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Traces of partition Eisenstein series and almost holomorphic modular forms

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Abstract

Recently, Amdeberhan, Griffin, Ono, and Singh started the study of “traces of partition Eisenstein series” and used it to give explicit formulas for many interesting functions. In this note we determine the precise spaces in which they lie, find modular completions, and show how they are related via operators.

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1 Introduction and statement of results

In [1] Amdeberhan–Griffin–Ono–Singh introduced the functions

$$e_k(\Lambda_\tau(s)) := \sum_{\substack{\omega_1, \dots, \omega_k \in \Lambda_\tau(s) \\ \text{distinct}}} \frac{1}{\omega_1 \cdots \omega_k},$$

for $k \in \mathbb{N}$ and $\tau \in \mathbb{H}$, where

$$\Lambda_\tau(s) := \{\omega^2 |\omega|^{2s} : \omega \in \Lambda_\tau \setminus \{0\}\}.$$

Here $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$. Moreover, let

$$e_k(\Lambda_\tau(0)) := \lim_{s \rightarrow 0^+} e_k(\Lambda_\tau(s)).$$

In Theorem 1.1 (1) of [1] it was shown that $e_k(\Lambda_\tau(0))$ are weight $2k$ non-holomorphic modular forms. In this note we make this statement precise and determine the space in which these lie and how they can be related via operators. From here on, write $\tau = u + iv$. Recall that the *lowering operator* is defined as $L := -2iv^2 \frac{\partial}{\partial \bar{\tau}}$. An *almost holomorphic modular form* $\hat{f}$ is a function on $\mathbb{H}$ that transforms like a modular form and is a polynomial in $\frac{1}{v}$ with holomorphic coefficients¹. The degree of the polynomial is called the *depth* of $\hat{f}$. The holomorphic part f of $\hat{f}$ is called a *quasimodular form*. Note that the quasimodular

¹See [6, Sect. 5.3] for an exposition on almost holomorphic modular forms.

form can be obtained from the almost holomorphic modular form by

$$f(\tau) = \lim_{\bar{\tau} \rightarrow -i\infty} \widehat{f}(\tau). \quad (1.1)$$

Here τ and $\bar{\tau}$ are considered as independent variables and we note that $\nu = \frac{\tau - \bar{\tau}}{2i}$. Often quasimodular forms are defined as constant terms in the expansions of the almost holomorphic modular form. The alternative definition, using (1.1), is borrowed from mathematical physics and turns out to be more useful in our setting.

Theorem 1.1 *Let $k \in \mathbb{N}$.*

- (1) *We have that $e_k(\Lambda_\tau(0))$ is an almost holomorphic modular form of weight $2k$ and depth k .*
- (2) *We have $e_0(\Lambda_\tau(0)) = 1$ and*

$$L(e_k(\Lambda_\tau(0))) = \frac{\pi}{2} e_{k-1}(\Lambda_\tau(0)).$$

- (3) *The (modified) generating function*

$$\sigma^\#(z; \tau) := e^{-\frac{\pi z^2}{2\nu}} \sum_{k \geq 0} (-1)^k e_k(\Lambda_\tau(0)) z^{2k+1}$$

is a Jacobi form of weight -1 and index $\frac{1}{2}$.

Next, we look at the Taylor coefficients of a meromorphic Jacobi form $\phi(z; \tau)$ (see Sect. 2.2 for the definition) with torsion divisors. The *divisor* of ϕ (with respect to τ) is the formal sum

$$\text{Div}_\tau(\phi) := \sum_{w \in \mathbb{C}/\Lambda_\tau} a_w \cdot [w],$$

where a_w is the order of $\phi_\tau(z) := \phi(z; \tau)$ at the point w and $[w]$ is treated as symbol representing the point w (see [10, Chap. II.3] for more details on divisors). Note that this is a finite sum. A point $z = a\tau + b \in \mathbb{C}$ is a *torsion point* if $a, b \in \mathbb{Q}$. Following [1], we define the *Eisenstein-theta function for the divisor $D = \text{Div}_\tau(\phi)$* as

$$G_{k,D}(\tau) := \sum_{w \in \mathbb{C}/\Lambda_\tau} a_w G_{k,w}(\tau).$$

Here for a non-zero torsion point w

$$G_{k,w}(\tau) := - \left[\left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^k \log \left(|\vartheta(z; \tau)|^2 \right) \right]_{z=w},$$

where the theta function ϑ is defined in Eq. (2.5). For convention let $G_{2k,0} := G_{2k}$ (see (2.1) for the definition of G_{2k}) and $G_{2k+1,0} := 0$. For a partition² $\lambda \vdash n$, define the *n -th partition Eisenstein-theta function for the divisor D* as

$$\mathcal{G}_{D,\lambda}(\tau) := G_{1,D}^{m_1}(\tau) G_{2,D}^{m_2}(\tau) \cdots G_{n,D}^{m_n}(\tau).$$

²Here and throughout we use $\lambda \vdash n$ for $(1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$.

