



Exploring strategies for the identification of triangles by students with learning difficulties: an eye-tracking study

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Abstract

Understanding geometric concepts is an important factor for coping with everyday life. This is also important for students with learning difficulties (LD)—students with significant and persistent difficulties in several school subjects, yet little is known about the understanding of geometric concepts of students with LD: The few previous studies have focused on their overall mathematical performance (usually looking at their products). This study focuses on the identification of triangles—an essential activity with respect to students’ understanding of geometric concepts. The aim of this study is to investigate if fifth grade students with LD ($n=20$) differ from students without LD ($n=165$) in the identification of triangles. We analyzed students’ error rates and use of strategies using eye tracking. We found that for representatives of triangles, students with LD used quick strategies, that is, strategies involving only few gazes, more often than students without LD and tended to make fewer errors. For non-representatives, however, we found that students with LD tended to use analytic strategies through attending to more parts of the shapes more often than students without LD but made more errors. The results indicate that students with LD differed from students without LD in the identification of triangles, and that students with LD not only have difficulties in mathematics, but also show certain strengths in geometry. However, the results also indicate that educators need to support students, especially with LD, in attending to critical attributes of triangles.

Keywords Eye tracking · Geometry · Identification of triangles · Learning difficulties · Strategy use

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Introduction

Geometry is one of the central mathematical subdomains in primary education (e.g., DfE, 2013; KMK, 2022b; NCTM, 2000). Basic geometric skills and an understanding of geometric concepts are of special importance in teaching mathematics as factors in solving everyday problems and challenges (Cawley et al., 2009; Hellmich, 2016). Geometry is important for all students, but little is known about geometry learning, particularly for students who have difficulties learning mathematics, for example, students with learning difficulties (LD). Students with LD generally show significant and persistent difficulties in several school subjects, including mathematics (e.g., Heimlich et al., 2016; OECD, 2007). Research on their learning is important to support them appropriately, as they seldom overcome their difficulties in learning without specific help (e.g., Heimlich, 2016). However, to date, little is known about the geometry learning of students with LD, and in particular, not much is known about specific or typical difficulties of children with LD and possible needs for support in geometry (Cawley et al., 2009; Hellmich, 2016).

This paper focuses on the geometry learning of students with LD, in particular on an essential activity to support and explore students' understanding of geometric concepts: identifying geometric shapes. The aim of this study is to investigate whether students with LD differ from students without LD in the identification of geometric shapes, particularly of triangles. In addition to analyzing students' responses, our study uses eye tracking (ET), which has been shown to provide insights into students' strategies in different mathematical domains, including geometry (e.g., Schindler & Lilienthal, 2019). As ET does not require students to verbalize the strategies, the use of ET provides a good opportunity to explore the mathematical learning of students with LD, who often have mathematical and language difficulties and difficulties with metacognitive reflection (e.g., Heimlich, 2016). We analyze ET videos of students with and without LD and qualitatively analyze their strategies for identifying triangles (see <https://youtu.be/1uiMSWK0PCE> for a video example).

Theoretical and empirical background

Understanding of triangles

Triangles and prototypes A central aim of primary school curricula in many countries is for students to develop an understanding of concepts of geometric shapes, such as triangles (e.g., DfE, 2013; KMK, 2022b; NCTM, 2000). For example, children are expected to be able to name geometric shapes and identify their properties (e.g., DfE, 2013; KMK, 2022b; NCTM, 2000). For geometric shapes, there are defining, critical attributes of being “closed,” and having a “particular number” of “straight sides” (Satlow & Newcombe, 1998, p. 548). Triangles, which are the focus of this study, can be described as a closed shape with three straight sides and three vertices. Although all representatives of triangles

contain all critical attributes (Hershkowitz, 1989), there may be differences between these representatives in visual appearance. Prototypical triangles contain non-defining, non-critical attributes that are dominant and attract the observer's attention (Hershkowitz, 1989; for prototype theory see Rosch, 1973) and are intuitively accepted as representatives of triangles (Tsamir et al., 2008). These non-critical attributes are a prototypical aspect ratio and symmetry, as in equilateral triangles or isosceles triangles (with an aspect ratio similar to an equilateral triangle), and orientation, that is, a horizontally aligned base. Non-prototypical triangles differ in appearance in these non-critical attributes, for example, through skewness of a shape (i.e., the asymmetry or obliquity of a shape) (Aslan & Aktaş Arnas, 2007; Clements et al., 1999).

Concept image and definition To take a closer look at students' understanding of triangles, it is important to distinguish between understanding an abstract mathematical definition of the concept of triangle and students' individual ideas that play a role in the development of an understanding of triangles. In this context, Tall and Vinner (1981) distinguish between "concept definition" and "concept image" (i.e., "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes", p. 152). Before children learn about the concept definition, they already have experiences that result in individual understandings and mental pictures of triangles (NCTM, 2000; Tall & Vinner, 1981). For these individual mental pictures, Gutiérrez (1996) uses the term *mental image*, which can be understood as "any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements" (p. 9). According to Presmeg (1986), there are different kinds of mental images, such as "concrete images" (i.e., pictures in the mind) and "dynamic images" (i.e., mental images with movement) (p. 43).

Levels of geometric thinking The development of students' understanding of triangles is a complex process (Tsamir et al., 2008), which can broadly be described in levels of geometric thinking, as presented by van Hiele (e.g., 1986) "from a Gestalt-like visual level through increasingly sophisticated levels of description, analysis, abstraction, and proof" (Clements & Battista, 1992, p. 426). This study focuses on the identification of triangles, which is covered by the first three levels of van Hiele's theory.¹ Following van Hiele (1986), these three levels are distinguished: (1) "visual": Learners perceive triangles as a visual whole, relying deeply on prototypes. (2) "descriptive": Learners perceive the properties of triangles and can use them for characterization. (3) "theoretical": Relationships between properties of a triangle and properties of related triangles can be recognized.²

¹ The higher levels of van Hiele's theory are not relevant for the identification of triangles; they will therefore not be discussed further.

² In more recent descriptions of the van Hiele levels, level 2 is also referred to as "descriptive/analytic" and level 3 as "abstract/relational" (see, e.g., Battista, 2007).

