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Johannes Hjelmslev – mathematician and mathematics educator

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ABSTRACT

Johannes Hjelmslev (1873 - 1950) was a renowned Danish mathematician and one of the most remarkable mathematicians of his time. A special feature is that he was active in two fields, mathematics (the foundations of geometry) and the didactics of mathematics (the didactics of geometry). He not only contributed original and fruitful ideas to both areas, but also had a significant impact on the concept of mathematics and school mathematics until the present day. While historical appraisals have already been published on individual topics of his research work - for example on his so-called geometry of reality (Lützen, 2021) - a summarizing presentation and appraisal of his work in both fields in a historical context is still lacking. This is the aim of this article, which, incidentally, is a response on the less appreciative assessment of Hjelmslev's achievements and his geometry of reality by Lützen (2021).

1. Introduction

Johannes Hjelmslev (1873–1950) is one of the best-known Danish mathematicians of the 20th century. He became famous for his *Neue Begründung der ebenen Geometrie* of Hilbert's plane absolute geometry (Hjelmslev, 1907). This is defined by the axiom system, which consists of the plane axioms of incidence, order and congruence from D. Hilbert's classic work *Grundlagen der Geometrie* (Hilbert, 1899). With his opus, Hjelmslev also laid the foundations of reflection geometry.

However, Hjelmslev became known not only as a mathematician but also as a didactician of mathematics. He designed a structure for school geometry in which it was systematically developed as an empirical (scientific) theory ((Hjelmslev, 1939–1940) and (Hjelmslev, 1940–1952)).

In his obituary of Hjelmslev, H. Bohr (1950, VIII) emphasizes another original aspect of Hjemslev's work: "He tried to bridge the gap from the practical geometry, e.g. of the drawing board with its "real" points, straight lines etc., to the theoretical plane, e.g., the analytical plane where the points are given by their coordinates." In his "Virkelighedsgeometrien"(i.e. geometry of reality) there are straight lines which have more than one point in common.

In his essay *Hjelmlev's geometry of reality*, which is well worth reading, J. Lützen (2021) deals with this geometry. In his "Conclusions", however, he comes to the—for some readers probably sobering—conclusion that Hjelmslev's ideas were not taken up by his contemporaries and successors. According to Lützen (2021, p. 114), Hjelmslev "continued to believe in his geometry" though (now quoting Fog (1950, p. 13)) "his geometry was not viable in the form he had given it". It seems that Hjelmslev has failed somewhat tragically.

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In our article, we attempt to place Hjelmslev's entire work in its historical context. To this end, we distinguish between the context of mathematics didactics in the 19th and early 20th centuries and the context of mathematics in this period, and consider, in addition, the international impact of Hjelmslev's research. In contrast to J. Lützen, who focuses on Hjelmslev's *geometry of reality* and, in our opinion, did not distinguish sufficiently between the different contexts, we conclude that Hjelmslev was very successful in both fields, geometry and geometry didactics, and that his ideas and approaches continue to influence the development of both fields on an international level today.

In the first part of our essay we describe Hjelmslev's work as a mathematics educator, in the second part as a mathematician. In order to adequately assess and evaluate Hjelmslev's work, we consider the historical context and circumstances in which Hjelmslev worked, both in mathematics education and in mathematics as a discipline. It turns out that Hjelmslev recognized the significant reform movements of his time in both areas and contributed substantially to their further development.

The article is structured as follows: We firstly deal with Hjelmslev work in mathematics education, which he developed in the 1890ths. To set the stage we describe the traditional teaching of geometry in the 19th century where geometry lessons were largely based on Euclid's *Elements*. The aim of the lessons was to provide the students with a "formal education", the power of abstract reasoning. We then outline the criticism of this traditional doctrine that arose as a result of the industrial revolution in Europe. The aim of the emerging educational reform movement was to provide pupils with a "material education" in which applications of mathematics are crucial. Against the backdrop of these developments, we will then present Hjelmslev's proposals and evaluate them.

Dealing with Hjelmslev as a mathematician, we firstly describe the historical starting point of Hjelmslev's mathematical activities, the traditional conception of mathematics and geometry, particularly in the 19th century. We will illustrate this using two examples from the works of A. F. Möbius and M. Pasch. In 1899 D. Hilbert presented in his book *Grundlagen der Geometrie* a new conception of mathematics, the now called "formalistic" view. Hilbert's perspective offers mathematicians considerably more freedom and possibilities in their scientific work. We show how Hjelmslev took advantage of these freedoms. 1907 he contributed to the new conception of mathematics with a now famous article about the *Neue Begründung der ebenen Geometrie*.

Subsequently we discuss Hjelmslev's *Allgemeine Kongruenzlehre* (1929), that is, the mathematical theory underlying his "geometry of reality". This work is a vast generalization of *Neue Begründung der ebenen Geometrie* since 'neighboured' points may have more than one joining line. We then study the reception of Hjelmslev's work and show that it had a considerable impact on various branches of geometry, including reflection geometry, incidence geometry, ring geometry and circle geometry, and that the underlying ideas are still being taken up today.

The article concludes with a final assessment of Hjelmslev's work in both areas, mathematics and mathematics education. What is striking is the different understanding of geometry that Hjelmslev advocates on the one hand in mathematics education, i.e., for schools, and on the other hand in mathematics as a scientific discipline. We discuss this discrepancy and show that it is only paradoxical at first glance. It turns out that Hjelmslev's overall work can be evaluated far more positively than Lützen (2021) suggests.

To sum up, the structure of the article is the following

1. Introduction
2. Hjelmslev as a math didactician
   2.1 The traditional teaching of geometry in the 19th century - the role of Euclid's *Elements*
   2.2 Criticism of Euclid and suggestions for improvement
      2.2.1 Social and educational developments in Europe in the 19th century
      2.2.2 The development in mathematics didactics - Klein's reform movement
   2.3 Hjelmslev's approach - his criticism and his proposals
   2.4 Evaluation of Hjelmslev's approach
3. Hjelmslev as a mathematician
   3.1 The historical starting point of Hjelmslev's mathematical activities
   3.2 Hjelmslev and Hilbert
   3.3 Hjelmslev's geometry of reality and its reception
   3.4 The geometric heritage of Hjelmslev
4. Conclusion

We close this introduction with a short biography of our protagonist:

Johannes Hjelmslev, originally called Johannes Trolle Petersen, was born in Horning near Aarhus (Hjelmslev district) in 1873. He began studying mathematics at the University of Copenhagen in 1890 and graduated with a master's degree in 1894. Three years later, he completed his doctorate with a thesis on descriptive geometry. During his doctorate, he taught at the *Borgerdydskolen* in Copenhagen for a total of ten years. In 1903 he was appointed as a lecturer, and two years later as the chair of descriptive geometry at *Den polytekniske Læreanstalt*. In 1904 Johannes Petersen changed his surname to Hjelmslev, the name of the area in which he was born. The reason for this was a frequent confusion between his birth name and one of his mathematics teachers at Copenhagen University, Julius Petersen (after whom the Petersen graph is named). Not only was the surname the same, but also the first letter of the first name, so that it was not clear who was meant by "J. Petersen". In 1917, Hjelmslev was appointed to a professorship at *Kobenhavns Univesitet*, where he taught and researched until his retirement in 1942. In 1928/29 he was rector of the university.

Hjelmslev died in Copenhagen in 1950. The well-known linguist Louis Hjelmslev was his son, born in 1899. Father and son were colleagues at the University of Copenhagen for five years.[1]

2. Hjelmslev as a math didactician

In order to understand Hjelmslev's didactic ideas and approaches, we will first describe the historical situation in which he made his suggestions in the 1890ths. In the first half of the 19th century, the teaching throughout Europe was largely determined by the *Elements* of Euclid, sometimes in a very rigid manner. We call this the *traditional teaching* of geometry and describe it in Section 2.1. In the subsequent section we present the criticisms that have been made of this approach and which gave rise to a *reform movement*. In Section 2.3, we describe Hjelmslev's ideas and his systematic approach against this background. We show that Hjelmslev's criticism of contemporary geometry teaching (and in particular the use of Euclid's *Elements* in the classroom) was very much in keeping with the spirit of the times, but that his systematic proposal for a geometry curriculum was quite innovative and that his basic idea has been implemented to this day. We discuss this in a Section 2.4.

2.1. The traditional teaching of geometry in the 19th century - the role of Euclid's Elements

Euclid's *Elements* played an outstanding and formative role in the teaching of mathematics in secondary schools in probably all European countries until the end of the 19th century.

For example, the regulations on the school-leaving examination of the Kingdom of Prussia from 1812 set out the content of geometry lessons at secondary schools (according to (Schimmack, 1911, p. 3)) with direct reference to Euclid: "… elementary geometry (insofar as it is presented in the first six and the 11th and 12th books of Euclides) and plane trigonometry". As late as 1875, J. Falke (1875) (quoted from (Schotten, 1893, p. 97)) described the situation in geometry lessons at secondary schools with the words: "For centuries, Euclid's work was probably the only textbook on planimetry; only a few years ago, it was introduced as a guide in many German schools … ". There was a reason for Euclid's paramount importance, which lay in the philosophy of education: in accordance with the neo-humanist aims of education in the 19th century, which had two politically influential proponents in Prussia, Karl August von Hardenberg and Wilhelm von Humboldt, '*formal education*' was to be taught in the humanist gymnasia. Euclid's works seemed predestined for this.

Treutlein—co-author of a textbook judged by F. Klein (1908, p. 261) to be "extremely noteworthy", which we will discuss in more detail later—characterizes the "principle of formal education"[2] as "the emphasis on the purely scientific, the withdrawal or disappearance of applications, and the turn to the more or completely abstract" (Treutlein, 1911, p. 40). The "almost sole task of mathematical teaching", he continues, is "to ensure logical rigor, above all to achieve logical training both through individual work and through the requirement of gaining clear insight into the context of all theorems of the systematically ordered lecture."[3]

In the last sentence, Treutlein addresses a methodological consequence for teaching that was derived from the curricular objectives at the time: the best way to present the logical structure is to present it clearly. This is how the "much praised … strictly dogmatic lecture" came about, as Treutlein wrote. This was supplemented, and sometimes replaced, by memorization by the students. In the foreword to his textbook for teaching geometry, J. K. Becker (1871, Vorwort) described his own experiences as follows:

> The basis and formative teacher was Brettner's textbook, which was widely used at the time, a textbook written according to the strict Euclidean method … From one lesson to the next we had to study a theorem from this book with proof, without instructions, and without any mention of it beforehand, and then reproduce what we had learned on the blackboard in the next lesson. Of course, this only worked as long as the memory faithfully reproduced what had been painstakingly memorized. If there was a stagnation, the Spanish reed[4] …[5]

F. Klein (1908) looked in his well-known book *Elementary Mathematics from a Higher Standpoint* not only at geometry teaching in Germany but also in England, France and Italy. The "English school-Euclid" (Klein, 1908, p. 232)[6] of the 19th century contains exactly the books of Euclid that were already listed in Prussia in 1812 (the planimetry in books 1-6 and the beginnings of spatial geometry and the exhaustion method). Klein added "and all in literal translation". According to Klein, Euclid's *Elements* were "a 'standard work' that every examinee knows".

