Definition 5.64. For subsets $U, V \subseteq A$ set

$$\Lambda_T[\zeta]/V := \Lambda_T[\zeta]/(\eta_a \mid a \in V), \quad \eta_a^U := \begin{cases} \eta_a, & \text{if } a \notin U, \\ 1, & \text{if } a \in U, \end{cases}$$

and define

$$\mathrm{MV}_V^U := D(\mathfrak{C}, \Lambda_T[\zeta]/V, (\eta_a^U)_{a\in A}).$$

Analogously,

$$\Lambda_{\overline{T}}[\zeta]/V := \Lambda_{\overline{T}}[\zeta]/(\overline{\eta}_a \mid a \in V), \quad \overline{\eta}_a^U := \begin{cases} \overline{\eta}_a, & \text{if } a \notin U, \\ 1, & \text{if } a \in U, \end{cases}$$

and

$$\overline{\mathrm{MV}}_V^U := D(\mathfrak{C}, \Lambda_{\overline{T}}[\zeta]/V, (\overline{\eta}_a^U)_{a\in A}).$$

If necessary, we denote the differentials of MV_V^U and $\overline{\mathrm{MV}}_V^U$ by d_V^U and $\overline{d}_V^U$, respectively.

Remark 5.65. The ring $\Lambda_T[\zeta]/V$ is a quotient of $\Lambda_T[\zeta]$ over an ideal generated by linear forms, hence a Noetherian integral domain, which is required in the definition $D(\mathfrak{C}, S)$, and even further, a UFD. Moreover, MV_V^U is naturally a $\Lambda_T[\zeta]$-module, and free as $\Lambda_T[\zeta]/V$-module.
The corresponding statements hold for $\Lambda_{\overline{T}}[\zeta]/V$ and $\overline{\mathrm{MV}}_V^U$.

Example 5.66. We abbreviate

$$\mathrm{MV}_V := \mathrm{MV}_V^{\varnothing}, \qquad \mathrm{MV}^U := \mathrm{MV}_{\varnothing}^U \quad U, V \subseteq A,$$

and similarly for $\overline{\mathrm{MV}}$.
As extreme cases,

$$\mathrm{MV}^A = 0, \quad \mathrm{MV}_A = C^*(\mathcal{N}(\mathfrak{C}), \Lambda_T[\zeta]/A),$$

where $\mathcal{N}(\mathfrak{C})$ is the nerve of the cover $\mathfrak{M}_{\mathfrak{C}}$.

Remark 5.67. The module MV^U coincides with the Mayer–Vietoris complex

$$\mathrm{MV}(T, P_{\mathfrak{C}\setminus U}, \mathfrak{M}_{\mathfrak{C}\setminus U}),$$

where $\mathfrak{C} \setminus U := (C_i \setminus U)_{i \in I}$.
Alternatively, we can consider the submodule

$$Z_V^U := \bigoplus_{\underline{i} \in \mathcal{I}} (\Lambda_T[\zeta]/V) \cdot \eta_{C_{\underline{i}} \setminus U} \subseteq \mathrm{MV}_V,$$

to obtain the description

$$\mathrm{MV}_V^U = \mathrm{MV}_V / Z_V^U.$$

Note that

$$Z_V^U \subseteq \mathrm{Ann}(\eta_U, \mathrm{MV}_V).$$

Analogously, we define

$$\overline{Z}_V^U := \bigoplus_{\underline{i} \in \mathcal{I}} (\Lambda_{\overline{T}}[\zeta]/V) \cdot \overline{\eta}_{C_{\underline{i}} \setminus U} \subseteq \overline{\mathrm{MV}}_V,$$

and obtain

$$\overline{\mathrm{MV}}_V^U = \overline{\mathrm{MV}}_V / \overline{Z}_V^U, \quad \overline{Z}_V^U \subseteq \mathrm{Ann}(\overline{\eta}_U, \overline{\mathrm{MV}}_V).$$

In the following we will consider *semiregular sequences* which are sequences defined in terms of partitions of their prefixes.

Definition 5.68. Let $a_1, \ldots, a_n$ be a sequence in A. For $1 \leq j \leq n$, we say that subsets $U, V \subseteq A$ *partition the jth prefix* $a_1, \ldots, a_{j-1}$ of $a_1, \ldots, a_n$ if

$$U \sqcup V = \{a_1, \ldots, a_{j-1}\}.$$

We further write

$$Uj := U \cup \{a_j\},$$

and use the convention that $\{a_1, \ldots, a_{j-1}\} = \varnothing$ for $j = 1$.
The sequence $a_1, \ldots, a_n$ is called *square-free* if for all $1 \leq j \leq n$ and all $a \in A \setminus \{a_1, \ldots, a_j\}$, the differences

$$\eta_{a_j} - \eta_a, \qquad \overline{\eta}_{a_j} - \overline{\eta}_a,$$

are nonzero in $\Lambda_T[\zeta]/(\eta_{a_1}, \ldots, \eta_{a_{j-1}})$ and $\Lambda_{\overline{T}}[\zeta]/(\eta_{a_1}, \ldots, \eta_{a_{j-1}})$, respectively.

Remark 5.69. A square-free sequence $a_1, \dots, a_n$ in A has the following property: for every $1 \le j \le n$ and every subset $V \subseteq \{a_1, \dots, a_j\}$, divisibility in the UFD $\Lambda_T[\zeta]/V$ (cf. Remark 5.65) satisfies

$$\eta_{a_j} \mid \eta_{\underline{i}} \Rightarrow \eta_{\underline{i}} = 0 \text{ or } a_j \in C_{\underline{i}} \quad \text{for all } \underline{i} \in \mathcal{I}.$$

The same statement holds in $\Lambda_{\overline{T}}[\zeta]/V$.

Remark 5.70. Given $1 \le j \le n$, let U, V be a partition of the jth prefix of $a_1, \dots, a_n$. We have seen in Remark 5.67 that

$$\mathrm{MV}_V^U = \mathrm{MV}_V / Z_V^U,$$

and further

$$\mathrm{MV}_V^{Uj} = \mathrm{MV}_V / Z_V^{Uj} = \left(\mathrm{MV}_V / Z_V^U\right) / \left(Z_V^{Uj} / Z_V^U\right) = \mathrm{MV}_V^U / \left(Z_V^{Uj} / Z_V^U\right).$$

If the sequence $a_1, \dots, a_n$ is square-free, then we see from Remark 5.69 and the definition of Z_V^U that

$$Z_V^{Uj} / Z_V^U = \mathrm{Ann}(\eta_{a_j}, \mathrm{MV}_V^U).$$

Consequently, the following is a short exact sequence of differential graded modules

$$0 \longrightarrow \mathrm{MV}_V^{Uj} \xrightarrow{\cdot \eta_{a_j}} \mathrm{MV}_V^U \xrightarrow{p} \mathrm{MV}_{Vj}^U \longrightarrow 0$$

where p is the projection onto the quotient MV_{Vj}^U.

The corresponding long exact cohomology sequence is

$$\dots \longrightarrow H^{p-1}(\mathrm{MV}_{Vj}^U) \xrightarrow{\delta^{p-1}} H^p(\mathrm{MV}_V^{Uj}) \xrightarrow{\cdot \eta_{a_j}} H^p(\mathrm{MV}_V^U) \xrightarrow{p} H^p(\mathrm{MV}_{Vj}^U) \xrightarrow{\delta^p} H^{p+1}(\mathrm{MV}_V^{Uj}) \longrightarrow \dots$$

and the connecting morphism is given by

$$\delta^p \colon H^p(\mathrm{MV}_{Vj}^U) \longrightarrow H^{p+1}(\mathrm{MV}_V^{Uj}), \qquad [\![x]\!] \longmapsto \left[\!\!\left[\frac{d_V^U(x)}{\eta_{a_j}} \right]\!\!\right], \quad x \in \ker d_{V_j}^U.$$

Since $x \in \ker d_{V_j}^U$ implies that $d_V^U(x) \in \eta_{a_j} \mathrm{MV}_V^U$, the notation $d_V^U(x)/\eta_{a_j}$ is justified.

The corresponding constructions and long exact sequence can be made for $\overline{\mathrm{MV}}_V^U$ as well.

Since $\overline{\mathrm{MV}} = \mathrm{MV} / \mathcal{R}\,\mathrm{MV}$, there is a natural surjective projection

$$\pi \colon \mathrm{MV} \longrightarrow \overline{\mathrm{MV}},$$

whose induced map in cohomology coincides with the restriction of tori $\hat{\theta}$. More generally, for subsets $U, V \subseteq A$ we obtain projections

$$\pi_V^U \colon \mathrm{MV}_V^U \longrightarrow \overline{\mathrm{MV}}_V^U,$$

which induce morphisms

$$\tilde{\pi}_V^U \colon H(\mathrm{MV}_V^U) \longrightarrow H(\overline{\mathrm{MV}}_V^U).$$

We denote by K_V^U the kernel of $\tilde{\pi}_V^U$. Explicitly,

$$K_V^U = \left(\ker d_V^U \cap \mathcal{R}\,\mathrm{MV}_V^U + \operatorname{im} d_V^U\right) / \operatorname{im} d_V^U.$$

Definition 5.71. We call a square-free sequence $a_1, \dots, a_n$ in A *semi-regular* if for every $1 \le j \le n$ and for every partition U, V of the jth prefix, the connecting morphism δ of Remark 5.70 satisfies

$$\delta^{-1}(K_V^{Uj}) \subseteq K_{Vj}^U + p(H(\mathrm{MV}_V^U)).$$

Example 5.72. In the setting of Definition 5.71, suppose that

$$H^p(\mathrm{MV}_V^{Uj}) = 0, \quad p > 0.$$

Then, by construction

$$\delta^{-1}(K_V^{Uj}) \subseteq \ker\delta = p(H(\mathrm{MV}_V^U)).$$

If moreover

$$\mathrm{MV}_{Vj}^U = C^*(\mathcal{I}, \Lambda_T[\zeta]/Vj),$$

i.e., it coincides with the cochain complex of $\mathcal{I}$ with coefficients in $\Lambda_T[\zeta]/Vj$, then

$$H^p(\mathrm{MV}_{Vj}^U) \cong \begin{cases} \Lambda_T[\zeta]/Vj, & \text{if } p = 0, \\ 0, & \text{if } p > 0, \end{cases}$$

and consequently

$$p(H(\mathrm{MV}_V^U)) = H(\mathrm{MV}_{Vj}^U).$$

Theorem 5.73. *Suppose $a_1, \dots, a_n$ is a semi-regular sequence in A and that*

$$\tilde{\pi}_V^U \colon H(\mathrm{MV}_V^U) \longrightarrow H(\overline{\mathrm{MV}}_V^U), \quad [\![x]\!] \longmapsto [\![x]\!],$$

is surjective for all partitions U, V of $a_1, \dots, a_n$. Then the restriction θ and extension Θ of tori are surjective.

Proof. Let $1 \le j \le n-1$ and let U, V be a partition of the jth prefix of $a_1, \dots, a_n$. The long exact sequences from Remark 5.70 for MV and $\overline{\mathrm{MV}}$ fit into a morphism of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{p-1}(\mathrm{MV}_{Vj}^U) & \xrightarrow{\delta^{p-1}} & H^p(\mathrm{MV}_V^{Uj}) & \xrightarrow{\cdot\eta_{a_j}} & H^p(\mathrm{MV}_V^U) & \xrightarrow{p} & H^p(\mathrm{MV}_{Vj}^U) & \xrightarrow{\delta^p} & H^{p+1}(\mathrm{MV}_V^{Uj}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \tilde{\pi}_{Vj}^U} & & \downarrow{\scriptstyle \tilde{\pi}_V^{Uj}} & & \downarrow{\scriptstyle \tilde{\pi}_V^U} & & \downarrow{\scriptstyle \tilde{\pi}_{Vj}^U} & & \downarrow{\scriptstyle \tilde{\pi}_V^{Uj}} & \\
\cdots \longrightarrow & H^{p-1}(\overline{\mathrm{MV}}_{Vj}^U) & \xrightarrow{\overline{\delta}^{p-1}} & H^p(\overline{\mathrm{MV}}_V^{Uj}) & \xrightarrow{\cdot\overline{\eta}_j} & H^p(\overline{\mathrm{MV}}_V^U) & \xrightarrow{\overline{p}} & H^p(\overline{\mathrm{MV}}_{Vj}^U) & \xrightarrow{\overline{\delta}^p} & H^{p+1}(\overline{\mathrm{MV}}_V^{Uj}) & \longrightarrow \cdots
\end{array}$$

By the definition of p,

$$p^{-1}(K_V^{Uj}) = K_V^U + \ker p,$$

and by assumption,

$$\delta^{-1}(K_V^{Uj}) \subseteq K_{Vj}^U + p(H(\mathrm{MV}_V^U)).$$

Therefore the sequence

$$H^p(\mathrm{MV}_V^{Uj})/(K_V^{Uj})^p \xrightarrow{\cdot\eta_{a_j}} H^p(\mathrm{MV}_V^U)/(K_V^U)^p \xrightarrow{p} H^p(\mathrm{MV}_{Vj}^U)/(K_{Vj}^U)^p \xrightarrow{\delta^p} H^{p+1}(\mathrm{MV}_V^{Uj})/(K_V^{Uj})^{p+1}$$

is exact. We thus obtain a commutative diagram

$$\begin{array}{ccccccc}
H^p(\mathrm{MV}_V^{Uj})/(K_V^{Uj})^p & \xrightarrow{\cdot\eta_{a_j}} & H^p(\mathrm{MV}_V^U)/(K_V^U)^p & \xrightarrow{p} & H^p(\mathrm{MV}_{Vj}^U)/(K_{Vj}^U)^p & \xrightarrow{\delta^p} & H^{p+1}(\mathrm{MV}_V^{Uj})/(K_V^{Uj})^{p+1} \\
\downarrow{\scriptstyle \tilde{\pi}_V^{Uj}} & & \downarrow{\scriptstyle \tilde{\pi}_V^U} & & \downarrow{\scriptstyle \tilde{\pi}_{Vj}^U} & & \downarrow{\scriptstyle \tilde{\pi}_V^{Uj}} \\
H^p(\overline{\mathrm{MV}}_V^{Uj}) & \xrightarrow{\cdot\overline{\eta}_j} & H^p(\overline{\mathrm{MV}}_V^U) & \xrightarrow{\overline{p}} & H^p(\overline{\mathrm{MV}}_{Vj}^U) & \xrightarrow{\overline{\delta}^p} & H^{p+1}(\overline{\mathrm{MV}}_V^{Uj})
\end{array}$$

with exact rows and injective vertical arrows. If $\tilde{\pi}^U_{Vj}$ and $\tilde{\pi}^{Uj}_V$ are surjective, then so is $\tilde{\pi}^U_V$ by the four-lemma.

Since surjectivity holds by assumption for all partitions of the full set $\{a_1, \dots, a_n\}$ it follows step by step for all smaller prefixes as well. Consequently, all morphisms $\tilde{\pi}^U_V$ are surjective for every $1 \leq j \leq n$ and every partition U, V of the jth prefix. In particular, since $\tilde{\pi}^{\varnothing}_{\varnothing} = \hat{\theta}$ is surjective, Remark 4.36 implies that θ itself, and hence also Θ is surjective. □

Example 5.74. Consider first the case

$$\mathrm{MV}^U_V = C^*(\mathcal{I}, \Lambda_T[\zeta]/V),$$

i.e., it is the cochain complex of $\mathcal{I}$ with coefficients in $\Lambda_T[\zeta]/V$. Then

$$\overline{\mathrm{MV}}^U_V = C^*(\mathcal{I}, \Lambda_{\overline{T}}[\zeta]/V),$$

and

$$\tilde{\pi}^U_V \colon H(\mathrm{MV}^U_V) \cong \Lambda_T[\zeta]/V \longrightarrow H(\overline{\mathrm{MV}}^U_V) \cong \Lambda_{\overline{T}}[\zeta]/V,$$

is surjective.

Similarly, if

$$\mathrm{MV}^U_V = D((C_1, C_2), \Lambda_T[\zeta]/V, (\eta^U_a)_{a\in A}),$$

is equal to a differential graded module $D(\mathfrak{C}, \Lambda_T[\zeta]/V)$ defined for a two-element cover, then also

$$\overline{\mathrm{MV}}^U_V = D((C_1, C_2), \Lambda_{\overline{T}}[\zeta]/V, (\overline{\eta}^U_a)_{a\in A}),$$

and since

$$H(\mathrm{MV}^U_V) \cong \{(f,g) \in (\Lambda_T[\zeta]/V)/(\eta_{C_1}) \oplus (\Lambda_T[\zeta]/V)/(\eta_{C_2}) \mid f - g \in (\eta_{C_1\cap C_2})\},$$

$$H(\overline{\mathrm{MV}}^U_V) \cong \{(f,g) \in (\Lambda_{\overline{T}}[\zeta]/V)/(\overline{\eta}_{C_1}) \oplus (\Lambda_{\overline{T}}[\zeta]/V)/(\overline{\eta}_{C_2}) \mid f - g \in (\overline{\eta}_{C_1\cap C_2})\},$$

we obtain again that

$$\tilde{\pi}^U_V \colon H(\mathrm{MV}^U_V) \longrightarrow H(\overline{\mathrm{MV}}^U_V),$$

is surjective (cf. Example 5.19).

We provide a more intrinsic characterization of semiregular sequences by considering annihilators inside the quotient modules $\mathrm{MV}^U_V / \operatorname{im} d^U_V$. Define

$$\mathcal{R}^U_V := \frac{\mathcal{R}\,\mathrm{MV}^U_V + \operatorname{im} d^U_V}{\operatorname{im} d^U_V},$$

so that

$$K^U_V = \frac{\ker d^U_V \cap \mathcal{R}\,\mathrm{MV}^U_V + \operatorname{im} d^U_V}{\operatorname{im} d^U_V} = \mathcal{R}^U_V \cap H(\mathrm{MV}^U_V).$$

Since $\mathcal{R}\,\mathrm{MV}^U_V$ is a differential graded submodule of MV^U_V, the inclusion induces a canonical morphism

$$H(\mathcal{R}\,\mathrm{MV}^U_V) \longrightarrow H(\mathrm{MV}^U_V) \cap \mathcal{R}^U_V.$$

Lemma 5.75. *Let $a_1, \dots, a_n$ be a square-free sequence in A. If for all $1 \le j \le n$ and all partitions U, V of the jth prefix*

$$\operatorname{Ann}(\eta_{a_j}, H(\mathrm{MV}_V^U) \cap \mathcal{R}_V^U) \cong \operatorname{Ann}(\eta_{a_j}, H(\mathcal{R}\,\mathrm{MV}_V^U)),$$

then $a_1, \dots, a_n$ is a semiregular sequence.

