Indefinite Theta Functions and Higher Depth Mock Modular Forms

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Kurzzusammenfassung

Diese Dissertation besteht aus Forschungsarbeiten über indefinite Thetafunktionen, Mock-Modulformen höherer Tiefe, Quanten-Modulformen höherer Tiefe und iterierte Eichler-Integrale. Zunächst betrachten wir eine explizite indefinite Thetafunktion der Signatur $(1, 3)$, die im Gromov-Witten-Potential einer elliptischen Orbifaltigkeit auftritt, und bestimmen ihre modulare Vervollständigung durch Methoden von Zwegers und Alexandrov, Banerjee, Manschot und Pioline, woraus folgt, dass sie Mock-Modulformen der Tiefe 3 sind. Weiter betrachten wir falsche Thetafunktionen höheren Ranges und zeigen, dass ihre asymptotische Entwicklung an rationalen Punkten übereinstimmt mit den Entwicklungen von iterierten Eichler-Integralen über definite Thetafunktionen. Die modularen Eigenschaften der Eichler-Integrale implizieren Quanten-Modularität höherer Tiefe für die falschen Thetafunktionen höheren Ranges. Außerdem zeigen wir, dass eines der auftretenden Eichler-Integrale gleichzeitig der rein nicht-holomorphe Teil einer indefiniten Thetareihe der Signatur $(2, 2)$ ist. Darüber hinaus beweisen wir, dass diese falschen Thetafunktionen unter der vollen Modulgruppe als vektorwertige Quanten-Modulformen höheren Tiefe transformieren und verallgemeinern eine Gleichung zwischen Eichler-Integralen und Mordell-Integralen zu einem zweidimensionalen Fall. Schließlich geben wir q-Reihen an, die die anderen Komponenten der vektorwertigen Quanten-Modulformen höherer Tiefe liefern sollten und zeigen dies in einem Fall.

Abstract

This thesis consists of research articles on indefinite theta functions, higher depth mock modular forms, higher depth quantum modular forms, and iterated Eichler integrals. First, we study an explicit indefinite theta function of signature $(1, 3)$ that occurs in the Gromov-Witten theory of elliptic orbifolds and determine its modular completion, showing that it is a mock modular form of depth 3 using methods of Zwegers and Alexandrov, Banerjee, Manschot, and Pioline. We continue by studying higher rank false theta functions and show that they have the same asymptotic behavior near rationals as iterated Eichler integrals of theta functions. The modular behavior of the Eichler integrals implies that the higher rank false theta functions are quantum modular forms of higher depth. Additionally, one of the occurring Eichler integrals is essentially the purely non-holomorphic part of an indefinite theta function of signature $(2, 2)$. Furthermore, we show that these false theta-functions actually satisfy higher depth vector-valued quantum modular behavior. We also generalize a connection between Eichler integrals and Mordell integrals to a two-dimensional case. Finally, we suggest q-series that should contribute the other components of the higher depth vector-valued quantum modular forms and show that they span a space that is essentially closed under modular transformation in a special case.

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Contents

Chapter I

Introduction and Statement of Objectives

This thesis consists mostly of the research articles [BKR, BKM1, BKM2, BKM3] that deal with indefinite theta functions and several related objects. In this chapter, we first recall the collected scientific context of these articles and then present their results, restating parts of their introductions and repeating the central theorems.

I.1 Definitions and previous results

I.1.1 Modular forms and Jacobi forms of matrix index

Modular forms and more general objects that behave nicely under the action of the modular group are fundamental objects in number theory. Their Fourier and Taylor coefficients, expansions and asymptotics have an abundance of connections to other fields in mathematics.

We begin by introducing some standard notation. Let $\mathbb{H} := \{\tau = u + iv \in \mathbb{C}; v > 0\}$ be the *complex upper half plane* and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); N|c \right\}$$

the *congruence subgroup of level* N of the *modular group* $\mathrm{SL}_2(\mathbb{Z})$. It acts on $\tau \in \mathbb{H}$ by *Möbius transformation*

$$M\tau := \frac{a\tau + b}{c\tau + d} \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Definition I.1.1. We call a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ *modular form of weight* $k \in \mathbb{Z}$ for $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ with character χ (and write $f \in M_k(\Gamma, \chi)$) if

$$f(M\tau) = \chi(d)(c\tau + d)^k f(\tau) \tag{I.1.1}$$

holds for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and f is bounded at all cusps $\Gamma_0(N) \setminus (\mathbb{Q} \cup \{\infty\})$ of $\Gamma_0(N)$. Holomorphic functions satisfying (I.1.1) that may have poles at the cusps are called *weakly holomorphic modular forms*. We call modular forms *cusp forms* and write $f \in S_k(\Gamma, \chi)$ if they vanish at all cusps.

Similarly one can define modular forms of half-integral weight $k \in \frac{1}{2} + \mathbb{Z}$ by replacing the transformation law with

$$f(M\tau) = \varepsilon_d^{-2k} \left(\frac{c}{d}\right) \chi(d)(c\tau + d)^k f(\tau)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, where $\varepsilon_d := i$ for $d \equiv 3 \pmod 4$ and $\varepsilon_d := 1$ otherwise.

Furthermore, we consider multivariate functions $\mathbb{C} \times \mathbb{H} \to \mathbb{C}, (z, \tau) \mapsto f(z; \tau)$ called *Jacobi forms* that satisfy modular transformation properties in τ, elliptic transformation properties in the z-variable (i.e., a simple behavior under shifts by $\mathbb{Z} + \tau\mathbb{Z}$) and certain growth conditions, such as the *Jacobi theta function* ($\zeta := e^{2\pi i z}$, $q := e^{2\pi i \tau}$, $z \in \mathbb{C}$, $\tau \in \mathbb{H}$)

$$\vartheta(z; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{\pi i n} q^{\frac{n^2}{2}} \zeta^n.$$

Since we are interested in generalizations in higher dimension, we refer to Eichler and Zagier [EZ] for the theory of classical Jacobi forms. We define *Jacobi forms of matrix index* as follows, where bold letters represent vectors throughout.

Definition I.1.2. Let $L_1, L_2 \subset \mathbb{Z}^N$ be lattices, $\nu_1 : \Gamma \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ a multiplier, $\nu_2 : L_1 \times L_2 \to S^1$ a homomorphism with finite image, $N \in \mathbb{N}$, and $A \in \frac{1}{4}\mathbb{Z}^{N \times N}$ with $A^T = A$ and $A_{j,j} \in \frac{1}{2}\mathbb{Z}$ for $j \in \{1, \ldots, N\}$. We call a meromorphic function $g : \mathbb{C}^N \times \mathbb{H} \to \mathbb{C}$ *Jacobi form of matrix index A and weight $k \in \frac{1}{2}\mathbb{Z}$ for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with respect to $L_1 \times L_2$ and ν_1,ν_2* if it satisfies the following transformation laws (for all $(\boldsymbol{z}, \tau) \in \mathbb{C}^N \times \mathbb{H}$):

1. For $\boldsymbol{m} \in L_1$, $\boldsymbol{\ell} \in L_2$ we have
$$g\left(\boldsymbol{z} + \boldsymbol{m}\tau + \boldsymbol{\ell}; \tau\right) = \nu_2(\boldsymbol{m}, \boldsymbol{\ell}) q^{-\boldsymbol{m}^T A \boldsymbol{m}} e^{-4\pi i \boldsymbol{m}^T A \boldsymbol{z}} g(\boldsymbol{z}; \tau).$$
2. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have
$$g\left(\frac{\boldsymbol{z}}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = \nu_1(M) \left(c\tau + d\right)^k e^{\frac{2\pi i c}{c\tau + d} \boldsymbol{z}^T A \boldsymbol{z}} g(\boldsymbol{z}; \tau).$$
3. For some $a > 0$, we have
$$g(\boldsymbol{z}; \tau) e^{-\frac{4\pi}{\mathrm{Im}(\tau)} \mathrm{Im}(\boldsymbol{z})^T A \mathrm{Im}(\boldsymbol{z})} \in O\left(e^{a\mathrm{Im}(\tau)}\right) \qquad \text{as } \mathrm{Im}(\tau) \to \infty.$$

Other authors refer to equivalent concepts as *Jacobi forms of lattice index*, see for example [Mo].

I.1.2 Definite theta functions

The definite theta function associated to a positive definite quadratic form given as $Q : \mathbb{Z}^N \to \mathbb{Z}, \boldsymbol{m} \mapsto \frac{1}{2}\boldsymbol{m}^T A\boldsymbol{m}$ and corresponding bilinear form $B(\boldsymbol{m}, \boldsymbol{n}) := \boldsymbol{m}^T A\boldsymbol{n}$ is defined as follows ($\boldsymbol{z} \in \mathbb{C}^N, \tau \in \mathbb{H}$)

$$\Theta_Q(\boldsymbol{z}; \tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^N} q^{Q(\boldsymbol{n})} e^{2\pi iB(\boldsymbol{z},\boldsymbol{n})}. \tag{I.1.2}$$

Schoeneberg showed in 1939 [Sc] that for even N, the function $\tau \mapsto \Theta_{Q,0}(0; \tau)$ is a *modular form* of weight $\frac{N}{2}$ for some subgroup $\Gamma_0(N_1)$ and character χ_Q, i.e., we have

$$\Theta_{Q,0}\left(0; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{\frac{N}{2}} \chi_Q(d)\Theta_{Q,0}(0; \tau) \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1).$$

One can easily show that definite theta functions also satisfy elliptic properties. They were the subject of influential research, for example by Eichler and Zagier, who developed the theory of classical Jacobi forms and used them to prove the Saito-Kurokawa conjecture [EZ, Z1].

I.1.3 Indefinite theta functions

Generalizing the study of theta functions to non-degenerate indefinite quadratic forms presents some difficulties. To begin with, the series in (I.1.2) does not converge for indefinite quadratic forms since arbitrarily large terms appear in the series since $|q^{Q(\ell\boldsymbol{n})} e^{2\pi iB(\boldsymbol{z},\ell\boldsymbol{n})}| \to \infty$ as $\ell \to \infty$ for $\boldsymbol{n} \in \mathbb{Z}^N$ with $Q(\boldsymbol{n}) < 0$. We will not discuss Siegel's results on dealing with this issue [Si1], but instead focus on the approach initiated by Zwegers in his doctoral thesis [Zw].

Convergent expressions can be obtained by restricting the summation to those lattice points in a suitable cone on which the quadratic form is positive and growing (i.e., it contains only finitely many lattice points with quadratic form below any fixed bound). Explicitly for signature $(r, 1)$, given two vectors $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{R}^N$ ($N = r + 1$) such that $Q(\boldsymbol{c}), Q(\boldsymbol{c}'), B(\boldsymbol{c}, \boldsymbol{c}') < 0$ one defines ($\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^N$, $\tau = u + iv \in \mathbb{H}$)

$$\Theta_{Q,(\boldsymbol{c},\boldsymbol{c}')}(\boldsymbol{z}; \tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left(\operatorname{sgn}\left(B\left(\boldsymbol{c}, \boldsymbol{n} + \frac{\boldsymbol{y}}{v}\right)\right) - \operatorname{sgn}\left(B\left(\boldsymbol{c}', \boldsymbol{n} + \frac{\boldsymbol{y}}{v}\right)\right)\right) q^{Q(\boldsymbol{n})} e^{2\pi iB(\boldsymbol{z},\boldsymbol{n})}. \tag{I.1.3}$$

Göttsche and Zagier first showed that holomorphic functions of this type are modular only in special cases [GZ]. In his doctoral thesis, Zwegers described the modularity properties of these indefinite theta functions of signature $(r, 1)$ in general, showing that they can be completed to a modular object of weight $\frac{N}{2}$ by adding an explicit real-analytic correction term [Zw]. While this completed modular form is not holomorphic, it is essentially a mock modular form, which are the holomorphic parts of harmonic Maass forms. We call a function that transforms as a modular form of weight k *harmonic Maass forms* if it is annihilated by the *weight k hyperbolic Laplacian*

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

and satisfies certain growth conditions (see [BFOR] for the theory and many applications of harmonic Maass forms).

By rewriting the hyperbolic Laplacian as $\Delta_k = -\xi_{2-k} \circ \xi_k$ with the *shadow operator* ξ_k given for real-analytic $f : \mathbb{H} \to \mathbb{C}$ as

$$\xi_k(f)(\tau) := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}} f(\tau)},$$

one can uniquely decompose a harmonic Maass form into an holomorphic part (called *mock modular form*) and a non-holomorphic part (called the *shadow* of the mock modular form). In other words, Zwegers showed that indefinite theta functions of signature $(r, 1)$ are essentially mock modular forms of weight $\frac{N}{2}$.

Furthermore, Zwegers showed that the mysterious "mock theta functions" appearing in Ramanujan's final letter to Hardy, such as the function

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)}, \tag{I.1.4}$$

fit in this framework.

Alexandrov, Banerjee, Manschot, and Pioline [ABMP] generalized Zwegers' results to quadratic forms of signature $(r, 2)$ in 2016, with the main innovation being the *generalized error functions* of the form

$$\mathrm{GL}_N(\mathbb{R}) \times \mathbb{R}^N \to \mathbb{R}, \quad (\mathcal{M}, \boldsymbol{x}) \mapsto \int_{\mathbb{R}^N} e^{-\pi(\boldsymbol{x}-\boldsymbol{y})^T(\boldsymbol{x}-\boldsymbol{y})} \prod_{j=1}^{N} \operatorname{sgn}((\mathcal{M}\boldsymbol{y})_j) d\boldsymbol{y}.$$

The name stems from the fact that for $N = 1$ this is essentially the well-known *error function* $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, which occurred in the case of signature $(r, 1)$ in [28]. An important tool in their proof of the modular properties was Vignéras' theorem in [Vi] which states that appropriately converging indefinite theta functions of the form

$$\sum_{\boldsymbol{n} \in \mathbb{Z}^N} p\left(\boldsymbol{n}\sqrt{v}\right) q^{Q(\boldsymbol{n})}$$

are modular forms of weight $\frac{N}{2}+\lambda \in \frac{1}{2}\mathbb{Z}$ if the function $p : \mathbb{R}^N \to \mathbb{R}$ satisfies the *Vignéras differential equation*

$$\left(\boldsymbol{x}^T \frac{\partial}{\partial \boldsymbol{x}} - \frac{1}{4\pi}\left(\frac{\partial}{\partial \boldsymbol{x}}\right)^T A^{-1}\frac{\partial}{\partial \boldsymbol{x}}\right) p(\boldsymbol{x}) = \lambda p(\boldsymbol{x}), \qquad \frac{\partial}{\partial \boldsymbol{x}} := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\right)^T.$$

Later in 2016, Nazaroglu [N] completed the proof for quadratic forms of arbitrary indefinite signature using the approach of Alexandrov, Banerjee, Manschot, and Pioline (although with somewhat restrictive conditions on the possible cones).

Related results on indefinite theta functions were also obtained by other authors. Kudla showed that indefinite theta functions can be viewed as integrals of Kudla-Millson theta series [Ku], Westerholt-Raum extended the approach to a more abstract geometric setting [WR] and Funke and Kudla used Kudla's approach to obtain general statements on modular completions of indefinite theta functions using so-called simplicial cones [FK].

I.1.4 Construction of the completion of an indefinite theta function

We continue by describing Nazaroglu's construction for the modular completion of indefinite theta functions in some detail [N]. For an indefinite quadratic form Q of signature (r, s), we require s pairs of vectors to generalize $(\boldsymbol{c}, \boldsymbol{c}')$ in I.1.3 and consider a product of differences of such sgn-functions. Specifically, write $C := (\boldsymbol{c}_1, \dots, \boldsymbol{c}_s, \boldsymbol{c}_1', \dots, \boldsymbol{c}_s') \in (\mathbb{R}^N)^{2s}$ and define the notation

$$\boldsymbol{c}_{j,S} := \begin{cases} \boldsymbol{c}_j' & \text{if } j \in S, \\ \boldsymbol{c}_j & \text{if } j \notin S \end{cases}$$

for $S \subseteq \{1, \dots, s\}$ to discuss the geometric conditions and to define the completion of the indefinite theta function. The conditions on C generalize those mentioned before (I.1.3), for example the spaces spanned by $C^{(S)} := (\boldsymbol{c}_{1,S}, \dots, \boldsymbol{c}_{s,S})$ should be negative definite of dimension s for $S \subseteq \{1, \dots, s\}$. The corresponding holomorphic theta function is

$$\begin{aligned} \Theta_{Q,C}(\boldsymbol{z};\tau) &:= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \prod_{j=1}^{s} \left(\operatorname{sgn}\left(B(\boldsymbol{c}_j, \boldsymbol{n})\right) - \operatorname{sgn}\left(B(\boldsymbol{c}_j', \boldsymbol{n})\right)\right) e^{2\pi i B(\boldsymbol{z},\boldsymbol{n})} q^{Q(\boldsymbol{n})} \qquad \text{(I.1.5)} \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \sum_{S\subseteq\{1,\dots,s\}} (-1)^{|S|} \left(\prod_{j=1}^{s} \operatorname{sgn}\left(B(\boldsymbol{c}_{j,S}, \boldsymbol{n})\right)\right) e^{2\pi i B(\boldsymbol{z},\boldsymbol{n})} q^{Q(\boldsymbol{n})}. \end{aligned}$$

Then one can obtain a modular (but not holomorphic) theta function by replacing each product of sign functions by a suitable generalized error function. Explicitly we let

$$\widehat{\Theta}_{Q,C}(\boldsymbol{z};\tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \sum_{S\subseteq\{1,\dots,s\}} (-1)^{|S|} E_s\left(C^{(S)}; \sqrt{2v}\boldsymbol{n}\right) e^{2\pi i B(\boldsymbol{z},\boldsymbol{n})} q^{Q(\boldsymbol{n})} \qquad \text{(I.1.6)}$$

with

$$E_s(C;\boldsymbol{x}) := \int_{\operatorname{span}(C)} e^{\pi Q(\pi_C(\boldsymbol{x})-\boldsymbol{y})} \prod_{j=1}^{N} \operatorname{sgn}(B(\boldsymbol{c}_j,\boldsymbol{y}))d\boldsymbol{y},$$

where $C = (\boldsymbol{c}_1,\ldots,\boldsymbol{c}_s)$ is a basis of an s-dimensional negative definite subspace of $\mathbb{R}^N$ with respect to Q and π_C the orthogonal projection to $\operatorname{span}(C)$. Nazaroglu showed that $\widehat{\Theta}_{Q,C}$ transforms like a vector-valued Jacobi form of weight $\frac{N}{2}$ if the conditions on the cone that ensure suitable convergence are satisfied [N, Theorem 4.1].

To show that the completed indefinite theta functions of signature (r,s) converge, it is helpful to decompose the $E_s(C;\boldsymbol{x})$ into functions

$$M_s(C;\boldsymbol{x}) := \left(\frac{i}{\pi}\right)^s \det\left(C^TAC\right)^{-1} \int_{\operatorname{span}(C)-i\pi_C(\boldsymbol{x})} \frac{e^{-\pi Q(\boldsymbol{w})-2\pi iB(\boldsymbol{x},\boldsymbol{w})}}{\prod_{j=1}^{s} B(D,\boldsymbol{z})} d\boldsymbol{z},$$

where the columns of $D \in \mathbb{R}^{N\times s}$ form a dual basis of C in $\operatorname{span}(C)$. Nazaroglu showed in [N, Proposition 3.15] that outside of a set with measure 0 we have

$$E_s(C;\boldsymbol{x}) = \sum_{S\subseteq\{1,\ldots s\}} M_{|S|}(C_S,\boldsymbol{x}) \prod_{j\notin S} \operatorname{sgn}\left(B\left(\pi_{\perp C_S}(c_j),\boldsymbol{x}\right)\right) \tag{I.1.7}$$

with $C_S := (c_{j_1}, c_{j_2},\ldots,c_{j_{|S|}})$ (writing $S = \{j_1,j_2,\ldots,j_{|S|}\}$ such that $j_1 < j_2 < \cdots < j_{|S|}$), $M_0 := 1$, and $\pi_{\perp C_S} := \operatorname{id}_{\operatorname{span}(C)} - \pi_{C_S}$ the orthogonal projection onto the orthogonal complement of $\operatorname{span}(C_S) \subseteq \operatorname{span}(C)$. Note that those terms with $S = \{\}$ correspond to the holomorphic part.

I.1.5 Eichler integrals and quantum modular forms

While derivatives of modular forms are not typically modular, differentiating a weight $2-k \in -\mathbb{N}$ modular form $(k-1)$ times returns a modular form of weight k, which is evident by Bol's identity (which states that the $(k-1)$-th power of the regular differential coincides with a modularity-preserving differential operator, see [LeZa, Z3]). Thus it is natural to consider, for a modular form $f(\tau) = \sum_{m\geq 1} c_f(m)q^m$ of weight k, the *Eichler integral*

$$\widetilde{f}(\tau) := \sum_{m\geq 1} \frac{c_f(m)}{m^{k-1}} q^m. \tag{I.1.8}$$

It is called Eichler integral since up to constants it equals

$$\int_{\tau}^{i\infty} f(w)(w-\tau)^{k-2}dw. \tag{I.1.9}$$

While in general $\widetilde{f}$ is not modular, its *error of modularity*

$$R_f(\tau) := \widetilde{f}(\tau) - \tau^{k-2}\widetilde{f}\left(-\frac{1}{\tau}\right) \tag{I.1.10}$$

can be shown to be a polynomial of degree $k-2$, the so called *period polynomial* of f. Up to constants R_f equals

$$\int_0^{i\infty} f(w)(w-\tau)^{k-2}dw.$$

While we cannot take $(k-1)$ derivatives for half-integral weight $k \in \frac{1}{2}+\mathbb{Z}$ modular forms, one can formally define the analogue of (I.1.8) in that case. Zagier first studied this in the context of *Kontsevich's strange function*

$$K(q) := \sum_{m\geq 0} (q;q)_m,$$

where $(a;q)_m := \prod_{j=0}^{m-1}(1-aq^j)$ denotes the *q-Pochhammer symbol* for $m \in \mathbb{N}_0 \cup \{\infty\}$ [Z2, Z4]. The function $K(q)$ does not converge on any open subset of $\mathbb{C}$, but collapses to a finite sum for roots of unity q. Zagier connected it with the weight $\frac{1}{2}$ *Dedekind eta function* $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty = \sum_{m\geq 1}(\frac{12}{m})q^{\frac{m^2}{24}}$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the extended Jacobi symbol, and its Eichler integral $\widetilde{\eta}(\tau) := \sum_{m\geq 1}\left(\frac{12}{m}\right)mq^{\frac{m^2}{24}}$. He showed that $\widetilde{\eta}(\tau)$ converges to $K(e^{2\pi i\frac{h}{k}})$ in the limit $\tau \to \frac{h}{k} \in \mathbb{Q}$ and that $\widetilde{\eta}$ has quantum modular properties. Zagier defined *quantum modular forms of weight* k to be functions $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subseteq \mathbb{Q}$), such that the error of modularity ($M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau+d)^{-k}f(M\tau)$$

is in some sense "nice", which is intentionally vague to capture different kinds of phenomena (for example, one could require continuity or analyticity) [Z4]. Additional examples appear in the study of limits of quantum invariants of 3-manifolds and knots [LaZa], Kashaev invariants of torus knots/links [HK, HL], and partial theta functions [FOR].

I.1.6 Characters of vertex algebras, false theta functions, and quantum modular forms

The characters of the atypical irreducible modules of the $(1,p)$-singlet vertex operator algebra $M_{1,s}$ for $1 \leq s \leq p-1$, which were studied in [BM1, CM1, CMW], are essentially the *false theta functions*

$$F_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \mathrm{sgn}\left(m+\frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}.$$

They are called "false" theta since removing the sgn-factor would give classical theta functions that are modular forms of weight $\frac{1}{2}$. They have also appeared in several other papers on vertex algebras [AM, GR, KW]. While they are not modular, Bringmann and Milas showed that the $F_{j,p}$ are quantum modular forms of weight $\frac{1}{2}$ by relating them to the non-holomorphic Eichler integrals

$$F_{j,p}^*(\tau) := -\sqrt{2}i \int_{-\overline{\tau}}^{i\infty} \frac{f_{j,p}(w)}{(-i(w+\tau))^{\frac{1}{2}}} dw,$$

where $f_{j,p}$ is the cuspidal theta function of weight $\frac{3}{2}$

$$f_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \left(m + \frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}.$$

Specifically, $F_{j,p}(\tau)$ agrees with $F_{j,p}^*(\tau)$ up to infinite order as $\tau \to \frac{h}{k}$ in vertical limits [BM1]. Bringmann and Milas showed that the error of modularity of $F_{j,p}^*$ is a period integral, which converges to an analytic function in the limit $\tau \to \frac{h}{k}$ (except at one point), providing quantum modularity of $F_{j,p}$. To determine modularity properties of $F_{j,p}^*$, note that it is also the "purely non-holomorphic part" of a non-holomorphic theta function corresponding to an indefinite quadratic form of signature $(1,1)$. Zwegers described their modular behavior in his thesis [Zw, Section 2.2].

I.1.7 Mordell integrals and quantum modular forms

Integrals such as

$$h(z) = h(z;\tau) := \int_{\mathbb{R}} \frac{\cosh(2\pi z w)}{\cosh(\pi w)} e^{\pi i \tau w^2} dw \tag{I.1.11}$$

were studied by many mathematicians including Kronecker, Lerch, Ramanujan, Riemann, Siegel, and Mordell [M1, M2, Si2]. They are called *Mordell integrals* since Mordell proved that a whole family of integrals reduces to (I.1.11) and they occur as the error of modularity of Lerch sums of the shape

$$\sum_{n\in\mathbb{Z}} \frac{q^{\frac{n^2+n}{2}} e^{2\pi i n z_1}}{1 - e^{2\pi i z_2} q^n} \quad (z_1, z_2 \in \mathbb{C} \setminus \{0\}).$$

Zwegers connected the Mordell integral with the theory of mock modular forms in his groundbreaking thesis [Zw] by writing the integrals in (I.1.11) as Eichler integrals, proving for $a, b \in (-\frac{1}{2}, \frac{1}{2})$ that

$$h(a\tau - b) = -q^{\frac{a^2}{2}} e^{-2\pi i a\left(b+\frac{1}{2}\right)} \int_0^{i\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(w)}{\sqrt{-i(w+\tau)}} dw, \tag{I.1.12}$$

where $g_{a,b}$ is the weight $\frac{3}{2}$ unary theta function defined by

$$g_{a,b}(\tau) := \sum_{n\in a+\mathbb{Z}} nq^{\frac{n^2}{2}} e^{2\pi ibn}.$$

I.2 Statement of objectives

I.2.1 Indefinite theta functions arising in Gromov-Witten theory of elliptic orbifolds

In the first project of this thesis presented in Chapter II, my advisor Kathrin Bringmann, Larry Rolen, and I study an indefinite theta function of signature $(1,3)$ that appears in a Gromov-Witten potential of an elliptic orbifold and determined its modular properties. Bringmann, Rolen, and Zwegers previously showed that some simpler parts of this potential are modular, mock modular, or products of mock modular forms [BRZ1]. We complete this analysis and prove that the remaining function is a higher depth mock modular form by explicitly constructing a modular completion by real-analytic functions. In total, this should help to provide a fuller picture of the mirror-symmetric properties of these orbifolds, which occur as natural geometric objects in Lagrangian Floer theory and mirror symmetry. The role that modularity plays in these geometric applications can be seen in [CHKL, CHL, LaZh], where other Gromov-Witten potentials containing simpler automorphic forms were discussed. Note that the indefinite theta function we studied could not have been treated purely by applying the results of [ABMP] or [N] due to unique features naturally arising here.

Chapter III gives further details for the proof of Proposition II.5.2 that are not mentioned in Chapter II.

I.2.2 Higher depth quantum modular forms, multiple Eichler integrals, and $\mathfrak{sl}_3$ false theta functions

Higher dimensional analogues of the false theta functions in Section I.1.6 appear in the characters of the vertex algebra $W^0(p)_{A_2}$ $(p \geq 2)$. They were thoroughly studied in [BM1, CM2] and showed up in [BM1] as constant terms of certain multivariable Jacobi forms and as characters of the zero weight space of the corresponding Lie algebra representation. For the simple Lie algebra $\mathfrak{sl}_3$, the following higher rank false theta functions appears

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right).$$

Chapter IV presents the following concepts and results from a joint project with Kathrin Bringmann and Antun Milas. Following some ideas of Section I.1.6, we find that the quantum modular properties of $F(e^{2\pi i\tau})$ can be studied by relating it to the "purely non-holomorphic" part of an indefinite theta function of signature $(2,2)$ after decomposing it as

$$F(q) = \frac{2}{p}F_1\left(q^p\right) + 2F_2\left(q^p\right). \tag{I.2.1}$$

These "purely non-holomorphic" parts can be written as iterated non-holomorphic Eichler integrals, taking for F_1 the shape

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{f(w_1,w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$

where $f\in S_{\frac{3}{2}}(\Gamma,\chi_1)\otimes S_{\frac{3}{2}}(\Gamma,\chi_2)$. Using the modularity of f one obtains that the error of modularity of the Eichler integral (and thus of the false theta function in the limit to rational points) is simpler than the original function, but this does not give quantum modularity in Zagier's original sense. Instead, we call the resulting functions higher depth quantum modular forms. In the simplest case, *depth two quantum modular forms of weight* $k\in\frac{1}{2}\mathbb{Z}$ satisfy the modular transformation property ($M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau)-(c\tau+d)^{-k}f(M\tau)\in\mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R)+\mathcal{O}(R)$$

for some $\kappa\in\frac{1}{2}\mathbb{Z}$, where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real analytic functions on $R\subset\mathbb{R}$. In short, we proved the following results.

