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RESEARCH ARTICLE

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Remarks on τ-tilted versions of the second Brauer–Thrall conjecture

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Abstract
In this short note, we state a stable and a τ-reduced version of the second Brauer–Thrall conjecture. The former is a slight strengthening of a brick version of the second Brauer–Thrall conjecture raised by Mousavand and Schroll–Treffinger–Valdivieso. The latter is stated in terms of Geiß–Leclerc–Schröer's generically τ-reduced components and provides a geometric interpretation of a question of Demonet. It follows that the stable second Brauer–Thrall conjecture implies our τ-reduced second Brauer–Thrall conjecture. Finally, we prove the reversed implication for the class of E-tame algebras recently introduced by Asai–Iyama.

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1 | INTRODUCTION

In 2014, an important new branch of Representation Theory emerged with Adachi–Iyama–Reiten's τ-tilting theory [1]. It generalizes the mutation theory of quivers with potentials [16] to arbitrary finite-dimensional algebras. This theory may be seen from two "Koszul dual" perspectives: There are τ-rigid modules central to τ-tilting theory [1] on the one hand and on the other there are bricks [14] and more specifically stable modules [9]. In particular, there is now a remarkable theory of τ-tilting finite algebras initiated in [14]. This motivates to go beyond the finite case just like Brauer and Thrall went beyond representation finite algebras a century ago. The famous

Brauer–Thrall conjectures were an important driving force of 20th century Representation Theory and stimulated the development of many key techniques to construct and classify modules over finite-dimensional algebras. These conjectures were first published in Jan's Thesis [26, chapter I.1] where he refers to unpublished notes of Brauer and Thrall who intended their conjectures as exercises for graduate students, see [35] for a thorough survey with historical remarks. The first Brauer–Thrall conjecture is now a well-established theorem due to Roiter and states

Brauer–Thrall I conjecture (Theorem due to [36]). *Let A be a representation infinite algebra. For every $d \geqslant 0$ exists an indecomposable $V \in \mathrm{mod}(A)$ with $\dim(V) \geqslant d$.*

Recently, a τ-tilted version of the first Brauer–Thrall conjecture in terms of bricks was proved by Schroll–Treffinger [37] (see also [30]). This motivates to seek for τ-tilted analogs of the much harder second Brauer–Thrall conjecture which classically states

Brauer–Thrall II conjecture. *Let A be a representation infinite algebra. For every $d_0 \geqslant 0$ exists $d \geqslant d_0$ and infinitely many pairwise nonisomorphic indecomposable $V \in \mathrm{mod}(A)$ with $\dim(V) = d$.*

It is still open over general fields. The first complete proof valid over algebraically closed fields was achieved by Bautista [7], see also [8] for a proof with remarks on its history and further references. The second conjecture is a bit too strong to allow for direct τ-tilted versions as the Kronecker algebra demonstrates. Fortunately, Smalø [39] provided an induction step which reduces Brauer–Thrall II to the a priori weaker

Brauer–Thrall II' conjecture. *Let A be a representation infinite algebra. There exist $d \geqslant 0$ and infinitely many pairwise nonisomorphic indecomposable $V \in \mathrm{mod}(A)$ with $\dim(V) = d$.*

In [29] and [38], a τ-tilted version of Brauer–Thrall II' in terms of bricks was proposed. We decided to strengthen their conjecture slightly in terms of stable modules, this is our Conjecture 5.2 and states

Stable Brauer– Thrall II' conjecture. *Let A be a τ-tilting infinite algebra. Then there exists a dimension vector $\mathbf{d} \in \mathrm{K}_0(A)^+$ and a weight $\theta \in \mathrm{K}_0(A)^*$ such that the dimension of the moduli space of θ-stable A-modules with dimension vector $\mathbf{d}$ is strictly positive.*

This conjecture was recently confirmed for the class of special biserial algebras in [38] (see Example 5.5) and we confirm it for the class of GLS algebras from [22] in [32] (see Example 5.4). On the other hand, elementary and well-known geometric considerations reveal that there can be only finitely isomorphism classes of τ-rigid modules of a fixed dimension. Thus, to state a τ-tilted version of Brauer–Thrall II, we invoke Geiß–Leclerc–Schröers τ-reduced components [21] as natural generalizations of τ-rigid modules and arrive at our Conjecture 4.2 which states

τ-Reduced Brauer–Thrall II' conjecture. *Let A be a τ-tilting infinite algebra. Then there exists a dimension vector $\mathbf{d} \in \mathrm{K}_0(A)$ and a generically τ-reduced and irreducible component of $\mathrm{Rep}(A, \mathbf{d})$ with generically at least one parameter.*

We use Plamondon's classification of τ-reduced components [33] to show that our τ-reduced Brauer–Thrall II' conjecture is in fact equivalent to a conjectural characterization of τ-tilting finite algebras in terms of their **g**-vector fans due to Demonet.

Demonet's conjecture. *[13, Question 3.49] A finite-dimensional algebra A is τ-tilting finite if and only if its* **g***-vector fan is rationally complete.*

With Jasso's τ-tilting reduction [27] and Brüstle–Smith–Treffingers semistability of τ-perpendicular subcategories [9], we observe that the stable Brauer– Thrall II' conjecture implies Demonet's conjecture. For the recently introduced class of E-tame algebras from [3] and [15] (see Definition 6.1), we are able to prove that all three conjectures are equivalent. This is the Main Theorem of our short note:

Main Theorem 1. *Let A be a finite-dimensional algebra. Consider the following statements.*

(i) *A satisfies Demonet's conjecture.*
(ii) *A satisfies the τ-reduced Brauer–Thrall II' conjecture.*
(iii) *A satisfies the stable Brauer– Thrall II' conjecture.*

Then the implications $(i) \Leftrightarrow (ii) \Leftarrow (iii)$ hold. If A is E-tame, then $(ii) \Rightarrow (iii)$ holds as well.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is Proposition 4.8. The implication $(iii) \Rightarrow (i)$ is Proposition 5.6. It remains to prove $(ii) \Rightarrow (i)$ for E-tame algebras. This is done in Section 7. □

2 | CONVENTIONS

Topology

All our topological spaces will be varieties over the field K with the Zariski topology. For a topological space $\mathcal{X}$ write $\dim(\mathcal{X})$ for its Krull dimension and given $x \in \mathcal{X}$ write $\dim_x(\mathcal{X})$ for the local dimension of $\mathcal{X}$ at x. For an irreducible topological space $\mathcal{Z}$ and a property P of points of $\mathcal{Z}$, we say

$$\text{“}\mathcal{Z}\ satisfies\ \mathsf{P}\ generically\text{” or “}the\ generic\ elements\ of\ \mathcal{Z}\ satisfy\ \mathsf{P}\text{” and so on,}$$

if there exists a nonempty open hence dense subset $\mathcal{U} \subseteq \mathcal{Z}$ such that every $z \in \mathcal{U}$ satisfies P. In particular, for a constructible map $f : \mathcal{Z} \to X$ to a set X we write $f(\mathcal{Z}) = x$ for some $x \in X$ if $f(z) = x$ for generic $z \in \mathcal{Z}$.