The n -th partition Eisenstein-theta trace for D and $\psi : \mathcal{P} \rightarrow \mathbb{C}$, a function on partition, is defined as

$$\mathrm{Tr}_n(D, \psi; \tau) := \sum_{\lambda \vdash n} \psi(\lambda) \mathcal{G}_{D, \lambda}(\tau). \quad (1.2)$$

In Theorem 1.4 of [1], it was shown that a meromorphic Jacobi form with torsion divisors D can be expressed in terms of these partition Eisenstein-theta traces for D . To state this result for a partition $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$, we let

$$\psi_J(\lambda) := \frac{(-1)^{\ell(\lambda)}}{\prod_{j=1}^n m_j! j^{m_j}},$$

where $\ell(\lambda) := \sum_{j=1}^n m_j$ is the length of λ .

Theorem 1.2 ([1, Theorem 1.4]) *Suppose that $\phi(z; \tau)$ is a meromorphic Jacobi form of weight k and index m , and torsion divisor D , and let a be the order of $\phi(z; \tau)$ at $z = 0$. Then there exists a unique weakly holomorphic modular form f_ϕ of weight³ $k + a$ for which, in a neighborhood around 0,*

$$\phi(z; \tau) = f_\phi(\tau) \sum_{n \geq 0} \mathrm{Tr}_n(D, \psi_J; \tau) (2\pi i z)^{n+a}.$$

We next determine the space in which $\mathrm{Tr}_n(D, \psi; \tau)$ lie. For this, and $n \in \mathbb{N}$, we define (using the notation from Theorem 1.2)

$$\widehat{\mathrm{Tr}}_n(D, \psi_J; \tau) := \sum_{0 \leq j \leq \frac{n}{2}} \frac{\left(\frac{\pi m}{v}\right)^j}{j!} (2\pi i)^{n-2j+a} \mathrm{Tr}_{n-2j}(D, \psi_J; \tau). \quad (1.3)$$

Theorem 1.3 (1) *The function $\widehat{\mathrm{Tr}}_n(D, \psi_J; \tau)$ is a weight n and depth at most $\lfloor \frac{n}{2} \rfloor$ almost holomorphic modular form with*

$$\lim_{\tau \rightarrow -i\infty} \widehat{\mathrm{Tr}}_n(D, \psi_J; \tau) = (2\pi i)^{n+a} \mathrm{Tr}_n(D, \psi_J; \tau).$$

(2) *We have*

$$\begin{aligned} \widehat{\mathrm{Tr}}_0(D, \psi_J; \tau) &= (2\pi i)^a, \quad \widehat{\mathrm{Tr}}_1(D, \psi_J; \tau) = -(2\pi i)^{1+a} G_{1,D}(\tau), \\ L\left(\widehat{\mathrm{Tr}}_n(D, \psi_J; \tau)\right) &= -\pi m \widehat{\mathrm{Tr}}_{n-2}(D, \psi_J; \tau). \end{aligned}$$

Next, we study the following two sequences of q -series considered by Ramanujan in his lost notebook [9, pp. 369] ($n \in \mathbb{N}$)

$$\begin{aligned} U_{2n}(q) &= \frac{1^{2n+1} - 3^{2n+1}q + 5^{2n+1}q^3 - 7^{2n+1}q^6 + 9^{2n+1}q^{10} - \dots}{1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \dots}, \\ V_{2n}(q) &= \frac{1^{2n} - 5^{2n}q - 7^{2n}q^2 + 11^{2n}q^5 + 13^{2n}q^7 - \dots}{1 - q - q^2 + q^5 + q^7 - \dots}. \end{aligned}$$

³Note that there is a typo in Theorem 1.4 of [1]: the weight there is $k - a$ instead of $k + a$.

Ramanujan offered many identities such as ($q := e^{2\pi i\tau}$)

$$\begin{aligned} U_0(q) &= 1, \quad U_2(q) = E_2(\tau), \quad U_4(q) = \frac{1}{3} (5E_2^2(\tau) - 2E_4(\tau)), \\ U_6(q) &= \frac{1}{9} (35E_2^3(\tau) - 42E_2(\tau)E_4(\tau) + 16E_6(\tau)), \dots, \end{aligned}$$

and

$$\begin{aligned} V_0(q) &= 1, \quad V_2(q) = E_2(\tau), \quad V_4(q) = 3E_2^2(\tau) - 2E_4(\tau), \\ V_6(q) &= 15E_2^3(\tau) - 30E_2(\tau)E_4(\tau) + 16E_6(\tau), \dots, \end{aligned}$$