"While the strictly conservative Englishman adheres to the old institutions", wrote F. Klein (1908, p. 236), "the Frenchman loves the new". According to Klein, this explains that criticism of Euclid arose in France as early as the 18th and 19th centuries and

[1] Obituaries for J. Hjelmslev were written by H. Bohr (1950), J. Nielsen (1950), D. Fog (1950) and B. Jessen (1950).

[2] Grundsatz der formalen Bildung.

[3] Das Betonen des rein Wissenschaftlichen, das Zurücktreten oder Verschwinden der Anwendungen, dafür die Hinwendung zum mehr oder völlig Abstrakten … fast einzige Aufgabe des mathematischen Unterrichtes, unbedingt für logische Strenge zu sorgen, vor allem logische Schulung zu erzielen sowohl durch Einzelarbeit wie durch die Zielforderung, … klare Einsicht zu gewinnen in den Zusammenhang sämtlicher Sätze des systematisch geordneten Vortrages.

[4] This instrument, the so-called 'spanische Rohr', was used for severe punishment, in which children were beaten with a stick.

[5] Als Grundlage und eigentlicher Lehrmeister diente das seiner Zeit weit verbreitete Lehrbuch von Brettner, ein nach streng euklidischer Methode geschriebenes Lehrbuch … Aus diesem Buche mußten wir von je einer Stunde zur andern jedesmal einen Lehrsatz mit Beweis, ohne alle Anleitung, und ohne daß vorher irgendwie davon die Rede war, einstudiren und das Gelernte dann in der nächsten Stunde an der Tafel reproduciren. Das ging nun natürlich nur so lange als das Gedächtnis das mühsam auswendig gelernte getreu wiedergab. Trat eine Stockung ein, so wurde das spanische Rohr …

[6] Here and in the following in the English translation of G. Schubring.

alternative textbooks on geometry were written. These include Clairaut's *Eléments de géométrie*, written in 1741, and in an "excellent style … in sharpest contrast to the uniform stereotyped 'Euclidean' style". An even more important textbook was Legendre's *Eléments de géométrie* from 1794. It had a dominant influence in France for a long time. During Legendre's lifetime, 15 editions were published. A. L. Crelle, known as the editor of the journal named after him, translated it into German in 1822 and began his foreword with the following sentences: "Legendre's textbook of geometry … is characterized by richness of content, clarity, order and consistency of presentation, accuracy and rigour of proofs. It approaches the Euclidean high standard" (translations by the authors). The model of Euclid is already recognizable in the title of the work. Legendre structured his book like Book I of Euclid's *Elements* and also wrote à la Euclid. Legendre's book represents a big step back to Euclid. However, it supplements the content of Euclid's work, for example, by dealing with plane and spherical trigonometry.

In Italy, many textbooks in the first half of the 19th century were influenced by Legendre's *Eléments de géométrie*, up to literal translations (cf. (Menghini, 1994)). In 1867, the well-known mathematician L. Cremona recommended the direct use of Euclid's *Elements* as a textbook for secondary school. He criticized Legendre's textbook for its excessive use of algebra. The minister responsible for education, Coppino, wrote in 1867 in a Law of Education (quoted from (Menghini, 2016, Section 2.1)):

> Mathematics in classical secondary schools is not to be regarded only as a set of propositions or theories, which are useful in themselves … to be applied to the needs of life; but primarily as a means of intellectual culture, as a gymnastics of the mind aimed to develop the power of reasoning, and to help that right and sane criterion needed to distinguish what is true …

Only Euclid's *Elements* could fulfill this task, a view that was still decisive for geometry teaching in almost all types of schools in Italy in the 20th century (Menghini, 1994, p. 143).

Euclid was also highly regarded in Denmark. The *Elements* were translated into Danish for the first time in 1744 (see (Ziegenbalg, 1744)), and a second translation was published in 1803. This translation was written by H. C. Linderup, a mathematics teacher at the renowned *Metropolitanskolen* in Copenhagen, who explicitly intended the translation—according to its title—for use in the classroom: *De förste sex Böger af Euclidis Elementer til Brug i de forandrede lærde Skoler*.[7] J. L. Heiberg, Professor of Classics at the University of Copenhagen, published a complete edition of Euclid's works together with H. Menge from 1896 to 1924 (Heiberg and Menge, 1883-1916). This edition of the *Elements* became the definitive text from which all later scholarly translations, such as Heath's into English, derived. A student, T. Eibe, translated the first six books of the *Elements* of Heiberg's edition from Latin into Danish (Eibe, 1897–1913).

2.2. Criticism of Euclid and suggestions for improvement

The criticism of Euclid-oriented geometry teaching was based on socio-political developments, which we therefore describe in the first section below. In the second section, we look at mathematics didactics in the larger European countries and describe how they are reacting to the socio-political changes. It becomes clear that a European movement had developed in mathematics didactics.

2.2.1. Social and educational developments in Europe in the 19th century

In European history, the 19th century is also known as the *age of the industrial revolution*. The country of origin of industrialization was England, where technical innovations such as the steam engine enabled the mechanical production of goods and thus a new form of economic growth as early as the 18th century. Most European countries were caught up in the process of industrialization between the middle of the 19th century and the beginning of the First World War in 1914. Germany went through an early industrial phase (ca. 1840–1870) and a highly industrialized phase (ca. 1870–1914) and became a modern industrial state during this period.

The term 'Industrial Revolution' refers to the historical development in which industrial production increasingly replaced agricultural and artisanal production. Factories were built and large-scale industry developed. The construction of machine tools for the mass production of tools allowed an enormous increase in production, and the building of railroads and steamships made it possible to supply cities with coal, i.e. energy, and gave many cities an economic upturn. - The main prerequisites for the industrial revolution were technology and science. These experienced an unexpected, rapid upswing.[8]

As a result, society's demand for school and university graduates with a good scientific and technical education increased considerably. Mathematics was of course also indispensable for the real sciences—but not (or not only) mathematics that imparted formal education (in the sense described above by Treutlein's "principle of formal education" and Coppino's Law of Education). Instead mathematics had to be taught that could be used to solve real problems. In view of this, the curriculum for secondary schools became business-oriented towards vocational training. The corresponding education was coined as *"material education"* - in contrast to "formal education".

In order to meet this social need, new types of schools were developed, in Prussia for example *Realschulen* and *Oberrealschulen*, whose names already indicate their special educational mission. The aim of Prussian secondary schools was "on the one hand to provide their pupils with a higher level of education than is the case in the multi-grade elementary schools, but on the other hand also to take into account the needs of economic life and the so-called middle class to a greater extent than can regularly be the case in secondary schools" (according to the "Allgemeine Bestimmungen" of 1872, quoted from (Herrlitz et al., 1986, p. 96)). - A

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[7] The first six books of Euclid's *Elements* for the use in secondary schools.

[8] A detailed description and discussion of the industrial revolution can be found in (Landes, 1969) and (Trebilcock, 1981).

similar development took place in the higher education sector. In addition to the traditional universities, new technical universities (universities of applied sciences) emerged from polytechnic schools. The *Polytechnische Schule* founded in Karlsruhe in 1825 became *Polytechnische Hochschule* in 1865 and *Technische Hochschule* in 1885. In 1855, the *Polytechnikum* was founded in Zurich, which was renamed the *Eidgenössische Technische Hochschule Zürich* in 1911. In Copenhagen, the *Den Polytekniske Læreanstalt* was founded in 1829—where Hjelmslev was appointed professor of descriptive geometry in 1905—and renamed *Danmarks tekniske Højskole* in 1933 (and *Danmarks Tekniske Universitet* in 1994).

The debate about formal education on the one hand and material education on the other was sometimes very heated between the supporters of humanist and realist Educational movements.[9] For Prussia, Schimmack (1911, p. 9) described how the “up-and-coming Realschulen struggled with the old Gymnasium for equal rights and equal prestige”. This battle was successful—even if it took until 1900 for the (Ober-)Realschulen to gain equal rights and for their graduates to be allowed to study all subjects offered at a university. This was a groundbreaking success for mathematics as a school subject.

In Felix Klein, the reform movement had a prominent supporter, who was also respected in the Prussian administration. Klein recognized the opportunities for mathematics if this subject were taught not only to impart formal education, but additionally to impart material education at secondary schools—and analogously as a fundamental subject at traditional universities and technical universities.

2.2.2. The development in mathematics didactics - Klein’s reform movement

In the discussion on educational philosophy described above, Euclid was criticized in various ways and a large number of suggestions for improvement were made.[10] We describe the innovations initiated by the European reform movement using the example of Germany, the country that most strongly influenced Denmark, Hjelmslev’s home country, and then briefly discuss other countries.

In Germany, the discussion about the role of science and mathematics in the various types of schools was given a new platform when the world’s first journal on the didactics of mathematics and science, the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht (ZMNU)*, was founded in 1869. In the words of its editor J. C. V. Hoffmann, see (Hoffmann, 1870, p. 6 ff.), the journal was intended to “provide information on the organization, methods and educational value of the exact sciences … ”. The term ‘exact sciences’ here refers to mathematics and the natural sciences, which is also indicated by the title of the journal. This shows the editor’s positioning in the direction of the new reform movement.

A little later, the editor addressed the teaching of geometry (Hoffmann, 1870, p. 490): “That rigid geometry … of Euclid is a point of view that has been overcome by every thinking teacher and it is precisely one of the main purposes of this journal to help establish a more rational method of teaching geometry.”[11] The lamented “rigidity” referred on the one hand to the formal structure of Euclid’s *Elements*, which was reproduced unchanged in lessons at the time by the strictly dogmatic lectures of the teachers, described in the section above. Humanist educational theorists regarded this as exemplary for the teaching of formal education. On the other hand, the criticism of “rigidity” referred to the content of Euclidean theory, which is a congruence geometry. Figures are depicted “rigidly” under congruence mappings; they do not change their shape or form.

