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Calculus of Variations

Quantitative stability for the complex Monge-Ampère equations II

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Abstract
This is a continuation of our previous work on quantitative stability for complex Monge-Ampère equation. In the recent paper [21], we treated the stability question for fixed cohomology classes and fixed prescribed singularity types. In this work, we establish quantitative stability estimates for complex Monge-Ampère equations when *both* the cohomology class and the prescribed singularity vary.

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Contents

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and let α be a big $(1, 1)$- cohomology class in X. Let θ be a closed smooth real $(1, 1)$-form in α and let μ be a non-pluripolar finite

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measure on X. Consider the complex Monge-Ampère equation

$$\theta_u^n = \mu, \tag{1.1}$$

where u is a θ-psh function, and $\theta_u := dd^c u + \theta$, and the left-hand side of (1.1) denotes the non-pluripolar self-product of θ_u (see [2, 5, 23, 46]). By monotonicity of non-pluripolar products (see [8, 46, 47]), if (1.1) has a solution, then it is necessary that $\mu(X) \le \mathrm{vol}(\alpha)$, where $\mathrm{vol}(\alpha)$ denotes the volume of the big class α. When $\mu(X) = \mathrm{vol}(\alpha)$, the equation (1.1) admits a unique solution by [5, 6, 13, 28, 49], and this solution is of minimal singularity in α if μ is sufficiently regular (for example, μ has a L^p $(p > 1)$ density with respect to a smooth volume form on X).

One expects that the regularity of solutions agrees well with that of the measure μ. This expectation is true at least for the following two classes of extreme regularities. The first one is the class of measures which are Hölder continuous as a linear functional on the space $\mathrm{PSH}_0(X, \omega)$ of ω-psh functions u with $\int_X u\omega^n = 0$ endowed with L^1-metric (we call such measures *Hölder continuous ones*). The second one is the class of measures of finite lower energy (*i.e,* non-pluripolar measures). These two classes are important because they are two regularities governing the range of measures where (1.1) is solvable (within the framework of the theory of non-pluripolar products of currents). We refer to [11, 16, 17, 19, 30, 32, 34, 36, 37, 39, 44, 45] and references therein for more informations in the setting where $\mu(X) = \mathrm{vol}(\alpha)$.

Consider now the case where the mass of μ is not necessarily equal to $\mathrm{vol}(\alpha)$, *i.e,* where $\mu(X) \le \mathrm{vol}(\alpha)$. In this case one can still solve (1.1) by putting it in the context of prescribed singularities and relative full mass potentials. Denote by $\mathrm{PSH}(X, \theta)$ the set of θ-psh functions and let $u_1, u_2 \in \mathrm{PSH}(X, \theta)$. One says that u_1 is more singular than u_2 if $u_1 \le u_2 + O(1)$, and u_1 is of the same singularity type as u_2 if $u_1 - u_2$ is bounded.

In the following, we recall the notions of model potentials and relative full mass potentials. Note that they are slightly different from the definitions in [8, 38], but the difference is not essential. For each $\phi \in \mathrm{PSH}(X, \theta)$, the roof-top envelope of ϕ is defined by the following formula (see [8]):

$$P_\theta[\phi] := \big(\sup\{\psi \in \mathrm{PSH}(X, \theta) : \psi \le 0,\ \psi \le \phi + O(1)\}\big)^*.$$

We say that ϕ is is a *model θ-psh function* if $\phi = P_\theta[\phi]$. By [8], the function $P_\theta[u]$ is a model one for every $u \in \mathrm{PSH}(X, \theta)$. If ϕ is model then we denote by $\mathcal{E}(X, \theta, \phi)$ the set of *ϕ-relative full mass potentials*, i.e., the set of θ-psh functions u with $u \le \phi$ and $\int_X \theta_u^n = \int_X \theta_\phi^n$. In the case where $\int_X \theta_\phi^n > 0$, it is well-known that $u \in \mathcal{E}(X, \theta, \phi)$ iff $P_\theta[u] = \phi$ (see [9]).

Let ϕ be now a model θ-psh function such that $\int_X \theta_\phi^n > 0$. Let μ be a non-pluripolar measure with $\mu(X) = \int_X \theta_\phi^n$. By [9] (see also [8, 19]), the equation

$$(dd^c u + \theta)^n = \mu, \tag{1.2}$$

admits a unique solution in $\mathcal{E}(X, \theta, \phi)$ satisfying $\sup_X u = 0$, and if μ has L^p density then the solution is of the same singularity type as ϕ. Furthermore a characterization of the class of measures μ where (1.2) admits a solution of finite pluricomplex energy was given in [19]. The hypothesis that ϕ is model is a minimal requirement so that (1.2) is solvable in a meaningful way; see [8] for an explanation about the nature of this assumption.

A θ-singularity type (in α) is an equivalence class of θ-psh functions of the same singularity type. The space of θ-singularity types is denoted by $\mathcal{S}(\theta)$(or $\mathcal{S}(\alpha)$ when θ is clear from the context). A natural pseudo-metric $d_{\mathcal{S}(\theta)}$ in $\mathcal{S}(\theta)$ was introduced in [10]. We refer to Section 2 for a recap of this pseudodistance. A model θ-singularity type is by definition the class of a model θ-psh function. By [8, Theorem 1.3], every model θ-singularity type contains a unique model θ-psh function. Hence there is a 1-1 correspondence between model θ-singularity types and model θ-psh functions. For $u \in \mathrm{PSH}(X, \theta)$, we denote by $[u]_\theta$ (or simply $[u]$ when θ is clear) the θ-singularity type of u. To ease the notation we will denote by $d_{\mathcal{S}(\theta)}(u, v)$ the distance $d_{\mathcal{S}(\theta)}\big([u]_\theta, [v]_\theta\big)$.

In Proposition 2.4 (in Section 2), we push further the study of metrics on the space of singularity types by observing that if we embed $\mathcal{S}(\theta)$ into a bigger space $\mathcal{S}(\theta')$ for $\theta' \geq \theta$ (notice that θ' is not necessarily in the cohomology class of θ), then the pseudodistance $d_{\mathcal{S}(\theta)}$ is actually comparable with that induced by $d_{\mathcal{S}(\theta')}$. This allows us to compare singularity types in different cohomology classes without changing the nature of the distance $d_{\mathcal{S}(\theta)}$. By this we will sometimes ignore θ and only write $d_{\mathcal{S}}$. In view of the resolution of (1.2), we are led to the following natural stability question. We fix a $\mathscr{C}^0$-norm on the space of smooth $(1, 1)$-forms on X.

Problem 1.1 *Let θ_1, θ_2 be closed smooth real $(1, 1)$-forms on X. Let ϕ_j be model θ_j-psh functions and μ_j be a non-pluripolar measure of mass equal to $\int_X \theta_{\phi_j}^n$ for $j = 1, 2$. Let u_j be the solution of (1.2) for μ_j, ϕ_j for $j = 1, 2$. Compare u_1 with u_2 in terms of $d_{\mathcal{S}}(\phi_1, \phi_2)$, $\|\theta_1 - \theta_2\|_{\mathscr{C}^0}$, and a suitable distance between μ_1, μ_2.*

Here by $d_{\mathcal{S}}(\phi_1, \phi_2)$, we mean $d_{\mathcal{S}(A\omega)}(\phi_1, \phi_2)$, where A is a big constant so that $\theta_j \leq A\omega$ for $j = 1, 2$. As discussed above, the condition that $d_{\mathcal{S}(A\omega)}(\phi_1, \phi_2)$ converges to 0 is independent of the choice of A. To get motivated about the above problem, let's consider the following simple situation. Let $(\alpha_j)_j$ be a sequence of Kähler $(1, 1)$-cohomology classes converging to a big class α_∞ as $j \to \infty$. We know that there exists a unique closed positive $(1, 1)$-current $T_j \in \alpha_j$ such that $T_j^n = \mathrm{vol}(\alpha_j)\omega^n / \int_X \omega^n$. One thus asks further: what can we say about the convergence of the sequence $(T_j)_j$? Even when α_∞ is also Kähler, it seems that known methods are not sufficient to deal with such a question.

There are two natural metrics on the space of probability measures on X. One of the most common ones is the metric induced by the mass norm of measures. The other metric, we will also study later, is the Kantorovich-Rubinstein one (or more generally Wasserstein metrics) metrizing the weak topology on the space of probability measures; see (1.6) below. We will study Problem 1.1 with both these metrics on measures. The results we obtained for the mass norm of measures hold under very general assumptions on ϕ_j. Although the mass norm of measures is quite natural to consider, it is not always easy to estimate this quantity in applications, while this is often the case for the Kantorovich-Rubinstein distance. For this reason, we will also treat the stability question for the Kantorovich-Rubinstein distance of measures.

The first stability result for varied prescribed singularities, which is not quantitative, was given in [10, Theorem 1.4]. Previously there were several stability results in the fixed prescribed singularity setting in the literature: some are quantitative and some are not. We refer to [3, 4, 15, 24, 29, 31, 34, 45] and references therein for more details. Key technical tools to obtain quantitative stability in these latter references are (variants of) Kołodziej's capacity method ([29]) and an integration by parts arguments originally in [4]. All of these cited results require the measures in the right-hand side of the Monge-Ampère equations to be sufficiently regular (to be more precise, measures must be at least the Monge-Ampère of θ-psh functions in $\mathcal{E}^1(X, \theta)$). A more robust method, which allows one to treat the (quantitative) stability for solutions of low energy in the same cohomology class and singularity type (*i.e,* $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$), was devised in [21]. In this paper we develop further this method and settle Problem 1.1 more or less satisfactorily.

In order to state our main results, let us now introduce some necessary notions. For every Borel set E in X, recall that the capacity of E is given by

$$\operatorname{cap}(E)=\operatorname{cap}_\omega(E):=\sup_{\{w\in \mathrm{PSH}(X,\omega):0\le w\le 1\}}\int_E \omega_w^n.$$

A sequence of Borel functions $(u_j)_j$ is said to *converge to a Borel function* u *in capacity* if for every constant $\epsilon>0$, we have that $\operatorname{cap}(\{|u_j-u|\ge\epsilon\})$ converges to 0 as $j\to\infty$. The convergence in capacity is of great importance in pluripotential theory in part because it implies the convergence of Monge-Ampère operators under reasonable circumstances. To study quantitatively the convergence in capacity, it is convenient to introduce the following distance function on $\mathrm{PSH}(X,\omega)$:

$$d_{\mathrm{cap}}(u,v):=\sup_{w\in \mathrm{PSH}(X,\omega):0\le w\le 1}\int_X |u-v|^{1/2}\omega_w^n$$

for every $u,v\in \mathrm{PSH}(X,\omega)$ (note that $d_{\mathrm{cap}}(u,v)<\infty$ thanks to the Chern-Levine-Nirenberg inequality [22, Proposition 3.1]). The number $\frac{1}{2}$ in the definition of d_{cap} can be replaced by any constant in $(0,1)$. One can see that for $u_j,u\in \mathrm{PSH}(X,\omega)$ for $j\in\mathbb{N}$, $d_{\mathrm{cap}}(u_j,u)\to 0$ if and only if $|u_j-u|\to 0$ in capacity.

For θ-psh functions u,v, we put

$$d_\theta(u,v):=2\int_X \theta^n_{\max\{u,v\}}-\int_X\theta_u^n-\int_X\theta_v^n.$$

The function d_θ is comparable to $d_{\mathcal{S}(\theta)}$ (see Proposition 2.4). For quantitative estimates, it is more convenient to use d_θ than $d_{\mathcal{S}(\theta)}$. It is perhaps worth noting that our method to prove the stability results below also implies that d_{cap} is bounded from above by a power of d_θ for model θ-potentials (see Proposition 3.13 for details).

Let $\mathcal{W}^-$ be the set of convex increasing functions $\chi:\mathbb{R}_{\le 0}\to\mathbb{R}_{\le 0}$ so that $\chi(0)=0$ and $\chi(-\infty)=-\infty$. It follows from [5, Proposition 3.2] that for every non-positive θ-psh function u, there exists $\chi\in\mathcal{W}^-$ and $C>0$ such that

$$-\int_X\chi(\psi)\,\theta_u^n\le C,$$

for every $\psi\in \mathrm{PSH}(X,\omega)$ with $\sup_X\psi=0$. The first main result of this paper is as follows:

Theorem 1.2 *Let θ be a closed smooth real $(1,1)$-form such that $\theta\le A\omega$ for a given constant $A\ge 1$. Let $u\in \mathrm{PSH}(X,\theta)$ such that $\sup_X u=0$ and $\int_X\theta_u^n:=2\delta>0$. Let $B\ge A$ and $\tilde\chi\in\mathcal{W}^-$ with $\tilde\chi(-1)=-1$ such that*

$$\int_X -\tilde\chi(\psi)\theta_u^n\le B\delta,$$

for every $\psi\in \mathrm{PSH}(X,(A+1)\omega)$ with $\sup_X\psi=0$. Let $h(s):=(-\tilde\chi(-s))^{1/2}$ for $s\le 0$. Then, for every constant $0<\gamma<1$, there exists a constant $C>0$ depending only on n,X,ω,A,B and γ such that

$$d_{cap}(u,v)\le C\left(h^{\circ(n)}\left(\frac{\delta}{\|\theta_u^n-\eta_v^n\|+A^n\|\theta-\eta\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u,v)}\right)\right)^{-\gamma/2},\quad (1.3)$$

for every closed smooth real $(1,1)$-form $\eta\le A\omega$ and for each $v\in \mathrm{PSH}(X,\eta)$ with $\sup_X v=0$.

Here, for finite measures μ, μ', we denote by $\|\mu - \mu'\|$ the mass norm of $\mu - \mu'$. The condition that $\tilde{\chi}(-1) = -1$ is merely a normalization one. For an arbitrary $\tilde{\chi} \in \mathcal{W}^-$, we can consider $\tilde{\chi}/|\tilde{\chi}(-1)|$ which satisfies the last requirement. Theorem 1.2 says that under a relatively weak assumption on a θ-psh function u with $\int_X \theta_u^n > 0$, one can bound reasonably the distance d_{cap} of u with *any other quasi-psh function* v. We give here a consequence of Theorem 1.2.

Theorem 1.3 *Let $(\theta_j)_{j \in \mathbb{N} \cup \{\infty\}}$ be a sequence of closed smooth real $(1,1)$-forms in X such that $\theta_j \to \theta_\infty$ in $\mathscr{C}^0$ topology as $j \to \infty$. Let ϕ_j be a model θ_j-psh function for $j \in \mathbb{N} \cup \{\infty\}$ such that*

$$d_{\mathcal{S}}(\phi_j, \phi_\infty) \to 0$$

as $j \to \infty$. Let μ_j be a non-pluripolar measure on X such that

$$\mu_j(X) = \int_X (dd^c \phi_j + \theta_j)^n$$

for every j and $\mu_j \to \mu_\infty$ in the mass norm. Let u_j be the θ-psh function satisfying

$$(dd^c u_j + \theta_j)^n = \mu_j, \quad \sup_X (u_j - \phi_j) = 0$$

for $j \in \mathbb{N} \cup \{\infty\}$. Then $u_j \to u_\infty$ in capacity as $j \to \infty$.

