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A class of globally analytic hypoelliptic operators on compact Lie groups

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Abstract

We obtain global analytic hypoellipticity for a class of differential operators that can be expressed as $P = \sum_{j=1}^{\nu} X_j^2 + X_0 + a$ with real-analytic coefficients on compact Lie groups such that the vector fields satisfy Hörmander's finite type condition; there exists a closed subgroup whose action leaves the vector fields invariant; and the operator must be elliptic in directions transversal to the action of the subgroup. This paves the way for further studies on the regularity of sums of squares on principal fiber bundles.

1 Introduction

We prove that a class of differential operators of the form

$$P = \sum_{j=1}^{\nu} X_j^2 + X_0 + a \tag{1}$$

with real coefficients on compact Lie groups is globally analytic hypoelliptic.

In order to precisely state our result, we first need some definitions and notations. The following is a notion of regularity of partial differential equations.

Definition 1 A differential operator P defined over a real-analytic manifold Ω is said to be *globally analytic hypoelliptic* (GAH) if for all distributions $u \in \mathcal{D}'(\Omega)$ such that $Pu \in C^{\omega}(\Omega)$ it holds that $u \in C^{\omega}(\Omega)$.

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Definition 2 Let Ω be a smooth manifold. We say that smooth vector fields $X_0, X_1, \dots, X_\nu$ defined on Ω satisfy the *Hörmander finite-type condition* (H) if the vector fields, together with their brackets of length at most r, span the tangent space at each point of Ω.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $K \subset G$ be a closed subgroup with Lie algebra $\mathfrak{k}$. We consider $X_0, X_1, \dots, X_\nu$, real-analytic K-invariant (real) vector fields on G satisfying the *Hörmander finite-type condition* (H), and such that

$$\operatorname{span}\{\pi_*(X_1(g)), \dots, \pi_*(X_\nu(g))\} = T_{\pi(g)}G/K, \forall g \in G, \tag{E}$$

with $\pi : G \longrightarrow G/K$ the projection.

Next is the main result of this paper. It generalizes the global analytic hypoellipticity result from [10] to the setting of compact Lie groups.

Theorem 1 *We consider the operator P as defined in (1) with real-analytic vector fields $X_0, X_1, \dots, X_\nu$. If we assume that the vector fields are K-invariant on G and satisfy conditions (H) and (E1), then the operator P is (GAH) on G for all $a \in C^\omega(G)$.*

Unlike [10], we do not assume that the ambient manifold is a product of manifolds in order to separate ellipticity in one variable and work to obtain the regularity in the other ones. We instead use the intrinsic geometrical properties of the Lie groups to obtain the global regularity result. Notice that, because of the zero-order term a, which does not need to be invariant under the action of K, the operator (1) does not commute with the Laplace-Beltrami operator on K coming from an ad-invariant metric. Also notice that, while the condition (H) is well-established in the literature, the condition regarding the existence of a closed subgroup requires some justification. For example, not only assuming K is closed is necessary for our technique, for example, in the statement of (E1), but also there are examples of order two partial differential operators that are not invariant under any subgroup and that are not (GAH). One of such examples can be found in [6]. We state the example here for the convenience of the reader.

Example 1 The Global Métivier operator

$$P = D_x^2 + (\sin x)^2 D_y^2 + (\sin y D_y)^2$$

defined on the 2-torus is not (GAH). A proof of this claim can be found in [6].

The following is a class of operators satisfying conditions of Theorem 1.

Example 2 Recall that the $\mathrm{SU}(2)$ is defined by

$$\mathrm{SU}(2) \doteq \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} : z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1 \right\}$$

with Lie algebra denoted by $\mathfrak{su}(2)$ and generated by

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The vectors are invariant under the right-action of G on itself, thus it is also invariant under the action of $K \doteq \exp(\text{span}_{\mathbb{R}}\{T\})$. The following relation between the brackets of X, Y and T

$$[T, X] = 2Y, \quad [T, Y] = -2X, \quad [X, Y] = 2T \tag{2}$$

guarantee that the operator $P = (fX)^2 + (gY)^2 + a$, with $a, f, g \in C^\omega(G)$ such that f, g are non-vanishing functions that are invariant under the action of K, satisfy condition (H) and clearly also satisfy $(E1)$.

We believe that our results open doors to further investigation of the regularity of certain operators on principal fiber bundles. The advantage of working first on Lie groups is that it is possible to give a global expression for the operators. The global expression makes it easy to discuss the commutation problem of the operator under study with the Laplacian, which would be more complicated in the case of fiber bundles in general.

Although the initial motivation for our research comes from the original work of Cordaro and Himonas [14, 15], we were motivated by several other papers in this and related topics, such as [2, 3, 5, 8, 12, 13, 16–19, 21, 26, 29].

For a comprehensive discussion of this problem in the local context, see [1, 9] and references therein. In the global case, on a compact manifold, an operator may fail to be locally analytic-hypoelliptic but still exhibit global analytic-hypoellipticity. Some examples of such operators can be found in the introduction of [14]. It is important to note that our family of operators includes a zero-order perturbation of a sum-of-squares operator.

Furthermore, sums of squares arise naturally when considering when studying involutive structures, for example in [11], and when considering the real part of the Hodge operator associated with involutive structures. Furthermore, in the cases in which the involutive structure is invariant under a group action, these sums of squares are themselves invariant. This makes them another natural operator to consider when studying the cohomology of invariant involutive structures, such as those in [4, 23–25].

The challenge of understanding the conditions that guarantee global analytic hypoellipticity of sum-of-squares operators on compact real-analytic manifolds remains an open and actively pursued area of research.