In ZMNU, projective geometry, which had been developed in the first half of the 19th century, was proposed as a new geometry to be taught in schools. This geometry was also referred to as “descriptive geometry” (A. Cayley), as “visual geometry” (F. Enriques) or simply as “newer geometry” (M. Pasch). As early as the 17th century, artists had already set themselves the problem of depicting objects as they are visually perceived by an observer, i.e., in a correct perspective reproduction. The resulting theory is projective geometry. The central tool of the theory at that time were central projections. These can be used to construct the image of a figure or body as an observer whose eye is in the center of the projection would see it. - A significant difference to Euclid’s theory is the following: Projective geometry had developed as a science of experience, based on the question of how to correctly depict figures and solids in perspective. Consequently the didactically problematic terms “infinitely distant straight line” and “infinitely distant points” were to be introduced in school lessons by referring to the horizon of the visual space ((Sturm, 1870)).[12]

The new, projective geometry was associated with ideas of mobility that made Euclidean geometry appear ‘rigid’. For example, conic sections can be transformed into each other almost continuously by changing the perspective. If you imagine the tip of the cone as the point of an observer’s eye, the plane sections of this cone are the images that the observer perceives. If you change the sectional plane, the conic sections change and an ellipse can first be transformed into a parabola and then into a hyperbola. In this way, relationships between figures became clear that Euclid’s structure of geometry does not allow.[13] - It should be noted that ideas of mobility of this kind were not only important for the education of pupils, but also of value for the scientific discipline of mathematics: In the 19th century, a number of prominent mathematicians were concerned with the question of whether Euclid’s fifth postulate, the parallel postulate, follows from the other axioms in the first book of Euclid, or whether it is possible to construct geometries in which the parallel postulate does not hold. Among these geometers were Gauss, Bolyai, Lobachevsky, and later Beltrami, Klein, Riemann,

[9] Hoffmann (1870, vol. 1, foreword) characterized the reputation that mathematics enjoyed, at least in part, at the humanistic Gymnasium with the following story: “A rector in Saxony ordered a pupil to put down his mathematical work during a lesson with the words: ‘What kind of barbarica are you doing?’”

[10] The emphasis on material education had a direct impact on the curriculum of secondary schools in Prussia: calculus was included in the curriculum, because basic knowledge in differential and integral calculus is essential for many applications of mathematics.

[11] Jene starre Geometrie … des Euklid ist für jeden denkenden Lehrer ein überwundener Standpunkt und es ist gerade ein Hauptzweck dieser Zeitschrift, eine rationellere Lehrmethode der Geometrie mit begründen zu helfen.

[12] For a more detailed description of the mathematics didactic discussion about projective geometry, we refer to (H. Struve, 1990).

[13] For a comparison of 19th-century textbooks based on Euclid and textbooks that teach the newer geometry of that time, see (H. Struve, 1994).

and Poincaré.[14] They constructed what we call today models of specific non-Euclidean geometries (e.g. the models of a hyperbolic plane named after Beltrami, Klein and Poincaré).

F. Klein was the first mathematician who not only considered individual examples of non-Euclidean geometries, but defined, following an idea of A. Cayley, the concept of a *non-Euclidean geometry* in a systematic way. These geometries are known today as *Cayley-Klein geometries*. He defined metrics (distances and angles) on the basis of a so-called *absolute figure* (briefly, the *Absolute*)[15] of a projective plane. While in the non-degenerate case these are classical conics, in the degenerate, for instance Euclidean, case F. Klein used notions of mobility to define the associated absolute figures (see (Klein, 1928) and (H. Struve and R. Struve, 2010)).

In the 1880s, J. Henrici and P. Treutlein (1881-1883) wrote a three-volume *Lehrbuch der Elementargeometrie*, which had several editions and was widely distributed. They took up the new ideas of the reform movement and showed how they could be implemented in the classroom. Geometry is structured with the help of the concept of mapping: The material is organized according to groups of geometric transformations and proofs are carried out with the help of mappings (cf. (Becker, 1994)). The book was written entirely in the "spirit of modern geometry", which F. Klein (1928, p. 228) understood as "the idea of the mobility of each figure", "by which … the general character of geometric figures can be grasped".[16]

The disputes between humanist and realist education advocates were not resolved by one group winning the day. The school system was too sluggish for that and both parties had convincing reasons for their respective positions. Treutlein, who had taught at both a humanist gymnasium and a Realgymnasium, made an interesting compromise proposal that was taken up in some federal states of Germany. He argued in favour of propaedeutic geometry lessons in the three lower grades and called for (see (Treutlein, 1911, 75); translation by the authors):

a) Geometry lessons in our secondary schools must be taught in two levels, a lower level and an upper level.[17]
b) Lessons in the lower school are *geometrischer Anschauungsunterricht*: the instruction is based on the observation of solids, derives the various geometric shapes from them, reshape them and create new ones, uses and promotes the pupils' own activity through estimating, measuring (also outdoors), drawing and modelling, cultivates inner visualisation ("innere Anschauung") and the imagination of space ("Raumvorstellung") and gradually lead from visual recognition to proving what has been recognized.[18]
c) The lessons in the upper school use the knowledge gained and, with constant reference to physical objects, establish elementary geometry as a theory and as a model of a deductive science.[19]

In contrast to Euclid's approach, three-dimensional solids were the starting point for the geometry lessons in the first level, i.e., objects from the pupils' everyday lives. The aim of the lessons was in particular to develop and promote spatial visualization skills, to train pupils' practical manual skills and to demonstrate the significance of what they have learned for 'practical life'. A new pedagogical perspective on the pupils was also adopted: In contrast to the Euclidean teaching method, which focused on exact repetition of the teacher's lecture, emphasis was placed on self-activity of the pupils in learning.

In other European countries, criticism of Euclid was along similar lines to that in Germany and various proposals were also made to reform the teaching of geometry:

In *England*, a reform movement of "one might even say, a revolutionary character" (Klein, 1908, p. 233) was initiated by J. Perry. He proposed that the subject matter should be taught with reference to its utility as well as being interesting to the pupils (Menghini, 2012). He thus introduces a new point into the discussion about geometry teaching, a *motivational aspect*: pupils' interest should be aroused by the usefulness of the subject matter for them.

Part of the *experimental geometry* or *practical geometry* propagated by Perry is that pupils fold and cut up paper, use tools such as right angles and check their suitability, draw with and without aids and take measurements, activities that does not appear anywhere in Euclid's *Elements*. These methods were also used to prove theorems, such as Pythagoras' theorem, by cutting and combining paper squares appropriately. - The experimental geometry of John Perry had a major influence at an international level, not only in Europe but also in the USA and even in Japan.[20]

In *Italy*, projective geometry was included in the curriculum for similar reasons as in Germany (see (Menghini, 2006)), but also with a view to national characteristics—in the words of Menghini: "Furthermore, the subject was not dominated by foreign authors and was in line with Italian research into geometry."

Analogies to Treutlein's approach of a propaedeutic geometry course can also be found in the Italian discussion. G. Veronese (1901), for example, propagated a "geometria intuitiva" for geometry lessons in the first grades (cf. (Menghini, 2010, p. 407)).

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[14] For the history of non-Euclidean geometry see (Bonola, 1955), (Gray, 1989) and (Gray, 2007).

[15] This terminology was coined by A. Cayley (1859). F. Klein (1928, p. 164) used in German the word "fundamentales Gebilde".

[16] Klein spricht vom "Geist der neueren Geometrie" und der "Idee der Beweglichkeit einer jeden Figur, durch die … der allgemeine Charakter der geometrischen Gebilde erfaßbar wird." Schubring tranlates "geometrische Gebilde" as "geometric structures". In our view "geometric figures" is a more appropriate translation.

[17] Der Geometrieunterricht unserer höheren Schulen muß in zwei Stufen erteilt werden, in einer Unterstufe und in einer Oberstufe.

[18] Der Unterricht der Unterstufe ist ein 'geometrischer Anschauungsunterricht': er lehnt sich an die Betrachtung von Körpern an, leitet daraus die verschiedenen geometrischen Gebilde ab, formt sie um und gestaltet neue, er benützt und fördert die Selbsttätigkeit der Schüler durch Schätzen, Messen (auch im Freien), Zeichnen und Modellieren, er pflegt die innere Anschauung und die Raumvorstellung und leitet allmählich das anschauliche Erkennen hin zum beweisenden Begründen des Erkannten.

[19] Der Unterricht der Oberstufe verwertet die gewonnenen Anschauungen und stellt unter stetem Bezug der Betrachtungen körperlicher Gebilde das Lehrgebäude der elementaren Geometrie auf das Muster einer deduktiven Wissenschaft.

[20] Cp. (Menghini, 2012, Section 6) and (Mock, 1963).

In *France* - “characteristically” as F. Klein (1908, p. 243) notes - the reform was initiated from the government, by the Paris Chamber of Deputies (French Chambre des députés). In 1898, it carried out a major survey on the state of secondary school education. The results led to a reform of the curriculum, which was implemented in the *plan d’etudes et programmes d’enseignement dans les lycées et collèges de garcons* (Paris, 1902). For mathematics the concept of function, graphical representation and the beginnings of calculus were imposed. A closer connection between arithmetic and geometry was also sought. In the textbooks by Meray (1874) and Borel (1905), the idea of a *transformation* was particularly emphasized.

Remark. It was in this intellectual environment of an educational reform movement that Hjelmslev developed his didactic ideas in the 1890s. For the interested reader, we will briefly outline how the reform movement developed over the next decades until the beginning of World War I.[21]

The reform movement was given an international institutional framework at the 1908 Congress of Mathematicians in Rome. The *Internationale Mathematische Unterrichtskommission (IMUK)* was founded there, also known as the *International Commission on Mathematical Instruction (ICMI)*. This gave the reform movement a strong international impetus. The new journal, *L’enseignement mathématique*, became the official organ of the IMUK. Its aim was to associate the world of teaching to the great international movements of the time and to promote the idea of internationalism in mathematics education.

Klein was elected as the first president of the ICMI. This was justified in particular by the fact that he was the driving force behind the reform movement in Germany. In 1903, at the annual meeting of the Society of German Natural Scientists and Physicians in Kassel, at Klein’s suggestion a commission was formed, which presented at the society’s subsequent meetings in Meran (1905) suggestions for the educaional reform. These became famous as “Meraner Reformvorschläge” (Meran reform suggestions) which formed the basis for the Commission’s work. Further development was abruptly stopped by the First World War.

2.3. Hjelmslev’s approach - his criticism and his proposals

Hjelmslev summarized and systematically published his didactic ideas in three articles in the journal *L’enseignement mathématique* ((Hjelmslev, 1939–1940) and (Hjelmslev, 1939–1940)) which he wrote three years before his retirement from the University of Copenhagen. The articles are in French and are entitled *La géométrie sensible*.