Representations

Throughout, we fix an algebraically closed field K. All our algebras $A = KQ/I$ are finite-dimensional associative and given by a quiver Q with admissible ideal $I \subseteq KQ$. All our modules are finite-dimensional left A-modules and we denote the category of modules by $\mathrm{mod}(A)$ with its Auslander–Reiten translation τ_A. We freely identify A-modules with K-linear representations of Q satisfying the relations in I. Let $\mathrm{K}_0(A)$ denote the Grothendieck group of $\mathrm{mod}(A)$ with standard basis given by the classes of simple A-modules S_i indexed by vertices $i \in Q_0$. More generally, define for any commutative ring R

$$\mathrm{K}_0(A)_R := \mathrm{K}_0(A) \otimes_{\mathbb{Z}} R, \qquad \mathrm{K}_0(A)^*_R := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), R).$$

Further, let $\mathrm{K}_0(A)^+ \subseteq \mathrm{K}_0(A)$ denote the submonoid of classes of A-modules. Given $\mathbf{d} \in \mathrm{K}_0(A)^+$ with $\mathbf{d} = (d_i)_{i \in Q_0}$, write $\mathrm{Rep}(A, \mathbf{d})$ for the affine variety of representations of A with dimension vector $\mathbf{d}$ and

$$\mathrm{GL}(K, \mathbf{d}) := \prod_{i \in Q_0} \mathrm{GL}(K, d_i)$$

for the structure group acting on $\mathrm{Rep}(A, \mathbf{d})$ via conjugation. We write $\mathrm{Irr}(A, \mathbf{d})$ for the set of irreducible components of $\mathrm{Rep}(A, \mathbf{d})$ and set

$$\mathrm{Irr}(A) := \bigsqcup_{\mathbf{d} \in \mathrm{K}_0(A)^+} \mathrm{Irr}(A, \mathbf{d}).$$

Denote the $\mathrm{GL}(K, \mathbf{d})$-orbit of a representation $V \in \mathrm{Rep}(A, \mathbf{d})$ by $\mathcal{O}(V)$. The *generic number of parameters* of $\mathcal{Z} \in \mathrm{Irr}(A)$ is

$$c_A(\mathcal{Z}) := \min\{\dim \mathcal{Z} - \dim \mathcal{O}(V) \mid V \in \mathcal{Z}\}.$$

The following functions are well-known to be upper semicontinuous for all $\mathbf{d}, \mathbf{d}' \in \mathrm{K}_0(A)^+$:

$$\begin{aligned}
\hom_A(-, ?) &: \mathrm{Rep}(A, \mathbf{d}) \times \mathrm{Rep}(A, \mathbf{d}') \to \mathbb{Z}, (V, W) \mapsto \dim_K \mathrm{Hom}_A(V, W),\\
\mathrm{ext}^1_A(-, ?) &: \mathrm{Rep}(A, \mathbf{d}) \times \mathrm{Rep}(A, \mathbf{d}') \to \mathbb{Z}, (V, W) \mapsto \dim_K \mathrm{Ext}^1_A(V, W),\\
\hom^\tau_A(-, ?) &: \mathrm{Rep}(A, \mathbf{d}) \times \mathrm{Rep}(A, \mathbf{d}') \to \mathbb{Z}, (V, W) \mapsto \dim_K \mathrm{Hom}_A(W, \tau_A(V)),
\end{aligned}$$

and so are their diagonal values

$$\begin{aligned}
\mathrm{end}_A(-) &: \mathrm{Rep}(A, \mathbf{d}) \to \mathbb{Z}, V \mapsto \hom_A(V, V),\\
\mathrm{ext}^1_A(-) &: \mathrm{Rep}(A, \mathbf{d}) \to \mathbb{Z}, V \mapsto \mathrm{ext}^1_A(V, V),\\
\hom^\tau_A(-) &: \mathrm{Rep}(A, \mathbf{d}) \to \mathbb{Z}, V \mapsto \hom^\tau_A(V, V);
\end{aligned}$$

see [12] and [20] for the proofs. Therefore, their generic values $\hom_A(\mathcal{Z}, \mathcal{Z}')$, $\mathrm{end}_A(\mathcal{Z})$, and so on, coincide with their minimal values on $\mathcal{Z} \times \mathcal{Z}'$, respectively, $\mathcal{Z}$ for $\mathcal{Z}, \mathcal{Z}' \in \mathrm{Irr}(A)$.

Presentations

The full exact subcategory of $\mathrm{mod}(A)$ of projective A-modules is denoted $\mathrm{proj}(A)$. Let $\mathrm{K}^b(A)$ be the bounded homotopy category of complexes of projective A-modules with Σ its shift functor. Let $\mathrm{K}^{\mathrm{proj}}_0(A)$ be the Grothendieck group of $\mathrm{proj}(A)$ with standard basis given by the classes of indecomposable projectives P_i indexed by $i \in Q_0$. Let $\mathrm{K}^{\mathrm{proj}}_0(A)^+$ be the submonoid of $\mathrm{K}^{\mathrm{proj}}_0(A)$

consisting of classes of projective A-modules. The *Euler pairing* is defined for $P \in \operatorname{proj}(A)$ and $V \in \operatorname{mod}(A)$ as

$$\langle -, ? \rangle_A : \ \mathrm{K}_0^{\mathrm{proj}}(A) \times \mathrm{K}_0(A) \to \mathbb{Z}, \quad \langle P, V \rangle_A := \dim_K \operatorname{Hom}_A(P, V).$$

As A is by our standing assumptions a split K-algebra, this induces an isomorphism

$$(-)^\vee : \ \mathrm{K}_0^{\mathrm{proj}}(A) \xrightarrow{\sim} \mathrm{K}_0(A)^*.$$

Given $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)^+$ written as $\boldsymbol{\gamma} = (\gamma_i)_{i \in Q_0}$, there is up to isomorphism a unique $P_{\boldsymbol{\gamma}} \in \operatorname{proj}(A)$ with $[P_{\boldsymbol{\gamma}}] = \boldsymbol{\gamma}$ in $\mathrm{K}_0^{\mathrm{proj}}(A)$ and we consider the smooth and irreducible open subvariety

$$\operatorname{Proj}(A, \boldsymbol{\gamma}) := \mathcal{O}(P_{\boldsymbol{\gamma}}) \subseteq \operatorname{Rep}(A, \underline{\mathbf{dim}}(P_{\boldsymbol{\gamma}})).$$

Given any $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$, set

$$\operatorname{Proj}(A, \boldsymbol{\gamma}) := \operatorname{Proj}(A, \boldsymbol{\gamma}_1) \times \operatorname{Proj}(A, \boldsymbol{\gamma}_0),$$

