Quantization in Cauchy-Riemann Geometry

INAUGURAL - DISSERTATION

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

CHIN-CHIA CHANG

aus Taipeh

Taiwan

Köln, 2026

Berichterstatter:	Prof. Dr. George Marinescu
	Prof. Dr. Duc-Viet Vu
	Prof. Dr. Rung-Tzung Huang
Vorsitzender:	Prof. Dr. Alexander Drewitz

Tag der mündlichen Prüfung: 21.04.2026

Zusammenfassung

Die Untersuchung von Bergman-Kernen und Szegő-Kernen stellt eine Verbindung zwischen den geometrischen Strukturen von Mannigfaltigkeiten und der Analysis von Funktionenräumen her, wobei Werkzeuge der mikrolokalen und semiklassischen Analysis verwendet werden. In diesem Zusammenhang verbindet die Theorie der Toeplitz-Operatoren auf Mannigfaltigkeiten die algebraische Struktur von Funktionenräumen auf einer gegebenen Mannigfaltigkeit mit ihrer Geometrie durch Quantisierung.

Im ersten Teil bestimmen wir den vierten Koeffizienten des Berezin–Toeplitz-Sternprodukts auf kompakten Kählerschen Mannigfaltigkeiten in dem Fall, dass die Volumenform nicht polarisiert ist.

Angeregt durch die Analysis der Szegő-Projektion ist es naheliegend, die Berezin–Toeplitz-Quantisierung auf Cauchy-Riemann-Mannigfaltigkeiten (CR-Mannigfaltigkeiten) zu erweitern und so ein Deformationsquantisierungsanalogon zur symplektischen Geometrie zu erhalten. Dieselbe Perspektive lässt sich auch auf Toeplitz-Operatoren anwenden, die über die Bergman-Projektion auf komplexen Mannigfaltigkeiten mit Rand definiert sind.

Im zweiten Teil dieser Arbeit zeigen wir, dass es zu jedem Pseudodifferentialoperator L auf einer streng pseudokonvexen CR-Mannigfaltigkeit X eine Folge glatter Funktionen auf X gibt, die L repräsentiert. Wir konstruieren eine Algebra solcher Folgen, die analog zur klassischen Deformationsquantisierung die Rolle der Algebra formaler Potenzreihen in einem Parameter mit glatten Funktionen als Koeffizienten übernimmt. Anschließend zeigen wir, dass diese Algebra eine Poisson-Struktur trägt und ein Sternprodukt zulässt. In demselben Sinn weisen wir nach, dass Toeplitz-Operatoren auf komplexen Mannigfaltigkeiten M mit streng pseudokonvexem CR-Rand X eine entsprechende Eigenschaft besitzen. Durch die Untersuchung des Poisson-Operators in Randnähe zeigen wir, dass bestimmte Pseudodifferentialoperatoren auf $\overline{M}$, die sich auf Operatoren am Rand einschränken lassen, durch Folgen harmonischer Funktionen auf $\overline{M}$ dargestellt werden können. Wir beweisen, dass diese Operatoren eine Poisson-Algebra bilden und ebenfalls ein Sternprodukt tragen.

Der dritte Teil ist eine gemeinsame Arbeit mit H. Herrmann und C.-Y. Hsiao und basiert auf neueren Resultaten von H. Herrmann, C.-Y. Hsiao, G. Marinescu und W.-C. Shen. Wir berechnen den zweiten Koeffizienten in der asymptotischen Entwicklung einer bestimmten Spektralprojektion semiklassischer Toeplitz-Operatoren auf kompakten, streng pseudokonvexen, einbettbaren CR-Mannigfaltigkeiten. Diese Operatoren entstehen durch Einführung einer charakteristischen Funktion χ sowie eines semiklassischen Parameters k mittels

Funktionalkalküls. Die Schwartz-Kerne dieser Operatoren besitzen eine vollständige asymptotische Entwicklung, was zu neuen Anwendungen in der CR-Geometrie führt. In diesem Teil erweitern wir zunächst die bisherigen Berechnungen des zweiten Koeffizienten des Szegő-Kerns auf den Fall einer allgemeinen Reeb-invarianten Volumenform. Anschließend untersuchen wir die Abhängigkeit von χ der Koeffizienten in der asymptotischen Entwicklung und bestimmen den zweiten Koeffizienten explizit.

Abstract

The study of Bergman kernels and Szegő kernels provides a link between the geometric structures of manifolds and the analysis of function spaces, using tools from microlocal and semi-classical analysis. In this framework, the theory of Toeplitz operators on manifolds connects the algebraic structure of function spaces on a given manifold with its geometry through quantization.

In the first part, we determine the fourth coefficient of the Berezin–Toeplitz star product on compact Kähler manifolds in the setting where the volume form is not polarized.

Motivated by the analysis of the Szegő projection, it is natural to extend Berezin–Toeplitz quantization to Cauchy–Riemann (CR) manifolds, giving a deformation quantization analogue of symplectic geometry. The same perspective also applies to Toeplitz operators defined via the Bergman projection on complex manifolds with boundary. In the second part of this thesis, we show that for any pseudodifferential operator L on a strictly pseudoconvex CR manifold X, there exists a sequence of smooth functions on X representing L. We construct an algebra of such sequences, which plays a role analogous to the algebra of formal power series in a parameter with smooth functions as coefficients, as in classical deformation quantization. We then prove that this algebra carries a Poisson structure and admits a star product. In the same spirit, we establish that Toeplitz operators on complex manifolds M with strictly pseudoconvex CR boundary X have a similar property. By examining the Poisson operator near the boundary, we show that certain pseudodifferential operators on $\overline{M}$, which reduce to operators on the boundary, can be represented by sequences of harmonic functions on $\overline{M}$. We prove that these operators form a Poisson algebra and also admit a star product.

The third part is joint work with H. Herrmann and C.-Y. Hsiao, building on recent results of H. Herrmann, C.-Y. Hsiao, G. Marinescu, and W.-C. Shen. We compute the second coefficient in the asymptotic expansion of a certain spectral projection of semi-classical Toeplitz operators on compact, strictly pseudoconvex, embeddable CR manifolds. These operators arise by introducing a characteristic function χ and a semi-classical parameter k via functional calculus, and their Schwartz kernels admit a complete asymptotic expansion, leading to new applications in CR geometry. In this part, we first extend existing computations of the second coefficient of the Szegő kernel to the case of a general Reeb-invariant volume form. We then analyze the dependence of the coefficients in the asymptotic expansion on χ and explicitly compute the second coefficient.

Acknowledgement

First, I would like to express my deepest gratitude to my advisor, Prof. George Marinescu. He provided a good research environment and supported me throughout my PhD. He offered many helpful suggestions and guidance in mathematics. I am truly grateful for his support and encouragement.

I would especially like to thank Prof. Chin-Yu Hsiao for the many mathematical discussions we had. These discussions were very important to my research and strongly motivated my work. We also worked together on a joint project with Hendrik Herrmann. I sincerely appreciate, from my heart, his unwavering encouragement and support throughout my PhD.

I am grateful to my collaborator, Hendrik Herrmann. I truly enjoyed working with him and discussing mathematics together. I also appreciate his helpful comments on my writing and his support in daily life in Germany.

I would like to thank my mentor, Prof. Duc-Viet Vu, for his guidance, advice, and for reviewing this thesis.

I especially thank Prof. Rung-Tzung Huang for reviewing my thesis.

I thank Prof. Xiaonan Ma, Nikhil Savale, Bingxiao Liu, Andrea Galasso, Ood Shabtai, and Wei-Chuan Shen for the discussions we had during my PhD. These conversations were very helpful to my research.

I am grateful to all the assistants and secretaries in the department who helped me during my PhD. Their support made my life in Germany and my activities in the department run smoothly.

I especially want to thank my best friend and roommate, Shuang Su. We spent many good times together in Cologne and shared many conversations about mathematics.

I am deeply grateful to my family for their emotional and practical support throughout these years.

Finally, I would like to thank my partner, Pei-Jun, for her constant support and companionship during these years.

I also thank my cat, Sîng-hông Noonnoon.

Contents

Chapter 1. Introduction 1
- 1.1. Background 1
- 1.2. Quantization 2
- 1.3. Main results 4
- 1.4. Organization of the thesis 6

Chapter 2. Preliminaries 7
- 2.1. Elements of microlocal and semi-classical analysis 7
- 2.2. Cauchy-Riemann geometry and Szegő kernels 11

Part 1. The Fourth Coefficient of Berezin-Toeplitz Star Product 15

Chapter 3. Berezin-Toeplitz quantization 16
- 3.1. Introduction and main results 16
- 3.2. Asymptotic expansion of Bergman kernels and Toeplitz operators 23
- 3.3. Proof of Theorem 3.8 27

Part 2. Quantization on CR Manifolds and Complex Manifolds with Boundaries 38

Chapter 4. Quantization on CR manifolds 39
- 4.1. Introduction and main results 40
- 4.2. Asymptotic expansion of pseudodifferential operators 43
- 4.3. Some operator spaces 48
- 4.4. Quantization of the operator space 51

Chapter 5. Quantization on complex manifolds with boundary 57
- 5.1. Introduction and main results 57
- 5.2. Poisson operator 63
- 5.3. Some operator spaces 82
- 5.4. Quantization of the operator space 90

Part 3. The Subleading term of semi-classical Toeplitz Operators 95

Chapter 6. Semi-classical Toeplitz operators 96
- 6.1. Introduction and main results 96

6.2. The second coefficient in Szegő kernel 99
6.3. Proof of Theorem 6.2 and 6.3 103
6.4. Relations between Bergman kernels 113

Bibliography 116

CHAPTER 1

Introduction

1.1. Background

The study of Bergman and Szegő projections using microlocal analysis and semi-classical methods originates from the work [4] of Boutet de Monvel and Sjöstrand. In their work, they analyzed Bergman and Szegő projections by microlocal techniques and established a Fourier integral operator framework for these projections. In particular, Bergman and Szegő projections can be realized as Fourier integral operators with complex phase.

Closely related developments in microlocal analysis and complex geometry appeared around the same period, notably in the works of Hörmander on Fourier integral operators and stationary phase methods, and in the work of Fefferman on boundary problems in complex analysis [2]. From the point of view of Fourier integral operator theory, geometric invariants of complex manifolds and CR manifolds appear naturally in the symbols of the associated operators. This observation connects analysis with geometry and provides a new analytic approach to problems in differential geometry.

Using the Fourier integral operator framework, one can study the asymptotic expansions of Bergman and Szegő kernels. In particular, the symbolic analysis of Fourier integral operators is very effective for computing coefficients in such expansions. This approach has been applied to the study of the Szegő kernel in the Catlin-Tian-Yau expansion and to the study of the Bergman kernel, with contributions by Hsiao [17] and others, where the existence of asymptotic expansions is established and leading coefficients are computed.

Another approach to Bergman and Szegő kernels is provided by heat kernel methods. Using heat kernel analysis, Ma and Marinescu established connections between the Bergman kernel and holomorphic Morse inequalities [25]. In related work, Hsiao used heat kernel techniques to prove the existence of asymptotic expansions for Bergman and Szegő kernels and to compute the first coefficient [17]. These results show that microlocal analysis and heat kernel methods both play important roles in the study of kernel asymptotics.

Starting from Bergman and Szegő projections, one can construct Toeplitz operators. From the perspective of Fourier integral operator theory, Toeplitz operators can be analyzed microlocally. In particular, one can study the spectrum of

Toeplitz operators and the symbolic behavior of their compositions. This naturally leads to the main theme of this dissertation, namely quantization.

More precisely, the central question is how far the Toeplitz operator symbol calculus for Bergman and Szegő projections extends beyond the classical compact Kähler setting. More specifically, can one use microlocal analysis of these projections to construct algebraic structures that reflect the underlying symplectic or contact geometry, not only on Kähler manifolds but also on strictly pseudoconvex CR manifolds and on complex manifolds with boundary? This problem motivates the developments in the following sections.

1.2. Quantization

There are many ways to interpret quantization. From an algebraic point of view, quantization concerns the problem of deforming a Poisson algebra into a noncommutative algebra in a way that reflects the underlying Poisson structure. A basic example comes from quantum mechanics. Given a physical system, its phase space is a symplectic manifold, typically the cotangent bundle T^*M of a configuration space M. Functions on T^*M form a commutative algebra under pointwise multiplication. In quantum mechanics, however, observables such as position and momentum are represented by operators acting on a Hilbert space, and these operators are in general noncommutative. This noncommutativity is the origin of the Heisenberg uncertainty principle and arises from the operator algebra structure itself.

From a geometric point of view, one asks whether the algebra of smooth functions on a symplectic manifold can be represented on a Hilbert space in such a way that the commutator of operators corresponds to the Poisson bracket of functions. This idea is the basic motivation of geometric quantization. When one considers a compact Kähler manifold M, the algebra of smooth functions on M can be quantized using Bergman projections. Assuming that M admits a pre-quantum line bundle L, one considers the tensor powers L^k and the Hilbert spaces of their holomorphic sections. Let B^k denote the Bergman projection onto these spaces. For a smooth function f on M, the Toeplitz operator T_f^k is defined by

$$T_f^k = B^k f B^k.$$

It was shown by Ma and Marinescu [26], Bordemann, Meinrenken, and Schlichenmaier [3] and others that, as $k \to \infty$, the composition of Toeplitz operators admits an asymptotic expansion

$$T_f^k \circ T_g^k \sim \sum_{j=0}^{\infty} T^k_{k^{-j} C_j(f,g)},$$

where C_j are bidifferential operators. This expansion defines a noncommutative product

$$f * g = \sum_{j=0}^{\infty} k^{-j} C_j(f, g),$$

called the Berezin-Toeplitz star product, giving a noncommutative multiplication on $\mathscr{C}^\infty(M)[\![k^{-1}]\!]$. The leading term $C_0(f, g)$ is the pointwise product fg, and the antisymmetric part of C_1 corresponds to the symplectic Poisson bracket of f and g on M. This semi-classical asymptotic construction is referred to as Berezin-Toeplitz quantization. The framework was later generalized by Ma and Marinescu to Toeplitz operators acting on sections of auxiliary vector bundles [26].

The geometric viewpoint leads to geometric quantization, while the algebraic viewpoint appears in the study of Toeplitz operators. Kernel asymptotics govern the symbol calculus of Toeplitz operators, which in turn underlies star products. The construction of star products is known as deformation quantization.

From the viewpoint of deformation quantization, this construction fits into a general framework developed by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [1], Fedosov [10], and De Wilde and Lecomte [7] where the algebra of smooth functions on a symplectic manifold is deformed using its Poisson structure. Later, Kontsevich extended this result to general Poisson manifolds [24]. A natural question in deformation quantization is how the coefficients of a star product reflect the geometric properties of the underlying manifold. In the Kähler setting, the coefficients of the Berezin-Toeplitz star product are known to involve geometric invariants such as curvature. Using microlocal analysis, Ma-Marinescu [26] and Hsiao [16] computed the first three coefficients of this star product, while Xu [32] obtained related results using combinatorial methods.

We use methods from analysis as a bridge to connect the algebraic and geometric viewpoints on quantization. More specifically, the main tools of this thesis are microlocal analysis and the theory of Fourier integral operators. By studying the asymptotic behavior of the Bergman kernel, the Szegő kernel, and Toeplitz operators, we construct star products, identify how their coefficients relate to geometric invariants, extend several classical quantization results, and provide a new perspective on deformation quantization.

In this thesis, we distinguish between several related notions of quantization. *Berezin–Toeplitz quantization* refers specifically to the semi-classical quantization on compact Kähler manifolds constructed from Bergman projections. More generally, in the CR and complex manifold with boundary settings, we study the algebraic structures generated by Toeplitz operators associated with Szegő or Bergman projections through analytic methods. We refer to this operator-level framework as *Toeplitz quantization*. Finally, *deformation quantization* is understood in the algebraic sense of constructing star products on suitable algebras.

1.3. Main results

Part I. This part of the dissertation computes the fourth coefficient $C_3(f,g)$ of the Berezin–Toeplitz star product on a compact Kähler manifold (Theorem 3.8) using a microlocal analysis of the Bergman kernel.

In contrast to the classical polarized setting, we do not assume that the volume form on M is induced by the polarization associated with the pre-quantum line bundle. Although this situation is equivalent, in the sense of [26], to working with a polarized metric and a determinant line bundle, our approach allows the geometric invariants of M to appear more directly in the coefficient $C_3(f,g)$. Moreover, the analysis applies to the twisted setting in which Toeplitz operators are defined via the Bergman projection onto holomorphic sections of $L^k \otimes E$. In particular, under the polarized metric assumption considered by Ma and Marinescu, the fourth coefficient $C_3(f,g)$ can be obtained within this framework (see Corollary 3.9).

Part II. On a strictly pseudoconvex CR manifold, Toeplitz operators defined via the Szegő projection interact naturally with the underlying contact geometry. Since the bracket induced by the contact form is a Jacobi bracket, the classical star product construction applies only to Reeb-invariant functions.

Motivated by this, we replace the usual formal deformation parameter by an elliptic pseudodifferential operator E. We prove that Toeplitz operators defined via a Szegő type projection admit asymptotic expansions in powers of E (Theorem 4.6). This leads to a closed algebra of pseudodifferential operators adapted to the CR structure and equipped with a star product at the operator level (Theorem 4.23).

We then consider complex manifolds with strictly pseudoconvex CR boundary. Using the factorization of the Bergman projection through the Poisson operator, we analyze the Poisson operator microlocally near the boundary (Proposition 5.11). We show that operators of the form $P^* f P$ are pseudodifferential operators of order -1 (Theorem 5.10), which allows us to reduce the quantization problem to the boundary and construct a deformation quantization in this setting (Theorem 5.26).

Part III. The final part of this dissertation studies semi-classical Toeplitz operators on strictly pseudoconvex CR manifolds, defined via functional calculus, following Herrmann, Hsiao, Marinescu, and Shen [13]. This part is based on [6] joint work with Herrmann and Hsiao. A semi-classical parameter is introduced through a family of cutoff functions, and the resulting operators are shown to admit asymptotic expansions similar to those of the Szegő kernel. We do not assume that the volume form is induced by the contact form, but only require it to be Reeb-invariant. Extending the approach of the work done by Hsiao and Shen [22], we compute the second coefficient of the Szegő kernel in this general setting (Theorem 6.7). As a consequence, we obtain the second coefficient of the associated semi-classical Toeplitz operators (Theorem 6.2). Moreover, in the case of a compact Kähler manifold, we show that these coefficients correspond to the coefficients of the Bergman kernel on the base manifold (see Sec. 6.4).

Summary and perspectives. This thesis establishes a deformation quantization framework on complex manifolds with strongly pseudoconvex CR boundary, based on the analysis of Bergman and Szegő type projections and the asymptotic structure of associated Toeplitz operators. A detailed analysis of the Poisson operator near the boundary allows us to determine subleading terms in its symbol expansion and the coefficients in the asymptotic expansion of the Bergman kernel of the complex manifold with boundary.

The present construction is formulated for a specific class of operators. It would be natural to further introduce a positive (pre-quantum) line bundle over the interior, which may lead to a semi-classical quantization scheme closer to the classical Berezin–Toeplitz construction. Another natural direction is the study of CR manifolds with boundary, where a more delicate analysis of the $\bar{\partial}_b$-Neumann problem could provide new insight into the relationship between Szegő and Bergman type Toeplitz quantizations.

Explicit formulas for these coefficients help clarify how geometric invariants like volume forms or curvatures appear in the asymptotics of Toeplitz operators. Higher order terms in the star product and kernel asymptotic expansions contain geometric information beyond the leading term. This may be useful for further study of spectral invariants and boundary phenomena in CR and complex geometry. For instance, in the CR setting, the assumption of a Reeb-invariant volume corresponds to a conformal change of the contact form, and it would be interesting to understand whether conformal invariants are reflected in the corresponding kernel coefficients. The semi-classical Toeplitz operators considered here may also serve as a framework for constructing Toeplitz quantization on CR manifolds, providing a setting in which spectral invariants can be studied in relation to geometric data.

1.4. Organization of the thesis

The thesis is organized as follows. For convenience, we also indicate the main results in each part.

Chapter 2 collects the analytic and geometric preliminaries used throughout the thesis.

Part I. Chapter 3 is devoted to the computation of the fourth coefficient of the Berezin–Toeplitz star product on compact Kähler manifolds.

- Theorem 3.8 determines the fourth coefficient $C_3(f, g)$.
- Corollary 3.9 treats the twisted line bundle case.

Part II. Chapters 4 and 5 develop Toeplitz quantization at the operator level on strictly pseudoconvex CR manifolds and on complex manifolds with strictly pseudoconvex CR boundary. The conceptual summary graphs for these two chapters are in Remarks 4.9 and 5.9.

- Theorem 4.6 establishes the asymptotic expansion of pseudodifferential operators in the CR setting.
- Proposition 4.22 shows that these operators form a Poisson algebra.
- Theorem 4.23 constructs the deformation quantization on CR manifolds.
- Proposition 5.11 shows that, near the boundary, the Poisson operator P admits a Fourier integral operator representation.
- Theorem 5.10 proves that $P^* f P$ is a pseudodifferential operator.
- Theorems 5.25 and 5.26 establish deformation quantization on complex manifolds with boundary.

Part III. Chapter 6 studies semi-classical Toeplitz operators and the computation of their second coefficients.

- Theorem 6.2 computes the second coefficient of the semi-classical Toeplitz operators.
- Theorem 6.3 describes the dependence on the derivatives of the cutoff function.
- Theorem 6.7 and 6.8 determine the second coefficient of the Szegő kernel under a general Reeb-invariant volume form.

CHAPTER 2

Preliminaries

We use the following notations throughout this thesis: $\mathbb{Z}$ is the set of integers, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers, and we put $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$; $\mathbb{R}$ is the set of real numbers. Also, $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set

$$
\begin{aligned}
&x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \\
&\partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}.
\end{aligned}
\tag{2.0.1}
$$

Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \dots, n$, be coordinates on $\mathbb{C}^n$. We write

$$
\begin{aligned}
&z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}, \quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1} \dots \overline{z}_n^{\alpha_n}, \\
&\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}}\Big), \quad \partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}}\Big), \\
&\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \dots \partial_{\overline{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha}.
\end{aligned}
\tag{2.0.2}
$$

For $j, s \in \mathbb{Z}$, set $\delta_{js} = 1$ if $j = s$, $\delta_{js} = 0$ if $j \neq s$.

Let M be a smooth paracompact manifold. We let TM and T^*M denote, respectively, the tangent bundle of M and the cotangent bundle of M. The complexified tangent bundle $TM \otimes \mathbb{C}$ of M will be denoted by $\mathbb{C}TM$, similarly we write $\mathbb{C}T^*M$ for the complexified cotangent bundle of M. Consider $\langle \cdot , \cdot \rangle$ to denote the pointwise duality between TM and T^*M; we extend $\langle \cdot , \cdot \rangle$ bi-linearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let $Y \subset M$ be an open set and let B be a vector bundle over Y. From now on, the spaces of distribution sections of B over Y and smooth sections of B over Y will be denoted by $\mathcal{D}'(Y, B)$ and $\mathscr{C}^\infty(Y, B)$, respectively. Let $\mathcal{E}'(Y, B)$ be the subspace of $\mathcal{D}'(Y, B)$ whose elements have compact support in Y. Let $\mathscr{C}_c^\infty(Y, B) := \mathscr{C}^\infty(Y, B) \cap \mathcal{E}'(Y, B)$.

2.1. Elements of microlocal and semi-classical analysis

We begin by introducing several notions of microlocal analysis that will be used here. Let $D \subset \mathbb{R}^N$ be an open set.

DEFINITION 2.1. For any $m \in \mathbb{R}$, set

$$S^m_{1,0}\big(D \times D \times \mathbb{R}^M\big)$$

to be the space of all $a(x,y,\eta) \in \mathscr{C}^\infty\big(D \times D \times \mathbb{R}^M\big)$ such that for every compact $K \Subset D \times D$, every $\alpha, \beta \in \mathbb{N}_0^N$ and $\gamma \in \mathbb{N}_0$, there exists a constant $C_{K,\alpha,\beta,\gamma} > 0$ satisfying

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\eta^\gamma a(x,y,\eta)\right| \leq C_{K,\alpha,\beta,\gamma}(1+|\eta|)^{m-|\gamma|}, \text{ for all } a(x,y,\eta) \in K \times \mathbb{R}^M. \tag{2.1.1}$$

We put $S^{-\infty}(D \times D \times \mathbb{R}^M) := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}^M)$.

Let $a_j \in S^{m_j}_{1,0}(D \times D \times \mathbb{R}^M)$, $j = 0,1,2,\ldots$ with $m_j \to -\infty$ as $j \to +\infty$. By the argument of Borel construction, there always exists $s \in S^{m_0}_{1,0}(D \times D \times \mathbb{R}^M)$ unique modulo $S^{-\infty}$ such that

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{m_\ell}_{1,0}(D \times D \times \mathbb{R}^M) \tag{2.1.2}$$

for all $\ell = 1,2,\ldots$. If a and a_j have the properties above, we write

$$a(x,y,\eta) \sim \sum_{j=0}^{+\infty} a_j(x,y,\eta) \text{ in } S^{m_0}_{1,0}(D \times D \times \mathbb{R}^M). \tag{2.1.3}$$

Also, we use the notation

$$a(x,y,\eta) \in S^m_{\mathrm{cl}}(D \times D \times \mathbb{R}^M) \tag{2.1.4}$$

if $a(x,y,\eta) \in S^m_{1,0}(D \times D \times \mathbb{R}^M)$ and we can find $a_j(x,y) \in \mathscr{C}^\infty(D \times D)$, $j \in \mathbb{N}_0$, such that

$$a(x,y,\lambda\eta) \sim \sum_{j=0}^{+\infty} \lambda^{m-j} a_j(x,y,\eta) \text{ in } S^m_{1,0}(D \times D \times \mathbb{R}^M). \tag{2.1.5}$$

For smooth paracompact manifolds M_1, M_2, we define the symbol spaces

$$S^m_{1,0}(M_1 \times M_2 \times \mathbb{R}^M), \quad S^m_{\mathrm{cl}}(M_1 \times M_2 \times \mathbb{R}^M)$$

and asymptotic sums thereof in the standard way.

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. Let E and F be vector bundles over W_1 and W_2, respectively. For any continuous operator $P : \mathscr{C}^\infty_c(W_2, F) \to \mathcal{D}'(W_1, E)$, we write $P(x,y)$ to denote the distribution kernel of P. A k-dependent continuous operator $A_k : \mathscr{C}^\infty_c(W_2, F) \to \mathcal{D}'(W_1, E)$ is called k-negligible on $W_1 \times W_2$ if, for k large enough, A_k is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial_x^\alpha \partial_y^\beta A_k(x,y)\right| \leq C_{K,\alpha,\beta,N} k^{-N} \text{ on } K, \ \forall k \gg 1.$$

In that case we write

$$A_k(x,y) = O(k^{-\infty}) \text{ on } W_1 \times W_2, \quad \text{or} \quad A_k = O(k^{-\infty}) \text{ on } W_1 \times W_2.$$

If $A_k, B_k : \mathscr{C}_c^\infty(W_2, F) \to \mathcal{D}'(W_1, E)$ are k-dependent continuous operators, we write $A_k = B_k + O(k^{-\infty})$ on $W_1 \times W_2$ or $A_k(x,y) = B_k(x,y) + O(k^{-\infty})$ on $W_1 \times W_2$ if $A_k - B_k = O(k^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometime write "on W".

Let M and M be smooth manifolds and let E and F be vector bundles over M and M, respectively. Let $A_k, B_k : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, E)$ be k-dependent smoothing operators. We write $A_k = B_k + O(k^{-\infty})$ on $M \times M$ if on every local coordinate patch D of M and local coordinate patch D_1 of M, $A_k = B_k + O(k^{-\infty})$ on $D \times D_1$. When $M = M$, we sometime write on M.

We recall the definition of the semi-classical symbol spaces

DEFINITION 2.2. Let W be an open set in $\mathbb{R}^N$. Let

$$S(1) = S(1;W) := \left\{a \in \mathscr{C}^\infty(W);\, \forall \alpha \in \mathbb{N}_0^N : \sup_{x\in W} |\partial^\alpha a(x)| < \infty\right\},$$

and let $S^0_{\rm loc}(1;W)$ be

$$\left\{(a(\cdot,k))_{k\in\mathbb{R}};\, \forall \alpha \in \mathbb{N}_0^N, \forall \chi \in \mathscr{C}_c^\infty(W) : \sup_{k\in\mathbb{R}, k\geq 1} \sup_{x\in W} |\partial^\alpha(\chi a(x,k))| < \infty\right\}.$$

Hence $a(\cdot,k) \in S^\ell_{\rm loc}(1;W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}_c^\infty(W)$, there exists $C_\alpha > 0$ independent of k, such that $|\partial^\alpha(\chi a(\cdot,k))| \leq C_\alpha k^\ell$ holds on W.

Consider a sequence $a_j \in S^{\ell_j}_{\rm loc}(1)$, $j \in \mathbb{N}_0$, where $\ell_j \searrow -\infty$, and let $a \in S^{\ell_0}_{\rm loc}(1)$. We say

$$a(\cdot,k) \sim \sum_{j=0}^\infty a_j(\cdot,k) \text{ in } S^{\ell_0}_{\rm loc}(1),$$

if, for every $N \in \mathbb{N}_0$, we have $a - \sum_{j=0}^N a_j \in S^{\ell_{N+1}}_{\rm loc}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\rm loc}(1) = S^{-\infty}_{\rm loc}(1;W) := \cap_\ell S^\ell_{\rm loc}(1)$.

Let $\ell \in \mathbb{R}$ and let

$$S^\ell_{\rm loc,cl}(1) := S^\ell_{\rm loc,cl}(1;W)$$

be the set of all $a \in S^\ell_{\rm loc}(1;W)$ such that we can find $a_j \in \mathscr{C}^\infty(W)$ independent of k, $j = 0, 1, \ldots$, such that

$$a(\cdot,k) \sim \sum_{j=0}^\infty k^{\ell-j} a_j(\cdot) \text{ in } S^{\ell_0}_{\rm loc}(1).$$

Similarly, we can define $S^\ell_{\rm loc}(1;Y)$ in the standard way, where Y is a smooth manifold.

DEFINITION 2.3. A function $\varphi(x,\theta) \in \mathscr{C}^\infty(D \times \dot{\mathbb{R}}^M)$ is called a phase function if, for all $(x,\theta) \in M \times \dot{\mathbb{R}}^M$, the following conditions are satisfied:

$$\operatorname{Im}\varphi(x,\theta) \geq 0, \quad \varphi \text{ is homogeneous of degree 1 in } \theta, \quad \text{and} \quad d\varphi \neq 0.$$

Let D_i be an open subset of $\mathbb{R}^{N_i}$ for $i = 1,2$, we define the concept of Fourier integral operators:

DEFINITION 2.4. Let $A : \mathscr{C}_0^\infty(D_2) \to \mathscr{D}'(D_1)$ be a continuous linear operator. A is called a Fourier integral operator if the distribution kernel of A can be formally expressed as

$$Au(x) = \iint e^{i\varphi(x,y,\eta)} a(x,y,\eta) u(y) dy d\eta, \quad u \in \mathscr{C}_0^\infty(D_2) \tag{2.1.6}$$

where $\varphi(x,y,\eta)$ is a phase function and $a \in S^m_{\mathrm{cl}}(D_1 \times D_2 \times \mathbb{R}^M)$.

We now state the classical version of the stationary phase formula due to Hörmander [14] which will be intensively applied in the proof of the composition of Fourier integral operators:

THEOREM 2.5. *Let D be an open set in $\mathbb{R}^n$, let $K \subset D$ be a compact set and $N \in \mathbb{N}$. Let u be a compactly supported smooth function on K and F a smooth function on D such that $\operatorname{Im} F \geq 0$ in D, $\operatorname{Im} F(0) = 0$, $F'(0) = 0$, $\det(F''(0)) \neq 0$ and $F'(x) \neq 0$ for $x \in K \setminus \{0\}$, then*

$$\begin{aligned} &\left| \int_D e^{i\lambda F(z)} u(z) dv(z) - e^{i\lambda F(0)} \det\left(\frac{\lambda F''(0)}{2i\pi}\right)^{-\frac{1}{2}} \sum_{j \leq N} \lambda^{-j} L_j u \right| \\ &\qquad\qquad\qquad\qquad \leq C k^{-N} \sum_{|\alpha| \leq 2N} \sup |\partial_x^\alpha u|, k > 0. \end{aligned} \tag{2.1.7}$$

Here C is bounded when F remains in a bounded set in $\mathscr{C}^\infty(D)$, and $\frac{|x|}{|F'(x)|}$ has a uniform bound. L_j is a differential operator defined by

$$L_j u = \sum_{\substack{\nu - \mu = j \\ 2\nu \geq 3\mu}} i^{-j} 2^{-\nu} \langle F''(0)^{-1} D, D \rangle^\nu \frac{(h^\mu V_\Theta u)(0)}{\nu! \mu!} \tag{2.1.8}$$

where

$$h(x) = F(x) - F(0) - \frac{1}{2}\langle F''(0)x, x\rangle \tag{2.1.9}$$

and

$$D = \begin{pmatrix} -i\partial_{x_1} \\ -i\partial_{x_2} \\ \vdots \\ -i\partial_{x_{2n}} \end{pmatrix}. \tag{2.1.10}$$

2.2. Cauchy-Riemann geometry and Szegő kernels

Let X be a smooth Riemannian manifold of dimension $2n+1$. Denote by TX its tangent bundle and by $\mathbb{C}TX := TX \otimes \mathbb{C}$ its complexified tangent bundle.

DEFINITION 2.6. Let X be a smooth Riemannian manifold of dimension $2n+1$. A Cauchy–Riemann (CR) structure on X is a pair (HX, J), where HX is a real subbundle of TX of rank $2m$, and $J \in \mathrm{End}(HX)$ is an endomorphism satisfying $J^2 = -\mathrm{id}_{HX}$. The pair (HX, J) is required to satisfy the integrability condition

$$[T^{1,0}X,\, T^{1,0}X] \subset T^{1,0}X, \tag{2.2.1}$$

where

$$T^{1,0}X := \{\, u - \sqrt{-1}\, Ju : u \in HX \,\} \subset \mathbb{C}TX.$$

We also define

$$\overline{T^{1,0}X} := T^{0,1}X := \{\, u + \sqrt{-1}\, Ju : u \in HX \,\}.$$

A smooth manifold X equipped with a CR structure (HX, J) is called a CR manifold. We shall denote such a CR manifold by the pair $(X, T^{1,0}X)$.

Let $(X, T^{1,0}X)$ be a compact orientable CR manifold of dimension $2n+1$, $n \geq 1$. Since X is orientable, there exists a global one-form $\xi \in \mathscr{C}^\infty(X, T^*X)$ such that $u \lrcorner \xi = 0$ for all $u \in T^{1,0}X$ and $\xi(x) \neq 0$ for all $x \in X$, where $\lrcorner$ denotes interior product. We call ξ a characteristic form on X.

DEFINITION 2.7. The Levi form $\mathcal{L}_x = \mathcal{L}_x^\xi$ of X at $x \in X$ associated to a characteristic form ξ is the Hermitian form on $T_x^{1,0}X$ given by

$$\mathcal{L}_x : T_x^{1,0}X \times T_x^{1,0}X \to \mathbb{C},\;\; \mathcal{L}_x(U, V) = \frac{1}{2i} d\xi(U, \overline{V}). \tag{2.2.2}$$

DEFINITION 2.8. A CR manifold $(X, T^{1,0}X)$ is said to be strictly pseudoconvex if there exists a characteristic 1-form ξ such that for every $x \in X$ the Levi form $\mathcal{L}_x^\xi$ is positive definite.

REMARK 2.9. On a strictly pseudoconvex CR manifold $(X, T^{1,0}X)$, the real part $\mathrm{Re}\, T^{1,0}X$ defines a contact structure on X. Any characteristic 1-form ξ is a contact form for this structure, since $\mathrm{Ker}\, \xi = \mathrm{Re}\, T^{1,0}X$ with $\dim_{\mathbb{R}} \mathrm{Re}\, T^{1,0}X = 2n$, $\dim_{\mathbb{R}} X = 2n+1$, and $\xi \wedge (d\xi)^n \neq 0$.

From now on, we assume that $(X, T^{1,0}X)$ is a compact strictly pseudoconvex CR manifold of dimension $2n+1$, $n \geq 1$, and we fix a characteristic one form ξ on X such that the associated Levi form $\mathcal{L}_x$ is positive definite at every point of $x \in X$.

DEFINITION 2.10. The Reeb vector field $\mathcal{T} \in \mathscr{C}^\infty(X, TX)$ with respect to ξ is the unique vector field such that $\mathcal{T} \lrcorner \xi \equiv 1$ and $\mathcal{T} \lrcorner d\xi \equiv 0$.

DEFINITION 2.11. The Levi metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ is the Hermitian metric induced by the Levi form $\mathcal{L}_x$ given by

$$\begin{aligned}
&\langle u \,|\, v \rangle := \mathcal{L}_x(u, \overline{v}), \ \ u, v \in T_x^{1,0}X, \\
&T^{1,0}X \perp T^{0,1}X, \\
&\mathcal{T} \perp (T^{1,0}X \oplus T^{0,1}X), \ \ \langle T \,|\, T \rangle = 1.
\end{aligned} \tag{2.2.3}$$

Let $T^{*1,0}X$ and $T^{*0,1}X$ be the subbundles of $\mathbb{C}T^*X$ annihilating $\mathbb{C}\mathcal{T} \oplus T^{0,1}X$ and $\mathbb{C}\mathcal{T} \oplus T^{1,0}X$, respectively. Define

$$T^{*p,q}X := (\wedge^p T^{*1,0}X) \wedge (\wedge^q T^{*0,1}X)$$

as the bundle of (p,q)-forms. The Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ induces a Hermitian metric, still denoted $\langle\cdot|\cdot\rangle$, on $\bigoplus_{p,q=0}^n T^{*p,q}X$. For $p, q = 0, 1, \ldots, n$, set $\Omega^{p,q}(X) := \mathscr{C}^\infty(X, T^{*p,q}X)$.

With respect to the given Hermitian metric $\langle \cdot | \cdot \rangle$, for $p, q \in \mathbb{N}_0$, $0 \leq p, q \leq n$, we consider the orthogonal projection

$$\pi^{(p,q)} : \Lambda^{p+q}\mathbb{C}T^*X \to T^{*p,q}X. \tag{2.2.4}$$

The tangential Cauchy–Riemann operator is defined to be

$$\overline{\partial}_b := \pi^{(0,q+1)} \circ d : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X). \tag{2.2.5}$$

Fix a volume form dV on X so that $\mathcal{L}_\mathcal{T} dV \equiv 0$, where $\mathcal{L}_\mathcal{T}$ is the Lie derivative of $\mathcal{T}$. Let $(\cdot | \cdot)$ be the L^2 inner product on $\Omega^{p,q}(X)$ induced by dV and $\langle \cdot | \cdot \rangle$, and let $L^2_{(p,q)}(X)$ be the completion of $\Omega^{p,q}(X)$ with respect to $(\cdot | \cdot)$ for $0 \leq p, q \leq n$. Set $L^2(X) := L^2_{(0,0)}(X)$.

Extend $\overline{\partial}_b$ to L^2 by

$$\overline{\partial}_b : \operatorname{Dom} \overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X),$$

where

$$\operatorname{Dom} \overline{\partial}_b := \{u \in L^2(X) \,;\, \overline{\partial}_b u \in L^2_{(0,1)}(X)\}.$$

Assume that X is embeddable, i.e., there exists a CR embedding

$$F : X \to \mathbb{C}^N$$

for some $N \in \mathbb{N}$. Moreover, X is embeddable if and only if $\overline{\partial}_b$ has closed range in L^2.

We recall some geometric quantities in pseudo-Hermitian geometry. The Hermitian metric $\langle\cdot|\cdot\rangle$ and volume form dV_ξ define an L^2 inner product $(\cdot|\cdot)_\xi$ on $\Omega^{p,q}(X)$ for $p, q \in \mathbb{N}_0$, $0 \leq p, q \leq n$. Let

$$\partial_b := \pi^{1,0} \circ d : \mathscr{C}^\infty(X) \to \Omega^{1,0}(X)$$

and let

$$d_b := \partial_b + \overline{\partial}_b : \mathscr{C}^\infty(X) \to \Omega^{0,1}(X) \oplus \Omega^{1,0}(X).$$

The *sublaplacian operator* $\Delta_b : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$ is defined by

$$(\Delta_b f \,|\, u)_\xi = \frac{1}{2}(d_b f \,|\, d_b u)_\xi, \quad f \in \mathscr{C}^\infty(X), u \in \mathscr{C}^\infty(X). \tag{2.2.6}$$

From 2.6, we can see that J is a complex structure on HX and HX can be expressed as

$$HX = \mathrm{Re}(T^{1,0}X \oplus T^{0,1}X).$$

DEFINITION 2.12. Given a contact form ξ on X, and J a complex structure on HX. There exists a unique affine connection $\nabla := \nabla^{\xi,J}$ with respect to ξ and J such that

(1) $\nabla_v \mathscr{C}^\infty(X, HX) \subset \mathscr{C}^\infty(X, HX)$ for any $v \in \mathscr{C}^\infty(X, TX)$,
(2) $\nabla \mathcal{T} = \nabla J = \nabla(d\xi) = 0$.
(3) The torsion τ of ∇ is defined by $\tau(u, u) = \nabla_u v - \nabla_v u - [u, v]$ and satisfies $\tau(u, v) = d\xi(u, v)\mathcal{T}$, $\tau(\mathcal{T}, Jv) = -J(\mathcal{T}, v)$ for any $u, v \in \mathscr{C}^\infty(X, TX)$,

Let $\{L_\alpha\}_{\alpha=1}^n$ be a local frame of $T^{1,0}X$. The dual frame of $\{L_\alpha\}_{\alpha=1}^n$ is denoted by $\{\xi^\alpha\}_{\alpha=1}^n$. The connection one form ω_α^β is defined by

$$\nabla L_\alpha = \omega_\alpha^\beta \otimes L_\beta.$$

Let $L_{\overline{\alpha}}$ and $\xi^{\overline{\alpha}}$ denote $\overline{L_\alpha}$ and $\overline{\xi^\alpha}$ respectively, we also have $\nabla L_{\overline{\alpha}} = \omega_{\overline{\alpha}}^{\overline{\beta}} \otimes L_{\overline{\beta}}$.

DEFINITION 2.13. The *Tanaka-Webster curvature two form* Θ_α^β is defined by

$$\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta. \tag{2.2.7}$$

We can directly express (2.2.7) as

$$\Theta_\alpha^\beta = R_{\alpha j \overline{k}}^\beta \xi^j \wedge \xi^{\overline{k}} + A_{\beta jk}^\alpha \xi^j \wedge \xi^k + B_{\beta jk}^\alpha \xi^{\overline{j}} \wedge \xi^{\overline{k}} + C \wedge \xi, \tag{2.2.8}$$

for some one-form C. The *pseudohermitian Ricci curvature* $R_{\alpha\overline{k}}$ is defined by

$$R_{\alpha\overline{k}} := \sum_{\beta=j=1}^n R_{\alpha j\overline{k}}^\beta.$$

Write $d\xi = ih_{\alpha\overline{\beta}}\xi^\alpha \wedge \xi^{\overline{\beta}}$ and let $h^{\overline{c}d}$ be the inverse matrix of $h_{a\overline{b}}$. The *Tanaka-Webster scalar curvature* R_{scal} is defined by

$$R_{\text{scal}} := \sum_{\alpha,k=1}^n h^{\alpha\overline{k}} R_{\alpha\overline{k}}. \tag{2.2.9}$$

DEFINITION 2.14 (Szegő projection). Let $H_b^0(X) := \mathrm{Ker}\, \overline{\partial}_b$, the orthogonal projection

$$\Pi : L^2(X) \to H_b^0(X)$$

is called the Szegő projection with respect to $(\cdot \,|\, \cdot)$.