The proof of the theorem uses two fundamental results: one from the theory of regularity of partial differential equations, specifically Theorem 1.1 of [7], stated here as Theorem 2 below, and another about the structure of Lie groups, namely Corollary 10.1.11 of [20], stated in the proof of Proposition 1. To state the regularity result, we need a few definitions.

For an open set $\Omega \subset \mathbb{R}^n$, we denote by $C^\omega(\Omega)$ the set of functions u in Ω such that, for every compact $K \subset \Omega$, there exists a constant $C_K > 0$ such that

$$\|D^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1}|\alpha|!$$

for any multi-index α.

Definition 3 Let $x_0 \in \Omega$, $u \in \mathcal{D}'(\Omega)$ and P be a differential operator of order m with coefficients in a manifold Ω; we say that u is an analytic vector for P at x_0 if there exist a cutoff function $\phi \in C_c^\infty(\Omega)$ with $\phi \equiv 1$ in an open neighborhood of x_0 and a constant $C > 0$ such that

$$\|\phi P^k u\|_0 \leq C^{k+1} k!^m, \qquad \forall k \in \mathbb{N}.$$

We say that u is an analytic vector for P if it is an analytic vector for P at x for every $x \in \Omega$.

In the following, we recall the definition of the principal symbol of a differential operator defined on a manifold. Let $(x, \xi) \in T^*\Omega$ and $\phi, f \in C^\infty(\Omega)$ with $\mathrm{d}\phi_x = \xi$. We define the principal symbol as

$$P_m(x, \xi) f(x) = \lim_{t \to \infty} t^{-m} \exp(-t\phi(x)) P(\exp(t\phi) f)(x).$$

The characteristic set of P, denoted $\mathrm{Char}(P)$, is the set of zeros of the principal symbol of P outside the zero section of the cotangent bundle:

$$\mathrm{Char}(P) \doteq \{(x, \xi) \in T^*(\Omega) \backslash \{0\};\ P_m(x, \xi) = 0\}.$$

For a detailed discussion of this definition and its invariance, we refer the reader to [22, pp. 151-152].

Moreover, in the next theorem, we use the notation $\mathrm{WF}_a(u)$ for the analytic wave front set and $\mathrm{singsupp}_a\,(u)$ for the analytic singular support of a distribution u. For precise definitions, we refer to [22, pp. 283] for the precise definition. The key property we use is that $\mathrm{WF}_a(u) =$ if and only if u is a real-analytic.

Theorem 2 (Theorem 1.1 of [7]) *Let P be an order m differential operator in a real-analytic manifold Ω with real-analytic coefficients and let u in $\mathcal{D}'(\Omega)$ be an analytic vector for P. Then:*

$$\mathrm{WF}_a(u) \subset \mathrm{Char}(P).$$

Furthermore, even though it is common to use even-order Sobolev spaces when using elliptic theory, we need to consider the fractional case, which we obtain by using interpolation theory. In order to keep the text self-contained, we include the results regarding interpolation in the appendix.

2 Preliminaries

In this section, we briefly recall facts about working with right invariant vector fields and how to use Fourier analysis on compact Lie groups. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We denote $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ as the set of all left and right invariant vector fields on G, *i.e.* denoting by L_h and R_h the left and right translations by $h \in G$, a vector field χ belongs to $\mathfrak{X}_L(G)$ if $(L_h)_* \chi|_x = \chi|_{hx}$ for every $x, h \in G$, and χ belongs to $\mathfrak{X}_R(G)$ if $(R_h)_* \chi|_x = \chi|_{xh}$ for every $x, h \in G$. If $X \in \mathfrak{g}$, we denote by X^L and X^R the left and right invariant vector fields given by

$$X^L f(x) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(x\exp(tX)) \quad \text{and} \quad X^R f(x) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\exp(tX)x),$$

where exp stands for the exponential function from $\mathfrak{g}$ to G. The application that maps $\mathfrak{g}$ to $\mathfrak{X}_L$ and $\mathfrak{X}_G$ is a Lie algebra isomorphism. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an Ad-invariant inner product on $\mathfrak{g}$, *i.e.*

$$\langle \mathrm{Ad}(x)X, \mathrm{Ad}(x)Y \rangle_{\mathfrak{g}} = \langle X, Y \rangle_{\mathfrak{g}}, \quad \forall x \in G, \forall X, Y \in \mathfrak{g},$$

where $\mathrm{Ad}(x) = (L_x \circ R_{x^{-1}})_*$. In particular, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is also ad-invariant, in other words,

$$\langle [X, Y], Z \rangle_{\mathfrak{g}} = \langle X, [Y, Z] \rangle_{\mathfrak{g}}, \quad \forall X, Y, Z \in \mathfrak{g}.$$

We consider $\langle \cdot, \cdot \rangle$ as the Riemannian metric on G given by left-translation of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, *i.e.* $\langle \cdot, \cdot \rangle_x = \langle (L_x)_* \cdot, (L_x)_* \cdot \rangle_{\mathfrak{g}}$, for every $x \in G$. Then the metric $\langle \cdot, \cdot \rangle$ is automatically left-invariant, and since $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is Ad-invariant, we also have that $\langle \cdot, \cdot \rangle$ is right-invariant, meaning that both $(L_x)_*$ and $(R_x)_*$ are isometries of $\langle \cdot, \cdot \rangle$ for every $x \in G$. In order to define a Laplace-Beltrami operator on G, we need to fix a measure on G. So let d_G be the bi-invariant Haar meadure in G (which exists because the group G is compact), and consider $L^2(G)$ the L^2-space associated with this measure. Let Δ_G be the Laplace-Beltrami operator associated with this metric, *i.e.* $\Delta_G = \mathrm{d}^*\mathrm{d}$, where d^* is the transpose of the de Rham operator with respect with the L^2-space, *i.e.*