Here by $d_S(\phi_j, \phi_\infty)$ we mean the pseudodistance $d_{\mathcal{S}(A\omega)}$ between the $(A\omega)$-singularity types of ϕ_j and ϕ_∞, where $A > 0$ is a big enough constant such that $\theta_j \le A\omega$ for every j. The property $d_{\mathcal{S}}(\phi_j, \phi_\infty) \to 0$ is independent of the choice of A. Moreover, as mentioned above when θ_j is equal to a fixed θ, the pseudometric $d_{\mathcal{S}(A\omega)}$ is comparable with $d_{\mathcal{S}(\theta)}$.

Theorem 1.3 considerably extends [24, Proposition A] (which treats the case where the cohomology class is fixed, $(\phi_j)_j$ is constant and of minimal singularity types, and only the convergence in L^1 was obtained) and [10, Theorem 1.4] which treats the case where again the cohomology class is fixed, and μ_j has L^p density with respect to ω^n; see also [12, Theorem 2.14] for a particular version of Theorem 1.3. The assumption that μ_j has L^p density is crucial in the plurisubharmonic envelope approach in [10, Theorem 1.4]. We note furthermore that in [41, 42] the convergence of solutions were also obtained under more restrictive conditions that the sequence of measures $(\mu_j)_j$ are of uniformly bounded E^1 energy (*i.e,* the corresponding solutions $(u_j)_j$ are of uniformly bounded χ-energy for $\chi(t) = t$, see below for the definition) and the sequence of prescribed singularities is totally ordered.

It is well-known that it is not possible to have $u_j \to u_\infty$ in L^1 if μ_j only converges weakly to μ_∞ in general (see [7, 24] and references therein for examples).

We now turn our attention to the class of Hölder continuous measures whose definition is recalled below. Let $\text{PSH}_0(X, \omega)$ be the set of ω-psh functions u with $\int_X u\omega^n = 0$. We endow $\text{PSH}_0(X, \omega)$ with the $L^1(\omega^n)$ distance. Let μ be a measure on X such that quasi-psh functions are μ-integrable. We say that μ *is Hölder continuous with Hölder constant A and Hölder exponent γ if it is so as a functional on* $\text{PSH}_0(X, \omega)$, in other words, for every $u_1, u_2 \in \text{PSH}_0(X, \omega)$, we have

$$\int_X |u_1 - u_2| d\mu \le A \|u_1 - u_2\|_{L^1(\omega^n)}^{\gamma}. \tag{1.4}$$

This notion was introduced in [17]. By expressing every ω-psh function u as $u = u - \int_X u\omega^n + \int_X u\omega^n$, we see that (1.4) implies that

$$\int_X |u_1 - u_2| d\mu \leq \big(A + \mu(X)\big) \max\{\|u_1 - u_2\|^{\gamma}_{L^1(\omega^n)}, \|u_1 - u_2\|_{L^1(\omega^n)}\} \tag{1.5}$$

for every ω-psh function u_1, u_2. Clearly the last inequality also yields that μ is Hölder continuous on $\mathrm{PSH}_0(X, \omega)$ with Hölder exponent γ and with Hölder constant $\lambda\big(A + \mu(X)\big)$, for some constant λ depending only on (X, ω). A measure is Hölder continuous if and only if it can be written as $(dd^c u + \omega)^n$ for some Hölder continuous ω-psh function u on X; see [17, Theorem 1.3] and also [11, 30]. We refer to these papers and [27, 33, 37, 44] for examples of Hölder continuous measures. Most basic examples are measures with L^p density or smooth volume forms of (immersed) generic (real) Cauchy-Riemann submanifolds on X.

Recall that, for every $0 \leq \delta < \infty$, the distance $\mathrm{dist}_{-\delta}$ on the set of Radon measures on X is defined as follows:

$$\mathrm{dist}_{-\delta}(\mu, \mu') := \sup_{\|v\|_{\mathscr{C}^\delta} \leq 1} \big|\langle \mu - \mu', v\rangle\big|, \tag{1.6}$$

where v is a smooth real function on X. The distance dist_{-1} is the Kantorovich-Rubinstein one in the theory of optimal transport, see [43, Remark 6.5]. Note that $\mathrm{dist}_{-\delta}$ induces the same weak topology when $\delta > 0$ (see [43, Theorem 6.9] or [18, Proposition 2.1.4]). When $\delta = 0$, it is the mass norm of $\mu_1 - \mu_2$. We also have the following interpolation inequality: for $0 \leq \beta_0 < \beta_1 < \beta_2$,

$$\mathrm{dist}_{-\beta_1} \leq \mathrm{dist}_{-\beta_0}^{\frac{\beta_2-\beta_1}{\beta_2-\beta_0}} \, \mathrm{dist}_{-\beta_2}^{\frac{\beta_1-\beta_0}{\beta_2-\beta_0}}. \tag{1.7}$$

We refer to [35, 40] for a proof (see also [44]). This estimate is very important in complex dynamics since the appearance of [18] where a more general version of (1.7) for currents was introduced.

Our next main result is as follows:

Theorem 1.4 *Let θ_1, θ_2 be closed smooth real $(1,1)$-forms and A be positive constant at least 1 such that $\theta_j \leq A\omega$ for $j = 1, 2$. Let $0 < \delta, \beta \leq 1$ and $M \geq 1$ be constants and $u_j \in \mathrm{PSH}(X, \theta_j)$ $(j = 1, 2)$ such that*

$$\sup_X u_j = 0, \quad \int_X \theta^n_{u_j} \geq \delta,$$

and $\mu_j := (\theta_j + dd^c u_j)^n$ $(j = 1, 2)$ are Hölder continuous measures on X with Hölder exponent β and with Hölder constant $M\delta$. Then, for every $0 < \gamma < 1$, there exists a constant $C > 0$ depending only on $n, X, \omega, A, M, \beta$ and γ such that

$$d_{cap}(u_1, u_2) \leq C \left(\frac{(\mathrm{dist}_{-1}(\mu_1, \mu_2))^{2\gamma\beta/(2\beta+1)} + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)}{\delta} \right)^{2^{-n-1}\gamma}.$$

By interpolation inequality (1.7), an analogous inequality also holds for $\mathrm{dist}_{-\beta}$ in place of dist_{-1} for any constant $\beta > 0$. Our last main result is a generalization of Cegrell-Kołodziej-Xing stability theorem ([7, 48]) which treated the case where $\theta = \omega$ (and only for the class of potentials of full Monge-Ampère mass) and a refinement of Dinew-Hiep [14, Theorem 3.4]. We also underline that the original result in [7, 48] is non-quantitative and Theorem 1.5 below already strengthens their results in their setting.

Theorem 1.5 *Let θ_1, θ_2 be closed smooth real $(1,1)$-forms and let A be positive constant at least 1 such that $\theta_j \leq A\omega$ for $j = 1, 2$. Let $0 < \delta \leq 1$ and $u_j \in \mathrm{PSH}(X, \theta_j)$ ($j = 1, 2$) such that $\sup_X u_j = 0$ and $\int_X \theta_{u_j}^n \geq \delta$. Assume that there exists a Radon measure μ on X such that μ vanishes on pluripolar sets and $(\theta_j + dd^c u_j)^n \leq \mu$ for $j = 1, 2$. Then, there exists a continuous increasing function $f_\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on n, X, ω, A, δ and μ such that $f(0) = 0$ and*

$$d_{cap}(u_1, u_2) \leq f_\mu \left(\mathrm{dist}_{-1}(\mu_1, \mu_2) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)\right),$$

where $\mu_j := (\theta_j + dd^c u_j)^n$ for $j = 1, 2$.

Theorem 1.5 implies particularly that for every model θ-psh function ϕ, the convergence in capacity or in L^1 and the weak convergence of Monge-Ampère measures are equivalent in the class of potentials in $\mathcal{E}(X, \theta, \phi)$ whose Monge-Ampère measures are bounded from above by a fixed non-pluripolar measure. This is more or less the original motivation of Cegrell-Kołodziej in [7]. We would also like to note that in the case of Kähler forms, sufficient conditions for the convergence in capacity in terms of Monge-Ampère operators were given in [14, Theorem 3.6] and [26, Lemma 2.3].

Finally we note that as an application of Theorem 1.5 or 1.4 (or rather our method), one can recover a main result in [10] that the pseudometric space of singularity types of volume bounded from below by a fixed positive constant is complete, we refer to Proposition 3.14 in Subsection 3.3 for details.

This paper is the second part of [20] (see [21] for the first part of [20]).

2 Pseudo-metric on the space of singularity types

We first recall some facts about the pseudo-metric on the space of singularity types. Let α be a big cohomology class and θ a smooth closed $(1,1)$-form in α. Let $\delta > 0$ be a constant. Let $\mathcal{S}(\theta)$ be the space of singularity types of θ-psh functions and

$$\mathcal{S}_\delta(\theta) := \{[u] \in \mathcal{S}(\theta) : \int_X \theta_u^n \geq \delta\}.$$

The pseudo-distance $d_{\mathcal{S}}$ on $\mathcal{S}$ was introduced in [10], and it satisfies

$$\begin{aligned} d_{\mathcal{S}(\theta)}([u],[v]) &\leq \sum_{j=0}^{n} \left(2\int_X \theta_{V_\theta}^j \wedge \theta_{\max\{u,v\}}^{n-j} - \int_X \theta_{V_\theta}^j \wedge \theta_u^{n-j} - \int_X \theta_{V_\theta}^j \wedge \theta_v^{n-j} \right) \\ &\leq C d_{\mathcal{S}(\theta)}([u],[v]), \end{aligned} \tag{2.1}$$

where $C > 1$ depends only on n. Here V_θ is the upper envelope of all non-positive θ-psh functions:

$$V_\theta := \sup\{\varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq 0 \quad \text{on} \quad X\}.$$

If θ' is another closed smooth form in α, then $\mathcal{S}_\delta(\theta)$ and $\mathcal{S}_\delta(\theta')$ are isometric under the map $u \mapsto u + \varphi$, where φ is a smooth function such that $dd^c\varphi = \theta' - \theta$. Hence in general in order to study singularity types in α, it is enough to fix a smooth form in α.

For all θ-psh functions u, v, we put

$$d_\theta(u, v) := 2\int_X \theta_{\max\{u,v\}}^n - \int_X \theta_u^n - \int_X \theta_v^n.$$

In particular, if $u \leq v$ then $d_\theta(u, v) = \int_X \theta_v^n - \int_X \theta_u^n$. By (2.1)and by monotonicity of non-pluripolar products (see [8, Theorem 1.1]), we have

$$d_\theta(u, v) \leq C d_{\mathcal{S}(\theta)}([u], [v]),$$

where $C = C(n) > 0$. Moreover, if $\theta = A\omega$ for some $A > 0$ then, we have

$$d_{\mathcal{S}(A\omega)}([u], [v]) \leq d_{2A\omega}(u, v),$$

for every $u, v \in \mathrm{PSH}(X, A\omega)$ by monotonicity of non-pluripolar products; see [8, 47]. In the sequel, we provide more properties of d_θ.

Lemma 2.1 *Let u_1, u_2 be θ-psh functions. Let θ' be a smooth real closed $(1, 1)$-form such that $\theta' \geq \theta$. Then*

$$d_\theta(u_1, u_2) \leq d_{\theta'}(u_1, u_2).$$

Proof By the fact $d_\eta(u_1, u_2) = d_\eta(u_1, \max\{u_1, u_2\}) + d_\eta(u_2, \max\{u_1, u_2\})$ for $\eta = \theta, \theta'$, the problem is reduced to the case $u_1 \leq u_2$. Then we have

$$d_\eta(u_1, u_2) = \int_X (\eta + dd^c u_2)^n - \int_X (\eta + dd^c u_1)^n,$$

for $\eta = \theta, \theta'$. Moreover,

$$(\theta' + dd^c u_j)^n - (\theta + dd^c u_j)^n = (\theta' - \theta) \wedge \sum_{l=0}^{n-1} (\theta' + dd^c u_j)^l \wedge (\theta + dd^c u_j)^{n-l-1},$$

for $j = 1, 2$. Hence

$$d_{\theta'}(u_1, u_2) - d_\theta(u_1, u_2) = \int_X (\theta' - \theta) \wedge T_2 - \int_X (\theta' - \theta) \wedge T_1,$$

where

$$T_j = \sum_{l=0}^{n-1} (\theta' + dd^c u_j)^l \wedge (\theta + dd^c u_j)^{n-l-1} \geq 0.$$

Thus, by the monotonicity of non-pluripolar products [8, Theorem 1.1] (see also Theorem 2.2 below), we obtain

$$d_{\theta'}(u_1, u_2) - d_\theta(u_1, u_2) \geq 0.$$

The proof is completed. □

Theorem 2.2 *(The monotonicity of non-pluripolar products) [8, Theorem 1.1] Let θ^j, $j \in \{1, 2, ..., n\}$, be smooth closed real $(1, 1)$-forms on X. Let u_j, v_j be θ^j-psh functions such that u_j is less singular than v_j for every j. Then*

$$\int_X \theta^1_{u_1} \wedge ... \wedge \theta^n_{u_n} \geq \int_X \theta^1_{v_1} \wedge ... \wedge \theta^n_{v_n}.$$

Lemma 2.3 *Let $\delta > 0$, $A \geq 1$ be constants. Let u, v be θ-psh functions such that $u \leq v$ and $\int_X \theta_u^n \geq \delta$. Let ψ be an η-psh function, where η is a closed smooth $(1, 1)$-form. Assume that $\theta \leq A\omega$, $\eta \leq A\omega$. Then there exists a constant C depending only on n, ω such that*

$$\left| \int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \right| \leq C A^n \left(\frac{d_\theta(u, v)}{\delta} \right)^{1/n}.$$

Proof This is essentially the proof of [10, Proposition 4.8]. Note that by monotonicity we have

$$d_\theta(u,v) = \int_X \theta_v^n - \int_X \theta_u^n, \quad \int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq \int_X \theta_u^m \wedge \eta_\psi^{n-m}.$$

Without loss of generality, we can assume $d_\theta(u,v) \leq \delta/2^{n+2}$. If $d_\theta(u,v) = 0$, then using $u \leq v$ and [8, Theorem 1.3 ($i \Leftrightarrow iii$)], we get $P_\theta[u] = P_\theta[v]$. In this case, the left-hand side of the desired inequality is also zero. Hence, from now on, we assume $d_\theta(u,v) > 0$.