Proof. Let $x \in \mathrm{MV}_V^U$ with

$$\delta([\![x]\!]) = \left[\!\!\left[\frac{d_V^U(x)}{\eta_{a_j}}\right]\!\!\right] \in K_V^{Uj} = \mathcal{R}_V^{Uj} \cap H(\mathrm{MV}_V^{Uj}).$$

Equivalently, there exist $r \in \mathcal{R}\,\mathrm{MV}_V^U \cap \ker d_V^U$, $y \in \mathrm{MV}_V^U$, and coefficients $r_{\underline{i}} \in \Lambda_T[\zeta]/V$ for $\underline{i} \in \mathcal{I}$ such that

$$\frac{d_V^U(x)}{\eta_{a_j}} = r + d_V^U(y) + \sum_{\underline{i} \in \mathcal{I}} r_{\underline{i}} \eta_{C_{\underline{i}} \setminus \{a_j\}} e_{\underline{i}},$$

where $e_{\underline{i}}$ denote the standard basic vectors of MV_V^U. Then

$$d_V^U(x - \eta_{a_j} y) = \eta_{a_j} r, \quad \text{in } \mathrm{MV}_V^U,$$

and therefore

$$[\![r]\!] \in \operatorname{Ann}(\eta_{a_j}, H(\mathrm{MV}_V^U)) \cap \mathcal{R}_V^U.$$

By the assumption, there exists $z \in \mathcal{R}\,\mathrm{MV}_V^U$ such that

$$d_V^U(x - \eta_{a_j} y) = \eta_{a_j} r = d_V^U(z).$$

Consequently,

$$x = \eta_{a_j} y + z + k,$$

for some $k \in \ker d_V^U$. Passing to $H(\mathrm{MV}_{Vj}^U)$, where $\ker d_V^U = p(H(\mathrm{MV}_V^U))$, we obtain

$$[\![x]\!] = [\![z + k]\!],$$

Thus $[\![x]\!]$ is of the required form, and the claim follows. □

Remark 5.76. We record a separate situation in which surjectivity of θ can be deduced recursively. We consider the union of a projective union $P_{\mathfrak{C}}$ with a projective space $P_B \subseteq \mathbb{P}$ in the case where their intersection is again a projective space, i.e. $P_{\mathfrak{C}} \cap P_B = P_{B'}$, $B' \subseteq B \subseteq A$.
The restriction of tori $\hat{\theta}$ is induced in cohomology from the morphism of Mayer–Vietoris complexes

$$\begin{array}{ccc} H_T^*(P_{\mathfrak{C}}) \oplus H_T^*(P_B) & \xrightarrow{r_{\mathfrak{C}} - r_B} & H_T^*(P_{B'}) \\ \downarrow{\scriptstyle (\theta_{\mathfrak{C}}, \theta_B)} & & \downarrow{\scriptstyle \theta_{B'}} \\ H_{\overline{T}}^*(P_{\mathfrak{C}}) \oplus H_{\overline{T}}^*(P_B) & \xrightarrow{\overline{r}_{\mathfrak{C}} - \overline{r}_B} & H_{\overline{T}}^*(P_{B'}) \end{array}$$

where

$$r_{\mathfrak{C}} \colon H_T^*(P_{\mathfrak{C}}) \to H_T^*(P_{B'}), \qquad r_B \colon H_T^*(P_B) \to H_T^*(P_{B'}),$$
$$\overline{r}_{\mathfrak{C}} \colon H_{\overline{T}}^*(P_{\mathfrak{C}}) \to H_{\overline{T}}^*(P_{B'}), \qquad \overline{r}_B \colon H_{\overline{T}}^*(P_B) \to H_{\overline{T}}^*(P_{B'}),$$

and $\theta_{\mathfrak{C}}, \theta_B, \theta_{B'}$ are the respective restrictions of tori.
Writing $H_T^*(P_B) = \Lambda_T[\zeta]/(\eta_B)$ and $H_T^*(P_{B'}) = \Lambda_T[\zeta]/(\eta_{B'})$, the kernels of the restrictions of tori are

$$\ker\theta_B = \mathcal{R}\Lambda_T[\zeta]/\mathcal{R}(\eta_B), \qquad \ker\theta_{B'} = \mathcal{R}\Lambda_T[\zeta]/\mathcal{R}(\eta_{B'}),$$

and hence

$$r_B^{-1}(\ker\theta_{B'}) \subseteq \ker\theta_B.$$

By Example 4.34 we know that θ_B and $\theta_{B'}$ are surjective. Since moreover r_B is surjective, we can argue as in Example 3.49 to conclude the following: if $\theta_{\mathfrak{C}}$ is surjective, then also

$$\theta\colon H_T^*(P_{\mathfrak{C}} \cup P_B) \longrightarrow H_{\overline{T}}^*(P_{\mathfrak{C}} \cup P_B)$$

is surjective.

Example 5.77. We now give an example where the restriction of tori

$$\hat{\theta}\colon G(H_T^*(P_{\mathfrak{C}})) \longrightarrow G(H_{\overline{T}}^*(P_{\mathfrak{C}}))$$

is not surjective.

Let $A = I(2,4)$, the set of strictly increasing pairs in $\{1,\ldots,4\}$. Besides the generic action of

$$T \cong (\mathbb{C}^\times)^{I(2,4)}$$

via its canonical characters $\mu_{(i,j)}$, we consider the action of

$$\overline{T} \cong (\mathbb{C}^\times)^4$$

on $\mathbb{P} = \mathbb{P}(\mathbb{C}^{I(2,4)})$ given by

$$\chi_{(i,j)} = e_i + e_j,$$

where $e_1,\ldots,e_4$ are the standard characters of $\overline{T}$. We will refer to this as the Grassmannian action of $\overline{T}$ in Chapter 6.2.

Let $P_{\mathfrak{C}}$ be the projective union defined by the collection

$$C_1 = \{(1,2),(1,3),(1,4)\}, \quad C_2 = \{(1,2),(2,3),(2,4)\},$$
$$C_3 = \{(1,3),(2,3),(3,4)\}, \quad C_4 = \{(1,4),(2,4),(3,4)\}.$$

In the Mayer–Vietoris complex $\overline{\mathrm{MV}}$ one finds that

$$(0,\ (e_2-e_1)(\zeta+e_1+e_2),\ (e_3-e_1)(\zeta+e_1+e_3),\ (e_4-e_1)(\zeta+e_1+e_4))$$

lies in $\ker\overline{d}^0$ and thus represents a nontrivial element of $H^0(\overline{\mathrm{MV}})$.

However, this class does not lie in the image of

$$\hat{\theta}\colon H^0(\mathrm{MV}) \longrightarrow H^0(\overline{\mathrm{MV}}),$$

and therefore $\hat{\theta}$ is not surjective in this case.

5.6 Projective Unions from Posets

There are many ways to define projective unions from combinatorial objects. In this chapter, once again motivated by the degenerated Grassmannian, we focus on one such construction, namely *poset unions*. Any projective union

$$P_{\mathfrak{C}} \subseteq \mathbb{P}(\mathbb{C}^A),$$

is fully determined (apart from its torus action) by the set A and the collection of subsets $\mathfrak{C}$. If A carries the structure of a poset, then a poset union is obtained by taking $\mathfrak{C}$ to be the collection of maximal chains in A, viewed as subsets.

After briefly recalling the definition of poset cohomology from [Wac07], we study the equivariant cohomology of such projective unions under a generic torus action. By considering the nerve of a suitable cover of the simplicial complexes Δ_S from Lemma 5.52, we obtain a dual description in terms of poset cohomology, formulated in Lemma 5.82. The remaining discussion serves as preparation for the poset-based description of the degenerated Grassmannian in Chapter 6.1.

Definition 5.78. A poset is a set P together with a partial order $\leq$. For $a, b \in P$, we say that b *covers* a if $a \leq b$ and there exists no $c \in P \setminus \{a, b\}$ with $a \leq c \leq b$. An element $x \in P$ is called the *least* (resp. *greatest*) element if $x \leq a$ (resp. $a \leq x$) holds for all $a \in P$.

Definition 5.79. Let A be a finite poset and $\mathfrak{C}$ the collection of its maximal chains, regarded as subsets of A. The associated projective union $P_{\mathfrak{C}} \subseteq \mathbb{P}(\mathbb{C}^A)$ is called a *poset union* and will be denoted by P_A.

Definition 5.80. The *order complex* of a poset P is the simplicial complex $\Delta(P)$ on P whose p-simplices are the p-chains in P, i.e., the totally ordered $(p+1)$-element subsets of P. The cohomology of P, denoted $H^*(P)$, is defined as the simplicial cohomology of $\Delta(P)$.

Any subset $A \subseteq P$ is again a poset under the restricted partial order, and its order complex $\Delta(A)$ is naturally included in $\Delta(P)$. This inclusion induces a restriction map $H^*(P) \to H^*(A)$.

Remark 5.81. In analogy with Remark 2.4, if $A, B \subseteq P$ are subposets, then the cohomology of $A \cup B$ is computed by the long exact sequence

$$\dots \longrightarrow H^p(A \cup B) \longrightarrow H^p(A) \oplus H^p(B) \longrightarrow H^p(A \cap B) \longrightarrow H^{p+1}(A \cup B) \longrightarrow \dots$$

Let A be a poset and suppose a torus T acts generically on P_A. By definition we then have $\overline{\mathfrak{C}} = A$. We continue to use the general notation of Chapter 5.4. In particular, we write $O = A \sqcup K$, assume that $\{\chi_a \mid a \in O\}$ forms a basis of M and regard the polynomial ring $\Lambda_T[\zeta]$ as polynomial ring in the variables

$$\zeta, \quad \eta_a = \zeta + \chi_a \quad (a \in O).$$

Lemma 5.82. *Assume that T acts generically on P_A. Then*

$$G(H_T^*(P_A)) \cong \Lambda_K[\zeta] \otimes \left(\bigoplus_{S \subseteq A} \Lambda_A^S \otimes H^*(S) \right).$$

Proof. By Lemma 5.52 we have

$$G(H_T^*(P_A)) \cong \Lambda_K[\zeta] \otimes H(\mathrm{GE}) = \Lambda_K[\zeta] \otimes \left(\bigoplus_{S \subseteq A} \Lambda_S \otimes H^*(\Delta_S) \right).$$

On the other hand,

$$\Delta_S = \bigcup_{a \in A \setminus S} \Delta_a,$$

where Δ_a denotes the simplex spanned by all chains $C \subseteq A$ with $a \in C$. By the Nerve Theorem (Theorem 2.6), the complex Δ_S is homotopy equivalent to the nerve of this covering, which is precisely the order complex of the subposet $A \setminus S$. Hence

$$H^*(\Delta_S) \cong H^*(A \setminus S),$$

and since taking the sum over all subsets of A is the same as taking the sum over all complements in A, the claim follows. □

From now on, we assume that R is a field, so that the cohomology of any simplicial complex with coefficients in R is a vector space.

For each subposet $S \subseteq A$, fix an index set J_S and a basis $\{g_S^i \mid i \in J_S\}$ of H^S. Then, by Lemma 5.82, every element $\underline{f} \in G(H_T^*(P_A))$ can be written uniquely in the form

$$\underline{f} = \sum_{S \subseteq A} \sum_{i \in J_S} \gamma_{S,i} g_S^i,$$

with coefficients $\gamma_{S,i} \in \Lambda_O^S[\zeta]$.

We now describe explicitly the $\Lambda_T[\zeta]$-module structure in this representation. Let $Z \subseteq S \subseteq A$ be subposets. Since $\Delta_Z \subseteq \Delta_S$, there is an induced restriction map

$$r_Z^S \colon H^*(S) \to H^*(Z).$$

In particular, for every $i \in J_S$ we have coefficients $a_{S,Z}^{i,j} \in R$ such that

$$r_Z^S(g_S^i) = \sum_{j \in J_Z} a_{S,Z}^{i,j} g_Z^j.$$

With this, multiplication by the generators of $\Lambda_T[\zeta]$ acts as follows:

$$\eta_a \cdot g_S^i = \begin{cases} \eta_a g_S^i, & a \in O \setminus S, \\ \sum_{j \in J_{S \setminus \{a\}}} a_{S,S\setminus\{a\}}^{i,j} g_{S\setminus\{a\}}^j, & a \in S, \end{cases} \qquad \zeta \cdot g_S^i = \zeta g_S^i.$$

Definition 5.83. Let $n_1, n_2 \in \mathbb{Z}_{>0}$. A *partial permutation matrix* is a matrix $M \in R^{n_1 \times n_2}$ with entries in $\{0,1\}$ such that in each row and each column there is at most one entry equal to 1.

Let X_1 and X_2 be finite-dimensional vector spaces over R, and fix bases of X_1 and X_2. An R-linear map

$$\varphi \colon X_1 \longrightarrow X_2,$$

is called a *partial permutation (with respect to the chosen bases)* if its matrix representation relative to these bases is a partial permutation matrix.

Remark 5.84. With fixed bases $\{e_j^1 \mid j \in J_1\}$ and $\{e_j^2 \mid j \in J_2\}$, a partial permutation $\varphi\colon X_1 \to X_2$ induces a bijection

$$\hat{\varphi}\colon D_1 \longrightarrow D_2$$

between subsets $D_1 \subseteq J_1$ and $D_2 \subseteq J_2$, determined by

$$\varphi(e_j^1) = \begin{cases} e^2_{\hat{\varphi}(j)}, & \text{if } j \in D_1, \\ 0, & \text{if } j \in J_1 \setminus D_1, \end{cases} \qquad j \in J_1.$$

In the following, we will also write φ for the induced map between sets, and use the shorthand

$$e^1_{\varphi(j)} \coloneqq \varphi(e_j^1), \quad j \in J_1.$$

Definition 5.85. Let $Z \subseteq S \subseteq A$ be subposets and fix bases

$$\{g_S^i, i \in J_S\}, \qquad \{g_Z^j, j \in J_Z\},$$

of $H^*(S)$ and $H^*(Z)$, respectively. We say that S *restricts simply to* Z with respect to these bases if the restriction

$$r_Z^S\colon H^*(S) \longrightarrow H^*(Z),$$

is a partial permutation.

Remark 5.86. Any linear map between finite-dimensional vector spaces becomes a partial permutation after a suitable choice of bases. Consequently, for subposets $Z \subseteq S \subseteq A$, it is always possible to choose suitable bases of $H^*(S)$ and $H^*(Z)$ such that S restricts simply to Z.

Example 5.87. Let $a \in S$ and $r\colon H^*(S) \to H^*(S \setminus \{a\})$ the restriction. If bases are chosen so that S restricts simply to $S \setminus \{a\}$, then

$$\eta_a \cdot g_S^i = g^{r(i)}_{S\setminus\{a\}},$$

where $r(i)$ denotes the image of i under the induced partial permutation on indices (as in Remark 5.84).

Remark 5.88. Let B be a subposet of A. As in Remark 5.54, we consider a basis $\{\chi_a \mid a \in O\}$ of M, indexed by

$$O = B \sqcup (A \setminus B) \sqcup K.$$

The poset union P_B is contained in P_A, and we may express the restriction map described from Remark 5.54 in terms of poset cohomology.

For $S \subseteq A$, we have an inclusion

$$\Delta^B_{B\setminus S} \hookrightarrow \Delta^A_{A\setminus S},$$

which induces a cochain map

$$C^*(\Delta^A_{A\setminus S}) \longrightarrow C^*(\Delta^B_{B\setminus S}),$$

and hence a restriction in poset cohomology

$$\hat{r}_S\colon H^*(S) \longrightarrow H^*(S \cap B).$$

Define

$$r_S\colon \Lambda_{A\setminus S} \otimes H^*(S) \longrightarrow \Lambda_{A\setminus B} \otimes \Lambda_{B\setminus S} \otimes H^*(S \cap B), \quad r_S(1 \otimes g) = \eta_{A\setminus(B\cup S)} \otimes 1 \otimes \hat{r}_S(g).$$

Then the global restriction

$$G(H_T^*(P_A)) \longrightarrow G(H_T^*(P_B)),$$

is expressed as the collection of the maps r_S, i.e.

$$r\colon \Lambda_K[\zeta] \otimes \left(\bigoplus_{S \subseteq A} \Lambda_A^S \otimes H^*(S) \right) \longrightarrow \Lambda_{K \cup A \setminus B}[\zeta] \otimes \left(\bigoplus_{S \subseteq B} \Lambda_B^S \otimes H^*(S \cap B) \right).$$

In particular, a basis element g_S^i in the poset description of P_A restricts as

$$r(g_S^i) = \sum_{j \in J_{S \cap B}} a_{S, S \cap B}^{i,j} \eta_{A \setminus (B \cup S)} g_{S \cap B}^j,$$

with $\eta_{A \setminus (B \cup S)} \in \Lambda_{K \cup A \setminus B}[\zeta] \otimes \Lambda_B^S$, as required.

6 Degenerated Grassmannians

The method of *Seshadri stratification* was introduced in [CFL23] as a generalization of Newton–Okounkov theory. Unlike the latter, which starts with a flag of subvarieties, a Seshadri stratification is based on a web of subvarieties together with a collection of homogeneous functions, both indexed by the same finite poset. This structural difference shapes the comparison in several respects: Newton–Okounkov theory yields a valuation on the coordinate ring of the embedded variety, while a Seshadri stratification gives a quasi-valuation. The former defines a monoid and leads to a toric degeneration, whereas the latter defines a fan of monoids and produces a semi-toric degeneration.

The simplest special fibers of semi-toric degenerations obtained from Seshadri stratifications arise when the stratification is of *Hodge type*, in which case each component is a weighted projective space corresponding to a maximal chain in the indexing poset ([CFL23, Chapter 12 and Chapter 16.1]). For Grassmannians, the natural choice of Schubert varieties as strata and Plücker coordinates as extremal functions provides a favorable example of such a stratification.

In this framework, the degenerated Grassmannian $P_{d,n}$ is precisely the special fiber of the semi-toric degeneration of the Grassmannian $\mathrm{Gr}(d,n)$ defined by its Hodge-type Seshadri stratification, inheriting the standard torus action as well as the structural richness of the Grassmannian. It was the motivating example for our study of projective unions and their torus-equivariant cohomology, and will now be treated as a special case in the final chapter of the thesis.

We define the degenerated Grassmannian $P_{d,n}$ as a poset union (Definition 6.1) and use the associated poset structure, under a generic torus action, in Chapter 6.1 to describe the image of the restriction map $P_{2,n} \to P_{2,n+1}$ between degenerated Grassmannians. Chapter 6.2 then considers the torus action induced by $\mathrm{Gr}(d,n)$ and formulates a combinatorial description of the first-column component $\nu(P_{d,n})$, which is the torsion-free, and therefore relevant part of cohomology for comparison with that of $\mathrm{Gr}(d,n)$.

Since the ordinary and the degenerated Grassmannian share the same combinatorial basis, the chapter begins with a brief introduction to the combinatorial data of the former. This both fixes notation for the study of the degenerated Grassmannian and provides context for a comparison of cohomologies at the end of Chapter 6.2.
For a detailed discussion of the ordinary Grassmannian we refer the reader to [Ful97].

For this section we assume that R is a field of characteristic zero.

