



Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra



Research Paper

Combinatorial Seshadri stratifications on normal toric varieties



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ARTICLE INFO

Article history:

Received 28 January 2025

Available online 13 June 2025

Communicated by David Hernandez

Keywords:

Toric varieties

Seshadri stratification

Quasi-valuation

Semi-toric degeneration

ABSTRACT

We apply the theory of Seshadri stratifications to embedded toric varieties $X_P \subseteq \mathbb{P}(V)$ associated with a normal lattice polytope P . The approach presented here is purely combinatorial and completely independent of [1]. In particular, we get a close connection between a certain class of triangulations of the polytope P , Seshadri stratifications of X_P arising from torus orbit closures, and the associated degenerate semi-toric varieties. In the last section we show that the approach here and the one in [1] produce the same quasi-valuations and hence the same degenerations of X_P .

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Introduction

Motivation. One of the aims of the theory of Seshadri stratifications [1] on embedded projective varieties $X \subseteq \mathbb{P}(V)$ is to use geometric data (subvarieties, vanishing order of functions) to construct a flat degeneration of X into a union of toric varieties X_0 . The construction is motivated by the fact that, though the degenerate variety X_0 is often more singular than X , its combinatorial structure makes it easier to understand, and information about X_0 can often be “lifted” to information on X .

In this article, we focus on the case where X is a toric variety. Let $T \simeq \mathbb{K}^n$ be a torus with character lattice M , where \mathbb{K} is an algebraically closed field of characteristic zero. Given a full dimensional normal lattice polytope $P \subseteq M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, we denote by $X_P \subseteq \mathbb{P}(V)$ the associated embedded toric variety (Section 1). The rich combinatorial structure of toric varieties makes it possible to present the theory of Seshadri stratifications on toric varieties in a way which uses merely the common combinatorial tools related to toric varieties: the polytope P , the weight monoid S of the homogeneous coordinate ring of X_P , and a certain class of triangulations of P . The exposition given here is completely independent of [1].

Combinatorial Seshadri stratification. Let A be the set of faces of P . For $\sigma \in A$ there is a unique T -orbits $O_{\sigma} \subseteq X_P$, denote by X_{σ} its Zariski-closure in X_P . For a collection of homogeneous T -eigenfunctions $f_{\sigma} \in \mathbb{K}[X_P] \setminus \{0\}$, indexed by $\sigma \in A$, let μ_{σ} be the weight of f_{σ} and set $m_{\sigma} = \deg f_{\sigma}$. A *combinatorial Seshadri stratification on X_P* is a collection of such pairs $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$, fulfilling the following compatibility condition:

for all $\sigma \in A$, the rational weight $-\frac{\mu_{\sigma}}{m_{\sigma}}$ lies in the relative interior of the face σ .

Given a combinatorial Seshadri stratification on X_P , for any fixed maximal chain \mathfrak{C} in A , the set $\mathbb{B}_{\mathfrak{C}} = \{(m_{\sigma}, \mu_{\sigma}) \mid \sigma \in \mathfrak{C}\}$ turns out to be a basis of $\mathbb{Q} \oplus M_{\mathbb{Q}}$. Just using linear algebra, we construct for every maximal chain $\mathfrak{C} \subseteq A$ a valuation $\nu_{\mathfrak{C}} : \mathbb{K}[X_P] \setminus \{0\} \rightarrow \mathbb{Q}^{\mathfrak{C}}$ as follows: for a homogeneous T -eigenfunction $f \in \mathbb{K}[X_P]$ of degree m_f and weight μ_f , the valuation $\nu_{\mathfrak{C}}(f)$ is given by the coefficients of the expression of (m_f, μ_f) as a \mathbb{Q} -linear combination of the basis $\mathbb{B}_{\mathfrak{C}}$. The quasi-valuation $\nu : \mathbb{K}[X_P] \setminus \{0\} \rightarrow \mathbb{Q}^A$ associated to the combinatorial Seshadri stratification is defined by

$$\nu(f) = \min_{>^t} \{ \nu_{\mathfrak{C}}(f) \mid \mathfrak{C} \text{ maximal chain in } A \} \subseteq \mathbb{Q}^A,$$

where, in order to define the minimum, a linearization $>^t$ of the partial order on A is fixed. This quasi-valuation induces a filtration on $\mathbb{K}[X_P]$ by ideals, let $\text{gr}_{\nu} \mathbb{K}[X_P]$ be the associated graded ring and set $X_0 = \text{Proj}(\text{gr}_{\nu} \mathbb{K}[X_P])$.

This construction raises many natural questions: Are there many combinatorial Seshadri stratifications on X_P ? What is the structure of X_0 ? How is the geometry of X_P related to the geometry of X_0 ? How to determine explicitly $\nu(f)$, etc...

Results. The aim of this article is to give answers to the above questions.

1. First of all, to achieve a classification, we introduce on the set of combinatorial Seshadri stratifications on X_P an equivalence relation (Definiton 2.5) which ensures $\text{gr}_\nu \mathbb{K}[X_P] \simeq \text{gr}_{\nu'} \mathbb{K}[X_P]$ if ν and ν' are quasi-valuations associated to equivalent combinatorial Seshadri stratifications. Recall that a flag of faces in A is chain, i.e. a totally ordered subset of A , see Section 3. A triangulation $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$ of P indexed by flags of faces is, roughly speaking, a marking $\{\mathbf{v}_\sigma\}_{\sigma \in A}$ of the faces of P by rational points in the relative interior of the faces. The simplices of the triangulation are given by: for a flag of faces C the simplex Δ_C is the convex hull of points $\{\mathbf{v}_\sigma\}_{\sigma \in C}$.

Theorem A. *There is a bijection between the set of equivalence classes of combinatorial Seshadri stratifications on X_P and triangulations \mathcal{T} of P indexed by flags of faces in A .*

2. The quasi-valuation can be completely expressed in terms of the triangulation: let $f \in \mathbb{K}[X_P] \setminus \{0\}$ be a homogeneous T -eigenfunction of degree m_f and weight μ_f , and let $\mathfrak{C} \subseteq A$ be a maximal chain. We show that the following statements are equivalent:

- (i) $\nu(f) = \nu_{\mathfrak{C}}(f)$;
- (ii) $-\frac{\mu_f}{m_f} \in \Delta_{\mathfrak{C}}$;
- (iii) $\nu_{\mathfrak{C}}(f)$ has only non-negative entries.

Being coordinates with respect to a basis, $\nu_{\mathfrak{C}}$ can be easily computed; and the quasi-valuation can be determined using the equivalence of (i) and (ii) above.

Moreover, if one of the equivalent conditions holds, then $\nu(f)$ is, up to rescaling, given by the coefficients of the expression of $-\frac{\mu_f}{m_f}$ as affine linear combination of the vertices of $\Delta_{\mathfrak{C}}$.

3. The next task is to describe the structure of $\text{gr}_\nu \mathbb{K}[X_P]$. *A priori*, one has a dependence on the choice of “ $>^t$ ”. But, by the result above, it turns out that the relevant properties of ν only depend on the triangulation \mathcal{T} and not on the choice of the linearization.

Let $S \subseteq \mathbb{Z} \oplus M$ be the weight monoid of the embedded toric variety. For a flag of faces $C \subseteq A$ let $K(\Delta_C) \subseteq \mathbb{R} \oplus M_{\mathbb{R}}$ be the cone over the simplex and set $S_C := S \cap K(\Delta_C)$. The union of the cones $K(\Delta_C)$ defines a fan of cones, where C is running over all flags of faces in A . In the same way, the union of the S_C defines a fan of monoids $S_{\mathcal{T}}$, where C is running over all flags of faces in A . We show in Section 4.4:

Theorem B. *Denote by $\Gamma = \{\nu(f) \mid f \in \mathbb{K}[X_P] \setminus \{0\}\} \subseteq \mathbb{Q}^A$ the image of the quasi-valuation ν .*

- (i) Γ is a fan of monoids, isomorphic to $S_{\mathcal{T}}$. In particular, Γ depends, up to isomorphism, only on the triangulation \mathcal{T} and is independent of the choice of the linearization $>^t$ of A .

- (ii) The associated graded algebra $\text{gr}_\nu \mathbb{K}[X_P]$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$. In particular, the algebra $\text{gr}_\nu \mathbb{K}[X_P]$ depends only on the triangulation \mathcal{T} .
- (iii) The variety $X_0 = \text{Proj}(\text{gr}_\nu \mathbb{K}[X_P])$ is reduced. It is the irredundant union of the toric varieties $\text{Proj}(\mathbb{K}[S_{\mathfrak{C}}])$, where \mathfrak{C} runs over all maximal chains in A . The variety X_0 is equidimensional, i.e. all irreducible components of X_0 have same dimension as X_P .

4. The next step is to view X_0 as the special fibre in a flat family. Using the connection between the quasi-valuations ν , monomial preorders and approximations by integral weight orders we show (Section 5.7):

Theorem C. *There exists a variety \check{X}_P together with a flat surjective morphism $\pi : \check{X}_P \rightarrow \mathbb{A}^1$ such that the fibre over $0 \in \mathbb{A}^1$ is isomorphic to X_0 , and π is trivial over $\mathbb{A}^1 \setminus \{0\}$ with fibre isomorphic to X_P .*

In Section 5.6 we provide another way to look at the degeneration in the above theorem: Indeed, there exists an embedding of X_P and X_0 into a weighted projective space $\mathbb{P}(m_1, \dots, m_r)$, endowed with the action of a one dimensional torus \mathbb{G}_m , such that one can view X_0 as a “limit variety”: $\lim_{s \rightarrow 0} s \cdot X_P = X_0$.

In particular, if all vertices of the triangulation \mathcal{T} are lattice points, then one can attach to each maximal chain $\mathfrak{C} \subseteq A$ a maximal simplex $\Delta_{\mathfrak{C}}$ and a toric variety $X_{\Delta_{\mathfrak{C}}} \subseteq \mathbb{P}(V)$. The one dimensional torus \mathbb{G}_m above acts also on the projective space $\mathbb{P}(V)$ we started with, and it makes sense to study the limit $\mathbb{X}_0 := \lim_{s \rightarrow 0} s \cdot X_P$ inside $\mathbb{P}(V)$, similar to what was done in [11]. We prove the following result in Section 6, which can be thought of as a special case of the results in [11]: The limit \mathbb{X}_0 inside $\mathbb{P}(V)$ is the union of the toric varieties $X_{\Delta_{\mathfrak{C}}}$, where \mathfrak{C} is running over all maximal chains in A .

The limit varieties $X_0 \subseteq \mathbb{P}(m_1, \dots, m_r)$ and $\mathbb{X}_0 \subseteq \mathbb{P}(V)$ are strongly connected. For example, the irreducible component $X_{\mathfrak{C}}$ of $X_0 \subseteq \mathbb{P}(m_1, \dots, m_r)$, where $\mathfrak{C} \subseteq A$ a maximal chain, is isomorphic to the normalization of $X_{\Delta_{\mathfrak{C}}}$, the corresponding irreducible component of \mathbb{X}_0 . The limit $\lim_{s \rightarrow 0} s \cdot I(X_P)$ of the vanishing ideal $I(X_P)$ of $X_P \subseteq \mathbb{P}(V)$ is in general not a radical ideal (see also [11]), for details about the radical see Theorem 6.2.

5. In the last section we compare the notion of a combinatorial Seshadri stratification in this article with the notion of a Seshadri stratification in [1]. We show that: A combinatorial Seshadri stratification $(X_\sigma, f_\sigma)_{\sigma \in A}$ on X_P is a Seshadri stratification on X_P in the sense of [1], which is equivariant with respect to the T -action on X_P . The quasi-valuation ν associated to a combinatorial Seshadri stratification $(X_\sigma, f_\sigma)_{\sigma \in A}$ is the same as the quasi-valuation \mathcal{V} associated to the Seshadri stratification in [1]. The associated semi-toric varieties are therefore the same.

Outlooks. In the case of toric varieties, the role played by the subvarieties in the usual framework of a Seshadri stratification is completely replaced by the triangulations indexed by flags of faces. In a forthcoming article we will generalize this approach and

consider, as it was done in the rank one case in [7], triangulations of the polytope P as a natural starting point to construct higher rank quasi-valuations on the homogeneous coordinate ring of an embedded normal toric variety, and to describe the corresponding degenerate variety.

Another line of generalization we are approaching is to replace toric varieties by spherical varieties. It is a class of varieties which is endowed with a large collection of combinatorial tools, and we plan to use this for a combinatorial description of the structure of the degenerate variety. But already the case of the flag variety (see [2,3]) shows that the transition from toric varieties to spherical varieties can not be done with ease.

Organization of the article. The article is structured as follows. In Section 1 we recall a few standard facts about normal toric varieties and we fix some notation. In Section 2 we introduce the notion of a combinatorial Seshadri stratification and in Section 3 we introduce the triangulations of P indexed by flags of faces. In Section 4 we define the quasi-valuation associated to a combinatorial Seshadri stratification on $X_P \subseteq \mathbb{P}(V)$ and prove a few first properties. In Section 5 we discuss a flat degeneration of X_P into X_0 given by a one parameter group, and in Section 6 we discuss the case where the vertices of the triangulation are all lattice points. In the last section, Section 7, we show that a combinatorial Seshadri stratification is indeed a Seshadri stratification as defined in [1].

Acknowledgments: The work of R.C. is partially supported by the Ministero dell’Università e della Ricerca, PRIN 2022 S8SSW2 “Algebraic and geometric aspects of Lie theory” (CUP I53D23002410006). The work of M.C.C. is funded by the Ministero dell’Università e della Ricerca, PRIN 2022 A7L229 “ALgebraic and TOPological combinatorics” (CUP J53D23003660006). The work of X.F. is funded by the Deutsche Forschungsgemeinschaft: “Symbolic Tools in Mathematics and their Application” (TRR 195, project-ID 286237555). The work of P.L. is partially supported by the Deutsche Forschungsgemeinschaft: “Symplektische Strukturen in Geometrie, Algebra und Dynamik” (TRR 191, project-ID 281071066).

The views expressed are purely those of the authors and may not in any circumstances be regarded as stating an official position of the European Research Council Executive Agency and the European Commission.

1. Polytopes and embedded normal toric varieties

We fix some notation and recall a few standard facts about embedded normal toric varieties. We denote by \mathbb{K} an algebraically closed field of characteristic zero. Let $T \simeq (\mathbb{K}^*)^n$ be a torus with character lattice M and dual lattice N . We write $\langle \cdot, \cdot \rangle$ for the non-degenerate pairing on $N \times M$ defined by: $\langle \eta, \mu \rangle$ is the unique integer such that $\mu(\eta(s)) = s^{\langle \eta, \mu \rangle}$ for all $s \in \mathbb{K}^*$.

Let $P \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be a full dimensional lattice polytope and set $\Lambda = P \cap M$. Let $\{e_{\chi} \mid \chi \in \Lambda\} \subseteq \mathbb{K}^{\Lambda}$ be the standard basis of $V = \mathbb{K}^{\Lambda}$.

Definition 1.1. ([4, Chapter 2]) The *embedded toric variety* $X_P \subseteq \mathbb{P}(V)$ is defined as the Zariski closure of the image of the map $\iota : T \rightarrow \mathbb{P}(V)$, $t \mapsto \left[\sum_{\chi \in \Lambda} \chi(t)e_\chi \right]$.

If P is a normal polytope, then $X_P \subseteq \mathbb{P}(V)$ is a projectively normal variety. For the rest of the article we assume that P is a normal polytope.

1.1. *Orbits and faces*

We fix the coordinates on $V = \mathbb{K}^\Lambda$ and write x_χ , $\chi \in \Lambda$, for the linear function on V dual to e_χ . Denote by $\hat{X}_P \subseteq V$ the affine cone over X_P . Let \hat{T} be the torus $\mathbb{K}^* \times T$ with character lattice $\hat{M} = \mathbb{Z} \oplus M$. Let $\hat{\iota} : \hat{T} \rightarrow V$ be the map $(c, t) \mapsto \sum_{\chi \in \Lambda} \chi(t)ce_\chi$, then \hat{X}_P is the closure of the image of $\hat{\iota}$ and $(x_\chi \circ \hat{\iota})(c, t) = c\chi(t)$ for $(c, t) \in \hat{T}$. One has a bijection between the T -orbits in X_P and the faces of P ([4, Section 2.3, Section 3.2]). Let σ be a face of P and set $\Lambda_\sigma = \Lambda \cap \sigma$. The T -orbit associated to a face σ of P is $O_\sigma = \{[\sum_{\chi \in \Lambda_\sigma} \chi(t)e_\chi] \mid t \in T\}$. In particular, the coefficients of the e_χ , $\chi \in \Lambda_\sigma$, are nonzero.

Definition 1.2. The Zariski closure $\overline{O_\sigma} \subseteq X_P$ of the orbit is denoted by X_σ .

The variety X_σ is a toric variety associated to the polytope σ , where we view the latter as a full dimensional lattice polytope in its affine span (see [4, Section 3.2]).

Let A be the set of faces of P . This set is partially ordered by the inclusion relations on the faces: we write $\tau \leq \sigma$ for $\sigma, \tau \in A$, if and only if $\tau \subseteq \sigma$. We have for the orbit closures: $X_\tau \subseteq X_\sigma$ if and only if $\tau \leq \sigma$.

1.2. *The homogeneous coordinate ring of X_P*

The homogeneous coordinate ring $\mathbb{K}[X_P]$ of $X_P \subseteq \mathbb{P}(V)$ and the coordinate ring $\mathbb{K}[\hat{X}_P]$ of the affine cone \hat{X}_P are the same rings. Since we often work with \hat{X}_P , we will use for the rest of the article the notation $\mathbb{K}[\hat{X}_P]$. This ring has a natural grading $\mathbb{K}[\hat{X}_P] = \bigoplus_{m \geq 0} \mathbb{K}[\hat{X}_P]_m$.

The action of \hat{T} on V induces a natural action of \hat{T} on $\mathbb{K}[V]$: for $\hat{t} = (c, t) \in \mathbb{K}^* \times T$ and $g \in \mathbb{K}[V]$ let $\hat{t} \cdot g$ be the regular function: $\hat{X} \rightarrow \mathbb{K}$, $x \mapsto g(t^{-1} \cdot (c^{-1}x))$. The coordinate functions x_χ , $\chi \in \Lambda$, are \hat{T} -eigenfunctions of weight $(-1, -\chi)$. Note that \hat{T} -eigenfunctions are automatically homogeneous, the degree is the absolute value of the first entry of the \hat{T} -weight. Since \hat{X}_P is \hat{T} -stable, we get an induced \hat{T} -action on $\mathbb{K}[\hat{X}_P]$.

Definition 1.3. Let Q be a rational polytope in $M_{\mathbb{R}}$. The *cone* $K(Q) \subseteq \mathbb{R} \oplus M_{\mathbb{R}}$ associated to Q is the subset $K(Q) := \{(c, cp) \mid c \in \mathbb{R}_{\geq 0}, p \in Q\}$.