Let $\Pi(x,y) \in \mathcal{D}'(X \times X)$ denote the distribution kernel of Π. We recall Boutet de Monvel–Sjöstrand's fundamental theorem [4] (see also [17, 20]) about the structure of the Szegő kernel. For any coordinate patch (D, x) on X, there is a smooth function $\varphi : D \times D \to \mathbb{C}$ with

$$\begin{aligned}
&\operatorname{Im} \varphi(x,y) \geq 0, \\
&\varphi(x,y) = 0 \text{ if and only if } y = x, \\
&d_x\varphi(x,x) = -d_y\varphi(x,x) = \lambda(x)\xi(x), \ \ \lambda(x) \in \mathscr{C}^\infty(D, \mathbb{R}_+),
\end{aligned} \tag{2.2.10}$$

such that we have on $D \times D$, the Szegő projector can be approximated by a Fourier integral operator

$$\Pi(x,y) = \int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt + F(x,y), \tag{2.2.11}$$

where $F(x,y) \in \mathscr{C}^\infty(D \times D)$ and $s(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ is a classical Hörmander symbol satisfying $s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-j}$ in $S^n_{1,0}(D \times D \times \mathbb{R}_+)$, $s_j(x,y) \in \mathscr{C}^\infty(D \times D)$, $j = 0, 1, \ldots$, and if $\lambda(x) = 1$,

$$s_0(x,x) = \frac{1}{2\pi^{n+1}} \frac{dV_\xi}{dV}(x), \tag{2.2.12}$$

where

$$dV_\xi := \frac{2^{-n}}{n!} \xi \wedge (d\xi)^n \,. \tag{2.2.13}$$

Part 1

The Fourth Coefficient of Berezin-Toeplitz Star Product

CHAPTER 3

Berezin-Toeplitz quantization

The path from classical mechanics to quantum mechanics involves different mathematical structures. For example, one can describe the phase space of a classical mechanics system as a symplectic manifold and solve the equations of motion by considering the flow of a Hamiltonian function. In quantum mechanics, the idea that the observables turn into operators inspires people to consider a corresponding vector space associated to a manifold. Such a process is called geometric quantization, and the classical observables are represented on a chosen Hilbert space. As for the representation of the algebra of the observables, it is not difficult to find that the representation is not commutative. For instance, the position function x and the momentum function p become operators $\hat{x}$ and $\hat{p}$, respectively, and satisfy the famous relation of the Heisenberg Uncertainty Principle, $[\hat{x}, \hat{p}] \neq 0$. Therefore, it is natural to understand the non-commutative structure of the representation of the algebra of classical observables. A way to investigate this nontrivial algebraic structure is to deform the algebra of smooth functions on the manifold $\mathscr{C}^\infty(M)$ into the formal power series $\mathscr{C}^\infty(M)[\![\lambda]\!]$, where λ is a formal parameter. The construction of a new product called *star product* on $\mathscr{C}^\infty(M)[\![\lambda]\!]$ is hereby presented. The star product (see Def 3.1) is helpful in understanding the relation of non-commutativity and the geometry of the underlying manifold. This process is called deformation quantization. It has been studied by [1, 7] on symplectic manifolds, and the star products on symplectic manifolds have been well classified in [23, 29]. On the other hand, [10] also constructs deformation quantization in the symplectic manifold using a different method. Using the Bergman kernel theory, the study of deformation quantization in the compact Kähler manifold is conducted by [3, 26, 31] through the process called the Berezin-Toeplitz quantification (see Theorem 3.7). The existence of a universal formula for the star product on a general Poisson manifold was proved in [24]. Similarly, one can consider the Toeplitz operators of Szegő projection to study the quantization on the CR manifolds [11, 18].

3.1. Introduction and main results

We begin with the definition of star product:

DEFINITION 3.1. Let M be a smooth manifold with the Poisson structure Π. For $f, g \in \mathscr{C}^\infty(M)$, a *star product* $*$ of f and g is of the form

$$f * g = \sum_{n=0}^{\infty} \lambda^n C_n(f, g),$$

where λ is a formal parameter, C_n is a bidifferential operator for all n and satisfies the following conditions:
If f, g and $h \in \mathscr{C}^\infty(M)$,

(1) $C_n(c, f) = C_n(f, c) = 0$ for all $c \in \mathbb{R}$, $n \geq 1$;
(2) $\sum_{a+b=n} C_a(C_b(f, g), h) = \sum_{a+b=n} C_a(f, C_b(g, h))$ for all $n \in \mathbb{N} \cup \{0\}$;
(3) $C_1(f, g) - C_1(g, f) = 2\lambda\Pi(f, g)$;
(4) $C_0(f, g) = f \cdot g = C_0(g, f)$.

The bidifferential operator $C_n(\cdot, \cdot)$ is called the coefficients of the star product $*$. It involves the geometric data of the given manifold. Therefore, it is natural to understand and compute the coefficients of the star products. In this part, we study the Berezin-Toeplitz star product on compact Kähler manifolds, defined via Toeplitz operators. Our main result is the computation of the fourth coefficient C_3 of this star product, adapting the methods of [16, 26].

We now formulate our main result. Let M be a compact Kähler manifold with a smooth Hermitian metric $\langle \cdot, \cdot \rangle$ on $T^{1,0}M$. Consider the complexified tangent bundle $\mathbb{C}TM$. By requiring the orthogonality of $T^{1,0}M$ and $T^{0,1}M$, we extend the smooth Hermitian metric to $\mathbb{C}TM$. Further, this metric also induces a Hermitian metric on $\Lambda^{p,q}T^*M \otimes \Lambda^{s,t}T^*M$. The Hermitian metric induces a real two form on M, denoted by Θ.

Let L be a positive Hermitian line bundle over M. We assume L has a smooth Hermitian fiber metric denoted by h^L. Let ϕ be a local weight of this metric. That is, if $s(x)$ is a local trivializing section of L on an open set $D \subset M$, then the pointwise norm is

$$|s(x)|^2 = |s(x)|^2_{h^L} = e^{-2\phi(x)}, \ \phi \in \mathscr{C}^\infty(D, \mathbb{R}).$$

The canonical curvature of the line bundle is denoted by R^L, we have

$$R^L = 2\bar{\partial}\partial\phi \tag{3.1.1}$$

in terms of local weight. Let $\dot{R}^L \in \mathrm{End}(T^{1,0}M)$ be the corresponding curvature operator of R^L, the line bundle L is positive is equivalent to say that the operator $\dot{R}^L$ is positive definite. The curvature induces Kähler form ω on M, satisfying

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L, \tag{3.1.2}$$

which induces a smooth Hermitian metric on $\mathbb{C}TM$ denoted by $\langle\cdot,\cdot\rangle_\omega$. The extension of this Hermitian metric to $\Lambda^{p,q}T^*M\otimes\Lambda^{s,t}T^*M$ also denoted by $\langle\cdot,\cdot\rangle_\omega$. We point out that we have two different metric on $T^{1,0}M$. More precisely, in local holomorphic coordinate $(z_1,\ldots,z_n)$, we have

$$\Theta=\sqrt{-1}\sum_{i,j}\Theta_{i,j}dz_i\wedge d\overline{z_j}, \tag{3.1.3}$$

$$\omega=\sqrt{-1}\sum_{i,j}\omega_{i,j}dz_i\wedge d\overline{z_j}, \tag{3.1.4}$$

where

$$\Theta_{i,j}=\langle\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z_j}}\rangle \tag{3.1.5}$$

and

$$\omega_{i,j}=\langle\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z_j}}\rangle_\omega. \tag{3.1.6}$$

Put $h=(h_{j,k})_{j,k=1}^n$, where $h_{j,k}=\omega_{k,j}$ for $j,k=1,\ldots,n$. On the other hand, $\langle dz_i,dz_j\rangle_\omega=h^{j,k}$, $h^{-1}=(h^{j,k})_{j,k=1}^n$ is the inverse matrix of h. The complex Laplacian with respect to ω is defined by

$$\Delta_\omega=-2\sum_{j,k}^n h^{j,k}\frac{\partial^2}{\partial z_j\partial\overline{z_k}}. \tag{3.1.7}$$

Besides, we put

$$\Delta_0=\sum_{j=1}^n\frac{1}{\lambda_j}\frac{\partial^2}{\partial z_j\partial\overline{z_j}}, \tag{3.1.8}$$

where $\lambda_1,\cdots,\lambda_n$ are eigenvalues of $\dot{R}^L$.

The determinants of the metrics are denoted by

$$V_\Theta=\det\left((\Theta_{i,j})_{i,j=1}^n\right),\ V_\omega=\det\left((\omega_{i,j})_{i,j=1}^n\right). \tag{3.1.9}$$

The scalar curvatures with respect to two metrics are

$$\hat{r}=\Delta_\omega\log V_\Theta,\ r=\Delta_\omega\log V_\omega. \tag{3.1.10}$$

The Ricci curvature of the determinant line bundle of $T^{1,0}M$ with respect to Θ is denoted by $R_\Theta^{\det}$. We have

$$R_\Theta^{\det}=-\bar\partial\partial\log V_\Theta. \tag{3.1.11}$$

Let $\theta = h^{-1}\partial h$ be the Chern connection matrix with respect to ω, where $\theta_{i,j} \in \Lambda^{1,0}T^*M$. The Chern curvature R^{TM}_ω is given by

$$\bar\partial\theta = (\bar\partial\theta_{i,j})_{i,j=1}^n \in \mathscr{C}^\infty\left(\Lambda^{1,1}T^*M \otimes \mathrm{End}(T^{1,0}M)\right).$$

Let $e_1, \ldots, e_n$ be an orthonormal frame of $T^{1,0}M$ with respect to ω, the Ricci curvature with respect to ω is defined by

$$Ric_\omega = -\sum_j \langle R^{TM}_\omega(\cdot, e_j)\cdot, e_j\rangle_\omega. \tag{3.1.12}$$

Let $D^{0,1}$ be the $(0,1)$-component of the Chern connection induced by ω.

$$D^{0,1} : \mathscr{C}^\infty(M, \Lambda^{0,1}T^*M) \longrightarrow \mathscr{C}^\infty(M, \Lambda^{0,1}T^*M \otimes \Lambda^{0,1}T^*M).$$

Let $z = (z_1, \ldots, z_n)$ be a local holomorphic coordinate, put $A = (a_{i,j})_{i,j=1}^n$ where $a_{i,j} = \langle d\overline{z_j}, d\overline{z_i}\rangle_\omega$ for $i, j = 1, \ldots, n$. Set $\mathcal{A} = A^{-1}\partial A = (\alpha_{i,j})_{i,j=1}^n$ where $\alpha_{i,j} \in \Lambda^{0,1}T^*M$. For $u = \sum_k u_k d\overline{z_k}$, we have

$$D^{0,1}u = \sum_j \bar\partial u_j \otimes d\overline{z_j} + \sum_{j,k} u_j \alpha_{k,j} \otimes d\overline{z_k}.$$

Notice that $D^{0,1}$ can be extended to

$$\mathscr{C}^\infty(M, (\Lambda^{0,1}T^*M)^{\otimes n}) \longrightarrow \mathscr{C}^\infty(M, (\Lambda^{0,1}T^*M)^{\otimes(n+1)})$$

by $D^{0,1}(u \otimes v) = (D^{0,1}u) \otimes v + u \otimes (D^{0,1}v)$ for $u, v \in \mathscr{C}^\infty(M, \Lambda^{0,1}T^*M)$ inductively. Similarly, the $(1,0)$-component of the Chern connection denoted by $D^{1,0}$ can be extended to $\mathscr{C}^\infty(M, (\Lambda^{1,0}T^*M)^{\otimes n})$. By choosing a Hermitian metric on $T^{1,0}M$, we obtain an isomorphism between the bundles $T^{1,0}M$ and $\Lambda^{0,1}T^*M$. Consequently, we have

$$\mathrm{End}(T^{1,0}M) \cong \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M.$$

We denote by $R \in \mathscr{C}^\infty\left(\Lambda^{1,1}T^*M \otimes \Lambda^{1,1}T^*M\right)$ the Chern curvature R^{TM}_ω identified under the isomorphism

$$\mathscr{C}^\infty(M, \Lambda^{1,1}T^*M \otimes \mathrm{End}(T^{1,0}M)) \cong \mathscr{C}^\infty(M, \Lambda^{1,1}T^*M \otimes \Lambda^{1,1}T^*M).$$

DEFINITION 3.2. Let $\tau : \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M \otimes \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M \to \Lambda^{1,0}T^*M \otimes \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M \otimes \Lambda^{0,1}T^*M$ be a bundle isomorphism, switch the two middle components. In matrix representation, τ acts as the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

on each fiber.

DEFINITION 3.3. Let $\mathbf{v} = v_1 \otimes v_2 \in \Lambda^{1,0}T^*M \otimes \Lambda^{1,0}T^*M$ and $\mathbf{w} = w_1 \otimes w_2 \in \Lambda^{0,1}T^*M \otimes \Lambda^{0,1}T^*M$. We define $\mathbf{v}\tilde{\otimes}\mathbf{w} := \tau(\mathbf{v} \otimes \mathbf{w})$.

EXAMPLE 3.4. Let f, g be smooth functions on M, with the notation we defined above, we have

$$D^{1,0}\partial f \in \Lambda^{1,0}T^*M \otimes \Lambda^{1,0}T^*M,$$

$$D^{0,1}\bar{\partial} g \in \Lambda^{0,1}T^*M \otimes \Lambda^{0,1}T^*M,$$

and

$$D^{1,0}\partial f \tilde{\otimes} D^{0,1}\bar{\partial} g \in \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M \otimes \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M.$$

Recall that $R^{\det}_{\Theta}$ and Ric_ω are $(1,1)$-forms on $T^{1,0}M$. Let $\{f^i\}_{i=1}^n$ and $\{\overline{f^i}\}_{i=1}^n$ be orthonormal frames of $\Lambda^{1,0}T^*M$ and $\Lambda^{0,1}T^*M$, respectively. With respect to these frames, the curvature forms can be written as

$$R^{\det}_{\Theta} = \sum_{i,j} [R^{\det}_{\Theta}]_{ij} \overline{f^i} \otimes f^j, \tag{3.1.13}$$

$$Ric_\omega = \sum_{i,j} [Ric_\omega]_{ij} \overline{f^i} \otimes f^j \tag{3.1.14}$$

for some coefficients $[R^{\det}_{\Theta}]_{ij}$ and $[Ric_\omega]_{ij}$. Identifying $\Lambda^{1,1}T^*M$ with $\mathrm{End}(T^{1,0}M)$, we define

$$(R^{\det}_{\Theta})^2 = \sum_{i,j,k} [R^{\det}_{\Theta}]_{ik} [R^{\det}_{\Theta}]_{kj} \overline{f^i} \otimes f^j, \tag{3.1.15}$$

$$(Ric_\omega)^2 = \sum_{i,j,k} [Ric_\omega]_{ik} [Ric_\omega]_{kj} \overline{f^i} \otimes f^j \tag{3.1.16}$$

which represent the $(1,1)$-forms corresponding to the compositions of these operators.

To establish our result of the computation of the Berezin-Toeplitz star product, we give a brief introduction of the Berezin-Toeplitz quantization in this section. Let $L^k := L^{\otimes k}$ be the k-th tensor power of L over M. The smooth Hermitian fiber metric h^L on L induces a smooth Hermitian fiber metric $h^{L^k} := (h^L)^{\otimes k}$ on L^k. That is, if $s(x)$ is a local trivializing section of L on $D \subset M$, then $s^k(x) := s(x)^{\otimes k}$ is a local trivializing section of L^k on D. The volume form on M induced by $\langle \cdot, \cdot \rangle$ on $\mathbb{C}TM$ is denoted by

$$dv_M = dv_M(x) := \frac{\Theta^n}{n!}.$$

Hence, we have natural inner products on $\mathscr{C}^\infty_0(M, L^k)$ and $\mathscr{C}^\infty_0(M)$ denoted by $(\cdot|\cdot)_k$ and $(\cdot|\cdot)$ respectively. That is, let s be a local trivializing section of L on an open set $D \subset M$, $|s|^2_{h^L} = e^{-2\phi}$. Note that we have the identification between $\mathscr{C}^\infty(D, L^k)$ and $\mathscr{C}^\infty(D)$:

$$\mathscr{C}^\infty(D, L^k) \longleftrightarrow \mathscr{C}^\infty(D)$$
$$u = s^k \widetilde{u} \longleftrightarrow \widehat{u}.$$

where $\widehat{u} := \widetilde{u}e^{-k\phi}$ (and similarly $\widehat{v} := \widetilde{v}e^{-k\phi}$), we have for $u = s^k\widetilde{u}, v = s^k\widetilde{v} \in \mathscr{C}^\infty_0(M, L^k)$,

$$(u|v)_k = (\widehat{u}|\widehat{v}) = \int_M \widetilde{u}\overline{\widetilde{v}}e^{-2k\phi}dv_M(x).$$

Let $L^2(M, L^k)$ be the completion of $\mathscr{C}^\infty_o(M, L^k)$ with respect to $(\cdot\,|\,\cdot)_k$. The Cauchy-Riemann operator with values in L^k is denoted by

$$\bar{\partial} : \mathscr{C}^\infty(M, L^k) \to \mathscr{C}^\infty\big(M, L^k \otimes \Lambda^{0,1}T^*M\big).$$

Let $H^0(M, L^k) := \{f \in \mathscr{C}^\infty(M, L^k) : \overline{\partial} f = 0\}$.

DEFINITION 3.5. The *Bergman projection of level k* is an orthogonal projection with respect to $(\cdot|\cdot)_k$ from $L^2(M, L^2)$ onto $H^0(M, L^k)$ denoted by $\Pi^{(k)}$.

DEFINITION 3.6. Let f be a smooth function on M, the *Toeplitz operator of level k with symbol f* is given by

$$T^{(k)}_f = \Pi^{(k)} \circ M_f \circ \Pi^{(k)} : L^2(M, L^k) \longrightarrow L^2(M, L^k), \tag{3.1.17}$$

where M_f is an multiplicative operator defined by $M_f(g)(x) = (fg)(x)$ for any $g \in L^2(M, L^k), x \in M$. Since $T^k_\bullet$ maps a function to an operator, we also called $T^{(k)}_f$ the *Berezin-Toeplitz quantization with symbol f*.

We state the classical well-known result of the deformation quantization on compact Kähler manifolds:

THEOREM 3.7. [3,31] *Let M be a compact Kähler manifold and L is a holomorphic positive line bundle over M. With the same notation above, there exists a star product* $*_{BT}$ *on M with* $f *_{BT} g = \sum_{i=0}^\infty k^{-i}C_i(f, g)$ *such that for any* $N \in \mathbb{N}$ *there exist* $C_N > 0$ *independent of k and*

$$\|T^{(k)}_f \circ T^{(k)}_g - \sum_{j=0}^N k^{-j}T^{(k)}_{C_j(f,g)}\| \le C_N k^{n-N-1}, \tag{3.1.18}$$

where $\|\cdot\|$ *is the operator norm on the space of bounded operators on* $\mathscr{C}^0(M, L^k)$.

We call $*_{BT}$ the Berezin-Toeplitz star product. For simplicity, one could write

$$T^{(k)}_f \circ T^{(k)}_g \sim T^{(k)}_{f *_{BT} g}$$

in the sense of k goes to infinity. In the form of Definition 3.1, we write

$$f *_{BT} g = \sum_{j=0}^{\infty} k^{-j} C_j(f,g), \tag{3.1.19}$$

where C_j satisfies the four conditions in Definition 3.1. Notice that the non-commutativity of the deformed algebra $\mathscr{C}^\infty(M)[\![k^{-1}]\!]$ reveals the algebraic structure of the Toeplitz operators. Let $T_f^{(k)}(x,y) \in \mathscr{C}^\infty(M \times M, \mathrm{End}(L_x^k, L_y^k))$ be the Schwartz kernel of $T_f^{(k)}$. By choosing a local trivializing section s of L on $D \subset M$, we have

$$T_f^{(k)}(x,y) = s(x)^k T_{f,s}^{(k)}(x,y) s^*(y)^k,$$

where $T_{f,s}^{(k)}(x,y) \in \mathscr{C}^\infty(D \times D)$. Hence, the action of Toeplitz operators on test functions can be realized locally by

$$\begin{aligned}
(T_f^{(k)} u)(x) &= s(x)^k \int_M T_{f,s}^{(k)}(x,y) \langle u(y), s^*(y)^k \rangle dv_M(y) \\
&= s(x)^k \int_M T_{f,s}^{(k)}(x,y) \widetilde{u}(y) dv_M(y),
\end{aligned}$$

where $u = s^k \widetilde{u}$ and $\widetilde{u} \in \mathscr{C}_0^\infty(D)$. If $x = y$, we have

$$T_f^{(k)}(x,x) = s(x)^k T_{f,s}^{(k)}(x,x) s^*(x)^k = T_{f,s}^{(k)}(x,x)$$

for $x \in D$. Therefore, $T_{f,s}^{(k)}(x,x)$ is independent of the choice of the local section $s \in \mathscr{C}^\infty(D, L^k)$. We call $T_{f,s}^{(k)}(x,x)$ the *kernel of Berezin-Toeplitz quantization on the diagonal with symbol* f and denote it by $T_f^{(k)}(x)$. The following is our main result in this part:

THEOREM 3.8. *The fourth coefficient of the Berezin-Toeplitz star product is given by*

(3.1.20)

$$\begin{aligned}
C_3(f,g) = &- \frac{1}{48\pi^3} \langle (D^{1,0})^2 \partial f, (D^{1,0})^2 \bar{\partial} \bar{g} \rangle_\omega - \frac{1}{32\pi^3} \langle R, D^{0,1} \bar{\partial} \bar{f} \widetilde{\otimes} D^{1,0} \partial \bar{g} \rangle_\omega \\
&- \frac{1}{8\pi^3} \sum_k \Big\{ \langle D^{1,0} \partial f(e_k) \otimes D^{0,1} \bar{\partial} g(\overline{e_k}), R_\Theta^{\det} \rangle_\omega \\
&\qquad\qquad + \langle R_\Theta^{\det}(\overline{e_k}) \wedge R_\Theta^{\det}(e_k), \bar{\partial} \bar{f} \wedge \partial \bar{g} \rangle_\omega \Big\} \\
&- \frac{1}{16\pi^3} \Big(\langle D^{0,1} \bar{\partial} g \otimes \partial f, D^{0,1} R_\Theta^{\det} \rangle_\omega + \langle D^{0,1} R_\Theta^{\det}, D^{0,1} \bar{\partial} \bar{f} \otimes \partial \bar{g} \rangle_\omega \Big),
\end{aligned}$$

where $\{e_k\}_{k=1}^n$ *is an orthonormal frame for* $T^{1,0}M$ *with respect to* $\langle \cdot, \cdot \rangle_\omega$.

Ma and Marinescu [26] compute the coefficients in the setting of the line bundle L tensor with a Hermitian vector bundle E. In this case, they computed $C_0(f,g)$ and $C_1(f,g)$ for $f,g \in \mathscr{C}^\infty(M, \mathrm{End}(E))$ and $C_2(f,g)$ for $f,g \in \mathscr{C}^\infty(M)$. They observed that when $\Theta \neq \omega$, one can reduce the case to the polarized case twisted with a trivial line bundle L' by choosing the local trivialization of L' by $|\mathbf{1}_{L'}|^2 = \frac{V_\Theta}{\omega^n}$. The result in this part is equivalent to the polarized case twisted with the determinant line bundle. Furthermore, we can generalize our computation on C_3 to the case of twisting any line bundle by the aforementioned observation:

COROLLARY 3.9. *Let M be a compact Kähler manifold and L be a positive line bundle over M with the same notation as above. Let E be another line bundle over M. For any $p \in M$, U is a neighborhood of p. Let $\varphi \in \mathscr{C}^\infty(U, \mathbb{R})$ be a smooth function such that $|\mathbf{1}_{E|_U}|^2 = e^{-\varphi}$. Then $\bar{\partial}\partial\varphi \in \bigwedge^{1,1} T^*M$ is the curvature of E denoted by Ric_E. Then the third coefficient of this setting is given by:*

$$\begin{aligned}
C_3(f,g) = &-\frac{1}{48\pi^3}\langle (D^{1,0})^2\partial f, (D^{1,0})^2\bar{\partial}\bar{g}\rangle_\omega - \frac{1}{32\pi^3}\langle R, D^{0,1}\bar{\partial}\bar{f}\widetilde{\otimes} D^{1,0}\partial\bar{g}\rangle_\omega \\
&-\frac{1}{8\pi^3}\sum_k \Big(\langle D^{1,0}\partial f(e_k) \otimes D^{0,1}\bar{\partial} g(\overline{e_k}), Ric_E + Ric_\omega\rangle_\omega \\
&+ \langle (Ric_E + Ric_\omega)(\overline{e_k}) \wedge (Ric_E + Ric_\omega)(e_k), \bar{\partial}\bar{f} \wedge \partial\bar{g}\rangle_\omega \Big) \\
&-\frac{1}{16\pi^3}\Big(\langle D^{0,1}\bar{\partial} g \otimes \partial f, D^{0,1}(Ric_E + Ric_\omega)\rangle_\omega \\
&+ \langle D^{0,1}(Ric_E + Ric_\omega), D^{0,1}\bar{\partial}\bar{f} \otimes \partial\bar{g}\rangle_\omega \Big).
\end{aligned} \tag{3.1.21}$$

The Corollary 3.9 is followed by choosing a specific metric Θ on $T^{1,0}M$. By (3.1.4), we define $\Theta_{i,j} = e^{\frac{-\varphi}{n}}\omega_{i,j}$.

3.2. Asymptotic expansion of Bergman kernels and Toeplitz operators

This section provides the preliminaries required for the proof of Theorem 3.8. We summarize some definitions and results following [16], which will be repeatedly used in the next section. For any $p \in M$, we can choose a local holomorphic coordinate $\{z_1, \ldots, z_n\}$ around $p \in D$, where D is an open neighborhood of p. Without lost of generality, we can identify p to the origin, that is $z(p) := (z_1(p), \ldots, z_n(p)) = 0$. By choosing a local trivializing section $s(z)$ for the positive line bundle L, there exist $\phi \in \mathscr{C}^\infty(D, \mathbb{R})$ such that $|s^k(z)|^2 = (s|s)_k(z) = e^{-2k\phi(z)}$. Therefore, we have the following setting proved by [30]:

$$\begin{aligned}
(3.2.1)\qquad & z(p) = 0,\\
& \phi(z) = \sum_{i=0}^{n} \lambda_i |z_i|^2 + \phi_1(z),\\
& \phi_1(z) = O(|z|^4),\ \frac{\partial^{\alpha+\beta}\phi_1}{\partial z^\alpha \partial \overline{z^\beta}}(0) = 0 \text{ if } |\alpha| \le 1 \text{ or } |\beta| \le 1,\\
& \Theta(z) = \sqrt{-1}\sum_{i=0}^{n} dz_i \wedge d\overline{z_i} + O(|z|).
\end{aligned}$$

By (3.1.1), let $\dot{R}^L \in \mathscr{C}^\infty(M, \mathrm{End}(T^{1,0}M))$ be the corresponding Hermitian matrix of the curvature R^L, we have $\frac{\lambda_i}{2}$ is the eigenvalue of $\dot{R}^L$ for $i = 1, \ldots, n$.

THEOREM 3.10. [16] *In the above setting, we have*

$$(3.2.2)\qquad e^{-k\phi(z)+k\phi(w)}\Pi_s^{(k)}(z,w) \equiv e^{ik\Psi(z,w)} b(z,w,k) \mod O(k^{-\infty}) \text{ on } D \times D,$$

where $b(z,w,k) \in \mathscr{C}^\infty(D \times D)$ *is properly supported and*

$$\begin{aligned}
(3.2.3)\qquad & b(z,w,k) \equiv \sum_{i=0}^{\infty} k^{n-i} b_i(z,w) \mod O(k^{-\infty}) \text{ on } D \times D,\\
& b_i(z,w) \in \mathscr{C}^\infty(D \times D) \text{ for } i \in \mathbb{N}_0,\\
& \overline{\partial_z} b_i(z,w) \text{ and } \partial_w b_i(z,w) \text{ vanish to infinite order at } z = w, \text{ for all } i \in \mathbb{N}_0,
\end{aligned}$$

$\Psi(z,w) \in \mathscr{C}^\infty(D \times D)$ *satisfies*

$$\begin{aligned}
(3.2.4)\qquad & \Psi(z,w) = -\overline{\Psi(w,z)},\\
& \operatorname{Im}\Psi(z,w) \ge c|z-w|^2 \text{ for some } c > 0,\\
& \Psi = 0 \text{ if and only if } z = w.
\end{aligned}$$

Furthermore, given point $p \in M$ *and a local holomorphic coordinate* $\{z_1, \ldots, z_n\}$ *with* $z(p) = 0$, *we have*

$$(3.2.5)\quad \Psi(z,w) = \sqrt{-1}\left(\phi(z) + \phi(w) - 2\sum_{\substack{\alpha,\beta\in\mathbb{N}_0\\ |\alpha|+|\beta|\le N,}} \frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^{|\alpha|}\partial \overline{z}^{|\beta|}}(0)\frac{z^\alpha}{|\alpha|!}\frac{\overline{z^\beta}}{|\beta|!}\right) + O(|(z,w)|^{N+1}),$$

for every $N \in \mathbb{N}_0$.

Let f be a smooth function on M. by (3.1.17) and Theorem 3.10, the Schwartz kernel of the Toeplitz operator can be written as follows:

$$
\begin{aligned}
(3.2.6)\quad & e^{-k\phi(z)+k\phi(w)}T^{(k)}_{f,s}(z,w) \\
&\equiv \int_D e^{ik(\Psi(z,u)+\Psi(u,w))}b(z,u,k)f(u)b(u,w,k)dV_M(u) \quad \mod O(k^{-\infty}).
\end{aligned}
$$

The integral in (3.2.6) is a Gaussian integral with a complex phase function $ik(\Psi(z,u)+\Psi(u,w))$. Let x_i, y_i, α_i for $i=1,\cdots,2n$ be the real coordinates defined as follows:

$$
\begin{aligned}
z_j &= x_{2j-1}+\sqrt{-1}x_{2j},\\
w_j &= y_{2j-1}+\sqrt{-1}y_{2j},\\
u_j &= \alpha_{2j-1}+\sqrt{-1}u_{2j},
\end{aligned}
$$

for $j=1,\cdots,n$. Define

$$F(x,\alpha,y) := ik\big(\Psi(z,u)+\Psi(u,w)\big).$$

By the properties of $\Psi(z,w)$ in (3.2.4), it satisfies $F(x,\alpha,x)\le 0$ and $F(x,x,x)=0$. From (3.2.5) we have

$$d_\alpha F(x,\alpha,y)|_{x=\alpha=y}=0,$$

and

$$\det\left(\frac{\partial^2 F}{\partial\alpha_j\partial\alpha_k}\right)^{2n}_{j,k=1}\Bigg|_{x=\alpha=y} = 2^{2n}(\det \dot{R}^L)^2\neq 0.$$

Thus $x=\alpha=y$ is a non-degenerate real critical point of $F(x,\alpha,y)$, and the stationary phase formula of Melin–Sjöstrand [27] applies to compute the integral (3.2.6), i.e. the kernel of the Toeplitz operators.

THEOREM 3.11. [16] *In the same setting with 3.10, we have*

$$(3.2.7)\qquad e^{-k\phi(z)+k\phi(w)}T^{(k)}_{f,s}(z,w)\equiv e^{ik\Psi(z,w)}b_f(z,w,k) \quad \mod O(k^{-\infty}) \text{ on } D\times D,$$

where $b_f(z,w,k)\in\mathscr{C}^\infty(D\times D)$ is properly supported and

$$
\begin{aligned}
(3.2.8)\quad & b_f(z,w,k)\equiv\sum_{i=0}^{\infty}k^{n-i}b_{i,f}(z,w) \quad \mod O(k^{-\infty}) \text{ on } D\times D,\\
& b_{i,f}(z,w)\in\mathscr{C}^\infty(D\times D) \text{ for } i\in\mathbb{N}_0,\\
& \overline{\partial_z}b_{i,f}(z,w) \text{ and } \partial_w b_{i,f}(z,w) \text{ vanish to infinite order at } z=w, \text{ for all } i\in\mathbb{N}_0,
\end{aligned}
$$

$\Psi(z,w)\in\mathscr{C}^\infty(D\times D)$ is same as (3.2.4).

In particular, we consider the diagonal part of (3.2.6) by choosing $z = w$, then we have:

$$
\begin{aligned}
(3.2.9)\quad T_f^{(k)}(z) &:= T_{f,s}^{(k)}(z,z) \\
&\equiv \int_D e^{ik(\Psi(z,u)+\Psi(u,z))} b(z,u,k) f(u) b(u,z,k) dV_M(u) \quad \mod O(k^{-\infty}).
\end{aligned}
$$

COROLLARY 3.12. *As the same setting above, let f and g be smooth functions on M. The integral kernel of the composition of Toeplitz operators is as following*

$$
\begin{aligned}
(3.2.10)\qquad & (T_f^{(k)} \circ T_g^{(k)})(z) \\
&\equiv \int_D e^{ik(\Psi(z,w)+\Psi(w,z))} b_f(z,w,k) b_g(w,z,k) dV_M(w) \quad \mod O(k^{-\infty}) \\
&\equiv \sum_{i=0}^{\infty} b_{i,f,g} k^{n-i} \quad \mod O(k^{-\infty}),
\end{aligned}
$$

where $b_{i,f,g}$ is smooth function on M for $i = 0, 1, \cdots$.

In the same setting as (3.2.1), we compute (3.2.9) at $z = 0$ using Theorem 2.5. The integral is

$$
\begin{aligned}
(3.2.11)\quad T_f^{(k)}(0) &:= T_{f,s}^{(k)}(0,0) \\
&\equiv \int_D e^{ik(\Psi(0,u)+\Psi(u,0))} b(0,u,k) f(u) b(u,0,k) dV_M(u) \quad \mod O(k^{-\infty}).
\end{aligned}
$$

Let $F(u) = \Psi(0,u) + \Psi(u,0)$ in (2.1.7), we have

$$
\begin{aligned}
(3.2.12)\quad T_f^{(k)}(0) &= (2\pi)^n (\det \dot{R}^L)^{-1} \times \\
&\sum_{j=0}^{N} k^{n-j} \left(\sum_{\substack{0 \le m \le j \\ \nu - \mu = m \\ \nu \ge 2\mu \\ s+t = j-m}} (-1)^{\mu} 2^{-m} \frac{\Delta_0^{\nu} (\phi_1^{\mu} V_{\Theta} f b_s(0,z) b_t(z,0))(0)}{\nu! \mu!} \right).
\end{aligned}
$$

By (3.2.8), we have the following recurrence relation of the coefficients of the asymptotic expansion of Toeplitz operators

THEOREM 3.13.

$$
(3.2.13)\quad b_{i,f}(0) = (2\pi)^n (\det \dot{R}^L)^{-1} \cdot \sum_{\substack{0 \le m \le i \\ \nu - \mu = m \\ \nu \ge 2\mu \\ s+t = i-m}} (-1)^{\mu} 2^{-m} \frac{\Delta_0^{\nu} (\phi_1^{\mu} V_{\Theta} f b_s(0,z) b_t(z,0))(0)}{\nu! \mu!}
$$

for all $i = 0, 1, \cdots$.

With Theorem 3.13, we can work on the composition of the Toeplitz operators similarly. Let $z = 0$ in Corollary 3.12, by applying Theorem 2.5 the coefficients of the expansion of the integral in (3.2.10) can be written as

COROLLARY 3.14.
(3.2.14)
$$b_{i,f,g}(0) = (2\pi)^n(\det \dot{R}^L)^{-1} \cdot \sum_{\substack{0\le m\le i\\ \nu-\mu=m\\ \nu\ge 2\mu\\ s+t=i-m}} (-1)^\mu 2^{-m} \frac{\Delta_0^\nu(\phi_1^\mu V_\Theta f b_{s,f}(0,z) b_{t,g}(z,0))(0)}{\nu!\mu!}$$
for all $i = 0, 1, \cdots$.

Using the estimate in (3.1.18) for compositions of Toeplitz operators and the Berezin–Toeplitz star product, the computation of its coefficients reduces to the following recurrence relation:

$$b_{0,C_n(f,g)} = b_{n,f,g} - b_{n,fg} - b_{n-1,C_1(f,g)} - \cdots - b_{1,C_{n-1}(f,g)}, \tag{3.2.15}$$
for every $n \in \mathbb{N}_0$.

3.3. Proof of Theorem 3.8

To derive (3.1.20) in Theorem 3.8, we compute each term in

$$b_{0,C_3(f,g)} = b_{3,f,g} - b_{3,fg} - b_{2,C_1(f,g)} - b_{1,C_2(f,g)}. \tag{3.3.1}$$
Using the properties of Toeplitz operators, we can simplify this computation. For any $p \in M$, choose local holomorphic coordinates $z = (z_1, \ldots, z_n)$ around p with $z(p) = 0$. Let $s(x)$ be a local trivializing section of the holomorphic line bundle (L, h^L) over M, and let $\phi(x) \in \mathscr{C}^\infty(D, \mathbb{R})$ be the local Kähler potential on a neighborhood D of p. By directly computing (3.1.6) and (3.1.7), we obtain:

LEMMA 3.15.
$$h^{ij} = \frac{\pi}{\lambda_i}\delta^{ij} + \sum_{k=1}^n \left(-\frac{\pi}{\lambda_k\lambda_j}\frac{\partial^2\phi_1}{\partial z_k \partial \overline{z_j}} + \frac{\pi}{\lambda_i\lambda_j\lambda_k}\frac{\partial^2\phi_1}{\partial z_i\partial\overline{z_j}}\frac{\partial^2\phi_1}{\partial z_k\partial\overline{z_i}} \right) + O(|z|^6) \tag{3.3.2}$$

$$\begin{aligned} \Delta_\omega = &\\ &-2\pi \sum_{j,k=1}^n \Big(\frac{1}{\lambda_j}\delta^{jk}\frac{\partial^2}{\partial z_j\partial\overline{z_k}} - \frac{1}{\lambda_j\lambda_k}\frac{\partial^2\phi_1}{\partial z_k\partial\overline{z_j}}\frac{\partial^2}{\partial z_j\partial\overline{z_k}} + \frac{1}{\lambda_i\lambda_j\lambda_k}\frac{\partial^2\phi_1}{\partial z_i\partial\overline{z_j}}\frac{\partial^2\phi_1}{\partial z_k\partial\overline{z_i}}\frac{\partial^2}{\partial z_j\partial\overline{z_k}} \Big) \\ &+ O(|z|^5). \end{aligned} \tag{3.3.3}$$

LEMMA 3.16.
$$[R_\Theta^{\det}]_{ij}(0) = -\frac{\partial^2 \log V_\Theta}{\partial\overline{z_i}\partial z_j}(0). \tag{3.3.4}$$

$$(3.3.5) \qquad [Ric_\omega]_{i\bar{j}}(0) = -\sum_{k=1}^{n} \frac{1}{\lambda_k} \frac{\partial^4 \phi}{\partial \overline{z_k} \partial z_k \partial \overline{z_i} \partial z_j}(0).$$

$$(3.3.6) \qquad [(R_\Theta^{\det})^2]_{i\bar{j}}(0) = \sum_{k=1}^{n} \frac{1}{\lambda_k} \frac{\partial^2 \log V_\Theta}{\partial \overline{z_i} \partial z_k} \frac{\partial^2 \log V_\Theta}{\partial \overline{z_k} \partial z_j}(0).$$

We also recall the off-diagonal expansion of the Bergman kernel.

LEMMA 3.17. *If χ_1 and χ_2 are smooth functions with compact supports on D such that* $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$, *then* $\chi_1 \Pi^{(k)} \chi_2 = O(k^{-\infty})$.

With the above lemma, we can prove the separable property of Berezin-Toeplitz star product:

LEMMA 3.18 (Separable property of Berezin-Toeplitz star product). *Let M be a compact Kähler manifold, and $p \in M$. Suppose f is anti-holomorphic near p, then for any $N \in \mathbb{N}$, $C_N(f,g)(p) = f(p)g(p)$ for any $g \in \mathscr{C}^\infty(M)$. That is, $(f * g)(p) = (fg)(p)$. Similarly, if g is holomorphic near p, then for any $N \in \mathbb{N}$, $C_N(h,g)(p) = h(p)g(p)$ for any $h \in \mathscr{C}^\infty(M)$, hence $(h * g)(p) = (hg)(p)$.*

PROOF. Suppose f is anti-holomorphic on $U \subset M$, where U is a neighborhood of p. We identify p as $0 \in \mathbb{C}^n$ and $U = \{z \in \mathbb{C}^n : |z| < \frac{\log k}{\sqrt{k}}\}$. Without loss of generality, let $g \in C_0^\infty(M)$ with $\operatorname{supp} g \subset U$, and let $u, v \in L_0^2(M, L^k)$. We aim to show that $(T_f^{(k)} T_g^{(k)} u \mid v)_k - (T_{fg}^{(k)} u \mid v)_k = O(k^{-\infty})$. Note that $(T_f^{(k)} \circ T_g^{(k)} u|v)_k$ can be written as

$$\begin{aligned}(\Pi^{(k)} \circ f \circ \Pi^{(k)} \circ g \circ \Pi^{(k)} u|v)_k &= (\Pi^{(k)} \circ f \circ (\Pi^{(k)} - I) \circ g \circ \Pi^{(k)} u|v)_k \\ &\quad + (\Pi^{(k)} \circ fg \circ \Pi^{(k)} u|v)_k.\end{aligned}$$

Since $\Pi^{(k)}$ is self-adjoint, we have

$$(\Pi^{(k)} \circ f \circ (\Pi^{(k)} - I) \circ g \circ \Pi^{(k)} u|v)_k = ((\Pi^{(k)} - I) \circ g \circ \Pi^{(k)} u|\overline{f} \circ \Pi^{(k)} v)_k.$$

Let $\chi \in \mathscr{C}_0^\infty(U)$ such that $\chi \equiv 1$ on $\operatorname{supp} g$ and $\widetilde{\chi} \in \mathscr{C}_0^\infty(U)$ such that $\widetilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. Then,

$$\begin{aligned}(\Pi^{(k)} - I) \circ g \circ \Pi^{(k)} u &= (\Pi^{(k)} - I) \circ \chi g \circ \Pi^{(k)} u \\ &= \widetilde{\chi}(\Pi^{(k)} - I) \circ \chi g \circ \Pi^{(k)} u + (1 - \widetilde{\chi})(\Pi^{(k)} - I) \circ \chi g \circ \Pi^{(k)} u.\end{aligned}$$

Let $R_k = (1 - \widetilde{\chi})(\Pi^{(k)} - I) \circ \chi g \circ \Pi^{(k)} u$, since $(1 - \widetilde{\chi})\chi = 0$ and by Lemma 3.17 we have $R_k = O(k^{-\infty})$. Then

$$(3.3.7) \qquad ((\Pi^{(k)} - I) \circ g \circ \Pi^{(k)} u|\overline{f} \circ \Pi^{(k)} v)_k = ((\Pi^{(k)} - I) \circ \chi g \circ \Pi^{(k)} u|\widetilde{\chi}\overline{f}\Pi^{(k)} v)_k + O(k^{-\infty}).$$

We claim that there exist a holomorphic section $\mathbf{b}$ such that $\mathbf{b} = \bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v + O(k^{-\infty})$.

$$|\bar{\partial}\left(\bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v\right)|^2 = \int_M |\bar{\partial}\left(\bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v\right)|^2 e^{-2k\phi(z)}d\lambda(z) \tag{3.3.8}$$
$$= \int_U |\bar{\partial}\left(\bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v\right)|^2 e^{-2k\phi(z)}d\lambda(z)$$
$$\leq \int_{U\cap\{z\in\mathbb{C}^n:|z|>\epsilon\}} |\bar{\partial}\bar{\tilde{\chi}}|^2|\bar{f}\Pi^{(k)}v|^2 e^{-2k\phi(z)}d\lambda(z)$$
$$\leq A_{f,\tilde{\chi}}\int_{C_1\frac{\log k}{\sqrt{k}}>|z|>C_2\frac{\log k}{\sqrt{k}}} |\Pi^{(k)}v|^2 e^{-2k\phi(z)}d\lambda(z)$$
$$\leq A_{f,\tilde{\chi}}\int_{C_1\frac{\log k}{\sqrt{k}}>|z|>C_2\frac{\log k}{\sqrt{k}}} k^n e^{-2k\left(\sum_{j=1}^n \lambda_j|z_j|^2+O(|z|^3)\right)}$$
$$\leq A'_{f,\tilde{\chi}}k^{n-\log k} = O(k^{-\infty}).$$

Let $\alpha = \bar{\partial}\left(\bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v\right) \otimes s^k \in \Omega^{0,1}(M, L^k)$. Since $\bar{\partial}\alpha = 0$, there exist β such that $\bar{\partial}\beta = \alpha$ and

$$||\beta||_k \leq \frac{C}{k}||\alpha||_k \leq \frac{1}{k}k^{n-\log k}. \tag{3.3.9}$$

Let $\mathbf{b} = \bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v \otimes s^k - \beta$. Then since $\bar{\partial}\mathbf{b} = 0$, we have $\mathbf{b} \in H^0(M, L^k)$. Therefore,

$$\bar{\tilde{\chi}}\bar{f}\Pi^{(k)}v = \mathbf{b} + r \tag{3.3.10}$$

where $||r|| \leq C_{f,\chi}k^{-N}$ for all $N \in \mathbb{N}$. Hence (3.3.7) vanishes up to $O(k^{-\infty})$. $\square$

REMARK 3.19. Note that with Lemma 3.18, one can deduce (3.3.1) by computing the terms that contain the holomorphic derivative of f and the anti-holomorphic derivative of g.