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0 - \boldsymbol{\gamma}_1$ for unique $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1 \in \mathrm{K}_0^{\mathrm{proj}}(A)^+$ with $\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_1 = 0$. Plamondon's *variety of reduced presentations* [33, section 2.4] is the variety of triples

$$\operatorname{Pres}(A, \boldsymbol{\gamma}) := \{\vec{P} = (p, P_1, P_0) \mid (P_1, P_0) \in \operatorname{Proj}(A, \boldsymbol{\gamma}) \text{ and } p \in \operatorname{Hom}_A(P_1, P_0)\}$$

upon which the group $\operatorname{GL}(K, \boldsymbol{\gamma}) := \operatorname{GL}(K, \underline{\mathbf{dim}}(P_{\boldsymbol{\gamma}_1})) \times \operatorname{GL}(K, \underline{\mathbf{dim}}(P_{\boldsymbol{\gamma}_0}))$ acts via conjugation. This is easily seen to be a $\operatorname{GL}(K, \boldsymbol{\gamma})$-equivariant vector bundle over the base $\operatorname{Proj}(A, \boldsymbol{\gamma})$. In particular, $\operatorname{Pres}(A, \boldsymbol{\gamma})$ is a smooth and irreducible variety. The following functions are well-known to be upper semicontinuous for every $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathrm{K}_0^{\mathrm{proj}}(A)$:

$$\begin{aligned}
h_A(-, ?) &: \ \operatorname{Pres}(A, \boldsymbol{\gamma}) \times \operatorname{Pres}(A, \boldsymbol{\gamma}') \to \mathbb{Z}, & (\vec{P}, \vec{Q}) &\mapsto \dim_K \operatorname{Hom}_{K^b(A)}(\vec{P}, \vec{Q});\\
e_A(-, ?) &: \ \operatorname{Pres}(A, \boldsymbol{\gamma}) \times \operatorname{Pres}(A, \boldsymbol{\gamma}') \to \mathbb{Z}, & (\vec{P}, \vec{Q}) &\mapsto \dim_K \operatorname{Hom}_{K^b(A)}(\vec{P}, \Sigma(\vec{Q}));\\
h_A(-) &: \ \operatorname{Pres}(A, \boldsymbol{\gamma}) \to \mathbb{Z}, & \vec{P} &\mapsto h_A(\vec{P}, \vec{P});\\
e_A(-) &: \ \operatorname{Pres}(A, \boldsymbol{\gamma}) \to \mathbb{Z}, & \vec{P} &\mapsto e_A(\vec{P}, \vec{P}),
\end{aligned}$$

where we naturally consider $\vec{P} = (P_1 \xrightarrow{p} P_0) \in \mathrm{K}^b(A)$ with P_0 in degree 0 and P_1 in degree -1. Therefore, their generic values $h_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$, $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$, $h_A(\boldsymbol{\gamma}), e_A(\boldsymbol{\gamma})$ coincide with their minimal values on $\operatorname{Pres}(A, \boldsymbol{\gamma}) \times \operatorname{Pres}(A, \boldsymbol{\gamma}')$, respectively, $\operatorname{Pres}(A, \boldsymbol{\gamma})$ for $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathrm{K}_0^{\mathrm{proj}}(A)$.

3 | DEMONET'S CONJECTURE AND g-VECTORS

The protagonists of Adachi–Iyama–Reiten's τ-tilting theory [1] are *τ-rigid modules* that is modules $V \in \operatorname{mod}(A)$ with $\operatorname{Hom}_A(V, \tau_A(V)) = 0$. More generally, a pair $(V, P) \in \operatorname{mod}(A) \times \operatorname{proj}(A)$ is *τ-rigid* if V is τ-rigid and $\operatorname{Hom}_A(P, V) = 0$. The **g**-*vector* of a module $V \in \operatorname{mod}(A)$ with minimal projective presentation $P_1 \to P_0 \to V \to 0$ and more generally of a pair (V, P) with $P \in \mathrm{proj}(A)$ is defined as the class

$$\mathbf{g}(V) := [P_0] - [P_1] \in \mathrm{K}_0^{\mathrm{proj}}(A), \qquad \mathbf{g}(V,P) := \mathbf{g}(V) - \mathbf{g}(P) \in \mathrm{K}_0^{\mathrm{proj}}(A).$$

The $\mathbf{g}$-vectors of indecomposable summands of τ-rigid pairs (V, P) span convex cones

$$\mathsf{C}(V,P) := \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{g}(V_i) - \sum_{j=1}^{m'} \beta_j \mathbf{g}(P_j) \;\middle|\; \forall i,j : \alpha_i, \beta_j \in \mathbb{R}_{\geqslant 0} \right\} \subseteq \mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{R}},$$

where $V = V_1 \oplus \cdots \oplus V_m$ and $P = P_1 \oplus \cdots \oplus P_{m'}$ are their direct sum decompositions into indecomposable summands. The $\mathbf{g}$-*vector fan* $\mathsf{Fan}(A) \subseteq \mathrm{K}_0(A)_{\mathbb{R}}^{\mathrm{proj}}$ is defined as the union of all cones of τ-rigid pairs. This is a rational nonsingular polyhedral fan in $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{R}}$ introduced and studied in [14].

It was realized by Brüstle–Smith–Treffinger [9] that τ-tilting theory is closely related to King's theory of semistability [28]. Given $\theta \in \mathrm{K}_0(A)_{\mathbb{R}}^*$, a module $V \in \mathrm{mod}(A)$ is θ-*semistable* if $\theta(V) = 0$ and for every proper nonzero submodule $U \subseteq V$ is $\theta(U) \leqslant 0$; and V is θ-*stable* if the inequality is always strict. We say that a module $V \in \mathrm{mod}(A)$ is *stable* if there exists a weight $\theta \in \mathrm{K}_0(A)_{\mathbb{R}}^*$ such that V is θ-stable. A bridge between stability and τ-tilting theory is provided by *Auslander–Reiten's* $\mathbf{g}$-*vector formula*:

$$\langle \mathbf{g}(V,P), \underline{\mathbf{dim}}(X) \rangle_A = \hom_A(V,X) - \hom_A(X, \tau_A(V)) - \hom_A(P,X) \tag{3.1}$$

for all $V, X \in \mathrm{mod}(A)$ and $P \in \mathrm{proj}(A)$ [6, Theorem 1.4]. Indeed, this formula allows us to identify parts of King's semistable subcategories

$$\mathcal{W}(\theta) := \{V \in \mathrm{mod}(A) \mid V \text{ is } \theta\text{-semistable}\}, \quad \text{for } \theta \in \mathrm{K}_0(A)^*$$

as Jasso's τ-*perpendicular subcategories* [27, Definition 3.3]

$$\mathcal{W}(V,P) := V^{\perp} \cap {}^{\perp}(\tau_A(V)) \cap P^{\perp}, \quad \text{for } (V,P) \in \mathrm{mod}(A) \times \mathrm{proj}(A).$$

Here we write $V^{\perp} := \{X \in \mathrm{mod}(A) \mid \mathrm{Hom}_A(V,X) = 0\}$ and ${}^{\perp}V := \{X \in \mathrm{mod}(A) \mid \mathrm{Hom}_A(X,V) = 0\}$ for $V \in \mathrm{mod}(A)$.

Lemma 3.1. *Let $\theta \in \mathrm{K}_0(A)^*$ and $(V,P) \in \mathrm{mod}(A) \times \mathrm{proj}(A)$ with $\mathbf{g}(V,P)^{\vee} = \theta$. Then*

$$\mathcal{W}(V,P) \subseteq \mathcal{W}(\theta)$$

is a Serre subcategory of $\mathcal{W}(\theta)$.