$$\langle \mathrm{d}f, \omega \rangle_{L^2} = \langle f, \mathrm{d}^*\omega \rangle_{L^2},$$

for every $f \in \mathcal{C}^\infty(G)$ and $\omega \in \mathcal{C}^\infty(G, \mathrm{T}^*(G))$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, and let $\chi_1, \ldots, \chi_n$ be the its dual basis. In view of d_G being bi-invariant, we have that X_j^L and X_j^R satisfies

$$\langle X_j^L f, g \rangle_{L^2} = -\langle f, X_j^L g \rangle_{L^2},$$

and

$$\langle X_j^R f, g \rangle_{L^2} = -\langle f, X_j^R g \rangle_{L^2},$$

for every $f, g \in \mathcal{C}^\infty(G)$. For each $j = 1, \ldots, n$ we set χ_j^L and the χ_j^R the following one forms:

$$\chi_j^L(Y)(x) = \chi_j((L_{x^{-1}})_* Y),$$

$$\chi_j^R(Y)(x) = \chi_j((R_{x^{-1}})_* Y),$$

for every $x \in G$ and $Y \in \mathrm{T}_x G$, and here we are identifying $\mathfrak{g}$ with $\mathrm{T}_e G$. It follows from this definitions that χ_j^L is left-invariant and χ_j^R is right-invariant, *i.e.* $(L_y)^*\chi_j^L = \chi_j^L$ and $(R_y)^*\chi_j$ for every $y \in G$. Since d_G and $\langle \cdot, \cdot \rangle$ are bi-invariant then we can write

$$\mathrm{d}f = \sum_{j=1}^n (X_j^L f)\chi_j^L,$$

and

$$\mathrm{d}f = \sum_{j=1}^{n} (X_j^R f)\chi_j^R,$$

for every $f \in \mathcal{C}^1(G)$. Therefore $\Delta_G = -\sum_{j=1}^{n}(X_j^L)^2 = -\sum_{j=1}^{n}(X_j^R)^2$.

Lemma 1 *For every $X \in \mathfrak{X}_L(G)$ and $Y \in \mathfrak{X}_R(G)$ we have that $[\Delta_G, X] = [\Delta_G, Y] = 0$.*

Proof Let $Y \in \mathfrak{X}_R(G)$. Since for every $j = 1, \ldots, n$, $[Y, X_j^R] \in \mathfrak{X}_R(G)$, we can write $[Y, X_j^R] = \sum_{k=1}^{n} a_{j,k} X_j^R$, where $a_{j,k} = \langle [Y, X_j^R], X_k^R \rangle \in \mathbb{R}$, so

$$\begin{aligned}
-[Y, \Delta_G] &= [Y, \sum_{j=1}^{n} (X_j^R)^2] \\
&= \sum_{j=1}^{n} [Y, X_j^R] X_j^R + X_j^R [Y, X_j^R] \\
&= \sum_{j,k=1}^{n} a_{j,k} X_k^R X_j^R + a_{j,k} X_j^R X_k^R \\
&= \sum_{j,k=1}^{n} (a_{j,k} + a_{k,j}) X_k^R X_j^R.
\end{aligned}$$

Since $\langle \cdot, \cdot \rangle$ is ad-invariant, we have that

$$a_{j,k} = \langle [Y, X_j^R], X_k^R \rangle = -\langle [X_j^R, Y], X_k^R \rangle = -\langle X_j^R, [Y, X_k^R] \rangle = -a_{k,j},$$

so $[Y, \Delta_G] = 0$. The case for left-invariant vector fields is analogous. □

Let $\hat{G}$ be the set of all classes of equivalences of irreducible, strongly continuous (s.c), unitary representations of G. Since G is compact, the irreducible s.c. unitary representations of G are finite dimensional. We recall that the Fourier transform of a distribution $u \in \mathcal{D}'(G)$ is given by

$$\mathfrak{F}_G[u](\xi) = \int_G u(x)\xi^*(x)\mathrm{d}_G(x), \qquad \xi \in \hat{G},$$

where d_G stands for the bi-invariant Haar measure in G. We can recover the distribution u with its Fourier transform in the following manner:

$$u(x) = \sum_{\xi \in \hat{G}} d_\xi Tr\big(\mathfrak{F}_G[u](\xi)\xi\big),$$

where d_ξ is the dimension of ξ, and we recall that when u is L^2 then the above series converges in the L^2 topology, and the same is valid replacing L^2 by $\mathcal{C}^\infty$, $\mathcal{C}^\omega$ and G^s. Furthermore, if $u \in L^2(G)$, then

$$\|u\|_{L^2}^2 = \sum_{\xi \in \hat{G}} d_\xi Tr\big(\mathfrak{F}_G[u](\xi)\mathfrak{F}_G[u](\xi)^*\big) = \sum_{\xi \in \hat{G}} d_\xi \|\mathfrak{F}_G[u](\xi)\|_{HS}^2.$$

The $\|\cdot\|_{HS}$ is the so-called Hilbert-Schmidt norm. More details about the HS-norm can be found in [28].