Recall the main equation we are computing is

$$b_{0,C_3(f,g)} = b_{3,f,g} - b_{3,fg} - b_{2,C_1(f,g)} - b_{1,C_2(f,g)}.$$

By (3.2.8), we have the following:

(3.3.11)

$$b_{3,fg}(0) = 2fg(z)(b_3(0)b_0(0) + b_2(0)b_1(0))$$

$$+\frac{1}{2}\Delta_0(V_\Theta fg(z)[b_0(0,z)b_2(z,0) + b_1(0,z)b_1(z,0) + b_2(0,z)b_0(z,0)]) \tag{3.3.12}$$

$$-\frac{1}{4}\Delta_0^2(\phi_1 V_\Theta fg(z)[b_0(0,z)b_2(z,0) + b_1(0,z)b_1(z,0) + b_2(0,z)b_0(z,0)]) \tag{3.3.13}$$

$$+\frac{1}{8}\Delta_0^2(V_\Theta fg(z)[b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)]) \tag{3.3.14}$$

$$-\frac{1}{24}\Delta_0^3(\phi V_\Theta fg(z)[b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)]) \tag{3.3.15}$$

$$+\frac{1}{192}\Delta_0^4(\phi^2 V_\Theta fg(z)[b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)]) \tag{3.3.16}$$

$$+\frac{1}{48}\Delta_0^3(V_\Theta fg(z)b_0(0,z)b_0(z,0)) \tag{3.3.17}$$

$$-\frac{1}{192}\Delta_0^4(\phi_1 V_\Theta fg(z)b_0(0,z)b_0(z,0)) \tag{3.3.18}$$

$$+\frac{1}{1920}\Delta_0^5(\phi_1^2 V_\Theta fg(z)b_0(0,z)b_0(z,0)) \tag{3.3.19}$$

$$-\frac{1}{34560}\Delta_0^6(\phi_1^3 V_\Theta fg(z)b_0(0,z)b_0(z,0)). \tag{3.3.20}$$

From (3.2.14), we know that $b_{3,f,g}$ only contains the terms that involving $\frac{\partial^\alpha f}{\partial z^\alpha}$ and $\frac{\partial^\beta g}{\partial z^\beta}$ for $\alpha, \beta \in \mathbb{N}_0^m$ for any $m \in \mathbb{N}$. Therefore, subtracting $b_{3,f,g}$ and $b_{3,fg}$, the only remains we have to concern are the six terms (3.3.12), (3.3.14), (3.3.15), (3.3.17), (3.3.18), (3.3.19), which become the following respectively:

$$(3.3.12) = \frac{1}{2\pi}\langle \partial f, \partial \bar{g}\rangle_\omega (2b_0b_2 + b_1^2). \tag{3.3.21}$$

(3.3.22)

$$
\begin{aligned}
(3.3.14) = \frac{1}{8}\Bigg\{ & \Delta_0^2(fg)\, 2b_1 \\
& + 2\sum \frac{1}{\lambda_j}\Bigg(\frac{\partial}{\partial z_j}\Delta_0(fg)\, \frac{\partial}{\partial \overline{z_j}}\Big(V_\Theta\big(b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)\big)\Big) \\
& \qquad + \frac{\partial}{\partial \overline{z_j}}\Delta_0(fg)\, \frac{\partial}{\partial z_j}\Big(V_\Theta\big(b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)\big)\Big)\Bigg) \\
& + 2\Delta_0(fg)\,\Delta_0\Big(V_\Theta\big(b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)\big)\Big) \\
& + 2\sum \frac{1}{\lambda_j\lambda_i}\,\frac{\partial^2(fg)}{\partial z_i \partial \overline{z_j}}\,\frac{\partial^2}{\partial \overline{z_i}\partial z_j}\Big(V_\Theta\big(b_1(0,z)b_0(z,0) + b_0(0,z)b_1(z,0)\big)\Big)\Bigg\}.
\end{aligned}
$$

(3.3.23)

$$
\begin{aligned}
(3.3.15) = \frac{-1}{24}\Bigg\{ & \Big(\frac{3}{2\pi^2}\hat{r} - \frac{3}{4\pi^2} r\Big)(\Delta_0^2\phi)(0)\langle \partial f, \partial \bar{g}\rangle_\omega \\
& - \frac{3}{2\pi^3}(2\hat{r} - r)\langle Ric_\omega, \bar{\partial}\bar{f} \wedge \partial \bar{g}\rangle_\omega \Bigg\}.
\end{aligned}
$$

$$(3.3.24)\quad (3.3.17) = \frac{1}{48}\Big\{ \Delta_0^3(fg)$$

$$+ 3\Big(\Delta_0^2(fg)\,\Delta_0\widetilde{V_\Theta} + \Delta_0(fg)\,\Delta_0^2\widetilde{V_\Theta}$$

$$+ 2\sum \frac{1}{\lambda_i\lambda_j}\,\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\Delta_0(fg)\,\frac{\partial^2}{\partial \overline{z_i}\partial z_j}\widetilde{V_\Theta}\Big)$$

$$+ 6\sum \frac{1}{\lambda_i\lambda_j}\,\frac{\partial^2(fg)}{\partial z_i \partial \overline{z_j}}\,\frac{\partial^2}{\partial \overline{z_i}\partial z_j}\Delta_0\widetilde{V_\Theta}$$

$$+ 6\sum \frac{1}{\lambda_i}\left(\frac{\partial}{\partial z_i}\Delta_0(fg)\,\frac{\partial}{\partial \overline{z_i}}\Delta_0\widetilde{V_\Theta} + \frac{\partial}{\partial \overline{z_i}}\Delta_0(fg)\,\frac{\partial}{\partial z_i}\Delta_0\widetilde{V_\Theta}\right)$$

$$+ 3\sum \frac{1}{\lambda_i\lambda_j\lambda_k}\left(\frac{\partial^3(fg)}{\partial z_i \partial z_j \partial \overline{z_k}}\,\frac{\partial^3}{\partial \overline{z_i}\partial \overline{z_j}\partial z_k}\widetilde{V_\Theta}\right.$$

$$\left. + \frac{\partial^3(fg)}{\partial \overline{z_i}\partial \overline{z_j}\partial z_k}\,\frac{\partial^3}{\partial z_i \partial z_j \partial \overline{z_k}}\widetilde{V_\Theta}\right)\Big\}.$$

(3.3.25)

$$(3.3.18) = \frac{-1}{192}\Big\{ 6\Delta_0^2\phi_1\,\Delta_0^2(fg\widetilde{V_\Theta})$$

$$+ 6\sum \frac{1}{\lambda_i\lambda_j\lambda_k\lambda_l}\,\frac{\partial^4\phi_1}{\partial z_i \partial \overline{z_j}\partial z_k \partial \overline{z_l}}\,\frac{\partial^4}{\partial \overline{z_i}\partial z_j \partial \overline{z_k}\partial z_l}(fg\widetilde{V_\Theta})$$

$$+ 24\sum \frac{1}{\lambda_i\lambda_j}\,\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\Delta_0\phi_1\,\frac{\partial^2}{\partial \overline{z_i}\partial z_j}\Delta_0(fg\widetilde{V_\Theta})$$

$$+ 4\Delta_0^3\phi_1\,\Delta_0(fg) + 12\sum \frac{1}{\lambda_i\lambda_j}\,\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\Delta_0^2\phi_1\,\frac{\partial^2(fg)}{\partial \overline{z_i}\partial z_j}$$

$$+ 12\sum \frac{1}{\lambda_i}\Big(\frac{\partial}{\partial z_i}\Delta_0^2\phi_1\,\frac{\partial}{\partial \overline{z_i}}\Delta_0(fg\widetilde{V_\Theta}) + \frac{\partial}{\partial \overline{z_i}}\Delta_0^2\phi_1\,\frac{\partial}{\partial z_i}\Delta_0(fg\widetilde{V_\Theta})\Big)$$

$$+ 12\sum \frac{1}{\lambda_i\lambda_j\lambda_k}\Big(\frac{\partial^3}{\partial z_i \partial z_j \partial \overline{z_k}}\Delta_0\phi_1\,\frac{\partial^3}{\partial \overline{z_i}\partial \overline{z_j}\partial z_k}(fg\widetilde{V_\Theta})$$

$$+ \frac{\partial^3}{\partial \overline{z_i}\partial \overline{z_j}\partial z_k}\Delta_0\phi_1\,\frac{\partial^3}{\partial z_i \partial z_j \partial \overline{z_k}}(fg\widetilde{V_\Theta})\Big)\Big\}.$$

$$(3.3.19) = \frac{1}{1920}\left\{\frac{5}{\pi}(\Delta_0^4\phi_1^2)\langle\partial f,\partial\bar{g}\rangle_\omega - \frac{120}{\pi^2}\Delta_0^2\phi_1\langle Ric_\omega,\bar{\partial}\bar{f}\wedge\partial\bar{g}\rangle_\omega \right. \tag{3.3.26}$$
$$\left. - \frac{240}{\pi^3}\langle R\circ Ric_\omega,\bar{\partial}\bar{f}\wedge\partial\bar{g}\rangle_\omega + \frac{120}{\pi^3}\langle R^2,\bar{\partial}\bar{f}\wedge\partial\bar{g}\rangle_\omega\right\},$$

where $\widetilde{V_\Theta}(z) = V_\Theta b_0(0,z)b_0(z,0)$. Computing the differential operator Δ_0, and taking the holomorphic derivative of f and the anti-holomorphic derivative of g into account, we obtain:

$$(3.3.12) = \frac{1}{2\pi}\langle\partial f,\partial\overline{g}\rangle_\omega(2b_0b_2 + b_1^2). \tag{3.3.27}$$

$$(3.3.14) = \frac{1}{8}\left\{\left(\frac{\hat{r}}{2\pi} - \frac{r}{4\pi}\right)\Delta_0^2(fg) + \frac{1}{\pi}\langle\partial f,\partial\overline{g}\rangle_\omega\left(-\frac{\hat{r}^2}{2\pi^2} + \frac{r\hat{r}}{4\pi^2}\right)\right. \tag{3.3.28}$$
$$- \left(\frac{\hat{r}}{\pi^3} - \frac{r}{2\pi^3}\right)\langle R_\Theta^{\det},\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega$$
$$+ \frac{1}{4\pi^2}\Big(\langle D^{1,0}\partial f(e_k)\otimes\overline{\partial}g(\overline{e_k}),\partial(2\bar{\hat{r}} - \bar{r})\rangle_\omega$$
$$\left. + \langle\partial f(e_k)\otimes D^{0,1}\overline{\partial}g(\overline{e_k}),\overline{\partial}(2\bar{\hat{r}} - \bar{r})\rangle_\omega\Big)\right\}.$$

(3.3.29)
$$(3.3.15) = \frac{-1}{24}\left\{\frac{6}{\pi}\Delta_0^2\phi\left(\frac{\hat{r}}{4\pi} - \frac{r}{8\pi}\right)\langle\partial f,\partial\overline{g}\rangle_\omega - \frac{3}{2\pi^2}(2\hat{r} - r)\langle Ric_\omega,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\right\}.$$

(3.3.30)

$$
\begin{aligned}
(3.3.17) = \frac{1}{48}\Big\{ & \Delta_0^3(fg) \\
& + 3\Big(-\frac{\hat{r}}{2\pi}\Delta_0^2(fg) \\
& \quad + \frac{1}{\pi}\langle \partial f, \partial \bar{g}\rangle_\omega \Big(\frac{\Delta_\omega \hat{r}}{4\pi^2} + \frac{1}{\pi^2}\langle R_\Theta^{\det}, Ric_\omega\rangle_\omega + \frac{\hat{r}^2}{4\pi^2} + \frac{1}{\pi^2}|R_\Theta^{\det}|_\omega^2 \Big) \\
& \quad - \frac{2}{\pi^3}\langle R_\Theta^{\det}, D^{0,1}\bar{\partial}\bar{g}(\bar{e}_j) \wedge D^{1,0}\partial \bar{f}(e_j)\rangle_\omega \Big) \\
& + 6\Big(-\frac{1}{2\pi^3}(\langle \bar{\partial}\partial \hat{r}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega + \langle \bar{\partial}\partial \hat{k}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega) \\
& \quad + \frac{\hat{r}}{2\pi^3}\langle R_\Theta^{\det}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega + \frac{1}{\pi^3}\langle (R_\Theta^{\det})^2, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega \Big) \\
& - \frac{3}{\pi^3}\Big(\langle \partial \hat{r}, D^{1,0}\partial \bar{g}(e_j) \otimes \bar{\partial}\bar{f}(e_j)\rangle_\omega + \langle \bar{\partial}\hat{r}, D^{0,1}\bar{\partial} f(e_j) \otimes \partial \bar{g}(e_j)\rangle_\omega \Big) \\
& - \frac{3}{\pi^3}\Big(\langle D^{0,1}R_\Theta^{\det}, D^{0,1}\bar{\partial}\bar{f} \otimes \partial \bar{g}\rangle_\omega + \langle D^{0,1}\bar{\partial} g \otimes \partial f, D^{0,1}R_\Theta^{\det}\rangle_\omega \Big) \Big\},
\end{aligned}
$$

where

$$
\hat{k}(z) := \hat{r}(z) + 2\pi\bigg(\sum_j \frac{1}{\lambda_j}\Big(-\left|\frac{\partial V_\Theta}{\partial z_j}\right|^2 + \frac{\partial^2 V_\Theta}{\partial \overline{z_j}\,\partial z_j} \Big) \bigg)(z).
$$

(3.3.31)

$$
\begin{aligned}
&(3.3.18) = \\
&\frac{-1}{192}\Big\{ 12\Delta_0^2\phi_1\Big(-\frac{\hat{r}}{2\pi^2}\langle \partial f, \partial \bar{g}\rangle_\omega - \frac{2}{\pi^2}\langle R_\Theta^{\det}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega + \Delta_0^2(fg)\Big) \\
&\quad + 6\Big(-\frac{4}{\pi^3}\langle R \circ R_\Theta^{\det}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega - \langle R, D^{0,1}\bar{\partial}\bar{g} \tilde{\otimes} D^{1,0}\partial \bar{f}\rangle_\omega\Big) \\
&\quad + 24\Big(\frac{\hat{r}}{2\pi^3}\langle Ric_\omega, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega + \frac{1}{\pi^3}\langle Ric_\omega, R_\Theta^{\det}\rangle_\omega \langle \partial f, \partial \bar{g}\rangle_\omega \\
&\qquad + \frac{2}{\pi^3}\langle Ric_\omega \circ R_\Theta^{\det}, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega \\
&\qquad\qquad - \langle Ric_\omega, D^{0,1}\bar{\partial}\bar{g}(\bar{e}_j) \wedge D^{1,0}\partial \bar{f}(e_j)\rangle_\omega\Big) \\
&\quad + \frac{4}{\pi}\Delta_0^3\phi_1\langle \partial f, \partial \bar{g}\rangle_\omega \\
&\quad + 12\Big(-\frac{1}{2\pi^3}\big(\langle \bar{\partial}\partial r, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega + \langle \bar{\partial}\partial k, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega\big) \\
&\qquad\qquad - \frac{3}{\pi^3}\langle R \circ Ric_\omega, \bar{\partial}\bar{g} \wedge \partial \bar{f}\rangle_\omega\Big) \\
&\quad + 12\Big(-\frac{1}{2\pi^3}\Big(\langle \partial r, D^{1,0}\partial \bar{g}(e_j) \otimes \bar{\partial}\bar{f}(\bar{e}_j)\rangle_\omega \\
&\qquad\qquad + \langle \bar{\partial} r, \partial \bar{g}(e_j) \otimes D^{0,1}\bar{\partial}\bar{f}(e_j)\rangle_\omega\Big) \\
&\qquad - \frac{1}{\pi^3}\Big(\langle D^{0,1}Ric_\omega, D^{0,1}\bar{\partial}\bar{f} \otimes \partial \bar{g}\rangle_\omega \\
&\qquad\qquad + \langle D^{0,1}\bar{\partial}\bar{g} \otimes \partial f, D^{0,1}Ric_\omega\rangle_\omega\Big)\Big)\Big\}.
\end{aligned}
$$

where

$$k(z) := r(z) + 2\pi\Delta_0^2\phi_1(z). \tag{3.3.32}$$

(3.3.33)

$$(3.3.19) = \frac{1}{1920}\Big\{\frac{5}{\pi}(\Delta_0^4\phi_1^2)\langle\partial f,\partial\bar{g}\rangle_\omega$$

$$+20\Big(-\frac{6}{\pi^2}(\Delta_0^2\phi_1)\langle Ric_\omega,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega-\frac{12}{\pi^3}\langle R\circ Ric_\omega,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega$$

$$+\frac{12}{\pi^3}\langle Ric_\omega^2,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega+\frac{6}{\pi^3}\langle R^2,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\Big)\Big\}.$$

Terms in $b_{1,C_2(f,g)}$ are the followings:

$$\frac{1}{8\pi^2}\left(\frac{\hat{r}}{4\pi}-\frac{r}{8\pi}\right)\langle D^{1,0}\partial f,D^{1,0}\partial\bar{g}\rangle_\omega$$

$$+\frac{1}{4\pi^2}\left(\frac{\hat{r}}{4\pi}-\frac{r}{8\pi}\right)\langle\bar{\partial}g\wedge\partial f,R_\Theta^{\det}\rangle_\omega$$

$$+\frac{1}{16}\Big\{\Delta_0^3(fg)+\frac{1}{\pi^3}\langle R^2,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega+\frac{2}{\pi^2}\langle R,D^{0,1}\bar{\partial}\bar{g}\tilde{\otimes}D^{1,0}\partial f\rangle_\omega$$

$$+\frac{2}{\pi^3}\langle D^{0,1}\bar{\partial}g(e_j)\wedge D^{1,0}\partial f(e_j),Ric_\omega\rangle_\omega\Big\}$$

(3.3.34)

$$+\frac{1}{16\pi^3}\Big\{\langle D^{0,1}Ric_\omega,D^{0,1}\bar{\partial}\bar{f}\otimes\partial\bar{g}\rangle_\omega+\langle D^{0,1}\bar{\partial}g\otimes\partial f,D^{0,1}Ric_\omega\rangle_\omega\Big\}$$

$$+\frac{1}{8}\Big\{\frac{1}{\pi^3}\langle D^{1,0}\partial f(e_k)\wedge D^{0,1}\bar{\partial}g(\bar{e_k}),R_\Theta^{\det}\rangle_\omega+\frac{2}{\pi^3}\langle Ric_\omega\circ R_\Theta^{\det},\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega$$

$$+\frac{1}{\pi^3}\Big(\langle D^{0,1}R_\Theta^{\det},D^{0,1}\partial\bar{f}\otimes\partial\bar{g}\rangle_\omega+\langle D^{0,1}\bar{g}\otimes\partial f,D^{0,1}R_\Theta^{\det}\rangle_\omega\Big)$$

$$+\frac{1}{2\pi^3}\langle\bar{\partial}\partial(\hat{r}-\hat{k}),\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\Big\}.$$

Terms in $b_{2,C_1(f,g)}$ are the followings:

$$\begin{aligned}
&-\frac{1}{2\pi}\left\{\frac{r^2}{128\pi^2}-\frac{r\hat{r}}{32\pi^2}+\frac{\hat{r}^2}{32\pi^2}-\frac{\Delta_\omega\hat{r}}{32\pi^2}+\frac{|R_\Theta^{\det}|_\omega^2}{8\pi^2}\right.\\
&\qquad\left.+\frac{\langle Ric_\omega,R_\Theta^{\det}\rangle_\omega}{8\pi^2}-\frac{\Delta_\omega r}{96\pi^2}-\frac{|Ric_\omega|_\omega^2}{24\pi^2}+\frac{|R_\omega^{TM}|_\omega^2}{96\pi^2}\right\}\langle\partial f,\partial\bar{g}\rangle_\omega\\
&+\frac{r-2\hat{r}}{32\pi^3}\left\{\langle D^{1,0}\partial f,D^{1,0}\partial\bar{g}\rangle_\omega+\langle Ric_\omega,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\right\}\\
&+\frac{1}{8\pi^3}\left\{\langle D^{1,0}\partial f(e_j)\wedge D^{0,1}\bar{\partial}(\bar{e}_j),R_\Theta^{\det}\rangle_\omega-\langle R\circ R_\Theta^{\det},\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\right\}\\
&-\frac{1}{16\pi^3}\langle D^{1,0}\partial f(e_j)\wedge D^{0,1}\bar{\partial}\bar{g}(\bar{e}_j),Ric_\omega\rangle_\omega-\frac{1}{16\pi^3}\langle R\circ Ric_\omega,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\\
&-\frac{1}{16}\left\{\Delta_0^3(fg)+\frac{3}{\pi^3}\langle Ric_\omega,D^{0,1}\bar{\partial}\bar{g}(\bar{e}_j)\wedge D^{1,0}\partial\bar{f}(e_j)\rangle_\omega\right.\\
&\qquad+\frac{2}{\pi^3}\langle R,D^{0,1}\bar{\partial}\bar{g}\otimes D^{1,0}\partial\bar{f}\rangle_\omega+\frac{1}{2\pi^3}\langle\bar{\partial}\partial(r-k),\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\\
&\qquad+\frac{2}{\pi^3}\langle R^2,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega+\frac{2}{\pi^3}\langle Ric_\omega^2,\bar{\partial}\bar{g}\wedge\partial\bar{f}\rangle_\omega\\
&\qquad\left.+\frac{2}{\pi^3}\Big(\langle D^{0,1}Ric_\omega,D^{0,1}\bar{\partial}\bar{f}\otimes\partial\bar{g}\rangle_\omega+\langle D^{0,1}\bar{\partial}\bar{g}\otimes\partial f,D^{0,1}Ric_\omega\rangle_\omega\Big)\right\}.
\end{aligned}\tag{3.3.35}$$

By substituting (3.3.27), (3.3.28), (3.3.29), (3.3.30), (3.3.31), (3.3.33), (3.3.34), and (3.3.35) in (3.3.1), we obtain (3.1.20).

Part 2

Quantization on CR Manifolds and Complex Manifolds with Boundaries

CHAPTER 4

Quantization on CR manifolds

The goal of this chapter is to build a star product on a strictly pseudoconvex Cauchy-Riemann (CR) manifold, providing the foundation for the later chapter on quantization for complex manifolds with boundaries. We first show that Toeplitz operators defined via the Szegő projector on strictly pseudoconvex CR manifolds give rise to a star product. For Kähler manifolds with boundary, we then reduce the composition of Toeplitz operators defined by the Bergman projection to the boundary using a Szegő-type projection. Finally, by studying the near-boundary behavior of the Poisson operator, we construct an algebra consisting of sequences of harmonic functions that carry a star product induced from the boundary.

On CR manifolds, the CR structure induces a contact form ξ. Since a contact manifold is the odd-dimensional analog of a symplectic manifold, it is natural to ask whether deformation quantization methods from symplectic geometry extend to contact geometry. From the viewpoint of complex geometry, the CR structure is the counterpart of a holomorphic structure. Motivated by Berezin–Toeplitz quantization, one can define Toeplitz operators on CR manifolds using the Szegő projection and compute their symbols to study the resulting algebraic structure, paralleling the Kähler case.

In Berezin–Toeplitz quantization on Kähler manifolds, the formal parameter corresponds to the tensor power of the prequantum line bundle in the semi-classical limit. On CR manifolds, the lack of such a semi-classical parameter is the first obstacle to constructing a star product. Motivated by [11], we propose replacing this parameter with a pseudodifferential operator E, so the star product of two smooth functions becomes a pseudodifferential operator rather than a smooth function. Moreover, constructing the star product in Theorem 4.23 on the space of pseudodifferential operators yields a closed algebra. The closed algebras of star products are the formal power series in the formal parameter (see Def. 3.1). In this part, we consider the algebra (Sec. 4.3) of pseudodifferential operators satisfying (see Sec. 4.2 for details)

$$T_L \equiv \sum_{j=0}^{\infty} T_{l_j E^{m-j}} \qquad \mathrm{mod}\ \Psi^{-\infty}(X).$$

4.1. Introduction and main results

We now introduce the key ingredient needed to construct the star product in this chapter:

Let X be a strictly pseudoconvex Cauchy-Riemann manifold of dimension $2n+1$. Let ξ be the characteristic one form on X, such that dV_ξ is the induced volume form on X. We denote by S the Szegő projection with respect to the inner product $(\cdot\,|\,\cdot)_\xi$, which has the same phase function φ as in (2.2.10). In this part, we work in a slightly more general setting:

For $m \leq 0$, we consider an elliptic classical pseudodifferential operator Q of order m on X such that for any $u \in \mathscr{C}^\infty(X)$ we have $(Qu|u)_\xi \geq 0$, and $(Qu|u) = 0$ if and only if $u = 0$. Let $(\cdot\,|\,\cdot)_Q$ be the L^2 inner product on $\mathscr{C}^\infty(X)$ given by

$$(u\,|\,v)_Q := (\,Qu\,|\,v\,), \quad u, v \in \mathscr{C}^\infty(X). \tag{4.1.1}$$

We now define a certain type of projection operator.

DEFINITION 4.1. Let $\tilde{S} : \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(X)$ be a linear operator. $\tilde{S}$ is called a *Szegő type projection* if it satisfies $\tilde{S}^2 \equiv \tilde{S}$, $\tilde{S}^\dagger \equiv \tilde{S}$, and $\overline{\partial_b}\tilde{S} \equiv 0$, where $\tilde{S}^\dagger$ is the formal adjoint of $\tilde{S}$ with respect to $(\cdot\,|\,\cdot)_Q$, and $\tilde{S}$ can be approximated by a Fourier integral operator whose kernel, in any coordinate patch (D, x) of X, has the form

$$\tilde{S}(x,y) \equiv \int_0^\infty e^{it\tilde{\varphi}(x,y)} a(x,y,t)\,dt, \tag{4.1.2}$$

where $\tilde{\varphi}(x,y)$ is the same phase function as in 2.2.10, $a(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$, and

$$a \sim \sum_{j=0}^{\infty} t^{n-j} a_j(x,y), \quad a_j(x,y) \in \mathscr{C}^\infty(D \times D), \quad a_0(x,x) \neq 0.$$

REMARK 4.2. We introduce the Szegő type projection as a Fourier integral operator whose principal term is nonvanishing, thereby excluding the degenerate case where the operator maps every element to zero. If a nondegenerate operator K satisfies $K^2 \equiv K$, $K^\dagger \equiv K$ and $\overline{\partial_b}K \equiv 0$, then by [4, 20], its integral kernel can always be written as in (4.1.2).

We denote the space of order m classical pseudodifferential operators on X by $\mathcal{L}^m(X)$ and $\mathcal{L}(X)$ is the space of classical pseudodifferential operators on X of integer order. We define a specific operator on X that is analogous to the formal parameter in the theory of deformation quantization. Let

$$E := (I + \Box_b + \mathcal{T}^*\mathcal{T})^{\frac{1}{2}} : L^2(X) \to L^2(X) \tag{4.1.3}$$

be the first-order pseudo-differential operator on X associated with the Reeb vector field $\mathcal{T}$, and $\mathcal{T}^*$ is the formal adjoint of $\mathcal{T}$ with respect to the inner product $(\cdot\,|\,\cdot)_{\xi}$. The principal symbol of E is $\sigma_E^0(x, \xi(x)) = 1$. We now define a class of Fourier integral operators on X:

DEFINITION 4.3. Let A be a Fourier integral operator on X of order m. A is said to be a *Szegő type* if, for any coordinate patch (D, x), there exists a phase function $\varphi \in \mathscr{C}^\infty(D \times D, \mathbb{C})$ satisfying (2.2.10) and $a(x, y, t) \in S^m_{cl}(D \times D \times \mathbb{R}_+)$ such that

$$A(x, y) \equiv \int_0^\infty e^{it\varphi(x,y)} a(x, y, t) dt \tag{4.1.4}$$

satisfying

$$\begin{aligned} &a(x, y, t) \sim \sum_{j=0}^{\infty} a_j(x, y) t^{m-j} \in S^m_{1,0}(D \times D \times \mathbb{R}_+), \\ &a_j(x, y) \in \mathscr{C}^\infty(D \times D), j = 0, 1, \ldots. \end{aligned} \tag{4.1.5}$$

We say $a_0(x, y)$ is the leading term of A. Let $\Psi^m(X)$ denote the space of Szegő type Fourier integral operators of order m on X, and $\Psi(X)$ be the space of Szegő type Fourier integral operators of integer order.

REMARK 4.4. Given a Szegő type projection $\tilde{S}$, we have $\tilde{S} \in \Psi^n(X)$.

DEFINITION 4.5. Let $L \in \mathcal{L}^m(X)$ and $\tilde{S} \in \Psi^n(X)$ with $\tilde{S}^2 = \tilde{S} = \tilde{S}^\dagger$. We say that L has an asymptotic sum with respect to $\tilde{S}$ if for every $N \in \mathbb{N}$ there exist unique $f_j \in \mathscr{C}^\infty(X)$, $j = 0, \ldots, N-1$, such that

$$\tilde{S} \circ \Big(L - \sum_{j=0}^{N-1} f_j E^{m-j}\Big) \circ \tilde{S} \in \Psi^{n+m-N}(X).$$

In this case we write

$$L \sim \sum_{j=0}^{\infty} f_j E^{m-j}. \tag{4.1.6}$$

For simplicity, we denote T_L the Toeplitz operator of L with respect to $\tilde{S}$ by the composition $\tilde{S} \circ L \circ \tilde{S}$. We now omit the composition notation to simplify the notation.

The first result in this chapter is:

THEOREM 4.6. *Let $(X, T^{1,0}X)$ be a compact orientable strictly pseudoconvex CR manifold, and $\tilde{S}$ is a Szegő type projection, then for any $L \in \mathcal{L}^m(X)$, L has the asymptotic sum with respect to $\tilde{S}$.*

Throughout this chapter, we fix a Szegő type projection $\tilde{S}$. By Theorem 4.6, for any $L \in \mathcal{L}^m(X)$ there is a unique sequence of smooth functions $l_j \in \mathscr{C}^\infty(X)$ such that L and the l_j satisfy (4.1.6). We call l_j the coefficients of the asymptotic sum of L. From the proof of Theorem 4.6 we obtain:

COROLLARY 4.7 (Corollary 4.15). *For any $A \in \Psi^{m+n}(X)$ with $\tilde{S}A \equiv A \equiv A\tilde{S}$ on X, there exist unique $a_j \in \mathscr{C}^\infty(X)$, $j \in \mathbb{N}_0$, such that for all $N \in \mathbb{N}$,*

$$A - \sum_{j=0}^{N-1} T_{a_j E^{m-j}} \in \Psi^{m+n-N}(X). \tag{4.1.7}$$

Similarly, we call a_j the coefficients of the asymptotic sum of A. Let $\mathcal{A}$ be the vector space of pseudodifferential operators on X in the form of an asymptotic sum with respect to S, that is

$$\mathcal{A} := \left\{ \sum_{j=0}^{\infty} l_j E^{m-j} : l_j \in \mathscr{C}^\infty(X) \text{ for } j \in \mathbb{N}_0, m \in \mathbb{Z} \right\}.$$

We will show in Sec. 4.3 that $\mathcal{A}$ carries a commutative Poisson algebraic structure, which in turn induces a commutative Poisson structure on the space of pseudodifferential operators $\mathcal{L}(X)$. That is, there exists a multiplication $\cdot$ and bilinear map $\{\,\cdot\,,\,\cdot\,\}$ on $\mathcal{L}(X)$ such that $(\mathcal{L}(X), \cdot, \{\,\cdot\,,\,\cdot\,\})$ forms a Poisson algebra. We define the product of pseudodifferential operators by (4.3.12); see Definition 4.21 for the Poisson bracket of pseudodifferential operators.

From Theorem 4.23 and Corollary 4.7, since the coefficients of the asymptotic sum are unique, we define maps

$$\begin{aligned} \mathsf{q} &: \mathcal{L}(X) \to \mathcal{A}, \\ \hat{\mathsf{q}} &: \tilde{\Psi}(X) \to \mathcal{A}, \end{aligned} \tag{4.1.8}$$

where $\tilde{\Psi}(X) := \{A \in \Psi(X) : \tilde{S}A \equiv A \equiv A\tilde{S} \text{ on } X\}$, such that

$$T_L \equiv T_{\mathsf{q}(L)}, \quad A \equiv T_{\hat{\mathsf{q}}(A)} \quad \text{on } X. \tag{4.1.9}$$

The main result of this chapter is the construction of a star product whose commutator reproduces the Poisson structure.

THEOREM 4.8. *Let $(X, T^{1,0}X)$ be a compact oriented strictly pseudoconvex CR manifold. Let L_1 and L_2 be pseudodifferential operators of order n_1 and n_2 respectively, then the star product of L_1 and L_2 is an element in $\mathcal{L}^{n_1+n_2}(X)$ defined by*

$$L_1 * L_2 := \hat{\mathsf{q}}(T_{\mathsf{q}(L_1)} \circ T_{\mathsf{q}(L_2)}). \tag{4.1.10}$$

The star product $*$ is associative, consequently, the pair $(\mathcal{L}(X), *)$ is an associative algebra. There exists $C'_j(L_1, L_2) \in \mathscr{C}^\infty(X)$ for $j = 0, 1, \ldots$ such that

$$L_1 * L_2 = \sum_{j=0}^{\infty} C'_j(L_1, L_2) E^{n_1+n_2-j}. \tag{4.1.11}$$

There also exists $C_j(L_1, L_2) \in \mathcal{L}^{n_1+n_2}(X)$ *for* $j = 0, 1, ...$

$$L_1 * L_2 = \sum_{j=0}^{\infty} C_j(L_1, L_2) E^{-j} \tag{4.1.12}$$

where

$$C_0(L_1, L_2) = L_1 \cdot L_2 \tag{4.1.13}$$

and

$$\big(C_1(L_1, L_2) - C_1(L_2, L_1)\big) E^{-1} = \{L_1, L_2\}. \tag{4.1.14}$$

The coefficients $C_j'(L_1, L_2)$ *and* $C_j(L_1, L_2)$ *satisfy the asymptotic relation*

$$\sum_{j=0}^{N} C_j'(L_1, L_2)\, E^{n_1+n_2-j} \;-\; \sum_{j=0}^{N} C_j(L_1, L_2)\, E^{-j} \;\in\; \mathcal{L}^{-N-1}(X) \tag{4.1.15}$$

for every $N \in \mathbb{N}_0$.

REMARK 4.9 (Conceptual summary). The construction described above can be summarized schematically as follows.

$$\mathcal{L}(X) \longrightarrow \mathcal{A} \longrightarrow L_1 * L_2 = \sum_{j=0}^{\infty} C_j(L_1, L_2)\, E^{-j}.$$

The first arrow represents Toeplitz quantization together with the asymptotic sum via Theorem 4.6, and the second arrow corresponds to the star product via Theorem 4.8.

4.2. Asymptotic expansion of pseudodifferential operators

In this section, we show that every pseudodifferential operator on X admits a sequence of smooth functions to represent with respect to E by proving Theorem 4.6. First, we show that the composition of Szegő type Fourier integral operators is still a Szegő type Fourier integral operator:

PROPOSITION 4.10. *For any* A_1 *and* A_2 *in* $\Psi^\bullet(X)$, $A_1 \circ A_2 \in \Psi^\bullet(X)$.

PROOF. Let A_1 and A_2 be Fourier integral operators of Szegő type of order n_1 and n_2 respectively. For any coordinate patch (D, x), we have

$$A_1(x, y) = \int_0^\infty e^{it\phi_1(x,y)} a_1(x, y, t) dt \tag{4.2.1}$$

and

$$A_2(x, y) = \int_0^\infty e^{it\phi_2(x,y)} a_2(x, y, t) dt \tag{4.2.2}$$

where

$$a_1(x,y,t) \sim \sum_{j=0}^{\infty} a_{1,j}(x,y)t^{n_1-j} \in S^{n_1}_{1,0}(D \times D \times \mathbb{R}_+) \tag{4.2.3}$$

and

$$a_2(x,y,t) \sim \sum_{j=0}^{\infty} a_{2,j}(x,y)t^{n_2-j} \in S^{n_2}_{1,0}(D \times D \times \mathbb{R}_+). \tag{4.2.4}$$

Since $A_1, A_2 \in \Psi(X)$, ϕ_1 and ϕ_2 can be chosen to be equivalent to φ in (2.2.10). Without lost of generality, we can assume ϕ_1 and ϕ_2 to be φ. Consider the composition $A_1 \circ A_2 \in \Psi^{n_1+n_2}(X)$, we compute the integral kernel of the operators $A_1 \circ A_2$ and show that it is also a Fourier integral operators of Szegő type. Consider the composition of the operators A_1 and A_2. Its Schwartz kernel is given by

$$(A_1 \circ A_2)(x,y) = \iiint_{D\times\mathbb{R}_+\times\mathbb{R}_+} e^{it\phi(x,z)+is\phi(z,y)}\, a_1(x,z,t)\, a_2(z,y,s)\, V(z)\, dz\, ds\, dt, \tag{4.2.5}$$

where $V(z)\,dz = dV_\xi$ denotes the volume form on X. Let $s = \sigma t$ then (4.2.5) becomes

$$\int_0^{\infty} \left(\iint_{D\times\mathbb{R}_+} e^{it(\phi(x,z)+\sigma\phi(z,y))} a_1(x,z,t) a_2(z,y,t\sigma) t V(z) dz d\sigma \right) dt \tag{4.2.6}$$

Let $F(x,y,z,\sigma) := \phi(x,z) + \sigma\phi(z,y)$. Note that $\operatorname{Im} F(x,y,z,\sigma) \geq 0$ and the first derivative of F with respect to the variables z and σ is

$$d_{z,\sigma}F(x,y,z,\sigma) = \left((\frac{\partial\phi(x,z)}{\partial z} + \sigma\frac{\partial\phi(z,y)}{\partial z})dz \qquad \phi_2(z,y)d\sigma \right). \tag{4.2.7}$$

By (2.2.10), $d_{z,\sigma}F(x,y,z,\sigma) = 0$ if and only if $x = y = z$ and $\sigma = 1$. By [22], the phase function ϕ can be written locally as

$$\phi(x,y) = -x_{2n+1} + y_{2n+1} + \frac{i}{2}\sum_{j=1}^{n} \left(|z_j - w_j|^2 + (\bar{z}_j w_j - z_j \bar{w}_j) \right) + O(|(x,y)|^4). \tag{4.2.8}$$

At $(x,y,z,\sigma) = (0,0,0,1)$ we have

$$\det \operatorname{Hess}_{z,\sigma} F(0,0,0,1) = \det \begin{pmatrix} 2i\,\mathrm{I}_{2n} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = 2^{2n}(-1)^{n+1}. \tag{4.2.9}$$

Therefore, there exists $(Z(x,y), \Sigma(x,y)) \in \mathbb{R}^{2n+1} \times \mathbb{R}$ near $(0,1)$ such that

$$d_z\tilde{F}(x,y,Z(x,y),\Sigma(x,y)) = d_\sigma\tilde{F}(x,y,Z(x,y),\Sigma(x,y)) = 0, \tag{4.2.10}$$

where $\tilde{F}$ is the almost extension of F with respect to z and σ. By stationary phase formula of Melin-Sjörstrand [27], we get

$$\int_0^\infty \left(\iint_{D\times\mathbb{R}_+} e^{it F(x,y,z,\sigma)} a_1(x,z,t) a_2(z,y,t\sigma) t V(z) dz d\sigma \right) dt \tag{4.2.11}$$

$$\equiv \int_0^\infty e^{it\tilde{F}(x,y,Z(x,y),\sigma(x,y))} b(x,y,t) dt \tag{4.2.12}$$

where

$$b(x,y,t) \sim \sum_{j=0}^{\infty} b_j(x,y) t^{n_1+n_2-n-j} \in S^{n_1+n_2-n}_{1,0}(D\times D\times \mathbb{R}_+)$$

and

$$b_j(x,y) \in \mathscr{C}^\infty(D\times D), \quad j\in \mathbb{N}_0.$$

Let $\Phi(x,y) = \tilde{F}(x,y,Z(x,y),\Sigma(x,y))$, we notice that $\Phi(x,y) = 0$ if and only if $x = y$, $d_x\Phi(x,x) = d_x\phi(x,x)$ and $d_y\Phi(x,x) = d_y\phi(x,x)$, hence Φ satisfies (2.2.10). $\square$

We define the concept of the leading term of an Szegő type Fourier integral operator:

DEFINITION 4.11. For any $m \in \mathbb{Z}$, let $\sigma^{0,m}_{\bullet}$ be a mapping from the space of Szegő type Fourier integral operator of order $m+n$ to the space of smooth functions on $X\times X$, such that for any $A \in \Psi^{m+n}(X)$ we write

$$\sigma^{0,m}_A(x,y) = a_0(x,y)$$

for the leading term in the expansion of (4.1.5).

Since the space $\Psi^\bullet(X)$ is filtered, with $\Psi^{m_1}(X) \subset \Psi^{m_2}(X)$ whenever $m_1 \le m_2$, the leading term has to be understood with respect to the chosen order. In particular, if an operator B of order $n+m-1$ is regarded as an element of $\Psi^{n+m}(X)$, then its leading term in $\Psi^{n+m}(X)$ is zero, that is, $\sigma^{0,m}_B(x,y) = 0$ for any $x,y\in X$.

To prove Theorem 4.6, we first recall a lemma in [11]:

LEMMA 4.12 ([11, Lemma 4.1]). *Let $A \in \Psi^m(X)$ with $\tilde{S}A \equiv A \equiv A\tilde{S}$ on X. If the leading term of A vanishes on the diagonal, then the leading term of A vanishes. That is $A \in \Psi^{m-1}(X)$.*

From the above lemma, we can show that

LEMMA 4.13. *For any $L \in \mathcal{L}^m(X)$, there exist $l_0 \in \mathscr{C}^\infty(X)$ such that*

$$T_L - T_{l_0 E^m} \in \Psi^{n+m-1}(X). \tag{4.2.13}$$

PROOF. Note that $T_L - T_{l_0E^m} \in \Psi^{n+m}(X)$ satisfies.

$$\tilde{S}(T_L - T_{l_0E^m}) \equiv T_L - T_{l_0E^m} \equiv (T_L - T_{l_0E^m})\tilde{S}. \tag{4.2.14}$$

Now we fix a coordinate patch (D,x) on X. Let $\sigma^0_L(x,\eta(x))$ be the principal symbol of L. It follows from 4.12 that SLS defines a Toeplitz operator on X with

$$\sigma^{0,m}_{SLS}(x,x) = \frac{1}{2\pi^{n+1}}|\det\mathcal{L}_X|\sigma^0_L(x,\xi(x)). \tag{4.2.15}$$

Since $\tilde{S}$ is of Szegő type, the first coefficient of $\tilde{S}$ equals S. Therefore, the first coefficient of $\tilde{S}L\tilde{S}$ equals SLS (by the stationary phase formula of Melin–Sjöstrand [27]), hence

$$\sigma^{0,m}_{\tilde{S}L\tilde{S}}(x,x) = \frac{1}{2\pi^{n+1}}|\det\mathcal{L}_X|\sigma^0_L(x,\xi(x)). \tag{4.2.16}$$

Let $l_0(x) = \sigma^0_L(x,\xi(x))$ then

$$\sigma^{0,m}_{T_{l_0E^m}}(x,x) = \frac{1}{2\pi^{n+1}}|\det\mathcal{L}_X|l_0(x) = \frac{1}{2\pi^{n+1}}|\det\mathcal{L}_X|\sigma^0_L(x,\xi(x)) = \sigma^{0,m}_{T_L}(x,x).$$

The leading term of $T_L - T_{l_0E^m}$ vanishes on the diagonal. Therefore, by Lemma 4.12, we have $T_L - T_{l_0E^m} \in \Psi^{n+m-1}(X)$. $\square$

Proof of Theorem 4.6. Fix a coordinate patch (D,x) on X and let $\sigma^0_L(x,\eta)$ denote the principal symbol of L. Set $f_0(x) := \sigma^0_L(x,\xi(x))$. This choice gives

$$\sigma^{0,m}_{\tilde{S}f_0E^m\tilde{S}}(x,x) = \sigma^{0,m}_{\tilde{S}L\tilde{S}}(x,x).$$

We now need the following lemma:

Lemma 4.14. *For any $A \in \Psi^{m+n}(X)$ such that $\tilde{S}A \equiv A \equiv A\tilde{S}$, there exists $a_0 \in \mathscr{C}^\infty(X)$ such that*

$$A - T_{a_0E^m} \in \Psi^{m+n-1}(X). \tag{4.2.17}$$

Proof of Lemma 4.14. Let $A \in \Psi^{m+n}(X)$, by (4.1.4), there exists $b(x,y,t)$ such that

$$\begin{aligned} & b(x,y,t) \sim \sum_{j=0}^{\infty} b_j(x,y)t^{m+n-j} \in S^m_{1,0}(D\times D\times\mathbb{R}_+), \\ & b_j(x,y) \in \mathscr{C}^\infty(D\times D), j=0,1,\ldots. \end{aligned} \tag{4.2.18}$$

and

$$A(x,y) \equiv \int_0^\infty e^{it\varphi(x,y)}b(x,y,t)dt \tag{4.2.19}$$

$$\equiv \int_0^\infty e^{it\varphi(x,y)}\left(t^{n+m}b_0(x,y) + t^{n+m-1}b_1(x,y) + O(t^{n+m-2})\right). \tag{4.2.20}$$

Let $a_0(x) = b_0(x,x)$, then by [11, Lemma 4.1], $T_{a_0E^m} \in \Psi^{m+n}(X)$ and

$$\sigma^{0,m}_{T_{a_0E^m}}(x,x) = a_0(x) = \sigma^{0,m}_A(x,x).$$

And since

$$\tilde{S}(A - T_{a_0E^m}) \equiv (A - T_{a_0E^m}) \equiv (A - T_{a_0E^m})\tilde{S}, \tag{4.2.21}$$

by Lemma 4.12 we have

$$A - T_{a_0E^m} \in \Psi^{m+n-1}(X). \tag{4.2.22}$$

That is, $\sigma^{0,m}_{A-T_{a_0E^m}}(x,y) = 0$. □

We have $T_{L-f_0E^m} := \tilde{S}(L - f_0E^m)\tilde{S} \in \Psi^{n+m}(X)$,

$$\tilde{S}T_{L-f_0E^m} \equiv T_{L-f_0E^m} \equiv T_{L-f_0E^m}\tilde{S},$$

and $\sigma^0_{T_{L-f_0E^m}}(x,x) = 0$. By Lemma 4.12, we have $T_{L-f_0E^m} \in \Psi^{n+m-1}(X)$. Hence, we get

$$T_{L-f_0E^m}(x,y) \equiv \int_0^\infty e^{it\varphi(x,y)} b(x,y,t)dt \tag{4.2.23}$$

where

$$\begin{gathered} b(x,y,t) \sim \sum_{j=0}^{\infty} b_j(x,y)t^{n+m-1-j} \in S^{n+m-1}_{1,0}(D \times D \times \mathbb{R}_+), \\ b_j(x,y) \in \mathscr{C}^\infty(D \times D), j = 0, 1, \ldots. \end{gathered} \tag{4.2.24}$$

From Lemma 4.14, there exists $f_1(x) = b_0(x,x)$, we have $f_1E^{m-1} \in \mathcal{L}^{m-1}(X)$, $\tilde{S}f_1E^{m-1}\tilde{S} \in \Psi^{n+m-1}(X)$ and

$$\sigma^0_{f_1E^{m-1}}(x,\xi(x)) = f_1(x) = b_0(x,x). \tag{4.2.25}$$

With the same process as above, we get

$$T_{L-f_0E^m-f_1E^{m-1}} \in \Psi^{n+m-2}(X). \tag{4.2.26}$$

Inductively, we can obtain $f_{j-1} \in \mathscr{C}^\infty(X)$ such that

$$T_{L-f_0E^m-f_1E^{m-1}-\cdots-f_{j-1}E^{m-j+1}} \in \Psi^{n+m-j}(X). \tag{4.2.27}$$

Therefore, there exists $f_j \in \mathscr{C}^\infty(X)$ for $j = 0, 1, \ldots$ such that

$$L \sim \sum_{j=0}^{\infty} f_jE^{m-j}. \tag{4.2.28}$$

□

From the Lemma 4.14, we also obtain the asymptotic expansion of certain Fourier integral operators of Szegő type.