Proof. Let $W \in \mathcal{W}(V,P)$ and $U \subseteq W$, then by (3.1)

$$\theta(U) = \langle \mathbf{g}(V,P), \underline{\mathbf{dim}}(U) \rangle_A = \hom_A(V,U) - \hom_A(U, \tau_A(V)) - \hom_A(P,U) \leqslant 0$$

because $\hom_A(V, U) = \hom_A(P, U) = 0$ for $W \in V^\perp \cap P^\perp$ and U is a submodule of W. Also, $\theta(W) = 0$ hence $W \in \mathcal{W}(\theta)$. If moreover $U \in \mathcal{W}(\theta)$, then $\theta(U) = 0$ hence we must have $\hom_A(U, \tau_A(V)) = 0$. Therefore $U \in \mathcal{W}(V, P)$. Similarly, every θ-semistable factor of W lies in $\mathcal{W}(V, P)$. Finally, it is clear that $\mathcal{W}(V, P) := V^\perp \cap {}^\perp(\tau_A V) \cap P^\perp$ is closed under extensions. □

This lemma can be found in [3, Lemma 3.8] in terms of presentations; as the proof is straight forward and our proof of the Main Theorem 1 heavily relies on it, we include it here for convenience. The proof strategy is the same as in [9, section 3.3] where they prove equality in Lemma 3.1 if θ lies in the relative interior of $\mathsf{C}(V, P)^\vee$ for a τ-rigid pair (V, P).

The outlined correspondences are particularly appealing in the τ-tilting finite case:

Theorem 3.2 ([4, 9, 14]). *For a finite-dimensional algebra A, the following statements are equivalent.*

- (i) *There are only finitely many isomorphism classes of indecomposable τ-rigid A-modules.*
- (ii) *There are only finitely many isomorphism classes of stable A-modules.*
- (iii) *The* **g***-vector fan of A is complete, that is,* $\mathsf{Fan}(A) = \mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{R}}$.

An algebra A is said to be *τ-tilting finite* if A satisfies any of the equivalent properties in Theorem 3.2. Of course, an algebra A is *τ-tilting infinite* if A is not τ-tilting finite. Demonet asks in [13, Question 3.49] whether it suffices for A to be τ-tilting finite that $\mathsf{Fan}(A)$ is "rationally complete" in the following sense:

Conjecture 3.3 (Demonet's conjecture). *A finite-dimensional algebra A is τ-tilting finite if and only if* $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{Z}} \subseteq \mathsf{Fan}(A)$.

The "only if" part follows immediately from Theorem 3.2. Conversely, if $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{Z}} \subseteq \mathsf{Fan}(A)$, we can only deduce that $\mathsf{Fan}(A)$ is dense in $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{R}}$ hence the complement may contain vectors with irrational coordinates. Therefore, the other implication in Demonet's conjecture does not follow immediately from Theorem 3.2. The Kronecker algebra is an example of a τ-tilting infinite algebra with a dense **g**-vector fan, but it does not satisfy $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{Z}} \subseteq \mathsf{Fan}(A)$.

4 | THE τ-REDUCED BRAUER–THRALL CONJECTURES

To illustrate the need of τ-reduced components in the formulation of a τ-tilted version of the second Brauer–Thrall conjecture, we start with an obvious τ-tilted version of the first Brauer–Thrall conjecture.

Proposition 4.1. *Let A be a τ-tilting infinite algebra. Then, for every $d \geqslant 0$ there exists an indecomposable τ-rigid $V \in \operatorname{mod}(A)$ with $\dim(V) \geqslant d$.*

Proof. Let $\mathbf{d} \in \mathrm{K}_0(A)^+$ be a fixed dimension vector and consider the variety of representations $\operatorname{Rep}(A, \mathbf{d})$. The orbit closure $\overline{\mathcal{O}(V)}$ of any τ-rigid module V in $\operatorname{Rep}(A, \mathbf{d})$ is an irreducible component of $\operatorname{Rep}(A, \mathbf{d})$ by Voigt's Isomorphism [18, section 1.1]. But $\operatorname{Rep}(A, \mathbf{d})$ has only finitely many irreducible components as an affine variety. This proves that there are only finitely many isomorphism classes of τ-rigid A-modules V with fixed dimension vector $\underline{\mathbf{dim}}(V) = \mathbf{d}$. By the pigeonhole principle, there must exist indecomposable τ-rigid A modules of arbitrarily large dimension if A is τ-tilting infinite. □

The proof shows that there cannot be infinitely many pairwise nonisomorphic τ-rigid modules of a fixed dimension. Therefore, we need to enlarge our class of τ-rigid modules. A natural geometric generalization is provided by Geiß–Leclerc–Schröer's generically τ-reduced components. Note that Voigt's Isomorphism [18, section 1.1] and Auslander–Reiten Duality [5] yield the inequalities

$$c_A(\mathcal{Z}) \leqslant \mathrm{ext}^1_A(\mathcal{Z}) \leqslant \hom^\tau_A(\mathcal{Z}).$$

A component $\mathcal{Z} \in \mathrm{Irr}(A)$ is *generically τ-reduced* if

$$c_A(\mathcal{Z}) = \hom^\tau_A(\mathcal{Z}, \mathcal{Z}).$$

Write $\mathrm{Irr}^\tau(A)$ for the set of generically τ-reduced components in $\mathrm{Irr}(A)$. They first appeared for certain Jacobi algebras in [21, section 1.5] under the name *strongly reduced components* and were defined and studied in full generality in [33]. Indeed, generically τ-reduced components are generically reduced. Further, there is a bijective correspondence

$$\{\text{Isomorphism classes of } \tau\text{-rigid } A\text{-modules } V\} \xrightarrow{1:1} \{\mathcal{Z} \in \mathrm{Irr}^\tau(A) \text{ with } c_A(\mathcal{Z}) = 0\}$$

given by sending a module V to the closure of its orbit $\overline{\mathcal{O}(V)}$. We refer to [33, section 2], [25, section 5], [19, sections 2–4], and [20] for further background. It is now natural to come up with the following τ-reduced version of the second Brauer–Thrall conjecture.

Conjecture 4.2 (τ-reduced Brauer–Thrall II'). *Let A be a τ-tilting infinite algebra. Then there exist $\mathbf{d} \in \mathrm{K}_0(A)^+$ and a generically indecomposable $\mathcal{Z} \in \mathrm{Irr}^\tau(A, \mathbf{d})$ with $c_A(\mathcal{Z}) \geqslant 1$.*

Example 4.3. Let $A = KQ$ be the path algebra of the Kronecker quiver $Q: 2 \rightrightarrows 1$. Then $\mathrm{Rep}(A, \mathbf{d})$ is irreducible and smooth for every $\mathbf{d} \in \mathrm{K}_0(A)^+$. As A is hereditary, it follows that $\mathrm{Rep}(A, \mathbf{d})$ is generically τ-reduced. It is well-known that $\mathrm{Rep}(A, \mathbf{d})$ is generically indecomposable with $c_A(\mathrm{Rep}(A, \mathbf{d})) \geqslant 1$ if and only if $\mathbf{d} = (1, 1)$. Therefore, A satisfies our τ-reduced Brauer–Thrall II' Conjecture 4.2 and provides a counterexample for a τ-reduced version of Brauer–Thrall II. More generally, every representation infinite path algebra satisfies Conjecture 4.2.

