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RESEARCH

Quantum theory of the Lemaître model for gravitational collapse

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Abstract

We investigate the quantum fate of the classical singularities that occur by gravitational collapse of a dust cloud. For this purpose, we address the quantization of a model first proposed by Georges Lemaître in 1933. We find that the singularities can generically be avoided. This is a consequence of unitary evolution in the quantum theory, whereby the quantum dust cloud collapses, bounces at a minimal radius and re-expands.

Keywords Quantum gravity · Gravitational collapse · Lemaître dust model

Contents

1 Introduction

One of George Lemaître's most important papers is *L'univers en expansion* published in 1933 [1]. As Andrzej Krasiński emphasizes in his editorial notes following the English translation of [1], this paper plays a pioneering role for various reasons [2].

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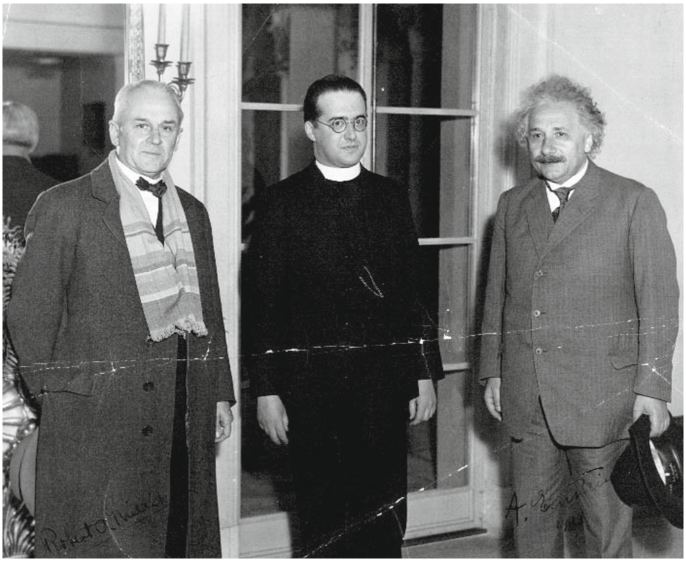

Fig. 1 Robert Millikan, Georges Lemaître, and Albert Einstein at California Institute of Technology, January 1933. . Figure credit: Wikimedia Commons

Perhaps the main reason is that the paper contains a derivation of a spherically-symmetric solution of inhomogeneous dust from Einstein's field equations. In this way, it generalizes Schwarzschild's solutions for both vacuum and homogeneous static dust. This solution plays a major role in our contribution. But there are at least three other important developments initiated in [1]. First, Lemaître suggests a possible mechanism to describe the formation of clusters of galaxies (called *nébuleuses* there). Second, the paper contains a proof that the Schwarzschild horizon at $r = 2GM/c^2$ is only a coordinate singularity. And third, it presents an introduction to the concept of Misner–Sharp mass thirty years before Misner and Sharp published their work.

Lemaître worked on his paper during his visit to the United States in 1932–33, in particular during his stay at the *Caltech* in Pasdadena from November 1932 to January 1933, where he also met Albert Einstein (Fig. 1).

The most intense interaction concering the topic of [1] during his stay at *Caltech* was with Richard Tolman, who himself wrote a paper on this topic in 1934. As discussed in [2], although Tolman gives explicit credit to Lemaître, the inhomogeneous dust solution became known as "Tolman model" or "Tolman–Bondi model" after a paper by Hermann Bondi on this topic had appeared in 1947. We shall follow here the more recent practice of calling this solution the Lemaître–Tolman–Bondi solution or LTB solution. For details on biographic aspects, we refer to [3].

The LTB model has since been used extensively in classical relativity, in particular in addressing questions of structure formation; see, for example, [4] for details of the classical theory. Here, instead, we use this model as a starting point for quantization, in order to get insights into how the classical picture of gravitational collapse may be modified in the quantum theory. This allows to address questions such as what is the fate of the black-hole singularity or what is the role of white holes.

Our paper is organized as follows. Before turning to our main topic, we shall briefly address the simpler situation of a thin null dust shell. Then we discuss the LTB model at the classical and at the quantum level. A major issue is to investigate the quantum version of the classical collapse scenario. We shall, in fact, see that the classical singularity can be avoided and that the initial collapse of a wave packet mimicking a shell in the dust cloud will be followed by its re-expansion. We then discuss the simpler case of the Oppenheimer–Snyder (OS) scenario, which is obtained from the LTB model in the limit of constant density [5]. This simplification allows the derivation of more explicit details. We shall end with a brief summary and reflections about future developments.