Let $b > 2$ be a constant such that $\delta/d_\theta(u,v) < 2b^n < 2\delta/d_\theta(u,v)$. We have

$$b^n \int_X \theta_u^n > (b^n - 1)\int_X \theta_v^n.$$

By this and [10, Lemma 4.3], we obtain $w_b := P_\theta(bu + (1-b)v) \in \mathrm{PSH}(X,\theta)$. Observe

$$b^{-1}w_b + (1-b^{-1})v \leq b^{-1}(bu + (1-b)v) + (1-b^{-1})v = u.$$

Combining this with monotonicity of non-pluripolar products gives

$$\int_X \theta_u^m \wedge \eta_\psi^{n-m} \geq \int_X \theta^m_{b^{-1}w_b+(1-b^{-1})v} \wedge \eta_\psi^{n-m} \geq (1-b^{-1})^m \int_X \theta_v^m \wedge \eta_\psi^{n-m}.$$

It follows that

$$\int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq -mb^{-1}\int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq -nb^{-1}A^n \int_X \omega^n$$

by monotonicity. Hence

$$\left| \int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \right| \leq Cb^{-1} \leq 2^{1/n} C \left(\frac{d_\theta(u,v)}{\delta} \right)^{1/n},$$

where $C := nA^n \int_X \omega^n$. This finishes the proof. □

By Lemma 2.3, we have

Proposition 2.4 *Let α, θ be as above. Then there exists a constant $C > 0$ such that*

$$C^{-1}\delta\, d_{\mathcal{S}(\theta)}([u],[v])^n \leq d_\theta(u,v) \leq C d_{\mathcal{S}(\theta)}([u],[v])$$

for every $[u],[v] \in \mathcal{S}_\delta(\alpha)$. Moreover if θ' is a smooth real closed $(1,1)$-form and A is a positive constant such that

$$\theta' \leq A\omega, \quad \theta \leq A\omega,$$

for some constant $A > 0$, then there exists a constant $C_1 > 0$ depending only on A, ω such that

$$\delta\big(d_{\theta'}(u,v)\big)^n \leq C_1 d_\theta(u,v),$$

for every $u, v \in \mathcal{S}_\delta(\alpha)$.

Proof The first desired assertion is clear from Lemma 2.3. Also by the same lemma, one gets

$$\delta\big(d_{A\omega}(u,v)\big)^n \leq C_1 d_\theta(u,v),$$

for every $u, v \in \mathcal{S}_\delta(\alpha)$, and some constant C_1 independent of u, v, δ. This coupled with Lemma 2.1 gives the last desired inequality. The proof is complete. □

3 Stability in a fixed cohomology class

In this section, we will study the stability question when solutions are in the same cohomology class.

3.1 Proof of Theorem 1.2 when $\eta = \theta$

Denote by $\widetilde{\mathcal{W}}^-$ the set of all convex, non-decreasing functions $\chi : \mathbb{R}_{\leq 0} \to \mathbb{R}_{\leq 0}$ such that $\chi(0) = 0$ and $\chi \not\equiv 0$. We stress that $\mathcal{W}^-$ is a proper subset of $\widetilde{\mathcal{W}}^-$, since there are functions $\chi \in \widetilde{\mathcal{W}}^-$ such that $\lim_{t\to-\infty} \chi(t) \neq -\infty$ (for example, $\chi(t) = \max\{t, -1\}$ belongs to $\widetilde{\mathcal{W}}^-$ and not to $\mathcal{W}^-$). Let ϕ be a model θ-psh function. For $\chi \in \widetilde{\mathcal{W}}^-$ and $u \in \mathrm{PSH}(X, \theta)$ with $u \leq \phi$, let

$$E_{\chi,\theta,\phi}(u) := -\int_X \chi(u-\phi)\theta_u^n.$$

We denote

$$\mathcal{E}_\chi(X,\theta,\phi) := \big\{u \in \mathcal{E}(X,\theta,\phi) : E_{\chi,\theta,\phi}(u) < \infty\big\}.$$

Put

$$I_\chi(u,v) := \int_{\{u<v\}} \chi(u-v)(\theta_v^n - \theta_u^n) + \int_{\{u>v\}} \chi(v-u)(\theta_u^n - \theta_v^n)$$

for $u, v \in \mathcal{E}_\chi(X,\theta,\phi)$. Note that each term in the sum defining $I_\chi(u,v)$ is nonnegative for $u, v \in \mathcal{E}_\chi(X,\theta,\phi)$ (hence $d_\theta(u,v) = 0$).

We will prove Theorem 1.2 by using some approximation lemmas and the following result:

Theorem 3.1 *[21, Theorem 4.2] Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$) and let ϕ be a model θ-psh function such that $\int_X \theta_\phi^n = \varrho > 0$. Let $B \geq A$, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E_{\tilde{\chi},\theta,\phi}(u_1) + E_{\tilde{\chi},\theta,\phi}(u_2) \leq B\varrho.$$

Denote $\chi(t) = \max\{t, -1\}$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1,u_2)^2 \leq C\,(A + |a_1 - a_2|)\left(|a_1 - a_2| + AB^2\lambda^\gamma\right), \tag{3.1}$$

where $a_j := \sup_X u_j$, $\lambda = \dfrac{1}{h^{\circ n}(\varrho/I_\chi(u_1,u_2))}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Let θ, η be closed smooth real $(1,1)$-forms representing big cohomology classes. For every $\chi \in \widetilde{\mathcal{W}}^-$ and $u \in \mathrm{PSH}(X,\theta)$, we denote

$$\tilde{E}_{\chi,\eta,\theta}(u) := \sup\left\{\int_X -\chi(\psi)\theta_u^n : \psi \in \mathrm{PSH}^-(X,\eta),\ \sup_X \psi = 0\right\}, \tag{3.2}$$

where we recall that $\mathrm{PSH}^-(X,\eta)$ is the space of negative η-psh functions on X. The quantity $\tilde{E}_{\chi,\eta,\theta}(u)$ is a sort of energy for u without reference to a model θ-psh function. This will be very important for us later because we will work in the situation where θ-psh functions in consideration are not in the same class $\mathcal{E}(X,\theta,\phi)$ for any model θ-psh function ϕ.

If χ is bounded then it is clear that $\tilde{E}_{\chi,\eta,\theta}(u) < \infty$ for every $u \in \mathrm{PSH}(X,\theta)$. Moreover, it follows from [5, Proposition 3.2] that for every $u \in \mathrm{PSH}(X,\theta)$, there exists $\chi \in \mathcal{W}^-$ such that $\tilde{E}_{\chi,\eta,\theta}(u) < \infty$.

For every constant $B > 0$ and for every $\chi \in \widetilde{\mathcal{W}}^-$, we define

$$\tilde{\mathcal{E}}_{\chi,\eta,B}(X,\theta) := \{u \in \mathrm{PSH}^-(X,\theta) : \tilde{E}_{\chi,\eta,\theta}(u) \leq B\}. \tag{3.3}$$

For the convenience, in the case $\eta = \theta$, we denote $\tilde{E}_{\chi,\theta}(u) := \tilde{E}_{\chi,\theta,\theta}(u)$ and $\tilde{\mathcal{E}}_{\chi,B}(X,\theta) = \tilde{\mathcal{E}}_{\chi,\theta,B}(X,\theta)$. For $u, v \in \mathrm{PSH}^-(X,\theta)$, we let

$$I_\chi(u,v) := \int_{\{u<v\}} -\chi(u-v)(\theta_u^n - \theta_v^n) + \int_{\{v<u\}} -\chi(v-u)(\theta_v^n - \theta_u^n). \tag{3.4}$$

We used this quantity before for $u, v \in \mathcal{E}(X,\theta,\phi)$ for some model θ-psh function ϕ, in which case, $I_\chi(u,v)$ is always non-negative. We underline that the number $I_\chi(u,v)$ may be negative in general (if $d_\theta(u,v) > 0$). Indeed, if $\theta = \omega$, $u = 0$ and $(\omega + dd^c v)^n = c\omega^n$ for some $0 < c < 1$ then $I_\chi(u,v) = (1-c)\int_X \chi(v)\omega^n < 0$. However, by the following series of Lemmas, if $\chi(t) \geq -1$ for every $t < 0$ then the value of $I_\chi(u,v)$ is always bounded from below by $-d_\theta(u,v)$ (see Remark 3.6 below).

Lemma 3.2 *Let $\chi \in \widetilde{\mathcal{W}}^-$. Assume that u, ϕ are negative θ-psh functions satisfying $u \leq \phi$. Denote $u_k = \max\{u, \phi - k\}$ for every $k > 0$. Then*

$$\int_X -\chi(u_k - \phi)\theta_{u_k}^n \leq \int_X -\chi(u-\phi)\theta_u^n - \chi(-k)d_\theta(u,\phi),$$

for every $k > 0$.

Proof Since $\theta_{u_k}^n = \theta_u^n$ in $\{u > \phi - k\}$ and $u_k = \phi - k$ in $\{u \leq \phi - k\}$, we have

$$\int_X -\chi(u_k - \phi)\theta_{u_k}^n = \int_{\{u\leq\phi-k\}} -\chi(-k)\theta_{u_k}^n + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta_u^n. \tag{3.5}$$

Since $\int_X \theta_\phi^n = \int_X \theta_{u_k}^n$, one obtains

$$\begin{aligned}\int_{\{u\leq\phi-k\}} -\chi(-k)\theta_{u_k}^n &= \int_X -\chi(-k)\theta_{u_k}^n + \int_{\{u>\phi-k\}} \chi(-k)\theta_{u_k}^n \\ &= \int_X -\chi(-k)\theta_\phi^n + \int_{\{u>\phi-k\}} \chi(-k)\theta_u^n.\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) gives

$$\begin{aligned}\int_X -\chi(u_k-\phi)\theta_{u_k}^n &= \int_X -\chi(-k)\theta_\phi^n + \int_{\{u>\phi-k\}} \chi(-k)\theta_u^n + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta_u^n \\ &= -\chi(-k)d_\theta(\phi,u) + \int_{\{u\leq\phi-k\}} -\chi(-k)\theta_u^n + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta_u^n \\ &= -\chi(-k)d_\theta(\phi,u) + \int_X -\chi(u_k-\phi)\theta_u^n \\ &\leq -\chi(-k)d_\theta(\phi,u) + \int_X -\chi(u-\phi)\theta_u^n\end{aligned}$$

because $u - \phi \leq u_k - \phi \leq 0$. The proof is completed. □

Lemma 3.3 *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^-$ such that $\inf_{\mathbb{R}_{<0}} \chi = -1$. Assume that u_1, u_2, u_3, ϕ are negative θ-psh functions satisfying $u_1 \leq u_2 \leq \phi$ and $u_1 \leq u_3 \leq \phi$. Denote $u_{j,k} = \max\{u_j, \phi - k\}$ for every $k > 1$ and $j = 1, 2, 3$. Then*

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \leq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + d_\theta(u_3, \phi) + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_3},$$

for every $k > 1$. Moreover, if additionally $u_3 = u_2$ on the set $\{u_1 < u_2\}$ then

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \leq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + \frac{1}{\tilde{\chi}(-k)} \int_{\{u_1 < u_2\}} \tilde{\chi}(u_1 - \phi)\theta^n_{u_2},$$

for every $k > 1$.

Proof Denote

$$A_k := \int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}}.$$

Since $\theta^n_{u_{3,k}} = \theta^n_{u_3}$ in $\{u_1 > \phi - k\} \subset \{u_3 > \phi - k\}$, we have

$$\begin{aligned} A_k &= \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + \int_{\{u_1 \leq \phi - k\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \\ &\leq \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + \int_{\{u_1 \leq \phi - k < u_2\}} -\chi(-k)\theta^n_{u_{3,k}}. \end{aligned}$$

Then, by the fact $\chi \geq -1$, we have

$$A_k \leq \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + \int_{\{u_1 \leq \phi - k < u_2\}} \theta^n_{u_{3,k}}. \tag{3.7}$$

Thus, by the fact $\int_X \theta^n_\phi = \int_X \theta^n_{u_{3,k}}$, we get

$$\begin{aligned} A_k &\leq \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + \int_X \theta^n_\phi - \int_{\{u_1 > \phi - k\}} \theta^n_{u_3} \\ &= \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + d_\theta(u_3, \phi) + \int_{\{u_1 \leq \phi - k\}} \theta^n_{u_3} \\ &\leq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + d_\theta(u_3, \phi) + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_3}. \end{aligned}$$

Thus the first desired inequality follows.

Now, consider the case where $u_3 = u_2$ on the set $\{u_1 < u_2\}$. By (3.7) and by the fact $\theta^n_{u_{3,k}} = \theta^n_{u_3} = \theta^n_{u_2}$ on $\{u_1 < u_2\} \cap \{\phi - k < u_2\}$, we have

$$\begin{aligned} A_k &\leq \int_{\{u_1 > \phi - k\}} -\chi(u_1 - u_2)\theta^n_{u_3} + \int_{\{u_1 \leq \phi - k < u_2\}} \theta^n_{u_2} \\ &\leq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + \frac{1}{\tilde{\chi}(-k)} \int_{\{u_1 < u_2\}} \tilde{\chi}(u_1 - \phi)\theta^n_{u_2}. \end{aligned}$$

The proof is completed. □

Lemma 3.4 *Let $\chi, \tilde{\chi}, u_j, u_{j,k}, \phi$ be as in Lemma 3.3 for $j = 1, 2, 3$. Then*

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_3},$$

for every $k > 1$. *Moreover, if additionally* $u_3 = u_2$ *on the set* $\{u_1 < u_2\}$ *then*

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_{\{u_1<u_2\}} \tilde{\chi}(u_1 - \phi)\theta^n_{u_2},$$

for every $k > 1$.

Proof Since $\theta^n_{u_{3,k}} = \theta^n_{u_3}$ in $\{u_1 > \phi - k\}$, we have

$$\begin{aligned}\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} &\geq \int_{\{u_1>\phi-k\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \\ &= \int_{\{u_1>\phi-k\}} -\chi(u_1 - u_2)\theta^n_{u_3} \\ &= \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \int_{\{u_1\leq\phi-k\}} -\chi(u_1 - u_2)\theta^n_{u_3} \\ &= \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \int_{\{u_1\leq\phi-k\}\cap U} -\chi(u_1 - u_2)\theta^n_{u_3},\end{aligned}$$

where $U = \{u_1 < u_2\}$. Then, by the fact $\chi \geq -1$, we have

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \int_{\{u_1\leq\phi-k\}\cap U} \theta^n_{u_3}. \tag{3.8}$$

Hence, by the monotonicity of $\tilde{\chi}$, we get

$$\begin{aligned}\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} &\geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_{\{u_1\leq\phi-k\}\cap U} \tilde{\chi}(u_1 - \phi)\theta^n_{u_3} \\ &\geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_3}.\end{aligned}$$

Now, consider the case where $u_3 = u_2$ on U. By (3.8) and by the fact $\theta^n_{u_3} = \theta^n_{u_2}$ on U, we have

$$\begin{aligned}\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} &\geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \int_{\{u_1\leq\phi-k\}\cap U} \theta^n_{u_2} \\ &\geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_{\{u_1\leq\phi-k\}\cap U} \tilde{\chi}(u_1 - \phi)\theta^n_{u_2} \\ &\geq \int_X -\chi(u_1 - u_2)\theta^n_{u_3} - \frac{1}{\tilde{\chi}(-k)} \int_U \tilde{\chi}(u_1 - \phi)\theta^n_{u_2}.\end{aligned}$$

This finishes the proof. □

Lemma 3.5 *Let* $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^-$ *such that* $\inf_{\mathbb{R}_{<0}} \chi = -1$. *Let* $B > 0$ *be a constant and let* $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi},B}(X, \theta)$ *with* $\sup_X u_1 = \sup_X u_2 = 0$. *Denote* $\phi := P_\theta[\max\{u_1, u_2\}]$ *and* $u_{j,k} := \max\{u_j, \phi - k\}$ *for every* $k > 1$ *and* $j = 1, 2$. *Then*

$$E_{\tilde{\chi},\theta,\phi}(u_{j,k}) \leq B - \tilde{\chi}(-k)d_\theta(u_j, \phi), \tag{3.9}$$

and

$$\frac{4B}{\tilde{\chi}(-k)} \leq I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) \leq -\frac{4B}{\tilde{\chi}(-k)} + d_\theta(u_1, u_2), \tag{3.10}$$

for every $k > 1$.