L’Enseignement mathématique, founded in 1899, was the official organ of the *ICMI*. Hjelmslev could therefore assume that by publishing three articles at once, his approach would reach a wider public—which sadly became obsolete with the outbreak of the Second World War. In keeping with the readership, the three articles are written systematically and convey a clear impression of Hjelmslev’s approach. We therefore base our description on them.[22]

At the beginning of the first article, Hjelmslev described his general approach. He began by discussing the achievements of the Greeks in antiquity and spook - also with reference to Euclid - of an “admirable scientific work”. He emphasized the methodical structure of geometry, the purely deductive system that Euclid created in such a form that it took more than two thousand years - he mentioned D. Hilbert - to close the gaps. Hjelmslev therefore by no means generally advocated an anti-Euclidean position, but rather held Euclid’s achievement in mathematics in very high esteem - a point that is not made clear in Lützen’s essay.

Hjelmslev then posed the question of the relationship of the Euclidean system to the sensible world (“monde sensible”) and to teaching.[23] For Hjelmslev, these two questions, one about the sensory world, the other about teaching, had the same answer: *La géométrie sensible*. This is the geometry that should be taught according to Hjelmslev. Therefore, the teaching of geometry should take other forms than those prevalent in the Euclidean tradition.

The starting point of Hjelmslev were two requirements, a normative curricular requirement and a requirement of philosophy of science, which is linked to the implementation of the first requirement. These are according to Hjelmslev (1939–1940, pp. 9–10) (translations by the authors):

Normative requirement (I): Geometry at school must deal with the sensible world. (“La Géométrie à l’école doit traiter du monde sensible.”)

Scientific-theoretical requirement (II): A geometry that describes the sensually perceptible world must be such that each of its statements always and in every case, without exception, contains a truth about sensually perceptible objects. (“Une Géométrie qui doit avoir avec le monde sensible un rapport tel que chacun de ses énoncés contienne toujours et sans exception une vérité sur des objets sensibles.”)

However, according to Hjelmslev, Euclid’s *Elements* are not suitable for this purpose for *two reasons*. The first reason relates to the *terms* Euclid uses, the second to the *postulates* (we would now say *axioms*) on which Euclid’s theory is based.

According to Hjelmslev, the objects of geometry should be “objects sensibles”, sensually perceptible objects. However, Euclid’s straight lines, as infinitely long objects, are not perceptible to the senses because our tactile drawing experiences and visual observation experiences are all finite. Statements about relations between infinitely straight lines such as parallelism, defined by the non-existence

[21] For a more detailed description of the development of mathematics education during this period, which is only outlined here, we refer to R. Tobies (2021) and for the times to come to H. Struve (2023). The work of Klein in the Göttingen tradition is presented in (Rowe, 2018).

[22] These three works are not mentioned in the otherwise extensive bibliography of J. Lützen (2021), presumably because the geometry of reality is not at their center. However, since the articles represent a systematic presentation of Hjelmslev’s approach, they are probably the most important source for an assessment of the significance of Hjelmslev’s work in mathematics education.

[23] Hjelmslev spoke of “mondes sensible” and “géométrie sensible” (French), of “Virkelighedsgeometrien” (Danish) and “Geometrie der Wirklichkeit” (German). Lützen translates this as “geometry of reality”.

of common points, therefore do not fulfill requirement (II). They are therefore not truths in the sense of (II). This also applies to certain postulates of Euclid in which "facts are established that contradict the sensually perceptible world". As an example, Hjelmslev cited a narrow rectangle whose sides have lengths of 4 dm and 4 mm. According to Hjelmslev, it can be observed that their diagonals intersect in an interval, i.e., in more than one point.

To support his argumentation, Hjelmslev referred to various well-known mathematicians and philosophers. In *Science et méthode* by H. Poincaré (1908) there is a chapter entitled *Mathematical definitions and education*. Hjelmslev began by quoting how Poincaré would define straight lines ("ligne droite"), namely "… simply start with the ruler and first show the student how to check a ruler by turning it around." Hjelmslev's intention was probably also to show that his famous French colleague also starts with operational definitions at school—and for good reason: "… in teaching, a good definition is one that is understood by the pupils."

Hjelmslev then showed that Poincaré also shared the problems he mentioned with the Euclidean concept of straight lines - at least for school lessons - and cited Poincaré (1908, p. 145): "Should we retain the classical definition of parallels and say that two straight lines are called such if they are in the same plane and do not meet, no matter how far they are extended? No, for this definition is not verifiable by experience, and therefore cannot be regarded as an immediate fact of intuition."

Finally, Hjelmslev quoted a pedagogical observation by Poincaré: "Under each word they wish to put a meaningful image." ("Sous chaque mot, ils veulent mettre une image sensible.") Hjelmslev could also see this as an argument for his claim (II).

In addition to Poincaré, Hjelmslev also referred to Rousseau, who criticizes the method used to teach Euclid's geometry. Rousseau (1762, Livre II) had written:

> I have said that geometry is not within the reach of children, but this is our mistake. Instead of letting us find the proofs, they are dictated to us; instead of teaching us to argue, the teacher argues for us and only trains our memory.

The same accusation—blunt memorization of Euclidean theorems and proofs—was made a hundred years later by J. K. Becker (1871), as we described above.

Instead, Rousseau called for "natural learning" in which children's observations take center stage:

> You will find all elementary geometry by going from observation to observation, without any talk of definitions, problems or any other form of reasoning than simple reasoning.[24]

This approach fits well with claim (II).

Finally, Hjelmslev referred to Clairaut, who, in his *Eléments de Géométrie* from 1741, blamed the logical structure of the *Elements* for the fact that "most beginners get tired and put off even before they have a clear idea of what they are being taught".

Roussseau and Clairaut thus contributed two further points of criticism of the use of Euclid's *Elements* in teaching, one methodological (memorization without understanding the content) and one motivational (boringness of the logical-deductive structure of the *Elements* for pupils).

After his discussion of the criticism of Euclid, Hjelmslev presented his own approach. He suggested introducing geometric concepts using objects that the students are familiar with from their everyday lives and then using drawing paper, often squared drawing paper, as the standard medium in class.

Hjelmslev introduced the basic geometric concepts in an operative way, i.e., in the same way that corresponding objects can be produced. Flat objects are determined using a three-plate method: Three plates that fit alternately into each other and can be moved in any direction must be flat. Two plates could have a concave and a convex spherical surface with the same radius, but the simultaneous fit with a third plate rules out this possibility. The rule for producing flat surfaces is therefore to grind three plates until they fit together. While this method was already known in the 19th century, Hjelmslev had the new idea of introducing the *orthogonality relation* in a similar way. To do this, he first defined *dihedral solids* Hjelmslev (1939–1940, p. 15):

> A body that has two flat surfaces that meet along an edge is called a dihedral solid (or dihedron). The composition of the remaining surface is irrelevant. Two dihedrons are called supplementary when they can be combined to form a flat dihedral, i.e. if they can rest on a plane with each having a flat face in that plane and fitting closely together along the faces not located in the plane. In other words, each of them must be able to fill the gap that forms next to the other when placed on a table. If the position of the planes is the same for the two supplementary dihedrons, they are called normal dihedrons.
>
> But how can we check if the two dihedrons are the same? We try to see if they can fill the same gap, thus being supplementary dihedrons to the same third. Using this method, we form normal dihedrons, three at a time, taking care that they can form additional dihedrons in pairs …

[24] Vous trouverez toute la Géomé trie élémentaire en marchant d'observation en observation, sans qu'il soit question ni de définitions, ni de problèmes, ni d'aucune autre forme de démonstration que la simple superposition.

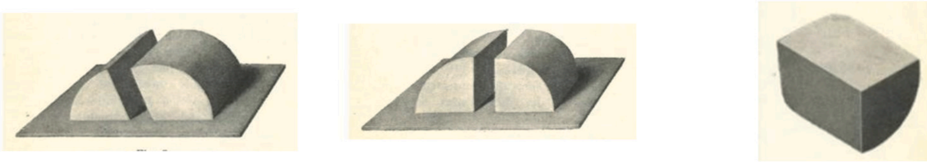

Fig. 1. Two dihedrons, two normal dihedrons and perpendicular planes.

The two boundary planes of a normal dihedron are called perpendicular to each other[25]; the edge of the normal dihedron is a straight line.[26]

In a schoolbook (Hjelmslev, 1916), that Hjelmslev wrote, this is illustrated by Fig. 1.

Subsequently, Hjelmslev introduced the drawing plane as a sheet of drawing paper lying on a flat table. The edge of a normal wedge defines a straight line on the drawing sheet, which can be traced with a ruler; because, according to Hjelmslev, a ruler is also a normal wedge, albeit a very flat one. Every straight line divides the drawing sheet into two parts, which overlap when the paper is folded along the straight line. Therefore, a *straight line* can also be created by simply folding a sheet of paper.

A straight line perpendicular to a straight line is obtained by folding the sheet of paper a second time so that the first straight line coincides with itself. You can also construct orthogonal straight lines with a set square.

Finally, Hjelmslev introduced tracing paper in order to experimentally prove symmetry properties of figures drawn on a sheet of paper. As an example, Hjelmslev considered the theorem that the diagonals in a rectangle are of equal length and have the same midpoint. To prove this, he constructed a rectangle on a sheet of drawing paper and made a copy of it on a sheet of tracing paper. He matched up the two copies of a rectangle. Then he turned the tracing paper over so that the two rectangles were congruent again. The diagonals were swapped and: “You can see that the diagonals have the same center and that each of them divides the rectangle into two equal triangles: One sees that they are equal.”

As a first conclusion, we note that, according to Hjelmslev, geometry should be taught at school as what is now called an *empirical theory*, i.e., as a theory of the natural sciences. Concepts are introduced *operationally* by specifying a construction or production rule, justifications (proofs) are given *experimentally* - i.e. in the same way as (experimental) physicists do. The topic of the lessons is figures that are constructed or folded on a drawing sheet.

It should be noted at this point that Hjelmslev’s proposal on how geometry should be taught at school, as presented so far, has been implemented with great success to this day. In almost all European countries geometry is taught in school lessons in the sense of Hjelmslev as a teaching of drawing figures, in which concepts are introduced operationally and proofs are of an experimental nature. Logical deductions serve to explain relationships. A detailed analysis of a textbook widely used in Germany in the 1960s and 1970s can be found in (H. Struve, 1990) and (H. Struve, 1996).