Theorem I.2.1. *For $p\geq 2$, the higher rank false theta function F can be written as the sum of two depth two quantum modular forms (with quantum set $\mathbb{Q}$) of weight one and two (F_j has weight j).*

Theorem I.2.2. *There exists an indefinite theta function of signature* $(2,2)$ *with "purely non-holomorphic" part $\Theta(\tau)\mathcal{E}_1(\tau)$, where Θ is a theta function of signature* $(2,0)$ *and $\mathcal{E}_1$ is the Eichler integral related to F_1.*

I.2.3 Vector-valued higher depth quantum modular forms and higher Mordell integrals

While Theorem I.2.1 describes quantum modular behavior under a congruence subgroup, the vector valued transformation under the full modular group was interesting from a vertex algebra standpoint. On one hand, it is expected that the S-transformation (with $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$) produces typical and atypical characters from a single atypical character. On the other hand, many important algebraic, analytic, and categorical properties of rational vertex algebras are encoded by the entries of the S-matrix, so their full asymptotic expansions should be relevant for irrational theories.

Chapter V presents the following results from a joint article with Kathrin Bringmann and Antun Milas [BKM2].

Theorem I.2.3. *The function F_1 is a component of a vector-valued depth two quantum modular form of weight one. The function F_2 is a component of a vector-valued quantum modular form of depth two and weight two.*

Furthermore, we consider higher-dimensional Mordell integrals, proving the following result.

Theorem I.2.4. *If $\alpha_j \notin \mathbb{Z}$, then we have, with $Q(\boldsymbol{w}) := 3w_1^2 + w_2^2 + 3w_1w_2$*

$$-\sqrt{3}\int_0^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) + \theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1 = \int_{\mathbb{R}^2} \cot(\pi i w_1 + \pi\alpha_1)\cot(\pi i w_2 + \pi\alpha_2)\, e^{2\pi i\tau Q(\boldsymbol{w})} dw_1 dw_2,$$

where the theta functions θ_1 and θ_2 are defined in (V.4.4) *and* (V.4.5).

We also prove such an equation when either $\alpha_1 \in \mathbb{Z}$ or $\alpha_2 \in \mathbb{Z}$. Furthermore, we prove a similar statement for the iterated Eichler integrals appearing for F_2, which is given in Theorem V.1.4.

I.2.4 Some examples of higher depth vector-valued quantum modular forms

In light of Chapter V, a natural question that arises is what the other components of the vector-valued forms are as q-series. To investigate this, we study the related series

$$\mathbb{F}_{\boldsymbol{s}}(q) := \sum_{\substack{m_1,m_2\geq 1 \\ m_1\equiv m_2 \pmod{3}}} \min(m_1,m_2) q^{\frac{p}{3}\left(\left(m_1-\frac{s_1}{p}\right)^2+\left(m_2-\frac{s_2}{p}\right)^2+\left(m_1-\frac{s_1}{p}\right)\left(m_2-\frac{s_2}{p}\right)\right)} \times \left(1 - q^{m_1s_1} - q^{m_2s_2} + q^{m_1s_1+(m_1+m_2)s_2} + q^{m_2s_2+(m_1+m_2)s_1} - q^{(m_1+m_2)(s_1+s_2)}\right)$$

for $1 \leq s_1, s_2 \leq p \in \mathbb{N}$ (note that $\mathbb{F}_{(1,1)} = F$). Chapter VI contains the following results and is joint work with Kathrin Bringmann and Antun Milas.

Theorem I.2.5. *Decomposing $\mathbb{F}_{\boldsymbol{s}}(q) = \frac{1}{p}\mathbb{F}_{1,\boldsymbol{s}}(q^p) + \mathbb{F}_{2,\boldsymbol{s}}(q^p)$, the functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$ are quantum modular forms (with respect to some subgroup) of weights one and weights two, respectively.*

As done for F in Chapter IV, we prove this by showing that $\mathbb{F}_{j,\boldsymbol{s}}(\tau)$ asymptotically agrees to infinite order with a certain Eichler integral $\mathcal{E}_{j,\boldsymbol{s}}(\frac{\tau}{p})$ defined in (VI.2.1) and (VI.2.2) and studying their modular behavior.

In the special case $p = 2$, we obtain more specific results. While for $p = 2$ all $\mathbb{F}_{2,\boldsymbol{s}}$ vanish (see Lemma VI.2.2), we obtain the following behavior under the full modular group for the companions of $\mathbb{F}_{1,\boldsymbol{s}}$.

Theorem I.2.6. *For $p = 2$, the space spanned by $\mathcal{E}_{1,(1,1)}$ and $\mathcal{E}_{1,(1,2)}$ is essentially invariant under modular transformations. By this we mean that the only terms appearing in the modular transformations which do not lie in the space are simpler (see* (VI.2.6) *and* (VI.2.7) *for the case of inversion).*

Furthermore, we determine the asymptotic behavior of $\mathcal{E}_{1,\boldsymbol{s}}(it)$ as $t \to 0^+$. To compute the asymptotics, we apply the S-transformation $\tau \mapsto -\frac{1}{\tau}$ and analyze the dominating term, which is a well-known technique for vector-valued modular forms. It is used to study quantum dimensions of modules of vertex algebras as their characters are often invariant under $\mathrm{SL}_2(\mathbb{Z})$. Because our functions transform with higher depth error terms, their asymptotics are more difficult to analyze. In the body of the paper, we show that it is enough to study the iterated Eichler integrals

$$\mathbb{E}_{1,(1,1)}(\tau) := 4I_{(1,3)}(\tau) \quad \text{and} \quad \mathbb{E}_{1,(1,2)}(\tau) := 2I_{(1,1)}(\tau) + 2I_{(1,5)}(\tau), \tag{I.2.2}$$

where the theta integrals $I_{\boldsymbol{k}}$ are defined in (VI.2.3). We obtain the following.

Theorem I.2.7. *We have, as $t \to 0^+$,*

$$\mathbb{E}_{1,(1,1)}(it) \sim \frac{1}{4}, \qquad \mathbb{E}_{1,(1,2)}(it) \sim \frac{3}{4}.$$

Chapter II

Indefinite theta functions arising in Gromov-Witten theory of elliptic orbifolds

This chapter is based on a manuscript published in *Cambridge Journal of Mathematics* (International Press of Boston, Inc.) and is joint work with Prof. Dr. Kathrin Bringmann and Prof. Dr. Larry Rolen [BKR].

II.1 Introduction and statement of results

Cho, Hong, and Lau [4] described open Gromov-Witten potentials for elliptic orbifolds (and homological mirror symmetry). Explicit expressions for these were computed by Cho, Hong, Kim, and Lau [3]. Recently, Lau and Zhou [11] investigated the modularity properties of some of these Gromov-Witten potentials. In the course of their work, they showed that several of them are essentially modular forms, a fact which they show closely related to their mirror-symmetric interpretation. More precisely, they considered the four elliptic $\mathbb{P}^1$ orbifolds denoted by $\mathbb{P}^1_a$ for $a \in \{(3,3,3),(2,4,4),(2,3,6),(2,2,2,2)\}$. For these choices of a, Cho, Hong, Kim, and Lau explicitly computed the open Gromov-Witten potentials $W_q(X,Y,Z)$ of $\mathbb{P}^1_a$, which are polynomials in the variables X,Y,Z. The reader is also referred to [3,5,6] for related results, as well as to Sections 2 and 3 of [11] for the definitions of the relevant geometric objects. These generating functions turn out to have quite natural modularity properties, as Lau and Zhou proved in Theorem 1.1 of [11].

Theorem (Lau, Zhou)**.** *For $a \in \{(3,3,3),(2,4,4),(2,2,2,2)\}$, the coefficients of $W_q(X,Y,Z)$ are essentially linear combinations of modular forms.*

Such results are particularly useful as they allow one to extend the domain of these potentials to global moduli spaces. In fact, this connection provides the geometric intuition for why modular or at least near-modular, behavior may be expected (cf. [3]). In the last case $a = (2,3,6)$, the relevant functions fail to be combinations of ordinary modular forms. However, it is natural to ask whether a suitably modified transformation still holds.

Question (Lau, Zhou)**.** *Can a simple description of the modular transformations of $W_q(X, Y, Z)$ be given for $a = (2, 3, 6)$?*

The first steps towards addressing Lau and Zhou's question were taken in [2], where new generalizations of mock modular forms were defined and utilized. However, several remaining pieces remained out of reach due to lack of theoretical structure for such series. In the meantime, important new work of Alexandrov, Banerjee, Manschot, and Pioline [1] has further explained and generalized the class of functions considered in [2], leading to a new theory of non-holomorphic modular objects extending those considered in Zwegers' seminal thesis [28]. The understanding of the structure of these non-holomorphic modular forms was furthered by Kudla in [9], where he showed that they can be viewed as integrals of Kudla-Millson theta series (cf. [10]). Further extensions to a general, geometric setting were given by Westerholt-Raum in [23]. The authors have also been informed that forthcoming work of Zagier and Zwegers will further fill in details of the general picture.

Here, we push things one step further by showing how to interpret the last pieces of Lau and Zhou's functions in terms of further classes of modular-type objects. As we shall see, these cannot be accounted for by the means presented in [1] due to unique features naturally arising in these functions which seem to deviate from the most basic higher type indefinite theta functions. In particular, we solve this question by providing "simpler" completion terms which combine with the coefficients of W_q to yield modular objects. These are, in the language of Zagier and Zwegers, known as *higher depth mock modular forms*, which are automorphic functions characterized and inductively defined by the key property that their images under "lowering operators" essentially lie in lower depth spaces (for example, in the case of classical "depth 1" mock modular forms, the Maass lowering operator essentially yields a classical modular form).

Theorem II.1.1. *The function c_Z is a higher depth mock modular form.*

Remark 1. The higher depth structure of c_Z may be deduced from its "shadow", the computation of which is discussed in the proof of Lemma II.4.2 (cf. the remark after Lemma II.2.3; the word "shadow" is justified since it is essentially the image under the Brunier-Funke operator ξ_k for classical harmonic Maass forms).

The answer to Lau and Zhou's original question about the modularity of c_Z can be directly read off of the transformation of the completed function in Theorem II.1.1. Although we do not explicitly write it down here, the interested reader can see (II.4.5) and the surrounding text for a discussion of how to determine it. After using explicit representations due to Lau and Zhou, the key step in the proof of Theorem II.1.1 is to understand how to complete a certain indefinite theta function of signature $(3, 1)$ (see (II.2.1) below). (Throughout, the second component denotes the number of negative eigenvalues).

The paper is organized as follows. In Section 2, we recall general indefinite theta series due to Vignéras and in particular give examples in signatures $(1, n)$ and $(2, n)$. In Section 3, we introduce the generalized error integrals which are our building blocks and investigate some of their properties. In Section 4, we rewrite certain generating functions in Gromov-Witten theory and start the investigation of their modularity properties. Section 5 is then devoted to modularity properties of a certain indefinite theta function of signature $(3, 1)$.

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II.2 Indefinite theta functions

II.2.1 Results of Vignéras

Let $B(n, m) := n^T Am$ be a symmetric non-degenerate bilinear form on $\mathbb{R}^N (N \in \mathbb{N})$ which takes integral values on a lattice $L \subset \mathbb{R}^N$ and set $Q(n) := \frac{1}{2}B(n, n)$. Further let $\mu \in L'/L$ (where L' is the dual lattice of L), $\lambda \in \mathbb{Z}$, and a function $p : \mathbb{R}^N \to \mathbb{C}$. Following Vignéras, we define the following indefinite theta function ($\tau = u + iv \in \mathbb{H}$, $z = x + iy \in \mathbb{C}^N$, $q := e^{2\pi i\tau}$)

$$\Theta_{\mu,L,A,p,\lambda}(z; \tau) := \Theta_\mu(z; \tau) := v^{-\frac{\lambda}{2}} \sum_{n\in\mu+L} p\left(\sqrt{v}\left(n + \frac{y}{v}\right)\right) q^{\frac{1}{2}n^T An} e^{2\pi iB(z,n)}. \quad \text{(II.2.1)}$$

Vignéras [22] gave conditions under which the indefinite theta series are in fact modular.

Theorem II.2.1 (Vignéras)**.** *Assuming the notation above, suppose that p satisfies the following conditions:*

1. *For any differential operator D of order 2 and only polynomial R of degree at most 2, $D(w)(p(w)e^{\pi Q(w)})$ and $R(w)p(w)e^{\pi Q(w)}$ belong to $\in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.*

2. Defining the Euler and Laplace operators ($w := (w_1, \ldots, w_N)^T$, $\partial_w := (\frac{\partial}{\partial w_1}, \ldots \frac{\partial}{\partial w_N})^T$)

$$\mathcal{E} := w^T \partial_w \qquad \text{and} \qquad \Delta = \Delta_{A^{-1}} := \partial_w^T A^{-1} \partial_w,$$

for some $\lambda \in \mathbb{Z}$ the Vignéras differential equation holds:

$$\left(\mathcal{E} - \frac{1}{4\pi} \Delta \right) p = \lambda p.$$

Then, assuming that Θ_μ is absolutely locally convergent, we have the following modular transformations:

$$\begin{aligned} \Theta_\mu \left(\frac{z}{\tau}; -\frac{1}{\tau} \right) &= \frac{(-i\tau)^{\lambda + \frac{N}{2}}}{\sqrt{|L'/L|}} e^{\frac{\pi i}{2} B\left(A^{-1}A^*, A^{-1}A^*\right)} \\ &\quad \times \sum_{\nu \in L'/L} e^{-2\pi i B(\mu,\nu) + \frac{2\pi i}{\tau} Q(z)} \Theta_\nu(z;\tau), \\ \Theta_\mu(z; \tau+1) &= e^{\pi i B\left(\mu + \frac{1}{2}A^{-1}A^*, \mu + \frac{1}{2}A^{-1}A^*\right)} \Theta_\mu(z;\tau), \end{aligned}$$

where $A^* := (A_{1,1}, \ldots, A_{N,N})^T$.

To simplify the calculations below, the following lemma allows us to restrict to specific diagonal matrices. In particular, writing $A = P^{-T}DP^{-1}$ with $P \in \mathrm{GL}_N(\mathbb{R})$ and $D := \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$ (with uniquely determined signs), we easily obtain the following.

Lemma II.2.2. *Assume the notation above. If $\widetilde{p}(x) := p(Px)$ satisfies Vigneras' differential equation for D, then $\Theta_{\mu,L,A,p,\lambda}$ transforms like a vector-valued Jacobi form.*

Remark 2. We frequently make use of the well-known fact that specializing the elliptic variable of Jacobi forms to torsion points yields modular forms or related objects. (See [7] for the classical one-dimensional case).

We next introduce a differential operator which, when applied to Vignéras' theta functions, often makes them simpler. Let

$$X_- := -2iv^2 \frac{\partial}{\partial \overline{\tau}} - 2iv \sum_{j=1}^{N} y_j \partial_{\overline{z_j}},$$

be the (multivariable) Maass lowering operator which decreases the weight of a (non-holomorphic) Jacobi form by 2. A direct calculation gives.

Lemma II.2.3. *We have*

$$X_-\left(\Theta_{\mu,L,A,p,\lambda}\right) = \Theta_{\mu,L,A,p_X,\lambda}$$

with

$$p_X(x) := \sum_{j=1}^{N} x_j \partial_{x_j}\left(p(x)\right).$$

Remark 3. We let $\Theta_{\mu,L,A,p,\lambda}$ be the holomorphic part of Θ (whenever this is well-defined as we comment on later) and call it *higher depth Jacobi form* with *shadow* $\Theta_{\mu,L,A,p_X,\lambda}$. Specializing to torsion points yields *higher depth mock modular forms.*

II.2.2 Examples of indefinite theta functions

Although Vignéras' beautiful theorem has a simple statement, it is far from obvious how one can find appropriate functions p such that the corresponding indefinite theta function converges and which has a fixed, desired "holomorphic part". In his celebrated thesis [28], Zwegers succeeded in doing this for quadratic forms of signature $(n, 1)$. In this case, the usual error function

$$E(w) := 2\int_0^w e^{-\pi t^2} dt$$

plays a vital role. For comparison with functions we shall need later, note that, as $w \to \pm\infty$,

$$E(w) \sim \operatorname{sgn}(w). \tag{II.2.2}$$

Moreover, we clearly find that

$$E'(w) = 2e^{-\pi w^2}.$$

Also note that E may be written as

$$E(w) = \int_{\mathbb{R}} \operatorname{sgn}(t) e^{-\pi(t-w)^2} dt.$$

To discuss Zwegers' breakthrough, we now fix a quadratic form Q of signature $(n, 1)$. We must first discuss a few preliminary geometric considerations to describe the full behavior. The set of vectors $c \in \mathbb{R}^N$ with $Q(c) < 0$ splits into two connected components. Two given vectors c_1 and c_2 lie in the same component if and only if $B(c_1, c_2) < 0$. We fix one of the components and denote it C_Q. Picking any vector $c_0 \in C_Q$, we then have

$$C_Q = \left\{c \in \mathbb{R}^N : Q(c) < 0,\ B(c, c_0) < 0\right\}.$$

Then the cusps are those vectors in the following set:

$$S_Q := \left\{c = (c_1, c_2, \dots, c_N) \in \mathbb{Z}^N : \gcd(c_1, c_2, \dots, c_N) = 1,\ Q(c) = 0,\ B(c, c_0) < 0\right\}.$$

A compactification of C_Q may be formed by taking the union $\overline{C}_Q := C_Q \cup S_Q$. Defining for any $c \in \overline{C}_Q$

$$\mathcal{R}(c) := \begin{cases} \mathbb{R}^N & \text{if } c \in C_Q, \\ \left\{a \in \mathbb{R}^N : B(c, a) \notin \mathbb{Z}\right\} & \text{if } c \in S_Q, \end{cases}$$

we set

$$D(c) := \left\{(z, \tau) \in \mathbb{C}^N \times \mathbb{H} : \frac{y}{v} \in \mathcal{R}(c)\right\}.$$

Zwegers' indefinite theta functions, which transform as modular forms and which are (almost always) non-holomorphic, are defined as follows. For $(z, \tau) \in D(c_1) \cap D(c_2)$, we consider the theta function

$$\theta(z; \tau) := \sum_{n \in \mathbb{Z}^N} \rho\left(n + \frac{y}{v}; \tau\right) q^{Q(n)} e^{2\pi i B(z,n)}, \qquad \text{where} \tag{II.2.3}$$

$$\rho(n; \tau) = \rho_Q^{c_1,c_2}(n; \tau) := \rho^{c_1}(n; \tau) - \rho^{c_2}(n; \tau) \qquad \text{with} \tag{II.2.4}$$

$$\rho^c(n; \tau) := \begin{cases} E\left(\frac{B(c,n)v^{\frac{1}{2}}}{\sqrt{-Q(c)}}\right) & \text{if } c \in C_Q, \\ \operatorname{sgn}(B(c, n)) & \text{if } c \in S_Q. \end{cases}$$

Here and throughout we use the usual convention that for $x \in \mathbb{R}$, $\operatorname{sgn}(0) := 0$ and $\operatorname{sgn}(x) = x/|x|$ for $x \in \mathbb{R} \setminus \{0\}$. Note that the cuspidal case, $c \in S_Q$, may be viewed as a limiting case of the general situation (for example by (II.2.2)).

Zwegers showed that (II.2.3) indeed converges. This is far from obvious, since the indefiniteness of Q implies that $q^{Q(n)}$ is unbounded for $n \in \mathbb{Z}^N$. In fact, as in our case, this is one of the more subtle and substantive aspects of his proof of modularity. The main reason for the interest in this theta function lies in the Jacobi transformation properties of θ, which are described using the following auxiliary set:

$$\mathcal{D}(c) := \{(a\tau + b, \tau) : \tau \in \mathbb{H}, a, b \in \mathbb{R}^r, B(c, a), B(c, b) \notin \mathbb{Z}\}.$$

Theorem II.2.4 (Zwegers)**.** *Assuming the notation above, the function θ satisfies the following transformations:*

1. *For all $\lambda \in \mathbb{Z}^N$ and $\mu \in A^{-1}\mathbb{Z}^N$, we have* $\left(e(x) := e^{2\pi i x}\right)$

$$\theta(z + \lambda\tau + \mu; \tau) = q^{-Q(\lambda)} e(-B(z, \lambda))\theta(z; \tau).$$

2. We have

$$\theta(z;\tau+1)=\theta\left(z+\frac{1}{2}A^{-1}A^*;\tau\right).$$

3. If $(z,\tau)\in\mathcal{D}(c_1)\cap\mathcal{D}(c_2)$*, then*

$$\theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)=\frac{i(-i\tau)^{\frac{N}{2}}}{\sqrt{-\det(A)}}\sum_{n\in A^{-1}\mathbb{Z}^N/\mathbb{Z}^N}e\left(\frac{Q(z+n\tau)}{\tau}\right)\theta(z+n\tau;\tau).$$

In a pathbreaking paper, Alexandrov, Banerjee, Manschot, and Pioline [1] then generalized Zwegers' construction to quadratic forms of signature $(n,2)$. We do not state their beautiful results as we do not require them for this paper. We only note in their setting E got replaced by ($\alpha\in\mathbb{R}$)

$$E_2(\alpha;w_1,w_2):=\int_{\mathbb{R}^2}\operatorname{sgn}(t_1)\operatorname{sgn}(t_2+\alpha t_2)e^{-\pi(w_1-t_1)^2-\pi(w_2-t_2)^2}dt_1dt_2.$$

We note for comparison that their notation slightly differs from ours. We have, as $\lambda\to\infty$,

$$E_2(\alpha;\lambda w_1,\lambda w_2)\sim\operatorname{sgn}(w_1)\operatorname{sgn}(w_2+\alpha w_2). \tag{II.2.5}$$

Moreover

$$\left(\partial_{w_1}^2+\partial_{w_2}^2+2\pi\left(w_1\partial_{w_1}+w_2\partial_{w_2}\right)\right)E_2\left(\alpha;w_1,w_2\right)=0, \tag{II.2.6}$$

$$\partial_{w_2}E_2\left(\alpha;w_1,w_2\right)=\frac{2}{\sqrt{1+\alpha^2}}e^{-\frac{\pi(w_2+\alpha w_1)^2}{1+\alpha^2}}E\left(\frac{\alpha w_2-w_1}{\sqrt{1+\alpha^2}}\right),\text{ and} \tag{II.2.7}$$

$$\partial_{w_1}E_2\left(\alpha;w_1,w_2\right)=2e^{-\pi w_1^2}E\left(w_2\right)+\frac{2\alpha}{\sqrt{1+\alpha^2}}e^{-\frac{\pi(w_2+\alpha w_1)^2}{1+\alpha^2}}E\left(\frac{\alpha w_2-w_1}{\sqrt{1+\alpha^2}}\right). \tag{II.2.8}$$

II.3 Generalized error integrals

In this section, we introduce higher-dimensional analogies of the error function, following ideas of [1,14].

II.3.1 Definitions and basic properties

The authors of [1] proposed a 3-dimensional analogue of E_2, namely

$$E_3^*(\alpha_1,\alpha_2,\alpha_3;w_1,w_2,w_3)$$

$$:= \int_{\mathbb{R}} \operatorname{sgn}(t_1 + \alpha_1\alpha_2 t_2 + \alpha_2 t_3) \operatorname{sgn}(t_2 + \alpha_2\alpha_3 t_3 + \alpha_3 t_1)$$
$$\times \operatorname{sgn}(t_3 + \alpha_3\alpha_1 t_1 + \alpha_1 t_2) e^{-\pi\left((w_1-t_1)^2+(w_2-t_2)^2+(w_3-t_3)^2\right)} dt_1 dt_2 dt_3.$$

Here we define a modified version which is convenient for our explicit functions. After finishing this article, Pioline pointed out to the authors that there is an explicit map from the suggested higher dimensional E_3 function of [1]; however, our function may also offer some advantages as it seems easier to directly work with in at least some examples. We define a generalized error function $E_N\colon R^{\frac{N(N-1)}{2}} \times \mathbb{R}^N \to \mathbb{R}$ by

$$E_N(\alpha; w) := \int_{\mathbb{R}^N} \operatorname{sgn}(t_1) \operatorname{sgn}(t_2 + \alpha_1 t_1) \tag{II.3.1}$$
$$\times \operatorname{sgn}(t_3 + \alpha_2 t_1 + \alpha_3 t_2) \cdots \operatorname{sgn}\left(t_N + \ldots + \alpha_{\frac{N(N-1)}{2}} t_{N-1}\right) e^{-\pi\|t-w\|_2^2} dt,$$

where $\|a\|_2 := \sqrt{a^T a}$ denotes the Euclidian norm. Note that we use $\|\cdot\|_2$
for different dimensions, the meaning being clear from context. Higher-dimensional E_N collapse to lower-dimensional ones if certain α_j are 0. For example,

$$E_3(\alpha, 0, 0; w) = E_2(\alpha; w_1, w_2) E(w_3), \tag{II.3.2}$$
$$E_3(0, \alpha, 0; w) = E(w_2) E_2(\alpha; w_1, w_3). \tag{II.3.3}$$

From now on, we restrict to $N = 3$, however most of our statements hold for general N. The following lemma, which generalizes (II.2.2) and (II.2.5), describes the asymptotic behavior of E_N, which is crucial in the construction of the appropriate completions of Lau and Zhou's functions.

Lemma II.3.1. *For any $\alpha = (\alpha_1, \alpha_2, \alpha_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$, we have, as $\lambda \to \infty$,*

$$E_N(\alpha; \lambda w) \sim \operatorname{sgn}(w_1) \operatorname{sgn}(w_2 + \alpha_1 w_1) \operatorname{sgn}(w_3 + \alpha_2 w_1 + \alpha_3 w_2).$$

Remark 4. Throughout the paper, we write $E_N(\alpha; w) \sim *$ to mean that $E_N(\alpha; \lambda w) \sim *$ as $\lambda \to \infty$.

Proof of Lemma II.3.1. Changing variables yields

$$E_3(\alpha; w_1, w_2, w_3) = \int_{\mathbb{R}^3} e^{-\pi t^T M t} \operatorname{sgn}(t_1 + w_1) \operatorname{sgn}(t_2 + v_2) \operatorname{sgn}(t_3 + v_3)\, dt,$$

where $t := (t_1, t_2, t_3)^T, v_2 := w_2 + \alpha_1 w_1, v_3 := w_3 + \alpha_2 w_1 + \alpha_3 w_2$, and

$$M := \begin{pmatrix} 1 + (\alpha_1\alpha_3 - \alpha_2)^2 + \alpha_1^2 & -\alpha_1 - \alpha_3(\alpha_1\alpha_3 - \alpha_2) & \alpha_1\alpha_3 - \alpha_2 \\ -\alpha_1 - \alpha_3(\alpha_1\alpha_3 - \alpha_2) & \alpha_3^2 + 1 & -\alpha_3 \\ \alpha_1\alpha_3 - \alpha_2 & -\alpha_3 & 1 \end{pmatrix}.$$

Note that $\det(M) = 1$ and that M is positive-definite.

Then consider the difference

$$\begin{aligned} &E_3(\alpha; \lambda w_1, \lambda w_2, \lambda w_3) - \operatorname{sgn}(w_1)\operatorname{sgn}(w_2 + \alpha_1 w_1)\operatorname{sgn}(w_3 + \alpha_3 w_2 + \alpha_2 w_1) \\ &= \int_{\mathbb{R}^3} e^{-\pi t^T M t}\big(\operatorname{sgn}(t_1 + \lambda v_1)\operatorname{sgn}(t_2 + \lambda v_2)\operatorname{sgn}(t_3 + \lambda v_3) \\ &\qquad\qquad - \operatorname{sgn}(v_1)\operatorname{sgn}(v_2)\operatorname{sgn}(v_3)\big)dt. \end{aligned}$$

It is easily checked that the integrand vanishes whenever $|t_j| < \lambda |v_j|$ for all j. Denote the complement of the cube given by these three inequalities for t by $\mathcal{B}(\lambda)$. Outside of $\mathcal{B}(\lambda)$ we bound the sum by 2 and obtain the following as an upper bound of the absolute value:

$$2\int_{\mathcal{B}(\lambda)} e^{-\pi t^T M t} dt.$$

Since M is positive-definite, this converges to 0 as $\lambda \to \infty$ whenever $v_j \neq 0$, proving the statement in this case. If (at least) one v_j vanishes, the integral expression for E_3 vanishes which may be seen by changing $t_j \mapsto -t_j$. □

Certain sign-factors that occur throughout our investigation turn out to not quite have the correct shape. For this, the following elementary lemma, whose proof we skip, is useful.

Lemma II.3.2. *For $a, b, c \in \mathbb{R}\backslash\{0\}$ and $\lambda_j \geq 0$ $(j = 1, 2, 3)$, $\varepsilon \in \{\pm 1\}$, we have (unless $(\lambda_1, \lambda_2) = (0, 0)$)*

$$\operatorname{sgn}(a)\operatorname{sgn}(b) = -\varepsilon + \operatorname{sgn}(\lambda_1 a + \lambda_2 \varepsilon b)(\varepsilon \operatorname{sgn}(a) + \operatorname{sgn}(b)) \tag{II.3.4}$$

and (unless $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$)

$$\begin{aligned} &\operatorname{sgn}(a)\operatorname{sgn}(b)\operatorname{sgn}(c) + \operatorname{sgn}(a) + \operatorname{sgn}(b) + \operatorname{sgn}(c) \\ &= \operatorname{sgn}(\lambda_1 a + \lambda_2 b + \lambda_3 c)(\operatorname{sgn}(a)\operatorname{sgn}(b) + \operatorname{sgn}(a)\operatorname{sgn}(c) + \operatorname{sgn}(b)\operatorname{sgn}(c) + 1). \end{aligned} \tag{II.3.5}$$

Remark 5. Applying Lemma II.3.2 to $E_2(\alpha; w)$ gives, for $\alpha \neq 0$,

$$E_2(\alpha; w_1, w_2) = E(w_1)E(w_2) - \operatorname{sgn}(\alpha)E_2\left(\alpha^{-1}; w_2, w_1\right) + \operatorname{sgn}(\alpha).$$

In the "cuspidal case" considered below, we must allow certain values to be 0. To do so, the following lemma turns out to be useful.