Proof By Lemma 3.2 and by the monotonicity of $\tilde{\chi}$, we have, for $j = 1, 2$,

$$\begin{aligned} E_{\tilde{\chi},\theta,\phi}(u_{j,k}) = \int_X -\tilde{\chi}(u_{j,k} - \phi)\theta^n_{u_{j,k}} &\leq \int_X -\tilde{\chi}(u_j - \phi)\theta^n_{u_j} - \tilde{\chi}(-k)d_\theta(u_j, \phi) \\ &\leq \int_X -\tilde{\chi}(u_j)\theta^n_{u_j} - \tilde{\chi}(-k)d_\theta(u_j, \phi) \\ &\leq B - \tilde{\chi}(-k)d_\theta(u_j, \phi). \end{aligned}$$

It remains to prove (3.10). For $j = 1, 2$ and $k > 1$, we denote

$$\begin{aligned} I_{1,j} &:= \int_{\{u_{1,k}<u_{2,k}\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{j,k}} + \int_{\{u_1<u_2\}} \chi(u_1 - u_2)\theta^n_{u_j} \\ &= \int_{\{u_1<u_2\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{j,k}} + \int_{\{u_1<u_2\}} \chi(u_1 - u_2)\theta^n_{u_j}, \end{aligned}$$

because $\{u_{1,k} < u_{2,k}\} \subset \{u_1 < u_2\}$. Similarly, we put

$$\begin{aligned} I_{2,j} &:= \int_{\{u_{2,k}<u_{1,k}\}} -\chi(u_{2,k} - u_{1,k})\theta^n_{u_{j,k}} + \int_{\{u_2<u_1\}} \chi(u_2 - u_1)\theta^n_{u_j} \\ &= \int_{\{u_2<u_1\}} -\chi(u_{2,k} - u_{1,k})\theta^n_{u_{j,k}} + \int_{\{u_2<u_1\}} \chi(u_2 - u_1)\theta^n_{u_j}. \end{aligned}$$

We have

$$I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) = (I_{1,1} - I_{1,2}) + (I_{2,2} - I_{2,1}) := I_1 + I_2. \tag{3.11}$$

We will estimate I_1 and I_2. Let $u' := \max\{u_1, u_2\}$. By [9, Theorem 2.1], we have

$$d_\theta(v, u') = d_\theta(v, P_\theta[u']) = d_\theta(v, \phi),$$

for every θ-psh function v. By this, Lemmas 3.3 and 3.4 (replace u_2, u_3 respectively, by u', u_1), we have

$$\begin{aligned} &\int_{\{u_1<u_2\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{1,k}} \\ &\quad = \int_X -\chi(u_{1,k} - u'_k)\theta^n_{u_{1,k}} \\ &\quad \leq \int_X -\chi(u_1 - u')\theta^n_{u_1} + d_\theta(u_1, \phi) + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_1} \\ &\quad = \int_{\{u_1<u_2\}} -\chi(u_1 - u_2)\theta^n_{u_1} + d_\theta(u_1, u') + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1 - \phi)\theta^n_{u_1} \\ &\quad \leq \int_{\{u_1<u_2\}} -\chi(u_1 - u_2)\theta^n_{u_1} + d_\theta(u_1, u') + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1)\theta^n_{u_1} \\ &\quad \leq \int_{\{u_1<u_2\}} -\chi(u_1 - u_2)\theta^n_{u_1} + d_\theta(u_1, u') - \frac{B}{\tilde{\chi}(-k)}, \end{aligned}$$

and

$$\begin{aligned}\int_{\{u_1<u_2\}} -\chi(u_{1,k}-u_{2,k})\theta^n_{u_{1,k}} &= \int_X -\chi(u_{1,k}-u'_k)\theta^n_{u_{1,k}} \\ &\geq \int_X -\chi(u_1-u')\theta^n_{u_1} - \frac{1}{\tilde{\chi}(-k)}\int_X \tilde{\chi}(u_1-\phi)\theta^n_{u_1} \\ &= \int_{\{u_1<u_2\}} -\chi(u_1-u_2)\theta^n_{u_1} - \frac{1}{\tilde{\chi}(-k)}\int_X \tilde{\chi}(u_1-\phi)\theta^n_{u_1} \\ &\geq \int_{\{u_1<u_2\}} -\chi(u_1-u_2)\theta^n_{u_1} - \frac{1}{\tilde{\chi}(-k)}\int_X \tilde{\chi}(u_1)\theta^n_{u_1} \\ &\geq \int_{\{u_1<u_2\}} -\chi(u_1-u_2)\theta^n_{u_1} + \frac{B}{\tilde{\chi}(-k)},\end{aligned}$$

where $u'_k := \max\{u', \phi - k\}$. Then

$$\frac{B}{\tilde{\chi}(-k)} \leq I_{1,1} \leq d_\theta(u_1, u') - \frac{B}{\tilde{\chi}(-k)}. \tag{3.12}$$

Using the last assertions of Lemmas 3.3 and 3.4 (replace u_2, u_3 by u'), we also get

$$\frac{B}{\tilde{\chi}(-k)} \leq I_{1,2} \leq -\frac{B}{\tilde{\chi}(-k)}. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\frac{2B}{\tilde{\chi}(-k)} \leq I_1 \leq -\frac{2B}{\tilde{\chi}(-k)} + d_\theta(u_1, u'). \tag{3.14}$$

Similar, we have

$$\frac{2B}{\tilde{\chi}(-k)} \leq I_2 \leq -\frac{2B}{\tilde{\chi}(-k)} + d_\theta(u_2, u'). \tag{3.15}$$

Combining (3.11), (3.14) and (3.15), we have

$$\frac{4B}{\tilde{\chi}(-k)} \leq I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) \leq -\frac{4B}{\tilde{\chi}(-k)} + d_\theta(u_1, u_2).$$

The proof is completed. □

Remark 3.6 For $u, v \in \mathrm{PSH}^-(X, \theta)$, let $\tilde{\chi} \in \mathcal{W}^-$ such that both $\tilde{E}_{\tilde{\chi},\theta}(u)$, $\tilde{E}_{\tilde{\chi},\theta}(v)$ are finite. By Lemma 3.5 we see that if $\inf_{\mathbb{R}_{<0}} \chi = -1$ then

$$I_\chi(u, v) \geq -d_\theta(u, v).$$

Indeed, by (3.10), we have

$$I_\chi(u, v) \geq I_\chi(u_k, v_k) + \frac{4B}{\tilde{\chi}(-k)} - d_\theta(u, v),$$

where $B > 0$ is an upper bound of $\tilde{E}_{\tilde{\chi},\theta}(u)$ and $\tilde{E}_{\tilde{\chi},\theta}(v)$. Note that $I_\chi(u_k, v_k) \geq 0$ since $u, v \in \mathcal{E}(X, \theta, \phi)$. Moreover, since $\tilde{\chi} \in \mathcal{W}^-$, we have $\lim_{k\to\infty} \tilde{\chi}(-k) = -\infty$. Hence, letting $k \to \infty$, we get $I_\chi(u, v) \geq -d_\theta(u, v)$.

The following theorem is the key step to prove the main results in the case of fixed cohomology:

Theorem 3.7 *Let $A \geq 1$, $0 < \delta < 1$, $B \geq A$ be constants. Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class. Let $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi},B\delta}(X,\theta)$ such that $\inf_{\mathbb{R}_{\leq 0}} \tilde{\chi} < \tilde{\chi}(-1) = -1$, $\sup_X u_1 = \sup_X u_2 = 0$ and $\int_X \theta_{u_1}^n + \int_X \theta_{u_2}^n \geq 2\delta$. Let $\epsilon > 0$ be a constant such that*

$$\inf \tilde{\chi} < \frac{-4B\delta}{\epsilon + d_\theta(u_1,u_2)}.$$

Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1,u_2)^2 \leq C(AB)^2 \left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)| + \epsilon + d_\theta(u_1,u_2)}\right)\right)^{-\gamma},$$

where $\chi(t) = \max\{t,-1\}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof Observe that h is increasing (and concave). In the case $\frac{4B\delta}{\epsilon + d_\theta(u_1,u_2)} < 1$, we have $\frac{\delta}{|I_\chi(u_1,u_2)|+\epsilon+d_\theta(u_1,u_2)} < \frac{1}{4B} < 1$, and then

$$\left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)| + \epsilon + d_\theta(u_1,u_2)}\right)\right)^{-\gamma} \geq \left(h^{\circ n}(1)\right)^{-\gamma} = 1.$$

By Chern-Levine-Nirenberg inequality [22, Proposition 3.1] (see also [22, Corollary 3.3]), if u and v are ω-psh functions satisfying $\sup_X u = 0$ and $-1 \leq v \leq 0$ then

$$\int_X |u|\omega_v^n \leq C, \tag{3.16}$$

where $C > 0$ is a constant depending only on n, X and ω. Using this fact for $u = u_j/A (j = 1, 2)$, we get $d_{cap}(u_1/A, u_2/A)^2 \leq C' := 2C \int_X \omega^n$. Hence,

$$d_{cap}(u_1,u_2)^2 \leq C'(AB)^2 \left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)| + \epsilon + d_\theta(u_1,u_2)}\right)\right)^{-\gamma}.$$

Then, without loss of generality, we can assume that

$$\frac{4B\delta}{\epsilon + d_\theta(u_1,u_2)} \geq 1. \tag{3.17}$$

Denote $\phi = P_\theta[\max\{u_1,u_2\}]$ and $u_{j,k} = \max\{u_j, \phi - k\}$ for every $k > 1$ and $j = 1, 2$. By Theorem 3.1 and Lemma 3.5, we get

$$d_{\text{cap}}(u_{1,k},u_{2,k})^2 \leq C_1 A^2 \left(B - \frac{\tilde{\chi}(-k) d_\theta(u_1,u_2)}{\delta}\right)^2 \times \left(h^{\circ n}\left(\frac{\delta}{I_\chi(u_1,u_2) - \frac{4B\delta}{\tilde{\chi}(-k)} + d_\theta(u_1,u_2)}\right)\right)^{-\gamma},$$

for every $k > 1$, where $C_1 > 0$ depends only on n, X, ω and γ. By (3.17), there exists $k_0 > 1$ such that

$$\tilde{\chi}(-k_0) = \frac{-4B\delta}{\epsilon + d_\theta(u_1,u_2)}.$$

Then, we have

$$d_{\mathrm{cap}}(u_{1,k_0}, u_{2,k_0})^2 \le 25C_1(AB)^2 \left(h^{\circ(n)}\left(\frac{\delta}{I_\chi(u_1,u_2)+\epsilon+2d_\theta(u_1,u_2)}\right)\right)^{-\gamma}. \quad (3.18)$$

On the other hand, for every $\varphi \in \mathrm{PSH}(X,\omega)$ with $0 \le \varphi \le 1$, we have,

$$\begin{aligned}\left(\int_X |u_j - u_{j,k_0}|^{1/2}\omega_\varphi^n\right)^2 &= \left(\int_{\{u_j<\phi-k_0\}} |u_j - u_{j,k_0}|^{1/2}\omega_\varphi^n\right)^2\\ &\le \left(\int_{\{u_j<\phi-k_0\}} |u_j|^{1/2}\omega_\varphi^n\right)^2\\ &\le \frac{1}{k_0}\left(\int_{\{u_j<\phi-k_0\}} |u_j|\omega_\varphi^n\right)^2\\ &\le \frac{C_2A^2}{k_0},\end{aligned}$$

for $j = 1, 2$, where $C_2 > 0$ depends only on X and ω. The last inequality is obtained by applying (3.16) to $u = u_j/A$ and $v = \varphi - 1$. Since $\tilde\chi \in \widetilde{\mathcal{W}}^-$ (i.e., $\tilde\chi$ is increasing, convex and $\tilde\chi(0) = 0$), we have

$$-\tilde\chi(-t) \ge h(t)h(1) = h(t) \text{ and } \tfrac{\tilde\chi(-t)}{-t} \le \tfrac{\tilde\chi(-1)}{-1} = 1,$$

for every $t \ge 1$. Then

$$t \ge -\tilde\chi(-t) \ge h(t),$$

for every $t \ge 1$. Hence, by the definition of d_{cap} and (3.17), we get

$$\begin{aligned}d_{\mathrm{cap}}(u_j, u_{j,k_0})^2 &\le \frac{C_2A^2}{k_0} \le \frac{C_2A^2(\epsilon + d_\theta(u_1,u_2))}{4B\delta} \quad (3.19)\\ &\le C_2A^2\left(h^{\circ n}\left(\frac{4B\delta}{\epsilon+d_\theta(u_1,u_2)}\right)\right)^{-1}\\ &\le C_2A^2\left(h^{\circ n}\left(\frac{\delta}{\epsilon+d_\theta(u_1,u_2)}\right)\right)^{-1}\end{aligned}$$

because $B \ge 1$ (observe that h is increasing). Using (3.18), (3.19) and the triangle inequality, we obtain

$$\begin{aligned}d_{\mathrm{cap}}(u_1,u_2)^2 &\le (d_{\mathrm{cap}}(u_1,u_{1,k_0}) + d_{\mathrm{cap}}(u_{1,k_0},u_{2,k_0}) + d_{\mathrm{cap}}(u_{2,k_0},u_2))^2\\ &\le 3(d_{\mathrm{cap}}(u_1,u_{1,k_0})^2 + d_{\mathrm{cap}}(u_{1,k_0},u_{2,k_0})^2 + d_{\mathrm{cap}}(u_{2,k_0},u_2)^2)\\ &\le C_3(AB)^2\left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)|+\epsilon+2d_\theta(u_1,u_2)}\right)\right)^{-\gamma}\\ &\le C_4(AB)^2\left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)|+\epsilon+d_\theta(u_1,u_2)}\right)\right)^{-\gamma},\end{aligned}$$

where $C_3, C_4 > 0$ depend only on n, X, ω and γ. The last inequality holds due to the fact that $\frac{h(t)}{\sqrt{t}}$ is decreasing on $(0,\infty)$.

The proof is completed. □

The following result implies Theorem 1.2 for the case of fixed cohomology (*i.e,* when $\eta = \theta$):

Theorem 3.8 *Let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class ($A \ge 1$). Let $u \in \mathrm{PSH}(X,\theta)$ such that $\sup_X u = 0$ and $\int_X \theta_u^n := 2\delta > 0$. Assume $u \in \tilde{\mathcal{E}}_{\tilde\chi, B\delta}(X,\theta)$, where $B \ge A$ is a given constant and $\tilde\chi \in \mathcal{W}^-$ with $\tilde\chi(-1) = -1$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that*

$$d_{cap}(u,v)^2 \le C(A\,B)^2 \left(h^{\circ n}\left(\frac{\delta}{\|\theta_u^n - \theta_v^n\| + d_\theta(u,v)}\right)\right)^{-\gamma},$$

for every $v \in \mathrm{PSH}(X,\theta)$ with $\sup_X v = 0$, where $h(s) = (-\tilde\chi(-s))^{1/2}$.