Findings on students' geometric development regarding the identification of triangles To investigate children's understanding of triangles, activities involving the identification of triangles (including representatives and non-representatives) are often studied. Studies on the identification of triangles address kindergarten children (e.g., Aslan & Aktaş Arnas, 2007; Clements et al., 1999; Hannibal, 1999; Maier & Benz, 2013; Satlow & Newcombe, 1998; Tsamir et al., 2008; Unterhauser, 2020; Yeşil Dağlı & Halat, 2016), as well as students in primary and secondary school (e.g., Burger & Shaughnessy, 1986; Ma et al., 2015; Satlow & Newcombe, 1998). The development of students' understanding of triangles begins in early childhood (e.g., NCTM, 2000). For example, infants play with building blocks or puzzles that represent or form geometric shapes. Thereby, they start to build their concept images by handling, manipulating, and naming these basic shapes (Hannibal, 1999). The understanding of the concept triangle in kindergarten children is primarily based on prototypes: Children base their considerations for identifying triangles mostly on the overall appearance of a shape (Aslan & Aktaş Arnas, 2007; Tsamir et al., 2008; Unterhauser, 2020; van Hiele, 1999; Yeşil Dağlı & Halat, 2016), which is characteristic for van Hiele level 1 (van Hiele, 1986). Whereas prototypical triangles are quickly identified as representatives of triangles, non-prototypical triangles are often mistakenly rejected: Children do not accept triangles as such, for example, when they consider them "too long" (non-prototypical aspect ratio) (Clements & Sarama, 2000, p. 483). The rejection of non-prototypical triangles shows that mathematically irrelevant, non-critical attributes influence children's considerations and cause them to make mistakes, indicating that children have not yet reached van Hiele level 2 (van Hiele, 1986). Children may also mistakenly accept non-triangles as triangles if non-triangles look similar to a prototypical triangle (in the following referred to as non-prototypical non-triangles³), such as if the non-triangles have a triangle-like shape but have curved sides or rounded vertices (Clements et al., 1999; Satlow & Newcombe, 1998; Tsamir et al., 2008; Unterhauser, 2020). The acceptance of non-prototypical non-triangles shows that mathematically relevant, critical attributes are not considered when the shape is triangle-like. As children get older and gain more experience with triangles (Hannibal, 1999), they tend to identify more shapes correctly: The appearance of a triangle becomes less important for the identification of triangles, and the proportion of property-based identification decisions increases (Aslan & Aktaş Arnas, 2007; Clements et al., 1999). This is also reflected in decreasing error rates, which vary depending on the selection of shapes presented: When presenting triangles and both non-prototypical non-triangles and prototypical non-triangles, such as squares, error rates range from 38% for 3-year-old children to 28% for 6-year-old children (Aslan & Aktaş Arnas, 2007). Presenting only triangles

³ We follow the classification of non-triangles by Tsamir and colleagues (2008). Prototypical non-triangles are other prototypical geometric shapes (e.g., circles) that are easily identified by children as non-triangles. Non-prototypical non-triangles have a triangle-like shape and are not as easily identified as non-triangles. As with the classification of triangles as prototypical, the classification of non-triangles as prototypical in this paper also refers to those non-representatives of triangles that children reliably identify as such.

and non-prototypical non-triangles, error rates range from 43% for 4-year-old children to 39% for 6-year-old children (Clements et al., 1999).

When continuously developing their understanding of triangles, students increasingly rely on critical attributes (Battista, 2007). However, identification of triangles can also still be prototypically shaped (Burger & Shaughnessy, 1986; Ma et al., 2015; Satlow & Newcombe, 1998). Ma et al. (2015) found most students in first and second grades to make decisions based on the appearance of the shapes (related to van Hiele level 1). Most students in third to sixth grades were able to analyze the properties, for example, number of vertices (related to van Hiele level 2), and in fifth and sixth grade, some students were found to be able to relate shapes and their properties (related to van Hiele level 3). However, Satlow and Newcombe (1998) found that fourth graders still reject 35% of non-prototypical triangles as triangles. The most difficult triangles appear to be scalene triangles, differing in symmetry from prototypical triangles, or isosceles triangles with an extreme acute or obtuse angle, differing in aspect ratio from prototypical triangles (Ma et al., 2015). Studies indicate that even students in grade 6 still have difficulty identifying triangles, which is reflected by a relatively high error rate for the identification of triangles of 19% (Clements & Battista, 1992; cited in Clements & Sarama, 2000). The above-mentioned studies illustrate the role of students' individual mental images (Gutiérrez, 1996) and the whole concept image (Tall & Vinner, 1981) in students' understanding of triangles. Even when correct concept definitions are known, decisions for the identification of geometric shapes may be prototypically shaped. Thus, when teaching geometry at school, students' mental images and the whole concept image of triangles need to be considered.

Learning difficulties

Definition of LD There are different definitions and associated understandings for describing difficulties in learning (Büttner & Hasselhorn, 2011; Lloyd et al., 2007; Tzouriadou, 2020). For example, the term “learning disability” is often used yet understood very differently (e.g., Grünke & Cavendish, 2016)—ranging from isolated disabilities in specific areas, such as reading, spelling, or arithmetic, “that cannot be explained by intelligence” (Büttner & Hasselhorn, 2011, p. 76) to what is commonly known as a “mild disorder of intellectual development”⁴ (e.g., Grünke & Cavendish, 2016).

In the following, we use the term students with “learning difficulties” (LD) to refer to students with special educational needs in the area of learning; that is, their learning is significantly, extensively, and persistently⁵ impaired (AO-SF NRW, 2022, §4(2), transl. by ALS; see also OECD, 2007) and who cannot overcome their

⁴ Wording according to ICD-11 (WHO, 2019).

⁵ Wording according to legislation in North Rhine-Westphalia (state in Germany with the most inhabitants). This need for special educational support must be determined by the education authority (§10(1), AO-SF NRW, 2022).

difficulties in learning without specific help (e.g., Heimlich, 2016). This understanding of LD has similarities with international understandings of difficulties in learning, where these difficulties are understood as significant and persistent difficulties in acquiring academic competence that are not caused by “other disabilities such as sensory impairments” or “disorder of intellectual development” (Lloyd et al., 2007, p. 159; see also APA, 2013; WHO, 2019). However, LD also need to be distinguished from other terms and understandings used in the context of difficulties in learning, such as difficulties that are temporary or occur only in one school subject (Heimlich et al., 2016). Students with LD can have difficulties in all school subjects (Grünke & Cavendish, 2016). Most often, they show particular difficulties in reading and writing and in mathematics as well as in “learning to learn” (Heimlich, 2016, p. 36; Pullen, 2016). Problems in learning to learn arise particularly in the control and reflection of the educational process, that is, in metacognitive abilities, as well as in the use of learning strategies (Heimlich, 2016). However, these struggles are not monocausal, but emerge from a complex set of conditions. The causes are to be found in an interplay of an inadequate fit between a student’s learning opportunities, the learning resources available at school, and the performance requirements formulated by the school (Dudley-Marling, 2004; Heimlich et al., 2016). Also, students with LD often enter school from difficult life situations (e.g., socio-economically difficult background; see Heimlich, 2016; Okoli et al., 2022).

Students with LD represent the largest subgroup among all students with special educational needs in many countries (e.g., Büttner & Hasselhorn, 2011; Pullen, 2016; Tzouriadou, 2020). In Germany, in 2022,⁶ more than one-third (39.47%) of all students with special educational needs were students with LD⁷ (KMK, 2022a). In the history of several educational systems, such as the German one, students with LD have been taught separately in special schools (Heimlich, 2016; Peetsma et al., 2001). Today in Germany, around half of the students with LD attend an inclusive school, while the other half attend a special school (KMK, 2022a).