Hjelmslev went on to make suggestions that did not catch on and which were probably based on his practical experience in descriptive geometry, in which he had completed his doctorate and which he taught as a professor at the Technical University in Copenhagen. Like Leibniz, he argued that a circle has more than just a single point in common with a tangent line, meaning that a circle can be understood as a polygon with infinitely many vertices.

In the second of the articles published in *L’enseignement*, Hjelmslev outlined the structure of a theory of plane geometry. He introduced vectors, discusses lengths and area measurement, and explained trigonometry. In the last section, entitled *Le plan arithmétique*, Hjelmslev defined the coordinate plane. After constructing a coordinate system on the drawing sheet, points and vectors can be characterized using two coordinates and straight lines can be described as sets of points that satisfy a linear equation. If you allow any real numbers as coordinates, you obtain the arithmetic plane, in which there are also infinite straight lines.

In the third essay, Hjelmslev focused on spatial geometry. First, he described how to represent three-dimensional solids with the help of projections, then he introduced a three-dimensional coordinate system and subsequently described the space with the help of coordinates.

2.4. Evaluation of Hjelmslev’s approach

Hjelmslev’s didactic ideas on mathematics were in line with the progressive ‘Zeitgeist’ of the 19th century to replace Euclid’s works in schools, as we described above. Hjelmslev was by no means criticizing Euclid’s achievements as a mathematician (on the contrary, he explicitly acknowledged them) but rather the use of Euclid’s works in schools.

[25] An algebraic argument for the orthogonality of the planes would be: Let α, β, γ be the angles which were formed by the planes of the three dihedrons. The condition that each two are supplementary (i.e., that the sums $\alpha + \beta$ and $\beta + \gamma$ and $\gamma + \alpha$ are equal to two right angles), imply $\alpha = \beta = \gamma$ and hence that α, β and γ are right angles.

[26] Nous appelons corps dièdre (ou dièdre) un corps qui a deux faces planes se rencontrant le long d’une arête. La composition du restant de la surface est sans importance. Deux dièdres s’appellent supplémentaires quand ils peuvent se compléter de façon à former un dièdre plat, c’est-à-dire s’ils peuvent reposer sur un plan en ayant chacun une face plane dans celui-ci et en s ajustant étroitement le long des faces non situées dans le plan. Autrement dit, chacun d’eux doit pouvoir remplir le vide qui se forme à côté de l’autre quand on le pose sur une table. Si la position des plans est la même pour les deux dièdres supplémentaires, on les appelle dièdres normaux. Mais comment vérifier si les deux dièdres sont pareils? On essaie s’ils peuvent remplir le même vide, donc être des dièdres supplémentaires à un même troisième. Par ce procédé on forme des dièdres normaux, trois à la fois, en prenant soin qu’ils puissent deux par deux composer des dièdres supplémentaires … On appelle les deux plans-limites d’un dièdre normal perpendiculaires l’un à l’autre; l’arête du dièdre normal est une ligne droite.

The systematic approach that Hjelmslev presents is remarkable. His basic didactic position is that geometry should be taught at school as an empirical (scientific) theory that examines "the sensually perceptible world". In doing so, he saw problems that are still essential for the teaching of geometry today: the introduction of concepts (such as the concept of the infinite straight line) and the meaning of proofs (experimental trials versus formal derivations). His suggestion that geometry should be conceived as a theory of figures constructed on a sheet of paper has survived to this day. It should also be noted that Hjelmslev's operational approach is being discussed again in modern mathematics didactics (see e.g. (Bender and Schreiber, 1985)).

Hjelmslev's suggestions concerning the intersection of straight lines and circles with straight lines were not taken up in mathematics didactics. The break with tradition would probably have been too great.[27]

All in all, we can say that Hjelmslev has given significant impetus to the development of mathematics didactics, even if not all of his suggestions have been implemented.

3. Hjelmslev as a mathematician

We describe Hjelmslev's contribution to geometry in two parts, one concerning his *Begründung der ebenen absoluten Geometrie* (Hjelmslev, 1907) and the other one concerning his *geometry of reality*. To set the stage for an appropriate valuation of Hjelmslev's work from 1907, we start in Section 3.1 with a description of the prevailing understanding of geometry in the 19th century. We do this by help of two examples, which can be found in the works of Möbius and Pasch. These examples illustrate the understanding of geometry that dominated in the 19th century and also reveal the inherent limitations. Hjelmslev grew up with this classical understanding of geometry, on the basis of which Möbius and Pasch practiced mathematics, and was then - at the end of his third decade of life - confronted with Hilbert's view. We present in Section 3.2 this modern formalistic view of mathematics, which Hilbert explicated in 1899 using the example of geometry, and place (Hjelmslev, 1907) in this context. In Section 3.3 we discuss the geometry of reality and the underlying mathematical theory which Hjelmslev described in *Einleitung in die allgemeine Kongruenzlehre* (1929). This work is a vast generalization of *Neue Begründung der ebenen Geometrie*, since Hjelmslev admitted that 'neighboured' points may have more than one joining line. We then study the reception of Hjelmslev's work and show that it had a considerable impact on the foundations of geometry and that the underlying ideas are still being taken up today. Section 3 concludes with a discussion of the geometric heritage of Hjelmslev (in Section 3.4).

3.1. The historical starting point of Hjelmslev's mathematical activities

The following two examples show how the 'classical' understanding of geometry shaped the ideas of two great 19th century geometers and characterize the starting point of Hjelmslev's perception of geometry. They also show how revolutionary Hilbert's ideas about geometry were.

(1) The example of Möbius

August Ferdinand Möbius (1790–1868) was one of the most famous mathematicians of the 19th century. The *Möbius strip*, an example of a one-sided surface, is named after him, as is *Möbius geometry*, the theory of circles in a Euclidean plane. His main mathematical work - Möbius was also a recognized astronomer - is *Der barycentrische Calcul* from 1827. In a letter to Schumacher from 1845 Gauss compared the value of this Calculus with Descartes' calculus of letters, the differential calculus and his own calculus with congruences (according to the preface to the first volume of the Collected Works of Möbius (1885, p. XII), which was formulated by R. Baltzer).

Two points should be emphasized in particular, one methodological and one substantive. On the one hand, the barycentric coordinates are the first homogeneous coordinates in the history of geometry. On the other hand, Möbius developed a theory of relationships ("Verwandtschaften") between geometric figures in which Felix Klein, the co-editor of Möbius' *Gesammelte Werke*, found with regard to the classification system developed by Möbius, his Erlangen program outlined - more than 10 years after he had established his program.[28]

Möbius dealt with five types of relationships between figures, which he called as follows: *Equality and similarity* (in today's terminology: congruence), *similarity*, *affinity*, *equality* (in the sense of area equality) and the *collineation relationship*. To define the first four of these relationships, Möbius used mappings whose domains and ranges are the figures under consideration. An example (see Möbius (1827, p. 169)):

[27] This also applies from a philosophical point of view. In his *Philosophy of Mathematics and Natural Science*, H. Weyl (1927, pp. 182–183) stated

> Recently the Danish geometer Hjelmslev has advocated a pure theory of approximation, … with the same arguments as Hume, who among other things observed that the certainty that 'two straight lines can have no distance in common, always presupposes a clearly perceptible inclination of the two lines towards each other, but that if the angle they form is very small, we lack the pattern of a straight line which would be sufficiently exact to enable us to recognize the truth of the proposition with certainty'. (Treatise Part III, Section 1)

Weyl spoke of *Hume-Hjelmslev sensualism* and criticizes it for focusing "far too much only on the figures drawn on the blackboard" and forgetting that geometry forms the foundation of physics, which is and must be based on an exact theory.

[28] Cp. F. Klein (1921, p. 497) and H. Wußing (1969, p. 24) who spoke of an anticipation of the Erlangen Program ("Vorwegnahme des Erlanger Programms"). While Möbius used invariants for his classification, Klein used transformations - whose invariants correspond exactly to those of Möbius.

> If in two figures each point of one figure corresponds to a point of the other, in such a way that the mutual distance of two points of one figure is equal to the mutual distance of the corresponding points of the other figure, then the figures are called equal and similar.

Möbius defined the fifth relationship, that of *collineation*, in a different way, namely by mapping the entire plane onto itself (Möbius (1827, p. 266)):

> The essence of this new relationship therefore consists in the fact that in two plane or physical spaces, each point of one space corresponds to a point in the other space in such a way that, if one draws any straight line in one space, of all the points that are met by this straight line (*collineantur*), the corresponding points in the other space can also be connected by a straight line. This relationship has therefore been called the relationship of *collineation*. Figures between which it takes place are called *collinearly related*, or *collinear figures* in general.

What is remarkable about this definition is that for the first time in the history of synthetic geometry, mappings were used with the entire plane or the entire space as the domain and range (cf. the report (Kötter, 1901, p. 194) on the historical development of synthetic geometry). In modern terminology, Möbius first defined the concept of *collineation* as a mapping between two planes or spaces and then the concept of *collinear relationship* of figures. Figures are collinearly related if there is a collineation of the plane or space that maps one figure onto the other. - This is exactly how it is done today.

But was it really necessary for Möbius to introduce the entire plane or the entire space as the domain and range of collineations? Möbius himself wrote (Möbius (1827, p. 10)): “I was led to the latter relationship of collineation by observing the perspective image of a plane figure.”

Möbius could therefore define that two plane figures are called *collinearly related* if there is a central projection which maps one figure onto the other—or in the style of the definition of his first four relationships between figures: if the two figures could be placed in two planes of space in such a way that each point of one figure corresponds to a point of the other figure, so that the connecting lines of corresponding points all pass through one and the same point. He explained why he did not do this as follows (Möbius (1827, p. 11)): “… tried to present the subject from an even more general point of view. And to extend the relationship that exists between a planar figure and its perspective image to figures in space.”

One year after the publication of the *Barycentric Calculus*, Möbius took up his idea to define the relationship of collineation by help of central projections again and wrote (Möbius, 1829, pp. 449–450): “But I have not made use of this in itself very fruitful view … since … this view is no longer readily applicable to figures in space.”

Why Möbius could not project spatial figures onto each other becomes clear in § 139 of the *Barycentric Calculus*. There the author discussed the relationship of *equality and similarity* (i.e. congruence) and drew attention to a difference between the planar and the spatial case (Möbius, 1827, p. 171): “It seems strange that with physical figures equality and similarity can take place without coincidence, whereas with figures in planes … equality and similarity are always connected with coincidence.”

By coincidence, Möbius meant that the figures could be made to come to cover each other in an empirical-geometric sense. He continued: “The reason for this may be found in the fact that beyond the physical space of three dimensions there is no other space, none of four dimensions … But since such a space cannot be conceived, coincidence is also not possible in this case.”

