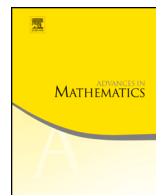




Contents lists available at ScienceDirect

## Advances in Mathematics

journal homepage: [www.elsevier.com/locate/aim](http://www.elsevier.com/locate/aim)

# $A_\infty$ deformations of extended Khovanov arc algebras and Stroppel's conjecture

Severin Barmeier<sup>a,b,\*</sup>, Zhengfang Wang<sup>c,d,\*</sup><sup>a</sup> Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany<sup>b</sup> Hausdorff Research Institute for Mathematics, Poppelsdorfer Allee 45, 53115 Bonn, Germany<sup>c</sup> Nanjing University, School of Mathematics, Nanjing 210093, Jiangsu, China<sup>d</sup> Universität Stuttgart, Institut für Algebra und Zahlentheorie, Pfaffenwaldring 57, 70569 Stuttgart, Germany

## ARTICLE INFO

*Article history:*

Received 15 December 2024

Received in revised form 12 August 2025

Accepted 19 November 2025

Available online 11 December 2025

Communicated by Pramod Achar

*MSC:*

primary 16E40

secondary 18G70, 16S80, 17B10, 57K18, 53D37

*Keywords:*

Extended Khovanov arc algebras

Koszul duality

 $A_\infty$  deformations

Reduction systems

Kazhdan–Lusztig polynomials

## ABSTRACT

Extended Khovanov arc algebras  $K_m^n$  are graded associative algebras which naturally appear in a variety of contexts, from knot and link homology, low-dimensional topology and topological quantum field theory to representation theory and symplectic geometry. C. Stroppel conjectured in her ICM 2010 address that the bigraded Hochschild cohomology groups of  $K_m^n$  vanish in a certain range, implying that the algebras  $K_m^n$  admit no nontrivial  $A_\infty$  deformations, in particular that the algebras are intrinsically formal.

Whereas Stroppel's conjecture is known to hold for the algebras  $K_m^1$  and  $K_1^n$  by work of Seidel and Thomas, we show that  $K_m^n$  does in fact admit nontrivial  $A_\infty$  deformations with nonvanishing higher products for all  $m, n \geq 2$ .

We describe both the extended Khovanov arc algebras  $K_m^n$  and their Koszul duals concretely as path algebras of quivers with relations. This allows us to give an explicit algebraic construction of  $A_\infty$  deformations of  $K_m^n$  by using the correspondence between  $A_\infty$  deformations of a Koszul

\* Corresponding authors.

E-mail addresses: [s.barmeier@gmail.com](mailto:s.barmeier@gmail.com) (S. Barmeier), [zhengfangw@nju.edu.cn](mailto:zhengfangw@nju.edu.cn) (Z. Wang).

algebra and filtered associative deformations of its Koszul dual.

© 2025 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

## Contents

|      |                                                                           |    |
|------|---------------------------------------------------------------------------|----|
| 1.   | Introduction . . . . .                                                    | 2  |
| 2.   | Extended Khovanov arc algebras . . . . .                                  | 5  |
| 2.1. | Diagrammatics and grading . . . . .                                       | 5  |
| 2.2. | Generators and relations . . . . .                                        | 9  |
| 3.   | The Koszul-dual picture . . . . .                                         | 20 |
| 3.1. | Generators and relations of the Koszul dual . . . . .                     | 21 |
| 3.2. | Reduction systems and Bergman's Diamond Lemma . . . . .                   | 23 |
| 3.3. | Irreducible paths and Kazhdan–Lusztig polynomials . . . . .               | 25 |
| 3.4. | Some useful relations in the Koszul dual . . . . .                        | 29 |
| 4.   | Hochschild cohomology and deformations for Koszul-dual algebras . . . . . | 34 |
| 4.1. | Bigraded Hochschild cohomology and Koszul duality . . . . .               | 35 |
| 4.2. | Associative and $A_\infty$ deformations under Koszul duality . . . . .    | 36 |
| 4.3. | Associative deformations via reduction systems . . . . .                  | 39 |
| 5.   | $A_\infty$ deformations and Stroppel's conjecture . . . . .               | 41 |
| 5.1. | The case $K_2^2$ . . . . .                                                | 43 |
| 5.2. | The general case $K_m^n$ . . . . .                                        | 46 |
|      | Acknowledgments . . . . .                                                 | 50 |
|      | Appendix A. Proof of Proposition 5.11 . . . . .                           | 50 |
| A.1. | The term $\delta(\psi)_{\bar{Y}_0 \bar{X}_0}$ . . . . .                   | 52 |
| A.2. | The term $\delta(\psi)_{\bar{Y}_1 \bar{X}_1}$ . . . . .                   | 54 |
| A.3. | The term $\delta(\psi)_{\bar{Y}_3 \bar{X}_3}$ . . . . .                   | 55 |
| A.4. | The term $\delta(\psi)_{\bar{X}_0 X_7}$ . . . . .                         | 56 |
| A.5. | The term $\delta(\psi)_{\bar{Y}_1 \bar{X}_3}$ . . . . .                   | 57 |
| A.6. | The term $\delta(\psi)_{\bar{X}_3 \bar{X}_5}$ . . . . .                   | 58 |
|      | References . . . . .                                                      | 61 |

## 1. Introduction

Khovanov arc algebras were introduced by Khovanov in his seminal work [29] on the categorification of the Jones polynomial of links. The Khovanov arc algebras are “diagrammatic” algebras whose basis is given by “arc diagrams” with composition defined by stacking the diagrams vertically and applying certain simplification rules obtained from a 2-dimensional TQFT (see §2 for more details). In [42] Stroppel introduced the *extended Khovanov arc algebras*  $K_m^n$  for  $m, n \geq 1$  as a generalisation, where the arc diagrams contain not only closed arcs (circles) but also open arcs (lines). The algebras  $K_m^n$  were also studied by Chen and Khovanov [14] by viewing them as subquotients of the classical Khovanov arc algebras. Brundan and Stroppel showed that the algebras  $K_m^n$  are quasi-hereditary covers of the classical Khovanov arc algebras and that they have many nice properties, for example they are Koszul, cellular and homologically smooth [9,10].

See also Stroppel's ICM 2022 address [44] for a recent account and further applications to link and tangle invariants.

Besides their role in link homology, the algebras  $K_m^n$  play a central role in representation theory and naturally appear in symplectic geometry. In [42] it is shown that the category  $\text{rep}(K_m^n)$  of finite-dimensional representations of  $K_m^n$  is equivalent to the category of perverse sheaves on the Grassmannian  $\text{Gr}(m, m+n)$  and also equivalent to the principal block of the parabolic Bernstein–Gelfand–Gelfand category  $\mathcal{O}$  associated to the parabolic subalgebra corresponding to  $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{gl}_{m+n}(\mathbb{C})$ , see Brundan and Stroppel [9–12], also for further applications in representation theory.

A symplectic analogue of classical Khovanov homology and the Jones polynomial was given by Seidel and Smith [40] via the Lagrangian Floer homology of certain nilpotent slices, described by Manolescu [34] as Hilbert schemes of surfaces. Abouzaid and Smith [1,2] proved this symplectic Khovanov homology to be isomorphic to Khovanov homology (after collapsing the bigrading of the latter). Similar results were recently obtained for the extended Khovanov arc algebras  $K_m^n$  by Mak and Smith [33] who showed that the DG category  $\text{perf}_{\text{dg}}(K_m^n)$  of perfect DG modules is quasi-equivalent to the Fukaya–Seidel  $A_\infty$  category  $\mathcal{FS}(X_m^n, \pi_m^n)$ , where  $X_m^n = \text{Hilb}^m(A_{m+n-1}) \setminus D$  is the affine complement of a certain divisor  $D \subset \text{Hilb}^m(A_{m+n-1})$  of the Hilbert scheme of  $m$  points on the Milnor fibre (the generic deformation) of an  $A_{m+n-1}$  surface singularity and  $\pi_m^n: X_m^n \rightarrow \mathbb{C}$  is a symplectic Lefschetz fibration. The key observation of their proof is that  $\mathcal{FS}(X_m^n, \pi_m^n)$  admits a generator whose endomorphism  $A_\infty$  algebra turns out to be formal and  $A_\infty$ -quasi-isomorphic to  $K_m^n$ . Whereas there is a natural “geometric” generator of the Fukaya–Seidel category  $\mathcal{FS}(X_m^n, \pi_m^n)$  given by the Lefschetz thimbles associated to vanishing cycles [39, §18], the endomorphism  $A_\infty$  algebra of this generator is *not* formal whenever  $m, n \geq 2$ . This is shown in [33, App. A] using the nontrivial  $A_\infty$  structure on the Ext-algebra of parabolic Verma modules given by Klamt and Stroppel [30]. The quasi-equivalence between  $\mathcal{FS}(X_m^n, \pi_m^n)$  and the DG category of perfect DG modules over  $K_m^n$  should therefore be understood as a formality result for  $\mathcal{FS}(X_m^n, \pi_m^n)$ . All of these results can be viewed as part of the continually growing dictionary between knot homology, symplectic geometry and representation theory.

In her ICM 2010 address, Stroppel stated the following conjecture about the bigraded Hochschild cohomology groups of  $K_m^n$ .

**Conjecture 1.1** ([43, Conj. 2.7]).  $\text{HH}_{i-2}^2(K_m^n, K_m^n) = 0$  if  $i \neq 0$ .<sup>1</sup>

Whereas the classical Khovanov arc algebras are symmetric algebras with large Hochschild (co)homology groups which can be described in terms of the Khovanov homology of torus links (see Rozansky [37, Thm. 6.9]), their quasi-hereditary covers  $K_m^n$  are homologically smooth and have much smaller Hochschild cohomology. Conjecture 1.1 implies in particular that (contrary to the classical arc algebras) the extended Khovanov arc

<sup>1</sup> What we denote by  $\text{HH}_{i-2}^2(A, A)$  in this article is denoted by  $\mathbb{H}^i(A)_{2-i}$  in [43].

algebras  $K_m^n$  are *intrinsically formal*, i.e. that every  $A_\infty$  structure on  $K_m^n$  is equivalent to the original associative structure. It was already known by earlier work of Seidel and Thomas [41, Lem. 4.21] (also for other choices of grading) that Conjecture 1.1 holds for  $K_m^n$  whenever  $\min(m, n) = 1$ . Moreover, Conjecture 1.1 would yield the formality result of Mak and Smith [33] mentioned above, as well as the formality result of Abouzaid and Smith [1] for the symplectic arc algebras.

In contrast, our main result is the following.

**Theorem** (Theorem 5.4 and Corollary 5.13). *For all  $m, n \geq 2$  we have the following:*

- (i)  $\mathrm{HH}_{i-2}^2(K_m^n, K_m^n)$  is 1-dimensional for  $i = 2mn - 4$ .
- (ii)  $K_m^n$  is not intrinsically formal.
- (iii)  $K_m^n$  admits an explicit nontrivial  $A_\infty$  deformation.

Although (i) settles Conjecture 1.1 negatively for  $m, n \geq 2$ , (ii) and (iii) establish the existence of interesting higher structures on the extended Khovanov arc algebras  $K_m^n$  whenever  $m, n \geq 2$ , see Corollary 5.13 and also Fig. 11 for a diagrammatic interpretation of the  $A_\infty$  deformations of  $K_2^2$ . Moreover, the  $A_\infty$  deformations of  $K_m^n$  can be viewed as deformations of the Fukaya–Seidel category  $\mathcal{FS}(X_m^n, \pi_m^n)$  (see Remark 5.15).

Our proof relies on a method of computing Hochschild cohomology and associative deformations developed in [4] and can be divided into three steps as follows. In the first step, we realise  $K_m^n$  as a quiver with relations, i.e. we give an algebra isomorphism  $K_m^n \simeq \mathbb{k}Q_m^n/I_m^n$  for some finite quiver  $Q_m^n$  with admissible ideal  $I_m^n$  which may be of independent interest. We show that the quiver  $Q_m^n$  is the double quiver of a bipartite graph  $\Gamma_m^n$  (Proposition 2.7) and that the ideal  $I_m^n$  is generated by three types of quadratic relations (Proposition 2.16).

In the second step, noting that  $K_m^n$  is a Koszul algebra [10, §5], we describe its Koszul dual as  $\overline{K}_m^n = \mathbb{k}\overline{Q}_m^n/\overline{I}_m^n$ , where  $\overline{Q}_m^n$  is the opposite quiver of  $Q_m^n$  and  $\overline{I}_m^n$  the ideal generated by the orthogonal quadratic relations (Proposition 3.3). We show that  $\overline{K}_m^n$  admits a natural reduction system  $\overline{R}_m^n$  satisfying the diamond condition by showing that the number of the associated irreducible paths can be described by the Kazhdan–Lusztig polynomials (see §3.3).

In the third step, we describe deformations of the reduction system  $\overline{R}_m^n$  to construct an explicit (nontrivial) first-order deformation of  $\overline{R}_m^n$  so that  $\mathrm{HH}_{2mn-6}^2(\overline{K}_m^n, \overline{K}_m^n)$  is 1-dimensional. By Keller’s duality theorem (Theorem 4.2) we obtain that  $\mathrm{HH}_{2mn-6}^2(K_m^n, K_m^n)$  is also 1-dimensional. The above construction in fact also gives a nontrivial (filtered) associative deformation of  $\overline{K}_m^n$ . The minimal model of the derived endomorphism algebra of its simples then gives an explicit  $A_\infty$  deformation of  $K_m^n$  (see Proposition 4.7 and Corollary 5.13).

The simplest example of a nontrivial  $A_\infty$  deformation of  $K_m^n$  can be found in §5.1, where the main results are proven for  $K_2^2$ . (See Fig. 10 for an illustration of the degree 0 and 1 arc diagrams in  $K_2^2$ .) In fact, §5.1 can also be viewed as a blueprint for the general

case which is technically more involved, but structurally similar. For degree reasons the algebra  $K_2^2$  in fact admits a *unique*  $A_\infty$  deformation which is given in Corollary 5.8. A diagrammatic description of this  $A_\infty$  deformation is given in Fig. 11.

## 2. Extended Khovanov arc algebras

### 2.1. Diagrammatics and grading

We first recall the diagrammatics for the extended Khovanov arc algebras. For more details we refer to [9,42].

Let  $m, n \geq 1$  be fixed natural numbers. We fix a horizontal line in the plane and  $m+n$  distinct points on it.

A *cup diagram* is a diagram consisting of  $k \leq \lfloor \frac{m+n}{2} \rfloor$  nested cups and  $m+n-2k$  half-lines which are attached to the  $m+n$  fixed points on the horizontal line. More precisely, the cups are nested lower semicircles lying below the horizontal line and joining pairs of points and the half-lines are attached to single points and extend downwards in such a way that cups and half-lines pairwise do not intersect (in particular, half-lines cannot be nested inside cups). For example,



is a cup diagram for  $m+n=5$ .

A *cap diagram* is the mirror image of a cup diagram obtained by reflecting all cups along the fixed horizontal line. For example,



is a cap diagram.

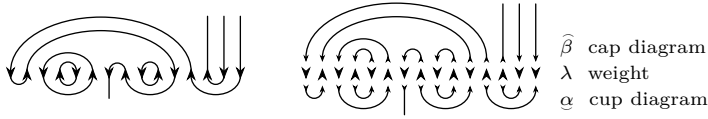
A *weight* of type  $\frac{n}{m}$  is a sequence of weight symbols  $\blacktriangledown$  and  $\blacktriangle$  placed at the  $m+n$  points, where  $m$  is the total number of  $\blacktriangledown$ 's and  $n$  is the total number of  $\blacktriangle$ 's. For example,



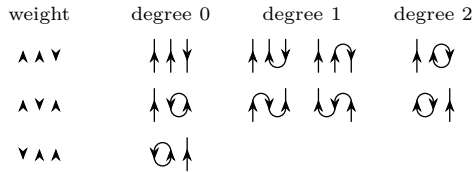
is a weight of type  $\frac{3}{2}$ . Let  $\Lambda_m^n$  denote the set of weights of type  $\frac{n}{m}$ . For example  $\Lambda_1^2 = \{\blacktriangle\blacktriangledown, \blacktriangledown\blacktriangle, \blacktriangledown\blacktriangle\blacktriangle\}$ . Note that  $\#\Lambda_m^n = \binom{m+n}{m}$ .

Given a weight  $\nu$ , let  $\underline{\nu}$  denote the unique cup diagram obtained by recursively connecting directly neighbouring  $\blacktriangledown\blacktriangle$  pairs in  $\nu$  with a cup, ignoring  $\blacktriangle$ 's and  $\blacktriangledown$ 's already connected in a previous step, drawing half-lines extending downwards for the remaining positions, and deleting the weight symbols  $\blacktriangledown$  and  $\blacktriangle$  in the end. For example,

$$\nu = \blacktriangle\blacktriangledown\blacktriangledown\blacktriangle\blacktriangle \rightarrow \blacktriangle\blacktriangledown\cup\blacktriangle\blacktriangle \rightarrow \blacktriangle\cup\cup\blacktriangle \rightarrow \uparrow\cup\cup\uparrow \rightarrow | \cup \cup = \underline{\nu}$$



**Fig. 1.** An arc diagram (left) of type  $\frac{7}{8}$  of degree 6 and its decomposition as a gluing of a cup diagram and a cap diagram along a weight (right).



**Fig. 2.** All arc diagrams for weights of type  $\frac{2}{1}$ .

(Note that  $\nu$  does not depend on the order of the neighbouring  $\blacktriangledown \blacktriangle$  pairs chosen in the recursion.)

Let  $\alpha, \beta, \lambda$  be weights of type  $\frac{n}{m}$ . The cup diagram  $\alpha$  and the cap diagram  $\beta$  can be glued along  $\lambda$  as follows. The weight symbols of  $\lambda$  are placed at the  $m + n$  fixed points and the cups, caps and half-lines in  $\alpha$  and  $\beta$  are glued together at these  $m + n$  points, giving rise to open and closed arcs in the plane passing through the  $\blacktriangledown$ 's and  $\blacktriangle$ 's of the weight  $\lambda$ . An *arc diagram* of type  $\frac{n}{m}$  is such a gluing, which we denote by  $\alpha \lambda \beta$ , satisfying the following constraints:

- (i) the  $\blacktriangle$ 's and  $\blacktriangledown$ 's of  $\lambda$  through which any single arc passes induce a well-defined orientation on the arc
- (ii) for any two half-lines extending both to the top or both to the bottom, their orientation must *not* be of the form  $\blacktriangledown \cdots \blacktriangle$ .

See Fig. 1 for an illustration of an arc diagram as a gluing of a cup and cap diagram along a weight and Fig. 2 for an illustration of all arc diagrams for all weights of type  $\frac{2}{1}$ . Note that the following diagrams are *not* arc diagrams of type  $\frac{2}{1}$



because they violate conditions (i), (ii) and (ii), respectively.

On the right of Fig. 1 we have also drawn the weight symbols of  $\alpha$  and  $\beta$  for reference, but in the resulting arc diagram, the weights  $\alpha$  and  $\beta$  are only reflected in the cups and caps, but not in the orientation of the arcs which are determined by the weight  $\lambda$ .

**Definition 2.1** ([9, §2]). The *degree* of an arc diagram is given by the number of clockwise caps and clockwise cups it contains. Note that for each weight  $\lambda$ , the arc diagram  $e_\lambda :=$

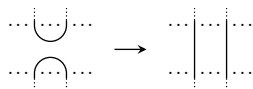
$\lambda\widehat{\lambda\lambda}$  is the unique arc diagram of degree 0, since the cups in  $\lambda$  and the caps in  $\widehat{\lambda\lambda}$  are all oriented counterclockwise in  $e_\lambda$ .

The number of (counterclockwise) circles in  $e_\lambda$  is called the *defect* of  $\lambda$  and is denoted by  $\text{def}(\lambda)$ .

For example, the arc diagram of Fig. 1 is of degree 6 as it contains four clockwise cups and two clockwise caps.

**Definition 2.2.** Let  $\mathbb{k}$  be a field of characteristic 0. The *extended Khovanov arc algebra*  $K_m^n$  is the  $\mathbb{N}$ -graded  $\mathbb{k}$ -algebra with basis given by the arc diagrams for weights of type  $\frac{n}{m}$ , graded by their degree, and multiplication given as follows.

For any two arc diagrams  $a = \alpha\lambda\widehat{\beta}$  and  $b = \gamma\mu\widehat{\delta}$ , their product  $ab$  is zero unless  $\beta = \gamma$  in which case  $ab$  is obtained by writing  $a$  below  $b$ , connecting the open arcs (lines) in the same position, and repeatedly performing the surgery



on the underlying (unoriented) arcs for a cup–cap pair (i.e. cup and cap correspond to each other in the reflection identifying  $\widehat{\beta}$  and  $\widehat{\beta}$ ) which can be connected without crossing other arcs. After each surgery (cutting the arcs open and reconnecting them as illustrated), the resulting arcs are reoriented by labelling the connected components containing the cup and the cap by one of 1,  $\epsilon$ , or  $\zeta$  according to whether the component is a closed arc oriented counterclockwise, a closed arc oriented clockwise, or an oriented open arc (line), respectively, and reorienting the resulting connected components according to the following rules:

- (i) If the surgery produces one connected component out of two, orient it according to

$$\begin{array}{llll} 1 \otimes 1 \mapsto 1 & 1 \otimes \epsilon \mapsto \epsilon & 1 \otimes \zeta \mapsto \zeta & \epsilon \otimes \zeta \mapsto 0 \\ \epsilon \otimes 1 \mapsto \epsilon & \epsilon \otimes \epsilon \mapsto 0 & \zeta \otimes 1 \mapsto \zeta & \zeta \otimes \epsilon \mapsto 0 \end{array}$$

where  $\mapsto 0$  indicates that the arc diagram is deleted and does not contribute to the product.

- (ii) If the surgery produces two connected components out of one, orient them according to

$$1 \mapsto 1 \otimes \epsilon + \epsilon \otimes 1 \quad \epsilon \mapsto \epsilon \otimes \epsilon \quad \zeta \mapsto \epsilon \otimes \zeta$$

where the first rule means that the surgery produces a sum of two diagrams by duplicating — in one diagram one of the two new closed arcs is oriented clockwise and the other counterclockwise and in the other diagram they are oriented in the

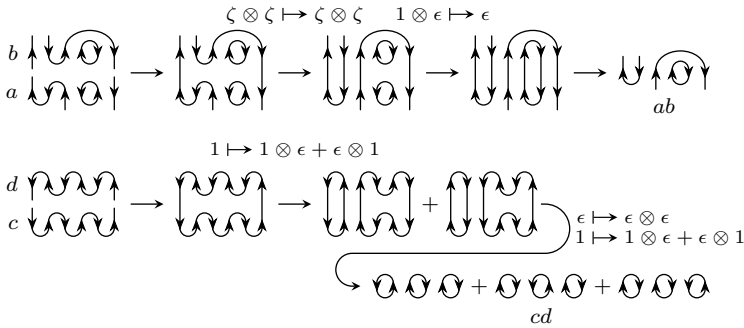


Fig. 3. Multiplications of arc diagrams in  $K_3^3$ .

opposite way (see the lower part of Fig. 3). The two diagrams should separately be simplified further if necessary.

- (iii) If the surgery produces two connected components out of two, then this only happens if both are open arcs (lines  $\zeta \otimes \zeta$ ) and one sets  $ab = 0$  unless both half-lines from one of the lines were oriented  $\blacktriangle$  and both half-lines from the other line were oriented  $\blacktriangledown$  and the orientation from the half-lines is carried over to the resulting lines (see the upper part of Fig. 3). Symbolically we write

$$\zeta \otimes \zeta \mapsto \begin{cases} 0 \\ \zeta \otimes \zeta. \end{cases}$$

Finally, if all cup–cap pairs have been removed through surgery in each summand, then we can identify the weights and obtain a sum of arc diagrams for some weights of the same type.

For example,  $\epsilon \mapsto \epsilon \otimes \epsilon$  indicates that if the surgery produces two closed arcs out of one clockwise closed arc ( $\epsilon$ ), then the resulting closed arcs are both oriented clockwise ( $\epsilon \otimes \epsilon$ ). See the lower part of Fig. 3 for an illustration.

**Remark 2.3** (*Relation to TQFT*). The explicit description of the multiplication in  $K_m^n$  (Definition 2.2) can be obtained by viewing  $K_m^n$  as a quotient of Khovanov’s original arc algebra  $H_{m+n}^{m+n}$  where all arc diagrams consist of (oriented) closed arcs. In  $H_k^k$ , the multiplication only uses the rules involving  $1, \epsilon$  and these rules are given by the multiplication and comultiplication in the Frobenius algebra  $\mathbb{k}[\epsilon]/(\epsilon^2)$  or, equivalently, by the algebraic structure in the corresponding 2-dimensional TQFT. This latter, topological viewpoint is useful for proving well-definedness and associativity of the multiplication as well as compatibility with the grading. In particular, the multiplication is independent of the order of the performed surgeries.

Note also that  $K_m^n$  is finite-dimensional, since there are finitely many arc diagrams for fixed  $m, n \geq 1$ . However, the dimension of  $K_m^n$  grows quite quickly in  $m$  and  $n$ . For

instance,  $\dim_{\mathbb{k}} K_l^1 = \dim_{\mathbb{k}} K_1^l = 4l + 1$  and  $\dim_{\mathbb{k}} K_l^2 = \dim_{\mathbb{k}} K_2^l = 8l^2 + 14l - 13$ , which may be computed using the Cartan matrix factorisation [10, (5.17)].

## 2.2. Generators and relations

In order to calculate the Hochschild cohomology of  $K_m^n$ , our first step is to translate the diagrammatics of  $K_m^n$  into generators and relations, i.e. we shall write  $K_m^n$  as the path algebra of a finite quiver  $Q_m^n$  modulo a two-sided ideal generated by three types of quadratic relations. This explicit algebraic description may be of independent interest.

Algorithmic descriptions of quivers with relations describing principal blocks of parabolic category  $\mathcal{O}$  for any parabolic subalgebra  $\mathfrak{p} \subset \mathfrak{g}$  in a semisimple complex Lie algebra were considered by Vybornov [45], but they are generally difficult to make explicit. An explicit description for  $K_m^n$  in terms of generators and relations was first given from the point of view of perverse sheaves by Braden [7] (see also [42, §5.6] for a comparison with the diagrammatic description), but the relations in [7] are not homogeneous with respect to the natural grading of  $K_m^n$ .

Our description is based on the diagrammatic description given by Brundan and Stropel [9] which can be directly related to describing parabolic category  $\mathcal{O}$  for parabolics associated to two-block partitions  $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{gl}_{m+n}(\mathbb{C})$  [11].

It follows from [5] and [10, §5] that  $K_m^n$  is a Koszul algebra generated in degree 1, so that we may take  $Q_m^n$  to be the quiver with vertices corresponding to arc diagrams of degree 0 and arrows corresponding to arc diagrams of degree 1. The vertices of  $Q_m^n$  are thus given by the degree 0 arc diagrams  $e_\lambda = \lambda\lambda\bar{\lambda}$  (see Definition 2.1) and there is a natural surjective algebra map

$$\rho: \mathbb{k}Q_m^n \twoheadrightarrow K_m^n \quad (2.4)$$

which sends the vertices  $e_\lambda$  to the degree 0 arc diagrams  $e_\lambda$  and sends arrows to the corresponding degree 1 arc diagrams.

We first describe  $Q_m^n$  as the double quiver of a certain bipartite graph  $\Gamma_m^n$ .

**Definition 2.5.** Let  $m, n \geq 1$  be fixed and let  $\Gamma_m^n$  be the (simple and undirected) graph defined as follows:

- **Vertices** The vertices  $V(\Gamma_m^n) = \{e_\lambda\}_{\lambda \in \Lambda_m^n}$  are given by the degree 0 arc diagrams.
- **Edges** There is an edge between two vertices  $e_\lambda$  and  $e_\mu$  if the weight of the one is obtained from the weight of the other by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in the same counterclockwise circle.

**Lemma 2.6.** The quiver  $Q_m^n$  for  $K_m^n$  is the double quiver of  $\Gamma_m^n$ , i.e. obtained by replacing each edge in  $\Gamma_m^n$  by two arrows in opposite directions.

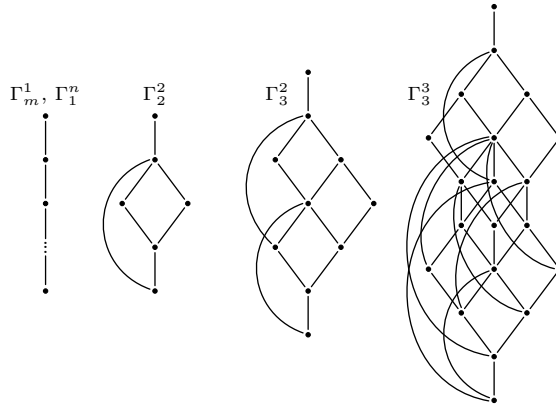


Fig. 4. Examples of  $\Gamma_m^n$  whose double quivers  $Q_m^n$  generate  $K_m^n$ .

**Proof.** Clearly,  $Q_m^n$  has the same vertices as  $\Gamma_m^n$ . If there is an edge between  $e_\lambda$  and  $e_\mu$  in  $\Gamma_m^n$  so that  $e_\mu$  is obtained from  $e_\lambda$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a counterclockwise circle of  $e_\lambda$ , then  $\lambda\mu$  (resp.  $\mu\widehat{\lambda}$ ) is a well-defined degree 1 oriented cup diagram (resp. cap diagram). As a result,  $\lambda\mu\widehat{\mu}$  (resp.  $\mu\widehat{\mu}\widehat{\lambda}$ ) is a degree 1 arc diagram from  $\lambda$  to  $\mu$  (resp. from  $\mu$  to  $\lambda$ ).

Conversely, if  $\lambda\mu\widehat{\mu}$  is a degree 1 arc diagram from  $\lambda$  to  $\mu$  then  $\lambda\mu$  is a degree 1 oriented cup diagram. This means that  $\lambda\mu$  is obtained from  $\lambda\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in a counterclockwise cup of  $\lambda\lambda$ . In other words,  $\mu$  can be obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in the counterclockwise cup. So there is an edge between  $e_\lambda$  and  $e_\mu$  in  $\Gamma_m^n$ .  $\square$

In order to draw  $\Gamma_m^n$ , we shall order the vertices by the *Bruhat order* (see [9, §2]), which is a partial order on  $\Lambda_m^n \simeq V(\Gamma_m^n)$ , such that  $\blacktriangle \cdots \blacktriangle \blacktriangledown \cdots \blacktriangledown$  is the highest weight and  $\blacktriangledown \cdots \blacktriangledown \blacktriangle \cdots \blacktriangle$  the lowest weight. Concretely, we may define the *height*  $|\lambda|$  of a weight  $\lambda$  by summing the number of  $\blacktriangle$ 's on the left of each  $\blacktriangledown$  in  $\lambda$ , for example

$$|\blacktriangle \blacktriangle \blacktriangle \blacktriangledown \blacktriangledown| = 3 + 3 = 6 \quad |\blacktriangle \blacktriangledown \blacktriangle \blacktriangledown \blacktriangle| = 1 + 2 = 3.$$

Permuting  $\blacktriangle$ 's to the left and  $\blacktriangledown$ 's to the right increases the height and the Bruhat order. Fig. 4 shows several graphs, where in each diagram the vertices aligned horizontally have the same height.