COROLLARY 4.15. *If $A \in \Psi^{m+n}(X)$ satisfies $\tilde{S}A \equiv A \equiv A\tilde{S}$, then there exist functions $a_j(x) \in \mathscr{C}^\infty(X)$ and $j = 0, 1, \ldots$ such that for any $N \in \mathbb{N}$*

$$A - \sum_{j=0}^{N-1} T_{a_j E^{m-j}} \in \Psi^{m+n-N}(X).$$

We say that A has an asymptotic sum with respect to $\tilde{S}$, denoted by $A \sim \sum_{j=0}^{\infty} a_j E^{m-j}$.

4.3. Some operator spaces

We denote the space of asymptotic sums of pseudodifferential operators of order m on X by

$$\mathcal{A}_m = \left\{ \sum_{j=0}^{\infty} l_j E^{m-j} : l_j \in \mathscr{C}^\infty(X), j \in \mathbb{N}_0 \right\}.$$

We define the multiplication

$$\cdot : \mathcal{A}_n \times \mathcal{A}_m \longrightarrow \mathcal{A}_{m+n} \tag{4.3.1}$$

$$\sum_{i=0}^{\infty} a_i E^{n-i} \cdot \sum_{j=0}^{\infty} b_j E^{m-j} \longmapsto \sum_{i,j=0}^{\infty} a_i b_j E^{n+m-i-j}. \tag{4.3.2}$$

The multiplication is commutative and hence forms a $\mathbb{Z}$-graded algebra

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n. \tag{4.3.3}$$

Let $\mathcal{L}(X) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n(X)$ denote the graded space of pseudodifferential operators on X. Using the asymptotic expansion of pseudodifferential operators, we define a linear map

$$\mathsf{q} : \mathcal{L}(X) \longrightarrow \mathcal{A} \tag{4.3.4}$$

$$L \longmapsto \sum_{j=0}^{\infty} l_j E^{\operatorname{ord}(L)-j}, \tag{4.3.5}$$

where $\operatorname{ord}(L)$ denotes the order of the operator L such that $T_L - T_{\mathsf{q(L)}} \in \Psi^{-\infty}(X)$.

DEFINITION 4.16. Let L_1 and L_2 be pseudodifferential operators on X. L_1 and L_2 are said to be equivalent if $T_{L_1} - T_{L_2} \in \Psi^{-\infty}(X)$. In this case, we denote the equivalence by $L_1 \sim L_2$.

It is straightforward to verify that $\sim$ defines an equivalence relation. We denote by $\mathbb{L} := \mathcal{L}(X)/\sim$ the quotient space of pseudodifferential operators modulo this equivalence relation.

PROPOSITION 4.17. *The vector space $\mathbb{L}$ is isomorphic to the vector space $\mathcal{A}$.*

PROOF. Let $\underline{q} : \mathbb{L} \to \mathcal{A}$ be the map induced by q such that $\underline{q}([L]) = q(L)$. We first check the well-definiteness of $\underline{q}$. If $[L_1] = [L_2]$, then there exists $l_{1,j}$ and $l_{2,k}$ smooth functions on X for $j, k = 0, 1, \dots$ such that $\underline{q}([L_1]) = \sum_{j=0}^{\infty} l_{1,j} E^{n_1 - j}$ and $\underline{q}([L_2]) = \sum_{k=0}^{\infty} l_{2,k} E^{n_2 - k}$. Since $L_i \sim q(L_i)$ for $i = 1, 2$, we have for any $N \in \mathbb{N}$

$$\sum_{j=0}^{N-1} T_{l_{1,j} E^{n_1 - j}} - \sum_{k=0}^{N-1} T_{l_{2,k} E^{n_2 - k}} \in \Psi^{n + \max\{n_1, n_2\} - N} \tag{4.3.6}$$

When $N = 1$, we have

$$T_{l_{1,0} E^{n_1} - l_{2,0} E^{n_2}} \in \Psi^{n + \max\{n_1, n_2\} - 1} \tag{4.3.7}$$

Since $l_{1,0} E^{n_1} - l_{2,0} E^{n_2} \in \mathcal{L}^{\max\{n_1, n_2\}}(X)$ and $l_{i,0}(x) \neq 0$ for $i = 1, 2.$, without loss of generality, we assume $n_1 \geq n_2$. Then

$$\sigma^{0,n_1}_{T_{l_{1,0} E^{n_1} - l_{2,0} E^{n_2}}}(x, x) = \frac{1}{2\pi^{n+1}} |\det \mathcal{L}_X| (l_{1,0} - l_{2,0})(x) = 0 \tag{4.3.8}$$

implies that $l_{1,0} = l_{2,0}$ and $n_1 = n_2$. Therefore,

$$q(L_1) - q(L_2) = \sum_{j=1}^{\infty} l_{1,j} E^{n_1 - j} - \sum_{k=1}^{\infty} l_{2,k} E^{n_1 - k} \in \mathcal{L}^{n_1}(X). \tag{4.3.9}$$

By (4.3.6), we can repeat the argument to show that $l_{1,j} = l_{2,j}$ for $j \in \mathbb{N}$. Hence, $q(L_1) = q(L_2)$ and $\underline{q}([L_1]) = \underline{q}([L_2])$.
To show the injectivity, let $\underline{q}([L_1]) = \underline{q}([L_2])$. With the same notation above, we have $l_{1,j} = l_{2,j}$ for $j = 0, 1, \dots$. By the definition of the map q, we have

$$T_{L_1} - T_{L_2} = T_{L_1} - T_{q(L_1)} + T_{q(L_1)} - T_{q(L_2)} + T_{q(L_2)} - T_{L_2} \in \Psi^{-\infty}(X). \tag{4.3.10}$$

Thus, $L_1 \sim L_2$.
The surjectivity follows directly from the fact that $q(\sum_{j=0}^{\infty} l_j E^{m-j}) = \sum_{j=0}^{\infty} l_j E^{m-j}$. Therefore, for any $a \in \mathcal{A}$, there exists $m \in \mathbb{Z}$ such that $a \in \mathcal{L}^m(X)$, and $\underline{q}([a]) = q(a) = a$. $\square$

Via the isomorphism $\underline{q}$, we endow $\mathbb{L}$ with a multiplicative structure by setting

$$[L_1] \cdot [L_2] := \underline{q}([L_1]) \cdot \underline{q}([L_2]), \tag{4.3.11}$$

which in turn induces a corresponding multiplicative structure on $\mathcal{L}(X)$:

$$L_1 \cdot L_2 := [L_1] \cdot [L_2] = \underline{q}([L_1]) \cdot \underline{q}([L_2]) = q(L_1) \cdot q(L_2). \tag{4.3.12}$$

We recall the transversal Poisson bracket on $\mathscr{C}^\infty(X)$. For any $f \in \mathscr{C}^\infty(X)$, let X_f be the unique vector field on X satisfying

$$X_f \lrcorner \xi = f \tag{4.3.13}$$

$$X_f \lrcorner d\xi = -(\mathcal{T} \lrcorner df)\xi - df. \tag{4.3.14}$$

We call X_f the Hamiltonian vector field associated to f on X.

DEFINITION 4.18. For any $f, g \in \mathscr{C}^\infty(X)$, the transversal Poisson bracket of f and g is defined by $d\xi(X_f, X_g)$, we denote it by $\{f, g\}_\xi$.

The transversal Poisson bracket induces a Poisson structure on $(\mathcal{L}(X), \cdot)$ in the sense of [28]:

DEFINITION 4.19. We define a bilinear operation

$$\{\,\cdot\,,\,\cdot\,\}_{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

by requiring that, for all $f, g \in \mathscr{C}^\infty(X)$ and $n_1, n_2 \in \mathbb{Z}$,

(1)
$$\{f, g\}_{\mathcal{A}} = i\{\, f\,,\, g\,\}_\xi E^{-1},$$

(2)
$$\{E^{n_1}, f\}_{\mathcal{A}} = n_1(\mathcal{T} f)E^{n_1-1} = -\{f, E^{n_1}\}_{\mathcal{A}},$$

(3)
$$\{E^{n_1}, E^{n_2}\}_{\mathcal{A}} = 0,$$

(4)
$$\{fE^{n_1}, g\}_{\mathcal{A}} = \{f, g\}_{\mathcal{A}} \cdot E^{n_1} + f\{E^{n_1}, g\}_{\mathcal{A}},$$

(5)
$$\{fE^{n_1}, E^{n_2}\}_{\mathcal{A}} = \{f, E^{n_2}\}_{\mathcal{A}} \cdot E^{n_1}.$$

The operation is extended to $\mathcal{A}$ by bilinearity and the Leibniz rule.

PROPOSITION 4.20. *The operation $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{A}}$ defines a Poisson bracket on the commutative algebra $(\mathcal{A}, \cdot)$.*

Via the isomorphism between $\mathcal{A}$ and $\mathcal{L}(X)/\!\sim$ established in Theorem 4.6, the Poisson bracket on $\mathcal{A}$ induces a Poisson bracket on $\mathcal{L}(X)$.

DEFINITION 4.21. Let L_1 and L_2 be pseudodifferential operators on X of orders n_1 and n_2, respectively. We then write

$$\mathrm{q}(L_1) = \sum_{j=0}^{\infty} l_{1,j} E^{n_1-j} \quad \text{and} \quad \mathrm{q}(L_2) = \sum_{k=0}^{\infty} l_{2,k} E^{n_2-k}.$$

The induced Poisson bracket is given by

(4.3.15)

$$\begin{aligned} &\{L_1, L_2\} := \{\mathrm{q}(L_1), \mathrm{q}(L_2)\}_{\mathcal{A}} \\ &= \sum_{j,k=0}^{\infty} \Big(i\{l_{1,j}, l_{2,k}\}_\xi + (n_1 - j)l_{1,j}(\mathcal{T} l_{2,k}) - (n_2 - k)(\mathcal{T} l_{1,j})l_{2,k} \Big) E^{n_1+n_2-j-k-1}. \end{aligned}$$

PROPOSITION 4.22. *The induced operation*

$$\{\,\cdot\,,\,\cdot\,\}\colon \mathcal{L}(X)\times\mathcal{L}(X)\longrightarrow\mathcal{L}(X)$$

defines a Poisson bracket on the commutative algebra $(\,\mathcal{L}(X)\,,\,\cdot\,)$. *Consequently,*

$$(\mathcal{L}(X)\,,\,\cdot\,,\,\{\,\cdot\,,\,\cdot\,\}\,)$$

is a Poisson algebra.

PROOF. This is a direct consequence of Proposition 4.20 together with the isomorphism between $\mathcal{A}$ and $\mathcal{L}(X)/\!\sim$. One readily checks that the Poisson bracket axioms are induced by the Poisson bracket on

$$\Sigma := \{(x, t\xi(x)) \in T^*X : t \in \mathbb{R}\setminus\{0\}\}.$$

□

Let $\tilde{\Psi}^\bullet(X)$ denote the space of Fourier integral operators of Szegő type associated with $\tilde{S}$, in the sense that

$$\tilde{\Psi}^\bullet(X) := \{\, A \in \Psi^\bullet(X) : \tilde{S}A \equiv A \equiv A\tilde{S} \text{ on } X\,\}. \tag{4.3.16}$$

From Corollary 4.15, we also define a map $\hat{\mathfrak{q}} : \tilde{\Psi}(X) \to \mathcal{L}(X)$. Let $A \in \Psi^\bullet(X)$. Then there exists an integer $m \in \mathbb{Z}$ such that $\sigma_A^{0,m}(x,y) \not\equiv 0$, while $\sigma_A^{0,m+j}(x,y) \equiv 0$ for all $j \in \mathbb{N}$. By Definition 4.11, this implies that $A \in \Psi^{n+m}(X)$. We then set

$$\hat{\mathfrak{q}}(A) := \sum_{j=0}^{\infty} a_j E^{m-j}, \tag{4.3.17}$$

where the coefficients a_j are chosen so that $A - T_{\hat{\mathfrak{q}}(A)} \in \Psi^{-\infty}(X)$.

4.4. Quantization of the operator space

THEOREM 4.23 (Theorem 4.8). *Let L_1 and L_2 be pseudodifferential operators of order n_1 and n_2 respectively, then the star product of L_1 and L_2 is an element in $\mathcal{L}^{n_1+n_2}(X)$ defined by*

$$L_1 * L_2 := \hat{\mathfrak{q}}(T_{\mathfrak{q}(L_1)} \circ T_{\mathfrak{q}(L_2)}). \tag{4.4.1}$$

The star product $*$ is associative, consequently, the pair $(\mathcal{L}(X), *)$ is an associative algebra. There exists $C'_j(L_1, L_2) \in \mathscr{C}^\infty(X)$ for $j = 0, 1, \ldots$ such that

$$L_1 * L_2 = \sum_{j=0}^{\infty} C'_j(L_1, L_2) E^{n_1+n_2-j}. \tag{4.4.2}$$

There also exists $C_j(L_1, L_2) \in \mathcal{L}^{n_1+n_2}(X)$ for $j = 0, 1, \ldots$

$$L_1 * L_2 = \sum_{j=0}^{\infty} C_j(L_1, L_2) E^{-j} \tag{4.4.3}$$

where

$$C_0(L_1, L_2) = L_1 \cdot L_2 \tag{4.4.4}$$

and

$$\big(C_1(L_1, L_2) - C_1(L_2, L_1)\big)E^{-1} = \{L_1, L_2\}. \tag{4.4.5}$$

The coefficients $C'_j(L_1, L_2)$ and $C_j(L_1, L_2)$ satisfy the asymptotic relation

$$\sum_{j=0}^{N} C'_j(L_1, L_2)\, E^{n_1+n_2-j} \;-\; \sum_{j=0}^{N} C_j(L_1, L_2)\, E^{-j} \;\in\; \mathcal{L}^{-N-1}(X) \tag{4.4.6}$$

for every $N \in \mathbb{N}_0$.

PROOF. Let L_1 and L_2 in $\mathcal{L}^{n_1}(X)$ and $\mathcal{L}^{n_2}(X)$ respectively. By Theorem 4.6, there exist $\{l_{1,j}\}_{j\in\mathbb{N}_0} \subset \mathscr{C}^\infty(X)$ and $\{l_{2,k}\}_{k\in\mathbb{N}_0} \subset \mathscr{C}^\infty(X)$ such that

$$\mathsf{q}(L_1) = \sum_{j=0}^{\infty} l_{1,j} E^{n_1-j}, \tag{4.4.7}$$

$$\mathsf{q}(L_2) = \sum_{k=0}^{\infty} l_{2,k} E^{n_2-k}. \tag{4.4.8}$$

By direct computation of the composition of Fourier integral operators, for any $N \in \mathbb{N}$ there exists

$$K_{j,k,p}\big(l_{1,j}E^{n_1-j},\, l_{2,k}E^{n_2-k}\big) \in \mathcal{L}^{n_1+n_2-j-k-p}, \qquad p = 0, 1, \dots, N-1,$$

such that

$$T_{l_{1,j}E^{n_1-j}} \circ T_{l_{2,k}E^{n_2-k}} - \sum_{p=0}^{N-1} T_{K_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})} \;\in\; \Psi^{n+n_1+n_2-j-k-N}(X). \tag{4.4.9}$$

Let $K_{j,k}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) := \sum_{p=0}^{\infty} K_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) \in \mathcal{L}^{n_1+n_2-j-k}$ then we have

$$T_{\mathsf{q}(L_1)} \circ T_{\mathsf{q}(L_2)} \equiv \sum_{j,k=0}^{\infty} T_{l_{1,j}E^{n_1-j}} \circ T_{l_{2,k}E^{n_2-k}} \tag{4.4.10}$$

$$\equiv \sum_{j,k=0}^{\infty} T_{K_{j,k}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})} \tag{4.4.11}$$

(4.4.12)

For each pair (j, k), there exists $\mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) \in \mathscr{C}^\infty(X)$, $p = 0, 1, \dots$ such that

$$K_{j,k}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) \sim \sum_{p=0}^{\infty} \mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) E^{n_1+n_2-j-k-p}. \tag{4.4.13}$$

Since
(4.4.14)
$$S \circ K_{j,k}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k}) \circ S \equiv \sum_{p=0}^{\infty} S \circ \mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k-p} \circ S$$

We have

$$\hat{\mathrm{q}}(T_{\mathrm{q}(L_1)} \circ T_{\mathrm{q}(L_2)}) \equiv \sum_{j,k=0}^{\infty} \hat{\mathrm{q}}(T_{K_{j,k}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})}) \tag{4.4.15}$$

$$= \sum_{p=0}^{\infty} \mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k-p} \tag{4.4.16}$$

Let

$$C_p(L_1, L_2) = \sum_{j,k=0}^{\infty} \mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k}. \tag{4.4.17}$$

By Lemma 4.12 and 4.14 , it is not difficult to see that

$$C_0(L_1, L_2) = \sum_{j,k=0}^{\infty} \mathscr{K}_{j,k,0}(l_{1,j}E^{n_1-j}, l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k} \tag{4.4.18}$$

$$= \sum_{j,k=0}^{\infty} l_{1,j}l_{2,k}E^{n_1+n_2-j-k}. \tag{4.4.19}$$

Now, we show that $*$ is associative: Let L_1, L_2, and L_3 in $\mathcal{L}^{\bullet}(X)$ with order n_1, n_2 and n_3 respectively. We recall that if $A \in \Psi^{n+m}(X)$, then there exists $a_j \in \mathscr{C}^{\infty}(X)$ for $j \in \mathbb{N}_0$ such that $\hat{\mathrm{q}}(A) = \sum_{j=0}^{\infty} a_j E^{m-j}$ satisfies $A - T_{\hat{\mathrm{q}}(A)} \in \Psi^{-\infty}(X)$.. Therefore, we have

$$\begin{aligned}(L_1 * L_2) * L_3 &= \hat{\mathrm{q}}(T_{\hat{\mathrm{q}}(T_{\mathrm{q}(L_1)} \circ T_{\mathrm{q}(L_2)})} \circ T_{\mathrm{q}(L_3)}) \\ &\equiv \hat{\mathrm{q}}(T_{\mathrm{q}(L_1)} \circ T_{\mathrm{q}(L_2)} \circ T_{\mathrm{q}(L_3)}) \\ &\equiv \hat{\mathrm{q}}(T_{\mathrm{q}(L_1)} \circ T_{\hat{\mathrm{q}}(T_{\mathrm{q}(L_1)} \circ T_{\mathrm{q}(L_3)})}) \\ &= L_1 * (L_2 * L_3).\end{aligned}$$

Now we compute $C_1(L_1, L_2) - C_1(L_2, L_1)$: with the same notation above, let $\mathrm{q}(L_i) = \sum_{j=0}^{\infty} l_{i,j}E^{n_i-j}$ where n_i is the order of L_i for $i = 1, 2$. Recall that

$$[T_{l_{1,j}E^{n_1-j}}, T_{l_{2,k}E^{n_2-k}}] \in \Psi^{n+n_1+n_2-j-k-1} \tag{4.4.20}$$

and from [28] and the computation in [21, Prop. 3.11], we get

(4.4.21)

$$\sigma^{0,n_1+n_2-j-k-1}_{[T_{l_{1,j}E^{n_1-j}},T_{l_{2,k}E^{n_2-k}}]}(x,x) = \frac{1}{2\pi^{n+1}}|\det \mathcal{L}_X|\big(i\{l_{1,j},l_{2,k}\}_\xi + (n_1-j)l_{1,j}(\mathcal{T}l_{2,k}) - (n_2-k)(\mathcal{T}l_{1,j})l_{2,k}\big)(x).$$

Therefore, there exists $A_0(x) = i\{l_{1,j},l_{2,k}\}_\xi + (n_1-j)l_{1,j}(\mathcal{T}l_{2,k}) - (n_2-k)(\mathcal{T}l_{1,j})l_{2,k}$ such that

$$\hat{\mathsf{q}}([T_{l_{1,j}E^{n_1-j}},T_{l_{2,k}E^{n_2-k}}]) = \sum_{i=0}^{\infty} A_i E^{n_1+n_2-j-k-1-i} \tag{4.4.22}$$

By (4.4.17), we have

$$C_1(L_1,L_2) = \sum_{j,k=0}^{\infty} \mathscr{K}_{j,k,1}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k}. \tag{4.4.23}$$

On the other hand, since $\mathsf{q}(fE^m) = fE^m$ for any $f \in \mathscr{C}^\infty(X)$ and $m \in \mathbb{Z}$, we have

$$\begin{aligned} l_{1,j}E^{n_1-j} * l_{2,k}E^{n_2-k} &= \hat{\mathsf{q}}(T_{l_{1,j}E^{n_1-j}} \circ T_{l_{2,k}E^{n_2-k}}) \\ &= \sum_{p=0}^{\infty} \mathscr{K}_{j,k,p}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k})E^{n_1+n_2-j-k-p}. \end{aligned} \tag{4.4.24}$$

By (4.4.20) and (4.4.22), we have

$$\begin{aligned} \mathscr{K}_{j,k,0}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k}) - \mathscr{K}_{j,k,0}(l_{2,j}E^{n_2-j},l_{1,k}E^{n_1-k}) &= 0, \\ \mathscr{K}_{j,k,1}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k}) - \mathscr{K}_{j,k,1}(l_{2,j}E^{n_2-j},l_{1,k}E^{n_1-k}) &= A_0. \end{aligned} \tag{4.4.25}$$

Since the expansions in (4.4.22) is unique and by the Definition 4.21, we obtain

(4.4.26)

$$\begin{aligned} &\big(C_1(L_1,L_2) - C_1(L_2,L_1)\big)E^{-1} \\ &\quad = \sum_{j,k=0}^{\infty} (\mathscr{K}_{j,k,1}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k}) - \mathscr{K}_{j,k,1}(l_{2,j}E^{n_2-j},l_{1,k}E^{n_1-k}))E^{n_1+n_2-j-k-1} \\ &\quad = \sum_{j,k=0}^{\infty} i\{l_{1,j},l_{2,k}\}_\xi E^{n_1+n_2-j-k-1} \\ &\quad = \{L_1,L_2\} \end{aligned}$$

which establishes the identity (4.4.5). From (4.4.16), we can see that for any $p \in \mathbb{N}_0$

$$C'_p(L_1,L_2) = \sum_{j+k+l=p} \mathscr{K}_{j,k,l}(l_{1,j}E^{n_1-j},l_{2,k}E^{n_2-k}). \tag{4.4.27}$$

Therefore, (4.4.6) follows immediately from (4.4.17) and (4.4.27). □

We conclude this chapter by identifying the analog of the formal parameter in Definition 3.1 and in the operator E:

REMARK 4.24 (Semi-classical meaning). Analogously to the Berezin–Toeplitz star product on compact Kähler manifolds, the formal deformation parameter of the star product plays the role of the tensor power of the prequantum line bundle (See [16,26,31] for details). We recall the result of Berezin-Toeplitz quantization on compact Kähler manifold M (See Sec. 3.1, [3,31] for details). Suppose (L,h) is a holomorphic positive line bundle over M, there exists a star product $*_{BT}$ on M with $f *_{BT} g = \sum_{i=0}^{\infty} k^{-i} C_i(f,g)$ such that for any $N \in \mathbb{N}$ there exist $C_N > 0$ independent of k and

$$\left\| T_f^{(k)} \circ T_g^{(k)} - \sum_{i=0}^{N} k^{-n} T^{(k)}_{C_i(f,g)} \right\| \leq C_N k^{n-N-1}, \tag{4.4.28}$$

where $\|\cdot\|$ is the operator norm on the space of bounded operators on $C^0(M, L^k)$.

From (4.4.28), we can see that the formal parameter comes from the tensor power of L. To connect the result in CR geometry, we need a formal parameter to bridge these two settings. In a certain case, this parameter coincides with the eigenvalue of a circle action:

Put

$$X := \{v \in L^* \mid |v|_{h^*} = 1\} \subset L, \;\; T^{1,0}X := \mathbb{C}TX \cap T^{1,0}L^*. \tag{4.4.29}$$

Here L^* denotes the dual line bundle with metric h^* induced by h. It is well known that X is equipped with a transversal CR S^1-action given by

$$S^1 \times X \ni (e^{i\theta}, v) \mapsto e^{i\theta} \cdot v := e^{i\theta} v \in X. \tag{4.4.30}$$

The generating vector field equals the Reeb vector field on X, we denote it by $\mathcal{T}$. Sections of L^k correspond to S^1-equivariant functions on X, giving the isomorphism

$$\mathscr{C}^\infty(M, L^k) \cong \mathscr{C}^\infty_k(X) := \{f \in \mathscr{C}^\infty(X) : f(e^{i\theta} \cdot x) = e^{ik\theta} f(x)\}. \tag{4.4.31}$$

Acting by $(\mathcal{T}^*\mathcal{T})^{1/2}$ yields eigenvalue k. The operator

$$(\mathcal{T}^*\mathcal{T})^{1/2} : \bigoplus_{k \neq 0} \mathscr{C}^\infty_k(X) \to \bigoplus_{k \neq 0} \mathscr{C}^\infty_k(X) \tag{4.4.32}$$

is bijective and has discrete spectrum $\mathrm{Spec}(\mathcal{T}^*\mathcal{T})^{1/2} = \mathbb{N}$. The Toeplitz quantization in (4.4.28) induces a star product via its asymptotic expansion in the limit of large k, which can equivalently be interpreted as in large eigenvalues of the operator $(\mathcal{T}^*\mathcal{T})^{\frac{1}{2}}$.

On CR manifolds, the operator $(\mathcal{T}^*\mathcal{T})^{1/2}$ need not behave like the generating vector field of a circle action, so we modify it to be elliptic by considering $(I + \Box_b + \mathcal{T}^*\mathcal{T})^{1/2}$ instead.

In our setting, we consider the large-eigenvalue limit of E, analogous to the semi-classical limit for high tensor powers in the Kähler case discussed above. By the definition of $E = (I + \Box_b + \mathcal{T}^*\mathcal{T})^{\frac{1}{2}}$ we have

$$\operatorname{Spec} E = \{0\} \cup \{\lambda_i | \lambda_i \in \mathbb{R},\, 0 < \lambda_i < \lambda_j \text{ if } i < j\}.$$

We define the eigenspace of the operator E associated with the eigenvalue λ by

$$\Gamma_{\lambda_i} := \{\, u \in \mathscr{C}^\infty(X) : Eu = \lambda_i u \,\}.$$

Then we have for any $N \in \mathbb{N}$, there exists $C_N \in \mathbb{R}$ such that

$$\left\| f * g - \sum_{j=0}^{N-1} C_j(f,g) E^{-j} \right\|_{\mathcal{L}(\Gamma_{\lambda_i}, L^2(X))} \leq C_N \lambda_i^{-N} \quad \text{as } i \to \infty. \tag{4.4.33}$$

CHAPTER 5

Quantization on complex manifolds with boundary

In this chapter, we study deformation quantization on a Kähler manifold with a strictly pseudoconvex CR boundary. We apply the Berezin–Toeplitz quantization method to manifolds with boundary. Building on Chapter 4, we show that one can construct an algebra on $\overline{M}$ that admits deformation quantization.

5.1. Introduction and main results

We first introduce the setting in this chapter. Let M be a relatively compact open subset with smooth boundary X of a Kähler manifold M' of real dimension $2n+2$. That is, (M', J, g, ω) is a compact Kähler manifold, where J denotes the complex structure, ω the Kähler form, and g the associated Riemannian metric. Assume that the boundary $X = \partial M$ is a compact, strictly pseudoconvex CR manifold.

Let $\rho \in \mathscr{C}^\infty(M')$ be the defining function of X such that ρ is real, $\rho = 0$ on X, $\rho < 0$ on M and $d\rho \neq 0$ near X. Let $h = g + i\omega$ be the Hermitian metric on $\mathbb{C}TM'$, in local coordinates $z = (z_1, \dots, z_{2n+2})$ we represent the Hermitian metric on $T^{1,0}M'$ by

$$(5.1.1) \qquad (\, u \,|\, v \,) = h(u, \bar{v}), \quad u, v \in T^{1,0}M', \quad h = \sum_{i,j=1}^{n+1} h_{i,j}(z) dz_i \otimes dz_j,$$

where the matrix $(h_{i,j}(z))_{i,j=1}^{2n+2}$ is Hermitian and positive definite. By requiring $T^{1,0}M'$ to be orthogonal to $T^{0,1}M'$ and $\overline{(u \,|\, v)} = (\bar{u} \,|\, \bar{v})$, we extend $(\cdot \,|\, \cdot)$ to $\mathbb{C}TM'$, and by duality also to $\mathbb{C}T^*M$. We also denote $(\,\cdot \,|\, \cdot\,)$ the Hermitian metric on $T^{*0,q}M'$ induced by (5.1.1). Let

$$dM' = \frac{\omega^{n+1}}{(n+1)!}$$

be the volume form and let $(\,\cdot \,|\, \cdot\,)_{M'}$ be the inner product on $\mathscr{C}^\infty(M')$ defined by

$$(\, f \,|\, h \,)_{M'} = \int_{M'} (\, f \,|\, h \,) dM'.$$

Let $(\cdot\,|\,\cdot)_M$ be the inner product on $\mathscr{C}^\infty(\overline{M})$ defined by

$$(f\,|\,h)_M = \int_M (f\,|\,h)dM'$$

for any $f,h \in \mathscr{C}^\infty(\overline{M})$. Since the inner product $(\cdot\,|\,\cdot)$ on $\mathbb{C}TM'$ induces a Hermitian metric on $\mathbb{C}TX$, we use the same notation $(\cdot\,|\,\cdot)$ for the induced structure on $\mathbb{C}TX$. For each point $z' \in X$, we then obtain the identification

$$\mathbb{C}T^*_{z'}X \;=\; \{\alpha \in \mathbb{C}T^*_{z'}M' : (\alpha\,|\,dr) = 0\}, \tag{5.1.2}$$

and for $(0,q)$ fomrs on X we have

$$T^{*0,q}_z X = \{\alpha \in T^{*0,q}_z M' : (\alpha\,|\,\bar\partial\rho(z)\wedge\gamma) = 0, \quad \forall\gamma \in T^{*0,q-1}_z(M')\}. \tag{5.1.3}$$

Let dX be the induced volume form on X and we define the inner product $(\cdot\,|\,\cdot)_X$ on X by

$$(f\,|\,h)_X = \int_X (f\,|\,h)dX$$

for any $f,h \in \mathscr{C}^\infty(X)$. Let $\bar\partial : \mathscr{C}^\infty(M') \to \mathscr{C}^\infty(M', T^{*0,1}M')$ be the exterior differential operator that maps smooth functions to $(0,1)$-forms. By the inner product on M' we define the adjoint operator

$$(\bar\partial f\,|\,g)_{M'} = (f\,|\,\bar\partial^* g)_{M'}.$$

Let $\square = \bar\partial^*\bar\partial + \bar\partial\bar\partial^* : \mathscr{C}^\infty(M', T^{*0,q}M') \to \mathscr{C}^\infty(M', T^{*0,q}M')$ denote the complex Laplace-Beltrami operator on $(0,q)$ forms, when restrict on smooth function we have

$$\square_0 : \bar\partial^*\bar\partial : \mathscr{C}^\infty(M') \to \mathscr{C}^\infty(M')$$

which is the same as the Laplace-Beltrami operator defined by the Riemannian metric

$$\Delta_g = \mathrm{div}_g\,\nabla : \mathscr{C}^\infty(M') \to \mathscr{C}^\infty(M').$$

Let $\gamma : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X)$ be the restriction operator, for any $f \in \mathscr{C}^\infty(\overline{M})$ and $x' \in X$,

$$(\gamma f)(x') = f(x').$$

Let $P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M})$ be the Poisson operator such that $\square Pu \equiv 0$ on $\mathscr{C}^\infty(\overline{M})$ and $\gamma P \equiv I$ on $\mathscr{C}^\infty(X)$ for any $u \in \mathscr{C}^\infty(X)$. The adjoint operator of P is defined by $P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X)$:

$$(P^*f\,|\,g)_X = (f\,|\,Pg)_M, \quad f \in \mathscr{C}^\infty(\overline{M}) \text{ and } g \in \mathscr{C}^\infty(X).$$

We define another inner product $[\cdot\,|\,\cdot]_X$ on X by

$$[u\,|\,v]_X = (Pu\,|\,Pv)_M = ((P^*P)f\,|\,g)_X, \quad u,v \in \mathscr{C}^\infty(X). \tag{5.1.4}$$

We recall here again the setting for Szegő projection on the boundary M: the tangential Cauchy-Riemann operator on X is defined by

$$\bar\partial_b := \pi^{(0,q+1)} \circ d : T^{*0,q}X \to T^{*0,q+1}X, \tag{5.1.5}$$

where $\pi^{(0,p)} : \bigwedge^p \mathbb{C}T^*X \to T^{*0,p}X$ is the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_X$. Let

$$S : L^2(X) \to \operatorname{Ker} \bar{\partial}_b$$

be the Szegő projection operator on X. Let $u \in \mathscr{C}^\infty(X)$, we have

$$(\,\bar{\partial}_b Su \,|\, v\,)_X = 0, \quad \text{for any } v \in \mathbb{C}T^*X.$$

Let

$$\begin{aligned} \bar{\partial}\rho^\wedge : \mathscr{C}^\infty(X, T^{*p,q}M') &\to \mathscr{C}^\infty(X, T^{*p,q+1}M') \\ \alpha &\mapsto (\bar{\partial}\rho) \wedge \alpha \end{aligned} \tag{5.1.6}$$

be the left exterior multiplication map. We denote by $\bar{\partial}\rho^{\wedge,*} : \mathscr{C}^\infty(X, T^{*p,q}M') \to \mathscr{C}^\infty(X, T^{*p,q-1}M')$ the adjoint operator of $\bar{\partial}\rho^\wedge$, which is defined by

$$(\,\bar{\partial}\rho^{\wedge,*}\alpha \,|\, \beta\,) := (\,\alpha \,|\, (\bar{\partial}\rho^\wedge)\beta\,) = (\,\alpha \,|\, \bar{\partial}\rho \wedge \beta\,), \tag{5.1.7}$$

where $\alpha \in \mathscr{C}^\infty(X, T^{*p,q}M')$ and $\beta \in \mathscr{C}^\infty(X, T^{*p,q-1}M')$. By (5.1.3), we see that $\operatorname{Ker}\,(\bar{\partial}\rho^{\wedge,*}|_{T^{*0,q}M'})$ is the space of $(0,q)$ forms on X.

Let $Q : \mathscr{C}^\infty(X, T^{*0,1}M') \to \operatorname{Ker} \bar{\partial}\rho^{\wedge,*}$ be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_X$. and $\hat{Q} : \mathscr{C}^\infty(X, T^{*0,1}M') \to \operatorname{Ker} \bar{\partial}\rho^{\wedge,*}$ be the orthogonal projection with respect to $[\,\cdot\,|\,\cdot\,]_X$. The tangential Cauchy-Riemann operator in (5.1.5) can be realized as

$$\bar{\partial}_b = Q \circ \gamma \circ \bar{\partial} \circ P. \tag{5.1.8}$$

We put

$$\bar{\partial}_\beta = \hat{Q} \circ \gamma \circ \bar{\partial} \circ P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X, T^{*0,1}X). \tag{5.1.9}$$

Let $\bar{\partial}_\beta^\dagger : \mathscr{C}^\infty(X, T^{*0,1}X) \to \mathscr{C}^\infty(X)$ be the formal adjoint of $\bar{\partial}_\beta$ with respect to $[\,\cdot\,|\,\cdot\,]_X$. Define

$$\Box_\beta = \bar{\partial}_\beta^\dagger \bar{\partial}_\beta : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X). \tag{5.1.10}$$

Let $\bar{\partial}_b^*$ be the formal adjoint of $\bar{\partial}_b$ with respect to $(\,\cdot\,|\,\cdot\,)_X$, and the Kohn Laplacian is defined by

$$\Box_b := \bar{\partial}_b^* \bar{\partial}_b : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X). \tag{5.1.11}$$

There exist pseudodifferential operators $\hat{S}, N$ and G such that

$$\begin{aligned} N\Box_\beta + \hat{S} &\equiv I \\ G\Box_b + S &\equiv I \\ \Box_\beta \hat{S} &\equiv 0. \end{aligned} \tag{5.1.12}$$

REMARK 5.1. From [4, Chapter 3] we have P^*P is a classic pseudodifferential operator of order -1. Hence, from (4.1.1) and Definition 2.14, we see that $\hat{S}$ is the Szegő type projection with respect to the inner product $[\,\cdot\,|\,\cdot\,]_X$.

The Bergman projection on M is the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$:

$$B : L^2(M) \longrightarrow \operatorname{Ker} \bar{\partial}.$$

By the results of Hsiao [17], the operator B can be decomposed in the form

$$B = P \circ \hat{S} \circ (P^*P)^{-1} \circ P^*. \tag{5.1.13}$$

For any $f \in \mathscr{C}^\infty(\overline{M})$, we define the associated Toeplitz operator by

$$T_f := B \circ M_f \circ B,$$

here M_f denotes the multiplication operator $(M_f u)(x) = f(x)u(x)$ for $x \in \overline{M}$ and $u \in \mathscr{C}^\infty(\overline{M})$. For brevity, we write f instead of M_f. Substituting the representation (5.1.13) of B into the definition of T_f, we obtain

$$T_f = P \circ \hat{S} \circ (P^*P)^{-1} \circ P^* \circ f \circ P \circ \hat{S} \circ (P^*P)^{-1} \circ P^*. \tag{5.1.14}$$

In order to analyze the asymptotic expansion of the integral kernel of T_f, it is necessary to apply Sjöstrand's stationary phase method to compute the composition of the operators appearing in (5.1.14). The first main result in this chapter is:

THEOREM 5.2. *Let M be a Kähler manifold of real dimension $2n+2$ with compact strictly pseudoconvex CR boundary X. Let $P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M})$ be the Poisson operator such that $\Box P u \equiv 0$ on $\mathscr{C}^\infty(\overline{M})$ and $\gamma P \equiv I$ on $\mathscr{C}^\infty(X)$ for any $u \in \mathscr{C}^\infty(X)$. Then for any $f \in \mathscr{C}^\infty(\overline{M})$, $\mathcal{P}_f := P^* \circ f \circ P$ is a pseudodifferential operator.*

To show Theorem 5.2, we first show that:

PROPOSITION 5.3. *There exists a Fourier integral operator $\widehat{P} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M})$ such that for any $u \in \mathscr{C}^\infty(X)$, for $x \in \overline{M}$ we have*

$$\begin{aligned}(\widehat{P}u)(x) &= \int_X \widehat{P}(x,y')u(y')dy' \\ &= \iint_{X\times\mathbb{R}^{2n+1}} e^{i\varphi(x,y',\eta')}a(x,y',\eta')u(y')d\eta' dy'\end{aligned} \tag{5.1.15}$$

where φ is a smooth function and homogeneous of degree 1 in η' and $a(x,y',\eta') \sim \sum_{j=0}^\infty a_j(x,y',\eta')$, such that $a_j(x,y',\eta')$ is a smooth function and homogeneous of degree $-j$ in η'. $\widehat{P}$ satisfies

$$\begin{aligned}\Box\widehat{P}(x,y') \equiv 0 \qquad &\operatorname{mod} \mathscr{C}^\infty(\overline{M}\times X) \\ \gamma\widehat{P} = I.&\end{aligned} \tag{5.1.16}$$

Let $\hat{\Psi}(X)$ be the space of Szegő type Fourier integral operators associated with $\hat{S}$, that is

$$\hat{\Psi}(X) := \{A \in \Psi(X) : \hat{S}A \equiv A \equiv A\hat{S} \text{ on } X\}. \tag{5.1.17}$$

Let $F : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M})$ be linear. We define $\mathcal{J}_F := (P^*P)^{-1}P^*FP$. We say F is reducible to the boundary if $\mathcal{J}_F \in \mathcal{L}(X)$. We call F a Fourier integral operator of Szegő type if it is reducible to the boundary and satisfies $\mathcal{J}_F \in \hat{\Psi}(X)$. Let

$$\Psi(\overline{M}) := \{F : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}) : \mathcal{J}_F \in \hat{\Psi}(X)\}$$

denote the space of Fourier integral operators of Szegő type. We also denote $\Psi^m(\overline{M}) := \{F : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}) : \mathcal{J}_F \in \hat{\Psi}^m(X)\}$.