2 Thin null dust shell

Before starting our discussion for the dust cloud in the LTB model, we shall briefly address the simpler case of a single self-gravitating *dust shell*. It has turned out that the case of a null dust shell is especially suitable for our purpose; see [6–8] and the references therein for details.

Classically, the shell either collapses to a black-hole singularity or expands from a white-hole singularity. A consistent quantum version can be obtained by the method of reduced quantization. The shell can there be represented by a quantum state $\Psi_{\kappa\lambda}(t, r)$, where t is the asymptotic (Killing) time, r the shell radius and κ (positive integer) and λ (positive number with dimension of a length) are two parameters characterizing the wave packet that is the quantum version of the shell. At $t = 0$, we choose the following family of wave packets in momentum space:

$$\psi_{\kappa\lambda}(p) := \frac{(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}}\, p^{\kappa+1/2}\mathrm{e}^{-\lambda p}. \tag{1}$$

By an appropriate choice of κ and λ, a narrow wave packet can be constructed. After an integral transform from the p- to the r-transformation, one can find an exact solution to the Schrödinger equation. It reads

$$\Psi_{\kappa\lambda}(t, r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{\mathrm{i}}{(\lambda + \mathrm{i}t + \mathrm{i}r)^{\kappa+1}} - \frac{\mathrm{i}}{(\lambda + \mathrm{i}t - \mathrm{i}r)^{\kappa+1}}\right]. \tag{2}$$

An important property of this solution is that the wave function vanishes at the position $r = 0$ of the classical singularity,

$$\lim_{r\to 0} \Psi_{\kappa\lambda}(t, r) = 0. \tag{3}$$

This means that the probability of finding the shell at vanishing radius is zero! In this sense, the singularity is avoided in the quantum theory. The quantum shell bounces and re-expands, and no event horizon forms.

From (2), we can find the expectation value

$$\langle R_0\rangle_{\kappa\lambda} := 2G\langle E\rangle_{\kappa\lambda} = (2\kappa+1)\frac{l_{\rm P}^2}{\lambda},$$

of the shell radius and its variance,

$$\Delta(R_0)_{\kappa\lambda} := 2G\Delta E_{\kappa\lambda} = \sqrt{2\kappa+1}\frac{l_{\rm P}^2}{\lambda},$$

where $l_{\rm P} = \sqrt{G\hbar}$ is the Planck length ($c = 1$ here). It thus turns out that the wave packet can be squeezed below its Schwarzschild radius if its energy is greater than the Planck energy—a genuine quantum effect!

In a sense, one can describe this scenario as a "superposition of black and white hole": the quantum solution contains information about the classical black hole *as well as* the classical white hole solution. The two together enable a singularity-free quantum state. Similar features were also found in loop quantum gravity [9] and in quantum cosmology [10].

This is an interesting result, but it emerges from a simple model that may not reflect the situation in the real world. The model can only be in accordance with observations if the timescale between collapse and re-expansion is sufficiently long, that is, if it is at least comparable with the age of the Universe (because we have so far no evidence for a reversal of collapsing stars). It is certainly imaginable that gravitational time delay is sufficiently long to guarantee this consistency. But explicit calculations are not simple, mainly due to the problem of *defining* an appropriate time delay between comoving and stationary observer (naively, for a stationary observer there is an infinite time delay). In fact, different results have appeared in the literature; see, for example, the review [11]. Timescales $\propto M^3$ or M^2, where M is the initial mass, may be sufficient for this purpose, but the issue is not settled. This question will also be of relevance for the LTB model.

Singularity avoidance in this model is reflected by the fact that the quantum state vanishes at the place of the classical singularity. This is here a consequence of the self-adjoint nature of the reduced Hamiltonian, which leads to a unitary time evolution. Unitarity prevents the wave packet from disappearing in a singularity – the packet must always be present somewhere. From this point of view, it is not surprising that the packet collapses, bounce, and re-expands. This is what we shall also find in the LTB model.

In quantum cosmology, one often imposes from the outset $\Psi = 0$ at places of classical singularities. This was already suggested by Bryce DeWitt in 1967 [12] and is called DeWitt criterion. While this is related in spirit to the vanishing of quantum states in our case, it is not equivalent because in quantum cosmology there is no asymptotic time and no unitarity – these concepts emerge there only in a semiclassical approximation [6].