The complex Grassmannian $\mathrm{Gr}(d,n)$ was introduced in Example 2.31 as the variety of d-dimensional subspaces of a complex vector space V of dimension n. It embeds into the projectivization $\mathbb{P}(\bigwedge^d V)$ of the d-th exterior power of V via the Plücker embedding, and thus can be realized as a smooth complex projective variety of dimension $d(n-d)$.
From both geometric and combinatorial perspectives, Schubert varieties form the fundamental building blocks in the structural study of the Grassmannian. There are several equivalent ways to index these varieties: by increasing tuples, by Young tableaux, or via certain words in the Coxeter group of type A_{n-1}. We restrict our attention to increasing tuples and Young diagrams, which are concrete and sufficient for treating degenerated Grassmannians in the Mayer–Vietoris context.

Let $I(d,n)$ denote the set of strictly increasing d-tuples with entries between 1 and n, i.e.,

$$I = (i_1, \ldots, i_d), \quad \text{with } i_1 < \ldots < i_d, \quad \text{and } i_j \in \{1, \ldots, n\}.$$

We define the *weight* of a d-tuple $I = (i_1, \ldots, i_d)$ to be equal to

$$\operatorname{wt}(I) \coloneqq \sum_{j=1}^{d} (i_j - j),$$

and equip $I(d,n)$ with a poset structure given by

$$I = (i_1, \ldots, i_d) \leq J = (i'_1, \ldots, i'_d) \text{ if and only if } i_j \leq i'_j, \text{ for all } 1 \leq j \leq d.$$

To introduce Schubert varieties, fix a complete flag

$$F_\bullet \colon \{0\} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = V,$$

$$\dim F_i = i, \quad i = 1, \ldots, n.$$

Then the Schubert cell associated to $I = (i_1, \ldots, i_d)$ is

$$\Sigma_I^\circ \coloneqq \{V \in \operatorname{Gr}(d,n) \mid \dim(V \cap F_{i_j}) = j \text{ for all } j, \text{ and } \dim(V \cap F_{i_j - 1}) = j - 1\},$$

and each Σ_I° is locally closed and isomorphic to the affine space $\mathbb{A}^{\operatorname{wt}(I)}$. The Schubert variety Σ_I is defined as the Zariski closure of Σ_I°:

$$\Sigma_I \coloneqq \overline{\Sigma_I^\circ}, \quad I \in I(d,n),$$

and the ordering of the poset $I(d,n)$ coincides with the Bruhat order on Schubert varieties in $\operatorname{Gr}(d,n)$, i.e.,

$$\Sigma_I \subseteq \Sigma_J, \text{ if and only if } I \leq J.$$

There is a bijection between the set of strictly increasing d-tuples $I(d,n)$ and the set of Young diagrams $\mathcal{Y}(d,n)$ fitting inside a box $d \times (n-d)$, given by

$$(i_1, \ldots, i_d) \longmapsto \text{diagram with } i_j - j \text{ boxes in row number } d - j + 1.$$

We define the weight of a diagram to be the number of its boxes and equip $\mathcal{Y}(d,n)$ with a partial order

$$D \leq D', \quad \text{if and only if} \quad D \text{ fits inside } D'.$$

With these definitions, the bijection between $I(d,n)$ and $\mathcal{Y}(d,n)$ preserves both the order and the weight, and we will use both indexing methods interchangeably as needed.

A natural approach to computing the cohomology of $\operatorname{Gr}(d,n)$ is to utilize its cellular decomposition into Schubert cells. Setting

$$X_\ell \coloneqq \bigcup_{\operatorname{wt}(I) \leq \ell} \Sigma_I,$$

we obtain a filtration

$$\varnothing \subseteq X_0 \subseteq \ldots \subseteq X_{d(n-d)} = \operatorname{Gr}(d,n),$$

by unions of affine cells, and by the classical cellular decomposition statement ([AF24, Proposition A.3.4]), it follows that $H^*(\operatorname{Gr}(d,n))$ is freely generated as an R-vector space by the Schubert classes, that is, the cohomology classes of the Schubert varieties,

$$[\Sigma_I] \in H^{2\operatorname{wt}(I)}(\operatorname{Gr}(d,n)).$$

In the equivariant setting, suppose a Lie group G acts on $\mathrm{Gr}(d,n)$ and leaves each Σ_I invariant. Then the equivariant analogue of the above statement holds ([AF24, Proposition 4.7.1]): the ring $H^*_G(\mathrm{Gr}(d,n))$ is freely generated over Λ_G by the equivariant Schubert classes

$$[\Sigma_I]^G \in H^{2\,\mathrm{wt}(I)}_G(\mathrm{Gr}(d,n)),$$

called the Schubert cycles. We denote these by σ_I in both ordinary and equivariant cohomology. The ring structure of $H^*_G(\mathrm{Gr}(d,n))$ is determined by the structure constants $c_{IJ}^K \in \Lambda_G$ appearing in

$$\sigma_I \cdot \sigma_J = \sum_{K \in I(d,n)} c_{IJ}^K \sigma_K.$$

Since the beginnings of Schubert calculus, obtaining a combinatorial description of these structure constants has been a central objective. Such formulas are commonly referred to as *Littlewood–Richardson rules*, and a selection of the latter can be found in [AF24, Chapter 9.8].

The ambient space for both $\mathrm{Gr}(d,n)$ and its degeneration is

$$\mathbb{P}_{d,n} := \mathbb{P}(\bigwedge^d V) \cong \mathbb{P}(\mathbb{C}^{I(d,n)}).$$

As in Chapter 5, we take equivariant cohomology with respect to a torus T acting diagonally on $\mathbb{C}^{I(d,n)}$ via characters χ_I for $I \in I(d,n)$.

Definition 6.1. The *degenerated Grassmannian* $P_{d,n}$ is defined as the poset union associated with the poset $I(d,n)$, i.e., as the projective union determined by the collection of subsets

$$\mathfrak{C}(d,n) := \{C \mid C \text{ is the set of a maximal chain in } I(d,n)\},$$

of $I(d,n)$.

Remark 6.2. As announced in the introduction, we will use the posets $I(d,n)$ and $\mathcal{Y}(d,n)$ interchangeably, since they are isomorphic. In particular, $P_{d,n}$ can be viewed as the poset union with respect to either $I(d,n)$ or $\mathcal{Y}(d,n)$, and $\mathfrak{C}(d,n)$ denotes the collection of maximal chains in these equivalent descriptions.

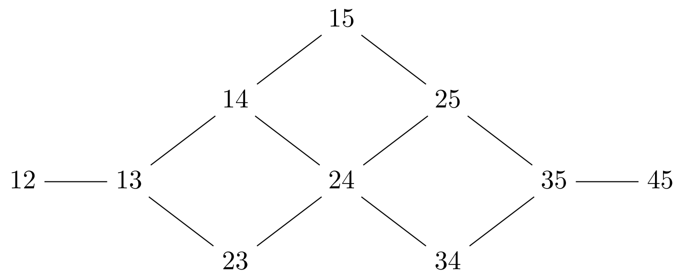
Example 6.3. The running example for this section will be the degenerated Grassmannian $P_{2,5}$. For an element $I = (i_1, i_2) \in I(2,5)$ we use the abbreviation

$$i_1 i_2 := (i_1, i_2).$$

The five maximal chains $C_0, \dots, C_4$ of $I(2,5)$ can be represented by their underlying sets of elements:

$$C_0 = \{12, 13, 14, 15, 25, 35, 45\}, \quad C_1 = \{12, 13, 14, 24, 25, 35, 45\}, \quad C_2 = \{12, 13, 14, 24, 34, 35, 45\},$$

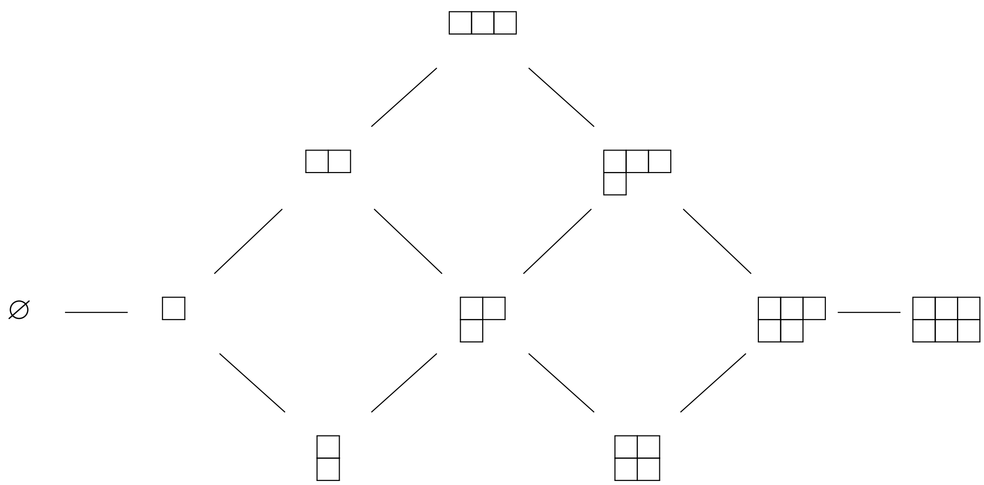
$$C_3 = \{12, 13, 23, 24, 25, 35, 45\}, \quad C_4 = \{12, 13, 23, 24, 34, 35, 45\}.$$



The degenerated Grassmannian $P_{2,5}$ is then defined as the union of the projective subspaces associated with these maximal chains,

$$P_{2,5} = P_{C_0} \cup \ldots \cup P_{C_4} \subseteq \mathbb{P}(\mathbb{C}^{I(2,5)}).$$

Example 6.4. The Hasse diagram of the poset $\mathcal{Y}(2,5)$ has the same form as in Example 6.3.



Remark 6.5. Let $\mathcal{SYT}(d,n)$ denote the set of standard Young tableaux of rectangular shape $d \times (n-d)$. There is a natural bijection between $\mathcal{SYT}(d,n)$ and the set $\mathfrak{C}(d,n)$, constructed as follows: The poset $\mathcal{Y}(d,n)$ of Young diagrams has as least element the empty diagram and as greatest element the full rectangle of shape $d \times (n-d)$ (see Definition 5.78).

Following a maximal chain in $\mathcal{Y}(d,n)$ corresponds to filling the empty rectangle of shape $d \times (n-d)$, starting from the top-left corner and successively adding boxes until the full rectangle is obtained. Given a maximal chain $C \in \mathfrak{C}(d,n)$ we enumerate the boxes of the full rectangle in the order in which they are added along C. By construction, this enumeration defines a standard Young tableau T_C of shape $d \times (n-d)$. Thus we obtain the desired bijection

$$\mathfrak{C}(d,n) \longrightarrow \mathcal{SYT}(d,n), \quad C \longmapsto T_C.$$

Example 6.6. In Example 6.3 we listed the maximal chains $C_0, \ldots, C_4$ of $I(2,5)$. Under the bijection of Remark 6.5, the corresponding standard Young tableaux $T_0, \ldots, T_4$ are given by

$$T_0 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \quad T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}$$

$$T_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \quad T_4 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

6.1 Poset Representation and a Restriction Map

The degenerated Grassmannian $P_{d,n}$ is defined as a poset union, and therefore, its equivariant cohomology with respect to a generic torus action can be described in terms of poset cohomology as stated in Lemma 5.82. As an application of this theory, we compute the restriction map

$$G(H_T^*(P_{2,n})) \longrightarrow G(H_T^*(P_{2,n+1})),$$

and obtain a compact characterization of its image.

While in principal, the same strategy can be applied to the describe the restriction from $P_{d,n+1}$ to $P_{d,n}$ arbitrary d, it is particularly well-structured for $d = 2$, so we restrict to this situation.

For the remainder of this subsection, let T be a torus with character lattice M acting generically on $\mathbb{P}_{d,n}$ by characters χ_I, for $I \in I(d,n)$. We continue with the notation from Chapter 5.4. In particular, we fix an index set

$$O = I(d,n) \sqcup K,$$

assume that $\{\chi_a \mid a \in O\}$ forms a basis of M, and regard the polynomial ring $\Lambda_T[\zeta]$ as polynomial ring in the variables

$$\zeta, \quad \eta_a = \zeta + \chi_a \quad (a \in O).$$

By Lemma 5.82, the *poset description* of the equivariant cohomology is then given by

$$G(H_T^*(P_{d,n})) \cong \Lambda_K[\zeta] \otimes \left(\bigoplus_{S \subseteq I(d,n)} \Lambda^S \otimes H^*(S) \right).$$

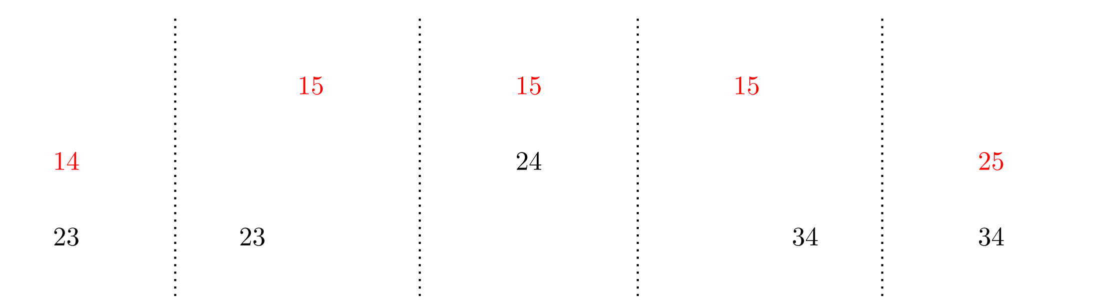
Example 6.7. We retain the notational convention of Example 6.3. Among all subposets of $I(2,5)$, the only ones with non-contractible order complexes are

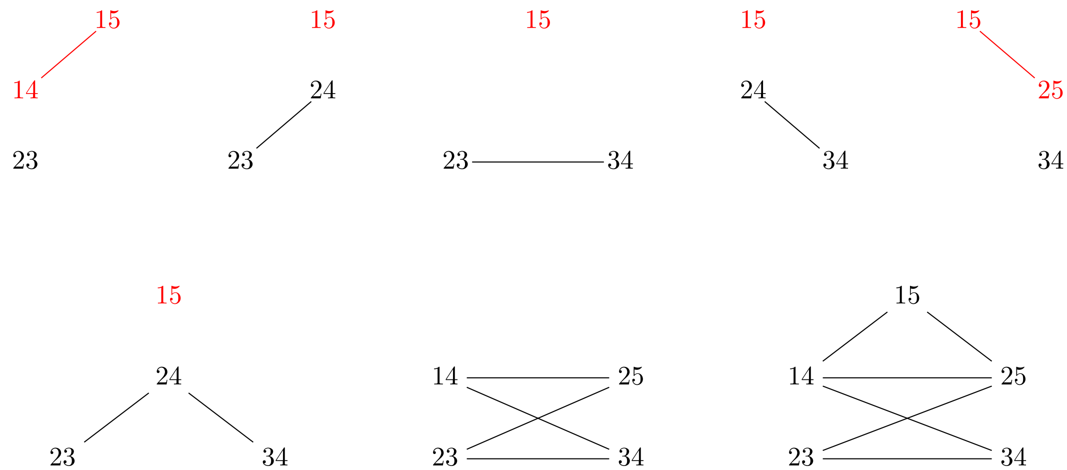
$$\{23,14\}, \{23,15\}, \{15,24\}, \{15,34\}, \{25,34\},$$

$$\{23,14,15\}, \{23,24,15\}, \{23,34,15\}, \{34,24,15\}, \{25,34,15\}, \{23,24,34,15\},$$

together with

$$B \coloneqq \{14,23,25,34\}, \quad B' \coloneqq \{14,23,25,34,15\}.$$





For any connected subposet S, there exists (up to scalars) a unique basis vector of $H^0(S)$, which we denote by g_S.

If S has two connected components, we fix basis vectors g_S and g'_S, where g_S is equal to one on each connected component and g'_S corresponds to the connected component highlighted in red in the above diagrams.

For the connected subposets B and B', we write g^1_B and $g^1_{B'}$ for the basis vectors of $H^1(B)$ and $H^1(B')$, respectively, each again unique up to scalar.

In this example we were able to choose basis vectors so that for any inclusion $Z \subseteq S$ of subposets, the restriction is simple. For instance, if S is connected and Z has two connected components, we still have

$$r^S_Z(g_S) = g_Z.$$

As a concrete example,

$$r^{B'}_B(g^1_{B'}) = g^1_B.$$

Consequently, the $\Lambda_T[\zeta]$-module structure of $G(H^*_T(P_{2,5}))$ is fully described by Remark 5.86.

For a fixed n, the poset $I(2,n)$ embeds into $I(2,n+1)$ as a subposet. Consequently both $\mathbb{P}_{2,n}$ and $P_{2,n}$ are naturally realized as subspaces of $\mathbb{P}_{2,n+1}$ and $P_{2,n+1}$, respectively. The corresponding restriction map r of projective unions was discussed in Remark 5.24. There we saw that the induced morphism between Mayer–Vietoris complexes,

$$r_1\colon \operatorname{MV}(P_{2,n+1}, \mathfrak{C}(2,n+1), T) \longrightarrow \operatorname{MV}(P_{2,n}, \mathfrak{C}(2,n), T),$$

is simply a projection map. Hence the associated morphism in equivariant cohomology admits the straightforward description

$$r\colon G(H^*_T(P_{2,n+1})) = H(\operatorname{MV}(P_{2,n+1}, \mathfrak{C}(2,n+1), T)) \longrightarrow G(H^*_T(P_{2,n})) = H(\operatorname{MV}(P_{2,n}, \mathfrak{C}(2,n), T)),$$

$$[\![x]\!] \longmapsto [\![x]\!],$$

identifying cosets in the respective quotient modules.

The task of determining whether a given element of $G(H_T^*(P_{2,n}))$ lies in the image of r, however, is more intricate and we develop a well-structured classification of the image by considering poset descriptions.

For the remainder of this subsection we adopt the abbreviations

$$I \coloneqq I(2,n), \quad I^+ \coloneqq I(2,n+1), \quad D \coloneqq I^+ \setminus I.$$

As in Remark 5.54 and Remark 5.88, we fix a basis $\{\chi_a \mid a \in O\}$ of M, indexed by

$$O = I^+ \sqcup K = I \sqcup \big(I^+ \setminus I\big) \sqcup K.$$

Furthermore, if for every subposet $S \subseteq I^+$ we fix an index set J_S and a basis $\{g_S^i \mid i \in J_S\}$ of $H^*(S)$, then an arbitrary element $\underline{f} \in G(H_T^*(P_{2,n+1}))$ can be expressed with respect to this basis as

$$\underline{f} = \sum_{S \subseteq I^+} \sum_{i \in J_S} \gamma_{S,i} g_S^i,$$

with coefficients $\gamma_{S,i} \in \Lambda_O^S[\zeta]$. By Remark 5.88, such an element restricts to

$$r(\underline{f}) = \sum_{S \subseteq I^+} \sum_{i \in J_S} \sum_{j \in J_{S \cap I}} a_{S,S\cap I}^{i,j} \gamma_{S,i} \eta_{D \setminus S} g_{S \cap I}^j.$$

For $0 \leq k \leq n$ we define the monomial

$$\eta_{\underline{k}} = \eta_{(k+1,n+1)} \cdot \ldots \cdot \eta_{(n,n+1)} \in \Lambda_T[\zeta], \quad \eta_{\underline{n}} \coloneqq 1.$$

The description of the image of r will be formulated in terms of certain integers $l_S(i)$, which will be defined in Definition 6.12 and record how far the basis element g_S^i survives under restriction.