Since $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is Ad-invariant, for every $\xi \in \hat{G}$ there exists λ_ξ, such that

$$\Delta_G \xi_{i,j} = \lambda_\xi \xi_{i,j}, \qquad i, j = 1, \ldots, d_\xi.$$

In view of Δ_G being an elliptic linear partial differential operator, by using standard arguments in the theory of linear PDEs, we have that, for every $k \in \mathbb{N}$, we can identify the Sobolev space $H^{2k}(G)$ with the space

$$\left\{u \in \mathcal{D}'(G) \ : \ \Delta_G^k u \in L^2(G)\right\},$$

which is the same as

$$\left\{u \in \mathcal{D}'(G) \ : \ \sum_{\xi \in \hat{G}} d_\xi \langle \xi \rangle^{4k} \|\mathfrak{F}_G[u](\xi)\|_{HS}^2 < \infty\right\},$$

where $\langle \xi \rangle = (1 + \lambda_\xi)^{\frac{1}{2}}$. Since $H^{-2k}(G)$ is the topological dual of $H^{2k}(G)$, we can also identify this space with

$$\left\{u \in \mathcal{D}'(G) \ : \ \sum_{\xi \in \hat{G}} d_\xi \langle \xi \rangle^{-4k} \|\mathfrak{F}_G[u](\xi)\|_{HS}^2 < \infty\right\},$$

Definition 4 We denote by $\mathcal{M}(\hat{G})$ the set of maps $F : \hat{G} \longrightarrow \cup_{m=1}^\infty U(\mathbb{C}^m)$ satisfying $F(\xi) \in U(\mathbb{C}^{d_\xi})$ for all ξ.

We note that the Fourier transform is an isometry between $H^{2k}(G)$, and the following weighted L^2 space:

$$\left\{F \in \mathcal{M}(\hat{G}) \ : \ \sum_{\xi \in \hat{G}} d_\xi \langle \xi \rangle^{4k} \|F(\xi)\|_{HS}^2 < \infty\right\}.$$

The same is valid for $H^{-2k}(G)$. We note that the Fourier transform in $\mathbb{R}^N$ is a isometry between $H^s(\mathbb{R}^N)$ and $L^2\big(\mathbb{R}^N; (1 + |\xi|^2)^s \xi\big)$. Therefore, in view of Proposition 3 and Corollary 1 from the Appendix, for every $s \in \mathbb{R}$, we have that $H^s(G)$ can be isometrically identified with

$$\left\{ u \in \mathcal{D}'(G) \ : \ \sum_{\xi \in \hat{G}} d_\xi \langle \xi \rangle^{2s} \|\mathfrak{F}_G[u](\xi)\|_{HS}^2 < \infty \right\},$$

with norm

$$\|u\|_s = \left(\sum_{\xi \in \hat{G}} d_\xi \langle \xi \rangle^{2s} \|\mathfrak{F}_G[u](\xi)\|_{HS}^2 \right)^{\frac{1}{2}}. \tag{3}$$

Having a global expression for the Sobolev norm is crucial in order to use interpolation techniques together with subelliptic estimates. Since the metric we are considering is ad-invariant, it remains ad-invariant when restricted to K. Therefore, let Δ_K be the Laplace-Beltrami operator associated with the restriction of $\langle \cdot, \cdot \rangle$ to K, so it commutes with all left and right invariant vector fields on K. In other words, $\Delta_k = -\sum_{j=1}^m (Y_j)_{R_K}^2$, where the Y_j's form a basis for the Lie algebra of K, with $(Y_j)_{R_K}$ the right-translations of Y_j by elements of K, so it is then a vector field on K. By taking right-translations on G, we can consider the vector fields $(Y_j)_R$ as vector fields on G, which are K-invariant, and then we can extend Δ_K to G simply by writing $\Delta_K = -\sum_{j=1}^m (Y_j)_R^2$. To simplify the notation, from now on we shall denote Δ_K by $\Delta_K = -\sum_{j=1}^m Y_j^2$.

Proposition 1 *The operator Δ_K commutes with K-invariant vector fields, and consequently with $P - a$ from (1). Furthermore, Char $(\Delta_K) \cap$ Char $(P) = \emptyset$.*

Proof It is enough to show the commutativity locally. Corollary 10.1.11 of [20] says that there is a covering $\mathfrak{U}$ of Ω by open sets such that for each $U \in \mathfrak{U}$, there is a smooth section $\sigma = \sigma_U$ of for $\pi : G \to \Omega$ such that

$$(u, k) \in U \times K \mapsto \Psi(u, k) \doteq \sigma(u)k \in \sigma(U)K \tag{4}$$

is a diffeomorphism onto an open subset $V = \sigma(U)K$ of G. By decreasing U if necessary, we assume that it has coordinates $(u_1, \ldots, u_m)$.

Let X be a K-invariant vector field on G. Next we show that $\left(\Psi^{-1}\right)_*$ maps the vector fields X into a vector fields of the form

$$L = \sum_{\ell=1}^n a_\ell(u) \frac{\partial}{\partial u_\ell} + \sum_{\ell=1}^m b_\ell(u) Y_\ell, \tag{5}$$

defined on $U \times K$ with Y_ℓ as before.

Writing $g = \Psi(u, k) = \sigma(u)k$, we have that $u = \pi(g)$. Since σ is a section of the projection map π on U, *i.e.* $\pi(\sigma(v)) = v$, for every $v \in U$, we have that, $\pi(g) = \pi(\sigma(u)) = u$, in other words, $g = \sigma(u)$, $g = \sigma(\pi(g))k$, for some $k \in K$, *i.e.*

$$k = \sigma(\pi(g))^{-1} g = \mathrm{i}(\sigma(\pi(g)))g = \mu(\mathrm{i}(\sigma(\pi(g))), g).$$

In conclusion, $\Psi^{-1}(g) = (\pi(g), \mu(\mathrm{i}(\sigma(\pi(g)), g)) \in U \times K$.