The edges of  $\Gamma_m^n$  only connect vertices corresponding to weights with height of different parity. This can be seen as follows. Assume that there is an edge between  $\lambda$  and  $\mu$  so that  $\mu$  is obtained from  $\lambda$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in some circle  $C$  of the arc diagram  $e_\lambda$ . Denote by  $k$  the number of circles inside  $C$  in  $e_\lambda$ . Then we have  $k$   $\blacktriangle$ 's and  $k$   $\blacktriangledown$ 's lying between the  $\blacktriangledown \cdots \blacktriangle$  pair being exchanged, whence  $|\mu| = |\lambda| + 2k + 1$ . This observation immediately yields the following result.

**Proposition 2.7.** *The graph  $\Gamma_m^n$  is bipartite when decomposing  $V(\Gamma_m^n)$  into vertices of even and odd heights. In particular, paths in  $\Gamma_m^n$  of even length can only connect weights whose heights are of the same parity and paths of odd length can only connect weights whose heights are of different parity.*

Since  $K_m^n$  is Koszul and hence quadratic, we also need a good understanding of the length 2 paths in  $\Gamma_m^n$  and we give a complete description in Proposition 2.12. The description is based on the following two lemmas.

**Lemma 2.8.** *Let  $m, n \geq 1$  and let  $\lambda, \mu \in \Lambda_m^n$  be two distinct weights such that  $|\lambda| \leq |\mu|$ .*

- (i) *There is no length 2 (undirected) path in  $\Gamma_m^n$  between  $\lambda$  and  $\mu$  passing through a vertex  $\nu$  with  $|\nu|$  equal to  $|\lambda|$  or  $|\mu|$ .*
- (ii) *The set of length 2 paths in  $\Gamma_m^n$  passing through vertices  $\nu$  satisfying  $|\nu| < |\lambda|$  contains at most one element.*
- (iii) *The set of length 2 paths in  $\Gamma_m^n$  passing through vertices  $\nu$  satisfying  $|\nu| > |\mu|$  contains at most one element.*

**Proof.** (i) follows directly from Proposition 2.7, since  $\nu$  is connected to  $\lambda$  and  $\mu$  by an edge whence  $|\nu|$  is different from  $|\lambda|$  and  $|\mu| \pmod 2$ .

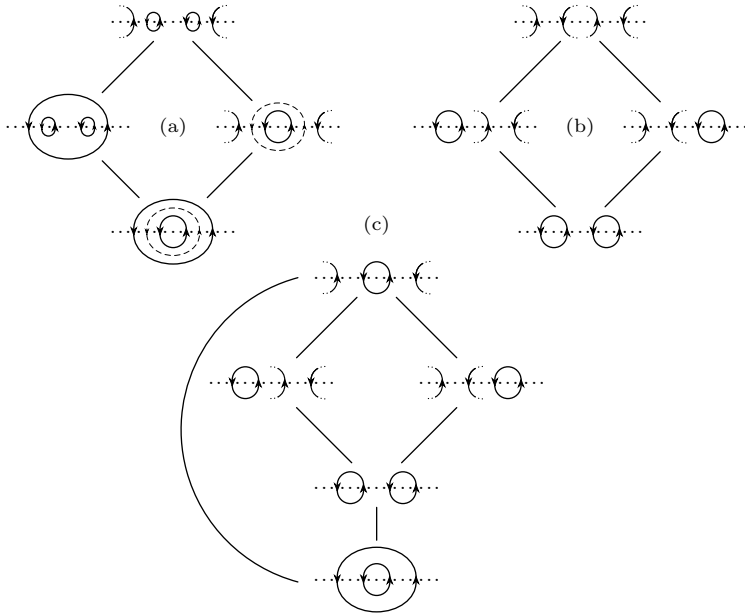
To prove (ii), let  $\nu, \nu'$  be two distinct weights both connected to  $\lambda$  and to  $\mu$  by an edge with  $|\nu|, |\nu'| < |\lambda| \leq |\mu|$ . Since  $\lambda$  and  $\mu$  are distinct, they are obtained from  $\nu$  (resp.  $\nu'$ ) by exchanging two  $\blacktriangledown \cdots \blacktriangle$  pairs lying in distinct circles of  $e_\nu$  (resp.  $e_{\nu'}$ ). These circles are either nested or not, so that  $e_\nu, e_{\nu'}$  are of the following forms

$$\cdots \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \quad \text{or} \quad \cdots \text{---} \bigcirc \text{---} \cdots \text{---} \bigcirc \text{---} \cdots \quad (2.9)$$

and  $\lambda, \mu, \nu, \nu'$  agree on all other positions represented by the dots. Note that there is an edge between  $\nu$  and  $\nu'$  since the diagram on the right of (2.9) may be obtained from the one on the left by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair in the inner circle, which implies that  $|\nu| \not\equiv |\nu'| \pmod 2$ . Since  $\nu$  and  $\nu'$  are connected to  $\lambda$  by an edge we must also have  $|\nu| \equiv |\nu'| \pmod 2$  by Proposition 2.7, giving a contradiction.

(iii) follows similarly, for if  $\nu, \nu'$  are two distinct weights both connected to  $\lambda$  and  $\mu$  by an edge with  $|\lambda| \leq |\mu| < |\nu|, |\nu'|$ , then  $\nu, \nu'$  are obtained from  $\lambda$  by exchanging two  $\blacktriangledown \cdots \blacktriangle$  pairs in distinct circles of  $e_\nu$ , and  $\nu, \nu'$  can also be obtained from  $\mu$  in the same way. Thus  $\lambda, \mu$  are both of the form (2.9) agreeing on all positions represented by the dots. We get a contradiction since  $|\lambda| \equiv |\mu| \pmod 2$  and yet, the two arc diagrams in (2.9) share an edge, whence their heights are different modulo 2.  $\square$

**Lemma 2.10.** *Let  $m, n \geq 2$ . There are no other paths of length 2 in  $\Gamma_m^n$  (except the illustrated ones) between any two distinct vertices of the square in Fig. 5 (a) or in Fig. 5 (c). Moreover, any square in  $\Gamma_m^n$  is of one of the forms appearing in Fig. 5.*



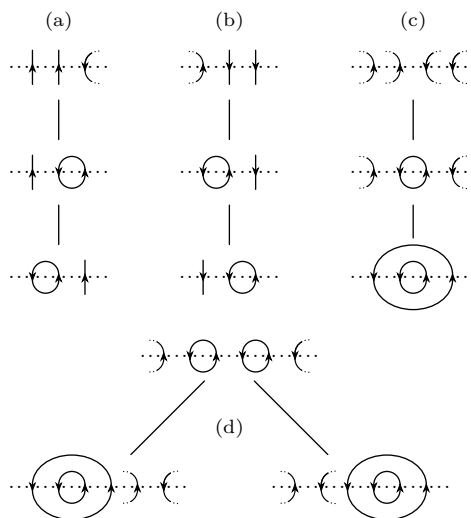
**Fig. 5.** Squares in  $\Gamma_m^n$  giving rise to commutative squares in  $K_m^n$ . In (a) the dashed circle indicates the innermost circle enclosing the smaller solid circle, which must exist and be different from the bigger solid circle for the square to exist (cf. Fig. 6 (c) for when such a circle does not exist).

**Remark 2.11.** Note that the top square in Fig. 5 (c) is a special case of the square in Fig. 5 (b). Namely, if the middle  $\cdots$  in the arc diagrams in Fig. 5 (b) is empty or consists of nested counterclockwise circles, then Fig. 5 (b) can be extended to Fig. 5 (c). So by a similar argument, the statement in Lemma 2.10 holds for the square (b) if it cannot be extended to Fig. 5 (c).

Note that in Fig. 5 (c) there are three squares, the obvious square at the top, and two further squares obtained by deleting the left or right vertex (see Remark 2.18 for illustrations).

**Proof of Lemma 2.10.** Let us first consider square (a). Denote the bottom, top, left and right vertices by  $\lambda, \mu, \nu$  and  $\nu'$ , respectively, and note that by Lemma 2.8 (i)  $|\lambda| < |\nu| < |\nu'| < |\mu|$ . If there is a path of length 2 between  $\nu$  and  $\nu'$  passing through a vertex  $\xi$  which is different from  $\lambda$  and  $\mu$ , then by Lemma 2.8 (ii) and (iii) we have  $|\nu| < |\xi| < |\nu'|$ . Note that  $\nu$  and  $\nu'$  are distinct in four positions and agree on all other positions. Since  $\xi$  is connected to  $\nu$  by an edge, we have that  $\xi$  and  $\nu$  are distinct in two positions and thus  $\xi$  must be obtained from  $\nu$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair which lies in two of the four positions and also in a circle of  $e_\nu$  (i.e. the outer one). This implies  $\xi = \mu$ , giving a contradiction.

If there is a path of length 2 between  $\lambda$  and  $\mu$  passing through a vertex  $\xi$  which is different from  $\nu$  and  $\nu'$ , then we have the following cases.



**Fig. 6.** Paths in  $\Gamma_m^n$  of length 2 without parallel length 2 paths. In (c) there is no circle between the two nested circles enclosing the smaller one (cf. Fig. 5 (a) for the case when such a circle does exist).

*Case  $|\xi| > |\mu|$ .* Since  $\lambda$  and  $\mu$  are distinct in four positions and agree on all other positions represented by the dots, we have that  $\xi$  is obtained from  $\lambda$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair which lies in two of the four positions and also in a circle of  $e_\lambda$  (resp.  $e_\mu$ ). Then  $\xi$  has to equal  $\nu$  or  $\nu'$  which contradicts the assumption.

*Case  $|\xi| < |\lambda|$ .* We have that  $\lambda$  is obtained from  $\xi$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair which lies in two of the four positions. But this cannot happen since in the four positions  $\lambda$  is of the form  $\cdots \blacktriangledown \cdots \blacktriangledown \cdots \blacktriangle \cdots \blacktriangle \cdots$  and thus is already as low as possible.

*Case  $|\lambda| < |\xi| < |\mu|$ .* By a similar reason we have that  $\xi$  is obtained from  $\lambda$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair which lies in two of the four positions and also lies in a circle of  $e_\lambda$ . So  $\xi$  has to equal  $\nu$  or  $\nu'$ .

Now let us consider square (c). Denote the vertices from bottom to top by  $\kappa, \lambda, \nu, \nu'$  and  $\mu$ . Note that  $\lambda$  (resp.  $\nu$ ) and  $\mu$  (resp.  $\nu'$ ) are distinct in four positions and agree on all other positions. Then by similar arguments as before, we have that there are no other paths of length 2 between  $\lambda$  (resp.  $\nu$ ) and  $\mu$  (resp.  $\nu'$ ). It thus suffices to prove the assertion for  $\kappa$  and  $\nu$  (resp.  $\nu'$ ). If there is a path of length 2 between  $\kappa$  and  $\nu$  passing through a vertex  $\xi$  which is different from  $\lambda$  and  $\mu$ , then by Lemma 2.8 (ii) we have  $|\xi| < |\nu|$ .

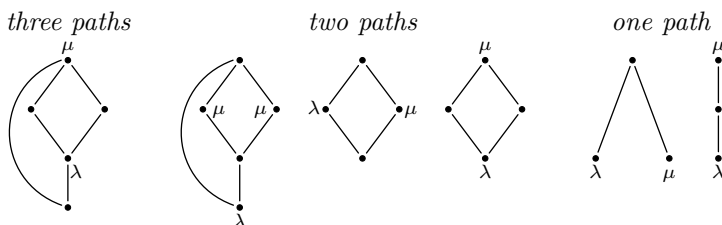
*Case  $|\kappa| < |\xi| < |\nu|$ .* Note that  $\kappa$  and  $\nu$  are distinct in two positions and agree on all other positions. Then  $\xi$  is obtained from  $\kappa$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle  $C$  of  $e_\kappa$  such that either the  $\blacktriangledown$  or  $\blacktriangle$  lies in one of the two positions (otherwise,  $\xi$  and  $\nu$  are distinct in four positions whence there are no edges between them). Thus,  $C$  has to be one of the nested circles in  $e_\kappa$  so that  $\xi = \lambda$ .

Case  $|\xi| < |\kappa|$ . We have that  $\kappa, \nu$  are obtained from  $\xi$  by exchanging two  $\blacktriangledown \cdots \blacktriangle$  pairs in distinct circles  $C, C'$  in  $e_\xi$ , so  $\kappa$  and  $\nu$  are distinct in four positions which contradicts the fact that  $\kappa$  and  $\nu$  are only distinct in two positions.

To obtain a square in Fig. 5, we may start with a vertex  $e_\lambda$  and choose two circles  $C_1$  and  $C_2$  in  $e_\lambda$  which correspond to two edges starting from  $e_\lambda$ . If  $C_1$  and  $C_2$  are nested such that there is a circle between them enclosing the smaller one then the two edges may be completed to the square (a), if they are nested such that there is no circle between them then we obtain the square (c), and if they are not nested then we obtain the square (b). Note that any square may be obtained in this way so that it is one of the forms in Fig. 5.  $\square$

We have the following complete description of the length 2 paths in  $\Gamma_m^n$ .

**Proposition 2.12.** *Let  $\lambda, \mu \in \Lambda_m^n$  be two distinct weights and assume that  $|\lambda| \leq |\mu|$ . Then there are at most three paths of length 2 between  $\lambda$  and  $\mu$ . Concretely, if there exists a length 2 path between  $\lambda$  and  $\mu$ , then we have the following cases*

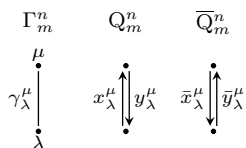


where

- (i) there are exactly three paths if  $\lambda$  and  $\mu$  are the two indicated vertices in Fig. 5 (c)
- (ii) there are exactly two paths if  $\lambda$  and  $\mu$  are the two indicated vertices of Fig. 5 (c) or the ones of Fig. 5 (a) or (b) (excluding the case appearing in (i))
- (iii) there is exactly one path if the vertices and edges are as in Fig. 6.

**Proof.** (i) and (ii) follow from Lemma 2.10 and (iii) follows by noting that the length 2 paths in Fig. 6 are precisely all the possible paths of length 2 which do not appear in any square. For instance, to see this for Fig. 6 (d), assume that there is a length 2 path between  $\lambda$  and  $\mu$  passing through a vertex  $\nu$  such that  $|\nu| > |\mu| \geq |\lambda|$ . Then  $\nu$  is obtained from  $\lambda$  (resp.  $\mu$ ) by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle of  $e_\lambda$  (resp.  $e_\mu$ ). Note that  $\lambda, \mu$  can differ in at most four positions. Since  $\lambda \neq \mu$  and  $|\lambda| \leq |\mu|$  we have the following three cases.

In the first case,  $\lambda$  and  $\mu$  are distinct in two positions and agree on all other positions. We claim this can only happen when  $\lambda, \mu, \nu$  are vertices in Fig. 5 (c) with  $\lambda$  the bottom vertex,  $\nu$  the top vertex and  $\mu$  either the left or the right vertex. Firstly, note that the



**Fig. 7.** An edge in the graph  $\Gamma_m^n$  between two weights  $\lambda, \mu \in \Lambda_m^n$  where  $\mu$  is obtained from  $\lambda$  so that  $|\lambda| < |\mu|$  and the corresponding arrows in the double quiver  $Q_m^n$  for  $K_m^n$  and in the opposite quiver  $\overline{Q}_m^n$  for the Koszul dual  $\overline{K}_m^n$ .

two positions in which  $\lambda, \mu$  differ cannot lie in a single circle of  $e_\lambda$ , as otherwise  $\mu$  would be obtained by changing the  $\blacktriangledown \cdots \blacktriangle$  pair lying in this circle, contradicting the fact that  $\lambda$  and  $\mu$  have the same height modulo 2. Since the two distinct positions should agree after exchanging a single  $\blacktriangledown \cdots \blacktriangle$  pair in both  $\lambda$  and in  $\mu$ , one finds that  $\lambda, \mu, \nu$  must be the claimed vertices in Fig. 5 (c).

In the second case,  $\lambda$  and  $\mu$  are distinct in four positions and  $\nu$  is of the form  $\cdots \blacktriangle \cdots \blacktriangledown \cdots \blacktriangle \cdots \blacktriangledown \cdots$  in these four positions. Then  $\nu$  has to be obtained from  $\lambda$  or  $\mu$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in the first and second position or lying in the third and fourth position in  $e_\lambda$  or  $e_\mu$ , whence  $\lambda, \mu, \nu$  may be completed into a square as in Fig. 5 (b).

In the third case,  $\lambda$  and  $\mu$  are distinct in four positions and  $\nu$  is of the form  $\cdots \blacktriangle \cdots \blacktriangle \cdots \blacktriangledown \cdots \blacktriangledown \cdots$ . If  $\nu$  is obtained from  $\lambda$  or  $\mu$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  lying in the first and fourth position or lying in the second and third position, then  $\lambda, \mu, \nu$  can be completed into a square as in Fig. 5 (a). If  $\nu$  is obtained from  $\lambda$  or  $\mu$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  lying in the first and third position or lying in the second and fourth position, then we obtain Fig. 6 (d) which by Lemma 2.10 cannot appear in any square.  $\square$

**Definition 2.13.** Let  $Q_m^n$  be the quiver obtained from  $\Gamma_m^n$  by replacing each edge by two arrows in opposite directions. Edges only connect vertices of different heights. For an edge  $\gamma_\lambda^\mu$  in  $\Gamma_m^n$  connecting two vertices  $\lambda, \mu \in \Lambda_m^n$  with  $|\lambda| < |\mu|$ , we denote the ascending arrow in  $Q_m^n$  (from  $\lambda$  to  $\mu$ ) by  $x_\lambda^\mu$  and the descending arrow (from  $\mu$  to  $\lambda$ ) by  $y_\lambda^\mu$  (see Fig. 7 and Fig. 10 for an illustration of  $Q_2^2$ ).

**Remark 2.14.** Note that the arrow  $x_\lambda^\mu$  (resp.  $y_\lambda^\mu$ ) corresponds to the degree 1 arc diagram  $\lambda\mu\widehat{\mu}$  (resp.  $\mu\mu\widehat{\lambda}$ ) in  $Q_m^n$ . The total number of ascending arrows starting from  $\lambda$  equals the number of circles in  $e_\lambda$ , i.e. the defect  $\text{def}(\lambda)$  (see Definition 2.1).

The following definition will be used to describe the quadratic relations of  $K_m^n$  and  $\overline{K}_m^n$ .

**Definition 2.15.** Fix a weight  $\lambda \in \Lambda_m^n$ . Let  $\mu$  be a weight obtained from  $\lambda$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle  $C$  of  $e_\lambda$ . Let  $\kappa$  be a weight such that  $\lambda$  is obtained from  $\kappa$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle  $D$  of  $e_\kappa$ . Then define an integer  $c_\kappa^\mu(\lambda)$  as follows

$$c_{\kappa}^{\mu}(\lambda) = \begin{cases} 1 & \text{if the circle } C \text{ does not appear in } e_{\kappa} \\ 0 & \text{if } C \text{ appears in } e_{\kappa} \text{ but does not enclose the circle } D \\ (-1)^{j-1}2 & \text{if } C \text{ in } e_{\kappa} \text{ is the } j\text{th circle from inner to outer enclosing } D. \end{cases}$$

Recall from (2.4) the surjective algebra map  $\rho: \mathbb{k}Q_m^n \rightarrow K_m^n$ .

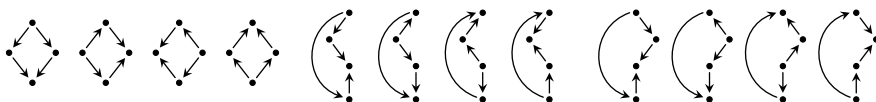
**Proposition 2.16.** *The map  $\rho$  sends the following quadratic relations to 0.*

- **Monomial relations.** *The paths between the top and bottom vertices in Fig. 6 (a) and (b) are zero respectively.*
- **Commutativity relations across all squares.** *All parallel paths of length 2 excluding 2-cycles in  $Q_m^n$  are equal.*
- **Relations at vertices.** *Fix  $\lambda \in \Lambda_m^n$ . Let  $x_{\lambda}^{\mu_i}$  be the ascending arrows in  $Q_m^n$  starting at  $e_{\lambda}$  for  $1 \leq i \leq \text{def}(\lambda)$ . Let  $x_{\kappa}^{\lambda}$  be any ascending arrow ending at  $e_{\lambda}$ . Then we have*

$$y_{\kappa}^{\lambda} x_{\kappa}^{\lambda} = \sum_{i=1}^{\text{def}(\lambda)} c_{\kappa}^{\mu_i}(\lambda) x_{\lambda}^{\mu_i} y_{\lambda}^{\mu_i} \quad (2.17)$$

where  $c_{\kappa}^{\mu_i}(\lambda)$  is given in Definition 2.15. In particular, if  $\lambda$  is the highest weight then  $y_{\kappa}^{\lambda} x_{\kappa}^{\lambda} = 0$  and if  $\lambda$  is the lowest weight then (2.17) is empty.

**Remark 2.18.** Note that by Proposition 2.12 the parallel length 2 paths appearing in the commutativity relations across squares are of the following forms



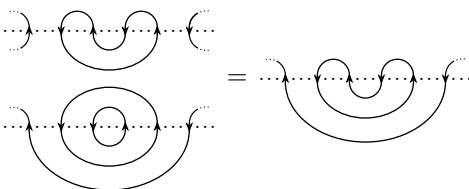
where the first four diagrams appear in Fig. 5 (a), (b) and (c) and the last eight in (c) only.

**Proof of Proposition 2.16.** The monomial relations stem from the vanishing of the following multiplications (using the rule  $\zeta \otimes \zeta \mapsto 0$  in Definition 2.2):

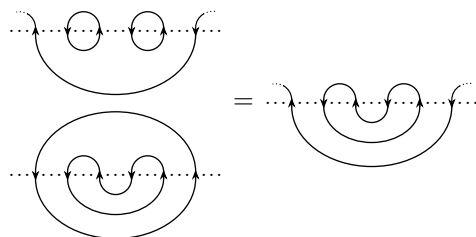
$$\begin{array}{cccc} \begin{array}{c} \cdots \uparrow \cdots \uparrow \cdots \\ \cdots \downarrow \cdots \downarrow \cdots \end{array} & = 0 & \begin{array}{c} \cdots \downarrow \cdots \downarrow \cdots \\ \cdots \uparrow \cdots \uparrow \cdots \end{array} & = 0 & \begin{array}{c} \cdots \uparrow \cdots \uparrow \cdots \\ \cdots \downarrow \cdots \downarrow \cdots \end{array} & = 0 & \begin{array}{c} \cdots \downarrow \cdots \downarrow \cdots \\ \cdots \uparrow \cdots \uparrow \cdots \end{array} & = 0. \end{array}$$

The commutativity relations across squares follow from the following *claim*: Let  $p$  be a length 2 path lying in a square in Fig. 5 from  $e_{\nu}$  to  $e_{\nu'}$  with  $\nu \neq \nu'$ . Then  $\rho(p) = \underline{\nu}\mu\nu'$ , where  $\mu$  is the top vertex in the square.

Let us prove the claim for the square (a) in Fig. 5. Denoting the bottom, top, left and right vertices in Fig. 5 (a) by  $\lambda$ ,  $\mu$ ,  $\nu$  and  $\nu'$  respectively, we have

$$\rho(x_\lambda^{\nu'} x_\nu^\mu) = \rho(x_\lambda^{\nu'}) \rho(x_\nu^\mu) =$$


using the rule  $1 \otimes 1 \mapsto 1$  twice. Similarly we have

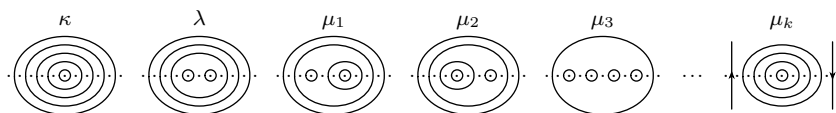
$$\rho(x_\lambda^{\nu'} x_{\nu'}^\mu) = \rho(x_\lambda^{\nu'}) \rho(x_{\nu'}^\mu) =$$


using  $1 \otimes * \mapsto *$  once and  $1 \otimes 1 \mapsto 1$  twice. This verifies that  $\rho(x_\lambda^{\nu'} x_\nu^\mu) = \lambda \mu \widehat{\mu} = \rho(x_\lambda^{\nu'} x_{\nu'}^\mu)$ . Using the involution in Remark 2.20 we have  $\rho(y_\nu^\mu y_\lambda^{\nu'}) = \mu \widehat{\mu} \lambda = \rho(y_\nu^\mu y_{\nu'}^{\nu'})$ . By a similar computation we have

$$\rho(x_\nu^\mu y_{\nu'}^{\nu'}) = \nu \widehat{\mu} \nu' = \rho(y_\lambda^{\nu'} x_\lambda^{\nu'}), \quad \rho(x_{\nu'}^\mu y_\nu^\mu) = \nu' \widehat{\mu} \nu = \rho(y_\lambda^{\nu'} x_\lambda^{\nu'}).$$

This proves the claim for Fig. 5 (a). The other cases may be proved in a similar way.

It remains to verify the relations at vertices. Note that  $\lambda$  is obtained from  $\kappa$  by exchanging a  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle  $D$  of  $e_\kappa$ . We have the following two cases. In the first case, there are circles in  $e_\lambda$  which appear in  $e_\kappa$  enclosing  $D$ . Then there are two circles (denoted by  $C_1$  and  $C_2$  from left to right) in  $e_\lambda$  which do not appear in  $e_\kappa$ . Note that the innermost circle in  $e_\kappa$  enclosing  $D$  does not appear in  $e_\lambda$ . Denote by  $C_i$  the  $(i-1)$ th circle in  $e_\kappa$  enclosing  $D$  for  $3 \leq i \leq k$  and denote by  $\mu_i$  the weights obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in  $C_i$ . Then  $e_\kappa, e_\lambda$  and  $e_{\mu_i}$  have the following forms



Using the rules  $1 \otimes 1 \mapsto 1$  and  $1 \mapsto 1 \otimes \epsilon + \epsilon \otimes 1$ , we have

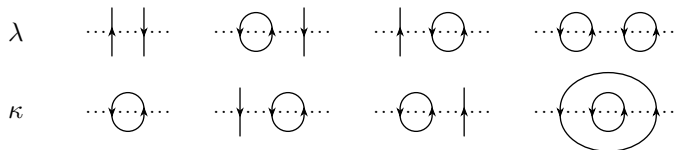
$$\begin{aligned} \rho(y_\kappa^\lambda) \rho(x_\kappa^\lambda) &= \text{diagram with two concentric circles, each with a dot on its boundary} + \text{diagram with two concentric circles, each with a dot on its boundary} = \lambda \mu_1 \widehat{\lambda} + \lambda \mu_2 \widehat{\lambda} \\ \rho(x_\lambda^{\mu_1}) \rho(y_\lambda^{\mu_1}) &= \text{diagram with two concentric circles, each with a dot on its boundary} + \text{diagram with two concentric circles, each with a dot on its boundary} = \lambda \mu_1 \widehat{\lambda} + \lambda \mu_3 \widehat{\lambda}. \end{aligned}$$

Similarly for  $2 \leq i < k$  we have  $\rho(x_\lambda^{\mu_i})\rho(y_\lambda^{\mu_i}) = \lambda_{\mu_i}\widehat{\lambda} + \lambda_{\mu_{i+1}}\widehat{\lambda}$  and  $\rho(x_\lambda^{\mu_k})\rho(y_\lambda^{\mu_k}) = \lambda_{\mu_k}\widehat{\lambda}$ . Combining the above equalities we obtain

$$\rho(y_\kappa^\lambda)\rho(x_\kappa^\lambda) = \rho(x_\lambda^{\mu_1}y_\lambda^{\mu_1}) + \rho(x_\lambda^{\mu_2}y_\lambda^{\mu_2}) - 2\rho(x_\lambda^{\mu_3}y_\lambda^{\mu_3}) + \cdots + (-1)^k 2\rho(x_\lambda^{\mu_k}y_\lambda^{\mu_k}).$$

This yields (2.17) since  $c_\kappa^{\mu_1}(\lambda) = 1 = c_\kappa^{\mu_2}(\lambda)$  and  $c_\kappa^{\mu_i}(\lambda) = (-1)^i 2$  for  $3 \leq i \leq k$ .

In the second case, there is no circle in  $e_\lambda$  which appears in  $e_\kappa$  enclosing  $D$ . Then  $\kappa, \lambda$  have the following subcases



Note that in the first subcase we have that  $c_\kappa^{\mu'}(\lambda) = 0$  where  $\mu'$  is any weight obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair in a circle of  $e_\lambda$  and that

$$\rho(y_\kappa^\lambda x_\kappa^\lambda) = \begin{array}{c} \cdots \downarrow \cdots \\ \cdots \uparrow \cdots \end{array} = 0.$$

For the second and third subcases we have  $\rho(y_\kappa^\lambda x_\kappa^\lambda) = \lambda_{\mu}\widehat{\lambda} = \rho(x_\lambda^{\mu}y_\lambda^{\mu})$  where  $\mu$  is obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in the (illustrated) circle of  $e_\lambda$ . Note that  $c_\kappa^{\mu}(\lambda) = 1$  and  $c_\kappa^{\mu'}(\lambda) = 0$  for any  $\mu' \neq \mu$ . For the fourth subcase we have

$$\rho(y_\kappa^\lambda x_\kappa^\lambda) = \cdots \circlearrowleft \cdots \circlearrowleft \cdots + \cdots \circlearrowright \cdots \circlearrowright \cdots = \rho(x_\lambda^{\mu_1}y_\lambda^{\mu_1}) + \rho(x_\lambda^{\mu_2}y_\lambda^{\mu_2})$$

where  $\mu_1, \mu_2$  are the weights obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pairs lying in the (illustrated) circles of  $e_\lambda$ . Note that  $c_\kappa^{\mu_1}(\lambda) = 1 = c_\kappa^{\mu_2}(\lambda)$ .  $\square$

Denote by  $I_m^n$  the two-sided ideal of  $\mathbb{K}Q_m^n$  generated by the above quadratic relations. Proposition 2.16 yields an algebra map (still denoted by  $\rho$ )

$$\rho: \mathbb{K}Q_m^n/I_m^n \longrightarrow K_m^n. \quad (2.19)$$

**Remark 2.20.** Note that  $\mathbb{K}Q_m^n/I_m^n$  admits a natural involution map

$$(-)^*: \mathbb{K}Q_m^n/I_m^n \longrightarrow \mathbb{K}Q_m^n/I_m^n \quad (2.21)$$

determined by  $(e_\lambda)^* = e_\lambda$  and  $(x)^* = y$  and  $(y)^* = x$  for a pair of opposite arrows  $x, y$  (cf. Fig. 7). We also recall the involution map of  $K_m^n$  defined in [9, (4.14)]

$$(-)^*: K_m^n \longrightarrow K_m^n, \quad \alpha\lambda\beta \longmapsto \beta\lambda\alpha.$$

Clearly  $\rho$  intertwines these two maps, i.e. we have  $\rho(a^*) = \rho(a)^*$  for any  $a \in \mathbb{K}Q_m^n/I_m^n$ .

In the rest of this section we will show that the map  $\rho$  in (2.19) is a bijection. As a result, the algebra  $K_m^n$  can be written as the path algebra  $\mathbb{k}Q_m^n$  of the quiver  $Q_m^n$  with relations  $I_m^n$ . Note that  $\rho$  is bijective in degrees 0 and 1, and the algebra  $K_m^n$  is Koszul and in particular quadratic. Therefore, it only remains to show that  $\rho$  is bijective in degree 2.