Mathematical learning of students with LD To date, it is not sufficiently clear if and how students with LD differ from students without LD in their mathematical learning. Studies looking at the mathematical learning of students with combined difficulties in mathematics and reading compared to students with isolated difficulties in mathematics and students without difficulties in mathematics (e.g., Hanich et al., 2001; Jordan et al., 2002; van der Sluis et al., 2004) indicate that students with combined difficulties in mathematics and reading encounter difficulties learning arithmetic in the same areas as students with isolated difficulties in mathematics (Hanich et al., 2001; van der Sluis et al., 2004), but sometimes to a greater extent (Jordan et al., 2002). For the subdomain of geometry, a study by Grobecker and De Lisi (2000) suggests that low-achieving students (in at least one subject) may also have difficulties or delayed development in spatial-geometrical understanding when

⁶ Latest available statistics.

⁷ This statistic is based on data from Germany. Since the concepts and definitions of LD vary greatly depending on the country, the number of children diagnosed with LD varies from country to country.

drawing and transforming squares and diamond shapes. For example, low-achieving students were found not to transform these shapes as efficiently as students without difficulties (Grobecker & De Lisi, 2000).

Studies examining mathematical learning of students with LD (according to the understanding of LD in this paper) have found that already upon entering primary school, students with LD appear to have difficulties in understanding basic arithmetic concepts and show a delay in mathematics, for example, knowledge of quantities and numbers, compared to students without LD (Moser Opitz, 2008). These difficulties appear to increase throughout the years. Students with LD in sixth grade were found not to have the mathematical competencies that students should have at the end of primary school (i.e., in fourth grade) (Werner et al., 2019). Analyses of PISA results in mathematics showed that around 90% of 15-year-old students with LD were at or below competence level I used in PISA (Gebhardt et al., 2015; Müller et al., 2017). Studies that examined mathematical competencies of students with LD at the end of school and at the transition to working life found that students with LD often still have difficulties with mathematical competencies at primary school level (Gebhardt et al., 2013, 2014; Lehmann & Hoffmann, 2009) and therefore often do not meet the requirements for mathematical competencies for working life (Lutz et al., 2023).

The existing studies point to difficulties and delays experienced by students with LD compared to children without LD. However, these studies tend to focus on students' products, such as responses to standardized mathematic tests, and often report results summarizing multiple mathematical domains. For the broad domain of geometry (see e.g., NCTM, 2000), not much is known about particular or typical difficulties that children with LD may encounter in geometry (Cawley et al., 2009; Hellmich, 2016). Since students with LD seldom overcome their difficulties in learning mathematics without specific help (e.g., Heimlich, 2016), research on their mathematical learning is important to support them appropriately.

Relevance of geometry for students with LD Geometry is of fundamental importance for students with LD. For students with LD, geometric skills and an understanding of geometric concepts are of particular importance for coping with everyday problems and challenges (Cawley et al., 2009; Hellmich, 2016). Other reasons also support the importance of teaching geometry to students with LD: Geometry can spark the interest of learners who tend to struggle and perform poorly in other mathematical subdomains, such as arithmetic, and teaching geometry can contribute to the success of students in mathematics who otherwise usually do not succeed (Jones, 2002).

Eye tracking

ET is understood as the recording of a person's eye movements (Duchowski, 2017; Holmqvist et al., 2011). In the human eye, visual acuity is best in the fovea centralis, the central 2° of the field of view (Holmqvist et al., 2011; Rayner, 1998). Therefore, the eyes need to move so that the objects of attention are perceived by foveal vision (Henderson, 2003; Rayner, 1998). These eye movements can be measured and

recorded using ET techniques (Duchowski, 2017; Holmqvist et al., 2011). Thus, by tracking eye movements, inferences can be made about visual attention and cognitive processes associated with the reception of visual information (Henderson, 2003; Holmqvist et al., 2011). The visual attention that can be captured by ET is so-called overt attention (Carrasco, 2011; Duchowski, 2017; Posner, 1980). So-called covert attention, that is, paying attention “to an area in the periphery without actually directing one’s gaze toward it” (Carrasco, 2011, p. 1487) can also be involved in the processing of information, since areas that are not seen foveally but are in the visual field can still be perceived. Peripheral vision, which takes place in extrafoveal areas, is not captured by ET (Duchowski, 2017; Klein & Ettinger, 2019).

With its potential to study visual attention and cognitive processes through the recording of foveal attention, ET is used in many research areas (Duchowski, 2017; Holmqvist et al., 2011). ET research often relies on the eye-mind hypothesis (Just & Carpenter, 1980), which implies that there is no meaningful difference between what the eyes fixate on and what the brain processes (Holmqvist et al., 2011). Originally, the eye-mind hypothesis was derived from reading research, and it is not clear how far it can be applied to mathematics (Schindler & Lilienthal, 2019). However, considering the experimental context, domain-specific interpretations of eye movements (i.e., taking the mathematical content into account) allow for inferences about mental processes (Hayhoe, 2004; Schindler & Lilienthal, 2019).

In recent years, there has been a growing interest in ET research within mathematics education, as reflected by an increasing number of studies using ET (e.g., Strohmaier et al., 2020). Most studies are conducted in mathematical content areas, mostly in the content area of numbers and operations (e.g., quantity comparison, e.g., see Pitta-Pantazi et al., 2024), but there are also studies that focus on mathematical processes (e.g., on creativity, e.g., see Bicer & Bicer, 2023). Within the content area of geometry, there are also an increasing numbers of ET studies (e.g., geometric problems, e.g., Heyd-Metzuyanım et al., 2023). Studies in the area of geometry (e.g., Merschmeyer-Brüwer, 2001; Schindler & Lilienthal, 2019) have demonstrated that the use of ET in this mathematical subdomain can provide insights into learners’ strategies. Simon et al. (2021) have shown that ET provides insights into students’ strategies for identifying triangles⁸: They found that strategy use differed between different types of triangles: for example, the identification of prototypical triangles often happened “at a glance” (p. 53). Shvarts et al. (2019) conducted an ET study on the identification of quadrilaterals by university students and showed similar results: Prototypical shapes (prototypical orientation) were often identified by extrafoveal vision—that is, there were no students’ gazes that fixated on these shapes—and rotated shapes (non-prototypical orientation) were more difficult to identify, resulting in more gazes that fixated on these shapes. Chumachenko et al. (2024) compared the identification of quadrilaterals in first-grade students’ and university students. They found that the number of fixations required to identify quadrilaterals decreases in university students (“experts”) compared to

⁸ The strategies in Simon et al.’s (2021) study are the same strategies analyzed in this study. Further information on the strategies can be found in the qualitative data analysis section.

first-grade students, and as expertise increases, foveal analysis is replaced by extrafoveal analysis (Chumachenko et al., 2024). In other research areas (e.g., chess, see Holmqvist et al., 2011), the use of fewer gazes is also associated with processing efficiency and expertise. “With increasing skill, more information is extracted around the point of fixation making eye movements overall more efficient” (Holmqvist et al., 2011, p. 383). There has also been ET research in mathematics education investigating the mathematical learning of students with special educational needs (e.g., students with down syndrome, Ranzato et al., 2020), and ET has shown to be a useful method in this research area. ET does not require verbalization by the students to investigate strategies.⁹ Since eye movements are not necessarily associated with conscious processes, ET can often be more informative than conscious reports by interviewees (Hayhoe, 2004), such as through thinking aloud (see Ericsson & Simon, 1980). Studies that compare, at a methodological level, insights about students’ strategies gained through ET and through thinking aloud found that ET partially revealed strategies that children did not report (for quantity recognition, see Schindler & Lilienthal, 2018; for number line estimation, see Simon & Schindler, 2020). ET was particularly informative for children with mathematical learning difficulties and other special educational needs (Schindler & Lilienthal, 2018; Simon & Schindler, 2020). Possible reasons for this could lie, for example, in children’s metacognitive and/or language abilities that influence their reports (Schindler, 2021; Schindler & Lilienthal, 2018). Given that children with LD often have mathematical and language difficulties and difficulties with metacognitive reflection (Heimlich, 2016), using ET is a promising approach to explore and to gain new insights into mathematical learning in such students.