Lemma II.3.3. *For $\alpha_1, \alpha_2, \alpha_3, w_1, w_2, w_3, w_4 \in \mathbb{R}$ with $\alpha_3, \alpha_1\alpha_3 - \alpha_2 \neq 0$, we have*

$$\begin{aligned}
&\mathcal{E}_3(\alpha; w_1, w_2, w_3) := \lim_{T\to\infty} E_3\left(\alpha_1, T\alpha_2, T\alpha_3; w_1, w_2, Tw_3\right) \\
&= \operatorname{sgn}(\alpha_3)E(w_1) + \delta\operatorname{sgn}(\alpha_3)E\left(\frac{w_2+\alpha_1 w_1}{\sqrt{\alpha_1^2+1}}\right) - \delta E\left(\frac{w_3+\alpha_2 w_1+\alpha_3 w_2}{\sqrt{\alpha_2^2+\alpha_3^2}}\right) \\
&\quad - \operatorname{sgn}\left(\alpha_3 w_3\right)\Bigg(1+\delta E_2\left(\alpha_1; w_1, w_2\right) + \delta E_2\left(\alpha_2\alpha_3^{-1}; w_1, \alpha_3^{-1}w_3 + w_2\right) \\
&\quad + \delta E_2\Bigg(\frac{\alpha_1\alpha_2+\alpha_3}{-\alpha_2+\alpha_1\alpha_3}; \frac{w_2+\alpha_1 w_1}{\sqrt{\alpha_1^2+1}}, \\
&\qquad\qquad \frac{\sqrt{\alpha_1^2+1}}{-\alpha_2+\alpha_1\alpha_3}\left(w_3 + \frac{(\alpha_1\alpha_3-\alpha_2)(\alpha_1 w_2 - w_1)}{\alpha_1^2+1}\right)\Bigg)\Bigg),
\end{aligned}$$

where $\delta := \operatorname{sgn}(\alpha_3(\alpha_2 - \alpha_1\alpha_3))$.

Proof. We directly compute that

$$\begin{aligned}
&E_3\left(\alpha_1, T\alpha_2, T\alpha_3; w_1, w_2, Tw_3\right) \\
&\overset{T\to\infty}{\longrightarrow} \int_{\mathbb{R}^3} e^{-\pi\|t\|_2^2}\operatorname{sgn}(t_1+w_1)\operatorname{sgn}(t_2+\alpha_1 t_1 + w_2 + \alpha_1 w_1) \\
&\qquad\qquad \times \operatorname{sgn}(\alpha_2 t_1 + \alpha_3 t_2 + w_3 + \alpha_2 w_1 + \alpha_3 w_2)dt.
\end{aligned}$$

The integral over t_3 may now be computed to be 1.

To determine the remaining two-dimensional integral, we use Lemma II.3.2, with $\lambda_1 = |\alpha_2 - \alpha_1\alpha_3|$, $\lambda_2 = |\alpha_3|$, $\lambda_3 = 1$, $a = \delta(t_1 - w_1)$, $b = t_2 + \alpha_1 t_1 + v_2$, and $c = \operatorname{sgn}(-\alpha_3)\alpha_2 t_1 - |\alpha_3|t_2 + \operatorname{sgn}(-\alpha_3)v_3$, where $v_2 := w_2 + \alpha_1 w_1$, $v_3 := w_3 + \alpha_2 w_1 + \alpha_3 w_2$, gives that the product of the signs equals (outside the zero set given by $abc = 0$)

$$\begin{aligned}
&\operatorname{sgn}(\alpha_3)\delta\Big(\operatorname{sgn}\left(\delta\left(t_1+w_1\right)\right) + \operatorname{sgn}\left(t_2+\alpha_1 t_1 + v_2\right) \\
&\qquad\qquad + \operatorname{sgn}\left(\operatorname{sgn}(-\alpha_3)\alpha_2 t_1 - |\alpha_3|t_2 + \operatorname{sgn}(-\alpha_3)v_3\right)\Big) \\
&- \operatorname{sgn}(\alpha_3 w_3)\Big(1 + \operatorname{sgn}\left(\delta\left(t_1+w_1\right)\right)\operatorname{sgn}\left(t_2+\alpha_1 t_1+v_2\right) \\
&\qquad + \operatorname{sgn}\left(\delta\left(t_1+w_1\right)\right)\operatorname{sgn}\left(\operatorname{sgn}(-\alpha_3)\alpha_2 t_1 - |\alpha_3|t_2 + \operatorname{sgn}(-\alpha_3)v_3\right) \\
&\qquad + \operatorname{sgn}\left(t_2+\alpha_1 t_1+v_2\right)\operatorname{sgn}\left(\operatorname{sgn}(-\alpha_3)\alpha_2 t_1 - |\alpha_3|t_2 + \operatorname{sgn}(-\alpha_3)v_3\right)\Big).
\end{aligned}$$

We compute all like integrals separately. The terms $-\operatorname{sgn}(\alpha_3 w_3)$ and $\operatorname{sgn}(\alpha_3) \times \operatorname{sgn}(t_1 + w_1)$ directly give $-\operatorname{sgn}(w_3\alpha_3)$ and $\operatorname{sgn}(\alpha_3)E(w_1)$, respectively.

To consider the contribution from $\operatorname{sgn}(\alpha_3)\delta\operatorname{sgn}(t_2+\alpha_1 t_1+v_2)$, we define the orthonormal matrix $M_1 := (\alpha_1^2+1)^{-\frac{1}{2}}\left(\begin{smallmatrix}\alpha_1 & 1\\ -1 & \alpha_1\end{smallmatrix}\right)$. Since M_1 is orthogonal, $\|M_1 t\|_2 = \|t\|_2$. Thus, the contribution from this term gives

$$\delta\operatorname{sgn}(\alpha_3)\int_{\mathbb{R}^2} e^{-\pi\|t\|_2^2}\operatorname{sgn}\left(t_1+\left(\alpha_1^2+1\right)^{-\frac{1}{2}}v_2\right)dt_1dt_2 = \delta\operatorname{sgn}(\alpha_3)E\left(\frac{v_2}{\sqrt{\alpha_1^2+1}}\right).$$

The contribution from $-\delta\operatorname{sgn}(\alpha_2 t_1+\alpha_3 t_2+v_3)$ is treated in exactly the same way, giving

$$-\delta E\left(\frac{v_3}{\sqrt{\alpha_2^2+\alpha_3^2}}\right).$$

Next, we consider the product of two sgn-factors. The terms $-\delta\operatorname{sgn}(\alpha_3 w_3)\times\operatorname{sgn}(t_1+w_1)\operatorname{sgn}(t_2+\alpha_1 t_1+v_2)$ and $-\delta\operatorname{sgn}(\alpha_3 w_3)\operatorname{sgn}(t_1+w_1)\operatorname{sgn}(\alpha_2\alpha_3^{-1}t_1+t_2+\alpha_3^{-1}v_3)$ yield the contributions $-\delta\operatorname{sgn}(\alpha_3 w_3)E_2(\alpha_1;w_1,w_2)$ and $-\delta\operatorname{sgn}(\alpha_3 w_3)E_2(\alpha_2\alpha_3^{-1};w_1,\alpha_3^{-1}v_3-\alpha_2\alpha_3^{-1}w_1)$, respectively.

Finally

$$\begin{aligned}
&\operatorname{sgn}(w_3)\int_{\mathbb{R}^2} e^{-\pi\|t\|_2^2}\operatorname{sgn}(t_2+\alpha_1 t_1+v_2)\operatorname{sgn}(\alpha_2 t_1+\alpha_3 t_2+v_3)\,dt\\
&=\operatorname{sgn}(w_3)\int_{\mathbb{R}^2} e^{-\pi\|t\|_2^2}\operatorname{sgn}\left((M_1t)_1\sqrt{\alpha_1^2+1}+v_2\right)\operatorname{sgn}(\alpha_2 t_1+\alpha_3 t_2+v_3)\,dt\\
&=\operatorname{sgn}(w_3)\int_{\mathbb{R}^2} e^{-\pi\|t\|_2^2}\operatorname{sgn}\left(t_1+\left(\alpha_1^2+1\right)^{-\frac{1}{2}}v_2\right)\\
&\qquad\qquad\times\operatorname{sgn}\left(\alpha_2\left(M^{-1}t\right)_1+\alpha_3\left(M^{-1}t\right)_2+v_3\right)dt\\
&=-\delta\operatorname{sgn}(\alpha_3 w_3)\\
&\quad\times E_2\left(\frac{\alpha_1\alpha_2+\alpha_3}{-\alpha_2+\alpha_1\alpha_3};\frac{v_2}{\sqrt{\alpha_1^2+1}},\frac{\sqrt{\alpha_1^2+1}}{-\alpha_2+\alpha_1\alpha_3}\left(v_3-\frac{\alpha_1\alpha_2+\alpha_3}{\alpha_1^2+1}v_2\right)\right).
\end{aligned}$$

□

We next show that the function E_3 satisfies a special differential equation.

Lemma II.3.4. *We have*

$$\sum_{j=1}^{3}\left(\partial_{w_j}^2+2w_j\partial_{w_j}\right)E_3(\alpha;w)=0.$$

Proof. We write

$$\begin{aligned}
&E_3(\alpha; w)\\
&= \int_{\mathbb{R}^2} \operatorname{sgn}(t_1)\, e^{-\pi(t_1-w_1)^2}\\
&\quad\times \int_{\mathbb{R}^2} \operatorname{sgn}(t_2) \operatorname{sgn}(t_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1 + \alpha_3 t_2)\, e^{-\pi\left((t_2-w_2-\alpha_1 t_1)^2+(t_3-w_3)^2\right)} dt\\
&= \int_{\mathbb{R}} \operatorname{sgn}(t_1) E_2\left(\alpha_3; w_2 + \alpha_1 t_1, w_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1\right) e^{-\pi(t_1-w_1)^2} dt_1.
\end{aligned}$$

Applying the operator on the left-hand-side gives

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{sgn}(t_1) \frac{\partial}{\partial t_1}\left((-2\pi t_1) e^{-\pi(t_1-w_1)^2}\right) E_2\left(\alpha_3; w_2 + \alpha_1 t_1, w_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1\right) dt_1\\
&\qquad + \int_{\mathbb{R}} \operatorname{sgn}(t_1)(-2\pi t_1) e^{-\pi(t_1-w_1)^2}\\
&\qquad \times \Big(\alpha_1 E_2^{(1,0)}\left(\alpha_3; w_2 + \alpha_1 t_1, w_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1\right)\\
&\qquad + (\alpha_2 - \alpha_1\alpha_3)\, E_2^{(0,1)}\left(\alpha_3; w_2 + \alpha_1 t_1, w_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1\right)\Big) dt_1\\
&= -2\pi \int_{\mathbb{R}} \operatorname{sgn}(t_1) \frac{\partial}{\partial t_1}\left(t_1 e^{-\pi(t_1-w_1)^2} E_2\left(\alpha_3; w_2 + \alpha_1 t_1, w_3 + (\alpha_2 - \alpha_1\alpha_3)\, t_1\right)\right) dt_1\\
&= 0,
\end{aligned}$$

by (II.2.6) and the chain rule. This gives the claim. □

Lemma II.3.5. *We have*

$$\begin{aligned}
&\partial w_3 E_3(\alpha; w) = \frac{2}{\sqrt{1+\alpha_3^2}} e^{-\frac{\pi\left(1+\alpha_3^2\right)}{1+\alpha_2^2+\alpha_3^2}(w_3+\alpha_3 w_2+\alpha_2 w_1)^2}\\
&\times E_2\left(\frac{\alpha_2\alpha_3 - \alpha_1\left(\alpha_3^2+1\right)}{\sqrt{\left(1+\alpha_3^2\right)\left(1+\alpha_2^2+\alpha_3^2\right)}}; \frac{(w_2+\alpha_3 w_2)\,\alpha_2 - \left(1+\alpha_3^2\right) w_1}{\sqrt{1+\alpha_2^2+\alpha_3^2}}, \frac{\alpha_3 w_3 - w_2}{\sqrt{1+\alpha_3^2}}\right),\\
&\partial w_2 E_3(\alpha; w) = \frac{2\alpha_1}{\sqrt{1+\alpha_3^2}} e^{-\frac{\pi\left(1+\alpha_3^2\right)}{1+\alpha_2^2+\alpha_3^2}(w_3+\alpha_3 w_2+\alpha_2 w_1)^2}\\
&\times E_2\left(\frac{\alpha_2\alpha_3 - \alpha_1\left(\alpha_3^2+1\right)}{\sqrt{\left(1+\alpha_3^2\right)\left(1+\alpha_2^2+\alpha_3^2\right)}}; \frac{(w_2+\alpha_3 w_2)\,\alpha_2 - \left(1+\alpha_3^2\right) w_1}{\sqrt{1+\alpha_2^2+\alpha_3^2}}, \frac{\alpha_3 w_3 - w_2}{\sqrt{1+\alpha_3^2}}\right)
\end{aligned}$$

$$+\frac{2}{\sqrt{1+\alpha_1^2}}e^{-\pi(\alpha_1 w_1-w_2)^2}E_2\left(\frac{\alpha_2-\alpha_1\alpha_3}{\sqrt{1+\alpha_1^2}};\frac{w_1-\alpha_1 w_2}{\sqrt{1+\alpha_1^2}},w_3\right),$$

$$\partial w_1 E_3(\alpha;w)=\frac{2\alpha_1\left(1+\alpha_2-\alpha_3\right)}{\sqrt{1+\alpha_3^2}}e^{-\frac{\pi\left(1+\alpha_3^2\right)}{1+\alpha_2^2+\alpha_3^2}(w_3+\alpha_3 w_2+\alpha_2 w_1)^2}$$

$$\times E_2\left(\frac{\alpha_2\alpha_3-\alpha_1\left(\alpha_3^2+1\right)}{\sqrt{\left(1+\alpha_3^2\right)\left(1+\alpha_2^2+\alpha_3^2\right)}};\frac{\left(w_2+\alpha_3 w_2\right)\alpha_2-\left(1+\alpha_3^2\right)w_1}{\sqrt{1+\alpha_2^2+\alpha_3^2}},\frac{\alpha_3 w_3-w_2}{\sqrt{1+\alpha_3^2}}\right)$$

$$+\frac{2\left(\alpha_2-\alpha_1\alpha_3\right)}{\sqrt{1+\alpha_1^2}}e^{-\pi(\alpha_1 w_1-w_2)^2}E_2\left(\frac{\alpha_2-\alpha_1\alpha_3}{\sqrt{1+\alpha_1^2}};\frac{w_1-\alpha_1 w_2}{\sqrt{1+\alpha_1^2}},w_3\right)$$

$$+2E_2\left(\alpha_2;w_2,w_3\right)e^{-\pi w_1^2}.$$

Proof. In the proof of Lemma II.3.4, we see that

$$E_3(\alpha;w)=\int_{\mathbb{R}}\operatorname{sgn}\left(t_1\right)e^{-\pi(t_1-w_1)^2}E_2\left(\alpha_3;w_2+\alpha_1 t_1,w_3+\left(\alpha_2-\alpha_1\alpha_3\right)t_1\right)dt_1.$$

We apply first (II.2.7) to get

$$\begin{aligned}
&\partial_{w_3}E_3(\alpha;w)\\
&=\int_{\mathbb{R}}\operatorname{sgn}\left(t_1\right)e^{-\pi(t_1-w_1)^2}\frac{2}{\sqrt{1+\alpha_3^2}}e^{-\frac{\pi}{1+\alpha_3^2}(w_3+\alpha_2 t_1+\alpha_3 w_2)^2}\\
&\qquad\times E\left(\frac{\alpha_3\left(w_3+\left(\alpha_2-\alpha_1\alpha_3\right)t_1\right)-\left(w_2+\alpha_1 t_1\right)}{\sqrt{1+\alpha_3^2}}\right)dt_1\\
&=\frac{2}{\sqrt{1+\alpha_3^2}}e^{-\frac{\left(1+\alpha_3^2\right)\pi}{1+\alpha_2^2+\alpha_3^2}(w_3+\alpha_3 w_2+\alpha_2 w_1)^2}\\
&\quad\times\int_{\mathbb{R}}\operatorname{sgn}\left(t_1\right)e^{-\pi\left(\sqrt{1+\alpha_2^2+\alpha_3^2}t_1+\frac{1}{\sqrt{1+\alpha_2^2+\alpha_3^2}}\left((w_3+\alpha_3 w_2)\alpha_2-\left(1+\alpha_3^2\right)w_1\right)^2\right)}\\
&\qquad\times E\left(\frac{\left(\alpha_2\alpha_3-\alpha_1\alpha_3^2-\alpha_1\right)t_1+\alpha_3 w_3-w_2}{\sqrt{1+\alpha_3^2}}\right)dt_1.
\end{aligned}\tag{II.3.6}$$

Making the change of variables $t_1\mapsto\frac{t_1}{\sqrt{1+\alpha_2^2+\alpha_3^2}}$ then gives the claim.

We next apply ∂_{w_2}. By (II.2.8), we get

$$\partial_{w_2}E_3(\alpha;w)$$

$$= \int_{\mathbb{R}} \operatorname{sgn}(t_1) e^{-\pi(t_1-w_1)^2} \big(\alpha_1 \partial_{w_3} \left(E_2\left(\alpha_3; w_2+\alpha_1 t_1, w_3+\left(\alpha_2-\alpha_1\alpha_3\right) t_1\right)\right) \\ + 2e^{-\pi(w_2+\alpha_1 t_1)^2} E\left(w_3+\left(\alpha_2-\alpha_1\alpha_3\right) t_1\right)\big) dt_1.$$

The first summand is computed as above. The second term gives the claimed contribution by simplifying and making the change of variables $t_1 \mapsto \frac{t_1}{\sqrt{1+\alpha_1^2}}$. Finally we apply ∂_{w_1} to give, using integration by parts,

$$\partial_{w_1} E_3\left(\alpha; w\right) = 2E_2\left(\alpha_2; w_2, w_3\right) e^{-\pi w_1^2} \\ + \int_{\mathbb{R}} \operatorname{sgn}\left(t_1\right) e^{-\pi(t_1-w_1)^2} \alpha_1 \frac{\partial}{\partial w_2} \left(E_2\left(\alpha_2; w_2+\alpha_1 t_1, w_3+\left(\alpha_2-\alpha_1\alpha_3\right) t_1\right)\right) \\ + \left(\alpha_2-\alpha_1\alpha_3\right) \frac{\partial}{\partial w_3} \left(E_2\left(\alpha_2; w_2+\alpha_1 t_1, w_3+\left(\alpha_2-\alpha_1\alpha_3\right) t_1\right)\right) dt_1.$$

From the above the result follows. $\square$

II.3.2 The function E_3 as a building block

The theta functions of interest in Gromov-Witten theory are indefinite theta functions in which the summation conditions may be written in terms of sgn-functions. The following proposition shows how to turn their sgn-factors into functions satisfying Vignéras differential equation.

Proposition II.3.6. *For $N \in \mathbb{N}_0$, let $A = P^{-T} \operatorname{diag}(I_N, -I_3) P^{-1} \in \operatorname{Mat}_{N+3}(\mathbb{R})$ be a symmetric matrix of signature $(N, 3)$, $P \in \operatorname{GL}_{N+3}(\mathbb{R})$, and assume that $a, b, c \in \mathbb{R}^{N+3}$ generate a 3-dimensional space of signature (N_+, N_-) with respect to the bilinear form $\langle \cdot, \cdot \rangle$ given by A^{-1}. Then there exist $d, e, f \in \mathbb{R}^{N+3}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ (determined explicitly in Lemma II.3.7 below) such that the following are true.*

1. *For $(N_+, N_-) = (0, 3)$ the map $X \mapsto E_3(\alpha_1, \alpha_2, \alpha_3; d^T PX, e^T PX, f^T PX$ satisfies Vignéras' differential equation for $\operatorname{diag}(I_N, -I_3)$, and for all $n \in \mathbb{R}^{N+3}$, we have*

$$E_3\left(\alpha_1, \alpha_2, \alpha_3; d^T n, e^T n, f^T n\right) \sim \operatorname{sgn}\left(a^T n\right) \operatorname{sgn}\left(b^T n\right) \operatorname{sgn}\left(c^T n\right).$$

2. *For $(N_+, N_-) = (0, 2)$ the map $X \mapsto \mathcal{E}_3\left(\alpha_1, \alpha_2, \alpha_3; d^T PX, e^T PX, f^T PX\right)$ satisfies Vignéras' differential equation for $\operatorname{diag}(I_N, -I_3)$ and for all $n \in \mathbb{R}^{N+3}$, we have*

$$\mathcal{E}_3\left(\alpha_1, \alpha_2, \alpha_3; d^T n, e^T n, f^T n\right) \sim \operatorname{sgn}\left(a^T n\right) \operatorname{sgn}\left(b^T n\right) \operatorname{sgn}\left(c^T n\right).$$

Before proving Proposition II.3.6, we require an auxiliary lemma.

Lemma II.3.7. *Assume that $a, b, c \in \mathbb{R}^{N+3}$ generate a 3-dimensional space of signature $(0,3)$ or $(0,2)$ with respect to a symmetric bilinear form $\langle\cdot,\cdot\rangle$ of signature $(N,3)$ on $\mathbb{R}^{N+3}$. Then there exist pairwise orthogonal vectors $d, e, f \in \mathbb{R}^{N+3}\backslash\{0\}$ and scalars $\lambda, \mu, \nu \in \mathbb{R}\backslash\{0\}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$*

$$d = \lambda a, \qquad e + \alpha_1 d = \mu b, \qquad f + \alpha_2 d + \alpha_3 e = \nu c \tag{II.3.7}$$

such that squares of the norms of d, e, f are $(-2,-2,-2)$ for signature $(0,3)$ and $(-2,-2,0)$ for signature $(0,2)$, respectively. Explicitly, they can be defined (after permuting a, b, c such that $\operatorname{span}\{a, b\}$ has signature $(0,2)$) as

$$\begin{aligned}
&\lambda := \sqrt{\frac{-2}{\|a\|^2}}, \qquad \mu := \sqrt{\frac{-2\|a\|^2}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}},\\
&\rho := -\frac{\|c\|^2}{2} + \frac{\langle a,c\rangle^2}{2\|a\|^2} - \frac{1}{4}\mu^2\left(\langle b,c\rangle - \frac{\langle a,b\rangle\langle a,c\rangle}{\|a\|^2}\right)^2, \ \nu := \begin{cases} |\rho|^{-\frac{1}{2}} & \text{if } \rho \neq 0,\\ 1 & \text{if } \rho = 0,\end{cases}\\
&\alpha_1 := -\frac{\langle a,b\rangle}{2}\lambda\mu, \quad \alpha_2 := -\frac{\langle a,c\rangle}{2}\lambda\nu, \quad \alpha_3 := -\frac{1}{2}\mu\nu\left(\langle b,c\rangle + \frac{\langle a,b\rangle\langle a,c\rangle\lambda^2}{2}\right).
\end{aligned}$$

Furthermore, $\rho \geq 0$ vanishes if and only if the signature is $(0,2)$.

Proof. Since by assumption a and b generate a negative-definite two-dimensional space, we have $\|a\|^2 < 0$. The definition of λ directly yields that $\|d\|^2 = -2$. Because the space spanned by a and b is negative-definite, we in particular have $\|a\|^2\|b\|^2 - \langle a,b\rangle^2 > 0$. Therefore, μ is a well-defined positive number. We then compute, using that $\|d\|^2 = -2$ and the definition of μ,

$$\begin{aligned}
\langle d,e\rangle &= \mu\langle d,b\rangle + 2\alpha_1 = \lambda\mu\langle a,b\rangle - \langle a,b\rangle\lambda\mu = 0,\\
\|e\|^2 &= \langle \mu b - \alpha_1 d, e\rangle = \mu\langle b, \mu b - \alpha_1\lambda a\rangle\\
&= \mu^2\|b\|^2 - \alpha_1\lambda\mu\langle a,b\rangle = \mu^2\left(\|b\|^2 + \frac{1}{2}\langle a,b\rangle^2\lambda^2\right) = -2.
\end{aligned}$$

Then by the choices of α_2 and α_3, we obtain that f, as defined in (II.3.7), is orthogonal to d and e, using $\langle d,e\rangle = 0$:

$$\begin{aligned}
\langle d,f\rangle &= \langle d, \nu c - \alpha_2 d - \alpha_3 e\rangle = \nu\langle d,c\rangle + 2\alpha_2 = \lambda\nu\langle a,c\rangle - \lambda\nu\langle a,c\rangle = 0,\\
\langle e,f\rangle &= \langle e, \nu c - \alpha_2 d - \alpha_3 e\rangle = \nu\langle e,c\rangle + 2\alpha_3\\
&= \nu\langle \mu b - \alpha_1\lambda a, c\rangle - \mu\nu\left(\langle b,c\rangle + \frac{\langle a,b\rangle\langle a,c\rangle\lambda^2}{2}\right)
\end{aligned}$$

$$= \frac{\langle a,b\rangle}{2}\lambda^2\mu\nu\langle a,c\rangle - \mu\nu\frac{\langle a,b\rangle\langle a,c\rangle\lambda^2}{2} = 0.$$

The above then yields

$$\|c\|^2 = \nu^{-2}\left(\|f\|^2 - 2\alpha_2^2 - 2\alpha_3^2\right).$$

Finally we rewrite

$$\begin{aligned}\rho &= -\frac{\|c\|^2}{2} + \frac{\langle a,c\rangle^2}{2\|a\|^2} + \frac{1}{4}\frac{2\|a\|^2}{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}\left(\langle b,c\rangle - \frac{\langle a,b\rangle\langle a,c\rangle}{\|a\|^2}\right)^2\\ &= -\frac{\|c\|^2}{2} + \frac{\langle a,c\rangle^2}{2\|a\|^2} - \frac{1}{4}\left(-\frac{\|a\|^2\|b\|^2 - \langle a,b\rangle^2}{2\|a\|^2}\right)^{-1}\left(\langle b,c\rangle - \frac{\langle a,b\rangle\langle a,c\rangle}{\|a\|^2}\right)^2\\ &= -\frac{\|c\|^2}{2} - \frac{\langle a,c\rangle^2}{4}\lambda^2 - \frac{1}{4}\mu^2\left(\langle b,c\rangle + \frac{\langle a,b\rangle\langle a,c\rangle\lambda^2}{2}\right)^2.\end{aligned}$$

Using the definition of ν then yields

$$\begin{aligned}\|f\|^2 &= \nu^2\|c\|^2 + 2\frac{\langle a,c\rangle^2}{4}\lambda^2\nu^2 + \frac{1}{2}\mu^2\nu^2\left(\langle b,c\rangle + \frac{\langle a,b\rangle\langle a,c\rangle\lambda^2}{2}\right)^2\\ &= -2\nu^2\rho = \begin{cases}0 & \text{if } \rho = 0,\\ -2\operatorname{sgn}(\rho) & \text{if } \rho \neq 0.\end{cases}\end{aligned}$$

This shows that ρ vanishes if and only if $\operatorname{span}\{d,e,f\} = \operatorname{span}\{a,b,c\}$ has signature $(0,2)$. Since $\operatorname{span}\{a,b,c\}$ is negative semi-definite, $\|f\|^2 \leq 0$ and thus $\|f\|^2 \in \{-2,0\}$. Therefore $\nu = \rho^{-\frac{1}{2}} \geq 0$ if $\rho \neq 0$ such that ν agrees with the definition in the proposition. ☐

We are now ready to prove Proposition II.3.6.

Proof of Proposition II.3.6. (1) We let $d,e,f \in \mathbb{R}^{N+3}$ and $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}$ be as in Lemma II.3.7. The definitions of $\alpha_1,\alpha_2,\alpha_3,d,e,f$, together with Lemma II.3.1, ensure that the asymptotics hold. Lemma II.3.7 also implies that P^Td, P^Te, P^Tf are pairwise orthogonal with squared norm -2 each (with respect to $D^{-1} = D := \operatorname{diag}(I_N, -I_3)$). Combining this with Lemma II.3.4 and the chain rule then gives the claimed satisfaction of Vignéras' differential equation.
(2) Lemma II.3.7 shows that $v,w \in \{d,e,f\}$ satisfy

$$-2\delta_{v,w}\left(1 - \delta_{f,v}\right) = \langle v,w\rangle = v^TA^{-1}w = v^TPD^{-1}P^Tw = \left(P^Tv\right)^TDP^Tw.$$

Therefore (P^Td, P^Te, P^Tf) forms an orthogonal basis with norms squared $(-2,-2,0)$ with respect to $D^{-1} = D$. Note that there exists a subspace of signature $(1,3)$ with orthogonal basis (d,e,f_+,f_-) such that $f = f_+ + f_-$. Setting (note that $\|f_-\| = \|f_+\|$)

$$w_\varepsilon := \frac{1}{\sqrt{2\|f_+\|^2\varepsilon}}\left((1-\varepsilon)f_+ + (1+\varepsilon)f_-\right),$$

we compute $||w_\varepsilon||^2 = -2$. Therefore $(P^Td, P^Te, P^Tw_\varepsilon)$ is an orthogonal basis with norms squared $(-2,-2,-2)$ with respect to D. Just like the previous case, applying Lemma II.3.4 and the chain rule shows that

$$X \mapsto E_3\left(\alpha_1, \frac{1}{\sqrt{2\|f_+\|^2\varepsilon}}\alpha_2, \frac{1}{\sqrt{2\|f_+\|^2\varepsilon}}\alpha_3; d^TPX, e^TPX, w_\varepsilon^TPX\right)$$

is a solution of Vigneras' differential equation. The scalar factors for α_2 and α_3 ensure that the function has the right asymtotic behaviour.