**Lemma 2.22.**  $\dim_{\mathbb{K}} e_{\lambda}(\mathbb{K}Q_m^n/I_m^n)_2 e_{\mu} \leq \begin{cases} 1 & \text{if } \lambda \neq \mu \\ \text{def}(\lambda) & \text{if } \lambda = \mu. \end{cases}$

**Proof.** Let  $\lambda \neq \mu$ . If  $\dim_{\mathbb{k}} e_{\lambda}(\mathbb{k}Q_m^n/I_m^n)_2 e_{\mu} > 1$  then there exist two distinct paths  $p, q$  of length 2 from  $e_{\lambda}$  to  $e_{\mu}$  in  $Q_m^n$  such that  $\rho(p) \neq \rho(q)$ , which contradicts the commutativity relations across squares. Thus,  $\dim_{\mathbb{k}} e_{\lambda}(\mathbb{k}Q_m^n/I_m^n)_2 e_{\mu} \leq 1$ .

If  $\lambda = \mu$ , from the relations at each vertex  $\lambda$  we note that  $e_\lambda(\mathbb{k}Q_m^n/I_m^n)_2e_\lambda$  is spanned by the set of 2-cycles at  $\lambda$  of the form  $xy$ . This set corresponds bijectively to the counterclockwise circles in  $e_\lambda$ , whence  $\dim e_\lambda(\mathbb{k}Q_m^n/I_m^n)_2e_\lambda \leq \text{def}(\lambda)$ .  $\square$

**Lemma 2.23.** *If  $\lambda \neq \mu$  then  $\dim_{\mathbb{K}} e_{\lambda}(\mathbb{K}Q_m^n/I_m^n)_2 e_{\mu} = \dim_{\mathbb{K}} e_{\lambda}(K_m^n)_2 e_{\mu}$ .*

**Proof.** Since  $\rho$  is surjective by Lemma 2.22 we have

$$1 \geq \dim_{\mathbb{K}} e_{\lambda}(\mathbb{K}Q_m^n/I_m^n)_{2e_{\mu}} \geq \dim_{\mathbb{K}} e_{\lambda}(K_m^n)_{2e_{\mu}}.$$

We claim that the second inequality is an equality. Indeed, if  $\dim_{e_\lambda}(\mathbb{k}Q_m^n/I_m^n)_{2e_\mu} = 0$  the claim holds since both sides have to be zero. If  $\dim_{e_\lambda}(\mathbb{k}Q_m^n/I_m^n)_{2e_\mu} = 1$  then there exists a length 2 path  $p$  in  $Q_m^n$  from  $e_\lambda$  to  $e_\mu$  such that  $[p] \neq 0$  in  $e_\lambda(\mathbb{k}Q_m^n/I_m^n)_{2e_\mu}$ . This can only happen in the following two cases.

In the first case, the path  $p$  appears in a square. Then it follows from the claim in the proof of Proposition 2.16 that  $\rho(p) = \lambda\xi\widehat{\mu}$ . So  $e_\lambda(K_m^n)_2e_\mu \neq 0$ .

In the second case, the path  $p$  does not lie in any square. Then  $p$  is the ascending path or the descending path of length 2 in Fig. 6 (c) or the length 2 path from left to right or from right to left in Fig. 6 (d). If  $p$  is the ascending path in Fig. 6 (c) then

$$\rho(p) = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---}$$

which is nonzero in  $K_m^n$ . Using the involution in Remark 2.20 we obtain an analogous statement for the descending path. If  $p$  is the path from left to right in Fig. 6 (d) then

which is nonzero in  $K_m^n$ . Again, using the involution the same holds for the path from right to left.  $\square$

**Proposition 2.24.** *The algebra map  $\rho: \mathbb{k}Q_m^n/I_m^n \rightarrow K_m^n$  is an isomorphism.*

**Proof.** As mentioned above, we need to show that  $\rho$  is bijective in degree 2. For this, it suffices to show that for any weights  $\lambda, \mu$  we have

$$\dim_{\mathbb{k}} e_{\lambda}(\mathbb{k}Q_m^n/I_m^n)_2 e_{\mu} = \dim_{\mathbb{k}} e_{\lambda}(K_m^n)_2 e_{\mu}. \quad (2.25)$$

The equality (2.25) holds for  $\lambda = \mu$  since

$$\text{def}(\lambda) \geq \dim_{\mathbb{k}} e_{\lambda}(\mathbb{k}Q_m^n/I_m^n)_2 e_{\lambda} \geq \dim_{\mathbb{k}} e_{\lambda}(K_m^n)_2 e_{\lambda} = \text{def}(\lambda)$$

where the first inequality follows from Lemma 2.22 and the second one follows from  $\rho$  being surjective, and the third equality follows from Remark 2.14. If  $\lambda \neq \mu$  then the equality (2.25) follows from Lemma 2.23.  $\square$

### 3. The Koszul-dual picture

Let  $A = \mathbb{k}Q/I$ , where  $Q$  is a finite quiver with vertex set  $Q_0$  and  $I \subset \mathbb{k}Q_{\geq 2}$  is a two-sided ideal of relations, where  $Q_{\geq 2}$  denotes the set of paths of length  $\geq 2$ . We say that  $A$  is *Koszul* in the sense of [5, 36] if it is graded by the path length such that the  $A$ -module  $\mathbb{k}Q_0$  admits a graded projective resolution

$$\cdots \rightarrow P_{-i} \rightarrow \cdots \rightarrow P_{-1} \rightarrow P_0 \twoheadrightarrow \mathbb{k}Q_0$$

with  $P_{-i}$  being generated in degree  $i$  as  $A$ -module. Then the ideal  $I$  is necessarily generated by quadratic relations  $I_2 \subset \mathbb{k}Q_2$  and the Koszul dual  $A^! = \text{Ext}_A^{\bullet}(\mathbb{k}Q_0, \mathbb{k}Q_0)$  (disregarding the grading of  $A$ ) is also a Koszul algebra (with elements in  $\text{Ext}_A^d(\mathbb{k}Q_0, \mathbb{k}Q_0)$  being of degree  $d$ ) such that  $(A^!)^! \simeq A$ .

The Koszul dual  $A^!$  is isomorphic to the linear dual algebra  $\mathbb{k}\overline{Q}/(I_2^{\perp})$  which we briefly recall from [5, §2]. Let  $\overline{Q}$  denote the opposite quiver of  $Q$ , i.e.  $\overline{Q}$  has the same vertices as  $Q$  and the opposite arrow  $\bar{a}$  for each arrow  $a$  in  $Q$ , so that  $\overline{\overline{Q}}$  can be naturally identified with  $Q$ . We have a natural nondegenerate  $\mathbb{k}$ -bilinear pairing  $\langle -, - \rangle: \mathbb{k}Q \times \mathbb{k}\overline{Q} \rightarrow \mathbb{k}$  which is uniquely determined by

$$\langle a_1 a_2 \cdots a_n, \bar{b}_n \cdots \bar{b}_2 \bar{b}_1 \rangle = \delta_{a_1, b_1} \cdots \delta_{a_n, b_n} \quad \text{for paths } a_1 \cdots a_n, b_1 \cdots b_n \text{ in } Q.$$

Then the quadratic relations for  $A^!$  are given by  $I_2^{\perp} = \{p \in \mathbb{k}\overline{Q}_2 \mid \langle I_2, p \rangle = 0\}$ .

### 3.1. Generators and relations of the Koszul dual

We now give a quiver description for the Koszul dual of the extended Khovanov arc algebras  $K_m^n$ .

**Notation 3.1.** In order to simplify notation, let us write

$$\overline{K}_m^n := (K_m^n)^\dagger \simeq \mathbb{k}\overline{Q}_m^n / \overline{I}_m^n$$

for the Koszul dual of  $K_m^n \simeq \mathbb{k}Q_m^n / I_m^n$ , where  $\overline{Q}_m^n$  is the opposite quiver of  $Q_m^n$  and  $\overline{I}_m^n = (I_m^n)^\perp$ . We write  $\bar{x}_\lambda^\mu, \bar{y}_\lambda^\mu$  for the Koszul duals of the generators  $x_\lambda^\mu, y_\lambda^\mu$  of  $K_m^n$ . Therefore,  $\bar{x}_\lambda^\mu$  is a descending arrow and  $\bar{y}_\lambda^\mu$  an ascending arrow (see Fig. 7).

**Remark 3.2.** Similar to Remark 2.20,  $\overline{K}_m^n$  also admits an involution

$$(-)^*: \overline{K}_m^n \longrightarrow \overline{K}_m^n$$

satisfying  $(\bar{x})^* = \bar{y}$  and  $(\bar{y})^* = \bar{x}$  for a pair of opposite arrows  $\bar{x}, \bar{y}$ .

**Proposition 3.3.** The ideal  $\overline{I}_m^n$  is generated by the following quadratic relations:

- **Monomial relations.** The paths of length 2 between the top and bottom vertices in Fig. 6 (c) and (d) are zero.
- **Anticommutativity and Plücker-type relations.** If there is more than one parallel path of length 2 (excluding 2-cycles), then the sum of all such parallel paths is 0.
- **Relations at vertices.** Fix  $\lambda \in \Lambda_m^n$ . Let  $\bar{y}_{\kappa_j}^\lambda$  be the ascending arrows in  $\overline{Q}_m^n$  ending at  $e_\lambda$ . Let  $\bar{y}_\lambda^\mu$  be any ascending arrow starting at  $e_\lambda$ . Then

$$\bar{y}_\lambda^\mu \bar{x}_\lambda^\mu = - \sum_j c_{\kappa_j}^\mu(\lambda) \bar{x}_{\kappa_j}^\lambda \bar{y}_{\kappa_j}^\lambda \quad (3.4)$$

where the coefficients  $c_{\kappa_j}^\mu(\lambda)$  are given in Definition 2.15.

**Proof.** We need to show that the above relations are orthogonal to the quadratic relations described in Proposition 2.16. The monomial relations are clear since by Proposition 2.16 the monomials do not appear in the quadratic relations  $(I_m^n)_2$ . The anticommutativity and Plücker-type relations involving length 2 paths (excluding 2-cycles) are clearly orthogonal to the commutativity relations across squares in  $I_m^n$ .

Let us check the relations at vertices. Fix the vertex  $e_\lambda$ . Let  $\mu_j$  be the weights obtained from  $\lambda$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in circles  $C_j$  of  $e_\lambda$  as in Proposition 2.16. Note that for any  $\kappa_k$  and  $\mu_l$  we have

$$\langle y_{\kappa_k}^\lambda x_{\kappa_k}^\lambda - \sum_i c_{\kappa_k}^{\mu_i} x_\lambda^{\mu_i} y_\lambda^{\mu_i}, \bar{y}_\lambda^{\mu_l} \bar{x}_\lambda^{\mu_l} + \sum_j c_{\kappa_j}^{\mu_l} \bar{x}_{\kappa_j}^\lambda \bar{y}_{\kappa_j}^\lambda \rangle = \sum_j \delta_{\kappa_k, \kappa_j} c_{\kappa_j}^{\mu_l} - \sum_i \delta_{\mu_i, \mu_l} c_{\kappa_k}^{\mu_i} = 0$$

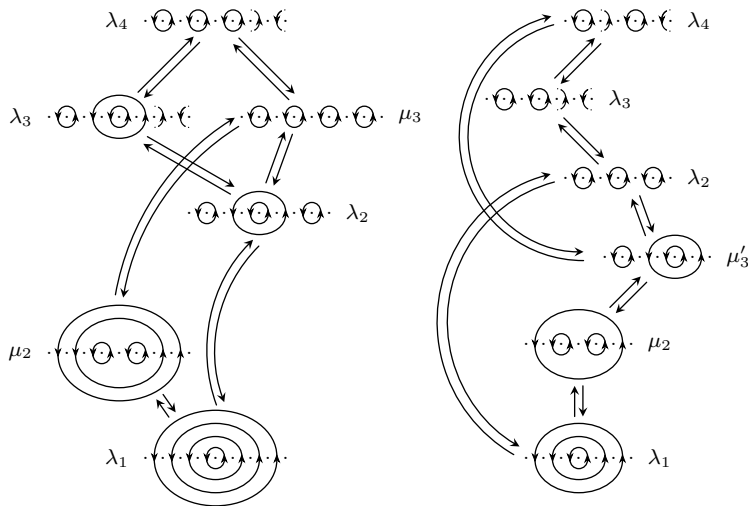


Fig. 8. Paths of length 3 in  $\overline{Q}_m^n$  which are equal in  $\overline{K}_m^n$ .

where we use  $\langle y_{\kappa_k}^\lambda x_{\kappa_k}^\lambda, \bar{x}_{\kappa_j}^\lambda \bar{y}_{\kappa_j}^\lambda \rangle = \delta_{\kappa_k, \kappa_j}$  and  $\langle x_\lambda^{\mu_i} y_\lambda^{\mu_i}, \bar{y}_\lambda^{\mu_l} \bar{x}_\lambda^{\mu_l} \rangle = \delta_{\mu_i, \mu_l}$ , and we denote  $c_\kappa^\mu(\lambda) = c_\kappa^\mu$ . This verifies the relations at vertex  $e_\lambda$ .  $\square$

The following definition will be useful later.

**Definition 3.5.** Let  $p = \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  be an ascending path of length  $k \geq 2$  in  $\overline{Q}_m^n$ . Assume that the weight  $\lambda_{i+1}$  is obtained from  $\lambda_i$  by exchanging the  $\blacktriangledown \dots \blacktriangle$  pair lying in a circle  $C_i$  of  $e_{\lambda_i}$  for  $1 \leq i \leq k$ .

We denote by  $h_i$  the integer such that  $C_i$  appears in  $e_{\lambda_{i-j}}$  for each  $0 \leq j \leq h_i$  but  $C_i$  does not appear in  $e_{\lambda_{i-h_i-1}}$ . For instance, if  $h_i = 0$  then  $C_i$  appears in  $e_{\lambda_i}$  but does not appear in  $e_{\lambda_{i-1}}$ . If  $h_k = k - 1$  then  $C_k$  appears in  $e_{\lambda_j}$  for all  $1 \leq j \leq k$ .

**Lemma 3.6.** Let  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  be a path of length  $k \geq 3$  such that  $\lambda_{i+1}$  is obtained from  $\lambda_i$  by exchanging the  $\blacktriangledown \dots \blacktriangle$  pair lying in a circle  $C_i$  of  $e_{\lambda_i}$  for  $1 \leq i \leq k$ . Assume that the following two conditions hold:

- (i) For each  $1 \leq i < k$  we have  $h_i = 0$  and  $h_k = k - 1$  (see Definition 3.5).
- (ii)  $C_k$  is enclosed in  $C_1$  in  $e_{\lambda_1}$  and lies on the left of  $C_i$  in  $e_{\lambda_i}$  for  $2 \leq i < k$ .

Denote by  $\mu_{i+1}$  the weight obtained from  $\lambda_i$  by exchanging the  $\blacktriangledown \dots \blacktriangle$  pair lying in  $C_k$  of  $e_{\lambda_i}$  for  $1 \leq i \leq k - 1$ . Then we have the following two statements.

If in  $e_{\lambda_1}$  there exists a circle between  $C_1$  and  $C_k$  which encloses  $C_k$  (see the left part of Fig. 8 for  $k = 3$ ) then

$$\begin{aligned} \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}} - (-1)^{k-1} \bar{y}_{\lambda_1}^{\mu_2} \bar{y}_{\mu_2}^{\mu_3} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} &\in \overline{I}_m^n \\ \bar{x}_{\lambda_k}^{\lambda_{k+1}} \dots \bar{x}_{\lambda_2}^{\lambda_3} \bar{x}_{\lambda_1}^{\lambda_2} - (-1)^{k-1} \bar{x}_{\mu_k}^{\lambda_{k+1}} \bar{x}_{\mu_{k-1}}^{\mu_k} \dots \bar{x}_{\mu_2}^{\mu_3} \bar{x}_{\lambda_1}^{\mu_2} &\in \overline{I}_m^n. \end{aligned}$$

Otherwise, we have (see the right part of Fig. 8 for  $k = 3$ )

$$\begin{aligned} \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}} - (-1)^{k-1} \bar{y}_{\lambda_1}^{\mu_2} \bar{y}_{\mu_2}^{\mu_3'} \bar{y}_{\mu_3'}^{\mu_4} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} &\in \bar{\Gamma}_m^n \\ \bar{x}_{\lambda_k}^{\lambda_{k+1}} \dots \bar{x}_{\lambda_2}^{\lambda_3} \bar{x}_{\lambda_1}^{\lambda_2} - (-1)^{k-1} \bar{x}_{\mu_k}^{\lambda_{k+1}} \bar{x}_{\mu_{k-1}}^{\mu_k} \dots \bar{x}_{\mu_3'}^{\mu_4} \bar{x}_{\mu_2}^{\mu_3'} \bar{x}_{\lambda_1}^{\mu_2} &\in \bar{\Gamma}_m^n \end{aligned}$$

where we note that there are two circles in  $e_{\mu_2}$  which do not appear in  $e_{\lambda_1}$  and  $\mu_3'$  is obtained from  $\mu_2$  by exchanging the  $\blacktriangledown \dots \blacktriangle$  pair lying in the left one of the two circles.

**Proof.** We only verify the statements for the ascending paths as the statements for descending paths follow by using the involution in Remark 3.2.

For the first case, note that the weight  $\mu_i$  may be also obtained from  $\mu_{i-1}$  by exchanging the  $\blacktriangledown \dots \blacktriangle$  pair lying in  $C_{i-1}$  so that the vertices  $\lambda_{i-1}, \lambda_i, \mu_i, \mu_{i+1}$  form a square in Fig. 5 (b) for  $2 < i \leq k$  and a square in Fig. 5 (a) for  $i = 2$ . Here, we write  $\mu_{k+1} = \lambda_{k+1}$ . So we have the anticommutativity relations for  $2 \leq i \leq k$

$$\bar{y}_{\lambda_{i-1}}^{\lambda_i} \bar{y}_{\lambda_i}^{\mu_{i+1}} + \bar{y}_{\lambda_{i-1}}^{\mu_i} \bar{y}_{\mu_i}^{\mu_{i+1}} = 0. \quad (3.7)$$

Applying these recursively to  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  we obtain the desired equality.

For the second case, the anticommutativity relations (3.7) still hold for  $i > 3$ . But for  $i = 3$  it is replaced by the Plücker-type relation

$$\bar{y}_{\lambda_2}^{\lambda_3} \bar{y}_{\lambda_3}^{\mu_4} + \bar{y}_{\lambda_2}^{\mu_3} \bar{y}_{\mu_3}^{\mu_4} + \bar{x}_{\mu_3'}^{\lambda_2} \bar{y}_{\mu_3'}^{\mu_4} = 0. \quad (3.8)$$

By the monomial relation  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\mu_3} = 0$  from Fig. 6 (c) in Proposition 3.3, we have

$$\begin{aligned} 0 &= -\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\mu_3} \bar{y}_{\mu_3}^{\mu_4} \bar{y}_{\mu_4}^{\mu_5} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} \\ &= \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \bar{y}_{\lambda_3}^{\mu_4} \bar{y}_{\mu_4}^{\mu_5} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} + \bar{y}_{\lambda_1}^{\lambda_2} \bar{x}_{\mu_3'}^{\lambda_2} \bar{y}_{\mu_3'}^{\mu_4} \bar{y}_{\mu_4}^{\mu_5} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} \\ &= (-1)^{k-1} \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \bar{y}_{\lambda_3}^{\lambda_4} \dots \bar{y}_{\lambda_{k-1}}^{\lambda_k} \bar{y}_{\lambda_k}^{\lambda_{k+1}} - \bar{y}_{\lambda_1}^{\mu_2} \bar{y}_{\mu_2}^{\mu_3'} \bar{y}_{\mu_3'}^{\mu_4} \bar{y}_{\mu_4}^{\mu_5} \dots \bar{y}_{\mu_{k-1}}^{\mu_k} \bar{y}_{\mu_k}^{\lambda_{k+1}} \end{aligned}$$

where the second equality follows from (3.8) and the third one from (3.7) and the relation  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{x}_{\mu_3'}^{\lambda_2} + \bar{y}_{\lambda_1}^{\mu_2} \bar{y}_{\mu_2}^{\mu_3'} = 0$ .  $\square$

### 3.2. Reduction systems and Bergman's Diamond Lemma

In this section we recall the notion of a *reduction system* and the closely related Diamond Lemma. Whereas  $K_m^n$  has a diagrammatic  $\mathbb{k}$ -basis given by arc diagrams, the Koszul dual  $\bar{K}_m^n$  is a priori only defined algebraically. Encoding the relations of  $\bar{K}_m^n$  into a reduction system gives a natural algebraic  $\mathbb{k}$ -basis of  $\bar{K}_m^n$  given by “irreducible” paths.

**Definition 3.9.** Let  $Q$  be a finite quiver and let  $S \subset Q_{\geq 2}$  be a subset of paths of length  $\geq 2$  such that for any  $s, s' \in S$ ,  $s$  is not a subpath of  $s'$ . We call a path in  $Q$  *irreducible*

(with respect to  $S$ ) if it does not contain elements in  $S$  as subpaths and we denote the set of all irreducible paths by  $\text{Irr}_S$ .

A *reduction system* is given by a set of pairs

$$R = \{(s, \varphi_s) \mid s \in S \text{ and } \varphi_s \in \mathbb{k}\text{Irr}_S\}$$

such that each path appearing in the linear combination  $\varphi_s$  is parallel to  $s$ .

Given a reduction system  $R = \{(s, \varphi_s)\}_{s \in S}$ , a *basic reduction*  $\mathbf{r}_{q,s,r}$  is a map  $\mathbb{k}Q \rightarrow \mathbb{k}Q$  determined by

$$\mathbf{r}_{q,s,r}(p) = \begin{cases} q\varphi_s r & \text{if } p = qsr \\ p & \text{otherwise} \end{cases}$$

for any paths  $p, q, r \in Q_\bullet$  and any  $s \in S$ . In other words,  $\mathbf{r}_{q,s,r}$  replaces  $s$  by  $\varphi_s$  in the path  $qsr$  and leaves all other paths invariant. A *reduction* is any (finite) composition of basic reductions.

**Definition 3.10.** A reduction system  $R = \{(s, \varphi_s)\}_{s \in S}$  is called

- *reduction-finite* if for any path  $p \in Q_\bullet$  and any infinite sequence  $(\mathbf{r}_1, \mathbf{r}_2, \dots)$  of reductions, there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have  $\mathbf{r}_n \circ \dots \circ \mathbf{r}_1(p) = \mathbf{r}_{n_0} \circ \dots \circ \mathbf{r}_1(p)$
- *reduction-unique* if it is reduction-finite and moreover for any path  $p$  and any reductions  $\mathbf{r}, \mathbf{r}'$  such that  $\mathbf{r}(p), \mathbf{r}'(p) \in \mathbb{k}\text{Irr}_S$ , we have that  $\mathbf{r}(p) = \mathbf{r}'(p)$ .

We say that  $R$  satisfies the *diamond condition* for a two-sided ideal  $I \subset \mathbb{k}Q$  if  $R$  is reduction-unique and  $I$  is equal to the ideal generated by the set  $\{s - \varphi_s\}_{s \in S}$ .

**Definition 3.11.** Let  $R = \{(s, \varphi_s)\}_{s \in S}$  be a reduction system which is reduction-finite. An *overlap ambiguity* of  $S$  is any path of the form  $pqr \in Q_{\geq 3}$  with  $p, q, r \in Q_{\geq 1}$  and  $pq, qr \in S$ .

An overlap ambiguity of  $S$  is said to be *resolvable* with respect to  $R$  if  $\mathbf{r}(p\varphi_{qr}) = \mathbf{r}'(\varphi_{pq}r)$  for some reductions  $\mathbf{r}, \mathbf{r}'$ .

The central result about reduction systems is the following Diamond Lemma.

**Theorem 3.12** ([6, Thm. 1.2]). Let  $R = \{(s, \varphi_s)\}_{s \in S}$  be a reduction system for  $\mathbb{k}Q$  and let  $I = \langle s - \varphi_s \rangle_{s \in S}$ . If  $R$  is reduction-finite, then the following are equivalent:

- $R$  is reduction-unique, i.e.  $R$  satisfies the diamond condition for  $I$ .
- All overlap ambiguities of  $S$  are resolvable with respect to  $R$ .

(iii) The image of the irreducible paths  $\text{Irr}_S$  under the projection  $\mathbb{k}Q \rightarrow \mathbb{k}Q/I$  forms a  $\mathbb{k}$ -basis of  $A = \mathbb{k}Q/I$ .

### 3.3. Irreducible paths and Kazhdan–Lusztig polynomials

In this subsection we will define a reduction system  $\overline{R}_m^n$  for  $\overline{K}_m^n$  and show that it satisfies the diamond condition, i.e. it is reduction-unique (cf. §4.3). In §4 we will see that deformations of  $\overline{R}_m^n$  describe  $A_\infty$  deformations of  $K_m^n$  and can also be used to compute the Hochschild cohomology of  $K_m^n$ .

**Definition 3.13** (*Reduction system for  $\overline{K}_m^n$* ). Let  $\overline{S}_m^n$  be the set of all paths in  $\overline{Q}_m^n$  of the following four types:

- (I) length 2 paths of the form  $\bar{y}'_\lambda \bar{x}'_\mu$  for some weights  $\lambda, \mu, \nu \in \Lambda_m^n$
- (II) length 2 paths of the form  $\bar{y}'_\lambda \bar{y}'_\nu$  or  $\bar{x}'_\nu \bar{x}'_\lambda$  in Fig. 6 (c)
- (III) length 2 paths of the form  $\bar{y}'_\lambda \bar{y}'_\nu$  or  $\bar{x}'_\nu \bar{x}'_\lambda$  in Fig. 5, where  $\nu$  is the right vertex in each figure
- (IV) length  $\geq 3$  paths of the form  $\bar{y}'_{\lambda_1} \bar{y}'_{\lambda_2} \cdots \bar{y}'_{\lambda_k} \bar{y}'_{\lambda_{k+1}}$  or  $\bar{x}'_{\lambda_k} \bar{x}'_{\lambda_{k+1}} \cdots \bar{x}'_{\lambda_2} \bar{x}'_{\lambda_1}$  as in Lemma 3.6

where in (I) we allow  $\lambda = \mu$ .

For each  $s \in \overline{S}_m^n$  let  $\varphi_s \in \mathbb{k}\text{Irr}_{\overline{S}_m^n}$  denote the linear combination of the irreducible paths so that  $s - \varphi_s \in \overline{I}_m^n$ . Then we define  $\overline{R}_m^n = \{(s, \varphi_s) \mid s \in \overline{S}_m^n\}$  (see §5.1 for a concrete description of  $\overline{R}_2^2$ ). Note that for any path  $s \in \overline{S}_m^n$  of type (II) we have  $\varphi_s = 0$  by the monomial relations in Proposition 3.3.

**Lemma 3.14.** *The reduction system  $\overline{R}_m^n$  is reduction-finite.*

**Proof.** It suffices to show that reduction-finiteness holds when acting on an arbitrary single path  $p$ . Let us prove this by induction on the path length of  $p$ . Clearly this holds if  $p$  is a vertex or an arrow.

Assume that it holds for all paths of length  $< m$ . Let us prove it for any path of length  $m$ . Let  $p = a_1 \cdots a_m$  be a length  $m$  path, where  $a_1, \dots, a_m$  are arrows in  $\overline{Q}_m^n$ , and denote by  $e_{\lambda_i}$  the starting vertex of  $a_i$  for  $1 \leq i \leq m$ . Let us call a vertex  $e_{\lambda_i}$  a *peak* along  $p$  if  $|\lambda_{i-1}| < |\lambda_i| > |\lambda_{i+1}|$ , i.e.  $a_{i-1} = \bar{y}'_{\lambda_{i-1}}$  is an ascending arrow but  $a_i = \bar{x}'_{\lambda_{i+1}}$  is descending. Denote by  $V$  the set of peaks along  $p$ . Note that peaks are only changed by reductions of type (I) and reductions of types (II)–(IV) leave the peaks along  $p$  invariant.

We have the following two cases. In the first case  $V$  is not empty. Let  $e_{\lambda_{i_0}} \in V$  be a peak whose height is not smaller than the height of any other peak. By induction, both  $a_1 \cdots a_{i_0-1}$  and  $a_{i_0} \cdots a_m$  are reduction-finite so at some point we have to use the reduction of type (I) on  $a'_{i_0-1} a'_{i_0}$  passing through  $e_{\lambda_{i_0}}$ , where  $a'_{i_0-1}$  (resp.  $a'_{i_0}$ ) is the last (resp. first) arrow obtained by performing some reductions on  $a_1 \cdots a_{i_0-1}$  (resp.  $a_{i_0} \cdots a_m$ ). (Note that  $a'_{i_0-1}$  and  $a'_{i_0}$  may or may not coincide with  $a_{i_0-1}$  and  $a_{i_0}$  but

$a'_{i_0-1}$  is necessarily ascending and  $a'_{i_0}$  descending.) This reduction causes the heights of the peaks to strictly decrease. Since the heights of vertices are bounded below by 0, it follows that  $p$  must be reduction-finite.

In the second case  $V$  is empty. Then we have the following three subcases. In the first subcase, the path  $p$  is of the form  $\bar{x}_1 \cdots \bar{x}_i \bar{y}_{i+1} \cdots \bar{y}_m$  for some  $1 \leq i \leq m-1$ . By induction both  $\bar{x}_1 \cdots \bar{x}_i$  and  $\bar{y}_{i+1} \cdots \bar{y}_m$  are reduction-finite, hence so is  $p$  since  $\bar{x}_i \bar{y}_{i+1}$  is irreducible. In the second subcase, the path  $p = \bar{y}_1 \cdots \bar{y}_m$  is an ascending path. For this, we may give an order on ascending paths as follows. Set  $\bar{y}'_\lambda \prec \bar{y}'_{\lambda'}$  whenever  $\nu, \nu'$  are obtained from  $\lambda$  by exchanging two different  $\blacktriangledown \cdots \blacktriangle$  pairs lying in circles  $C, C'$  of  $e_\lambda$ , respectively, and either  $C, C'$  are nested and  $C$  is contained in  $C'$  or  $C, C'$  are disjoint and  $C$  lies to the right of  $C'$ . Extend  $\prec$  by the degree-lexicographic order for longer ascending paths. Then reductions of types (II)–(IV) respect this order in the sense that reductions strictly decrease the order, so that  $p$  is reduction-finite since the set of length  $m$  paths is finite. Note that a descending arrow may appear in a summand  $p'$  when performing the reduction of type (III) from the Plücker-type relations on the ascending path  $p$ . Then there is a peak along  $p'$  so that  $p'$  is reduction-finite by the first case. In the third subcase,  $p$  is a descending path. Then  $p$  is reduction-finite by the second subcase and the fact that the reduction system is invariant under involution.  $\square$

**Remark 3.15.** If  $m = 1$  or  $n = 1$  then  $\bar{S}_m^n$  only consists of paths  $\bar{y}'_\lambda \bar{x}'_\lambda$  of type (I). The paths of type (IV) only exist when  $m \geq 2$  and  $n > 2$ . In other words,  $\bar{S}_m^2$  consists only of paths of types (I)–(III) so that  $\bar{R}_m^2$  is quadratic.

The following lemma completely describes the ascending irreducible paths in  $\bar{K}_m^n$ .