Theorem 6.8. *We can choose bases of $H^*(S)$, for $S \subseteq I$, such that an element*

$$\underline{f} = \sum_{S,i} \beta_{S,i} g_S^i \in G(H_T^*(P_{2,n}))$$

lies in the image of r if and only if

$$\beta_{S,i} \text{ is divisible by } \eta_{\underline{l_S(i)}},$$

for every $S \subseteq I$ and every $i \in J_S$.

Note that

$$D = \{(1,(n+1)), \ldots, (n,(n+1))\},$$

and given $S \subseteq I, 1 \leq k \leq n$, and $L \subseteq \{1, \ldots, k-1\}$, we define subposets of I^+ by

$$S_k \coloneqq S \cup \{(1,(n+1)), \ldots, (k,(n+1))\}, \quad S_k^L \coloneqq S_k \setminus \{(\ell,(n+1)) \mid \ell \in L\}.$$

Lemma 6.9. *Let $S \subseteq I, 0 \le k \le n$ and $L \subseteq \{1, \dots, k-1\}$. Then*

$$H^*(S_k^L) \cong H^*(S_k).$$

Proof. Let $\mathfrak{L}_k$ denote the set of chains in S_k^L that pass through (and hence terminate in) $(k, n+1)$. Note that every chain in S_k^L is either a subset of S or belongs to $\mathfrak{L}_k$. Then

$$\mathfrak{L}_k \cap S = \{C \cap S \mid C \in \mathfrak{L}_k\}$$

is precisely the set of chains in S ending at one of the elements covered by $(k, n+1)$. In particular, this set depends only on k and not on the choice of L.
Consider

$$\overline{\mathfrak{L}}_k \coloneqq \bigcup_{C \in \mathfrak{L}_k} C.$$

By the poset Mayer–Vietoris sequence (Remark 5.81), the cohomology $H^*(S_k^L)$ can be computed from the cohomologies of $\overline{\mathfrak{L}}_k$, of S, and of their intersection $\overline{\mathfrak{L}}_k \cap S$. Each of these R-modules depends only on k, not on L. Consequently, $H^*(S_k^L)$ depends only on k, and we conclude that

$$H^*(S_k^L) \cong H^*(S_k).$$

□

We now specify the choice of bases required for Theorem 6.8.

Lemma 6.10. *For every subposet $S \subseteq I$ and every $0 \le k \le n$, one can fix bases of $H^*(S_k)$ such that S_k restricts simply to S_t for all $0 \le t \le k \le n$.*

Proof. For $1 \le m \le n$, let

$$t_m \colon H^*(S_m) \longrightarrow H^*(S_{m-1})$$

denote the restriction. Given $0 \le p < k \le n$, set

$$s_p^k \coloneqq t_k \circ t_{k-1} \circ \dots \circ t_{p+1} \colon H^*(S_k) \longrightarrow H^*(S_p), \quad Z_p^k \coloneqq \ker s_p^k.$$

For fixed m, the kernels form a chain

$$Z_{m-1}^m \subseteq Z_{m-2}^m \subseteq \dots \subseteq Z_0^m,$$

and moreover $t_m(Z_p^m) \subseteq Z_p^{m-1}$. Hence we may choose bases so that the induced maps

$$Z_0^n \xrightarrow{t_n} Z_0^{n-1} \xrightarrow{t_{n-1}} \dots \longrightarrow Z_0^1 \xrightarrow{t_1} 0$$

are partial permutations.
Now choose, for each $0 \le m \le n$, a complement R_m of Z_0^m in $H^*(S_m)$. Define subspaces $Q_m \subseteq R_m$ inductively by requiring

$$Q_m \oplus \bigoplus_{k=m+1}^{n} s_m^k(Q_k) = R_m, \quad Q_n = R_n.$$

This choice allows us to select bases in such a way that the restrictions in

$$Q_m \xrightarrow{t_m} s^m_{m-1}(Q_m) \xrightarrow{t_{m-1}} \dots \xrightarrow{t_2} s^m_1(Q_m) \xrightarrow{t_1} s^m_0(Q_m)$$

are partial permutations. Consequently, the restrictions in the full chain

$$R_n \xrightarrow{t_n} R_{n-1} \xrightarrow{t_{n-1}} \dots \xrightarrow{t_2} R_1 \longrightarrow R_0 = H^*(S)$$

are also partial permutations with respect to these bases. Since we already arranged the same property for the chain of kernels, we can combine the two choices of bases, obtaining for each $H^*(S_m)$ a basis in which every restriction $H^*(S_k) \to H^*(S_t)$ is a partial permutation. □

Remark 6.11. Lemma 6.10 allows us to choose bases of $H^*(S_k)$ $(0 \leq k \leq n)$ so that the restriction maps are partial permutations. In particular, the restrictions identify basis vectors across different S_k. We may therefore fix a common index set $\hat{J}_S$ containing all J_{S_k}, such that each restriction satisfies

$$s^k_p(g^i_{S_k}) = \begin{cases} g^i_{S_p}, & i \in J_{S_p}, \\ 0, & \text{otherwise.} \end{cases}$$

For the remainder of this subsection, and for every $S \subseteq I$, we fix bases of $H^*(S_k)$ $(0 \leq k \leq n)$ according to Lemma 6.10, together with the common index set $\hat{J}_S$ from Remark 6.11. Moreover, if $S \subseteq I$, $0 \leq k \leq n$, and $L \subseteq \{1, \dots, k-1\}$, we choose the basis of $H^*(S^L_k)$ to coincide with that of $H^*(S_k)$ (cf. Lemma 6.9).

Definition 6.12. For $S \subseteq I$ and $i \in J_S$ we define

$$l_S(i) \coloneqq \max(k \mid i \in J_{S_k}).$$

Proof of Theorem 6.8. Let $\underline{f} \in G(H^*_T(P_{2,n+1}))$. Expanding with respect to the chosen basis gives

$$\underline{f} = \sum_{\hat{S} \subseteq I^+} \sum_{i \in J_{\hat{S}}} \gamma_{\hat{S},i} g^i_{\hat{S}} = \sum_{\substack{S \subseteq I \\ Z \subseteq D}} \sum_{i \in J_{S \cup D}} \gamma_{S \cup D,i} g^i_{S \cup D}$$

$$= \sum_{S \subseteq I} \sum_{k=0}^{n} \sum_{L \subseteq \{1,\dots,k-1\}} \sum_{i \in J_{S^L_k}} \gamma_{S^L_k,i} g^i_{S^L_k}.$$

By our convention on bases and Remark 5.86, the terms corresponding to a fixed pair (S, k) combine as

$$\sum_{L \subseteq \{1,\dots,k-1\}} \gamma_{S^L_k,i} g^i_{S^L_k} = \left(\sum_{L \subseteq \{1,\dots,k-1\}} \gamma_{S^L_k,i} \eta_L \right) g^i_{S_k}.$$

Define

$$\gamma^k_{S,i} \coloneqq \sum_{L \subseteq \{1,\dots,k-1\}} \gamma_{S^L_k,i} \eta_L,$$

so that

$$\underline{f} = \sum_{S \subseteq I} \sum_{k=0}^{n} \sum_{i \in J_{S_k}} \gamma^k_{S,i} g^i_{S_k}.$$

The restriction of a basis element is determined by Remark 6.11 and Remark 5.88:

$$r(g^i_{S_k}) = \begin{cases} \eta_{\underline{k}} g^i_S, & \text{if } l_S(i) \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$r(\underline{f}) = \sum_{S \subseteq I} \sum_{i \in J_S} \left(\sum_{k=0}^{l_S(i)} \gamma^k_{S,i} \eta_{\underline{k}} \right) g^i_S = \sum_{S \subseteq I} \sum_{i \in J_S} \left(\sum_{k=0}^{l_S(i)} \gamma^k_{S,i} \frac{\eta_{\underline{k}}}{\eta_{\underline{l}_S(i)}} \right) \eta_{\underline{l}_S(i)} g^i_S.$$

Setting

$$\rho_{S,i} := \sum_{k=0}^{l_S(i)} \gamma^k_{S,i} \frac{\eta_{\underline{k}}}{\eta_{\underline{l}_S(i)}},$$

we obtain

$$r(\underline{f}) = \sum_{S \subseteq I} \sum_{i \in J_S} \rho_{S,i} \eta_{\underline{l}_S(i)} g^i_S, \quad \rho_{S,i} \in \Lambda^S_O[\zeta],$$

showing that all restrictions have the required form.

For the converse, fix $S \subseteq I, i \in J_S$ and let $\rho_{S,i} \in \Lambda^S_O[\zeta]$ be divisible by $\eta_{\underline{l}_S(i)}$. It suffices to treat the case where $\rho_{S,i}$ is a monomial. Define

$$k := \min\{t \mid \eta_{\underline{t}} \text{ divides } \rho_{S,i}\} \leq l_S(i),$$

and let L be the maximal subset of $\{1, \dots, k-1\}$ such that η_L divides $\rho_{S,i}$. Then

$$\frac{\rho_{S,i}}{\eta_L \eta_{\underline{k}}} \in \Lambda^{S^L_k}_O[\zeta],$$

and

$$r\left(\frac{\rho_{S,i}}{\eta_L \eta_{\underline{k}}} g^i_{S^L_k} \right) = \rho_{S,i} g^i_S.$$

This shows that every term of the required form arises as a restriction, and hence the claim follows. □

Example 6.13. In order to determine which elements of $G(H^*_T(P_{2,5}))$ lie in the image of r it is sufficient by Theorem 6.8 to determine the integers $l_S(i)$ for all $S \subseteq I(2,5)$ and $i \in J_S$. We will do so in the example cases of two posets.
For $B = \{14, 25, 23, 34\}$ the posets B_5, B_4 and B_3 are contractible. The posets B_2 and B_1 are connected and have cohomology of rank one in degree 1. This means, with notation as in Example 6.7, that,

$$l_B(g_B) = 5, \quad l_B(g^1_B) = 2.$$

Given the poset $S = \{14, 23\}$ we see that $S_5, \dots, S_2$ are contractible and S_1 keeps two connected components, i.e.,

$$l_S(g_S) = 5, \quad l_S(g'_S) = 1.$$

6.2 Grassmannian Torus Action and the First-Column Component

While the original motivation for studying projective unions was the degenerated Grassmannian, the motivation for considering equivariant cohomology arose from the observation that the action of the diagonal torus $T \subset \mathrm{GL}_n$ on $\mathrm{Gr}(d,n)$ extends to the degeneration. This induced action, referred to as the *Grassmannian torus action*, provides the natural torus action with respect to which the cohomologies of the degenerated and the original Grassmannian can be compared.

As noted in the introduction to this chapter, the Grassmannian is equivariantly formal with respect to this action: its torus-equivariant cohomology is free over Λ_T and completely determined by the localization image. For a meaningful comparison between the two cohomologies, attention must therefore be directed to the torsion-free part of $H_T^*(P_{d,n})$.

After defining a slightly generalized version of the Grassmannian torus action, we present an example calculation of $H_T^*(P_{2,5})$ in Example 6.19, using semi-regular sequences and the method of Theorem 5.73. We then analyze the localization image and the first-column component of $P_{d,n}$, which yield an equivalent description of the torsion-free part of $H_T^*(P_{d,n})$ (see Corollaries 5.37 and 5.38), and conclude with a brief comparison to the cohomology of $\mathrm{Gr}(d,n)$.

Let T be a torus acting on $\mathbb{C}^n$ via characters $\psi_1, \dots, \psi_n$, as in Examples 2.31 and 2.40, which however in this setting are not required to be distinct. This action extends to the ambient projective space $\mathbb{P}_{d,n} = \mathbb{P}(\bigwedge^d \mathbb{C}^n)$ with characters

$$\chi_I \coloneqq \sum_{j \in I} \psi_j, \quad I \in I(d,n).$$

Definition 6.14. Let T be a torus acting on the complex vector space $\mathbb{C}^n$ via characters $\psi_1, \dots, \psi_n$. The associated *Grassmannian torus action* on the degenerated Grassmannian $P_{d,n}$ is the action induced from the T-action on $\mathbb{P}_{d,n}$ through the characters

$$\chi_I = \sum_{j \in I} \psi_j, \quad I \in I(d,n).$$

Remark 6.15. The set $\mathrm{Gr}(d,n)^T$ is finite precisely when the characters $\psi_1, \dots, \psi_n$ are pairwise distinct (see Example 2.31). Likewise, $P_{d,n}^T$ is finite precisely when

$$\chi_I - \chi_{I'} \neq 0$$

for all $I \leq I'$, that is, for all d-tuples I, I' lying in a common maximal chain of $I(d,n)$ (see Remark 5.33). Note moreover that finiteness of $P_{d,n}^T$ implies finiteness of $\mathrm{Gr}(d,n)^T$, while the converse does not hold.

Remark 6.16. The Grassmannian $\mathrm{Gr}(d,n)$ is a GKM-variety if and only if for any fixed $I \in I(d,n)$ the characters

$$\psi_i - \psi_j, \quad i \in I, j \notin I,$$

are pairwise relatively prime (see Example 2.40).

The cover $\mathfrak{M}_{\mathfrak{C}(d,n)}$ of $P_{d,n}$ consists of GKM-varieties if and only if

$$\chi_{I_1} - \chi_{I_2} \text{ is relatively prime to } \chi_{I_3} - \chi_{I_4},$$

for all $I_1, \dots, I_4 \in I(d,n)$ that are pairwise comparable, by construction and Example 2.39.

As in Remark 6.15, if $P_{d,n}$ is covered by GKM-varieties, then $\mathrm{Gr}(d,n)$ is a GKM-variety, while the converse does not hold.

Example 6.17. Let T be the diagonal torus in GL_n acting on $\mathbb{C}^n$ by its standard characters $e_1, \dots, e_n$. With respect to the induced Grassmannian torus action, both the cover $\mathfrak{M}_{\mathfrak{C}(d,n)}$ of $P_{d,n}$ consists of GKM-varieties and $\mathrm{Gr}(d,n)$ is a GKM-variety. In particular, both $\mathrm{Gr}(d,n)$ and $P_{d,n}$ contain finitely many T-fixed points.

In Chapter 5.5, we outlined an approach to computing equivariant cohomology for non-generic torus actions by comparison with the cohomology of a generic torus action via a torus change. The suitable torus with generic action, chosen in Remark 5.61, is

$$\hat{T} \cong (\mathbb{C}^\times)^{I(d,n)},$$

acting on $\mathbb{P}_{d,n}$ by its canonical characters μ_I, $I \in I(d,n)$. The morphism of character lattices

$$\varphi^* \colon \hat{M} \longrightarrow M, \quad \mu_I \longmapsto \chi_I,$$

between $\hat{T}$ and T defines a compatible morphism $\varphi \colon T \to \hat{T}$, and the induced restriction and extension of tori

$$\theta \colon H^*_{\hat{T}}(P_{\mathfrak{C}}) \longrightarrow H^*_T(P_{\mathfrak{C}}), \quad \Theta \colon \Lambda_T \otimes_{\Lambda_{\hat{T}}} H^*_{\hat{T}}(P_{\mathfrak{C}}) \longrightarrow H^*_T(P_{\mathfrak{C}}),$$

relate the Grassmannian action to the generic one.

Remark 6.18. Since R is a field, φ introduces relations $\mathcal{R}$ on $\Lambda_{\hat{T}}$ (Lemma 4.17). These relations $\mathcal{R}$ are defined as the kernel of the morphism $\hat{\varphi} : \Lambda_{\hat{T}} \to \Lambda_T$. As such, they are generated as $\Lambda_{\hat{T}}$-module by

$$\mu_I + \mu_{I'} - \mu_J - \mu_{J'},$$

for all $I, I', J, J' \in I(d,n)$, that satisfy

$$I \cup I' = J \cup J' \text{ and } I \cap I' = J \cap J',$$

as sets.

For this setting, we developed in Lemma 5.62 a criterion for short exact sequences describing

$$\hat{\Theta} \colon \Lambda_T \otimes_{\Lambda_{\hat{T}}} G(H^*_{\hat{T}}(P_{\mathfrak{C}})) \longrightarrow G(H^*_T(P_{\mathfrak{C}})),$$

and in Theorem 5.73 a recursive method based on semi-regular sequences to prove surjectivity of torus change. We now apply the latter method to the example of $P_{2,5}$.

Example 6.19. For this example only, let

$$T = (\mathbb{C}^\times)^{I(2,5)}$$

denote the torus with a generic action on $P_{2,5}$, and let

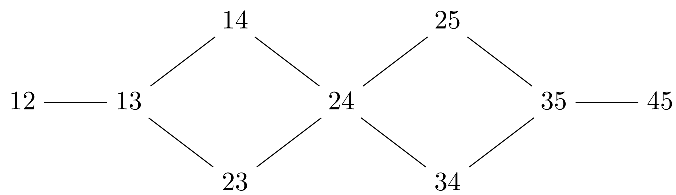
$$\overline{T} = (\mathbb{C}^\times)^5$$

be the torus acting on $\mathbb{C}^5$ via characters $e_1, \dots, e_5$, which induce the Grassmannian action on $P_{2,5}$.

The most natural situation is that $e_1, \dots, e_5$ are taken to be the canonical characters of $\overline{T}$, and this will be our standing assumption throughout the example. In this case, note that both $P_{2,5}^{T}$ and $P_{2,5}^{\overline{T}}$ are finite (see Remark 6.15).

In the first part, we consider the subposet

$$J(2,5) \coloneqq I(2,5) \setminus \{(1,5)\} \subseteq I(2,5),$$



with maximal chains given by

$$\begin{aligned}
C_1 &= \{(1,2),(1,3),(1,4),(2,4),(2,5),(3,5),(4,5)\},\\
C_2 &= \{(1,2),(1,3),(1,4),(2,4),(3,4),(3,5),(4,5)\},\\
C_3 &= \{(1,2),(1,3),(2,3),(2,4),(2,5),(3,5),(4,5)\},\\
C_4 &= \{(1,2),(1,3),(2,3),(2,4),(3,4),(3,5),(4,5)\}.
\end{aligned}$$

The poset union $P_{J(2,5)}$ is the projective union

$$P_{\mathfrak{C}}, \qquad \mathfrak{C} = (C_i \mid i \in I), \quad I = \{1,2,3,4\}.$$

It is contained in $P_{2,5}$ and we consider it with the induced torus actions by T and $\overline{T}$. Our aim is to show that the restriction of tori

$$\theta\colon H_T^*(P_{J(2,5)}) \longrightarrow H_{\overline{T}}^*(P_{J(2,5)})$$

is surjective, using Theorem 5.73.