Let $K(P) \subseteq \mathbb{R} \oplus M_{\mathbb{R}}$ be the cone over P . Denote by $S \subseteq K(P)$ the submonoid generated by the elements $(1, \chi)$, $\chi \in \Lambda$. The assumption that P is a full dimensional

normal lattice polytope implies: $S = K(P) \cap (\mathbb{Z} \times M)$ ([4, Lemma 2.2.14]). The monoid S is called the *weight monoid* associated to $X_P \subseteq \mathbb{P}(V)$.

A monomial $\prod_{\chi \in \Lambda} x_\chi^{a_\chi} \in \mathbb{K}[V]$ is a \hat{T} -eigenvector, its \hat{T} -weight is $(-m, -\eta)$, where $(m, \eta) = \sum_{\chi \in \Lambda} a_\chi(1, \chi)$ is an element in the weight monoid S . The algebra $\mathbb{K}[\hat{X}_P]$ is linearly spanned by the restrictions $\prod_{\chi \in \Lambda} x_\chi^{a_\chi}|_{\hat{X}_P}$ of these monomials.

For every $(m, \eta) \in S$, we fix a decomposition $(m, \eta) = \sum_{\chi \in \Lambda} a_\chi(1, \chi)$, and set $f_{m, \eta} = \prod_{\chi \in \Lambda} x_\chi^{a_\chi}|_{\hat{X}_P}$. The following is well known:

Lemma 1.4. *The function $f_{m, \eta}$ depends only on (m, η) and not on the choice of the decomposition of (m, η) . The set $\{f_{m, \eta} \mid (m, \eta) \in S\}$ is a \mathbb{K} -basis for $\mathbb{K}[\hat{X}_P]$.*

The subspaces of homogeneous functions are given by $\mathbb{K}[\hat{X}_P]_m = \bigoplus_{\chi \in (mP \cap M)} \mathbb{K}f_{m, \chi}$, $m \in \mathbb{N}$ (see [4, Example 4.3.7]).

2. Combinatorial Seshadri stratifications and valuations

Let $X_P \subseteq \mathbb{P}(V)$ be an embedded toric variety as in Section 1 and recall that A denotes the set of faces of the polytope P . Let X_σ , $\sigma \in A$, be the collection of T -orbit closures in X_P and let $f_\sigma \in \mathbb{K}[\hat{X}_P]$, $\sigma \in A$, be a collection of homogeneous T -eigenfunctions of degree $\deg f_\sigma \geq 1$. Denote by $\mu_\sigma = (-\deg f_\sigma, \tilde{\mu})$, $\tilde{\mu} \in M$, the \hat{T} -weight of f_σ . In the following we identify $P \subseteq M_\mathbb{R}$ with the polytope in the affine hyperplane $\{(1, \eta) \mid \eta \in M_\mathbb{R}\} \subset \hat{M}_\mathbb{R}$ obtained as the convex hull of the points $(1, \chi)$, $\chi \in \Lambda$.

Definition 2.1. The collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of T -orbits closures $X_\sigma \subseteq X_P$ and \hat{T} -eigenfunctions $f_\sigma \in \mathbb{K}[\hat{X}_P]$ is called a *combinatorial Seshadri stratification* on X_P if the \hat{T} -weights μ_σ of the functions f_σ , $\sigma \in A$, satisfy the following condition:

$$\forall \sigma \in A, \frac{-\mu_\sigma}{\deg f_\sigma} \text{ is a point in the relative interior } \sigma^\circ \text{ of } \sigma. \tag{1}$$

Example 2.2. A natural choice of the functions f_σ is the product of the linear functions associated to vertices of σ .

Let $T = (\mathbb{K}^*)^2$ be the torus. The character lattice of T is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $\{e_1, e_2\}$ is the natural basis of \mathbb{R}^2 . Let P be the square with set of vertices $E = \{v_0 = (0, 0), v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, 1)\}$. Then $\Lambda = E$.



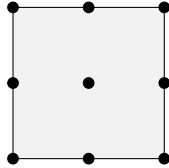
The faces of P are: the vertices in E , the 4 edges $e_{(i,j)}$ connecting v_i with v_j where $(i, j) \in \{(0,1), (0,2), (1,3), (2,3)\}$, and the whole square P . The homogeneous \hat{T} -eigenfunctions

$$f_{v_i} = x_{v_i}, f_{e_{(i,j)}} = x_{v_i}x_{v_j}, f_P = x_{v_0}x_{v_1}x_{v_2}x_{v_3}$$

satisfy the property (1).

Example 2.3. Suppose P has enough lattice points in the following sense: for every face $\sigma \in A$ there exists a weight $\chi_\sigma \in \Lambda \cap \sigma^\circ$ in the relative interior; fix a collection of such weights $\chi_\sigma, \sigma \in A$. The linear functions $f_\sigma = x_{\chi_\sigma}, \sigma \in A$, satisfy the condition on the weights, so the collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of subvarieties and \hat{T} -eigenfunctions is a combinatorial Seshadri stratification.

Example 2.4. Let T and M be as in Example 2.2. Let P the square with set of vertices $E = \{v_0 = (0,0), v_1 = (2,0), v_2 = (0,2), v_3 = (2,2)\}$. Then $\Lambda = E \cup \{(1,0), (0,1), (1,1), (2,1), (1,2)\}$.



The polytope P has enough lattice points. So the functions f_σ corresponding to the coordinate associated with the weights in $\Lambda \cap \sigma$, satisfy the property (1).

By a *marking* \mathbf{m} of the faces of the polytope P we mean a collection $\mathbf{m} = (u_\sigma)_{\sigma \in A}$ of points such that u_σ is a rational point in the relative interior σ° of the face $\sigma \subseteq P$. The collection of weights in (1) defines such a marking of the faces.

Definition 2.5. Let $(X_\sigma, f_\sigma)_{\sigma \in A}$ be a combinatorial Seshadri stratification on X_P and denote by μ_σ the \hat{T} -weight of the function $f_\sigma, \sigma \in A$. We call $\mathbf{m}_f = (\frac{-\mu_\sigma}{\deg f_\sigma})_{\sigma \in A}$ the *associated marking of the faces*. Two combinatorial Seshadri stratifications $(X_\sigma, f_\sigma)_{\sigma \in A}$ and $(X_\sigma, h_\sigma)_{\sigma \in A}$ are called *equivalent* if the associated markings are equal: $\mathbf{m}_f = \mathbf{m}_h$.

Remark 2.6. Let $(X_\sigma, f_\sigma)_{\sigma \in A}$ be a combinatorial Seshadri stratification on $X_P \subseteq \mathbb{P}(V)$. For $\sigma \in A$ let $M_{\mathbb{R},\sigma} \subseteq M_{\mathbb{R}}$ be the affine span of σ . Set $M_\sigma = M \cap M_{\mathbb{R},\sigma}$. So σ is a full dimensional lattice polytope in $M_{\mathbb{R},\sigma}$; set $V_\sigma = \langle \hat{X}_\sigma \rangle_{\mathbb{K}} \subseteq V$ and $A_\sigma = \{\kappa \in A \mid \kappa \leq \sigma\}$. The collection of varieties $X_\tau \subseteq X_\sigma$ and functions $f_\tau|_{X_\sigma}, \tau \in A_\sigma$, defines a combinatorial Seshadri stratification for the normal embedded toric variety $X_\sigma \hookrightarrow \mathbb{P}(V_\sigma) \subseteq \mathbb{P}(V)$.

2.1. First properties

Let $(X_\sigma, f_\sigma)_{\sigma \in A}$ be a combinatorial Seshadri stratification on X_P and denote by μ_σ the \hat{T} -weights of the extremal functions $f_\sigma, \sigma \in A$.

Lemma 2.7. For $\sigma, \tau \in A$, the restriction $f_\sigma|_{\hat{X}_\tau} \equiv 0$ is identically zero if and only if $\tau \not\geq \sigma$.

Proof. Since $\frac{-\mu_\sigma}{\deg f_\sigma} \in \sigma^o$ is an element in the relative interior of σ , it follows that f_σ can be written as the restriction of a monomial in the x_χ , $\chi \in \Lambda_\sigma$. All the coefficients of the e_χ in the expression $y = [\sum_{\chi \in \Lambda_\tau} a_\chi e_\chi] \in O_\tau$ are nonzero (Section 1.1), so the restriction of the function to X_τ is not identically zero if $\tau \geq \sigma$ because $\Lambda_\sigma \subseteq \Lambda_\tau$.

If $\tau \not\geq \sigma$, then σ is not a face of τ . Since $\frac{-\mu_\sigma}{\deg f_\sigma} \in \sigma^o$, there exists at least one $\chi \in \Lambda_\sigma \setminus \Lambda_\tau$ such that x_χ is a factor of f_σ , and hence the restriction of f_σ to O_τ is identically zero. \square

Lemma 2.8. The map Ψ , which associates to a combinatorial Seshadri stratification $(X_\sigma, f_\sigma)_{\sigma \in A}$ the marking $\mathbf{m}_f = (\frac{-\mu_\sigma}{\deg f_\sigma})_{\sigma \in A}$, induces a bijection between the set of equivalence classes of combinatorial Seshadri stratifications on X_P and the set of markings of the faces of P . In every equivalence class of combinatorial Seshadri stratifications there exists an element $(X_\sigma, f_\sigma)_{\sigma \in A}$ such that all other combinatorial Seshadri stratifications in the same equivalence class are equal to $(X_\sigma, c_\sigma f_\sigma^{\ell_\sigma})_{\sigma \in A}$ for some $c_\sigma \in \mathbb{K}^*$, $\ell_\sigma \in \mathbb{N}_{>0}$, $\sigma \in A$.

Proof. The map Ψ is well defined and injective on the equivalence classes. Let $\mathbf{m} = (u_\sigma)_{\sigma \in A}$ be a marking of the faces. For $\sigma \in A$, the set $\{n \in \mathbb{Z} \mid nu_\sigma \in \hat{M}\} \subseteq \mathbb{Z}$ is an ideal, let $q_\sigma > 0$ be a generator of the ideal. For $\sigma \in A$ set $\lambda_\sigma = -q_\sigma u_\sigma \in \hat{M}$, this is the weight of a \hat{T} -eigenfunction f_σ of degree q_σ , and $(X_\sigma, f_\sigma)_{\sigma \in A}$ is a combinatorial Seshadri stratification with associated marking $\mathbf{m}_f = (u_\sigma)_{\sigma \in A}$. It follows that Ψ is a bijection, we fix these functions f_σ , $\sigma \in A$, for the rest of the proof.

A \hat{T} -eigenfunction h_σ of weight ν_σ with $\frac{-\nu_\sigma}{\deg h_\sigma} = u_\sigma$ is equal to $h_\sigma = c_\sigma f_\sigma^{\ell_\sigma}$ for some $c_\sigma \in \mathbb{K}^*$, $\ell_\sigma \in \mathbb{N}_{>0}$. It follows: the equivalence class of $(X_\sigma, f_\sigma)_{\sigma \in A}$ consists of combinatorial Seshadri stratifications of the form $(X_\sigma, c_\sigma f_\sigma^{\ell_\sigma})_{\sigma \in A}$, $c_\sigma \in \mathbb{K}^*$, $\ell_\sigma \in \mathbb{N}_{>0}$, $\sigma \in A$. \square

Lemma 2.9. If $\mathbf{m} = (u_\sigma)_{\sigma \in A}$ is a marking of the faces of P , then, for every maximal chain $\mathfrak{C} \subseteq A$, the rational weights $\{u_\sigma\}_{\sigma \in \mathfrak{C}}$ form a \mathbb{Q} -basis for $\hat{M}_\mathbb{Q}$.

Proof. By assumption, u_σ is an element of the relative interior of σ . So $u_{\sigma_1} - u_{\sigma_0}$, $u_{\sigma_2} - u_{\sigma_1}$, \dots , $u_{\sigma_r} - u_{\sigma_{r-1}}$ are linearly independent and hence form a basis for $M_\mathbb{Q}$, which implies: $\{u_\sigma\}_{\sigma \in \mathfrak{C}}$ is a basis for $\hat{M}_\mathbb{Q}$. \square

2.2. Generalities on valuations

Definition 2.10. A quasi-valuation on $\mathbb{K}[\hat{X}_P]$ with values in a totally ordered abelian group G is a map $\nu : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow G$ which

- (i) has the minimum property, i.e., $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$;

- (ii) is not affected by scalar multiplication: $\nu(cf) = \nu(f)$ for all $c \in \mathbb{K}^*$;
- (iii) is quasi-additive, i.e. $\nu(fg) \geq \nu(f) + \nu(g)$.

We assume in all cases $f, g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$, and if appropriate, $f + g \neq 0$. The quasi-valuation ν is called *homogeneous* if $\nu(f^p) = p\nu(f)$ for all $p \in \mathbb{N}$ and $f \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$. The quasi-valuation ν is called a *valuation* if it is additive, i.e., $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$.

Given a quasi-valuation ν and $\underline{a} \in G$, the subset

$$\mathbb{K}[\hat{X}_P]_{\geq \underline{a}} = \{h \in \mathbb{K}[\hat{X}_P] \setminus \{0\} \mid \nu(h) \geq \underline{a}\}$$

is an ideal. The ideal $\mathbb{K}[\hat{X}_P]_{> \underline{a}}$ is defined similarly. The quotient $F_{\underline{a}} = \mathbb{K}[\hat{X}_P]_{\geq \underline{a}} / \mathbb{K}[\hat{X}_P]_{> \underline{a}}$ is called a *leaf* of the quasi-valuation. The direct sum of the leaves: $\bigoplus_{\underline{a} \in G} F_{\underline{a}}$ inherits an algebra structure and is called the *associated graded algebra*, denote it by $\text{gr}_{\nu} \mathbb{K}[\hat{X}_P]$.

Let $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$ be a combinatorial Seshadri stratification and fix a maximal chain $\mathfrak{C} = \{\sigma_r > \dots > \sigma_0\}$ in A . Let μ_r, \dots, μ_0 be the weights of the extremal functions $f_{\sigma_r}, \dots, f_{\sigma_0}$. By Lemma 2.9, these weights form a \mathbb{Q} -basis for $\hat{M}_{\mathbb{Q}}$. Let $\{e_{\sigma_r}, \dots, e_{\sigma_0}\}$ be the standard basis of $\mathbb{Q}^{\mathfrak{C}}$. We endow $\mathbb{Q}^{\mathfrak{C}}$ with the lexicographic order, i.e. $(a_r, \dots, a_0) > (b_r, \dots, b_0)$ if $a_r > b_r$ or $a_r = b_r$ and $a_{r-1} > b_{r-1}$ and so on. In this way $\mathbb{Q}^{\mathfrak{C}}$ becomes a totally ordered abelian group.

Definition 2.11. For a \hat{T} -eigenfunction $g \in \mathbb{K}[\hat{X}_P]$ of weight λ_g , let $\lambda_g = a_r \mu_r + \dots + a_0 \mu_0$ be the uniquely determined expression of λ as a linear combination in terms of the basis given by the weights $\{\mu_r, \dots, \mu_0\}$. We set $\nu_{\mathfrak{C}}(g) = a_r e_{\sigma_r} + \dots + a_0 e_{\sigma_0}$. For an arbitrary function $g \in \mathbb{K}[\hat{X}_P]$, let $g = g_{\eta_1} + \dots + g_{\eta_t}$ be a decomposition of g into \hat{T} -eigenfunctions of pairwise distinct weights $\eta_i, i = 1, \dots, t$. We define a map $\nu_{\mathfrak{C}} : \mathbb{K}[\hat{X}_P] \rightarrow \mathbb{Q}^{\mathfrak{C}}$ by:

$$\nu_{\mathfrak{C}}(g) = \min\{\nu_{\mathfrak{C}}(g_{\eta_i}) \mid 1 \leq i \leq t\}.$$

Example 2.12. If $\sigma \in \mathfrak{C}$, then $\nu_{\mathfrak{C}}(f_{\sigma}) = e_{\sigma}$.

Remark 2.13. The \hat{T} -weight spaces are one dimensional. If g is a \hat{T} -eigenfunction, then the value $\nu_{\mathfrak{C}}(g)$ on a \hat{T} -eigenvector depends only on the \hat{T} -weight of the function, and vice versa, the weight of g can be reconstructed from $\nu_{\mathfrak{C}}(g)$.

Lemma 2.14. The map $\nu_{\mathfrak{C}} : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow \mathbb{Q}^{\mathfrak{C}}$ is a valuation with at most one-dimensional leaves.

Proof. The map $\nu_{\mathfrak{C}}$ has the minimum property by construction, and it is not affected by non-zero scalar multiplication. If $g = g_{\eta_0} + \dots + g_{\eta_t}$ and $h = h_{\omega_0} + \dots + h_{\omega_s}$ are decompositions of g and h into pairwise distinct \hat{T} -eigenfunctions of weight $\eta_i, i = 0, \dots, t$, respectively $\omega_j, j = 0, \dots, s$, such that $\nu_{\mathfrak{C}}(g) = \nu_{\mathfrak{C}}(g_{\eta_0})$ and $\nu_{\mathfrak{C}}(h) = \nu_{\mathfrak{C}}(h_{\omega_0})$,

then gh is a sum of \hat{T} -eigenfunctions: $g_{\eta_i}h_{\nu_j}$, $i = 0, \dots, t$, $j = 0, \dots, s$, and $\nu_{\mathfrak{C}}(g_{\eta_i}h_{\omega_j}) = \nu_{\mathfrak{C}}(g_{\eta_i}) + \nu_{\mathfrak{C}}(h_{\omega_j})$. It follows: $\nu_{\mathfrak{C}}(gh) = \nu_{\mathfrak{C}}(g_{\eta_0}) + \nu_{\mathfrak{C}}(h_{\omega_0}) = \nu_{\mathfrak{C}}(g) + \nu_{\mathfrak{C}}(h)$, so the map is additive and hence a valuation. Positive dimensional leaves are just \hat{T} -weight spaces by Remark 2.13, so they are one-dimensional. \square

We extend the valuation to $\nu_{\mathfrak{C}} : \mathbb{K}(\hat{X}) \setminus \{0\} \rightarrow \mathbb{Q}^{\mathfrak{C}}$ by $\nu_{\mathfrak{C}}(\frac{g}{h}) = \nu_{\mathfrak{C}}(g) - \nu_{\mathfrak{C}}(h)$. Denote by $F_{\mathfrak{C}}$ the product $\prod_{\sigma \in \mathfrak{C}} f_{\sigma}$ and let $\mathbb{K}[\hat{X}_P]_{F_{\mathfrak{C}}} \subseteq \mathbb{K}(\hat{X})$ be the localization of the homogeneous coordinate ring. It is the coordinate ring of $U_{F_{\mathfrak{C}}} = \{x \in \hat{X}_P \mid F_{\mathfrak{C}}(x) \neq 0\}$, an affine toric \hat{T} -variety. Note that the valuation may have negative entries.

