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BOOK REVIEW

K. Cieliebak, Y. Eliashberg, N. Mishachev: "Introduction to the h-Principle"

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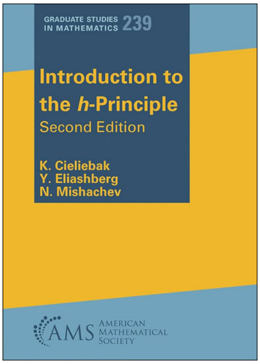

Imagine, if you will, the problem of walking up a steep hill, without expending too much effort at any given moment. The solution, of course, is to choose a serpentine path (as C^0-close to the straight way up as you like), which allows you to follow a gently sloping route at the price of lengthening your hike. This, in a nutshell, is the idea of proving an h-principle via the method of holonomic approximation.

As a second example, think of a motion in 3-space with some restriction on the directions you may take—in practice, such a problem might arise in robotics. If we require, say, that the velocity lie in a neighbourhood of the cone $\{x^2+y^2-z^2=0\}$, we can still reach any point in 3-space. In order to move in the z-direction, we choose a helicoidal path of slope close to 1; in order to move in a horizontal direction, we follow a sawtooth path of slope ± 1, slowing down to zero at the non-smooth points of the sawtooth. This example contains the idea of the convex integration method for establishing an h-principle: by integrating over the velocities—this produces the path one follows—one can realise any direction in the convex hull of the cone, which is all of 3-space.

The h-principle or *homotopy principle* made its first appearance, *avant la lettre*, in the early 1960s in the work of Whitney, Smale, Hirsch, Poénaru and Phillips on immersions and submersions. These are also the simplest cases for explaining the formal set-up. Let me restrict attention to the case of immersions. Suppose we are given differential manifolds M, N of dimension m, n, respectively, with $m \le n$. An *immersion* $M \to N$ is a smooth map f whose differential Tf, in local coordinates simply given by the Jacobian of f, has maximal rank m everywhere, i.e., the differential is pointwise an injective linear map $T_pf\colon\ T_pM \to T_{f(p)}N$ on tangent spaces.

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These maps fit into a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{\;Tf\;} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{\;f\;} & N \end{array}$$

of tangent bundles. When $F\colon\ TM \to TN$ is a fibrewise injective continuous bundle map covering a merely continuous map f, meaning that with F in place of Tf the diagram is still commutative, the pair (f, F) is called a *formal immersion*. Rather amazingly, if $m < n$, or if $m = n$ and M is open, i.e., every component of M is noncompact or has boundary, every formal immersion (f_0, F_0) can be continuously deformed into ('is homotopic to') an honest immersion $(f_1, F_1 = Tf_1)$. Thus, the solution to a purely homotopical problem (existence of a bundle map)—which usually can be found using algebraic topological methods—suffices for finding a solution to a geometric problem formulated in terms of differential equations or inequalities (here: a rank condition on the differential). In modern parlance, this statement would be shortened to saying that immersions, under the stated assumptions on M, satisfy an h-principle.

In his dissertation, Gromov [6] formalised the ideas behind the immersion theory into a general method of *continuous sheaves*. The *holonomic approximation* method developed in the first edition of the book under review is a geometrically more intuitive realisation of Gromov's ideas. (For brief introductions to holonomic approximation, see [2] or [1].) The term 'h-principle' or rather 'w.h.e. principle' for *weak homotopy equivalence principle* first appeared in Gromov's ICM talk [7] and in his paper [8], which introduced the *convex integration* method. The name of the principle derives from the fact that, topologically speaking, it stipulates that the inclusion map from honest solutions (e.g., immersions) to the space of formal solutions (e.g., formal immersions) be a weak homotopy equivalence, i.e., a map that induces isomorphisms on all homotopy groups. In my example above, saying that every formal immersion is homotopic to an actual immersion is really just the statement that this inclusion map is surjective on π_0, the space of path components. Injectivity on π_0 translates into the statement that any two immersions that are homotopic as formal immersions are also homotopic via honest immersions.

In 1986, Gromov collected his deep ideas on the h-principle into the tome [9], written in a rather terse style. The book remains a treasure-trove to this day. At a point early in my mathematical career, together with Jesús Gonzalo I approached Eliashberg for suggestions how to handle a question about contact forms we were pondering. He pointed us to a specific page of Gromov's book. It took us about a month to unravel what was written on that page, but indeed it led to a solution of our problem via convex integration.