Parallel to quantization on CR manifolds, we obtain the result for Toeplitz operators that allows one to apply the machinery of Berezin-Toeplitz quantization to obtain the deformation quantization on $\overline{M}$:

THEOREM 5.4. *Let M be a Kähler manifold of real dimension $2n+2$ with compact strictly pseudoconvex CR boundary X. Let $f \in \mathscr{C}^\infty(\overline{M})$ and B be the Bergman projection on M, then the Toeplitz operator $T_f = B \circ f \circ B$ is a Fourier integral operator and $T_f \in \Psi(\overline{M})$. Consequently, if $g \in \mathscr{C}^\infty(\overline{M})$, then $T_f T_g \in \Psi(\overline{M})$.*

In Chapter 4, we have defined

$$E = (I + \Box_b + \mathcal{T}^*\mathcal{T})^{\frac{1}{2}} \in \mathcal{L}^1(X).$$

Now we put

$$\hat{E} = PE(P^*P)^{-1}P^*. \tag{5.1.18}$$

Let

$$\mathcal{H} := \{f \in \mathscr{C}^\infty(\overline{M}) : \Box f = 0\}. \tag{5.1.19}$$

be the space of harmonic functions on $\overline{M}$. We have the following proposition that is analogous to the CR setting

PROPOSITION 5.5. *For any $f \in \mathscr{C}^\infty(\overline{M})$, there exists $\hat{f}_j \in \mathcal{H}$ for $j = 0, 1, 2, \cdots$ such that*

$$T_f - \sum_{j=0}^{N-1} T_{\hat{f}_j \hat{E}^{-j}} \in \Psi^{n-N}(\overline{M}). \tag{5.1.20}$$

This can be further generalized to

COROLLARY 5.6. *Let $L \in \mathcal{L}^m(X)$ and define*

$$\hat{L} := PL(P^*P)^{-1}P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}).$$

Then there exist functions $\hat{l}_j \in \mathcal{H}$ such that, for every $N \in \mathbb{N}$,

$$T_{\hat{L}} - \sum_{j=0}^{N-1} T_{\hat{l}_j \hat{E}^{m-j}} \in \Psi^{n+m-N}(\overline{M}). \tag{5.1.21}$$

We define

$$\hat{\mathcal{H}} := \left\{ \hat{L} : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}) : (P^*P)^{-1}P^*\hat{L}P \in \mathcal{L}(X) \right\}, \tag{5.1.22}$$

the space of operators on $\overline{M}$ for which $\mathcal{J}_{\hat{L}}$ is a pseudodifferential operator on X. Set

$$\hat{\mathcal{Q}} := \left\{ \hat{L} \in \hat{\mathcal{H}} : \hat{L} = \sum_{j=0}^{\infty} \hat{l}_j \hat{E}^{m-j},\ \hat{l}_j \in \mathcal{H}\, \forall j \in \mathbb{N}_0,\ m \in \mathbb{Z} \right\}.$$

$\hat{\mathcal{Q}}$ can be viewed as the duality part of $\mathcal{A}$ (see Sec. 4.3) in $\overline{M}$ (see Proposition 5.24.) Let $\mathscr{Q}$ be the map

$$\begin{aligned} \mathscr{Q} : \hat{\mathcal{H}} &\to \hat{\mathcal{Q}} \\ \hat{L} &\mapsto \sum_{j=0}^{\infty} \hat{l}_j \hat{E}^{\operatorname{ord}(\mathcal{J}_L)-j}, \end{aligned} \tag{5.1.23}$$

where $\hat{l}_j$ is given by Corollary 5.6. We obtain the quantization result on $\overline{M}$:

THEOREM 5.7. *Let F, $G \in \hat{\mathcal{Q}}$, the star product $*_{\mathcal{H}}$ of F and G is defined by*

$$F *_{\mathcal{H}} G = \mathscr{Q}\left(\widehat{\mathcal{J}_F * \mathcal{J}_G}\right), \tag{5.1.24}$$

where $\widehat{\mathcal{J}_F * \mathcal{J}_G} = P \circ (\mathcal{J}_F * \mathcal{J}_G) \circ (P^*P)^{-1}P^* \in \hat{\mathcal{H}}$. The product $*_{\mathcal{H}}$ is associative, and if $F = \sum_{j=0}^{\infty} \hat{f}_j \hat{E}^{m_1-j}$ and $G = \sum_{k=0}^{\infty} \hat{g}_k \hat{E}^{m_2-k}$ there exists $\hat{C}_j(F,G) \in \mathcal{H}$ such that

(1) $F *_{\mathcal{H}} G = \sum_{j=0}^{\infty} \hat{C}_j(F,G)\hat{E}^{m_1+m_2-j}$

(2) $\hat{C}_0(F,G) = \hat{C}_0(G,F) = h(\hat{f}_0, \hat{g}_0)$,

(3) $\left(\hat{C}_1(F,G) - \hat{C}_1(G,F)\right)\hat{E}^{-1} = \{\hat{f}_0, \hat{g}_0\}^{\mathcal{H}}$,

If $F = f$ and $G = g$ in $\mathscr{C}^\infty(\overline{M})$, we deform the algebra $(\mathcal{H}, h(\,\cdot\,,\,\cdot\,), \{\,\cdot\,,\,\cdot\,\}^{\mathcal{H}})$ inside $\hat{\mathcal{Q}}$. Analogously to Theorem 4.23, we introduce an algebraic product and a Poisson bracket on $\hat{\mathcal{Q}}$ ((5.3.32) and (5.3.33)). For the deformation quantization on $(\hat{\mathcal{Q}}, h(\,\cdot\,,\,\cdot\,), \{\,\cdot\,,\,\cdot\,\}^{\mathcal{H}})$, we then obtain that the commutator recovers the underlying Poisson structure:

THEOREM 5.8. *For any $F, G \in \hat{\mathcal{Q}}$ satisfying (5.3.31), there exists a sequence $\hat{C}_j(F,G) \in \hat{\mathcal{Q}}$ with $\operatorname{ord}(\mathcal{J}_{\hat{C}_j(F,G)}) = m_1 + m_2$ for all j, such that for any $N \in \mathbb{N}$,*

$$T_{F *_{\mathcal{H}} G} - \sum_{p=0}^{N-1} T_{\hat{C}_j(F,G)\hat{E}^{-j}} \in \Psi^{n+m_1+m_2-N}(\overline{M}), \tag{5.1.25}$$

where

$$\begin{aligned} \hat{C}_0(F,G) &= h(F,G), \\ \left(\hat{C}_1(F,G) - \hat{C}_1(G,F)\right)\hat{E}^{-1} &= \{\, F\,,\, G\,\}^{\mathcal{H}}. \end{aligned} \tag{5.1.26}$$

REMARK 5.9 (Conceptual summary). The overall construction on complex manifolds with boundary can be summarized as follows.

$$
\begin{array}{ccc}
P \text{ is FIO near } X & \xrightarrow{\text{Thm. 5.2}} & P^* f P \text{ is } \Psi\text{DO} \\
 & & \Downarrow \text{ Thm. 5.4} \\
 & & B = P\hat{S}(P^*P)^{-1}P^* \longrightarrow BfB \text{ is FIO} \\
 & & \Downarrow \text{ Chp. 4} \\
\text{Reduction to boundary via } P & \xrightarrow{\text{Thm. 5.8}} & \hat{Q} \text{ carries a star product.}
\end{array}
$$

5.2. Poisson operator

From (5.1.14), we can see that the computation of the symbol of T_f requires a detailed analysis of the Poisson operator. Consequently, in this section we present a step-by-step proof of Theorem 5.4.

To show 5.4, we first prove that

THEOREM 5.10 (Theorem 5.2). *$\mathcal{P}_f := P^* \circ f \circ P$ is a pseudodifferential operator.*

To prove Theorem 5.10 we have to study the Poisson operator P and its integral kernel. For the point near the boundary, we have the following result for the Poisson operator:

PROPOSITION 5.11 (Proposition 5.3). *There exists a Fourier integral operator $\widehat{P} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M})$ such that for any $u \in \mathscr{C}^\infty(X)$, for $x \in \overline{M}$ we have*

$$
\begin{aligned}
(\widehat{P}u)(x) &= \int_X \widehat{P}(x,y')u(y')dy' \\
&= \iint_{X\times\mathbb{R}^{2n+1}} e^{i\varphi(x,y',\eta')}a(x,y',\eta')u(y')d\eta'dy'
\end{aligned}
\tag{5.2.1}
$$

where φ is a smooth function and homogeneous of degree 1 in η' and $a(x,y',\eta') \sim \sum_{j=0}^\infty a_j(x,y',\eta')$, such that $a_j(x,y',\eta')$ is a smooth function and homogeneous of degree $-j$ in η'. $\widehat{P}$ satisfies

$$
\begin{aligned}
\Box\widehat{P}(x,y') &\equiv 0 \qquad \text{mod } \mathscr{C}^\infty(\overline{M}\times X) \\
\gamma\widehat{P} &= I.
\end{aligned}
\tag{5.2.2}
$$

PROOF. The proof strategy is to construct the phase function and symbols locally via Borel's lemma, then patch these local constructions using Lemma 5.12.

Step 1: local construction. First, we work in a collar neighborhood of the boundary $X = \partial M$. Let $r = \mathrm{dist}_g(\cdot, X)$ denote the g-distance to the boundary. Then there exist $\varepsilon > 0$ and a diffeomorphism

$$\Phi : X \times [0, \varepsilon) \to \widetilde{X} \subset M, \qquad \Phi(x, r) = \exp_x(r\nu_x),$$

where ν is the outward unit normal vector field along X. We refer to $\widetilde{X}$ as a collar neighborhood of X in M. In the induced coordinate system $(x, r) \in X \times [0, \varepsilon)$, the Riemannian metric takes the form

$$g = dr^2 + g_r, \tag{5.2.3}$$

where $\{g_r\}_{r\in[0,\varepsilon)}$ denotes a smooth one-parameter family of Riemannian metrics on X. Furthermore, the radial vector field ∂_r is everywhere orthogonal to the tangential directions, that is,

$$g(\partial_r, V) = 0 \quad \text{for all } V \in T_x X,$$

and in particular,

$$g(\partial_r, \partial_r) = 1 \quad \text{on } \widetilde{X}.$$

Let $X = \bigcup_{i=1}^N D_i$ be a finite covering of X by coordinate patches, where each $D_i \subseteq X$ is open. Without loss of generality, we may assume that $\widetilde{X} = \bigcup_{i=1}^N U_i$, where each U_i is given by

$$U_i = D_i \times [0, \varepsilon)$$

for some fixed $\varepsilon > 0$. For $x \in U_i$ and $y' \in D_i$, let $x = (x_1, x_2, \ldots, x_{2n+1}, r) = (x', r)$ and $y' = (y_1, \ldots, y_{2n+1})$ be the local coordinates. First, we define $\varphi_0(x', y', \eta') = \langle x' - y', \eta' \rangle$. Consider the equation $\Delta_g P = 0$, in the local coordinates, the Laplacian can be written as

$$\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right).$$

On $U_i \subset \widetilde{X}$, by (5.2.3) we have

$$\Delta_g = \frac{\partial^2}{\partial r^2} + \sum_{i,j=1}^{2n+1}\left(g_r^{ij}\frac{\partial^2}{\partial x_i \partial x_j}\right) + b(x', r)\frac{\partial}{\partial r} + \sum_{i=1}^{2n+1}\left(c^i(x', r)\frac{\partial}{\partial x_i}\right), \tag{5.2.4}$$

where $b(x', r) = \frac{1}{2}\frac{\partial}{\partial r}\log|g_r|$ and $c^i(x', r) = \sum_{j=1}^{2n+1}\left(\frac{\partial}{\partial x_j}g_r^{ji} + \frac{1}{2}g_r^{ji}\frac{\partial}{\partial x_j}(\log|g_r|)\right)$.

We will show below that there exist $\tilde{\varphi}(x, y', \eta')$ and $\tilde{a}(x, y', t')$ such that $\tilde{\varphi}$ is smooth and homogeneous with degree 1 in η' and $\tilde{a} \sim \sum_{j=0}^{\infty}\tilde{a}_j(x, y', \eta')$ where $\tilde{a}_j$ is smooth and homogeneous with degree $-j$ in η' and such that they satisfy

$$\Delta_g\left(e^{i\tilde{\varphi}(x,y',\eta')}\tilde{a}(x, y', \eta')\right) \equiv 0. \tag{5.2.5}$$

Which is equivalent to

(5.2.6)
$$\sum_{i,j=1}^{2n+1} g_r^{ij}\left[\left(-\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{\varphi}}{\partial x_j}+\sqrt{-1}\frac{\partial^2\tilde{\varphi}}{\partial x_i\partial x_j}\right)\tilde{a}+2\sqrt{-1}\left(\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{a}}{\partial x_j}\right)+\frac{\partial^2\tilde{a}}{\partial x_i\partial x_j}\right]$$
$$+\left[\left(-\frac{\partial\tilde{\varphi}}{\partial r}\frac{\partial\tilde{\varphi}}{\partial r}+\sqrt{-1}\frac{\partial^2\tilde{\varphi}}{\partial r\partial r}\right)\tilde{a}+2\sqrt{-1}\left(\frac{\partial\tilde{\varphi}}{\partial r}\frac{\partial\tilde{a}}{\partial r}\right)+\frac{\partial^2\tilde{a}}{\partial r\partial r}\right]$$
$$+\sqrt{-1}\left(b(x',r)\frac{\partial\tilde{\varphi}}{\partial r}+\sum_{i=1}^{2n+1}c^i(x',r)\frac{\partial\tilde{\varphi}}{\partial x_i}\right)\tilde{a}+b(x',r)\frac{\partial\tilde{a}}{\partial r}+\sum_{i=1}^{2n+1}c^i(x',r)\frac{\partial\tilde{a}}{\partial x_i}=0.$$

The equation (5.2.6) can be separated according to the homogeneous degree of η'. We first consider the part of homogeneous degree 2. Therefore,

$$\sum_{i,j=1}^{2n+1} g_r^{ij}\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{\varphi}}{\partial x_j}+\left(\frac{\partial\tilde{\varphi}}{\partial r}\right)^2=0. \tag{5.2.7}$$

Let

$$\Phi((x',r))=\sum_{i=0}^{\infty}\tilde{\varphi}_i(x',y',\eta')\,r^i \tag{5.2.8}$$

be a formal power series solution of (5.2.7). This is equivalent to constructing the coefficients $\tilde{\varphi}_j$ so that they satisfy

$$\sum_{p=0}^{\infty}\sum_{i,j=1}^{2n+1} g_{r,p}^{ij}r^p\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}+\left(\frac{\partial\Phi}{\partial r}\right)^2=f_0+f_1r+f_2r^2+\cdots, \tag{5.2.9}$$

under the assumption that $f_j=0$ for all j. Here, $g_{r,p}^{ij}$ denotes the (i,j)-component of the $(0,2)$-tensor appearing as the coefficient of r^p in the Taylor expansion of the metric g_r with respect to the normal coordinate r at $r=0$ along the boundary. Explicitly,

$$g_r^{ij}=g_{r,0}^{ij}+g_{r,1}^{ij}\,r+g_{r,2}^{ij}\,r^2+\cdots. \tag{5.2.10}$$

By choosing normal coordinates, we obtain $g_{r,0}^{ij}=\delta^{ij}$. We first define

$$\tilde{\varphi}_0(x',y',\eta')\;=\;\varphi_0(x',y',\eta')\;=\;\langle x'-y',\eta'\rangle.$$

In the case $f_0=0$, the relation

$$\sum_{i,j=1}^{2n+1} g_{r,0}^{ij}\,\eta_i'\eta_j'\;+\;\tilde{\varphi}_1^2\;=\;0$$

holds, which implies that

$$\tilde{\varphi}_1(x',y',\eta')\;=\;\sqrt{-1}\,|\eta'|,$$

where we have introduced the notation

$$\sum_{i,j=1}^{2n+1} \delta^{ij}\, \eta_i' \eta_j' \;=\; |\eta'|^2.$$

Similarly, in the case $f_1 = 0$, the relation

$$\sum_{i,j=1}^{2n+1} (g_{r,1}^{ij} \eta_i' \eta_j') + 4\tilde{\varphi}_1 \tilde{\varphi}_2 = 0$$

holds, which implies that

$$\tilde{\varphi}_2(x', y', \eta') = -\sum_{i,j=1}^{2n+1} \frac{g_{r,1}^{ij} \eta_i' \eta_j'}{2\tilde{\varphi}_1}.$$

Inductively, in the case $f_m = 0$, we have the relation

(5.2.11)
$$\sum_{i,j=0}^{2n+1} \sum_{p=0}^{m} \sum_{k=0}^{m-p} \left(g_{r,p}^{ij} \frac{\partial \tilde{\varphi}_k}{\partial x_i} \frac{\partial \tilde{\varphi}_{m-p-k}}{\partial x_j} \right) + 2(m+1)\tilde{\varphi}_1 \tilde{\varphi}_{m+1} + \sum_{k=1}^{m-1} (m-k+1)(k+1)\tilde{\varphi}_{k+1} \tilde{\varphi}_{m-k+1} = 0,$$

which implies that

(5.2.12)
$$\tilde{\varphi}_{m+1}(x', y', \eta') = \frac{\sum_{i,j=0}^{2n+1} \sum_{p=0}^{m} \sum_{k=0}^{m-p} \left(g_{r,p}^{ij} \frac{\partial \tilde{\varphi}_k}{\partial x_i} \frac{\partial \tilde{\varphi}_{m-p-k}}{\partial x_j} \right) + \sum_{k=1}^{m-1} (m-k+1)(k+1)\tilde{\varphi}_{k+1} \tilde{\varphi}_{m-k+1}}{-2(m+1)\tilde{\varphi}_1}.$$

Therefore, the coefficients of Φ in (5.2.8) can be solved recursively. By Borel's lemma, given a sequence of smooth functions $\{\tilde{\varphi}_i(x', y', \eta')\}_{i \in \mathbb{N}_0} \subset \mathscr{C}^\infty(D_i \times D_i \times \mathbb{R}^{2n+1})$, there exists a function $\tilde{\varphi}((x', r), y', \eta') \in \mathscr{C}^\infty(U_i \times D_i \times \mathbb{R}^{2n+1})$ with $d\tilde{\varphi} \neq 0$ and $\operatorname{Im} \tilde{\varphi} \geq 0$ such that

$$\frac{\partial^n \tilde{\varphi}}{\partial r^n}((x', 0), y', \eta') = \tilde{\varphi}_n(x', y', \eta') \quad \text{for all } n \in \mathbb{N}_0.$$

By examining the Taylor expansion of $\tilde{\varphi}$ at $r = 0$, one verifies that $\tilde{\varphi}$ satisfies equation (5.2.7). Our next step is to consider the homogeneous degree 1 part of

(5.2.6) which is

(5.2.13)
$$\sum_{i,j=1}^{2n+1} g_r^{ij}\sqrt{-1}\left(\frac{\partial^2\tilde{\varphi}}{\partial x_i\partial x_j}\tilde{a}_0+2\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{a}_0}{\partial x_j}\right)$$
$$+\sqrt{-1}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}\tilde{a}_0+2\frac{\partial\tilde{\varphi}}{\partial r}\frac{\partial\tilde{a}_0}{\partial r}+b(x',r)\frac{\partial\tilde{\varphi}}{\partial r}\tilde{a}_0+\sum_{i=0}^{2n+1}c^i(x',r)\frac{\partial\tilde{\varphi}}{\partial x_i}\tilde{a}_0\right)=0.$$

Let $b_p, c_p^i \in \mathscr{C}^\infty(U_i)$ for $p \in \mathbb{N}_0$ denote the p-th Taylor coefficients in the expansion of $b(x',r)$ and $c^i(x',r)$ with respect to the variable r at $r=0$. Similarly, we consider the formal power series

$$A_0(x',r)=\sum_{i=0}^{\infty}\tilde{a}_{0,i}(x',y',\eta')r^i \tag{5.2.14}$$

which satisfies

(5.2.15)
$$\sum_{m=0}^{\infty} g_{0,m}r^m=\sum_{p=0}^{\infty}\sum_{i,j=1}^{2n+1} g_{r,p}^{ij}r^p\sqrt{-1}\left(\frac{\partial^2\tilde{\varphi}}{\partial x_i\partial x_j}A_0+2\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial A_0}{\partial x_j}\right)$$
$$+\sqrt{-1}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}A_0+2\frac{\partial\tilde{\varphi}}{\partial r}\frac{\partial A_0}{\partial r}+\sum_{p=0}^{\infty}b_pr^p\frac{\partial\tilde{\varphi}}{\partial r}A_0+\sum_{p=0}^{\infty}\sum_{i=0}^{2n+1}c_p^ir^p\frac{\partial\tilde{\varphi}}{\partial x_i}A_0\right)$$

under the assumption that $g_{0,m}=0$ for all $m \in \mathbb{N}_0$. We first define

$$\tilde{a}_{0,0}(x',y',\eta')=1, \tag{5.2.16}$$

by assuming $g_{0,0}=0$, we have the relation

$$2\tilde{\varphi}_2\tilde{a}_{0,0}+2\tilde{\varphi}_1\tilde{a}_{0,1}+b_0\tilde{\varphi}_1\tilde{a}_{0,0}+\sum_{i=0}^{2n+1}c_0^i\eta_i'=0.$$

Therefore, we get

$$\tilde{a}_{0,1}(x',y',\eta')=-\frac{b_0\tilde{\varphi}_1+2\tilde{\varphi}_2+\sum_{i=1}^{2n+1}c_0^i\eta_i'}{2\tilde{\varphi}_1}. \tag{5.2.17}$$

Similarly, in the case $g_{0,1}=0$ we have the relation

$$\begin{aligned}&\sum_{i=1}^{2n+1}\left(\frac{\partial^2\tilde{\varphi}_1}{\partial x_i^2}+2\eta_i'\frac{\partial\tilde{a}_{0,1}}{\partial x_i}\right)+6\tilde{\varphi}_3+6\tilde{\varphi}_2\tilde{a}_{0,1}\\&\quad+4\tilde{a}_{0,2}\tilde{\varphi}_1+b_0(2\tilde{\varphi}_2+\tilde{\varphi}_1\tilde{a}_{0,1})+b_1\tilde{\varphi}_1\\&\quad+\sum_{i=1}^{2n+1}\left(c_0^i\left(\frac{\partial\tilde{\varphi}_1}{\partial x_i}+\eta_i'\tilde{a}_{0,1}\right)+c_1^i\eta_i'\right)=0\end{aligned} \tag{5.2.18}$$

which implies

(5.2.19)
$$\tilde{a}_{0,2} = -\frac{1}{4\tilde{\varphi}_1} \cdot \Big(\sum_{i=1}^{2n+1} \left(\frac{\partial^2 \tilde{\varphi}_1}{\partial x_i^2} + 2\eta_i' \frac{\partial \tilde{a}_{0,1}}{\partial x_i} \right) + 6\tilde{\varphi}_3 + 6\tilde{\varphi}_2 \tilde{a}_{0,1} + b_0(2\tilde{\varphi}_2 + \tilde{\varphi}_1 \tilde{a}_{0,1}) + b_1 \tilde{\varphi}_1 + \sum_{i=1}^{2n+1} \left(c_0^i \left(\frac{\partial \tilde{\varphi}_1}{\partial x_i} + \eta_i' \tilde{a}_{0,1} \right) + c_1^i \eta_i' \right) \Big).$$

Similarly, in the case $g_{0,m} = 0$ we have the relation

(5.2.20)
$$0 = \sum_{p=0}^{m} \sum_{i,j=1}^{2n+1} \sum_{k=0}^{m-p} g_{r,p}^{ij} \Big(\frac{\partial^2 \tilde{\varphi}_{m-p-k}}{\partial x_i \partial x_j} \tilde{a}_{0,k} + 2 \frac{\tilde{\varphi}_{m-p-k}}{\partial x_i} \frac{\partial \tilde{a}_{0,k}}{\partial x_j} \Big)$$
$$+ \sum_{k=0}^{m} \Big((m+2-k)(m+1-k) \tilde{\varphi}_{m+2-k} \tilde{a}_{0,k} + 2(m+1-k)(k+1) \tilde{\varphi}_{m+1-k} \tilde{a}_{0,k+1} \Big)$$
$$+ \sum_{p=0}^{m} \sum_{k=0}^{m-p} b_p (m+1-k) \tilde{\varphi}_{m+1-k} \tilde{a}_{0,k} + \sum_{p=0}^{m} \sum_{i=1}^{2n+1} \sum_{k=0}^{m-p} c_p^i \frac{\partial \tilde{\varphi}_{m-k}}{\partial x_i} \tilde{a}_{0,k}$$

which implies

$$\begin{aligned} \tilde{a}_{0,m+1} = & -\frac{1}{2(m+1)\tilde{\varphi}_1} \cdot \\ & \Big(\sum_{p=0}^{m} \sum_{i,j=1}^{2n+1} \sum_{k=0}^{m-p} g_{r,p}^{ij} \Big(\frac{\partial^2 \tilde{\varphi}_{m-p-k}}{\partial x_i \partial x_j} \tilde{a}_{0,k} + 2 \frac{\tilde{\varphi}_{m-p-k}}{\partial x_i} \frac{\partial \tilde{a}_{0,k}}{\partial x_j} \Big) \\ & + \sum_{k=0}^{m} \Big((m+2-k)(m+1-k) \tilde{\varphi}_{m+2-k} \tilde{a}_{0,k} \Big) \\ & + 2 \sum_{k=0}^{m-1} \Big((m+1-k)(k+1) \tilde{\varphi}_{m+1-k} \tilde{a}_{0,k+1} \Big) \\ & + \sum_{p=0}^{m} \sum_{k=0}^{m-p} b_p (m+1-k) \tilde{\varphi}_{m+1-k} \tilde{a}_{0,k} \\ & + \sum_{p=0}^{m} \sum_{i=1}^{2n+1} \sum_{k=0}^{m-p} c_p^i \frac{\partial \tilde{\varphi}_{m-k}}{\partial x_i} \tilde{a}_{0,k} \Big). \end{aligned} \tag{5.2.21}$$

Therefore, we can obtain $\tilde{a}_{0,m}$ recursively for all $m \in \mathbb{N}_0$. By Borel's lemma, there exists a smooth function $\tilde{a}_0((x',r),y',\eta') \in \mathscr{C}^\infty(U_i \times D_i \times \mathbb{R}^{2n+1})$ such that

$$\frac{\partial^n \tilde{a}_0}{\partial r^n}((x',0),y',\eta') = \tilde{a}_{0,n}(x',y',\eta') \quad \text{for all } n \in \mathbb{N}_0.$$

By examining the Taylor expansion of $\tilde{\varphi}$ at $r=0$, it follows that $\tilde{\varphi}$ satisfies equation (5.2.13). Note that through (5.2.21), we can see that $\tilde{a}_{0,j}$ does not depend on y', hence we conclude that $\tilde{a}_0((x',r),\eta') \in \mathscr{C}^\infty(U_i \times \mathbb{R}^{2n+1})$ We apply the same method above to solve for $\tilde{a}_l$ be considering the homogeneous degree $1-l$ part of (5.2.6). For $l \geq 1$, we always define $\tilde{a}_{l,0}(x',y',\eta') = 0$. We have the recurence relation for $\tilde{a}_{l,m+1}$ for every $l,m \in \mathbb{N}_0$:

(5.2.22)

$$\begin{aligned}
\tilde{a}_{l,m+1} = &-\frac{1}{2(m+1)\tilde{\varphi}_1}\cdot \\
&\Bigg\{ \sum_{p=0}^{m} \sum_{i,j=1}^{2n+1} \sum_{k=0}^{m-p} g^{ij}_{r,p} \Big[\sqrt{-1} \big(\frac{\partial^2 \tilde{\varphi}_{m-p-k}}{\partial x_i \partial x_j} \tilde{a}_{l,k} + 2 \frac{\partial \tilde{\varphi}_{m-p-k}}{\partial x_i} \frac{\partial \tilde{a}_{l,k}}{\partial x_j} \big) + \frac{\partial^2 \tilde{a}_{l-1,k-p}}{\partial x_i \partial x_j} \Big] \\
&+ \sqrt{-1} \Big(\sum_{k=0}^{m} (m+2-k)(m+1-k) \tilde{\varphi}_{m+2-k} \tilde{a}_{l,k} \\
&\qquad + \sum_{k=0}^{m-1} 2(m+1-k)(k+1) \tilde{\varphi}_{m+1-k} \tilde{a}_{l,k+1} + (m+2)(m+1) \tilde{a}_{l-1,m+2} \Big) \\
&+ \sqrt{-1} \sum_{p=0}^{m} \big(\sum_{k=0}^{m-p} \tilde{a}_{l,k} \Big(b_p (m-p+1-k) \tilde{\varphi}_{m-p+1-k} + \sum_{i=1}^{2n+1} c^i_p \frac{\partial \tilde{\varphi}_{m-p-k}}{\partial x_i} \big) \Big) \\
&+ \sum_{p=0}^{m} \Big(b_p (m+p-1) \tilde{a}_{l-1,m+1-p} + \sum_{i=1}^{2n+1} c^i_p \frac{\partial \tilde{a}_{l-1,m-p}}{\partial x_i} \Big) \Bigg\}.
\end{aligned}$$

Since $\tilde{a}_0$ does not depend on the variable y', we can see that $\tilde{a}_j$ is also independent to y' for every j. Therefore we have

$$\tilde{a}_j \in \mathscr{C}^\infty(U_i \times \mathbb{R}^{2n+1})$$

and since $\tilde{a}_j(x,\lambda\eta') = \lambda^{-j}\tilde{a}_j(x,\eta')$, there exist $\tilde{a} \in \mathscr{C}^\infty(U_i \times \mathbb{R}^{2n+1})$ such that $\tilde{a} \sim \sum_{j=0}^\infty \tilde{a}_j$.

We define $\tilde{P}^{(i)} : \mathscr{C}^\infty(D_i) \to \mathscr{C}^\infty(U_i)$ by

$$\tilde{P}^{(i)}u(x) = \iint_{D_i \times \mathbb{R}^{2n+1}} e^{i\tilde{\varphi}(x,y',\eta')} \tilde{a}(x,\eta') u(y') d\eta' dv(y'). \tag{5.2.23}$$

By the construction of $\tilde{a}$ and $\tilde{\varphi}$, we have $\Box \tilde{P}^{(i)} \equiv 0$ on $\mathscr{C}^\infty(U_i)$. We can see that if $w' \in D_i \subset X$, since $\tilde{a}_{0,0} = 1$, $\tilde{a}_{k,0} = 0$ for all $k \geq 1$ and $\tilde{\varphi}(w',y',\eta') = \langle w'-y',\eta' \rangle$, we have

$$\tilde{P}^{(i)}u(w') = \iint_{D_i\times\mathbb{R}^{2n+1}} e^{i\langle w'-y',\eta'\rangle}u(y')dv(y')d\eta' = u(w'). \tag{5.2.24}$$

Therefore, $\gamma\tilde{P}^{(i)} = I$ on $\mathscr{C}^\infty(D_i)$.

Step 2: extend to $\tilde{X}$. Let $\chi_i \in \mathscr{C}_0^\infty(X)$, $i = 1,\dots,N$, be the partition of unity subordinate to the cover $\{D_i\}$ satisfying χ_i. Furthermore, for each index i let $\tilde{\chi}_i \in \mathscr{C}^\infty(X)$ be such that $\operatorname{supp}\tilde{\chi}_i \subset D_i$ and $\tilde{\chi}_i = 1$ on $\operatorname{supp}\chi_i$. We extend $\tilde{\chi}_i$ to U_i by defining a function $\tilde{\chi}_i' \in \mathscr{C}_0^\infty(\overline{M})$ satisfying $\operatorname{supp}\tilde{\chi}_i' \subset U_i$

$$\tilde{\chi}_i'(x',r) = \tilde{\chi}_i(x') \quad \text{for } r \in [0,\tfrac{\varepsilon}{3}), \text{ and} \tag{5.2.25}$$

$$\operatorname{supp}\tilde{\chi}_i' = \operatorname{supp}\tilde{\chi}_i \times [0,\tfrac{\varepsilon}{2}]. \tag{5.2.26}$$

We have the lemma:

LEMMA 5.12. *The operator*

$$\widetilde{P} = \sum_{i=1}^N \tilde{\chi}_i'\tilde{P}^{(i)}\chi_i : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\tilde{X})$$

satisfies

$$\begin{aligned} &\Box\widetilde{P} \equiv 0 \qquad \operatorname{mod} \mathscr{C}^\infty(\overline{M}), \textit{ and} \\ &\gamma\widetilde{P} = I. \end{aligned} \tag{5.2.27}$$

Step 3: extend to $\overline{M}$. Let $\hat{\chi} \in \mathscr{C}_0^\infty(\overline{M})$ with $\operatorname{supp}\hat{\chi} \subset \tilde{X}$ and $\hat{\chi}(x) = 1$ if $\operatorname{dist}_g(x,X) \le \frac{\varepsilon}{2}$. Let

$$\widehat{P} = \hat{\chi}\widetilde{P} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M}), \tag{5.2.28}$$

by repeating the proof of Lemma 5.12, we have $\Box\widehat{P} \equiv 0$ modulo a smoothing operator and $\gamma\widehat{P} = I$. By the theory of partial differential operators [15], there exists a smoothing operator $N : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M})$ such that $N\Box + P\gamma \equiv I$. From

$$N\Box\widehat{P} + P\gamma\widehat{P} \equiv \widehat{P} \tag{5.2.29}$$

we can see that

$$\widehat{P} \equiv P. \tag{5.2.30}$$

$\square$

PROOF OF LEMMA 5.12. For $x \in \tilde{X}$ and $y' \in X$, let $\widetilde{P}(x,y')$ denote the kernel of $\widetilde{P}$. We first show that

$$\gamma\widetilde{P} = I. \tag{5.2.31}$$

Let $u \in \mathscr{C}^\infty(X)$, then $(\gamma \widetilde{P}u)(x) = (\sum_{i=1}^N \gamma\tilde{\chi}_i'\tilde{P}^{(i)}\chi_i u)(x)$ for any $x \in \tilde{X}$. Since $\tilde{\chi}_i'|_X = \tilde{\chi}_i$ and $\tilde{P}^{(i)}(\chi_i u)|_X = \chi_i u$, we have

$$\gamma\widetilde{P}u = \sum_{i=1}^N \tilde{\chi}_i\chi_i u = u. \tag{5.2.32}$$

By applying Δ_g to $\widetilde{P}(x,y')$, we could assume $x \in U_i$ and $y' \in D_i$ for some i. Since $\operatorname{supp} d\tilde{\chi}_i' \cap \operatorname{supp}\chi_i = \emptyset$, it is suffices to consider the case $x'-y' \geq \delta > 0$ for some δ. Suppose $x = (x',r)$ for $r \geq r_0 > 0$ for some r_0, since $\tilde{a}(x,\eta') \in S^0_{1,0}(U_i \times \mathbb{R}^{2n+1})$, for any $\alpha \in \mathbb{N}_0^{2n+2}$ and $\beta \in \mathbb{N}_0^{2n+1}$ there exists $C_{\alpha,\beta} \in \mathbb{R}$ such that

$$\left|\partial_x^\alpha\partial_{y'}^\beta\tilde{\chi}_i'(x)e^{i\varphi(x,y',\eta')}\tilde{a}(x,\eta')\chi_i(y')\right| \leq C_{\alpha,\beta}e^{-|\eta'|r_0}(1+|\eta'|)^{m_\alpha}|\eta'|^{m_\beta} \tag{5.2.33}$$

for some $m_\alpha, m_\beta \in \mathbb{N}_0$ such that $m_\alpha \leq |\alpha|$ and $m_\beta \leq |\beta|$.
Assume that $r_0 \geq r > 0$, and define $\hat{\varphi}$ such that

$$\varphi(x,y',\eta') = \langle x'-y',\eta'\rangle + i\,|\eta'|\,r + \hat{\varphi}(x,y',\eta')\,r^2.$$

Then one has

$$D_{\eta'}\varphi = x'-y' + i\,\frac{\eta'}{|\eta'|}\,r + O(r^2).$$

By choosing $r > 0$ sufficiently small, we can ensure that

$$|D_{\eta'}\varphi| \geq \frac{\delta}{2}.$$

Let

$$L = \frac{1}{i|D_{\eta'}\varphi|^2}D_{\eta'}\varphi\cdot D_{\eta'},$$

then $L(e^{i\varphi}) = e^{i\varphi}$. We have

$${}^tL(\tilde{a}) = -\left(D_{\eta'}\left(\frac{D_{\eta'}\varphi}{i|D_{\eta'}\varphi|^2}\right)\tilde{a} + \frac{D_{\eta'}\varphi}{i|D_{\eta'}\varphi|^2}\cdot D_{\eta'}\tilde{a}\right). \tag{5.2.34}$$

Since $\varphi(x,y',\eta')$ is homogeneous in η' of degree 1, we have $D_{\eta'}\left(\frac{D_{\eta'}\varphi}{i|D_{\eta'}\varphi|^2}\right)$ is homogeneous in η' of degree -1. We also have

$$|D_{\eta'}\tilde{a}| \leq C_1(1+|\eta'|)^{-1}$$

for some $C_1 \in \mathbb{R}$. Therefore, we have for any $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{2n+2}$ and $\beta \in \mathbb{N}_0^{2n+1}$ there exists $C_{\alpha,\beta} \in \mathbb{R}$ such that

$$\left|\partial_x^\alpha\partial_{y'}^\beta e^{i\varphi(x,y',\eta')}\,({}^tL)^N(\tilde{a})(x,\eta')\right| \leq C_{\alpha,\beta}(1+|\eta'|)^{-(N-m_\alpha-m_\beta)} \tag{5.2.35}$$

for some $m_\alpha, m_\beta \in \mathbb{N}_0$ such that $m_\alpha \leq |\alpha|$ and $m_\beta \leq |\beta|$. From

$$\int_{\mathbb{R}^{2n+1}} e^{i\varphi(x,y',\eta')}\tilde{a}(x,\eta')d\eta' = \int_{\mathbb{R}^{2n+1}} e^{i\varphi(x,y',\eta')}\,({}^tL)^N(\tilde{a})(x,\eta')d\eta', \tag{5.2.36}$$

and (5.2.33), (5.2.35) we conclude that

$$\Delta_g \widetilde{P}(x,y') \in \mathscr{C}^\infty(\tilde{X} \times X).$$

Hence we have $\Box\widetilde{P} \equiv 0$ modulo a smoothing operator. $\square$

Now we are ready to prove Theorem 5.10:

PROOF OF THEOREM 5.10. The proof strategy is to apply Proposition 5.11 and work locally, then integrate the $(2n+2)$-nd variable to obtain the symbols via Lemma 5.13, and finally apply the stationary phase formula to get the symbol of $\mathcal{P}_f$.

Step 1: near boundary consideration. Let $f \in \mathscr{C}^\infty(\overline{M})$. From (5.2.30), we have $\mathcal{P}_f \equiv \widehat{P}f\widehat{P}$. The kernel of $\widehat{P}$ can be written as

$$\widehat{P}(z,x') = \hat{\chi}(z)\sum_{i=1}^{N}\tilde{\chi}_i'(z)\tilde{P}^{(i)}(z,x')\chi_i(x'). \tag{5.2.37}$$

For $z \in U_i$, we introduce the following notation:

$$A(z,\eta',\gamma') = \bar{\tilde{a}}(z,\eta')\,\tilde{a}(z,\gamma'), \tag{5.2.38}$$

$$A_j(z,\eta',\gamma') = \sum_{l=0}^{\infty}\bar{\tilde{a}}_l(z,\eta')\,\tilde{a}_{j-l}(z,\gamma'), \tag{5.2.39}$$

$$\Phi(x',y',z,\eta',\gamma') = -\bar{\tilde{\varphi}}(z,x',\eta') + \tilde{\varphi}(z,y',\gamma'), \tag{5.2.40}$$

$$\Phi_j(x',y',z,\eta',\gamma') = -\bar{\tilde{\varphi}}_j(z',x',\eta') + \tilde{\varphi}_j(z',y',\gamma'), \tag{5.2.41}$$

where the functions $\tilde{a}$ and $\tilde{\varphi}$ are defined in (5.2.23). Let $\chi \in \mathscr{C}_0^\infty(\overline{M})$ such that $\chi \equiv 1$ on $\tilde{X}$. The integral kernel of $\mathscr{P}_f$ can be written as

$$\begin{aligned}\mathcal{P}_f(x',y') &= \int_{\overline{M}} \overline{\widehat{P}(z,x')}f(z)\widehat{P}(z,y')dv(z)\\ &= \int_{\overline{M}} \overline{\widehat{P}(z,x')}\chi(z)f(z)\widehat{P}(z,y')dv(z)\\ &\qquad + \int_{\overline{M}} \overline{\widehat{P}(z,x')}(1-\chi(z))f(z)\widehat{P}(z,y')dv(z)\end{aligned} \tag{5.2.42}$$

Let

$$\mathcal{P}_f^1(x',y') = \int_{\overline{M}} \overline{\widehat{P}(z,x')}\chi(z)f(z)\widehat{P}(z,y')dv(z) \tag{5.2.43}$$

$$\mathcal{P}_f^0(x',y') = \int_{\overline{M}} \overline{\widehat{P}(z,x')}(1-\chi(z))f(z)\widehat{P}(z,y')dv(z). \tag{5.2.44}$$

Then we can split (5.2.42) into

$$\mathcal{P}_f(x',y') = \mathcal{P}_f^1(x',y') + \mathcal{P}_f^0(x',y'), \tag{5.2.45}$$

From (5.2.28) and $\operatorname{supp}(1-\chi) \cap \operatorname{supp}\hat{\chi} = \emptyset$, it is straightforward to see that $\mathcal{P}_f^0(x',y') = 0$ for every $x',y' \in X$. To obtain $\mathcal{P}_f^1(x',y')$, work locally on $D_i \subset X$ and assume $x',y' \in D_i$. Then, from (5.2.37), we have

(5.2.46)

$$\begin{aligned}
&(\widehat{P}^*\chi f\widehat{P})(x',y') \\
&= \sum_{i,j=1}^{N} \chi_j(x')\chi_i(y') \int_M \hat{\chi}^2(z)\overline{\tilde{P}^{(j)}(z,x')}\tilde{\chi}_i'(z)f(z)\tilde{\chi}_j'(z)\tilde{P}^{(i)}(z,y')dv(z) \\
&= \sum_{i,j=1}^{N} \chi_j(x')\chi_i(y')\cdot \\
&\qquad \iiint_{(\operatorname{supp}\tilde{\chi}_i'\cap\operatorname{supp}\tilde{\chi}_j')\times\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}} e^{i\Phi(x',y',z,\eta',\gamma')}\hat{\chi}^2(z)\tilde{\chi}_i'(z)f(z)\tilde{\chi}_j'(z)A(z,\eta',\gamma')dv(z)d\gamma'd\eta'.
\end{aligned}$$

From (5.2.25) and (5.2.26), there exists a compact set $K_i \subset D_i$ such that

$$\operatorname{supp}\tilde{\chi}_i' \cap \operatorname{supp}\tilde{\chi}_j' = K_i \times [0,\tfrac{\varepsilon}{2}].$$

For simplicity, set $\tilde{K}_i = K_i \times [0,\frac{\varepsilon}{2}]$. We first consider the integral on $\tilde{K}_i$ and let

(5.2.47)

$$\begin{aligned}
\beta_{(i)}(x',y',\eta',\gamma') &= \int_{\tilde{K}_i} e^{i\Phi(x',y',z,\eta',\gamma')}\hat{\chi}^2(z)\tilde{\chi}_i'(z)\tilde{\chi}_j'(z)f(z)A(z,\eta',\gamma')dv(z) \\
&= \int_{K_i}\int_0^{\frac{\varepsilon}{2}} e^{i\Phi(x',y',z,\eta',\gamma')}\hat{\chi}^2(z)\tilde{\chi}_i'(z)\tilde{\chi}_j'(z)f(z)A(z,\eta',\gamma')drdv(z').
\end{aligned}$$

We consider the Taylor expansion of Φ with respect to r. By Proposition 5.11, on $U_i \times D_i$ we have

$$\begin{aligned}
\Phi(x',y',(z',r),\eta',\gamma') &= \sum_{j=0}^{\infty}(-\bar{\varphi}_j(z',x',\eta') + \varphi_j(z',y',\gamma'))r^j \\
\Phi_0(x',y',z',\eta',\gamma') &= \langle z'-y',\eta'\rangle - \langle z'-x',\gamma'\rangle \\
\Phi_1(x',y',z',\eta',\gamma') &= \sqrt{-1}(|\eta'|+|\gamma'|)
\end{aligned} \tag{5.2.48}$$

We also define

$$\hat{\Phi}(x',y',(z',r),\eta',\gamma')r^2 := \Phi - \Phi_0 - \Phi_1 r.$$

So (5.2.47) can be written as
(5.2.49)

$$\int_{K_i} e^{i(\langle z'-y',\eta'\rangle-\langle z'-x',\gamma'\rangle)} \int_0^{\frac{\varepsilon}{2}} e^{i\Phi_1 r} e^{\hat{\Phi} r^2} \tilde{\chi}^2(z)\tilde{\chi}_i'(z)\tilde{\chi}_j'(z) f(z) A(z,\eta',\gamma') dr dv(z')$$

Step 2: integrate over r. We denote the integral with respect to r in (5.2.49) by

$$b_{(i)}(x',y',z',\eta',\gamma') := \int_0^{\frac{\varepsilon}{2}} e^{i\Phi_1 r} e^{\hat{\Phi} r^2} \tilde{\chi}^2(z)\tilde{\chi}_i'(z)\tilde{\chi}_j'(z) f(z) A(z,\eta',\gamma') dr \tag{5.2.50}$$

To obtain the integral of (5.2.50), we need a lemma:

LEMMA 5.13. *There exist* $b_{(i),j}(x',z',(\eta',\gamma')) \in S^{-1-j}_{1,0}(D_i \times K_i \times \mathbb{R}^{4n+2})$ *such that* $b_{(i)} \sim \sum_j b_{(i),j}$ *and*

$$b_{(i),j}(x',y',z',(\lambda\eta',\lambda\gamma')) = \lambda^{-1-j} b_{(i),j}(x',y',z',(\eta',\gamma')). \tag{5.2.51}$$

PROOF OF LEMMA 5.13. Let $G(r,\eta',\gamma') = e^{\hat{\Phi} r^2}\tilde{\chi}^2(z)\tilde{\chi}_i'(z)\tilde{\chi}_j'(z) f(z) A(z,\eta',\gamma')$, we consider the Taylor expansion of G_j with respect to r. We first observe that for any N,

$$G(r,\eta',\gamma') - \sum_{l=0}^{N} G^{(l)}(0,\eta',\gamma') r^j = O(r^{N+1})$$

and the integral
(5.2.52)

$$\begin{aligned}\int_0^{\epsilon} e^{i\Phi_1 r} r^{N+1} dr &= \frac{(N+1)!}{(|\eta'|+|\gamma'|)^{N+2}} \left(1 - e^{-\epsilon(|\eta'|+|\gamma'|)} \sum_{k=0}^{N+1} \frac{(\epsilon(|\eta'|+|\gamma'|))^k}{k!}\right) \\ &= \frac{(N+1)!}{(|\eta'|+|\gamma'|)^{N+2}} + O\left((|\eta'|+|\gamma'|)^{-\infty}\right)\end{aligned}$$

as $(|\eta'|+|\gamma'|) \to \infty$, where $\epsilon = \frac{\varepsilon}{2}$. Therefore, we have for any $N \in \mathbb{N}_0$

$$\int_0^{\epsilon} G(r,\eta',\gamma') dr - \sum_{l=0}^{N} \int_0^{\epsilon} G^{(l)}(0,\eta',\gamma') r^l dr = O((|\eta'|+|\gamma'|)^{-N-2}). \tag{5.2.53}$$