**Lemma 3.16.** Let  $p = \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  be a path of length  $k \geq 2$  such that  $\lambda_{i+1}$  is obtained from  $\lambda_i$  by exchanging the  $\blacktriangledown \cdots \blacktriangle$  pair lying in a circle  $C_i$  of  $e_{\lambda_i}$ .

Then  $p$  is irreducible if and only if for any  $i$  with  $h_i \neq 0$  (Definition 3.5), the circle  $C_i$  either lies on the left of  $C_{i-j}$  or encloses  $C_{i-j}$  in  $e_{\lambda_{i-j}}$  for each  $0 < j \leq h_i$ . In particular, if  $h_i = 0$  for all  $1 \leq i \leq k$  then  $p$  is irreducible.

**Proof.** Let us prove the “if” part. If  $p$  is not irreducible then by definition it contains elements in  $\bar{S}_m^n$  as subpaths. We have the following two cases.

In the first case, there is a subpath  $\bar{y}_{\lambda_{i-1}}^{\lambda_i} \bar{y}_{\lambda_i}^{\lambda_{i+1}}$  of length 2 belonging to  $\bar{S}_m^n$ . Then  $\bar{y}_{\lambda_{i-1}}^{\lambda_i} \bar{y}_{\lambda_i}^{\lambda_{i+1}}$  is in Fig. 5, so that  $\lambda_i$  is the right vertex in each figure. Note that  $C_i$  appears in  $e_{\lambda_{i-1}}$  and it lies on the right of  $C_{i-1}$  for the squares (b) and (c) and is enclosed in  $C_{i-1}$  for the square (a), giving a contradiction.

In the second case, there is a subpath  $\bar{y}_{\lambda_j}^{\lambda_{j+1}} \bar{y}_{\lambda_{j+1}}^{\lambda_{j+2}} \cdots \bar{y}_{\lambda_i}^{\lambda_{i+1}}$  of length  $i - j + 1 > 2$  belonging to  $\bar{S}_m^n$ . Then by the assumption in Lemma 3.6 we have that  $C_i$  appears in  $e_{\lambda_j}$  and is enclosed in  $C_j$  in  $e_{\lambda_j}$ , giving a contradiction.

Let us prove the “only if” part. Namely, we need to prove that if there exists  $1 \leq i_0 \leq k$  with  $h_{i_0} \neq 0$  so that one of the following conditions holds

- (i)  $C_{i_0}$  is on the right of  $C_{i_0-j}$  for some  $0 < j \leq h_{i_0}$
- (ii)  $C_{i_0}$  is enclosed in  $C_{i_0-j}$  for some  $0 < j \leq h_{i_0}$

then  $p$  is not irreducible. We will prove this by induction on  $k$ . Clearly it holds for  $k = 2$  by the definition of  $\bar{S}_m^n$ . Assume  $k > 2$ . If  $i_0 < k$  then consider the subpath  $p' = \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_{k-1}}^{\lambda_k}$  of length  $k - 1$ . Note that the above conditions (i) and (ii) still hold for  $p'$ . It follows by induction that  $p'$  is not irreducible, so neither is  $p$ . If  $i_0 = k$  but  $h_k < k - 1$  then consider  $p'' = \bar{y}_{\lambda_2}^{\lambda_3} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$ . Similarly, the conditions (i) and (ii) still hold for  $p''$ . By induction  $p''$  is not irreducible, so neither is  $p$ .

If  $i_0 = k$  and  $h_k = k - 1$ , by the induction hypothesis (on the subpath  $\bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$ ) we may assume that  $C_k$  either lies on the left of  $C_i$  or encloses  $C_i$  in  $e_{\lambda_i}$  for each  $2 \leq i < k$ . Consider the first condition (i), namely  $C_k$  is on the right of  $C_1$  in  $e_{\lambda_1}$ . Since by assumption  $C_2$  is either on the right of  $C_k$  or enclosed in  $C_k$ , it follows that  $C_2$  has to appear in  $e_{\lambda_1}$  and lie on the right of  $C_1$ , whence  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3}$  is not irreducible so that neither is  $p$ .

Consider (ii), namely  $C_k$  is enclosed in  $C_1$ . Since by assumption  $C_2$  is either on the right of  $C_k$  or enclosed in  $C_k$  in  $e_{\lambda_2}$ , we may then further assume that  $C_2$  does not appear in  $e_{\lambda_1}$ , i.e.  $h_2 = 0$  (otherwise  $C_2$  must be either enclosed in  $C_1$  or on the right of  $C_1$ , whence  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3}$  is not irreducible and neither is  $p$ ). Similarly, we may also assume that  $h_3 = 0$ . Otherwise  $C_3$  must be either enclosed in  $C_2$  or on the right of  $C_2$  in  $e_{\lambda_2}$ , whence  $\bar{y}_{\lambda_2}^{\lambda_3} \bar{y}_{\lambda_3}^{\lambda_4}$  is not irreducible and neither is  $p$ . By a similar argument we may assume that  $h_i = 0$  for all  $2 \leq i < k$ . Then the path  $p$  satisfies the assumption in Lemma 3.6 so that  $p \in \bar{S}_m^n$  which is not irreducible.  $\square$

The diamond condition for  $\bar{R}_m^n$  will follow from properties of the *Kazhdan–Lusztig polynomials* which were first defined geometrically by Kazhdan and Lusztig [25] with a closed formula for Grassmannians given by Lascoux and Schützenberger [31], see also [10, §5]. The Kazhdan–Lusztig polynomials for Grassmannians can be defined in terms of the quiver  $\bar{Q}_m^n$  for  $\bar{K}_m^n$  as follows.

**Definition 3.17.** Let  $\lambda, \mu \in \Lambda_m^n$  be two weights. For each  $k \geq 0$ , denote by  $a_k$  the number of *ascending* irreducible paths of length  $k$  from  $\lambda$  to  $\mu$ . Define a polynomial

$$P_{\lambda, \mu}(q) = \sum_{k \geq 0} a_k q^k$$

where we set  $a_k = 0$  if there are no ascending irreducible paths of length  $k$ .

Clearly,  $P_{\lambda, \mu}(1)$  is the total number of ascending irreducible paths from  $\lambda$  to  $\mu$ . Note that  $a_k$  also equals the number of descending irreducible paths of length  $k$  from  $\mu$  to  $\lambda$ .

**Proposition 3.18.** *The following properties uniquely determine  $P_{\lambda, \mu}(q)$ .*

- (i) If  $\lambda = \mu$  then  $P_{\lambda,\mu}(q) = 1$  and if  $|\lambda| \geq |\mu|$  for  $\lambda \neq \mu$  then  $P_{\lambda,\mu}(q) = 0$ .
- (ii) If  $|\lambda| < |\mu|$  denote by  $C$  the rightmost circle in  $e_\lambda$  which does not enclose any other circles. Let  $\lambda'$  and  $\mu'$  be the weights in  $\Lambda_{m-1}^{n-1}$  obtained from  $\lambda$  and  $\mu$ , respectively, by deleting the vertices in the position of  $C$  and let  $\lambda''$  be the weight in  $\Lambda_m^n$  obtained from  $\lambda$  by exchanging the  $\blacktriangledown\blacktriangleright$  pair in  $C$ . Then

$$P_{\lambda,\mu}(q) = \begin{cases} P_{\lambda',\mu'}(q) + qP_{\lambda'',\mu}(q) & \text{if } e_\mu \text{ contains the circle } C \\ qP_{\lambda'',\mu}(q) & \text{otherwise.} \end{cases} \quad (3.19)$$

As a result,  $P_{\lambda,\mu}(q)$  coincides with the Kazhdan–Lusztig polynomials in [10, §5].

**Proof.** The first assertion is clear since if  $|\lambda| \geq |\mu|$  then there are no ascending paths from  $\lambda$  to  $\mu$ , except for the “lazy path” of length 0 at  $\lambda = \mu$ .

Let us prove the second assertion. Assume that  $e_\mu$  contains the circle  $C$ . Let  $p = \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \dots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  be an ascending irreducible path from  $\lambda$  to  $\mu$  such that  $\lambda_{i+1}$  be obtained from  $\lambda_i$  by exchanging the  $\blacktriangledown \dots \blacktriangleright$  pair lying in a circle  $C_i$  of  $e_{\lambda_i}$  (denote  $\lambda_1 = \lambda$  and  $\lambda_{k+1} = \mu$ ). Then the path is of one of the following two forms:

In the first one, the vertex  $e_{\lambda_2}$  contains the circle  $C$ . Then we claim that  $e_{\lambda_j}$  must also contain the circle  $C$  for all  $1 \leq j \leq k+1$ . Indeed, if  $j_0 > 1$  is the smallest integer so that  $e_{\lambda_{j_0+1}}$  does not contain  $C$  then  $\lambda_{j_0+1}$  is obtained from  $\lambda_{j_0}$  by exchanging the  $\blacktriangledown\blacktriangleright$  pair in  $C$ , whence  $C_{j_0} = C$  and  $h_{j_0} = j_0 - 1 \neq 0$ . Since  $C$  is the rightmost circle in  $e_{\lambda_1}$  and in particular  $C_{j_0}$  is on the right of  $C_1$ , it follows by Lemma 3.16 that the path  $p$  is not irreducible. This proves the claim. Thus, the ascending irreducible paths of this form bijectively correspond to the ascending irreducible paths from  $\lambda'$  to  $\mu'$ , since removing  $C$  from all  $\lambda_i$  the irreducible path  $p$  induces an irreducible path in  $\bar{K}_{m-1}^{n-1}$  by Lemma 3.16 again.

In the second one,  $e_{\lambda_2}$  does not contain  $C$  (i.e.  $\lambda_2$  is obtained from  $\lambda_1$  by exchanging the  $\blacktriangledown\blacktriangleright$  pair in  $C$ ) then  $\lambda_2 = \lambda''$ . Since  $C$  is the rightmost circle, by Lemma 3.16 the irreducible paths of this form bijectively correspond to the ascending irreducible paths from  $\lambda''$  to  $\mu$  by composing with  $\bar{y}_{\lambda_1}^{\lambda_2}$ .

So putting these two forms together we have  $P_{\lambda,\mu}(q) = P_{\lambda',\mu'}(q) + qP_{\lambda'',\mu}(q)$ .

Assume that  $e_\mu$  does not contain  $C$ . Then we claim that  $\lambda_2 = \lambda''$ . Indeed, if  $\lambda_2 \neq \lambda''$  then  $C$  appears in  $e_{\lambda_2}$ . Since  $e_\mu$  does not contain  $C$  there exists  $j_0 \leq k$  such that  $C$  appears in  $e_{\lambda_j}$  for each  $j \leq j_0$  but does not appear in  $e_{\lambda_{j_0+1}}$ . That is,  $\lambda_{j_0+1}$  is obtained from  $\lambda_{j_0}$  by exchanging the  $\blacktriangledown \dots \blacktriangleright$  pair lying in  $C$ , so  $C_{j_0} = C$  and  $h_{j_0} = j_0 - 1$ . Since  $C$  is the rightmost circle in  $e_{\lambda_1}$  it follows that  $C_{j_0}$  is on the right of  $C_1$  in  $e_{\lambda_1}$  whence  $p$  is not irreducible by Lemma 3.16. This proves the claim. Note that ascending irreducible paths from  $\lambda''$  to  $\mu$  bijectively induce ascending irreducible paths from  $\lambda$  to  $\mu$  by composing with  $\bar{y}_{\lambda_1}^{\lambda_2}$ . Therefore, we have  $P_{\lambda,\mu}(q) = qP_{\lambda'',\mu}(q)$ .

Note that the recursive formulae for  $P_{\lambda,\mu}(q)$  proved above coincide with the ones for the (combinatorial) Kazhdan–Lusztig polynomials (see [10, Lem. 5.2]).  $\square$

**Lemma 3.20.** *The number of irreducible paths from  $\lambda$  to  $\mu$  is equal to*

$$\sum_{\kappa \in \Lambda_m^n} P_{\kappa, \lambda}(1) P_{\kappa, \mu}(1).$$

**Proof.** Note that any irreducible path can be written as  $\bar{x}_1 \bar{x}_2 \cdots \bar{x}_k \bar{y}_l \cdots \bar{y}_2 \bar{y}_1$  where  $\bar{x}_1 \bar{x}_2 \cdots \bar{x}_k$  (resp.  $\bar{y}_l \cdots \bar{y}_2 \bar{y}_1$ ) is a descending (resp. ascending) irreducible path from  $\lambda$  to  $\kappa$  (resp. from  $\kappa$  to  $\mu$ ) for some weight  $\kappa$ .  $\square$

**Theorem 3.21.** *The irreducible paths form a basis of  $\bar{K}_m^n$ . As a result, the reduction system  $\bar{R}_m^n$  is reduction-unique and therefore satisfies the diamond condition.*

**Proof.** By [9, Cor. 5.9] and Proposition 3.18 we have

$$\dim_{\mathbb{k}} e_{\lambda} \bar{K}_m^n e_{\mu} = \sum_{\kappa \in \Lambda_m^n} P_{\kappa, \lambda}(1) P_{\kappa, \mu}(1).$$

By Lemma 3.20 the right-hand side is equal to the number of irreducible paths from  $\lambda$  to  $\mu$ , whence the irreducible paths form a basis of  $\bar{K}_m^n$ . By the Diamond Lemma (Theorem 3.12) it follows that  $\bar{R}_m^n$  is reduction-unique.  $\square$

**Remark 3.22.** Denote by  $(\bar{K}_m^n)_i$  the image of the length  $i$  paths under the projection  $\mathbb{k}\bar{Q}_m^n \rightarrow \bar{K}_m^n$  so that  $\bar{K}_m^n = \bigoplus_i (\bar{K}_m^n)_i$ . Let  $\lambda, \mu \in \Lambda_m^n$  be any two weights. Then  $e_{\lambda}(\bar{K}_m^n)_i e_{\mu}$  is one dimensional if  $i = |\lambda| + |\mu|$  and is zero if  $i > |\lambda| + |\mu|$ . In particular,  $(\bar{K}_m^n)_i = 0$  if either  $i < 0$  or  $i > 2mn$  and  $(\bar{K}_m^n)_{2mn}$  is of dimension 1 spanned by the irreducible path  $\bar{x}_0 \bar{x}_1 \cdots \bar{x}_{mn-1} \bar{y}_{mn-1} \cdots \bar{y}_1 \bar{y}_0$  in Fig. 12 which is parallel to the vertex corresponding to the highest weight.

The following observation is useful. Let  $p$  be any path of length  $|\lambda| + |\mu|$  from  $e_{\lambda}$  to  $e_{\mu}$ . Then  $\bar{y}p = 0$  in  $\bar{K}_m^n$  for any ascending arrow  $\bar{y}$ , since  $\bar{y}p \in e_{\kappa}(\bar{K}_m^n)_{|\lambda|+1} e_{\mu} = 0$ , where  $e_{\kappa}$  is the starting vertex of  $\bar{y}$  and note that  $|\kappa| < |\lambda|$ . Similarly, we have  $p\bar{x} = 0$  in  $\bar{K}_m^n$  for any descending arrow  $\bar{x}$ .

**Remark 3.23.** It would be very interesting to provide a geometric or diagrammatic description of the Koszul dual  $\bar{K}_m^n$ , analogous to the diagrammatic description of  $K_m^n$ . We refer to Webster [46, Thm. 3.7] for an explicit construction of diagrammatic algebras which are Morita equivalent to  $\bar{K}_m^n$ .

### 3.4. Some useful relations in the Koszul dual

Since the multiplication in  $\bar{K}_m^n$  has no straightforward diagrammatic description, computing the multiplication of arbitrary elements in  $\bar{K}_m^n$  is rather complicated. In this section we record some formulae for “long” irreducible paths in  $\bar{Q}_m^n$  which are used in the proof of our main theorem (Theorem 5.4).

By [33, Proof of Cor. 1.5] we have  $K_m^n \simeq (K_n^m)^{\text{op}}$  as graded algebras and thus  $\overline{K}_m^n \simeq (\overline{K}_n^m)^{\text{op}}$ . In the remainder of this section let us therefore assume that  $m \geq n$ .

**Notation 3.24.** Denote by  $\bar{x}_0 \bar{x}_1 \cdots \bar{x}_{mn-1} \bar{y}_{mn-1} \cdots \bar{y}_1 \bar{y}_0$  the longest irreducible path in  $\overline{K}_m^n$  (see Fig. 12). For simplicity, we shall often use the short-hand notation

$$\bar{x}_{0\dots mn-1} = \bar{x}_0 \bar{x}_1 \cdots \bar{x}_{mn-1} \quad \text{and} \quad \bar{y}_{mn-1\dots 0} = \bar{y}_{mn-1} \cdots \bar{y}_1 \bar{y}_0$$

and similar for other sequences of descending or ascending arrows in Fig. 12.

We have the following relations at vertices (see Proposition 3.3)

$$\begin{aligned} \bar{y}_{ni+j} \bar{x}_{ni+j} + \bar{x}_{ni+j+1} \bar{y}_{ni+j+1} &= 0 \\ \bar{y}_{ni+n-1} \bar{x}_{ni+n-1} &= 0 \\ \bar{y}_{ni} \bar{x}_{ni} + \bar{x}_{ni+1} \bar{y}_{ni+1} + \bar{x}_{ni+1}' \bar{y}_{ni+1}' &= 0 \\ \bar{y}_{n(m-1)} \bar{x}_{n(m-1)} + \bar{x}_{n(m-1)+1} \bar{y}_{n(m-1)+1} &= 0 \end{aligned} \tag{3.25}$$

where  $0 \leq i \leq m-1$ ,  $1 \leq j \leq n-2$  and  $\bar{x}_{ni+1}'$ ,  $\bar{y}_{ni+1}'$  are depicted in Fig. 9.

**Lemma 3.26.**

(i) For each  $0 \leq i \leq m-1$  and  $1 \leq j \leq k \leq n-1$  we have

$$\bar{y}_{ni+j} \bar{x}_{ni+j} \dots \bar{x}_{ni+k} = \begin{cases} 0 & \text{if } k = n-1 \\ (-1)^{k-j+1} \bar{x}_{ni+j+1} \dots \bar{x}_{ni+k+1} \bar{y}_{ni+k+1} & \text{otherwise.} \end{cases}$$

(ii) For each  $1 \leq k \leq n$  we have

$$\bar{y}_{n(m-1)} \bar{x}_{n(m-1)} \dots \bar{x}_{nm-k} = \begin{cases} 0 & \text{if } k = 1 \\ (-1)^{n-k+1} \bar{x}_{n(m-1)+1} \dots \bar{x}_{nm-k+1} \bar{y}_{nm-k+1} & \text{if } k > 1. \end{cases}$$

**Proof.** Let us prove the first assertion. By (3.25) the desired equation holds if  $j = k$ . The general case follows by induction on  $j$  since for  $j < k$  we have

$$\bar{y}_{ni+j} \bar{x}_{ni+j} \dots \bar{x}_{ni+k} = -\bar{x}_{ni+j+1} \bar{y}_{ni+j+1} \bar{x}_{ni+j+1} \dots \bar{x}_{ni+k}.$$

For the second assertion, by (3.25) we have

$$\bar{y}_{n(m-1)} \bar{x}_{n(m-1)} \dots \bar{x}_{nm-k} = -\bar{x}_{n(m-1)+1} \bar{y}_{n(m-1)+1} \bar{x}_{n(m-1)+1} \dots \bar{x}_{nm-k}.$$

Then the desired identity follows from (i).  $\square$

**Lemma 3.27.** For each  $0 \leq i \leq m-2$  we have

$$\begin{aligned}\bar{y}_{ni}\bar{x}_{ni\dots nm-3} &= -(-1)^{n+m-i}\bar{x}_{ni+1\dots nm-2}\bar{y}_{nm-2} \\ &\quad - \bar{x}_{ni+3\dots n(i+1)+1}^2\bar{x}_{n(i+1)+2\dots nm-1}\bar{y}_{nm-1}\bar{y}_{nm-2}.\end{aligned}$$

As a result, we have that  $\bar{y}_{ni}\bar{x}_{ni\dots nm-1} = 0$  and

$$\bar{y}_{ni}\bar{x}_{ni\dots nm-2} = (-1)^{n+m-i}\bar{x}_{ni+1\dots nm-1}\bar{y}_{nm-1}$$

**Proof.** By the third equality in (3.25) we obtain

$$\begin{aligned}\bar{y}_{ni}\bar{x}_{ni\dots nm-3} &= -\bar{x}_{ni+1}\bar{y}_{ni+1}\bar{x}_{ni+1\dots nm-3} - \bar{x}_{ni+1}^{1'}\bar{y}_{ni+1}^{1'}\bar{x}_{ni+1\dots nm-3} \\ &= -\bar{x}_{ni+1}^{1'}\bar{y}_{ni+1}^{1'}\bar{x}_{ni+1\dots nm-3} \\ &= (-1)^n\bar{x}_{ni+1}^{1'}\bar{x}_{ni+2\dots n(i+1)}^1\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots nm-3}\end{aligned}\tag{3.28}$$

where the second equality follows from Lemma 3.26 (i) and the third one uses the anticommutativity relations  $\bar{y}_{ni+j}^{1'}\bar{x}_{ni+j} = -\bar{x}_{ni+j+1}^1\bar{y}_{ni+j+1}^{1'}$  for  $1 \leq j \leq n-1$  (see Fig. 9). Here we denote  $\bar{y}_{n(i+1)}^{1'} = \bar{y}_{n(i+1)}$  when  $j = n-1$ . Since  $\bar{x}_{ni+1}^{1'}\bar{x}_{ni+2}^1$  lies in a square as in Fig. 5 (c), it satisfies the Plücker-type relation  $\bar{x}_{ni+1}^{1'}\bar{x}_{ni+2}^1 = -\bar{x}_{ni+1}\bar{x}_{ni+2}^{1'} - \bar{x}_{ni+3}^2\bar{y}_{ni+3}^{2'}$ . Substituting this into (3.28) we obtain

$$\begin{aligned}\bar{y}_{ni}\bar{x}_{ni\dots nm-3} &= (-1)^{n+1}\bar{x}_{ni+1}\bar{x}_{ni+2}^{1'}\bar{x}_{ni+3\dots n(i+1)}^1\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots nm-3} \\ &\quad + (-1)^{n+1}\bar{x}_{ni+3}^2\bar{y}_{ni+3}^{2'}\bar{x}_{ni+3\dots n(i+1)}^1\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots nm-3} \\ &= -\bar{x}_{ni+1\dots n(i+1)}\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots nm-3} \\ &\quad - \bar{x}_{ni+3\dots n(i+1)+1}^2\bar{y}_{n(i+1)+1}\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots nm-3}\end{aligned}\tag{3.29}$$

where the second equality uses the anticommutativity relations (see Fig. 9)

$$\bar{x}_{ni+j}^{1'}\bar{x}_{ni+j+1}^1 = -\bar{x}_{ni+j}\bar{x}_{ni+j+1}^{1'} \quad \text{and} \quad \bar{y}_{ni+j+1}^{2'}\bar{x}_{ni+j+1}^1 = -\bar{x}_{ni+j+2}^2\bar{y}_{ni+j+2}^{2'}$$

where  $2 \leq j \leq n-1$  and denote  $\bar{x}_{n(i+1)}^{1'} = \bar{x}_{n(i+1)}$  and  $\bar{y}_{n(i+1)+1}^{2'} = \bar{y}_{n(i+1)+1}$ .

Take  $i = m-2$  in (3.29) and apply the identity in Lemma 3.26 (ii) to obtain

$$\begin{aligned}\bar{y}_{n(m-2)}\bar{x}_{n(m-2)\dots nm-3} &= (-1)^{n+1}\bar{x}_{n(m-2)+1\dots nm-2}\bar{y}_{nm-2} \\ &\quad + (-1)^{n+1}\bar{x}_{n(m-2)+3\dots n(m-1)+1}^2\bar{y}_{n(m-1)+1}\bar{x}_{n(m-1)+1\dots nm-2}\bar{y}_{nm-2}.\end{aligned}$$

Then applying Lemma 3.26 (i) to the term  $\bar{y}_{n(m-1)+1}\bar{x}_{n(m-1)+1\dots nm-2}$  we obtain the desired identity.

For  $i < m - 2$  from (3.29) again we have

$$\begin{aligned}
 & \bar{y}_{ni}\bar{x}_{ni\dots nm-3} \\
 &= -(-1)^{n+m-i}\bar{x}_{ni+1\dots nm-2}\bar{y}_{nm-2} \\
 & \quad + \bar{x}_{ni+1\dots n(i+1)}\bar{x}_{n(i+1)+3\dots n(i+2)+1}^2\bar{x}_{n(i+2)+2\dots nm-1}\bar{y}_{nm-1}\bar{y}_{nm-2} \\
 & \quad - (-1)^{n+m-i}\bar{x}_{ni+3\dots n(i+1)+1}^2\bar{y}_{n(i+1)+1}\bar{x}_{n(i+1)+1\dots nm-2}\bar{y}_{nm-2} \\
 & \quad + \bar{x}_{ni+3\dots n(i+1)+1}^2\bar{y}_{n(i+1)+1}\bar{x}_{n(i+1)+3\dots n(i+2)+1}^2\bar{x}_{n(i+2)+2\dots nm-1}\bar{y}_{nm-1}\bar{y}_{nm-2} \\
 &= -(-1)^{n+m-i}\bar{x}_{ni+1\dots nm-2}\bar{y}_{nm-2} \\
 & \quad - \bar{x}_{ni+3\dots n(i+1)+1}^2\bar{x}_{n(i+1)+2\dots nm-1}\bar{y}_{nm-1}\bar{y}_{nm-2}
 \end{aligned}$$

where the first equality uses the induction (we assume it holds for  $i + 1$  and then prove it for  $i$ ). Let us explain the second equality. Using  $\bar{x}_{n(i+1)}\bar{x}_{n(i+1)+3}^2 = 0$  and Lemma 3.26 (i) we see that the second and third summands vanish. For the fourth summand, we are abusing notation since  $\bar{x}_{n(i+1)+3}^2$  denotes the arrow which has the same starting vertex as  $\bar{x}_{n(i+1)+1}$  (just similar to the arrows  $\bar{x}_{ni+3}^2$  and  $\bar{x}_{ni+1}$ ). For this summand, we use the following identity (see Fig. 9 where we shift  $i$  by  $i + 1$ )

$$\bar{y}_{n(i+1)+1}\bar{x}_{n(i+1)+3\dots n(i+2)+1}^2\bar{x}_{n(i+2)+2\dots nm-1} = -\bar{x}_{n(i+1)+2\dots nm-1} \quad (3.30)$$

which may be obtained by using the anticommutativity relations iteratively ( $2n-4$  times) and the Plücker-type relation  $\bar{x}_{n(i+1)+3}^2\bar{x}_{n(i+1)+4}^2 = \bar{x}_{n(i+1)+3}^1\bar{x}_{n(i+1)+4}^2 + \bar{x}'\bar{y}''$ . Note that the summand corresponding to the term  $\bar{x}'\bar{y}''$  vanishes since  $\bar{y}''\bar{x}_{n(i+1)+5\dots n(i+2)+1}^2\bar{x}_{n(i+2)+2\dots nm-1} = 0$  by the observation in Remark 3.22.  $\square$

**Lemma 3.31.** For each  $0 \leq i \leq m - 2$  we have (see Fig. 12)

$$\bar{y}_{ni}\bar{x}_{ni\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0 = \bar{x}_{ni+1\dots nm-2}\bar{y}_{nm-2}'.$$

As a result we have

$$\begin{aligned}
 & \bar{y}_{ni}\bar{x}_{ni\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0\bar{y}_{nm-3\dots n(m-1)-1}^0\bar{y}_{n(m-1)-2\dots n(m-2)} \\
 &= -(-1)^n\bar{x}_{ni+1\dots nm-2}\bar{y}_{nm-2\dots n(m-2)}.
 \end{aligned}$$

**Proof.** Let us first consider the case of  $i = m - 2$ . We have

$$\bar{y}_{n(m-2)}\bar{x}_{n(m-2)\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0$$

$$\begin{aligned}
&= -\bar{x}_{n(m-2)+1}\bar{y}_{n(m-2)+1}\bar{x}_{n(m-2)+1\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&\quad - \bar{x}_{n(m-2)+1}^{1'}\bar{y}_{n(m-2)+1}^{1'}\bar{x}_{n(m-2)+1\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&= (-1)^{n-1}\bar{x}_{n(m-2)+1\dots n(m-1)-1}\bar{y}_{n(m-1)-1}\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&\quad + (-1)^{n-1}\bar{x}_{n(m-2)+1}^{1'}\bar{x}_{n(m-2)+2\dots n(m-1)-1}^1\bar{y}_{n(m-1)-1}^0\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&= \bar{x}_{n(m-2)+1}\bar{x}_{n(m-2)+2\dots nm-2}\bar{y}_{nm-2}'
\end{aligned}$$

where the first equality follows from (3.25) and the second one from Lemma 3.26 (i) and  $\bar{y}_{n(m-2)+j}^{1'}\bar{x}_{n(m-2)+j} = -\bar{x}_{n(m-2)+j+1}^{1'}\bar{y}_{n(m-2)+j+1}^{1'}$  for  $1 \leq j \leq n-2$  (denote  $\bar{y}_{n(m-1)-1}^{1'} = \bar{y}_{n(m-1)-1}^0$  for  $j = n-2$ ), and the third one from the anticommutativity relations involving  $\bar{y}_{n(m-1)-j}\bar{x}_{n(m-1)-j}^0$  ( $n-3$  times) and the following identity

$$\bar{y}_{n(m-1)-1}^0\bar{x}_{n(m-1)-1\dots nm-3}^0 = (-1)^{n-2}\bar{x}_{n(m-1)\dots nm-3}^0\bar{y}_{nm-3}^0\bar{x}_{nm-3}^0 = 0$$

which uses  $\bar{y}_{n(m-1)+j-1}^0\bar{x}_{n(m-1)+j-1}^0 = -\bar{x}_{n(m-1)+j}^0\bar{y}_{n(m-1)+j}^0$  for  $0 \leq j \leq n-3$  and  $\bar{y}_{nm-3}^0\bar{x}_{nm-3}^0 = 0$ .

For  $i < m-2$  we may proceed by induction since similar to (3.29) we have

$$\begin{aligned}
&\bar{y}_{ni}\bar{x}_{ni\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&= -\bar{x}_{ni+1\dots n(i+1)}\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0 \\
&\quad - \bar{x}_{ni+3\dots n(i+1)+1}^2\bar{y}_{n(i+1)+1}\bar{y}_{n(i+1)}\bar{x}_{n(i+1)\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0
\end{aligned}$$

where we also need to use Lemma 3.26 (i) to show that the second summand vanishes. This proves the first desired identity.