Proposition 2.15. *Let $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$. For $\nu_{\mathfrak{C}}(g) = (a_{\sigma})_{\sigma \in \mathfrak{C}}$ let $k > 0$ be a positive integer such that $ka_{\sigma} \in \mathbb{Z}$, $\sigma \in \mathfrak{C}$. In $\mathbb{K}[\hat{X}_P]_{F_{\mathfrak{C}}}$ one has: $g^k = c \prod_{\sigma \in \mathfrak{C}} f_{\sigma}^{ka_{\sigma}} + h$, where $c \in \mathbb{K}^*$ and either $h = 0$, or $\nu_{\mathfrak{C}}(h) > (ka_{\sigma})_{\sigma \in \mathfrak{C}}$.*

Proof. Let $g = g_{\eta_0} + \dots + g_{\eta_t}$ be a decomposition of g into \hat{T} -eigenfunctions of pairwise different weights η_i , $i = 0, \dots, t$. Without loss of generality we assume $\nu_{\mathfrak{C}}(g) = \nu_{\mathfrak{C}}(g_{\eta_0})$. It follows that $\nu_{\mathfrak{C}}(g^k) = k\nu_{\mathfrak{C}}(g_{\eta_0})$. This implies that $g_{\eta_0}^k$ and $\prod_{\sigma \in \mathfrak{C}} f_{\sigma}^{ka_{\sigma}}$ are \hat{T} -eigenfunctions with the same valuation and hence the same \hat{T} -weight. So in $\mathbb{K}[\hat{X}_P]_{F_{\mathfrak{C}}}$ they must be nonzero scalar multiples of each other. The value of $\nu_{\mathfrak{C}}$ on the remaining summands in $g^k = (\sum_{i=0}^t g_{\eta_i})^k$ is strictly larger than $\nu_{\mathfrak{C}}(g_{\eta_0}^k)$, which proves the claim. \square

As an immediate consequence of Proposition 2.15 we see:

Corollary 2.16. *Let $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ be a \hat{T} -eigenfunction of \hat{T} -weight λ_g and suppose $\nu_{\mathfrak{C}}(g) = (a_{\sigma})_{\sigma \in \mathfrak{C}}$. If k is as in Proposition 2.15, then $g^k = c \prod_{\sigma \in \mathfrak{C}} f_{\sigma}^{ka_{\sigma}}$ in $\mathbb{K}[\hat{X}_P]_{F_{\mathfrak{C}}}$ for some $c \in \mathbb{K}^*$, $\deg g = \sum_{\sigma \in \mathfrak{C}} a_{\sigma} \deg f_{\sigma}$ and $\lambda_g = \sum_{\sigma \in \mathfrak{C}} a_{\sigma} \mu_{\sigma}$.*

We get also some information about the value of $\nu_{\mathfrak{C}}(f_{\tau})$ for $\tau \notin \mathfrak{C}$:

Corollary 2.17. *For $\tau \in A$, $\tau \notin \mathfrak{C}$, let $\nu_{\mathfrak{C}}(f_{\tau}) = (a_{\sigma_i})_{0 \leq i \leq r}$. If $\sigma_j \in \mathfrak{C}$ is minimal such that $\sigma_j > \tau$, then $a_{\sigma_j} > 0$ and $a_{\sigma_i} = 0$ for $i > j$.*

Proof. Let $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$. By assumption, we have $\frac{-\mu_{\tau}}{\deg f_{\tau}} \in \tau^{\circ} \subseteq \sigma_j$, which implies: $\frac{-\mu_{\tau}}{\deg f_{\tau}}$ is in the linear \mathbb{Q} -span of the $\{\frac{\mu_{\sigma_i}}{\deg f_{\sigma_i}} \mid i = 0, \dots, j\}$, and hence $a_{\sigma_i} = 0$ for $i > j$. If $a_{\sigma_j} \leq 0$, then let $k > 0$ be as in Corollary 2.16 and set $N \gg 0$. By *ibid.*, the regular functions $f_{\tau}^k f_{\sigma_j}^{|ka_{\sigma_j}|} (\prod_{i=0, \dots, j-1} f_{\sigma_i})^N$ and $\prod_{i=0, \dots, j-1} f_{\sigma_i}^{ka_{\sigma_i} + N}$ have the same \hat{T} -weight, so they are equal in $\mathbb{K}[\hat{X}_P]$ up to some nonzero scalar multiple. But this is not possible: the first function vanishes identically on $X_{\sigma_{j-1}}$ because $\sigma_{j-1} \not\geq \tau$ (Lemma 2.7), but the second not. It follows: $a_{\sigma_j} > 0$. \square

2.3. The valuation monoid

The additivity of $\nu_{\mathfrak{C}}$ implies that

$$\mathbb{V}_{\mathfrak{C}}(\hat{X}_P) = \{\nu_{\mathfrak{C}}(h) \mid h \in \mathbb{K}[\hat{X}_P] \setminus \{0\}\} \subseteq \mathbb{Q}^{\mathfrak{C}}$$

is a submonoid of $\mathbb{Q}^{\mathfrak{C}}$, called the *valuation monoid*. For an arbitrary embedded projective variety $Y \subseteq \mathbb{P}(V)$ a valuation monoid $\mathbb{V}(\hat{Y})$ is typically used to construct (if possible) a flat degeneration of Y into a toric variety, having the valuation monoid as weight monoid. So it is no surprise that in the case where $Y = X_P$ is a toric variety we do not get anything really new. By Lemma 1.4 and Corollary 2.16 we see:

Corollary 2.18. *Let μ_r, \dots, μ_0 be the weights of the extremal functions $f_{\sigma_r}, \dots, f_{\sigma_0}$. The map $\psi_{\mathfrak{C}} : \mathbb{V}_{\mathfrak{C}}(\hat{X}_P) \rightarrow S, (a_r, \dots, a_0) \mapsto \sum_{j=0}^r a_j \mu_j$, is an isomorphism of monoids.*

A new point of view comes in Section 4. Let us just remark that for every maximal chain we have a special submonoid: $\mathbb{V}_{\mathfrak{C}}(\hat{X}_P)^+ = \mathbb{V}_{\mathfrak{C}}(\hat{X}_P) \cap \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$, the intersection of the valuation monoid with the positive orthant. We will see that the images $\psi_{\mathfrak{C}}(\mathbb{V}_{\mathfrak{C}}(\hat{X}_P)^+) \subseteq S$ define a decomposition of S as \mathfrak{C} is running over all maximal chains in A .

Example 2.19. Let X_P the toric variety with the combinatorial Seshadri stratification of Example 2.2. Then, using Corollary 2.18, is easy to see that, for every maximal chain \mathfrak{C} , the valuation monoid $\mathbb{V}_{\mathfrak{C}}(\hat{X}_P)$ is the monoid

$$\{(a_2, a_1, a_0) \in \mathbb{Q}^3 \mid 2a_2 \in \mathbb{Z}, a_1 \in \mathbb{Z}, a_0 \in \mathbb{Z}\}.$$

Example 2.20. Let X_P the toric variety with the combinatorial Seshadri stratification of Example 2.4. By Corollary 2.18 one can see that, for every maximal chain \mathfrak{C} , the monoid $\mathbb{V}_{\mathfrak{C}}(\hat{X}_P)$ is \mathbb{Z}^3 .

3. Markings and triangulations of P indexed by flags

A flag of faces in A is a chain in A , i.e. a totally ordered subset of the form $\sigma_1 \subsetneq \dots \subsetneq \sigma_s$, where the $\sigma_i \in A, i = 1, \dots, s$, are faces of P . Let $\mathcal{F}(A)$ be the set of all flags in A , i.e. the set of all totally ordered subsets of A .

Definition 3.1. A *triangulation of P indexed by flags of faces* is a triangulation $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$ of P with rational vertices and simplices Δ_C indexed by $\mathcal{F}(A)$, such that

- (1) the relative interior of every face $\sigma \in A$ contains exactly one vertex $\mathbf{v}_{\sigma} \in M_{\mathbb{Q}}$ of \mathcal{T} ,
- (2) Δ_C is the convex hull of the $\mathbf{v}_{\sigma}, \sigma \in C$.

Example 3.2. The barycentric subdivision of P is a triangulation of P indexed by flags.

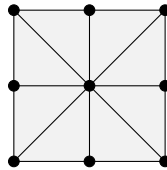
Given a triangulation indexed by flags $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$, the collection $(\mathbf{v}_\sigma)_{\sigma \in A}$ of vertices defines a marking $\mathbf{m}_{\mathcal{T}}$ of the faces of P . Vice versa, let $\mathbf{m} = (u_\sigma)_{\sigma \in A}$ be a marking of the faces. For a flag $C \in \mathcal{F}(A)$ let $\Delta_C \subseteq P$ be the convex hull of the u_σ , $\sigma \in C$. By Lemma 2.9, the u_σ , $\sigma \in C$, are linearly independent and hence Δ_C is a simplex of dimension $|C| - 1$. We attach to a marking of the faces $\mathbf{m} = (u_\sigma)_{\sigma \in A}$ the collection of simplices $\mathcal{T}_{\mathbf{m}} = (\Delta_C)_{C \in \mathcal{F}(A)}$.

Lemma 3.3. *The collection of simplices $\mathcal{T}_{\mathbf{m}} = (\Delta_C)_{C \in \mathcal{F}(A)}$ is a triangulation of P .*

Proof. This is evident for $r = 0$. The general case follows by induction on the dimension.

For each facet of P , we have a marking that, by induction, gives a triangulation of the facet. The collection of simplices $\mathcal{T}_{\mathbf{m}}$ is the triangulation given by the cones with vertex u_P over these triangulations of the facets. \square

Example 3.4. Let P be the polytope as in Example 2.4. Let \mathbf{m} be the marking associated with the extremal functions given in Example 2.4. The triangulation associated with \mathbf{m} is



Summarizing we get together with Lemma 2.8:

Theorem 3.5. *There exists a bijection between the set of equivalence classes of combinatorial Seshadri stratifications and the set of triangulations of P indexed by flags of faces.*

4. A higher rank quasi-valuation

Let $(X_\sigma, f_\sigma)_{\sigma \in A}$, be a combinatorial Seshadri stratification on $X_P \subseteq \mathbb{P}(V)$ and denote by $\mathcal{F}_{\max}(A)$ the set of all maximal chains in A . Let \mathbb{Q}^A be the vector space with the standard basis $\{e_\tau \mid \tau \in A\}$. Given a maximal chain $\mathfrak{C} \in \mathcal{F}_{\max}(A)$, we identify the vector space $\mathbb{Q}^{\mathfrak{C}}$ with the subspace of \mathbb{Q}^A spanned by the basis elements e_τ , $\tau \in \mathfrak{C}$. So we view the valuation $\nu_{\mathfrak{C}}$ defined in Definition 2.11 as a map $\nu_{\mathfrak{C}} : \mathbb{K}(\hat{X}_P) \setminus \{0\} \rightarrow \mathbb{Q}^A$, such that the image lies in the subspace $\mathbb{Q}^{\mathfrak{C}} \subseteq \mathbb{Q}^A$.

We fix on A a linearization “ $>^t$ ” of the partial order on A , i.e. “ $>^t$ ” is a total order on A such that $\tau > \sigma$ for $\tau, \sigma \in A$ implies $\tau >^t \sigma$. We get on \mathbb{Q}^A a total order by taking the induced lexicographic order, which makes \mathbb{Q}^A into a totally ordered abelian group.

It is well known that the minimum function applied to a finite family of quasi-valuations is a quasi-valuation (see, for example, [6]).

Definition 4.1. The quasi-valuation $\nu : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow \mathbb{Q}^A$ associated to the combinatorial Seshadri stratification $(X_\sigma, f_\sigma)_{\sigma \in A}$ and the fixed total order $>^t$ on A is the map defined by:

$$g \mapsto \nu(g) := \min\{\nu_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{F}_{\max}(A)\}.$$

Since $\nu_{\mathfrak{C}}(g^k) = k\nu_{\mathfrak{C}}(g)$ for all $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$, $k \in \mathbb{N}$, and all maximal chains in A , the quasi-valuation ν is *homogenous*: $\nu(g^k) = k\nu(g)$ for $k \in \mathbb{N}$.

Remark 4.2. The quasi-valuation ν depends on the choice of the linearization “ $>^t$ ”, so it would be more apt to write $\nu_{>^t}$ and to add “ $>^t$ ” to the objects defined in the following. To avoid an excess of indexing we stick to the notation above. In addition, many of these objects turn out to be essentially independent of the choice of “ $>^t$ ”.

4.1. First properties

Let $\Gamma := \{\nu(g) \mid g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}\} \subseteq \mathbb{Q}^A$ be the image of the quasi-valuation. For $\underline{a} = \sum_{\tau \in A} a_\tau e_\tau \in \mathbb{Q}^A$, denote by $\text{supp } \underline{a} := \{\tau \in A \mid a_\tau \neq 0\}$ the support of \underline{a} . By construction, the support of $\underline{a} \in \Gamma$ is always contained in some maximal chain \mathfrak{C} .

Lemma 4.3.

- a) If $g = g_{\eta_1} + \dots + g_{\eta_q}$ is a decomposition of $g \in \mathbb{K}[\hat{X}_P]$ into \hat{T} -eigenvectors, then $\nu(g) = \min\{\nu(g_{\eta_j}) \mid j = 1, \dots, q\}$.
- b) $\Gamma := \{\nu(g) \mid g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}, g \text{ is a } \hat{T}\text{-eigenfunction}\}$.
- c) $\nu(f_\tau) = e_\tau$ for all extremal functions f_τ , $\tau \in A$, of the Seshadri stratification, independent of the choice of the linearization “ $>^t$ ”.

Proof. Let $g = g_{\eta_1} + \dots + g_{\eta_q}$ be a decomposition of $g \in \mathbb{K}[\hat{X}_P]$ into \hat{T} -eigenvectors. By definition we have

$$\nu(g) = \min\{\min\{\nu_{\mathfrak{C}}(g_{\eta_i}) \mid i = 1, \dots, t\} \mid \mathfrak{C} \in \mathcal{F}_{\max}(A)\} = \min\{\nu(g_{\eta_j}) \mid j = 1, \dots, q\},$$

which proves a) and b). By Example 2.12 we know $\nu_{\mathfrak{C}}(f_\tau) = e_\tau$ if $\tau \in \mathfrak{C}$, and Corollary 2.17 implies $\nu_{\mathfrak{C}}(f_\tau) >^t e_\tau$ if $\tau \notin \mathfrak{C}$, independent of the choice of the linearization. \square

Lemma 4.4. For a product $\prod_{\sigma \in A} f_\sigma^{m_\sigma}$ of extremal functions we have $\nu(\prod_{\sigma \in A} f_\sigma^{m_\sigma}) \geq^t \sum_{\sigma \in A} m_\sigma e_\sigma$, where equality holds if and only if there exists a maximal chain $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$ such that $\{\sigma \in A \mid m_\sigma > 0\} \subseteq \mathfrak{C}$. If $\nu(\prod_{\sigma \in A} f_\sigma^{m_\sigma}) = \sum_{\sigma \in A} m_\sigma e_\sigma$, then the equality holds independent of the choice of the linearization “ $>^t$ ”.

Proof. The quasi-additivity and the homogeneity of a quasi-valuation implies immediately $\nu(\prod_{\sigma \in A} f_\sigma^{m_\sigma}) \geq^t \sum_{\sigma \in A} m_\sigma \nu(f_\sigma) = \sum_{\sigma \in A} m_\sigma e_\sigma$. So this is a lower bound for

$\nu_{\mathfrak{C}}(\prod_{\sigma \in A} f_{\sigma}^{m_{\sigma}})$ for all maximal chains \mathfrak{C} , and this bound is independent of the choice of the linearization.

If there exists a maximal chain such that $\{\sigma \in A \mid m_{\sigma} > 0\} \subseteq \mathfrak{C}$, then the additivity of a valuation implies: $\nu_{\mathfrak{C}}(\prod_{\sigma \in A} f_{\sigma}^{m_{\sigma}}) = \sum_{\sigma \in \mathfrak{C}} m_{\sigma} \nu_{\mathfrak{C}}(f_{\sigma}) = \sum_{\sigma \in \mathfrak{C}} m_{\sigma} e_{\sigma}$, and hence we have equality also for the quasi-valuation, independent of the choice of the linearization.

Now let $\mathfrak{C}' = (\tau_r, \dots, \tau_0)$ be a maximal chain such that $\{\sigma \in A \mid m_{\sigma} > 0\} \not\subseteq \mathfrak{C}'$. We proceed by induction on $\#\{\sigma \in A \setminus \mathfrak{C}' \mid m_{\sigma} > 0\}$. If the number is equal to zero, there is nothing to prove: $\nu_{\mathfrak{C}'}(\prod_{\sigma \in A} f_{\sigma}^{m_{\sigma}}) = \sum_{\sigma \in \mathfrak{C}'} m_{\sigma} e_{\sigma}$.

Suppose now $\#\{\sigma \in A \setminus \mathfrak{C}' \mid m_{\sigma} > 0\} = q \geq 1$ and let $\kappa \in A \setminus \mathfrak{C}'$ be such that $m_{\kappa} > 0$. We assume by induction $\nu_{\mathfrak{C}'}(\prod_{\sigma \in A \setminus \{\kappa\}} f_{\sigma}^{m_{\sigma}}) \geq^t \sum_{\sigma \in A \setminus \{\kappa\}} m_{\sigma} e_{\sigma}$. The additivity of a valuation, induction and Corollary 2.17 implies:

$$\begin{aligned} \nu_{\mathfrak{C}'}(\prod_{\sigma \in A} f_{\sigma}^{m_{\sigma}}) &= \nu_{\mathfrak{C}'}(\prod_{\sigma \in A \setminus \{\kappa\}} f_{\sigma}^{m_{\sigma}}) + \nu_{\mathfrak{C}'}(f_{\kappa}^{m_{\kappa}}) \\ &\geq^t \sum_{\sigma \in A \setminus \{\kappa\}} m_{\sigma} e_{\sigma} + \nu_{\mathfrak{C}'}(f_{\kappa}^{m_{\kappa}}) \\ &>^t \sum_{\sigma \in A} m_{\sigma} e_{\sigma}. \end{aligned}$$

It follows, independently of the choice of the linearization: if $\{\sigma \in A \mid m_{\sigma} > 0\} \not\subseteq \mathfrak{C}'$, then $\nu_{\mathfrak{C}'}(\prod_{\sigma \in A} f_{\sigma}^{m_{\sigma}}) >^t \sum_{\sigma \in A} m_{\sigma} e_{\sigma}$, which finishes the proof of the lemma. \square

4.2. The quasi-valuation and weight combinatorics

Let $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$ be a combinatorial Seshadri stratification on $X_P \subseteq \mathbb{P}(V)$. Denote by $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$ the triangulation of P indexed by flags associated to the equivalence class of $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$ (Theorem 3.5). Given a \hat{T} -eigenfunction $g \in \mathbb{K}[X]$ of weight λ_g , the triangulation suggests to us a preferred class of maximal chains in A : the maximal chains \mathfrak{C} such that $\frac{-\lambda_g}{\deg g} \in \Delta_{\mathfrak{C}}$. Indeed:

Proposition 4.5. *Let $g \in \mathbb{K}[X_P] \setminus \{0\}$ be a \hat{T} -eigenfunction of weight λ_g and let $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$ be a maximal chain in A . The following are equivalent:*

- i) $\nu(g) = \nu_{\mathfrak{C}}(g)$;
- ii) $\frac{-\lambda_g}{\deg g} \in \Delta_{\mathfrak{C}}$;
- iii) $\nu_{\mathfrak{C}}(g) \in \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$.

In particular, $\nu(g) \in \mathbb{Q}_{\geq 0}^A$, and $\nu(g)$ is independent of the choice of the linearization “ $>^t$ ”.