where E_{2k} are the normalized Eisenstein series defined in (2.4). Ramanujan further claimed that for $n \in \mathbb{N}$, U_{2n} and V_{2n} can be written in terms of E_2 , E_4 , and E_6 . Berndt et al. [3, 4] proved this claim. However, their result was not explicit. This was recently made explicit in Theorem 1.2 of [2] by writing U_{2n} and V_{2n} as partition Eisenstein trace. Define $\psi_1, \psi_2 : \mathcal{P} \rightarrow \mathbb{R}$, for a partition $\lambda \vdash n$ by

$$\begin{aligned} \psi_1(\lambda) &:= 4^n (2n+1)! \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{B_{2k}}{2k(2k)!} \right)^{m_k}, \\ \psi_2(\lambda) &:= 4^n (2n)! \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{(4^k - 1) B_{2k}}{2k(2k)!} \right)^{m_k}. \end{aligned}$$

In [2, Theorem 1.2] it was shown that

$$U_{2n}(q) = \text{Tr}_n(\psi_1; \tau), \quad V_{2n}(q) = \text{Tr}_n(\psi_2; \tau), \quad (1.4)$$

where, for any $\psi : \mathcal{P} \rightarrow \mathbb{C}$, we define

$$\text{Tr}_n(\psi; \tau) := \sum_{\lambda \vdash n} \psi(\lambda) E_2^{m_1}(\tau) E_4^{m_2}(\tau) \cdots E_{2n}^{m_n}(\tau).$$

Remark The partition Eisenstein series considered here should not be confused with the beautiful functions of the same name studied by Just and Schneider [8]. In their work the partition Eisenstein series are defined as lattice sums that are dictated by partitions.

As our next result, we determine the space in which U_{2n} and V_{2n} lie. Define

$$\widehat{U}_{2n}(\tau) := 2i \sum_{0 \leq j \leq n} (-1)^{n+j} \frac{\pi^{2n-j+1} U_{2n-2j}(q)}{(2v)^j j! (2n-2j)!}, \quad (1.5)$$

$$\widehat{V}_{2n}(\tau) := \sum_{0 \leq j \leq n} (-1)^{n+j} \frac{3^j \pi^{2n-j} V_{2n-2j}(q)}{(2v)^j j! (2n-2j)!}. \quad (1.6)$$

Theorem 1.4 (1) The functions $\widehat{U}_{2n}$ and $\widehat{V}_{2n}$ are almost holomorphic modular forms of weight $2n$ and depth n .

(2) We have $\widehat{U}_0 = 2\pi i$, $\widehat{V}_0 = 1$, and

$$L\left(\widehat{U}_{2n}\right) = -\frac{\pi}{2} \widehat{U}_{2n-2}, \quad L\left(\widehat{V}_{2n}\right) = -\frac{3\pi}{2} \widehat{V}_{2n-2}.$$

(3) We have

$$\lim_{\bar{\tau} \rightarrow -i\infty} \widehat{U}_{2n}(\tau) = \frac{2(\pi i)^{2n+1} U_{2n}(q)}{(2n+1)!}, \quad \lim_{\bar{\tau} \rightarrow -i\infty} \widehat{V}_{2n}(\tau) = \frac{(\pi i)^{2n}}{(2n)!} V_{2n}(q).$$

The paper is organized as follows. In Sect. 2, we recall some basic facts about Eisenstein series, Jacobi forms, and Pólya cycle index polynomials. In Sects. 3, 4, and 5, we prove Theorems 1.1, 1.3, and 1.4, respectively.

2 Preliminaries

2.1 Eisenstein series

The *Eisenstein series of weight k* are defined as

$$G_{2k}(\tau) := -\frac{B_{2k}}{2k} + 2 \sum_{n \geq 1} \sigma_{2k-1}(n) q^n, \quad (2.1)$$

where B_ℓ denotes the ℓ -th Bernoulli number and $\sigma_\ell(n) := \sum_{d|n} d^\ell$ the ℓ -th divisor sum. For $k > 1$ the Eisenstein series are modular forms whereas G_2 can be completed by adding a simple term. That is defining

$$\widehat{G}_2(\tau) := G_2(\tau) + \frac{1}{4\pi\tau} \quad (2.2)$$

we have for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, for $k \geq 2$,

$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_{2k}(\tau), \quad \widehat{G}_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \widehat{G}_2(\tau). \quad (2.3)$$