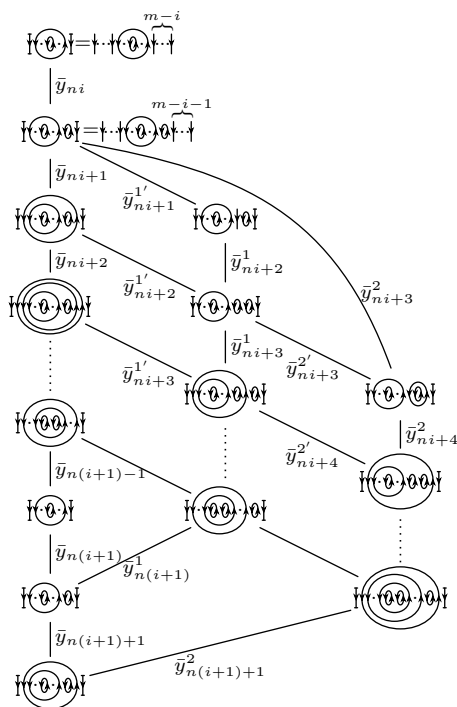
To prove the second desired identity, it suffices to prove the following one

$$\bar{y}_{nm-2\dots n(m-1)-1}\bar{y}_{n(m-1)-2\dots n(m-2)} = (-1)^{n-1}\bar{y}_{nm-2\dots n(m-2)}.$$

For this, by the Plücker-type relation  $\bar{y}_{nm-2}'\bar{y}_{nm-3}^0 = -\bar{y}_{nm-2}\bar{y}_{nm-3}' - \bar{x}_{nm-1}\bar{y}'''$ , where  $\bar{y}'''$  is the missing arrow connecting the bottom vertex with the ending vertex of  $\bar{y}_{nm-3}^0$  in Fig. 12, it follows that

$$\begin{aligned}
&\bar{y}_{nm-2}'\bar{y}_{nm-3\dots n(m-1)-1}^0\bar{y}_{n(m-1)-2\dots n(m-2)} \\
&= -\bar{y}_{nm-2}\bar{y}_{nm-3}'\bar{y}_{nm-4\dots n(m-1)-1}^0\bar{y}_{n(m-1)-2\dots n(m-2)} \\
&\quad - \bar{x}_{nm-1}\bar{y}'''\bar{y}_{nm-4\dots n(m-1)-1}^0\bar{y}_{n(m-1)-2\dots n(m-2)} \\
&= (-1)^{n-1}\bar{y}_{nm-2\dots n(m-2)}
\end{aligned}$$

where the second equality repeatedly uses the anticommutativity relations involving  $\bar{y}_{nm-i}'\bar{y}_{nm-i-1}^0$  for  $3 \leq i \leq n$ . Note that  $\bar{y}'''\bar{y}_{nm-4\dots n(m-1)-1}^0\bar{y}_{n(m-1)-2}$  is in  $\bar{S}_m^n$  and using Lemma 3.6 iteratively ( $n-2$  times for  $k = n, n-1, \dots, 3$ ) we obtain

Fig. 9.  $n-1 \leq i \leq m-2$ .

$$\begin{aligned} \bar{x}_{nm-1} \bar{y}''' \bar{y}_{nm-4 \dots n(m-1)-1}^{0-} \bar{y}_{n(m-1)-2 \dots n(m-2)} \\ = (-1)^{\frac{(n-2)(n+1)}{2}} \bar{x}_{nm-1} \bar{y}_{nm-1} \bar{y}'_{nm-2} \cdots \bar{y}_{n(m-2)+3}^2 \bar{y}_{n(m-2)} = 0 \end{aligned}$$

where we do not need the explicit arrows appearing in  $\cdots$  since the second equality already follows from  $\bar{y}_{n(m-2)+3}^2 \bar{y}_{n(m-2)} = 0$ .  $\square$

#### 4. Hochschild cohomology and deformations for Koszul-dual algebras

Let  $A$  be a graded algebra. The Hochschild cochain complex  $C^\bullet(A, A)$  is the product total complex of the double complex

$$A \xrightarrow{\delta^0} \text{Hom}(A, A) \rightarrow \cdots \rightarrow \text{Hom}(A^{\otimes p}, A) \xrightarrow{\delta^p} \text{Hom}(A^{\otimes p+1}, A) \rightarrow \cdots \quad (4.1)$$

where the horizontal differential  $\delta^p$  is the Hochschild differential and each term  $\text{Hom}(A^{\otimes i}, A) = \bigoplus_j \text{Hom}(A^{\otimes i}, A)^j$  is viewed as a graded vector space with trivial (vertical) differential, where  $\text{Hom}(A^{\otimes i}, A)^j$  is the set of graded  $\mathbb{k}$ -linear maps of degree  $j$  from  $A^{\otimes i}$  to  $A$  so that we have  $C^k(A, A) = \prod_i \text{Hom}(A^{\otimes i}, A)^{k-i}$ .

#### 4.1. Bigraded Hochschild cohomology and Koszul duality

Let  $A = \mathbb{k}Q/I$  be a Koszul algebra and  $A^! = \mathbb{k}\overline{Q}/(I_2^\perp)$  be its Koszul dual. In this subsection we recall from Keller [27] the isomorphism between the (bigraded) Hochschild cohomology of  $A$  and  $A^!$ .

##### 4.1.1. Bigraded vector spaces

Consider bigraded vector spaces  $V = \bigoplus_{p,q \in \mathbb{Z}} V_q^p$  where the superscript  $p$  denotes the (cohomological) *differential grading* and the subscript  $q$  denotes the *Adams grading* which is viewed as an additional grading. Let  $(V, d)$  be a *differential bigraded vector space*, where  $d: V \rightarrow V$  is a differential of bidegree  $(1, 0)$  giving for each fixed  $q \in \mathbb{Z}$  a cochain complex  $V_q^\bullet$

$$\cdots \rightarrow V_q^p \xrightarrow{d} V_q^{p+1} \xrightarrow{d} V_q^{p+2} \rightarrow \cdots$$

For any two differential bigraded vector spaces  $(U, d_U), (V, d_V)$ , consider the differential bigraded Hom-space  $\text{Hom}(U, V) = \bigoplus_{p,q \in \mathbb{Z}} \text{Hom}(U, V)_q^p$  where

$$\text{Hom}(U, V)_q^p := \prod_{i,j \in \mathbb{Z}} \text{Hom}(U_j^i, V_{q+j}^{p+i})$$

with differential  $\delta(f) = d_V \circ f - (-1)^p f \circ d_U$  for any  $f \in \text{Hom}(U, V)_q^p$ . We may also consider the bigraded tensor product  $U \otimes V = \bigoplus_{p,q \in \mathbb{Z}} (U \otimes V)_q^p$  where

$$(U \otimes V)_q^p := \bigoplus_{i,j \in \mathbb{Z}} U_j^i \otimes V_{q-j}^{p-i}$$

with differential  $d(u \otimes v) = d_U(u) \otimes v + (-1)^k u \otimes d_V(v)$  for any  $u \otimes v \in U_l^k \otimes V_n^m$ .

Let  $A = \bigoplus_{k \in \mathbb{Z}} A^k$  be a graded algebra with additional Adams grading so that  $A^k = \bigoplus_{l \in \mathbb{Z}} A_l^k$ . We denote the Hochschild cochain complex with this additional grading by  $C_\bullet^*(A, A)$  where  $C_q^p(A, A) = \prod_{i \geq 0} \text{Hom}(A^{\otimes i}, A)_{q-i}^{p-i}$  with the induced differential. For each  $q$  let  $\text{HH}_q^\bullet(A, A)$  denote the cohomology of the complex  $C_q^\bullet(A, A)$ . The  $B_\infty$  structure on  $C^\bullet(A, A)$  can be restricted to the subspace

$$\bigoplus_{q \in \mathbb{Z}} C_q^\bullet(A, A) \subset C^\bullet(A, A)$$

but the inclusion is not necessarily a (quasi-)isomorphism of complexes, even if  $A$  is finite-dimensional (see Remark 4.3 below).

##### 4.1.2. Hochschild cohomology under Koszul duality

Let  $A = \mathbb{k}Q/I$  be a Koszul algebra. We view  $A$  as bigraded by assigning any arrow  $a \in Q$  the bidegree  $(0, 1)$ . Recall that the Koszul dual  $A^!$  is isomorphic to  $\mathbb{k}\overline{Q}/(I_2^\perp)$ . It also admits a bigrading by assigning any arrow  $\bar{a} \in \overline{Q}$  the bidegree  $(1, -1)$ .

**Theorem 4.2** ([27, §3.5]). *There is an isomorphism in the homotopy category of (bi-graded)  $B_\infty$  algebras*

$$C_\bullet^*(A, A) \longrightarrow C_\bullet^*(A^!, A^!).$$

*In particular, we have  $\mathrm{HH}_q^p(A^!, A^!) \simeq \mathrm{HH}_q^p(A, A)$  for all  $p, q$ .*

**Remark 4.3.** The additional Adams grading is crucial in Theorem 4.2, as otherwise there may be some issues with completions as discussed in [28, 35]. For instance, let  $A = \mathbb{k}[x]$  be the polynomial algebra. Its Koszul dual  $A^! = \mathbb{k}[y]/(y^2)$  is the 2-dimensional algebra of dual numbers. Take the differential gradings  $|x| = 0$  and  $|y| = 1$ . Then  $\mathrm{HH}^0(A, A) \simeq \mathbb{k}[z]$  whereas  $\mathrm{HH}^0(A^!, A^!) \simeq \mathbb{k}[[z]]$  which are not isomorphic (see [28, §1]). However, the Adams grading gives isomorphisms  $\mathrm{HH}_q^0(A, A) \simeq \mathbb{k}z^q \simeq \mathrm{HH}_q^0(A^!, A^!)$  for each  $q \geq 0$ . In fact, we have

$$\mathrm{HH}^0(A, A) = \bigoplus_{q \in \mathbb{Z}} \mathrm{HH}_q^0(A, A) \quad \text{and} \quad \mathrm{HH}^0(A^!, A^!) \simeq \prod_{q \in \mathbb{Z}} \mathrm{HH}_q^0(A^!, A^!).$$

**Remark 4.4.** As mentioned in [27] the isomorphism between Hochschild cohomology of  $A$  and  $A^!$  as graded algebras was announced by Buchweitz [13]. See [15] for the isomorphism preserving the Batalin–Vilkovisky structures for a Koszul Calabi–Yau algebra  $A$ .

#### 4.2. Associative and $A_\infty$ deformations under Koszul duality

For a graded algebra we may consider the following two different types of deformations.

**Definition 4.5.** Let  $A = (\bigoplus_{k \geq 0} A^k, \mu)$  be a (non-negatively) graded algebra.

(i) A *filtered deformation* of  $A$  is an associative algebra  $(A, \tilde{\mu})$  such that

$$\begin{aligned} \tilde{\mu}(a \otimes b) - \mu(a \otimes b) &= 0 && \text{if } a \in A^0 \text{ or } b \in A^0 \\ \tilde{\mu}(a \otimes b) - \mu(a \otimes b) &\in A^{>(|a|+|b|)} && \text{for homogeneous } a, b \in A^{>0}. \end{aligned}$$

(ii) An  $A_\infty$  *deformation* of  $A$  is an  $A_\infty$  algebra  $(A, m_1, m_2, m_3, \dots)$  such that  $m_1 = 0$  and  $m_2 = \mu$ .

**Remark 4.6.** The two types of deformations in Definition 4.5 are inherently very different: a filtered deformation is strictly associative, whereas an  $A_\infty$  deformation is generally only associative up to higher homotopies. However, from the point of view of bigraded algebras, these two types of deformations naturally appear as two sides of the same coin, the two sides being related by Koszul duality. For a graded algebra  $A = (\bigoplus_{k \geq 0} A^k, \mu)$ , we may equip  $A$  with two different natural bigradings. The first bigrading assigns each element  $a \in A^k$  the bidegree  $(0, k)$ . Then the cocycles of  $\mathrm{HH}_q^2(A, A)$  lie in

$$C_q^2(A, A) = \prod_{i \geq 0} \operatorname{Hom}(A^{\otimes i}, A)_q^{2-i} = \operatorname{Hom}(A^{\otimes 2}, A)_q^0$$

where  $\operatorname{Hom}(A^{\otimes i}, A)_q^{2-i} = 0$  for  $i \neq 2$  (since  $A$  is trivially graded with respect to the differential grading whence the same is true for  $\operatorname{Hom}(A^{\otimes i}, A)$ ). Therefore, if  $A^0$  is semisimple then  $\operatorname{HH}_q^2(A, A)$  corresponds to (equivalence classes of) first-order filtered deformations  $(A[t]/(t^2), \tilde{\mu})$  of  $A$  such that  $\tilde{\mu}(a \otimes b) - \mu(a \otimes b) \in A^{|a|+|b|+q}t$  for any homogeneous elements  $a, b \in A^{>0}$ .

The second bigrading assigns each element  $a \in A^k$  the bidegree  $(k, -k)$ . The cocycles of  $\operatorname{HH}_q^2(A, A)$  then lie in

$$C_q^2(A, A) = \prod_{i \geq 0} \operatorname{Hom}(A^{\otimes i}, A)_q^{2-i} = \operatorname{Hom}(A^{\otimes q+2}, A)_q^{-q}$$

where  $\operatorname{Hom}(A^{\otimes i}, A)_q^{2-i} = 0$  if  $i \neq q+2$  (since the total degree of  $A$  is zero whence the same is true for  $\operatorname{Hom}(A^{\otimes i}, A)$ ). Therefore,  $\operatorname{HH}_q^2(A, A)$  corresponds to first-order  $A_\infty$  deformations of  $A$  with the only nontrivial higher product being  $m_{q+2}$ . Moreover, it is known from [24, Cor. 4] and [41, Thm. 4.7] that if  $\operatorname{HH}_q^2(A, A) = 0^2$  for each  $q > 0$  then  $A$  is *intrinsically formal*, i.e. any  $A_\infty$  deformation of  $A$  is  $A_\infty$ -quasi-isomorphic to the underlying graded algebra  $A$ .

Let  $A = \mathbb{k}Q/I$  be an algebra such that  $I \subset \mathbb{k}Q_{\geq 2}$ . By [23] the graded space  $\operatorname{Ext}_A^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$  admits a natural minimal  $A_\infty$  algebra structure, which is the *minimal model* of the derived endomorphism algebra of  $\mathbb{k}Q_0$ . Moreover, the minimal model is formal (i.e. quasi-isomorphic to the underlying graded associative algebra which is  $\operatorname{Ext}_A^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$  with the cup product) if and only if  $A$  is Koszul (see [26, §2.2] and [19, Cor. V.0.6]).

Now let  $A$  be Koszul and let  $A^\dagger = \mathbb{k}\overline{Q}/(I_2^\perp)$  be its Koszul dual. As before, assign any arrow  $a \in Q$  the bidegree  $(0, 1)$  and any arrow  $\bar{a} \in \overline{Q}$  the bidegree  $(1, -1)$ . Then by Remark 4.6 we may consider filtered deformations of  $A$  and  $A_\infty$  deformations of  $A^\dagger$ .

Theorem 4.2 and Remark 4.6 motivate the following result.

**Proposition 4.7.** *Let  $A = \mathbb{k}Q/I$  be a Koszul algebra and let  $A^\dagger = \mathbb{k}\overline{Q}/(I_2^\perp)$  be its Koszul dual. View  $A$  as an ungraded algebra and  $A^\dagger$  as a graded algebra. Then for any nontrivial filtered deformation  $B = (A, \tilde{\mu})$  of  $A$ , the minimal model of the derived endomorphism algebra of  $\mathbb{k}Q_0$  is a nontrivial  $A_\infty$  deformation of  $A^\dagger$ .*

**Proof.** Let us first show that the  $A_\infty$  structure on  $\operatorname{Ext}_B^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$ , obtained as the minimal model of the derived endomorphism algebra of  $\mathbb{k}Q_0$ , is indeed an  $A_\infty$  deformation of  $A^\dagger$ , i.e. the underlying graded algebra is isomorphic to  $A^\dagger$ .

<sup>2</sup> This is denoted by  $\operatorname{HH}^{2+q, -q}(A, A)$  in [24] and [41].

Recall the  $A$ - $A$ -bimodule Koszul resolution  $K_\bullet(A)$  of  $A$  where  $K_0(A) = A \otimes A$ ,  $K_{-1}(A) = A \otimes \mathbb{k}Q_1 \otimes A$  and

$$K_{-n}(A) = A \otimes \left( \bigcap_{i=0}^n \mathbb{k}Q_i \otimes I_2 \otimes \mathbb{k}Q_{n-2-i} \right) \otimes A \quad \text{for } n \geq 2$$

where we write  $\otimes = \otimes_{\mathbb{k}Q_0}$ . Its differential is given by

$$d(1 \otimes \sum_i (a_1^i \otimes \cdots \otimes a_n^i) \otimes 1) = \sum_i a_1^i \otimes a_2^i \otimes \cdots \otimes a_n^i \otimes 1 - \sum_i 1 \otimes a_1^i \otimes \cdots \otimes a_{n-1}^i \otimes a_n^i$$

for any  $\sum_i a_1^i \otimes \cdots \otimes a_n^i \in \bigcap_{k=0}^n \mathbb{k}Q_k \otimes I_2 \otimes \mathbb{k}Q_{n-2-k}$  with  $a_1^i, \dots, a_n^i \in Q_1$ . Since  $K_\bullet(A)$  is exact in negative degrees we may choose a homotopy  $\rho: K_{-n}(A) \rightarrow K_{-n-1}(A)$  so that  $\rho$  is a left  $A$ -module morphism which preserves the path length. We may also assume that  $\rho(a \otimes \sum_i (a_1^i \otimes \cdots \otimes a_n^i) \otimes b) = 0$  if  $b = 1$ .

Let  $\text{Bar}_\bullet(A)$  denote the normalised bar resolution of  $A$ , i.e.  $\text{Bar}_{-n}(A) = A \otimes \bar{A}^{\otimes n} \otimes A$  where  $\bar{A} = A/(\mathbb{k}Q_0)$  is the quotient  $\mathbb{k}Q_0$ - $\mathbb{k}Q_0$ -bimodule. Using the same construction as in [4, §5.1] we obtain a deformation retract of  $A$ - $A$ -bimodules

$$K_\bullet(A) \xrightleftharpoons[G_\bullet]{F_\bullet} \text{Bar}_\bullet(A) \xhookrightarrow{h_\bullet} \quad (4.8)$$

namely  $G_n F_n = \text{id}$  and  $F_n G_n - \text{id} = h_{n-1} d_n + d_{n+1} h_n$ . In particular,  $F_\bullet$  is the natural embedding

$$F_n(1 \otimes \sum_i (a_1^i \otimes \cdots \otimes a_n^i) \otimes 1) = 1 \otimes \sum_i (a_1^i \otimes \cdots \otimes a_n^i) \otimes 1.$$

Since  $\rho$  preserves the path length so do the maps  $G_\bullet$  and  $h_\bullet$  by construction. Applying  $\text{Hom}_{A^{\text{op}}}(\mathbb{k}Q_0 \otimes_A -, \mathbb{k}Q_0)$  to (4.8) we obtain a new deformation retract

$$(A^!, 0) \xrightleftharpoons[G^\bullet]{F^\bullet} (\text{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0), \partial) \xhookrightarrow{h^\bullet} \quad (4.9)$$

Here we use the natural isomorphisms  $\text{Hom}_{A^{\text{op}}}(\mathbb{k}Q_0 \otimes_A K_\bullet(A), \mathbb{k}Q_0) \simeq A^!$  and  $\text{Hom}_{A^{\text{op}}}(\mathbb{k}Q_0 \otimes_A \text{Bar}_n(A), \mathbb{k}Q_0) \simeq \text{Hom}(\bar{A}^{\otimes n}, \mathbb{k}Q_0)$ . Note that  $\text{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0)$  is a DG algebra whose cohomology is isomorphic to the graded algebra  $A^!$ , see [17, §7.1].

Let  $B = (A, \tilde{\mu})$  be a filtered deformation of  $A$ . Clearly, the DG algebra  $\text{Hom}(\bar{B}^{\otimes \bullet}, \mathbb{k}Q_0)$ , which computes  $\text{Ext}_B^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$ , has the same underlying graded space as  $\text{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0)$ . The differential  $\tilde{\partial}$  in the former may be viewed as a perturbed differential by the perturbation  $\delta = \tilde{\partial} - \partial$ . Note that  $\text{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0)$  has an additional grading (filtration) induced by the path length, i.e. the dual of a path  $p$  is of degree  $-|p|$ , so that  $\text{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0)$  is in negative degrees. Note that  $\delta$  strictly increases this grading and it follows that for any  $f \in \text{Hom}(\bar{A}^{\otimes n}, \mathbb{k}Q_0)$  we have  $(\delta h^\bullet)^i(f) = 0$  for  $i > n$ , i.e.  $\delta$  is (locally) small. Applying

the homological perturbation lemma (see e.g. [22, §1]) to  $\delta$  we have that the homotopy deformation retract (4.10) induces a new one

$$(A^!, d_\infty) \xrightleftharpoons[G_\infty^\bullet]{F_\infty^\bullet} (\mathrm{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0), \tilde{\partial}) \hookrightarrow h_\infty^\bullet \quad (4.10)$$

where

$$d_\infty = \sum_{n \geq 0} F^\bullet(\delta h^\bullet)^n \delta G^\bullet, \quad F_\infty^\bullet = \sum_{n \geq 0} F^\bullet(\delta h^\bullet)^n, \quad G_\infty^\bullet = G^\bullet + \sum_{n \geq 0} h^\bullet(\delta h^\bullet)^n \delta G^\bullet.$$

Since  $\delta$  is locally small the above sums are well-defined. We claim that  $\delta G^\bullet = 0$ . Indeed, for any  $f \in \mathrm{Hom}_{A^{\mathrm{op}}}(\mathbb{k}Q_0 \otimes_A K_n(A), \mathbb{k}Q_0)$  we have

$$\delta G^\bullet(f)(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{n+1}) = \sum_{i=1}^n (-1)^i f G_\bullet(1 \otimes \bar{a}_1 \otimes \cdots \otimes \tilde{a}_{i,i+1} \otimes \cdots \otimes \bar{a}_{n+1} \otimes 1)$$

where  $\bar{a}_j \in \bar{A}$  for  $1 \leq j \leq n+1$  and  $\tilde{a}_{i,i+1} := \tilde{\mu}(a_i \otimes a_{i+1}) - a_i a_{i+1}$ . Since the path length of  $\tilde{a}_{i,i+1}$  is strictly bigger than 2, the one of  $p := a_1 \otimes \cdots \otimes \tilde{a}_{i,i+1} \otimes \cdots \otimes a_{n+1}$  is bigger than  $n+1$  so is the one of  $G_\bullet(p)$ . It follows that the right-hand side of the equality vanishes since the path length of elements in  $\mathbb{k}Q_0 \otimes_A K_n(A)$  is  $n$ . This proves the claim and shows that  $d_\infty = 0$  and  $G_\infty^\bullet = G^\bullet$ .

Applying the homotopy transfer theorem we obtain an  $A_\infty$  algebra  $(A^!, m_2, m_3, \dots)$  which is the minimal model  $\mathrm{Ext}_B^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$ . Using the path length as above, we may show that  $m_2 = F_\infty^\bullet \circ m'_2 \circ (G^\bullet \otimes G^\bullet) = F^\bullet \circ m'_2 \circ (G^\bullet \otimes G^\bullet)$  and the latter coincides with the underlying product of  $A^!$ , since the product  $m'_2$  in  $\mathrm{Hom}(\bar{A}^{\otimes \bullet}, \mathbb{k}Q_0)$  does not depend on the product of  $A$ , see [17, (7.1)].

If  $\mathrm{Ext}_B^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$  is  $A_\infty$ -quasi-isomorphic to the underlying graded algebra  $A^!$ , then  $B$  is Koszul so that  $B^! \simeq A^!$ . Thus, we have  $B \simeq B^! \simeq A^! \simeq A$ , i.e.  $B$  is a trivial associative deformation of  $A$ .  $\square$

**Remark 4.11.** Proposition 4.7 should be well-known to experts as it may be viewed as a deformation-theoretical interpretation of Theorem 4.2. Note that Braverman and Gaitsgory [8] give a deformation-theoretical construction of PBW deformations of quadratic Koszul algebras, which are filtered deformations in the “opposite” sense, using shorter instead of longer paths, and Fløystad and Vatne [20, Thm. 2.1] give a correspondence between PBW deformations of an  $N$ -Koszul algebra and certain  $A_\infty$  deformations of its Koszul dual.

#### 4.3. Associative deformations via reduction systems

By the results of §4.2,  $A_\infty$  deformations of a Koszul algebra correspond to (filtered) associative deformations of its Koszul dual. In [4] we show how to describe the latter

concretely as deformations of a reduction system (cf. §3.2). The main result of [4] is the following.

**Theorem 4.12** ([4, Thm. 7.1]). *Let  $Q$  be any finite quiver, let  $I \subset \mathbb{k}Q$  be any two-sided ideal and let  $R$  be any reduction system satisfying the diamond condition for  $I$ . Then there is an equivalence of deformation problems between*

- (i) *deformations of the associative algebra  $A = \mathbb{k}Q/I$*
- (ii) *deformations of the ideal  $I$*
- (iii) *deformations of the reduction system  $R$ .*

This result can be obtained by “replacing” the bar resolution of  $A$ , used in the classical formulation of associative deformations in terms of the Hochschild cochain complex equipped with the Gerstenhaber bracket [21], by the Bardzell–Chouhy–Solotar resolution [3, 18] associated to a reduction system and studying the deformation theory on the resulting smaller cochain complex. Theorem 4.12 gives a practicable description of the full formal deformation theory of  $A = \mathbb{k}Q/I$ . A special case, corresponding to the case of first-order deformations, is the following result.

**Corollary 4.13** ([4, Cor. 7.44]). *Let  $A = \mathbb{k}Q/I$  and  $R$  be any reduction system satisfying the diamond condition for  $I$ . Then  $\mathrm{HH}^2(A, A)$  is isomorphic to the space of first-order deformations of  $R$  modulo equivalence.*

Here a *first-order deformation* (over  $\mathbb{k}[t]/(t^2)$ ) of a reduction system  $R = \{(s, \varphi_s)\}_{s \in S}$  is given by

$$\tilde{R} = \{(s, \varphi_s + \tilde{\varphi}_s t)\}_{s \in S}$$

where  $\tilde{\varphi}_s \in \mathbb{k}\mathrm{Irr}_S$  such that  $\tilde{R}$  is reduction-unique when viewed as a reduction system for  $\mathbb{k}Q \otimes_{\mathbb{k}} \mathbb{k}[t]/(t^2)$ . Here,  $\tilde{\varphi}_s$  can be viewed as the image of  $s$  under a map  $\tilde{\varphi} \in \mathrm{Hom}_{\mathbb{k}Q_0^c}(\mathbb{k}S, \mathbb{k}\mathrm{Irr}_S)$  sending  $s \mapsto \tilde{\varphi}_s$ . Note that  $\mathrm{Hom}_{\mathbb{k}Q_0^c}(\mathbb{k}S, \mathbb{k}\mathrm{Irr}_S)$  is the space of 2-cochains in the cochain complex associated to the Bardzell–Chouhy–Solotar resolution associated to  $R$  [18] and  $\tilde{\varphi}$  defines a first-order deformation of  $R$  if and only if it is a 2-cocycle in this complex [4, §7.A].

Two first-order deformations  $\tilde{R}$  and  $\tilde{R}' = \{(s, \varphi_s + \tilde{\varphi}'_s t)\}_{s \in S}$  of  $R$  are called *equivalent* if there exists  $\psi \in \mathrm{Hom}_{\mathbb{k}Q_0^c}(\mathbb{k}Q_1, \mathbb{k}\mathrm{Irr}_S)$  such that  $\delta(\psi)_s = \tilde{\varphi}_s - \tilde{\varphi}'_s$  for any  $s = s_1 \cdots s_m \in S$  with  $s_i \in Q_1$ . Here  $\delta$  is the map

$$\delta: \mathrm{Hom}_{\mathbb{k}Q_0^c}(\mathbb{k}Q_1, \mathbb{k}\mathrm{Irr}_S) \longrightarrow \mathrm{Hom}_{\mathbb{k}Q_0^c}(\mathbb{k}S, \mathbb{k}\mathrm{Irr}_S) \quad (4.14)$$

defined by  $\delta(\psi)(s) = T(\psi)(s - \varphi_s)$  and  $T(\psi): \mathbb{k}Q \rightarrow \mathbb{k}\mathrm{Irr}_S$  is the  $\mathbb{k}$ -linear map

$$T(\psi)(a_1 a_2 \cdots a_m) = \sum_{i=1}^m \sigma \pi(a_1 \cdots a_{i-1} \psi(a_i) a_{i+1} \cdots a_m)$$

where  $a_1, \dots, a_m$  are arrows in  $Q$ ,  $\pi: \mathbb{k}Q \rightarrow \mathbb{k}Q/I$  is the natural projection and  $\sigma: \mathbb{k}Q/I \xrightarrow{\sim} \mathbb{k}\text{Irr}_S$  is the inverse of the restriction  $\pi|_{\mathbb{k}\text{Irr}_S}$ . In other words, for any path  $p$  in  $Q$ ,  $\sigma\pi(p)$  may be obtained by performing reductions (with respect to  $R$ ) on  $p$  until all elements are irreducible. Here  $\delta$  coincides with the differential of the cochain complex associated to the Bardzell–Chouhy–Solotar resolution and two first-order deformations of  $R$  are equivalent if and only if the corresponding 2-cocycles  $\tilde{\varphi}$  and  $\tilde{\varphi}'$  are cohomologous [4, §7.A].

Corollary 4.13 gives a straightforward method to compute  $\text{HH}^2(A, A)$  and in §5 we will apply this to the algebra  $\bar{K}_m^n = \mathbb{k}\bar{Q}_m^n/\bar{I}_m^n$  with the reduction system  $\bar{R}_m^n$ .

## 5. $A_\infty$ deformations and Stroppel's conjecture

We now apply the general theory in §4 to the extended Khovanov arc algebras, viewing  $K_m^n \simeq \mathbb{k}Q_m^n/I_m^n$  and its Koszul dual  $\bar{K}_m^n \simeq \mathbb{k}\bar{Q}_m^n/\bar{I}_m^n$  as bigraded algebras by assigning any arrow  $a \in Q_m^n$  the bidegree  $(1, -1)$  and any arrow  $\bar{a} \in \bar{Q}_m^n$  the bidegree  $(0, 1)$ .<sup>3</sup> Theorem 4.2 gives for each  $q \in \mathbb{Z}$  an isomorphism

$$\text{HH}_q^\bullet(K_m^n, K_m^n) \simeq \text{HH}_q^\bullet(\bar{K}_m^n, \bar{K}_m^n) \quad (5.1)$$

whence Stroppel's Conjecture 1.1 is equivalent to

$$\text{HH}_{i-2}^2(\bar{K}_m^n, \bar{K}_m^n) = 0 \quad \text{if } i \neq 0.$$

By §4.3 these cohomology groups may be computed as certain equivalence classes of first-order deformations of the reduction system  $\bar{R}_m^n$ .

The following vanishing result holds for degree reasons.

**Lemma 5.2.** *If either  $i$  is odd,  $i < 0$  or  $i > 2mn - 2$  then*

$$\text{HH}_{i-2}^2(K_m^n, K_m^n) \simeq \text{HH}_{i-2}^2(\bar{K}_m^n, \bar{K}_m^n) = 0.$$

**Proof.** Using the Koszul resolution of  $\bar{K}_m^n$  (see the proof of Proposition 4.7), the 2-cocycles of  $\text{HH}_{i-2}^2(\bar{K}_m^n, \bar{K}_m^n)$  lie in  $\text{Hom}_{\mathbb{k}Q_0^n}((\bar{I}_m^n)_2, \bar{K}_m^n)_{i-2}^0 = \text{Hom}_{\mathbb{k}Q_0^n}((\bar{I}_m^n)_2, (\bar{K}_m^n)_i)$  (see Remark 4.6). By Proposition 2.7 parallel paths have the same parity, so if  $i$  is odd then  $\text{Hom}_{\mathbb{k}Q_0^n}((\bar{I}_m^n)_2, (\bar{K}_m^n)_i) = 0$  whence  $\text{HH}_{i-2}^2(\bar{K}_m^n, \bar{K}_m^n) = 0$ .