□

II.4 Lau and Zhou's explicit Gromov-Witten potential and simplifications for the proof of Theorem II.1.1

In this section, we explicitly recall the functions arising in Gromov-Witten theory, which were studied by Lau and Zhou in [11], as well the explicit summation formulas for them by Cho, Hong, Kim, and Lau [3], and we start the investigation of their modularity properties. We assume throughout that $\mathbf{a} = (2,3,6)$ and study the function $W_q(2,3,6)$ defined in [11]. Namely, noting that in the notation of [11] we have $q = q_d^{48}$, and writing the resulting coefficients as functions of τ, by (3.29) of [11], $W_q(2,3,6)$ can be expanded as

$$\begin{aligned} &W_q(2,3,6)\\ &\quad = q^{\frac{1}{8}}X^2 - q^{\frac{1}{48}}XYZ + c_Y(\tau)Y^3 + c_Z(\tau)Z^6 + c_{YZ2}(\tau)Y^2Z^2 + c_{YZ4}(\tau)YZ^4, \end{aligned} \tag{II.4.1}$$

where

$$c_Y(\tau) := q^{\frac{3}{16}}\sum_{n\geq 0}(-1)^{n+1}(2n+1)q^{\frac{n(n+1)}{2}}, \tag{II.4.2}$$

$$\begin{aligned} c_{YZ2}(\tau) := q^{-\frac{1}{12}}\sum_{n\geq a\geq 0}\Big((-1)^{n+a}(6n-2a+8)q^{\frac{(n+2)(n+1)}{2}-\frac{a(a+1)}{2}}\\ +(2n+4)q^{n+an+1-a^2}\Big), \end{aligned} \tag{II.4.3}$$

$$c_{YZ4}(\tau) := q^{-\frac{17}{48}}\sum_{\substack{a,b\geq 0\\ n\geq a+b}}(-1)^{n+a+b}(6n-2a-2b+7)q^{\frac{(n+1)(n+2)}{2}-\frac{a(a+1)}{2}-\frac{b(b+1)}{2}}, \tag{II.4.4}$$

$$c_Z(q) := q^{-\frac{5}{8}} \sum_{(n,a,b,c)\in T_1\cup T_2\cup T_3\cup T_6} (-1)^{n+a+b+c} \frac{(6n-2a-2b-2c+6)}{\eta(n,a,b,c)} q^{A(n,a,b,c)}, \qquad \text{(II.4.5)}$$

where

$$\begin{aligned}
&A(n,a,b,c) := \binom{n+2}{2} - \binom{a+1}{2} - \binom{b+1}{2} - \binom{c+1}{2},\\
&T_6 := \{(3a,a,a,a) : a \in \mathbb{N}_0\},\\
&T_3 := \{(3a+k,a,a,a) : a \in \mathbb{N}_0, k \in \mathbb{N}\},\\
&T_2 := \{(a+b+c,a,b,c) : a,b,c \in \mathbb{N}_0 \text{ such that } a < \min(b,c) \text{ or } a = c < b\},\\
&T_1 := \{(a+b+c+k,a,b,c) : k \in \mathbb{N}, a,b,c \in \mathbb{N}_0\\
&\qquad\qquad\qquad\qquad \text{such that } a < \min(b,c) \text{ or } a = c < b\},\\
&\eta(n,a,b,c) := j \text{ if } (n,a,b,c) \in T_j.
\end{aligned}$$

Note that the authors in [3] and [11] have an extra condition "distinct" in T_1. This turns out to just be a typo.

In [2] modularity properties of c_Y, c_{YZ2}, and c_{YZ4} were laid out and proven. We are thus left to investigate the hardest piece c_Z. The following lemma decomposes c_Z into 3 simpler pieces.

Lemma II.4.1. *We have*

$$q^{\frac{5}{4}} c_Z(\tau) = F_1(\tau) - F_2(\tau) - \frac{2}{3}F_3(\tau), \qquad \textit{where}$$

$$F_1(\tau) := \left(\sum_{\substack{a,b,c\geq 0\\ k>0\\ a<\min(b,c)}} - \sum_{\substack{a,b,c<0\\ k\leq 0\\ a\geq \max(b,c)}} \right) (-1)^k (3k+2a+2b+2c+3) \times q^{\frac{k^2}{2}+\frac{3k}{2}+ab+ac+ak+bc+bk+ck+a+b+c+1}, \qquad \text{(II.4.6)}$$

$$F_2(\tau) := \frac{1}{2}\left(\sum_{a,b\geq 0} - \sum_{a,b<0} \right) (6a+2b+3) q^{3a^2+2ab+3a+b+2}, \qquad \text{(II.4.7)}$$

$$F_3(\tau) := \frac{3}{4}\left(\sum_{a,k\geq 0} - \sum_{a,k<0} \right) (-1)^k \left(2a+k+1\right) q^{3a^2+3ak+3a+\frac{k^2}{2}+\frac{3k}{2}+1}. \qquad \text{(II.4.8)}$$

Proof. Let

$$f_j(\tau) := \frac{1}{j} \sum_{(n,a,b,c)\in T_j} \widetilde{g}(n,a,b,c;\tau),$$
$$\widetilde{g}(n,a,b,c;\tau) = (-1)^{n+a+b+c}(6n-2a-2b-2c+6)q^{A(n,a,b,c)}$$

and $g(k,a,b,c;\tau) := \widetilde{g}(a+b+c+k,a,b,c;\tau)$ and split

$$f_1(\tau) = f_{11}(\tau) + f_{12}(\tau), \qquad f_2(\tau) = f_{21}(\tau) + f_{22}(\tau)$$

with

$$f_{11}(\tau) := \sum_{\substack{a,b,c\geq 0, k>0\\ a<\min(b,c)}} g(k,a,b,c;\tau), \qquad f_{12}(\tau) := \sum_{\substack{a,b,c\geq 0, k>0\\ c=a<b}} g(k,a,b,c;\tau),$$
$$f_{21}(\tau) := \frac{1}{2} \sum_{\substack{a,b,c\geq 0, k=0\\ a<\min(b,c)}} g(k,a,b,c;\tau), \qquad f_{22}(\tau) := \frac{1}{2} \sum_{\substack{a,b,c\geq 0, k=0\\ c=a<b}} g(k,a,b,c;\tau).$$

Note that

$$\widetilde{g}\left(-n-3,-a-1,-b-1,-c-1\right) = \widetilde{g}\left(n,a,b,c\right),$$
$$g\left(-k,-a-1,-b-1,-c-1\right) = g(k,a,b,c),$$

which we use repeatedly. We now compute

$$f_1(\tau) = \frac{1}{2}\left(\sum_{\substack{a,b,c\geq 0, k>0\\ a<\min(b,c)}} - \sum_{\substack{a,b,c,k<0\\ a>\max(b,c)}}\right) g(k,a,b,c;\tau)$$
$$= \frac{1}{2}\left(\sum_{\substack{a,b,c\geq 0, k>0\\ a<\min(b,c)}} - \sum_{\substack{a,b,c<0, k\leq 0\\ a>\max(b,c)}}\right) g(k,a,b,c;\tau) - f_{21}(\tau),$$

and similarly

$$f_{12}(\tau) + 2f_{22}(\tau)$$
$$= \sum_{\substack{a,b,c\geq 0, k>0\\ c=a<b}} g(k,a,b,c;\tau) + \sum_{\substack{a,b,c\geq 0, k=0\\ c=a<b}} g(k,a,b,c;\tau) = \sum_{\substack{a,b,c\geq 0, k\geq 0\\ c=a<b}} g(k,a,b,c;\tau)$$
$$= \frac{1}{2} \sum_{\substack{a,b,c,k\geq 0\\ a=\min(b,c)}} g(k,a,b,c;\tau) - \frac{1}{2} \sum_{\substack{a,b,c,k\geq 0\\ c=a=b}} g(k,a,b,c;\tau)$$

$$= -\frac{1}{2} \sum_{\substack{a,b,c<0,k\leq 0 \\ a=\max(b,c)}} g(k,a,b,c;\tau) - \frac{3}{2} f_3(\tau) - \frac{6}{2} f_6(\tau).$$

Therefore

$$\begin{aligned}
&f_1(\tau) + f_2(\tau) + f_{22}(\tau) + \frac{3}{2} f_3(\tau) + 3f_6(\tau) \\
&= f_{11}(\tau) + f_{12}(\tau) + f_{21}(\tau) + 2f_{22}(\tau) + \frac{3}{2} f_3(\tau) + 3f_6(\tau) \\
&= \frac{1}{2} \left(\sum_{\substack{a,b,c\geq 0,k>0 \\ a<\min(b,c)}} - \sum_{\substack{a,b,c<0,k\leq 0 \\ a>\max(b,c)}} \right) g(k,a,b,c;\tau) - \frac{1}{2} \sum_{\substack{a,b,c<0,k\leq 0 \\ a=\max(b,c)}} g(k,a,b,c;\tau) \\
&= \frac{1}{2} \left(\sum_{\substack{a,b,c\geq 0,k>0 \\ a<\min(b,c)}} - \sum_{\substack{a,b,c<0,k\leq 0 \\ a\geq\max(b,c)}} \right) g(k,a,b,c;\tau) = F_1(\tau).
\end{aligned}$$

For f_{22}, we find that

$$\begin{aligned}
f_{22}(\tau) = \frac{1}{2} \sum_{\substack{a,b,c\geq 0 \\ c=a<b}} g(0,a,b,c;\tau) &= \frac{1}{4} \left(\sum_{\substack{a,b,c\geq 0 \\ c=a<b}} - \sum_{\substack{a,b,c<0 \\ c=a>b}} \right) g(0,a,b,c;\tau) \\
&= \frac{1}{4} \left(\sum_{\substack{a,b,c\geq 0 \\ c=a\leq b}} - \sum_{\substack{a,b,c<0 \\ c=a>b}} \right) g(0,a,b,c;\tau) - \frac{3}{2} f_6(\tau).
\end{aligned}$$

Making the change of variables $b \mapsto b + a$, we obtain that the first sum equals

$$\begin{aligned}
&\frac{1}{4} \left(\sum_{a,b\geq 0} - \sum_{a,b<0} \right) g(0,a,a+b,a;\tau) \\
&= \frac{1}{4} \left(\sum_{a,b\geq 0} - \sum_{a,b<0} \right) (12a + 4b + 6) q^{3a^2+2ab+3a+b+2} = F_2(\tau).
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
&3f_3(\tau) + 3f_6(\tau) \\
&= \sum_{\substack{a=b=c\geq 0 \\ k>0}} g(k,a,b,c) + \frac{1}{2} \sum_{\substack{a=b=c\geq 0 \\ k=0}} g(k,a,b,c;\tau)
\end{aligned}$$

$$= \frac{1}{2}\left(\sum_{\substack{a=b=c\geq 0\\ k>0}} - \sum_{\substack{a=b=c<0\\ k<0}}\right) g(k,a,b,c;\tau) + \frac{1}{2}\sum_{\substack{a=b=c>0\\ k=0}} g(k,a,b,c;\tau)$$
$$= \frac{1}{2}\left(\sum_{\substack{a=b=c\geq 0\\ k\geq 0}} - \sum_{\substack{a=b=c<0\\ k<0}}\right) g(k,a,b,c;\tau)$$
$$= \frac{1}{2}\left(\sum_{a,k\geq 0} - \sum_{a,k<0}\right) (-1)^k(12a+6k+6)q^{3a^2+3ak+3a+\frac{k^2}{2}+\frac{3k}{2}+1} = 4F_3(\tau).$$

Combining completes the proof. □

Lemma II.4.2. *The functions F_2 and F_3 have modular completions.*

Proof. We view F_2 as derivatives of indefinite theta series with additional Jacobi variables (where $\zeta_j := e^{2\pi i z_j}$)

$$F_2(z_1,z_2;\tau) := \zeta_2^{\frac{3}{2}}\left(\sum_{a,b\geq 0} - \sum_{a,b<0}\right)(-1)^a\zeta_1^a\zeta_2^b q^{\frac{3a(a+1)}{2}+ab}.$$

Then

$$F_2(\tau) = \frac{q^{\frac{1}{2}}}{4}\left[\left(12\zeta_1\frac{\partial}{\partial\zeta_1} + 4\zeta_2\frac{\partial}{\partial\zeta_2} + 3\right)F_2(z_1,z_2;\tau)\right]_{\substack{\zeta_1=-1\\ \zeta_2=q\\ q\mapsto q^2}}.$$

Define

$$\widehat{F}_2(z_1,z_2;\tau) := F_2(z_1,z_2;\tau) + \frac{i}{2}\sum_{k=0}^{2}\zeta_2^k\vartheta(z_1+k\tau;3\tau)R(3z_2-z_1-k\tau;3\tau),$$

where (with $z = x+iy$)

$$\vartheta(z;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{2\pi i n\left(z+\frac{1}{2}\right)}q^{\frac{n^2}{2}}$$
$$= -iq^{\frac{1}{8}}e^{-\pi i z}\prod_{n\geq 1}(1-q^n)\left(1-e^{2\pi i z}q^{n-1}\right)\left(1-e^{-2\pi i z}q^n\right),$$
$$R(z;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}}\left(\operatorname{sgn}(n) - E\left(\left(n+\frac{y}{v}\right)\sqrt{2v}\right)\right)(-1)^{n-\frac{1}{2}}q^{-\frac{n^2}{2}}e^{-2\pi i n z}.$$

Setting

$$\widehat{F}_2(\tau) := \frac{q^{\frac{1}{2}}}{4}\left[\left(12\zeta, \frac{\partial}{\partial\zeta_1} + 4\zeta_2\frac{\partial}{\partial\zeta_2} + 3\right)\widehat{F}\left(z_1, z_2; \tau\right)\right]_{\substack{\zeta_1=1\\ \zeta_2=q\\ q\mapsto q^2}}$$

we have $\widehat{F}_2^+ = F_2$. Using [14] we see that we have, for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $n_1, n_2, m_1, m_2 \in \mathbb{Z}$,

$$\widehat{F}_2\left(\frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)e^{\frac{\pi ic\left(-3z_2^2+2z_1z_2\right)}{c\tau+d}}\widehat{F}_2\left(z_1, z_2; \tau\right),$$
$$\widehat{F}_2\left(z_1+n_1\tau+m_1, z_2+n_2\tau+m_2\right) = (-1)^{n_1+m_1}\zeta_2^{3n_2-n_1}\zeta_1^{-n_2}q^{\frac{3n_2^2}{2}-n_1n_2}\widehat{F}_2\left(z_1, z_2; \tau\right).$$

From this one can then derive the transformation law of $F_2(z_1+\frac{1}{2}, z_2+\frac{\tau}{2}; \tau)$. Taking the appropriate derivatives with respect to z_1 and z_2 then gives additional terms involving $1, \tau, \tau^2$ which can be removed with the help of $1/v$-terms (or using powers of the weight 2 Eisenstein series). This yields the modular completion. The shadow of $F_2(\tau)$ can be determined using (II.2.7) and (II.2.8) for $\widehat{F}_3\left(z_1, z_2; \tau\right)$ and then applying the appropriate Jacobi derivatives.

We next turn to F_3 and define the Jacobi version of F_3

$$F_3(z_1, z_2; \tau) := \left(\sum_{a,b\geq 0} - \sum_{a,b<0}\right)\zeta_1^a\zeta_2^b q^{3a^2+\frac{b^2}{2}+3ab}.$$

Then

$$F_3(\tau) = \frac{3q}{4}\left[\left(2\zeta_1\frac{\partial}{\partial\zeta_1} + \zeta_2\frac{\partial}{\partial\zeta_2} + 1\right)F_3(z_1, z_2; \tau)\right]_{\substack{\zeta_1=q^3\\ \zeta_2=-q^{\frac{3}{2}}}}.$$

Writing

$$F_3(z_1, z_2; \tau) = \frac{1}{2}\sum_{a,b\in\mathbb{Z}}\left(\operatorname{sgn}\left(a+\frac{1}{2}\right) + \operatorname{sgn}\left(b+\frac{1}{2}\right)\right)\zeta_1^a\zeta_2^b q^{\frac{1}{2}Q(a,b)}$$

with $Q(a,b) := 6a^2+b^2+6ab$, we obtain the completion

$$\widehat{F}_3(z_1, z_2; \tau) := \frac{1}{2}\sum_{a,b\in\mathbb{Z}}\left(E\left(\sqrt{v}\left(\sqrt{6}a+\frac{1}{2}+y_1\right)\right) - E\left(\sqrt{v}\left(b+\frac{1}{2}+y_2\right)\right)\right)$$
$$\times\ \zeta_1^a\zeta_2^b q^{\frac{1}{2}Q(a,b)}.$$

The proof then follows as before. □

II.5 An indefinite theta function of signature $(1, 3)$

A key in understanding F_1 is the following indefinite theta function

$$\Theta(z;\tau) := \Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}(z;\tau),$$

where $A := \left(\begin{smallmatrix} 1&1&1&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&1&1&0 \end{smallmatrix}\right)$, and

$$\begin{aligned}
\widehat{p}(\ell) &:= -E(\ell_1) - E\left(\sqrt{2}\ell_2\right) + E(\ell_3-\ell_1)\, E_2\left(\frac{1}{\sqrt{3}}; \ell_1, \frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4)\right)\\
&+ E_3\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}; \ell_3-\ell_2, \frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4), \sqrt{\frac{2}{3}}(\ell_2+\ell_3+\ell_4)\right)\\
&+ \operatorname{sgn}(\ell_1+\ell_2+\ell_3)\\
&\times \Big(E(\ell_1)\, E(\ell_3-\ell_2) - E_2(1;\ell_1,-\ell_1-2\ell_2) + E_2(-1;\ell_3-\ell_2,\ell_2+\ell_3) + 1\Big)\\
&+ \operatorname{sgn}(\ell_1+\ell_2+\ell_4)\\
&\times \Big(E(\ell_1)\, E(\ell_4-\ell_2) - E_2(1;\ell_1,-\ell_1-2\ell_2) + E_2(-1;\ell_4-\ell_2,\ell_2+\ell_4) + 1\Big).
\end{aligned}$$

Theorem II.5.1. *The function Θ transforms like a vector-valued Jacobi form.*

There are two main steps that have to be made: convergence and showing that $\widehat{p}$ satisfies Vignéras' differential equation.

Proposition II.5.2. *The theta series $\Theta_{0,\mathbb{Z}^4,A,p,0}(z;\tau)$, as well as its modular completion $\Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}(z;\tau)$ converges absolutely and uniformly on compact subsets of $\{(z,\tau)\in\mathbb{C}^n\times\mathbb{C} : B(c_j,\frac{y}{v}) \notin \mathbb{Z}\ (j\in\{0,1,2,3\})\}$. Here*

$$\begin{aligned}
p(\ell) &:= \Big(\operatorname{sgn}\left(c_0^T\ell\right) + \operatorname{sgn}\left(c_1^T\ell\right)\Big)\Big(\operatorname{sgn}\left(c_0^T\ell\right) + \operatorname{sgn}\left(c_2^T\ell\right)\Big)\Big(\operatorname{sgn}\left(c_1^T\ell\right) + \operatorname{sgn}\left(c_3^T\ell\right)\Big)\\
&= (\operatorname{sgn}(\ell_1)+\operatorname{sgn}(\ell_2))(\operatorname{sgn}(\ell_2)+\operatorname{sgn}(\ell_3-\ell_2))(\operatorname{sgn}(\ell_1)+\operatorname{sgn}(\ell_4-\ell_2))
\end{aligned}$$

with $c_0 := (0,1,0,0)^T, c_1 := (1,0,0,0)^T, c_2 := (0,-1,1,0)^T$ and $c_3 := (0,-1,0,1)^T$.

Proof. We begin by proving (absolute local uniform) convergence of the holomorphic theta series $\Theta_{0,\mathbb{Z}^4,A,p,0}$, so that it suffices to additionally prove the convergence of $\Theta_{0,\mathbb{Z}^4,A,\widehat{p}-p,0} = \Theta_{0,\mathbb{Z}^4,A,\widehat{p},0} - \Theta_{0,\mathbb{Z}^4,A,p,0}$. We also note that for (z,τ) lying in the stated range, we have $B(c_j, n+\frac{y}{v}) \neq 0$ for all $n\in\mathbb{Z}$.

The proof of the convergence of the holomorphic theta function is a straightforward generalization of the proof by Zwegers [28] and Alexandrov, Banerjee, Manschot, and Pioline [1]. We rewrite

$$\Big(\operatorname{sgn}\left(c_k^T \ell\right) + \operatorname{sgn}\left(c_j^T \ell\right) \Big) = \Big(\operatorname{sgn}\left((A^{-1}c_k)^T A\ell\right) + \operatorname{sgn}\left((A^{-1}c_j)^T A\ell\right) \Big)$$

in p and observe that

$$\begin{aligned} \left| q^{Q(n)} \zeta^{An} \right| &= \exp\Big(-\pi v n^T A n - 2\pi y^T A n \Big) \\ &= \exp\left(-2\pi Q\left(\sqrt{v}\left(n + \frac{y}{v} \right) \right) \right) \exp\left(\frac{2\pi Q(y)}{v} \right), \end{aligned}$$

so that

$$\sum_{n \in \mu + L} \left| p\left(\sqrt{v}\left(n + \frac{y}{v} \right) \right) q^{Q(n)} \zeta^{An} \right| = e^{\frac{2\pi Q(y)}{v}} \sum_{n \in \mu + L} \left| p\left(\sqrt{v}\left(n + \frac{y}{v} \right) \right) \right| e^{-2\pi Q\left(\sqrt{v}\left(n + \frac{y}{v}\right)\right)}.$$

Therefore we need to investigate

$$\sum_{m \in \Lambda} p(m) e^{-2\pi Q(m)}$$

for Λ some lattice. Note that $B(c_j, m) = c_j^T A m = (A c_j)^T m$. Since the $\operatorname{sgn}(c_j^T A m)$ do not vanish, $p(m) \neq 0$ only if all $\operatorname{sgn}(c_j^T A m)$ are equal. Therefore we obtain $B(c_j, m) B(c_k, m) > 0$ for all $j, k \in \{0, 1, 2, 3\}$ whenever $p(m) \neq 0$. The matrix

$$\begin{pmatrix} B(m,m) & B(c_0,m) & B(c_1,m) & B(c_2,m) & B(c_3,m) \\ B(c_0,m) & B(c_0,c_0) & B(c_0,c_1) & B(c_0,c_2) & B(c_0,c_3) \\ B(c_1,m) & B(c_0,c_1) & B(c_1,c_1) & B(c_1,c_2) & B(c_1,c_3) \\ B(c_2,m) & B(c_0,c_2) & B(c_1,c_2) & B(c_2,c_2) & B(c_2,c_3) \\ B(c_3,m) & B(c_0,c_3) & B(c_1,c_3) & B(c_2,c_3) & B(c_3,c_3) \end{pmatrix}$$

has determinant zero. We refer to the bottom right 4×4 block as G, which has negative determinant. Both of these statements come from the fact that $\operatorname{span}\{c_0, c_1, c_2, c_3\}$ has signature $(1, 3)$. Furthermore, a Laplace expansion along the first column and then another along the first row lets us write the determinant as

$$\begin{aligned} 0 \geq B(m,m) \det(G) &- \sum_{k=0}^{3} B(c_k, m)^2 \det((G_{p,q})_{p,q \neq k}) \\ &- 2 \sum_{0 \leq j < k \leq 3} (-1)^{k+j} B(c_j, m) B(c_k, m) \det\left((G_{p,q})_{p \neq k, q \neq j} \right). \end{aligned}$$

Now note that $\det((G_{p,q})_{p,q\neq k}) \leq 0$ since the space $\operatorname{span}\{c_p; p \neq k\}$ has signature $(0,3)$ or $(0,2)$ and the $\operatorname{sgn}(\det((G_{p,q})_{p\neq k, q\neq j})) = (-1)^{k+j+1}$ by direct calculation. Therefore, whenever it is $B(c_j, m)B(c_k, m) \geq 0$ for all $j, k \in \{0,1,2,3\}$ we obtain

$$\begin{aligned}
B(m,m) &= \det(G)^{-1}\Bigg(\sum_{k=0}^{3} B(c_k,m)^2 \det((G_{p,q})_{p,q\neq k}) \\
&\qquad - 2\sum_{0\leq j<k\leq 3} B(c_j,m)B(c_k,m)\left|\det\left((G_{p,q})_{p\neq k,q\neq j}\right)\right|\Bigg) \\
&= |\det(G)|^{-1}\Bigg(\sum_{k=0}^{3} B(c_k,m)^2 |\det((G_{p,q})_{p,q\neq k})| \\
&\qquad + 2\sum_{0\leq j<k\leq 3} \left|B(c_j,m)B(c_k,m)\det\left((G_{p,q})_{p\neq k,q\neq j}\right)\right|\Bigg) \\
&=: K(m) \geq 0.
\end{aligned}$$

If $p(m) \neq 0$, then at most two $B(c_j, m)$ vanish, such that some terms in the second sum do not vanish and the inequality is strict. Therefore, with $p(m) \leq 8$, we obtain

$$\sum_{m\in\Lambda} p(m)e^{-2\pi Q(m)} \leq 8\sum_{m\in\Lambda} e^{-2\pi K(m)}.$$

Now since none of the determinants in the second sum of $K(m)$ vanishes and the $B(c_j, m)$ do not vanish, we obtain for some constant $c > 0$ that

$$K(m) \geq c\min(|B(c_j,m)|),$$

which yields exponential decay of $e^{-2\pi K(m)}$ as $\|m\| \to \infty$ and thus the convergence (uniform on compact sets with respect to translations of Λ) of

$$\sum_{m\in\Lambda} p(m)e^{-2\pi Q(m)} \leq 8\sum_{m\in\Lambda} e^{-2\pi K(m)} < \infty.$$

Next we treat the difference between the holomorphic part and the modular completion. By multiplying out p and coupling the terms of p and $\widehat{p}$ appropriately, we get a sum of series over terms which have either the shape

$$\left(-E_3\left(\alpha; B(d_1,m), B(d_2,m), B(d_3,m)\right) + \prod_{j\in\{0,1,2,3\}\setminus\{\ell\}} \operatorname{sgn}(B(c_j,m))\right) q^{Q(n)}e^{2\pi i B(z,n)},$$

where d_1, $d_2 + \alpha_1 d_1$, $d_3 + \alpha_2 d_1 + \alpha_3 d_2$ are the c_j appearing in the product or the same shape with E_3 replaced by $\mathcal{E}_3$ or

$$(\operatorname{sgn}(B(c_\ell, m)) - E(B(c_\ell, m))) \operatorname{sgn}(B(c_k, m))^2 q^{Q(n)} e^{2\pi i B(z,n)}$$

for some $(k, \ell) \in \{(0,1), (0,3), (1,0), (1,2)\}$. In a similar fashion as in the proof of Theroem 4.2 of [1], one can decompose each of these terms into a sum of integrals decaying square-exponentially in some directions $\{c_{j_1}, \ldots, c_{j_k}\}$ (i.e., it grows like $e^{-\pi \sum_k B(c_{j_k}, m)^2}$ in additionto the general factor $e^{-\pi 2Q(m)}$). By combining the integrals of the same decay from different terms, one obtains cancellation of the sign-terms whenever the integrals times do not decay. This gives convergence of the the theta function. Further details can also be found in the second author's doctoral thesis [8]. Therefore the theta series $\Theta_{0,\mathbb{Z}^4,A,p-\widehat{p},0}$ and $\Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}$ converge. □

We next turn to proving that Vignéras' differential equation is satisfied in our situtaion.

Proposition II.5.3. *The function $\ell \mapsto \widehat{p}(P\ell)$ is a solution of Vignéras' differential equation with respect to $D := \operatorname{diag}(1, -1, -1, -1)$. It approximates p if $\ell_1, \ell_2 \neq 0$.*

Proof. Our approach is to split $p(\ell)$ into various terms, which we treat separately using Proposition II.3.6 and Lemma II.2.2. Multiplying the product of signs out, we obtain, using that ℓ_1 and ℓ_2 do not vanish

$$\begin{aligned}
&\operatorname{sgn}(\ell_1) + \operatorname{sgn}(\ell_2) + \operatorname{sgn}(\ell_3 - \ell_2) \\
&+ \operatorname{sgn}(\ell_4 - \ell_2) + \operatorname{sgn}(\ell_2) \operatorname{sgn}(\ell_3 - \ell_2) \operatorname{sgn}(\ell_4 - \ell_2) \\
&+ \operatorname{sgn}(\ell_1) \operatorname{sgn}(\ell_2) \operatorname{sgn}(\ell_3 - \ell_2) + \operatorname{sgn}(\ell_1) \operatorname{sgn}(\ell_2) \operatorname{sgn}(\ell_4 - \ell_2) \\
&+ \operatorname{sgn}(\ell_1) \operatorname{sgn}(\ell_3 - \ell_2) \operatorname{sgn}(\ell_4 - \ell_2).
\end{aligned}$$

We first compute $A^{-1} = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$.

The single sign factors are treated as in Zwegers' thesis [28] (see the description in Section II.2.2), yielding the following function

$$\begin{aligned}
E(\ell_1) + E\left(\sqrt{2}\ell_2\right) + E(\ell_3 - \ell_2) + E(\ell_4 - \ell_2)& \\
\sim \operatorname{sgn}(\ell_1) + \operatorname{sgn}(\ell_2) + \operatorname{sgn}(\ell_3 - \ell_2) + \operatorname{sgn}(\ell_4 - \ell_2)&,
\end{aligned}$$

where each of the summands on the left satisfies Vignéras differential equation in $(\ell_1, \ell_2, \ell_3, \ell_4)$ with respect to $\operatorname{diag}(1, -I_3)$ as can be verified directly. Thus we are left to consider 3 sign factors.

We start with $\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)$ and set

$$v_1 := (1,0,0,0)^T, \qquad v_2 := (0,-1,1,0)^T, \qquad v_3 := (0,-1,0,1)^T.$$

Then, with $\langle a,b\rangle = a^T A^{-1} b$, we obtain

$$\|v_j\|^2 = -2, \qquad \langle v_1,v_2\rangle = \langle v_1,v_3\rangle = 0, \qquad \langle v_2,v_3\rangle = -1.$$

This easily gives that the corresponding signature is $(0,3)$. We plug these into (II.3.7) to obtain

$$\lambda = 1, \qquad \mu = 1, \qquad \nu = \frac{2}{\sqrt{3}}, \qquad \alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = \frac{1}{\sqrt{3}},$$

$$d = v_1, \qquad e = v_2, \qquad f = \frac{2}{\sqrt{3}}v_3 - \frac{1}{\sqrt{3}}w_2.$$

Lemma II.3.1 then yields that

$$E_3\left(0, \frac{1}{\sqrt{3}}, 0; \ell_1, \ell_3-\ell_2, \frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4)\right) \sim \operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)$$

and $X \mapsto E_3(0, \frac{1}{\sqrt{3}}, 0; v_1^T P, v_2^T XP, 3^{-\frac{1}{2}}(2v_3^T - v_2^T)P)$. The claim then follows using (II.3.3).