Proof. If $\frac{-\lambda_g}{\deg g} \in \Delta_{\mathfrak{C}}$, then λ_g is a non-negative linear combination of the weights μ_{σ_i} , $i = 0, \dots, r$, which implies $\nu_{\mathfrak{C}}(g) \in \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$. If $\nu_{\mathfrak{C}}(g) = (a_r, \dots, a_0) \in \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$, then by Corollary 2.16:

$$\frac{-\lambda_g}{\deg g} = \frac{-1}{\sum_{j=0}^r a_j \deg f_{\sigma_j}} \left(\sum_{i=0}^r a_i \mu_{\sigma_i} \right) = \sum_{i=0}^r \left(\frac{a_i \deg f_{\sigma_i}}{\sum_{j=0}^r a_j \deg f_{\sigma_j}} \right) \frac{-\mu_{\sigma_i}}{\deg f_{\sigma_i}} \in \Delta_{\mathfrak{C}},$$

which shows the equivalence of *ii*) and *iii*).

Given a \hat{T} -eigenfunction $g \in \mathbb{K}[X_P] \setminus \{0\}$ of weight λ_g , fix a maximal chain $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$ such that $\frac{-\lambda_g}{\deg g} \in \Delta_{\mathfrak{C}}$. It follows that $\nu_{\mathfrak{C}}(g) = (a_r, \dots, a_0)$ consists only of non-negative numbers. So if we fix k as in Corollary 2.16, then the equality $g^k = c f_{\sigma_r}^{ka_r} \dots f_{\sigma_0}^{ka_0}$ for some $c \in \mathbb{K}^*$ holds in $\mathbb{K}[\hat{X}_P]$, and hence

$$\nu(g) = \frac{1}{k} \nu(f_{\sigma_r}^{ka_r} \dots f_{\sigma_0}^{ka_0}) = \sum_{j=0}^r a_j e_{\sigma_j} = \nu_{\mathfrak{C}}(g)$$

by Lemma 4.4. This proves *ii*) implies *i*), and $\nu(g) \in \mathbb{Q}_{\geq 0}^A$ is independent of the choice of “ $>^t$ ”. Suppose $\mathfrak{C} = (\tau_r, \dots, \tau_0)$ is a maximal chain such that $\nu(g) = \nu_{\mathfrak{C}}(g)$. By the above we know $\nu(g) \in \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$, and hence $\nu_{\mathfrak{C}}(g) \in \mathbb{Q}_{\geq 0}^{\mathfrak{C}}$ which shows *i*) implies *iii*). \square

Let $g \in \mathbb{K}[X_P] \setminus \{0\}$ be a \hat{T} -eigenfunction of weight λ_g and let $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$ be a maximal chain in A such that $\frac{-\lambda_g}{\deg g} \in \Delta_{\mathfrak{C}}$. The value of the quasi-valuation in g is completely determined by its \hat{T} -weight:

Corollary 4.6. *If $\frac{-\lambda_g}{\deg g} = \sum_{\sigma \in \mathfrak{C}} a_{\sigma} \frac{-\mu_{\sigma}}{\deg f_{\sigma}}$ is an expression of $\frac{-\lambda_g}{\deg g}$ as a convex linear combination of the vertices of $\Delta_{\mathfrak{C}}$, then $\nu(g) = \sum_{\sigma \in \mathfrak{C}} \frac{a_{\sigma} \deg g}{\deg f_{\sigma}} e_{\sigma}$*

One can go the other way round too: the quasi-valuation determines the weight and the degree. Let $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ be a \hat{T} -eigenfunction of weight λ_g . Since $\nu(g) = \nu_{\mathfrak{C}}(g)$ for some maximal chain, we can apply Corollary 2.16 and get:

Corollary 4.7. *If $\nu(g) = (a_{\sigma})_{\sigma \in A}$, then $\deg g = \sum_{\sigma \in A} a_{\sigma} \deg f_{\sigma}$ and $\lambda_g = \sum_{\sigma \in A} a_{\sigma} \mu_{\sigma}$, and there exists a positive integer $k > 0$ and $c \in \mathbb{K}^*$ such that $g^k = c \prod_{\sigma \in A} f_{\sigma}^{ka_{\sigma}}$.*

4.3. Two fans of monoids

The explicit description of the quasi-valuation in Corollary 4.6 in terms of weight combinatorics and triangulations makes it possible to give a similar description of the image of the quasi-valuation $\Gamma := \{\nu(g) \mid g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}\} \subseteq \mathbb{Q}^A$. A quasi-valuation is only quasi-additive: $\nu(gh) \geq^t \nu(g) + \nu(h)$, so, in general, Γ is not a monoid.

Lemma 4.8. *If $g, h \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ are \hat{T} -eigenfunctions of weight λ_g respectively λ_h , then $\nu(gh) = \nu(g) + \nu(h)$ if and only if there exists a maximal chain \mathfrak{C} such that $\frac{-\lambda_g}{\deg g}, \frac{-\lambda_h}{\deg h} \in \Delta_{\mathfrak{C}}$.*

Proof. The quasi-valuation is homogeneous, so we can replace g and h by a k -th power, where k is chosen as in Corollary 4.7, i.e. we can replace without loss of generality g and h by products of extremal functions, say $g = c_1 \prod_{\sigma \in \text{supp } \nu(g)} f_{\sigma}^{m_{\sigma}}$, $h = c_2 \prod_{\tau \in \text{supp } \nu(h)} f_{\tau}^{n_{\tau}}$. Here $c_1, c_2 \in \mathbb{K}^*$.

Now the condition $\frac{-\lambda_h}{\deg h}, \frac{-\lambda_g}{\deg h} \in \Delta_{\mathfrak{C}}$ is by Proposition 4.5 equivalent to the existence of a maximal chain \mathfrak{C} such that $\text{supp } \nu(g), \text{supp } \nu(h) \subseteq \mathfrak{C}$, so we can apply Lemma 4.4 to the product of extremal functions, which finishes the proof. \square

Let $g, h \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$. After rewriting both as a sum of \hat{T} -eigenvectors, one concludes by Lemma 4.8:

Corollary 4.9. $\nu(gh) = \nu(g) + \nu(h)$ if and only if there exists a maximal chain \mathfrak{C} such that $\text{supp } \nu(g), \text{supp } \nu(h) \subseteq \mathfrak{C}$.

Recall that $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$ denotes the triangulation of P indexed by flags associated to the equivalence class of $(X_\sigma, f_\sigma)_{\sigma \in A}$. For a chain $C \in \mathcal{F}(A)$ let $K(\Delta_C)$ be the cone over Δ_C and set $S_C = S \cap K(\Delta_C)$. The union of the cones $K(\Delta_C)$, $C \in \mathcal{F}(A)$, (together with the origin $\{0\}$ as cone over Δ_C for C the empty chain) form a fan. In the same way the union of the S_C , $C \in \mathcal{F}(A)$, forms a fan of monoids, i.e. for all $C, C', C'' \in \mathcal{F}(A)$ one has: $S_C \subseteq S_{C'}$ if and only if $C \subseteq C'$, and $S_C \cap S_{C'} = S_{C''}$, where $C'' = C \cap C'$. We write $S_{\mathcal{T}}$ for this fan of monoids. As a set one has $S_{\mathcal{T}} = S$, but as operation $\hat{+}$ we have: $\lambda \hat{+} \eta = \lambda + \eta$ if there exists a chain such that $\lambda, \eta \in S_C$, and $\lambda \hat{+} \eta$ is not defined otherwise.

The quasi-valuation provides a similar construction in \mathbb{Q}^A . For a chain $C \in \mathcal{F}(A)$ we replace the cone $K(\Delta_C) \subseteq \hat{M}_{\mathbb{Q}}$ by the cone $K_C \subseteq \mathbb{R}^A$ spanned by the basis vectors $\{e_\sigma \mid \sigma \in C\}$. The collection of cones $\{K_C \mid C \in \mathcal{F}(A)\}$ defines a fan in \mathbb{R}^A . Its maximal cones are the cones $K_{\mathfrak{C}}$ associated to the maximal chains \mathfrak{C} in A . For a chain $C \in \mathcal{F}(A)$ denote by Γ_C the subset $\Gamma_C = \{\underline{a} \in \Gamma \mid \text{supp } \underline{a} \subseteq C\} \subseteq K_C$. By Lemma 4.8, Γ_C has a natural structure as a monoid, which makes $\Gamma = \bigcup_{C \in \mathcal{F}(A)} \Gamma_C$ into a fan of monoids.

Theorem 4.10.

- i) For all $C \in \mathcal{F}(A)$, Γ_C is a finitely generated monoid.
- ii) The image of the quasi-valuation $\Gamma := \{\nu(g) \mid g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}\}$ is a fan of finitely generated monoids.
- iii) Γ is, as a fan of monoids, isomorphic to $S_{\mathcal{T}}$. Γ is independent of the choice of “ $>^t$ ”, and equivalent Seshadri stratifications yield isomorphic fans of monoids.

Proof. The natural map $\Gamma \rightarrow S$, which sends a tuple $\underline{a} \in \Gamma$ to the weight $\sum_{\sigma \in A} a_\sigma \mu_\sigma$, is by Corollary 4.6 and Corollary 4.7 a bijection. By Lemma 4.8, it is a morphism between the fans of monoids Γ and $S_{\mathcal{T}}$. Since $S_{\mathcal{T}}$ depends only on the triangulation \mathcal{T} , this proves *iii)* by Theorem 3.5.

Since $K(\Delta_C)$ is a rational polyhedral cone, the monoid $S_C = K(\Delta_C) \cap \hat{M} = K(\Delta_C) \cap S$ is finitely generated (by Gordan’s Lemma). The isomorphism above sends Γ_C onto S_C , which implies that Γ_C is finitely generated, which proves *i)* and *ii)*. \square

4.4. The associated graded algebra and the fan algebra

Definition 4.11. The fan algebra $\mathbb{K}[\Gamma]$ associated to the fan of monoids Γ is defined as $\mathbb{K}[\Gamma] := \mathbb{K}[y_{\underline{a}} \mid \underline{a} \in \Gamma]/I(\Gamma)$, where $I(\Gamma)$ is the ideal generated by the following elements:

- a) $y_{\underline{a}} \cdot y_{\underline{b}} - y_{\underline{a+b}}$ if there exists a chain $C \subset A$ such that $\underline{a}, \underline{b} \in K_C \subseteq \mathbb{Q}^A$,
- b) $y_{\underline{a}} \cdot y_{\underline{b}}$ if there exists no such a chain.

To simplify the notation, we will write $y_{\underline{a}}$ also for its class in $\mathbb{K}[\Gamma]$. For a chain C let $\mathbb{K}[\Gamma_C]$ be the subalgebra: $\mathbb{K}[\Gamma_C] := \bigoplus_{\underline{a} \in \Gamma_C} \mathbb{K}y_{\underline{a}} \subseteq \mathbb{K}[\Gamma]$. The algebra $\mathbb{K}[\Gamma_C]$ is naturally isomorphic to the usual semigroup algebra associated to the monoid Γ_C .

We endow the algebra $\mathbb{K}[\Gamma]$ with a \mathbb{N} -grading inspired by Corollary 4.7: for $\underline{a} \in \mathbb{Q}^A$, the degree of $y_{\underline{a}}$ is defined by: $\deg y_{\underline{a}} = \sum_{\sigma \in A} a_{\sigma} \deg f_{\sigma}$.

The quasi-valuation defines a filtration on $\mathbb{K}[\hat{X}_P]$ given by ideals. We set for $\underline{a} \in \Gamma$:

$$\mathbb{K}[\hat{X}_P]_{\geq t_{\underline{a}}} = \{g \in \mathbb{K}[\hat{X}_P] \mid \nu(g) \geq t_{\underline{a}}\}, \quad \mathbb{K}[\hat{X}_P]_{> t_{\underline{a}}} = \{g \in \mathbb{K}[\hat{X}_P] \mid \nu(g) > t_{\underline{a}}\}.$$

Denote by $\text{gr}_{\nu} \mathbb{K}[\hat{X}_P] = \bigoplus_{\underline{a} \in \Gamma} \mathbb{K}[\hat{X}_P]_{\geq t_{\underline{a}}}/\mathbb{K}[\hat{X}_P]_{> t_{\underline{a}}}$ the associated graded algebra.

Theorem 4.12. The associated graded algebra $\text{gr}_{\nu} \mathbb{K}[\hat{X}_P]$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$. In particular, it is independent of the choice of the linearization and it depends, up to isomorphism, only on the triangulation \mathcal{T} associated to the equivalence class of the Seshadri stratification $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$.

The variety $X_0 = \text{Proj}(\mathbb{K}[\Gamma])$ is reduced, it is the irredundant union of the toric varieties $X_{\mathfrak{C}} = \text{Proj}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$, where \mathfrak{C} is running over the set of all maximal chains in A . The variety is equidimensional, all irreducible components of X_0 have same dimension as X_P .

Proof. The classes $\{\bar{f}_{m,\eta} \mid (m,\eta) \in S\}$ of the basis elements of $\mathbb{K}[\hat{X}_P]$ (Lemma 1.4) form a basis for the associated graded algebra $\text{gr}_{\nu} \mathbb{K}[\hat{X}_P]$. We have a natural map π between the basis of $\mathbb{K}[\Gamma]$ and the basis of $\text{gr}_{\nu} \mathbb{K}[\hat{X}_P]$: it sends $y_{\underline{a}}, \underline{a} \in \Gamma$ to $\bar{f}_{m,\eta}$, where $(m,\eta) = \sum_{\sigma \in A} a_{\sigma} \mu_{\sigma}$. This map extends linearly to a vector space isomorphism $\pi : \mathbb{K}[\Gamma] \rightarrow \text{gr}_{\nu} \mathbb{K}[\hat{X}_P]$, which by Lemma 4.8 is an algebra isomorphism.

The algebra $\mathbb{K}[\Gamma]$ has no nilpotent elements, so $X_0 = \text{Proj}(\mathbb{K}[\Gamma])$ is reduced. Set $y_{\mathfrak{C}} = \prod_{\sigma \in \mathfrak{C}} y_{e_{\sigma}}$ and let $I_{\mathfrak{C}}$ be the annihilator of $y_{\mathfrak{C}}$ in $\mathbb{K}[\Gamma]$. The quotient $\mathbb{K}[\Gamma]/I_{\mathfrak{C}}$ is isomorphic to $\mathbb{K}[\Gamma_{\mathfrak{C}}]$, an algebra which has no zero-divisors. Hence $I_{\mathfrak{C}}$ is a prime ideal. It also follows that the intersection $\bigcap_{\mathfrak{C} \in \mathcal{F}_{\max}(A)} I_{\mathfrak{C}} = (0)$ is the zero ideal.

The ideal $I_{\mathfrak{C}}$ is a minimal prime ideal: suppose $I \subsetneq I_{\mathfrak{C}}$ is an ideal. Then $\mathbb{K}[\Gamma]/I$ contains an element $\bar{g} \neq 0$ such that $\nu(g) \notin \Gamma_{\mathfrak{C}}$, and hence $y_{\mathfrak{C}}\bar{g} = 0$ by Corollary 4.9. So the quotient has a zero divisor and I is hence not a prime ideal. It follows that $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}} = (0)$ is the minimal prime decomposition of the zero ideal in $\mathbb{K}[\Gamma]$. For a maximal chain \mathfrak{C}' , $y_{\mathfrak{C}'}$ is a non-zero element in the intersection $\bigcap_{\mathfrak{C} \neq \mathfrak{C}'} I_{\mathfrak{C}}$. This shows that the intersection $\bigcap_{\mathfrak{C}} I_{\mathfrak{C}}$ is non-redundant.

An irreducible component of X_0 is hence isomorphic to $X_{\mathfrak{C}} = \text{Proj}(\mathbb{K}[\Gamma_{\mathfrak{C}}])$ for some maximal chain $\mathfrak{C} = (\tau_r, \dots, \tau_0)$. By definition, the functions $y_{e_{\tau_i}}, 0 \leq i \leq r$, are algebraically independent and all other functions $y_{\underline{a}}, \underline{a} \in \Gamma_{\mathfrak{C}}$ depend algebraically on these functions. It follows $\dim X_{\mathfrak{C}} = \dim \text{Proj}(\mathbb{K}[\Gamma_{\mathfrak{C}}]) = r = \dim X_P$. \square

5. A flat degeneration induced by a \mathbb{G}_m -action

Let $\mathbf{G}_1 = \{f_{1,\chi_1}, \dots, f_{1,\chi_r}\}$ be the degree 1 elements in the basis (see Lemma 1.4) of $\mathbb{K}[\hat{X}_P]$. They generate $\mathbb{K}[\hat{X}_P]$, but, in general, their classes $\bar{f}_{1,\chi_1}, \dots, \bar{f}_{1,\chi_r}$ do not generate $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$. So to describe a flat degeneration $X_P \rightsquigarrow X_0$, we replace the given embedding $\hat{\iota} : \hat{X}_P \hookrightarrow V$ (see Section 1) by an embedding $\hat{X}_P \hookrightarrow V \oplus U$ into a larger space.

Example 5.1. We take the same polytope $P \subseteq \mathbb{R}^2$ and lattice M as in Example 2.4, with the same marking except for the edge σ joining the vertices $(0, 2)$ and $(2, 2)$, here we take as marking the point $(\frac{4}{3}, 2)$. Let \mathcal{T}_m be the associated triangulation, let \mathfrak{C} be the maximal chain starting with the vertex $(2, 2)$, the edge joining $(0, 2)$ and $(2, 2)$ and P as maximal element. Denote by $\Delta_{\mathfrak{C}}$ the corresponding simplex, the vertices are $(2, 2), (\frac{4}{3}, 2)$ and $(1, 1)$. The points $(2, 3, 4)$ and $(3, 4, 6)$ are elements in $S_{\mathfrak{C}} = S \cap K(\Delta_{\mathfrak{C}})$, but they are not elements of the submonoid generated by $\{(1, a, b) \mid (a, b) \in \Delta_{\mathfrak{C}} \cap M\} = \{(1, 1, 1), (1, 2, 2)\}$. So by the multiplication rules in $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$ (see Definition 4.11, Theorem 4.10 and Theorem 4.12), to get a generating system for $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$, one has to add at least the classes $\bar{f}_{(2,3,4)}$ and $\bar{f}_{(3,4,6)}$.

We add to \mathbf{G}_1 some higher degree elements $\mathbf{G} = \mathbf{G}_1 \cup \{f_{m_{r+1},\chi_{r+1}}, \dots, f_{m_p,\chi_p}\}$ taken from the basis (Lemma 1.4) so that $\bar{\mathbf{G}} = \{\bar{f}_{m,\chi} \mid f_{m,\chi} \in \mathbf{G}\}$ is a generating system for $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$.