Since  $(\bar{K}_m^n)_i = 0$  if  $i < 0$  or  $i > 2mn$  by Remark 3.22, we obtain  $\text{HH}_{i-2}^2(\bar{K}_m^n, \bar{K}_m^n) = 0$  in this case. Let us consider  $i = 2mn$ . Note that the longest irreducible path is of

<sup>3</sup> In the notation of §4, we are considering filtered associative deformations of  $A = \bar{K}_m^n$  (so that  $Q = \bar{Q}_m^n$ ) and  $A_\infty$  deformations of  $A^! = K_m^n$ .

length  $2mn$  and parallel to the vertex  $e_\lambda$  where  $\lambda$  is the highest weight and elements in  $(\bar{I}_m^n)_2$  are not parallel to  $e_\lambda$ , i.e. there are no quadratic relations at  $e_\lambda$ , so that  $\text{Hom}_{\mathbb{k}Q_0^s}((\bar{I}_m^n)_2, (\bar{K}_m^n)_{2mn}) = 0$  whence  $\text{HH}_{2mn-2}^2(\bar{K}_m^n, \bar{K}_m^n) = 0$ .  $\square$

The next vanishing result holds by direct computation.

**Lemma 5.3.**  $\text{HH}_{2mn-4}^2(K_m^n, K_m^n) \simeq \text{HH}_{2mn-4}^2(\bar{K}_m^n, \bar{K}_m^n) = 0$ .

**Proof.** We apply Corollary 4.13. Note that an element  $\tilde{\varphi}$  in  $\text{Hom}_{\mathbb{k}Q_0^s}(\mathbb{k}\bar{S}_m^n, \bar{K}_m^n)_{2mn-2}$  has the following general form

$$\tilde{\varphi}_{\bar{y}_0\bar{x}_0} = \alpha\bar{x}_{1\dots mn-1}\bar{y}_{mn-1\dots 1}$$

for  $\alpha \in \mathbb{k}$  and  $\tilde{\varphi}_s = 0$  for all other  $s \in \bar{S}_m^n$ . Denote  $\tilde{R} = \{(s, \varphi_s + \tilde{\varphi}_s t)\}_{s \in \bar{S}_m^n}$ . Observe that the irreducible paths parallel to each overlap are of length smaller than  $2mn - 2$ , so all overlaps have to reduce to zero. It follows that all overlaps are resolvable in  $\tilde{R}$ , i.e.  $\tilde{R}$  is a first-order deformation of  $\bar{R}_m^n$ .

We claim that  $\tilde{R}$  is equivalent to the trivial deformation. Indeed, consider  $\psi \in \text{Hom}_{\mathbb{k}Q_0^s}(\mathbb{k}(\bar{Q}_m^n)_1, \bar{K}_m^n)_{2mn-2}$  given by

$$\psi_{\bar{x}_0} = (-1)^{n-m}\alpha\bar{x}_{0\dots mn-2}\bar{y}_{mn-2\dots 1}$$

and  $\psi_{\bar{x}} = 0 = \psi_{\bar{y}}$  for all the other arrows  $\bar{x}$  and  $\bar{y}$ . Then we have

$$\begin{aligned} \delta(\psi)_{\bar{y}_0\bar{x}_0} &= \bar{y}_0\psi_{\bar{x}_0} = (-1)^{n-m}\alpha\bar{y}_0\bar{x}_{0\dots mn-2}\bar{y}_{mn-2\dots 1} \\ &= \alpha\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots 1} \end{aligned}$$

where the last equality follows from Lemma 3.27.  $\square$

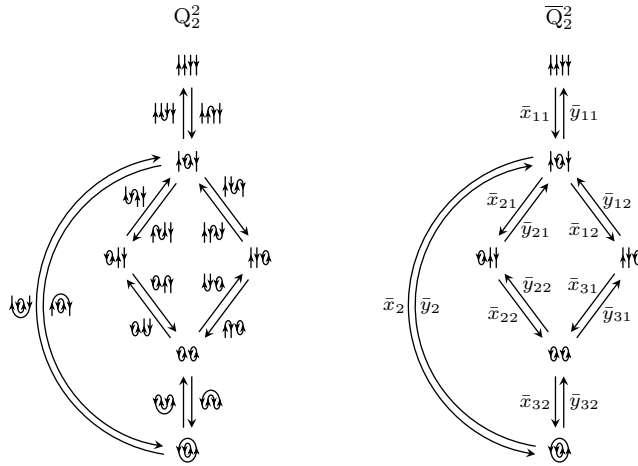
The following theorem shows that, contrary to Conjecture 1.1,  $\text{HH}^2$  also has nonvanishing components  $\text{HH}_i^2$  for  $i \neq 0$ .

**Theorem 5.4.** *Let  $m, n \geq 2$ . Then*

$$\dim_{\mathbb{k}} \text{HH}_{2mn-6}^2(K_m^n, K_m^n) \simeq \dim_{\mathbb{k}} \text{HH}_{2mn-6}^2(\bar{K}_m^n, \bar{K}_m^n) = 1.$$

*As a result,  $K_m^n$  is not intrinsically formal.*

In the rest of this section we give a proof of this theorem by using the results of §4.3 to calculate the Hochschild cohomology  $\text{HH}_{2mn-6}^2(\bar{K}_m^n, \bar{K}_m^n)$  as equivalence classes of certain first-order deformations of the reduction system  $\bar{R}_m^n$ . From now on we assume that  $m \geq n$  since by [33, Proof of Cor. 1.5] we have  $K_m^n \simeq (K_n^m)^{\text{op}}$ , so that it follows from [16, Prop. 6.4] (cf. [32, E.2.1.4]) that  $K_m^n$  and  $K_n^m$  have the same (bigraded) Hochschild cohomology and the same deformation theory.



**Fig. 10.** The quiver  $Q_2^2$  for  $K_2^2$  with all arc diagrams of degrees 0 (vertices) and 1 (arrows) and the opposite quiver  $\overline{Q}_2^2$  for  $\overline{K}_2^2$ .

### 5.1. The case $K_2^2$

We first give a proof of Theorem 5.4 for the case  $(m, n) = (2, 2)$  which also serves as a blueprint for the general case  $m, n \geq 2$ . An illustration of the degree 0 and 1 arc diagrams of  $K_2^2$  and the vertices and arrows of the Koszul dual  $\overline{K}_2^2$  are given in Fig. 10.

The reduction system  $\overline{R}_2^2$  (see §3.3) for  $\overline{K}_2^2$  is given by the following set of pairs

$$\begin{array}{cccc} (\bar{y}_2 \bar{x}_2, 0) & (\bar{y}_{11} \bar{x}_{11}, -\bar{x}_{21} \bar{y}_{21} - \bar{x}_{12} \bar{y}_{12}) & (\bar{y}_2 \bar{x}_{21}, -\bar{y}_{32} \bar{y}_{22}) & (\bar{y}_{22} \bar{x}_{22}, -\bar{x}_{32} \bar{y}_{32}) \\ (\bar{y}_2 \bar{y}_{11}, 0) & (\bar{y}_{31} \bar{y}_{12}, -\bar{y}_{22} \bar{y}_{21} - \bar{x}_{32} \bar{y}_2) & (\bar{y}_{21} \bar{x}_{12}, -\bar{x}_{22} \bar{y}_{31}) & (\bar{y}_{12} \bar{x}_2, -\bar{x}_{31} \bar{x}_{32}) \\ (\bar{x}_{11} \bar{x}_2, 0) & (\bar{x}_{12} \bar{x}_{31}, -\bar{x}_{21} \bar{x}_{22} - \bar{x}_2 \bar{y}_{32}) & (\bar{y}_{12} \bar{x}_{21}, -\bar{x}_{31} \bar{y}_{22}) & (\bar{y}_2 \bar{x}_{12}, -\bar{y}_{32} \bar{y}_{31}) \\ (\bar{y}_{21} \bar{x}_{21}, 0) & (\bar{y}_{12} \bar{x}_{12}, 0) & (\bar{y}_{32} \bar{x}_{32}, 0) & (\bar{y}_{31} \bar{x}_{31}, -\bar{x}_{32} \bar{y}_{32}) & (\bar{y}_{21} \bar{x}_2, -\bar{x}_{22} \bar{x}_{32}) \end{array}$$

so that

$$\begin{aligned} \overline{S}_2^2 = \{ & \bar{y}_2 \bar{x}_2, \bar{y}_{11} \bar{x}_{11}, \bar{y}_2 \bar{x}_{21}, \bar{y}_{22} \bar{x}_{22}, \bar{y}_2 \bar{y}_{11}, \bar{y}_{31} \bar{y}_{12}, \bar{y}_{21} \bar{x}_{12}, \bar{y}_{12} \bar{x}_2, \\ & \bar{x}_{11} \bar{x}_2, \bar{x}_{12} \bar{x}_{31}, \bar{y}_{12} \bar{x}_{21}, \bar{y}_2 \bar{x}_{12}, \bar{y}_{21} \bar{x}_{21}, \bar{y}_{12} \bar{x}_{12}, \bar{y}_{32} \bar{x}_{32}, \bar{y}_{31} \bar{x}_{31}, \bar{y}_{21} \bar{x}_2 \}. \end{aligned}$$

We thus have eight overlaps

$$\begin{array}{cccc} \bar{y}_2 \bar{x}_{12} \bar{x}_{31}, & \bar{y}_{12} \bar{x}_{12} \bar{x}_{31}, & \bar{y}_{21} \bar{x}_{12} \bar{x}_{31}, & \bar{y}_{11} \bar{x}_{11} \bar{x}_2, \\ \bar{y}_{31} \bar{y}_{12} \bar{x}_2, & \bar{y}_{31} \bar{y}_{12} \bar{x}_{12}, & \bar{y}_{31} \bar{y}_{12} \bar{x}_{21}, & \bar{y}_2 \bar{y}_{11} \bar{x}_{11}. \end{array}$$

**Proposition 5.5.**  $\dim_{\mathbb{k}} \mathrm{HH}_2^2(K_2^2, K_2^2) = \dim_{\mathbb{k}} \mathrm{HH}_2^2(\overline{K}_2^2, \overline{K}_2^2) = 1$ .

**Proof.** Let  $\{(s, \varphi_s + \tilde{\varphi}_s t)\}_{s \in \overline{S}_2^2}$  be a first-order deformation of  $\overline{R}_2^2$  corresponding to a cocycle in  $\mathrm{HH}_2^2(\overline{K}_2^2, \overline{K}_2^2)$  as in §4.3. Here,  $\tilde{\varphi}_s$  is the image of  $s$  under the cochain  $\tilde{\varphi} \in \mathrm{Hom}_{\mathbb{K}Q_6^e}(\mathbb{K}\overline{S}_2^2, \overline{K}_2^2)_2$ . Note that  $\tilde{\varphi}_s$  has the following form

$$\begin{aligned} \tilde{\varphi}_{\bar{y}_{11}\bar{x}_{11}} &= \alpha_1 \bar{x}_{21} \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} + \alpha_2 \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} \bar{y}_2 + \alpha_3 \bar{x}_2 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21} \\ \tilde{\varphi}_{\bar{y}_{21}\bar{x}_{21}} &= \alpha_4 \bar{x}_{22} \bar{x}_{32} \bar{y}_{32} \bar{y}_{22} & \tilde{\varphi}_{\bar{y}_{12}\bar{x}_{12}} &= \alpha_5 \bar{x}_{31} \bar{x}_{32} \bar{y}_{32} \bar{y}_{31} \\ \tilde{\varphi}_{\bar{x}_{11}\bar{x}_2} &= \alpha_6 \bar{x}_{11} \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} & \tilde{\varphi}_{\bar{y}_2 \bar{y}_{11}} &= \alpha_7 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21} \bar{y}_{11} \\ \tilde{\varphi}_{\bar{y}_{21}\bar{x}_{12}} &= \alpha_8 \bar{x}_{22} \bar{x}_{32} \bar{y}_{32} \bar{y}_{31} & \tilde{\varphi}_{\bar{y}_{12}\bar{x}_{21}} &= \alpha_9 \bar{x}_{31} \bar{x}_{32} \bar{y}_{32} \bar{y}_{22} \\ \tilde{\varphi}_{\bar{x}_{12}\bar{x}_{31}} &= \alpha_{10} \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} \bar{y}_{32} & \tilde{\varphi}_{\bar{y}_{31}\bar{y}_{12}} &= \alpha_{11} \bar{x}_{32} \bar{y}_{32} \bar{y}_{22} \bar{y}_{21} \end{aligned} \quad (5.6)$$

for some coefficients  $\alpha_1, \dots, \alpha_{11} \in \mathbb{K}$  and  $\tilde{\varphi}_s = 0$  for all other  $s \in \overline{S}_2^2$ . Note that none of the overlaps has a parallel irreducible path of length  $\geq 5$ , whence they are resolvable in the first-order deformation, i.e. any cochain  $\tilde{\varphi}$  is a cocycle.

Next we compute the coboundaries. An element  $\psi \in \mathrm{Hom}_{\mathbb{K}Q_6^e}(\mathbb{K}(\overline{Q}_2^2)_1, \overline{K}_2^2)_2$  is of the form

$$\begin{aligned} \psi_{\bar{x}_{11}} &= \mu_1 \bar{x}_{11} \bar{x}_{21} \bar{y}_{21} + \mu_2 \bar{x}_{11} \bar{x}_{12} \bar{y}_{12} & \psi_{\bar{y}_{11}} &= \nu_1 \bar{x}_{21} \bar{y}_{21} \bar{y}_{11} + \nu_2 \bar{x}_{12} \bar{y}_{12} \bar{y}_{11} \\ \psi_{\bar{x}_{21}} &= \mu_3 \bar{x}_{21} \bar{x}_{22} \bar{y}_{22} + \mu_4 \bar{x}_2 \bar{y}_{32} \bar{y}_{22} & \psi_{\bar{y}_{21}} &= \nu_3 \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} + \nu_4 \bar{x}_{22} \bar{x}_{32} \bar{y}_2 \\ \psi_{\bar{x}_{12}} &= \mu_5 \bar{x}_{21} \bar{x}_{22} \bar{y}_{31} + \mu_6 \bar{x}_2 \bar{y}_{32} \bar{y}_{31} & \psi_{\bar{y}_{12}} &= \nu_5 \bar{x}_{31} \bar{y}_{22} \bar{y}_{21} + \nu_6 \bar{x}_{31} \bar{x}_{32} \bar{y}_2 \\ \psi_{\bar{x}_{22}} &= \mu_7 \bar{x}_{22} \bar{x}_{32} \bar{y}_{32} & \psi_{\bar{y}_{22}} &= \nu_7 \bar{x}_{32} \bar{y}_{32} \bar{y}_{22} \\ \psi_{\bar{x}_{31}} &= \mu_8 \bar{x}_{31} \bar{x}_{32} \bar{y}_{32} & \psi_{\bar{y}_{31}} &= \nu_8 \bar{x}_{32} \bar{y}_{32} \bar{y}_{31} \\ \psi_{\bar{x}_2} &= \mu_9 \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} & \psi_{\bar{y}_2} &= \nu_9 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21} \end{aligned}$$

and  $\psi_{\bar{x}_{32}} = 0 = \psi_{\bar{y}_{32}}$ , where  $\mu_1, \dots, \mu_9, \nu_1, \dots, \nu_9 \in \mathbb{K}$ . If  $\delta(\psi)_s = \tilde{\varphi}_s$  for each  $s \in \overline{S}_m^n$  then using (4.14) we obtain the following eleven equations

$$\begin{aligned} -\mu_1 - \nu_1 - \mu_2 - \nu_2 + \mu_3 + \nu_3 - \mu_5 - \nu_5 &= \alpha_1 \\ -\nu_1 - \mu_2 + \nu_4 - \mu_5 - \nu_6 &= \alpha_2 & -\mu_1 - \nu_2 + \mu_4 - \nu_5 - \mu_6 &= \alpha_3 \\ -\nu_4 - \mu_4 &= \alpha_4 & \nu_5 + \mu_5 - \nu_6 - \mu_6 &= \alpha_5 \\ -\mu_1 + \mu_2 + \mu_9 &= \alpha_6 & -\nu_1 + \nu_2 + \nu_9 &= \alpha_7 \\ \nu_3 - \nu_4 - \mu_6 + \mu_7 + \nu_8 &= \alpha_8 & \mu_3 - \mu_4 - \nu_6 + \nu_7 + \mu_8 &= \alpha_9 \\ -\mu_3 - \mu_5 + \mu_7 - \mu_8 + \mu_9 &= \alpha_{10} & -\nu_3 - \nu_5 + \nu_7 - \nu_8 + \nu_9 &= \alpha_{11}. \end{aligned}$$

Let us check the first three equations. For this, we have

$$\begin{aligned} \delta(\psi)_{\bar{y}_{11}\bar{x}_{11}} &= \psi_{\bar{y}_{11}} \bar{x}_{11} + \bar{y}_{11} \psi_{\bar{x}_{11}} + \psi_{\bar{x}_{21}} \bar{y}_{21} + \bar{x}_{21} \psi_{\bar{y}_{21}} + \psi_{\bar{x}_{12}} \bar{y}_{12} + \bar{x}_{12} \psi_{\bar{y}_{12}} \\ &= \nu_1 \bar{x}_{21} \bar{y}_{21} \bar{y}_{11} \bar{x}_{11} + \nu_2 \bar{x}_{12} \bar{y}_{12} \bar{y}_{11} \bar{x}_{11} + \mu_1 \bar{y}_{11} \bar{x}_{11} \bar{x}_{21} \bar{y}_{21} + \mu_2 \bar{y}_{11} \bar{x}_{11} \bar{x}_{12} \bar{y}_{12} \end{aligned}$$

$$\begin{aligned}
 & + \mu_3 \bar{x}_{21} \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} + \mu_4 \bar{x}_2 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21} + \nu_3 \bar{x}_{21} \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} + \nu_4 \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} \bar{y}_2 \\
 & + \mu_5 \bar{x}_{21} \bar{x}_{22} \bar{y}_{31} \bar{y}_{12} + \mu_6 \bar{x}_2 \bar{y}_{32} \bar{y}_{31} \bar{y}_{12} + \nu_5 \bar{x}_{12} \bar{x}_{31} \bar{y}_{22} \bar{y}_{21} + \nu_6 \bar{x}_{12} \bar{x}_{31} \bar{x}_{32} \bar{y}_2 \\
 & = \beta_1 \bar{x}_{21} \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} + \beta_2 \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} \bar{y}_2 + \beta_3 \bar{x}_2 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21}
 \end{aligned}$$

where  $\beta_1 := -\mu_1 - \nu_1 - \mu_2 - \nu_2 + \mu_3 + \nu_3 - \mu_5 - \nu_5$ ,  $\beta_2 := -\nu_1 - \mu_2 + \nu_4 - \mu_5 - \nu_6$  and  $\beta_3 := -\mu_1 - \nu_2 + \mu_4 - \nu_5 - \mu_6$ , and the third equality follows by performing reductions (with respect to  $\overline{R}_2^2$ ) on each summand. For instance, we have

$$\begin{aligned}
 \bar{x}_{21} \bar{y}_{21} \bar{y}_{11} \bar{x}_{11} &= -\bar{x}_{21} \bar{y}_{21} \bar{x}_{21} \bar{y}_{21} - \bar{x}_{21} \bar{y}_{21} \bar{x}_{12} \bar{y}_{12} = \bar{x}_{21} \bar{x}_{22} \bar{y}_{31} \bar{y}_{12} \\
 &= -\bar{x}_{21} \bar{x}_{22} \bar{y}_{22} \bar{y}_{21} - \bar{x}_{21} \bar{x}_{22} \bar{x}_{32} \bar{y}_2
 \end{aligned}$$

where the first equality follows by performing reductions on  $\bar{y}_{11} \bar{x}_{11}$ , the second one on  $\bar{y}_{21} \bar{x}_{21}$  and  $\bar{y}_{21} \bar{x}_{12}$ , and the third one on  $\bar{y}_{31} \bar{y}_{12}$ . Then by  $\delta(\psi)_{\bar{y}_{11} \bar{x}_{11}} = \varphi_{\bar{y}_{11} \bar{x}_{11}}$  we obtain the first three equations. Similarly we may verify the other equations.

From the above eleven equations we observe that

$$-\alpha_2 + \alpha_3 - \alpha_6 + \alpha_7 - \alpha_8 + \alpha_9 + \alpha_{10} - \alpha_{11} = 0. \quad (5.7)$$

Moreover, writing the coefficients as an  $(18 \times 11)$ -matrix, the rank of this matrix is seen to be 10, i.e. (5.7) turns out to be the unique constraint for  $\tilde{\varphi}$  to be a coboundary. Thus,  $\mathrm{HH}_2^2(\overline{K}_2^2, \overline{K}_2^2)$  is 1-dimensional. By (5.7) an explicit nontrivial cocycle can be given by setting  $\alpha_2 = 1$  and  $\alpha_i = 0$  for  $i \neq 2$ .  $\square$

**Corollary 5.8.** *We have that  $\dim_{\mathbb{k}}(\bigoplus_{i \neq 0,2} \mathrm{HH}_{i-2}^2(K_2^2, K_2^2)) = 1$ . Moreover,  $K_2^2$  admits a unique  $A_\infty$  deformation (up to  $A_\infty$  isomorphism) given on arrows by setting*

$$m_4(y_2 \otimes x_{32} \otimes x_{22} \otimes x_{21}) = x_{11} y_{11}$$

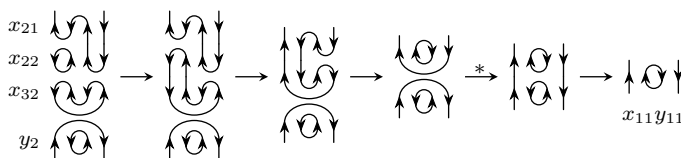
and  $m_4 = 0$  when restricting to other arrows and setting  $m_i = 0$  for  $i \neq 2, 4$  for all arrows.

**Proof.** The first assertion follows from Proposition 5.5 and Lemmas 5.2 and 5.3.

For the second assertion, there is a nontrivial cocycle  $\tilde{\varphi}$  in  $\mathrm{HH}_2^2(\overline{K}_2^2, \overline{K}_2^2)$  given by  $\alpha_2 = 1$  and  $\alpha_i = 0$  for  $i \neq 2$ , as in the proof of Proposition 5.5. Note that  $\{(s, \varphi_s + \tilde{\varphi}_s)\}_{s \in \overline{S}_2^2}$  is an actual deformation of  $\overline{R}_2^2$ , i.e. it is reduction-finite and reduction-unique so that  $\overline{K}_2^2$  admits a filtered deformation by changing the relation  $\bar{y}_{11} \bar{x}_{11} + \bar{x}_{21} \bar{y}_{21} + \bar{x}_{12} \bar{y}_{12} = 0$  into

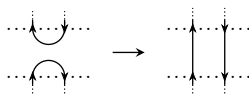
$$\bar{y}_{11} \bar{x}_{11} + \bar{x}_{21} \bar{y}_{21} + \bar{x}_{12} \bar{y}_{12} = \bar{x}_2 \bar{y}_{32} \bar{y}_{22} \bar{y}_{21}.$$

By Proposition 4.7, the minimal model of the derived endomorphism algebra of  $\mathbb{k}(\overline{Q}_m^n)_0$  is an  $A_\infty$  deformation of  $K_2^2$ .  $\square$



**Fig. 11.** A diagrammatic illustration of the unique  $A_\infty$  deformation of  $K_2^2$  given by  $m_4(y_2 x_{32} x_{22} x_{21}) = x_{11} y_{11}$  where the arrow marked with  $*$  is of degree  $-2$  and breaks the usual diagrammatic multiplication rule of  $K_2^2$ .

See Fig. 11 for a diagrammatic interpretation of the  $A_\infty$  deformation of  $K_2^2$  given in Corollary 5.8. (A diagrammatic description of the arrows in  $K_2^2$  is given in Fig. 10.) This  $A_\infty$  deformation appears to correspond to using the usual surgery/reorientation rules for  $K_m^n$ , but breaking the rule  $\zeta \otimes \zeta \mapsto 0$  exactly once by using  $\zeta \otimes \zeta \mapsto \zeta \otimes \zeta$  of degree  $-2$  of the form



(For higher  $m, n$  also other rules appear to be broken, but we do not give a full description here.) It would also be interesting to compare this to a diagrammatic interpretation of the deformations of  $\bar{K}_m^n$  (cf. Remark 3.23).

**Remark 5.9.** We also have  $\mathrm{HH}_0^2(K_2^2, K_2^2) \neq 0$ . For this, we show that there is a nontrivial cocycle given by  $\tilde{\varphi} \in \mathrm{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}\bar{S}_2^2, \bar{K}_2^2)_0 = \mathrm{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}\bar{S}_2^2, (\bar{K}_2^2)_2)$  such that

$$\tilde{\varphi}_{\bar{y}_{11}\bar{x}_{11}} = \bar{x}_2\bar{y}_2$$

and  $\tilde{\varphi}_s = 0$  for all the other  $s \in \bar{S}_2^2$ . The overlaps are resolvable in the first-order deformation  $\{(s, \varphi_s + \tilde{\varphi}_s t)\}_{s \in \bar{S}_2^2}$  as follows. The involved overlaps are  $\bar{y}_{11}\bar{x}_{11}\bar{x}_2$  and  $\bar{y}_2\bar{y}_{11}\bar{x}_{11}$ . Let us verify the claim for  $\bar{y}_{11}\bar{x}_{11}\bar{x}_2$ . We have

$$\begin{aligned} \bar{y}_{11}\bar{x}_{11}\bar{x}_2 &\mapsto -\bar{x}_{21}\bar{y}_{21}\bar{x}_2 - \bar{x}_{12}\bar{y}_{12}\bar{x}_2 + \bar{x}_2\bar{y}_2\bar{x}_2 t \mapsto 0 \\ \bar{y}_{11}\bar{x}_{11}\bar{x}_2 &\mapsto 0 \end{aligned}$$

where we use  $\bar{y}_2\bar{x}_2 = 0$  and  $\bar{x}_{11}\bar{x}_2 = 0$ . Let us show that  $\tilde{\varphi}$  is not a coboundary. Note that an element  $\psi \in \mathrm{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}(\bar{Q}_2^2)_1, (\bar{K}_2^2)_1)$  has the following general form  $\psi(a) = \beta_a a$  for each arrow  $a \in \bar{Q}_2^2$  where  $\beta_a \in \mathbb{k}$ . Since  $\bar{x}_2\bar{y}_2$  does not appear in  $\bar{R}_2^2$  it follows that  $\delta(\psi)_{\bar{y}_{11}\bar{x}_{11}}$  cannot equal  $\bar{x}_2\bar{y}_2$ . This shows that  $K_2^2$  also admits a nontrivial (graded) associative deformation.

## 5.2. The general case $K_m^n$

The case  $(m, n) = (2, 2)$  was proved in Proposition 5.5. We thus assume that  $(m, n) \neq (2, 2)$ . Although  $\bar{K}_m^n$  is generally much more complicated than  $\bar{K}_2^2$  (cf. Fig. 4), we may

exploit the fact that the long irreducible paths in  $\bar{K}_m^n$  have a structure quite similar to those of  $\bar{K}_2^2$ . (By Lemmas 5.2 and 5.3,  $\mathrm{HH}_{2mn-6}^2$  appearing in Theorem 5.4 is the top nonvanishing component of  $\mathrm{HH}^2$  with respect to the lower index corresponding to the Adams grading.) The computation of  $\mathrm{HH}_{2mn-6}^2(\bar{K}_m^n, \bar{K}_m^n)$  for general  $(m, n)$  then parallels the proof for  $(m, n) = (2, 2)$  and is for the most part an exercise in labelling the arrows appearing in long paths and keeping track of the indices which is why we relegate the technical computations to the appendix. The arrows appearing in the computation are illustrated in Fig. 12.

Note that any element  $\tilde{\varphi} \in \mathrm{Hom}_{\mathbb{k}Q_0^n}(\mathbb{k}\bar{S}_m^n, \bar{K}_m^n)_{2mn-6}$  (cf. (5.6)) is of the form

$$\begin{aligned}\tilde{\varphi}_{\bar{Y}_0\bar{X}_0} &= \alpha_1\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 1} + \alpha_2\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 3}^2 \\ &\quad + \alpha_3\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 1} \\ \tilde{\varphi}_{\bar{Y}_1\bar{X}_1} &= \alpha_4\bar{x}_{2\dots nm-1}\bar{y}_{nm-1\dots 2} \\ \tilde{\varphi}_{\bar{Y}_3\bar{X}_3} &= \alpha_5\bar{x}_{2\dots n}^1\bar{x}_{n+1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1 \\ \tilde{\varphi}_{\bar{X}_0\bar{X}_7} &= \alpha_6\bar{x}_{0\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 4}^2 \\ \tilde{\varphi}_{\bar{Y}_7\bar{Y}_0} &= \alpha_7\bar{x}_{4\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 0} \\ \tilde{\varphi}_{\bar{Y}_1\bar{X}_3} &= \alpha_8\bar{x}_{2\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1 \\ \tilde{\varphi}_{\bar{Y}_2\bar{X}_1} &= \alpha_9\bar{x}_{2\dots n}^1\bar{x}_{n+1\dots nm-1}\bar{y}_{nm-1\dots 2} \\ \tilde{\varphi}_{\bar{X}_3\bar{X}_5} &= \alpha_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1 \\ \tilde{\varphi}_{\bar{Y}_5\bar{Y}_3} &= \alpha_{11}\bar{x}_{3\dots n}^1\bar{x}_{n+1\dots nm-1}\bar{y}_{nm-1\dots 1}\end{aligned}\tag{5.10}$$

and  $\tilde{\varphi}_s = 0$  for all the other  $s \in \bar{S}_m^n$ , where  $\alpha_1, \dots, \alpha_{11} \in \mathbb{k}$ . The arrows appearing in (5.10) are illustrated in Fig. 12. The space  $\mathrm{Hom}_{\mathbb{k}Q_0^n}(\mathbb{k}\bar{S}_m^n, \bar{K}_m^n)_{2mn-6}$  is thus of dimension 11.

Let  $w$  be any overlap of  $\bar{S}_m^n$  starting from  $e_\lambda$  and ending at  $e_\mu$  for some weights  $\lambda, \mu$ . Observe that  $|\lambda| + |\mu| \leq 2mn - 4$  (for instance, the equality only holds for  $\bar{y}_0\bar{x}_0\bar{x}_3^2$  and  $\bar{y}_3^2\bar{y}_0\bar{x}_0$ ). By Remark 3.22, there are no irreducible paths of length  $\geq 2mn - 3$  parallel to  $w$ . So all overlaps will reduce to zero after using  $\tilde{\varphi}_s$ , whence  $\tilde{R} = \{(s, \varphi_s + \tilde{\varphi}_s t)\}_{s \in \bar{S}_m^n}$  is vacuously reduction-unique — even when evaluating  $t$  to any constant in  $\mathbb{k}$ .

**Proposition 5.11.** *The element  $\tilde{\varphi}$  in (5.10) induces a trivial first-order deformation of  $R$  if and only if the coefficients  $\alpha_1, \dots, \alpha_{11}$  satisfy*

$$-(-1)^n\alpha_2 + (-1)^n\alpha_3 - (-1)^n\alpha_6 + (-1)^n\alpha_7 - \alpha_8 + \alpha_9 + \alpha_{10} - \alpha_{11} = 0.\tag{5.12}$$

The proof, which relies on some long and technical computations, can be found in the appendix. Proposition 5.11 directly yields our main result Theorem 5.4.