Adopting the notation from Chapter 5.5, we set

$$\mathrm{MV} = \mathrm{MV}(T, P_{J(2,5)}, \mathfrak{C}), \qquad \overline{\mathrm{MV}} = \mathrm{MV}(\overline{T}, P_{J(2,5)}, \mathfrak{C}).$$

and begin by showing that the sequence $a_1 = (1,4), a_2 = (2,4)$ in $I(2,5)$ is square-free:
for $j = 1$, the differences

$$\eta_{(1,4)} - \eta_L, \qquad \overline{\eta}_{(1,4)} - \overline{\eta}_L, \quad L \in I(2,5) \setminus \{(1,4)\},$$

are nonzero in $\Lambda_T[\zeta]$ and $\Lambda_{\overline{T}}[\zeta]$. For $j = 2$, one checks that

$$\eta_{(2,4)} - \eta_L \notin (\eta_{(1,4)}), \qquad \overline{\eta}_{(2,4)} - \overline{\eta}_L \notin (\overline{\eta}_{(1,4)}),$$

for all $L \neq (1,4), (2,4)$.

To verify semi-regularity, we examine the cohomology of MV_V^U. The relevant partitions are

$$(U,V) = (\varnothing, \varnothing) \quad \text{for } j = 1,$$

and

$$(U,V) = (\{(1,4)\}, \varnothing), \quad (U,V) = (\varnothing, \{(1,4)\}) \quad \text{for } j = 2.$$

Observe the following:

1. $\mathrm{MV}^{(1,4)}_{(2,4)}$ and $\mathrm{MV}_{\{(1,4),(2,4)\}}$ are cochain complexes on the full simplex $\mathcal{I}$:

$$\mathrm{MV}^{(1,4)}_{(2,4)} = C^*(\mathcal{I}, \Lambda_T[\zeta]/(\eta_{(2,4)})), \qquad \mathrm{MV}_{\{(1,4),(2,4)\}} = C^*(\mathcal{I}, \Lambda_T[\zeta]/(\eta_{(1,4)}, \eta_{(2,4)})).$$

2. $\mathrm{MV}^{(1,4)}$ and $\mathrm{MV}^{\{(1,4),(2,4)\}}$ coincide with Mayer–Vietoris complexes of a projective union of two subspaces (see Remark 5.67).

By Example 5.72 we obtain

$$\delta^{-1}(K^{(1,4)}) \subseteq K_{(1,4)} + p(H(\mathrm{MV})), \qquad \delta^{-1}(K^{\{(1,4),(2,4)\}}) \subseteq K^{(1,4)}_{(2,4)} + p(H(\mathrm{MV}^{(1,4)})),$$

$$\delta^{-1}(K^{(2,4)}_{(1,4)}) \subseteq K_{\{(1,4),(2,4)\}} + p(H(\mathrm{MV}_{(1,4)})),$$

so $(1,4),(2,4)$ is indeed a semi-regular sequence.

Theorem 5.73 further requires surjectivity of

$$\tilde{\pi}^{(1,4)}_{(2,4)}, \quad \tilde{\pi}^{(2,4)}_{(1,4)}, \quad \tilde{\pi}^{\{(1,4),(2,4)\}}, \quad \tilde{\pi}_{\{(1,4),(2,4)\}},$$

and by Example 5.74, only $\tilde{\pi}^{(2,4)}_{(1,4)}$ requires checking.
Write

$$\tilde{\pi} := \tilde{\pi}^{(2,4)}_{(1,4)}, \qquad S := \Lambda_T[\zeta]/(\eta_{(1,4)}), \qquad \overline{S} := \Lambda_{\overline{T}}[\zeta]/(\overline{\eta}_{(1,4)}).$$

The map $\tilde{\pi}$ is induced by the morphism of differential graded modules

$$\pi\colon \mathrm{MV}^{(2,4)}_{(1,4)} = D(\mathfrak{C}, S, (\eta^{(2,4)}_L)_{L\in I(2,5)}) \longrightarrow \overline{\mathrm{MV}}^{(2,4)}_{(1,4)} = D(\mathfrak{C}, \overline{S}, (\overline{\eta}^{(2,4)}_L)_{L\in I(2,5)}),$$

with differentials denoted s and $\overline{s}$. Since the computation is identical in all degrees, we restrict to $\tilde{\pi}^0$ for brevity.
Let

$$\eta_0 := \eta_{(1,2)}\eta_{(1,3)}\eta_{(3,5)}\eta_{(4,5)}.$$

Then

$$\begin{aligned} s^0\colon & S \oplus S \oplus S/(\eta_0\eta_{(2,3)}\eta_{(2,5)}) \oplus S/(\eta_0\eta_{(2,3)}\eta_{(3,4)}) \\ & \longrightarrow S \oplus S/(\eta_0\eta_{(2,5)}) \oplus S/(\eta_0) \oplus S/(\eta_0) \oplus S/(\eta_0\eta_{(2,3)}) \oplus S/(\eta_0\eta_{(3,4)}), \end{aligned}$$

$$(t_1, t_2, t_3, t_4) \longmapsto (t_1 - t_2, t_1 - t_3, t_1 - t_4, t_2 - t_3, t_2 - t_4, t_3 - t_4).$$

Thus, elements of $\ker s^0$ are precisely the tuples

$$(t,\ t,\ t + \alpha\,\eta_0\eta_{(2,5)},\ t + \beta\,\eta_0\eta_{(2,3)}),$$

with $t, \alpha, \beta \in S$ satisfying $\alpha\eta_{(2,5)} - \beta\eta_{(2,3)} \in (\eta_{(3,4)})$, viewed as elements of

$$S \oplus S \oplus S/(\eta_0\eta_{(2,3)}\eta_{(2,5)}) \oplus S/(\eta_0\eta_{(2,3)}\eta_{(3,4)}).$$

An entirely analogous computation yields

$$\ker \overline{s}^0 = \{(t,\, t,\, t + \alpha\, \overline{\eta}_0 \overline{\eta}_{(2,5)},\, t + \beta\, \overline{\eta}_0 \overline{\eta}_{(2,3)})\},$$

where $t, \alpha, \beta \in \overline{S}$ and $\alpha\overline{\eta}_{(2,5)} - \beta\overline{\eta}_{(2,3)} \in (\overline{\eta}_{(3,4)})$, considered as elements of

$$\overline{S} \oplus \overline{S} \oplus \overline{S}/(\overline{\eta}_0 \overline{\eta}_{(2,3)} \overline{\eta}_{(2,5)}) \oplus \overline{S}/(\overline{\eta}_0 \overline{\eta}_{(2,3)} \overline{\eta}_{(3,4)}).$$

Since

$$\mathrm{Syz}_S(\eta_{(2,3)}, \eta_{(2,5)}, \eta_{(3,4)}) \longrightarrow \mathrm{Syz}_{\overline{S}}(\overline{\eta}_{(2,3)}, \overline{\eta}_{(2,5)}, \overline{\eta}_{(3,4)})$$

is surjective, so is

$$\hat{\pi}^0 \colon \ker s^0 \longrightarrow \ker \overline{s}^0.$$

All conditions of Theorem 5.73 are therefore met, and

$$\theta \colon H^*_T(P_{J(2,5)}) \longrightarrow H^*_{\overline{T}}(P_{J(2,5)})$$

is surjective.

Finally, note that

$$P_{2,5} = P_{J(2,5)} \cup P_{C_0},$$

where

$$C_0 = \{(1,2), (1,3), (1,4), (1,5), (2,5), (3,5), (4,5)\}$$

is the unique maximal chain in $I(2,5)$ through $(1,5)$. The intersection

$$P_{J(2,5)} \cap P_{C_0} = P_{C_0'}, \qquad C_0' = \{(1,2), (1,3), (1,4), (2,5), (3,5), (4,5)\},$$

is a projective space, and hence, by Remark 5.76, the restriction

$$\theta \colon H^*_T(P_{2,5}) \longrightarrow H^*_{\overline{T}}(P_{2,5})$$

is also surjective.

The major obstacle in computing the cohomology for a non-generic torus action lies in describing the additional cocycles in the Mayer–Vietoris complex. By showing that θ, and in particular the morphism

$$\hat{\theta} \colon G(H^*_T(P_{2,5})) = H(\mathrm{MV}(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)})) \longrightarrow G(H^*_{\overline{T}}(P_{2,5})) = H(\mathrm{MV}(\overline{T}, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)})),$$

$$[\![x]\!] \longmapsto [\![x]\!],$$

is surjective, this part of the problem is resolved. To obtain a complete description of $G(H^*_{\overline{T}}(P_{2,5}))$, it now suffices to consider the kernel of $\hat{\theta}$.

We write d and $\overline{d}$ for the differentials of the respective Mayer–Vietoris complexes. Generators of $H(\mathrm{MV}(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}))$ over $\Lambda_T[\zeta]$ can be obtained from the poset description in Example 6.7. To determine $H^0(\mathrm{MV}(\overline{T}, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}))$, we need to consider the quotient

$$H^0(\mathrm{MV}(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)})) = \ker d^0$$

by the kernel of $\hat{\theta}$, which in this degree is simply

$$\ker \hat{\theta}^0 = \ker d^0 \cap \big(\mathcal{R} \cdot \mathrm{MV}^0(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)})\big),$$

see Remark 4.39.

As generators of $H^1(\mathrm{MV}(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}))$, Example 6.7 provides the two elements

$$g^1_B = \eta_0\eta_{(1,5)}e_{C_1\cap C_2}, \qquad g^1_{B'} = \eta_0 e_{C_1\cap C_2}.$$

Thus $H^1(\mathrm{MV}(T, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}))$ is generated over $\Lambda_T[\zeta]$ by $g^1_{B'}$, while $H^1(\mathrm{MV}(\overline{T}, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}))$ is generated over $\Lambda_{\overline{T}}[\zeta]$ by

$$\overline{g}^1_{B'} \coloneqq \overline{\eta}_0 e_{C_1\cap C_2}.$$

By directly computing the image $\operatorname{im} \overline{d}^0$, we can show that $\overline{g}^1_{B'}$ is non-zero in both

$$\mathrm{MV}^1(\overline{T}, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)}) \quad \text{and} \quad H^1(\mathrm{MV}(\overline{T}, P_{2,5}, \mathfrak{M}_{\mathfrak{C}(2,5)})).$$

Consequently, both $G(H^*_T(P_{2,5}))$ and $G(H^*_{\overline{T}}(P_{2,5}))$ share a non-trivial torsion part generated by $g^1_{B'}$ and $\overline{g}^1_{B'}$, respectively(see Corollary 5.38).

We continue this chapter with constructing a combinatorial description for the first-column component of $P_{d,n}$ and consequently obtaining an alternative description of $\iota(P_{d,n})$ in the case that $P^T_{d,n}$ is finite.

For an arbitrary projective union, we gave the description

$$\nu(P_{\mathfrak{C}}) = \{(f_i)_{i\in I} \in \mathrm{MV}^0(T, P_{\mathfrak{C}}, \mathfrak{M}_{\mathfrak{C}}) \mid f_i - f_j \in (\eta_{ij})\},$$

in Remark 5.23. For degenerated Grassmannians, we can exploit the additional structure of the set $\mathcal{SYT}(d, n)$ indexing the cover (see Remark 6.5), together with its connection to the action of the symmetric group.

In the following, we will abbreviate $r \coloneqq d(n-d)$.

Remark 6.20. We may view $\mathcal{SYT}(d, n)$ as a subset of $\mathcal{T}(d, n)$, the set of rectangles of shape $d\times(n-d)$ filled with the numbers $1, \dots, r$ in an unordered way. The symmetric group S_r acts transitively and faithfully on $\mathcal{T}(d, n)$ by permuting the labels of the boxes. For $1 \le i \le j \le r$, let $\tau(i, j)$ denote the transposition swapping i and j. The *simple transpositions* are those of the form $\tau(i, i+1)$.

For later computations with standard Young tableaux it will be convenient to adopt an equivalent combinatorial model.

Definition 6.21. An *unordered lattice word of type* $d \times (n-d)$ is a word

$$w = w_1 w_2 \dots w_r,$$

of length r containing exactly $(n-d)$ copies of each integer $1 \le i \le d$.

For $1 \le k \le r$ and $1 \le i \le d$, we write $c_i(k, w)$ for the number of occurrences of i in the prefix $w_1 \dots w_k$. The set of unordered lattice words of type $d \times (n-d)$ is denoted by $\mathcal{ULW}(d, n)$.

A *lattice word of type* $d \times (n-d)$ is an unordered lattice word w such that for every $1 \le k \le r$, the prefix condition

$$c_i(k, w) \ge c_{i+1}(k, w), \quad \text{for all } 1 \le i \le d-1,$$

holds. We denote the set of lattice words of type $d \times (n-d)$ by $\mathcal{LW}(d, n)$.

Remark 6.22. There is a natural bijection $T(d,n) \to \mathcal{ULW}(d,n)$: an unordered tableau T is mapped to the unordered lattice word w in which the j-th letter records the row number of the box containing j in T. This restricts to a bijection

$$\mathcal{SYT}(d,n) \longleftrightarrow \mathcal{LW}(d,n),$$

under which the action of S_r on $\mathcal{ULW}(d,n)$ corresponds to permuting the positions of the word. In particular, a simple transposition swaps two neighboring entries w_i and w_{i+1}.

By Remark 6.5, we obtain an induced bijection between lattice words and maximal chains $\mathfrak{C}(d,n)$ in $\mathcal{Y}(d,n)$. Explicitly, for $w \in \mathcal{LW}(d,n)$ with associated chain $C_w \in \mathfrak{C}(d,n)$, one constructs C_w by successively adding boxes according to the letters of w: after reading the prefix $w_1 \dots w_k$, the corresponding Young diagram in the chain has $c_i(k,w)$ boxes in row i. Hence each prefix of w corresponds uniquely to a Young diagram in the maximal chain C_w.

Definition 6.23. Let $G_{d,n}$ be the graph whose vertices are the elements of $\mathcal{SYT}(d,n)$ (equivalently, $\mathcal{LW}(d,n)$). Two distinct vertices are joined by an edge if the corresponding tableaux (or lattice words) differ by a simple transposition.

Lemma 6.24. *If $w = w_1 w_2 \dots w_r$ is a lattice word with $w_j > w_{j+1}$ for some $1 \le j \le r-1$, then the word obtained by swapping the neighbors,*

$$\tau(j,j+1).w = w_1 \dots w_{j+1} w_j \dots w_r,$$

is again a lattice word.

Proof. The lattice condition only needs to be checked for the prefix $w_1 \dots w_{j-1} w_{j+1}$ of $\tau(j,j+1).w$. Since w itself is a lattice word, we have

$$c_{w_{j+1}}(w, j+1) \le c_{w_j}(w, j+1).$$

Because $w_j > w_{j+1}$, this inequality ensures that after swapping the two letters, the prefix of length j in $\tau(j,j+1).w$ still satisfies the lattice condition. All other prefixes remain unchanged. Hence $\tau(j,j+1).w$ is a lattice word. □

Given a string $w = w_1 \dots w_k$ of integers, let $s(w) = s(w)_1 s(w)_2 \dots s(w)_k$ denote the non-decreasing rearrangement of w, i.e.

$$s(w)_1 \le s(w)_2 \le \dots \le s(w)_k, \quad \{w_1, \dots, w_k\} = \{s(w)_1, \dots, s(w)_k\}.$$

For two lattice words $v, w \in \mathcal{LW}(d,n)$, define

$$e(v,w) := \min(i \mid v_i \ne w_i),$$

and let $t(v,w)$ be the minimal integer greater than $e(v,w)$ such that

$$s(v_{e(v,w)} \dots v_{t(v,w)}) = s(w_{e(v,w)} \dots w_{t(v,w)}).$$

Then define $r(v,w) \in \mathcal{ULW}(d,n)$ by

$$r(v,w) := v_1 \dots v_{e(v,w)-1} \tilde{s}_1 \dots \tilde{s}_{t(v,w)-e(v,w)+1} v_{t(v,w)+1} \dots v_r,$$

where $\tilde{s} = s(v_{e(v,w)} \dots v_{t(v,w)}) = s(w_{e(v,w)} \dots w_{t(v,w)})$.
In words: we discard the initial segment on which v and w agree, then take the shortest following segment such that the two words contain exactly the same letters (necessarily in different order). We then reorder this segment in increasing order and reattach the common prefix.
Starting from $v^{(0)} = v$ and $w^{(0)} = w$, define recursively for $i \geq 0$:

$$v^{(i+1)} = r(v^{(i)}, w^{(i)}), \quad w^{(i+1)} = r(w^{(i)}, v^{(i)}).$$

By construction, the first index of disagreement strictly increases:

$$e(v^{(i+1)}, w^{(i+1)}) > e(v^{(i)}, w^{(i)}),$$

and hence for some ℓ one eventually has $e(v^{(\ell)}, w^{(\ell)}) = r + 1$, i.e.

$$v^{(\ell)} = w^{(\ell)} = v^{(\ell+1)} = w^{(\ell+1)}.$$

Definition 6.25. For two lattice words $v, w \in \mathcal{LW}(d, n)$, the *parent word* is the common word

$$p(v, w) = v^{(\ell)} = w^{(\ell)}.$$

Remark 6.26. For every $0 \leq i \leq \ell - 1$, there exists a sequence of simple transpositions $\tau_1, \dots, \tau_s$, such that

$$v^{(i+1)} = \tau_s \dots \tau_1.v^{(i)}.$$

Moreover, each intermediate word stays in $\mathcal{LW}(d, n)$, i.e.,

$$\tau_{s'} \dots \tau_1.v^{(i)} \in \mathcal{LW}(d, n), \quad \text{for all } 1 \leq s' \leq s,$$

by Lemma 6.24.

Lemma 6.27. *For $v, w \in \mathcal{LW}(d, n)$, the parent word $p(v, w)$ is uniquely determined, lies in $\mathcal{LW}(d, n)$, and is connected to both v and w in $G_{d,n}$.*

Proof. The definition of $p(v, w)$ is constructive, hence $p(v, w)$ is uniquely determined. Inductively applying Remark 6.26 shows that $p(v, w)$ is a lattice word and that both v and w are connected to $p(v, w)$ in $G_{d,n}$. □

Corollary 6.28. *The graph $G_{d,n}$ is simple and connected.*

Proof. Simplicity follows from the construction of $G_{d,n}$. By Lemma 6.27, any two vertices $v, w \in G_{d,n}$ are both connected to their parent word $p(v, w)$, and hence to each other. □

Remark 6.29. An alternative argument for the proof of Corollary 6.28 is that the set of standard Young tableaux can be seen as the linear extension of the poset defined on the set of boxes in the same shape ordered from left to right from top to bottom. The statement is then equivalent to the fact that the linear extension graph of a finite poset is connected (proof of [Mas09, Lemma 2.1]).