Note that for all maximal chains \mathfrak{C} holds: $\bar{\mathfrak{G}}_{\mathfrak{C}} = \{\bar{f}_{m,\chi} \mid f_{m,\chi} \in \mathbf{G}, \text{supp}\nu(f_{m,\chi}) \subseteq \mathfrak{C}\}$ generates the subalgebra $\mathbb{K}[\Gamma_{\mathfrak{C}}] \subseteq \mathbb{K}[\Gamma] \simeq \text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$. By construction, one has hence for the algebra $\mathbb{K}[V \oplus U] = \mathbb{K}[x_1, \dots, x_p]$ two surjective algebra morphisms:

$$\begin{aligned} \bar{\theta} : \mathbb{K}[x_1, \dots, x_p] &\rightarrow \text{gr}_{\nu}\mathbb{K}[\hat{X}_P]; & \theta : \mathbb{K}[x_1, \dots, x_p] &\rightarrow \mathbb{K}[\hat{X}_P]; \\ \forall i = 1, \dots, p : x_i &\mapsto \bar{f}_{m_i,\chi_i}; & \forall i = 1, \dots, p : x_i &\mapsto f_{m_i,\chi_i}, \end{aligned} \tag{2}$$

and corresponding embeddings $\bar{\Theta} : \hat{X}_0 = \text{Spec}(\mathbb{K}[\Gamma]) \rightarrow V \oplus U$ and $\Theta : \hat{X}_P \rightarrow V \oplus U$. Since \hat{X}_P is already embedded in V , here is another description of the morphism Θ :

$$\Theta : \hat{X}_P \rightarrow V \oplus U, \quad x \mapsto (x, f_{m_{r+1},\chi_{r+1}}(x), \dots, f_{m_p,\chi_p}(x)). \tag{3}$$

5.1. Weighted projective varieties

We recall some notation, for more details we refer to the notes [8]. We endow the polynomial ring $\mathbb{K}[x_1, \dots, x_p]$ with a \mathbb{N} -grading defined by $\deg_{\underline{m}} x_i = m_i$ (in particular $m_1 =$

$\dots = m_r = 1$). We denote the “Proj” of the $\text{deg}_{\underline{m}}$ -graded ring by $\text{Proj}_{\underline{m}}\mathbb{K}[x_1, \dots, x_p]$ to avoid confusion with the standard projective space $\mathbb{P}(\mathbb{K}^p) = \text{Proj}(\mathbb{K}[x_1, \dots, x_p])$.

We denote the “Proj” of the $\text{deg}_{\underline{m}}$ -graded ring by $\mathbb{P}(m_1, \dots, m_p)$ and call it *weighted projective space*. Denote by $\text{Gr}_{\underline{m}} \simeq \mathbb{K}^*$ the grading group acting on \mathbb{K}^p by

$$\xi \cdot (a_1, \dots, a_p) = (\xi^{m_1} a_1, \dots, \xi^{m_p} a_p)$$

for $\xi \in \text{Gr}_{\underline{m}}$. A more geometric description of $\mathbb{P}(m_1, \dots, m_p)$ is given as a quotient by the action of the grading group: $\mathbb{P}(m_1, \dots, m_p) = (\mathbb{K}^p \setminus \{0\})/\text{Gr}_{\underline{m}}$.

If $f \in \mathbb{K}[x_1, \dots, x_p]$ is $\text{deg}_{\underline{m}}$ -homogeneous, i.e.

$$f(\xi^{m_1} a_1, \dots, \xi^{m_p} a_p) = \xi^d f(a_1, \dots, a_p),$$

where $d = \text{deg}_{\underline{m}} f$, then $V(f) = \{[v] \in \mathbb{P}(m_1, \dots, m_p) \mid f(v) = 0\}$ is well defined.

For a $\text{deg}_{\underline{m}}$ -homogeneous ideal $I \subset \mathbb{K}[x_1, \dots, x_p]$ let $\hat{Y} = V(I) \subseteq \mathbb{K}^p$ be the vanishing set of I . We denote by $Y = V(I) \subseteq \mathbb{P}(m_1, \dots, m_p)$ the vanishing set of all $\text{deg}_{\underline{m}}$ -homogeneous f in I . We call Y the *weighted algebraic set* associated to I , and \hat{Y} is called the *affine quasi-cone over Y* , note that $Y = (\hat{Y} \setminus \{0\})/\text{Gr}_{\underline{m}}$. If Y is irreducible, then Y is called a *weighted projective variety*. If I is a radical ideal, then the graded ring $\mathbb{K}[x_1, \dots, x_p]/I$ is called the homogeneous coordinate ring $\mathbb{K}[\hat{Y}]$ of $Y = V(I)$.

The morphisms $\bar{\theta}$ and θ defined in (2) send monomials of $\text{deg}_{\underline{m}}$ -degree d to monomials in the \bar{f}_{m_i, χ_i} (respectively f_{m_i, χ_i}) of the same degree with respect to the standard grading on $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$ (respectively on $\mathbb{K}[\hat{X}_P]$). Both rings, $\text{gr}_{\nu}\mathbb{K}[\hat{X}_P]$ and $\mathbb{K}[\hat{X}_P]$, are reduced, so $\ker \bar{\theta}$, as well as $\ker \theta$, are $\text{deg}_{\underline{m}}$ -homogeneous radical ideals and define weighted algebraic sets in $\mathbb{P}(m_1, \dots, m_p)$, isomorphic to X_0 respectively X_P . Since $\ker \theta$ is a prime ideal, note that $X_P = V(\ker \theta) \subseteq \mathbb{P}(m_1, \dots, m_p)$ is a weighted projective variety.

For a polynomial $g(x_1, \dots, x_p) = \sum_{\alpha} b_{\alpha} x^{\alpha}$ we use the multi-index notation, i.e. $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$ and $x^{\alpha} = x_1^{\alpha_1} \cdots x_p^{\alpha_p}$, and for the polynomial ring $\mathbb{K}[x_1, \dots, x_p]$ we just write $\mathbb{K}[\underline{x}]$. Note that for a monomial x^{α} we have: $\text{deg}_{\underline{m}} x^{\alpha} = \text{deg} \theta(x^{\alpha})$.

We endow \mathbb{K}^p with a \hat{T} -action. Let $\{e_1, \dots, e_p\}$ be the standard basis of \mathbb{K}^p , we set for $\hat{t} = (c, t) \in \hat{T}$: $\hat{t} \cdot e_i = c^{m_i} \chi_i(t) e_i$, $i = 1, \dots, p$. We get an induced T -action on the corresponding weighted projective space $\mathbb{P}(m_1, \dots, m_p)$.

The polynomial ring $\mathbb{K}[\underline{x}]$ gets endowed with an induced \hat{T} -action by algebra isomorphisms, here x_i becomes a \hat{T} -eigenfunction of weight $(-m_i, -\chi_i)$, $i = 1, \dots, p$. So the morphisms $\bar{\theta}$ and θ are surjective, \hat{T} -equivariant and $\text{deg}_{\underline{m}}$ -preserving morphisms.

5.2. A global monomial preorder

We define a total order on $\mathbb{N} \times \mathbb{Q}_{\geq 0}^A$ by: $(m', \underline{a}') \preceq (m, \underline{a})$ if $m' < m$ or $m' = m$, $\underline{a}' \succeq \underline{a}$. We now endow the polynomial ring $\mathbb{K}[\underline{x}]$ with an $\mathbb{N} \times \mathbb{Q}_{\geq 0}^A$ -grading.

Definition 5.2. The $\mathbb{N} \times \mathbb{Q}_{\geq 0}^A$ -grading is defined by $\deg_A x_i = (m_i, \nu(f_{m_i, \chi_i}))$, $i = 1, \dots, p$, and $\deg_A 1 = 0$. We introduce on the set of all monomials in $\mathbb{K}[\underline{x}]$ a binary relation: $x^\alpha \succ_A x^\beta$ if $\deg_A x^\alpha \succ \deg_A x^\beta$.

Since “ \succ ” defines a total order on $\mathbb{N} \times \mathbb{Q}_{\geq 0}^A$, the induced binary relation “ \succ_A ” on the set of monomials is a *weak order* (or *total preorder*). By definition, \deg_A is additive, i.e. $\deg_A x^\alpha x^\beta = \deg_A x^\alpha + \deg_A x^\beta$. The additivity of \deg_A implies that “ \preceq_A ” is compatible and cancellative with the multiplication, i.e. if $x^\alpha, x^\beta, x^\gamma$ are monomials, then

$$x^\alpha \succ_A x^\beta \Leftrightarrow x^\alpha x^\gamma \succ_A x^\beta x^\gamma.$$

It follows that “ \prec_A ” is a *monomial preorder* (see [9]). The total degree part of the order ensures in addition that $1 \prec_A x_i$ for all $i = 1, \dots, p$, and if x^α, x^β are monomials such that $x^\alpha \neq 1$, then $x^\alpha x^\beta \succ_A x^\beta$. So we have a *global monomial preorder*, see [9].

Definition 5.3. The *initial term* $\text{in}_\nu g$ of a non-zero polynomial $g \in \mathbb{K}[\underline{x}]$ is the sum of the greatest terms of g with respect to the global monomial preorder “ \succ_A ”. If $I \subseteq \mathbb{K}[\underline{x}]$ is an ideal, then denote by $\text{in}_\nu I$ the ideal generated by the elements $\text{in}_\nu g$, $g \in I$.

Remark 5.4. If f is $\deg_{\underline{m}}$ -homogeneous, then so is $\text{in}_\nu f$. In particular, if the ideal I is $\deg_{\underline{m}}$ -homogeneous, then so is the ideal $\text{in}_\nu I$.

5.3. Minimal lifts

Let \mathcal{I} be the subset

$$\{(m, \underline{a}) \in \mathbb{N} \times \mathbb{Q}_{\geq 0}^A \mid \underline{a} \in \Gamma, m = \sum_{\sigma \in A} a_\sigma \deg f_\sigma\} \subseteq \mathbb{N} \times \mathbb{Q}_{\geq 0}^A.$$

Definition 5.5. The map $\text{val}_\nu : \mathbb{K}[\underline{x}] \setminus \{0\} \rightarrow \mathcal{I}$ is defined for a monomial $g = x^\alpha$ by $\text{val}_\nu g := (\deg_{\underline{m}} g, \nu(\theta(g))) \in \mathcal{I}$. For a polynomial $g = \sum_\alpha b_\alpha x^\alpha$ we define $\text{val}_\nu g$ to be the maximum of the values of the summands: $\text{val}_\nu g = \max_{\succeq} \{\text{val}_\nu(x^\alpha) \mid b_\alpha \neq 0\}$.

Definition 5.6. A monomial $\prod_{i=1}^p x_i^{\ell_i} \in \mathbb{K}[\underline{x}]$ is called a *minimal monomial* if there exists a maximal chain \mathfrak{C} in A such that $\{\nu(\theta(x_i)) \mid 1 \leq i \leq p \text{ and } \ell_i > 0\} \subseteq \Gamma_{\mathfrak{C}}$. We call such a maximal chain a *support chain* for the minimal monomial.

By the properties of a quasi-valuation we know $\nu(\theta(x^\alpha)) \geq^t \sum_{i=1}^p \alpha_i \nu(\theta(x_i))$ and hence

$$\text{val}_\nu x^\alpha = (\deg_{\underline{m}} x^\alpha, \nu(\theta(x^\alpha))) \preceq \left(\deg_{\underline{m}} x^\alpha, \sum_{i=1}^p \alpha_i \nu(\theta(x_i)) \right) = \deg_A x^\alpha. \tag{4}$$

Together with Corollary 4.9 one has:

Lemma 5.7. For a monomial $x^\alpha \in \mathbb{K}[\underline{x}] \setminus \{0\}$ holds: $\text{val}_\nu x^\alpha \preceq \text{deg}_A x^\alpha$, and we have equality: $\text{val}_\nu x^\alpha = \text{deg}_A x^\alpha$ if and only if the monomial x^α is minimal.

Consider for $(m, \eta) \in S$ the function $f_{m,\eta} \in \mathbb{K}[\hat{X}_P]$. We know $\nu(f_{m,\eta}) \in \Gamma_{\mathfrak{C}}$ for some maximal chain \mathfrak{C} . Let $1 \leq i_1 < \dots < i_\ell \leq p$ be such that $\bar{\mathfrak{G}}_{\mathfrak{C}} = \{\bar{f}_{m_{i_1}, \chi_{i_1}}, \dots, \bar{f}_{m_{i_\ell}, \chi_{i_\ell}}\}$. By the assumptions made at the beginning of this section, we can write $\nu(f_{m,\eta})$ as a \mathbb{N} -linear combination: $\nu(f_{m,\eta}) = \sum_{j=1}^\ell b_j \nu(f_{m_{i_j}, \chi_{i_j}})$. By taking the coefficients as exponents, we find a monomial $\mathbf{f}_{m,\eta} := x_{i_1}^{b_{i_1}} \dots x_{i_\ell}^{b_{i_\ell}} \in \mathbb{K}[\underline{x}]$ with the following property: it is a minimal monomial such that $\bar{\theta}(\mathbf{f}_{m,\eta}) = \bar{f}_{m,\eta} \in \text{gr}_\nu \mathbb{K}[\hat{X}_P]$ and $\theta(\mathbf{f}_{m,\eta}) = f_{m,\eta} \in \mathbb{K}[\hat{X}_P]$.

Definition 5.8. We fix for all $(m, \eta) \in S$ such a lift for $f_{m,\eta}$: $\mathbf{f}_{m,\eta} = x_{i_1}^{b_{i_1}} \dots x_{i_\ell}^{b_{i_\ell}} \in \mathbb{K}[\underline{x}]$, called the fixed minimal lift for $f_{m,\eta} \in \mathbb{K}[\hat{X}_P]$, $(m, \eta) \in S$.

5.4. A basis compatible with $\ker \bar{\theta}$

Let $\bar{\mathbb{B}}_1$ be the set of all monomials in $\mathbb{K}[\underline{x}]$ which are not minimal, and let $\bar{\mathbb{B}}_3$ be the set of all fixed minimal lifts: $\bar{\mathbb{B}}_3 = \{\mathbf{f}_{m,\eta} \mid (m, \eta) \in S\}$. Finally, we set

$$\bar{\mathbb{B}}_2 = \left\{ x^\alpha - \mathbf{f}_{m,\eta} \mid \begin{array}{l} (m,\eta) \in S, \bar{\theta}(x^\alpha) = \bar{f}_{m,\eta}, \mathbf{f}_{m,\eta} \neq x^\alpha \\ x^\alpha \text{ minimal monomial} \end{array} \right\}.$$

Lemma 5.9. The union $\bar{\mathbb{B}} = \bar{\mathbb{B}}_1 \cup \bar{\mathbb{B}}_2 \cup \bar{\mathbb{B}}_3$ is a basis for $\mathbb{K}[\underline{x}]$ which is homogeneous with the $\text{deg}_{\underline{m}}$ -grading and the deg_A -grading. In addition, $\bar{\mathbb{B}}_1 \cup \bar{\mathbb{B}}_2$ is a basis for $\ker \bar{\theta}$, and the image of $\bar{\mathbb{B}}_3$ is a basis for $\text{gr}_\nu \mathbb{K}[\hat{X}_P] \simeq \mathbb{K}[\underline{x}] / \ker \bar{\theta}$.

Proof. The union $\bar{\mathbb{B}} = \bar{\mathbb{B}}_1 \cup \bar{\mathbb{B}}_2 \cup \bar{\mathbb{B}}_3$ clearly is a basis for $\mathbb{K}[\underline{x}]$. By Theorem 4.12 (for $\bar{\mathbb{B}}_1$) and construction (for $\bar{\mathbb{B}}_2$), $\bar{\mathbb{B}}_1 \cup \bar{\mathbb{B}}_2 \subseteq \ker \bar{\theta}$, whereas $\bar{\mathbb{B}}_3$ is mapped by $\bar{\theta}$ bijectively onto a basis of $\text{gr}_\nu \mathbb{K}[\hat{X}_P]$ (see *ibidem*). It follows that $\bar{\mathbb{B}}_1 \cup \bar{\mathbb{B}}_2$ is a basis for $\ker \bar{\theta}$.

The elements in $\bar{\mathbb{B}}_2$ are val_ν -homogeneous, i.e. for $\mathbf{f}_{m,\eta} - x^\alpha \in \bar{\mathbb{B}}_2$ one has $\text{val}_\nu(\mathbf{f}_{m,\eta}) = \text{val}_\nu(x^\alpha)$ because $\theta(\mathbf{f}_{m,\eta}) = \theta(x^\alpha)$. Since both monomials are minimal by assumption, one has in addition $\text{deg}_A \mathbf{f}_{m,\eta} = \text{val}_\nu(\mathbf{f}_{m,\eta}) = \text{val}_\nu(x^\alpha) = \text{deg}_A x^\alpha$, so the elements are also deg_A -homogeneous. The elements in $\bar{\mathbb{B}}_1$ and $\bar{\mathbb{B}}_3$ are just monomials, so the basis is compatible with the $\text{deg}_{\underline{m}}$ -grading and the deg_A -grading. \square

5.5. A basis compatible with $\ker \theta$

To get a basis of $\mathbb{K}[\underline{x}]$ compatible with the morphism $\theta : \mathbb{K}[\underline{x}] \rightarrow \mathbb{K}[\hat{X}_P]$ we slightly change $\bar{\mathbb{B}}_1$ and set:

$$\mathbb{B}_1 := \{x^\alpha - \mathbf{f}_{m,\eta} \mid x^\alpha \in \bar{\mathbb{B}}_1, \theta(x^\alpha) = f_{m,\eta}, \mathbf{f}_{m,\eta} \text{ fixed minimal lift}\}, \mathbb{B}_2 := \bar{\mathbb{B}}_2, \mathbb{B}_3 := \bar{\mathbb{B}}_3.$$

Lemma 5.10. The union $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3$ is a basis for $\mathbb{K}[\underline{x}]$ such that $\mathbb{B}_1 \cup \mathbb{B}_2$ is a basis for $\ker \theta$, which is compatible with the $\text{deg}_{\underline{m}}$ -grading, and the image of \mathbb{B}_3 is a basis for $\mathbb{K}[\hat{X}_P] \simeq \mathbb{K}[\underline{x}] / \ker \theta$.

Proof. For $x^\alpha \in \overline{\mathbb{B}}_1$ one has $\text{val}_\nu x^\alpha \prec \text{deg}_A x^\alpha$ (Lemma 5.7). So if $x^\alpha - \mathbf{f}_{m,\eta} \in \mathbb{B}_1$, then, by the minimality of $\mathbf{f}_{m,\eta}$, we have

$$\text{deg}_A \mathbf{f}_{m,\eta} = \text{val}_\nu \mathbf{f}_{m,\eta} = \text{val}_\nu x^\alpha \prec \text{deg}_A x^\alpha. \tag{5}$$

The switch from $\overline{\mathbb{B}}_1$ to \mathbb{B}_1 can be viewed as a triangular base change, and hence \mathbb{B} is a basis. By construction, $\theta(f) = 0$ for all $f \in \mathbb{B}_1 \cup \mathbb{B}_2$, whereas \mathbb{B}_3 is mapped by θ onto a basis of $\mathbb{K}[\hat{X}_P]$ (Lemma 1.4). As a consequence we have: $\mathbb{B}_1 \cup \mathbb{B}_2$ is a basis for $\ker \theta$.