**Proof of Theorem 5.4.** It follows from Proposition 5.11 that the coboundary space is of dimension 10. Thus,  $\mathrm{HH}_{2mn-6}^2(\bar{K}_m^n, \bar{K}_m^n)$  is 1-dimensional.  $\square$

**Corollary 5.13.** *For  $m, n \geq 2$ , the algebra  $K_m^n$  admits a unique  $A_\infty$  deformation such that the only nontrivial higher product restricted to arrows is given by*

$$m_{2mn-4}(y_1 \otimes \cdots \otimes y_{nm-1} \otimes x_{nm-1} \otimes \cdots \otimes x_{n+2} \otimes x_{n+1}^2 \otimes \cdots \otimes x_3^2) = x_0 y_0$$

where the arrows in this formula are illustrated in Fig. 12.

**Proof.** Consider  $\tilde{\varphi}$  in (5.10) and take  $\alpha_2 = 1$  and  $\alpha_i = 0$  for  $i \neq 2$ . By Proposition 5.11 this  $\tilde{\varphi}$  gives a nontrivial cocycle of  $\mathrm{HH}_{2mn-6}^2(\overline{K}_m^n, \overline{K}_m^n)$ . As in the proof of Corollary 5.8 this induces a filtered deformation of  $\overline{K}_m^n$ , namely the algebra

$$B := \mathbb{k}\overline{Q}_m^n / (s - \varphi_s - \tilde{\varphi}_s)_{s \in \overline{S}_m^n}$$

is a filtered deformation of  $\overline{K}_m^n$ . By Proposition 4.7 we obtain that the minimal model  $\mathrm{Ext}_B^\bullet(\mathbb{k}Q_0, \mathbb{k}Q_0)$  is an  $A_\infty$  deformation of  $K_m^n$ .  $\square$

See Fig. 11 for a diagrammatic interpretation of this deformation for the case  $m = n = 2$ .

**Remark 5.14** (The case  $m \geq n = 1$ ). If  $m \geq n = 1$  then by [41, Lem. 4.21] one has

$$\mathrm{HH}_{i-2}^2(K_m^1, K_m^1) \simeq \mathrm{HH}_{i-2}^2(\overline{K}_m^1, \overline{K}_m^1) = 0 \quad \text{for } i > 0$$

i.e. Stroppel's Conjecture 1.1 holds. Using the language of reduction systems, this can be seen as follows. The reduction system  $\overline{R}_m^1$  for  $\overline{K}_m^1$  (Definition 3.13) is

$$\{(\bar{y}_k \bar{x}_k, -\bar{x}_{k+1} \bar{y}_{k+1})\}_{0 \leq k \leq m-2} \cup \{(\bar{y}_{m-1} \bar{x}_{m-1}, 0)\}.$$

By Lemma 5.2 it suffices to prove  $\mathrm{HH}_{2i-2}^2(\overline{K}_m^1, \overline{K}_m^1) = 0$  for  $1 \leq i \leq m-1$ . For any fixed  $i$ , an element  $\tilde{\varphi} \in \mathrm{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}\overline{S}_m^1, \overline{K}_m^1)_{2i-2}$  has the following general form

$$\tilde{\varphi}_{\bar{y}_k \bar{x}_k} = \begin{cases} \alpha_k \bar{x}_{k+1 \dots k+i} \bar{y}_{k+i \dots k+1} & \text{if } 0 \leq k < m-i \\ 0 & \text{otherwise.} \end{cases}$$

Since there are no overlaps, any such  $\tilde{\varphi}$  is a cocycle. We claim that it is also a coboundary. Indeed, consider  $\psi \in \mathrm{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}(\overline{Q}_m^1)_1, \overline{K}_m^1)_{2i-2}$  given by

$$\psi_{\bar{x}_k} = \beta_k \bar{x}_{k \dots k+i-1} \bar{y}_{k+i-1 \dots k+1}$$

for each  $0 \leq k < m-i$ , where  $\beta_k := \sum_{j=0}^{m-i-1-k} (-1)^{(i+1)j} \alpha_{k+j}$ . Then we have

$$\delta(\psi)_{\bar{y}_k \bar{x}_k} = \bar{y}_k \psi_{\bar{x}_k} + \psi_{\bar{x}_{k+1}} \bar{y}_{k+1} = ((-1)^i \beta_k + \beta_{k+1}) \bar{x}_{k+1 \dots k+i} \bar{y}_{k+i \dots k+1}.$$

Noting that  $(-1)^i \beta_k + \beta_{k+1} = \alpha_k$  this shows that  $\delta(\psi) = \tilde{\varphi}$ .

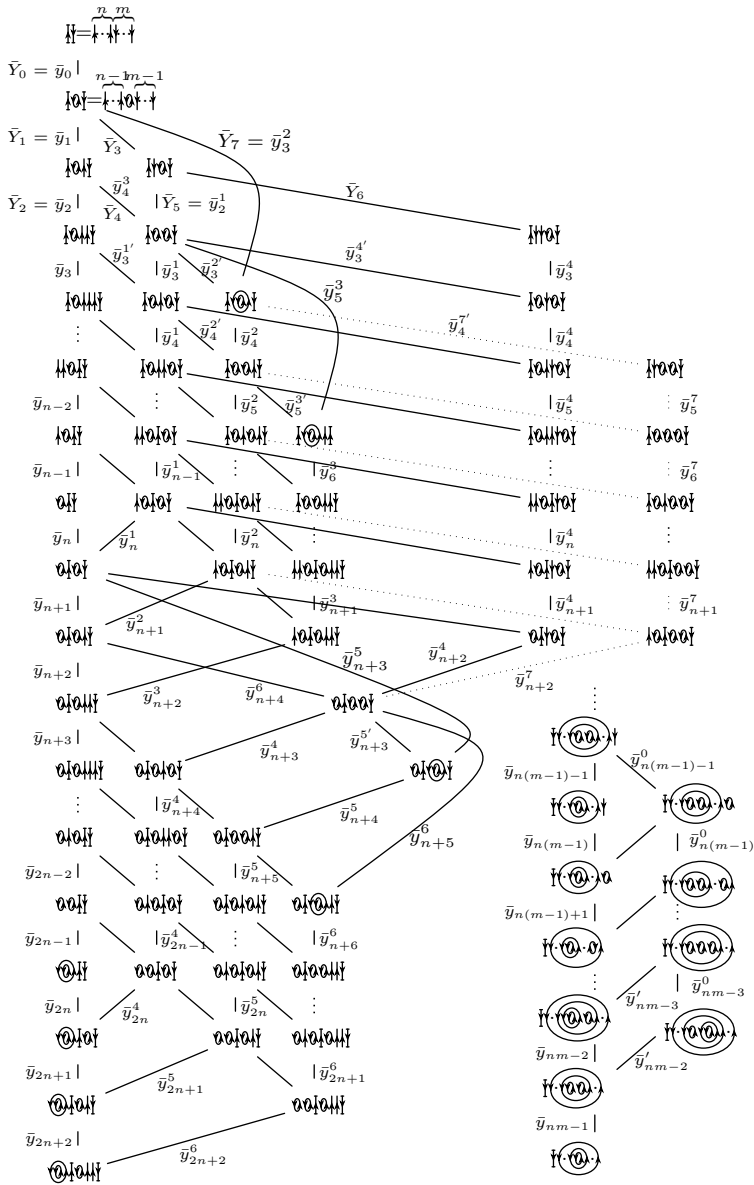


Fig. 12. A part of  $\bar{Q}_m^n$  used in the proof of Theorem 5.4 where for simplicity the pairs of opposite arrows are drawn by an unoriented edge and only the label of the ascending arrow is given.

**Remark 5.15.** For  $\mathbb{k} = \mathbb{C}$  it follows from [33] that there is a quasi-equivalence between the DG category  $\text{perf}_{\text{dg}}(K_m^n)$  and the Fukaya–Seidel category  $\mathcal{FS}(X_m^n, \pi_m^n)$  of a symplectic Lefschetz fibration

$$\pi_m^n : X_m^n \longrightarrow \mathbb{C}$$

where  $X_m^n = \text{Hilb}^m(A_{m+n-1}) \setminus D$  is the affine complement of an ample divisor  $D$  on the Hilbert scheme of  $m$  points of the Milnor fibre of the surface singularity of type  $A_{m+n-1}$ . (See [1,2,34,40] for a similar collection of results for the classical Khovanov arc algebras.) The  $A_\infty$  deformations of  $K_m^n$  can thus be viewed as  $A_\infty$  deformations of the Fukaya–Seidel category  $\mathcal{FS}(X_m^n, \pi_m^n)$ . That is, our main algebraic result (Theorem 5.4) shows that  $\mathcal{FS}(X_m^n, \pi_m^n)$  is not intrinsically formal. In light of Seidel’s ICM 2002 address [38], the algebraic deformations may correspond geometrically to partial compactifications of the affine variety  $X_m^n = \text{Hilb}^m(A_{m+n-1}) \setminus D \subset \text{Hilb}^m(A_{m+n-1})$ . We leave a geometric interpretation of these deformations to future work.

## Acknowledgments

We express our gratitude to Catharina Stroppel for introducing us to the extended Khovanov arc algebras and for her suggestion to apply the combinatorial method developed in [4] to study their deformations. We would also like to thank Bernhard Keller, Mikhail Khovanov, Steffen Koenig, Julian Külshammer, Yanbo Li, Ivan Smith, Jim Stasheff and Ben Webster for many helpful comments and discussions and the anonymous referee for their suggestions for improvements.

Part of this work was carried out during the Junior Trimester Program “New Trends in Representation Theory” at the Hausdorff Research Institute for Mathematics in Bonn, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2047/1 – 390685813 whose support we gratefully acknowledge. The first-named author also warmly thanks the Institute of Algebra and Number Theory at the University of Stuttgart for the hospitality while working on this article. The second-named author was also supported by the Alexander von Humboldt-Stiftung, the Fundamental Research Funds for the Central Universities (No. 020314380037), the National Key R&D Program of China (2024YFA1013803) and by the NSFC (Nos. 13004005 and 12371043).

## Appendix A. Proof of Proposition 5.11

In this appendix we give a proof of Proposition 5.11 which is the key ingredient in the proof of our main result Theorem 5.4 for the case  $m \geq n \geq 2$  and  $(m, n) \neq (2, 2)$ .

An element  $\psi \in \text{Hom}_{\mathbb{k}Q_0^e}(\mathbb{k}(\overline{Q}_m^n)_1, (\overline{K}_m^n)_{2mn-5})$  has the following general form

$$\begin{aligned} \psi_{\bar{X}_0} &= \mu_1 \bar{x}_{0\dots nm-3} \bar{y}_{nm-3\dots 1} \\ &\quad + \mu_2 \bar{x}_{0\dots n(m-1)-2} \bar{x}_n^0 \bar{y}_{nm-3\dots n(m-1)-1} \bar{y}_{n(m-1)-2\dots 1} \\ &\quad + \mu_3 \bar{x}_{0\dots nm-2} \bar{y}_{nm-2\dots n+2} \bar{y}_{n+1\dots 3}^2 \\ &\quad + \mu_4 \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{y}_{n+1\dots 3}^2 \\ \psi_{\bar{X}_1} &= \mu_5 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots 2} + \mu_6 \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+3} \bar{y}_{n+2\dots 4}^3 \end{aligned}$$

$$\begin{aligned}
& + \mu_7 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots 2} \\
\psi_{\bar{X}_2} &= \mu_8 \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots 3} \\
\psi_{\bar{X}_3} &= \mu_9 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{y}_{n\dots 2}^1 \\
& + \mu_{10} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^5 \bar{y}_{n\dots 2}^1 \\
& + \mu_{11} \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \\
\psi_{\bar{X}_4} &= \mu_{12} \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 3}^1 \\
\psi_{\bar{X}_5} &= \mu_{13} \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 3}^1 \\
\psi_{\bar{X}_6} &= \mu_{14} \bar{x}_{2\dots n}^1 x_{n+1\dots nm-1} \bar{y}_{nm-1\dots 2n+1} \bar{y}_{2n\dots 3}^4 \\
\psi_{\bar{X}_7} &= \mu_{15} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 4}^2
\end{aligned}$$

where  $\psi_{\bar{Y}_i}$  is involutive to  $\psi_{\bar{X}_i}$  with coefficients  $\nu_1, \dots, \nu_{15}$ . Note that  $\psi_{\bar{X}_i} = 0 = \psi_{\bar{Y}_i} = 0$  for all other arrows. It follows that the dimension of the vector space  $\text{Hom}_{\mathbb{K}Q_6^e}(\mathbb{K}(\bar{Q}_m^n)_1, (\bar{K}_m^n)_{2mn-5})$  is 30. If  $n = 2$ , the terms with coefficient  $\mu_4$  and  $\nu_4$  do not exist, i.e.  $\text{Hom}_{\mathbb{K}Q_6^e}(\mathbb{K}(\bar{Q}_m^2)_1, (\bar{K}_m^2)_{4m-5})$  is of dimension 28.

Recall that any element  $\tilde{\varphi} \in \text{Hom}_{\mathbb{K}Q_6^e}(\mathbb{K}\bar{S}_m^n, \bar{K}_m^n)_{2mn-4}$  has the form as in (5.10). Also recall the differential  $\delta$  from (4.14).

**Lemma A.1.**  $\delta(\psi) = \tilde{\varphi}$  if and only if the following eleven equations hold

$$\begin{aligned}
\alpha_1 &= -(-1)^{n+m}(\mu_1 + \nu_1) - (-1)^n(\mu_2 + \nu_2) + (\mu_5 + \nu_5) - (-1)^n(\mu_9 + \nu_9) \\
\alpha_2 &= (-1)^{n-m}\mu_3 - (-1)^n\mu_6 + (-1)^n\mu_{10} - \nu_1 + \nu_7 - (-1)^n\nu_{11} \\
\alpha_3 &= (-1)^{n-m}\nu_3 - (-1)^n\nu_6 + (-1)^n\nu_{10} - \mu_1 + \mu_7 - (-1)^n\mu_{11} \\
\alpha_4 &= -(\mu_7 + \nu_7) + (\mu_8 + \nu_8) \\
\alpha_5 &= (-1)^m(\mu_9 + \nu_9) - (-1)^n(\mu_{11} + \nu_{11}) + (\mu_{14} + \nu_{14}) \\
\alpha_6 &= -\mu_1 - (-1)^{n-m}\mu_3 + \mu_{15} \\
\alpha_7 &= -\nu_1 - (-1)^{n-m}\nu_3 + \nu_{15} \\
\alpha_8 &= -\mu_{11} + \mu_{12} + (-1)^m\nu_5 - (-1)^n\nu_7 + (-1)^n\nu_{13} \\
\alpha_9 &= -\nu_{11} + \nu_{12} + (-1)^m\mu_5 - (-1)^n\mu_7 + (-1)^n\mu_{13} \\
\alpha_{10} &= -(-1)^m\mu_5 - \mu_6 + \mu_{10} + \mu_{12} - (-1)^n\mu_{13} + (-1)^n\mu_{15} \\
\alpha_{11} &= -(-1)^m\nu_5 - \nu_6 + \nu_{10} + \nu_{12} - (-1)^n\nu_{13} + (-1)^n\nu_{15}.
\end{aligned}$$

This lemma immediately yields Proposition 5.11.

**Proof of Proposition 5.11.** From Lemma A.1 it follows that if  $\tilde{\varphi}$  is a coboundary then equation (5.12) holds. It not difficult to see that (5.12) is the only constraint since the rank of the coefficient matrix is 10.  $\square$

The remaining part of this appendix is thus devoted to the proof of Lemma A.1. Note that the equations involving  $\alpha_3, \alpha_7, \alpha_9, \alpha_{11}$  may be obtained from those involving  $\alpha_2, \alpha_6, \alpha_8, \alpha_{10}$  by exchanging  $\mu_i$  and  $\nu_i$ , and the others involving  $\alpha_1, \alpha_4, \alpha_5$  are invariant under this operation. (This is the case because the reduction system  $\bar{R}_m^n$  is invariant under the involution in Remark 3.2.)

The proof of Lemma A.1 is divided into several computations which are performed by “chasing the diagram” in Fig. 12, i.e. by repeatedly using the anticommutativity and Plücker-type relations for  $\bar{K}_m^n$  given in Proposition 3.3 for the paths appearing in the components of  $\delta(\psi)$ . In this process, some of the terms appearing from applying the Plücker-type relations vanish due to the following observation.

**Observation A.2.** Let  $p = \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}$  be any ascending path in  $\bar{Q}_m^n$  such that  $\lambda_1$  is the lowest weight (i.e.  $|\lambda_1| = 0$ ) and  $|\lambda_{k+1}| = k$ . Assume that  $\bar{y}_{\lambda_{i-1}}^{\lambda_i} \bar{y}_{\lambda_i}^{\lambda_{i+1}}$  lies in Fig. 5 (c) for some  $1 < i \leq k$  such that  $\lambda_i$  is the right vertex in the figure. Denote by  $\lambda'_i$  the left vertex. Then

$$p = -\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_{i-1}}^{\lambda'_i} \bar{y}_{\lambda'_i}^{\lambda_{i+1}} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}.$$

A similar statement holds for descending paths.

**Proof.** We have  $\bar{y}_{\lambda_{i-1}}^{\lambda_i} \bar{y}_{\lambda_i}^{\lambda_{i+1}} = -\bar{y}_{\lambda_{i-1}}^{\lambda'_i} \bar{y}_{\lambda'_i}^{\lambda_{i+1}} - \bar{x}_{\kappa}^{\lambda_{i-1}} \bar{y}_{\kappa}^{\lambda_{i+1}}$  by the Plücker-type relation given in Proposition 3.3, where  $\kappa$  is the bottom vertex in Fig. 5 (c) whence we have

$$p = -\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_{i-1}}^{\lambda'_i} \bar{y}_{\lambda'_i}^{\lambda_{i+1}} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}} - \bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{x}_{\kappa}^{\lambda_{i-1}} \bar{y}_{\kappa}^{\lambda_{i+1}} \cdots \bar{y}_{\lambda_k}^{\lambda_{k+1}}.$$

The desired equality then follows since  $\bar{y}_{\lambda_1}^{\lambda_2} \bar{y}_{\lambda_2}^{\lambda_3} \cdots \bar{y}_{\lambda_{i-2}}^{\lambda_{i-1}} \bar{x}_{\kappa}^{\lambda_{i-1}} = 0$  by Remark 3.22.  $\square$

A.1. The term  $\delta(\psi)_{\bar{Y}_0 \bar{X}_0}$

The following lemma proves  $\frac{3}{11}$  of Lemma A.1. The computations for the remaining  $\frac{8}{11}$  are very similar and are completed in the following subsections.

**Lemma A.3.**  $\delta(\psi)_{\bar{Y}_0 \bar{X}_0} = \tilde{\varphi}_{\bar{Y}_0 \bar{X}_0}$  if and only if the first three equations in Lemma A.1 involving  $\alpha_1, \alpha_2, \alpha_3$  hold.

**Proof.** Since  $\bar{Y}_0 \bar{X}_0 + \bar{X}_1 \bar{Y}_1 + \bar{X}_3 \bar{Y}_3 = 0$ , we have

$$\delta(\psi)_{\bar{Y}_0 \bar{X}_0} = \psi_{\bar{Y}_0} \bar{X}_0 + \bar{Y}_0 \psi_{\bar{X}_0} + \psi_{\bar{X}_1} \bar{Y}_1 + \bar{X}_1 \psi_{\bar{Y}_1} + \psi_{\bar{X}_3} \bar{Y}_3 + \bar{X}_3 \psi_{\bar{Y}_3}$$

where  $\psi_{\bar{Y}_0} \bar{X}_0$  is the product of  $\psi_{\bar{Y}_0}$  and  $\bar{X}_0$  in  $\bar{K}_m^n$  and similarly for the other terms. Note that  $\psi_{\bar{Y}_0} \bar{X}_0, \bar{X}_1 \psi_{\bar{Y}_1}, \bar{X}_3 \psi_{\bar{Y}_3}$  may be obtained from  $\bar{Y}_0 \psi_{\bar{X}_0}, \psi_{\bar{X}_1} \bar{Y}_1, \psi_{\bar{X}_3} \bar{Y}_3$ , respectively, by taking the involution in Remark 3.2 (and exchanging  $\mu_i$  and  $\nu_i$ ). Therefore, it suffices to compute the latter ones. We first claim

$$\begin{aligned}\bar{Y}_0\psi_{\bar{X}_0} &= (-(-1)^{n+m}\mu_1 - (-1)^n\mu_2)\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 1} \\ &\quad - \mu_1\bar{x}_{3\dots n+2}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 1} \\ &\quad + (-1)^{n+m}\mu_3\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 3}^2.\end{aligned}\tag{A.4}$$

To see this, note that  $\bar{Y}_0 = \bar{y}_0$  (see Fig. 12), whence by definition of  $\psi_{\bar{X}_0}$  we have

$$\begin{aligned}\bar{Y}_0\psi_{\bar{X}_0} &= \mu_1\bar{y}_0\bar{x}_{0\dots nm-3}\bar{y}_{nm-3\dots 1} \\ &\quad + \mu_2\bar{y}_0\bar{x}_{0\dots n(m-1)-2}\bar{x}_{n(m-1)-1\dots nm-3}^0\bar{y}_{nm-3\dots n(m-1)-1}\bar{y}_{n(m-1)-2\dots 1} \\ &\quad + \mu_3\bar{y}_0\bar{x}_{0\dots nm-2}\bar{y}_{nm-2\dots n+2}\bar{y}_{n+1\dots 3}^2 \\ &\quad + \mu_4\bar{y}_0\bar{x}_{0\dots nm-1}\bar{y}_{nm-1\dots 2n+3}\bar{y}_{2n+2\dots n+4}^6\bar{y}_{n+1\dots 3}^2.\end{aligned}$$

Applying Lemma 3.31 to the second summand and Lemma 3.27 (with  $i = 0$ ) to the other three summands we obtain (A.4).

We also claim that

$$\begin{aligned}\psi_{\bar{X}_1}\bar{Y}_1 &= \mu_5\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 1} \\ &\quad + (-1)^{n-1}\mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 3}^2 \\ &\quad + \mu_7\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 1}.\end{aligned}\tag{A.5}$$

Indeed, note that  $\bar{Y}_1 = \bar{y}_1$  so that by definition of  $\psi_{\bar{X}_1}$  we have

$$\begin{aligned}\psi_{\bar{X}_1}\bar{Y}_1 &= \mu_5\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 1} + \mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 4}^3\bar{y}_1 \\ &\quad + \mu_7\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 1}.\end{aligned}$$

It suffices to perform reductions on the second summand. For this, we have

$$\begin{aligned}\mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 4}^3\bar{y}_1 &= \mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 6}^3\bar{y}_5^{3'}\bar{y}_4^2\bar{y}_3^2 \\ &= (-1)^{n-1}\mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 3}^2\end{aligned}$$

where the first equality uses the relation  $\bar{y}_5^3\bar{y}_4^3\bar{y}_1 = \bar{y}_5^{3'}\bar{y}_4^2\bar{y}_3^2$  in Lemma 3.6, the second one uses the Plücker-type relation  $\bar{y}_6^3\bar{y}_5^{3'} = -\bar{y}_6^{3'}\bar{y}_5^2 - \bar{x}'\bar{y}''$  and the anticommutativity relations  $\bar{y}_{i+1}^3\bar{y}_i^{3'} = -\bar{y}_{i+1}^{3'}\bar{y}_i^2$  for  $6 \leq i \leq n+1$  (denote  $\bar{y}_{n+2}^{3'} = \bar{y}_{n+2}$ ). Here, Observation A.2 shows that the summand given by  $\bar{x}'\bar{y}''$  vanishes. This gives (A.5).

Now we claim

$$\begin{aligned}\psi_{\bar{X}_3}\bar{Y}_3 &= (-1)^{n-1}\mu_9\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 1} \\ &\quad + (-1)^n\mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 3}^2 \\ &\quad - (-1)^n\mu_{11}\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 1}.\end{aligned}\tag{A.6}$$

Indeed, we first note that

$$\bar{y}_{n\dots 2}^1 \bar{Y}_3 = (-1)^{n-1} \bar{y}_{n\dots 1} + (-1)^{n-1} \bar{x}_{n+1} \bar{y}_{n+1\dots 3}^2 \quad (\text{A.7})$$

where we use the Plücker-type relation  $\bar{y}_2^1 \bar{Y}_3 = -\bar{y}_4^3 \bar{y}_1 - \bar{x}_2'^2 \bar{y}_3^2$  and the anticommutativity relations  $\bar{y}_i^1 \bar{y}_{i-1}^{1'} = -\bar{y}_i^{1'} \bar{y}_{i-1}$  and  $\bar{y}_i^1 \bar{x}_i^{2'} = -\bar{x}_{i+1}^{2'} \bar{y}_{i+1}^2$  for  $3 \leq i \leq n$  (where  $\bar{y}_2^{1'} = \bar{y}_4^3$ ,  $\bar{y}_n^{1'} = \bar{y}_n$  and  $\bar{x}_{n+1}^{2'} = \bar{x}_{n+1}$ ). Then we have

$$\begin{aligned} \psi_{\bar{X}_3} \bar{Y}_3 &= \mu_9 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{y}_{n\dots 2}^1 \bar{Y}_3 \\ &\quad + \mu_{10} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^5 \bar{y}_{n\dots 2}^1 \bar{Y}_3 \\ &\quad + \mu_{11} \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \bar{Y}_3 \\ &= (-1)^{n-1} \mu_9 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots 1} \\ &\quad + (-1)^n \mu_{10} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 3}^2 \\ &\quad + (-1)^{n-1} \mu_{11} \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots 1} \end{aligned}$$

where in the second equality we first use (A.7) and then for the summands with coefficients  $\mu_9$  and  $\mu_{11}$  we use Lemma 3.26 (i), and for the two summands with coefficient  $\mu_{10}$  we use  $\bar{y}_{n+3}^5 \bar{y}_n = 0$  and respectively

$$\begin{aligned} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^5 \bar{x}_{n+1} &= -\bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+4}^5 \bar{y}_{n+3}^{5'} \bar{y}_{n+4}^6 \\ &= -\bar{y}_{nm-1\dots n+2}. \end{aligned}$$

Here in the first equality, we use  $\bar{y}_{n+3}^5 \bar{x}_{n+1} = \bar{y}_{n+3}^{5'} \bar{y}_{n+4}^6$  which follows from the anticommutativity relation across a square (cf. the eighth diagram in Remark 2.18 viewed as a diagram in  $\overline{\mathcal{Q}}_n^n$ ) which may be verified by using the Plücker-type relation  $\bar{y}_{n+4}^5 \bar{y}_{n+3}^{5'} = -\bar{y}_{n+4}^{5'} \bar{y}_{n+3}^4 - \bar{x}_{n+5}^{6'} \bar{y}_{n+5}^6$  and then the anticommutativity relations  $\bar{y}_{n+1+j}^5 \bar{y}_{n+j}^{5'} = -\bar{y}_{n+1+j}^{5'} \bar{y}_{n+j}^4$  for  $4 \leq j \leq n+1$  (where  $\bar{y}_{2n+1}^{5'} = \bar{y}_{2n+1}$ ). Here, the extra term  $\bar{x}_{n+5}^{6'} \bar{y}_{n+5}^6$  in the Plücker-type relation vanishes by Observation A.2. This proves (A.6).

Combining (A.4), (A.5) and (A.6) and comparing the coefficients of each summand in  $\delta(\psi)_{\bar{Y}_0 \bar{X}_0} = \tilde{\varphi}_{\bar{Y}_0 \bar{X}_0}$  completes the proof.  $\square$

## A.2. The term $\delta(\psi)_{\bar{Y}_1 \bar{X}_1}$

**Lemma A.8.**  $\delta(\psi)_{\bar{Y}_1 \bar{X}_1} = \tilde{\varphi}_{\bar{Y}_1 \bar{X}_1}$  if and only if the equation in Lemma A.1 involving  $\alpha_4$  holds.

**Proof.** We have  $\delta(\psi)_{\bar{Y}_1 \bar{X}_1} = \psi_{\bar{Y}_1} X_1 + \bar{Y}_1 \psi_{\bar{X}_1} + \psi_{\bar{X}_2} \bar{Y}_2 + \bar{X}_2 \psi_{\bar{Y}_2}$ , where  $\psi_{\bar{Y}_1} X_1, \bar{X}_2 \psi_{\bar{Y}_2}$  are involutive to  $\bar{Y}_1 \psi_{\bar{X}_1}, \psi_{\bar{X}_2} \bar{Y}_2$  by exchanging  $\mu_i$  to  $\nu_i$ . Since  $\bar{Y}_1 = \bar{y}_1$  we have

$$\begin{aligned} \bar{Y}_1 \psi_{\bar{X}_1} &= \mu_5 \bar{y}_1 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots 2} + \mu_6 \bar{y}_1 \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+3} \bar{y}_{n+2\dots 4}^3 \\ &\quad + \mu_7 \bar{y}_1 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots 2} \end{aligned}$$

$$= -\mu_7 \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots 2}$$

where by Lemma 3.26 (i) the first two summands vanish and the third one uses

$$\bar{y}_1 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} = -\bar{x}_4^3 \bar{x}_3^{2'} \bar{x}_{4\dots n+1}^2 \bar{x}_{n+2\dots nm-1} = -\bar{x}_{2\dots nm-1}. \quad (\text{A.9})$$

Here, the first equality uses an anticommutativity relation across a square and the second equality uses the Plücker-type relation  $\bar{x}_3^{2'} \bar{x}_4^2 = -\bar{x}_3^1 \bar{x}_4^{2'} - \bar{x}_5^3 \bar{y}_5^{3'}$  and then the anticommutativity relations ( $2n - 5$  times). Again, the summand given by  $\bar{x}_5^{3'} \bar{y}_5^3$  vanishes by Observation A.2.

Note that  $\bar{Y}_2 = \bar{y}_2$ . Then  $\psi_{\bar{X}_2} \bar{Y}_2 = \mu_8 \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots 2}$  is already irreducible. Comparing the coefficients we complete the proof.  $\square$

### A.3. The term $\delta(\psi)_{\bar{Y}_3 \bar{X}_3}$

**Lemma A.10.**  $\delta(\psi)_{\bar{Y}_3 \bar{X}_3} = \tilde{\varphi}_{\bar{Y}_3 \bar{X}_3}$  if and only if the equation in Lemma A.1 involving  $\alpha_5$  holds.