Even though ET has received increasing attention in mathematics education research in recent years and there have been an increasing number of studies in the field of geometry in recent years, there have been only a few studies focusing on the mathematical learning of children with special educational needs (see Strohmaier et al., 2020). To date, to the best of our knowledge, there is no ET study on the identification of triangles of students with LD.

Aim and research question

The purpose of this paper is to contribute to the research on geometry learning of students with LD and thus contribute to the knowledge about the mathematical learning of students with LD in general. The specific aim of this study is to investigate if students with LD differ from students without LD in the identification of triangles. We ask the research question, “*Do students with and without LD differ in the identification of triangles?*” For this purpose, we conducted an ET study. The potential of ET to provide insights into learners’ strategies has been demonstrated in several studies in mathematics education in different subdomains, including

⁹ The studies on the identification of triangles presented earlier (e.g., Clements et al., 1999) used student interviews.

Table 1 Participants of the study

	With LD ($n = 20$)	Without LD ($n = 165$)
Participant information		
Age: mean (standard deviation)	11.7 (0.7)	10.7 (0.6)
Gender: n_{girls} (% $_{\text{girls}}$)	10 (50.00)	83 (50.30)
Mathematical abilities		
BASIS-MATH-G 4 ⁺ Mean t-value (standard deviation)	33.50 (6.12)	54.63 (8.74)
Mathematical difficulties: n (%)	20 (100.00)	51 (30.91)

geometry (Merschmeyer-Brüwer, 2001; Schindler & Lilienthal, 2019; Shvarts et al., 2019; Simon et al., 2021). ET can provide insights into learners' strategies without requiring learners to verbalize their strategies. Especially for children with LD, who often have mathematical and language difficulties and difficulties with metacognitive reflection (Heimlich, 2016), the use of ET offers a good opportunity to explore mathematical learning.

The study

Participants

The sample consisted of 20 fifth grade students from a German (state: North Rhine-Westphalia) special school for LD from three different classes (students per class: a: $n = 11$, b: $n = 6$, c: $n = 3$) (in the following referred to as “with LD”) and 165 fifth grade students from a German (state: North Rhine-Westphalia) inclusive comprehensive school from six different classes (students per class: a: $n = 21$, b: $n = 29$, c: $n = 24$; d: $n = 28$, e: $n = 32$, f: $n = 31$) (in the following referred to as “without LD”)—resulting in a total sample of 185 fifth graders (for more information regarding age and gender, see Table 1). We asked even more fifth graders from the special school and the inclusive comprehensive school to participate in the study: a total of more than 200 students. The 185 students were the students who wanted to participate in the study and whose parents or legal guardians agreed to their participation. All students with LD had officially diagnosed special educational needs in the area of learning; that is, they have significant and persistent difficulties in learning, including learning mathematics (AO-SF NRW, §4(2)).

The study took place at the beginning of the school year, which means that students were in transition from primary to secondary school.¹⁰ We chose fifth grade students, even though the identification of triangles is addressed in mathematics education in the first 2 years of primary school (KMK, 2022b; NCTM, 2000), because it

¹⁰ In North Rhine-Westphalia, children attend primary school for 4 years and usually go on to secondary school from the fifth school year onwards.



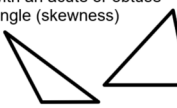
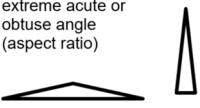

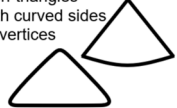

	Prototypical	Non-prototypical		
Triangles		Rotated triangles with a prototypical aspect ratio (orientation) 	Scalene triangles with an acute or obtuse angle (skewness) 	Isosceles triangles with an extreme acute or obtuse angle (aspect ratio) 
#	3	7	4	4
Non-triangles		Non-triangles with a gap 	Non-triangles with curved sides or vertices 	Non-triangles with more than three vertices 
#	0	4	6	6

Fig. 1 Selected items used in this study. Note that prototypical non-triangles (like squares) were not used in this study, which is why this cell remains empty in this table. Note. # = number of items

is not trivial for children to identify triangles, even in grade 6 (Clements & Battista, 1992; cited in Clements & Sarama, 2000; Satlow & Newcombe, 1998). Moreover, we could be sure that all students in grade 5—including those with LD—had dealt with the identification of triangles in previous school years.

To gain insights into the mathematical performance of the students with LD and without LD, we conducted a standardized mathematics test, BASIS-MATH-G 4⁺ (Moser Opitz et al., 2016), prior to the ET study. The test examines central abilities of primary school arithmetic, for example, understanding of numbers and operations. According to BASIS-MATH-G 4⁺, students with *t*-value ≤ 49 and corresponding percentile rank ≤ 46 (i.e., 46% of the norm sample performed equally or worse) are considered to have mathematical difficulties. Students with *t*-value ≥ 55 and corresponding percentile rank ≥ 69 are considered to not have mathematical difficulties. For students who fall between these values, mathematical difficulties cannot be ruled out. Results showed that all students with LD and about a third of the students without LD had difficulties with the tasks of the standardized mathematics test (*t*-value ≤ 49) (see Table 1). The BASIS-MATH-G 4⁺ provides insights into students’ general mathematical abilities without focusing on geometry.

Materials

The items (i.e., geometric shapes) were developed following studies by Aslan and Aktaş Arnas (2007), Clements et al. (1999), Satlow and Newcombe (1998), and Tsamir et al. (2008). The classification of items into prototypical and non-prototypical triangles and non-prototypical non-triangles was based on Tsamir et al. (2008).¹¹ The prototypical triangles are the equilateral and isosceles triangles (orientation and aspect ratio as shown in Fig. 1). The non-prototypical triangles differ from the prototypical triangles

¹¹ Tsamir et al. (2008) use the term “intuitive”.

in non-critical attributes, that is, in orientation, skewness, and aspect ratio (as marked in Fig. 1). We decided not to consider the non-critical attribute size in the selection of items because modifications of triangles in size have little effect on the identification of triangles, even in kindergarten children (e.g., Aslan & Aktaş Arnas, 2007). The non-prototypical non-triangles have a resemblance to triangles, that is, a similar shape to triangles, but are characterized by violating critical attributes of a triangle (i.e., a *closed shape with three straight sides and three vertices*) (see Fig. 1). The consideration of prototypical non-triangles, that is, other prototypical geometric shapes, such as squares, was omitted, since these are already reliably identified as non-triangles even by kindergarteners (Aslan & Aktaş Arnas, 2007; Tsamir et al., 2008).