By applying the product rule of the derivative on G, we have

$$G^{(l)}(0,\eta',\gamma') = \sum_{k=0}^{l} \binom{l}{k} \frac{k!}{l!} \sum_{j_1+2j_2+\cdots+kj_k=k} \prod_{i=1}^{k} \frac{\mathrm{a}_i^{j_i}}{i!^{j_i} j_i!} (fA)^{(l-k)}(z',0), \tag{5.2.54}$$

where $(fA)^{(l-k)}(z',0)$ denotes the $l-k$-th derivative of $f(z)A(z,\eta',\gamma')$ with respect to r and evaluate at $r=0$ and $\mathrm{a}_n := (\hat{\Phi} r^2)^{(n)}|_{r=0}$ for $n \geq 1$. We remind that for $k \geq 1$

$$(e^{\hat{\Phi}r^2})^{(k)}|_{r=0} = \sum_{j_1+2j_2+\cdots+kj_k=k} \prod_{i=1}^{k} \frac{\mathrm{a}_i^{j_i}}{i!^{j_i} j_i!} \tag{5.2.55}$$

By direct computation, for $n \geq 2$ we have

$$\mathrm{a}_n = n(n-1)\hat{\Phi}^{(n-2)}|_{r=0} = n!\Phi_n = n!(-\bar{\varphi}_n(z', x', \eta') + \varphi_n(z', y', \gamma')), \tag{5.2.56}$$

and $\mathrm{a}_1 = 0$. We further define $\mathrm{a}_0 = 1$ so that (5.2.55) is valid for $k = 0$.

$$(e^{\hat{\Phi}r^2})|_{r=0} = 1. \tag{5.2.57}$$

By (5.2.39), we see that

$$A_j(z', \lambda\eta', \lambda\gamma') = \lambda^{-j} A_j(z', \eta', \gamma'),$$

and for any $N' \in \mathbb{N}_0$ as $(|\eta'| + |\gamma'|) \to \infty$ we have

$$A(z, \eta', \gamma') - \sum_{j=0}^{N'} A_j(z, \eta', \gamma') = O((|\eta'| + |\gamma'|^{-N'-1})). \tag{5.2.58}$$

We substitute A by $\sum_{j=0}^{\infty} A_j$ in (5.2.54); the resulting expression preserves the form given in (5.2.53). By the construction of a_j we see that the Taylor expansion of A_j with respect to r also has the following property

$$A_j(z, \eta', \gamma') - \sum_{l=0}^{M} A_j^{(l)}(z', \eta', \gamma') r^l = O(r^{M+1}), \tag{5.2.59}$$

where

$$A_j^{(l)}(z', \eta', \gamma') = l! \cdot \sum_{i=0}^{j} \sum_{k=0}^{l} \cdot \bar{a}_{i,k}(z', \eta') a_{j-i,l-k}(z', \gamma'). \tag{5.2.60}$$

By applying (5.2.59), we substitute the Taylor expansion of A_j into (5.2.54). The resulting expression remains valid, and thus we can rearrange the terms according to their homogeneous degree in (η', γ'). Hence we have

(5.2.61)

$$\int_0^{\epsilon} G(r, \eta', \gamma') dr =$$

$$\sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k} k! \sum_{j_1+2j_2+\cdots+kj_k=k} \prod_{i=1}^{k} \frac{\mathrm{a}_i^{j_i}}{i!^{j_i} j_i!} (fA_j)^{(l-k)}(z', 0) \frac{1}{(|\eta'| + |\gamma'|)^{l+1}}$$

$$+ O((|\eta'| + |\gamma'|)^{-\infty})$$

as $(|\eta'|+|\gamma'|)\to\infty$. When $l=k=j=0$, we can see that the highest degree term is

$$b_{(i),0}=\frac{f(z',0)}{|\eta'|+|\gamma'|}. \tag{5.2.62}$$

When $(j,l,k)=(0,1,0)$, $(0,2,2)$, and $(1,0,0)$ we have the homogeneous degree -2 terms

$$b_{(i),1}=\frac{(fA_0)^{(1)}(z',0)}{(|\eta'|+|\gamma'|)^2}+\frac{\mathrm{a}_2(fA_0)(z',0)}{(|\eta'|+|\gamma'|)^3}+\frac{fA_1(z',0)}{|\eta'|+|\gamma'|}. \tag{5.2.63}$$

Hence, by (5.2.61) we deduce that there exist coefficients $b_{(i),j}$, homogeneous of degree $-1-j$, such that

$$b_{(i),j}\in S^{-1-j}_{1,0}(D_i\times D_i\times K_i\times\mathbb{R}^{4n+2}).$$

Moreover, by (5.2.50) we obtain the asymptotic expansion

$$b_{(i)}\sim\sum_{j=0}^{\infty}b_{(i),j}.$$

$\square$

Step 3: apply stationary phase formula. We now return to the proof of Theorem 5.10. Consider the integral

$$\iint_{K_i\times\mathbb{R}^{2n+1}}e^{i(\langle z'-y',\eta'\rangle-\langle z'-x',\gamma'\rangle)}b_{(i)}(x',y',z',\eta',\gamma')\,dv(z')\,d\gamma'. \tag{5.2.64}$$

Let $\lambda=|\eta'|$ and $\lambda\tilde{\gamma}'=\gamma$, by changing the variable we get

$$\iint_{K_i\times\mathbb{R}^{2n+1}}e^{i\lambda(\langle z'-y',\frac{\eta'}{|\eta'|}\rangle-\langle z'-x',\tilde{\gamma}'\rangle)}b_{(i)}(x',y',z',\eta',\tilde{\gamma}')\lambda^{2n+1}dv(z')d\tilde{\gamma}'. \tag{5.2.65}$$

Let $F(z',\tilde{\gamma}')=\langle z'-y',\frac{\eta'}{|\eta'|}\rangle-\langle z'-x',\tilde{\gamma}'\rangle$ we have

$$d_{z',\tilde{\gamma}'}F=\begin{pmatrix}\frac{\eta'}{|\eta'|}-\tilde{\gamma}' & -z'+x'\end{pmatrix}. \tag{5.2.66}$$

The critical point is $(z',\tilde{\gamma}')=(x',\frac{\eta'}{|\eta'|})$, i.e.

$$d_{z',\tilde{\gamma}'}F(x',\frac{\eta'}{|\eta'|})=0.$$

The Hessian matrix of F with respect to the variables z' and $\tilde{\gamma}'$, evaluated at the critical point, is given by

$$\operatorname{Hess}F\left(x',\frac{\eta'}{|\eta'|}\right)=\begin{pmatrix}0 & -I_{2n+1}\\ -I_{2n+1} & 0\end{pmatrix}. \tag{5.2.67}$$

Since

$$\det \operatorname{Hess} F\Big(x', \frac{\eta'}{|\eta'|}\Big) = -1 \neq 0,$$

this critical point is non-degenerate. For notational convenience, we denote $\operatorname{Hess} F\Big(x', \frac{\eta'}{|\eta'|}\Big)$ by $F''\Big(x', \frac{\eta'}{|\eta'|}\Big)$.

We apply the stationary phase formula of Hörmander in Theorem 2.5 to the integral (5.2.65). There exists an operator P_j such that for any $u \in \mathscr{C}_0^\infty(U_i)$

$$P_j u = \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} \langle d^2 F(x', \frac{\eta'}{|\eta'|})^{-1} D, D\rangle^\nu (\frac{h^\mu u}{\nu! \mu!})(x', \frac{\eta'}{|\eta'|}), \tag{5.2.68}$$

where

$$\begin{aligned} h(z', \tilde{\gamma}') :&= F(z', \tilde{\gamma}') - F(x', \frac{\eta'}{|\eta'|}) - \frac{1}{2}\langle d^2 F(x', \frac{\eta'}{|\eta'|}) \begin{pmatrix} z' \\ \tilde{\gamma}' \end{pmatrix}, \begin{pmatrix} z' \\ \tilde{\gamma}' \end{pmatrix} \rangle \\ &= \langle z' - y', \frac{\eta'}{|\eta'|}\rangle - \langle z' - x', \tilde{\gamma}'\rangle - \langle x' - y', \frac{\eta'}{|\eta'|}\rangle + \langle z', \tilde{\gamma}'\rangle \\ &= \langle z' - x', \frac{\eta'}{|\eta'|}\rangle + \langle x', \tilde{\gamma}'\rangle. \end{aligned} \tag{5.2.69}$$

and $D = \begin{pmatrix} -i\partial_{z_1'} & \cdots - i\partial_{z_{2n+1}'} & -i\partial_{\tilde{\gamma}_1'} & \cdots - i\partial_{\tilde{\gamma}_{2n+1}'} \end{pmatrix}^T$ such that

$$\begin{aligned} &\iint e^{i\lambda(\langle z'-y', \frac{\eta'}{|\eta'|}\rangle - \langle z'-x', \tilde{\gamma}'\rangle)} b(x', y', z', \eta', \tilde{\gamma}') \lambda^{2n+1} dv(z') d\tilde{\gamma}' \\ &\qquad \sim e^{i\langle x'-y', \eta'\rangle} (2\pi)^{2n+1} \sum_{j=0}^{\infty} \lambda^{-j} P_j(b_{(i)})|_{(z', \tilde{\gamma}') = (x', \frac{\eta'}{|\eta'|})}. \end{aligned} \tag{5.2.70}$$

Let Δ_X be the real Laplacian on X and let $\sqrt{-\Delta_X}$ be the square root of $-\Delta_X$. The leading term in (5.2.70) is

$$\begin{aligned} e^{i\langle x'-y', \eta'\rangle} (2\pi)^{2n+1} P_0(b_0)|_{(z', \tilde{\gamma}') = (x', \frac{\eta'}{|\eta'|})} &= e^{i\langle x'-y', \eta'\rangle} (2\pi)^{2n+1} \frac{f(z', 0)}{2|\eta'|} \\ &= e^{i\langle x'-y', \eta'\rangle} (2\pi)^{2n+1} \frac{f|_X}{\sigma_{(2\sqrt{-\Delta_X})^{-1}}}, \end{aligned} \tag{5.2.71}$$

where $\sigma_{(2\sqrt{-\Delta_X})^{-1}}$ denotes the principal symbol of $(2\sqrt{-\Delta_X})^{-1}$. We denote

$$\beta_{f,(i),0}(x', y', \eta') = \frac{f|_X}{\sigma_{(2\sqrt{-\Delta_X})^{-1}}} \tag{5.2.72}$$

and note that $\beta_{f,(i),0}(x',y',\eta') \in S^{-1}(D_i \times D_i \times \mathbb{R}^{2n+1})$. By considering the homogeneous degree of b_j and the operator $\lambda^{-j}P_j$, we obtain

$$\beta_{f,(i),j}(x',y',\eta') = \sum_{k=0}^{j} \lambda^{-k} P_k(b_{(i),j-k})|_{(z',\tilde{\gamma}')=(x',\frac{\eta'}{|\eta'|})}. \tag{5.2.73}$$

Hence,

$$\mathcal{P}_{f,(i)}(x',y') := \int e^{i\langle x'-y',\eta'\rangle} \beta_{f,(i)}(x',y',\eta')\, d\eta' \tag{5.2.74}$$

defines a pseudodifferential operator on the coordinate patch D_i. By applying a partition of unity and the global theory of Fourier integral operators together, we obtain a corresponding global result on X. Consequently, we deduce that $\mathcal{P}^1_f$ is a pseudodifferential operator on X of order -1.

□

We recall that P^*P is an elliptic pseudodifferential operator of order -1 (see [4]), and the principal symbol $\sigma_{P^*P} = \sigma_{(2\sqrt{-\Delta_X})^{-1}}$; therefore, $(P^*P)^{-1}$ is a pseudodifferential operator of order 1 with the principal symbol $\sigma_{(P^*P)^{-1}} = \sigma_{(2\sqrt{-\Delta_X})}$. Then

$$\mathcal{J}_f := (P^*P)^{-1} \circ \mathcal{P}_f \tag{5.2.75}$$

is a pseudodifferential operator on X of order 0 with the principal symbol

$$\sigma_{\mathcal{J}_f}(x',\eta') = f|_X$$

for any $f \in \mathscr{C}^\infty(\overline{M})$.

Let $S^\dagger$ be the formal adjoint of the Szegő projection S with respect to $[\,\cdot\,|\,\cdot\,]_X$. We have the following result:

Proposition 5.14. *For any coordinate patch $D \in X$, there exist $\phi \in \mathscr{C}^\infty(D \times D, \mathbb{C})$ satisfying (2.2.10) such that we have on $D \times D$, $S^\dagger$ can be approximated by a Fourier integral operator*

$$S^\dagger(x,y) \equiv \int_0^\infty e^{it\phi(x,y)} s^\dagger(x,y,t)dt \tag{5.2.76}$$

where $s^\dagger(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ is a classical Hörmander symbol satisfying

$$s^\dagger(x,y,t) \sim \sum_{j=0}^{+\infty} s^\dagger_j(x,y)t^{n-j} \text{ in } S^n_{1,0}(D \times D \times \mathbb{R}_+),$$

$s^\dagger_j(x,y) \in \mathscr{C}^\infty(D \times D)$, $j = 0,1,\ldots$. Moreover, $S^\dagger$ and S can be chosen to have the same phase function, and $s^\dagger_0(x,y) = s_0(x,y)$ and $s^\dagger_1(x,x) = s_1(x,x)$.

PROOF. First, we note that

$$
\begin{aligned}
[\, Su \,|\, v \,]_X &= [\, u \,|\, S^{\dagger} v \,]_X \\
&= (\, PSu \,|\, Pv \,)_M \\
&= (\, Su \,|\, P^* P v \,)_X \\
&= (\, u \,|\, SP^* P v \,)_X \\
&= [\, u \,|\, (P^* P)^{-1} SP^* P v \,]_X .
\end{aligned}
\tag{5.2.77}
$$

Hence, by the uniqueness of the adjoint operator with respect to the inner product $[\,\cdot\,|\,\cdot\,]_X$, we conclude that

$$S^{\dagger} = (P^* P)^{-1} SP^* P.$$

From

$$(S^{\dagger})^2 = (P^* P)^{-1} S^2 P^* P = (P^* P)^{-1} SP^* P = S^{\dagger}$$

we can see that $S^{\dagger}$ is a projection operator. By composition of a Szegő type Fourier integral operator with a pseudodifferential operator, we have SP^*P as a Szegő type Fourier integral operator with the same phase function, and the leading term is $s_0(x,y)(2|d_x\varphi(x,x)|)^{-1}$. By composing $(P^*P)^{-1}$, we get a Szegő type Fourier integral operator with the same phase function, and the leading term is

$$(2|d_x\varphi(x,x)|)s_0(x,y)(2|d_x\varphi(x,x)|)^{-1} = s_0(x,y). \tag{5.2.78}$$

Therefore, we conclude that $S^{\dagger}$ is a Szegő-type Fourier integral operator with the same phase function as S, and that its leading term coincides with that of S. To compute the subleading term of $s^{\dagger}(x,y,t)$, we apply the stationary phase method and repeat exactly the same computation as in [22]. From this, we obtain

$$s_1^{\dagger}(x,x) = s_1(x,x).$$

$\square$

From the properties of the adjoint operator $S^{\dagger}$, we further obtain the following result for the Szegő type Fourier integral operator $\hat{S}$:

PROPOSITION 5.15. *$\hat{S} \in \Psi^n(X)$ has the same leading term as S, i.e., $\sigma^{0,0}_{\hat{S}}(x',y') = \sigma^{0,0}_{S}(x',y')$.*

PROOF. From (5.1.12), we have $S \equiv \hat{S}S$ and $\hat{S} \equiv S\hat{S}$, hence $S^{\dagger} \equiv S^{\dagger}(\hat{S})^{\dagger} \equiv S^{\dagger}\hat{S}$. Therefore

$$
\begin{aligned}
\hat{S} \equiv S\hat{S} &\equiv (S^{\dagger} + S - S^{\dagger})\hat{S} \\
&\equiv S^{\dagger}\hat{S} + (S - S^{\dagger})\hat{S} \\
&\equiv S^{\dagger} + (S - S^{\dagger})\hat{S}.
\end{aligned}
\tag{5.2.79}
$$

Put $R = S - S^{\dagger} \in \Psi^{n-1}(X)$, we can see that $R^N \in \Psi^{n-N}(X)$ for any $N \in \mathbb{N}_0$. By [9], there exists $\mathcal{R} \in \Psi^n(X)$ such that

$$\mathcal{R} - \sum_{j=0}^{N-1} R^j \in \Psi^{n-N}(X), \quad \text{for any } N \in \mathbb{N}_0.$$

We then have $\mathcal{R}(1-R) \equiv I$. Therefore,

$$\hat{S} \equiv \mathcal{R}S^{\dagger} \in \Psi^n(X). \tag{5.2.80}$$

From (5.2.80), it follows that $\hat{S} - S^{\dagger} \in \Psi^{n-1}(X)$. Consequently, we deduce that their leading terms coincide. □

By the work in [11], let L be a pseudodifferential operator on X of order m, and S be the Szegő projection on X. then SLS is a Fourier integral operator of order $m+n$. Since $\hat{S}$ satisfies Definition 4.1, it follows that $\hat{S}L\hat{S} \in \Psi^{n+m}(X)$. From [27], we have $\hat{S}L\hat{S}(P^*P)^{-1}$ is a Fourier integral operator of Szegő type.

PROPOSITION 5.16. *Let $Z \in \Psi^m(X)$, such that in local coordinates $D \in X$, we have*

$$Z(x', y') = \int_0^{\infty} e^{it\varphi(x',y')} a(x', y', t)dt$$

$$a(x,y,t) \sim \sum_{j=0}^{\infty} a_j(x,y)t^{m-j} \in S^m_{1,0}(D \times D \times \mathbb{R}_+),$$

$$a_j(x,y) \in \mathscr{C}^{\infty}(D \times D), j = 0, 1, \ldots.$$

there exists $\hat{\varphi}(x,y) \in \mathscr{C}^{\infty}(\overline{M} \times \overline{M})$ such that $\hat{\varphi}(x', y') = \varphi(x', y')$ for $x', y' \in X$, and $\hat{a}(x,y,t) \in S^m_{1,0}(\overline{M} \times \overline{M} \times \mathbb{R}_+)$ such that

$$\begin{aligned} &\hat{a}(x,y,t) \sim \sum_{j=0}^{\infty} \hat{a}_j(x,y)t^{m-j} \in S^m_{1,0}(\overline{M} \times \overline{M} \times \mathbb{R}_+), \\ &\hat{a}_j(x,y) \in \mathscr{C}^{\infty}(\overline{M} \times \overline{M}), j = 0, 1, \ldots. \end{aligned} \tag{5.2.81}$$

and

$$\begin{aligned} \hat{a}(x', y', t) &= a(x', y', t), \\ \hat{a}_j(x', y') &= a_j(x', y') \quad \text{for } j \in \mathbb{N}_0. \end{aligned} \tag{5.2.82}$$

PROOF. From Proposition 5.11, composing the Poisson operator P with Z shows that PZ is a Fourier integral operator whose kernel satisfies

$$(PZ)(x, y') \equiv (\hat{P}Z)(x, y') \qquad \text{mod } \mathscr{C}^{\infty}(\overline{M} \times X), \tag{5.2.83}$$

and

$$\begin{aligned} &\Box_x (PZ)(x, y') = 0, \\ &(\gamma PZ)(x, y') = Z(x', y'). \end{aligned} \tag{5.2.84}$$

Repeating the construction in the proof of Proposition 5.11 obtains

$$b_j(x,y') \in \mathscr{C}^\infty(\overline{M} \times X), \text{ and}$$

$$\varphi'(x,y') \in \mathscr{C}^\infty(\overline{M} \times X)$$

such that

$$(PZ)(x,y') \equiv \int_0^\infty e^{it\varphi(x,y')} b(x,y',t)\, dt \qquad \text{mod } \mathscr{C}^\infty(\overline{M} \times X), \tag{5.2.85}$$

where $b(x,y',t) \sim \sum_{j=0}^\infty b_j(x,y')t^{n-j} \in S^n_{1,0}(\overline{M} \times X \times \mathbb{R}_+)$, with

$$\begin{aligned} b_j(\cdot,y')|_X &= a_j(\cdot,y'), \\ \varphi'(\cdot,y')|_X &= \varphi(\cdot,y'). \end{aligned} \tag{5.2.86}$$

By taking the adjoint, we have $(PZ)^* = Z^*P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X)$, and

$$(PZ)^*(y',z) = \overline{(PZ)(z,y')}. \tag{5.2.87}$$

From Proposition 5.11, composing the Poisson operator P with $(PZ)^*$ again shows that PZ^*P^* is a Fourier integral operator whose kernel satisfies

$$(PZ^*P^*)(x,z) \equiv (\hat{P}Z^*P^*)(x,z) \qquad \text{mod } \mathscr{C}^\infty(\overline{M} \times \overline{M}), \tag{5.2.88}$$

and

$$\begin{aligned} \Box_x(PZ^*P^*)(x,z) &= 0, \\ (\gamma PZ^*P^*)(x,z) &= \overline{(PZ)(z,x')}. \end{aligned} \tag{5.2.89}$$

By (5.2.85) and (5.2.86), repeating the construction in Proposition 5.11 obtains $\tilde{b}_j(x,y) \in \mathscr{C}^\infty(\overline{M} \times \overline{M})$ and $\tilde{\varphi}'(x,y) \in \mathscr{C}^\infty(\overline{M} \times \overline{M})$ such that

$$PZ^*P^*(x,y) \equiv \int_0^\infty e^{it\tilde{\varphi}'(x,y)} \tilde{b}(x,y,t)dt \qquad \text{mod } \mathscr{C}^\infty(\overline{M} \times \overline{M}), \tag{5.2.90}$$

where $\tilde{b}(x,y,t) \sim \sum_{j=0}^\infty \tilde{b}_j(x,y)t^n \in S^n_{1,0}(\overline{M} \times \overline{M})$ with

$$\begin{aligned} \tilde{b}_j(\cdot,z)|_X &= \overline{b_j(z,\cdot)}, \\ \tilde{\varphi}'(\cdot,z)|_X &= -\overline{\varphi'(z,\cdot)}. \end{aligned} \tag{5.2.91}$$

By replacing Z with Z^* in (5.2.83), we obtain the desired result of (5.2.81) and (5.2.82). ☐

From Proposition 5.16, we obtain Theorem 5.4.

5.3. Some operator spaces

In this section, we define the operator space on $\overline{M}$ and show that the Toeplitz operators admit an asymptotic expansion, paralleling the technique in Sec. 4.2. Recall that

$$E = (1 + \Box_b + \mathcal{T}^*\mathcal{T})^{\frac{1}{2}} \in \mathcal{L}^1(X),$$

we put

$$\hat{E} = PE(P^*P)^{-1}P^*. \tag{5.3.1}$$

Let

$$\mathcal{H} := \{f \in \mathscr{C}^\infty(\overline{M}) : \Box f = 0\}. \tag{5.3.2}$$

be the space of harmonic functions on $\overline{M}$. We introduce the following notation to avoid any ambiguity regarding operators defined on X and $\overline{M}$. For $L \in \mathcal{L}^m(X)$, the Toeplitz operator associated with L with respect to $\hat{S}$ is defined by

$$\mathscr{T}_L := \hat{S}L\hat{S}. \tag{5.3.3}$$

We recall that there exist $l_j \in \mathscr{C}^\infty(X)$ such that $\mathscr{T}_L - \mathscr{T}_{\mathsf{q}(L)} \equiv 0 \mod \mathcal{L}^{-\infty}(X)$, where

$$\mathsf{q}(L) = \sum_{j=0}^{\infty} l_j E^{-j},$$

and

$$l_0(x) = \sigma_L^0(x, \xi(x)).$$

Proposition 5.17 (Proposition 5.5). *For any $f \in \mathscr{C}^\infty(\overline{M})$, there exists $\hat{f}_j \in \mathcal{H}$ for $j = 0, 1, 2, \cdots$ such that*

$$T_f - \sum_{j=0}^{N-1} T_{\hat{f}_j \hat{E}^{-j}} \in \Psi^{n-N}(\overline{M}). \tag{5.3.4}$$

Proof. Let $f \in \mathscr{C}^\infty(\overline{M})$, we have $\mathcal{J}_f \in \mathcal{L}^0(X)$, and the principal symbol of $\mathcal{J}_f \in \mathcal{L}^0(X)$ is $f|_X$. From 4.6, there exists $f_j \in \mathscr{C}^\infty(X)$ for $j \in \mathbb{N}_0$ such that

$$\mathsf{q}\left(\mathcal{J}_f\right) = \sum_{j=0}^{\infty} f_j E^{-j}. \tag{5.3.5}$$

Clearly, $f_0 = f|_X \in \mathscr{C}^\infty(X)$. Let $\hat{f}_0 := P(f_0)$. Then there exist $g_{0,j} \in \mathscr{C}^\infty(X)$ for $j \in \mathbb{N}_0$ such that

$$\mathsf{q}(\mathcal{J}_{\hat{f}_0}) = \sum_{j=0}^{\infty} g_{0,j} E^{-j}.$$

Note that $g_{0,0} = \sigma^0_{\mathcal{J}_{\hat{f}_0}}(x', \xi(x')) = \hat{f}_0|_X = f_0$, where $x' \in X$. Therefore, we get

(5.3.6) $$\mathfrak{q}\left(\mathcal{J}_f\right) - \mathfrak{q}(\mathcal{J}_{\hat{f}_0}) \in \mathcal{L}^{-1}(X).$$

Let $\hat{f}_1 := P(f_1 - g_{0,1})$, then $\mathcal{J}_{\hat{f}_1} E^{-1} \in \mathcal{L}^{-1}(X)$. There exists $g_{1,j} \in \mathscr{C}^\infty(X)$ for $j \in \mathbb{N}_0$ such that

(5.3.7) $$\mathfrak{q}(\mathcal{J}_{\hat{f}_1} E^{-1}) = \sum_{j=0}^{\infty} g_{1,j} E^{-1-j},$$

where $g_{1,0} = \sigma^0_{\mathcal{J}_{\hat{f}_1} E^{-1}}(x', \xi(x')) = \hat{f}_1|_X = f_1 - g_{0,1}$, where $x' \in X$. Thus, we have

(5.3.8) $$\mathfrak{q}\left(\mathcal{J}_f\right) - \mathfrak{q}(\mathcal{J}_{\hat{f}_0}) - \mathfrak{q}(\mathcal{J}_{\hat{f}_1} E^{-1}) \in \mathcal{L}^{-2}(X).$$

Inductively, let

(5.3.9) $$\hat{f}_k = P\left(f_k - \sum_{l=0}^{k-1} g_{l,k-l}\right),$$

then $\mathcal{J}_{\hat{f}_k} E^{-k} \in \mathcal{L}^{-k}(X)$, there exists $g_{k,j} \in \mathscr{C}^\infty(X)$ for $j \in \mathbb{N}_0$ such that

(5.3.10) $$\mathfrak{q}(\mathcal{J}_{\hat{f}_k} E^{-k}) = \sum_{j=0}^{\infty} g_{k,j} E^{-k-j},$$

where $g_{k,0} = f_k - \sum_{l=0}^{k-1} g_{l,k-l}$. Therefore, we have for any $N \in \mathbb{N}_0$

(5.3.11) $$\mathfrak{q}(\mathcal{J}_f) - \sum_{j=0}^{N} \mathfrak{q}(\mathcal{J}_{\hat{f}_j} E^{-j}) \in \mathcal{L}^{-N-1}(X).$$

Let $\mathfrak{q}_N(\mathcal{J}_f) := \sum_{j=0}^{N} f_j E^{-j}$, from

(5.3.12) $$\begin{aligned} &\hat{S}\, \mathcal{J}_f\, \hat{S} - \hat{S}\, \mathfrak{q}_N(\mathcal{J}_f)\, \hat{S} \\ =&\hat{S}\, \mathcal{J}_f\, \hat{S} - \hat{S}\, \sum_{j=0}^{N} \mathfrak{q}(\mathcal{J}_{\hat{f}_j} E^{-j})\, \hat{S} + \hat{S}\, \left(\sum_{j=0}^{N} \mathfrak{q}(\mathcal{J}_{\hat{f}_j} E^{-j}) - \mathfrak{q}_N(\mathcal{J}_f)\right) \hat{S} \end{aligned}$$

We can see that

(5.3.13) $$\hat{S}\, \mathcal{J}_f\, \hat{S} - \hat{S}\, \sum_{j=0}^{N} \mathfrak{q}(\mathcal{J}_{\hat{f}_j} E^{-j})\, \hat{S} \in \Psi^{n-N-1}(X).$$

Since $\hat{E} = PE(P^*P)^{-1}P^*$, we have $E^{-j} = (P^*P)^{-1}P^*\hat{E}^{-j}P$ for every $j \in \mathbb{Z}$. Recall that for any $L \in \mathcal{L}(X)$ we have $\hat{S}\,L\,\hat{S} \equiv \hat{S}\,\mathrm{q}(L)\,\hat{S}$. Thus, we get

$$
\begin{aligned}
& T_f - \sum_{j=0}^{N} P\hat{S}\,\mathrm{q}(\mathcal{J}_{\hat{f}_j}E^{-j})\,\hat{S}(P^*P)^{-1}P^* \\
\equiv & T_f - \sum_{j=0}^{N} P\hat{S}\,\mathcal{J}_{\hat{f}_j}E^{-j}\,\hat{S}(P^*P)^{-1}P^* \\
= & T_f - \sum_{j=0}^{N} P\hat{S}\,(P^*P)^{-1}P^*\hat{f}_jPE^{-j}\,\hat{S}(P^*P)^{-1}P^* \\
= & T_f - \sum_{j=0}^{N} P\hat{S}\,(P^*P)^{-1}P^*\hat{f}_jPE^{-j}(P^*P)^{-1}P^*P\,\hat{S}(P^*P)^{-1}P^* \\
= & T_f - \sum_{j=0}^{N} P\hat{S}\,(P^*P)^{-1}P^*\hat{f}_j\hat{E}^{-j}P\,\hat{S}(P^*P)^{-1}P^* \\
= & T_f - \sum_{j=0}^{N} T_{\hat{f}_j\hat{E}^{-j}}.
\end{aligned}
\tag{5.3.14}
$$

From (5.3.13), we deduce that

$$
T_f - \sum_{j=0}^{N} P\hat{S}\,\mathrm{q}(\mathcal{J}_{\hat{f}_j}E^{-j})\,\hat{S}(P^*P)^{-1}P^* \in \Psi^{n-N-1}(\overline{M}).
$$

Therefore, we have $T_f - \sum_{j=0}^{N} T_{\hat{f}_j\hat{E}^{-j}} \in \Psi^{n-N-1}(\overline{M})$. □

Let $f, g \in \mathscr{C}^\infty(\overline{M})$,

$$
T_f\,T_g = P\mathscr{T}_{\mathcal{J}_f}\mathscr{T}_{\mathcal{J}_g}(P^*P)^{-1}P^*. \tag{5.3.15}
$$

From Theorem 4.6, there exists $f_j, g_j \in \mathscr{C}^\infty(X)$ such that

$$
\begin{aligned}
\mathrm{q}(\mathcal{J}_f) &= \sum_{j=0}^{\infty} f_jE^{-j} \quad \text{and} \\
\mathrm{q}(\mathcal{J}_g) &= \sum_{j=0}^{\infty} g_jE^{-j},
\end{aligned}
\tag{5.3.16}
$$

together with Theorem 4.23 and (4.4.2), we have the following:

THEOREM 5.18. *For any $f, g \in \mathscr{C}^\infty(\overline{M})$, there exists $\hat{C}_j(f,g) \in \mathcal{H}$ such that for any $N \in \mathbb{N}$*

$$T_f T_g - \sum_{j=0}^{N-1} T_{\hat{C}_j(f,g)\hat{E}^{-j}} \in \Psi^{n-N}(\overline{M}). \tag{5.3.17}$$

PROOF. Let

$$C'_j(\mathcal{J}_f, \mathcal{J}_g) \in \mathscr{C}^\infty(X)$$

satisfy (4.4.2). Let $\hat{C}_0(f,g) = P(C'_0(\mathcal{J}_f, \mathcal{J}_g)) \in \mathcal{H}$. From Theorem 4.6, there exists $k_{0,i} \in \mathscr{C}^\infty(X)$ such that

$$\mathfrak{q}(\mathcal{J}_{\hat{C}_0(f,g)}) = \sum_{i=0}^{\infty} k_{0,i} E^{-i}. \tag{5.3.18}$$

We can see that $k_{0,0} = \hat{C}_0(f,g)|_X = C'_0(\mathcal{J}_f, \mathcal{J}_g) \in \mathscr{C}^\infty(X)$. Therefore, we have

$$\mathscr{T}_{\mathcal{J}_{\hat{C}_0(f,g)}} - \mathscr{T}_{C'_0(\mathcal{J}_f, \mathcal{J}_g)} \in \Psi^{n-1}(X) \tag{5.3.19}$$

Let $\hat{C}_1(f,g) = P\left(C'_1(\mathcal{J}_f, \mathcal{J}_g) - k_{0,1}\right)$, then there exists $k_{1,i} \in \mathscr{C}^\infty(X)$ such that

$$\mathfrak{q}(\mathcal{J}_{\hat{C}_1(f,g)} E^{-1}) = \sum_{j=0}^{\infty} k_{1,j} E^{-1-j}. \tag{5.3.20}$$

We have $k_{1,0} = \hat{C}_1(f,g)|_X = C'_1(\mathcal{J}_f, \mathcal{J}_g) - k_{0,1}$, hence

$$\mathscr{T}_{(\mathcal{J}_{\hat{C}_0(f,g)} + \mathcal{J}_{\hat{C}_1(f,g)} E^{-1})} - \mathscr{T}_{(C'_0(\mathcal{J}_f, \mathcal{J}_g) + C'_1(\mathcal{J}_f, \mathcal{J}_g) E^{-1})} \in \Psi^{n-2}(X). \tag{5.3.21}$$

Inductively, let $\hat{C}_j(f,g) := P\left(C'_j(\mathcal{J}_f, \mathcal{J}_g - \sum_{l=0}^{j-1} k_{l,j-l})\right)$, we get for any $N \in \mathbb{N}$

$$\sum_{j=0}^{N-1} \mathscr{T}_{(\mathcal{J}_{\hat{C}_j(f,g)} E^{-j})} - \sum_{i=0}^{N-1} \mathscr{T}_{C'_i(\mathcal{J}_f, \mathcal{J}_g) E^{-i}} \in \Psi^{n-N}(X). \tag{5.3.22}$$

Therefore, from Theorem 4.23 and (5.3.22), we have

$$\mathscr{T}_{\mathcal{J}_f} \mathscr{T}_{\mathcal{J}_g} - \sum_{j=0}^{N-1} \mathscr{T}_{(\mathcal{J}_{\hat{C}_j(f,g)} E^{-j})} \in \Psi^{n-N}(X) \tag{5.3.23}$$

for any $N \in \mathbb{N}$. Since

$$\begin{aligned} \mathscr{T}_{\mathcal{J}_{\hat{C}_j(f,g)} E^{-j}} &= \hat{S}(P^*P)^{-1} P^* \hat{C}_j(f,g) P E^{-j} \hat{S} \\ &= \hat{S}(P^*P)^{-1} P^* \hat{C}_j(f,g) P E^{-j} (P^*P)^{-1} P^* P \hat{S} \\ &= \hat{S}(P^*P)^{-1} P^* \hat{C}_j(f,g) \hat{E}^{-j} P \hat{S} \\ &= \mathscr{T}_{\mathcal{J}_{\hat{C}_j(f,g)\hat{E}^{-j}}} \end{aligned} \tag{5.3.24}$$

we conclude that for any $N \in \mathbb{N}$

$$T_f T_g - \sum_{j=0}^{N-1} T_{\hat{C}_j(f,g)\hat{E}^{-j}} \in \Psi^{n-N}(X). \tag{5.3.25}$$

□

From the proofs of Proposition 5.17 and Theorem 5.18 we obtain the following more general statement.

COROLLARY 5.19 (Corollary 5.6). *Let $L \in \mathcal{L}^m(X)$ and define*

$$\hat{L} := PL(P^*P)^{-1}P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}).$$

Then there exist functions $\hat{l}_j \in \mathcal{H}$ such that, for every $N \in \mathbb{N}$,

$$T_{\hat{L}} - \sum_{j=0}^{N-1} T_{\hat{l}_j \hat{E}^{m-j}} \in \Psi^{n+m-N}(\overline{M}). \tag{5.3.26}$$

We now introduce the space of operators that will be subject to quantization. It is well known that the space of harmonic functions on $\overline{M}$ is not closed under the standard pointwise multiplication. We therefore construct a natural product structure induced from the boundary as follows: Let

$$\begin{aligned} h : \mathcal{H} \times \mathcal{H} &\to \mathcal{H} \\ (f,g) &\mapsto h(f,g) := P((fg)|_X) \end{aligned} \tag{5.3.27}$$

be a bilinear map on $\mathcal{H}$. We have

PROPOSITION 5.20. *$(\mathcal{H}, h(\cdot\,,\cdot))$ is an associative commutative algebra.*

PROOF. For any $f_i \in \mathcal{H}, \quad j = 1,2,3.$, it is obvious that

$$h(f_1, f_2) = P((f_1 f_2)|_X) = P((f_2 f_1)|_X) = h(f_2, f_1).$$

Since $P(f|_X)|_X = f|_X$, for any $f \in \mathscr{C}^\infty(\overline{M})$. It is straightforward to see that

$$h(h(f_1, f_2), f_3) = h(f_1, h(f_2, f_3)).$$

□

Let

$$\hat{\mathcal{H}} := \left\{\hat{L} : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}) : (P^*P)^{-1}P^*\hat{L}P \in \mathcal{L}(X)\right\} \tag{5.3.28}$$

denote the space of operators on $\overline{M}$ such that $\mathcal{J}_{\hat{L}}$ defines a pseudodifferential operator on X. It is clear that $\mathcal{H} \subset \hat{\mathcal{H}}$. Recall that the contact form ξ defines a transversal Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_\xi$ on X. Let

$$\begin{aligned} \{\,\cdot\,,\,\cdot\,\}^{\mathcal{H}} : \mathcal{H} \times \mathcal{H} &\to \hat{\mathcal{H}} \\ (f,g) &\mapsto \{\,f\,,\,g\,\}^{\mathcal{H}} := P(i\{\,f|_X\,,\,g|_X\,\}_\xi)\hat{E}^{-1}. \end{aligned} \tag{5.3.29}$$

be a bilinear map on $\mathcal{H}$. Let

$$\hat{\mathcal{Q}} := \left\{ L \in \hat{\mathcal{H}} : L = \sum_{j=0}^{\infty} \hat{l}_j \hat{E}^{m-j},\ \hat{l}_j \in \mathcal{H}\ \forall j \in \mathbb{N}_0,\ m \in \mathbb{Z} \right\}. \tag{5.3.30}$$

The algebraic structure of $\mathcal{H}$ induces a commutative multiplication on $\hat{\mathcal{Q}}$ as follows. Let $F, G \in \hat{\mathcal{Q}}$ be given by

$$F = \sum_{j=0}^{\infty} \hat{f}_j \hat{E}^{m_1-j}, \qquad G = \sum_{k=0}^{\infty} \hat{g}_k \hat{E}^{m_2-k}, \tag{5.3.31}$$

where $\hat{f}_j, \hat{g}_k \in \mathcal{H}$. We define

$$h(F,G) := \sum_{j,k=0}^{\infty} h(\hat{f}_j, \hat{g}_k)\, \hat{E}^{m_1+m_2-j-k}. \tag{5.3.32}$$

In analogy with Def. 4.19, we extend the bracket to operators of the form $\sum_{j=0}^{\infty} f_j \hat{E}^{m-j}$, where each $f_j \in \mathcal{H}$ and $m \in \mathbb{Z}$. In other words, we extend the bracket to $\hat{\mathcal{Q}}$, using the same notation as before, for $f, g \in \mathcal{H}$ and $n_1, n_2 \in \mathbb{Z}$, the bracket $\{\cdot, \cdot\}^{\mathcal{H}}$ is determined by the following rules:

(1) $\{\hat{E}^{n_1}, f\}^{\mathcal{H}} = P(n_1 \mathcal{T} f|_X)\hat{E}^{n_1-1} = -\{f, \hat{E}^{n_1}\}^{\mathcal{H}}$.
(2) $\{\hat{E}^{n_1}, \hat{E}^{n_2}\}^{\mathcal{H}} = 0$.
(3) $\{f\hat{E}^{n_1}, g\}^{\mathcal{H}} = \{f, g\}^{\mathcal{H}}\hat{E}^{n_1} + h\big(f, P(n_1 \mathcal{T} g|_X)\big)\hat{E}^{n_1-1}$.
(4) $\{f\hat{E}^{n_1}, \hat{E}^{n_2}\}^{\mathcal{H}} = \{f, \hat{E}^{n_2}\}^{\mathcal{H}}\hat{E}^{n_1}$.

PROPOSITION 5.21. *The bilinear operation $\{\cdot, \cdot\}^{\mathcal{H}}$ is anti-symmetric, satisfies the Jacobi identity, and acts as a derivation in each argument. Consequently, it endows $\hat{\mathcal{Q}}$ with the structure of a Poisson algebra, i.e., it defines a Poisson bracket on $\hat{\mathcal{Q}}$. With the same notation in (5.3.31), the Poisson bracket of F and G is*

$$\{F, G\}^{\mathcal{H}} := \sum_{j,k=0}^{\infty} \{\hat{f}_j \hat{E}^{m_1-j}, \hat{g}_k \hat{E}^{m_2-k}\}^{\mathcal{H}}. \tag{5.3.33}$$

PROOF. Let $f, g \in \mathcal{H}$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$. We have

$$\{f\hat{E}^{n_1}, g\hat{E}^{n_2}\}^{\mathcal{H}} = P\big(i\{f|_X, g|_X\}_{\xi} + n_1 f|_X \mathcal{T} g|_X - n_2 g|_X \mathcal{T} f|_X\big)\hat{E}^{n_1+n_2-1}.$$

The Jacobi identity and the Leibniz rule are obtained immediately from the Poisson structure on Σ, by applying the same argument and calculation as in Prop. 4.22. □

Notice that for any $\hat{L} \in \hat{\mathcal{H}}$, from Corollary 5.19, suppose $\operatorname{ord}(\mathcal{J}_{\hat{L}}) = m$, there exists $\hat{l}_j \in \mathcal{H}$ such that

$$T_{\hat{L}} \equiv \sum_{j=0}^{\infty} T_{\hat{l}_j \hat{E}^{m-j}},$$

the explicit formula for $\hat{l}_j$ is given by

$$\hat{l}_j = P\left(l_j - \sum_{k=0}^{l-1} a_{k,l-k}\right), \tag{5.3.34}$$

where l_j and $a_{k,l-k}$ is given by

$$\begin{aligned} \mathsf{q}(\mathcal{J}_{\hat{L}}) &= \sum_{j=0}^{\infty} l_j E^{m-j} \\ \mathsf{q}(\mathcal{J}_{\hat{l}_j} E^{m-j}) &= \sum_{k=0}^{\infty} a_{j,k} E^{m-j-k}. \end{aligned} \tag{5.3.35}$$

Conversely, let $\hat{f}_j \in \mathcal{H}$ for $j \in \mathbb{N}_0$, then $F := \sum_{j=0}^{\infty} \hat{f}_j \hat{E}^{m-j}$ defines an operator in $\hat{\mathcal{H}}$. From Theorem 5.18 and Corollary 5.19, we obtain the result for the composition of Toeplitz operators:

COROLLARY 5.22. *Let $L_1, L_2 \in \hat{\mathcal{H}}$, there exists $\hat{C}_j(L_1, L_2) \in \mathcal{H}$ such that for any $N \in \mathbb{N}$*

$$T_{L_1} \circ T_{L_2} - \sum_{j=0}^{N-1} T_{\hat{C}_j(L_1,L_2)\hat{E}^{m_1+m_2-j}} \in \Psi^{n+m_1+m_2-N}(\overline{M}), \tag{5.3.36}$$

where $m_i = \operatorname{ord}(\mathcal{J}_{L_i})$ for $i = 1, 2$.

Keeping in mind that in (5.3.30) the space $\hat{\mathcal{Q}}$ consists of reducible operators of asymptotic summand type, and recalling that $\mathcal{A}$ denotes the vector space of asymptotic expansions of pseudodifferential operators on X (see (4.3.3)), we now determine how $\hat{\mathcal{Q}}$ is related to $\mathcal{A}$.