**Proof.** We have  $\delta(\psi)_{\bar{Y}_3 \bar{X}_3} = \psi_{\bar{Y}_3} \bar{X}_3 + \bar{Y}_3 \psi_{\bar{X}_3} + \psi_{\bar{X}_6} \bar{Y}_6 + \bar{X}_6 \psi_{\bar{Y}_6}$ , where  $\psi_{\bar{Y}_3} \bar{X}_3, \bar{X}_6 \psi_{\bar{Y}_6}$  are involutive to  $\bar{Y}_3 \psi_{\bar{X}_3}, \psi_{\bar{X}_6} \bar{Y}_6$  by exchanging  $\mu_i$  to  $\nu_i$ . We claim that

$$\bar{Y}_3 \psi_{\bar{X}_3} = ((-1)^m \mu_9 - (-1)^n \mu_{11}) \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1.$$

Indeed, we have

$$\begin{aligned} \bar{Y}_3 \psi_{\bar{X}_3} &= \mu_9 \bar{Y}_3 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{y}_{n\dots 2}^1 \\ &\quad + \mu_{10} \bar{Y}_3 \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^1 \bar{y}_{n\dots 2}^1 \\ &\quad + \mu_{11} \bar{Y}_3 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \\ &= (-1)^m \mu_9 \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \\ &\quad + (-1)^{n-1} \mu_{11} \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \end{aligned}$$

where in the second equality we use  $\bar{Y}_3 \bar{x}_{1\dots nm-1} = 0$  by Remark 3.22 to obtain that the summand with  $\mu_{10}$  vanishes, and we also use the following relations

$$\begin{aligned} \bar{Y}_3 \bar{x}_{1\dots nm-2} &= (-1)^{n-1} \bar{x}_{2\dots n}^1 \bar{y}_n \bar{x}_{n\dots nm-2} \\ &= (-1)^m \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \\ \bar{Y}_3 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} &= -\bar{x}_2^1 \bar{x}_3^{2'} \bar{x}_{4\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \\ &= (-1)^{n-1} \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \end{aligned}$$

which may be verified by using the anticommutativity and Plücker-type relations (and applying Observation A.2). We have

$$\begin{aligned}\psi_{\bar{X}_6} \bar{Y}_6 &= \mu_{14} \bar{x}_{2\dots n} x_{n+1\dots nm-1} \bar{y}_{nm-1\dots 2n+1} \bar{y}_{2n\dots 3}^4 \bar{Y}_6 \\ &= \mu_{14} \bar{x}_{2\dots n}^1 \bar{x}_{n+1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1\end{aligned}$$

where similarly the second equality uses the anticommutativity relations ( $2n - 3$  times) and the Plücker-type relation once (also using Observation A.2).  $\square$

A.4. The term  $\delta(\psi)_{\bar{X}_0 X_7}$

**Lemma A.11.**  $\delta(\psi)_{\bar{X}_0 X_7} = \tilde{\varphi}_{\bar{X}_0 X_7}$  if and only if the equation in Lemma A.1 involving  $\alpha_6$  holds.

**Proof.** Note that  $\delta(\psi)_{\bar{X}_0 X_7} = \psi_{\bar{X}_0} X_7 + \bar{X}_0 \psi_{\bar{X}_7}$ . We claim that

$$\psi_{\bar{X}_0} X_7 = (-\mu_1 - (-1)^{n-m} \mu_3) \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 4}^2.$$

Indeed, since  $X_7 = \bar{x}_3^2$  we have

$$\begin{aligned}\psi_{\bar{X}_0} X_7 &= \mu_1 \bar{x}_{0\dots nm-3} \bar{y}_{nm-3\dots 1} \bar{x}_3^2 \\ &\quad + \mu_2 \bar{x}_{0\dots n(m-1)-2} \bar{x}_{n(m-1)-1\dots nm-3}^0 \bar{y}_{nm-3\dots n(m-1)-1} \bar{y}_{n(m-1)-2\dots 1} \bar{x}_3^2 \\ &\quad + \mu_3 \bar{x}_{0\dots nm-2} \bar{y}_{nm-2\dots n+2} \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 \\ &\quad + \mu_4 \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 \\ &= (-\mu_1 - (-1)^{n-m} \mu_3) \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 4}^2\end{aligned}$$

where the summands with coefficients  $\mu_3$  and  $\mu_4$  are computed in Lemma A.16 below. Let us compute the first two summands with coefficients  $\mu_1$  and  $\mu_2$ . For this, we apply the relation  $\bar{y}_{n-1\dots 1} \bar{x}_3^2 = -\bar{x}_n \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2$  (which uses the anticommutativity relations  $2n - 3$  times) to the two summands and obtain

$$\begin{aligned}-\mu_1 \bar{x}_{0\dots nm-3} \bar{y}_{nm-3\dots n} \bar{x}_n \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2 \\ + (-1)^n \mu_2 \bar{x}_{0\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2\end{aligned}\quad (\text{A.12})$$

where the second summand also uses the involutive version of Lemma 3.31. The second summand of (A.12) vanishes by Lemma 3.26 (i) and the first one equals

$$\begin{aligned}-\mu_1 \bar{x}_{0\dots nm-3} \bar{y}_{nm-3\dots n} \bar{x}_n \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2 \\ = -(-1)^{n+m} \mu_1 \bar{x}_{0\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2 \\ + \mu_1 \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^5 \bar{x}_{n+1} \bar{y}_{n+1\dots 4}^2 \\ = -\mu_1 \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 4}^2\end{aligned}$$

where the first equality follows from Lemma 3.27, in the second equality, the first summand vanishes by Lemma 3.26 (i) and for the second summand we use the anticommutativity relations ( $2n - 4$  times) and the Plücker-type relation (involving  $\bar{y}_{n+4}^5 \bar{y}_{n+3}^{5'}$ ) once (applying Observation A.2 to see the vanishing of the extra term). This proves the claim.

Since  $\bar{X}_0 = \bar{x}_0$  we have that  $\bar{X}_0 \psi_{\bar{X}_7} = \mu_{15} \bar{x}_{0\dots nm-1} \bar{y}_{nm-1\dots n+2} y_{n+1\dots 4}^2$  is already irreducible. We complete the proof by comparing the coefficients.  $\square$

#### A.5. The term $\delta(\psi)_{\bar{Y}_1 \bar{X}_3}$

**Lemma A.13.**  $\delta(\psi)_{\bar{Y}_1 \bar{X}_3} = \tilde{\varphi}_{\bar{Y}_1 \bar{X}_3}$  if and only if the equation in Lemma A.1 involving  $\alpha_8$  holds.

**Proof.** We have  $\delta(\psi)_{\bar{Y}_1 \bar{X}_3} = \psi_{\bar{Y}_1} \bar{X}_3 + \bar{Y}_1 \psi_{\bar{X}_3} + \psi_{\bar{X}_4} \bar{Y}_5 + \bar{X}_4 \psi_{\bar{Y}_5}$ . We claim that

$$\psi_{\bar{Y}_1} \bar{X}_3 = ((-1)^m \nu_5 - (-1)^n \nu_7) \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1.$$

Indeed, we have

$$\begin{aligned} \psi_{\bar{Y}_1} \bar{X}_3 &= \nu_5 \bar{x}_{2\dots nm-2} \bar{y}_{nm-2\dots 1} \bar{X}_3 + \nu_6 \bar{x}_{4\dots n+2}^3 \bar{x}_{n+3\dots nm-1} \bar{y}_{nm-1\dots 1} \bar{X}_3 \\ &\quad + \nu_7 \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 3}^2 \bar{X}_3 \\ &= -(-1)^n \nu_5 \bar{x}_{2\dots nm-2} \bar{y}_{nm-2\dots n} \bar{x}_n \bar{y}_{n\dots 2}^1 \\ &\quad - (-1)^n \nu_7 \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \\ &= ((-1)^m \nu_5 - (-1)^n \nu_7) \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \end{aligned}$$

where in the second equality, the summand with coefficient  $\nu_6$  vanishes by Remark 3.22, the summand with  $\nu_5$  uses the anticommutativity relations ( $n - 1$  times), and the summand with  $\nu_7$  uses the anticommutativity relations ( $n - 4$  times) and the Plücker-type relation  $\bar{x}_3^2 \bar{x}_4^2 = -\bar{x}_3^1 \bar{x}_4^{2'} - \bar{x}_5^3 \bar{y}_5^{3'}$  once. The term  $\bar{x}_5^3 \bar{y}_5^{3'}$  vanishes by Observation A.2. The third equality follows from Lemma 3.27.

Since  $\bar{Y}_1 = \bar{y}_1$ , we have

$$\begin{aligned} \bar{Y}_1 \psi_{\bar{X}_3} &= \mu_9 \bar{y}_1 \bar{x}_{1\dots nm-2} \bar{y}_{nm-2\dots n+1} \bar{y}_{n\dots 2}^1 \\ &\quad + \mu_{10} \bar{y}_1 \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots 2n+2} \bar{y}_{2n+1\dots n+3}^5 \bar{y}_{n\dots 2}^1 \\ &\quad + \mu_{11} \bar{y}_1 \bar{x}_{3\dots n+1}^2 \bar{x}_{n+2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \\ &= -\mu_{11} \bar{x}_{2\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 2}^1 \end{aligned}$$

where the first two summands vanish by Lemma 3.26 (i) and the third one follows from (A.9). Since  $\bar{X}_4 = \bar{x}_4^3$  we have

$$\begin{aligned}\bar{X}_4\psi_{\bar{Y}_5} &= \nu_{13}\bar{x}_4^3\bar{x}_{3\dots n}^1\bar{x}_{n+1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1 \\ &= (-1)^n\nu_{13}\bar{x}_{2\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1\end{aligned}$$

where the second equality uses the anticommutativity relations ( $n-2$  times).

Since  $\bar{Y}_5 = \bar{y}_2^1$ , the term  $\psi_{\bar{X}_4}\bar{Y}_5 = \mu_{12}\bar{x}_{2\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1$  is already irreducible. We complete the proof by comparing the coefficient of each summand.  $\square$

#### A.6. The term $\delta(\psi)_{\bar{X}_3\bar{X}_5}$

**Lemma A.14.**  $\delta(\psi)_{\bar{X}_3\bar{X}_5} = \tilde{\varphi}_{\bar{X}_3\bar{X}_5}$  if and only if the equation in Lemma A.1 involving  $\alpha_{10}$  holds.

**Proof.** We have  $\delta(\psi)_{\bar{X}_3\bar{X}_5} = \psi_{\bar{X}_3}\bar{X}_5 + \bar{X}_3\psi_{\bar{X}_5} + \psi_{\bar{X}_1}\bar{X}_4 + \bar{X}_1\psi_{\bar{X}_4} + \psi_{\bar{X}_7}\bar{y}_3^{2'}$ . Then we claim that

$$\psi_{\bar{X}_3}\bar{X}_5 = \mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1.$$

Indeed, since  $\bar{X}_5 = \bar{x}_2^1$  we have

$$\begin{aligned}\psi_{\bar{X}_3}\bar{X}_5 &= \mu_9\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 2}^1\bar{x}_2^1 \\ &\quad + \mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots 2n+2}\bar{y}_{2n+1\dots n+3}^1\bar{y}_{n\dots 2}^1\bar{x}_2^1 \\ &\quad + \mu_{11}\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1\bar{x}_2^1 \\ &= \mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1\end{aligned}\tag{A.15}$$

where the third summand vanishes since  $\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 2}^1\bar{x}_2^1 = 0$  by Remark 3.22.

We first show that the first summand involving  $\mu_9$  in (A.15) vanishes. Note that  $\bar{y}_2^1\bar{x}_2^1 = -\bar{x}_3^1\bar{y}_3^1 - \bar{x}_3^{2'}\bar{y}_3^{2'} - \bar{x}''\bar{y}''$  where  $\bar{x}''$  is the other longer arrow (which is missing in Fig. 12) with the same starting vertex as  $\bar{x}_3^1$ . So we have

$$\begin{aligned}\mu_9\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 2}^1\bar{x}_2^1 \\ = -\mu_9\bar{x}_{1\dots nm-2}(\bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 3}^1\bar{x}_3^1\bar{y}_3^1 + \bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 3}^1\bar{x}_3^{2'}\bar{y}_3^{2'}) = 0\end{aligned}$$

where the first equality uses  $\bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 3}^1\bar{x}'' = 0$  by Remark 3.22. In the second equality, the first summand vanishes by using  $\bar{y}_i^1\bar{x}_i^1 = -\bar{x}_{i+1}^1\bar{y}_{i+1}^1$  for  $3 \leq i \leq n-1$  and  $\bar{y}_n^1\bar{x}_n^1 = 0$  and the second summand vanishes since we have

$$\bar{y}_{nm-2\dots n+1}\bar{y}_{n\dots 3}^1\bar{x}_3^{2'} = (-1)^n\bar{y}_{nm-2\dots n+1}\bar{x}_{n+1}\bar{y}_{n+1\dots 4}^2 = 0.$$

Here, the first equality uses the anticommutativity relations and the second one follows from Lemma 3.26 (i).

We now compute the summand with  $\mu_{10}$  in (A.15). We have

$$\begin{aligned}\mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots 2n+2}\bar{y}_{2n+1\dots n+3}^5\bar{y}_{n\dots 2}^1\bar{x}_2^1 \\ &= (-1)^{n-1}\mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots 2n+2}\bar{y}_{2n+1\dots n+3}^5\bar{x}_{n+1}\bar{y}_{n+1\dots 4}^2\bar{y}_3^{2'} \\ &= \mu_{10}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1\end{aligned}$$

where the first equality is similar to the above computation, and the second equality uses the Plücker-type relations twice (involving  $\bar{y}_4^2\bar{y}_3^{2'}$  and  $\bar{y}_{n+4}^2\bar{y}_{n+3}^{2'}$ ) and the anticommutativity relations ( $3n-7$  times). This proves the claim.

Using (A.7) and Lemma 3.26 (i) we obtain

$$\begin{aligned}\bar{X}_3\psi_{\bar{X}_5} &= \mu_{13}\bar{X}_3\bar{x}_{2\dots n}^1\bar{x}_{n+1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1 \\ &= (-1)^{n-1}\mu_{13}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1.\end{aligned}$$

Since  $\bar{X}_4 = \bar{x}_4^3$  we have

$$\begin{aligned}\psi_{\bar{X}_1}\bar{X}_4 &= \mu_5\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots 2}\bar{x}_4^3 + \mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 4}^3\bar{x}_4^3 \\ &\quad + \mu_7\bar{x}_{3\dots n+1}^2\bar{x}_{n+2\dots nm-1}\bar{y}_{nm-1\dots 2}\bar{x}_4^3 \\ &= (-1)^n\mu_5\bar{x}_{1\dots nm-2}\bar{y}_{nm-2\dots n}\bar{x}_n\bar{y}_{n\dots 3}^1 \\ &\quad - \mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1 \\ &= ((-1)^{m-1}\mu_5 - \mu_6)\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1\end{aligned}$$

where the summand with  $\mu_7$  vanishes since  $\bar{y}_{nm-1\dots 2}\bar{x}_4^3 = 0$  by Remark 3.22.

To obtain the summand with coefficient  $\mu_5$  we use the anticommutativity relations and Lemma 3.27. To obtain the summand with  $\mu_6$ , we apply the relation  $\bar{y}_4^3\bar{x}_4^3 = -\bar{x}_3^{2'}\bar{y}_3^{2'} - \bar{x}_3^{4'}\bar{y}_3^{4'} - \bar{x}_3^3\bar{y}_5^3$  and obtain

$$\begin{aligned}\mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 4}^3\bar{x}_4^3 \\ &= -\mu_6\bar{x}_{1\dots nm-1}(\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 5}^3\bar{x}_3^{2'}\bar{y}_3^{2'} + \bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 5}^3\bar{x}_3^{4'}\bar{y}_3^{4'}) \\ &= -\mu_6\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+1}\bar{y}_{n\dots 3}^1\end{aligned}$$

where the first equality uses  $\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 5}^3\bar{x}_5^3 = 0$  by Remark 3.22 and in the second equality we note that  $\bar{y}_5^3, \bar{x}_3^{4'}$  lie in a square so that  $\bar{y}_5^3\bar{x}_3^{4'} = -\bar{x}'\bar{y}''$  for some  $\bar{x}', \bar{y}''$ , then  $\bar{y}_{nm-1\dots n+3}\bar{y}_{n+2\dots 5}^3\bar{x}' = 0$  by Remark 3.22 and thus the second summand vanishes. For the first summand we use the anticommutativity relations ( $2n-5$  times) and the Plücker-type relation once (involving  $\bar{y}_4^2\bar{y}_3^{2'}$ ).

Similar to the second equality in (A.9) we have

$$\psi_{\bar{X}_7}\bar{y}_3^{2'} = \mu_{15}\bar{x}_{1\dots nm-1}\bar{y}_{nm-1\dots n+2}\bar{y}_{n+1\dots 4}^2\bar{y}_3^{2'}$$

$$= (-1)^n \mu_{15} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 3}^1.$$

Note that  $\bar{X}_1 \psi_{\bar{X}_4} = \mu_{12} \bar{x}_{1\dots nm-1} \bar{y}_{nm-1\dots n+1} \bar{y}_{n\dots 3}^1$  is already irreducible.  $\square$

The following lemma was used in Lemma A.11.

**Lemma A.16.** *We have*

$$\begin{aligned} \mu_3 \bar{y}_{nm-2\dots n+2} \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 &= -(-1)^{n-m} \mu_3 \bar{y}_{nm-1\dots n+2} \bar{y}_{n+1\dots 4}^2 \\ \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 &= 0. \end{aligned}$$

**Proof.** Let us prove the first identity. Note that we have

$$\begin{aligned} \mu_3 \bar{y}_{nm-2\dots n+2} \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 &= \mu_3 \bar{y}_{nm-2\dots n+2} (\bar{y}_{n+1\dots 5}^2 \bar{x}_5^2 \bar{y}_5^2 \bar{y}_4^2 + \bar{y}_{n+1\dots 5}^2 \bar{x}_5^{3'} \bar{y}_5^{3'} \bar{y}_4^2 - \bar{y}_{n+1\dots 4}^2 \bar{x}_4^{7'} \bar{y}_4^{7'}) \\ &= -\mu_3 \bar{y}_{nm-2\dots 2n} \bar{x}_{2n} \bar{y}_{2n\dots n+2} \bar{y}_{n+1\dots 4}^2 \end{aligned}$$

where the first equality uses  $\bar{y}_3^2 \bar{x}_3^2 = -\bar{x}_4^2 \bar{y}_4^2 - \bar{x}_4^{7'} \bar{y}_4^{7'}$  and then  $\bar{y}_4^2 \bar{x}_4^2 = -\bar{x}_5^2 \bar{y}_5^2 - \bar{x}_5^{3'} \bar{y}_5^{3'}$ . Let us explain the second equality: the first summand vanishes by using  $\bar{y}_i^2 \bar{x}_i^2 = -\bar{x}_{i+1}^2 \bar{y}_{i+1}^2$  for  $5 \leq i \leq n$  and  $\bar{y}_{n+1}^2 \bar{x}_{n+1}^2 = 0$ ; the second summand vanishes since by the anticommutativity relations  $\bar{y}_i^2 \bar{x}_i^{3'} = -\bar{x}_{i+1}^{3'} \bar{y}_{i+1}^3$  for  $5 \leq i \leq n+1$  (denote  $\bar{x}_{n+2}^{3'} = \bar{x}_{n+2}$ ) we have

$$\bar{y}_{nm-2\dots n+2} \bar{y}_{n+1\dots 5}^2 \bar{x}_5^{3'} = -(-1)^n \bar{y}_{nm-2\dots n+2} \bar{x}_{n+2} \bar{y}_{n+2\dots 6}^3 = 0$$

where the second equality follows from Lemma 3.26 (i), and for the third summand we use the anticommutativity relations (4n - 7 times) and the Plücker-type relation  $\bar{y}_5^7 \bar{y}_4^{7'} = -\bar{y}_5^{7'} \bar{y}_4^2 - \dots$  (Observation A.2 is applied to kill the extra term here). Then the desired equality follows from Lemma 3.27.

Let us prove the second identity. Note that we have

$$\begin{aligned} \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{y}_{n+1\dots 3}^2 \bar{x}_3^2 &= (-1)^{n+1} \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 (\bar{x}_{n+2} \bar{y}_{n+2\dots 6}^3 \bar{y}_5^{3'} \bar{y}_4^2 + \bar{x}_{n+4} \bar{y}_{n+2\dots 5}^7 \bar{y}_4^{7'}) \\ &= \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 (\bar{x}_{n+2} \bar{y}_{n+2} \bar{y}_{n+1\dots 4}^2 - \bar{x}_{n+4}^6 \bar{y}_{n+4}^6 \bar{y}_{n+1\dots 4}^2) \end{aligned} \quad (\text{A.17})$$

where the first equality follows from a similar computation as above, the second one uses the relations  $\bar{y}_i^3 \bar{y}_{i-1}^{3'} = -\bar{y}_i^{3'} \bar{y}_{i-1}^2$  and  $\bar{y}_j^7 \bar{y}_{j-1}^{7'} = -\bar{y}_j^{7'} \bar{y}_{j-1}^2$  for  $6 \leq i \leq n+2$  and  $5 \leq j \leq n+2$  (denote  $\bar{y}_{n+2}^{3'} = \bar{y}_{n+2}$  and  $\bar{y}_{n+2}^{7'} = \bar{y}_{n+4}^6$ ).

The first summand on the right-hand side of the second equality in (A.17) equals

$$\begin{aligned} \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{x}_{n+2} \bar{y}_{n+2} \bar{y}_{n+1\dots 4}^2 \\ = \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+6}^6 \bar{y}_{n+5}^{6'} \bar{y}_{n+4}^{5'} \bar{y}_{n+3}^{4'} \bar{y}_{n+2} \bar{y}_{n+1\dots 4}^2 \end{aligned}$$

where we first use  $\bar{y}_{n+4}^6 \bar{x}_{n+2} = -\bar{x}_{n+3}^4 \bar{y}_{n+3}^{4'}$  and then  $\bar{y}_{n+5}^6 \bar{x}_{n+3}^4 = -\bar{y}_{n+5}^{6'} \bar{y}_{n+4}^{5'}$ .

Let us compute the second summand in (A.17). Note that  $\bar{y}_{n+4}^6 \bar{x}_{n+4}^6 = -\bar{x}_{n+3}^{5'} \bar{y}_{n+3}^{5'} - \bar{x}_{n+5}^6 \bar{y}_{n+5}^6 - \bar{x}' \bar{y}'$  where  $\bar{x}'$  is the arrow (which is missing in Fig. 12) with the same starting vertex as  $\bar{x}_{n+5}^6$ . Then we have

$$\begin{aligned} \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+4}^6 \bar{x}_{n+4}^6 \bar{y}_{n+4}^6 \bar{y}_{n+1\dots 4}^2 \\ = -\mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+5}^6 (\bar{x}_{n+3}^{5'} \bar{y}_{n+3}^{5'} + \bar{x}_{n+5}^6 \bar{y}_{n+5}^6 + \bar{x}' \bar{y}') \bar{y}_{n+4}^6 \bar{y}_{n+1\dots 4}^2 \\ = -\mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+5}^6 \bar{x}_{n+3}^{5'} \bar{y}_{n+3}^{5'} \bar{y}_{n+4}^6 \bar{y}_{n+1\dots 4}^2 \\ = \mu_4 \bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+6}^6 \bar{y}_{n+5}^{6'} \bar{y}_{n+4}^{5'} \bar{y}_{n+3}^{4'} \bar{y}_{n+2} \bar{y}_{n+1\dots 4}^2. \end{aligned} \quad (\text{A.18})$$

To see the second equality of (A.18) follows since the second summand vanishes using  $\bar{y}_{n+5}^6 \bar{x}_{n+5}^6 = 0$  and the third summand vanishes, since  $\bar{y}_{n+5}^6 \bar{x}'$  lies in a square so that  $\bar{y}_{n+5}^6 \bar{x}' = -\bar{x}'' \bar{y}'''$  for some arrows  $\bar{x}'', \bar{y}'''$ . We then have

$$\bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+5}^6 \bar{x}' = -\bar{y}_{nm-1\dots 2n+3} \bar{y}_{2n+2\dots n+6}^6 \bar{x}'' \bar{y}''' = 0$$

where the second equality follows from Remark 3.22. For the third equality of (A.18) we use  $\bar{y}_{n+5}^6 \bar{x}_{n+3}^{5'} = -\bar{y}_{n+5}^{6'} \bar{y}_{n+4}^{5'}$ ,  $\bar{y}_{n+4}^{5'} \bar{y}_{n+3}^{5'} = -\bar{y}_{n+4}^{5'} \bar{y}_{n+3}^{4'} - \dots$  and then  $\bar{y}_{n+3}^{4'} \bar{y}_{n+4}^6 = -\bar{y}_{n+3}^{4'} \bar{y}_{n+2}^6$  (the extra term  $\dots$  vanishes due to Observation A.2).

Substituting the above equalities into (A.17) we obtain the desired equality.  $\square$

## References

- [1] M. Abouzaid, I. Smith, The symplectic arc algebra is formal, *Duke Math. J.* 165 (2016) 985–1060.
- [2] M. Abouzaid, I. Smith, Khovanov homology from Floer cohomology, *J. Am. Math. Soc.* 32 (2019) 1–79.
- [3] M.J. Bardzell, The alternating syzygy behavior of monomial algebras, *J. Algebra* 188 (1997) 69–89.
- [4] S. Barmeier, Z. Wang, Deformations of path algebras of quivers with relations, *Astérisque*, arXiv: 2002.10001, 2020, forthcoming.
- [5] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, *J. Am. Math. Soc.* 9 (1996) 473–527.
- [6] G.M. Bergman, The Diamond Lemma for ring theory, *Adv. Math.* 29 (1978) 178–218.
- [7] T. Braden, Perverse sheaves on Grassmannians, *Can. J. Math.* 54 (2002) 493–532.
- [8] A. Braverman, D. Gaitsgory, Poincaré–Birkhoff–Witt theorem for quadratic algebras of Koszul type, *J. Algebra* 181 (1996) 315–328.
- [9] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra I: cellularity, *Mosc. Math. J.* 11 (2011) 685–722.
- [10] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra II: Koszulity, *Transform. Groups* 15 (2010) 1–45.
- [11] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra III: category  $\mathcal{O}$ , *Represent. Theory* 15 (2011) 170–243.
- [12] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup, *J. Eur. Math. Soc.* 14 (2012) 373–419.

- [13] R.-O. Buchweitz, Hochschild cohomology of Koszul algebras, in: *Conference on Representation Theory*, Canberra, 2003.
- [14] Y. Chen, M. Khovanov, An invariant of tangle cobordisms via subquotients of arc rings, *Fundam. Math.* 225 (2014) 23–44.
- [15] X. Chen, S. Yang, G. Zhou, Batalin–Vilkovisky algebras and the noncommutative Poincaré duality of Koszul Calabi–Yau algebras, *J. Pure Appl. Algebra* 220 (2016) 2500–2532.
- [16] X.-W. Chen, H. Li, Z. Wang, Leavitt path algebras,  $B_\infty$ -algebras and Keller’s conjecture for singular Hochschild cohomology, [arXiv:2007.06895](https://arxiv.org/abs/2007.06895), 2020.
- [17] X.-W. Chen, Z. Wang, The dg Leavitt algebra, singular Yoneda category and singularity category, with an appendix by Bernhard Keller and Yu Wang, *Adv. Math.* 440 (2024) 109541.
- [18] S. Chouhy, A. Solotar, Projective resolutions of associative algebras and ambiguities, *J. Algebra* 432 (2015) 22–61.
- [19] A.B. Conner,  $A_\infty$ -structures, generalized Koszul properties, and combinatorial topology, PhD thesis, available at <https://hdl.handle.net/1794/11559>, University of Oregon, 2011.
- [20] G. Fløystad, J.E. Vatne, PBW-deformations of  $N$ -Koszul algebras, *J. Algebra* 302 (2006) 116–155.
- [21] M. Gerstenhaber, On the deformation of rings and algebras, *Ann. Math.* 79 (1964) 59–103.
- [22] J. Huebschmann, T. Kadeishvili, Small models for chain algebras, *Math. Z.* 207 (1991) 245–280.
- [23] T.V. Kadeishvili, The algebraic structure in the homology of an  $A(\infty)$ -algebra (Russian), *Soobshch. Akad. Nauk Gruz. SSR* 108 (1982) 249–252.
- [24] T.V. Kadeishvili, The structure of the  $A(\infty)$ -algebra, and the Hochschild and Harrison cohomologies, *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruz. SSR* 91 (1988) 19–27.
- [25] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* 53 (1979) 165–184.
- [26] B. Keller, Introduction to  $A$ -infinity algebras and modules, *Homol. Homotopy Appl.* 3 (2001) 1–35.
- [27] B. Keller, Derived invariance of higher structures on the Hochschild complex, available at <https://webusers.imj-prg.fr/~bernhard.keller/publ/dih.pdf>, 2003.
- [28] B. Keller, A remark on Hochschild cohomology and Koszul duality, in: *Advances in Representation Theory of Algebras, ARTA VII (Mexico City 2018)*, American Mathematical Society, Providence, RI, 2021, pp. 131–136.
- [29] M. Khovanov, A categorification of the Jones polynomial, *Duke Math. J.* 101 (2000) 359–426.
- [30] A. Klamt, C. Stroppel, On the Ext algebras of parabolic Verma modules and  $A_\infty$ -structures, *J. Pure Appl. Algebra* 216 (2012) 323–336.
- [31] A. Lascoux, M.-P. Schützenberger, Polynômes de Kazhdan et Lusztig pour les Grassmanniennes, *Astérisque* 87–88 (1981) 249–266.
- [32] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, Berlin, 1992.
- [33] C.Y. Mak, I. Smith, Fukaya–Seidel categories of Hilbert schemes and parabolic category  $\mathcal{O}$ , *J. Eur. Math. Soc.* 24 (2022) 3215–3332.
- [34] C. Manolescu, Nilpotent slices, Hilbert schemes, and the Jones polynomial, *Duke Math. J.* 132 (2006) 311–369.
- [35] L. Positselski, Two kinds of derived categories, Koszul duality, and comodule–contramodule correspondence, *Mem. Am. Math. Soc.* 212 (2011) 996.
- [36] S.B. Priddy, Koszul resolutions, *Trans. Am. Math. Soc.* 152 (1970) 39–60.
- [37] L. Rozansky, A categorification of the stable  $SU(2)$  Witten–Reshetikhin–Turaev invariant of links in  $S^2 \times S^1$ , [arXiv:1011.1958](https://arxiv.org/abs/1011.1958), 2010.
- [38] P. Seidel, Fukaya categories and deformations, in: *Proceedings of the International Congress of Mathematicians (Beijing 2002)*, vol. II, Higher Education Press, Beijing, 2002, pp. 351–360.
- [39] P. Seidel, *Fukaya Categories and Picard–Lefschetz Theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2008.
- [40] P. Seidel, I. Smith, A link invariant from the symplectic geometry of nilpotent slices, *Duke Math. J.* 134 (2006) 453–514.
- [41] P. Seidel, R. Thomas, Braid group actions on derived categories of coherent sheaves, *Duke Math. J.* 108 (2001) 37–108.
- [42] C. Stroppel, Parabolic category  $\mathcal{O}$ , perverse sheaves on Grassmannians, Springer fibres and Khovanov homology, *Compos. Math.* 145 (2009) 954–992.
- [43] C. Stroppel, Schur–Weyl dualities and link homologies, in: *Proceedings of the International Congress of Mathematicians (Hyderabad 2010)*, vol. III, Hindustan Book Agency, New Delhi, 2010, pp. 1344–1365.
- [44] C. Stroppel, Categorification: tangle invariants and TQFTs, in: *Proceedings of the International Congress of Mathematicians (Virtual ICM 2022)*, vol. II, EMS Press, Berlin, 2023, pp. 1312–1353.

- [45] M. Vybornov, Perverse sheaves, Koszul IC-modules, and the quiver for the category  $\mathcal{O}$ , *Invent. Math.* 167 (2007) 19–46.
- [46] B. Webster, Tensor product algebras, Grassmannians and Khovanov homology, in: *Physics and Mathematics of Link Homology*, Montreal, 2013, American Mathematical Society, Providence, RI, 2016, pp. 23–58.