We next turn to the case $\operatorname{sgn}(\ell_2)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)$ and set

$$v_1 := (0,-1,1,0)^T, \quad v_2 := (0,-1,0,1)^T, \quad v_3 := (0,1,0,0)^T.$$

Then

$$\|v_1\|^2 = \|v_2\|^2 = -2, \quad \|v_3\|^2 = \langle v_1,v_2\rangle = -1, \quad \langle v_1,v_3\rangle = \langle v_2,v_3\rangle = 1.$$

We plug these into Lemma II.3.7 to obtain

$$\lambda = 1, \quad \mu = \frac{2}{\sqrt{3}}, \quad \nu = \sqrt{6}, \quad \alpha_1 = \frac{1}{\sqrt{3}}, \quad \alpha_2 = -\sqrt{\frac{3}{2}}, \quad \alpha_3 = -\frac{1}{\sqrt{2}},$$

and

$$d = v_1, \qquad e = \frac{1}{\sqrt{3}}(0,-1,-1,2)^T, \qquad f = \sqrt{\frac{2}{3}}(0,1,1,1)^T.$$

Thus we obtain the completion

$$E_3\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}; \ell_3-\ell_2, \frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4), \sqrt{\frac{2}{3}}(\ell_2+\ell_3+\ell_4)\right)$$

$$\sim \operatorname{sgn}(\ell_2)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2).$$

We now turn to the case $\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_2)\operatorname{sgn}(\ell_3-\ell_2)$ and set

$$v_1 := (1,0,0,0)^T, \quad v_2 := (0,-1,1,0)^T, \quad v_3 := (0,1,0,0)^T.$$

We compute

$$||v_1||^2 = ||v_2||^2 = -2, \quad ||v_3||^2 = -1, \quad \langle v_1, v_2\rangle = 0, \quad \langle v_1, v_3\rangle = \langle v_2, v_3\rangle = 1.$$

We plug these into the (II.3.7) to obtain

$$\lambda = 1, \qquad \mu = 1, \qquad \nu = 1, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{1}{2}, \qquad \alpha_3 = -\frac{1}{2},$$

$$d = v_1, \qquad e = v_2, \qquad f = \frac{1}{2}(1,1,1,0)^T.$$

We use the second part of Proposition II.3.6 to obtain the differential equation for $\mathcal{E}_3$ and then use Lemma II.3.3 to obtain explicitely

$$\begin{aligned}&-E(\ell_1) - E(\ell_3-\ell_2) - E\left(\sqrt{2}\ell_2\right) + \operatorname{sgn}(\ell_1+\ell_2+\ell_3)\\ &\quad\times\left(E_2(0;\ell_1,\ell_3-\ell_2) - E_2(1;\ell_1,-\ell_1-2\ell_2) + E_2(-1;\ell_3-\ell_2,\ell_2+\ell_3) + 1\right).\end{aligned}$$

In the same way, replacing ℓ_3 by ℓ_4 yields the completion for $\operatorname{sgn}(\ell_1)$, $\operatorname{sgn}(\ell_2)$, $\operatorname{sgn}(\ell_4-\ell_2)$

$$\begin{aligned}&-E(\ell_1) - E(\ell_4-\ell_2) - E\left(\sqrt{2}\ell_2\right) + \operatorname{sgn}(\ell_1+\ell_2+\ell_4)\\ &\quad\times\Big(E_2(0;\ell_1,\ell_4-\ell_2) - E_2(1;\ell_1,-\ell_1-2\ell_2) + E_2(-1;\ell_4-\ell_2,\ell_2+\ell_4) + 1\Big).\end{aligned}$$

Combining all terms then gives the claim.

□

Corollary II.5.4. *The function F_1 has a modular completion.*

Proof. Define

$$F(z_1,z_2,z_3,z_4;\tau) := \left(\sum_{\substack{n_2,n_3,n_4\geq 0\\ n_1>0\\ n_2<\min(n_3,n_4)}} - \sum_{\substack{n_2,n_3,n_4<0\\ n_1\leq 0\\ n_2\geq\max(n_3,n_4)}}\right)$$

$$\times (-1)^{n_1} q^{\frac{n_1^2}{2}+n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4} e^{2\pi i B(z,n)}.$$

Noting $e^{2\pi i B(z,n)} = \zeta_1^{n_1+n_2+n_3+n_4}\zeta_2^{n_1+n_3+n_4}\zeta_3^{n_1+n_2+n_4}\zeta_4^{n_1+n_2+n_3}$, we obtain that F_1 equals

$$\lim_{w\to 0^+}\left[\left(3+\frac{\partial}{2\pi i\partial w}\right) F\left(-w^2\tau, 2w\tau + w^2\tau + \frac{\tau}{2}, 2w\tau + \frac{\tau}{2}, 2w\tau + \frac{\tau}{2}; \tau\right)\right]$$

With $z(w) := \tau(-w^2, 2w + w^2 + \frac{1}{2}, 2w + \frac{1}{2}, 2w + \frac{1}{2})$ we write for $w > 0$ small enough

$$F(z(w);\tau) = \sum_{n\in\mathbb{Z}^4} \frac{1}{8} p\left(n + \frac{\mathrm{Im}(z(w))}{v}\right)$$
$$\times (-1)^{n_1} q^{\frac{n_1^2}{2}+n_1n_2+n_1n_3+n_1n_4+n_2n_3+n_2n_4+n_3n_4} e^{2\pi i B(z(w),n)},$$

since

$$p\left(n + \left(-w^2, 2w + w^2 + \frac{1}{2}, 2w + \frac{1}{2}, 2w + \frac{1}{2}\right)^T\right)$$
$$= \begin{cases} 8 & \text{if } n_1 > 0, n_2 \geq 0, n_3 > n_2, n_4 > n_2, \\ -8 & \text{if } n_1 \leq 0, n_2 < 0, n_3 \leq n_2, n_4 \leq n_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p(x) = p(\sqrt{v}x)$, we obtain

$$\begin{aligned} 8F(z(w);\tau) &= \sum_{n\in\mathbb{Z}^4} p\left(n + \frac{\mathrm{Im}(z(w))}{v}\right)(-1)^{n_1} q^{\frac{1}{2}n^TAn} e^{2\pi i B(z,n)} \\ &= \sum_{n\in\mathbb{Z}^4} p\left(n + \frac{\mathrm{Im}(z(w))}{v}\right) q^{\frac{1}{2}n^TAn} e^{2\pi i B\left(z + A^{-1}\left(\frac{1}{2},0,0,0\right), n\right)} \\ &= \Theta_0\left(z(w) + \frac{1}{2}A^{-1}(1,0,0,0)^T; \tau\right). \end{aligned}$$

By Lemma II.2.2 and Proposition II.5.3, $\widehat{\Theta}_0 = \Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}$ is the modular completion of $\Theta_0 = \Theta_{0,\mathbb{Z}^4,A,p,0}$. Note that for $w > 0$ small enough, we have $z(w) \in \{(z,\tau) \in \mathbb{C}^n \times \mathbb{C} : B(c_j, \frac{y}{v}) \notin \mathbb{Z}\ (j \in \{0,1,2,3\})\}$ such that with Proposition II.5.2 and Proposition II.5.3,

$$\frac{1}{8}\lim_{w\to 0^+}\left[\left(3+\frac{\partial}{2\pi i\partial w}\right)\widehat{\Theta}_0\left(z(w) + \frac{1}{2}A^{-1}(1,0,0,0)^T; \tau\right)\right]$$

is the modular completion of

$$\begin{aligned} F_1(\tau) &= \lim_{w\to 0^+}\left[\left(3+\frac{\partial}{2\pi i\partial w}\right)F(z(w);\tau)\right] \\ &= \frac{1}{8}\lim_{w\to 0^+}\left[\left(3+\frac{\partial}{2\pi i\partial w}\right)\Theta_0\left(z(w)+\frac{1}{2}A^{-1}(1,0,0,0)^T;\tau\right)\right], \end{aligned}$$

which proves the claim.

$\square$

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Chapter III

Additional details for "Indefinite theta functions arising in Gromov-Witten Theory of elliptic orbifolds"

In this chapter, we give some technical details for the interested reader that were omitted in the proof of Proposition II.5.2 in [BKR] and Chapter II. Specifically, we show that the completed indefinite theta function $\Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}(z;\tau)$ converges.

Note that there are two typos in the definition of $\widehat{p}$ in [BKR]. The second and third argument of E_3 are falsely exchanged and the

$$E\left(\ell_3-\ell_1\right)E_2\left(\frac{1}{\sqrt{3}};\ell_1,\frac{1}{\sqrt{3}}\left(-\ell_2-\ell_3+2\ell_4\right)\right)$$

should be

$$E\left(\ell_1\right)E_2\left(\frac{1}{\sqrt{3}};\ell_3-\ell_2,\frac{1}{\sqrt{3}}\left(-\ell_2-\ell_3+2\ell_4\right)\right),$$

which came from a typo in the proof of Proposition II.5.3. Therefore we have

$$\begin{aligned}\widehat{p}(\ell)=&-E\left(\ell_1\right)-E\left(\sqrt{2}\ell_2\right)+E\left(\ell_1\right)E_2\left(\frac{1}{\sqrt{3}};\ell_3-\ell_2,\frac{1}{\sqrt{3}}\left(-\ell_2-\ell_3+2\ell_4\right)\right)\\&+E_3\left(\frac{1}{\sqrt{3}},-\sqrt{\frac{3}{2}},-\frac{1}{\sqrt{2}};\ell_3-\ell_2,\frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4),\sqrt{\frac{2}{3}}(\ell_2+\ell_3+\ell_4)\right)\\&+\sum_{j\in\{3,4\}}\operatorname{sgn}\left(\ell_1+\ell_2+\ell_j\right)\\&\quad\times\Big(E\left(\ell_1\right)E\left(\ell_j-\ell_2\right)-E_2\left(1;\ell_1,-\ell_1-2\ell_2\right)+E_2\left(-1;\ell_j-\ell_2,\ell_2+\ell_j\right)+1\Big).\end{aligned}$$

Note that the convergence of the completed indefinite theta function actually requires that the Jacobi variable is not contained in more hyperplanes than indicated in Proposition II.5.2 (compare with Lemma III.1.1).

III.1 Additional details for the proof of Proposition II.5.2

We will show the following.

Lemma III.1.1. *The theta series $\Theta_{0,\mathbb{Z}^4,A,\widehat{p},0}(z;\tau)$ converges absolutely and uniformly on compact subsets of*

$$\left\{(z,\tau)\in\mathbb{C}^4\times\mathbb{H} : B\left(c_j,\frac{y}{v}\right)\notin\mathbb{Z}\ (j\in\{0,1,2,3\}),\frac{y_2+y_3+y_4}{v},\frac{3y_1-y_2-y_3-y_4}{v}\notin\mathbb{Z}\right.$$
$$\left.\frac{y_2+y_k}{v},\frac{y_1+y_2+y_k}{v},\frac{-y_2-y_3-y_4+3y_k}{v}\notin\mathbb{Z}\ (k\in\{3,4\})\right\}.$$

Proof. We begin by decomposing all E_j functions appearing in $\widehat{p}$ into M_j-functions and sgn-functions using the equations (that hold when none of the appearing arguments containing w_j vanish) $E(w)=M(w)+\operatorname{sgn}(w)$,

$$\begin{aligned}E_2(\kappa;w)=M_2(\kappa;w)&+\operatorname{sgn}(w_2)M(w_1)\\&+\operatorname{sgn}(w_1-\kappa w_2)M\left(\frac{w_2+\kappa w_1}{\sqrt{1+\kappa^2}}\right)+\operatorname{sgn}(w_1)\operatorname{sgn}(w_2+\kappa w_1),\end{aligned}\tag{III.1.1}$$

which is given in IV.2.4, and

$$\begin{aligned}E_3(\alpha;w)&=M_3(\alpha;w)+\operatorname{sgn}(w_1)\operatorname{sgn}(w_2+\alpha_1w_1)\operatorname{sgn}(w_3+\alpha_2w_1+\alpha_3w_2)\\&+M_2(\alpha_1;w_1,w_2)\operatorname{sgn}(w_3)+M_2\left(\frac{\alpha_2}{\sqrt{1+\alpha_3^2}};w_1,\frac{w_2\alpha_3+w_3}{\sqrt{1+\alpha_3^2}}\right)\operatorname{sgn}(w_2-\alpha_3w_3)\\&+M_2\left(\kappa;\frac{w_2+\alpha_1w_1}{\sqrt{1+\alpha_1^2}},\frac{w_1(\alpha_2-\alpha_1\alpha_3)+w_2(\alpha_1^2\alpha_3-\alpha_1\alpha_2)+w_3(1+\alpha_1^2)}{\sqrt{(\alpha_2-\alpha_1\alpha_3)^2+(\alpha_1^2\alpha_3-\alpha_1\alpha_2)^2+(1+\alpha_1^2)^2}}\right)\\&\quad\times\operatorname{sgn}(w_1-\alpha_1w_2+(\alpha_1\alpha_3-\alpha_2)w_3)+M(w_1)\operatorname{sgn}(w_2)\operatorname{sgn}(w_3+\alpha_3w_2)\\&+M\left(\frac{w_2+\alpha_1w_1}{\sqrt{1+\alpha_1^2}}\right)\operatorname{sgn}(w_1-\alpha_1w_2)\operatorname{sgn}\left((w_1-\alpha_1w_2)(\alpha_2-\alpha_1\alpha_3)+\left(1+\alpha_1^2\right)w_3\right)\\&+M\left(\frac{w_3+\alpha_2w_1+\alpha_3w_2}{\sqrt{1+\alpha_2^2+\alpha_3^2}}\right)\operatorname{sgn}\left(w_1-\alpha_2\frac{w_3+\alpha_3w_2}{1+\alpha_3^2}\right)\\&\quad\times\operatorname{sgn}\left(w_1\left(\alpha_1-\alpha_2\alpha_3+\alpha_1\alpha_3^2\right)+w_2\left(1+\alpha_2^2-\alpha_1\alpha_2\alpha_3\right)-w_3\left(\alpha_1\alpha_2+\alpha_3\right)\right)\end{aligned}$$

with

$$\kappa:=\frac{(\alpha_1\alpha_2+\alpha_3)\sqrt{(\alpha_2-\alpha_1\alpha_3)^2+(\alpha_1^2\alpha_3-\alpha_1\alpha_2)^2+(1+\alpha_1^2)^2}}{\sqrt{1+\alpha_1^2}\left(1+\alpha_1^2+\alpha_2^2+\alpha_1^2\alpha_3^2-2\alpha_1\alpha_2\alpha_3\right)}.$$

One can verify the identity for E_3 using Proposition 3.11 of [N] since $E_3((\alpha_1, \alpha_2, \alpha_3), w)$ corresponds to Nazaroglus $E_3(\left(\begin{smallmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_3 \\ 0 & 0 & 1 \end{smallmatrix}\right); w)$.

Applying these identities for all functions in $\widehat{p}$, we obtain

$$\begin{aligned}
\widehat{p}(\ell) =& - E(\ell_1) - E\left(\sqrt{2}\ell_2\right) + E(\ell_1) E_2\left(\frac{1}{\sqrt{3}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4)\right) \\
&+ E_3\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4), \sqrt{\frac{2}{3}}(\ell_2 + \ell_3 + \ell_4)\right) \\
&+ \sum_{j \in \{3,4\}} \operatorname{sgn}(\ell_1 + \ell_2 + \ell_j) \\
&\quad \times \Big(E(\ell_1) E(\ell_j - \ell_2) - E_2(1; \ell_1, -\ell_1 - 2\ell_2) + E_2(-1; \ell_j - \ell_2, \ell_2 + \ell_j) + 1\Big) \\
=& - M(\ell_1) - \operatorname{sgn}(\ell_1) - M\left(\sqrt{2}\ell_2\right) - \operatorname{sgn}\left(\sqrt{2}\ell_2\right) + (M(\ell_1) + \operatorname{sgn}(\ell_1)) \\
&\quad \times \Big(M_2\left(\frac{1}{\sqrt{3}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4)\right) + \operatorname{sgn}(-\ell_2 - \ell_3 + 2\ell_4) M(\ell_3 - \ell_2) \\
&\qquad + \operatorname{sgn}(-\ell_2 + 2\ell_3 - \ell_4) M(\ell_4 - \ell_2) + \operatorname{sgn}(\ell_3 - \ell_2)\operatorname{sgn}(\ell_4 - \ell_2)\Big) \\
&+ M_3\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4), \sqrt{\frac{2}{3}}(\ell_2 + \ell_3 + \ell_4)\right) \\
&+ \operatorname{sgn}(\ell_2)\operatorname{sgn}(\ell_3 - \ell_2)\operatorname{sgn}(\ell_4 - \ell_2) \\
&+ M_2\left(\frac{1}{\sqrt{3}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4)\right)\operatorname{sgn}(\ell_2 + \ell_3 + \ell_4) \\
&+ M_2(-1; \ell_3 - \ell_2, \ell_2 + \ell_3)\operatorname{sgn}(\ell_4) + M_2(-1; \ell_4 - \ell_2, \ell_2 + \ell_4)\operatorname{sgn}(\ell_3) \\
&+ M(\ell_3 - \ell_2)\operatorname{sgn}(2\ell_4 - \ell_2 - \ell_3)\operatorname{sgn}(\ell_2 + \ell_3) \\
&+ M(\ell_4 - \ell_2)\operatorname{sgn}(2\ell_3 - \ell_2 - \ell_4)\operatorname{sgn}(\ell_2 + \ell_4) + M\left(\sqrt{2}\ell_2\right)\operatorname{sgn}(\ell_3)\operatorname{sgn}(\ell_4) \\
&+ \sum_{j \in \{3,4\}} \operatorname{sgn}(\ell_1 + \ell_2 + \ell_j)\Big(1 + (M(\ell_1) + \operatorname{sgn}(\ell_1))(M(\ell_j - \ell_2) + \operatorname{sgn}(\ell_j - \ell_2)) \\
&\qquad - M_2(1; \ell_1, -\ell_1 - 2\ell_2) - \operatorname{sgn}(-\ell_1 - 2\ell_2)M(\ell_1) - \operatorname{sgn}(\ell_1 + \ell_2) M\left(-\sqrt{2}\ell_2\right) \\
&\qquad - \operatorname{sgn}(\ell_1)\operatorname{sgn}(-\ell_2) + M_2(-1; \ell_j - \ell_2, \ell_2 + \ell_j) + \operatorname{sgn}(\ell_2 + \ell_j) M(\ell_j - \ell_2) \\
&\qquad + \operatorname{sgn}(\ell_j) M\left(\sqrt{2}\ell_2\right) + \operatorname{sgn}(\ell_j - \ell_2)\operatorname{sgn}(\ell_2)\Big).
\end{aligned}$$

Now we group these terms such that we obtain a sum of multiple convergent series. The terms consisting only of sgn-functions give the holomorphic part, which we show using the second part of Lemma II.3.2

$$\begin{aligned}
&-\operatorname{sgn}(\ell_1)-\operatorname{sgn}(\ell_2)+\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)\\
&+\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)\operatorname{sgn}(\ell_2)\\
&+\sum_{j\in\{3,4\}}\operatorname{sgn}(\ell_1+\ell_2+\ell_j)\left(1+\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_j-\ell_2)-\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_2)+\operatorname{sgn}(\ell_j-\ell_2)\operatorname{sgn}(\ell_2)\right)\\
=&-\operatorname{sgn}(\ell_1)-\operatorname{sgn}(\ell_2)+\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)\\
&+\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)\operatorname{sgn}(\ell_2)\\
&+\sum_{j\in\{3,4\}}\Big(\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_2)\operatorname{sgn}(\ell_j-\ell_2)+\operatorname{sgn}(\ell_1)+\operatorname{sgn}(\ell_2)+\operatorname{sgn}(\ell_j-\ell_2)\Big)=p(\ell).
\end{aligned}$$

The convergence of $\Theta_{0,\mathbb{Z}^4,A,p,0}(z;\tau)$ is shown in the proof of II.5.2 in Chapter II.

Next, the terms containing only $M(\ell_1)$ are (when none of the sgn-functions vanish)

$$\begin{aligned}
&M(\ell_1)\Big(-1+\operatorname{sgn}(\ell_3-\ell_2)\operatorname{sgn}(\ell_4-\ell_2)\\
&\qquad\qquad+\sum_{j\in\{3,4\}}\underbrace{\operatorname{sgn}(\ell_1+\ell_2+\ell_j)\left(\operatorname{sgn}(\ell_j-\ell_2)+\operatorname{sgn}(\ell_1+2\ell_2)\right)}_{=1+\operatorname{sgn}(\ell_j-\ell_2)\operatorname{sgn}(\ell_1+2\ell_2)}\Big)\\
&=M(\ell_1)\left(\operatorname{sgn}(\ell_1+2\ell_2)+\operatorname{sgn}(\ell_3-\ell_2)\right)\left(\operatorname{sgn}(\ell_1+2\ell_2)+\operatorname{sgn}(\ell_4-\ell_2)\right)\\
&=M(\ell_1)\times\begin{cases}(\pm 4) & \text{if } \operatorname{sgn}(\ell_1+2\ell_2)=\operatorname{sgn}(\ell_3-\ell_2)=\operatorname{sgn}(\ell_4-\ell_2)=\pm 1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

To see that this contributes a convergent series, note that

$$\begin{aligned}
\left|M\left(\sqrt{v}\left(n_1+\frac{y_1}{v}\right)\right)q^{Q(n)}\zeta^{An}\right| &\le 2e^{-\pi v\left(n_1+\frac{y_1}{v}\right)^2}e^{2\pi\frac{Q(y)}{v}}e^{-2\pi vQ\left(n+\frac{y}{v}\right)} \qquad \text{(III.1.2)}\\
&=2e^{2\pi\frac{Q(y)}{v}}e^{-2\pi vQ_1\left(n+\frac{y}{v}\right)}
\end{aligned}$$

with $Q_1(\ell):=\frac{1}{2}\ell^T\begin{pmatrix}2&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}\ell$ of signature $(2,2)$. Now

$$2Q_1(\ell)=(\ell_1+\ell_2)^2+(\ell_1+2\ell_2)^2+\ell_2^2+2\left(\ell_1+2\ell_2\right)\left(\ell_3-\ell_2+\ell_4-\ell_2\right)+2\left(\ell_3-\ell_2\right)\left(\ell_4-\ell_2\right)$$

is positive and growing when $\operatorname{sgn}(\ell_1+2\ell_2)=\operatorname{sgn}(\ell_3-\ell_2)=\operatorname{sgn}(\ell_4-\ell_2)=\pm 1$ and $\ell_3-\ell_2,\ell_4-\ell_2\neq 0$. Therefore these terms contribute a converging series. The terms with only the factors $M(\sqrt{2}\ell_2)$, $M(\ell_3-\ell_2)$ or $M(\ell_4-\ell_2)$ can be treated in the same way.

Next, we consider the terms containing $M_2(\frac{1}{\sqrt{3}};\ell_3-\ell_2,\frac{1}{\sqrt{3}}(-\ell_2-\ell_3+2\ell_4))$ as a factor, which are

$$M_2\left(\frac{1}{\sqrt{3}};\ell_3-\ell_2,\frac{1}{\sqrt{3}}\left(-\ell_2-\ell_3+2\ell_4\right)\right)\left(\operatorname{sgn}(\ell_1)+\operatorname{sgn}(\ell_2+\ell_3+\ell_4)\right).$$

Now we use $|M_2(\alpha;x)| \leq 2e^{-\pi x^T x}$ as a special case of [N, Proposition 3.8] and obtain

$$\begin{aligned}
&\left| M_2\left(\frac{1}{\sqrt{3}}; \sqrt{v}\left(n_3 - n_2 + \frac{y_3 - y_2}{v}\right), \frac{\sqrt{v}}{\sqrt{3}}\left(2n_4 - n_2 - n_3 + \frac{2y_4 - y_2 - y_3}{v}\right)\right) q^{Q(n)}\zeta^{An}\right| \\
&\leq 2e^{-\pi v\left(\left(n_3 - n_2 + \frac{y_3-y_2}{v}\right)^2 + \frac{1}{3}\left(2n_4 - n_2 - n_3 + \frac{2y_4 - y_2 - y_3}{v}\right)^2\right)} e^{2\pi\frac{Q(y)}{v}} e^{-2\pi v Q\left(n + \frac{y}{v}\right)} \\
&= 2e^{2\pi\frac{Q(y)}{v}} e^{-2\pi v Q_2\left(n + \frac{y}{v}\right)}
\end{aligned}$$

with $Q_2(\ell) := \frac{1}{6}\ell^T \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 4 & 1 & 1 \\ 3 & 1 & 4 & 1 \\ 3 & 1 & 1 & 4 \end{pmatrix} \ell$ of signature $(3,1)$, which is positive and growing when $(\mathrm{sgn}(\ell_1) + \mathrm{sgn}(\ell_2 + \ell_3 + \ell_4)) \neq 0$. The other M_2-terms and the product of two M-functions can be treated the same way.

Finally, using $|M_3(\alpha;x)| \leq 3!e^{-\pi x^T x}$ as a special case of [N, Proposition 3.8] we see that the single M_3 term is simply decaying fast enough such that

$$\begin{aligned}
&\left| M_3\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}; \ell_3 - \ell_2, \frac{1}{\sqrt{3}}(-\ell_2 - \ell_3 + 2\ell_4), \sqrt{\frac{2}{3}}(\ell_2 + \ell_3 + \ell_4)\right)_{\ell = n + \frac{y}{v}} q^{Q(n)}\zeta^{An}\right| \\
&\leq 6e^{-\pi v\left(n_3 - n_2 + \frac{y_3 - y_2}{v}\right)^2 + \frac{1}{3}\left(2n_4 - n_2 - n_3 + \frac{2y_4 - y_2 - y_3}{v}\right)^2 + \frac{1}{3}\left(n_2 + n_3 + n_4 + \frac{y_2 + y_3 + y_4}{v}\right)^2} \\
&\qquad\qquad \times e^{2\pi\frac{Q(y)}{v}} e^{-2\pi v Q\left(n + \frac{y}{v}\right)} \\
&= 6e^{2\pi\frac{Q(y)}{v}} e^{-2\pi v Q_3\left(n + \frac{y}{v}\right)}
\end{aligned}$$

with $Q_3(\ell) := \frac{1}{2}\ell^T \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & -1 \\ 1 & -1 & -1 & 4 \end{pmatrix} \ell$ of signature $(3,0)$, and requiring $3y_1 - y_2 - y_3 - y_4 \notin v\mathbb{Z}$ ensures that the series over these terms converges as well. The product of an M-function with an M_2-function can be handled the same way (this term gives a positive definite quadratic form using the same bound for M_3).

$\square$

Chapter IV

Higher depth quantum modular forms, multiple Eichler integrals and $\mathfrak{sl}_3$ false theta functions

This chapter is based on a manuscript published in *Research in the Mathematical Sciences* and is joint work with Prof. Dr. Kathrin Bringmann and Prof. Dr. Antun Milas [BKM1].

IV.1 Introduction and statement of results

In this paper, we study higher depth quantum modular forms which occur as rank two false theta functions coming from characters of the vertex algebra $W^0(p)_{A_2}$ for $p \geq 2$. Via asymptotic expansions we relate these to double Eichler integrals which may be viewed as purely non-holomorphic parts of indefinite theta functions.

Let us first recall the classical rank one case. Note that the derivative of a modular form is typically not a modular form (only a so-called quasi-modular form). However, thanks to Bol's identity, differentiating a weight $2-k \in -\mathbb{N}$ modular form $k-1$ times returns a modular form of weight k. Thus it is natural to consider holomorphic Eichler integrals. That is, if $f(\tau) = \sum_{m\geq 1} c_f(m)q^m$ ($q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$ throughout) is a cusp form of weight k, then set

$$\widetilde{f}(\tau) := \sum_{m\geq 1} \frac{c_f(m)}{m^{k-1}} q^m. \tag{IV.1.1}$$

It easily follows, by Bol's identity and the modularity of f, that the following function is annihilated by differentiating $k-1$ times

$$R_f(\tau) := \widetilde{f}(\tau) - \tau^{k-2}\widetilde{f}\left(-\frac{1}{\tau}\right). \tag{IV.1.2}$$

This yields that R_f is a polynomial of degree $k-2$ (R_f is the so called *period polynomial* of f). So in particular R_f is much simpler than the starting function $\widetilde{f}$. Note that $\widetilde{f}$ may also be written as an integral, namely, up to constants it equals

$$\int_\tau^{i\infty} f(w)(w-\tau)^{k-2}dw. \tag{IV.1.3}$$

Similarly R_f has an integral representation, namely up to constants it equals

$$\int_0^{i\infty} f(w)(w-\tau)^{k-2}dw.$$

A similar construction works for *weakly holomorphic modular forms*, i.e., those meromorphic modular forms that only have poles at the cusp $i\infty$ and not in $\mathbb{H}$. In this situation, (IV.1.3) needs to be regularized since the integral does not converge. Moreover, there is a "companion integral" (again regularized)

$$\int_{-\overline{\tau}}^{i\infty} g(w)(w+\tau)^{k-2}dw, \tag{IV.1.4}$$

where g is a certain weakly holomorphic modular form related to f in the sense that the corresponding period polynomial, defined analogously to (IV.1.2), basically agrees with R_f.