Let $\phi(g) \doteq i(\sigma \circ \pi(g))g$ be defined for $g \in \pi^{-1}(U)$ and consider $g = \sigma(u)k$ and $g' = gk' = \sigma(u)kk'$ for some $k' \in K$.

Notice that $\pi(g) = gK = gkk'K = \pi(g')$ and so we have that

$$k = i(\sigma(\pi(g)))g = i(\sigma(\pi(g')))g = i(\sigma(\pi(g')))g'k'^{-1} = R_{k'^{-1}} \circ i(\sigma(\pi(g')))g'$$

which implies that $\phi(R_{k'}(g)) = R_{k'} \circ \phi(g)$ for all $g \in \pi^{-1}(U)$. This means that, if a vector field is invariant by the right-action of K in G, then it descends into a right-invariant vector field in K via ϕ_*. Now since

$$\left(\Psi^{-1}\right)_*(X|_{(u,k)}) = \pi_*\Big(X\big|_{\Psi(u,k)}\Big) + \phi_*\Big(X\big|_{\Psi(u,k)}\Big),$$

and $\pi_* R_{k'} = \pi_*$, we have that $\pi_*\Big(X\big|_{\Psi(u,k)}\Big)$ is independent of k, and therefore $\left(\Psi^{-1}\right)_*(X)$ is of the form (5).

Now, because Δ_K does not act on the t directions and commutes with Y_ℓ in view of Lemma 1, we have that

$$[X, \Delta_K](f) = \Psi_*\left\{[L_j, \Delta_K]\right\}(f \circ \Psi^{-1}) = 0, \qquad \forall f \in \mathcal{C}_c^\infty(\Psi(U \times K)),$$

which is enough to conclude that $[\Delta_K, P-a] = 0$. For the second part, it is also enough to prove that $\mathrm{Char}\,(\Delta_K)\Big|_{\Psi(U\times K)} \bigcap \mathrm{Char}\,(P)\Big|_{\Psi(U\times K)} = \emptyset$. In view of condition (E), we have that $\mathrm{Char}\,(P)\Big|_{\Psi(U\times K)} \subset \{0\} \times T^*K$, and since $\mathrm{Char}\,(\Delta_K)\Big|_{\Psi(U\times K)} \subset T^*U \times \{0\}$, we have that $\mathrm{Char}\,(\Delta_K)\Big|_{\Psi(U\times K)} \bigcap \mathrm{Char}\,(P)\Big|_{\Psi(U\times K)} = \emptyset$, since we exclude the zero section in the characteristic set.

□

3 Proof of Theorem 1

Consider the vector fields $X_0, X_1, \ldots, X_\nu$ and the operator P satisfying the conditions of Theorem 1. Assuming that $Pu \in C^\omega(G)$ and we want to show that $u \in C^\omega(G)$. In other words, we wish to prove that $\mathrm{singsupp}_a\,(u) = \emptyset$, or equivalently, that $\mathrm{WF}_a(u) = \emptyset$.

In view of Hörmander's Theorem, we already have that $u \in \mathcal{C}^\infty(G)$. Furthermore, Hormander's Theorem, as discussed in [27], provides us with local subelliptic estimates. In other words, for every $x \in G$ there exist an $\varepsilon > 0$ and $V \subset G$, an open neighborhood of x, such that the following property holds true:

$$u \in L^2_{\mathrm{loc}}(V),\ Pu \in L^2_{\mathrm{loc}}(V) \Rightarrow u \in H^\varepsilon_{\mathrm{loc}}(V).$$

Since G is compact, there exists $\varepsilon > 0$ such that

$$u \in L^2(G),\ Pu \in L^2(G) \Rightarrow u \in H^\varepsilon(G).$$

By the Closed Graph Theorem, there exists a constant $C > 0$ such that

$$\|u\|_\varepsilon \leq C(\|Pu\| + \|u\|),$$

Therefore, in view of the explicit formula for the Sobolev norm (3) and a simple interpolation argument based on the Young's inequality for products, one can find a constant $C > 0$ such that

$$\|u\|_\varepsilon \leq C(\|Pu\| + \|u\|_{-2}), \tag{6}$$

for every $u \in \mathcal{C}^\infty(G)$. Before we proceed, we would also like to point out that since Δ_K is a second-order partial differential operator, it is bounded from $L^2(G)$ to $H^{-2}(G)$. In particular, there exists a constant $\gamma > 0$ such that $\|\Delta_K v\|_{-2} \leq \gamma \|v\|$, for every $v \in C^\infty(G)$.

Proof of Theorem 1 Let $u \in \mathcal{C}^\infty(G)$ be such that $Pu = f$, with $f \in \mathcal{C}^\omega(G)$. It is enough to prove that u is an analytic-vector for Δ_K, *i.e.* that there exists a constant $B > 0$ such that, for every $j \in \mathbb{N}$, the following estimate holds true:

$$\|\Delta_K^j u\| \leq B^{j+1} j!^2. \tag{7}$$

Indeed, if u is an analytic-vector for Δ_K, then by Theorem 2 we have that $\mathrm{WF}_a(u) \subset \mathrm{Char}(\Delta_K)$. But since $Pu \in C^\omega(G)$, by the elliptic regularity Theorem, we also have that $\mathrm{WF}_a(u) \subset \mathrm{Char}(P)$. Now, in view of Propostion 1, the set $\mathrm{Char}(\Delta_K) \cap \mathrm{Char}(P)$ is empty, therefore $u \in C^\omega(G)$.