The tasks were presented on a 24" full HD screen (refresh rate of 60 Hz; resolution of 1920 × 1080 pixels). The size of the prototypical triangle in Fig. 1 was 680 × 589 pixels in width and height. The distance between the students' heads and the monitor was about 60 cm.

Data collection

The data collection took place in individual sessions in a quiet room at the students' school. Each student was shown a series of 34 shapes (three prototypical triangles, 15 non-prototypical triangles, and 16 non-prototypical non-triangles). The tasks were arranged in a random order that was the same for each student. Before each shape was presented, the students looked at a fixation star displayed at the left side of the screen to ensure a clear transition from one task to another and to ensure that all students' gazes on the task started from the same place. Students were asked if the presented shape was a triangle and were instructed to answer "yes" or "no" as correctly as possible as soon as they knew the answer. They did not receive feedback on their answers. Students' oral responses were recorded using an audio recorder in order to examine error rates. Students' eye movements were recorded with the Tobii Pro X3-120 screen-based eye tracker (binocular, 120 Hz) (Tobii, 2019). The average ET accuracy in our study was 0.9°, which means that the deviation between the gaze position recorded by the eye tracker and the actual gaze position on the screen amounted to about 0.9 cm on average (with a distance of 60 cm between the students' eyes and the screen). Our study covered a total of 6290 trials (185 students × 34 shapes). Due to student-related data loss, where students did not answer the task (i.e., instead of answering incorrectly, the students did not answer at all), or technology-related data loss, where the eye tracker did not function properly and no valid recordings of students' eye movements were available, 41 trials (0.65%) were excluded, resulting in a final analysis of 6249 trials (553 trials for prototypical triangles, 2754 trials for non-prototypical triangles, and 2942 trials for non-prototypical non-triangles).

Qualitative data analysis

For the analysis of student strategies, we used gaze-overlaid videos provided by the software Tobii Pro Lab. The gaze-overlaid videos show students' gazes represented as a semitransparent dot (see <https://youtu.be/1uiMSWK0PCE> for an example).

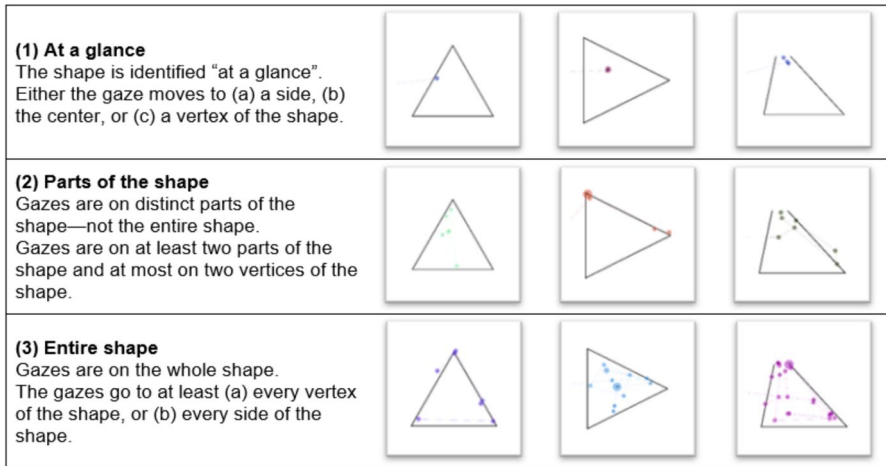


Fig. 2 Categories of student strategies and examples of students' gazes

This allowed us to account for all eye movements to infer strategies. Analysis of gaze-overlaid videos has already proven useful in analyzing student strategies in a detailed way (e.g., Schindler et al., 2020). This has also been shown previously for the analysis of student strategies for identifying triangles (Simon et al., 2021).

We categorized the students' gazes deductively, based on an afore developed category system. The category system was developed inductively in a previous study with kindergarten children by Schindler and colleagues.¹² All videos were coded by the first author of this paper. To investigate the reliability of the coding process, about 20% of the data (1258 trials) were coded a second time by the first author of this paper more than 15 weeks after the first coding of the data (intra-rater reliability). The large amount of data together with the time interval between the coding processes ensured that the rater did not remember the codings of the items assigned during the first coding. We calculated intra-rater reliability using Cohen's kappa (Cohen, 1960). The agreement was 0.92, which can be considered almost perfect (Landis & Koch, 1977). The category system (see Fig. 2) includes three categories, that is, strategies for identifying triangles. For visualizing gaze patterns corresponding to the different strategies, we use gaze plots (i.e., static visualization of scan-paths), although we used gaze-overlaid videos (i.e., dynamic visualization of scan-paths) for the analysis of strategies.

Due to our study design and participant group, we chose to focus only on ET and error rates and refrained from asking students to verbalize their strategies while thinking aloud (see Ericsson & Simon, 1980). In concurrent thinking aloud (introspective), the additional cognitive effort can be immense (e.g., Schindler & Lilienthal, 2018), which can interfere with task processing, especially for students with LD, for whom the task itself can be a major cognitive effort. Additionally, students

¹² The study has not been published, yet, but information can be provided upon request.

with LD may have difficulty verbalizing their strategies, as eye movements are not necessarily associated with conscious processes (Hayhoe, 2004). Retrospective thinking aloud also has aspects that may affect verbalization (in students with LD), such as metacognitive reflection or verbalization problems (Schindler & Lilienthal, 2018). We also refrained from using stimulated recall interviews with videos of students' eye movements as stimuli ("memory aid"): Our study included many tasks with short processing times for each task (Simon et al., 2022). In an exploratory study in which we used stimulated recall interviews, we found that even students without LD in grade 6 could not recall or interpret their eye movements in this domain; that is, the students were overexerted with the interpretation of these fast sequences and could not remember strategies (anymore) based on the short videos. Thus, we used ET data in combination with students' answers (to the question if the presented shape was a triangle) to analyze how students identify triangles.

Statistical analysis

For the statistical analysis, we used the software IBM SPSS 28. For comparing students with and without LD with respect to error rates for non-prototypical non-triangles, we used the Welch's test (corrected t-test for unequal variances), and calculated effect sizes using Cohen's d (Cohen, 1988). For comparing the two groups with respect to error rates for prototypical triangles and non-prototypical triangles, we used the Mann–Whitney U test as a non-parametric test because of not normally distributed error rates for the group of students with LD (Shapiro–Wilk $p < 0.05$). We calculated effect sizes using r (Cohen, 1988). Because of the different numbers of items grouped under the different types of triangles, the relative frequencies of errors were used. For comparing students with and without LD with respect to strategy use for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles, we used chi-squared tests based on the total number of strategies used by the students. We calculated effect sizes using Cramér's V (Cohen, 1988). Bonferroni–Holm adjusted p -values are reported.

Results

Error rates

The total number of errors was relatively high. Out of 6249 trials, the students made 1188 errors (19.01%) overall. The error rate of students without LD was 18.41% (1025 errors/5569 trials), and students with LD had an error rate of 23.97% (163 errors/680 trials), which are relatively high given that the probability of guessing correctly was 50%.