Proof Put

$$t_0 = \|\theta_u^n - \theta_v^n\| + d_\theta(u,v).$$

If $t_0 \ge \delta$ then $h^{\circ n}\left(\frac{\delta}{t_0}\right) \le h^{\circ n}(1) = 1$, and the desired property is trivial. Hence, without loss of generality, we can assume that $t_0 < \delta$. Denote

$$M = \frac{5B\delta}{t_0}, \quad \tilde\chi_M(-s) = \max\{\tilde\chi(-s), -M\} \quad \text{and} \quad h_M(s) = (-\tilde\chi_M(-s))^{1/2},$$

for every $s \ge 0$. We have $Mt_0 = 5B\delta$. Furthermore for $\psi \in \mathrm{PSH}(X,\theta)$ with $\sup_X \psi = 0$, we get

$$\begin{aligned}\int_X -\tilde\chi_M(\psi)\theta_v^n &= \int_X -\tilde\chi_M(\psi)\theta_u^n + \int_X -\tilde\chi_M(\psi)(\theta_v^n - \theta_u^n)\\ &\le B\delta + Mt_0,\end{aligned}$$

because of the hypothesis on u and the choice of t_0, χ_M. Consequently, we obtain that $v \in \tilde{E}_{\tilde\chi_M, B\delta + Mt_0}$. Since $\inf \tilde\chi_M = -M < \dfrac{-4B\delta}{\|\theta_u^n - \theta_v^n\| + d_\theta(u,v)}$, it follows from Theorem 3.7 that

$$d_{\mathrm{cap}}(u,v)^2 \le C_1(AB)^2 \left(h_M^{\circ n}\left(\frac{\delta}{|I_\chi(u,v)| + \|\theta_u^n - \theta_v^n\| + d_\theta(u,v)}\right)\right)^{-\gamma},$$

where $\chi(s) = \max\{s, -1\}$ and $C_1 > 0$ depends only on n, X, ω and γ. Since $|I_\chi(u,v)| \le \|\theta_u^n - \theta_v^n\|$, it follows that

$$d_{\mathrm{cap}}(u,v)^2 \le C_1(A\,B)^2 \left(h_M^{\circ n}\left(\frac{\delta}{2\|\theta_u^n - \theta_v^n\| + d_\theta(u,v)}\right)\right)^{-\gamma}. \tag{3.20}$$

Since $\tilde\chi_M$ is convex, we have h_M is concave, and then $(h_M)^{\circ n}$ is concave. In particular,

$$\frac{h_M^{\circ n}(t_1)}{t_1} = \frac{h_M^{\circ n}(t_1) - h_M^{\circ n}(0)}{t_1 - 0} \le \frac{h_M^{\circ n}(t_2) - h_M^{\circ n}(0)}{t_2 - 0} = \frac{h_M^{\circ n}(t_2)}{t_2},$$

for every $t_1 > t_2 > 0$. Hence, by (3.20), we have

$$d_{\mathrm{cap}}(u,v)^2 \le C_2(A\,B)^2 \left(h_M^{\circ n}\left(\frac{\delta}{\|\theta_u^n - \theta_v^n\| + d_\theta(u,v)}\right)\right)^{-\gamma}, \tag{3.21}$$

where $C_2 = 2^\gamma C_1$. Since $\tilde{\chi}_M$ is convex and $\tilde{\chi}_M(0) = 0$, we have $\frac{\tilde{\chi}_M(-t)}{-t} \leq \frac{\tilde{\chi}_M(-1)}{-1} = 1$, for every $t > 1$. As a consequence, for every $1 < t < M$, we have $\tilde{\chi}_M(-t) = \tilde{\chi}(-t) > -M$, and then $h_M(t) = h(t)$. Hence, by (3.20), we obtain

$$d_{\text{cap}}(u, v)^2 \leq C_2(A\,B)^2 \left(h^{\circ n} \left(\frac{\delta}{\|\theta_u^n - \theta_v^n\| + d_\theta(u, v)} \right) \right)^{-\gamma}.$$

The proof is completed. □

3.2 Proof of Theorems 1.4 and 1.5 when $\theta_1 = \theta_2$

In order to prove the next main results when $\theta_1 = \theta_2$, we need several auxiliary lemmas.

Lemma 3.9 *Let $u : 2\mathbb{B} := \{z \in \mathbb{C}^n : |z| < 2\} \to [-\infty, 0]$ be a subharmonic function such that $\int_{2\mathbb{B}} \Delta u := A < \infty$. Assume that h is a non-negative radial smooth function on $\mathbb{C}^n$ satisfying $\int_{\mathbb{C}^n} h dV = 1$ and $\text{Supp}(h) \subset \mathbb{B}$. For every $0 < \epsilon < 1$, we denote $h_\epsilon(z) = \frac{1}{\epsilon^{2n}} h\left(\frac{z}{\epsilon}\right)$. Then, there exists $C_1 > 0$ depending only on n and A such that*

$$\int_{\mathbb{B}} (u * h_\epsilon - u) dV \leq C_1 \epsilon^2, \tag{3.22}$$

for every $0 < \epsilon < 1$. Moreover, if $u \geq -M$ for some $M > 0$ then there exists a constant $C_2 > 0$ depending only on n and h such that

$$\|u * h_\epsilon\|_{\mathscr{C}^1(\mathbb{B})} \leq \frac{C_2 M}{\epsilon}. \tag{3.23}$$

Although this lemma is elementary, we could not find any reference to it. Therefore, we write in detail for readers' convenience.

Proof We note that $\|u\|_{C^0(2\mathcal{B})} \leq M$ since $-M \leq u \leq 0$. By the fact $D(u * h_\epsilon) = u * Dh_\epsilon$, we also have

$$|D(u * h_\epsilon)(z)| = \left| \int_{\mathbb{B}} u(z - w) Dh_\epsilon(w) dV(w) \right| \leq C_1 \|u\|_{\mathscr{C}^0(2\mathcal{B})} \|Dh_\epsilon\|_{L^1(\mathcal{B})} \leq \frac{C_2 M}{\epsilon},$$

where $C_1, C_2 > 0$ are constants depending only on n and h. Hence,

$$\|u * h_\epsilon\|_{\mathscr{C}^1(\mathbb{B})} \leq \frac{C_3 M}{\epsilon},$$

where $C_3 > 0$ is a constant depending only on n and h.

It remains to prove (3.22). After approximating u by a decreasing sequence of smooth subharmonic functions, the problem is reduced to the case where u is smooth. By Jensen formula [1, page 36], for every $z \in \mathbb{B}$ and $0 < \epsilon < 1$, we have

$$\int_{\mathbb{B}} (\bar{u}_\epsilon - u) dV = \int_{\mathbb{B}} \int_0^\epsilon \int_{\{|\xi| \leq t\}} \Delta u(z + \xi) dV(\xi) \frac{dt}{t^{2n-1}} dV(z),$$

where $\bar{u}_t(z)$ is the average of u over $\partial B_t(z) := z + t\partial\mathbb{B}$. Therefore, by Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{B}} (\bar{u}_\epsilon - u)(z)dV &= \int_0^\epsilon \int_{\{|\xi|\le t\}} \int_{\mathbb{B}} \Delta u(z+\xi)dV(z)dV(\xi)\frac{dt}{t^{2n-1}} \\ &\le A\int_0^\epsilon \int_{\{|\xi|\le t\}} dV(\xi)\frac{dt}{t^{2n-1}} \\ &= S_{2n}A\int_0^\epsilon t dt \\ &= \frac{S_{2n}A\epsilon^2}{2},\end{aligned}$$

where S_{2n} is the volume of the unit ball $\mathbb{B}$. Moreover, by using the polar coordinates and the monotonicity of $\bar{u}_t(z)$ with respect to t, we get

$$\begin{aligned}u * h_\epsilon(z) = 2nS_{2n}\int_0^\epsilon \bar{u}_t(z)H_\epsilon(t)t^{2n-1}dt &\le 2nS_{2n}\int_0^\epsilon \bar{u}_\epsilon(z)H_\epsilon(t)t^{2n-1}dt \\ &= \bar{u}_\epsilon(z)\int_{\epsilon\mathbb{B}} h_\epsilon(\xi)dV(\xi) \\ &= \bar{u}_\epsilon(z),\end{aligned}$$

where $H_\epsilon(t) = h_\epsilon(z)$ for $t = |z|$. Hence, we obtain

$$\int_{\mathbb{B}} (u * h_\epsilon - u)(z)dV \le \int_{\mathbb{B}} (\bar{u}_\epsilon - u)(z)dV \le \frac{S_{2n}A\epsilon^2}{2}.$$

The proof is completed. □

Lemma 3.10 *Let (X, ω) be a compact Kähler manifold and let μ be a Radon measure on X such that*

$$\int_X \min\{|u_1 - u_2|, 1\}d\mu \le H(\|u_1 - u_2\|_{L^1(X)}),$$

for all $u_1, u_2 \in \mathrm{PSH}(X, \omega)$, where $H : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is an increasing function. Let $\Omega \subset X$ such that there exists a biholomorphic mapping $\varphi : 2\mathbb{B} \to \Omega$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^n$. Then there exists a constant $C \ge 1$ depending only on X, ω, Ω and φ such that

$$\int_{\varphi(\mathbb{B})} |(u - v)\circ\varphi^{-1}|d\mu \le CM\, H(\|(u - v)\circ\varphi^{-1}\|_{L^1(\varphi(2\mathbb{B}))}).$$

for every $M \ge 1$ and for all psh functions $u, v \in \mathrm{PSH}(2\mathbb{B})$ with $-M \le u, v \le 0$.

Proof Let $u' := \max\{u, M|z|^2 - 2M\}$. We have $u = u'$ on $\mathbb{B}$ and $u' = M|z|^2 - 2M$ outside $\sqrt{2}\mathbb{B}$. Let $\phi \in \mathscr{C}_0^\infty(2\mathbb{B})$ such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on $\frac{3}{2}\mathbb{B}$. Then $\tilde{u} = (\phi u')\circ\varphi^{-1}$ is a $(CM\omega)$-psh function on X, where $C \ge 1$ depends only on X, ω, φ and ϕ. We do similarly for v to obtain $\tilde{v}$. Observe that $-2M \le \tilde{u} - \tilde{v} \le 2M$. We have $\frac{\tilde{u}}{(C+2)M}$ and $\frac{\tilde{v}}{(C+2)M}$ are two ω-psh functions whose difference is at most 1. By the assumption, we get

$$\int_X \frac{|\tilde{u} - \tilde{v}|}{(C+2)M}d\mu \le H\left(\frac{\|\tilde{u} - \tilde{v}\|_{L^1(X)}}{(C+2)M}\right) \le H\left(\|\tilde{u} - \tilde{v}\|_{L^1(X)}\right)$$

because H is increasing. Since $\tilde{u} = \tilde{v}$ outside $\varphi(\sqrt{2}\mathbb{B})$, we have

$$\|\tilde{u} - \tilde{v}\|_{L^1(X)} = \|\tilde{u} - \tilde{v}\|_{L^1(\varphi(\sqrt{2}\mathbb{B}))} \le \|(u - v)\circ\varphi^{-1}\|_{L^1(\varphi(2\mathbb{B}))},$$

Thus, we obtain

$$\int_{\varphi(\mathbb{B})} |(u-v)\circ\varphi^{-1}|d\mu \le \int_X |\tilde{u}-\tilde{v}|d\mu \le (C+2)M\, H(\|(u-v)\circ\varphi^{-1}\|_{L^1(\varphi(2\mathbb{B}))}).$$

The proof is completed. □

Lemma 3.11 *Let (X,ω) be a compact Kähler manifold and let μ be a Radon measure on X such that*

$$\int_X \min\{|u_1-u_2|,1\}d\mu \le H(\|u_1-u_2\|_{L^1(X)}),$$

for all $u_1,u_2\in \mathrm{PSH}(X,\omega)$, where $H:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is an increasing function. Assume that Ω is an open subset of X and K is a compact subset of Ω. Then, there exists a constant $C>0$ depending only on n, X, ω, K and Ω such that

$$\int_K \min\{|u-v|,1\}d\mu \le CM\, H\left(\|u-v\|_{L^1(\Omega)}\right) + \int_{K\cap\{u<-M\}} d\mu + \int_{K\cap\{v<-M\}} d\mu,$$

for all negative ω-psh function u, v in Ω and for every $M\ge 1$.

Proof By using a suitable open cover of Ω, we can assume that there exists a biholomorphic mapping $\varphi : 2\mathbb{B}\to\Omega$ such that $K\Subset\varphi(\mathbb{B})$. Put $u_M := \max\{u,-M\}$ and $v_M := \max\{v,-M\}$. We have

$$\begin{aligned}\min\{|u-v|,1\} &\le |u_M-v_M| + \min\{|u-u_M|,1\} + \min\{|v-v_M|,1\}\\ &\le |u_M-v_M| + |u_{M+1}-u_M| + |v_{M+1}-v_M|.\end{aligned}$$

Then

$$\int_K \min\{|u-v|,1\}d\mu \le \int_K |u_M-v_M|d\mu + \int_K |u_{M+1}-u_M|d\mu + \int_K |v_{M+1}-v_M|d\mu.$$

By Lemma 3.10, we have

$$\int_K |u_M-v_M|d\mu \le CM\, H\left(\|u-v\|_{L^1(\Omega)}\right),$$

where $C\ge 1$ depends only on n, X, ω and φ.

Moreover

$$\int_K |u_{M+1}-u_M|d\mu \le \int_{K\cap\{u<-M\}} d\mu,$$

and

$$\int_K |v_{M+1}-v_M|d\mu \le \int_{K\cap\{v<-M\}} d\mu,$$

Therefore

$$\int_K \min\{|u-v|,1\}d\mu \le CM\, H\left(\|u-v\|_{L^1(\Omega)}\right) + \int_{K\cap\{u<-M\}} d\mu + \int_{K\cap\{v<-M\}} d\mu,$$

This finishes the proof. □

The following result generalizes both Theorems 1.4 and 1.5 when $\theta_1=\theta_2$ (see the next section, especially Lemma 4.2, for more details about this point).

Theorem 3.12 *Let θ be a closed smooth real $(1,1)$-form such that $\theta \leq A\omega$ for a given constant $A \geq 1$. Let $0 < \delta \leq 1$, $B \geq A$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi},B\delta}(X,\theta)$ such that $\tilde{\chi}(-1) = -1$, $\sup_X u_1 = \sup_X u_2 = 0$ and $\int_X \theta_{u_1}^n + \int_X \theta_{u_2}^n \geq 2\delta$. Assume that there exists a concave increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for $j = 1, 2$,*

$$\int_X \min\{|\psi_1 - \psi_2|, 1\}\theta_{u_j}^n \leq H(\|\psi_1 - \psi_2\|_{L^1(X)}), \tag{3.24}$$

for every $\psi_1, \psi_2 \in \mathrm{PSH}(X,\omega)$. Then, for every $0 < a, b, \gamma < 1$ and $m > 0$, there exists $C > 0$ depending only on $n, X, A, \omega, H(1), a, b, \gamma$ and m such that

$$d_{cap}(u_1, u_2)^2 \leq C\, B^2 \left(h^{\circ n}\left(\frac{\delta}{G(x) + d_\theta(u_1, u_2)}\right)\right)^{-\gamma},$$

where

$$x := \mathrm{dist}_{-1}(\theta_{u_1}^n, \theta_{u_2}^n), \quad G(x) = H^a(x^{2(1-b)}) + H(H^m(x^{1-b})) + x^{ab},$$

and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof Without loss of generality, we can assume that $0 < x := \mathrm{dist}_{-1}(\theta_{u_1}^n, \theta_{u_2}^n) < 1$. Denote $\chi(t) = \max\{t, -1\}$. Since $\inf \tilde{\chi} = -\infty$, it follows from Theorem 3.7 that

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_0(A\,B)^2 \left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1, u_2)| + d_\theta(u_1, u_2)}\right)\right)^{-\gamma}, \tag{3.25}$$

where $C_0 > 0$ depends only on n, X, ω and γ. We will estimate $|I_\chi(u_1, u_2)|$.