Remark 6.30. By Remark 6.22 there is a bijection

$$\mathfrak{C}(d, n) \longleftrightarrow \mathcal{LW}(d, n)$$

between maximal chains in $\mathcal{Y}(d,n)$ and lattice words. Each prefix of a lattice word corresponds to a Young diagram in the associated chain, and the diagram depends only on the sorted prefix. We therefore set

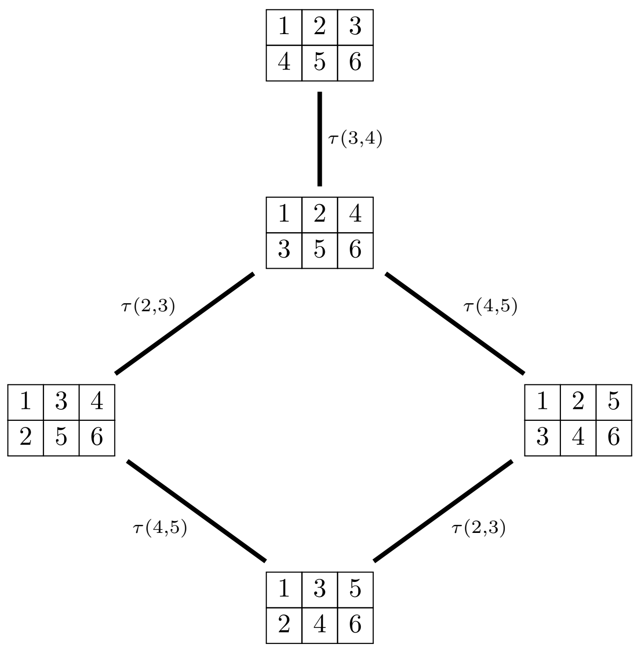
$$\mathrm{Pref}(w) := \{\, s(w') \mid w' \text{ is a prefix of } w \,\},$$

and view $\mathrm{Pref}(w)$ as the corresponding subset of $\mathcal{Y}(d,n)$.

For lattice words w_1, w_2 with corresponding chains C_1, C_2, each pair of prefixes of w_1 and w_2 that agree after sorting determines a unique Young diagram contained in both chains. Hence,

$$w_1 \cap w_2 := \mathrm{Pref}(w_1) \cap \mathrm{Pref}(w_2) = C_1 \cap C_2.$$

Example 6.31. For $P_{2,5}$ the graph $G_{2,5}$ is given below.



Theorem 6.32. *A tuple* $(f_C)_{C\in\mathfrak{C}(d,n)} \in \mathrm{MV}^0(T, P_{d,n}, \mathfrak{M}_{\mathfrak{C}(d,n)})$ *is contained in* $\nu(P_{d,n})$ *if and only if*

$$f_C - f_D \in (\eta_{C\cap D}),$$

for all C, D *that share an edge in* $G_{d,n}$.

Proof. In this proof we index the tuple $(f_C)_{C\in\mathfrak{C}(d,n)}$ by the corresponding lattice words.

By Remark 5.23, it remains to prove sufficiency. Assume that $(f_v)_{v\in\mathcal{LW}(d,n)}$ satisfies $f_v - f_w \in (\eta_{v\cap w})$ whenever v, w share an edge in $G_{d,n}$.

Fix arbitrary lattice words $v, w \in \mathcal{LW}(d,n)$ and let $v^{(i)}, w^{(i)}$ denote the intermediate words arising in the definition of the parent word $p(v,w)$ (see Definition 6.25). By construction,

$$v^{(i)} \cap w^{(i)} \subseteq v^{(i+1)} \cap w^{(i+1)},$$

for all $0 \leq i \leq \ell - 1$, where the process terminates at some finite ℓ with $v^{(\ell)} = w^{(\ell)} = p(v,w)$. Moreover, by Remark 6.26, $v^{(i+1)}$ is obtained from $v^{(i)}$ by a sequence of simple transpositions $\tau_1, \dots, \tau_s$, and for all $1 \leq k \leq s$ one has

$$v^{(i)} \cap w^{(i)} \subseteq (\tau_{k-1} \dots \tau_1.v^{(i)}) \cap (\tau_k \dots \tau_1.v^{(i)}).$$

For each i,

$$f_{v^{(i)}} - f_{v^{(i+1)}} = \sum_{j=1}^{s} \Big(f_{\tau_{j-1}\dots\tau_1.v^{(i)}} - f_{\tau_j\dots\tau_1.v^{(i)}} \Big),$$

and since $\tau_{j-1} \dots \tau_1.v^{(i)}$ and $\tau_j \dots \tau_1.v^{(i)}$ are connected by an edge in $G_{d,n}$, it follows by the inclusion above that each difference is divisible by $\eta_{v^{(i)} \cap w^{(i)}}$, and hence also by $\eta_{v \cap w}$. Consequently the same holds for

$$f_{v^{(i)}} - f_{v^{(i+1)}},$$

and therefore also for

$$f_v - f_w = \sum_{j=0}^{\ell} \Big((f_{v^{(j)}} - f_{v^{(j+1)}}) - (f_{w^{(j)}} - f_{w^{(j+1)}}) \Big).$$

Thus $(f_v)_{v \in \mathcal{LW}(d,n)} \in \nu(P_{d,n})$, again by Remark 5.23. □

The relevance of the first-column component was discussed in Chapter 4.1, where its close relation to the torsion-free part of the cohomology module was emphasized. More precisely, for an arbitrary projective union $P_{\mathfrak{C}}$ one has

$$\operatorname{tor}(\Lambda_T, H_T^*(P_{\mathfrak{C}})) = \tau(P_{\mathfrak{C}}), \quad H_T^*(P_{\mathfrak{C}})/\operatorname{tor}(\Lambda_T, H_T^*(P_{\mathfrak{C}})) \cong \iota(P_{\mathfrak{C}}),$$

by Corollary 5.37. In general, the torsion is non-trivial (see Example 5.22). If, in addition, $P_{\mathfrak{C}}^T$ is finite, then

$$\iota(P_{\mathfrak{C}}) \cong \nu(P_{\mathfrak{C}}),$$

by Corollary 5.38.

Thus, the torsion-free part of $H_T^*(P_{d,n})$ is given by $\iota(P_{d,n})$, which coincides with $\nu(P_{d,n})$ whenever the fixed point set is finite. After describing the first-column component in Theorem 6.32, we turn to the presentation of the localization image provided in Theorem 4.10, under the assumption that $P_{d,n}$ is covered by GKM-varieties.

Corollary 6.33. *If $P_{d,n}$ is a union of GKM-varieties, then*

$$\iota(P_{d,n}) = \{(u_I)_{I \in I(d,n)} \mid u_I - u_{I'} \text{ is divisible by } \chi_I - \chi_{I'} \text{ for all } I \leq I'\}.$$

Proof. By Lemma 5.44, $P_{d,n}$ is determined by its localization image. Moreover, an edge between I and I' in the moment graph $\Gamma_{P_{d,n}}$ occurs precisely when I and I' correspond to coordinates of the same projective component, that is, when they lie in a common chain of $I(d,n)$ and are comparable. □

The motivating example throughout this thesis has been the Grassmannian action induced by the diagonal torus $T \subseteq \mathrm{GL}_n$ acting on $\mathbb{C}^n$ by its standard characters. We summarize the results for the torsion-free part of the equivariant cohomology for this specific torus action. Recall that for a subset $J \subseteq I(d,n)$ the polynomial η_J is a completely reducible polynomial in $\Lambda_T[\zeta]$ with factors corresponding to the elements in J (see the beginning of Chapter 5.2).

Corollary 6.34. *Consider the Grassmannian torus action of the diagonal torus $T \subseteq \mathrm{GL}_n$ induced by its action via standard characters $e_1, \ldots, e_n$ on $\mathbb{C}^n$. The torsion-free part of $H_T^*(P_{d,n})$ is isomorphic to the localization image of $P_{d,n}$. Further, we obtain as equivalent descriptions*

$$\iota(P_{d,n}) = \{(u_I)_{I \in I(d,n)} \in \bigoplus_{I(d,n)} \Lambda_T \mid u_I - u_{I'} \text{ is divisible by } \chi_I - \chi_{I'} \text{ for all } I \leq I'\},$$

where $\chi_I = \sum_{j \in I} e_j$, and

$$\iota(P_{d,n}) \cong \nu(P_{d,n})$$

$$= \left\{(f_C)_{C \in \mathfrak{C}(d,n)} \in \bigoplus_{C \in \mathfrak{C}(d,n)} H_T^*(P_C) \mid f_C - f_D \in (\eta_{C \cap D}) \text{ for all } C, D \text{ that share an edge in } G_{d,n}\right\}.$$

Proof. In Example 6.17 it was highlighted that $P_{d,n}$ is covered by GKM-varieties and contains finitely many fixed points with respect to the chosen torus action. The Corollary therefore follows by combining the results from Corollary 5.38, Corollary 6.33 and Theorem 6.32. □

Example 6.35. In this example, we compute the first-column component of $P_{2,5}$ with respect to the Grassmannian torus action induced by the standard characters of $T = (\mathbb{C}^\times)^5$ utilizing Corollary 6.34.

The graph $G_{2,5}$ was presented in Example 6.31 and the first-column component of $P_{2,5}$ consists of tuples

$$(f_0, \ldots, f_4) \in \bigoplus_{i=0}^{4} \Lambda_T[\zeta]/(\eta_i),$$

where we write $\mathfrak{C}(2,5) = \{C_0, \ldots, C_4\}$ as in Example 6.3. By the above Corollary, the condition for $(f_0, \ldots, f_4)$ to lie in $\nu(P_{2,5})$ is

$$\begin{aligned} f_0 - f_1 &\in (\eta_{01}), \\ f_1 - f_2 &\in (\eta_{12}), \\ f_1 - f_3 &\in (\eta_{13}), \\ f_2 - f_4 &\in (\eta_{24}), \\ f_3 - f_4 &\in (\eta_{34}), \end{aligned}$$

which is in particular satisfied if there exists an $f \in \Lambda_T[\zeta]$ such that

$$f_i = f \mod \eta_i, \quad i = 0, \ldots, 4. \tag{1}$$

To compute the remaining tuples $(f_0, \ldots, f_4)$, we can consequently fix $f_1 = 0$, and by the divisibility conditions in Corollary 6.34, such a tuple is then contained in the first-column component if there exist coefficients $c_0, c_2, c_3, c_4, c_4' \in \Lambda_T[\zeta]$, such that

$$\begin{aligned} f_0 &= c_0(\zeta + e_1 + e_4)(\zeta + e_2 + e_5)\eta \qquad \mod \eta_0, \\ f_2 &= c_2(\zeta + e_1 + e_4)(\zeta + e_2 + e_4)\eta \qquad \mod \eta_2, \\ f_3 &= c_3(\zeta + e_2 + e_4)(\zeta + e_2 + e_5)\eta \qquad \mod \eta_3, \\ f_4 &= c_2(\zeta + e_1 + e_4)(\zeta + e_2 + e_4)\eta + c_4(\zeta + e_2 + e_4)(\zeta + e_3 + e_4)\eta \qquad \mod \eta_4, \end{aligned}$$

$$f_4 = c_3(\zeta + e_2 + e_4)(\zeta + e_2 + e_5)\eta + c_4'(\zeta + e_2 + e_3)(\zeta + e_2 + e_4)\eta \mod \eta_4,$$

where

$$\eta \coloneqq (\zeta + e_1 + e_2)(\zeta + e_1 + e_3)(\zeta + e_3 + e_5)(\zeta + e_4 + e_5).$$

The conditions imposed on the coefficients simplify to

$$c_2(\zeta + e_1 + e_4) + c_4(\zeta + e_2 + e_4) = c_3(\zeta + e_2 + e_5) + c_4'(\zeta + e_2 + e_3), \tag{2}$$

modulo $\eta_4 / ((\zeta + e_2 + e_4)\eta)$. The only solution to this equation are the elementary syzygies (see Remark 5.26). In other words, the solutions (c_2, c_3, c_4, c_4') form the submodule of $(\Lambda_T[\zeta])^4$ spanned by

$$(\zeta + e_2 + e_4, 0, -\zeta - e_1 - e_4, 0), \quad (\zeta + e_2 + e_5, \zeta + e_1 + e_4, 0, 0), \quad (\zeta + e_2 + e_3, 0, 0, \zeta + e_1 + e_4),$$

$$(0, \zeta + e_2 + e_4, \zeta + e_2 + e_5, 0), \quad (0, 0, \zeta + e_2 + e_3, \zeta + e_2 + e_4), \quad (0, \zeta + e_2 + e_3, 0, -\zeta - e_2 - e_5).$$

In summary, the tuples in $\nu(P_{2,5})$ are $\Lambda_T[\zeta]$-linear combinations of the constant tuples (see (1)) and the tuples corresponding to the solutions of (2), which are generated by the above elements.

We conclude this subsection with a brief comparison of the localization images of $\mathrm{Gr}(d,n)$ and $P_{d,n}$, as well as of their first-column components.

Recall that, since R is a field, Example 2.31 shows that $\mathrm{Gr}(d,n)$ is equivariantly formal, i.e., a free Λ_T-module isomorphic to its localization image $\iota(\mathrm{Gr}(d,n))$ (see Theorem 2.38).

Remark 6.36. In the case that $\mathfrak{M}_{\mathfrak{C}(d,n)}$ is a cover by GKM-varities, the localization images $\iota(\mathrm{Gr}(d,n))$ and $\iota(P_{d,n})$ can be compared by considering the moment graphs of $\Gamma_{\mathrm{Gr}(d,n)}$ and $\Gamma_{P_{d,n}}$, respectively. Both graphs share the same set of vertices $I(d,n)$, but $\Gamma_{\mathrm{Gr}(d,n)}$ contains strictly fewer edges than $\Gamma_{P_{d,n}}$.

Indeed, as noted in Example 2.40, there is an edge between I and J in $\Gamma_{\mathrm{Gr}(d,n)}$ if and only if I can be obtained from J by swapping a single element. In this situation I and J are also comparable in the poset order on $I(d,n)$, and hence there is an edge between them in $\Gamma_{P_{d,n}}$. Consequently,

$$\iota(P_{d,n}) \subseteq \iota(\mathrm{Gr}(d,n)) \subseteq \Lambda_T^{I(d,n)}.$$

Remark 6.37. The first-column component was originally defined for covered spaces. An analogue for the ordinary Grassmannian can be established by considering the union of its T-curves.

Let $\mathfrak{K}$ denote the collection of all subsets of $I(d,n)$ of the form

$$\{I, J\}, \quad \text{where } I \setminus \{i\} = J \setminus \{j\} \text{ for some } 1 \leq i < j \leq n,$$

and set

$$\nu(\mathrm{Gr}(d,n)) \coloneqq \nu(P_{\mathfrak{K}}).$$

This definition is valid for arbitrary choices of characters $\psi_1, \ldots, \psi_n$ by which T acts on $\mathbb{C}^n$.

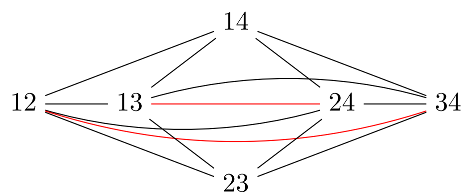
If $\mathrm{Gr}(d,n)$ is a GKM-variety, then $P_{\mathfrak{K}}$ coincides with the union of the finitely many T-curves between fixed points. In this case, $P_{\mathfrak{K}}$ is covered by GKM-varieties and described by its moment graph (see Theorem 4.10), which by construction is equal to the moment graph of $\mathrm{Gr}(d,n)$. Consequently,

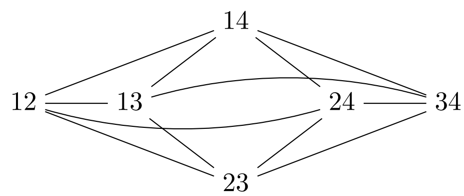
$$\nu(\mathrm{Gr}(d,n)) = \nu(P_{\mathfrak{K}}) \cong \iota(P_{\mathfrak{K}}) = \iota(\mathrm{Gr}(d,n)),$$

as expected for the first-column component in the case of finitely many fixed points (cf. Corollary 5.38 for projective unions).

Example 6.38. If we consider the Grassmannian torus action of the diagonal torus $T \subseteq \mathrm{GL}_n$ induced by its action via standard characters $e_1, \ldots, e_n$ on $\mathbb{C}^n$, then $\mathrm{Gr}(d,n)$ is a GKM-variety and $P_{d,n}$ is a union of GKM-varieties by Example 6.17. As a consequence, we can compare the torsion-free part of the respective equivariant cohomologies by either comparing the moment graphs as in Remark 6.36, or the first-column components as described in Remark 6.37.

Example 6.39. Below we display the moment graphs of $P_{2,4}$ and $\mathrm{Gr}(2,4)$, respectively, with the additional edges present in $\Gamma_{P_{d,n}}$ but not in $\Gamma_{\mathrm{Gr}(d,n)}$ highlighted in red.





A Spectral Sequences

This appendix provides an introduction to spectral sequences tailored to the needs of the thesis. The exposition mainly follows [God73, Chapter 1.4] and [McC01, Chapters 1–3].
Let R be a ring and K an R-module with (decreasing non-negative) filtration of submodules

$$K = K_0 \supseteq K_1 \supseteq \dots .$$

The *associated graded module* is given by

$$G(K) = \bigoplus K_p/K_{p+1},$$

with the grading $G(K)^p = K_p/K_{p+1}$. If $K = \bigoplus_{q\geq 0} K^q$ itself is graded, and the grading is compatible with the filtration of K, i.e., if

$$K_p = \bigoplus_{q\geq 0} K_p \cap K^q,$$

we say that K is a *filtered graded module*. In this situation, the associated graded module obtains a double grading

$$G(K)^{p,q} = (K^{p+q} \cap K_p)/(K^{p+q} \cap K_{p+1}).$$

Example A.1. For a bigraded R-module $K = \bigoplus_{i,j} K^{i,j}$, there are two natural filtrations given by

$${}'K_p = \bigoplus_{i\geq p} K^{i,j} \quad \text{and} \quad {}''K_p = \bigoplus_{j\geq p} K^{i,j},$$

called the *row-wise filtration* and the *column-wise filtration*, respectively. Both of these are compatible with both canonical gradings and the total grading on K.

Definition A.2. A *differential R-module* (K, d) is an R-module K together with an endomorphism d verifying $d^2 = 0$. We write

$$Z(K) = \ker(d), \qquad B(K) = \operatorname{im}(d),$$

and

$$H(K) = H(K, d) = Z(K)/B(K),$$

for the *derived module* of K.

We call K a *filtered differential module* if $d(K_p) \subseteq K_p$ for all p, and write

$$Z_r^p = Z(K_p \mod K_{p+r}),$$

for the set of elements $x \in K_p$ with $dx \in K_{p+r}$. Evidently, Z_r^p contains both Z_{r-1}^{p+1} as well as dZ_{r-1}^{p-r+1}, so we can form the quotient

$$E_r^p \coloneqq Z_r^p/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}), \quad \text{and} \quad E_r \coloneqq \bigoplus_{p\geq 0} E_r^p.$$

Theorem A.3. *For each $r \geq 0$, the endomorphism d_r on E_r induced by d makes E_r into a differential module, and E_{r+1} is canonically isomorphic to $H(E_r)$ as a graded module.*

We define the *spectral sequence of the filtered differential module* K as the collection of differential graded modules $\{(E_r, d_r)\}_{r\geq 0}$. For brevity, we sometimes write $\{E_r\}$ or $\{E_r, d_r\}$. The module E_r is referred to as the *rth page* of the spectral sequence.