Möbius therefore did not define the collinear relationship of figures with the help of central projections, since the generalization to spatial figures would presuppose a four-dimensional space in which the figures could be projected—and a space with more than three dimensions did not exist for Möbius.

We state that it was evident to Möbius that there is no space with more than three dimensions. This was not just a marginal point for him, but had far-reaching consequences for the structure of his theory of relationships. He was the first geometer to introduce mappings within synthetic geometry whose domain and range are not just individual figures but the entire plane respectively space. This idea was later enthusiastically taken up by his contemporaries and successors and became, for example, an essential basis for F. Klein’s *Erlangen Programme*. In this program, geometries are characterized and ordered by their groups of automorphisms, which are mappings of the entire space to itself and preserve geometric relations such as incidence, order or orthogonality.

Both the view that there is no space of more than three dimensions and the intention to largely avoid mappings of the entire plane or the entire space show that Möbius’ understanding of geometry was largely shaped by empirical science.

(2) The example of Pasch

Moritz Pasch became famous in the history of geometry because he was the first mathematician to complete Euclid’s system of axioms (Gray, 2007, p. 247). Euclid had aspired to derive all propositions from his axioms - which he still referred to as postulates - without any gaps. However, he did not succeed in this: Without realizing it, he adopted intuitive ordering properties - with regard to the relation of betweenness - from the drawing sheet without founding them axiomatically. Already the proof of the first proposition, which describes the construction of an equilateral triangle from a given line, is incomplete in this sense. It was not until 1882 that Pasch succeeded in closing this axiomatic gap in his *Vorlesungen über neuere Geometrie*. His axioms of order served Hilbert as a model for his axiom system in *Grundlagen der Geometrie* (Hilbert, 1899).

Moritz Pasch’s view of geometry is clearly expressed in the preface (Pasch, 1882, p. III) and (Pasch, 1882, p. 3):

> … the successful application that geometry continually experiences in the natural sciences and in practical life is in any case only based on the fact that the geometric concepts originally corresponded exactly to the empirical objects … geometry retains the character of natural science …
>
> … the point of view which we intend to adhere to strictly in the following, and according to which we see nothing more in geometry than a part of natural science.

This view point of Pasch had consequences for the conceptual structure of his empirically conceived geometry. In (Pasch, 1882, p. 4) he describes what he wants to understand by a straight line:

> It is said that a straight line can be drawn through two points. But the line can be bounded in different ways; the indeterminacy of the boundary has led to the straight line being said not to be bounded, it must be 'imagined' without limit, in infinite extension. This demand does not correspond to any perceptible object; rather, only the well-limited straight line, the straight path between two points, the limited straight line ("gerade Strecke") is directly perceptible. We use this concept …

At the end of (Pasch, 1882, §1), Pasch once again explained why he has chosen limited straight line ("Strecke") as the basic concept and not the unlimited straight line ("Gerade"). If an observer moves along a closed line and can only see and traverse a small part of the line at a time and has therefore not yet recognized the line as closed, he will regard the order of the points on this line as linear, although it is cyclic. (Pasch may have been thinking in this example of great circles of the globe). - This example by Pasch once again clearly shows that he is concerned with an appropriate description of empirically given geometry.

Pasch thus made the choice of the basic concepts of his theory highly dependent on his understanding of geometry.

3.2. Hjelmslev and Hilbert

David Hilbert (1862–1943) developed the modern formalistic view of mathematics. In 1899 he wrote the *Festschrift zur Feier der Enthüllung des Gauß-Weber-Denkmals in Göttingen*. The great importance of the work - from the 2nd edition onwards entitled *Grundlagen der Geometrie* - can be seen from the fact that a French translation was produced as early as 1900 and an English translation in 1902. To date, fourteen editions have been printed in German.

The new conception of mathematics that Hilbert[29] developed using the example of geometry is already expressed in the first paragraph entitled *The Elements of Geometry and the Five Groups of Axioms* (in the English translation by Townsend 1902). It begins with the words

> Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C, …; those of the second, we will call *straight lines* and designate them by the letters a, b, c, …; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ, … The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*.
>
> We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between", "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms may be arranged in five groups.

The five groups of axioms Hilbert called axioms of *connections*, *order*, *parallels*, *congruence* and *continuity*.

The difference in the conceptions of geometry between Pasch and Hilbert is enormous. At the beginning of his *Vorlesungen über neuere Geometrie*, Pasch described which phenomena of reality he wanted to describe and explain - namely physical-geometric phenomena - and outlined which terms he wanted to use for this, so that the reader knows what they refer to. Pasch was anxious to derive his fundamental notions from experience and to postulate no more than experience seems to grant - therefore he does not use the infinitely extended straight line ("Gerade") as a basic concept but only the finitely extended straight line ("Strecke").

Hilbert's points, lines and planes, on the other hand, have no reference objects, but are *only regarded* as elements ("things") of arbitrary sets ("systems"). They *are* not points, lines or planes, they are *just called that*. The terms are *variables*, i.e., they can be interpreted differently (in the logical sense). The relations in which they stand can therefore not be 'read off' in reality, but are only thought and defined by the axioms. Just like the propositions of the theory, the axioms are propositional forms (i.e., formulas of the underlying predicate-logical language, which can only be assigned a truth value through an interpretation of the relations and an assignment of the variables). If one interprets the terms, one obtains models of the theory. Hilbert's theory has models both within mathematics and outside mathematics. An example of the first kind (a model of the plane axioms of Hilbert's system) is the real coordinate plane, in which 'points' are pairs of real numbers, 'lines' are sets of pairs of real numbers that satisfy a linear equation, and 'incidence' is the ϵ-relation. An example of a model of the second kind is the drawing plane, in which 'points' are little marks on a sheet of paper, 'lines' are drawn on a sheet of paper using a ruler and 'incidence' is the apparent coincidence of a 'point' and a 'line' on the sheet of paper. In Hilbert's formalistic view, the *truth* of a statement no longer means correspondence with the phenomena that one wishes to describe, as Pasch supposed, but only validity with every interpretation of the variables.

[29] For a discussion of the historical development of Hilbert's conception of geometry see (Pambuccian, 2013).

Through Hilbert, mathematics became an independent science, a study of abstract structures that is independent of empiricism. Hans Freudenthal (1961, p. 14) described Hilbert's approach with the words: "the bond with reality is cut".[30] Bertrand Russell (1918) characterized the special character of mathematics in a pointed way as follows: "Mathematics is the subject in which we never know what we are talking about nor whether what we are saying is true."

The new formalistic view of mathematics was, of course, discussed intensively and controversially by Hilbert's contemporaries. One representative of the classical view of geometry was Gottlob Frege (1848–1925). The correspondence between Frege and Hilbert, in which the different positions are clearly expressed, has become famous (see also (Tapp, 2013, p. 56 ff.)). This will be briefly explained by the following two examples.

Firstly, the different understanding of terms and their definitions is pointedly expressed in Frege's question as to whether his pocket watch was a point. For Frege, this would be an absurd idea; for Hilbert, it was an unanswerable question as long as Frege did not say what the other basic terms such as "straight line" and "incidence" were supposed to mean in his proposed model of a Euclidean plane.

Secondly, while for Frege the axioms of a theory are true statements about phenomena of reality, for Hilbert they were propositional forms that have no truth value (in themselves). In principle, they could be freely chosen. Therefore, for Frege, the question of the non-contradiction of axiomatically given theories was irrelevant, or, more precisely, meaningless: Axioms, such as those of Euclidean geometry, are true statements—in the sense of correctly describing real (physical) geometry—and are hence non-contradictory due to their nature.

Hilbert's formalism was attractive to young mathematicians - Hjelmslev was 26 years old when *Grundlagen der Geometrie* were published - for various reasons. For one thing, certain philosophical problems that were difficult to solve were no longer relevant to Hilbert's mathematics: One example is Möbius' question of whether there is a space of more than three dimensions. Of course, Hilbert would say, such a space can be defined axiomatically. Whether it exists in the physical sense is a completely different question and belongs to physics, not mathematics. Another example is Pasch's question as to whether infinitely extended straight lines exist. Hilbert uses the concept of straight lines (and planes) as a basic concept in his axiom system of Euclidean geometry as a matter of course. For him, concepts have no reference objects; Hilbert is not bothered by Pasch's scruples.

On the other hand, Hilbert's view gives mathematicians considerably more freedom in their scientific work. Axioms can be freely chosen and have no longer to be true or evident. Concepts can be introduced *implicitly* through their relations to other concepts and do not need to have reference objects in reality. - Non-Euclidean geometries are an impressive and representative example of this. According to Hilbert's formalistic view of mathematics, non-Euclidean geometries belong to mathematics – a position that was not shared by all mathematicians in the 19th century. In his lectures on non-Euclidean geometry, F. Klein (1928) differentiated between projective metrics ("projektive Maßbestimmungen") and non-Euclidean geometries ("nicht euklidischen Geometrien"). The former are logically possible, but the latter "permit" "practical applications in the external world around us". According to Klein, therefore, of the seven projective metrics, only the elliptic and the hyperbolic ones define a non-Euclidean geometry; because only these, like our universe, are uniform in all directions. - Klein recognized the problematic nature of his differentiation at the latest when he noticed that the space-time of Einstein's special theory of relativity, which today is referred to as Minkowskian, corresponds to his pseudo-Euclidean projective metric. - For Hilbert, this discussion about the *concept of geometry* was irrelevant.

The freedoms of Hilbert's approach are used by (almost) all mathematicians today. At the beginning of the 20th century, many of Hilbert's contemporaries had an inkling of the new possibilities that this approach could open up, but initially these were only hopes. One of Hjelmslev's main achievements is the provision of a "much admired" example (according to F. Bachmann (1989, Preface)) that demonstrates the fruitfulness of Hilbert's approach.

Hilbert himself had already shown the directions in which the foundations of geometry could develop. He had divided the axioms of Euclidean geometry into groups that suggested a step-by-step structure of geometry: Incidence, order, congruence, existence of parallels, continuity. Hilbert had already shown in *Grundlagen der Geometrie* that plane Euclidean geometry can be developed even without axioms of continuity (cf. (Hilbert, 1903, p. 137)). By 'plane Euclidean geometry', of course, we no longer mean real Euclidean geometry but a generalized one. Hilbert represents these planes as (affine) coordinate planes over ordered fields - as we would say today.[31]

Hilbert had thus paved the way for two new and promising research directions. On the one hand, there is the possibility of modifying axiom systems by omitting axioms in order to show the independence of axioms, and to investigate, on the other hand, whether meaningful structures arise in this way. In the case of Hilbert's axiom system of Euclidean geometry, "meaningful" means whether the new structures can still be adequately described algebraically.[32] Omitting axioms was a revolutionary idea for representatives of the classical view of geometry, such as Pasch. Axioms are true statements about an area of reality that the theory is supposed to describe.