Since all the basis elements are homogeneous with respect to the $\text{deg}_{\underline{m}}$ -grading, the basis is compatible with the $\text{deg}_{\underline{m}}$ -grading. \square

Lemma 5.11. *The map $b \mapsto \text{in}_\nu b$, which sends an element $b \in \mathbb{B}$ to its initial term, induces a bijection $\mathbb{B} \rightarrow \overline{\mathbb{B}}$ such that $\text{in}_\nu g \in \overline{\mathbb{B}}_j$ for $g \in \mathbb{B}_j$, $j = 1, 2, 3$.*

Proof. The elements in $\mathbb{B}_2 \cup \mathbb{B}_3$ are deg_A -homogenous and hence one has for $b \in \mathbb{B}_i$, $i = 2, 3$: $\text{in}_\nu b = b \in \overline{\mathbb{B}}_i$, so in_ν is the identity map on $\mathbb{B}_2 = \overline{\mathbb{B}}_2$ and $\mathbb{B}_3 = \overline{\mathbb{B}}_3$.

Given $b \in \mathbb{B}_1$, say $b = x^\alpha - \mathbf{f}_{m,\eta}$, its initial term is by (5): $\text{in}_\nu b = x^\alpha \in \overline{\mathbb{B}}_1$. By the construction of \mathbb{B}_1 , the map $\nu : \mathbb{B}_1 \rightarrow \overline{\mathbb{B}}_1$ is a bijection. \square

Lemma 5.12. *We have $\text{in}_\nu(\ker \theta) = \ker \bar{\theta}$.*

Proof. By Lemma 5.11, one has $\ker \bar{\theta} \subseteq \text{in}_\nu(\ker \theta)$. Let $g \in \ker \theta$, we write $g = \sum_{j \in \mathcal{S}} c_j b_j$ as a linear combination of elements $b_j \in \mathbb{B}_1 \cup \mathbb{B}_2$, where \mathcal{S} is some finite indexing set. Since the $\text{in}_\nu b$, $b \in \mathbb{B}$, are linearly independent, one has $\text{in}_\nu g = \text{in}_\nu(\sum_{j \in \mathcal{S}} c_j \text{in}_\nu b_j) = \sum_{j \in \mathcal{S}'} c_j \text{in}_\nu b_j$, where $\mathcal{S}' \subseteq \mathcal{S}$ is the subset of indices such that $c_j \neq 0$ and $\text{in}_\nu b_j$ is of maximal deg_A -degree. In particular, $\text{in}_\nu g \in \ker \bar{\theta}$, and hence $\text{in}_\nu(\ker \theta) = \ker \bar{\theta}$. \square

5.6. An approximation by a weight function

As in the case of monomial orders, global monomial preorders can be approximated by integral weight orders, see, for example, [9], [10] and references therein.

For $\alpha, \beta \in \mathbb{Z}^p$ let $\alpha \cdot \beta = \sum_{i=1}^p \alpha_i \beta_i$. For $\lambda \in \mathbb{Z}^p$ let “ \succ_λ ” be the corresponding integral weight order on $\mathbb{K}[\underline{x}]$ defined by $x^\alpha \succeq_\lambda x^\beta$ if $\lambda \cdot \alpha \geq \lambda \cdot \beta$. The initial term $\text{in}_\lambda(g)$ of a nonzero polynomial and the initial ideal $\text{in}_\lambda I$ are defined as in Definition 5.3: $\text{in}_\lambda(g)$ is the sum of the greatest nonzero terms of g with respect to the weight order “ \succ_λ ”, and $\text{in}_\lambda I$ is the ideal generated by the elements $\text{in}_\nu g$, $g \in I$. Note if the ideal I is $\text{deg}_{\underline{m}}$ -homogeneous, then so is the ideal $\text{in}_\lambda I$. The following theorem holds for monomial preorders, we formulate it here just for the monomial preorder induced by the quasi-valuation ν .

Theorem 5.13. [9, Theorem 3.2] *There exists an integral vector $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{Z}^p$ such that $\text{in}_\nu(\ker \theta) = \text{in}_\lambda(\ker \theta)$.*

This integral vector λ can be used to define a linear \mathbb{G}_m -action on \mathbb{K}^p : $s \cdot (\sum_{i=1}^p c_i e_i) = \sum_{i=1}^p s^{\lambda_i} c_i e_i$ for $s \in \mathbb{K}^*$. For the corresponding \mathbb{G}_m -action on $\mathbb{K}[\underline{x}]$ by algebra homomorphisms we have for a monomial: $s \cdot x^\alpha = s^{-\lambda \cdot \alpha} x^\alpha$. The \mathbb{G}_m -action on \mathbb{K}^p commutes with the grading action of \mathbf{Gr}_m on \mathbb{K}^p ; so we get an induced action on $\mathbb{P}(m_1, \dots, m_r)$.

For a non-zero polynomial $f = \sum_{\alpha} c_{\alpha} x^{\alpha} \in \mathbb{K}[\underline{x}]$ set $\text{deg}_{\lambda} f = \max\{\lambda \cdot \alpha \mid c_{\alpha} \neq 0\}$. Let \mathcal{S} be the finite index set of α such that $c_{\alpha} \neq 0$, and set $\mathcal{S}' = \{\alpha \in \mathcal{S} \mid \lambda \cdot \alpha < \text{deg}_{\lambda} f\}$. We get

$$s^{\text{deg}_{\lambda} f} (s \cdot f) = \text{in}_{\lambda} f + \sum_{\alpha \in \mathcal{S}'} s^{\text{deg}_{\lambda} f - \lambda \cdot \alpha} c_{\alpha} x^{\alpha}. \tag{6}$$

By the definition of an initial ideal with respect to the weight order “ \succ_{λ} ” it follows hence that such an ideal is generated by \mathbb{G}_m -eigenfunctions. In particular, the ideal $\ker \bar{\theta} = \text{in}_{\nu}(\ker \theta) = \text{in}_{\lambda}(\ker \theta)$ (Lemma 5.12, Theorem 5.13) is generated by \mathbb{G}_m -eigenfunctions and hence:

Lemma 5.14. *$X_0 \subseteq \mathbb{P}(m_1, \dots, m_r)$ is a \mathbb{G}_m -stable subvariety.*

For a subvariety $Y \subseteq \mathbb{P}(m_1, \dots, m_r)$ denote by $I(Y)$ the $\text{deg}_{\underline{m}}$ -homogeneous vanishing ideal, and for a $\text{deg}_{\underline{m}}$ -homogeneous ideal $I \subseteq \mathbb{K}[\underline{x}]$ let $V(I) \subseteq \mathbb{P}(m_1, \dots, m_r)$ be the vanishing set of the ideal. For $s \in \mathbb{G}_m$ we have $I(s \cdot Y) = s \cdot I(Y)$ and $s \cdot V(I) = V(s \cdot I)$.

Let I be a $\text{deg}_{\underline{m}}$ -homogeneous ideal. The ideal $\lim_{s \rightarrow 0} s \cdot I$ is the ideal in $\mathbb{K}[\underline{x}]$ generated by the limit of the rescaled function $\lim_{s \rightarrow 0} s^{\text{deg}_{\lambda} f} (s \cdot f)$, $f \in I$. Equation (6) implies: $\lim_{s \rightarrow 0} s \cdot I = \text{in}_{\lambda} I$. This ideal is again $\text{deg}_{\underline{m}}$ -homogeneous and hence we have by Lemma 5.12 and Theorem 5.13:

$$\lim_{s \rightarrow 0} s \cdot \ker \theta = \text{in}_{\lambda} \ker \theta = \text{in}_{\nu} \ker \theta = \ker \bar{\theta}. \tag{7}$$

Definition 5.15. For a weighted projective subvariety $Y \subseteq \mathbb{P}(m_1, \dots, m_r)$ denote by $I(Y)$ the $\text{deg}_{\underline{m}}$ -homogeneous vanishing ideal. We say that the weighted algebraic set $Y_0 \subseteq \mathbb{P}(m_1, \dots, m_r)$ is a *toric degeneration of Y inside $\mathbb{P}(m_1, \dots, m_r)$* and write $\lim_{s \rightarrow 0} s \cdot Y = Y_0$, if $Y_0 = V(\text{in}_{\lambda} I(Y))$.

Summarizing we have for $X_P = V(\ker \theta)$ and $X_0 = V(\ker \bar{\theta}) = V(\text{in}_{\lambda}(\ker \theta))$:

Theorem 5.16. *The variety $X_0 \subseteq \mathbb{P}(m_1, \dots, m_r)$ is a toric degeneration of X_P inside the weighted projective space $\mathbb{P}(m_1, \dots, m_r)$: $\lim_{s \rightarrow 0} s \cdot X_P = X_0$.*

5.7. Homogenization and a flat degeneration

A more formal way to look at the results in Section 5.6 is to use the λ -homogenization of an ideal (see, for example, [5, Section 15.8] or [10, Section 4.3]). We have the affine quasi-cone $\hat{X}_P \subseteq \mathbb{K}^p$ embedded in \mathbb{K}^p , together with a \mathbb{G}_m -action on \mathbb{K}^p . We add a

variable u and extend the action of \mathbb{G}_m to $\mathbb{K}^p \oplus \mathbb{K}$ by $s \cdot (\sum_{i=1}^p c_i e_i, c) = (\sum_{i=1}^p s^{\lambda_i} c_i e_i, sc)$ for $s \in \mathbb{K}^*$.

We extend the \mathbb{N} -grading to $\mathbb{K}[\mathbb{K}^p \oplus \mathbb{K}] = \mathbb{K}[\underline{x}, u]$ by setting $\deg_{\underline{m}} u = 0$. And we extend the action of the grading group $\mathbf{Gr}_{\underline{m}} \simeq \mathbb{K}^*$ to $\mathbb{K}^p \oplus \mathbb{K}$ by letting $\mathbf{Gr}_{\underline{m}}$ act trivially on \mathbb{K} . The action of \mathbb{G}_m on $\mathbb{K}^p \oplus \mathbb{K}$ induces an action on $\mathbb{K}[\underline{x}, u]$ by algebra isomorphisms and, since the action preserves the $\deg_{\underline{m}}$ -grading, we get an induced action on $\text{Proj}_{\underline{m}}(\mathbb{K}[\underline{x}, u])$.

The inclusion $\mathbb{K}[u] \hookrightarrow \mathbb{K}[\underline{x}, u]$ induces a morphism $\pi : \text{Proj}_{\underline{m}}(\mathbb{K}[\underline{x}, u]) \rightarrow \mathbb{A}^1$ which is \mathbb{G}_m -equivariant with respect to the \mathbb{G}_m -action on $\text{Proj}_{\underline{m}}(\mathbb{K}[\underline{x}, u])$ and the \mathbb{G}_m -action on \mathbb{K} by multiplication.

Definition 5.17. For a polynomial $f = \sum_{\alpha} a_{\alpha} x^{\alpha} \in \mathbb{K}[\underline{x}]$ set $\deg_{\lambda} f = \max\{\lambda \cdot \alpha \mid a_{\alpha} \neq 0\}$. We define a new function $\check{f} \in \mathbb{K}[x_1, \dots, x_p, u]$, called the λ -homogenization of f :

$$\check{f} = u^{\deg_{\lambda} f} f(u^{-\lambda_1} x_1, \dots, u^{-\lambda_p} x_p) \in \mathbb{K}[x_1, \dots, x_p, u]. \tag{8}$$

For a $\deg_{\underline{m}}$ -homogeneous ideal $I \subseteq \mathbb{K}[\underline{x}]$ denote by $\check{I} \subseteq \mathbb{K}[\underline{x}, u]$ the ideal generated by all the elements $\check{f}, f \in I$.

For the \mathbb{G}_m -action we get: $s \cdot \check{f} = s^{-\deg_{\lambda} f} \check{f}$, so the function \check{f} is \mathbb{G}_m -homogeneous. Note that $\check{f} = \text{in}_{\lambda} f + uh$, where $h \in \mathbb{K}[\underline{x}, u]$. Moreover, if f is $\deg_{\underline{m}}$ -homogeneous in $\mathbb{K}[\underline{x}]$, then so is \check{f} in $\mathbb{K}[\underline{x}, u]$. We apply this homogenization procedure to the elements of the basis \mathbb{B} of $\mathbb{K}[\underline{x}]$ to get $\check{\mathbb{B}} = \{\check{b} \mid b \in \mathbb{B}\}$. It is easy to see:

Lemma 5.18. $\check{\mathbb{B}} = \check{\mathbb{B}}_1 \cup \check{\mathbb{B}}_2 \cup \check{\mathbb{B}}_3$ is a basis of $\mathbb{K}[\underline{x}, u]$ as a $\mathbb{K}[u]$ -module, where $\check{\mathbb{B}}_2 = \mathbb{B}_2$, $\check{\mathbb{B}}_3 = \mathbb{B}_3$ and $\check{\mathbb{B}}_1 = \{x^{\alpha} - u^{\ell_{m,\eta}} \mathbf{f}_{m,\eta} \mid x^{\alpha} \in \mathbb{B}_1, \theta(x^{\alpha}) = f_{m,\eta}\}$, where $\ell_{m,\eta} = \lambda \cdot \alpha - \deg_{\lambda} \mathbf{f}_{m,\eta} > 0$.

We apply the homogenization procedure to the ideal $J = \ker \theta$. Let $\check{\mathfrak{X}}_P \subseteq \mathbb{K}^p \oplus \mathbb{K}$ be the affine algebraic set obtained as the zero set $V(\check{J})$ of the ideal \check{J} . Since \check{J} is generated by \mathbb{G}_m -eigenfunctions, $\check{\mathfrak{X}}_P$ is a \mathbb{G}_m -stable subset of $\mathbb{K}^p \oplus \mathbb{K}$, and we have a \mathbb{G}_m -equivariant morphism $\tilde{\pi} : \check{\mathfrak{X}}_P \rightarrow \mathbb{A}^1$.

Lemma 5.19. $\check{\mathfrak{X}}_P \subseteq \mathbb{K}^p \oplus \mathbb{K}$ is an affine variety with coordinate ring $\mathbb{K}[\underline{x}, u]/\check{J}$. The variety is stable under the action of the grading group $\mathbf{Gr}_{\underline{m}}$. The union $\check{\mathbb{B}}_1 \cup \check{\mathbb{B}}_2$ is a basis for \check{J} as $\mathbb{K}[u]$ -module, and $\mathbb{K}[\underline{x}, u]/\check{J}$ is a free $\mathbb{K}[u]$ -module with basis the image of $\check{\mathbb{B}}_3$.

Proof. The ideal $J = \ker \theta$ is a prime ideal, which implies by [10, Proposition 4.3.10] that \check{J} is a prime ideal and hence $\check{\mathfrak{X}}_P \subseteq \mathbb{K}^p \oplus \mathbb{K}$ is an affine variety with coordinate ring $\mathbb{K}[\underline{x}, u]/\check{J}$. The ideal $J = \ker \theta$ is $\deg_{\underline{m}}$ -homogeneous, hence so is \check{J} , and $\check{\mathfrak{X}}_P$ is thus stable under the action of the grading group $\mathbf{Gr}_{\underline{m}}$.

The union $\check{\mathbb{B}}_1 \cup \check{\mathbb{B}}_2$ is contained in \check{J} by construction, so the image of $\check{\mathbb{B}}_3$ in $\mathbb{K}[\underline{x}, u]/\check{J}$ is a generating system over the ring $\mathbb{K}[u]$. Since $J \subset \mathbb{K}[\underline{x}]$ is a proper prime ideal, one knows that u is not a zero divisor in $\mathbb{K}[\underline{x}, u]/\check{J}$ [10, Proposition 4.3.5 e)]. So given a

linear dependence relation between elements in the image of $\check{\mathbb{B}}_3$ with coefficients in $\mathbb{K}[u]$, one may assume without loss of generality that at least one coefficient has a non-zero constant term. But this would give at $u = 0$ a non-trivial linear dependence relation between the elements in $\overline{\mathbb{B}}_3$, which would be a contradiction. So the image of $\check{\mathbb{B}}_3$ is a $\mathbb{K}[u]$ -basis for $\mathbb{K}[\underline{x}, u]/\check{J}$. \square

It follows that $\check{\mathcal{X}}_P$ is an affine variety with coordinate ring $\mathbb{K}[\underline{x}, u]/\check{J}$, and the \mathbb{G}_m -equivariant morphism $\tilde{\pi} : \check{\mathcal{X}}_P \rightarrow \mathbb{A}^1$ is a flat morphism. Since \check{J} is $\text{deg}_{\underline{m}}$ -homogeneous, we get for $\check{\mathcal{X}}_P = (\check{\mathcal{X}}_P \setminus \{0\})/\text{Gr}_{\underline{m}}$ an induced morphism $\pi : \check{\mathcal{X}}_P \rightarrow \mathbb{A}^1$.

Theorem 5.20.

- i) The morphism $\pi : \check{\mathcal{X}}_P \rightarrow \mathbb{A}^1$ is flat.*
- ii) The fibre over 0 is isomorphic to X_0 .*
- iii) π is trivial over $\mathbb{A}^1 \setminus \{0\}$ with fibre isomorphic to X_P .*

Proof. Since $\mathbb{K}[\underline{x}, u]/\check{J}$ is a free module over $\mathbb{K}[u]$, it is in particular a flat module, which implies by $\text{deg}_{\underline{m}}$ -homogeneity the first claim. Part *ii)* and *iii)* follows by [9, Section 3], where it has been shown (here applied to the case $J = \ker \theta$): $\mathbb{K}[\underline{x}, u]/(\check{J}, u) \simeq \mathbb{K}[\underline{x}]/\text{in}_\lambda J$, and $(\mathbb{K}[\underline{x}, u]/\check{J})[u^{-1}] \simeq (\mathbb{K}[\underline{x}]/J)[u, u^{-1}]$, which finishes the proof. \square

6. The integral case

In this section we assume the combinatorial Seshadri stratification $(X_\sigma, f_\sigma)_{\sigma \in A}$ arises from a situation as in Example 2.3, i.e. for all $\sigma \in A$: the extremal function f_σ is of degree one, and the \hat{T} -weight μ_σ of f_σ is a lattice point in the relative interior of the face σ . The associated triangulation \mathcal{T} has hence lattice points as vertices.

In particular, for every maximal chain $\mathfrak{C} \subseteq A$ we have in the triangulation \mathcal{T} a simplex $\Delta_{\mathfrak{C}}$ with lattice points as vertices, and hence a toric variety $X_{\Delta_{\mathfrak{C}}} \subseteq \mathbb{P}(\mathbb{K}^{\Lambda_{\mathfrak{C}}})$, where $\Lambda_{\mathfrak{C}} = \Lambda \cap \Delta_{\mathfrak{C}}$. Via the inclusion $\mathbb{K}^{\Lambda_{\mathfrak{C}}} \hookrightarrow V = \mathbb{K}^\Lambda$ we view the (not necessarily normal) toric varieties $X_{\Delta_{\mathfrak{C}}}$ as being embedded in $\mathbb{P}(V)$. The following example shows a normal polytope with an integral triangulation having a non-normal simplex.

Example 6.1. Let $P \subseteq \mathbb{R}^3$ be the polytope with vertices $v_0 = (0, 0, 0)$, $v_1 = (6, 0, 0)$, $v_2 = (0, 6, 0)$ and $v_3 = (0, 0, 6)$. By [4, Theorem 2.2.11], the polytope P is normal.