We also emphasize that in our case there is no ambiguity arising from the measure of the inner product for wave function. The reason is that we shall construct a self-adjoint

Hamiltonian *with respect to* a given inner product with a given measure (more precisely, due to factor-ordering ambiguities, we have a class of self-adjoint Hamiltonians).

3 LTB model

In this section, we shall briefly introduce the classical LTB model. It describes a spherically-symmetric solution of Einstein's equations with non-rotating dust of energy (mass) density $\epsilon(\rho)$ as its source (where ρ denotes the radial coordinate), see (4) below. For constant energy density, we arrive at the Oppenheimer–Snyder (OS) scenario, which provided the first example of a solution describing the gravitational collapse to what was later called a black hole [5]. In the case of dust (no pressure or viscosity), we can interpret the cloud as consisting of infinitely many *independent shells*. This will be mandatory for developing the formalism of quantization.

The line element of the LTB solution can be written in the form

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + \frac{R'}{1+2f}\mathrm{d}\rho^2 + R^2\mathrm{d}\Omega^2, \tag{4}$$

$$\frac{8\pi G}{c^2}\epsilon = \frac{F'}{R^2R'}, \tag{5}$$

$$\frac{\dot{R}^2}{c^2} = \frac{F}{R} + 2f, \tag{6}$$

where τ is the dust proper time and a prime denotes a derivative with respect to the radial variable ρ that labels the dust shells comprising the dust cloud; $F(\rho)$ is twice the active gravitational mass (the Misner–Sharp mass M) inside the shell with label ρ.[1] We restrict ourselves to vanishing cosmological constant. The variable f is a measure of the curvature of the subspaces with constant time; below we restrict ourselves to the marginally bound (flat) case $f = 0$.

An important quantity is $R(\tau, \rho)$, which is the curvature radius of the shell labelled by ρ at time τ. It describes how the shell collapses or expands. A central singularity forms at $R = 0$. There are also singularities from shell crossings happening when two dust shells occupy the same radius. In the simplified setting below, these do not occur.

Let us now address the quantization of this model and its consequences for the collapse situation. One possibility is to apply canonical (Wheeler–DeWitt) quantization. This has led to interesting results concerning Hawking radiation and black-hole entropy, but did not allow finding exact quantum states [6]. Exact results can be obtained if one restricts oneself to self-gravitating dust clouds without introducing additional quantum fields, that is, without the possibility to implement Hawking radiation. This is what we shall review here. In this, we shall follow [13] to which we refer the reader for more details.

As for the case of the thin shell above, we employ here the method of *reduced quantization*. We assume that the infinitely many different shells comprising the cloud decouple, so we can focus on a single shell, here: the outermost shell. This simplifies

[1] Inserting the gravitational constant G and the speed of light c, we have $F = 2GM/c^2$.

the calculations drastically. An early analysis along these lines was presented by Fernando Lund [14], who obtained qualitative results about singularity avoidance without calculating exact quantum wave packets. In order to derive exact solutions, we start with the Hamiltonian for the outermost shell (with radius R_o) given by

$$H = -\frac{P_o^2}{2R_o}, \tag{7}$$

which is the negative of the ADM energy. (P_o is the momentum conjugate to R_o.) As mentioned above, restriction is made here to the marginally bound case of the LTB solution ($f = 1$ in (4)).

The fact that the Hamiltonian (7) is negative might seem surprising. But this is not unusual for gravitational systems because it reflects the attractivity of gravity [15]. The physical (ADM) energy is positive.

As in the case of the collapsing shell, we seek for a unitary evolution (here with respect to the dust proper time τ). In the Schrödinger representation, we have

$$P_o \to \hat{P}_o = -\mathrm{i}\hbar \frac{d}{dR_o}.$$

The operator $\hat{R}_o$ acts by multiplication. (In the following, we shall suppress the subscript o.) The Hamilton operator is then given by

$$\hat{H} = \frac{\hbar^2}{2} R^{-1+a+b} \frac{d}{dR} R^{-a} \frac{d}{dR} R^{-b},$$

where a and b encode factor ordering ambiguities. The Schrödinger equation reads

$$\mathrm{i}\hbar \frac{\partial \Psi(R, \tau)}{\partial \tau} = \hat{H} \Psi(R, \tau). \tag{8}$$

We impose square-integrability on wave functions and let them evolve unitarily according to a self-adjoint Hamiltonian. This corresponds to enforcing probability conservation in dust proper time. The dynamics of the resulting wave packets will be presented in the next section.