Note that taking $\mathbf{g}$-vectors defines constructible maps $\mathbf{g}: \mathrm{Rep}(A, \mathbf{d}) \to \mathrm{K}_0^{\mathrm{proj}}(A)$ for every $\mathbf{d} \in \mathrm{K}_0(A)^+$ hence any $\mathcal{Z} \in \mathrm{Irr}(A)$ possesses a *generic* $\mathbf{g}$*-vector* $\mathbf{g}(\mathcal{Z}) \in \mathrm{K}_0^{\mathrm{proj}}(A)$. Plamondon proves that τ-reduced components are determined by their *generic* $\mathbf{g}$*-vector*: Let $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$. Building upon Palu's constructability of cokernels [31, Lemma 2.3], Plamondon constructs in [33, Lemma 2.11] a regular map on an open dense subset $\mathcal{U} \subseteq \mathrm{Pres}(A, \boldsymbol{\gamma})$

$$\Phi: \mathcal{U} \to \mathrm{Rep}(A, \mathbf{d})$$

for some $\mathbf{d} \in \mathrm{K}_0(A)^+$ such that $\Phi(\vec{P}) \cong \mathrm{Cok}(p)$ for all $\vec{P} = (p, P_1, P_0) \in \mathcal{U}$. He then considers the irreducible and closed subset

$$\mathcal{P}(\boldsymbol{\gamma}) := \overline{\Phi(\mathcal{U})} \subseteq \mathrm{Rep}(A, \mathbf{d}).$$

Theorem 4.4 ([33, Theorem 1.2]). *The maps*

$$\operatorname{Irr}^{\tau}(A) \underset{\mathcal{P}}{\overset{\mathbf{g}}{\rightleftarrows}} \mathrm{K}_0^{\mathrm{proj}}(A)$$

satisfy

(i) *for all* $\mathcal{Z} \in \operatorname{Irr}^{\tau}(A)$ *is* $\mathcal{P}(\mathbf{g}(\mathcal{Z})) = \mathcal{Z}$;
(ii) *for all* $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$ *is* $-\boldsymbol{\delta} := \mathbf{g}(\mathcal{P}(\boldsymbol{\gamma})) - \boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$ *with* $e_A(\mathbf{g}(\mathcal{P}(\boldsymbol{\gamma})), \boldsymbol{\delta}) = e_A(\boldsymbol{\delta}, \mathbf{g}(\mathcal{P}(\boldsymbol{\gamma}))) = 0$.

The bridge between presentations and τ-tilting theory is provided by a precursor of the Auslander–Reiten formula:

Lemma 4.5 ([33, Lemma 2.6]). *Let* $V, W \in \operatorname{mod}(A)$ *with minimal projective presentations* $\vec{P}(V)$ *and* $\vec{P}(W)$. *Then*

$$\operatorname{Hom}_{K^b(A)}(\vec{P}(V), \Sigma(\vec{P}(W))) \cong \mathrm{D}\operatorname{Hom}_A(W, \tau_A(V)).$$

Lemma 4.6. *Let* $\mathcal{Z}, \mathcal{Z}' \in \operatorname{Irr}^{\tau}(A)$. *Then*

$$\hom_A^{\tau}(\mathcal{Z}, \mathcal{Z}') = e_A(\mathbf{g}(\mathcal{Z}), \mathbf{g}(\mathcal{Z}')) \qquad \hom_A^{\tau}(\mathcal{Z}) = e_A(\mathbf{g}(\mathcal{Z})).$$

Proof. Set $\boldsymbol{\gamma} := \mathbf{g}(\mathcal{Z})$ and $\boldsymbol{\gamma}' := \mathbf{g}(\mathcal{Z}')$. Let $\mathcal{V} \subseteq \mathcal{Z}$ and $\mathcal{V}' \subseteq \mathcal{Z}'$ be open dense subsets with $\hom_A^{\tau}(V, V') = \hom_A^{\tau}(\mathcal{Z}, \mathcal{Z}')$ for all $V \in \mathcal{V}$ and $V' \in \mathcal{V}'$. Let $\mathcal{U} \subseteq \operatorname{Pres}(A, \boldsymbol{\gamma})$ and $\mathcal{U}' \subseteq \operatorname{Pres}(A, \boldsymbol{\gamma}')$ be open dense subsets with $e_A(\vec{P}, \vec{P}') = e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}')$ for all $\vec{P} \in \operatorname{Pres}(A, \boldsymbol{\gamma})$ and $\vec{P}' \in \operatorname{Pres}(A, \boldsymbol{\gamma}')$. For $\vec{P} \in \mathcal{U} \cap \Phi^{-1}(\mathcal{V})$ and $\vec{P'} \in \mathcal{U}' \cap \Phi^{-1}(\mathcal{V}')$ we find

$$e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}') = e_A(\vec{P}, \vec{P'}) = \hom_A^{\tau}(V, V') = \hom_A^{\tau}(\mathcal{Z}, \mathcal{Z}'),$$

where the second equality holds by Lemma 4.5. Similarly, one shows $e_A(\boldsymbol{\gamma}) = \hom_A^{\tau}(\mathcal{Z})$. □

Lemma 4.7. *For* $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$, *the following are equivalent:*

(i) $\boldsymbol{\gamma} \in \mathsf{Fan}(A)$;
(ii) $e_A(\boldsymbol{\gamma}) = 0$;
(iii) $c_A(\mathcal{P}(\boldsymbol{\gamma})) = 0$.

Proof. By definition of the $\mathbf{g}$-vector fan, we have $\boldsymbol{\gamma} \in \mathsf{Fan}(A)$ if and only if there exists a τ-rigid pair (V, P) with $\mathbf{g}(V, P) = \boldsymbol{\gamma}$.

$(i) \Rightarrow (ii), (iii)$: Let (V, P) be a τ-rigid pair with $\mathbf{g}(V, P) = \boldsymbol{\gamma}$. Take a minimal projective presentation $\vec{P}'$ of V and consider $\vec{P} := \vec{P}' \oplus \Sigma(P) \in \operatorname{Pres}(A, \boldsymbol{\gamma})$. Then $e_A(\vec{P}) = e_A(\vec{P}') = \hom_A^{\tau}(V) = 0$. This shows $e_A(\boldsymbol{\gamma}) = 0$. Moreover, $\overline{\mathcal{O}(V)}$ is a τ-reduced component with $\mathbf{g}(V) = \mathbf{g}(\mathcal{P}(\boldsymbol{\gamma}))$, hence $\mathcal{P}(\boldsymbol{\gamma}) = \overline{\mathcal{O}(V)}$ by Theorem 4.4. This shows $c_A(\mathcal{P}(\boldsymbol{\gamma})) = 0$.

$(ii) \Rightarrow (i)$: Let $\vec{P} \in \operatorname{Pres}(A, \boldsymbol{\gamma})$ with $e_A(\vec{P}) = 0$. Write $\vec{P} = \vec{P}' \oplus \Sigma(P)$ for some $P \in \operatorname{proj}(A)$ and $\vec{P}$ a minimal projective presentation of some $V \in \operatorname{mod}(A)$. Then $\hom_A^{\tau}(V, V) = e_A(\vec{P}', \vec{P}') \leqslant e_A(\vec{P}, \vec{P}) = 0$ hence (V, P) is a τ-rigid pair with $\mathbf{g}(V, P) = \mathbf{g}(\vec{P}) = \boldsymbol{\gamma}$.