In those days, probably the best way to start learning about the h-principle were Haefliger's lectures [10], to which I added my own exposition [5]. The first edition of the book under review, which appeared in 2002, was really the first monograph on the h-principle that was comprehensive as well as readable at the level of a graduate course. It deals with both methods for proving the h-principle alluded to above. The book starts out with a whole part (Chapters 1 to 5 in the second edition) on holonomic approximation, which constitutes a self-contained course in itself, leading to some spectacular applications such as the sphere and cone eversion, of which more later. Chapter 5 is an addition to the second edition and contains an interesting extension of holonomic approximation that produces global approximation results at the expense of working with multivalued functions, i.e., functions that take values in a cover of the target space.

Part 2 (Chapters 6 to 10 in the second edition) then introduces the language of differential relations and the general philosophy of Gromov's h-principle. Newly added to the second edition is Chapter 10 on foliations. The second edition as a whole has a stronger emphasis on aspects of foliation theory both as a tool and in applications, where the h-principle has demonstrated its powers in recent years.

It is this language of differential relations that allows one to establish h-principles of great generality, so here is a précis of the central ideas. One considers a fibre bundle $\pi\colon\ E \to M$ over a manifold M. In the example of the h-principle for immersions, this would simply be the product bundle $M \times N \to M$. A differential relation (of first order, to keep things simple) is a subset $\mathcal{R}$ of the 1-jet bundle E^1. (A 1-jet at a point $p \in M$ is an equivalence class of local C^1-sections of E near p having the same value and the same derivative at p. The space E^1 of 1-jets fibres in a natural way over M.) Any C^1-section of E, i.e., a map $\sigma\colon\ M \to E$ with $\pi \circ \sigma = \mathrm{id}_M$, gives rise to a continuous section $j^1\sigma$ of E^1, which locally is simply given by σ and its first order partial derivatives. We write Γ^i for the space of C^i-sections of the bundle (or the subset of a bundle) in question.

A *solution* of the relation $\mathcal{R}$ is a section $\sigma \in \Gamma^1(E)$ with $j^1\sigma \in \Gamma^0(\mathcal{R})$. So the minimal requirement for the existence of solutions is that $\mathcal{R} \subset E^1$ admit continuous sections. We write $\Gamma^1_0(E)$ for the space of solutions. However, one may also consider arbitrary continuous sections of $\mathcal{R} \subset E^1$, which is like saying one prescribes the coefficients of a first order Taylor polynomial subject to the restrictions imposed by $\mathcal{R}$ (e.g., derivatives close to zero in our very first example). This gives the space $\Gamma^0(\mathcal{R})$ of formal solutions. There is an obvious map

$$\begin{array}{ccc} \Gamma^1_0(E) & \longrightarrow & \Gamma^0(\mathcal{R}) \\ \sigma & \longmapsto & j^1\sigma. \end{array}$$

One says that $\mathcal{R}$ satisfies the (multiparametric) h-principle if this map is a weak homotopy equivalence. Once such an h-principle has been established, it suffices to find a formal solution (a mere bundle-theoretic problem) in order to guarantee the existence of an honest solution.

The method of holonomic approximation allows one to establish the h-principle for quite general classes of differential relations, for instance, relations that are open and, in a suitably defined sense, invariant under the action of local diffeomorphisms of M. In the case of immersions, these general concepts are realised as follows:

– Sections of $E^1 = (M \times N)^1$ are pairs (f, F) made up of a continuous map $f\colon\ M \to N$ and a continuous bundle map $F\colon\ TM \to TN$ covering f.

- The immersion relation $\mathcal{R}$ consists of the pairs (f, F) with F fibrewise injective (which is an open condition).
- A local diffeomorphism φ of M acts in the obvious way on a section σ of $E = M \times N$ via $\sigma \mapsto (\varphi \times \mathrm{id}) \circ \sigma \circ \varphi^{-1}$, and this extends to an action on 1-jets. This action preserves the fibrewise injectivity.

And indeed, such open and invariant relations do satisfy the h-principle, provided that M is open. The latter condition can be replaced by a dimension condition, because one may then pass from an arbitrary manifold to an open manifold $M \times \mathbb{R}^k$. The holonomic approximation method goes beyond this simplest of general h-principles: while it may not be possible to find a solution even over a submanifold of positive codimension (as in the path up a hill in our first example), it allows one to find solutions over a perturbed submanifold (our serpentine path).