[30] English translation from Streefland (1994).

[31] M. Dehn (1900), a student of Hilbert, showed that over a non-Archimedean ordered field there exist models of Hilbert's axioms of incidence, order and congruence whose metric is Euclidean (i.e., every quadrilateral with three right angles is a rectangle) but which do not satisfy the Euclidean parallel axiom (of Hilbert's *Grundlagen der Geometrie* or of Euclid's *Elements*).

[32] Hilbert's approach thus became an important impetus for the development of modern algebra. By modifying geometric axiom systems, new "Zahlbereiche" (a term used by Hilbert for sets of numbers) were obtained, which could be used to analytically describe the models of newly developed theories. This made the - axiomatic - definition of new algebraic structures necessary, such as the concept of ordered fields mentioned above. Fields, rings, vector spaces, quasifields, near-fields, semifields, ternary rings, but also the general concept of an affine plane and a projective plane were coined (see also Pickert (1955)). The first textbook of the resulting "new school of abstract algebra" was van der Waerden's *Moderne Algebra* from 1931. It should be noted that an important problem of the new algebra was to determine all models of an axiomatically defined algebraic structure, such as e.g. all fields or all groups. This classification problem is also a problem that cannot be formulated at all without the new Hilbertian conception of mathematics.

What sense should it make to omit axioms from such an empirical theory? For Pasch, this would be absurd (at least assuming the independence of the given axioms), because the phenomenal domain that was to be described and explained by the theory would be completely unclear. - But for Hjelmslev this was a challenging question!

In *Neue Begründung der Bolyai-Lobatschefskyschen Geometrie*, Hilbert (1903) showed that a (plane) hyperbolic geometry can be obtained by replacing in the axiom system of *Grundlagen der Geometrie* the Euclidean parallel axiom with a hyperbolic one—even without continuity axioms. To this end, Hilbert showed that every hyperbolic plane can be embedded in a projective coordinate plane. Hjelmslev now posed a much more difficult question:

Is it possible to dispense with every parallel axiom when setting up plane geometry?

In his (1907) *Neue Begründung der ebenen Geometrie*, Hjelmslev showed that one can actually do without any form of a parallel axiom (and without continuity axioms) and stated that even "the order axioms … are only rarely used". M. Dehn (1926, p. 236) cites Hjelmslev's setting up of plane geometry "as the highest point that modern mathematics has reached in the foundation of elementary geometry beyond Euclid".

Not only was the result of Hjelmslev's work new, but also his methodical approach. "Begründung" of a geometry means to embed a given plane in a projective coordinate plane. The given geometry can then be represented algebraically with coordinates of the embedding projective plane.

Hjelmslev's approach was as follows: Starting from a plane geometry that satisfies Hilbert's axioms (without continuity axioms and without parallel axioms), he defined "new" points and straight lines, so-called *ideal points* and *ideal lines*, which turn the given plane into a projective plane in which Pappus' theorem holds. This theorem ensures that the points and lines of the projective plane can be described algebraically as (homogeneous) triples of elements of a field and the lines as sets of points that satisfy a linear equation. But how can a given incidence structure of points and lines be extended?

This question alone is a challenge for a representative of the traditional view of geometry such as Frege: points and straight lines are something given by the phenomena that the theory is supposed to explain. Points cannot be "defined", they are found in reality. - This did not apply to Hilbert and therefore also not to Hjelmslev. The latter defines so-called ideal points ("uneigentliche Punkte") as sets of straight lines that satisfy a certain relationship—the product of the reflections in three of these lines is again a reflection in one of the lines. For the definition of ideal lines Hjelmslev introduced a new type of mappings, which he called *half-rotations* ("Halbdrehungen"; (Ewald, 2013) speaks of "snail maps").

The approach and ideas of Hjelmslev were taken up by G. Thomsen (1933), E. Podehl and K. Reidemeister (1934) and A. Schmidt (1943), to name but a few, and led to the establishment of reflection geometry (by F. Bachmann, H. Karzel, R. Lingenberg).[33] An axiom system without assumptions about order and continuity, and also without free mobility, can be found in (Bachmann, 1973). On the basis of his axiom system, Bachmann also succeeded in describing the metric of the projective ideal plane constructed by Hjelmslev as a projective metric in the sense of Cayley and Klein. An axiomatic characterization of all Cayley-Klein planes is given by R. Struve (2016). - The difficulty of the task that Hjelmslev had set himself is shown by the fact that it was not until 1961 that W. Pejas, a student of Bachmann, succeeded in characterizing the models of Hilbert's axiom system algebraically. Hjelmslev showed in 1907 that the models can be embedded in a Pappian projective plane, while Pejas showed more than 50 years later what the corresponding subplanes look like (see Pejas (1961)).

3.3. Hjelmslev's geometry of reality and its reception

Hjelmslev's *Einleitung in die allgemeine Kongruenzlehre* (1929), briefly denoted by *AKL*, provides a large generalization of his paper *Neue Begründung der ebenen Geometrie* (1907). He studied the "Begründung" of plane geometry without any assumptions about continuity, order or any version of a parallel axiom. Moreover, Hjelmslev admitted that "neighboured" points may have more than one joining line.[34] As a model of such geometries Hjelmslev explicitly mentions his geometry of reality. Hjelmslev (1929, first announcement, pp. 3–4) wrote:

> The path was paved by the study of the geometry of reality, where we have the closest example of a geometry belonging here.[35]

Hence we can state: *The mathematical theory underlying the geometry of reality is described by Hjelmslev in AKL*. The critique of Lützen that "Hjelmslev never succeeded in formulating a fixed axiomatic foundation" (Lützen, 2021, p. 113) obviously ignores the *AKL*.

As to be expected after his fundamental paper *Neue Begründung der ebenen Geometrie* (1907), Hjelmslev developed in *AKL* an axiom system entirely in the sense of Hilbert's conception of geometry, thus meeting also today's standards. The critique of an "unsatisfactory formulation of his geometry of reality" or that Hjelmslev's geometry "was not viable in the form he had given it" (see (Lützen, 2021, p. 114)) in no way concerns the mathematical theory behind Hjelmslev's geometry of reality.

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[33] For a more detailed bibliography see Bachmann (1973).

[34] By the way, the discussion about distinct points with several joining lines may be traced back to antiquity. Euclid says in his second axiom that any segment can be extended to a line. But he does not postulate the uniqueness of this extension. And already some of the commentators of Euclid and later geometers as Saccheri tried to prove the uniqueness (see (Heath, 1926)).

[35] Der Weg wurde angebahnt durch die Studien der Geometrie der Wirklichkeit, wo man das nächstliegende Beispiel einer hierher gehörigen Geometrie vor Augen hat.

AKL consists of six announcements ("Mitteilungen") published between 1929 and 1949. In the first and second one, Hjelmslev showed that his axiom system allows a 'Begründung' of plane geometry if, in addition, the uniqueness of joining lines is assumed. Thus he succeeded in a foundation of plane absolute geometry without the assumption of order axioms. This is an essential enhancement of his paper (Hjelmslev, 1907) and an important result with respect to the foundations of geometry.

In the third announcement of *AKL*, Hjelmslev studied geometries in which two points may have several joining lines. Points with more than one joining line are called *neighboured*. The classes of neighboured points are called *great-points* and these are regarded as points of a *great-geometry*. Hjelmslev showed that under certain additional conditions which include Hilbert's axioms of order (Hjelmslev, 1929, third announcement, p. 35) "… all axioms of the original geometry are also valid in the associated great-geometry … and moreover, the uniqueness axiom is fulfilled."

In other words, the associated great-geometry is a model of Hilbert's plane absolute geometry (given by the plane axioms of incidence, order and congruence of Hilbert's *Grundlagen der Geometrie*) for which a foundation was presented in the first and second announcement of *AKL*.

Hjelmslev then studied the geometry within a class of neighboured points, i.e., within a great-point, and started with the case where the geometry within a great-point is *Euclidean*.

As a model from analytic geometry Hjelmslev provided the affine coordinate plane over the ring $R(\varepsilon)$ of dual numbers $u+\varepsilon v$ with real numbers u, v and $\varepsilon^2=0$ (see (Hjelmslev, 1929, first announcement, p. 7) and (Hjelmslev, 1923, p. 12)). If $I=\varepsilon R$ is the ideal of zero divisors then points are represented by pairs (x, y) of real numbers (or equivalently by elements $x+\varepsilon y \in R(\varepsilon)$) and lines as homogeneous[36] triples $[u, v, w]$ with $u, v, w \in R(\varepsilon)$ and $u \notin I$ or $v \notin I$. A point (x, y) is incident with a line $[u, v, w]$ if the equation $x \cdot u+y \cdot v+w=0$ is satisfied. Obviously, the points $(0,0)$ and $(0, \varepsilon)$ have an infinite number of joining lines, namely $[u, \varepsilon, w]$ with real numbers u and w. The orthogonality of lines and the reflections in points and lines are defined by the same formulas as in real Euclidean geometry.

In the fifth announcement of *AKL*, Hjelmslev generalized his investigations about the Euclidean case and studies *singular* great-geometries, i.e., geometries where the fourth angle in a quadrilateral with three right angles is also a right angle. For these geometries he provided a foundation by generalizing the segment calculus of Hilbert (1899) (under the assumption of order axioms). Hjelmslev (1929, sixth announcement, p. 3) noticed: "This shows that the main theorems of Euclidean geometry are indeed independent of the uniqueness axiom."

For *non-Euclidean* great-geometries, Hjelmslev presented corresponding results in the sixth announcement. There he made use of an "axiomatic transformation", that is, he introduced a singular pseudo-metric which turns a given non-Euclidean geometry into a Euclidean one. Since for Euclidean geometries a foundation was already presented, this provides a foundation for non-Euclidean geometries. Hjelmslev (1929, sixth announcement, p. 27) noticed: "The specified axiomatic transformation of the various types of geometries into a geometry of the Euclidean type will undoubtedly lead to a simplification in all applications."

We now come to a discussion of the *reception of Hjelmslev's geometry*. Hjelmslev's *Neue Begründung der ebenen Geometrie* from 1907 was published in the renowned journal *Mathematische Annalen* and hence was accessible to all geometers interested in the foundations of geometry. This has been even more true since the work was mentioned in Hilbert's *Grundlagen der Geometrie* from the 4th edition onwards. Hjelmslev himself noticed that among his contemporaries F. Schur (1902) and M. Dehn (1931) referred to his paper, and it is also treated in detail in the fundamental books *Aufbau der Geometrie aus dem Spiegelungsbegriff* (Bachmann, 1973) and *Geschichte der Geometrie seit Hilbert* (Karzel and Kroll, 1988, Chap. III, §5).