(iii) ⇒ *(ii)*: Let $\gamma \in \mathrm{K}_0^{\mathrm{proj}}(A)$ and set $\gamma' := \mathbf{g}(\mathcal{P}(\gamma))$. Write $\gamma = \gamma' + \delta$ as in Plamondon's Theorem 4.4. With Lemma 4.6, we find:

$$e_A(\gamma) = e_A(\gamma') + e_A(\delta) + e_A(\gamma', \delta) + e_A(\delta, \gamma') = e_A(\gamma') = c_A(\mathcal{P}(\gamma)) = 0 \qquad \square$$

Proposition 4.8. *Let A be a finite-dimensional algebra. The following statements are equivalent.*

(i) *A satisfies Demonet's Conjecture 3.3.*
(ii) *A satisfies the τ-reduced Brauer–Thrall II' Conjecture 4.2.*

Proof. This is an immediate consequence of Lemmas 4.7, 4.6, and Plamondon's Theorem 4.4. □

5 | THE STABLE BRAUER–THRALL CONJECTURES

With similar geometric arguments as in the proof of Proposition 4.1, Mousavand–Paquette [30, Theorem 6.2] prove a stable analog of the first Brauer–Thrall conjecture. Another proof using **c**- and **g**-vectors, thus valid over not necessarily algebraically closed fields, is due to Schroll–Treffinger [37, Theorem 1.1]. Both work with bricks instead of stables, but their bricks are well-known to be stable by [9, Proposition 3.13].

Proposition 5.1 (Stable Brauer– Thrall I). *Let A be a τ-tilting infinite algebra. Then, for every $d \geqslant 0$ there exists a stable $V \in \mathrm{mod}(A)$ with $\dim(V) \geqslant d$.*

A brick version of Brauer–Thrall II' was formulated in [29, Conjecture 1.3.(2)] and [38, Conjecture 2]. We strengthen their conjecture slightly to a stable version. Given a dimension vector $\mathbf{d} \in \mathrm{K}_0(A)^+$ and a weight $\theta \in \mathrm{K}_0(A)^*$, denote by $\mathrm{Rep}(A, \mathbf{d}, \theta)^{\mathrm{st}}$ the open subvariety of $\mathrm{Rep}(A, \mathbf{d})$ consisting of θ-stable representations. King [28] constructs a fine moduli space of θ-stable A-modules in dimension $\mathbf{d}$. This is a quasi-projective variety $\mathrm{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}}$ together with a surjective map of varieties

$$\pi : \ \mathrm{Rep}(A, \mathbf{d}, \theta)^{\mathrm{st}} \twoheadrightarrow \mathrm{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}}$$

such that $\pi(V) = \pi(W)$ if and only if $V \cong W$.

Conjecture 5.2 (Stable Brauer– Thrall II'). *Let A be a τ-tilting infinite algebra. Then there exist $\mathbf{d} \in K_0(A)^+$ and $\theta \in \mathrm{K}_0(A)^*$ such that* $\dim \mathrm{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}} \geqslant 1$.

Example 5.3. Let $A = KQ$ be a representation infinite path algebra. Then there exists an ideal $I \subseteq A$ generated by idempotents and arrows such that $A/I \cong KQ'$ for a quiver Q' of affine type. Thus, we may assume that Q is of affine type. The homogeneous quasi-simple regular A-modules are a 1-parameter family of stables with respect to the defect. Therefore, A satisfies Conjecture 5.2.

Example 5.4. Generalizing Example 5.3, let $A = H(C, D, \Omega)$ be a GLS algebra [22] associated to a generalized Cartan matrix C with symmetrizer D and orientation Ω. Geiß–Leclerc–Schröer prove in [24] that H is τ-tilting finite if and only if C is of finite type. Assume that C is of affine type. In [32], we provide a generic classification of locally free representations of H. In the course of our proof of this generic classification, we construct a 1-parameter family of stable representations of H. In particular, GLS algebras satisfy Conjecture 5.2.

Example 5.5. Let A be a τ-tilting infinite special biserial algebra. It is proved in [38, Theorem 1.1] that there exists a family of pairwise nonisomorphic band modules V_λ for $\lambda \in K^\times$ which are all bricks. But as band modules they satisfy $\tau_A(V_\lambda) \cong V_\lambda$ by the discussion in [10, pp. 165–166]. Thus V_λ must be stable with respect to the weight $\mathbf{g}(V_\lambda)^\vee$ by Auslander–Reiten's $\mathbf{g}$-vector formula (3.1).

Proposition 5.6. *Let A be a finite-dimensional algebra. Consider the following statements.*

(i) *A satisfies the stable Brauer– Thrall II' Conjecture 5.2.*
(ii) *A satisfies Demonet's Conjecture 3.3.*

Then the implication $(i) \Rightarrow (ii)$ holds.

Proof. Let $\boldsymbol{\gamma} \in \mathsf{Fan}(A) \cap \mathrm{K}_0^{\mathrm{proj}}(A)$ and $\theta := \boldsymbol{\gamma}^\vee$ We find $\boldsymbol{\gamma} = \mathbf{g}(V, P)$ for a τ-rigid pair (V, P). By [9, Proposition 3.13], we have $\mathcal{W}(\theta) = \mathcal{W}(V, P)$. By [27, Theorem 3.8], we have $\mathcal{W}(V, P) \simeq \mathrm{mod}(B)$ for a finite-dimensional algebra B. In particular, there are only finitely many simple objects in $\mathcal{W}(\theta)$. But θ-stable modules are precisely simple objects in $\mathcal{W}(\theta)$. Therefore, there are only finitely many θ-stable modules. In other words $\dim \mathrm{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}} = 0$ for all $\mathbf{d} \in \mathrm{K}_0(A)^+$. □

6 | *E*-TAME ALGEBRAS

Recently, different notions of "tameness" in τ-tilting theory gained some interest, see, for example, [9, section 3.3], [2, 34], and [3, section 6]. They also play a role in understanding the τ-tilted Brauer–Thrall II conjectures. It is, for example, enough to prove Demonet's conjecture for the class of $\mathbf{g}$-*tame algebras*, those algebras A whose $\mathbf{g}$-vector fan $\mathsf{Fan}(A)$ is dense in $\mathrm{K}_0^{\mathrm{proj}}(A)_{\mathbb{R}}$. On the other hand, the main result of this note is that all our τ-tilted versions of Brauer–Thrall II are equivalent for the class of E-tame algebras.

Definition 6.1 (see [3, Definition 6.3] and [15, Definition 4.6]). A finite-dimensional algebra A is *E-tame* if $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 0$ for all $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$.

First, note that it is enough to check E-tameness for generically indecomposable τ-reduced components; we include a proof for convenience:

Lemma 6.2. *The following are equivalent for a finite-dimensional algebra A.*

(i) *A is E-tame.*
(ii) *Every $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$ with generically indecomposable* $\mathrm{Pres}(A, \boldsymbol{\gamma})$ *satisfies* $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 0$.
(iii) *Every $\mathcal{Z} \in \mathrm{Irr}^\tau(A)$ satisfies* $\hom_A^\tau(\mathcal{Z}, \mathcal{Z}) = 0$.
(iv) *Every generically indecomposable $\mathcal{Z} \in \mathrm{Irr}^\tau(A)$ satisfies* $\hom_A^\tau(\mathcal{Z}, \mathcal{Z}) = 0$.