We also require the *normalized Eisenstein series*

$$E_{2k}(\tau) := -\frac{2k}{B_{2k}} G_{2k}(\tau). \quad (2.4)$$

2.2 Jacobi forms

Definition 2.1 A holomorphic Jacobi form of weight k and index m ($k \in \mathbb{N}$, $m \in \frac{1}{2}\mathbb{N}$) is a holomorphic function $\phi : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ which, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$, satisfies ($\zeta := e^{2\pi iz}$):

- (1) $\phi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}} \phi(z; \tau)$,
- (2) $\phi(z + \lambda\tau + \mu; \tau) = (-1)^{2m\mu + \lambda\varepsilon} e^{-2\pi im(\lambda^2\tau + 2\lambda z)} \phi(z; \tau)$, for some $\varepsilon \in \{0, 1\}$.
- (3) The function ϕ has a Fourier expansion of the shape

$$\phi(z; \tau) = \sum_{n \geq 0, r \in \mathbb{Z}} c(n, r) q^n \zeta^r,$$

with $c(n, r) = 0$ unless $r^2 \leq 4nm$.

Moreover, we call a Jacobi form ϕ *meromorphic* if together with (1) and (2), we allow it to have poles in z , and where the expansions in (3) allow for finite order poles in τ at cusps. In this case k can also be non-positive.

The *Jacobi theta function* is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ (under a slightly modified definition)

$$\vartheta(z; \tau) := \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{2\pi i n(z + \frac{1}{2})} q^{\frac{n^2}{2}}. \quad (2.5)$$

To be more precise, it satisfies the elliptic transformation $(\lambda, \mu \in \mathbb{Z})$

$$\vartheta(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} \zeta^{-\lambda} \vartheta(z; \tau) \quad (2.6)$$

and the modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\vartheta\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = v_\eta^3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (c\tau + d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau + d}} \vartheta(z; \tau). \quad (2.7)$$

Here $v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the multiplier of the *Dedekind η -function*

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n),$$

i.e.,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (c\tau + d)^{\frac{1}{2}} \eta(\tau). \quad (2.8)$$

Note that $z \mapsto \vartheta(z; \tau)$ has roots precisely for $z \in \mathbb{Z} + \mathbb{Z}\tau$ and these are simple.

2.3 Pólya cycle index polynomials

We bring a result for Pólya cycle index polynomials in the case of symmetric groups that is required to prove our results; see [11] for more details on cycle index polynomials. Let S_n be the symmetric group; the set of permutations of the symbols $x_1, x_2, \dots, x_n$. The *cycle index polynomial* for S_n is given by

$$Z(S_n) := \sum_{\lambda \vdash n} \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{x_k}{k}\right)^{m_k}.$$

Lemma 2.2 (Example 5.2.10 of [11]) We have

$$\sum_{n \geq 0} Z(S_n) w^n = \exp\left(\sum_{k \geq 1} x_k \frac{w^k}{k}\right),$$

as a formal power series in w .

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 (1) The modular transformation property follows directly from Theorem 1.1 (1) of [1]. For the readers convenience however we give a complete proof. We have

$$e_k(\Lambda_\tau(0)) = \lim_{s \rightarrow 0^+} e_k(\Lambda_\tau(s)).$$

Consider the generating function (see e.g. (3.1) of [1])

$$\begin{aligned} \sigma^*(z; \tau) &:= \sum_{k \geq 0} (-1)^k e_k(\Lambda_\tau(0)) z^{2k+1} = \lim_{s \rightarrow 0^+} \sum_{k \geq 0} (-1)^k e_k(\Lambda_\tau(s)) z^{2k+1} \\ &= \lim_{s \rightarrow 0^+} \sigma_s(z; \tau) = z \exp \left(-\widehat{G}_2(\tau) \frac{(2\pi iz)^2}{2} - \sum_{k \geq 2} G_{2k}(\tau) \frac{(2\pi iz)^{2k}}{(2k)!} \right), \end{aligned} \quad (3.1)$$

where the last equality comes from [1, Eq. (3.3)]. Using equation (2.3) gives that

$$\sigma^* \left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{-1} \sigma^*(z; \tau).$$

Thus we have

$$\sum_{k \geq 1} (-1)^k e_k \left(\Lambda_{\frac{a\tau+b}{c\tau+d}}(0) \right) \left(\frac{z}{c\tau + d} \right)^{2k+1} = \frac{1}{c\tau + d} \sum_{k \geq 1} (-1)^k e_k(\Lambda_\tau(0)) z^{2k+1}.$$

Comparing coefficients yields that

$$e_k \left(\Lambda_{\frac{a\tau+b}{c\tau+d}}(0) \right) = (c\tau + d)^{2k} e_k(\Lambda_\tau(0)),$$

which is the claimed transformation law. The claim on the depth follows from part (2).

(2) The fact that $e_0(\Lambda_\tau(0)) = 1$ is direct (for example by comparing coefficients in (3.1)).