4 Singularity avoidance

To solve the Schrödinger equation, an initial quantum state (e.g. at $\tau = 0$) must be specified. In order to find the quantum version of the classical collapse, we choose a narrow initial wave packet that mimicks the classical shell. A detailed investigation shows that for a wide class of wave packets, the probability for the outermost dust shell to be in the classically singular configuration $R = 0$ is *zero* [13]. To give one

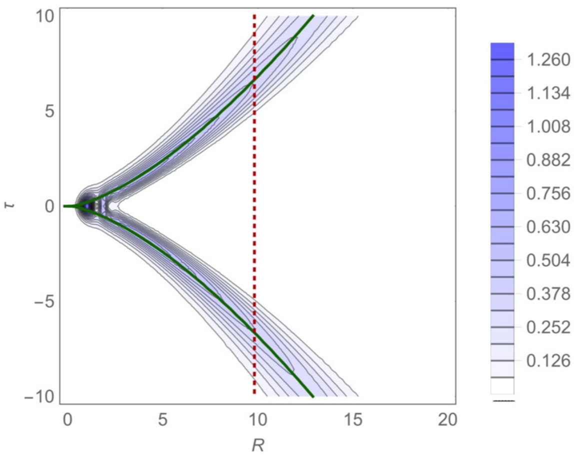

Fig. 2 Probability amplitude for R as given by $R^{1-a-2b}\,|\Psi(R,\tau)|^2$, see (9), compared to the classical trajectories (full green line) and the exterior apparent horizon (dotted red line), with $a=2$ and $b=1$, and $\lambda=2.2$, $\kappa=0.96$. Reproduced from [13] with kind permission by the American Physical Society

example for an exact solution of (8):

$$\Psi(R,\tau)=\sqrt{3}\left(\frac{\sqrt{2}}{3}\right)^{\frac{1}{3}|1+a|+1}\frac{\Gamma\left(\frac{1}{6}|1+a|+\frac{\kappa}{2}+1\right)}{\sqrt{\Gamma(\kappa+1)}\Gamma\left(\frac{1}{3}|1+a|+1\right)}\,R^{\frac{1}{2}(1+a+|1+a|+2b)}$$
$$\times\frac{\lambda^{\frac{1}{2}(\kappa+1)}}{(\frac{\lambda}{2}-\mathrm{i}\tau)^{\frac{1}{6}|1+a|+\frac{\kappa}{2}+1}}$$
$${}_1F_1\left(\frac{1}{6}|1+a|+\frac{\kappa}{2}+1;\frac{1}{3}|1+a|+1;-\frac{2R^3}{9(\frac{\lambda}{2}-\mathrm{i}\tau)}\right),\qquad(9)$$

where ${}_1F_1$ denotes Kummer's confluent hypergeometric function. The parameters κ and λ have the same meaning as in the thin-shell case above; see Eq. (1). This solution is plotted in Fig. 2, where it is also compared with the classical trajectory.

We recognize that the packet first follows the infalling classical trajectory up to some minimal radius R and then makes a transition to the outgoing classical trajectory. The behaviour of the wave packet thus resembles the behaviour for the thin shell described above. The packet first collapses, enters the apparent horizon, but then bounces and re-epands. This is, again, a consequence of the unitary evolution which follows from having a self-adjoint reduced Hamiltonian. In this sense it is different from imposing the DeWitt criterion of vanishing wave function from the outset. The measure in the inner product for wave functions is directly taken into account, see [13], and there is thus no ambiguity in applying the DeWitt criterion.

As for the thin shell above, a crucial issue is the lifetime of the bouncing cloud (or "Planck star"), that is, the elapsed time between collapse and re-expansion. For this purpose, two observers are introduced, one at a fixed physical radius outside the object (stationary observer), one comoving with the cloud. These observers meet twice – first during collapse and second during re-expansion. Applying the method for calculating the lifetime in a similar situations for spinfoams [16], one finds here a lifetime $\propto M^3$, which would be long enough to be in accordance with the observational non-evidence for such objects. In fact, a lifetime $\propto M^3$ is distinguished because it coincides with the lifetime of black holes due to emission of Hawking radiation and also with the spreading time for wave packets in models of stationary quantum black holes [17].

5 Oppenheimer–Snyder model

An important special case of the LTB model is the Oppenheimer–Snyder (OS) model [5]. Here, the collapsing dust cloud is homogeneous. The geometry is then described in the exterior by the Schwarzschild metric and in the interior by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. As in cosmology, one can distinguish between flat, closed, and open FLRW metric.