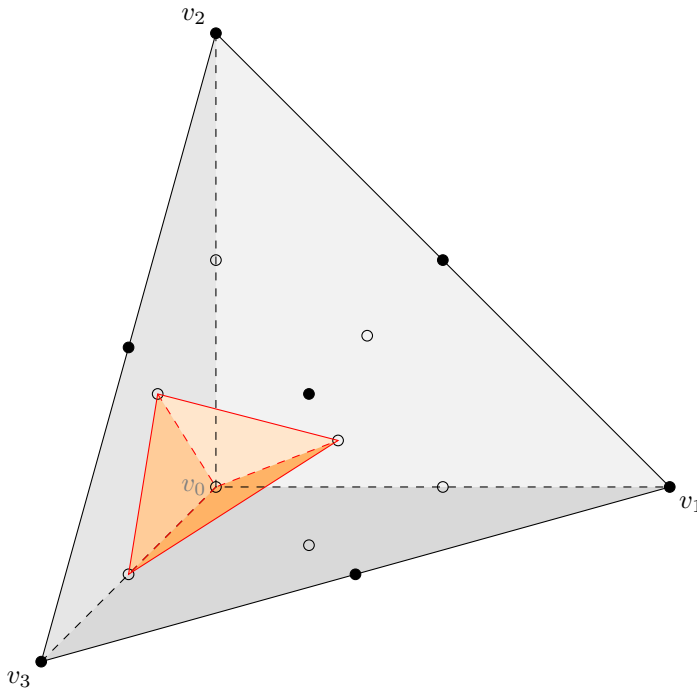
We consider the triangulation of P whose vertices are: the point $(2, 1, 1)$ and the barycenters of the proper faces.

The simplex Q having vertices $v_0 = (0, 0, 0)$, $(2, 1, 1)$, $(0, 2, 2)$ and $(0, 0, 3)$ is not normal. We checked this using Macaulay2 with the following code:

```
A = transpose matrix {{0,0,0}, {2,1,1}, {0,2,2}, {0,0,3}}
Q = convexHull A
isNormal Q
```

However, not all the simplices of the triangulation are non-normal; for example, the one with vertices $(0, 0, 6)$, $(0, 3, 3)$, $(2, 2, 2)$ and $(2, 1, 1)$ is normal.

Here is a picture of the polytope P with the non-normal simplex Q in orange (for interpretation of the colors in the text, the reader is referred to the web version of this article).



6.1. A shadow

The subspace $V \subseteq V \oplus U$ is stable with respect to the \hat{T} -, the \mathbb{G}_m - and the \mathbf{Gr}_m -action on $V \oplus U$ (see Section 5.1), and the projection $\hat{\phi} : V \oplus U \rightarrow V$ is equivariant with respect to these actions. Recall that \mathbf{Gr}_m acts on V by scalar multiplication, so $\hat{O} = \{(v, u) \mid v \in V, u \in U, v \neq 0\}$ is an open and dense subset of $V \oplus U$, stable with respect to the actions by \hat{T} -, \mathbb{G}_m - and \mathbf{Gr}_m -. We get hence a \hat{T} - and \mathbb{G}_m -equivariant rational map $\phi : \mathbb{P}(m_1, \dots, m_p) \dashrightarrow \mathbb{P}(V)$, which is well defined on the open and dense subset $O = \{[v, u] \in \mathbb{P}(m_1, \dots, m_p) \mid v \neq 0\}$.

We write $X_P^V \subseteq \mathbb{P}(V)$ to emphasize the fixed embedding of X_P and to not confuse the embedding with $X_P \hookrightarrow \mathbb{P}(m_1, \dots, m_p)$. Via the rational morphism ϕ we can see the family of varieties $\{s \cdot X_P^V \mid s \in \mathbb{G}_m\} \subseteq \mathbb{P}(V)$ as a shadow of the family $\{s \cdot X_P \mid s \in \mathbb{G}_m\} \subseteq O$. From this point of view we may think of

$$\mathbb{X}_0 := \lim_{s \rightarrow 0} s \cdot X_P^V \subseteq \mathbb{P}(V)$$

as a shadow of the limit $X_0 = \lim_{s \rightarrow 0} s \cdot X_P$ in $\mathbb{P}(m_1, \dots, m_p)$.

The embedding $X_P^V \subseteq \mathbb{P}(V)$ (Definition 1.1) induces a surjective algebra homomorphism $\theta_V : \mathbb{K}[V] \rightarrow \mathbb{K}[X_P]$. We view $V = \mathbb{K}^\Lambda \simeq \mathbb{K}^r$ as a subspace of $V \oplus U$ and write $\mathbb{K}[x_1, \dots, x_r]$ for $\mathbb{K}[V]$. The ideal $\ker \theta_V$ is the vanishing ideal for $\hat{X}_P^V \subseteq \mathbb{P}(V)$, it is equal to the intersection $\ker \theta \cap \mathbb{K}[x_1, \dots, x_r]$.

The integral vector $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{Z}^p$ (Theorem 5.13) induces also a weight order on $\mathbb{K}[x_1, \dots, x_r]$. Let $\lambda^\dagger = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$ be the truncated vector, we define for $\alpha, \beta \in \mathbb{Z}^r$: $x^\alpha \succeq_{\lambda^\dagger} x^\beta$ if and only if $\lambda^\dagger \cdot \alpha \geq \lambda^\dagger \cdot \beta$. Note that $\lambda^\dagger \cdot \alpha = \lambda \cdot \alpha$ for all $x^\alpha \in \mathbb{K}[x_1, \dots, x_r] \subseteq \mathbb{K}[x_1, \dots, x_p]$. Being translated into ideals, the task is to study $\lim_{s \rightarrow 0} s \cdot \ker \theta_V = \text{in}_{\lambda^\dagger} \ker \theta_V$. Theorem 6.2 below recovers for this special situation a result by Zhu [11]:

Theorem 6.2.

- i) We have $\sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V} = \ker \bar{\theta} \cap \mathbb{K}[x_1, \dots, x_r]$.
- ii) If we consider the toric degeneration of X_P induced by the \mathbb{G}_m -action on $\mathbb{P}(V)$, then we get $\mathbb{X}_0 = \lim_{s \rightarrow 0} s \cdot X_P = \bigcup X_{\Delta_{\mathfrak{C}}}$, where the union runs over all maximal chains \mathfrak{C} in A .

Before we come to the proof, we want to point out that the integrality condition on the vertices of the triangulation \mathcal{T} ensures that \mathbb{X}_0 is not too different from X_0 . For a flag $C \subseteq A$ let $K(\Delta_C)$ be the cone over the simplex Δ_C and let $S_C = K(\Delta_C) \cap S$ be the associated monoid in the fan of monoids $S_{\mathcal{T}}$.

Definition 6.3. We denote by $S_{\mathfrak{C}}^1 \subseteq S_{\mathfrak{C}}$ the submonoid $S_{\mathfrak{C}}^1 = \langle (1, \eta) \mid \eta \in \Delta_C \cap M \rangle_{\mathbb{N}}$ generated by the degree one elements. Let S^1 be the fan of monoids obtained as the union $\bigcup_C S_C^1$, where C is running over all flags $C \subseteq A$.

Let \mathfrak{C} be a maximal chain in A . Note that $S_{\mathfrak{C}}^1$ is the weight monoid of the embedded toric variety $X_{\Delta_{\mathfrak{C}}} \subseteq \mathbb{P}(V)$. So one can attach to a maximal chain \mathfrak{C} two affine toric varieties: $\hat{X}_{\Delta_{\mathfrak{C}}} = \text{Spec } \mathbb{K}[S_{\mathfrak{C}}^1] \subseteq V$, which is the affine cone over $X_{\Delta_{\mathfrak{C}}}$, and the weighted affine cone $\hat{X}_{\mathfrak{C}} = \text{Spec } \mathbb{K}[S_{\mathfrak{C}}] \subseteq V \oplus U$ over the irreducible component $X_{\mathfrak{C}} \subseteq X_0$.

Proposition 6.4. For all maximal chains \mathfrak{C} in A , the morphism $\hat{X}_{\mathfrak{C}} \rightarrow \hat{X}_{\Delta_{\mathfrak{C}}}$, induced by the inclusion of monoids $S_{\mathfrak{C}}^1 \subseteq S_{\mathfrak{C}}$, is the normalization morphism.

Proof. The irreducible components $X_{\mathfrak{C}} \subseteq X_0$ are normal because $S_{\mathfrak{C}}$ is saturated. Let \mathfrak{A} be the set $\{f_{m_i, \eta_i} \in G \mid r + 1 \leq i \leq p, \nu(f_{m_i, \eta_i}) \in \mathfrak{C}\}$ and denote by \mathfrak{a} the set $\{\nu(f_{m, \eta}) \mid f_{m, \eta} \in \mathfrak{A}\}$.

Fix $k > 0$ such that $k\nu(f_{m, \eta}) \in \mathbb{N}^A$ for all $f_{m, \eta} \in \mathfrak{A}$. By Corollary 4.7 we know for $f_{m, \eta} \in \mathfrak{A}$: $f_{m, \eta}^k$ is a product of the extremal weight vectors f_σ , $\sigma \in \mathfrak{C}$. Since the $(1, \mu_\sigma)$, $\sigma \in A$, are elements in $S_{\mathfrak{C}}^1$, it follows that $k\nu(f_{m, \eta}) \in S^1$, and hence: every element in $S_{\mathfrak{C}}$

can be written as a linear combination of elements in $S_{\mathfrak{C}}^1$ and elements in \mathfrak{a} , with non-negative integer coefficients, but where the coefficients of the elements in \mathfrak{a} are bounded by k . It follows that $\mathbb{K}[S_{\mathfrak{C}}]$ is a finite $\mathbb{K}[S_{\mathfrak{C}}^1]$ -module, and hence $\mathbb{K}[S_{\mathfrak{C}}]$ is integral over $\mathbb{K}[S_{\mathfrak{C}}^1]$, which finishes the proof. \square

6.2. Proof of Theorem 6.2

Proof. If $(m, \eta) \in S_{\mathfrak{C}}^1$ for some maximal chain \mathfrak{C} , then, by the definition of $S_{\mathfrak{C}}^1$ (see Definition 6.3) and Proposition 4.5, one can find a minimal lift (in the sense of Definition 5.8) which is an element in $\mathbb{K}[x_1, \dots, x_r]$. In the following we assume without loss of generality that we have fixed such a minimal lift $\mathbf{f}_{m,\eta} \in \mathbb{K}[x_1, \dots, x_r]$ for all $(m, \eta) \in S_{\mathfrak{C}}^1$.

Since $\text{in}_{\lambda^\dagger} f = \text{in}_\lambda f$ for $f \in \mathbb{K}[x_1, \dots, x_r] \subseteq \mathbb{K}[x_1, \dots, x_p]$, it follows for the initial ideals: $\text{in}_{\lambda^\dagger} \ker \theta_V \subseteq \text{in}_\lambda \ker \theta \cap \mathbb{K}[x_1, \dots, x_r]$ and hence $\text{in}_{\lambda^\dagger} \ker \theta_V \subseteq \ker \bar{\theta} \cap \mathbb{K}[x_1, \dots, x_r]$. Moreover, since $\ker \bar{\theta}$ is a radical ideal: $\sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V} \subseteq \ker \bar{\theta} \cap \mathbb{K}[x_1, \dots, x_r]$.

Let $x^\alpha \in \mathbb{K}[x_1, \dots, x_r]$ be a monomial which is not minimal, so $x^\alpha \in \overline{\mathbb{B}}_1 \cap \mathbb{K}[x_1, \dots, x_r]$. Let $x^\alpha - \mathbf{f}_{m,\eta} \in \mathbb{K}[x_1, \dots, x_p]$ be the corresponding element in \mathbb{B}_1 . If $(m, \eta) \in S^1$, then $\mathbf{f}_{m,\eta} \in \mathbb{K}[x_1, \dots, x_r]$ and hence $x^\alpha - \mathbf{f}_{m,\eta} \in \mathbb{K}[x_1, \dots, x_r]$. It follows

$$\text{in}_{\lambda^\dagger}(x^\alpha - \mathbf{f}_{m,\eta}) = \text{in}_\lambda(x^\alpha - \mathbf{f}_{m,\eta}) = x^\alpha \in \text{in}_{\lambda^\dagger} \ker \theta_V.$$

If $(m, \eta) \notin S^1$, then let $k > 0$ be an integer such that $k\nu(x^\alpha) \in \mathbb{N}^A$. It follows that $x^{k\alpha}$ is still an element of $\overline{\mathbb{B}}_1$, but by Corollary 4.7 (and Corollary 2.16), $\theta(x^{k\alpha})$ is equal to a product of extremal functions which are of degree one, and all of them have support in the same maximal chain. So if $x^{k\alpha} - \mathbf{f}_{km,k\eta}$ is the corresponding element in \mathbb{B}_1 , then $(km, k\eta) \in S^1$, and hence $x^{k\alpha} \in \text{in}_{\lambda^\dagger} \ker \theta_V$, which implies $x^\alpha \in \sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$. In other words: $\overline{\mathbb{B}}_1 \cap \mathbb{K}[x_1, \dots, x_r] \subseteq \sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$.

Denote by

$$\overline{\mathbb{B}}_2^1 = \mathbb{B}_2^1 = \{x^\alpha - \mathbf{f}_{m,\eta} \in \mathbb{B}_2 \mid x^\alpha \in \mathbb{K}[x_1, \dots, x_n], (m, \eta) \in S^1\}.$$

By assumption, $\mathbf{f}_{m,\eta} \in \mathbb{K}[x_1, \dots, x_n]$, and hence

$$\text{in}_{\lambda^\dagger}(x^\alpha - \mathbf{f}_{m,\eta}) = \text{in}_\lambda(x^\alpha - \mathbf{f}_{m,\eta}) = x^\alpha - \mathbf{f}_{m,\eta} \in \ker \theta_V.$$

It follows: $\overline{\mathbb{B}}_2^1 \subseteq \sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$.

To prove part *i*) of the Theorem, let $f \in \ker \bar{\theta} \cap \mathbb{K}[x_1, \dots, x_r]$. Since $\bar{\theta}$ is \hat{T} -equivariant, one can assume without loss of generality that f is a \hat{T} -eigenfunction. So there exist an element $(m, \eta) \in S$ such that $f = \sum_{\alpha \in \mathcal{S}} c_\alpha x^\alpha$ is a finite linear combination of monomials $x^\alpha \in \mathbb{K}[x_1, \dots, x_r]$ such that $\bar{\theta}(x^\alpha) = \bar{f}_{m,\eta}$ for all α . Here we assume that \mathcal{S} is a finite index system and $c_\alpha \neq 0$ for all $\alpha \in \mathcal{S}$.

Since all non-minimal monomials are in $\overline{\mathbb{B}}_1 \cap \mathbb{K}[x_1, \dots, x_r]$ and hence in $\sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$ as well as in $\ker \bar{\theta}$, one can assume in addition that $c_\alpha \neq 0$ implies x^α is a minimal

monomial. So either $f = 0$, which finishes the proof, or necessarily $(m, \eta) \in S^1$. Rewrite f as

$$f = c_\alpha \mathbf{f}_{m,\eta} + \sum_{\beta \in \mathcal{S} \setminus \{\alpha\}} c_\beta x^\beta \text{ and set } \tilde{f} = \sum_{\beta \in \mathcal{S} \setminus \{\alpha\}} c_\beta (x^\beta - \mathbf{f}_{m,\eta}).$$

By construction, \tilde{f} is a linear combination of elements in $\mathbb{B}_{2,V}^1$, hence $\tilde{f} \in \sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$ and $f, \tilde{f} \in \ker \bar{\theta} \cap \mathbb{K}[x_1, \dots, x_r]$. This implies $f = \tilde{f}$ because otherwise $\mathbf{f}_{m,\eta} \in \ker \bar{\theta}$, which is not possible, hence $f \in \sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$, which finishes the proof of part *i*).

To prove *ii*), note that we have just shown: $(\bar{\mathbb{B}}_1 \cap \mathbb{K}[x_1, \dots, x_r]) \cup \mathbb{B}_{2,V}^1$ is a vector space basis for $\sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V}$ of \hat{T} -eigenfunctions. If we add to this set $\mathbb{B}_{3,V}^1 = \{\mathbf{f}_{m,\eta} \mid (m, \eta) \in S^1\}$, then we have a basis for $\mathbb{K}[x_1, \dots, x_r]$: a monomial in this ring is either not minimal, and hence an element of $\bar{\mathbb{B}}_1 \cap \mathbb{K}[x_1, \dots, x_r]$; or it is minimal, and then it is an element in the linear span of $\mathbb{B}_{2,V}^1 \cup \mathbb{B}_{3,V}^1$. It follows that the zero set $V(\sqrt{\text{in}_{\lambda^\dagger} \ker \theta_V})$ is the union of toric varieties, where the irreducible components are indexed by maximal chains $\mathcal{C} \subseteq A$ and the associated weight monoid is $S_{\mathcal{C}}^1$, which finishes the proof: $\mathbb{X}_0 = \lim_{s \rightarrow 0} s \cdot X_P = \bigcup X_{\Delta_{\mathcal{C}}}$. \square

7. A geometric interpretation

In this section we compare the construction of a combinatorial Seshadri stratification in this article with the construction of a Seshadri stratification in [1]. We first recall the definition of a Seshadri stratification on an embedded projective variety $X \subseteq \mathbb{P}(V)$.

7.1. Seshadri stratifications

Let V be a finite dimensional vector space over \mathbb{K} . The vanishing set of a homogeneous function $f \in \mathbb{K}(V^*)$ will be denoted by $\mathcal{H}_f := \{[v] \in \mathbb{P}(V) \mid f(v) = 0\}$. For an embedded projective subvariety $X \subseteq \mathbb{P}(V)$, we let \hat{X} denote its affine cone in V .

Let $X_p, p \in A$, be a finite collection of projective subvarieties of X and $f_p \in \mathbb{K}[X]$, $p \in A$, be homogeneous functions of positive degrees. The index set A inherits a poset structure by requiring: for $p, q \in A$, $p \geq q$ if and only if $X_p \supseteq X_q$. We assume that there exists a unique maximal element $p_{\max} \in A$ with $X_{p_{\max}} = X$. We say that $\tau > \sigma$ is a covering relation if $\tau \geq \tau' > \sigma$ implies $\tau = \tau'$.

Definition 7.1 ([1]). The collection of subvarieties X_p and homogeneous functions f_p for $p \in A$ is called a *Seshadri stratification* on X , if the following conditions are fulfilled:

- (S1) the projective subvarieties $X_p, p \in A$, are smooth in codimension one; if $q < p$ is a covering relation in A , then X_q is a codimension one subvariety in X_p ;
- (S2) for $p, q \in A$ with $q \not\leq p$, the function f_q vanishes on X_p ;

(S3) for $p \in A$, it holds set-theoretically

$$\mathcal{H}_{f_p} \cap X_p = \bigcup_{q \text{ covered by } p} X_q.$$

The functions f_p will be called *extremal functions*.

Remark 7.2. A Seshadri stratification of an embedded variety $X \subseteq \mathbb{P}(V)$ is compatible with the subvarieties realizing the stratification: for $p \in A$, the poset $A_p = \{q \in A \mid q \leq p\}$ has a unique maximal element. The collection of varieties $X_q \subseteq X_p$, $q \in A_p$, and the extremal functions $f_q|_{X_p}$, $q \in A_p$, satisfies the conditions (S1)-(S3), and hence defines a Seshadri stratification for $X_p \hookrightarrow \mathbb{P}(V)$.