The proof of (7) will be carried by induction j and relies on Lemma 4.2 and on Lemma 4.3 of [10]. For clarity, the statements of the two lemmas are included below. Recall that $Y_1, \ldots, Y_m$ is a basis for the Lie algebra of K, $\Delta_K = -\sum_{j=1}^m Y_m^2$, and if I is a multi-index of length $q = |I|$, we define the order q differential operator $Y_I = Y_{i_1} Y_{i_2} \cdots Y_{i_q}$. Also recall that $[\Delta_K, Y_\ell] = 0$, for every $\ell = 1, \ldots, m$.

Lemma 2 *If* (7) *holds for every* $j = 1, \ldots k-1$, *for fixed* B *and* k *then*

$$\|Y_I \Delta_K^{j-|I|} u\| \leq B^{j-|I|/2+1} j!(j-|I|)! \tag{8}$$

is valid for $j \leq k-1$ *and for all sequences* $I = (i_1, \ldots, i_\ell)$ *of elements in* $\{1, \ldots, m\}$, *with length* $|I| \doteq \ell \leq j$. *Morover, if* $0 < |I| \leq k$, *then*

$$\|Y_I \Delta_K^{k-|I|} u\| \leq B^{k-|I|/2}(k-1)!(k-|I|+1)! \tag{9}$$

Proof of Lemma 2 First note that for every $i = 1, \ldots, m$, and every $v \in C^\infty(G)$, since $\langle Y_i v, Y_i v\rangle = -\langle Y_i^2 v, v\rangle$, we have that $0 \leq |\langle Y_i^2 v, v\rangle| = -\langle Y_i^2 v, v\rangle \leq \langle -(Y_1^2 + \cdots + Y_m^2)v, v\rangle = \langle \Delta_K v, v\rangle$. So let $I = (i_1, \ldots, i_\ell)$ and $j \leq k$. Then

$$\begin{aligned}\|Y_I \Delta_K^{j-|I|} u\|^2 &= \langle Y_{i_1} \ldots Y_{i_\ell} \Delta_K^{j-\ell} u, Y_{i_1} \ldots Y_{i_\ell} \Delta_K^{j-\ell} u\rangle \\ &= \left|\langle Y_{i_1}^2 Y_{i_2} \ldots Y_{i_\ell} \Delta_K^{j-\ell} u, Y_{i_2} \ldots Y_{i_\ell} \Delta_K^{j-\ell} u\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\langle Y_{i_2}\dots Y_{i_\ell}\Delta_K^{j-\ell+1}u, Y_{i_2}\dots Y_{i_\ell}\Delta_K^{j-\ell}u\rangle\right| \\
&\leq \left|\langle Y_{i_\ell}\Delta_K^{j-1}u, Y_{i_\ell}\Delta_K^{j-\ell}u\rangle\right| \\
&\leq \left|\langle \Delta_K^{j-1}u, \Delta_K^{j-\ell-1}u\rangle\right| \\
&\leq \|\Delta_K^{j-1}u\|\|\Delta_K^{j-\ell-1}u\| \\
&\leq B^{2j-\ell}(j-1)!^2(j-\ell-1)!^2
\end{aligned}
$$

□

Lemma 3 *If* $f, g \in C^\infty(G)$ *and* $k \in \mathbb{Z}_+$ *then*

$$\Delta_K^k(fg) = \sum_{\ell=0}^{k} 2^\ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} (Y_I\Delta_K^j f)(Y_I\Delta_K^{k-\ell-j} g).$$

So assume that (7) is valid for $j = 1, 2, \dots, k-1$. We first select a constant $A > 0$ such that

$$\|Y_I\Delta_G^k f\| \leq A^{k+|I|/2+1}k!^2|I|!, \qquad \|X_I\Delta_G^k a\|_{L^\infty} \leq A^{k+|I|/2+1}k!^2|I|!$$

for all $k \in \mathbb{Z}_+$ and $I = (i_1, \dots, i_{|I|})$. In view of (6) we have that

$$\|\Delta_K^k u\|_\varepsilon \leq C\left(\|\Delta_K^k f\| + \|[\Delta_K^k, a]u\| + \|\Delta_K^k u\|_{-2}\right). \tag{10}$$

Now Lemma 3 gives the estimate

$$
\begin{aligned}
\|[\Delta_K^k, a]u\| &= \|\Delta_K^k(au) - a\Delta_K^k u\| \\
&\leq \sum_{\ell=1}^{k} 2^\ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} \|Y_I\Delta_K^j a\|_{L^\infty}\|Y_I\Delta_K^{k-\ell-j}u\| \\
&\quad + \sum_{j=1}^{k} \binom{k}{j} \|\Delta_K^j a\|_{L^\infty}\|\Delta_K^{k-j}u\| \doteq S_1 + S_2.
\end{aligned}
$$

Firstly we observe that by the inductive hypothesis we can estimate

$$S_2 \leq \sum_{j=1}^{k} \binom{k}{j} A^{j+1} j!^2 B^{k-j+1}(k-j)!^2 \leq AB^{k+1}k!^2 \sum_{j=1}^{k}\left(\frac{A}{B}\right)^j.$$

For the other term we use again the inductive hypothesis in conjunction with Lemma 2. We obtain

$$S_1 \leq \sum_{\ell=1}^{k} 2^\ell \binom{k}{\ell} \sum_{j=1}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} A^{j+\frac{\ell}{2}+1} j!^2 \ell! B^{k-\frac{\ell}{2}-j+1}(k-j)!(k-j-\ell)!$$

$$
\begin{aligned}
&+\sum_{\ell=1}^{k} 2^{\ell}\binom{k}{\ell} \sum_{|I|=\ell} A^{\frac{\ell}{2}+1} \ell! B^{k-\frac{\ell}{2}}(k-1)!(k-\ell-1)! \\
&\leq A B^{k+1} k!^{2} \sum_{\ell=1}^{k} 2^{\ell} m^{\ell} \sum_{j=0}^{k-\ell}\left(\frac{A}{B}\right)^{j+\ell/2} .
\end{aligned}
$$