We next compute, using (3.1),

$$\begin{aligned} L(\sigma^*(z; \tau)) &= L \left(z \exp \left(-\frac{\widehat{G}_2(\tau)}{2} (2\pi iz)^2 - \sum_{k \geq 2} \frac{G_{2k}(\tau)}{(2k)!} (2\pi iz)^{2k} \right) \right) \\ &= \sigma^*(z; \tau) \frac{-(2\pi iz)^2}{2} L(\widehat{G}_2(\tau)) = -\frac{\pi z^2}{2} \sigma^*(z; \tau), \end{aligned}$$

where the last equality follows using (2.2) and that

$$L\left(\frac{1}{v}\right) = -1. \quad (3.2)$$

Thus, we have, plugging into (3.1),

$$\sum_{k \geq 0} (-1)^k L(e_k(\Lambda_\tau(0))) z^{2k+1} = -\frac{\pi}{2} \sum_{k \geq 0} (-1)^k e_k(\Lambda_\tau(0)) z^{2k+3}.$$

Comparing coefficients gives

$$L(e_k(\Lambda_\tau(0))) = \frac{\pi}{2} e_{k-1}(\Lambda_\tau(0)),$$

as claimed.

(3) We use (3.1), (3.2), and then [12, Eq. (7)] to write

$$\begin{aligned} \sigma^\#(z; \tau) &= e^{-\frac{\pi z^2}{2v}} z \exp\left(-\widehat{G}_2(\tau) \frac{(2\pi iz)^2}{2} - \sum_{k \geq 2} G_{2k}(\tau) \frac{(2\pi iz)^{2k}}{(2k)!}\right) \\ &= z \exp\left(-G_2(\tau) \frac{(2\pi iz)^2}{2} - \sum_{k \geq 2} G_{2k}(\tau) \frac{(2\pi iz)^{2k}}{(2k)!}\right) = \frac{\vartheta(z; \tau)}{-2\pi \eta^3(\tau)}. \end{aligned} \quad (3.3)$$

From equation (2.6) we obtain, for $\lambda, \mu \in \mathbb{Z}$,

$$\sigma^\#(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} \zeta^{-\lambda} \sigma^\#(z; \tau).$$

Next, using (2.7) and (2.8) gives, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\sigma^\# \left(\frac{z}{c\tau + b}; \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{-1} e^{\frac{\pi icz^2}{c\tau + d}} \sigma^\#(z; \tau). \quad \square$$

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3.

Proof of Theorem 1.3 (1) Let $\phi(z; \tau)$ be a meromorphic Jacobi form of weight k , index m , and torsion divisor D , and let a be the order of $\phi(z; \tau)$ at $z = 0$. Then by Theorem 1.2, there exist a weight $k + a$ modular form f_ϕ and Eisenstein-theta traces $\mathrm{Tr}_n(D, \psi_J; \tau)$ such that

$$\phi(z; \tau) = f_\phi(\tau) \sum_{n \geq 0} \mathrm{Tr}_n(D, \psi_J; \tau) (2\pi iz)^{n+a}$$

in a neighbourhood of $z = 0$. We let

$$\phi^*(z; \tau) := e^{\frac{\pi m z^2}{v}} \phi(z; \tau). \quad (4.1)$$

Then, in a neighbourhood of 0, we have

$$\begin{aligned} \phi^*(z; \tau) &= f_\phi(\tau) \sum_{j \geq 0} \frac{\left(\frac{\pi m z^2}{v}\right)^j}{j!} \sum_{\ell \geq 0} \mathrm{Tr}_\ell(D, \psi_J; \tau) (2\pi iz)^{\ell+a} \\ &= f_\phi(\tau) \sum_{n \geq 0} z^{n+a} \sum_{\substack{j, \ell \geq 0 \\ n=2j+\ell}} \frac{\left(\frac{\pi m}{v}\right)^j}{j!} (2\pi i)^\ell \mathrm{Tr}_\ell(D, \psi_J; \tau) \\ &= f_\phi(\tau) \sum_{n \geq 0} c_n(\tau) z^{n+a}, \end{aligned} \quad (4.2)$$

where

$$c_n(\tau) := \sum_{0 \leq j \leq \frac{n}{2}} \frac{\left(\frac{\pi m}{v}\right)^j}{j!} (2\pi i)^{n-2j+a} \text{Tr}_{n-2j}(D, \psi_J; \tau) = \widehat{\text{Tr}}_n(D, \psi_J; \tau), \quad (4.3)$$

by definition (1.3).