In contrast, for half-integral weight modular forms there is no half-derivative and thus Bol's identity does not apply. However, one can formally define the analogue of (IV.1.1) for theta functions. This was first investigated by Zagier [26, 27] in connection to Kontsevich's "strange" function

$$K(q) := \sum_{m\geq 0} (q;q)_m,$$

where for $m \in \mathbb{N}_0 \cup \{\infty\}$, $(a;q)_m := \prod_{j=0}^{m-1}(1-aq^j)$ denotes the usual *q-Pochhammer symbol*. The function $K(q)$ does not converge on any open subset of $\mathbb{C}$, but converges as a finite sum for q a root of unity. Zagier's study of K depends on the identity

$$\sum_{m\geq 0}\left(\eta(\tau) - q^{\frac{1}{24}}(q;q)_m\right) = \eta(\tau)D(\tau) + \frac{1}{2}\widetilde{\eta}(\tau), \tag{IV.1.5}$$

with $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty = \sum_{m\geq 1}(\frac{12}{m})q^{\frac{m^2}{24}}$, $D(\tau) := -\frac{1}{2} + \sum_{m\geq 1}\frac{q^m}{1-q^m}$ and $\widetilde{\eta}(\tau) := \sum_{m\geq 1}\left(\frac{12}{m}\right) mq^{\frac{m^2}{24}}$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the extended Jacobi symbol. The key observation of Zagier is that in (IV.1.5), the functions $\eta(\tau)$ and $\eta(\tau)D(\tau)$ vanish of infinite order as $\tau \to \frac{h}{k} \in \mathbb{Q}$. So at a root of unity ζ, $K(\zeta)$ is essentially the limiting value of the Eichler integral of η, which Zagier showed has quantum modular properties. Roughly speaking, Zagier defined "quantum modular forms" to be functions $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subseteq \mathbb{Q}$), such that the error of modularity ($M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \tag{IV.1.6}$$

is "nice". The definition is intentionally vague to include many examples; in this paper we require (V.1.2) to be real-analytic. For example, $\widetilde{f}$ (recall $k \in \mathbb{Z}$ in this case) is a quantum modular form, since R_f is a polynomial and thus real-analytic. Additional examples appear in the study of limits of quantum invariants of 3-manifolds and knots [27], Kashaev invariants of torus knots/links [14, 15], and partial theta functions [11].

Motivated in part by vertex operator algebra theory, further (but similar) examples of quantum modular forms were investigated in the setup of characters of vertex algebra modules in [4] and [9]. These examples are given by characters of $M_{r,s}$, the atypical irreducible modules of the $(1, p)$-singlet algebra for $p \geq 2$ [4, 7]. For $r = 1$ and $1 \leq s \leq p-1$, they take the particularly nice shape

$$\mathrm{ch}_{M_{1,s}}(\tau) = \frac{F_{p-s,p}(p\tau)}{\eta(\tau)},$$

where

$$F_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \mathrm{sgn}\left(m + \frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}$$

is a *false theta function*. The function $F_{j,p}$ is called "false theta" since getting rid of the sgn-factor yields the theta function $\sum_{m\in\mathbb{Z}} q^{(m+\frac{j}{2p})^2}$, which is a modular form of weight $\frac{1}{2}$. The quantum modularity of $F_{j,p}$ is now given by relating it to a non-holomorphic Eichler integral, as in (IV.1.4). To be more precise, set (correcting a typographical error in [4])

$$F^*_{j,p}(\tau) := -\sqrt{2}i \int_{-\overline{\tau}}^{i\infty} \frac{f_{j,p}(w)}{(-i(w+\tau))^{\frac{1}{2}}} dw,$$

where $f_{j,p}$ is the cuspidal theta function of weight $\frac{3}{2}$

$$f_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \left(m + \frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}.$$

One can show that $F_{j,p}(\tau)$ agrees for $\tau = \frac{h}{k}$ with $F^*_{j,p}(\tau)$ up to infinite order [4]. Quantum modularity then follows by the (mock) modular transformation of $F^*_{j,p}$ which we recall in Lemma IV.2.5 below. By "mock-modular", we mean that the occurence of the extra term $r_{f,\frac{d}{c}}$ in Lemma IV.2.5 prevents the function from being modular. However, there exists a "modular completion" in the sense that after multiplying it with a theta function, $F^*_{j,p}$ is the "purely non-holomorphic part" of a non-holomorphic theta function corresponding to an indefinite quadratic form (of signature $(1,1)$). Its modularity now can be proven by using results of Zwegers [28, Section 2.2]. The functions $\tau \mapsto F_{j,p}(p\tau)$, especially for $p = 2$, have appeared in several studies of vertex algebras from different standpoints [3, 7, 12, 13].

In this paper we investigate higher-dimensional analogues. For this we consider certain q-series appearing in representation theory of vertex algebras and W-algebras. They are sometimes called *higher rank false theta functions* and are thoroughly studied in [4, 8]. They appear from extracting the constant term of certain multivariable Jacobi forms [4]. The constant term can be interpreted as the character of the zero weight space of the corresponding Lie algebra representation. In the case of the simple Lie algebra $\mathfrak{sl}_3$, the false theta function takes the following shape ($p \in \mathbb{N}$, $p \geq 2$)

$$F(q) := \sum_{\substack{m_1, m_2 \geq 1 \\ m_1 \equiv m_2 \pmod 3}} \min(m_1, m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right). \tag{IV.1.7}$$

Below we decompose this function as $F(q) = \frac{2}{p}F_1(q^p) + 2F_2(q^p)$ with F_1 and F_2 defined in (IV.3.1) and (IV.3.2), respectively. The function F_1 and F_2 turn out to have generalized quantum modular properties. This connection goes via an analouge of (IV.1.1). For instance, we show that F_1 asymptotically agrees with an integral of the shape

$$\int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f(w_1, w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

where $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$ (χ_j are multipliers and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$). Modular properties follow from the modularity of f which in turn gives quantum modular properties of F_1. The idea is that here the error of modularity (V.1.2) is less complicated than the original function. We call the resulting functions higher depth quantum modular forms (see Definition V.2.2 for a precise definition). Roughly speaking (see Definition V.2.2 for a precise definition), depth two quantum modular forms of weight $k \in \frac{1}{2}\mathbb{Z}$ satisfy, in the simplest case, the modular transformation property ($M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R)$$

for some $\kappa \in \frac{1}{2}\mathbb{Z}$, where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real analytic functions on $R \subset \mathbb{R}$. Clearly, we can construct examples of depth two simply by multiplying two (depth one) quantum modular forms. Non-trivial examples arise from F (see Theorem IV.1.1 for precise statement).

Theorem IV.1.1. *For $p \geq 2$, the higher rank false theta function F can be written as the sum of two depth two quantum modular forms (with quantum set $\mathbb{Q}$) of weight one and two.*

It is worth noting that all of our examples of quantum modular forms, including those studied in [4], have $\mathbb{Q}$ as quantum set. Even though this feature is rare, a possible explanation is that vertex algebra characters are generally better behaved functions and are expected to combine into vector-valued families under the full modular group. Thus in our future work [4] we explore a vector-valued generalization of this theorem and its consequences to representation theory.

Zwegers [28] found an important connection between the error term of the Eichler integral (as in Lemma IV.2.5) and classical Mordell integrals. This result applied to the case of $F^*_{j,p}$ leads to an elegant expression for the error term as a Mordell integral

$$\int_{\mathbb{R}} \cot\left(\pi i w + \frac{\pi j}{2p}\right) e^{2\pi i p w^2 \tau} dw.$$

In this work we encounter error terms for iterated (double) Eichler integrals, so it is natural to attempt to extend Zwegers' result to two dimensions. In [4] we solve this problem in several special cases. In particular, we find that relevant integrals for the weight one component $\mathcal{E}_1$ (cf. Lemma IV.5.2) take the form

$$\int_{\mathbb{R}^2} \cot\left(\pi i w_1 + \pi \alpha_1\right) \cot\left(\pi i w_2 + \pi \alpha_2\right) e^{2\pi i \left(3w_1^2 + 3w_1 w_2 + w_2^2\right)\tau} dw_1 dw_2,$$

for some scalars α_1, α_2. This is what we call a *double Mordell* integral. We next turn to the modular completion of these Eichler integrals (see Propostiton IV.8.1 for a more precise version). For theta functions associated to indefinite quadratic forms, the reader is referred to [1, 17, 20, 23].

Theorem IV.1.2. *There exists an indefinite theta function, defined via* (IV.8.1)*, of signature* $(2, 2)$ *with "purely non-holomorphic" part* $\Theta(\tau)\mathcal{E}_1(\tau)$ *where* Θ *is a theta function of signature* $(2, 0)$ *and the Eichler integral* $\mathcal{E}_1$ *is defined in* (IV.5.5)*.*

The paper is organized as follows. In Section 2, we review basic results on special functions, non-holomorphic Eichler integrals, and "double error" functions. We also recall the notion of quantum modular forms and introduce higher depth quantum modular forms. In Section 3, the $\mathfrak{sl}_3$ higher rank false theta function $F(q) = \frac{2}{p}F_1(q^p) + 2F_2(q^p)$ is introduced. In Section 4, we determine the asymptotic behavior of F_1 and F_2 at roots of unity. In Section 5, we introduce multiple Eichler integrals and prove modular transformation formulas for the double Eichler integrals. We also study certain linear combinations of double Eichler integrals associated to F_j. In Section 6, we express special double Eichler integrals as pieces of indefinite theta series. Based on results in this section, in Section 7, we prove the main result, Theorem 1.1, on the quantum modularity of F. Section 8 deals with the completion of certain indefinite theta functions of signature $(2, 2)$ associated to the companions of F_j proving Theorem IV.1.2. We conclude in Section 9 with several questions.

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IV.2 Preliminaries

IV.2.1 Special functions

Define, for $u \in \mathbb{R}$,

$$E(u) := 2\int_0^u e^{-\pi w^2} dw.$$

This function is essentially the error function and its derivative is $E'(u) = 2e^{-\pi u^2}$. We have the representation

$$E(u) = \operatorname{sgn}(u)\left(1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right)\right), \tag{IV.2.1}$$

where $\Gamma(\alpha, u) := \int_u^\infty e^{-w} w^{\alpha-1} dw$ is the *incomplete gamma function* and where for $u \in \mathbb{R}$, we set

$$\operatorname{sgn}(u) := \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0. \end{cases}$$

We also require the functional equation of the incomplete Γ-function with $\alpha = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}, u\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}, u\right) + \frac{1}{\sqrt{u}} e^{-u}. \tag{IV.2.2}$$

Moreover, for $u \neq 0$, set

$$M(u) := \frac{i}{\pi} \int_{\mathbb{R}-iu} \frac{e^{-\pi w^2 - 2\pi i u w}}{w} dw.$$

We have

$$M(u) = E(u) - \operatorname{sgn}(u).$$

Thus, by (IV.2.1)

$$M(u) = -\frac{\operatorname{sgn}(u)}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \pi u^2\right). \tag{IV.2.3}$$

This implies that the following bound holds

$$|M(u)| \leq 2e^{-\pi u^2}.$$

We next turn to two-dimensional analogues, following [1] (using slightly different notation). Define $E_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by (throughout we use bold letters for vectors and denote their components using subscripts)

$$E_2(\kappa; \boldsymbol{u}) := \int_{\mathbb{R}^2} \operatorname{sgn}(w_1) \operatorname{sgn}(w_2 + \kappa w_1) e^{-\pi\left((w_1-u_1)^2 + (w_2-u_2)^2\right)} dw_1 dw_2.$$

Note that

$$E_2(\kappa; -\boldsymbol{u}) = E_2(\kappa; \boldsymbol{u}).$$

Moreover, also following [1], for $u_2, u_1 - \kappa u_2 \neq 0$ we set

$$M_2(\kappa; \boldsymbol{u}) := -\frac{1}{\pi^2} \int_{\mathbb{R}-iu_2} \int_{\mathbb{R}-iu_1} \frac{e^{-\pi w_1^2 - \pi w_2^2 - 2\pi i(u_1 w_1 + u_2 w_2)}}{w_2(w_1 - \kappa w_2)} dw_1 dw_2.$$

Then we have

$$\begin{aligned} M_2(\kappa;\boldsymbol{u}) &= E_2(\kappa;\boldsymbol{u}) - \operatorname{sgn}(u_2)\, M(u_1) \\ &\quad - \operatorname{sgn}(u_1-\kappa u_2)\, M\left(\frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}}\right) - \operatorname{sgn}(u_1)\operatorname{sgn}(u_2+\kappa u_1). \end{aligned} \tag{IV.2.4}$$

Note that (IV.2.4) extends the definition of M_2 to $u_2 = 0$ or $u_1 = \kappa u_2$. With $x_1 := u_1 - \kappa u_2$, $x_2 := u_2$, a direct calculation shows that

$$\begin{aligned} M_2(\kappa;\boldsymbol{u}) &= E_2(\kappa; x_1+\kappa x_2, x_2) + \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) \\ &\quad - \operatorname{sgn}(x_2)E(x_1+\kappa x_2) - \operatorname{sgn}(x_1)E\left(\frac{\kappa x_1}{\sqrt{1+\kappa^2}} + \sqrt{1+\kappa^2}x_2\right). \end{aligned}$$

We have the first partial derivatives

$$M_2^{(0,1)}(\kappa;\boldsymbol{u}) = \frac{2}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right), \tag{IV.2.5}$$

$$M_2^{(1,0)}(\kappa;\boldsymbol{u}) = 2e^{-\pi u_1^2}M(u_2) + \frac{2\kappa}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right), \tag{IV.2.6}$$

and the limiting behavior (cf. [1, Proposition 3.3, iii])

$$M_2(\kappa;\lambda\boldsymbol{u}) \sim -\frac{e^{-\pi\lambda^2\left(u_1^2+u_2^2\right)}}{\lambda^2\pi^2 u_2(u_1-\kappa u_2)} \qquad (\text{as} \quad \lambda\to\infty). \tag{IV.2.7}$$

Lemma IV.2.1. *For $u_3, u_4+\kappa u_3 \neq 0$, we have the following limits*

$$\lim_{\varepsilon\to 0^+} E_2\left(\varepsilon\kappa; u_1, \varepsilon u_2 + \varepsilon^{-1}u_3\right) = \operatorname{sgn}(u_3)E(u_1),$$
$$\lim_{\varepsilon\to 0^+} E_2\left(\kappa; \varepsilon u_1 + \varepsilon^{-1}u_3, \varepsilon u_2 + \varepsilon^{-1}u_4\right) = \operatorname{sgn}(u_3)\operatorname{sgn}(u_4+\kappa u_3).$$

Proof. We only prove the first statement, the second follows analogously. We may compute the limit inside the integral due to the convergence of the dominating integral $\int_{\mathbb{R}^2} e^{-\pi(w_1^2+w_2^2)}dw = 1$ to obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} E_2\left(\varepsilon\kappa; u_1, \varepsilon u_2 + \varepsilon^{-1}u_3\right) \\ &\quad = \int_{\mathbb{R}^2} e^{-\pi\left(w_1^2+w_2^2\right)} \operatorname{sgn}(w_1+u_1) \lim_{\varepsilon\to 0^+} \operatorname{sgn}\left(u_3 + \varepsilon\left(w_2+\varepsilon\kappa w_1+\varepsilon u_2+\varepsilon\kappa u_1\right)\right) dw_2 dw_1 \\ &\quad = \int_{\mathbb{R}} e^{-\pi w_1^2}\operatorname{sgn}(w_1+u_1)\int_{\mathbb{R}} e^{-\pi w_2^2}\operatorname{sgn}(u_3)\, dw_2 dw_1 = \operatorname{sgn}(u_3)E(u_1). \end{aligned}$$

□

IV.2.2 Euler-Maclaurin summation formula

We now state a special case of the Euler-Maclaurin summation formula. We only give it in the two-dimensional case; the one-dimensional case can be concluded by viewing the second variable as constant.

Let $B_m(x)$ be the m-th Bernoulli polynomial defined by $\frac{te^{xt}}{e^t-1} =: \sum_{m\geq 0} B_m(x)\frac{t^m}{m!}$. We also require

$$B_m(1-x) = (-1)^m B_m(x).$$

The Euler-Maclaurin summation formula implies that, for $\boldsymbol{\alpha} \in \mathbb{R}^2$, $F : \mathbb{R}^2 \to \mathbb{R}$ a C^∞-function which has rapid decay, we have (generalizing a result of [25] to include shifts by α)

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^2} F((\boldsymbol{n}+\boldsymbol{\alpha})t) \sim \frac{\mathcal{I}_F}{t^2} - \sum_{n_2\geq 0} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} \int_0^\infty F^{(0,n_2)}(x_1,0)dx_1 t^{n_2-1}$$
$$- \sum_{n_1\geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \int_0^\infty F^{(n_1,0)}(0,x_2)dx_2 t^{n_1-1} + \sum_{n_1,n_2\geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} F^{(n_1,n_2)}(0,0)t^{n_1+n_2}, \tag{IV.2.8}$$

where $\mathcal{I}_F := \int_0^\infty \int_0^\infty F(\boldsymbol{x})dx_1dx_2$. Here by $\sim$ we mean that the difference between the left- and the right-hand side is $O(t^N)$ for any $N \in \mathbb{N}$.

IV.2.3 Shimura's theta functions

We require transformation laws of certain theta functions studied, for example, by Shimura [11]. For $\nu \in \{0,1\}$, $h \in \mathbb{Z}$, $N, A \in \mathbb{N}$, with $A|N$, $N|hA$, define

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z} \\ m\equiv h \pmod{N}}} m^\nu q^{\frac{Am^2}{2N^2}}. \tag{IV.2.9}$$

Recall the following modular transformation

$$\Theta_\nu(A,h,N;M\tau) = e\left(\frac{abAh^2}{2N^2}\right)\left(\frac{2Ac}{d}\right)\varepsilon_d^{-1}(c\tau+d)^{\frac{1}{2}+\nu}\Theta_\nu(A,ah,N;\tau) \tag{IV.2.10}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2N)$ with $2|b$. Here $e(x) := e^{2\pi i x}$, for odd d, $\varepsilon_d = 1$ or i, depending on whether $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$. Also note that if $h_1 \equiv h_2 \pmod N$, then we have

$$\Theta_\nu(A,h_1,N;\tau) = \Theta_\nu(A,h_2,N;\tau), \qquad \Theta_\nu(A,-h,N;\tau) = (-1)^\nu\Theta_\nu(A,h,N;\tau).$$

IV.2.4 Indefinite theta functions

We begin by defining (possibly indefinite) theta functions.

Definition IV.2.2. Let $A \in M_m(\mathbb{Z})$ be a non-singular symmetric $m \times m$ matrix, $P : \mathbb{R}^m \to \mathbb{C}$ and $\boldsymbol{a} \in \mathbb{Q}^m$. We define the associated theta function by ($\tau = u + iv$)

$$\Theta_{A,P,\boldsymbol{a}}(\tau) := \sum_{\boldsymbol{n} \in \boldsymbol{a}+\mathbb{Z}^m} P\left(\sqrt{v}\boldsymbol{n}\right) q^{\frac{1}{2}\boldsymbol{n}^T A \boldsymbol{n}}.$$

The following theorem shows that under certain conditions $\Theta_{A,P,\boldsymbol{a}}$ is modular.

Theorem IV.2.3 (Vignéras, [22])**.** *Suppose that $A \in M_m(\mathbb{Z})$ is non-singular and that P satisfies the following conditions:*

1. *For any differential operator D of order two and any polynomial R of degree at most two, we have that $D(\boldsymbol{w})(P(\boldsymbol{w})e^{\pi Q(\boldsymbol{w})})$ and $R(\boldsymbol{w})P(\boldsymbol{w})e^{\pi Q(\boldsymbol{w})}$ belong to $L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.*

2. *For some $\lambda \in \mathbb{Z}$ the Vignéras differential equation holds:*

$$\left(\mathcal{D} - \frac{1}{4\pi}\Delta\right) P = \lambda P.$$

Here we define the Euler and Laplace operators ($\boldsymbol{w} := (w_1, \ldots, w_m)$, $\partial_{\boldsymbol{w}} := (\frac{\partial}{\partial w_1}, \ldots \frac{\partial}{\partial w_m})^T$)

$$\mathcal{D} := \boldsymbol{w}\partial_{\boldsymbol{w}} \qquad \text{and} \qquad \Delta = \Delta_{A^{-1}} := \partial_{\boldsymbol{w}}^T A^{-1} \partial_{\boldsymbol{w}}.$$

Then, assuming that $\Theta_{A,P,\boldsymbol{a}}$ is absolutely locally convergent, $\Theta_{A,P,\boldsymbol{a}}$ is modular of weight $\lambda + \frac{m}{2}$ for some subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

IV.2.5 Quantum modular forms

We already motivated quantum modular forms in the introduction. The formal definition is as follows [27].

Definition IV.2.4. A function $f : \mathcal{Q} \to \mathbb{C}$ (here $\mathcal{Q} \subseteq \mathbb{Q}$) is called a *quantum modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier χ for a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and quantum set $\mathcal{Q}$* if for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the function

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k} f(M\tau)$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there. We denote the vector space of such forms by $\mathcal{Q}_k(\Gamma, \chi)$.

Remark 6. Zagier also considered *strong quantum modular forms.* Here one is looking at asymptotic expansions instead of just values.

The introduction already gives examples of quantum modular forms. As mentioned there, the functions $F_{j,p}$ satisfy modular type transformations making them quantum modular forms. More generally, for $f \in S_k(\Gamma, \chi)$, the space of cusp forms of weight k transforming as

$$f(M\tau) = (c\tau + d)^k \chi(M) f(\tau)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and χ some multiplier, we set, for $\frac{d}{c} \in \mathbb{Q}$,

$$I_f(\tau) := \int_{-\overline{\tau}}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw, \qquad r_{f,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw. \quad \text{(IV.2.11)}$$

For weight $k = \frac{1}{2}$, we allow $f \in M_{\frac{1}{2}}(\Gamma, \chi)$, the space of holomorphic modular forms of weight $\frac{1}{2}$. To state the modularity properties of I_f, we let $\Gamma^* := P\Gamma P^{-1}$, where $P := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The proof of the following lemma follows along the same lines as the proof of Theorem IV.5.1 below.

Lemma IV.2.5. *We have the transformation, for $M \in \Gamma^*$,*

$$I_f(\tau) - \chi^{-1}(M^*)(c\tau + d)^{k-2} I_f(M\tau) = r_{f,\frac{d}{c}}(\tau).$$

The function I_f is defined on $\mathbb{H} \cup \mathbb{Q}$ whereas $r_{f,\frac{d}{c}}$ exists on all of $\mathbb{R} \setminus \{-\frac{d}{c}\}$ and is real-analytic there. If $f \in S_k(\Gamma, \chi)$, then $r_{f,\frac{d}{c}}$ exists on $\mathbb{R}$.

IV.2.6 Higher Depth Quantum modular forms

We next turn to generalizations of quantum modular forms.

Definition IV.2.6. A function $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subset \mathbb{Q}$) is called a *quantum modular form of depth $N \in \mathbb{N}$, weight $k \in \frac{1}{2}\mathbb{Z}$, multiplier χ, and quantum set $\mathcal{Q}$ for* Γ if for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k} f(M\tau) \in \bigoplus_j \mathcal{Q}_{\kappa_j}^{N_j}(\Gamma, \chi_j)\mathcal{O}(R),$$

where j runs through a finite set, $\kappa_j \in \frac{1}{2}\mathbb{Z}$, $N_j \in \mathbb{N}$ with $\max_j(N_j) = N - 1$, the χ_j are characters, $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$ which contains an open subset of $\mathbb{R}$, $\mathcal{Q}_k^1(\Gamma, \chi) := \mathcal{Q}_k(\Gamma, \chi)$, $\mathcal{Q}_k^0(\Gamma, \chi) := 1$, and $\mathcal{Q}_k^N(\Gamma, \chi)$ denotes the space of quantum modular forms of weight k, depth N, multiplier χ for Γ.

Remark 7. Again one can consider *higher depth strong quantum modular forms* by looking at asymptotic expansions instead of values. The examples of this paper satisfiy this stronger property.

Example IV.2.7. *For $f_1 \in \mathcal{Q}^1_{k_1}(\Gamma_1, \chi_1)$ and $f_2 \in \mathcal{Q}^1_{k_2}(\Gamma_2, \chi_2)$, we have that $f_1 f_2 \in Q^2_{k_1+k_2}(\Gamma_1 \cap \Gamma_2, \chi_1\chi_2)$.*

IV.3 A rank two false theta function

We briefly recall a construction from [5, 8, 10]. For $p \in \mathbb{N}_{\geq 2}$, there is a vertex operator algebra $W(p)_{A_2}$ associated to the simple Lie algebra $\mathfrak{sl}_3$ (more precisely, to its root lattice of type A_2). The character formula of $W(p)_Q$, where Q is any ADE root lattice, was proposed in [10] (note that some arguments in [10] are not completely rigorous) and further studied in [5, 8, 10]; see also [2]. Letting $\zeta_j := e^{2\pi i z_j}$, we have [5, 8]

$$\eta(\tau)^2 \mathrm{ch}[W(p)_{A_2}](\tau, \boldsymbol{z}) = \sum_{m_1, m_2 \in \mathbb{Z}} \frac{q^{p\left(\left(m_1 - \frac{1}{p}\right)^2 + \left(m_2 - \frac{1}{p}\right)^2 - \left(m_1 - \frac{1}{p}\right)\left(m_2 - \frac{1}{p}\right)\right)}}{\left(1 - \zeta_1^{-1}\right)\left(1 - \zeta_2^{-1}\right)\left(1 - \zeta_1^{-1}\zeta_2^{-1}\right)}$$
$$\times \left(\zeta_1^{m_1-1}\zeta_2^{m_2-1} - \zeta_1^{-m_1+m_2-1}\zeta_2^{m_2-1} - \zeta_1^{m_1-1}\zeta_2^{-m_2+m_1-1}\right.$$
$$\left. + \zeta_1^{-m_2-1}\zeta_2^{-m_2+m_1-1} + \zeta_1^{-m_1+m_2-1}\zeta_2^{-m_1-1} - \zeta_1^{-m_2-1}\zeta_2^{-m_1-1}\right).$$

The six term expression in the numerator comes from the summation over the Weyl group W of $\mathfrak{sl}_3$ which is isomorphic to S_3. Thanks to Weyl's character formula, the rational $\boldsymbol{z}$-part is in fact a Laurent polynomial. There are two important operations on this character:

(1) taking the limit $\boldsymbol{z} = (z_1, z_2) \to (0, 0)$, yielding a modular form [5];

(2) taking the constant term

$$\mathrm{ch}[W^0(p)_{A_2}](\tau) := \mathrm{CT}_{\zeta_1, \zeta_2} \mathrm{ch}[W(p)_{A_2}](\tau, \boldsymbol{z}),$$

which computes the character of another vertex algebra. It was shown in [5] that

$$\mathrm{ch}[W^0(p)_{A_2}](\tau) = \frac{F(q)}{\eta(\tau)^2}.$$

Note that formulas like $\eta(\tau)^{\mathrm{rank}(Q)} \mathrm{ch}[W^0(p)_Q](\tau)$, where Q is any root lattice, are of interest beyond vertex algebra theory [5, 8]. The coefficients appearing in the q-expansion are essentially dimensions of the zero weight spaces of finite-dimensional irreducible representations of simple Lie algebras (for the recent progress in understanding these numbers see [18]).

Remark 8. Modular-type properties of regularized (or Jacobi) characters, in particular $\mathrm{ch}[W^0(p)^{\varepsilon}_{A_2}](\tau)$, were investigated in [8] (see also [7]). There are two important differences between the current work and [8]. In this paper, the value of the Jacobi parameter ε is always zero whereas in [8] it is necessarily non-zero. Secondly, there seems to be no clear connection between transformation formulas appearing in [8] and mock modular forms. On the other hand, here we make this connection quite explicit by virtue of generalized Eichler integrals (see Section 5).