Example A.4. If $r \leq 0$, then $Z_r^p = K_p$. For example, we have

$$Z_0^p = K_p, \quad Z_{-1}^{p+1} = K_{p+1}, \quad dZ_{-1}^{p+1} \subseteq K_{p+1},$$

and therefore

$$E_0^p = K_p/K_{p+1}, \text{ as well as } E_1^p = H(K_p/K_{p+1}).$$

Set $K_\infty = 0$ and $K_{-\infty} = K$, and further define

$$Z_\infty^p := Z(K_p \mod K_\infty), \qquad B_\infty^p = K_p \cap dK_{-\infty},$$

as the cycles and boundaries of K_p in K. The *infinity page* of the spectral sequence is given by the graded module

$$E_\infty^p := Z_\infty^p/(B_\infty^p + Z_\infty^{p+1}), \qquad E_\infty := \bigoplus_{p\geq 0} E_\infty^p.$$

If we define more generally that

$$B_r^p := K_p \cap dK_{p-r},$$

we obtain the description

$$E_r^p = Z_r^p/(B_{r-1}^p + Z_{r-1}^{p+1}).$$

The filtration on K induces a filtration on $H(K)$, defined by

$$H(K)_p := \operatorname{im}\big(H(K_p) \to H(K)\big).$$

Theorem A.5. *There is a canonical isomorphism*

$$E_\infty \cong G(H(K)),$$

of graded modules.

Now, assume that K is a *filtered complex*, i.e., that K is a filtered differential module with a compatible grading and a differential d that is homogeneous of degree 1. Then, also the grading and the filtration on $H(K)$ are compatible, and we obtain the double grading

$$G(H(K))^{p,q} = H^{p+q}(K)_p/H^{p+q}(K)_{p+1},$$

where $H^{p+q}(K)_p := H^{p+q}(K) \cap H(K)_p$, on the associated graded module. If we introduce the corresponding bigrading on E_r by setting

$$Z_r^{p,q} = Z_r^p \cap K^{p+q}, \quad B_r^{p,q} = B_r^p \cap K^{p+q},$$

and

$$E_r^{p,q} = Z_r^{p,q}/(B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}),$$

for all $r \geq 0$ and $r = \infty$, then the isomorphisms in Theorem A.3 and Theorem A.5 are isomorphisms of bigraded modules. More specifically,

$$E_\infty^{p,q} \cong H^{p+q}(K)_p/H^{p+q}(K)_{p+1},$$

for $p, q \geq 0$.

Remark A.6. For each $r \geq 0$, the differential d_r on the differential bigraded module $E_r^{*,*}$ has bidegree $(r, 1-r)$, that is,

$$d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

Definition A.7. We call a filtration on a filtered complex *regular* if for any q we have an integer $n(q)$ with

$$K_p \cap K^q = 0 \quad \text{for } p > n(q).$$

If the filtration on K is regular, then the differential d_r vanishes on $E_r^{p,q}$ for sufficiently large r, and hence there are epimorphisms

$$\theta_s^r \colon E_r^{p,q} \longrightarrow E_s^{p,q},$$

for $s > r > n(p+q+1) - p$, which allow us to define an inductive limit of $E_r^{p,q}$ as $r \to \infty$. This limit is, in fact, the module $E_\infty^{p,q}$.

Remark A.8. In the above case, we say that the spectral sequence *converges* to E_∞, and by virtue of Theorem B.1, also to $H(K)$, even though the actual isomorphism is between E_∞ and $G(H(K))$. If there further exists an r_0 such that $E_r = E_s$ for all $r_0 \leq r \leq s$, then $E_{r_0} = E_\infty$, and the spectral sequence is said to *collapse* at the r_0th page.

Example A.9. Assume that $E_2^{p,q} = 0$ for $p > 1$. The spectral sequence collapses due to the bidegree of the differential on E_2 (see Remark A.6), and we obtain short exact sequences

$$0 \longrightarrow E_2^{1,q-1} \longrightarrow H^q(K) \longrightarrow E_2^{0,q} \longrightarrow 0,$$

for all $q \geq 0$.

Remark A.10. Suppose that for some $r_0, p_0 \geq 0$, we have $E_{r_0}^p = 0$ for all $p > p_0$. Then $E_r^p = 0$ for all $r \geq r_0$ and $p > p_0$, and consequently $H(K)_p = 0$ for all $p > p_0$.

The most extreme case is when there exists an $r_0 \geq 1$ such that $E_{r_0}^p = 0$ for all $p > 0$. In this situation, the spectral sequence collapses at the r_0th page. Moreover, the filtration is trivial, i.e.,

$$H(K)_1 = H(K)_2 = \ldots = 0 \text{ and } G(H(K)) = H(K).$$

We take a closer look at the differential d_1: we know that

$$Z_0^p = K_p, \quad Z_{-1}^{p+1} = K_{p+1}, \quad dZ_{-1}^{p+1} \subseteq K_{p+1},$$

and consequently

$$E_0^p = K_p / K_{p+1}.$$

On the first page of the spectral sequence, we get

$$E_1^p = H(K_p / K_{p+1}),$$

and as differential

$$d_1 \colon E_1^p = H(K_p/K_{p+1}) \longrightarrow H(K_{p+1}/K_{p+2}) = E_1^{p+1},$$
$$z \in Z(K_p \mod K_{p+1}) \longmapsto dz \in Z(K_{p+1} \mod K_{p+2}).$$

Remark A.11. The differential d_1 can be obtained as the *operator* of the short exact sequence of differential graded modules

$$0 \longrightarrow K_{p+1}/K_{p+2} \longrightarrow K_p/K_{p+2} \longrightarrow K_p/K_{p+1} \longrightarrow 0,$$

meaning the natural morphism

$$\delta\colon H(K_p/K_{p+1}) \longrightarrow H(K_{p+1}/K_{p+2}),$$

obtained by applying the snake lemma.

A further notion we require is that of a morphism of spectral sequences.

Definition A.12. Given two spectral sequences $\{(E_r, d_r)\}_{r\geq 0}$ and $\{(\overline{E}_r, \overline{d}_r)\}_{r\geq 0}$, we define a *morphism of spectral sequences* as a collection $\{f_r\}_{r\geq 0}$ of morphisms of bigraded modules $f_r\colon E_r \to \overline{E}_r$, commuting with the differentials, and such that f_{r+1} is induced by f_r on cohomology.

Remark A.13. A morphism of filtered complexes

$$\phi\colon (K, d) \longrightarrow (\overline{K}, \overline{d}),$$

gives rise to a morphism of spectral sequences

$$\phi_r\colon E_r \longrightarrow \overline{E}_r, \quad r \geq 0,$$

and $\phi_\infty\colon G(H(K)) \to G(H(\overline{K}))$, is induced by

$$H(\phi)\colon H(K) \longrightarrow H(\overline{K}).$$

Theorem A.14. *Let*

$$\phi\colon (K, d) \longrightarrow (\overline{K}, \overline{d}),$$

be a morphism of filtered complexes with associated spectral sequences $\{E_r\}$ and $\{\overline{E}_r\}$. If for some n, $\phi_n\colon E_n \to \overline{E}_n$ is an isomorphism of bigraded modules, then ϕ_r is an isomorphism for all $n \leq r \leq \infty$. If the filtrations are regular, then ϕ induces an isomorphism of graded modules

$$H(\phi)\colon H(K, d) \longrightarrow H(\overline{K}, \overline{d}).$$

Proof. [McC01, Theorem 3.5]. □

Remark A.15. The proof of Theorem A.14 relies on the isomorphism ϕ_∞, together with an inductive application of the Five Lemma, to conclude that $H(\phi)$ is an isomorphism. This reasoning extends beyond the context of spectral sequences and applies more generally: given a morphism of filtered graded modules

$$\phi\colon A \longrightarrow \overline{A},$$

if the induced morphism on the associated graded modules

$$\hat{\phi}\colon G(A) \longrightarrow G(\overline{A})$$

is an isomorphism (respectively, an epimorphism or a monomorphism) of bigraded modules, and both filtrations are regular, then ϕ itself is an isomorphism (respectively, an epimorphism or a monomorphism).

Finally, we adapt the results introduced above to the case that we work with an R-algebra instead of an R-module.

Definition A.16. Suppose K is a complex with a graded product. We call K a *differential graded algebra* if the product satisfies the graded Leibniz rule. If K is, in addition, a filtered complex and

$$K_s \cdot K_r \subseteq K_{s+r},$$

for all $s, r \geq 0$ then we call K a *filtered differential graded algebra.*

Remark A.17. We can proceed as before to obtain a *spectral sequence of algebras* $\{E_r, d_r\}_{r\geq 0}$, i.e., a spectral sequence where each page E_r is a differential bigraded algebra, and E_{r+1} is canonically isomorphic to $H(E_r)$ as bigraded algebra. The isomorphism in Theorem A.5 is an isomorphism of bigraded algebras. A morphism of filtered differential graded algebras

$$\phi\colon (K, d) \longrightarrow (\overline{K}, \overline{d}),$$

induces a morphism of spectral sequences of algebras, and the isomorphism in Theorem A.14 is an isomorphism of graded algebras.

Remark A.18. The above discussion applies without change to the case that

$$R = \bigoplus_{t\geq 0} R^t,$$

is a graded ring. If K is a differential graded algebra over R, then the differential d is assumed to be graded R-linear, i.e.,

$$d(r \cdot x) = (-1)^t r \cdot d(x), \quad \text{for } r \in R^t, x \in K.$$

This is a special case of the situation in which R itself is a differential graded algebra and K is a differential graded R-algebra via a morphism of differential graded algebras $R \to K$ (see, for example [Wei94, Chapter 9.9]).

B The Spectral Sequence of a Double Complex

This section follows the construction of a spectral sequence from a double complex as presented in [God73, Chapter 1.4.8], complemented by [McC01], in particular Chapter 2.4.
Let

$$K = \sum_{p,q\geq 0} K^{p,q},$$

be a double complex with differentials

$$d'\colon K^{p,q} \longrightarrow K^{p+1,q}, \quad d''\colon K^{p,q} \longrightarrow K^{p,q+1},$$

satisfying $d'd'' + d''d' = 0$. There is an associated filtered complex, the *total complex*

$$\text{tot}(K)^n = \sum_{p+q=n} K^{p,q},$$

with differential $d = d' + d''$, and either of the two filtrations presented in Example A.1. Since we assume that $K^{p,q} = 0$ if either $p < 0$ or $q < 0$, both filtrations on $\operatorname{tot}(K)$ are regular.
The first page of the spectral sequence associated to the row-wise filtration is

$$'E_1^p = H('K_p/'K_{p+1}),$$

and we can identify $'K_p/'K_{p+1}$, together with its differential d_0, with the complex K^{p*} and the differential d''. This leads to

$$'E_1^p = H(K^{p,*}, d'').$$

We compute d_1 as the operator of the short exact sequence

$$0 \longrightarrow K^{p+1,*} \longrightarrow K^{p,*} + K^{p+1,*} \longrightarrow K^{p,*} \longrightarrow 0$$

in the fashion of Remark A.11. In accordance with the previous identification, the differential on the first and last module is d'', whereas the middle module has d acting on K^{p*} as well as d'' on $K^{p+1,*}$. It follows that the operator of this exact sequence is induced by d'. Thus, if we equip the module $H(K, d'')$ with the natural grading

$$H^p(K, d'') = H(K^{p,*}, d''),$$

and with the differential induced by d', we obtain

$$'E_2^p = H^p(H(K, d''), d').$$

Including the second grading, we have that $'E_1^{p,q}$ is equal to

$$'E_1^{p,q} = H^q(K^{p,*}, d''),$$

the cohomology group of degree q with respect to the second differential and second grading of K. Hence, we arrive at

$$'E_2^{p,q} = H^p(H^q(K, d''), d').$$

Theorem B.1. *Let K be a double complex with $K^{p,q} = 0$ if $p < 0$ or $q < 0$, and suppose that*

$$H^p(H^q(K, d''), d') = 0, \text{ for } q \geq 1.$$

Let L be the R-submodule of K formed by the sums of elements $x^{p0} \in K^{p0}$ satisfying $d''x^{p0} = 0$, then the injection $L \hookrightarrow K$ induces an isomorphism in cohomology.

Proof. We consider L as double complex and compare the spectral sequences $\{E'_r(L)\}$ and $\{E'_r(K)\}$ induced by the row-wise filtration. By definition of L, we have

$$'E_2^{p,q}(L) = 0, \text{ for } q \neq 0, \text{ and } 'E_2^{p,0} = H^p(L).$$

On the other hand, from the assumption on K, it follows that

$$'E_2^{p,0}(K) = H^p(H^0(K, d''), d'), \text{ and } H^0(K, d'') = L,$$

which leads to

$$'E_2^{p,0}(K) = H^p(L).$$

Since the embedding of L in K induces an isomorphism of bigraded modules on the second page of the first spectral sequence and all relevant filtrations are regular, the claim is a consequence of Theorem A.14. □

Remark B.2. If $\mathrm{tot}(K)$ is a filtered differential graded algebra with respect to both the row-wise and column-wise filtration, then L defined as in Theorem B.1 is a graded subalgebra and the isomorphism is an isomorphism of graded algebras.

In Appendix A and Appendix B, up to this point, we restricted our attention to the setting of a decreasing non-negative filtration, a non-negative grading and a differential of degree one on K. The resulting spectral sequence $E_r^{p,q}$ is a *first-quadrant spectral sequence*, a name justified by the fact that $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$.
There are analogous definitions for spectral sequences corresponding to the more general situation that K is $\mathbb{Z}$-graded, and the filtration is either decreasing or increasing with indices in $\mathbb{Z}$. This generalization is not included in this thesis and once again we refer to [McC01] for a rigorous treatment.
Nonetheless, we want to give an example of a *second-quadrant* spectral sequence, the *Künneth spectral sequence*.
The version of the Künneth spectral sequence we consider can be found in [McC01, Chapter 2.4]. We follow the introduction presented there.
Let L^* and T^* be differential graded R-modules with differentials of degree one and T^* flat over R. The target of the Künneth spectral sequence is the cohomology $H^*(T^* \otimes L^*)$ of the tensored complexes. This goal is approached by considering a *proper projective resolution* of L^*:
Suppose we have an exact sequence of differential graded R-modules

$$\dots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow L \longrightarrow 0.$$

For any differential graded R-module M we write d_M for its differential as well as

$$Z^n(M) \coloneqq \ker d_M \colon M^n \longrightarrow M^{n+1}, \quad \text{and} \quad H^n(M) \coloneqq Z^n(M)/d_M(M^{n-1}).$$

The derived module $H^*(M)$ is considered as differential graded module by equipping it with the zero differential. We call the exact sequence above a *proper projective resolution* of L^* if for each n, the following are projective resolutions

$$C_1: \qquad \dots \longrightarrow P^{-2,n} \longrightarrow P^{-1,n} \longrightarrow P^{0,n} \longrightarrow L^n \longrightarrow 0,$$

$$C_2: \qquad \dots \longrightarrow Z^n(P^{-2}) \longrightarrow Z^n(P^{-1}) \longrightarrow Z^n(P^0) \longrightarrow Z^n(L) \longrightarrow 0,$$

$$C_3: \qquad \dots \longrightarrow H^n(P^{-2}) \longrightarrow H^n(P^{-1}) \longrightarrow H^n(P^0) \longrightarrow H^n(L) \longrightarrow 0.$$

There exists a proper projective resolution for every differential graded module ([McC01, Chapter 2.4, Lemma 2.19]).

Suppose we are given such a proper projective resolution for L^*, as shown in the diagram below.

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\longrightarrow & & P^{-2,2} & \longrightarrow & P^{-1,2} & \longrightarrow & P^{0,2} & \longrightarrow & L^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\longrightarrow & & P^{-2,1} & \longrightarrow & P^{-1,1} & \longrightarrow & P^{0,1} & \longrightarrow & L^1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_1} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_L} & & \\
\longrightarrow & & P^{-2,0} & \xrightarrow{d_P} & P^{-1,0} & \xrightarrow{d_P} & P^{0,0} & \xrightarrow{d_P} & L^0 & \longrightarrow & 0.
\end{array}$$

The relevant double complex is then given by

$$K^{p,q} := \bigoplus_{s+t=q} T^s \otimes P^{p,t},$$

with differentials

$$d' = \sum (-1)^q 1 \otimes d_P \colon K^{p,q} \longrightarrow K^{p+1,q},$$

and

$$d'' = \left(\sum d_T \otimes 1 + \sum (-1)^s 1 \otimes d_p\right) : K^{p,q} \longrightarrow K^{p,q+1}.$$

As opposed to the double complexes $K^{*,*} = \bigoplus K^{p,q}$ considered in the first part of this chapter, in this case, the degree q is non-negative and the degree p is *non-positive*. In other words, $K^{*,*}$ lies in the second quadrant of $\mathbb{Z}\times\mathbb{Z}$. Nevertheless, and in the same way as before, we can associate spectral sequences to K and its canonical filtrations, and by a similar procedure as for the Mayer–Vietoris spectral sequence in Chapter 3.2, we obtain the following result.

Theorem B.3. *Let L^* and T^* be differential graded R-modules with differentials of degree one and T^* flat over R. There is a spectral sequence with second page given by*

$$E_2^{p,q} = \bigoplus_{s+t=q} \operatorname{Tor}^R_{-p}(H^s(T), H^t(L)).$$

If $\{E_r^{*,*}\}$ converges, then it converges to

$$H(T^* \otimes L^*, d_T \otimes d_L).$$

Here, $\operatorname{Tor}_i^R(-,-)$ denotes the i-th left derived functor of the tensor product over R.

Proof. [McC01, Chapter 2.4, Theorem 2.20]. □

Remark B.4. Until now, we have relied on the assumption that the filtration is regular in order to guarantee convergence of the spectral sequence. The filtration

$$\ldots \supseteq K_{-2} \supseteq K_{-1} \supseteq K_0,$$

associated to the Künneth spectral sequence, where

$$K_p = \bigoplus_{r \geq p} K^{r,s},$$

is, in general, not regular. As a consequence, convergence becomes a more intricate matter and depends on the related notion of the filtration being *weakly convergent*. For a precise definition and further discussion see [McC01, Chapter 3.1].