We next prove modular transformation properties of $\widehat{\text{Tr}}_n(D, \psi_J; \tau)$. It can be shown that $\phi^*(z; \tau)$ satisfies the modular transformation formula of a weight k and index 0 Jacobi form (see the proof of [5, Proposition 3.1]). We also have that f_ϕ is a modular form of weight $k + a$. Thus we obtain, from (4.2),

$$\begin{aligned} \phi^*\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) &= f_\phi\left(\frac{a\tau + b}{c\tau + d}\right) \sum_{n \geq 0} c_n\left(\frac{a\tau + b}{c\tau + d}\right) \left(\frac{z}{c\tau + d}\right)^{n+a} \\ &= (c\tau + d)^{k+a} f_\phi(\tau) \sum_{n \geq 0} c_n\left(\frac{a\tau + b}{c\tau + d}\right) \left(\frac{z}{c\tau + d}\right)^{n+a} \\ &= (c\tau + d)^k \phi^*(z; \tau) = (c\tau + d)^k f_\phi(\tau) \sum_{n \geq 0} c_n(\tau) z^{n+a}. \end{aligned}$$

Thus

$$c_n\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n c_n(\tau).$$

Using (4.3) this gives

$$\widehat{\text{Tr}}_n\left(D, \psi_J; \frac{a\tau + b}{c\tau + d}\right) = c_n\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n c_n(\tau) = (c\tau + d)^n \widehat{\text{Tr}}_n(D, \psi_J; \tau).$$

Since by definition $\widehat{\text{Tr}}_n$ is a polynomial in $\frac{1}{v}$ with holomorphic coefficients and degree at most $\lfloor \frac{n}{2} \rfloor$, $\widehat{\text{Tr}}_n$ is an almost holomorphic modular form of weight n and depth at most $\lfloor \frac{n}{2} \rfloor$. By (1.3), we have that

$$\lim_{\tau \rightarrow -i\infty} \widehat{\text{Tr}}_n(D, \psi_J; \tau) = (2\pi i)^{n+a} \text{Tr}_n(D, \psi_J; \tau).$$

This proves (1).

(2) Using (4.3) and the definition of $\text{Tr}_n(D, \psi_J; \tau)$ for $n = 0, 1$, we get

$$\widehat{\text{Tr}}_0(D, \psi_J; \tau) = (2\pi i)^a, \quad \widehat{\text{Tr}}_1(D, \psi_J; \tau) = -(2\pi i)^{1+a} G_{1,D}(\tau).$$

We next compute the action of the lowering operator on ϕ^* [7]. We use (4.2), (4.1), and (3.2) to get

$$f_\phi(\tau) \sum_{n \geq 0} L(c_n(\tau)) z^{n+a+2} = L(\phi^*(z; \tau)) = -\pi m \phi^*(z; \tau) z^2 = -\pi m f_\phi(\tau) \sum_{n \geq 0} c_n(\tau) z^{n+a+2}.$$

Thus

$$L(c_n(\tau)) = -\pi m c_{n-2}(\tau).$$

Thus for $n \geq 2$, we have, using (4.3),

$$L(\widehat{\text{Tr}}_n(D, \psi_f; \tau)) = L(c_n(\tau)) = -\pi m c_{n-2}(\tau) = -\pi m \widehat{\text{Tr}}_{n-2}(D, \psi_f; \tau).$$

This completes the proof of (2). $\square$

5 Proof of Theorem 1.4

We next construct almost holomorphic modular forms from U_{2n} and V_{2n} .

Proof of Theorem 1.4 We first prove the results on U_{2n} . We use (1.4) to write

$$\begin{aligned} U(z; \tau) &:= \sum_{n \geq 0} (-1)^n U_{2n}(\tau) \frac{(\pi z)^{2n+1}}{(2n+1)!} \\ &= \pi z \sum_{n \geq 0} \sum_{\lambda \vdash n} \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{B_{2k} E_{2k}(\tau)}{2k(2k)!} \right)^{m_k} (2\pi iz)^{2n}. \end{aligned} \quad (5.1)$$

If we take $x_k = \frac{B_k E_k(\tau)}{k!}$ and $w = 2\pi iz$ in Lemma 2.2, then we get

$$\begin{aligned} \sum_{n \geq 0} \sum_{\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})} \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{B_k E_k(\tau)}{k \cdot k!} \right)^{m_k} (2\pi iz)^n \\ = \exp \left(\sum_{n \geq 1} \frac{B_{2n} E_{2n}(\tau)}{2n} \frac{(2\pi iz)^{2n}}{(2n)!} \right), \end{aligned} \quad (5.2)$$

where we use that $E_{2n+1}(\tau) = 0$. We next also use this fact for the left-hand side. Note that we have that $\prod_{k=1}^n \frac{1}{m_k!} \left(\frac{B_k E_k(\tau)}{k \cdot k!} \right)^{m_k} = 0$ if $m_k \neq 0$ for some odd k in the product, and in particular if n is odd. Hence we can write the left-hand side as

$$\sum_{n \geq 0} \sum_{\lambda = (1^0, 2^{m_2}, 3^0, 4^{m_4}, \dots, (2n)^{m_{2n}})} \prod_{k=1}^n \frac{1}{m_{2k}!} \left(\frac{B_{2k} E_{2k}(\tau)}{2k(2k)!} \right)^{m_{2k}} (2\pi iz)^{2n}.$$