LEMMA 5.23. *If $\hat{f} \in \mathcal{H}$, let $f = \hat{f}|_X$, then $\mathcal{J}_{\hat{f}} = f$.*

PROOF. It is known that $H := P(P^*P)^{-1}P^*$ is the orthogonal projection from $\mathscr{C}^\infty(\overline{M})$ onto $\mathcal{H}$ (see [4]). Since $\hat{f} \in \mathcal{H}$,

$$H\hat{f} = \hat{f} = P\gamma\hat{f},$$

where $\gamma : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X)$ is the restriction map. Thus

$$P(P^*P)^{-1}P^*\hat{f}P = \hat{f}P = P\gamma\hat{f}P,$$

so

$$(P^*P)^{-1}P^*\hat{f}P = \gamma\hat{f}P = f \in \mathscr{C}^\infty(X) \subset \mathcal{L}^0(X).$$

$\square$

PROPOSITION 5.24. *Suppose $\hat{L} \in \hat{\mathcal{Q}}$ such that*

$$\hat{L} = \sum_{j=0}^{\infty} \hat{l}_j \hat{E}^{m-j}$$

and

$$\mathsf{q}(\mathcal{J}_{\hat{L}}) = \sum_{j=0}^{\infty} l_j E^{m-j},$$

then $l_j = \hat{l}_j|_X$ for every j.

PROOF. From Theorem 4.6, for any $N \in \mathbb{N}$, we have

$$\mathscr{T}_{\mathcal{J}_{\hat{L}}} - \sum_{j=0}^{N-1} \mathscr{T}_{l_j E^{m-j}} \in \Psi^{n+m-N}(X). \tag{5.3.37}$$

We also have

$$\mathscr{T}_{\mathcal{J}_{\hat{L}}} - \sum_{j=0}^{N-1} \mathscr{T}_{\mathcal{J}_{\hat{l}_j} E^{m-j}} \in \Psi^{n+m-N}(X). \tag{5.3.38}$$

For each j, from Theorem 4.6 there exist $a_{j,k} \in \mathscr{C}^\infty(X)$ such that

$$\mathscr{T}_{\mathcal{J}_{\hat{l}_j} E^{m-j}} - \sum_{k=0}^{N-1} \mathscr{T}_{a_{j,k} E^{m-j-k}} \in \Psi^{n+m-j-N}(X). \tag{5.3.39}$$

From Lemma 5.23, $a_{j,k} = 0$ for $k \geq 1$. We have known that

$$a_{j,0}(x) = \sigma^0_{\mathcal{J}_{\hat{l}_j} E^{m-j}}(x, \xi(x)) = \hat{l}_j|_X. \tag{5.3.40}$$

From (5.3.37), (5.3.38), and (5.3.39), let $N_j := N - j - 1$ we can see that

$$\sum_{j=0}^{N-1} \left(\sum_{k=0}^{N_j} \left(\mathscr{T}_{a_{j,k} E^{m-k}} \right) - \mathscr{T}_{l_j E^{m-j}} \right) \in \Psi^{n+m-N}(X). \tag{5.3.41}$$

From (5.3.40), (5.3.41) and the uniqueness of the asymptotic expansion, we get

$$l_j = \sum_{k=0}^{j} a_{k,j-k} = a_{j,0}. \tag{5.3.42}$$

From (5.3.34) and (5.3.35), we have

$$\hat{l}_j = P(l_j - \sum_{k=0}^{j-1} a_{k,j-k}) = P(a_{j,0}) = P(l_j). \tag{5.3.43}$$

□

We define a map

$$\mathscr{Q} : \hat{\mathcal{H}} \to \hat{\mathcal{Q}}$$

$$L \mapsto \sum_{j=0}^{\infty} l_j \hat{E}^{\operatorname{ord}(\mathcal{J}_L)-j}. \tag{5.3.44}$$

5.4. Quantization of the operator space

The presence of the coefficients $\hat{C}_j(\,\cdot\,,\cdot\,)$ provides a deformation quantization on the harmonic functions on $\overline{M}$, in the sense that the star product of any two harmonic functions is realized as an element in $\hat{\mathcal{Q}}$.

THEOREM 5.25 (Theorem 5.7). *Let F, $G \in \hat{\mathcal{Q}}$, the star product $*_{\mathcal{H}}$ of F and G is defined by*

$$F *_{\mathcal{H}} G = \mathscr{Q}(\widehat{\mathcal{J}_F * \mathcal{J}_G}), \tag{5.4.1}$$

where $\widehat{\mathcal{J}_F * \mathcal{J}_G} = P \circ (\mathcal{J}_F * \mathcal{J}_G) \circ (P^*P)^{-1}P \in \hat{\mathcal{H}}$. The product $*_{\mathcal{H}}$ is associative, and if $F = \sum_{j=0}^{\infty} \hat{f}_j \hat{E}^{m_1-j}$ and $G = \sum_{k=0}^{\infty} \hat{g}_k \hat{E}^{m_2-k}$ there exists $\hat{C}_j(F,G) \in \mathcal{H}$ such that

(1) $F *_{\mathcal{H}} G = \sum_{j=0}^{\infty} \hat{C}_j(F,G)\hat{E}^{m_1+m_2-j}$,
(2) $\hat{C}_0(F,G) = \hat{C}_0(G,F) = h(\hat{f}_0, \hat{g}_0)$,
(3) $\big(\hat{C}_1(F,G) - \hat{C}_1(G,F)\big)\hat{E}^{-1} = \{\hat{f}_0, \hat{g}_0\}^{\mathcal{H}}$,

PROOF. From Theorem 5.18, we can see that

$$\hat{C}_0(F,G) = P(C_0'(\mathcal{J}_F, \mathcal{J}_G)). \tag{5.4.2}$$

There exists f_j, $g_k \in \mathscr{C}^\infty(X)$ for j, $k \in \mathbb{N}_0$ such that

$$\mathsf{q}(\mathcal{J}_F) = \sum_{j=0}^{\infty} f_j E^{m_1-j}, \text{ and}$$
$$\mathsf{q}(\mathcal{J}_G) = \sum_{k=0}^{\infty} g_k E^{m_2-k}.$$

From the equalities

$$\mathsf{q}(\mathcal{J}_F) = \sum_{j=0}^{\infty} \mathsf{q}(\mathcal{J}_{\hat{f}_j} E^{m_1-j}), \text{ and}$$
$$\mathsf{q}(\mathcal{J}_G) = \sum_{k=0}^{\infty} \mathsf{q}(\mathcal{J}_{\hat{g}_k} E^{m_2-k}) \tag{5.4.3}$$

we have $f_0 = \hat{f}_0|_X$ and $g_0 = \hat{g}_0|_X$. From Theorem 4.23, we have

$$C_0'(\mathcal{J}_F, \mathcal{J}_G) = \sigma^0_{\mathcal{J}_F}(x, \xi(x))\sigma^0_{\mathcal{J}_G}(x, \xi(x)) = f_0 \cdot g_0 = (\hat{f}_0 \cdot \hat{g}_0)|_X. \tag{5.4.4}$$

Thus, we get

$$\hat{C}_0(F,G) = P(C_0'(\mathcal{J}_F, \mathcal{J}_G)) = P((\hat{f}_0\hat{g}_0)|_X) = h(\hat{f}_0, \hat{g}_0). \tag{5.4.5}$$

Since $\mathcal{H}$ is commutative, we have $\hat{C}_0(F,G) = \hat{C}_0(G,F)$. Let $a_{0,j}(\mathcal{J}_F, \mathcal{J}_G) \in \mathscr{C}^\infty(X)$ be the smooth function such that

$$\mathfrak{q}(\mathcal{J}_{\hat{C}_0(F,G)} E^{m_1+m_2}) = \sum_{j=0}^{\infty} a_{0,j}(\mathcal{J}_F, \mathcal{J}_G) E^{m_1+m_2-j}. \tag{5.4.6}$$

Note that $a_{0,j}(\mathcal{J}_F, \mathcal{J}_G) = a_{0,j}(\mathcal{J}_G, \mathcal{J}_F)$ for every $j \in \mathbb{N}$ because of (5.4.5). From Theorem 5.18, we have

$$\hat{C}_1(F,G) = P(C_1'(\mathcal{J}_F, \mathcal{J}_G) - a_{0,1}(\mathcal{J}_F, \mathcal{J}_G)). \tag{5.4.7}$$

The commutator is

$$\begin{aligned} \hat{C}_1(F,G) - \hat{C}_1(G,F) &= P\left(C_1'(\mathcal{J}_F, \mathcal{J}_G) - C_1'(\mathcal{J}_G, \mathcal{J}_F) - a_{0,1}(\mathcal{J}_F, \mathcal{J}_G) + a_{0,1}(\mathcal{J}_G, \mathcal{J}_F)\right) \\ &= P(C_1'(\mathcal{J}_F, \mathcal{J}_G) - C_1'(\mathcal{J}_G, \mathcal{J}_F)). \end{aligned} \tag{5.4.8}$$

From Theorem 4.23 and (4.4.25), we have

$$\begin{aligned} &C_1'(\mathcal{J}_F, \mathcal{J}_G) \\ &\quad = \mathscr{K}_{0,0,1}(f_0E^{m_1}, g_0E^{m_2}) + \mathscr{K}_{1,0,0}(f_1E^{m_1-1}, g_0E^{m_2}) + \mathscr{K}_{0,1,0}(f_0E^{m_1}, g_1E^{m_2-1}), \\ &C_1'(\mathcal{J}_G, \mathcal{J}_F) \\ &\quad = \mathscr{K}_{0,0,1}(g_0E^{m_2}, f_0E^{m_1}) + \mathscr{K}_{1,0,0}(g_1E^{m_2-1}, f_0E^{m_1}) + \mathscr{K}_{0,1,0}(g_0E^{m_2}, f_1E^{m_1-1}), \end{aligned} \tag{5.4.9}$$

and

$$\begin{aligned} &\mathscr{K}_{0,0,1}(f_0E^{m_1}, g_0E^{m_2}) - \mathscr{K}_{0,0,1}(g_0E^{m_2}, f_0E^{m_1}) = i\{\, f_0\,,\, g_0\,\}_\xi, \\ &\mathscr{K}_{1,0,0}(f_1E^{m_1-1}, g_0E^{m_2}) - \mathscr{K}_{1,0,0}(g_1E^{m_2-1}, f_0E^{m_1}) = 0, \\ &\mathscr{K}_{0,1,0}(f_0E^{m_1}, g_1E^{m_2-1}) - \mathscr{K}_{0,1,0}(g_0E^{m_2}, f_1E^{m_1-1}) = 0. \end{aligned} \tag{5.4.10}$$

Thus, we get

$$\begin{aligned} (\hat{C}_1(F,G) - \hat{C}_1(G,F))\hat{E}^{-1} &= P(C_1'(\mathcal{J}_F, \mathcal{J}_G) - C_1'(\mathcal{J}_G, \mathcal{J}_F))\hat{E}^{-1} \\ &= P(i\{\, f_0\,,\, g_0\,\}_\xi)\hat{E}^{-1} \\ &= P(i\{\, \hat{f}_0|_X\,,\, \hat{g}_0|_X\,\}_\xi)\hat{E}^{-1} \\ &= \{\, \hat{f}_0\,,\, \hat{g}_0\,\}^{\mathcal{H}}. \end{aligned} \tag{5.4.11}$$

We show the associativity of $*_{\mathcal{H}}$. Let F, G, and H in $\hat{\mathcal{Q}}$. From Theorem 4.23, we have

$$\mathscr{T}_{\mathcal{J}_F}\mathscr{T}_{\mathcal{J}_G} \equiv \mathscr{T}_{\mathcal{J}_F * \mathcal{J}_G} \mod \Psi^{-\infty}(X). \tag{5.4.12}$$

Since $*$ is associative, we have

$$
\begin{aligned}
(\mathscr{T}_{\mathcal{J}_F}\mathscr{T}_{\mathcal{J}_G})\mathscr{T}_{\mathcal{J}_H} &\equiv \mathscr{T}_{\mathcal{J}_F*\mathcal{J}_G}\mathscr{T}_{\mathcal{J}_H} \\
&\equiv \mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_F*\mathcal{J}_G)}}}\mathscr{T}_{\mathcal{J}_H} \\
&\equiv \mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_F*\mathcal{J}_G)}}*\mathcal{J}_H} \\
&\equiv \mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_F*\mathcal{J}_G)}}*\mathcal{J}_{\mathcal{H}})}}} \qquad \bmod \Psi^{-\infty}(X).
\end{aligned}
\tag{5.4.13}
$$

Similarly, we have

$$
\mathscr{T}_{\mathcal{J}_F}(\mathscr{T}_{\mathcal{J}_G}\mathscr{T}_{\mathcal{J}_H}) \equiv \mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_{\mathcal{F}}*\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_G*\mathcal{J}_H)}})}}} \qquad \bmod \Psi^{-\infty}(X).
\tag{5.4.14}
$$

Thus, we get

$$
\mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_F*\mathcal{J}_G)}}*\mathcal{J}_{\mathcal{H}})}}} \equiv \mathscr{T}_{\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_{\mathcal{F}}*\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_G*\mathcal{J}_H)}})}}} \qquad \bmod \Psi^{-\infty}(X).
\tag{5.4.15}
$$

From the uniqueness of the asymptotic expansion in 4.6 and Proposition 5.24, we can see that

$$
\widehat{\mathscr{Q}(\mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_F * \mathcal{J}_G)}} * \mathcal{J}_H)} = \widehat{\mathscr{Q}(\mathcal{J}_F * \mathcal{J}_{\widehat{\mathscr{Q}(\mathcal{J}_G * \mathcal{J}_H)}})}
\tag{5.4.16}
$$

which implies

$$
(F *_{\mathcal{H}} G) *_{\mathcal{H}} H = F *_{\mathcal{H}} (G *_{\mathcal{H}} H).
\tag{5.4.17}
$$

□

In Theorem 5.25, the expansion is carried out at the level of function coefficients.In direct analogy with Theorem 4.23, we construct operator-valued coefficients $\hat{\mathcal{C}}_j \in \hat{\mathcal{Q}}$, which then yield a deformation quantization of the Poisson algebra $(\hat{\mathcal{Q}}, h(\cdot\,,\cdot), \{\,\cdot\,,\cdot\,\}^{\mathcal{H}})$.

THEOREM 5.26 (Theorem 5.8). *For any $F, G \in \hat{\mathcal{Q}}$ satisfying (5.3.31), there exists a sequence $\hat{\mathcal{C}}_j(F,G) \in \hat{\mathcal{Q}}$ with $\mathrm{ord}(\mathcal{J}_{\hat{\mathcal{C}}_j(F,G)}) = m_1 + m_2$ for all j, such that for any $N \in \mathbb{N}$,*

$$
T_{F *_{\mathcal{H}} G} - \sum_{p=0}^{N-1} T_{\hat{\mathcal{C}}_j(F,G)\hat{E}^{-j}} \in \Psi^{n+m_1+m_2-N}(\overline{M}),
\tag{5.4.18}
$$

where

$$
\begin{aligned}
\hat{\mathcal{C}}_0(F,G) &= h(F,G), \\
(\hat{\mathcal{C}}_1(F,G) - \hat{\mathcal{C}}_1(G,F))\hat{E}^{-1} &= \{\,F\,,\,G\,\}^{\mathcal{H}}.
\end{aligned}
\tag{5.4.19}
$$

PROOF. From Proposition 5.24, we have

$$
\begin{aligned}
\mathsf{q}(\mathcal{J}_F) &= \sum_{j=0}^{\infty} f_j E^{m_1-j} \\
\mathsf{q}(\mathcal{J}_G) &= \sum_{k=0}^{\infty} g_k E^{m_2-k},
\end{aligned} \tag{5.4.20}
$$

where $f_j = \hat{f}_j|_X$ and $g_j = \hat{g}_j|_X$ for every $j \in \mathbb{N}_0$. From Theorem 4.23 and (4.4.17), we have

$$
\mathcal{J}_F * \mathcal{J}_G = \hat{\mathsf{q}}(\mathscr{T}_{\mathsf{q}(\mathcal{J}_F)}\mathscr{T}_{\mathsf{q}(\mathcal{J}_G)}) = \sum_{j,k,p=0}^{\infty} \mathscr{K}_{j,k,p}(f_j E^{m_1-j}, g_k E^{m_2-k}) E^{m_1+m_2-j-k-p}. \tag{5.4.21}
$$

From Theorem 5.25, there exists $\hat{C}_p(F,G) \in \mathcal{H}$ such that

$$
\begin{aligned}
F *_{\mathcal{H}} G &= \sum_{p=0}^{\infty} \hat{C}_p(F,G)\hat{E}^{m_1+m_2-p} \\
&= \mathscr{Q}\left(\widehat{\sum_{j,k,p=0}^{\infty} \mathscr{K}_{j,k,p}(f_j E^{m_1-j}, g_k E^{m_2-k}) E^{m_1+m_2-j-k-p}}\right).
\end{aligned} \tag{5.4.22}
$$

From Proposition 5.24 and the uniqueness of the asymptotic expansion in Theorem 4.6, we have

$$
\hat{C}_p(F,G) = \sum_{j+k+l=p} P\big(\mathscr{K}_{j,k,l}(f_j E^{m_1-j}, g_k E^{m_2-k})\big). \tag{5.4.23}
$$

Let

$$
\hat{\mathcal{C}}_p(F,G) = \sum_{j,k=0}^{\infty} P\big(\mathscr{K}_{j,k,p}(f_j E^{m_1-j}, g_k E^{m_2-k})\big)\hat{E}^{m_1+m_2-j-k}, \tag{5.4.24}
$$

as in (4.4.27). Then

$$
\sum_{p=0}^{N-1} \mathcal{J}_{\hat{C}_p(F,G)\hat{E}^{m_1+m_2-p}} - \sum_{p=0}^{N-1} \mathcal{J}_{\hat{\mathcal{C}}_p(F,G)\hat{E}^{-p}} \in \mathcal{L}^{m_1+m_2-N}(X), \tag{5.4.25}
$$

which implies (5.4.18). Since

$$
\mathscr{K}_{j,k,0}(f_j E^{m_1-j}, g_k E^{m_2-k}) = f_j g_k
$$

for all $j,k \in \mathbb{N}_0$ and from (4.4.25)

$$
\begin{aligned}
&\mathscr{K}_{j,k,1}(f_j E^{m_1-j}, g_k E^{m_2-k}) - \mathscr{K}_{j,k,1}(g_k E^{m_2-k}, f_j E^{m_1-j}) \\
&\quad = i\{\, f_j\,,\, g_k \,\}_{\xi} + (m_1-j) f_j \mathcal{T} g_k - (m_2-k) g_k \mathcal{T} f_j,
\end{aligned} \tag{5.4.26}
$$

from (5.4.24) we get

$$\hat{\mathcal{C}}_0(F,G) = \sum_{j,k=0}^{\infty} h(\hat{f}_j, \hat{g}_k)\hat{E}^{m_1+m_2-j-k} \tag{5.4.27}$$

and

$$\begin{aligned} \big(\hat{\mathcal{C}}_1(F,G) - \hat{\mathcal{C}}_1(G,F)\big)\hat{E}^{-1} &= \sum_{j,k=0}^{\infty} \{\, \hat{f}_j \hat{E}^{m_1-j} \,,\, \hat{g}_k \hat{E}^{m_2-k} \,\}^{\mathcal{H}} \\ &= \{\, F \,,\, G \,\}^{\mathcal{H}} \end{aligned} \tag{5.4.28}$$

which implies (5.4.19). □

Part 3

The Subleading term of semi-classical Toeplitz Operators

CHAPTER 6

Semi-classical Toeplitz operators

In this part, we return to the CR manifold to study semi-classical Toeplitz operators based on the published article [6], joint work with Chin-Yu Hsiao and Hendrik Herrmann.

Throughout this part, we always assume the CR manifold $(X, T^{1,0}X)$ is a compact, orientable, strictly pseudoconvex embeddable CR manifold of dimension $2n+1$, $n \geq 1$ without further notice. We always fix a global one form $\xi \in \mathscr{C}^\infty(X, T^*X)$ such that for any $x \in X$ we have $\xi(x) \neq 0$, $\operatorname{Ker} \xi(x) = \operatorname{Re} T_x^{1,0}X$ and the respective Levi form $\mathcal{L}_x$ is positive definite. Denote by $\mathcal{T}$ the Reeb vector field associated to ξ, choose a volume form dV on X with $\mathcal{L}_{\mathcal{T}} dV \equiv 0$.

We now formulate the main object in this part: Let Π be the Szego projection in Def. 2.14, we define the Toeplitz operator A associated with a Reeb vector field $\mathcal{T} \in \mathscr{C}^\infty(X, TX)$ by

$$A := \Pi(-i\mathcal{T})\Pi : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X).$$

We extend A to L^2 space:

$$A := \Pi(-i\mathcal{T})\Pi : \operatorname{Dom} A \subset L^2(X) \to L^2(X), \tag{6.0.1}$$

where $\operatorname{Dom} A := \left\{u \in L^2(X);\ Au \in L^2(X)\right\}$.

6.1. Introduction and main results

It was shown in [13, Theorem 3.3] that A is a self-adjoint operator. We consider the operator $\chi_k(A)$ defined by the functional calculus of A, where χ is a smooth function with compact support in the positive real line and $\chi_k(\lambda) := \chi(k^{-1}\lambda)$. Let $\chi_k(A)(x, y)$ denote the distribution kernel of $\chi_k(A)$. We now restate the following consequence of the result proved in [13].

THEOREM 6.1. *Let $(X, T^{1,0}X)$ be a compact orientable strictly pseudoconvex embeddable CR manifold of dimension $2n+1$, $n \geq 1$. Fix a global one form $\xi \in \mathscr{C}^\infty(X, T^*X)$ such that for any $x \in X$ we have $\xi(x) \neq 0$, $\operatorname{Ker} \xi(x) = \operatorname{Re} T_x^{1,0}X$ and the respective Levi form $\mathcal{L}_x$ is positive definite. Denote by $\mathcal{T}$ the Reeb vector field associated to ξ, choose a volume form dV on X with $\mathcal{L}_{\mathcal{T}} dV \equiv 0$ and consider the operator A given by (6.0.1). Let (D, x) be any coordinate patch and let $\varphi : D \times D \to \mathbb{C}$ be any phase function satisfying (2.2.10) and (2.2.11). Then, for any $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ putting*

$\chi_k(\lambda) := \chi(k^{-1}\lambda)$, $k, \lambda \in \mathbb{R}_+$, one has for the distributional kernel $\chi_k(A)(x,y)$ of $\chi_k(A)$ that

$$\chi_k(A)(x,y) = \int_0^{+\infty} e^{ikt\varphi(x,y)} b^\chi(x,y,t,k)dt + O\left(k^{-\infty}\right) \text{ on } D \times D, \tag{6.1.1}$$

where $b^\chi(x,y,t,k) \in S^{n+1}_{\rm loc}(1; D \times D \times \mathbb{R}_+)$,

$$\begin{aligned} & b^\chi(x,y,t,k) \sim \sum_{j=0}^{+\infty} b^\chi_j(x,y,t)k^{n+1-j} \text{ in } S^{n+1}_{\rm loc}(1; D \times D \times \mathbb{R}_+), \\ & b^\chi_j(x,y,t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+),\ j = 0,1,2,\ldots, \\ & b^\chi_0(x,x,t) = \frac{1}{2\pi^{n+1}}\frac{dV_\xi}{dV}(x)\,\chi(t)\,t^n \not\equiv 0,\ \text{ for every } x \in D \text{ with } \lambda(x) = 1, \end{aligned} \tag{6.1.2}$$

with dV_ξ given by (2.2.13) and for $I^\chi := \operatorname{supp}\chi$,

$$\operatorname{supp}_t b^\chi(x,y,t,k),\ \operatorname{supp}_t b^\chi_j(x,y,t) \subset I^\chi,\quad j = 0,1,2,\ldots\ . \tag{6.1.3}$$

Here, supp_t is defined as follows. Let U be an open subset of $\mathbb{R}^N$, and let

$$F : U \times \mathbb{R} \to \mathbb{R},\ (x,t) \mapsto F(x,t)$$

be a function on $U \times \mathbb{R}$, for a subset $I \subset \mathbb{R}$, we write $\operatorname{supp}_t F(x,t) \subset I$ if for any $(x,t) \in U \times \mathbb{R}$ with $t \notin I$ $F(x,t) = 0$.
In particular, we have

$$\chi_k(A)(x,x) \sim \sum_{j=0}^{+\infty} b^\chi_j(x)k^{n+1-j} \text{ in } S^{n+1}_{\rm loc}(1; X), \tag{6.1.4}$$

where $b^\chi_j(x) = \int b^\chi_j(x,x,t)dt$, $b^\chi_j(x,y,t)$ is as in (6.1.2), $j = 0,1,\ldots$.

The expansion (6.1.1) can be viewed as a semi-classical analogue of Boutet de Monvel–Sjöstrand's result on strictly pseudoconvex CR manifolds (see (2.2.11)). A natural and fundamental problem is to compute $b^\chi_1(x)$ in (6.1.4). This calculation can be regarded as a local index theorem in CR geometry. In complex geometry, the second coefficient in the Bergman kernel asymptotic expansion plays a key role in Kähler geometry (see [25]). We expect that determining b_1 will be important for the CR Donaldson program [8] and related topics in CR geometry. In this part, we obtain the coefficient b_1. We also generalize the result in [22] by considering a general Reeb-invariant volume, which plays an analogous role in computing the second coefficient of the Bergman kernel in [16]. We now formulate the main result of this part:

THEOREM 6.2. *Under the same assumptions and notations as in Theorem 6.1, we have*

$$b^\chi_1(x) = \frac{1}{2\pi^{n+1}} e^{-2(n+1)f(x)}\left(\frac{1}{2}R_{scal}(x) + (n+1)\Delta_b f(x)\right)\int \chi(t)t^{n-1}dt, \tag{6.1.5}$$

where $f(x) := \frac{1}{2(n+1)} \log \frac{dV(x)}{dV_\xi(x)} \in \mathscr{C}^\infty(X)$ *with* dV_ξ *given by* (2.2.13), R_{scal} *is the Tanaka-Webster scalar curvature with respect to the contact form* ξ *(see* (2.2.9)*) and* $\Delta_b : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$ *denotes the CR sublaplacian operator (see* (2.2.6)*).*

To prove Theorem 6.2, we need to understand how the symbol $b_j(x,y,t)$ in (6.1.2) depends on χ and t. We have the following.

THEOREM 6.3. *Under the same assumptions and notations as in Theorem 6.1, let* (D,x) *be any coordinate patch and let* $\varphi : D \times D \to \mathbb{C}$ *be any phase function satisfying* (2.2.10) *and* (2.2.11)*. Then there exist smooth functions*

$$a_{j,s}(x,y) \in \mathscr{C}^\infty(D \times D), \quad j,s = 0,1,\ldots, \tag{6.1.6}$$

such that for any $\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+)$ *we can take* $b^\chi_j(x,y,t)$ *in* (6.1.2)*,* $j = 0,1,\ldots$*, so that*

$$b^\chi_j(x,y,t) = \sum_{s=0}^{+\infty} a_{j,s}(x,y)\chi^{(s)}(t)t^{n+s-j}, \quad j = 0,1,\ldots, \tag{6.1.7}$$

where $\chi^{(s)} := (\partial/\partial t)^s\chi$ *and the infinite sum in* (6.1.7) *converges uniformly in* $\mathscr{C}^\infty(K \times K \times I)$ *topology, for any compact subset* $K \subset D$*,* $I \subset \mathbb{R}$*, for all* $j = 0,1\ldots.$

Assume in addition that $\mathcal{T}$ *is a CR vector field, that is,* $[\mathcal{T}, \mathscr{C}^\infty(X, T^{1,0}X)] \subset \mathscr{C}^\infty(X, T^{1,0}X)$*. Then, there exists a phase function* $\varphi : D \times D \to \mathbb{C}$ *satisfying* (2.2.10) *and* (2.2.11) *such that* $\mathcal{T}_x\varphi(x,y) \equiv 1$*,* $\mathcal{T}_y\varphi(x,y) \equiv -1$*, and we can take the* $a_{j,s}(x,y)$ *in* (6.1.6)*,* $j,s = 0,1,\ldots$*, such that for all* $j \geq 0$ *we have*

$$\begin{aligned} &a_{j,s}(x,y) = 0, \quad s = 1,2,\ldots, \\ &\mathcal{T}_x a_{j,0}(x,y) = \mathcal{T}_y a_{j,0}(x,y) = 0. \end{aligned} \tag{6.1.8}$$

By Theorems 6.2 and 6.3, applying integration by parts in t yields:

THEOREM 6.4. *Under the same assumptions and notations as in Theorem 6.1, there exist smooth functions* $\{a_j\}_{j\geq 0} \subset \mathscr{C}^\infty(X)$ *such that for any* $\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+)$ *we have*

$$\chi_k(A)(x,x) \sim \sum_{j=0}^{+\infty} k^{n+1-j} a_j(x) \int \chi(t)t^{n-j}dt \ \text{ in } S^{n+1}_{\mathrm{loc}}(1;X). \tag{6.1.9}$$

Furthermore, we have

$$\begin{aligned} a_0(x) &= \frac{1}{2\pi^{n+1}} \frac{dV_\xi}{dV}(x), \\ a_1(x) &= \frac{1}{2\pi^{n+1}} e^{-2(n+1)f(x)} \left(\frac{1}{2} R_{scal}(x) + (n+1)\Delta_b f(x)\right), \end{aligned} \tag{6.1.10}$$

where $f(x) := \frac{1}{2(n+1)} \log \frac{dV(x)}{dV_\xi(x)} \in \mathscr{C}^\infty(X)$.

6.2. The second coefficient in Szegő kernel

The goal of this section is to prove the preparation theorem used to derive Theorem 6.2. Our observation is that $\chi_k(A)$ behaves like the Szegő projection: they share the same phase function as in (6.1.1), up to a factor k. Moreover, the leading terms of their kernel asymptotic expansions agree up to the smooth function χ. Notice that the second coefficient of the Szegő kernel was computed in [22] by assuming $dV = dV_\xi$. We first generalize the result by assuming dV is an arbitrary Reeb invariant volume form. To begin with, we recall the result in [22]:

THEOREM 6.5. *Under the same assumptions and notations as in Theorem 6.1, assume $dV_\xi = dV$. Recall that we work with the assumption that X is embeddable. Let (D, x) be any coordinate patch and let $\varphi : D \times D \to \mathbb{C}$ be any phase function satisfying (2.2.10) with $\lambda(x) \equiv 1$ and (2.2.11). Suppose that*

$$\overline{\partial}_{b,x}\varphi(x,y) \text{ vainishes to infinite order on } x = y. \tag{6.2.1}$$

Suppose further that

$$\mathcal{T}_y s(x,y,t) = 0, \ \ \mathcal{T}_y s_j(x,y) = 0, \ \ j = 0, 1, \ldots, \tag{6.2.2}$$

where $s(x,y,t)$, $s_j(x,y)$, $j = 0, 1, \ldots$, are as in (2.2.11). Then, $s_0(x,y)$ has unique Taylor expansion at $x = y$, $s_1(x,x)$ is a global defined as a smooth function on X,

$$s_0(x,y) - \tfrac{1}{2\pi^{n+1}} \text{ vanishes to infinite order at } x = y, \text{ for every } x \in D, \tag{6.2.3}$$

and

$$s_1(x,x) = \tfrac{1}{4\pi^{n+1}} R_{\mathrm{scal}}(x), \text{ for every } x \in D, \tag{6.2.4}$$

where R_{scal} is the Tanaka–Webster scalar curvature on X (see (2.2.9)).

It is known that (see [22, (3.99), Proposition 3.2]), there is an open local coordinate patch D of X with local coordinates $x = (x_1, \ldots, x_{2n+1})$, $x(p) = 0$, $\mathcal{T} = \frac{\partial}{\partial x_{2n+1}}$ on D and a phase $\varphi : D \times D \to \mathbb{C}$ satisfying (2.2.10) with $\lambda(x) = 1 + O(|x|^3)$ and (2.2.11) such that

$$\overline{\partial}_{b,x}(\varphi(x,y)) \text{ vanishes to infinite order at } x = y, \tag{6.2.5}$$

and

$$\varphi(x,y) = x_{2n+1} - y_{2n+1} + \frac{i}{2}\sum_{j=1}^{n}\left[|z_j - w_j|^2 + (\overline{z}_j w_j - z_j \overline{w}_j)\right] + O\left(|(x,y)|^4\right), \tag{6.2.6}$$

$$\begin{aligned}\sum_{\ell,j=1}^{n}\frac{\partial^4\phi}{\partial z_j\partial z_\ell\partial\overline{z}_j\partial\overline{z}_\ell}(0,0)&=-\frac{i}{2}R_{\rm scal}(0),\\ \sum_{\ell,j=1}^{n}\frac{\partial^4\phi}{\partial w_j\partial w_\ell\partial\overline{w}_j\partial\overline{w}_\ell}(0,0)&=-\frac{i}{2}R_{\rm scal}(0),\end{aligned}\tag{6.2.7}$$

where $\frac{\partial}{\partial w_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}-i\frac{\partial}{\partial y_{2j}})$, $\frac{\partial}{\partial\overline{w}_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}+i\frac{\partial}{\partial y_{2j}})$, $j=1,\ldots,n$.

Now, we work in the local coordinate patch (D,x) and we assume that φ satisfies (2.2.10) with $\lambda(x)=1+O(|x|^3)$, (2.2.11), (6.2.5), (6.2.6) and (6.2.7). Recall that dV_ξ is the volume form induced by the contact form ξ on X. We have the expression

$$dV_\xi(x)=V_\xi(x)dx_1\cdots dx_{2n+1}=\frac{1}{n!}\left(\frac{d\xi}{2}\right)^n\wedge\xi,$$

$V_\xi(x)\in\mathscr{C}^\infty(D)$. There exists a function $f\in\mathscr{C}^\infty(X)$ satisfying $\mathcal{T}f\equiv 0$ on X such that

$$dV(x)=e^{2(n+1)f(x)}V_\xi(x)dx_1\cdots dx_{2n+1}=:V(x)dx_1\cdots dx_{2n+1}.\tag{6.2.8}$$

We need

LEMMA 6.6. *With the notations used above, for $s_0(x,y)$ in (2.2.11), we can take $s_0(x,y)$ so that $s_0(x,y)$ is independent of y_{2n+1},*

$$s_0(x,y)=s_0(x,y'),\ \ \textit{for all } (x,y)\in D\times D,\tag{6.2.9}$$

where $y'=(y_1,\ldots,y_{2n})$,

$$s_0(0,0)=\frac{1}{2\pi^{n+1}}\frac{dV_\xi}{dV}(0)=\frac{1}{2\pi^{n+1}}(e^{-2(n+1)f})(0),\tag{6.2.10}$$

$$\begin{aligned}&\frac{\partial s_0}{\partial\overline{z}_j}(0,0)=\frac{\partial s_0}{\partial w_j}(0,0)=0,\ \ j=1,\ldots,n,\\ &\frac{\partial s_0}{\partial z_j}(0,0)=\frac{1}{2\pi^{n+1}}\frac{\partial}{\partial z_j}(e^{-2(n+1)f})(0),\ \ j=1,\ldots,n,\\ &\frac{\partial s_0}{\partial\overline{w}_j}(0,0)=\frac{1}{2\pi^{n+1}}\frac{\partial}{\partial\overline{w}_j}(e^{-2(n+1)f})(0),\ \ j=1,\ldots,n,\\ &\frac{\partial^2 s_0}{\partial\overline{z}_j\partial z_\ell}(0,0)=\frac{\partial^2 s_0}{\partial\overline{w}_j\partial w_\ell}(0,0)=0,\ \ j,\ell=1,\ldots,n,\end{aligned}\tag{6.2.11}$$

and

$$(\mathcal{T}_x s_0)(0,0)=(\mathcal{T}_y s_0)(0,0)=0,\tag{6.2.12}$$

where for $j = 1, \ldots, n$,

$$\frac{\partial}{\partial w_j} = \frac{1}{2}(\frac{\partial}{\partial y_{2j-1}} - i\frac{\partial}{\partial y_{2j}}),$$
$$\frac{\partial}{\partial \overline{w}_j} = \frac{1}{2}(\frac{\partial}{\partial y_{2j-1}} + i\frac{\partial}{\partial y_{2j}}), \textit{ and}$$
$$f(x) := \frac{1}{2(n+1)} \log \frac{dV(x)}{dV_{\xi}(x)} \in \mathscr{C}^\infty(X).$$

PROOF. Applying integration by parts in t, we can choose s_0 so that it does not depend on y_{2n+1}. Since both s_0 and φ are independent of y_{2n+1}, $\overline{\partial}_{b,x}\varphi(x,y)$ vanishes to infinite order at $x = y$. Consequently,

(6.2.13) $\overline{\partial}_{b,x}s_0(x,y')$ vanishes to infinite order at $x = y$.

Therefore,

$$\frac{\partial s_0}{\partial \overline{z}_j}(0,0) = 0, \ \ j = 1, \ldots, n. \tag{6.2.14}$$

From $\overline{\partial}_{b,x}\int e^{-it\overline{\varphi}(y,x)}\overline{s}(y,x,t)dt \equiv 0$, we get

(6.2.15) $\overline{\partial}_{b,x}(-\overline{\varphi}(y,x)) + g(x,y)\overline{\varphi}(y,x)$ vanishes to infinite order at $(0,0)$,

where $s(x,y,t)$ is as in (2.2.11) and g is a smooth function defined near $(0,0)$. From (6.2.6) and (6.2.14), we can check that

$$g(x,y) = O(|(x,y)|^2). \tag{6.2.16}$$

By using integration by parts in t, we get

$$\begin{aligned}&\overline{\partial}_{b,x}\int e^{-it\overline{\varphi}(y,x)}\overline{s}(y,x,t)dt \\ &= \int e^{-it\overline{\varphi}(y,x)}(-(n+1)g(x,y)\overline{s}_0(y,x) + \overline{\partial}_{b,x}\overline{s}_0(y,x))t^n + O(t^{n-1})dt.\end{aligned} \tag{6.2.17}$$

From (6.2.17), we obtain

$$(-(n+1)g(x,y)\overline{s}_0(y,x) + \overline{\partial}_{b,x}\overline{s}_0(y,x)) - h(x,y)\overline{\varphi}(y,x) \tag{6.2.18}$$

which vanishes to infinite order at $(0,0)$, where h is smooth near $(0,0)$. Combining (6.2.16) and (6.2.18), we deduce

$$\frac{\partial s_0}{\partial w_j}(0,0) = \frac{\partial^2 s_0}{\partial w_j \partial \overline{w}_j}(0,0) = 0, \ \ j = 1, \ldots, n. \tag{6.2.19}$$

From (6.2.14) and (6.2.19), we obtain the first equation in (6.2.11).

Using (6.2.14), (6.2.19) and the fact that $s_0(x,x') = \frac{1}{2\pi^{n+1}}(e^{-2(n+1)f})(x)$, we derive the second and third equation in (6.2.11).

From (6.2.13) together with (6.2.19), we obtain the last equation in (6.2.11).

Finally, using $s_0(x,x') = \frac{1}{2\pi^{n+1}}(e^{-2(n+1)f})(x)$ and the $\mathcal{T}$-invariance of f, we deduce (6.2.12). $\square$

We define a new one-form $\widehat{\xi} = e^{2f}\xi$. One can easily verify that $\widehat{\xi}$ is again a contact form, and that it induces the volume form dV, namely,

$$dV = \frac{1}{n!}\left(\frac{d\widehat{\xi}}{2}\right)^n \wedge \widehat{\xi}.$$

With this volume, we have the following theorem:

THEOREM 6.7. *With the same notation above, the second coefficient of the Szegő kernel is*

$$s_1(0,0) = \frac{1}{2\pi^{n+1}}e^{-2(n+1)f(0)}\left(\frac{1}{2}R_{\mathrm{scal}}(0) + (n+1)\Delta_b f(0)\right),$$

where R_{scal} is the Tanaka-Webster scalar curvature with respect to the contact form ξ (see (2.2.9)) and Δ_b denotes the CR sublaplacian operator (see (2.2.6)).

PROOF. By replacing the volume dV_ξ with dV in the proof of [22, Section 3.3], we have

$$s_1(0,0) = -L^{(1)}_{(x,\sigma)}(s_0(0,x)s_0(x,0)e^{2(n+1)f(x)}\sigma^n)|_{(x,\sigma)=(0,1)} \tag{6.2.20}$$

where $L^{(1)}$ is the partial differential operator in the stationary phase formula of Hörmander in Theorem 2.5. In present case,

(6.2.21)

$$\begin{aligned}
&L^{(1)}_{(x,\sigma)}(s_0(0,x)s_0(x,0)e^{2(n+1)f(x)}\sigma^n)\\
&=\sum_{\mu=0}^{2}\frac{i^{-1}}{\mu!(\mu+1)!}\left(\sum_{j=1}^{n} i\frac{\partial^2}{\partial z_j\partial\overline{z_j}} + \frac{\partial^2}{\partial x_{2n+1}\partial\sigma}\right)^{\mu+1}(G^\mu s_0(0,x)s_0(x,0)e^{2(n+1)f(x)}\sigma^n),
\end{aligned}$$

where

$$G(x,\sigma) = F(x,\sigma) - F(0,1) - \frac{1}{2}\left\langle \mathrm{Hess}(F)(p,1)\begin{pmatrix} x\\ \sigma-1\end{pmatrix}, \begin{pmatrix} x\\ \sigma-1\end{pmatrix}\right\rangle,$$

and

$$F(x,\sigma) = \sigma\varphi(0,x) + \varphi(x,0).$$

From (6.2.10) and (6.2.11), we have

(6.2.22)

$$\begin{aligned}
s_1(0,0) =& \frac{-1}{2\pi^{n+1}}e^{-2(n+1)f(0)}\cdot\\
&\left(2(n+1)\sum_{i=1}^{n}\frac{\partial^2 f}{\partial z_i\partial\overline{z}_i}(0)V_\xi(0) + \sum_{i=1}^{n}\frac{\partial^2 V_\xi}{\partial z_i\partial\overline{z}_i}(0) + i\sum_{j,k=1}^{n}\frac{\partial^4\varphi}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k}(0)\right).
\end{aligned}$$

From (6.2.7), we see that

$$i\sum_{j,k=1}^{n}\frac{\partial^4\varphi}{\partial z_j\partial\bar{z}_j\partial z_k\partial\bar{z}_k}(0)=\frac{1}{2}R_{\mathrm{scal}}(0). \tag{6.2.23}$$

From [22, (3.101)],

$$\sum_{i=1}^{n}\frac{\partial^2 V_\xi}{\partial z_i\partial\bar{z}_i}(0)=-R_{\mathrm{scal}}(0). \tag{6.2.24}$$

By the definition of Δ_b,

$$\sum_{i=1}^{n}\frac{\partial^2 f}{\partial z_i\partial\bar{z}_i}(0)=-\frac{1}{2}(\Delta_b f)(0). \tag{6.2.25}$$

Using (6.2.22), (6.2.23), (6.2.24) and (6.2.25), we obtain

$$s_1(0,0)=\frac{-1}{2\pi^{n+1}}e^{-2(n+1)f(0)}\left(-(n+1)\Delta_b f(0)-\frac{1}{2}R_{scal}\right). \tag{6.2.26}$$

This proves the theorem. □

From Theorem 6.7, we generalize the result in [22] to any Reeb-invariant volume form. Recall that we work with the assumption that X is embeddable, the following theorem shows that we obtain globally defined functions:

THEOREM 6.8. *Let (D,x) be any coordinate patch and let $\varphi: D\times D\to\mathbb{C}$ be any phase function satisfying (2.2.10) with $\lambda(x)\equiv 1$ and (2.2.11). Suppose that*

$$\bar{\partial}_{b,x}\varphi(x,y)\ \textit{vainishes to infinite order on}\ x=y. \tag{6.2.27}$$

Suppose further that

$$\mathcal{T}_y s(x,y,t)=0,\ \ \mathcal{T}_y s_j(x,y)=0,\ \ j=0,1,\ldots, \tag{6.2.28}$$

where $s(x,y,t)$, $s_j(x,y)$, $j=0,1,\ldots$, are as in (2.2.11). Then, $s_1(x,x)$ is well-defined as a smooth function on X and we have

$$s_1(x,x)=\tfrac{-1}{2\pi^{n+1}}e^{-2(n+1)f(x)}\left(-(n+1)\Delta_b f(x)-\tfrac{1}{2}R_{scal}(x)\right),\ \textit{for every}\ x\in D, \tag{6.2.29}$$

where $f(x):=\frac{1}{2(n+1)}\log\frac{dV(x)}{dV_\xi(x)}\in\mathscr{C}^\infty(X)$.

6.3. Proof of Theorem 6.2 and 6.3

Fix $p\in D$, we take local coordinates $x=(x_1,\ldots,x_{2n+1})$ defined on D so that $x(p)=0$ and $\mathcal{T}=\frac{\partial}{\partial x_{2n+1}}$. Until further notice, we work in D and work with the local coordinates $x=(x_1,\ldots,x_{2n+1})$. First, we show how the coefficients b_j^χ depend on χ:

THEOREM 6.9. *Let $\varphi : D \times D \to \mathbb{C}$ be any phase function satisfying (2.2.10) and (2.2.11). Then there exist smooth functions*

$$a_{j,s}(x,y) \in \mathscr{C}^\infty(D \times D), \quad j,s = 0,1,\ldots, \tag{6.3.1}$$

such that for any $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ we can take $b_j^\chi(x,y,t)$ in (6.1.2), $j = 0,1,\ldots$, so that

$$b_j^\chi(x,y,t) = \sum_{s=0}^{+\infty} a_{j,s}(x,y)\chi^{(s)}(t)t^{n+s-j}, \quad j = 0,1,\ldots, \tag{6.3.2}$$

where $\chi^{(s)} := (\partial/\partial t)^s\chi$ and the infinite sum in (6.1.7) converges uniformly in $\mathscr{C}^\infty(K \times K \times I)$ topology, for any compact subsets $K \subset D$, $I \subset \mathbb{R}$, for all $j = 0,1\ldots$.