Finally, if $\gamma > 0$ is such that $\|\Delta_K v\|_{-2} \leq \gamma \|v\|$, $v \in C^{\infty}(\Omega)$, gathering together in (10) all the information obtained, and assuming $A \ll B$, we get

$$
\begin{aligned}
\|\Delta_G^k u\| &\leq C B^{k+1} k!^2 \left(\frac{A^{k+1}}{B^{k+1}} + A \sum_{j=1}^{k} \left(\frac{A}{B}\right)^{j} + A \sum_{\ell=1}^{k} 2^{\ell} m^{\ell} \sum_{j=0}^{k-\ell} \left(\frac{A}{B}\right)^{j+\ell/2} + \frac{\gamma}{Bk^2} \right) \\
&\leq C B^{k+1} k!^2 \left(\frac{A}{B} + \frac{A^2}{B-A} + \frac{8mA}{B^{1/2}} + \frac{\gamma}{B} \right).
\end{aligned}
$$

If we choose $B > 0$ so large such that the term between parentheses is $\leq 1/C$ our inductive argument proves that (7) indeed holds for $j \leq k$, and every k, so u is an analytic-vector for Δ_K. □

4 Appendix

In this section, we recall some results and definitions regarding the theory of interpolation spaces and the K-Method. Let $\mathcal{E}$ be a topological vector space, and let E_0, E_1 be two normed spaces, continuously embedded in $\mathcal{E}$. We equip $E_0 \cap E_1$ and $E_0 + E_1$ with the following norms:

$$
\begin{aligned}
\|a\|_{E_0 \cap E_1} &\doteq \max\{\|a\|_{E_0}, \|a\|_{E_1}\}, \\
\|a\|_{E_0 + E_1} &\doteq \inf\{\|a_0\|_{E_0} + \|a_1\|_{E_1} \ : \ a = a_0 + a_1, a_0 \in E_0, a_1 \in E_1\}.
\end{aligned}
$$

To simplify the notation, we write the infimum in the definition of $\|a\|_{E_0+E_1}$ as $\inf_{a=a_0+a_1} \left(\|a_0\|_{E_0} + \|a_1\|_{E_1}\right)$.

Definition 5 We say that a normed space E is an intermediate space between E_0 and E_1 if $E_0 \cap E_1 \subset E \subset E_0 + E_1$, with continuous inclusions.

Definition 6 We say that a normed space E is an interpolation space between E_0 and E_1 if it is an intermediate space and if it has the following property: if $T : E_0 + E_1 \longrightarrow E_0 + E_1$ is a linear map, and $T \in \mathcal{L}(E_0, E_0) \cap \mathcal{L}(E_1, E_1)$, then $T \in \mathcal{L}(E)$. The space E is said to be of exponent $0 < \theta < 1$ if there exists a constant $C > 0$ such that

$$
\|T\|_{\mathcal{L}(E,E)} \leq C \|T\|_{\mathcal{L}(E_0,E_0)}^{1-\theta} \|T\|_{\mathcal{L}(E_1,E_1)}^{\theta}, \qquad \forall T \in \mathcal{L}(E_0, E_0) \cap \mathcal{L}(E_1, E_1).
$$

In the following, we describe a method to construct such interpolation spaces, it is called the K-Method. For every $a \in E_0 + E_1$ and every $t > 0$, we define

$$K_2(t;a) \doteq \inf_{a=a_0+a_1} \left(\|a_0\|^2_{E_0} + t^2\|a_1\|^2_{E_1}\right)^{\frac{1}{2}}.$$

For every $0 < \theta < 1$, we define

$$\left(E_0, E_1\right)_{\theta,2} \doteq \left\{a \in E_0 + E_1 \ : \ t^{-\theta} K_2(t;a) \in L^2\left(\mathbb{R}_+; \frac{\mathrm{d}t}{t}\right)\right\},$$

with norm

$$\|a\|_{(E_0,E_1)_{\theta,2}} \doteq \|t^{-\theta} K_2(t;a)\|_{L^2\left(\mathbb{R}_+;\frac{\mathrm{d}t}{t}\right)}.$$

Proposition 2 *Let F_0 and F_1 be normed spaces with the same properties as E_0 and E_1 before. If $T : E_0 + E_1 \longrightarrow F_0 + F_1$ is a linear map, such that $T \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$, then $T \in \mathcal{L}((E_0, E_1)_{\theta,2}, (F_0, F_1)_{\theta,2})$ for every $0 < \theta < 1$, and $\|T\|_{\mathcal{L}((E_0,E_1)_{\theta,2},(F_0,F_1)_{\theta,2})} \leq \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)}\|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}$.*