Here is one of the most surprising applications of the h-principle for immersions, the *Smale paradox*, where we consider immersions of the 2-sphere S^2 in $\mathbb{R}^3$. We take $M = S^2 \times \mathbb{R}$ and $N = \mathbb{R}^3$. The tangent bundle of $S^2 \times \mathbb{R}$ is trivial (think of $S^2 \times \mathbb{R}$ embedded in $\mathbb{R}^3$ as a neighbourhood of $S^2 \subset \mathbb{R}^3$), so the relation of fibrewise injective and orientation-preserving maps of tangent bundles can then be thought of as the space of maps

$$\mathcal{R} = \bigl\{(f, F)\colon\ S^2 \times \mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathrm{GL}^+(3, \mathbb{R})\bigr\}$$

The set $\pi_0(\mathcal{R})$ of path components equals the set $\pi_2\bigl(\mathrm{GL}^+(3, \mathbb{R})\bigr) = \pi_2\bigl(\mathrm{SO}(3)\bigr)$ of homotopy classes of maps $S^2 \to \mathrm{SO}(3)$. The special orthogonal group $\mathrm{SO}(3)$ is a manifold diffeomorphic to $\mathbb{R}\mathrm{P}^3 = S^3/\pm\mathrm{id}$, whose second homotopy group vanishes, hence $\mathcal{R}$ is path-connected.

Now we consider two orientation-preserving immersions of $S^2 \times \mathbb{R}$ into $\mathbb{R}^3$: (i) the map $S^2 \times \mathbb{R} \to \mathbb{R}^3$ given by the standard embedding $S^2 \times \{0\} = S^2 \subset \mathbb{R}^3$, with the $\mathbb{R}$-direction corresponding to the *outward* normal; and (ii) the immersion given by the antipodal map of S^2, followed by the inclusion in $\mathbb{R}^3$, with the $\mathbb{R}$-direction corresponding to the *inward* normal. Our considerations above tell us that these two immersions are homotopic via immersions—the 2-sphere can be turned inside out without creating singularities.

Even when the h-principle guarantees the existence of a solution to a geometric problem, it can still be quite a challenge to find one explicitly. In the case of sphere eversion, beautiful illustrations of actual solutions exist both in print and on video [3, 12, 13].

A related paradox is that of *cone eversion*: The functions $x \mapsto |x|$ and $x \mapsto -|x|$ on $\mathbb{R}^2 \setminus \{0\}$ can be joined by a continuous family of C^1-functions with nonvanishing gradient. Here one can write down a solution in terms of elementary functions; a beautiful explanation how to reach this solution is given in [4, Lecture 27].

Returning to the book under review, the main addition to the second edition is Part 3 on singularities and wrinkling. Here the multivalued holonomic approximation theorem is used, amongst other things, to construct maps with prescribed fold singularities.

Part 4 deals with h-principles in symplectic and contact geometry. In this area, the progress in establishing new h-principles has been phenomenal in recent years, no doubt triggered by the impetus the first edition of this monograph provided. One of the major breakthroughs is the h-principle for overtwisted contact structures in all (odd) dimensions, proved by Borman–Eliashberg–Murphy in 2015. The authors have wisely decided to report on most of these developments in a separate forthcoming monograph on *Symplectic Topology and the h-Principle*.

Part 5 deals with the convex integration method for proving h-principles. The highlight is another apparent paradox, the Nash–Kuiper theorem on isometric C^1-immersions, proved in the final chapter. An isometric C^2-immersion of S^2 in $\mathbb{R}^3$ preserves the Gauß curvature; in particular, one cannot isometrically immerse S^2 into a small ball. In fact, the sphere is rigid: any isometric C^2-immersion $S^2 \to \mathbb{R}^3$ is congruent to the standard embedding.

However, the notion of isometric immersion only requires C^1-smoothness of the map, where the notion of curvature breaks down. And indeed, Nash (for codimension ≥ 2) and Kuiper (who extended the theorem to codimension 1) showed that isometric C^1-immersions of a Riemannian manifold M into $\mathbb{R}^n$ satisfy the h-principle for $\dim M < n$. In particular, S^2 can be isometrically C^1-immersed (even embedded) into an arbitrarily small ball in $\mathbb{R}^3$.

Similarly, the flat 2-torus can be isometrically C^1-embedded in $\mathbb{R}^3$. The web page [11] contains links to various relevant papers and amazing visualisations of such isometric C^1-embeddings.

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