In the following, error rates are analyzed separately for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles. For prototypical triangles, "error" means that prototypical shapes were incorrectly rejected as triangles. For non-prototypical triangles, "error" means that shapes were incorrectly rejected

as triangles; that is, students considered mathematically irrelevant, non-critical attributes in their identification decisions. For non-prototypical non-triangles (i.e., with a triangle-like shape), “error” means that shapes were incorrectly accepted as triangles; that is, students did not consider mathematically relevant, critical attributes in their decisions.

For *prototypical triangles*, both students with LD (1.67%) and students without LD (2.84%) showed low error rates ($U = 1592.50$, $Z = -0.54$, $p = 0.591$). For *non-prototypical triangles*, students with LD did not make more errors than students without LD; in fact, they tended to make fewer errors for triangles (21.33%) than students without LD (29.18%). However, there was no significant group difference in error rates for non-prototypical triangles ($U = 1352.50$, $Z = -1.34$, $p = 0.364$). For *non-prototypical non-triangles*, there was a significant group difference with a large effect ($t(20.70) = 3.56$, $p = 0.006$, $d = 1.24$); that is, students with LD made significantly more errors on non-prototypical non-triangles (30.63%) than students without LD (11.25%). Figure 3 shows the error rates of students with and without LD for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles.

Students' strategy use

We also investigated strategy use of students with and without LD separately for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles. For *prototypical triangles*, students with LD tended to use *Strategy 1* (at a glance) more often and *Strategy 3* (entire shape) less often than students without LD. This means that students with LD tended to look at fewer parts of the presented shapes more often than students without LD. Yet, there was no significant group difference for prototypical triangles ($\chi^2(2) = 5.73$, $p = 0.114$). For *non-prototypical triangles*, chi-squared tests revealed significant differences in the strategy use between students with and without LD ($\chi^2(2) = 8.70$, $p = 0.039$, $V = 0.06$). Cell tests for the strategy use for non-prototypical triangles revealed that students with LD used *Strategy 1* (at a glance) significantly more often than

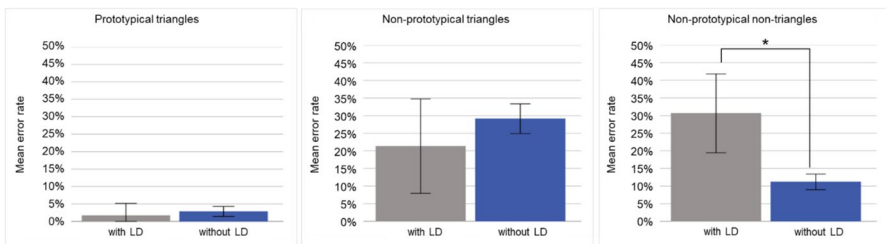


Fig. 3 Error rates of students with and without LD for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles. (For better comparability of the error rates of the two groups, we visualize the mean error rates for different types of triangles in Fig. 3, even though the Mann–Whitney U test is a rank-sum test and not based on mean comparisons.) Note. Vertical bars denote 95% confidence interval; * $p < 0.05$

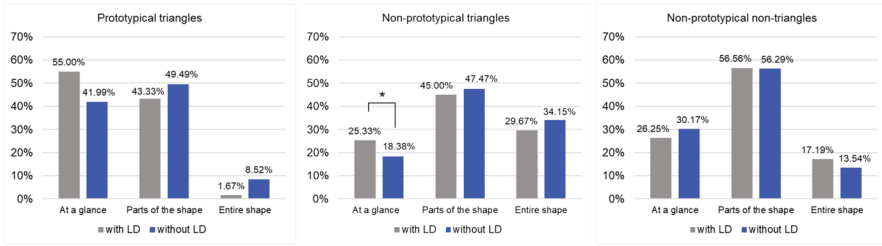


Fig. 4 Students’ strategy use for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles. Note. * $p < 0.05$

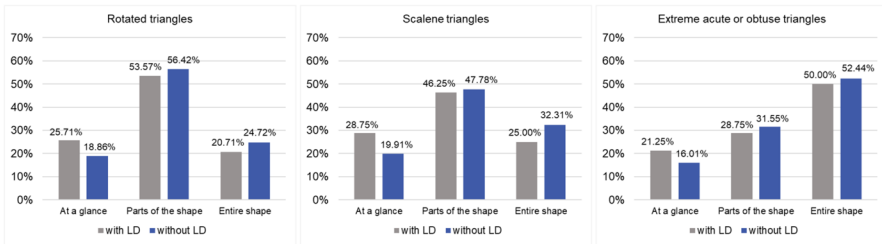


Fig. 5 Students’ strategy use for different types of non-prototypical triangles

students without LD with a small effect ($\chi^2(1) = 8.36, p = 0.015, V = 0.06$). This means that students with LD looked at fewer parts of the presented shapes more often than students without LD. For *non-prototypical non-triangles*, students with LD tended to use *Strategy 1* (at a glance) less often and *Strategy 3* (entire shape) more often than students without LD. This means that students with LD tended to look at more parts of the presented shapes more often than students without LD. Yet, there was no significant group difference for non-prototypical non-triangles ($\chi^2(2) = 4.20, p = 0.123$). Figure 4 shows the use of strategies of students with and without LD for prototypical triangles, non-prototypical triangles, and non-prototypical non-triangles.

Since our study included only a small number of items for the different types of non-prototypical triangles and non-prototypical non-triangles and a relatively small number of students with LD, we have not conducted further statistical analysis. Instead, we investigated strategy use for different types of non-prototypical triangles and non-prototypical non-triangles descriptively, which will be presented in the following sections.

Non-prototypical triangles in this study include rotated triangles with a prototypical aspect ratio, scalene triangles with an acute or obtuse angle, and isosceles triangles with an extreme acute or obtuse angle. For all three types of non-prototypical triangles, we found a similar trend: Students with LD tended to use *Strategy 1* (at a glance) more often, while they tended to use *Strategy 2* (parts of the shape) and *Strategy 3* (entire shape) less often than students without LD (Fig. 5). This means that students with LD tended to look at fewer parts of the shapes more often than students without LD for all types of non-prototypical triangles. Figure 5 shows the

use of strategies of students with and without LD for different types of non-prototypical triangles, that is, rotated, scalene, and extreme acute or obtuse triangles.

Non-prototypical non-triangles in this study include non-triangles with a gap, non-triangles with curved sides or rounded vertices, and non-triangles with more than three vertices. For the different types of non-prototypical non-triangles, a diverse picture emerged. The most divergent distribution in strategy use was found for non-triangles with more than three vertices (Fig. 6): Students with LD tended to use *Strategy 1* (at a glance) less often, while they tended to use *Strategy 3* (entire shape) more often than students without LD. This means that students with LD tended to look at more parts of the shapes more often than students without LD. Figure 6 shows the use of strategies of students with and without LD for different types of non-prototypical non-triangles, that is, non-triangles with a gap, with curved sides or vertices, and with more than three vertices.