Proof. The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are obvious. The reversed implications $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$ follow immediately from the corresponding generic Krull–Remak–Schmidt decompositions, see [15, Theorem 4.4] and [33, Theorem 2.7] for presentations and [25, Theorem 5.11] for generically τ-reduced components.

$(i) \Rightarrow (iii)$: Let $\mathcal{Z} \in \mathrm{Irr}^\tau(A)$ with generic $\mathbf{g}$-vector $\boldsymbol{\gamma} := \mathbf{g}(\mathcal{Z})$. We have seen in Lemma 4.6 that $\hom_A^\tau(\mathcal{Z}, \mathcal{Z}) = e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma})$ and $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 0$ because A is assumed to be E-tame.

$(iii) \Rightarrow (ii)$: Let $\boldsymbol{\gamma} \in \mathrm{K}_0^{\mathrm{proj}}(A)$ be generically indecomposable. Then $\boldsymbol{\gamma} = \mathbf{g}(\mathcal{Z})$ for a generically τ-reduced component or $\boldsymbol{\gamma} = -\mathbf{g}(P)$ for a projective $P \in \mathrm{proj}(A)$ by Plamondon's Theorem 4.4. In the former case, we have $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = \hom_A^\tau(\mathcal{Z}, \mathcal{Z}) = 0$ by Lemma 4.6 and assumption. In the latter case, we obviously have $e_A(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 0$. □

To motivate the concept of E-tameness, we introduce a more natural notion of generically τ-reduced tameness. Note that for a representation tame algebra A the generic number of parameters of a generically indecomposable $\mathcal{Z} \in \mathrm{Irr}(A)$ satisfies the bound $c_A(\mathcal{Z}) \leqslant 1$. Hence, we arrive at the following definition.

Definition 6.3. A finite-dimensional algebra A is *generically τ-reduced tame* if $c_A(\mathcal{Z}) \leqslant 1$ for all generically indecomposable $\mathcal{Z} \in \mathrm{Irr}^\tau(A)$.

Of course, there is a corresponding reformulation in terms of varieties of reduced presentations but we leave this to the reader. The relation with representation and E-tameness is a consequence of well-known facts:

Proposition 6.4. *Let A be a finite-dimensional algebra. Consider the following statements.*

(i) *A is representation tame.*
(ii) *A is generically τ-reduced tame.*
(iii) *A is E-tame.*

Then the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold.

Proof. $(i) \Rightarrow (ii)$: We already noted that $c_A(\mathcal{Z}) \leqslant 1$ for generically indecomposable $\mathcal{Z} \in \mathrm{Irr}(A)$ (see [11, Lemma 3] for a proof). In particular, A is generically τ-reduced tame.

$(ii) \Rightarrow (iii)$: Let $\mathcal{Z} \in \mathrm{Irr}^\tau(A)$, we may assume that $\mathcal{Z}$ is generically indecomposable by Lemma 6.2. Nothing is to show if $0 = c_A(\mathcal{Z}) = \hom_A^\tau(\mathcal{Z})$. If $c_A(\mathcal{Z}) = 1$, then $\hom_A^\tau(\mathcal{Z}, \mathcal{Z}) < \hom_A^\tau(\mathcal{Z}) = c_A(\mathcal{Z})$ by [20, Theorem 1.5]. Thus, $\hom_A^\tau(\mathcal{Z}, \mathcal{Z}) = 0$. □

Let us next provide two examples. The former shows that the implication $(iii) \Rightarrow (ii)$ in the previous proposition does not hold. The latter is a τ-tilting infinite counterexample for the implication $(ii) \Rightarrow (i)$.

Example 6.5. Let $n \geqslant 1$ and $H_n = KQ/I_n$ the algebra given by

$$Q: \quad d \circlearrowright 2 \overset{a}{\underset{b}{\rightrightarrows}} 1 \circlearrowleft c \qquad\qquad I_n = \langle c^n, d^n, ca - ad, cb - bd \rangle.$$

This is the Kronecker algebra for $n = 1$, thus representation tame hence generically τ-reduced tame and E-tame by Proposition 6.4. For general $n \geqslant 1$, this is the GLS algebra $H_n = H(C, nD, \Omega)$ from [22] associated to

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad nD = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \qquad \Omega = \{(1,2)\}.$$

Note that $c + d$ lies in the center of H_n and $H_n/\langle c + d\rangle \cong H_1$. By Eisele–Janssens–Raedschelders' reduction [17, Equivalence (4.2)] H_n is E-tame as H_1 is E-tame. Consider the unique irreducible component $\mathcal{Z} \in \mathrm{Irr}(H_n, (n, n))$ whose generic element is *locally free*, that means c and d act on generic $V \in \mathcal{Z}$ by linear operators of maximal rank $n - 1$; see [23, Proposition 3.1]. For $\lambda \in K^n$, consider the representation

$$V_\lambda : \qquad N_n \circlearrowright K^n \underset{L_\lambda}{\overset{\mathbb{1}_n}{\rightrightarrows}} K^n \circlearrowleft N_n$$

where $\mathbb{1}_n$ is the identity matrix of size n, N_n is the lower triangular nilpotent Jordan block of size n and L_λ is the lower triangular matrix with entries

$$(L_\lambda)_{ij} := \begin{cases} \lambda_{i-j+1} & \text{if } i \geqslant j \\ 0 & \text{else.} \end{cases} \qquad\qquad L_\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ \lambda_2 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-1} & \lambda_{n-2} & & \lambda_1 & 0 \\ \lambda_n & \lambda_{n-1} & \cdots & \lambda_2 & \lambda_1 \end{pmatrix}.$$

Then $V_\lambda \in \mathcal{Z}$ and they satisfy

$$\hom_{H_n}(V_\lambda, V_\mu) = \begin{cases} 0 & \text{if } \lambda_1 \neq \mu_1, \\ n & \text{if } \lambda = \mu, \end{cases}$$

for all $\lambda, \mu \in K^n$. By comparing dimensions one finds

$$\mathcal{Z} = \overline{\bigcup_{\lambda \in K^n} \mathcal{O}(V_\lambda)}.$$

Moreover, the generic element of $\mathcal{Z}$ has projective dimension at most 1 by [22, Theorem 1.2]. Therefore,

$$c_{H_n}(\mathcal{Z}) = \hom^\tau_{H_n}(\mathcal{Z}) = \mathrm{ext}^1_{H_n}(\mathcal{Z}) = \mathrm{end}_{H_n}(\mathcal{Z}) = n,$$

the first equality holds because $\mathcal{Z}$ is generically τ-reduced, the second follows from Auslander–Reiten duality, and the third holds by [22, Proposition 4.1].

Example 6.6. Consider the algebra $H = KQ/I$ given by

$$Q: \; 2 \xrightarrow{\alpha} 1 \circlearrowleft \varepsilon \qquad\qquad I := \langle \varepsilon^4 \rangle.$$

This is the GLS algebra $H = H(C, D, \Omega)$ [22] associated to

$$C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Omega = \{(1, 2)\}.$$

The generalized Cartan matrix C is of affine type. We show in [32] that H is generically τ-reduced tame. On the other hand, its Galois-covering has a convex hypercritical subcategory of double extended type $\widetilde{\widetilde{\mathrm{D}}}_5$ thus H is representation wild.