7.2. Seshadri stratifications on toric varieties which are T -equivariant

In the case of an embedded toric variety $X_P \subseteq \mathbb{P}(V)$ as in Section 1, it makes sense to consider only T -stable subvarieties and homogeneous T -eigenfunctions. Denote by A the set of faces of the polytope P . Recall that this set is partially ordered.

Definition 7.3. A Seshadri stratification on X_P is called T -equivariant if

- (E1) the collection of projective subvarieties of the Seshadri stratification consists of the T -orbit closures X_σ , $\sigma \in A$,
- (E2) the collection of homogeneous functions of the Seshadri stratification f_σ , $\sigma \in A$, consists of homogeneous T -eigenfunctions.

In the case of toric varieties, the usual expectation is that all “ T -equivariant” conditions on the variety and properties of the variety can be rephrased in terms of weight combinatorics. This holds also in the case of T -equivariant Seshadri stratifications, we recover here the condition on the weights of the extremal functions in Definition 2.1:

Theorem 7.4. Let $X_P \subseteq \mathbb{P}(V)$ be an embedded toric variety as in Section 1 and denote by A the set of faces of the polytope P .

Let X_σ , $\sigma \in A$, be the collection of T -orbit closures in X_P and let f_σ , $\sigma \in A$, be a collection of \hat{T} -eigenfunctions $f_\sigma \in \mathbb{K}[\hat{X}_P]$ of degree $\deg f_\sigma \geq 1$. Denote by μ_σ the \hat{T} -weight of f_σ . The following are equivalent:

- The collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of subvarieties and homogeneous functions defines a Seshadri stratification on X_P which is T -equivariant in the sense of Definition 7.3.
- The collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of subvarieties and homogeneous functions defines a combinatorial Seshadri stratification in the sense of Definition 2.1.

The proof of Theorem 7.4 is divided into several steps. We start by proving:

Lemma 7.5. *The collection X_σ , $\sigma \in A$, of T -orbit closures in X_P satisfies the condition (S1) for a Seshadri stratification.*

Proof. Since P is a normal polytope, the variety X_P is a normal toric variety. And, by the general theory of toric varieties, so are the orbit closures $X_\sigma = \overline{O_\sigma}$ for $\sigma \in A$. In particular, the varieties X_σ are smooth in codimension one. The condition on the cover relations is satisfied by the fact that in the case of toric varieties, the complement of an orbit O_σ in its closure is the union of the orbit closures of the orbits of codimension 1 in X_σ . \square

In the following, we assume always that the collection of subvarieties X_σ , $\sigma \in A$, is given by the orbit closures.

Lemma 7.6. *A collection of homogeneous T -eigenfunctions $f_\sigma \in \mathbb{K}[\hat{X}_P]$, $\sigma \in A$, $\deg f_\sigma \geq 1$, has property (1) in Definition 2.1 if and only if it satisfies the condition (S3).*

Proof. Let σ be a face of P . If (S3) is satisfied by the collection of functions, then f_σ , $\sigma \in A$, does not vanish on X_σ , but f_σ vanishes on X_τ for τ a proper face of σ . A homogeneous T -eigenfunction in $\mathbb{K}[\hat{X}_P]$ can be written (up to a non-zero scalar factor) as the restriction of a monomial in the $x_\chi, \chi \in \Lambda$. Now a coordinate function x_χ vanishes on X_σ unless $\chi \in \sigma$. So f_σ can be written as the restriction of a monomial in the $x_\chi, \chi \in \Lambda_\sigma$, and hence the weight μ_σ of f_σ has the property: $-\mu_\sigma / \deg f_\sigma$ is an affine convex combination of the $\chi \in \Lambda_\sigma$. In particular: $\frac{-\mu_\sigma}{\deg f_\sigma} \in \sigma$. A face σ is the disjoint union of the relative interiors its faces. So let $\tau \leq \sigma$ be the unique face such that $\frac{-\mu_\sigma}{\deg f_\sigma} \in \tau$ is in the relative interior of τ . If $\tau \neq \sigma$, then f_σ must be a monomial in the $x_\chi, \chi \in \Lambda_\tau$, and hence f_σ is not identically zero on X_τ . So (S3) implies $\tau = \sigma$, and hence (S3) implies: $\frac{-\mu_\sigma}{\deg f_\sigma} \in \sigma^\circ$.

Vice versa, if $\frac{-\mu_\sigma}{\deg f_\sigma} \in \sigma^\circ$ is an element in the relative interior of σ , then the proof of Lemma 2.7 implies (S3). \square

Proof of Theorem 7.4. If the collection of subvarieties X_σ and functions f_σ , $\sigma \in A$, defines Seshadri stratification which is T -equivariant in the sense of Definition 7.3, then the collection of functions f_σ , $\sigma \in A$, satisfies by Lemma 7.6 also the condition (1). Hence it is also a combinatorial Seshadri stratification in the sense of Definition 2.1.

Vice versa, suppose the collection of subvarieties X_σ and functions f_σ , $\sigma \in A$, defines combinatorial Seshadri stratification in the sense of Definition 2.1. So the subvarieties are given by the T -orbit closures in X_P and the extremal functions are \hat{T} -eigenfunctions, hence the conditions (E1) and (E2) are satisfied. The conditions (S1) and (S3) for a Seshadri stratification are automatically satisfied by Lemma 7.5 and Lemma 7.6, and (S2) follows by Lemma 2.7. So the collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of subvarieties and homogeneous functions defines a Seshadri stratification on X_P which is T -equivariant. \square

7.2.1. The valuations

In [1], we use a Seshadri stratification to define for every maximal chain \mathfrak{C} in A a valuation $\mathcal{V}_{\mathfrak{C}} : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow \mathbb{Q}^{\mathfrak{C}}$, using renormalized successive vanishing multiplicities. Before showing that the valuation $\nu_{\mathfrak{C}}$ defined in Definition 2.11 is equal to the one defined in [1], we recall quickly the construction of $\mathcal{V}_{\mathfrak{C}}$ and some of its properties.

We add to the set A the element $\{0\}$, i.e. $\hat{A} = A \cup \{0\}$. The variety $\hat{X}_0 = \{0\}$ is just the origin in V and hence contained in all the affine varieties \hat{X}_{τ} , $\tau \in A$. So it makes sense to extend the partial order from A to \hat{A} by: $\tau > 0$ for all $\tau \in A$.

7.2.2. The one-step case

Let $\tau > \sigma$ be a covering in \hat{A} . Since \hat{X}_{τ} is smooth in codimension one, we have a well defined valuation $\mathcal{V}_{\tau,\sigma} : \mathbb{K}(\hat{X}_{\tau}) \setminus \{0\} \rightarrow \mathbb{Z}$, which associates to $g \in \mathbb{K}(\hat{X}_{\tau}) \setminus \{0\}$ its vanishing multiplicity on \hat{X}_{σ} .

For $\tau \in A$ let f_{τ} be the fixed extremal function associated to τ and denote by $b_{\tau,\sigma} = \mathcal{V}_{\tau,\sigma}(f_{\tau})$ the vanishing multiplicity of $f_{\tau}|_{X_{\tau}}$ on \hat{X}_{σ} . We associate to $g \in \mathbb{K}(\hat{X}_{\tau})$ a new rational function on \hat{X}_{σ} as follows: set

$$g' := \frac{g^{b_{\tau,\sigma}}}{f_{\tau}^{\mathcal{V}_{\tau,\sigma}(g)}|_{X_{\tau}}}. \tag{9}$$

By construction, g' is a well defined rational function on \hat{X}_{τ} . It has been shown in [1] (in a more general context):

Lemma 7.7 ([1], Lemma 4.1). *The restriction $g'|_{\hat{X}_{\sigma}}$ is a well-defined, non-zero rational function on \hat{X}_{σ} .*

Suppose now $g \in \mathbb{K}(\hat{X}_{\tau})$ is a \hat{T} -eigenfunction of weight λ_g . Recall that f_{τ} is a \hat{T} -eigenfunction, denote by μ_{τ} its character. As a quotient of \hat{T} -eigenfunctions, the function g' itself is a \hat{T} -eigenfunction. By construction we see:

Lemma 7.8. *If $g \in \mathbb{K}(\hat{X}_{\tau})$ is a \hat{T} -eigenfunction of character λ_g , then $g'|_{\hat{X}_{\sigma}} \in \mathbb{K}(\hat{X}_{\sigma})$ is a \hat{T} -eigenfunction of character $\lambda_{g'} = b_{\tau,\sigma}\lambda_g - \mathcal{V}_{\tau,\sigma}(g)\mu_{\tau}$.*

7.2.3. The valuation associated to a maximal chain $\mathfrak{C} \subseteq A$

Let $\mathfrak{C} : \tau_r = P > \tau_{r-1} > \dots > \tau_0$ be a maximal chain in A . We endow $\mathbb{Q}^{\mathfrak{C}}$ with the associated lexicographic order and define a $\mathbb{Q}^{\mathfrak{C}}$ -valued valuation on $\mathbb{K}(\hat{X}_P)$ as follows:

To simplify the notation, we write just \mathcal{V}_i instead of $\mathcal{V}_{\tau_i,\tau_{i-1}}$ for the valuation associated to the cover $\tau_i > \tau_{i-1}$ in A . The element τ_0 is a minimal element in A , we write ν_0 for the valuation given by the vanishing multiplicity of a rational function $g \in \mathbb{K}(\mathbb{A}^1) \setminus \{0\}$ in the origin. We simplify in the same way the notation for the vanishing multiplicity $b_{\tau_i,\tau_{i-1}}$ of $f_{\tau_i}|_{\hat{X}_{\tau_{i-1}}}$ on $\hat{X}_{\tau_{i-1}}$, we write just b_i instead.

We associate to a rational function $g \in \mathbb{K}(\hat{X}_P)$ a sequence of rational functions on the subvarieties corresponding to the elements in the fixed maximal chain: $g_{\mathcal{C}} = (g_r, \dots, g_1, g_0)$, where $g_r = g$, and then we repeat the procedure in Lemma 7.7: $g_{r-1} = g'_r|_{X_{\tau_{r-1}}}$, $g_{r-2} = g'_{r-1}|_{X_{\tau_{r-2}}}$, $g_{r-3} = g'_{r-2}|_{X_{\tau_{r-3}}}$ and so on.

Definition 7.9. Let $\{e_{\tau_i} \mid i = 0, \dots, r\}$ be the standard basis for $\mathbb{Q}^{\mathcal{C}}$. We set:

$$\mathcal{V}_{\mathcal{C}} : \mathbb{K}(\hat{X}_P) \setminus \{0\} \rightarrow \mathbb{Q}^{\mathcal{C}}, \quad g \mapsto \frac{\mathcal{V}_r(g_r)}{b_r} e_{\tau_r} + \frac{\mathcal{V}_{r-1}(g_{r-1})}{b_r b_{r-1}} e_{\tau_{r-1}} + \dots + \frac{\mathcal{V}_0(g_0)}{b_r \dots b_0} e_{\tau_0}. \quad (10)$$

It has been proved in [1]:

Proposition 7.10 ([1], Proposition 6.10). *The map $\mathcal{V}_{\mathcal{C}}$ is a $\mathbb{Q}^{\mathcal{C}}$ -valued valuation.*

Remark 7.11. Using an inductive procedure, one gets the following formula for $j = 0, \dots, r - 1$:

$$\begin{aligned} g_j &= g_{j+1}^{b_{j+1}} f_{\tau_{j+1}}^{-\nu_{j+1}(g_{j+1})} |_{\hat{X}_{\tau_j}} \\ &= \dots = g^{b_r \dots b_{j+1}} f_{\tau_r}^{-b_{r-1} \dots b_{j+1} \nu_r(g_r)} \dots f_{\tau_{j+2}}^{-b_{j+1} \nu_{j+2}(g_{j+2})} f_{\tau_{j+1}}^{-\nu_{j+1}(g_{j+1})} |_{\hat{X}_{\tau_j}}. \end{aligned}$$

Our aim is to show that the two valuations $\mathcal{V}_{\mathcal{C}}$ and $\nu_{\mathcal{C}}$ coincide. The following Lemma is a first step:

Lemma 7.12. *If $g \in \mathbb{K}(\hat{X}_{\tau})$ is a \hat{T} -eigenfunction of character λ_g and $\mathcal{V}_{\mathcal{C}}(g) = (a_r, \dots, a_0)$, then $\lambda_g = a_r \mu_{\sigma_r} + \dots + a_0 \mu_{\sigma_0}$. In particular, $\mathcal{V}_{\mathcal{C}}(g) = \nu_{\mathcal{C}}(g)$.*

Proof. We know by Lemma 7.8 that $g = g_r, \dots, g_0$ are \hat{T} -eigenfunctions, and the corresponding characters can be calculated by the formula $\lambda_{g_{j-1}} = b_j \lambda_{g_j} - \mathcal{V}_j(g_j) \mu_{\tau_j}$. So inductively we get:

$$\lambda_g = \lambda_{g_r} = \frac{\mathcal{V}_r(g_r)}{b_r} \mu_{\tau_r} + \frac{\lambda_{g_{r-1}}}{b_r} = \dots = a_r \mu_{\tau_r} + \dots + a_0 \mu_0,$$

which finishes the proof. \square

The next step is to reduce the problem to the case of \hat{T} -eigenfunctions.

Lemma 7.13.

- i) $\mathcal{V}_{\mathcal{C}}$ is \hat{T} -invariant, i.e. $\mathcal{V}_{\mathcal{C}}(g) = \mathcal{V}_{\mathcal{C}}(t \cdot g)$ for $g \in \mathbb{K}(\hat{X}_P) \setminus \{0\}$ and $t \in \hat{T}$.
- ii) For $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ let $g = g_{\eta_1} + \dots + g_{\eta_q}$ be a decomposition of g as a linear combination of \hat{T} -eigenfunctions g_{η_i} , $\eta_i \neq \eta_j$ for all $i \neq j$, $\eta_i \in \hat{M}$, $i = 1, \dots, q$. Then $\mathcal{V}_{\mathcal{C}}(g) = \min\{\mathcal{V}_{\mathcal{C}}(g_{\eta_j}) \mid j = 1, \dots, q\}$.

Lemma 7.12, the second part of Lemma 7.13 together with Definition 2.11 implies:

Corollary 7.14. For all $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ and $\mathfrak{C} \in \mathcal{F}_{\max}(A)$ one has: $\mathcal{V}_{\mathfrak{C}}(g) = \nu_{\mathfrak{C}}(g)$.

Proof of Lemma 7.13. The action of \hat{T} stabilizes the divisor $\hat{X}_{\tau_{j-1}}$ in \hat{X}_{τ_j} for $j = 1, \dots, r$. The associated algebra automorphisms stabilize hence the associated maximal ideal in the local ring $\mathcal{O}_{\hat{X}_{\tau_j}, \hat{X}_{\tau_{j-1}}} \subset \mathbb{K}(\hat{X}_{\tau_j})$ for $j = 1, \dots, r$. So for all $g \in \mathbb{K}(\hat{X}_{\sigma_j})$: the vanishing multiplicities of g respectively $t \cdot g$ on $\hat{X}_{\sigma_{j-1}}$ are the same.

To prove *i)*, for $g \in \mathbb{K}[\hat{X}_P]$ let $g_{\mathfrak{C}} = (g_r, \dots, g_0)$ be the associated sequence of rational functions. We can use the \hat{T} -action to construct two new tuples: for $t \in \hat{T}$ consider the t -twisted tuple $(t \cdot g_r, \dots, t \cdot g_0)$ obtained by twisting component-wise each of the rational functions in the sequence associated to g . And we have the sequence (g'_r, \dots, g'_0) associated to the function $t \cdot g$.

The functions $\{f_{\sigma} \mid \sigma \in A\}$ are T -eigenfunctions, so by Remark 7.11 the j -th component $t \cdot g_j$ in the t -twisted sequence and the j -th component g'_j in the sequence associated to $t \cdot g$ differ only by a nonzero scalar multiple. It follows: $\mathcal{V}_{\mathfrak{C}}(t \cdot g) = \mathcal{V}_{\mathfrak{C}}(g)$ for $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$.

To prove *ii)*, fix $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ and let $g = g_{\eta_1} + \dots + g_{\eta_q}$ be a decomposition into \hat{T} -eigenfunctions, we suppose the characters are pairwise different. We know: $\mathcal{V}_{\mathfrak{C}}(g) \geq \min\{\mathcal{V}_{\mathfrak{C}}(g_{\eta_j}) \mid 1 \leq j \leq q\}$ by the minimum property of a valuation. The assumption on the characters to be pairwise different implies that one find $t \in \hat{T}$ such that $t \cdot g_{\eta_j} = c_j g_{\eta_j}$ for pairwise distinct $c_1, \dots, c_q \in \mathbb{K}^*$. It follows that the linear span of the following functions coincides:

$$\langle g_{\eta_1}, \dots, g_{\eta_q} \rangle_{\mathbb{K}} = \langle g, t \cdot g, \dots, t^{q-1} \cdot g \rangle_{\mathbb{K}}.$$

So one can express the \hat{T} -eigenfunction g_{η_j} as a linear combination of $g, t \cdot g, \dots, t^{q-1} \cdot g$. Now part *i)* of Lemma 7.13 implies: for all $j = 1, \dots, q$,

$$\mathcal{V}_{\mathfrak{C}}(g_{\eta_j}) \geq \min\{\mathcal{V}_{\mathfrak{C}}(t^i \cdot g) \mid i = 0, \dots, q - 1\} = \mathcal{V}_{\mathfrak{C}}(g),$$

and hence $\mathcal{V}_{\mathfrak{C}}(g) = \min\{\mathcal{V}_{\mathfrak{C}}(g_{\eta_j}) \mid j = 1, \dots, q\}$. \square

7.3. The quasi-valuations

In [1] we have used the valuations $\mathcal{V}_{\mathfrak{C}}$ to define a quasi-valuation. Rephrased in terms of an embedded toric variety $X_P \hookrightarrow \mathbb{P}(V)$ the definition in [1] reads as:

Definition 7.15. The quasi-valuation $\mathcal{V} : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow \mathbb{Q}^A$ associated to the T -equivariant Seshadri stratification $(X_{\sigma}, f_{\sigma})_{\sigma \in A}$ and the fixed total order $>^t$ on A is the map defined by:

$$g \mapsto \nu(g) := \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{F}_{\max}(A)\}.$$

Corollary 7.16. *Let ν be the quasi-valuation on $\mathbb{K}[\hat{X}_P]$ defined in Definition 4.1 and let \mathcal{V} be the quasi-valuation $\mathbb{K}[\hat{X}_P]$ defined in [1]. For all $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ one has: $\mathcal{V}(g) = \nu(g)$.*

Proof. For $g \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ one has by definition and by Corollary 7.14:

$$\mathcal{V}(g) = \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{F}_{\max}(A)\} = \min\{\nu_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{F}_{\max}(A)\} = \nu(g). \quad \square$$

Data availability

No data was used for the research described in the article.

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