For every $k > 0$ and $j = 1, 2$, it follows from the Skoda integrability theorem that

$$\int_{\{u_j < -k\}} \omega^n \leq \int_X \exp\left(\frac{-c_0(u_j + k)}{A}\right)\omega^n \leq C_1 \exp\left(\frac{-c_0 k}{A}\right), \tag{3.26}$$

where $c_0, C_1 > 0$ depend only on X and ω.

Let $\{U_l\}_{1 \leq l \leq l_0}$ be a finite cover of X such that

- for every l, there exists a biholomorphic function $\varphi_l : 2\mathbb{B} \to U_l$, where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^n$;
- the family $\{\varphi_l(\mathbb{B})\}_{1 \leq l \leq l_0}$ is also a finite cover of X.

For the convenience, we denote $V_l := \varphi\left(\frac{3}{2}\mathbb{B}\right)$ and $W_l := \varphi(\mathbb{B})$. Let ρ_l be a smooth function on $\varphi_l(2\mathbb{B})$ such that $dd^c\rho = A\omega \geq 0$ on $\varphi_l(2\mathbb{B})$. Since $\max\{u_j, -k\}$ is $A\omega$-psh, we have $\max\{u_j, -k\} \circ \varphi_l + \rho_l \circ \varphi_l$ is psh on $2\mathbb{B}$. Moreover, for every $0 < \epsilon < 1/2$, $\max\{u_j, -k\} \circ \varphi_l + \rho_l \circ \varphi_l$ is dominated on $\frac{3}{2}\mathbb{B}$ by $(\max\{u_j, -k\} \circ \varphi_l + \rho_l \circ \varphi_l) * h_\epsilon$ (which is also psh), where $h_\epsilon(z) = \frac{1}{\epsilon^{2n}} h\left(\frac{z}{\epsilon}\right)$, h is a non-negative radial smooth function on $\mathbb{C}^n$ satisfying $\int_{\mathbb{C}^n} h dV = 1$ and $\mathrm{Supp}(h) \subset \mathbb{B}$. For every $0 < \epsilon < 1/2$, $1 \leq l \leq l_0$, $k > 0$ and $j = 1, 2$, we denote

$$u_{j,k,l,\epsilon}(z) = \left((\max\{u_j, -k\} \circ \varphi_l) * h_\epsilon\right) \circ \varphi_l^{-1}(z), \quad z \in V_l.$$

Then, $u_{j,k,l,\epsilon} + (\rho_l \circ \varphi_l * h_\epsilon) \circ \varphi_l^{-1}$ is a psh function on V_l dominating $\max\{u_j, -k\} + \rho_l$. In particular, there exists $C_2 > 0$ depending only on $A\omega$ such that $u_{j,k,l,\epsilon} \in \mathrm{PSH}(V_l, C_2\omega)$ and $\max\{u_j, -k\} - C_2\epsilon \leq u_{j,k,l,\epsilon} \leq 0$ on V_l. Moreover, by Lemma 3.9, there exists $C_3 > 0$ depending only on n, h, X and $A\omega$ such that

$$\|u_{j,k,l,\epsilon}\|_{\mathscr{C}^1(V_l)} \leq \frac{C_3 k}{\epsilon}, \tag{3.27}$$

and

$$\|u_{j,k,l,\epsilon} - \max\{u_j, -k\}\|_{L^1(V_l)} \leq C_3\epsilon^2, \tag{3.28}$$

Let $\{\psi_l\}$ be a partition of unity subordinate to $\{W_l\}$ and denote

$$u_{j,k,\epsilon} = \sum_{l=1}^{l_0} \psi_l u_{j,k,l,\epsilon}.$$

Put $\mu = \theta_{u_1}^n + \theta_{u_2}^n$. For every $k, M \geq 1$, $j = 1, 2$ and $0 < \epsilon < 1$, we have

$$\begin{aligned}
\int_X \min\{|u_j - u_{j,k,\epsilon}|, 2\} d\mu &\leq \int_X \min\{|\max\{u_j, -k-2\} - u_{j,k,\epsilon}|, 2\} d\mu \\
&\leq \int_X \min\{\sum_{l=1}^{l_0} \psi_l |\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}|, 2\} d\mu \\
&\leq \sum_{l=1}^{l_0} \int_{W_l} \min\{|\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}|, 2\} d\mu \\
&\leq C_4 A M \sum_{l=1}^{l_0} H\left(\|\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}\|_{L^1(V_l)}\right) \\
&+ 4\sum_{l=1}^{l_0} \int_{\overline{W_l} \cap \{u_j, u_{j,k,\epsilon} < -M\}} d\mu \\
&\leq C_4 A M \sum_{l=1}^{l_0} H\left(\|\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}\|_{L^1(V_l)}\right) \\
&+ 4 l_0 \int_{\{u_j < -M+1\}} d\mu,
\end{aligned}$$

where $C_4 > 0$ is a constant depending only on $n, X, \{V_l\}, \{W_l\}$ and $A\omega$, and the 4th inequality holds due to Lemma 3.11. Moreover, by the fact

$$\begin{aligned}
|\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}| \leq\ & |\max\{u_j, -k\} - u_{j,k,l,\epsilon}| \\
& + |\max\{u_j, -k-2\} - \max\{u_j, -k\}|,
\end{aligned}$$

we have

$$\begin{aligned}
& H\left(\|\max\{u_j, -k-2\} - u_{j,k,l,\epsilon}\|_{L^1(V_l)}\right) \\
& \quad \leq H\left(\|\max\{u_j, -k\} - u_{j,k,l,\epsilon}\|_{L^1(V_l)} + 2\int_{V_l \cap \{u_j < -k\}} \omega^n\right).
\end{aligned}$$

Then the previous estimates yield

$$\int_X \min\{|u_j - u_{j,k,\epsilon}|, 2\} d\mu$$
$$\leq C_4 AM \sum_{l=1}^{l_0} H\left(\| \max\{u_j, -k\} - u_{j,k,l,\epsilon} \|_{L^1(V_l)} + 2 \int_{V_l \cap \{u_j < -k\}} \omega^n \right)$$
$$+ 4l_0 \int_{\{u_j < -M+1\}} d\mu.$$

Hence, by (3.26) and (3.28), we have

$$\int_X \min\{|u_j - u_{j,k,\epsilon}|, 2\} d\mu \leq C_5 AM\, H\left(\epsilon^2 + k \exp\left(\frac{-c_0 k}{A}\right)\right) + 4l_0 \int_{\{u_j < -M+1\}} d\mu, \tag{3.29}$$

where $C_5 > 0$ depends on C_1, C_3 and C_4. Using the facts $u_j \in \mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$ and $\mathbf{1}_{\{u_j < -M+1\}} \leq \max\{u_j, -M+2\} - \max\{u_j, -M+1\} \leq \mathbf{1}_{\{u_j < -M+2\}}$, we have

$$\begin{aligned}
\int_{\{u_j < -M+1\}} d\mu &\leq \int_X |\max\{u_j, -M+1\} - \max\{u_j, -M+2\}| d\mu \\
&= A \int_X \left| \frac{\max\{u_j, -M+1\}}{A} - \frac{\max\{u_j, -M+2\}}{A} \right| (\theta_{u_1}^n + \theta_{u_2}^n) \\
&\leq 2A\, H\left(\frac{\| \max\{u_j, -M+1\} - \max\{u_j, -M+2\} \|_{L^1(X)}}{A} \right) \\
&\leq 2A\, H\left(\int_{\{u_j < -M+2\}} \frac{\omega^n}{A} \right) \\
&\leq 2A\, H\left(\int_{\{u_j < -M+2\}} \omega^n \right).
\end{aligned}$$

Then, by (3.26), we get

$$4l_0 \int_{\{u_j < -M+1\}} d\mu \leq C_6 A\, H\left(\exp\left(\frac{-c_0 M}{A} \right) \right), \tag{3.30}$$

where $C_6 > 0$ depends only on X, l_0 and ω. Combining (3.29) and (3.30), we get

$$\int_X \min\{|u_j - u_{j,k,\epsilon}|, 2\} d\mu \leq C_5 AM\, H\left(\epsilon^2 + k \exp\left(\frac{-c_0 k}{A}\right)\right) + C_6 A\, H\left(\exp\left(\frac{-c_0 M}{A} \right) \right). \tag{3.31}$$

Recall that

$$\begin{aligned}
I_\chi(u_1,u_2) &= \int_{\{u_1<u_2\}} \min\{|u_1-u_2|,1\}(\theta^n_{u_1}-\theta^n_{u_2}) + \int_{\{u_2<u_1\}} \min\{|u_1-u_2|,1\}(\theta^n_{u_2}-\theta^n_{u_1})\\
&= \int_{\{u_1<u_2\}} \min\{u_2-u_1,1\}(\theta^n_{u_1}-\theta^n_{u_2}) + \int_{\{u_2<u_1\}} \min\{u_1-u_2,1\}(\theta^n_{u_2}-\theta^n_{u_1})\\
&= \int_{\{u_1<u_2\}} \min\{u_2-u_1,1\}(\theta^n_{u_1}-\theta^n_{u_2}) + \int_{\{u_2<u_1\}} \max\{u_2-u_1,-1\}(\theta^n_{u_1}-\theta^n_{u_2})\\
&= \int_{\{u_1<u_2\}} \max\{\min\{u_2-u_1,1\},-1\}(\theta^n_{u_1}-\theta^n_{u_2})\\
&\quad + \int_{\{u_2<u_1\}} \max\{\min\{u_2-u_1,1\},-1\}(\theta^n_{u_1}-\theta^n_{u_2})\\
&= \int_X \max\{\min\{u_2-u_1,1\},-1\}(\theta^n_{u_1}-\theta^n_{u_2}).
\end{aligned}$$

By the fact that

$$\max\{t_1,t_3\}-\max\{t_2,t_3\}=\min\{-t_2,-t_3\}-\min\{-t_1,-t_3\}\le \max\{t_1-t_2,0\},$$

we have

$$|\max\{\min\{u_2-u_1,1\},-1\}-\max\{\min\{u_{2,k,\epsilon}-u_{1,k,\epsilon},1\},-1\}|\le |u_2-u_1-u_{2,k,\epsilon}+u_{1,k,\epsilon}|,$$

for every $k>0$ and $0<\epsilon<1$. Since the LHS of the last inequality is bounded by 2, it follows that

$$|\max\{\min\{u_2-u_1,1\},-1\}-\max\{\min\{u_{2,k,\epsilon}-u_{1,k,\epsilon},1\},-1\}|\le \Psi_{k,\epsilon},$$

where

$$\Psi_{k,\epsilon}=\min\{|u_2-u_1-u_{2,k,\epsilon}+u_{1,k,\epsilon}|,2\}.$$

Therefore

$$|I_\chi(u_1,u_2)|\le \left|\int_X \Phi_{k,\epsilon}(\theta^n_{u_1}-\theta^n_{u_2})\right| + \int_X \Psi_{k,\epsilon}(\theta^n_{u_1}+\theta^n_{u_2}), \tag{3.32}$$

where $\Phi_{k,\epsilon}=\max\{\min\{u_{2,k,\epsilon}-u_{1,k,\epsilon},1\},-1\}$. By (3.27), we have

$$\left|\int_X \Phi_{k,\epsilon}(\theta^n_{u_1}-\theta^n_{u_2})\right| \le \frac{C_3k}{\epsilon}\,\mathrm{dist}_{-1}(\theta^n_{u_1},\theta^n_{u_2})=\frac{C_3k\,x}{\epsilon}. \tag{3.33}$$

Combining (3.31), (3.32), (3.33), we get

$$|I_\chi(u_1,u_2)|\le C_5AM\,H\left(\epsilon^2+k\,\exp\left(\frac{-c_0k}{A}\right)\right)+C_6A\,H\left(\exp\left(\frac{-c_0M}{A}\right)\right)+\frac{C_3k\,x}{\epsilon}.$$

Now, choosing $\epsilon=x^{1-b}$, $M=-\dfrac{Am\log(H(\epsilon)/H(1))}{c_0}$ and $k=-\dfrac{3A\log\epsilon}{c_0}$, we have

$$|I_\chi(u_1,u_2)|\le C_7\left(H^a(x^{2(1-b)})+H(H^m(x^{1-b}))+x^{ab}\right), \tag{3.34}$$

where $C_7>1$ depends only on $A, C_3, C_5, C_6, a, b, m$ and $H(1)$. Combining (3.25) and (3.34), we obtain the desired inequality.

The proof is completed. □

3.3 Application to the space of singularity types

In this part we apply quantitative stability theorems in the previous subsection to deduce some properties of the pseudometric space of singularity types in a big cohomology class.

Proposition 3.13 *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class ($A \geq 1$). Assume that u_1 and u_2 are model θ-psh functions such that $\int_X \theta_{u_1}^n + \int_X \theta_{u_2}^n \geq 2\delta > 0$, where $\delta > 0$ is a constant. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that*

$$d_{cap}(u_1, u_2)^2 \leq C \frac{A^{2n+4}}{\delta^2} \left(\frac{d_\theta(u_1, u_2)}{\delta} \right)^{2^{-n}\gamma}.$$

The above result implies in particular that for model potentials, the convergence in $d_{\mathcal{S}}$ is stronger than that in capacity. This non-quantitative fact follows also from [10, Theorem 5.6].

Proof By [8, Theorem 3.8], we have

$$\theta_{u_j}^n \leq \mathbf{1}_{\{u_j=0\}} \theta^n \leq A^n \omega^n,$$

for $j = 1, 2$. Therefore, there exists $C_1 > 0$ depending only on X and ω such that

$$\int_X (-\psi) \theta_{u_j}^n \leq C_1 A^{n+1},$$

for every $\psi \in \mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$ with $\sup_X \psi = 0$. Using Theorem 3.7 for $\tilde{\chi}(t) = t$, we get

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_2 \left(A \frac{C_1 A^{n+1}}{\delta} \right)^2 \left(\frac{|I_\chi(u_1, u_2)| + d_\theta(u_1, u_2)}{\delta} \right)^{2^{-n}\gamma}, \tag{3.35}$$

where $\chi(t) = \max\{t, -1\}$ and $C_2 > 0$ is a constant depending only on n, X, ω and γ. Since $\theta_{u_j}^n \leq \mathbf{1}_{\{u_j=0\}} \theta^n$, we have

$$\int_{\{u_1 < u_2\}} -\chi(u_1 - u_2) \theta_{u_1}^n = \int_{\{u_2 < u_1\}} -\chi(u_2 - u_1) \theta_{u_2}^n = 0.$$

Therefore

$$I_\chi(u_1, u_2) \leq 0. \tag{3.36}$$

Moreover, it follows from Lemma 3.5 that

$$I_\chi(u_1, u_2) \geq -d_\theta(u_1, u_2). \tag{3.37}$$

Combining (3.35), (3.36) and (3.37), we obtain

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_3 \frac{A^{2n+4}}{\delta^2} \left(\frac{d_\theta(u_1, u_2)}{\delta} \right)^{2^{-n}\gamma},$$

where $C_3 > 0$ depends only on n, X, ω and γ. The proof is completed. □

By using Proposition 3.13, we recover the following result which is obtained in [10] (with a different proof).