PROOF. From [13, []Lemma 4.2, Lemma 4.3, Theorem 4.11], there exists a phase $\varphi_1(x,y) \in \mathscr{C}^\infty(D \times D)$ satisfying (2.2.10) and (2.2.11) such that

$$\chi_k(A)(x,y) = \int_0^{+\infty} e^{ikt\varphi_1(x,y)}\hat{b}^\chi(x,y,t,k)\,dt + O\big(k^{-\infty}\big) \quad \text{on } D \times D, \tag{6.3.3}$$

where $\hat{b}^\chi(x,y,t,k) \in S^{n+1}_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+)$ and

$$\begin{aligned} &\hat{b}^\chi(x,y,t,k) \sim \sum_{j=0}^{+\infty} \hat{b}_j^\chi(x,y,t)k^{n+1-j} \quad \text{in } S^{n+1}_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+),\\ &\hat{b}_j^\chi(x,y,t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+),\ j = 0,1,2,\ldots, \end{aligned} \tag{6.3.4}$$

and for some bounded open interval $I^\chi \Subset \mathbb{R}_+$,

$$\mathrm{supp}_t\, \hat{b}^\chi(x,y,t,k),\ \mathrm{supp}_t\, \hat{b}_j^\chi(x,y,t) \subset I^\chi,\ j = 0,1,2,\ldots \tag{6.3.5}$$

and

$$\begin{aligned} &\hat{b}_j^\chi(x,y,t) = \sum_{s=0}^{N_j} \hat{a}_{j,s}(x,y)\chi^{(s)}(t)t^{n+s-j},\ j = 0,1,\ldots,\\ &\hat{a}_{j,s}(x,y) \in \mathscr{C}^\infty(D \times D),\ s = 0,1,\ldots,N_j,\ j = 0,1,\ldots, \end{aligned} \tag{6.3.6}$$

where $N_j \in \mathbb{N}$, $j \in \mathbb{N}_0$, and the coefficients $\hat{a}_{j,s}(x,y)$, $s = 0,1,\ldots,N_j$, $j \in \mathbb{N}_0$, are independent of χ. Since φ satisfies (2.2.10) and (2.2.11), from [20, Theorem 5.4], there exists $g(x,y)$, $f(x,y) \in \mathscr{C}^\infty(D \times D)$, $f(x,y)$ vanishes to infinite order at $x = y$ such that

$$\varphi(x,y) = g(x,y)\varphi_1(x,y) + f(x,y). \tag{6.3.7}$$

By the Malgrange preparation theorem, in a neighborhood of $(0,0)$ we can write

$$\begin{aligned} \varphi(x,y) &= \alpha(x,y)\big(-y_{2n+1} + \hat{\varphi}(x,y')\big),\\ \varphi_1(x,y) &= \beta(x,y)\big(-y_{2n+1} + \hat{\varphi}_1(x,y')\big), \end{aligned} \tag{6.3.8}$$

where $y' = (y_1, \ldots, y_{2n})$ and $\alpha, \beta, \hat{\varphi}, \hat{\varphi}_1$ are smooth near $(0,0)$. Assume that (6.3.8) holds on $D \times D$, so $\alpha, \beta, \hat{\varphi}, \hat{\varphi}_1 \in \mathscr{C}^\infty(D \times D)$. Shrinking D if necessary, we may assume $\alpha(x,y) \neq 0$ and $\beta(x,y) \neq 0$ on $D \times D$. Moreover,

$$\begin{aligned} \operatorname{Im} \alpha(x,y) &= O(|x-y|), \\ \operatorname{Im} \beta(x,y) &= O(|x-y|). \end{aligned} \tag{6.3.9}$$

From (6.3.7) and (6.3.8), we obtain

$$\hat{\varphi}(x,y') - \hat{\varphi}_1(x,y') = O(\left|x'-y'\right|^N) \quad \text{for all } N \in \mathbb{N}. \tag{6.3.10}$$

To simplify notation, we write $\hat{b}(x,y,t,k)$, $\hat{b}_j(x,y,t)$, and I instead of $\hat{b}^\chi(x,y,t,k)$, $\hat{b}^\chi_j(x,y,t)$, and I^χ, with the understanding that $\hat{b}(x,y,t,k)$, $\hat{b}_j(x,y,t)$, and I still depend on χ. Let $\widetilde{\hat{b}}(x,y,\widetilde{t},k) \in \mathscr{C}^\infty(D \times D \times I^{\mathbb{C}})$ be an almost analytic extension of $\hat{b}(x,y,t,k)$ in the t-variable, where $I^{\mathbb{C}} \subset \mathbb{C}$ is a bounded open set with $I^{\mathbb{C}} \cap \mathbb{R} = I$. Recall that I is a bounded open interval as in (6.3.5). We take $\widetilde{\hat{b}}(x,y,\widetilde{t},k)$ so that

$$\operatorname{supp}_{\widetilde{t}} \widetilde{\hat{b}}(x,y,\widetilde{t},k) \subset I^{\mathbb{C}}, \;\; j = 0,1,2,\ldots, \tag{6.3.11}$$

and

$$\widetilde{\hat{b}}(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k) \sim \sum_{j=0}^{+\infty} \widetilde{\hat{b}}_j(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t)k^{n+1-j} \;\text{ in } S^{n+1}_{\mathrm{loc}}(1; D \times D \times \mathbb{R}), \tag{6.3.12}$$

where $\widetilde{\hat{b}}_j(x,y,\widetilde{t}) \in \mathscr{C}^\infty(D \times D \times I^{\mathbb{C}})$ denotes an almost analytic extension of $\hat{b}_j(x,y,t)$ in t variable, $j = 0,1,\ldots$, so that $\operatorname{supp}_{\widetilde{t}} \widetilde{\hat{b}}_j(x,y,\widetilde{t}) \subset I^{\mathbb{C}}$, $j = 0,1,2,\ldots$, and

$$\widetilde{\hat{b}}_j(x,y,\widetilde{t}) = \sum_{s=0}^{N_j} \hat{a}_{j,s}(x,y)(\frac{\partial^s}{\partial \widetilde{t}^s}\widetilde{\chi})(\frac{\alpha(x,y)}{\beta(x,y)}t)(\frac{\alpha(x,y)}{\beta(x,y)}t)^{n+s-j}, \tag{6.3.13}$$

$\widetilde{\chi}$ is an almost analytic extension of χ with $\operatorname{supp} \widetilde{\chi} \subset I^{\mathbb{C}}$ and $\hat{a}_{j,s}(x,y)$ is independent of χ for all $j \in \mathbb{N}_0$. From (6.3.9), applying Stokes' theorem to (6.3.3) yields

(6.3.14)

$$\chi_k(A)(x,y) = \int_0^{+\infty} e^{ikt\alpha(x,y)(-y_{2n+1}+\hat{\varphi}_1(x,y'))}\widetilde{\hat{b}}(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k)\frac{\alpha(x,y)}{\beta(x,y)}dt + O\left(k^{-\infty}\right) \text{ on } D \times D.$$

From (6.3.10) and the fact that $\operatorname{Im}\hat{\varphi}_1(x,y') \geq C\,|x'-y'|^2$ for some constant $C > 0$, we can replace $\hat{\varphi}_1(x,y')$ in (6.3.14) with $\hat{\varphi}(x,y')$ and obtain

(6.3.15)
$$\chi_k(A)(x,y) = \int_0^{+\infty} e^{ikt\varphi(x,y))}\widetilde{\hat{b}}\Big(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k\Big)\frac{\alpha(x,y)}{\beta(x,y)}dt + O\big(k^{-\infty}\big) \text{ on } D\times D.$$

From Borel construction, we can find $b^\chi(x,y,t,k) \in S^{n+1}_{\mathrm{loc}}(1; D\times D\times\mathbb{R}_+)$, such that $b^\chi(x,y,t,k) - \frac{\alpha(x,y)}{\beta(x,y)}\widetilde{\hat{b}}(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k)$ vanishes to infinite order at $x=y$,

$$\begin{aligned} &b^\chi(x,y,t,k) \sim \sum_{j=0}^{+\infty} b^\chi_j(x,y,t)k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1; D\times D\times\mathbb{R}_+), \\ &b^\chi_j(x,y,t) \in \mathscr{C}^\infty(D\times D\times\mathbb{R}_+),\ j=0,1,2,\ldots, \end{aligned} \tag{6.3.16}$$

and $\operatorname{supp}_t b^\chi(x,y,t,k) \subset I^\chi$, $\operatorname{supp}_t b^\chi_j(x,y,t) \subset I^\chi$, $j=0,1,2,\ldots$, and

$$\begin{aligned} &b^\chi_j(x,y,t) = \sum_{s=0}^{+\infty} a_{j,s}(x,y)\chi^{(s)}(t)t^{n+s-j}, \quad j=0,1,\ldots, \\ &a_{j,s}(x,y) \in \mathscr{C}^\infty(D\times D),\ \ j,s=0,1,\ldots, \end{aligned} \tag{6.3.17}$$

where the infinite sum in (6.3.17) converges uniformly in $\mathscr{C}^\infty(K\times K\times I)$ topology, for any compact subsets $K\subset D$, $I\subset\mathbb{R}$ and the $a_{j,s}(x,y)$ is independent of χ for all $j\in\mathbb{N}_0$. Since $b^\chi(x,y,t,k) - \frac{\alpha(x,y)}{\beta(x,y)}\widetilde{\hat{b}}(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k)$ vanishes to infinite order at $x=y$, we may replace $\frac{\alpha(x,y)}{\beta(x,y)}\widetilde{\hat{b}}(x,y,\frac{\alpha(x,y)}{\beta(x,y)}t,k)$ by $b^\chi(x,y,t,k)$ in (6.3.15). This completes the proof. □

In the rest of this section, we assume that φ satisfies (2.2.10) with $\lambda(x) = 1+O(|x|^3)$, (2.2.11), (6.2.5), (6.2.7), and has the form

$$\begin{aligned} &\varphi(x,y) = -y_{2n+1} + x_{2n+1} + \hat{\varphi}(x,y'), \quad y' = (y_1,\cdots,y_{2n}), \\ &\hat{\varphi}(x,y') \in \mathscr{C}^\infty(D\times D), \\ &\hat{\varphi}(x,y') = \frac{i}{2}\sum_{j=1}^{n}\Big(|z_j-w_j|^2 + (\overline{z}_jw_j - z_j\overline{w}_j)\Big) + O\big(|(x,y')|^4\big). \end{aligned} \tag{6.3.18}$$

This is always possible (see [22, (3.99), Proposition 3.2]). By integration by parts, we can choose $b^\chi_j(x,y,t)$ in (6.1.2), $j=0,1,\ldots$, so that

$$\begin{aligned} &b^\chi_j(x,y,t) = b^\chi_j(x,y',t) = \sum_{s=0}^{+\infty} a_{j,s}(x,y')\chi^{(s)}(t)t^{n+s-j}, \quad j=0,1,\ldots, \\ &a_{j,s}(x,y') \in \mathscr{C}^\infty(D\times D),\ \ j,s=0,1,\ldots \end{aligned} \tag{6.3.19}$$

and the $a_{j,s}(x,y')$ are independent of χ. Assume $D=\widehat{D}\times(-\epsilon,\epsilon)$, $\widehat{D}$ is an open set of $\mathbb{R}^{2n}$ of $0\in R^{2n}$, $\varepsilon>0$ is a constant. Fix $\tau\in\mathscr{C}^\infty_c((-\epsilon,\epsilon)), \tau\equiv 1$ near $0\in\mathbb{R}$ and $m_0\in\mathbb{R}$. Consider the map

$$\begin{aligned}\widetilde{\chi_k(A)}:\mathscr{C}^\infty_c(\hat{D})&\longrightarrow\mathscr{C}^\infty(D),\\ u&\longmapsto\frac{k}{2\pi}\int\chi_k(A)(x,y)e^{ikm_0y_{2n+1}}\tau(y_{2n+1})u(y')dy'dy_{2n+1}.\end{aligned}$$

This distribution kernel of $\widetilde{\chi_k(A)}$ is of the form:

$$\begin{aligned}&\widetilde{\chi_k(A)}(x,y')\\ &\equiv\frac{k}{2\pi}\int e^{ikt\varphi(x,y)+ikm_0y_{2n+1}}b^\chi(x,y',t,k)\tau(y_{2n+1})dy_{2n+1}dt\\ &\equiv\frac{k}{2\pi}\int e^{ikt(-y_{2n+1}+\widehat{\varphi}(x,y'))+ikm_0y_{2n+1}}b^\chi(x,y',t,k)\tau(y_{2n+1})dy_{2n+1}dt\\ &\equiv\frac{k}{2\pi}\int e^{ikt(-y_{2n+1}+\widehat{\varphi}(x,y'))+ikm_0y_{2n+1}}b^\chi(x,y',t,k)dy_{2n+1}dt\\ &\equiv e^{ikm_0\widehat{\varphi}(x,y')}b^\chi(x,y',m_0,k)\qquad\mod O(k^{-\infty}).\end{aligned}\tag{6.3.20}$$

The uniqueness of the expansion is given by the following:

THEOREM 6.10. *With the notations and assumptions above, assume*

$$\int e^{it\varphi(x,y)}a(x,y',t,k)dt=\int e^{it\varphi(x,y)}\widehat{a}(x,y',t,k)dt+O(k^{-\infty})\quad on\ D\times D,$$

where φ satisfies (6.3.18), $a(x,y',t,k)$, $\hat{a}(x,y',t,k)\in S^{n+1}_{\mathrm{loc}}(1;D\times D\times\mathbb{R})$,

$$\begin{aligned}&a(x,y',t,k)\sim\textstyle\sum_{j=0}^\infty a_j(x,y',t)k^{n+1-j}\ in\ S^{n+1}_{\mathrm{loc}}(1;D\times D\times\mathbb{R}),\\ &\widehat{a}(x,y',t,k)\sim\textstyle\sum_{j=0}^\infty \widehat{a}_j(x,y',t)k^{n+1-j}\ in\ S^{n+1}_{\mathrm{loc}}(1;D\times D\times\mathbb{R}),\\ &a_j,\hat{a}_j\in\mathscr{C}^\infty(D\times D\times\mathbb{R}),\ j=0,1,\ldots,\\ &\mathrm{supp}_t\ a\subset I,\ \mathrm{supp}_t\ \widehat{a}\subset I,\ \mathrm{supp}_t\ a_j\subset I,\ \mathrm{supp}_t\ \widehat{a}_j\subset I, j=0,1,\ldots,\end{aligned}$$

$I\Subset\mathbb{R}$ is a bounded interval. Then

$$a_j(x,y',t)=\widehat{a}_j(x,y',t)+O(|(x'-y')|^N),\quad\forall N,\forall t\in I,\forall j=0,1,\ldots.$$

PROOF. Fix $m_0\in I$. Repeating the argument in (6.3.20) gives

$$e^{ikm_0\widehat{\varphi}(x,y')}a(x,y',m_0,k)=e^{ikm_0\widehat{\varphi}(x,y')}\widehat{a}(x,y',m_0,k)+F_k(x,y'),\quad F_k(x,y')=O(k^{-\infty}).$$

Thus

$$a(x,y',m_0,k)-\widehat{a}(x,y',m_0,k)=e^{-ikm_0\widehat{\varphi}(x,y')}F_k(x,y').\tag{6.3.21}$$

Hence

$$
\begin{aligned}
a_0(x,x',m_0)-\widehat{a}_0(x,x',m_0) &= \lim_{k\to\infty}\frac{1}{k^{n+1}}e^{-ikm_0\widehat{\varphi}(x,x')}F_k(x,x')\\
&= \lim_{k\to\infty}\frac{1}{k^{n+1}}e^{ikm_0x_{2n+1}}F_k(x,x')=0.
\end{aligned}
$$

Similarly,

$$
\left(\partial_x^\alpha\partial_{y'}^\beta(a_0-\widehat{a}_0)\right)(x,x',m_0)=\lim_{k\to\infty}\frac{1}{k^{n+1}}\partial_x^\alpha\partial_{y'}^\beta\left(e^{-ikm_0\widehat{\varphi}(x,y')}F_k(x,y')\right)|_{(x,x')}=0,
$$

for all $\alpha\in\mathbb{N}_0^{2n+1}$, $\beta\in\mathbb{N}_0^{2n}$. In the same way, we obtain

$$
a_j(x,y',t)=\widehat{a}_j(x,y',t)+O(|x'-y'|^N),\quad \forall\, N\in\mathbb{N},\ j=1,2,\ldots.
$$

□

From now on, assume $b_j^\chi(x,y,t)$, $j=0,1,\ldots,n$, satisfy (6.3.19). Then we can show that b_1^χ is independent of derivatives of χ of order ≥ 1:

THEOREM 6.11. *With the notations and assumptions used above, recall that φ satisfies (6.2.5), (6.2.7), (6.3.18), and $b_j^\chi(x,y,t)$, $j\in\mathbb{N}_0$ in (6.1.2), satisfy (6.3.19). Then, for all $x\in D$, $t\in\mathbb{R}$ and $\chi\in\mathscr{C}_c^\infty(\mathbb{R}_+)$,*

$$
\begin{aligned}
&b_0^\chi(0,0,t)=s_0(0,0)\chi(t)t^n,\\
&\frac{\partial b_0^\chi}{\partial x_j}(0,0,t)=\frac{\partial s_0}{\partial x_j}(0,0)\chi(t)t^n,\ \ j=1,\ldots,2n+1,\\
&\frac{\partial b_0^\chi}{\partial y_j}(0,0,t)=\frac{\partial s_0}{\partial y_j}(0,0)\chi(t)t^n,\ \ j=1,\ldots,2n+1,\\
&\frac{\partial^2 b_0^\chi}{\partial\overline{z}_j\partial z_\ell}(0,0,t)=\frac{\partial^2 b_0^\chi}{\partial\overline{w}_j\partial w_\ell}(0,0,t)=0,\ \ j,\ell=1,\ldots,n,
\end{aligned}
\tag{6.3.22}
$$

and

$$
a_{1,s}(p,p)=0,\quad s=1,2,\cdots,
\tag{6.3.23}
$$

where $s_0(x,y)$ is as in (2.2.11) and satisfies (6.2.9), (6.2.10), (6.2.11), (6.2.12), and $a_{1,s}(x,y')$ is as in (6.3.19) for $s=1,2,\cdots$.

PROOF. The proof of (6.3.22) is the same as those of (6.2.11) and (6.2.12).

From (2.2.11) and (6.3.18), we can check that

$$
\begin{aligned}
&((-i)\mathcal{T}\Pi)\,(x,y) = \int e^{it\varphi(x,y)} g(x,y',t)dt, \\
&(\Pi(-i\mathcal{T}))\,(x,y) = \int e^{it\varphi(x,y)} \hat{g}(x,y',t)dt, \\
&g(x,y',t) \sim \sum_{j=0}^{\infty} t^{n+1-j} g_j(x,y') \ \text{ in } S^{n+1}_{1,0}(D\times D\times\mathbb{R}), \\
&\hat{g}(x,y',t) \sim \sum_{j=0}^{\infty} t^{n+1-j} \hat{g}_j(x,y') \ \text{ in } S^{n+1}_{1,0}(D\times D\times\mathbb{R}), \\
&g_j(x,y'), \hat{g}_j(x,y') \in \mathscr{C}^\infty(D\times D), \ \ j=0,1,\ldots,
\end{aligned}
$$

$$
g_0(x,y') - \hat{g}_0(x,y') = O(|x-y|^3). \tag{6.3.24}
$$

Let $\widetilde{\chi_k(A)}(x,y')$ be as in (6.3.20). We have

$$
\begin{aligned}
&\frac{1}{k}(-i\mathcal{T})\Pi\widetilde{\chi_k(A)}(p,p) = k^{n+1}m_0 b^\chi_0(p,p',m_0) + k^n m_0 b^\chi_1(p,p',m_0) + O(k^{n+1}) \\
&= k^{n+1}m_0^{n+1}\chi(m_0)s_0(p,p') + \sum_{s=0}^{\infty} k^n a_{1,s}(p,p')\chi^{(s)}(m_0)m_0^{n+s} + O(k^{n-1}).
\end{aligned} \tag{6.3.25}
$$

Let $\widehat{\chi}(t) = t\chi(t)$. We have $\frac{1}{k}\Pi(-i\mathcal{T})\widetilde{\chi_k(A)} = \widetilde{\widehat{\chi}_k(A)}$. From this observation, we get

$$
\begin{aligned}
&\widetilde{\widehat{\chi}_k(A)}(p,p) \\
&= k^{n+1}m_0^{n+1}\chi(m_0)s_0(p,p') + \sum_{s=0}^{\infty} k^n a_{1,s}(p,p')(t\chi(t))^{(s)}|_{t=m_0} m_0^{n-1+s} + O(k^{n-1}).
\end{aligned} \tag{6.3.26}
$$

From (6.3.24) and stationary phase formula of Hörmander, we can check that

$$
\left(\frac{1}{k}(-i\mathcal{T})\Pi\widetilde{\chi_k(A)}\right)(p,p) - \left(\frac{1}{k}\Pi(-i\mathcal{T})\widetilde{\chi_k(A)}\right)(p,p) = O(k^{n-1}). \tag{6.3.27}
$$

From (6.3.25), (6.3.26) and (6.3.27), we get

$$
\sum_{s=0}^{\infty} a_{1,s}(p,p')\chi^{(s)}(m_0)m_0^{n+s} = \sum_{s=0}^{\infty} a_{1,s}(p,p')(t\chi(t))^{(s)}|_{t=m_0} m_0^{n-1+s}. \tag{6.3.28}
$$

Take χ so that $\chi(m_0) \neq 0$ and $\chi^{(s)}(m_0) = 0$, for all $s \geq 1$. From (6.3.28), we get

$$
a_{1,0}(p,p')\chi(m_0)m_0^n = a_{1,0}(p,p')\chi(m_0)m_0^n + a_{1,1}(p,p')\chi(m_0)m_0^n.
$$

Thus, $a_{1,1}(p,p')=0$. Suppose $a_{1,s}(p,p')=0$, for all $1\leq s\leq N_0$, for some $N_0\in\mathbb{N}$. Take χ so that $\chi^{(N_0)}(m_0)\neq 0$, $\chi^{(s)}(m_0)=0$, $s>N_0$. By induction assumption and (6.3.28), we get

$$a_{1,0}(p,p')\chi(m_0)m_0^n=a_{1,0}(p,p)\chi(m_0)m_0^n+a_{1,N_0+1}(p,p')\chi^{(N_0)}(m_0)m_0^{n+N_0}.$$

Thus, $a_{1,N_0+1}(p,p')=0$. By induction we get (6.3.23). □

PROOF OF THEOREM 6.2. Let $\chi,\tau\in\mathscr{C}_c^\infty(\mathbb{R}_+)$, $\tau\equiv 1$ on $\operatorname{supp}\chi$. By Theorem 6.9, Theorem 6.11, we have the following expansion:

$$\begin{aligned}
&\chi_k(T_P)(x,y)=\int e^{ikt\varphi(x,y)}\left(k^{n+1}\hat{b}_0(x,y',t,\chi)+k^n\hat{b}_1(x,y',t,\chi)+O(k^{n-1})\right)dt,\\
&\tau_k(T_P)(x,y)=\int e^{ikt\varphi(x,y)}\left(k^{n+1}\hat{b}_0(x,y',t,\tau)+k^n\hat{b}_1(x,y',t,\tau)+O(k^{n-1})\right)dt,
\end{aligned}\tag{6.3.29}$$

where

$$\begin{aligned}
&\hat{b}_0(x,y',t,\chi)=\hat{b}_0(x,y,t,\chi)=b_0^\chi(x,y,t)=\sum_{s=0}^{+\infty}a_{0,s}(x,y')\chi^{(s)}(t)t^{n+s},\\
&\hat{b}_1(x,y',t,\chi)=\hat{b}_1(x,y,t,\chi)=b_1^\chi(x,y,t)=\sum_{s=0}^{+\infty}a_{1,s}(x,y')\chi^{(s)}(t)t^{n+s-1},\\
&a_{j,s}(x,y')\in\mathscr{C}^\infty(D\times D),\ \ j,s=0,1,\ldots,\\
&a_{1,s}(p,p)=0,\ \ s=1,2,\ldots,
\end{aligned}\tag{6.3.30}$$

where $b_0^\chi(x,y,t)$, $b_1^\chi(x,y,t)$ are as in (6.1.2). By the relation,

$$\tau_k(T_P)\circ\chi_k(T_P)=(\tau\chi)_k(T_P)=\chi_k(T_P),\tag{6.3.31}$$

and complex stationary phase formula, we have

(6.3.32)

$$
\begin{aligned}
&\tau_k(T_P)\circ\chi_k(T_P)(x,y)\\
&=\int e^{ik(s\varphi(x,u)+t\varphi(u,y))}V(u)\Big(k^{2n+2}\hat{b}_0(x,u,s,\tau)\hat{b}_0(u,y,t,\chi)\\
&\quad+k^{2n+1}\big(\hat{b}_1(x,u,s,\tau)\hat{b}_0(u,y,t,\chi)+\hat{b}_0(x,u,s,\tau)\hat{b}_1(u,y,t,\chi)\big)+O(k^{2n})\Big)dudsdt\\
&=\int e^{ik(t(\sigma\varphi(x,u)+\varphi(u,y))}\Big(k^{2n+2}\hat{b}_0(x,u,t\sigma,\tau)\hat{b}_0(u,y,t,\chi)\\
&\qquad\qquad+k^{2n+1}\big(\hat{b}_1(x,u,t\sigma,\tau)\hat{b}_0(u,y,t,\chi)\\
&\qquad\qquad+\hat{b}_0(x,u,t\sigma,\tau)\hat{b}_1(u,y,t,\chi)\big)+O(k^{2n})\Big)tV(u)dud\sigma dt\\
&=\int e^{ikt\varphi(x,y)}\Big(k^{n+1}\beta_0^{\chi,\tau}(x,y',t)+k^n\beta_1^{\chi,\tau}(x,y',t)+O(k^{n-1})\Big)dt,
\end{aligned}
$$

where $\beta_0^{\chi,\tau}(x,y',t),\beta_1^{\chi,\tau}(x,y',t)\in\mathscr{C}^\infty(D\times D\times I)$. From Theorem 6.10 we have uniqueness of the expansion, and from (6.3.30) we obtain

$$
\begin{aligned}
\beta_0^{\chi,\tau}(p,p,t)&=\hat{b}_0(p,p,t,\chi)=b_0^{\chi}(p,p,t)=s_0(p,p)\chi(t)t^n,\\
\beta_1^{\chi,\tau}(p,p,t)&=\hat{b}_1(p,p,t,\chi)=a_{1,1}(p,p)\chi(t)t^{n-1}.
\end{aligned}
\tag{6.3.33}
$$

From (6.3.33) and Theorem 2.5, we have

$$
\begin{aligned}
&k^na_{1,0}(p,p)t^{n-1}\chi(t)\\
&=k^nt^{-n-1}L^{(1)}_{(u,\sigma)}\big(\hat{b}_0(p,u,\sigma t,\tau)\hat{b}_0(u,p,t,\chi)e^{2(n+1)f(u)}\big)|_{(u,\sigma)=(p,1)}\\
&+(2\pi^{n+1})2k^na_{1,1}(p,p)t^{n-1}\chi(t)s_0(p,p)V(p)\chi(t),
\end{aligned}
$$

where $L^{(1)}_{(u,\sigma)}$ is as in (6.2.21). From (6.3.22) and notice that

$$
s_0(p,p)=\frac{1}{2\pi^{n+1}}\left(\frac{dV_\xi}{dV}\right)(p),
$$

we can check that

$$
\begin{aligned}
a_{1,0}(p,p)t^{n-1}\chi(t)&=-\,L^{(1)}_{(u,\sigma)}\big(s_0(p,u)s_0(u,p)e^{2(n+1)f(u)}\tau(\sigma t)\sigma^n\big)|_{(u,\sigma)=(p,1)}t^{n-1}\chi(t)\\
&=-\,L^{(1)}_{(u,\sigma)}\big(s_0(p,u)s_0(u,p)e^{2(n+1)f(u)}\sigma^n\big)|_{(u,\sigma)=(p,1)}t^{n-1}\chi(t).
\end{aligned}
\tag{6.3.34}
$$

Therefore, by (6.2.22), we have

$$
a_{1,0}(p,p)=s_1(p,p).
\tag{6.3.35}
$$

If we assume further $dV = dV_\xi$, then $f \equiv 0$, and thus

$$a_{1,0}(p,p) = \frac{1}{4\pi^{n+1}} R_{\mathrm{scal}}(p). \tag{6.3.36}$$

□

PROOF OF THEOREM 6.3. We now prove Theorem 6.3. The first part of the χ-dependence of b_j^χ is proved in Theorem 6.9, it remains to prove (6.1.8). Therefore, we assume that $\mathcal{T}$ is a CR vector field. We work on the local coordinate D where $\mathcal{T} = \frac{\partial}{\partial x_{2n+1}}$. Since $\mathcal{T}$ is a CR vector field, we can choose $\varphi : D \times D \to \mathbb{C}$ satisfying (2.2.10), (2.2.11), and

$$\varphi(x,y) = x_{2n+1} - y_{2n+1} + \hat{\varphi}(x',y'), \quad \hat{\varphi}(x',y') \in \mathscr{C}^\infty(D \times D). \tag{6.3.37}$$

We take $b_j^\chi(x,y,t)$ in (6.1.2), $j = 0,1,\ldots$, so that (6.3.19) hold. Let $\widetilde{\chi_k(A)}$(x,y′) be as in (6.3.20). We have

$$\frac{1}{k}\widetilde{\chi_k(A)(-i\mathcal{T})}(x,x') = \sum_{j=0}^{N} k^{n+1-j} m_0 b_j^\chi(x,x',m_0) + O(k^{n-N}), \tag{6.3.38}$$

for every $N \in \mathbb{N}$, for every $x \in D$. Let $\widehat{\chi}(t) = t\chi(t)$. We have

$$\frac{1}{k}\widetilde{\chi_k(A)(-i\mathcal{T})} = \widetilde{\widehat{\chi}_k(A)}.$$

From this observation and (6.3.38), similar to the proof of (6.3.23) we deduce that

$$a_{j,s}(x,x') = 0, \text{ for all } s \in \mathbb{N}_0, \text{ for all } j \in \mathbb{N}_0, \tag{6.3.39}$$

where $a_{j,s}$, $j,s \in \mathbb{N}_0$, are as in (6.3.19). Thus,

$$b_j^\chi(x,x',t) = a_{j,0}(x,x')\chi(t)t^{n-j}, \quad j \in \mathbb{N}_0. \tag{6.3.40}$$

Observe that $a_{j,0}$ is independent of the choice of χ for every $j \in \mathbb{N}_0$, and the map $x \mapsto \chi_k(A)(x,x)$ is $\mathcal{T}$-invariant. Combining this observation and (6.3.40), we can see that $b_j(x,x',t)$ is independent of x_{2n+1} for all $j \in \mathbb{N}_0$. Consequently,

$$b_j^\chi(x,x',t) = b_j^\chi(x',x',t) = a_{j,0}(x',x')\,\chi(t)\,t^{n-j}, \quad j \in \mathbb{N}_0. \tag{6.3.41}$$

Since $\overline{\partial}_{b,x}\varphi(x,y)$ and $\overline{\partial}_{b,x}\overline{\varphi}(y,x)$ vanish to infinite order at $x = y$, it follows directly that

$$\overline{\partial}_{b,x} b_j(x,y',t) \text{ and } \overline{\partial}_{b,x}\overline{b}_j(y,x',t) \text{ also vanish to infinite order at } x = y. \tag{6.3.42}$$

From (6.3.41) and (6.3.42), repeating the proof of [11, Lemma 3.1] with minor modifications yields functions $\alpha_j(x',y') \in \mathscr{C}^\infty(D \times D)$, $j \in \mathbb{N}_0$, such that

$$b_j^\chi(x,y',t) - \alpha_j(x',y')\chi(t)t^{n-j} \text{ vanishes to infinite order at } x = y,\ j \in \mathbb{N}_0. \tag{6.3.43}$$

Thus we may replace $b_j^{\chi}(x,y',t)$ by $\alpha_j(x',y')\chi(t)t^{n-j}$ for all $j \in \mathbb{N}_0$, and the theorem follows. $\square$

6.4. Relations between Bergman kernels

In this section, we formulate the relation between a_j in Theorem (6.4) and the coefficients of the asymptotic expansion of the Bergman kernel. Let M be a compact Kähler manifold of dimension n as in Sec. 3.1. Let $(L,h) \to M$ be a positive holomorphic line bundle with a Hermitian metric h over M. Fix any Hermitian metric $\langle \cdot \mid \cdot \rangle_M$ on $\mathbb{C}TM$ and let Θ be the real two form on M induced by $\langle \cdot \mid \cdot \rangle_M$. Let dV_Θ be the volume form on M induced by Θ. For every $k \in N$, let $(L^k, h^k) \to M$ be the k-th power of (L,h), where h^k is the Hermitian metric on L^k induced by h as in Sec. 3.1. Let $(\cdot \mid \cdot)_k$ be the L^2 inner product on $L^2(M,L^k)$ induced by dV_Θ and h^k. Let $H^0(M,L^k)$ be the space of global holomorphic sections with values in L^k. Let $\{f_1,\ldots,f_{d_k}\}$ be an orthonormal basis of $H^0(M,L^k)$ with respect to $(\cdot \mid \cdot)_k$. The Bergman kernel function is given by

$$B_k(x) := \sum_{j=1}^{d_k} \left|f_j(x)\right|^2_{h^k} \in \mathscr{C}^\infty(M). \tag{6.4.1}$$

It is well-known that B_k has an asymptotic expansion (see [5], [19], [25], [33])

$$\begin{aligned} &B_k(x) \sim \sum_{j=0}^{+\infty} b_j(x)k^{n-j} \ \text{ in } S^n_{\mathrm{loc}}(1;M),\\ &b_j(x) \in \mathscr{C}^\infty(M), \ \ j \in \mathbb{N}_0. \end{aligned} \tag{6.4.2}$$

Put

$$X := \{v \in L^* \mid |v|_{h^*} = 1\} \subset L, \ \ T^{1,0}X := \mathbb{C}TX \cap T^{1,0}L^*. \tag{6.4.3}$$

Here L^* denotes the dual line bundle with the metric h^* induced by h. It is well known that $(X, T^{1,0}X)$ is a compact, strictly pseudoconvex CR manifold with a transversal CR S^1-action given by

$$S^1 \times X \ni (\lambda, v) \mapsto \lambda v \in X. \tag{6.4.4}$$

Let $\mathcal{T}$ be the infinitesimal generator of this action, and let $\xi \in \mathscr{C}^\infty(X, T^*X)$ be the unique real one-form satisfying $\mathcal{T} \lrcorner \xi \equiv 1$ and $\xi(T^{1,0}X) = 0$. Let $\pi \colon X \to M$ be the projection. With $dV_X = \xi \wedge \pi^* dV_M$, the assumptions of Theorem 6.4 hold. Thus, there are smooth functions $a_j \in \mathscr{C}^\infty(X,\mathbb{R})$ and $j \geq 0$, such that for any $\chi \in \mathscr{C}^\infty_c(X, \mathbb{R}_+)$,

$$\chi_k(A)(x,x) \sim \sum_{j=0}^{+\infty} k^{n+1-j} a_j(x) \int \chi(t) t^{n-j} dt \quad \text{in } S^{n+1}_{\mathrm{loc}}(1;X).$$

It was shown in [13] that $a_j = \frac{1}{2\pi} b_j \circ \pi$, $j \geq 0$.

Now we show how the auxiliary choice of Hermitian volume form on M affects the coefficients a_1 on X: Let us consider the associated circle bundle $\pi\colon X \to M$ in (6.4.3). Let $\mathcal{T} \in \mathscr{C}^\infty(X, TX)$ be the vector field on X induced by the S^1 action acting on the fiber of L^* in (6.4.4). Take

$$dV := \frac{1}{n!}\xi \wedge \Theta^n \tag{6.4.5}$$

and let Π be the Szegő projection associated with dV. Let s_1 be as in Theorem 6.8. We have

$$s_1(x,x) = \frac{1}{2\pi} b_1(\pi(x)), \ \text{ for all } x \in X, \tag{6.4.6}$$

where $\pi : X \to M$ is the natural projection. From Theorem 6.8 and (6.4.6), we get

(6.4.7)

$$b_1(\pi(x)) = \frac{-(2\pi)}{2\pi^{n+1}} e^{-2(n+1)f(x)} \left(-(n+1)\Delta_b f(x) - \tfrac{1}{2}R_{scal}(x)\right), \text{ for every } x \in X.$$

We now reformulate (6.4.7) in geometric invariance on M. We can check that

$$d\xi(x) = \sqrt{-1}R^L(\pi(x)) \tag{6.4.8}$$

and

$$dV_\xi(x) = \left(\sqrt{-1}R^L(\pi(x))\right)^n \frac{2^{-n}}{n!}. \tag{6.4.9}$$

For every $x \in M$, let

$$\dot{R}^L(x) : T_x^{1,0}M \to T_x^{1,0}M$$

be the linear map given by $\langle \dot{R}^L(x)U \,|\, V \rangle_M = \langle R^L(x), U \wedge \overline{V} \rangle$, $U, V \in T_x^{1,0}M$. For every $x \in M$, let $\det \dot{R}^L(x) = \lambda_1(x) \cdots \lambda_n(x)$, where $\lambda_j(x)$, $j = 1, \ldots, n$, be the eigenvalues of $\dot{R}^L(x)$. From (6.4.8) and (6.4.9) we have

$$\det \dot{R}^L(\pi(x)) = 2^n \frac{dV_\xi(x)}{dV(x)}, \ \text{ for every } x \in X, \tag{6.4.10}$$

and

(6.4.11)

$$f(x) = \frac{1}{2(n+1)} \log \frac{dV(x)}{dV_\xi(x)} = \frac{1}{2(n+1)} \log\left(\left(\det \dot{R}^L(\pi(x))\right)^{-1} 2^n\right), \ \text{ for every } x \in X.$$

Let $\omega := \frac{\sqrt{-1}}{2\pi} R^L$ be the Kähler form on M as in Sec. 3.1, the induced L^2 inner product on $\Lambda^{0,q}T^*M$ is denoted by $(\cdot \,|\, \cdot)_\omega$. The complex Laplacian with respect to ω is the operator $\Delta_\omega : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ as in (3.1.7) given by

$$(\Delta_\omega f \,|\, u)_\omega = (df \,|\, du)_\omega, \quad f \in \mathscr{C}^\infty(M), u \in \mathscr{C}^\infty(M). \tag{6.4.12}$$

Let $g \in \mathscr{C}^\infty(M)$ and let $\hat{g} \in \mathscr{C}^\infty(X)$ be given by $\widehat{g}(x) := g(\pi(x))$ for all $x \in X$. We have

$$\Delta_b \widehat{g}(x) = \frac{1}{2\pi}(\Delta_\omega g)(\pi(x)), \text{ for all } x \in X. \tag{6.4.13}$$

Let r be the scalar curvature on M with respect to $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ (see3.1.2, (3.1.10) and [16, (1.8)]). It was shown in [12, Theorem 3.5] that

$$r(\pi(x)) = 4\pi R_{\mathrm{scal}}(x), \text{ for every } x \in X. \tag{6.4.14}$$

From (6.4.7), (6.4.11), (6.4.13) and (6.4.14), we get

$$b_1(\pi(x)) = (2\pi)^{-n} \det \dot{R}^L(\pi(x)) \Big(\frac{1}{8\pi} r(\pi(x)) + (n+1)(\Delta_b f)(x) \Big). \tag{6.4.15}$$

From (6.4.11) and (6.4.13), we obtain

$$(n+1)(\Delta_b f)(x) = \frac{1}{4\pi} \big(\hat{r}(\pi(x)) - r(\pi(x)) \big), \tag{6.4.16}$$

for all $x \in X$, where $\hat{r}$ is defined in [16, (1.8)]. Combining (6.4.15) and (6.4.16) yields

$$b_1(x) = (2\pi)^{-n} \det \dot{R}^L(x) \Big(\frac{1}{4\pi} \hat{r}(x) - \frac{1}{8\pi} r(x) \Big), \tag{6.4.17}$$

for all $x \in M$, which agrees with [16, Theorem 1.4].

The computation of Bergman kernel coefficients in a more general setting can be found in [26]. See also [12] for the coefficients b_0, b_1, and b_2 on quasi-regular Sasakian manifolds.

Bibliography

[1] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Phys. **111** (1978), no. 1, 61–110.

[2] M. Beals, C. Fefferman, and R. Grossman, *Strictly pseudoconvex domains in* $\mathbf{C}^n$, Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 2, 125–322.

[3] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* $\mathfrak{gl}(n)$, $n \to \infty$ *limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.

[4] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Journées: Équations aux Dérivées partielles de Rennes (1975), 1976, pp. 123–164.

[5] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), 1999, pp. 1–23.

[6] C.-C. Chang, H. Herrmann, and C.-Y. Hsiao, *On the second coefficient in the semi-classical expansion of Toeplitz operators*, Anal. Math. Phys. **15** (2025), no. 4, Paper No. 112, 32.

[7] M. De Wilde and P. B. A. Lecomte, *Existence of star-products on exact symplectic manifolds*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 2, 117–143.

[8] S. K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), no. 3, 479–522.

[9] J. J. Duistermaat and L. Hörmander, *Fourier integral operators. II*, Acta Math. **128** (1972), no. 3-4, 183–269.

[10] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), no. 2, 213–238.

[11] A. Galasso and C.-Y. Hsiao, *Toeplitz operators on CR manifolds and group actions*, J. Geom. Anal. **33** (2023), no. 1, Paper No. 21.

[12] H. Herrmann, C.-Y. Hsiao, and X. Li, *Szegő kernel asymptotic expansion on strongly pseudoconvex CR manifolds with* S^1 *action*, Internat. J. Math. **29** (2018), no. 9, 1850061, 35.

[13] H. Herrmann, C.-Y. Hsiao, G. Marinescu, and W.-C. Shen, *Semi-classical spectral asymptotics of Toeplitz operators on CR manifolds*, 2023. arXiv:2303.17319 [math.CV].

[14] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III: Pseudo-Differential Operators*, Classics in Mathematics, Springer, Berlin, 2007. Reprint of the 1994 edition.

[16] C.-Y. Hsiao, *On the coefficients of the asymptotic expansion of the kernel of Berezin–Toeplitz quantization*, Ann. Global Anal. Geom. **42** (2012), no. 2, 207–245.

[17] C.-Y. Hsiao, *Projections in several complex variables*, Mémoires de la Société Mathématique de France, Société Mathématique de France, 2010. 131 pages.

[18] C.-Y. Hsiao, X. Ma, and G. Marinescu, *Geometric quantization on CR manifolds*, Commun. Contemp. Math. **25** (2023), no. 10, Paper No. 2250074, 73.

[19] C.-Y. Hsiao and G. Marinescu, *Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles*, Comm. Anal. Geom. **22** (2014), no. 1, 1–108.

[20] ———, *On the singularities of the Szegő projections on lower energy forms*, J. Differential Geom. **107** (2017), no. 1, 83–155.

[21] C.-Y. Hsiao and O. Shabtai, *On the distribution kernels of Toeplitz operators on CR manifolds*, 2025. arXiv:2512.16506 [math.CV].

[22] C.-Y. Hsiao and W.-C. Shen, *On the second coefficient of the asymptotic expansion of Boutet de Monvel–Sjöstrand*, Bull. Inst. Math. Acad. Sin. (N.S.) **15** (2020), no. 4, 329–365.

[23] S. Jansen, N. Neumaier, G. Schaumann, and S. Waldmann, *Classification of invariant star products up to equivariant Morita equivalence on symplectic manifolds*, Lett. Math. Phys. **100** (2012), no. 2, 203–236.

[24] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.

[25] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser, Basel, 2007.

[26] ———, *Berezin-Toeplitz quantization on Kähler manifolds*, J. Reine Angew. Math. **662** (2012), 1–56.

[27] A. Melin and J. Sjöstrand, *Fourier integral operators with complex-valued phase functions*, Fourier integral operators and partial differential equations (colloq. internat., univ. nice, nice, 1974), 1974, pp. 120–223.

[28] L. B. D. MONVEL and V. GUILLEMIN, *The Spectral Theory of Toeplitz Operators. (am-99)*, Princeton University Press, 1981.

[29] T. Reichert, *Classification and Reduction of Equivariant Star Products on Symplectic Manifolds*, ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–Bayerische Julius-Maximilians-Universität Würzburg (Germany).

[30] W. Ruan, *Canonical coordinates and Bergman metrics*, Comm. Anal. Geom. **6** (1998), no. 3, 589–631.

[31] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin–Toeplitz quantization*, Conférence Moshé Flato 1999, vol. II (Dijon), 2000, pp. 289–306.

[32] H. Xu, *A closed formula for the asymptotic expansion of the Bergman kernel*, Comm. Math. Phys. **314** (2012), no. 3, 555–585.

[33] S. Zelditch, *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices **6** (1998), 317–331.