Let $n_1 = m_1 - m_2, n_2 = m_2$ in (IV.1.7) and then change $n_1 \mapsto 3n_1$. Then we have, with F given in (IV.1.7),

$$\frac{1}{2}F(q) = f_1(q) + f_2(q) + f_3(q),$$

where, with $Q(\boldsymbol{x}) := 3x_1^2 + 3x_1x_2 + x_2^2$, we define

$$\begin{aligned} f_1(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})} \left(q^{-3n_1-2n_2} - q^{3n_1+2n_2}\right), \\ f_2(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})} \left(q^{n_2} - q^{-n_2}\right), \\ f_3(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})} \left(q^{3n_1+n_2} - q^{-3n_1-n_2}\right). \end{aligned}$$

Here $\sum^*$ means that the $n_1 = 0$ term is weighted by $\frac{1}{2}$. We then rewrite

$$\begin{aligned} f_1(q) = &- \sum_{n_1,n_2\geq 0} \left(n_2 + \frac{1}{p}\right) q^{pQ\left(n_1+1,n_2+\frac{1}{p}\right)} + \sum_{n_1,n_2\geq 0} \left(n_2 + 1 - \frac{1}{p}\right) q^{pQ\left(n_1,n_2+1-\frac{1}{p}\right)} \\ &+ \frac{1}{p} \sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1,n_2+\frac{1}{p}\right)} + \frac{1}{p} \sum_{n_1,n_2\geq 0} q^{pQ\left(n_1,n_2+1-\frac{1}{p}\right)} - \frac{1}{2}\sum_{m\geq 0} \left(m + \frac{1}{p}\right) q^{p\left(m+\frac{1}{p}\right)^2} \\ &- \frac{1}{2}\sum_{m\geq 0} \left(m + 1 - \frac{1}{p}\right) q^{p\left(m+1-\frac{1}{p}\right)^2} + \frac{1}{2p}\sum_{m\geq 0} q^{p\left(m+\frac{1}{p}\right)^2} - \frac{1}{2p}\sum_{m\geq 0} q^{p\left(m+1-\frac{1}{p}\right)^2}, \\ f_2(q) = &\sum_{n_1,n_2\geq 0} \left(n_2 + \frac{2}{p}\right) q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{2}{p}\right)} - \sum_{n_1,n_2\geq 0} \left(n_2 + 1 - \frac{2}{p}\right) q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{2}{p}\right)} \\ &- \frac{2}{p} \sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{2}{p}\right)} - \frac{2}{p} \sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{2}{p}\right)} \\ &+ \frac{q^{\frac{3}{4p}}}{2} \sum_{m\geq 1} m q^{p\left(m-\frac{1}{2p}\right)^2} + \frac{q^{\frac{3}{4p}}}{2} \sum_{m\geq 1} m q^{p\left(m+\frac{1}{2p}\right)^2}, \end{aligned}$$

$$
\begin{aligned}
f_3(q) = &\sum_{n_1,n_2\geq 0}\left(n_2+1-\frac{1}{p}\right) q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{1}{p}\right)} - \sum_{n_1,n_2\geq 0}\left(n_2+\frac{1}{p}\right) q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{1}{p}\right)}\\
&+\frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{1}{p}\right)} + \frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{1}{p}\right)}\\
&-\frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} mq^{p\left(m+\frac{1}{2p}\right)^2} - \frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} mq^{p\left(m-\frac{1}{2p}\right)^2}.
\end{aligned}
$$

We thus obtain

$$F(q) = \frac{2}{p}F_1\left(q^p\right) + 2F_2\left(q^p\right)$$

with

$$F_1(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})} + \frac{1}{2}\sum_{m\in\mathbb{Z}} \operatorname{sgn}\left(m+\frac{1}{p}\right) q^{\left(m+\frac{1}{p}\right)^2}, \tag{IV.3.1}$$

where

$$\mathscr{S} := \left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(\frac{1}{p},1-\frac{2}{p}\right),\left(1,\frac{1}{p}\right)\left(0,1-\frac{1}{p}\right),\left(\frac{1}{p},1-\frac{1}{p}\right),\left(1-\frac{1}{p},\frac{1}{p}\right)\right\},$$

and for $\boldsymbol{\alpha}\pmod{\mathbb{Z}^2}$, we set

$$\varepsilon(\boldsymbol{\alpha}) := \begin{cases} -2 & \text{if } \boldsymbol{\alpha}\in\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(\frac{1}{p},1-\frac{2}{p}\right)\right\},\\ 1 & \text{otherwise.}\end{cases}$$

Moreover

$$F_2(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \eta(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2 q^{Q(\boldsymbol{n})} - \frac{1}{2}\sum_{m\in\mathbb{Z}} \left|m+\frac{1}{p}\right| q^{\left(m+\frac{1}{p}\right)^2}, \tag{IV.3.2}$$

where for $\boldsymbol{\alpha}\pmod{\mathbb{Z}^2}$, we let

$$\eta(\boldsymbol{\alpha}) := \begin{cases} 1 & \text{if } \boldsymbol{\alpha}\in\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(0,1-\frac{1}{p}\right),\left(\frac{1}{p},1-\frac{1}{p}\right)\right\},\\ -1 & \text{otherwise.}\end{cases}$$

IV.4 Asymptotic behavior of F_1 and F_2

In this section we determine the asymptotic behavior of $F(e^{2\pi i\frac{h}{k}-t})$ ($h,k\in\mathbb{Z}$ with $k>0$ and $\gcd(h,k)=1$) as $t\to 0^+$ and in particular show that the limit exists.

IV.4.1 The function F_1

We decompose

$$F_1(q) = F_{1,1}(q) + F_{1,2}(q),$$

where

$$F_{1,1}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})}, \qquad F_{1,2}(q) := \frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{Z}} \operatorname{sgn}(m) q^{m^2}.$$

We first study the asymptotic behavior of $F_{1,1}$, rewriting it in a shape in which we can apply the Euler-Maclaurin formula (IV.2.8). For this, let $\boldsymbol{n} \mapsto \boldsymbol{\ell} + \boldsymbol{n}\frac{kp}{\delta}$ with $\boldsymbol{n} \in \mathbb{N}_0^2$, $0 \le \boldsymbol{\ell} \le \frac{kp}{\delta} - 1$, where $\delta := \gcd(h,p)$. Here by the inequality we mean that it should hold componentwise. It is not hard to see that, with $\mathcal{F}_1(\boldsymbol{x}) := e^{-Q(\boldsymbol{x})}$,

$$F_{1,1}\left(e^{2\pi i\frac{h}{k}-t}\right) = \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2} \mathcal{F}_1\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right).$$

The main term in (IV.2.8) is then

$$\frac{\delta^2}{k^2p^2t}\mathcal{I}_{\mathcal{F}_1} \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}. \tag{IV.4.1}$$

It is not hard to see that one may let $\boldsymbol{\ell}$ run modulo $\frac{kp}{\delta}$ (again meant componentwise). We write $\boldsymbol{\ell} = \boldsymbol{N} + k\boldsymbol{\nu}$ with $\boldsymbol{N}$ running modulo k, $\boldsymbol{\nu}$ modulo $\frac{p}{\delta}$, and $\boldsymbol{a} \in \{(-1,2),(1,-2),(0,1),(0,-1),(1,-1),(-1,1)\}$ such that $\boldsymbol{\alpha} - \frac{\boldsymbol{a}}{p} \in \mathbb{Z}^2$. We then compute that the sum over $\boldsymbol{\ell}$ in (IV.4.1) equals (since $Q(\boldsymbol{a}) = 1$)

$$e^{2\pi i\frac{h}{p^2k}} \sum_{\boldsymbol{N}\pmod{k}} e^{\frac{2\pi ih}{pk}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)}$$
$$\times \sum_{\boldsymbol{\nu}\ \left(\operatorname{mod}\frac{p}{\delta}\right)} e^{\frac{2\pi ih/\delta}{p/\delta}((6a_1+3a_2)\nu_1+(2a_2+3a_1)\nu_2)}.$$

Since $\gcd(\frac{h}{\delta},\frac{p}{\delta}) = 1$, the inner sum vanishes unless $\frac{p}{\delta}|\, 3(2a_1+a_2)$ and $\frac{p}{\delta}|\, (2a_2+3a_1)$. If $3|\frac{p}{\delta}$, then in particular $3 \mid a_2$. This is however not satisfied for elements in $\mathscr{S}$. If $3 \nmid \frac{p}{\delta}$, then we easily obtain that $a_1 \equiv a_2 \equiv 0 \pmod{\frac{p}{\delta}}$, implying that $\frac{p}{\delta} = 1$. We are thus left to show that ($\frac{p}{\delta} = 1$)

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{N}\pmod{k}} e^{\frac{2\pi ih/\delta}{k}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)} = 0. \tag{IV.4.2}$$

Changing $\boldsymbol{N} \mapsto \boldsymbol{N} - \boldsymbol{a}\overline{p}$, with $\overline{p}$ the inverse of p modulo k (note that $\frac{p}{\delta} = 1$ implies that $\gcd(p,k) = 1$), the sum on $\boldsymbol{N}$ equals

$$e^{-\frac{2\pi i \overline{p} h/\delta}{k}} \sum_{\boldsymbol{N} \pmod k} e^{\frac{2\pi i h}{k} Q(\boldsymbol{N})},$$

which is independent of $\boldsymbol{a}$. Thus (IV.4.2) holds.

The second term in (IV.2.8) is

$$-\sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n_2\ge0} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!} \int_0^\infty \mathcal{F}_1^{(0,n_2)}(x_1,0)dx_1 \left(\frac{kp\sqrt{t}}{\delta}\right)^{n_2-1}. \tag{IV.4.3}$$

We claim that the contribution from those n_2 which are even vanishes. This follows, once we show that, for $\boldsymbol{\alpha} \in \mathscr{S}$,

$$\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} \left(e^{2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} B_{2n_2+1}\left(\frac{\delta\left(\ell_2+\alpha_2\right)}{kp}\right) + e^{2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{1}-\boldsymbol{\alpha})} B_{2n_2+1}\left(\frac{\delta\left(\ell_2+1-\alpha_2\right)}{kp}\right)\right) = 0.$$

This is seen to be true by the change of variables $\boldsymbol{\ell} \mapsto -\boldsymbol{\ell} + (-1 + \frac{kp}{\delta})\boldsymbol{1}$ for the second term.

Arguing in the same way for the contribution from n_2 odd, we obtain that (IV.4.3) equals

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n_2\ge0} \frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+2)!} \int_0^\infty \mathcal{F}_1^{(0,2n_2+1)}(x_1,0)dx_1 \left(\frac{k^2p^2}{\delta^2}t\right)^{n_2},$$

where

$$\mathscr{S}^* := \left\{ \left(1-\frac{1}{p}, \frac{2}{p}\right), \left(0, 1-\frac{1}{p}\right), \left(\frac{1}{p}, 1-\frac{1}{p}\right) \right\}.$$

The third term in (IV.2.8) is treated in the same way, yielding the contribution

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n_1\ge0} \frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+2)!} \int_0^\infty \mathcal{F}_1^{(2n_1+1,0)}(0,x_2)dx_2 \left(\frac{k^2p^2}{\delta^2}t\right)^{n_1}.$$

The final term in (IV.2.8) equals

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}$$
$$\times \sum_{n_1,n_2\ge0} \frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!} \mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

Arguing in the same way as before this equals

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{0\le\ell\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}$$
$$\times \sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2 \pmod 2}} \frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!} \mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

The function $F_{1,2}$ is treated similarly, yielding, with $\mathcal{F}_2(x) := e^{-x^2}$,

$$-\sum_{0\le r\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2} \sum_{m\ge0} \frac{B_{2m+1}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+1)!} \mathcal{F}_2^{(2m)}(0)\left(\frac{k^2p^2}{\delta^2}t\right)^m.$$

IV.4.2 The function F_2

Since the calculations are similar to those for F_1, we skip some of the details. Decompose

$$F_2(q) = F_{2,1}(q) + F_{2,2}(q),$$

with

$$F_{2,1}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \eta(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2 q^{Q(\boldsymbol{n})}, \qquad F_{2,2}(q) := -\frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{Z}} |m| q^{m^2}.$$

We first study the asymptotic behavior of $F_{2,1}$. Arguing as for $F_{1,1}$, we have

$$F_2\left(e^{2\pi i\frac{h}{k}-t}\right) = \frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha}) \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2} \mathcal{G}_1\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),$$

with $\mathcal{G}_1(\boldsymbol{x}) := x_2\mathcal{F}_1(\boldsymbol{x})$. The Euler-Maclaurin main term is

$$\frac{1}{t^{\frac{3}{2}}}\left(\frac{\delta}{kp}\right)^2 \mathcal{I}_{\mathcal{G}_1} \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \eta(\boldsymbol{\alpha}) \sum_{\boldsymbol{\ell} \pmod{\frac{kp}{\delta}}} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}.$$

As in Subsection IV.4.1, one can show that this vanishes.

The second term in the Euler-Maclaurin summation formula is

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge1}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!} \times\int_0^\infty \mathcal{G}_1^{(0,2n_2)}(x_1,0)dx_1\left(\frac{kp}{\delta}\right)^{2n_2-1} t^{n_2-1}, \tag{IV.4.4}$$

again pairing $\boldsymbol{\alpha}$ and $\mathbf{1}-\boldsymbol{\alpha}$ and using that $\mathcal{G}_1(x_1,0)=0$.

In the same way we obtain that the third term in the Euler-Maclaurin summation formula is

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!} \times\int_0^\infty \mathcal{G}_1^{(2n_1,0)}(0,x_2)dx_2\left(\frac{kp}{\delta}\right)^{2n_1-1} t^{n_1-1}. \tag{IV.4.5}$$

The final term in Euler-Maclaurin evaluates as

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \times\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}\mathcal{G}_1^{(n_1,n_2)}(0,0)\left(\frac{kp}{\delta}\right)^{n_1+n_2} t^{\frac{n_1+n_2-1}{2}},$$

again pairing $\boldsymbol{\alpha}$ with $\mathbf{1}-\boldsymbol{\alpha}$.

We next determine those terms of $F_{2,1}$ that grow as $t\to0^+$. Inspecting the terms above we see that this comes from the $n_1=0$ term of (IV.4.5) and is given by

$$-\frac{2\delta}{kpt}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\int_0^\infty \mathcal{G}_1(0,x_2)dx_2. \tag{IV.4.6}$$

Using that $\mathcal{G}_1(0,x_2)=x_2e^{-x_2^2}=:\mathcal{G}_2(x_2)$, we obtain that (IV.4.6) equals

$$-\frac{2\delta}{kpt}\mathcal{I}_{\mathcal{G}_2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}.$$

Turning to $F_{2,2}$, its Euler-Maclaurin main term is

$$-\frac{\delta}{kpt}\mathcal{I}_{\mathcal{G}_2}\sum_{r\pmod{\frac{kp}{\delta}}} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}. \tag{IV.4.7}$$

Arguing as before, the second term in the Euler-Maclaurin summation formula equals

$$\sum_{0\le r\le\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge0}\frac{B_{2m+2}\left(\frac{\delta(\ell+\frac{1}{p})}{kp}\right)}{(2m+2)!}\mathcal{G}_2^{(2m+1)}(0)\left(\frac{kp}{\delta}\right)^{2m+1}t^m.$$

To see that all terms that grow as $t\to0^+$ cancel, we need to prove that

$$\sum_{r\pmod{\frac{kp}{\delta}}} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2} = -2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}. \tag{IV.4.8}$$

To show (IV.4.8), we first assume that $\frac{p}{\delta}\notin\{1,2\}$. Writing $\boldsymbol{\ell}=\boldsymbol{N}+k\boldsymbol{\nu}$, $0\le\boldsymbol{N}<k$, $0\le\boldsymbol{\nu}<\frac{p}{\delta}$ and $\boldsymbol{a}=p\boldsymbol{\alpha}$, we obtain that the sum on $\boldsymbol{\ell}$ equals

$$e^{2\pi i\frac{h}{p^2k}Q(\boldsymbol{a})}\sum_{0\le\boldsymbol{N}<k} e^{\frac{2\pi ih}{pk}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)} \tag{IV.4.9}$$
$$\times\sum_{0\le\boldsymbol{\nu}<\frac{p}{\delta}} B_1\left(\frac{\delta\left(N_1+k\nu_1+\frac{a_1}{p}\right)}{kp}\right)e^{2\pi i\frac{h/\delta}{p/\delta}((6a_1+3a_2)\nu_1+(2a_2+3a_1)\nu_2)}.$$

The sum on ν_2 vanishes unless $\frac{p}{\delta}|(2a_2+3a_1)$. It is not hard to see that (under the assumption that $\frac{p}{\delta}\notin\{1,2\}$) this is not satisfied for elements in $p\mathscr{S}^*$.

We next assume that $\frac{p}{\delta}=1$. It is not hard to see that

$$e^{2\pi i\frac{h}{k}Q\left(k-\ell_1-1+1-\frac{1}{p},\ell_2+3\ell_1+1+\frac{2}{p}\right)} = e^{2\pi i\frac{h}{k}Q\left(\ell_1+\frac{1}{p},\ell_2+1-\frac{1}{p}\right)}. \tag{IV.4.10}$$

This then implies that the contribution of the first and third element in $\mathscr{S}^*$ cancel due to a negative sign from the Bernoulli polynomial and we can shift the sum in ℓ_2 by integers. Thus the right-hand side of (IV.4.8) becomes

$$-2\sum_{0\le\boldsymbol{\ell}<k} B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}. \tag{IV.4.11}$$

Now one can show that

$$e^{2\pi i\frac{h}{k}Q\left(k-\ell_1,\ell_2+3\ell_1+1-\frac{1}{p}\right)}=e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}. \tag{IV.4.12}$$

To finish the claim (IV.4.8), we assume, without loss of generality, that k is odd. We split the sum in (IV.4.11), substitute $(\ell_1,\ell_2)\mapsto(k-\ell_1,\ell_2+3\ell_1)$ in the second part and use (IV.4.12) to obtain

$$\begin{aligned}
&-2\sum_{0\le\boldsymbol{\ell}<k}B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}\\
&=-2\left(\sum_{\substack{0\le\ell_1\le\frac{1}{2}(k-1)\\ \ell_2\pmod{k}}}+\sum_{\substack{\frac{1}{2}(k+1)\le\ell_1<k\\ \ell_2\pmod{k}}}\right)B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}\\
&=-2\sum_{\substack{0\le\ell_1\le\frac{1}{2}(k-1)\\ \ell_2\pmod{k}}}B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}-2\sum_{\substack{0<\ell_1\le\frac{1}{2}(k-1)\\ \ell_2\pmod{k}}}B_1\left(1-\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}\\
&=-2B_1(0)\sum_{\ell_2\pmod{k}}e^{2\pi i\frac{h}{k}Q\left(0,\ell_2+1-\frac{1}{p}\right)}=\sum_{\ell_2\pmod{k}}e^{2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2}.
\end{aligned}$$

The case $\frac{p}{\delta}=2$ is done similarly.

IV.5 Companions in the lower half plane

In this section we investigate multivariable Eichler integrals.

IV.5.1 Multiple Eichler integrals

Let $f_j\in S_{k_j}(\Gamma,\chi_j)$; if $k_j=\frac{1}{2}$ we also allow $f_j\in M_{\frac{1}{2}}(\Gamma,\chi_j)$. Define the *double Eichler integral*

$$I_{f_1,f_2}(\tau):=\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}}dw_2dw_1,$$

and the *multiple error of modularity*

$$r_{f_1,f_2,\frac{d}{c}}(\tau):=\int_{\frac{d}{c}}^{i\infty}\int_{w_1}^{\frac{d}{c}}\frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}}dw_2dw_1.$$

Theorem IV.5.1. *We have, for* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^*$,

$$I_{f_1,f_2}(\tau) - \chi_1^{-1}(M^*)\,\chi_2^{-1}(M^*)\,(c\tau+d)^{k_1+k_2-4} I_{f_1,f_2}(M\tau) = r_{f_1,f_2,\frac{d}{c}}(\tau) + I_{f_1}(\tau) r_{f_2,\frac{d}{c}}(\tau). \tag{IV.5.1}$$

Moreover $r_{f_1,f_2,\frac{d}{c}} \in \mathcal{O}(\mathbb{R}\setminus\{-\frac{d}{c}\})$. *If* $f_j \in S_{k_j}(\Gamma,\chi_j)$ *(for* $j=1,2$*), then* $r_{f_1,f_2,\frac{d}{c}} \in \mathcal{O}(\mathbb{R})$.

Proof of Theorem IV.5.1. For simplicity, we assume that $\frac{1}{2} \leq k_j \leq 2$ and that f_1, f_2 are cuspidal. The proof in the case that f_1 or f_2 are not cuspidal and of weight $\frac{1}{2}$ is basically the same; we then require the bound

$$f_j\left(iw_j + \frac{d}{c}\right) \ll 1 + w_j^{-\frac{1}{2}}.$$

A direct calculation gives that, for $M \in \Gamma^*$,

$$I_{f_1,f_2}(M\tau) = \chi_1(M^*)\,\chi_2(M^*)\,(c\tau+d)^{4-k_1-k_2} \times \int_{-\overline{\tau}}^{\frac{d}{c}} \int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}} dw_2 dw_1.$$

The transformation (IV.5.1) now follows by splitting

$$\int_{-\overline{\tau}}^{\frac{d}{c}} \int_{w_1}^{\frac{d}{c}} = \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} + \int_{\frac{d}{c}}^{i\infty} \int_{\frac{d}{c}}^{w_1} - \int_{-\overline{\tau}}^{i\infty} \int_{\frac{d}{c}}^{i\infty}.$$

Using Lemma IV.2.5, we are left to show that $r_{f_1,f_2,\frac{d}{c}}$ is real-analytic on $\mathbb{R}$ which follows once we prove that the following function is real-analytic

$$\int_0^\infty \int_0^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right) f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1 - i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1} \left(w_2 - i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1. \tag{IV.5.2}$$

We use that for $w_j \geq 1$

$$f_j\left(iw_j + \frac{d}{c}\right) \ll e^{-a_j w_j} \qquad a_j \in \mathbb{R}^+, \tag{IV.5.3}$$

and for $0 < w_j \leq 1$ (the implied constant and b_j may depend on c)

$$f_j\left(iw_j + \frac{d}{c}\right) \ll w_j^{-k_j} e^{-\frac{b_j}{w_j}} \qquad b_j \in \mathbb{R}^+. \tag{IV.5.4}$$

To show real-analycity of (IV.5.2) on $\mathbb{R}$, we split it into 3 pieces. Firstly, set

$$I_1 := \int_1^\infty \int_1^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right) f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1.$$

Using (IV.5.3) and that $w_1 \geq 1$ easily gives the locally uniform bound

$$I_1 \ll \int_1^\infty \frac{e^{-a_1 w_1}}{w_1^{2-k_1}} dw_1 \int_1^\infty \frac{e^{-a_2 w_2}}{w_2^{2-k_2}} dw_2 \ll 1.$$

Next consider

$$I_2 := \int_0^1 \int_0^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right) f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1.$$

Using (IV.5.4) gives that

$$I_2 \ll \int_0^1 \frac{e^{-\frac{b_1}{w_1}}}{w_1^2} dw_1 \int_0^1 \frac{e^{-\frac{b_2}{w_2}}}{w_2^2} dw_2 \ll 1.$$

Finally, we set

$$I_3 := \int_1^\infty \frac{f_1\left(iw_1+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}} dw_1 \int_0^1 \frac{f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}} dw_2.$$

Combining the above bounds gives again $I_3 \ll 1$.

$\square$

IV.5.2 Special multiple Eichler integrals of weight one

Define for $\boldsymbol{\alpha} \in \mathscr{S}^*$

$$\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) := -\frac{\sqrt{3}}{4} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

with

$$\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (2n_1+n_2)n_2 e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}},$$

$$\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (3n_1+2n_2)n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.$$

Moreover set

$$\mathcal{E}_1(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha})\mathcal{E}_{1,\boldsymbol{\alpha}}(p\tau), \tag{IV.5.5}$$

$$\Gamma_p := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(12p) : b \equiv 0 \pmod{4p}, d \equiv \pm 1 \pmod{2p}\right\}.$$

Remark 9. Note that $\Gamma_p^* = \Gamma_p$.

Remark 10. One can show that

$$\mathcal{E}_1(\tau) = -\frac{\sqrt{3}}{4p} \sum_{\delta\in\{0,1\}} I_{\Theta_1(2p,1+p\delta,2p;\,\cdot\,),\Theta_1(6p,3+3p\delta,6p;\,\cdot\,)}(\tau).$$

However, as this representation is not required for the remainder of the paper, we do not provide a proof of this identity.

Proposition IV.5.2. *We have, for* $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_p$,

$$\mathcal{E}_1(\tau) - \left(\frac{-3}{d}\right)(c\tau+d)^{-1}\mathcal{E}_1(M\tau) = \sum_{j=1}^{12}\left(r_{f_j,g_j,\frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j,\frac{d}{c}}(\tau)\right),$$

where f_j, g_j *are cusp forms of weight* $\frac{3}{2}$ *(with some multiplier).*

Proof. To use Theorem IV.5.1, we write θ_j in terms of Shimura's theta functions (IV.2.9). For θ_1, we set $\nu_1 := 2n_1 + n_2$, $\nu_2 := n_2$. Then $\nu_1 \in 2\alpha_1 + \alpha_2 + \mathbb{Z}$, $\nu_2 \in \alpha_2 + \mathbb{Z}$, and $\nu_1 - \nu_2 \in 2\alpha_1 + 2\mathbb{Z}$ and we obtain

$$\begin{aligned}\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) &= \sum_{\substack{\boldsymbol{\nu}\in(2\alpha_1+\alpha_2,\alpha_2)+\mathbb{Z}^2\\ \nu_1-\nu_2\in 2\alpha_1+2\mathbb{Z}}} \nu_1\nu_2 e^{\frac{3\pi i\nu_1^2 w_1}{2}+\frac{\pi i \nu_2^2 w_2}{2}} \\ &= \sum_{\varrho\in\{0,1\}} \sum_{\nu_1\in 2\alpha_1+\alpha_2+\varrho+2\mathbb{Z}} \nu_1 e^{\frac{3\pi i\nu_1^2 w_1}{2}} \sum_{\nu_2\in\alpha_2+\varrho+2\mathbb{Z}} \nu_2 e^{\frac{\pi i\nu_2^2 w_2}{2}}.\end{aligned}$$

Summing then easily gives

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha})\,\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) &= \frac{1}{p^2}\sum_{\boldsymbol{A}\in\mathcal{A}} \varepsilon_1(\boldsymbol{A}) \sum_{\nu_1\equiv A_1 \pmod{2p}} \nu_1 e^{\frac{3\pi i\nu_1^2 w_1}{2p^2}} \sum_{\nu_2\equiv A_2 \pmod{2p}} \nu_2 e^{\frac{\pi i\nu_2^2 w_2}{2p^2}} \\ &= \frac{1}{p^2}\sum_{\boldsymbol{A}\in\mathcal{A}} \varepsilon_1(\boldsymbol{A})\,\Theta_1\left(2p, A_1, 2p; \frac{3w_1}{p}\right)\Theta_1\left(2p, A_2, 2p; \frac{w_2}{p}\right)\end{aligned}$$

with

$$\begin{aligned}
\mathcal{A} :=& \{(0,2), (p,p+2), (p-1,p-1), (-1,-1), (p+1,p-1), (1,-1)\},\\
\varepsilon_1(\boldsymbol{A}) :=& \varepsilon\left(\frac{A_1-A_2}{2p}, \frac{A_2}{p}\right).
\end{aligned}$$

For θ_2, we proceed similarly. Set $\nu_1 = 3n_1 + 2n_2$, $\nu_2 = n_1$. Then $\nu_1 \in 3\alpha_1 + 2\alpha_2 + \mathbb{Z}$, $\nu_2 \in \alpha_1 + \mathbb{Z}$, and $\nu_1 - 3\nu_2 \in 2\alpha_2 + 2\mathbb{Z}$ and we obtain

$$\begin{aligned}
\theta_2(\boldsymbol{\alpha}; \boldsymbol{w}) &= \sum_{\substack{\boldsymbol{\nu}\in(3\alpha_1+2\alpha_2,\alpha_1)+\mathbb{Z}^2\\ \nu_1-3\nu_2\in 2\alpha_2+2\mathbb{Z}}} \nu_1\nu_2 e^{\frac{\pi i\nu_1^2 w_1}{2}+\frac{3\pi i\nu_2^2 w_2}{2}}\\
&= \sum_{\varrho\in\{0,1\}} \sum_{\nu_1\in 3\alpha_1+2\alpha_2+\varrho+2\mathbb{Z}} \nu_1 e^{\frac{\pi i\nu_1^2 w_1}{2}} \sum_{\nu_2\in\alpha_1+\varrho+2\mathbb{Z}} \nu_2 e^{\frac{3\pi i\nu_2^2 w_2}{2}}.
\end{aligned}$$

Summing gives

$$\begin{aligned}
\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha})\,\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) &= \frac{1}{p^2}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B}) \sum_{\nu_1\equiv B_1 \pmod{2p}} \nu_1 e^{\frac{\pi i\nu_1^2 w_1}{2p^2}} \sum_{\nu_2\equiv B_2 \pmod{2p}} \nu_2 e^{\frac{3\pi i\nu_2^2 w_2}{2p^2}}\\
&= \frac{1}{p^2}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B})\,\Theta_1\left(2p, B_1, 2p; \frac{w_1}{p}\right)\Theta_1\left(2p, B_2, 2p; \frac{3w_2}{p}\right)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{B} :=& \{(p+1,p-1), (1,-1), (p+2,p), (2,0), (1,1), (p+1,p+1)\},\\
\varepsilon_2(\boldsymbol{B}) :=& \varepsilon\left(\frac{B_2-3B_1}{2p}, \frac{B_1}{p}\right).
\end{aligned}$$

Combining the above yields that

$$\begin{aligned}
\mathcal{E}_1(\tau) = &-\frac{\sqrt{3}}{4p}\sum_{\boldsymbol{A}\in\mathcal{A}} \varepsilon_1(\boldsymbol{A}) \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(2p, A_1, 2p; 3w_1)\Theta_1(2p, A_2, 2p; w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1\\
&-\frac{\sqrt{3}}{4p}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B}) \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(2p, B_1, 2p; w_1)\,\Theta_1(2p, B_2, 2p; 3w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1.
\end{aligned}$$

For $M \in \Gamma_p$, we have, using (IV.2.9) and (IV.2.10),

$$\Theta_1(2p, A, 2p; \ell M\tau) = \pm\left(\frac{\ell pc}{d}\right)\varepsilon_d^{-1}(c\tau+d)^{\frac{3}{2}}\Theta_1(2p, A, 2p; \ell\tau).$$

Theorem IV.5.1 then finishes the claim using that $\varepsilon_d^2 = (\frac{-1}{d})$.

∎

IV.5.3 Special multiple Eichler integrals of weight two

Define for $\boldsymbol{\alpha} \in \mathscr{S}^*$

$$\begin{aligned}\mathcal{E}_{2,\boldsymbol{\alpha}}(\tau) :=& \frac{\sqrt{3}}{8\pi} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{2\theta_3(\boldsymbol{\alpha};\boldsymbol{w}) - \theta_4(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1 \\ &+ \frac{\sqrt{3}}{8\pi} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_5(\boldsymbol{\alpha};\boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}} dw_2 dw_1\end{aligned}$$

with

$$\begin{aligned}\theta_3(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{n\in\alpha+\mathbb{Z}^2} (2n_1+n_2) e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}}, \\ \theta_4(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{n\in\alpha+\mathbb{Z}^2} (3n_1+2n_2) e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}, \\ \theta_5(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{n\in\alpha+\mathbb{Z}^2} n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.\end{aligned}$$

We then set

$$\mathcal{E}_2(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \mathcal{E}_{2,\boldsymbol{\alpha}}(p\tau).$$

Remark 11. Similarly as for $\mathcal{E}_1$, one can simplify $\mathcal{E}_2$ as

$$\mathcal{E}_2(\tau) = -\frac{\sqrt{3}}{8\pi} \sum_{\boldsymbol{B}\in\mathcal{B}} I_{\Theta_1(2p,B_1,2p;\,\cdot\,),\Theta_0(6p,3B_2,6p;\,\cdot\,)}(\tau).$$

This function again transforms as a depth two quantum modular.