7 | PROOF OF THE MAIN THEOREM

In this final section, we finish our proof of the Main Theorem 1 of the present short note. It remains to consider E-tame algebras thus assume from now on that A is E-tame and τ-tilting infinite. By (ii), there exists $\mathcal{Z} \in \operatorname{Irr}^\tau(A)$ with $c_A(\mathcal{Z}) \geqslant 1$. As A is assumed to be E-tame, we must have $\hom^\tau_A(\mathcal{Z}, \mathcal{Z}) = 0$ according to Lemma 6.2. Our final Lemma 7.1 constructs infinitely many pairwise nonisomorphic stable A-modules as factors of generic elements of $\mathcal{Z}$. This then finishes our proof of Theorem 1.

Lemma 7.1. *Let* $\mathcal{Z} \in \operatorname{Irr}^\tau(A)$ *with* $\hom^\tau_A(\mathcal{Z}, \mathcal{Z}) = 0$ *and* $c_A(\mathcal{Z}) \geqslant 1$. *Set* $\theta := \mathbf{g}(\mathcal{Z})^\vee \in \mathrm{K}_0(A)^*$. *Then there exists* $\mathbf{d} \in \mathrm{K}_0(A)^+$ *such that* $\dim \operatorname{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}} \geqslant 1$.

Proof. Let $\mathcal{U}' \subseteq \mathcal{Z} \times \mathcal{Z}$ be the nonempty open subset such that

$$0 = \operatorname{Hom}_A(W, \tau_A(V)),$$

and $\mathbf{g}(V) = \mathbf{g}(W) = \mathbf{g}(\mathcal{Z})$ for all $(V, W) \in \mathcal{U}'$. Let $p_1, p_2 : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ denote the canonical projections onto the first, respectively, second factor. As projections are open, $p_1(\mathcal{U}')$ and $p_2(\mathcal{U}')$ are nonempty open subsets of $\mathcal{Z}$. But $\mathcal{Z}$ is irreducible, hence $\mathcal{U} := p_1(\mathcal{U}') \cap p_2(\mathcal{U}')$ is nonempty and still open in $\mathcal{Z}$. Let $V_0 \in \mathcal{U}$, then there is an open subset $\mathcal{U}_0 \subset \mathcal{Z}$ such that $\operatorname{Hom}_A(V_0, \tau_A(W)) = \operatorname{Hom}_A(W, \tau_A(V_0)) = 0$ for all $W \in \mathcal{U}_0$ (in particular $W \ncong V_0$ because $\hom^\tau_A(\mathcal{Z}) = c_A(\mathcal{Z}) \neq 0$). Let $V_1 \in \mathcal{U} \cap \mathcal{U}_0$, then there exists an open subset $\mathcal{U}_1 \subseteq \mathcal{Z}$ with $\operatorname{Hom}_A(V_1, \tau_A(W)) = \operatorname{Hom}_A(W, \tau_A(V_1)) = 0$ for all $W \in \mathcal{U}_1$. Let $V_2 \in \mathcal{U} \cap \mathcal{U}_0 \cap \mathcal{U}_1$. Inductively, construct an infinite family $\mathcal{V} := \{V_i \mid i \geqslant 0\} \subset \mathcal{Z}$ with $\operatorname{Hom}_A(V_i, \tau_A(V_j)) = \operatorname{Hom}_A(V_j, \tau_A(V_i)) = 0$ for all $i, j \geqslant 0$ with $i \neq j$.

Next, we want to define for each $i \geqslant 0$ a θ-stable factor $V_i \twoheadrightarrow X_i$ with $X_i \in \mathcal{W}(V_j)$ for all $j \neq i$. We then get $\operatorname{Hom}_A(X_i, X_j) = 0$ for all $i, j \geqslant 0$ with $i \neq j$ because $X_i \in \operatorname{fac}(V_i) \subseteq {}^\perp(V_i^\perp) \subseteq {}^\perp\mathcal{W}(V_i)$ and $X_j \in \mathcal{W}(V_i)$ by construction. Note that the dimension of the X_i for $i \geqslant 0$ is bounded by the dimension of elements in $\mathcal{Z}$. Thus, there is a dimension vector $\mathbf{d} \in \mathrm{K}_0(A)^+$ such that $X_i \in \operatorname{Rep}(A, \mathbf{d}, \theta)^{\mathrm{st}}$ for infinitely many $i \geqslant 0$. Therefore, $|\operatorname{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}}| = \infty$, but $\operatorname{Mod}(A, \mathbf{d}, \theta)^{\mathrm{st}}$ is a quasi-projective variety hence its dimension must be at least 1.

It remains to construct such stable factors: Fix $i \geqslant 0$ and take any $j \geqslant 0$ with $j \neq i$. Consider the canonical short exact sequence

$$0 \to U_j \to V_i \to W_j \to 0$$

with $U_j \in {}^\perp(V_j^\perp)$ and $W_j \in V_j^\perp$. As by choice, $V_i \in {}^\perp(\tau_A(V_j))$ we do find $W_j \in \mathcal{W}(V_j)$. Even more, we have $\operatorname{Hom}_A(W_j, \tau_A(V_i)) \neq 0$ because $\operatorname{Hom}_A(V_i, \tau_A(V_i)) \neq 0$ and $\operatorname{Hom}_A(U_j, \tau_A(V_i)) = 0$ for $\tau_A(V_i) \in V_j^\perp$ by choice. In particular, $W_j \neq 0$.

Let $k \geqslant 0$ with $k \neq i, j$. Consider the canonical short exact sequence

$$0 \to U_k \to W_j \to W_k \to 0$$

with $U_k \in {}^\perp(V_k^\perp)$ and $W_k \in V_k^\perp$. As before, we find $W_k \in \mathcal{W}(V_k)$. Furthermore, $W_k \in \operatorname{fac}(\mathcal{W}(V_j))$ but $\mathcal{W}(V_j)$ is a Serre subcategory of $\mathcal{W}_\theta$ by Lemma 3.1 and $W_k \in \mathcal{W}(V_k) \subseteq \mathcal{W}_\theta$ hence we still have $W_k \in \mathcal{W}(V_j)$. Also, $\operatorname{Hom}_A(W_k, \tau_A(V_i)) \neq 0$ because $\operatorname{Hom}_A(U_k, \tau_A(V_i)) = 0$ and we know from before that $\operatorname{Hom}_A(W_j, \tau_A(V_i)) \neq 0$.

We can proceed like this for all indices different from i, to obtain a nonzero factor module $V_i \twoheadrightarrow X'_i$ with $X'_i \in \mathcal{W}(V_j)$ for all $j \neq i$. Finally, take X_i to be a θ-stable factor of X'_i. Again, $X_i \in \mathcal{W}(V_j)$ for all $j \neq i$ because $\mathcal{W}(V_j)$ is a Serre subcategory of $\mathcal{W}_\theta$ for all $j \geqslant 0$ by Lemma 3.1. □

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