Dirac quantization of this model for the flat FLRW case (with curvature parameter $k = 0$) is attempted in [18]. In contrast to the above discussion, the whole cloud is treated, not just the outermost shell as a representative of the cloud. Using Brown-Kuchař dust for matter and employing the standard Kuchař decomposition in the canonical formulation [6], one arrives at a form of the Hamiltonian constraint that is of a multivalued nature and looks too complicated for a direct application of Dirac quantization. Nevertheless, some preliminary results can be obtained [18].

A more promising method is affine coherent state quantization (ACSQ) [19]. Coherent state quantization relies on the identification of the classical phase space with a Lie group. One can then consider a unitary irreducible representation of the group on a suitable Hilbert space, letting it act on a fixed state. This allows constructing a family of coherent states in the quantum theory. The affine group comes into play because phase space here corresponds to a half line. If the phase-space function is semi-bounded, the resulting operator after quantization is self-adjoint. A brief self-contained introduction to affine quantization can be found in [19].

Using again the flat FLRW case for the OS dust cloud, the authors of [19] find that both the comoving and the stationary observers see a bounce: the OS cloud collapses to a minimal radius outside the photon sphere and then re-expands. Because of this large minimal radius, one cannot even speak of a black hole. The lifetime seen by the comoving observer is again proportional to M, but it was not possible to define a suitable lifetime for the stationary observer. The question of whether this scenario reflects features of the real world thus remains unanswered.

The closed and open OS models, also called nonmarginal models (curvature parameters $k = 1$ and $k = -1$), are discussed in [20]. There again the method of ACSQ is used. The discussion is more involved than in the flat case, but one finds again that under some conditions there is a bounce of the collapsing cloud, as seen both from the viewpoint of the comoving and of the stationary observers. An interesting special scenario for the closed case (particular choice of parameters) is shown in Fig. 3.

A comoving observer obtains the following picture. Inside the horizon, the curve $R(\tau)$ exhibits oscillations. This is different from the flat case and means that the cloud collapses, experiences a bounce and oscillates until it reaches an equilibrium. This looks as if from the outside this situation could not be distinguished from a Schwarzschild black hole. Unfortunately, it turns out that a horizon never forms from

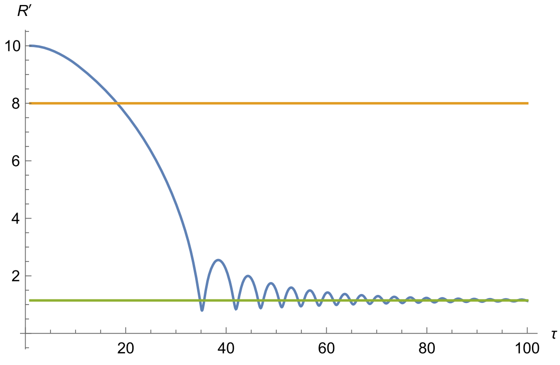

Fig. 3 Graph in the $R-\tau$ space. The orange line represents the Schwarzschild radius $R=2GM/c^2$, the green line the location where equilibrium is reached after oscillations, where $\dot{R}=\ddot{R}=0$. Reproduced from [20] with kind permission by the American Physical Society

the viewpoint of the stationary observer and that therefore the object does not resemble a black hole. This apparent conflict remains an open issue and is subject for further discussion.

6 Summary and outlook

We hope we have convinced the reader that Lemaître's model from 1933 [1] is not only well suitable for problems in classical cosmology, but also for addressing fundamental issues in quantum gravity. As we have reviewed here, one can construct quantum models for gravitational collapse which are *singularity-free*, that is, both the classical black-hole as well as the classical white-hole singularities can be avoided. There is, in fact, a unitary evolution for the quantum state from a collapsing wave packet to a bounce and a re-expanding packet. In a sense, the quantum theory knows of both black holes and white holes, even if classically white holes are irrelevant. If generally true, this would solve the cosmic-censorship problem because singularities are fundamentally absent.

In spite of these promising results, there remain open problems. First, these results were found for spherically-symmetric systems only. Real black holes are described by the Kerr metric, but an analysis similar to the one presented here seems, at present, impossible for the rotating case. Second, there remains the problem of how the lifetime of these collapsing and re-expanding wave packets can be properly defined for stationary observers. There will be no conflict with observation only if the time delay between collapse and expansion is at least of the order of the age of our Universe. And third, these models should be extended by taking into account appropriate quantum matter fields so that the issues of Hawking radiation and information loss can be discussed at an exact quantum gravity level. Perhaps the groundbreaking work of Georges Lemaître will continue to guide us towards that goal.

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