Proof Let $a \in E_0 + E_1$. For each decomposition $a = a_0 + a_1$, we have that

$$\begin{aligned}
K_2(t;Ta) &\leq \left(\|Ta_0\|^2_{F_0} + t^2\|Ta_1\|^2_{F_1}\right)^{\frac{1}{2}} \\
&\leq \left(\|T\|^2_{\mathcal{L}(E_0,F_0)}\|a_0\|^2_{E_0} + t^2\|T\|^2_{\mathcal{L}(E_1,F_1)}\|a_1\|^2_{E_1}\right)^{\frac{1}{2}} \\
&= \|T\|_{\mathcal{L}(E_0,F_0)}\left(\|a_0\|^2_{E_0} + t^2\frac{\|T\|^2_{\mathcal{L}(E_1,F_1)}}{\|T\|^2_{\mathcal{L}(E_0,F_0)}}\|a_1\|^2_{E_1}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, $K_2(t;Ta) \leq \|T\|_{\mathcal{L}(E_0,F_0)} K_2\left(t\frac{\|T\|_{\mathcal{L}(E_1,F_1)}}{\|T\|_{\mathcal{L}(E_0,F_0)}}; a\right)$. To conclude, we note that

$$\begin{aligned}
\|Ta\|_{(F_0,F_1)_{\theta,2}} &= \left(\int_0^\infty t^{-2\theta} K_2(t;Ta)^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\infty t^{-2\theta}\|T\|^2_{\mathcal{L}(E_0,F_0)} K_2\left(t\frac{\|T\|_{\mathcal{L}(E_1,F_1)}}{\|T\|_{\mathcal{L}(E_0,F_0)}}; a\right)^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \\
&\underset{t=\frac{\|T\|_{\mathcal{L}(E_0,F_0)}}{\|T\|_{\mathcal{L}(E_1,F_1)}}s}{=} \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)}\|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}\left(\int_0^\infty s^{-2\theta} K_2(s;a)^2\frac{\mathrm{d}s}{s}\right)^{\frac{1}{2}} \\
&= \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)}\|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}\|a\|_{(E_0,E_1)_{\theta,2}}.
\end{aligned}$$

$\square$

Corollary 1 *Let F_0 and F_1 be normed spaces with the same properties as E_0 and E_1 before. If $T : E_0+E_1 \longrightarrow F_0+F_1$ is a linear map, such that T is a linear isomorphism* *between E_0 and F_0, and between E_1 and F_1, then T is a linear isomorphism between $\big(E_0, E_1\big)_{\theta,2}$ and $\big(F_0, F_1\big)_{\theta,2}$ for every $0 < \theta < 1$.*

We restrict our attention to the case in which E_0 and E_1 are weighted L^2 spaces. Let $\Omega \subset \mathbb{R}^N$ be an open set, and let ω be a positive measurable function defined on Ω. We define

$$E(\omega) \doteq \left\{ u : \Omega \longrightarrow \mathbb{C} \ : \ u \text{ is measurable}\,, \int_\Omega |u(x)|^2 \omega(x) \mathrm{d}x < \infty \right\},$$

and

$$\|u\|_\omega \doteq \left(\int_\Omega |u(x)|^2 \omega(x) \mathrm{d}x \right)^{\frac{1}{2}}.$$

With the introduced definitions and notation, we can easily prove the following proposition.

Proposition 3 *Let ω_0 and ω_1 be two positive measurable functions on Ω. Then, for every $0 < \theta < 1$,*

$$\big(E(\omega_0), E(\omega_1)\big)_{\theta,2} = E(\omega_\theta),$$

where $\omega_\theta = \omega_0^{1-\theta} \omega_1^{\theta}$.

Proof We have an explicit formula for $K_2(t; a) = \inf_{a=a_0+a_1} \left(\|a_0\|^2_{\omega_0} + t^2 \|a_1\|^2_{\omega_1} \right)^{\frac{1}{2}}$. Let $a \in E(\omega_0) + E(\omega_1)$, and let $x \in \Omega$. Consider the following optimization problem: find $\lambda \in \mathbb{C}$ that minimizes

$$|\lambda|^2 \omega_0(x) + t^2 |a(x) - \lambda|^2 \omega_1(x).$$

Such lambda is obtained by solving the following equations:

$$\begin{aligned} \lambda \omega_0(x) - t^2 (a(x) - \lambda) \omega_1(x) = 0, \\ \overline{\lambda} \omega_0(x) - t^2 \overline{(a(x) - \lambda)} \omega_1(x) = 0, \end{aligned}$$

thus $\lambda = \frac{t^2 a(x) \omega_1(x)}{\omega_0(x) + t^2 \omega_1(x)}$. Therefore

$$\begin{aligned} K_2(t; a)^2 &= \int_\Omega \frac{t^4 \omega_1(x)^2 + t^2 \omega_0(x)^2}{(\omega_0(x) + t^2 \omega_1(x))^2} |a(x)|^2 \mathrm{d}x \\ &= \int_\Omega \frac{t^2 \omega_0(x) \omega_1(x)}{\omega_0(x) + t^2 \omega_1(x)} |a(x)|^2 \mathrm{d}x, \end{aligned}$$

and this implies that

$$\|t^{-\theta} K_2(t; a)\|^2_{L^2\left(\mathbb{R}_+; \frac{\mathrm{d}t}{t}\right)} = \int_0^\infty \int_\Omega t^{-2\theta} \frac{t^2 \omega_0(x) \omega_1(x)}{\omega_0(x) + t^2 \omega_1(x)} |a(x)|^2 \mathrm{d}x \frac{\mathrm{d}t}{t}$$

$$
\begin{aligned}
&= \int_\Omega \int_0^\infty t^{-2\theta} \frac{t^2 \omega_0(x)\omega_1(x)}{\omega_0(x) + t^2\omega_1(x)} |a(x)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}x \\
&\underset{t=\sqrt{\frac{\omega_0(x)}{\omega_1(x)}}s}{=} \int_\Omega \int_0^\infty \frac{s^{2-2\theta}}{1+s} \frac{\omega_0(x)\omega_1(x)}{\omega_0(x)^\theta \omega_1(x)^{1-\theta}} |a(x)|^2 \frac{\mathrm{d}s}{s} \mathrm{d}x \\
&= \left(\int_0^\infty \frac{s^{1-2\theta}}{1+s} \mathrm{d}s \right) \|a\|_{\omega_\theta}^2 .
\end{aligned}
$$

□

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