Proposition 3.14 *Let $\delta > 0$ be a constant. Let $\mathcal{S}_\delta(\theta)$ be the subset of $\mathcal{S}(\theta)$ consisting of $[u] \in \mathcal{S}_\theta$ such that $\int_X \theta_u^n \geq \delta$. Then $(\mathcal{S}_\delta(\theta), d_{\mathcal{S}})$ is a complete (pseudo)-metric space.*

Proof Let $([u_j])_j$ be a Cauchy sequence in $\mathcal{S}_\delta(\theta)$ (recall that $[u_j]$ denotes the singularity type of a θ-psh function u_j with $\sup_X u_j = 0$), *i.e,* for every constant $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that $d_\theta(u_j, u_k) \leq \epsilon$ for every $j \geq k_\epsilon$, and $k \geq k_\epsilon$. We need to prove that there exists a class $[u_\infty] \in \mathcal{S}_\delta(\theta)$ so that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$. By a contradiction argument, it suffices to prove it for some subsequence of $(u_j)_j$. Hence we can assume that

$$d_\theta(u_j, u_{j+1}) \leq 4^{-n2^{n+1}},$$

because one can always extract a subsequence of $(u_j)_j$ with that property.

Since $d_\theta\big(u, P_\theta[u]\big) = 0$ for every $u \in \mathrm{PSH}(X, \theta)$, without loss of generality, we can assume that $u_j = P_\theta[u_j]$ for every $j \in \mathbb{N}$, in other words, u_j's are model θ-psh functions. Consequently, by Proposition 3.13 (with $\gamma = 1/2$), we get

$$d_{\mathrm{cap}}(u_j, u_{j+1}) \leq 2^{-n} C_1,$$

for every j, where $C_1 > 0$ is a constant depending only on n, X, ω, θ and δ. Therefore, there exists a θ-psh function u_∞ such that u_j converges to u_∞ in capacity as $j \to \infty$.

Moreover, it follows from [8, Theorem 3.8] that

$$\theta_{u_j}^n \leq \mathbf{1}_{\{u_j=0\}} \theta^n \leq C_2 \omega^n, \tag{3.38}$$

for some constant $C_2 > 0$ independent of j. This coupled with Lemma 3.15 below yields that

$$\theta_{u_j}^n \to \theta_{u_\infty}^n, \tag{3.39}$$

as $j \to \infty$. It is clear that $\int_X \theta_{u_\infty}^n \geq \delta$. It remains to show that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$.

Since $u_k \to u_\infty$ in capacity, we have $\max\{u_j, u_k\} \to \max\{u_j, u_\infty\}$ for every $j > 0$, and then it follows from [8, Theorem 2.3] that

$$\liminf_{k\to\infty} \int_X \theta_{\max\{u_j,u_k\}}^n \geq \int_X \theta_{\max\{u_j,u_\infty\}}^n. \tag{3.40}$$

Recall that

$$d_\theta(u_j, u_k) = 2\int_X \theta_{\max\{u_j,u_k\}}^n - \int_X \theta_{u_j}^n - \int_X \theta_{u_k}^n.$$

Using (3.39) and (3.40), one gets

$$\liminf_{k\to\infty} d_\theta(u_j, u_k) \geq 2\int_X \theta_{\max\{u_j,u_\infty\}}^n - \int_X \theta_{u_j}^n - \int_X \theta_{u_\infty}^n = d_\theta(u_j, u_\infty).$$

It follows that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$. In other words, $[u_j] \to [u_\infty]$ in the topology induced by the pseudo-metric $d_{\mathcal{S}}$ (we note that $[u_\infty]$ might not be unique, but the singularity type of its envelope $P_\theta[u_\infty]$ is unique).

□

The following lemma is a corollary of [8, Theorem 2.3].

Lemma 3.15 *Let u_j be a sequence of θ-psh functions satisfying*

$$\int_E \theta_{u_j}^n \leq F(cap_\omega(E)),$$

for every $j \in \mathbb{Z}^+$ and for every Borel set $E \subset X$, where $F : [0,\infty) \to [0,\infty)$ is a continuous function with $F(0) = 0$. Assume that u_j converges in capacity to $u \in \mathrm{PSH}(X,\theta)$. Then for every positive bounded quasi-continuous function $\psi : X \to [0,\infty)$, we have

$$\lim_{j\to\infty}\int_X \psi\theta^n_{u_j} = \int_X \psi\theta^n_u.$$

Proof Put $M = \sup_X \psi$. By [8, Theorem 2.3], we have

$$\liminf_{j\to\infty}\int_X \psi\theta^n_{u_j} \geq \int_X \psi\theta^n_u \quad\text{and}\quad \liminf_{j\to\infty}\int_X \theta^n_{u_j} \geq \int_X \theta^n_u,$$

and

$$M\limsup_{j\to\infty}\int_X \theta^n_{u_j} - \limsup_{j\to\infty}\int_X \psi\theta^n_{u_j} \geq \liminf_{j\to\infty}\int_X (M-\psi)\theta^n_{u_j} \geq \int_X (M-\psi)\theta^n_u.$$

Therefore

$$\liminf_{j\to\infty}\int_X \psi\theta^n_{u_j} \geq \int_X \psi\theta^n_u \geq \limsup_{j\to\infty}\int_X \psi\theta^n_{u_j} + M\left(\int_X \theta^n_u - \limsup_{j\to\infty}\int_X \theta^n_{u_j}\right).$$

Hence, the problem is reduced to show that

$$\limsup_{j\to\infty}\int_X \theta^n_{u_j} \leq \int_X \theta^n_u.$$

For $k > 0$, we denote $u_j^k = \max\{u_j, V_\theta - k - 1\}$, $u^k = \max\{u, V_\theta - k - 1\}$ and $\varphi_k = \max\{\min\{u - V_\theta + k + 1, 1\}, 0\}$. Then $\int_X \theta^n_{u_j^k} = \int_X \theta^n_{u^k} = \int_X \theta^n_{V_\theta}$ for every j, k, and, by the quasi-continuity of φ_k,

$$\liminf_{j\to\infty}\int_X (1-\varphi_k)\theta^n_{u_j^k} \geq \int_X (1-\varphi_k)\theta^n_{u^k}.$$

Therefore, we have

$$\int_X \varphi_k\theta^n_{u^k} \geq \limsup_{j\to\infty}\int_X \varphi_k\theta^n_{u_j^k}. \tag{3.41}$$

Since $\theta^n_{u^k} = \theta^n_u$ on $\{u^k > V_\theta - k - 1\} \supset \{\varphi_k > 0\}$, we have

$$\int_X \varphi_k\theta^n_{u^k} = \int_X \varphi_k\theta^n_u \leq \int_X \theta^n_u. \tag{3.42}$$

Put $E_{j,k} = \{|u_j - u| \geq 1\} \cup \{u \leq V_\theta - k\} \supset \{u_j \leq V_\theta - k - 1\} \cup \{\varphi_k < 1\}$. We have $\theta^n_{u_j^k} = \theta^n_{u_j}$ on $X \setminus E_{j,k}$ and

$$\int_X \varphi_k\theta^n_{u_j^k} \geq \int_{X\setminus E_{j,k}} \theta^n_{u_j} \geq \int_X \theta^n_{u_j} - F(\mathrm{cap}_\omega(E_{j,k})). \tag{3.43}$$

Combining (3.41), (3.42) and (3.43), we have

$$\begin{aligned}\int_X \theta^n_u &\geq \limsup_{j\to\infty}\int_X \theta^n_{u_j} - \limsup_{j\to\infty} F(\mathrm{cap}_\omega(E_{j,k}))\\ &\geq \limsup_{j\to\infty}\int_X \theta^n_{u_j} - \limsup_{j\to\infty} F(\mathrm{cap}_\omega(\{u \leq V_\theta - k\})).\end{aligned}$$

Letting $k \to \infty$, we obtain the desired inequality. □

4 Stability when cohomology classes vary

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2 Put $\epsilon = \|\theta - \eta\|_{\mathscr{C}^0}$. Then, there exists a constant $C_1 \geq 1$ depending only on X and ω such that

$$\theta \leq \eta + C_1\epsilon\omega \leq \theta + 2C_1\epsilon\omega. \tag{4.1}$$

Note that, by Chern-Levine-Nirenberg inequality [22, Corollary 3.3] and by the compactness of $\{w \in \mathrm{PSH}(X, \omega) : \sup_X w = 0\}$ in $L^1(X)$, there exists a constant $C_\omega > 0$ depending only on X and ω such that

$$d_{\mathrm{cap}}(u, v)^2 \leq C_\omega A \lesssim \left(h^{\circ n}(2C_1)\right)^{-\gamma}.$$

Therefore, if $0 < \dfrac{\delta}{2C_1} \leq \epsilon$ then the desired inequality (1.3) holds. Hence, without loss of generality, we can assume that

$$\epsilon < \frac{\delta}{2C_1},$$

and, as a consequence, we have $\tilde{\theta} := \theta + C_1\epsilon\omega \leq (A+1)\omega$.

It follows from [9, Theorem 4.7] that there exists a unique $\tilde{u} \in \mathcal{E}(X, \tilde{\theta}, P_{\tilde{\theta}}[u])$ such that

$$\begin{cases} \tilde{\theta}^n_{\tilde{u}} = c\theta^n_u, \\ \sup_X \tilde{u} = 0, \end{cases} \tag{4.2}$$

where $c = \dfrac{\int_X \tilde{\theta}^n_u}{\int_X \theta^n_u} \geq 1$. By the assumption, we have

$$\int_X -\tilde{\chi}(\psi)\theta^n_u \leq B\delta,$$

for every $\psi \in \mathrm{PSH}(X, \tilde{\theta}) \subset \mathrm{PSH}(X, (A+1)\omega)$. Since $\tilde{\theta}^n_{\tilde{u}} = c\theta^n_u$, it follows that

$$\int_X -\tilde{\chi}(\psi)\tilde{\theta}^n_{\tilde{u}} \leq Bc\delta,$$

for every $\psi \in \mathrm{PSH}(X, \tilde{\theta})$. Hence, $\tilde{u} \in \tilde{\mathcal{E}}_{\tilde{\chi}, Bc\delta}(X, \tilde{\theta})$. Observe that $\int_X \tilde{\theta}^n_{\tilde{u}} \geq c\delta$. It follows from Theorem 3.8 that

$$d_{\mathrm{cap}}(\tilde{u}, u)^2 \leq C_2(A+1)^2B^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_u\| + d_{\tilde{\theta}}(\tilde{u}, u)}\right)\right)^{-\gamma}, \tag{4.3}$$

and

$$d_{\mathrm{cap}}(\tilde{u}, v)^2 \leq C_2(A+1)^2B^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_v\| + d_{\tilde{\theta}}(\tilde{u}, v)}\right)\right)^{-\gamma}, \tag{4.4}$$

where $C_2 > 0$ depends only on n, X, ω and γ. Since $P_{\tilde{\theta}}[u] = P_{\tilde{\theta}}[\tilde{u}]$, we have

$$d_{\tilde{\theta}}(\tilde{u}, u) = 0 \quad \text{and} \quad d_{\tilde{\theta}}(\tilde{u}, v) = d_{\tilde{\theta}}(u, v). \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we get

$$d_{\mathrm{cap}}(u,v)^2 \le C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde\theta^n_{\tilde u}-\tilde\theta^n_u\|+\|\tilde\theta^n_{\tilde u}-\tilde\theta^n_v\|+d_{\tilde\theta}(u,v)}\right)\right)^{-\gamma}, \tag{4.6}$$

where $C_3>0$ depends only on n, X, ω and γ. By (4.1), we have

$$\theta^n_u \le \tilde\theta^n_u \le \theta^n_u + C_4\epsilon(\theta_u+\omega)^n,$$

and

$$\eta^n_v \le \tilde\theta^n_v \le (\eta_v + 2C_1\epsilon\omega)^n \le \eta^n_v + C_4\epsilon(\eta_v+\omega)^n,$$

where $C_4>0$ depends only on X and ω. Therefore

$$\|\theta^n_u-\tilde\theta^n_u\|+\|\eta^n_v-\tilde\theta^n_v\| \le C_5(A+1)^n\,\mathrm{vol}(X)\epsilon, \tag{4.7}$$

where $C_5>0$ depends only on X and ω. Moreover,

$$\|\theta^n_u-\tilde\theta^n_{\tilde u}\| = (c-1)\int_X\theta^n_u = \int_X(\tilde\theta^n_u-\theta^n_u) \le \|\theta^n_u-\tilde\theta^n_u\|. \tag{4.8}$$

Combining (4.7) and (4.8), we get

$$\begin{aligned}\|\tilde\theta^n_{\tilde u}-\tilde\theta^n_u\|+\|\tilde\theta^n_{\tilde u}-\tilde\theta^n_v\| &\le \|\theta^n_u-\tilde\theta^n_u\|+2\|\theta^n_u-\tilde\theta^n_{\tilde u}\|+\|\theta^n_u-\eta^n_v\|+\|\eta^n_v-\tilde\theta^n_v\|\\ &\le 3\|\theta^n_u-\tilde\theta^n_u\|+\|\eta^n_v-\tilde\theta^n_v\|+\|\theta^n_u-\eta^n_v\|\\ &\le 3C_5(A+1)^n\,\mathrm{vol}(X)\epsilon+\|\theta^n_u-\eta^n_v\|.\end{aligned}$$

Hence, by (4.6), we obtain

$$\begin{aligned}d_{\mathrm{cap}}(u,v)^2 &\le C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{3C_5(A+1)^n\,\mathrm{vol}(X)\epsilon+\|\theta^n_u-\eta^n_v\|+d_{\tilde\theta}(u,v)}\right)\right)^{-\gamma}\\ &\le C_6(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A^n\epsilon+\|\theta^n_u-\eta^n_v\|+d_{(A+1)\omega}(u,v)}\right)\right)^{-\gamma},\end{aligned}$$

where $C_6>0$ depends only on n, X, ω and γ. Here we use the facts $d_{\tilde\theta}(u,v)\le d_{(A+1)\omega}(u,v)$ (see Lemma 2.1) and $h(t)\le h(Mt)\le M\,h(t)$ for every $M\ge 1$ and $t>0$. The proof is completed. □

Proof of Theorem 1.3 Since $\theta_j\to\theta_\infty$ in $\mathscr{C}^0$-norm, there exists a constant $A\ge 1$ so that $\theta_j\le A\omega$ for every $j\in\mathbb{N}\cup\{\infty\}$. By [5, Proposition 3.2], there exists $\tilde\chi\in\mathcal{W}^-$ such that

$$\sup_{\psi\in\mathrm{PSH}(X,(A+1)\omega):\sup_X\psi=0}\int_X -\tilde\chi(\psi)d\mu_\infty<\infty.$$

By considering $\tilde\chi/|\tilde\chi(-1)|$ instead of $\tilde\chi$, we can assume that $\tilde\chi(-1)=-1$. This allows us to apply Theorem 1.2 to $u:=u_\infty$, $v:=u_j$, $\theta:=\theta_\infty$, and $\eta:=\theta_j$, and we note that

$$d_{(A+1)\omega}(u,v)=d_{(A+1)\omega}(u_j,u_\infty)=d_{(A+1)\omega}(\phi_j,\phi_\infty)\to 0$$

as $j\to\infty$ by the hypothesis. We thus obtain $d_{\mathrm{cap}}(u_j,u_\infty)\to 0$ as $j\to\infty$. The desired convergence hence follows. The proof is finished. □

In the sequel, we will proceed to prove Theorems 1.4 and 1.5.