Discussion

The aim of this study was to investigate whether students with LD differ from students without LD in the identification of triangles. We analyzed students' strategies and error rates for identifying triangles, using prototypical triangles, non-prototypical triangles (each representatives), and non-prototypical non-triangles (non-representatives) based on ET videos of 185 students with LD ($n = 20$) and without LD ($n = 165$). Our results indicate that students with LD differed from students without LD on the important geometric topic of the identification of triangles in an unexpected way, in that they partially performed better or equally well, as is detailed in the following sub-section.

Empirical findings

Our study indicates that, in some cases, students with LD used more quick strategies and tended to make fewer errors than their peers without LD, depending on the types of triangles. For *prototypical triangles*, students with LD tended to identify triangles more often at a glance, whereas students without LD tended to look at the entire shape more often. However, there was no significant difference in the use of strategies between both groups of students—possibly related to the small number of trials

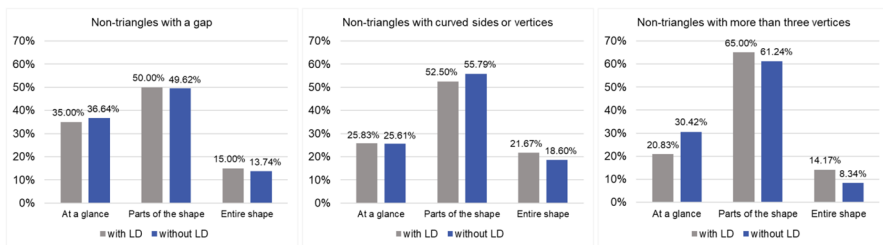


Fig. 6 Students' strategy use for different types of non-prototypical non-triangles

considered. Error rates for prototypical triangles were low for both students with and without LD. For *non-prototypical triangles*, there was a significant difference in strategy use between students of both groups: Students with LD looked at fewer parts of the presented shapes more often than students without LD. This means that students with LD used quick strategies, that is, strategies involving only few gazes significantly more often than students without LD. This finding is interesting, since the use of fewer gazes is associated with processing efficiency and expertise. “With increasing skill, more information is extracted around the point of fixation making eye movements overall more efficient” (Holmqvist et al., 2011, p. 383). This has also been shown for the identification of geometric shapes: With increasing expertise, foveal analysis is replaced by extrafoveal analysis (Chumachenko et al., 2024). Error rates showed that the use of more quick strategies by students with LD for non-prototypical triangles was also related to fewer errors by students with LD, although there was no significant group difference possibly due to the relatively small sample size of students with LD and a large variance in error rates within the group of students with LD. Nevertheless, students with LD did not make more errors but even tended to make less errors than students without LD. Regarding strategy use, effect sizes were small.

The fact that students with LD used more quick strategies and did not make more errors in identifying non-prototypical triangles is an interesting and even surprising finding, considering that students with LD typically tend to be slower at learning and to perform lower in mathematics than their peers without LD (Gebhardt et al., 2015; Werner et al., 2019). Also, considering that students with LD might have difficulties in spatial-geometric understanding, such as in transforming geometric shapes (Grobeck & De Lisi, 2000), this result is surprising. For example, mental transformations of non-prototypical triangles, such as stretching a triangle with an extreme acute angle to obtain a more prototypical aspect ratio or rotating a triangle to obtain a horizontal base, might have played a role in children’s identification decisions. However, students with LD appeared to have less difficulties in identifying non-prototypical triangles as compared to students without LD. This suggests that children with LD were well able to identify triangles that differ in appearance in non-critical attributes (e.g., skewness) from prototypical triangles. This indicates that mathematically irrelevant, non-critical attributes did not influence the considerations for the identification of triangles by students with LD as much as might have been expected, and not as much as for students without LD. Although the effect sizes are small, our findings suggest that students with LD did not appear to have difficulty identifying non-prototypical triangles as compared to students without LD. They did not perform worse, but rather appeared to use more quick strategies and tended to make fewer errors than their peers without LD. Similarly, for prototypical triangles, although there was no significant group difference, students with LD did not perform worse, but tended to perform better than students without LD.

The differences found between students with and without LD are also interesting against the background that the students with LD had performed lower than the students without LD in the standardized mathematics test conducted before the ET study. The differences in performance in arithmetic tasks between students with and without LD found in this test were not found in the geometric activities in this ET

study. It appears that, at least for the domain of geometry (e.g., Cawley et al., 2009), children with LD partially perform better or at least perform not worse than children without LD. When interpreting these results, the students' teaching must also be taken into account: It is possible that the children with LD, who attended a special school, were taught differently than their peers, and that the teachers' focus was, for example, more on showing, experiencing, and handling examples, including non-prototypical examples of triangles, than in the teaching of students without LD.

On the other hand, when identifying *non-prototypical non-triangles*, students with LD tended to look at more parts of the shapes more often, and they made significantly more errors than students without LD. This means that students with LD tended to use less quick strategies; that is, they tended to look at the entire shape more often, whereas students without LD tended to identify non-prototypical non-triangles more often at a glance. Looking at more parts of the shape could also mean that students with LD paid more attention to the properties of the shapes. However, results indicate that students with LD needed more gazes and mistakenly accepted non-triangles that looked similar to a prototypical triangle (e.g., having a triangle-like shape but curved sides or more than three vertices) more often than students without LD. This shows that students with LD oftentimes—almost in one-third of the tasks—did not consider mathematically relevant, critical attributes in their identification decisions when non-triangles had triangle-like shapes. Regarding error rates, we found a large effect size. One factor that may have played a role in the differences between students with and without LD in the identification of non-prototypical non-triangles could be differences in teaching, and that children with LD who are considered to have difficulties in mathematics had fewer opportunities to learn about these more complex shapes than students without LD.

Our results suggest that students with LD were well able to identify triangles with both prototypical and non-prototypical appearances and thus did not orient solely to the (prototypical) appearance of a triangle and non-critical attributes, but were often able to consider critical attributes, that is, properties of triangles, which are distinctive of the “descriptive” level in the van Hiele theory (van Hiele, 1986). The high frequency of acceptance of non-triangles with a triangle-like shape indicates a high orientation to the appearance of shapes for non-prototypical non-triangles, which is distinctive of the “visual” level in the van Hiele theory (van Hiele, 1986). However, the use of strategies by students with LD and the fact that they looked intensively at non-prototypical non-triangles indicate that students with LD took an analytic approach but made incorrect identification decisions due to missing content knowledge.

In general, the identification of prototypical triangles was easiest for students with and without LD. This is in line with previous ET studies, which also have shown that the identification of prototypical shapes is easiest and requires few to no gazes on the shape (i.e., identification of shapes by extrafoveal vision) (Shvarts et al., 2019; Simon et al., 2021). The relatively high number of shapes that required extensive student gazes and the relatively high overall error rate (19.01%)—consistent with the error rates of students in grade six (Clements & Battista, 1992; cited in Clements & Sarama, 2000)—indicate that identifying triangles was generally not easy for the fifth graders in our study, regardless of diagnosed LD.