We have that $\lambda = (1^0, 2^{m_2}, 3^0, 4^{m_4}, \dots, (2n)^{m_{2n}}) \vdash 2n$ is in one to one correspondence with $(1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n}) \vdash n$ with $\mu_j = m_{2j}$, hence the above can be written as

$$\sum_{n \geq 0} \sum_{\lambda = (1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n})} \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{B_{2k} E_{2k}(\tau)}{2k(2k)!} \right)^{m_k} (2\pi iz)^{2n}. \quad (5.3)$$

Combining (5.1), (5.2), and (5.3) gives that

$$U(z; \tau) = \pi z \exp \left(\sum_{n \geq 1} \frac{B_{2n} E_{2n}(\tau)}{2n} \frac{(2\pi iz)^{2n}}{(2n)!} \right).$$

Using (3.3), this equals

$$\frac{\vartheta(z; \tau)}{-2\eta^3(\tau)},$$

which is shown in the proof of Theorem 1.1 (3) to be a weight -1 and index $\frac{1}{2}$ Jacobi form. Next we prove that

$$U(z; \tau) = \frac{1}{2i} \sum_{n \geq 0} \text{Tr}_n([0], \psi_J; \tau) (2\pi iz)^{n+1}. \quad (5.4)$$

Recall that we have $G_{2k,0}(\tau) = G_{2k}(\tau)$ and $G_{2k+1,0}(\tau) = 0$ by definition. By (1.2) we have

$$\begin{aligned} \text{Tr}_{2n}([0], \psi_J; \tau) &= \sum_{\lambda=(1^{m_1}, 2^{m_2}, \dots, (2n)^{m_{2n}}) \vdash 2n} \psi_J(\lambda) \mathcal{G}_{[0], \lambda}(\tau) \\ &= \sum_{\lambda=(1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n}) \vdash n} \frac{(-G_2(\tau))^{\mu_1} (-G_4(\tau))^{\mu_2} \cdots (-G_{2n}(\tau))^{\mu_n}}{\prod_{j=1}^n \mu_j! (2j)^{\mu_j}}, \end{aligned}$$

where we use that $\lambda = (1^0, 2^{m_2}, 3^0, 4^{m_4}, \dots, (2n)^{m_{2n}}) \vdash 2n$ is in bijection with $(1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n}) \vdash n$ with $\mu_j = m_{2j}$. We use (2.4) to write this as

$$\sum_{\lambda \vdash n} \prod_{j=1}^n \frac{(B_{2j} E_{2j}(\tau))^{m_j}}{(2j)^{m_j} m_j! (2j)^{m_j}} = \frac{\text{Tr}_n(\psi_1; \tau)}{4^n (2n+1)!} = \frac{U_{2n}(q)}{4^n (2n+1)!},$$

where the first equality comes from the definition of $\text{Tr}_n(\psi_1, \tau)$ and the second from (1.4). Plugging this back into the definition of U in (5.1) gives (5.4). Hence we are in the setting of Theorem 1.2 with $D = [0]$, $a = 1$, and $f_\phi = \frac{1}{2i}$. A direct calculation shows that

$$\widehat{U}_{2n}(\tau) = \widehat{\text{Tr}}_{2n}([0], \psi_J; \tau).$$

By Theorem 1.3 (1), $\widehat{U}_{2n}$ is a weight $2n$ and depth at most n almost holomorphic modular form satisfying

$$\lim_{\tau \rightarrow -i\infty} \widehat{U}_{2n}(\tau) = 2i(-1)^n \frac{\pi^{2n+1} U_{2n}(q)}{(2n+1)!}.$$

In this case the depth is exactly n as by (1.5) the coefficient of v^{-n} is $\frac{\pi^{n-1} i U_0(q)}{2^{n-1} n!} \neq 0$ since $U_0(q) = 1$. This proves (1) and (3) for U_{2n} . By Theorem 1.3 (2), we have that $\widehat{U}_0 = 2\pi i$ and $L(\widehat{U}_{2n}) = -\frac{\pi}{2} \widehat{U}_{2n-2}$ which proves (2) for U_{2n} .