Theorem 4.1 *Let $\theta_1, \theta_2 \leq A\omega$ be closed smooth real $(1,1)$-forms ($A \geq 1$). Let $0 < \delta \leq 1$, $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_j \in \tilde{\mathcal{E}}_{\tilde{\chi},(A+1)\omega,B\delta}(X, \theta_j)$ ($j = 1, 2$) such that $\tilde{\chi}(-1) = -1$, $\sup_X u_j = 0$ and $\int_X \theta_{u_j}^n \geq \delta$. Assume that there exists a concave increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for $j = 1, 2$,*

$$\int_X \min\{|\psi_1 - \psi_2|, 1\}(\theta_j + dd^c u_j)^n \leq H(\|\psi_1 - \psi_2\|_{L^1(X)}), \tag{4.9}$$

for every $\psi_1, \psi_2 \in \mathrm{PSH}(X, \omega)$. Then, for every $0 < a, b, \gamma < 1$ and $m > 0$, there exists $C > 0$ depending only on $n, X, A, \omega, H(1), a, b, \gamma$ and m such that

$$d_{cap}(u_1, u_2)^2 \leq C\, B^2 \left(h^{\circ n} \left(\frac{\delta}{G(\tau) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)} \right) \right)^{-\gamma},$$

where $\tau = \mathrm{dist}_{-1}((\theta_1 + dd^c u_1)^n, (\theta_2 + dd^c u_2)^n)$, $G(\tau) = H^a(A\tau^{2(1-b)}) + H(H^m(\tau^{1-b})) + \tau^{ab}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof Without loss of generality, we can assume that $\int_X (\theta_2 + dd^c u_2)^n \geq \int_X (\theta_1 + dd^c u_1)^n$. Denote $\mu_1 = (\theta_1 + dd^c u_1)^n$, $\mu_2 = (\theta_2 + dd^c u_2)^n$ and $c = \dfrac{\mu_1(X)}{\mu_2(X)} \leq 1$. It follows from [9, Theorem 4.7] that there exists a unique $u_3 \in \mathcal{E}(X, \theta_1, P_{\theta_1}[u_1])$ such that

$$\begin{cases} (\theta_1 + dd^c u_3)^n = c\mu_2, \\ \sup_X u_3 = 0. \end{cases} \tag{4.10}$$

By Theorem 3.12, we have

$$d_{\mathrm{cap}}(u_1, u_3)^2 \leq C_1\, B^2 \left(h^{\circ n} \left(\frac{\delta}{G(x) + d_{\theta_1}(u_1, u_3)} \right) \right)^{-\gamma}, \tag{4.11}$$

where $x := \mathrm{dist}_{-1}(\mu_1, c\mu_2)$ and $C_1 > 0$ depends only on $n, X, A, \omega, H(1), a, b, \gamma$ and m.

By Theorem 1.2, we have

$$d_{\mathrm{cap}}(u_2, u_3)^2 \leq C_2 (A\, B)^2 \left(h^{\circ n} \left(\frac{\delta}{(1-c)\|\mu_2\| + A^n \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_2, u_3)} \right) \right)^{-\gamma} \tag{4.12}$$

where $C_2 > 0$ depends only on n, X, ω and γ.

Combining (4.11), (4.12) and using the fact $d_{\theta_1}(u_1, u_3) = d_{(A+1)\omega}(u_1, u_3) = 0$, we get

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_3\, B^2 \left(h^{\circ n} \left(\frac{\delta}{G(x) + (1-c)\|\mu_2\| + R} \right) \right)^{-\gamma}, \tag{4.13}$$

where $R = A^n \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)$ and $C_3 > 0$ is a constant depending only on $n, X, A, \omega, H(1), a, b, \gamma$ and m.

Note that

$$(1-c)\|\mu_2\| = \int_X d\mu_2 - \int_X d\mu_1 \leq \mathrm{dist}_{-1}(\mu_1, \mu_2) = \tau. \tag{4.14}$$

Then

$$x = \mathrm{dist}_{-1}(\mu_1, c\mu_2) \leq \mathrm{dist}_{-1}(\mu_1, \mu_2) + (1-c)\|\mu_2\| \leq 2\, \mathrm{dist}_{-1}(\mu_1, \mu_2) = 2\tau. \tag{4.15}$$

Combining (4.13), (4.14) and (4.15), we get

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C_4\, B^2\left(h^{\circ n}\left(\frac{\delta}{G(\tau)+\|\theta_1-\theta_2\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u_1,u_2)}\right)\right)^{-\gamma},$$

where $C_4>0$ depends only on n, X, A, ω, $H(1)$, a, b, γ and m.

This finishes the proof. □

Proof of Theorem 1.4 By the assumption and by [17, Lemma 3.3], we have $\mu_j := (\theta_j + dd^c u_j)^n$ satisfies (4.9) for $H(t)=\tilde{M}\delta t^\beta$ and $j=1,2$, where $\tilde{M}>0$ is a constant depending only on X,ω and M. Moreover, it follows from [17, Proposition 4.4] that, for every $\psi\in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$,

$$\int_X -\psi\mu_j \le B\delta,$$

where $B>0$ depends on X,ω, M and β. Hence, by using Theorem 4.1 (choose $a=\gamma$, $b=\dfrac{2\beta}{2\beta+1}$ and $m=\dfrac{2a}{\beta}$), we have

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C\left(\frac{\tau^{2\gamma\beta/(2\beta+1)}+\|\theta_1-\theta_2\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u_1,u_2)}{\delta}\right)^{2^{-n}\gamma},$$

where $\tau = \mathrm{dist}_{-1}(\mu_1,\mu_2)$ and $C>0$ is a constant depending only on n, X, ω, A, M, γ and β.

The proof is completed. □

In order to prove Theorem 1.5, we need again several auxiliary lemmas.

Lemma 4.2 *Let μ be a Radon measure on X vanishing on every pluripolar set. Assume that u_j, $j\in\mathbb{N}\cup\{\infty\}$, are negative θ-psh functions satisfying $u_j\to u_\infty$ in $L^1(X)$ as $j\to\infty$. Then*

$$\int_X \min\{|u_j-u_\infty|,1\}d\mu \to 0,$$

as $j\to\infty$.

Proof Denote $B=\sup_j\|u_j\|_{L^1}$. By Chern-Levine-Nirenberg inequality [22, Proposition 3.1], there exists $C>0$ such that

$$\mathrm{cap}\{u_j<-k\} \le \frac{B\,C}{k},$$

for every $j\in\mathbb{N}\cup\{\infty\}$ and $k>0$. Since μ vanishes on pluripolar sets, by [23, Lemma 4.5], there exists $w\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ such that $\mu = f\omega_w^n$ for some nonnegative function $f\in L^1(\omega_w^n)$. Let $M>0$ be a big enough constant such that

$$\int_{\{f>M\}} d\mu < \epsilon/6.$$

We have

$$\begin{aligned}\mu(\{u_j < -k\}) &= \int_{\{f>M\}\cap\{u_j<-k\}} d\mu + \int_{\{f>M\}\cap\{u_j<-k\}} d\mu \\ &\leq \int_{\{f\leq M\}\cap\{u_j<-k\}} d\mu + \int_{\{f>M\}} d\mu \\ &\leq M(\sup_X w - \inf_X w)\mathrm{cap}\{u_j < -k\} + \epsilon/6.\end{aligned}$$

It follows that for each $\epsilon > 0$, there exists $k_0 \geq 1$ such that

$$\mu(\{u_j < -k\}) \leq \epsilon/3 \tag{4.16}$$

for every $j \in \mathbb{N}\cup\{\infty\}$ and $k \geq k_0$. Denote $u_{j,k} = \max\{u_j, -k\}$ and $v_{j,k} = \max\{u_{j,k}, u_{\infty,k}\}$. Thus for every k, we have $u_{j,k} \to u_{\infty,k}$ in $L^1(X)$ and $v_{j,k} \to u_{\infty,k}$ in capacity as $j \to \infty$. It follows from [25, Lemma 11.5] that

$$\int_X \max\{u_{j,k} - u_{\infty,k}, 0\}d\mu = \int_X (v_{j,k} - u_{\infty,k})d\mu \to 0,$$

and

$$\int_X (u_{j,k} - u_{\infty,k})d\mu \to 0,$$

as $j \to \infty$. Combining the last two convergences gives

$$\int_X |u_{j,k} - u_{\infty,k}|d\mu \to 0,$$

as $j \to \infty$. Choose j_0 such that

$$\int_X |u_{j,k_0} - u_{\infty,k_0}|d\mu < \frac{\epsilon}{3},$$

for every $j > j_0$. Using the last inequality and (4.16), we have

$$\begin{aligned}\int_X \min\{|u_j - u_\infty|, 1\}d\mu &\leq \int_{\{u_j,u_\infty\geq -k_0\}} |u_j - u_\infty|d\mu + \mu(\{u_j < -k\}) + \mu(\{u_\infty < -k\}) \\ &\leq \int_{\{u_j,u_\infty\geq -k_0\}} |u_{j,k_0} - u_{\infty,k_0}|d\mu + \frac{2\epsilon}{3} \leq \epsilon,\end{aligned}$$

for every $j > j_0$. Thus $\int_X \min\{|u_{a_j} - u_\infty|, 1\}d\mu \to 0$ as $j \to \infty$. □

Lemma 4.3 *Let μ be a Radon measure on X vanishing on every pluripolar set. Then, there exists a concave, non-decreasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $H(0) = 0$ such that*

$$\int_X \min\{|u - v|, 1\}d\mu \leq H\left(\|u - v\|_{L^1(X)}\right),$$

for every $u, v \in \mathrm{PSH}(X, \omega)$.

Proof For every $t > 0$, we denote

$$h(t) = \sup\left\{\int_X \min\{|u - v|, 1\}d\mu : u, v \in \mathrm{PSH}(X, \omega), \|u - v\|_{L^1(X)} \leq t\right\}.$$

Then h is non-decreasing. We will show that

$$\lim_{t\to 0^+} h(t) = 0. \tag{4.17}$$

Indeed, if $\lim_{t\to 0^+} h(t) = 2\epsilon > 0$ then there exist sequences $u_j, v_j \in \mathrm{PSH}(X,\omega)$ such that $\|u_j - v_j\|_{L^1(X)} \to 0$ as $j \to \infty$ and

$$\int_X \min\{|u_j - v_j|, 1\} d\mu \geq \epsilon, \tag{4.18}$$

for every j. Without loss of generality, we can assume that $c_j := \sup_X v_j \leq \sup_X u_j = 0$

By the compactness of $\mathrm{PSH}_{\sup}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) : \sup_X \varphi = 0\}$ in $L^1(X)$, we can assume that $u_j, v_j - c_j \to w \in \mathrm{PSH}_{\sup}(X,\omega)$ as $j \to \infty$. In particular,

$$c_j \int_X \omega^n \leq \|u_j - v_j\|_{L^1(X)} + \|u_j - w\|_{L^1(X)} + \|v_j - c_j - w\|_{L^1(X)} \stackrel{j\to\infty}{\longrightarrow} 0. \tag{4.19}$$

Moreover, it follows from Lemma 4.2 that

$$\lim_{j\to\infty} \int_X \min\{|u_j - w|, 1\} d\mu = \lim_{j\to\infty} \int_X \min\{|v_j - c_j - w|, 1\} d\mu = 0,$$

and it follows that

$$\lim_{j\to\infty} \int_X \min\{|u_j - v_j - c_j|, 1\} d\mu = 0. \tag{4.20}$$

Combining (4.19) and (4.20), we get

$$\lim_{j\to\infty} \int_X \min\{|u_j - v_j|, 1\} d\mu = 0.$$

This contradicts with (4.18). Hence, (4.17) is true.

Now, we put

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } \ 0 < t < 1, \\ \int_X d\mu & \text{if } \ t \geq 1. \end{cases}$$

For every $m > 1$, we also define

$$k_m = \sup\left\{ \frac{\tilde{h}(s)}{s} : \frac{1}{m} \leq t \right\} \qquad \text{and} \qquad H_m(t) = k_m t + h(1/m).$$

Then $H_m(t) \geq h(t)$ for every $t \geq 0$ and $\lim_{t\to 0^+} H_m(t) = h(1/m)$. Set $H(t) = \inf_{m>1} H_m(t)$. We have H is a concave, non-decreasing function satisfying $H(0) = 0$ and $H \geq h$. In particular,

$$\int_X \min\{|u - v|, 1\} d\mu \leq H\left(\|u - v\|_{L^1(X)}\right),$$

for every $u, v \in \mathrm{PSH}(X,\omega)$.

The proof is completed. □

Proof of Theorem 1.5 By Lemma 4.3, there exists a concave, non-decreasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on μ, X and ω such that $H(0) = 0$ and

$$\int_X \min\{|\psi_1 - \psi_2|, 1\} d\mu \leq H\left(\|\psi_1 - \psi_2\|_{L^1(X)}\right),$$

for every $\psi_1, \psi_2 \in \mathrm{PSH}(X, \omega)$.

Moreover, it follows from [5, Proposition 3.2] that there exist a constant $B > 0$ and a function $\tilde{\chi} \in \mathcal{W}^-$ depending on X, ω and μ such that

$$\int -\tilde{\chi}(\psi) d\mu \leq B,$$

for every $\psi \in \mathrm{PSH}(X, \omega)$ with $\sup_X \psi = 0$. In particular, $u_j \in \tilde{\mathcal{E}}_{\tilde{\chi},(A+1)\omega,(A+1)B}(X, \theta_j)$ for $j = 1, 2$. Hence, by Theorem 4.1, there exists $C > 0$ depending only on n, X, A, ω and $H(1)$ such that

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq \frac{C B^2}{\delta^2} \left(h^{\circ n} \left(\frac{\delta}{G(\tau) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)} \right) \right)^{-1/2},$$

where $\tau = \mathrm{dist}_{-1}(\mu_1, \mu_2)$, $G(\tau) = H^{1/2}(\tau) + H(H(\tau^{1/2})) + \tau^{1/4}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$. Denote

$$f(t) = \frac{C B^2}{\delta^2} \left(h^{\circ n} \left(\frac{\delta}{G(t) + t} \right) \right)^{-1/2}.$$

We obtain

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq f\left(\mathrm{dist}_{-1}(\mu_1, \mu_2) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2) \right).$$

The proof is completed. □

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Declarations

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