Pedagogical implications

For educational practice, our results indicate that students with LD in fifth grade were well able to identify triangles with prototypical appearances. Furthermore, students with LD were also quite capable of correctly assessing non-critical attributes for validity as representatives of triangles—even better than students without LD. Therefore, mathematically irrelevant, non-critical attributes did not seem to negatively influence triangle identification considerations. However, the high frequency of acceptance of non-prototypical non-triangles as triangles also indicates that critical attributes of triangles (e.g., three vertices) were often not considered. A possible explanation for the differences between children with and without LD in the identification of triangles could be the previous teaching in mathematics. Even though students with LD in North Rhine-Westphalia are supposed to be taught closely to the curriculum of primary or lower secondary school, it could be that their teaching focused more on triangles and less on non-prototypical non-triangles and, accordingly, less on critical attributes of triangles.

Considering the above-mentioned difficulties, especially for children with LD, educators need to help students understand the critical attributes, such as the number of sides and vertices of a triangle (Satlow & Newcombe, 1998), and address the critical attributes of a triangle more extensively in their teaching. Geometry instruction, especially for students with LD, should include activities that explicitly focus on distinguishing between critical attributes and non-critical attributes, for example, by representing or creating a variety of triangles: triangles in different orientations and with different sizes of angles (non-critical attributes: skewness and aspect ratio) and shapes that are triangle-like but violate critical attributes of triangles. The development of a formal understanding of what constitutes a triangle could be fostered through exposure to unfamiliar shapes (Edwards & Harper, 2010) and by focusing on distinguishing and identifying examples and non-examples of triangles. To foster students' ability to assign mathematical meaning to the critical attributes, for example, transforming non-examples into examples of triangles or transforming prototypical examples into non-prototypical examples could be a helpful activity (e.g., in “dynamic geometry environments”; e.g., Kaur, 2015). It may be beneficial to explore and discuss with the students why a triangle remains a triangle when a prototypical triangle is transformed into a non-prototypical triangle, and how a triangle can be transformed into a non-triangle and vice versa. Students' understanding may be deepened by encouraging them to describe the properties of triangles and to categorize shapes based on these properties. This could be supported by discussing the meaning of necessary terms related to triangles.

In general, appropriate activities should represent a variety of geometric shapes, paying attention to formal concept definitions of geometric shapes (Hannibal, 1999). Following Tsamir and colleagues (2008), “one of our major aims, as educators, is to bring our students to use only critical attributes as the deciding factor in identifying examples and forming geometric concepts” (p. 83).

Methodological implications

Our results suggest that in geometry, for our tasks with short processing times (Simon et al., 2022), it is valuable to use ET in combination with error rates. Based on the category system, we were able to code and interpret the children's gazes for the identification of triangles relatively easily. This is in line with studies in other mathematical subdomains in which student strategies in tasks with short processing times could be reliably analyzed based on developed category systems (e.g., for quantity recognition, Schindler et al., 2020; for quantity comparison and pattern extension, Pitta-Pantazi et al., 2024). Interpretation of students' identification decisions in relation to the strategy use showed that identifying triangles *at a glance* was associated with a higher rate of correct responses, whereas looking at the *entire shape* was associated with a higher number of errors. The latter suggests that looking at the entire shape in detail was often not associated with students scanning the shapes for violations of critical attributes of triangles. Rather, looking at the entire shape appeared to indicate difficulties in identifying triangles. These results are in line with findings on the identification of quadrilaterals by Shvarts and colleagues (2019): Easy-to-identify prototypical shapes were often identified by extrafoveal vision, and more-difficult-to-identify non-prototypical shapes required more gazes at these shapes. Also, Chumachenko and colleagues (2024) found that efficient identification of geometric shapes increases with growing expertise. Our ET study connects to other studies that have shown the potential of analyzing strategies based on student gazes in the field of mathematics education research (e.g., Schindler & Lilienthal, 2018; Simon & Schindler, 2020).

Limitations and implications for future studies

One limitation of this study is the sample. There is a relatively large difference in sample size between the group of students with LD and the group of students without LD. Furthermore, the group of students with LD is relatively heterogeneous in terms of error rates. This is associated with limitations for statistical analysis, which were addressed in the statistical analyses performed. Nonetheless, this study had a relatively large sample size for an ET study with school students (for an overview of sample sizes of other ET studies, see Strohmaier et al., 2020) and provides insight into strategy use for the identification of triangles for many students with and without LD. In general, it is important to say that it cannot be assumed that groups of students with LD are homogeneous groups in terms of their mathematical learning. Furthermore, it must be noted that even though we found statistically significant differences in strategy use between students with and without LD, the effect sizes were small, indicating practically small differences. However, our findings show an interesting trend: Students with LD did not perform worse than students without LD in an essential activity for students' understanding of geometric concepts: identifying geometric shapes. Future research should attempt to involve larger groups of students with LD and achieve

similarly large group sizes for better comparability, especially with regard to statistical analyses of students with and without LD. In addition, further studies should include more students with LD, also from inclusive schools, to obtain more representative samples and to investigate if this study's trends can be confirmed when the students with LD have received teaching at an inclusive school, together with their peers without LD.

Our results must be interpreted against the background of various factors that have an influence on LD: the learning opportunities of the students, the learning resources available at the school, and the performance requirements formulated by the school (Dudley-Marling, 2004; Heimlich et al., 2016). We do not have detailed data on the previous mathematics teaching of the participating students with LD. It is important to say that we cannot draw any conclusions about the general ability of students with and without LD to identify triangles. The only conclusions we can draw are about the students' identification strategies and performance at the time our study took place, which was influenced by their previous teaching.

Another possible factor influencing students' response behavior is that we did not include prototypical non-triangles (e.g., squares or circles) in our study. In studies investigating the identification of triangles by children of kindergarten age (e.g., Aslan & Aktaş Arnas, 2007; Clements et al., 1999), children's considerations for identifying triangles have been shown to be influenced depending on the shapes shown. However, because our study involved children in grade 5, we estimate the impact on students' decisions for the identification of triangles—a topic in mathematics education in the first 2 years of primary school (e.g., NCTM, 2000)—to be small. Nevertheless, a similar study including prototypical non-triangles could provide a valuable comparison for this study.

Conclusion

Our study adds to the—so far rare—research findings about the mathematical learning of students with LD. We believe that these results contribute to research on mathematical learning of children with LD: Students with LD appear not to just have difficulties in this aspect of geometric learning but also particular strengths as compared to students without LD. This suggests that a more sophisticated picture of geometric learning for children with LD emerges than previous depictions of LD suggest (see, e.g., Grünke & Cavendish, 2016, p. 3).

Author contribution All authors contributed to the study conception and design. Material preparation, data collection, and analysis were performed by Anna L. Simon and Maïke Schindler. The first draft of the manuscript was written by Anna L. Simon and Maïke Schindler, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Data Availability The data that support the findings of this study are available from the corresponding author, ALS, upon reasonable request.

Declarations

Ethical approval and consent to participate Ethical review and approval were not required for the study on human participants in accordance with the local legislation and institutional requirements. Written informed consent was obtained from all legal guardians of the students participating in the study.

Conflict of interest The authors declare no competing interests.

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