Next we show the results on $V_{2n}(q)$. We use (1.4) to write

$$\begin{aligned} V(z; \tau) &:= \sum_{n \geq 0} (-1)^n V_{2n}(q) \frac{(\pi z)^{2n}}{(2n)!} \\ &= \sum_{n \geq 0} \sum_{\lambda \vdash n} \prod_{k=1}^n \frac{1}{m_k!} \left(\frac{(4^k - 1) B_{2k} E_{2k}(\tau)}{2k(2k)!} \right)^{m_k} (2\pi iz)^{2n} \\ &= \exp \left(\sum_{n \geq 1} (4^n - 1) \frac{B_{2n} E_{2n}(\tau)}{2n} \frac{(2\pi iz)^{2n}}{(2n)!} \right) = \frac{\vartheta(2z; \tau)}{2\vartheta(z; \tau)}, \end{aligned} \quad (5.5)$$

taking $x_k = \frac{(2^k - 1) B_k E_k(\tau)}{k!}$ and $w = 2\pi iz$ in Lemma 2.2 and applying a similar argument as for U_{2n} and using (3.3).

Next we determine the transformation laws for $\frac{\vartheta(2z;\tau)}{2\vartheta(z;\tau)}$. We start with the modular transformation. Using (2.7), we have for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\frac{\vartheta\left(\frac{2z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right)}{2\vartheta\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right)} = e^{\frac{3\pi icz^2}{c\tau+d}} \frac{\vartheta(2z; \tau)}{2\vartheta(z; \tau)}.$$

Next, we consider at the elliptic transformation. Using (2.6), we have for $\lambda, \mu \in \mathbb{Z}$

$$\frac{\vartheta(2(z + \lambda\tau + \mu); \tau)}{2\vartheta(z + \lambda\tau + \mu; \tau)} = (-1)^{\lambda+\mu} q^{-\frac{3\lambda^2}{2}} \zeta^{-3\lambda} \frac{\vartheta(2z; \tau)}{2\vartheta(z; \tau)}.$$

So $\frac{\vartheta(2z;\tau)}{2\vartheta(z;\tau)}$, and hence $V(z; \tau)$, is a Jacobi form of weight 0 and index $\frac{3}{2}$. Using (5.5), we write

$$\begin{aligned} V^*(z; \tau) &:= e^{\frac{3\pi z^2}{2v}} V(z; \tau) = \sum_{j \geq 0} \frac{\left(\frac{3\pi z^2}{2v}\right)^j}{j!} \sum_{n \geq 0} (-1)^n V_{2n}(q) \frac{(\pi z)^{2n}}{(2n)!} \\ &= \sum_{n \geq 0} \widehat{V}_{2n}(\tau) z^{2n}, \end{aligned} \quad (5.6)$$

where the last equality follows from definition (1.6). It can be shown that $V^*(z; \tau)$ satisfies the modular transforms like a weight 0 and index 0 Jacobi form (see the proof of [5, Proposition 3.1]). Using (5.6), this gives

$$\sum_{n \geq 0} \widehat{V}_{2n} \left(\frac{a\tau + b}{c\tau + d} \right) \left(\frac{z}{c\tau + d} \right)^{2n} = V^* \left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right) = V^*(z; \tau) = \sum_{n \geq 0} \widehat{V}_{2n}(\tau) z^{2n}.$$

Comparing the $(2n)$ -th coefficients gives that

$$\widehat{V}_{2n} \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{2n} \widehat{V}_{2n}(\tau).$$

Also by definition $\widehat{V}_{2n}(\tau)$ is a polynomial in $\frac{1}{v}$ with holomorphic coefficients and degree n . This is true because the term corresponding to v^{-n} is $\frac{3^n \pi^n V_0(q)}{2^n n!} \neq 0$ since $V_0(q) = 1$. Hence $\widehat{V}_{2n}$ is an almost holomorphic modular form of weight $2n$ and depth n . This proves part (1) for V . To prove part (2) for V , first note that by definition $\widehat{V}_0 = 1$. We use (3.2) to get

$$L(V^*(z; \tau)) = L\left(e^{\frac{3\pi z^2}{2v}} V(z; \tau)\right) = -\frac{3\pi}{2} z^2 e^{\frac{3\pi z^2}{2v}} V(z; \tau) = -\frac{3\pi}{2} z^2 V^*(z; \tau).$$

Combining this with (5.6) then gives

$$\sum_{n \geq 0} L(\widehat{V}_{2n}(\tau)) z^{2n} = -\frac{3\pi}{2} \sum_{n \geq 0} \widehat{V}_{2n}(\tau) z^{2n+2}.$$

Comparing the $(2n)$ -th coefficients gives part (2) for V . Part (3) for V follows by taking the limit $\bar{\tau} \rightarrow i\infty$ in the definition of $\widehat{V}_{2n}$. $\square$

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