Example B.5. Suppose T^* is a flat differential graded R-module and $L = R/I$ is a quotient of R, considered as differential graded R-module with zero differential. In this case, any projective resolution

$$\ldots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow R/I,$$

of R/I is automatically proper, and the double complex used to construct the spectral sequence in Theorem B.3 reduces to

$$K^{p,q} = T^q \otimes P^p.$$

In particular, if R/I has finite projective dimension, then both filtrations on $\mathrm{tot}(K)$ are regular and we are in a situation analogous to that in Appendix A with regard to convergence.
In the special case where R is Noetherian and $I = (\eta)$ is a principal ideal, we know that R/I has finite projective dimension if η is not a zero divisor in R. Moreover, the Tor-groups take the form

$$\mathrm{Tor}^R_{-p}(R/(\eta), H^q(T)) = \begin{cases} H^q(T)/\left(\eta \cdot H^q(T)\right), & p = 0, \\ \mathrm{Ann}(\eta, H^q(T)), & p = -1, \\ 0, & \text{else}. \end{cases}$$

where

$$\mathrm{Ann}(\eta, H^q(T)) \coloneqq \{x \in H^q(T) \mid \eta \cdot x = 0\},$$

denotes the annihilator of η in $H^q(T)$. Thus we are in the situation of Example A.9 translated into the setting of a second-quadrant spectral sequence. In other words, the Künneth spectral sequence collapses at the second page, and we obtain short exact sequences

$$0 \longrightarrow H^q(T)/\left(\eta \cdot H^q(T)\right) \longrightarrow H^q(T/\eta \cdot T) \longrightarrow \mathrm{Ann}(\eta, H^{q+1}(T)) \longrightarrow 0,$$

for all $q \geq 0$.

List of Notation

Δ_p	set of p-simplices of a simplicial complex Δ
$\underline{v} = (v_0, \ldots, v_p)$	p-simplex in a simplicial complex
$\langle \mathfrak{D} \rangle$	simplicial complex whose simplices are all subsets of elements of $\mathfrak{D}$
Λ_G	equivariant coefficient ring of a Lie group G
pt	topological space consisting of a single point
ζ	Chern class of the line bundle dual to the tautological line bundle $\mathscr{O}(-1)$ on a projective space
$T_{M'}$	subgroup of a torus T corresponding to a sublattice $M' \subseteq M$, where M is the character lattice of T
$M_{T'}$	sublattice of the character lattice M of T corresponding to a subgroup $T' \subseteq T$
$\iota(X), \tau(X)$	image and kernel of the localization map ι^* for a T-space X
$\mathrm{tor}(S, M)$	submodule of the S-module M consisting of elements that are torsion over S
Γ_X	moment graph of a T-space X
$\mathfrak{M} = (M_i)_{i \in I}$	cover by topological subspaces
I	from Chapter 3.2 onward, indexing set for subspaces in the cover $\mathfrak{M}$
$\mathcal{I}$	full simplex on the set I
$M_{\underline{i}}$	intersection $M_{i_0} \cap \ldots \cap M_{i_p}$ corresponding to a p-simplex $\underline{i} = (i_0, \ldots, i_p)$ in $\mathcal{I}$
$\{E_r, d_r\}$	spectral sequence expressed as collection of its pages and their differentials
$G(H^*(K)), G(H^*_G(X)), \ldots$	graded module associated to a filtration
$\mathrm{MV}, \mathrm{MV}(G, X, \mathfrak{M})$	Mayer–Vietoris complex of a G-space X with cover $\mathfrak{M}$
$\nu(X), \nu(G, X, \mathfrak{M})$	first-column component of a G-space X (with cover $\mathfrak{M}$)
GCov	category with objects given by triplets $(G, X, \mathfrak{M})$, i.e., G-spaces with a cover $\mathfrak{M}$ such that the Mayer–Vietoris spectral sequence computes the G-equivariant cohomology of X

f^*	morphism in cohomology induced by a morphism f between topological spaces
$\hat{f}$	morphism between associated graded algebras induced by a morphism between objects in **GCov**
f_i	restriction of the morphism f in **GCov** to the corresponding component of the cover
$f_{\underline{i}}$	restriction of the morphism f in **GCov** to the corresponding intersection of components in the cover
$S(M')$	saturation of a sublattice M'
$\mathcal{R}, \mathcal{R}_\varphi$	relations introduced by a morphism between tori φ
θ, θ_φ	restriction of tori corresponding to a morphism between tori φ
Θ, Θ_φ	extension of tori corresponding to a morphism between tori φ
$\mathbb{P} = \mathbb{P}(\mathbb{C}^A)$	ambient projective space with coordinates indexed by an index set A
P_J	coordinate subspace of $\mathbb{P}$ defined by the vanishing of coordinates not contained in $J \subseteq A$
$\mathfrak{C}$	set of subsets of A indexed by the index set I
$P_{\mathfrak{C}}$	projective union corresponding to the coordinate subspaces defined by the sets in $\mathfrak{C}$
P_i	coordinate subspace defined by C_i, for $i \in I$
$P_{\underline{i}}$	coordinate subspace defined by $C_{\underline{i}} = C_{i_0} \cap \ldots \cap C_{i_p}$, where $\underline{i} = (i_0, \ldots, i_p) \in \mathcal{I}_p$
$\overline{\mathfrak{C}}$	union of all sets in $\mathfrak{C}$
$\underline{\mathfrak{C}}$	intersection of all sets in $\mathfrak{C}$
$\mathfrak{M}_{\mathfrak{C}}$	cover of $P_{\mathfrak{C}}$ formed by $(P_i)_{i \in I}$
η_J	completely reducible polynomial in $\Lambda_T[\zeta]$ with factors corresponding to the elements in $J \subseteq A$
Ω_J	equivariant cohomology ring of P_J
$\eta_i, \eta_{\underline{i}}, \Omega_i, \Omega_{\underline{i}}$	as above, abbreviations for $\eta_{C_i}, \eta_{C_{\underline{i}}}, \Omega_{C_i}, \Omega_{C_{\underline{i}}}$
$\iota^{\mathfrak{C}}_{\mathfrak{D}}$	restriction in cohomology between projective unions $P_{\mathfrak{C}}$ and $P_{\mathfrak{D}}$
g^i_S	a basis element in the cohomology ring of the poset S
r^S_Z	restriction map between the poset cohomologies of posets $Z \subseteq S$
$I(d,n)$	poset of strictly increasing d-tuples with entries between 1 and n
$\mathbb{P}_{d,n}$	ambient projective space with coordinates indexed by $I(d,n)$
$\mathfrak{C}(d,n)$	set of maximal chains of $I(d,n)$
$P_{d,n}$	degeneration of the Grassmannian $\mathrm{Gr}(d,n)$, i.e., the projective union associated to $\mathfrak{C}(d,n)$
$G_{d,n}$	graph with vertex set $\mathfrak{C}(d,n)$ and edges indicating maximal intersection

References

[AB84] Michael F. Atiyah and Raoul Bott. "The moment map and equivariant cohomology". English. In: *Topology* 23 (1984), pp. 1–28. ISSN: 0040-9383. DOI: 10.1016/0040-9383(84)90021-1.

[AF24] David Anderson and William Fulton. *Equivariant cohomology in algebraic geometry.* English. Vol. 210. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2024. ISBN: 978-1-00-934998-7; 978-1-00-934997-0; 978-1-00-934999-4. DOI: 10.1017/9781009349994.

[AFP14] Christopher Allday, Matthias Franz, and Volker Puppe. "Equivariant cohomology, syzygies and orbit structure". English. In: *Trans. Am. Math. Soc.* 366.12 (2014), pp. 6567–6589. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2014-06165-5.

[Ahn24] Jonghyeon Ahn. *S^1-equivariant relative symplectic cohomology and relative symplectic capacities.* Preprint, arXiv:2410.01977 [math.SG] (2024). 2024. URL: https://arxiv.org/abs/2410.01977.

[Bjö95] Anders Björner. "Topological methods". English. In: *Handbook of combinatorics. Vol. 1-2.* Amsterdam: Elsevier (North-Holland); Cambridge, MA: MIT Press, 1995, pp. 1819–1872. ISBN: 0-444-88002-X; 0-444-82346-8; 0-444-82351-4; 0-262-07169-X; 0-262-07170-3; 0-262-07171-1.

[BL00] Sara Billey and V. Lakshmibai. *Singular loci of Schubert varieties.* English. Vol. 182. Prog. Math. Boston, MA: Birkhäuser, 2000. ISBN: 3-7643-4092-4.

[Bor60] Armand Borel. *Seminar on transformation groups. With contributions by G. Bredon, E. E. Floyd, D. Montgomery and R. Palais.* English. Vol. 46. Ann. Math. Stud. Princeton University Press, Princeton, NJ, 1960. DOI: 10.1515/9781400882670.

[Bre74] Glen E. Bredon. "The free part of a torus action and related numerical equalities". English. In: *Duke Math. J.* 41 (1974), pp. 843–854. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-74-04184-2.

[Bri98] Michel Brion. "Equivariant cohomology and equivariant intersection theory. (Notes by Alvaro Rittatore)". English. In: *Representation theories and algebraic geometry. Proceedings of the NATO Advanced Study Institute, Montreal, Canada, July 28–August 8, 1997.* Dordrecht: Kluwer Academic Publishers, 1998, pp. 1–37. ISBN: 0-7923-5193-2.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology.* English. Vol. 82. Grad. Texts Math. Springer, Cham, 1982.

[CFL23] Rocco Chirivı, Xin Fang, and Peter Littelmann. "Seshadri stratifications and standard monomial theory". English. In: *Invent. Math.* 234.2 (2023), pp. 489–572. ISSN: 0020-9910. DOI: 10.1007/s00222-023-01206-4.

[CG97] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry.* English. Boston, MA: Birkhäuser, 1997. ISBN: 0-8176-3792-3.

[CHN23] Marc Chardin, Rafael Holanda, and José Naéliton. *Homology of multiple complexes and Mayer-Vietoris spectral sequences.* Preprint, arXiv:2306.02119 [math.AC] (2023). 2023. DOI: 10.1090/proc/16551. URL: https://arxiv.org/abs/2306.02119.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. English. Vol. 124. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2011. ISBN: 978-0-8218-4819-7.

[CS74] Theodore Chang and Tor Skjelbred. "The Topological Schur Lemma and Related Results". In: *Annals of Mathematics* 100.2 (1974), pp. 307–321. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1971074 (visited on 08/02/2025).

[DEP82] Corrado De Concini, David Eisenbud, and Claudio Procesi. *Hodge algebras*. English. Vol. 91. Astérisque. Société Mathématique de France (SMF), Paris, 1982.

[DP95] C. De Concini and C. Procesi. "Wonderful models of subspace arrangements". English. In: *Sel. Math., New Ser.* 1.3 (1995), pp. 459–494. ISSN: 1022-1824. DOI: 10.1007/BF01589496.

[Eis05] David Eisenbud. *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*. English. Vol. 229. Grad. Texts Math. New York, NY: Springer, 2005. ISBN: 0-387-22232-4; 0-387-22215-4. DOI: 10.1007/b137572.

[Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry*. English. Vol. 150. Grad. Texts Math. Berlin: Springer-Verlag, 1995. ISBN: 3-540-94269-6; 3-540-94268-8.

[ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. English. Vol. 15. Princeton Math. Ser. Princeton University Press, Princeton, NJ, 1952.

[FM19] Roberto Frigerio and Andrea Maffei. *A remark on the Mayer-Vietoris double complex for singular cohomology*. Preprint, arXiv:1912.07736 [math.AT] (2019). 2019. URL: https://arxiv.org/abs/1912.07736.

[FN24] Naoki Fujita and Yuta Nishiyama. "Combinatorics of semi-toric degenerations of Schubert varieties in type C". English. In: *Math. Z.* 307.4 (2024). Id/No 69, p. 24. ISSN: 0025-5874. DOI: 10.1007/s00209-024-03531-7.

[FP07] Matthias Franz and Volker Puppe. "Exact cohomology sequences with integral coefficients for torus actions". English. In: *Transform. Groups* 12.1 (2007), pp. 65–76. ISSN: 1083-4362. DOI: 10.1007/s00031-005-1127-0.

[FP08] Matthias Franz and Volker Puppe. "Freeness of equivariant cohomology and mutants of compactified representations". English. In: *Toric topology. International conference, Osaka, Japan, May 28–June 3, 2006*. Providence, RI: American Mathematical Society (AMS), 2008, pp. 87–98. ISBN: 978-0-8218-4486-1.

[FP11] Matthias Franz and Volker Puppe. "Exact sequences for equivariantly formal spaces". English. In: *C. R. Math. Acad. Sci., Soc. R. Can.* 33.1 (2011), pp. 1–10. ISSN: 0706-1994.

[Fra17] Matthias Franz. "A quotient criterion for syzygies in equivariant cohomology". English. In: *Transform. Groups* 22.4 (2017), pp. 933–965. ISSN: 1083-4362. DOI: 10.1007/s00031-016-9408-3.

[Fra24] Matthias Franz. "The Chang-Skjelbred lemma and generalizations". English. In: *Group actions and equivariant cohomology. AMS special session on equivariant cohomology, virtual, March 19–20, 2022*. Providence, RI: American Mathematical Society (AMS), 2024, pp. 103–112. ISBN: 978-1-4704-7180-4; 978-1-4704-7723-3. DOI: 10.1090/conm/808/16182.

[Ful93] William Fulton. *Introduction to toric varieties. The 1989 William H. Roever lectures in geometry*. English. Vol. 131. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 1993. ISBN: 0-691-00049-2. DOI: 10.1515/9781400882526.

[Ful97] William Fulton. *Young tableaux. With applications to representation theory and geometry*. English. Vol. 35. Lond. Math. Soc. Stud. Texts. Cambridge: Cambridge University Press, 1997. ISBN: 0-521-56724-6.

[FY19] Matthias Franz and Hitoshi Yamanaka. "Graph equivariant cohomological rigidity for GKM graphs". English. In: *Proc. Japan Acad., Ser. A* 95.10 (2019), pp. 107–110. ISSN: 0386-2194. DOI: 10.3792/pjaa.95.107.

[FZ00] Eva Maria Feichtner and Günter M. Ziegler. "On cohomology algebras of complex subspace arrangements". English. In: *Trans. Am. Math. Soc.* 352.8 (2000), pp. 3523–3555. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-00-02537-X.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". English. In: *Invent. Math.* 131.1 (1998), pp. 25–83. ISSN: 0020-9910. DOI: 10.1007/s002220050197.

[God73] Roger Godement. *Topologie algébrique et théorie des faisceaux. 3e éd. revue et corrigée*. French. Vol. 13. Publ. Inst. Math. Univ. Strasbourg. Hermann, Paris, 1973.

[Hat02] Allen Hatcher. *Algebraic topology*. English. Cambridge: Cambridge University Press, 2002. ISBN: 0-521-79540-0.

[Hat03] Allen Hatcher. *Vector Bundles and K-Theory*. Available at https://pi.math.cornell.edu/~hatcher/VBKT/VBKT.pdf. 2003. URL: https://pi.math.cornell.edu/~hatcher/VBKT/VBKT.pdf.

[HK15] Megumi Harada and Kiumars Kaveh. "Integrable systems, toric degenerations and Okounkov bodies". English. In: *Invent. Math.* 202.3 (2015), pp. 927–985. ISSN: 0020-9910. DOI: 10.1007/s00222-014-0574-4.

[Hu08] Yi Hu. "Toric degenerations of GIT quotients, Chow quotients, and $\mathrm{M}_{0,n}$". English. In: *Asian J. Math.* 12.1 (2008), pp. 47–53. ISSN: 1093-6106. DOI: 10.4310/AJM.2008.v12.n1.a3.

[Hum81] James E. Humphreys. *Linear algebraic groups. Corr. 2nd printing*. English. Vol. 21. Grad. Texts Math. Springer, Cham, 1981.

[HW18] Tara S. Holm and Gareth Williams. "Mayer-Vietoris sequences and equivariant K-theory rings of toric varieties". English. In: *Homology Homotopy Appl.* 21.1 (2018), pp. 375–401. ISSN: 1532-0073. DOI: 10.4310/HHA.2019.v21.n1.a18.

[JOS94] Ken Jewell, Peter Orlik, and Boris Z. Shapiro. "On the complements of affine subspace arrangements". English. In: *Topology Appl.* 56.3 (1994), pp. 215–233. ISSN: 0166-8641. DOI: 10.1016/0166-8641(94)90076-0.

[JT24] Freya Jensen and Álvaro Torras-Casas. *Distributed Persistent Homology for 2D Alpha Complexes*. Preprint, arXiv:2403.00445 [math.AT] (2024). 2024. URL: https://arxiv.org/abs/2403.00445.

[KT03] Allen Knutson and Terence Tao. "Puzzles and (equivariant) cohomology of Grassmannians". English. In: *Duke Math. J.* 119.2 (2003), pp. 221–260. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-03-11922-5.

[Kui65] N. H. Kuiper. "The homotopy type of the unitary group of Hilbert space". English. In: *Topology* 3 (1965), pp. 19–30. ISSN: 0040-9383. DOI: 10.1016/0040-9383(65)90067-4.

[LR25] Thomas J. X. Li and Christian Reidys. "On weighted simplicial homology". English. In: *Rocky Mt. J. Math.* 55.2 (2025), pp. 463–481. ISSN: 0035-7596. DOI: 10.1216/rmj.2025.55.463. URL: projecteuclid.org/journals/rocky-mountain-journal-of-mathematics/volume-55/issue-2/ON-WEIGHTED-SIMPLICIAL-HOMOLOGY/10.1216/rmj.2025.55.463.full.

[LSV11] David Lipsky, Primoz Skraba, and Mikael Vejdemo-Johansson. *A spectral sequence for parallelized persistence.* 2011. arXiv: 1112.1245 [cs.CG]. URL: https://arxiv.org/abs/1112.1245.

[Mas09] Mareike Massow. "Linear Extension Graphs and Linear Extension Diameter". Diplomarbe it at TU Berlin; PDF available from author's page. Diplomarbeit. Berlin, Germany: Technische Universität Berlin, 2009.

[McC01] John McCleary. *A user's guide to spectral sequences.* English. 2nd ed. Vol. 58. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2001. ISBN: 0-521-56759-9.

[Nic11] Liviu I. Nicolaescu. *The Generalized Mayer–Vietoris Principle and Spectral Sequences.* https://www3.nd.edu/~lnicolae/TopicsFall2011-MV.pdf. Lecture notes, University of Notre Dame. 2011.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes.* English. Vol. 300. Grundlehren Math. Wiss. Berlin: Springer-Verlag, 1992. ISBN: 3-540-55259-6.

[Pas24] Abraham Pascoe. "Local Cohomology of Subspace Arrangements and Simplicial Homology". PhD thesis. University of Kansas, 2024.

[Tor23] Álvaro Torras-Casas. "Distributing persistent homology via spectral sequences". English. In: *Discrete Comput. Geom.* 70.3 (2023), pp. 580–619. ISSN: 0179-5376. DOI: 10.1007/s00454-023-00549-2.

[Wac07] Michelle L. Wachs. "Poset topology: tools and applications". English. In: *Geometric combinatorics.* Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Studies, 2007, pp. 497–615. ISBN: 978-0-8218-3736-8.

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* English. Vol. 38. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 1994. ISBN: 0-521-43500-5.