The situation seems to be different with *AKL*. A first indication of this is that *AKL* is not mentioned at all in Karzel and Kroll (1988). A second observation is that no part of *AKL* has ever been published in a mathematical journal (but, instead, in the physically and technically oriented *Announcements of the Royal Danish Academy of Sciences*). This is all the more surprising as Hjelmslev had previously published papers in the journal *Mathematische Annalen*. One reason for this may have been the Second World War, which, perhaps, made this seem inopportune.

It should be mentioned, however, that there were several personal contacts. In 1922, Hjelmslev gave a series of lectures on *Die natürliche Geometrie* (Hjelmslev, 1923) at the *Mathematical Seminar of the University of Hamburg*, and M. Dehn visited Copenhagen in 1931 and gave a lecture on some new research in the Foundations of Geometry (Dehn, 1931) in the presence of Hjelmslev. In addition Hjelmslev has reported on his investigations at several Scandinavian mathematicians congresses. Thus it can be assumed that the existence of *AKL* and Hjelmslev's ideas and intentions were known at least to a small circle of experts. *AKL* is also mentioned in papers of F. Bachmann, A. Schmidt and F. Schur, which were published in *Mathematische Annalen*.

After Hjelmslev's death in 1950, his mathematical ideas were taken up in various ways, particularly by F. Bachmann and his students at the University of Kiel, however, not out of interest in Hjelmslev's geometry of reality but, instead, in the underlying geometry of reflections, as F. Bachmann (1971, p. 44) notices: "I am not so much interested in natural geometry, but I am interested in the natural generality of the calculus of reflections."

For a remarkable example, where the ideas of Hjelmslev were taken up, we refer to W. Klingenberg (1954b). In this paper about Euclidean planes with neighboured elements he referred to *AKL* and wrote: "… then some questions remain unanswered in the investigations of J. Hjelmslev. Hjelmslev does show that analytic geometry over certain rings, which are generalizations of the dual numbers, satisfies his axioms; but he does not investigate whether each of his axiomatically defined planes can be described in this way, and this is not correct either, as our investigations will show."

[36] $[u, v, w]$ and $r[u, v, w]$ with $r \in R(\varepsilon) \setminus I$ represent the same line.

AKL, the mathematical theory behind Hjelmslev's geometry of reality, had a considerable impact on the Foundations of Geometry. The following list presents some of the many further developments of Hjelmslev's geometry by successors in the 20th and 21st centuries, who explicitly refer to Hjelmslev's work. The list illustrates that the underlying ideas are still being taken up today.[37]

- A. Schmidt (1943) and F. Bachmann (1973) replaced the axioms of Hjelmslev (1907) with a *purely group-theoretic axiom system*[38] and provided a *Begründung of plane absolute geometry* without order and free mobility.[39]
- W. Pejas (1961) succeeded in describing all *models of Hilbert's plane absolute geometry* algebraically. His main tools were the reflection geometric methods from Bachmann (1973).
- F. Bachmann (1971) and (1989) generalized the axiomatic approach of *AKL* by substituting the existence and uniqueness of joining lines in his textbook (1973) by the existence and uniqueness of perpendiculars. Since no assumptions were made about order, free mobility and the existence of joining lines, this *theory of Hjelmslev groups* is an essential extension of *AKL*. For a detailed list of literature, we refer to V. Pambuccian, H. Struve and R. Struve (2017).
- W. Klingenberg (1954a), (1954b), (1956) and Knüppel and Kunze (1979) applied Hjelmslev's idea of a *great-geometry* to affine, projective and metric planes. The transition from a given geometry to a great-geometry is made precise by the introduction of homomorphisms between geometries.
- R. Lingenberg (1958) and E. Salow (1973) toke up Hjelmslev's idea of a *singular pseudometric* for an algebraization of non-Euclidean geometries where points can have several joining lines.
- M. Kunze (1981) described all *models of Hjelmslev's AKL* (with order and free mobility) algebraically.
- R. Struve (2016) modified the *AKL* approach by specifying an axiom system for *Cayley-Klein geometries* in which the existence of joining lines is not assumed, but the uniqueness is retained. In this way, a reflection geometric characterization is provided which includes not only the three classical geometries (that is, Euclidean, hyperbolic and elliptic geometry) but also Galilean, Minkowskian and all of the dual geometries.
- R. Struve and H. Struve (2019) showed recently that the correspondence between metric geometries and their group of motions (generated by reflections)[40] can be precisely stated in a framework of first-order logic.
- E. Molnár (1981) used the reflection-geometric methods of Hjelmslev for an investigation of *circle geometries* and states the influence of Hjelmslev's work explicitly in (Molnár, 2016).[41]
- Finally, J. Hjelmslev was a pioneer with respect to *geometries over rings* (for example, over the ring of dual numbers). There is a considerable number of geometers who follow him. For a detailed bibliography we refer to (Keppens, 2019).

3.4. *The geometric heritage of Hjelmslev*

The geometric heritage of Hjelmslev is diverse and broad. We briefly mention a few points:

- Hjelmslev was a convinced advocate of Hilbert's *conception of geometry*, which he closely follows in his papers on the foundation of plane absolute geometry.
- *Mathematical theories* are associated with Hjelmslev, such as the theory of *Hjelmslev planes* (i.e., incidence structures where two distinct points may have several joining lines) and the theory of *Hjelmslev groups* (see Bachmann (1989)).
- *Geometric theorems* bear his name, for example, *Hjelmslev's theorem* (about the midpoints of congruent rows of points) in Coxeter (1968) or *Hjelmslev's Lotensatz* in Bachmann (1973).
- *Geometric methods* are associated with Hjelmslev, particularly the reflection geometric one, which allows to *formulate geometry in the group of motions*: Points and lines correspond to elements of the group of motions (namely to the reflections in points and lines) and geometric relations (such as incidence and orthogonality) to group-theoretical ones. By this correspondence geometric theorems can be proved by group-theoretical calculations, both, for Euclidean and non-Euclidean geometries.
- The associated group-theoretical calculus is called the *calculus of reflections*. Hjelmslev developed for this calculus his own *Zeichensprache*[42] which he used in *AKL* with virtuosity and which is still today the standard notation. In this context it should also be noted that the calculus of reflections satisfies Leibniz's requirement for Descates' analytic geometry, namely to calculate directly with geometric entities (such as points and lines), instead of dealing with derived coordinates (which are not genuine geometric elements).
- Finally, Hjelmslev is a pioneer of *ring geometry*, which he proposed as a new field of research, and which is still taken up by many geometers today. This is also demonstrated by the fact that in the Mathematical Subject Classification System 2020 (MSC2020)

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[37] The judgement of Lützen (2021, p. 113) that Hjelmslev had "limited success" and that he "seems to have had few followers" does not apply to Hjelmslev as a mathematician.

[38] With very basic axioms about line-reflections stating, essentially, that there is exactly one line incident with two distinct points, and that the three-reflection theorem holds, i.e., that the composition of three reflections in lines which have a common point or a common perpendicular must be a line-reflection.

[39] This includes not only the embedding of the incidence structure of the models of the axiom system in a projective plane but also of the metric in a projective metric (which has never been carried out by Hjelmslev).

[40] Summarized by Bachmann by "The group of motions contains an image of the properties of the plane" and "Geometry can be formulated in the group of motions"; see H. Behnke et al. (1974, p. 129, p. 134).

[41] In this paper (to the memory of Gyula Strommer) Hjelmslev is called "the second Euclides Danicus", which indicates a high appreciation of his work.

[42] See Hjelmslev (1929, first announcement, §2); Hjelmslevs *Zeichensprache* is, obviously, more effective than the notations introduced by G. Thomsen (1933, §3).

of the American Mathematical Society in 2020, the field of geometry (51) is divided into 14 subclasses, of which one class relates directly to Hjelmslev (*51Cxx Ring Geometry (Hjelmslev, Barbilian, etc.)*).

Our short list illustrates that Hjelmslev achieved impressive results in all of these areas and left behind a significant geometric heritage.[43]

4. Conclusion

Looking back on the work of Hjelmslev in the two different fields of didactics of mathematics and mathematics as science one thing may be surprising, at least at first glance. Hjelmslev advocated for two different understandings of geometry, on the one hand for an empirical understanding in mathematics education, i.e. for schools, and on the other hand for a formalistic understanding in mathematics as a scientific discipline. One may ask, "What was geometry for Hjelmslev ultimately?" Was geometry for him an empirical science or a formalistic science - two at first glance mutually exclusive views? The answer can be formulated by help of Hilbert's concept of formalism. The drawing plane, described in Section 3.2 is an (empirical) model of the formalistic geometry Hilbert defined and described in his *Grundlagen der Geometrie* ("model" in the logical sense, described in Section 3.2).[44] Hjelmslev advocated for a formalistic conception of geometry in the mathematical sciences, and he advocated to teach a model of the axiom system presented in *AKL*, the drawing plane (see Section 3.2), in school. There is no contradiction in this attitude.

In summary, it can be said that Hjelmslev was a modern geometer (from today's perspective) with a modern view on didactics, who knew how to interpret and further develop the avant-garde trends of his time.

He was well aware of his achievements and their significance, as is evident from his own assessment of his work, which he gave at the Scandinavian Congress of Mathematicians in Trondheim in 1952 (six months before he died). If one differentiates between the two fields of Hjelmslev's research, geometry and the didactics of geometry, and if the historical context is borne in mind, then it becomes clear that Hjelmslev made fundamental contributions to both of these fields. He further developed the concept of geometry (in Hilbert's formalistic sense) and the conception of school geometry (in the sense of an empirical theory) and provided in both areas paradigmatic examples, combined with a systematic and consistent methodical approach. Given the political and social turbulences of his time, this is an outstanding achievement. Hjelmslev was certainly one of the most remarkable mathematicians of his time.

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No data was used for the research described in the article.

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[43] Which in no way gives a reason to speak of "limited success" (Lützen, 2021, p. 113).

[44] Worth reading classical descriptions of the relationship between formalistic and empirical geometry can be found in Hempel (1945) and Einstein (1921). Einstein characterizes the epistemological difference between these two kinds of theories with the now famous words: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

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A recent joint publication of the two, together with H. J. Burscheid, in the history of mathematics deals with the principle of Cavalieri and the application of indivisibles by Torricelli and de Sluse (see Mathematische Semesterberichte, Volume 72, pages 11–28 (2025)).