Proposition IV.5.3. *We have, for $M \in \Gamma_p$,*

$$\mathcal{E}_2(\tau) - \left(\frac{3}{d}\right)(c\tau+d)^{-2}\mathcal{E}_2(M\tau) = \sum_{j=1}^{18} \left(r_{f_j,g_j,\frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j,\frac{d}{c}}(\tau) \right),$$

where f_j and g_j are holomorphic modular forms of weight $\frac{1}{2}$ or cusp forms of weight $\frac{3}{2}$.

Proof. As in the proof of Proposition IV.5.2, we obtain

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^* \\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}} (2n_1+n_2) e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}}$$

$$= \frac{1}{p}\sum_{\boldsymbol{A}\in\mathcal{A}} \Theta_1\left(2p, A_1, 2p; \frac{3w_1}{p}\right)\Theta_0\left(2p, A_2, 2p; \frac{w_2}{p}\right),$$

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^*\\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}} (3n_1+n_2)e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2 w_2}{2}}$$

$$= \frac{1}{p}\sum_{\boldsymbol{B}\in\mathcal{B}} \Theta_1\left(2p, B_1, 2p; \frac{w_1}{p}\right)\Theta_0\left(2p, B_2, 2p; \frac{3w_2}{p}\right),$$

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^*\\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}} n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i}{2}n_1^2 w_2} = \frac{1}{p}\sum_{\boldsymbol{B}\in\mathcal{B}} \Theta_0\left(2p, B_1, 2p; \frac{w_1}{p}\right)\Theta_1\left(2p, B_2, 2p; \frac{3w_2}{p}\right).$$

The claim now again follows from Theorem IV.5.1 using (IV.2.9) and (IV.2.10). □

IV.5.4 More on double Eichler integrals

We have an obvious map $S_k(\Gamma,\chi)\to\mathcal{Q}_{2-k}(\Gamma^*,\chi^*)$, where $\chi^*(M):=\chi(M^*)$, which assigns to $f\in S_k(\Gamma,\chi)$ its Eichler integral I_f, defined in (IV.2.11). Clearly, we also have a map from $S_k(\Gamma,\chi)\otimes S_k(\Gamma,\chi)$, actually from its symmetric square, to $(\mathcal{Q}_{2-k}(\Gamma^*,\chi^*))^2$, by mapping $f_1\otimes f_2$ to $I_{f_1}I_{f_2}$. The double Eichler integral construction I_{f_1,f_2} gives rise to a map

$$\Lambda^2\left(S_k(\Gamma,\chi)\right)\longrightarrow \mathcal{Q}^2_{4-2k}\left(\Gamma^*,{\chi^*}^2\right)\Big/\left(\mathcal{Q}_{2-k}\left(\Gamma^*,\chi^*\right)\right)^2,$$

where $\Lambda^2(S_{2-k}(\Gamma,\chi))$ is the second exterior power of $S_{2-k}(\Gamma,\chi)$. To see this, it suffices to observe the simplest *shuffle* relation for iterated integrals

$$I_{f_1,f_2}+I_{f_2,f_1}=I_{f_1}I_{f_2}.$$

Remark 12. It is now straightforward to consider even more general iterated Eichler integrals ($r\in\mathbb{N}$):

$$I_{f_1,\ldots,f_r}:=\int_{-\overline{\tau}}^{i\infty}\int_{w_{r-1}}^{i\infty}\cdots\int_{w_2}^{i\infty}\prod_{j=1}^{r}\frac{f_j(w_j)}{(-i(w_j+\tau))^{2-k_j}}dw_1\cdots dw_r,$$

where the f_j are cusp forms of weight $k_j\geq\frac{1}{2}$ (or possibly holomorphic forms for weight $\frac{1}{2}$). We do not pursue their (mock/quantum) modular properties here – we will address this in our future work [4] (see also Section 9 for related comments).

IV.6 Indefinite theta functions

We next realize the double Eichler integrals studied in Section 5 as pieces of indefinite theta functions.

IV.6.1 The function $\mathcal{E}_1$ as an indefinite theta function

The next lemma rewrites $\mathbb{E}_1(\tau) := \mathcal{E}_1(\frac{\tau}{p})$ in a shape to which one can apply the Euler-Maclaurin summation formula.

Lemma IV.6.1. *We have*

$$\mathbb{E}_1(\tau) = \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} M_2\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(\boldsymbol{n})}.$$

Proof. The claim follows, once we prove that

$$\begin{aligned}
&M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)\\
&\quad=-\frac{\sqrt{3}}{2}(2n_1+n_2)n_2q^{Q(\boldsymbol{n})}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi in_2^2w_2}{2}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1\\
&\qquad-\frac{\sqrt{3}}{2}(3n_1+2n_2)n_1q^{Q(\boldsymbol{n})}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{3\pi in_1^2w_2}{2}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1.
\end{aligned}\tag{IV.6.1}$$

For simplicity we only show (IV.6.1) for $n_1\neq 0$. Since, by (IV.2.7),

$$\lim_{\lambda\to\infty}M_2(\kappa;\lambda u_1,\lambda u_2)=0,$$

we obtain, using (IV.2.5) and (IV.2.6),

$$\begin{aligned}
&M_2(\kappa;u_1,u_2) = -\int_1^\infty\frac{\partial}{\partial w_1}M_2(\kappa;w_1u_1,w_1u_2)dw_1\\
&=-\int_1^\infty\left(u_1M_2^{(1,0)}(\kappa;w_1u_1,w_1u_2)+u_2M_2^{(0,1)}(\kappa;w_1u_1,w_1u_2)\right)dw_1\\
&=-2\int_1^\infty\left(u_1e^{-\pi u_1^2w_1^2}M(u_2w_1)+\frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}}e^{-\frac{\pi(u_2+\kappa u_1)^2w_1^2}{1+\kappa^2}}M\left(w_1\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right)\right)dw_1\\
&=-\int_1^\infty\left(u_1e^{-\pi u_1^2w_1}M\left(u_2\sqrt{w_1}\right)+\frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}}e^{-\frac{\pi(u_2+\kappa u_1)^2w_1}{1+\kappa^2}}M\left(\sqrt{w_1}\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right)\right)\frac{dw_1}{\sqrt{w_1}}
\end{aligned}\tag{IV.6.2}$$

$$\begin{aligned}
&=\frac{i}{\sqrt{2}}\int_{-\overline{\tau}}^{i\infty}\left(\frac{u_1}{\sqrt{v}}e^{\frac{\pi iu_1^2w_1}{2v}}q^{\frac{u_1^2}{4v}}M\left(\sqrt{\frac{-i(w_1+\tau)}{2v}}u_2\right)\right.\\
&\left.+\frac{u_2+\kappa u_1}{\sqrt{(1+\kappa^2)v}}e^{\frac{\pi i(u_2+\kappa u_1)^2w_1}{2(1+\kappa^2)v}}q^{\frac{(u_2+\kappa u_1)^2}{4(1+\kappa^2)v}}M\left(\sqrt{\frac{-i(w_1+\tau)}{2}}\frac{u_1-\kappa u_2}{\sqrt{(1+\kappa^2)v}}\right)\right)\frac{dw_1}{\sqrt{-i(w_1+\tau)}}.
\end{aligned}$$

Now write for $N \in \mathbb{R}^+$

$$M\left(\sqrt{\frac{-i(w_1+\tau)}{2}}N\right) = \frac{iN}{\sqrt{2}} q^{\frac{N^2}{4}} \int_{w_1}^{i\infty} e^{\frac{\pi i N^2 w_2}{2}} \frac{dw_2}{\sqrt{-i(w_2+\tau)}}.$$

Plugging this into (IV.6.2) easily yields that

$$\begin{aligned} M_2(\kappa; u_1, u_2) &= -\frac{u_1}{2\sqrt{v}} \frac{u_2}{\sqrt{v}} q^{\frac{u_1^2}{4v}+\frac{u_2^2}{4v}} \int_{-\overline{\tau}}^{i\infty} \frac{e^{\frac{\pi i u_1^2 w_1}{2v}}}{\sqrt{-i(w_1+\tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i u_2^2 w_2}{2v}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1 \\ &- \frac{u_2+\kappa u_1}{2\sqrt{(1+\kappa^2)v}} \frac{u_1-\kappa u_2}{\sqrt{(1+\kappa^2)v}} q^{\frac{(u_2+\kappa u_1)^2}{4(1+\kappa^2)v}+\frac{(u_1-\kappa u_2)^2}{4(1+\kappa^2)v}} \\ &\times \int_{-\overline{\tau}}^{i\infty} \frac{e^{\frac{\pi i (u_2+\kappa u_1)^2 w_1}{2(1+\kappa^2)v}}}{\sqrt{-i(w_1+\tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i (u_1-\kappa u_2)^2 w_2}{2(1+\kappa^2)v}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1. \end{aligned}$$

From this it is not hard to conclude (IV.6.1). □

IV.6.2 The function $\mathcal{E}_2$ as an indefinite theta function

We next write $\mathbb{E}_2(\tau) := \mathcal{E}_2(\frac{\tau}{p})$ as a piece of a derivative of an indefinite theta function, having an extra Jacobi variable.

Lemma IV.6.2. *We have*

$$\begin{aligned} \mathbb{E}_2(\tau) = \frac{1}{4\pi i} &\sum_{\boldsymbol{\alpha} \in \mathscr{S}^*} \sum_{\boldsymbol{n} \in \boldsymbol{\alpha}+\mathbb{Z}^2} \\ &\times \left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right)\right]_{z=0} q^{-Q(\boldsymbol{n})}. \end{aligned}$$

Proof. We first compute

$$\begin{aligned} &\frac{1}{2\pi i}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3}; \sqrt{3v}\,(2n_1+n_2)\,, \sqrt{v}\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right)\right]_{z=0} \\ &\qquad = n_2 M_2\left(\sqrt{3}; \sqrt{3v}\,(2n_1+n_2)\,, \sqrt{v}n_2\right) + \frac{1}{2\pi\sqrt{v}} e^{-\pi(3n_1+2n_2)^2 v} M\left(\sqrt{3v}n_1\right). \end{aligned} \tag{IV.6.3}$$

We show below that

$$n_2 M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}n_2\right)$$

$$= -\frac{\sqrt{3}}{2\pi}(2n_1+n_2)\int_{2v}^{\infty}\frac{e^{-\frac{3\pi}{2}(2n_1+n_2)^2w_1}}{\sqrt{w_1}}\int_{w_1}^{\infty}\frac{e^{-\frac{\pi n_2^2w_2}{2}}}{w_2^{\frac{3}{2}}}dw_2dw_1 \tag{IV.6.4}$$

$$+\frac{\sqrt{3}}{4\pi}(3n_1+2n_2)\int_{2v}^{\infty}\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{\sqrt{w_1}}\int_{w_1}^{\infty}\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{w_2^{\frac{3}{2}}}dw_2dw_1$$

$$-\frac{1}{2\pi\sqrt{v}}e^{-\pi(3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)-\frac{\sqrt{3}n_1}{4\pi}\int_{2v}^{\infty}\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^{\infty}\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{w_2^{\frac{1}{2}}}dw_2dw_1.$$

Since the third term cancels the second term on the right-hand side of (IV.6.3) this then implies the claim, using that

$$\int_{2v}^{\infty}\frac{e^{-2\pi M^2w_1}}{w_1^{\frac{1}{2}}}\int_{w_1}^{\infty}\frac{e^{-2\pi N^2w_2}}{w_2^{\frac{3}{2}}}dw_2dw_1 = -q^{M^2+N^2}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi iM^2w_1}}{(-i(w_1+\tau))^{\frac{1}{2}}}\int_{w_1}^{i\infty}\frac{e^{2\pi iN^2w_2}}{(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1,$$

$$\int_{2v}^{\infty}\frac{e^{-2\pi M^2w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^{\infty}\frac{e^{-2\pi N^2w_2}}{w_2^{\frac{1}{2}}}dw_2dw_1 = -q^{N^2+M^2}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi iM^2w_1}}{(-i(w_1+\tau))^{\frac{3}{2}}}\int_{w_1}^{i\infty}\frac{e^{2\pi iN^2w_2}}{(-i(w_2+\tau))^{\frac{1}{2}}}dw_2dw_1.$$

To prove (IV.6.4), we again, for simplicity, restrict to $n_1\neq 0$.

Plugging in (IV.6.2) yields

$$M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right) = -\int_1^{\infty}\left(\sqrt{3v}(2n_1+n_2)e^{-3\pi v(2n_1+n_2)^2w_1}M\left(\sqrt{vw_1}n_2\right)\right.$$
$$\left.+\sqrt{v}(3n_1+2n_2)e^{-\pi v(3n_1+2n_2)^2w_1}M\left(\sqrt{3vw_1}n_1\right)\right)\frac{dw_1}{\sqrt{w_1}}. \tag{IV.6.5}$$

Using (IV.2.3) and (IV.2.2) the first term in (IV.6.5) multiplied by n_2 gives

$$-\frac{\sqrt{3v}}{2\sqrt{\pi}}|n_2|(2n_1+n_2)\int_1^{\infty}e^{-3\pi v(2n_1+n_2)^2w_1}\Gamma\left(-\frac{1}{2},\pi vn_2^2w_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$+\frac{\sqrt{3}}{\pi}(2n_1+n_2)\int_1^{\infty}e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}. \tag{IV.6.6}$$

For the second term in (IV.6.5), we split

$$n_2=\frac{1}{2}(3n_1+2n_2)-\frac{3}{2}n_1. \tag{IV.6.7}$$

The n_1-term contributes to n_2M_2 as

$$\frac{3}{4}\sqrt{\frac{v}{\pi}}|n_1|(3n_1+2n_2)\int_1^\infty e^{-\pi v(3n_1+2n_2)^2w_1}\Gamma\left(-\frac{1}{2},3\pi vn_1^2w_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$-\frac{\sqrt{3}}{2\pi}(3n_1+2n_2)\int_1^\infty e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}. \quad \text{(IV.6.8)}$$

We next use that for $N\in\mathbb{N}_0$, $M\in\mathbb{N}$

$$\int_1^\infty e^{-4\pi N^2vw_1}\Gamma\left(-\frac{1}{2},4\pi vM^2w_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$=\frac{1}{2\sqrt{\pi v}|M|}\int_{2v}^\infty\frac{e^{-2\pi N^2w_1}}{\sqrt{w_1}}\int_{w_1}^\infty\frac{e^{-2\pi M^2w_2}}{w_2^{\frac{3}{2}}}dw_2dw_1.$$

We use this to rewrite the first terms in (IV.6.6) and (IV.6.8). The first term in (IV.6.6) is the first term on the right-hand side of (IV.6.4). Similarly, since $n_1\neq 0$, the first term in (IV.6.8) equals the second term in (IV.6.4). Now we combine the second terms in (IV.6.6) and (IV.6.8), to get

$$\frac{\sqrt{3}n_1}{2\pi}\int_1^\infty e^{-4\pi vQ(\boldsymbol{n})w_1}\frac{dw_1}{w_1}. \quad \text{(IV.6.9)}$$

Next we compute the contribution from the first term in (IV.6.7),

$$-\frac{\sqrt{v}}{2}(3n_1+2n_2)^2\int_1^\infty e^{-\pi v(3n_1+2n_2)^2w_1}M\left(\sqrt{3vw_1}n_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$=\frac{1}{2\pi\sqrt{v}}\int_1^\infty\frac{\partial}{\partial w_1}\left(e^{-\pi v(3n_1+2n_2)^2w_1}\right)\frac{M\left(\sqrt{3vw_1}n_1\right)}{\sqrt{w_1}}dw_1.$$

Using integration by parts, this becomes

$$-\frac{1}{2\pi\sqrt{v}}e^{-\pi v(3n_1+2n_2)^2}M\left(\sqrt{3v}n_1\right)-\frac{\sqrt{3}n_1}{2\pi}\int_1^\infty e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}$$
$$+\frac{1}{4\pi\sqrt{v}}\int_1^\infty e^{-\pi v(3n_1+2n_2)^2w_1}\frac{M\left(\sqrt{3vw_1}n_1\right)}{w_1^{\frac{3}{2}}}dw_1. \quad \text{(IV.6.10)}$$

The second term now cancels (IV.6.9) and the first term equals the third term in (IV.6.4).

To rewrite the final term in (IV.6.10), we use that for $M,N\in\mathbb{Z}$ with $N\neq 0$

$$\int_1^\infty e^{-4\pi vM^2w_1}\frac{M\left(2\sqrt{vw_1}N\right)}{w_1^{\frac{3}{2}}}dw_1=-2\sqrt{v}N\int_{2v}^\infty\frac{e^{-2\pi M^2w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^\infty\frac{e^{-2\pi N^2w_2}}{w_2^{\frac{1}{2}}}dw_2dw_1.$$

Thus the last term in (IV.6.10) gives the final term in (IV.6.4).

□

IV.7 Asymptotic behavior of multiple Eichler integrals and proof of Theorem IV.1.1

In this section, we asymptotically relate F_j and $\mathbb{E}_j$.

IV.7.1 Asymptotic behavior of $\mathbb{E}_1$

Write

$$F_1\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} a_{h,k}(m)t^m \quad \left(t\to 0^+\right).$$

The goal of this subsection is to prove the following.

Theorem IV.7.1. *We have, for $h,k\in\mathbb{Z}$ with $k>0$ and $\gcd(h,k)=1$,*

$$\mathbb{E}_1\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{-h,k}(m)(-t)^m \quad \left(t\to 0^+\right).$$

Proof. We use Lemma IV.6.1 and the fact that M_2 is an even function, to rewrite

$$\begin{aligned}\mathbb{E}_1(\tau) = &\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(\boldsymbol{n})}\\ &+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2\left(\sqrt{3};\sqrt{v}\left(-2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(-n_1,n_2)},\end{aligned}$$

where

$$\widetilde{\mathscr{S}} := \left\{(1-\alpha_1,\alpha_2):\boldsymbol{\alpha}\in\mathscr{S}\right\},\quad \widetilde{\varepsilon}(\boldsymbol{\alpha}) := \varepsilon(1-\alpha_1,\alpha_2).$$

To apply the Euler-Maclaurin summation formula directly, we turn every sgn into sgn*, where $\operatorname{sgn}^*(x):=\operatorname{sgn}(x)$ for $x\neq 0$ and $\operatorname{sgn}^*(0):=1$. To be more precise, we set

$$\begin{aligned}M_2^*\left(\sqrt{3};\sqrt{3}(2x_1+x_2),x_2\right) := &\operatorname{sgn}^*(x_1)\operatorname{sgn}^*(x_2)+E_2\left(\sqrt{3};\sqrt{3}(2x_1+x_2),x_2\right)\\ &-\operatorname{sgn}^*(x_2)E\left(\sqrt{3}(2x_1+x_2)\right)-\operatorname{sgn}^*(x_1)E(3x_1+2x_2).\end{aligned} \tag{IV.7.1}$$

Using that

$$M_2\left(\sqrt{3};\sqrt{3}x_2,x_2\right)-\lim_{x_1\to 0^+}M_2^*\left(\sqrt{3};\sqrt{3}\left(\pm 2x_1+x_2\right),x_2\right)=\pm M(2x_2), \tag{IV.7.2}$$

we then split

$$\mathbb{E}_1(\tau)=\mathcal{E}_1^*(\tau)+H_1(\tau)$$

with

$$\begin{aligned}\mathcal{E}_1^*(\tau) &:= \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2^*\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(\boldsymbol{n})}\\ &\quad+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2^*\left(\sqrt{3};\sqrt{v}\left(-2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(-n_1,n_2)},\\ H_1(\tau) &:= -\frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{N}_0} M\left(2\sqrt{v}m\right)q^{-m^2}+\frac{1}{2}\sum_{m\in 1-\frac{1}{p}+\mathbb{N}_0} M\left(2\sqrt{v}m\right)q^{-m^2}.\end{aligned}$$

Note that for $n_1=0$ we take the limit $n_1\to 0$ in the M_2^*-functions.

We proceed as in Subsection IV.4.1 to determine the asymptotic behavior of $\mathcal{E}_1^*$ and H_1. Firstly we rewrite

$$\begin{aligned}\mathcal{E}_1^*\left(\frac{h}{k}+\frac{it}{2\pi}\right) &= \sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{F}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)\\ &\quad+\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\alpha)\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(-(\ell_1+\alpha_1),\ell_2+\alpha_2)}\sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{F}}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),\end{aligned}$$

where

$$\mathcal{F}_3(\boldsymbol{x}) := \frac{1}{2}M_2^*\left(\sqrt{3};\frac{1}{\sqrt{2\pi}}\left(\sqrt{3}\left(2x_1+x_2\right),x_2\right)\right)e^{Q(\boldsymbol{x})},\qquad \widetilde{\mathcal{F}}_3(\boldsymbol{x}) := \mathcal{F}_3(-x_1,x_2).$$

The contribution from the $\mathcal{F}_3$ term to the first term in (IV.2.8) is

$$\frac{\delta^2}{k^2p^2t}\mathcal{I}_{\mathcal{F}_3}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}=0,$$

conjugating (IV.4.2). In the same way the main term coming from $\widetilde{\mathcal{F}_3}$ is shown to vanish.

The contribution to the second term of Euler-Maclaurin is

$$\begin{aligned}&-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge 0}\frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+2)!}\\ &\qquad\times\int_0^\infty\left(\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)\right)dx_1\left(\frac{k^2p^2t}{\delta^2}\right)^{n_2}.\end{aligned}$$

We now claim that

$$\int_0^\infty \left(\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)\right)dx_1=(-1)^{n_2}\int_0^\infty \mathcal{F}_1^{(0,2n_2+1)}(x_1,0)dx_1. \tag{IV.7.3}$$

Firstly the right-hand side of (IV.7.3) equals

$$\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\int_0^\infty \mathcal{F}_1(x_1,x_2)dx_1\right]_{x_2=0}=\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\int_0^\infty e^{-3\left(x_1+\frac{x_2}{2}\right)^2}dx_1\right)\right]_{x_2=0}. \tag{IV.7.4}$$

Now the integral in (IV.7.4) evaluates as

$$\sqrt{\frac{\pi}{3}}\int_{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}}^\infty e^{-\pi x_1^2}dx_1=\frac{\sqrt{\pi}}{2\sqrt{3}}\left(1-E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right).$$

Thus (IV.7.4) becomes

$$\begin{aligned}\frac{\sqrt{\pi}}{2\sqrt{3}}&\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\left(1-E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right)\right]_{x_2=0} \qquad \text{(IV.7.5)}\\ &=-\frac{\sqrt{\pi}}{2\sqrt{3}}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right]_{x_2=0}.\end{aligned}$$

To compute the left-hand side of (IV.7.3), we decompose, according to (IV.7.1),

$$\begin{aligned}&M_2^*\left(\sqrt{3},\sqrt{3}(2x_1+x_2),x_2\right)\\ &\qquad=\operatorname{sgn}^*(x_1)\operatorname{sgn}^*(x_2)+h_1(\boldsymbol{x})-\operatorname{sgn}^*(x_2)h_2(\boldsymbol{x})-\operatorname{sgn}^*(x_1)h_3(\boldsymbol{x}),\end{aligned}$$

where

$$h_1(\boldsymbol{x}):=E_2\left(\sqrt{3};\sqrt{3}\left(2x_1+x_2\right),x_2\right),\quad h_2(\boldsymbol{x}):=E\left(\sqrt{3}\left(2x_1+x_2\right)\right),$$
$$h_3(\boldsymbol{x}):=E\left(3x_1+2x_2\right).$$

Setting

$$a_0(\boldsymbol{x}):=e^{Q(\boldsymbol{x})},\qquad a_j(\boldsymbol{x}):=h_j\left(\frac{1}{\sqrt{2\pi}}(\boldsymbol{x})\right)e^{Q(\boldsymbol{x})},$$

we then obtain

$$\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(a_0^{(0,2n_2+1)}(x_1,0) + a_1^{(0,2n_2+1)}(x_1,0) - a_2^{(0,2n_2+1)}(x_1,0) - a_3^{(0,2n_2+1)}(x_1,0)\right) \\
&\quad + \frac{1}{2}\left(-a_0^{(0,2n_2+1)}(-x_1,0) + a_1^{(0,2n_2+1)}(-x_1,0) - a_2^{(0,2n_2+1)}(-x_1,0) + a_3^{(0,2n_2+1)}(-x_1,0)\right) \\
&= a_0^{(0,2n_2+1)}(x_1,0) - a_2^{(0,2n_2+1)}(x_1,0),
\end{aligned}
$$

using that a_0 and a_1 are even and a_2 and a_3 are odd. Plugging in the definition of a_0 and a_2, we need to consider

$$
-\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{x_2^2}\int_0^\infty e^{3x_1^2+3x_1x_2} M\left(\sqrt{\frac{3}{2\pi}}(2x_1+x_2)\right) dx_1\right)\right]_{x_2=0}. \tag{IV.7.6}
$$

Changing variables $w := \sqrt{\frac{3}{2\pi}}(2x_1+x_2)$, the function in (IV.7.6) before differentiation is

$$
-\sqrt{\frac{\pi}{6}} e^{\frac{x_2^2}{4}} \int_{\sqrt{\frac{3}{2\pi}}x_2}^\infty M(w) e^{\frac{\pi w^2}{2}} dw = -\sqrt{\frac{\pi}{6}} e^{\frac{x_2^2}{4}} \left(\int_0^\infty M(w) e^{\frac{\pi w^2}{2}} dw - \int_0^{\sqrt{\frac{3}{2\pi}}x_2} M(w) e^{\frac{\pi w^2}{2}} dw\right).
$$

The first integral vanishes upon differentiating an odd number of times and then setting $x_2 = 0$. In the second integral we decompose $M(w) = E(w) - 1$. The contribution of the E-function vanishes, since E is an odd function. We are left with

$$
\begin{aligned}
&-\sqrt{\frac{\pi}{6}}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{\frac{x_2^2}{4}}\int_0^{\sqrt{\frac{3}{2\pi}}x_2} e^{\frac{\pi w^2}{2}} dw\right)\right]_{x_2=0} \\
&\qquad = -\sqrt{\frac{\pi}{6}}\, i^{-2n_2-1}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\int_0^{\sqrt{\frac{3}{2\pi}}x_2 i} e^{\frac{\pi w^2}{2}} dw\right)\right]_{x_2=0}.
\end{aligned}
$$

The integral equals

$$
i\sqrt{2}\int_0^{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}} e^{-\pi w^2} dw = \frac{i}{\sqrt{2}} E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right).
$$

Thus we obtain

$$
\frac{\sqrt{\pi}}{2\sqrt{3}}(-1)^{n_2+1}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}} E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right]_{x_2=0},
$$

as claimed, by comparing with (IV.7.5).

In the same way one can show that the third term in Euler-Maclaurin equals

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\\ \times\frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+2)!}\int_0^\infty\mathcal{F}_1^{(2n_1+1,0)}(0,x_2)dx_2\left(-\frac{k^2p^2t}{\delta^2}\right)^{n_1}.$$

The contribution to the final term is, pairing as in Section 4

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2\pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_1+1)!}\\ \times\left(\mathcal{F}_3^{(\boldsymbol{n})}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(\boldsymbol{n})}(\boldsymbol{0})\right)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

We next show that

$$\mathcal{F}_3^{(n_1,n_2)}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(n_1,n_2)}(\boldsymbol{0})=i^{n_1+n_2}\mathcal{F}_1^{(n_1,n_2)}(\boldsymbol{0}).$$

For this, we compute

$$\mathcal{F}_3^{(n_1,n_2)}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(n_1,n_2)}(\boldsymbol{0})=a_0^{(n_1,n_2)}(\boldsymbol{0})-a_3^{(n_1,n_2)}(\boldsymbol{0}).$$

Since $a_3(-x_1,-x_2)=-a_3(\boldsymbol{x})$, we obtain

$$a_3^{(n_1,n_2)}(\boldsymbol{0})=(-1)^{n_1+n_2+1}a_3^{(n_1,n_2)}(\boldsymbol{0}).$$

Because in the sums of interest $n_1\equiv n_2\pmod 2$, the contribution of a_3 vanishes. As claimed, we are left with

$$a_0^{(n_1,n_2)}(\boldsymbol{0})=i^{n_1+n_2}\left[\frac{\partial^{n_1}}{\partial x_1^{n_1}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}e^{-Q(\boldsymbol{x})}\right]_{\boldsymbol{x}=0}=i^{n_1+n_2}\mathcal{F}_1^{(\boldsymbol{n})}(\boldsymbol{0}).$$

Finally, the contribution from H_1 gives, observing that the Euler-Maclaurin main term vanishes,

$$\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+1)!}\mathcal{F}_4^{(2m)}(0)\left(\frac{k^2p^2}{\delta^2}t\right)^m$$

with $\mathcal{F}_4(x) := M(\sqrt{\frac{2}{\pi}}x)e^{x^2}$. The claim then follows, observing that

$$\mathcal{F}_4^{(2m)}(0) = (-1)^{m+1}\left[\frac{\partial^{2m}}{\partial x^{2m}}e^{-x^2}\right]_{x=0} = (-1)^{m+1}\mathcal{F}_2^{(2m)}(0).$$

□

IV.7.2 Asymptotics of $\mathcal{E}_2$

We write

$$F_2\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} b_{h,k}(m)t^m \qquad \left(t\to 0^+\right),$$

Theorem IV.7.2. *We have, for $h,k\in\mathbb{Z}$ with $k>0$ and $\gcd(h,k)=1$,*

$$\mathbb{E}_2\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} b_{-h,k}(m)(-t)^m \qquad \left(t\to 0^+\right).$$

Proof. We write, using Lemma IV.6.2 and (IV.6.3)

$$\mathbb{E}_2(\tau) = \mathcal{E}_{2,1}(\tau)+\mathcal{E}_{2,2}(\tau),$$

where

$$\begin{aligned}\mathcal{E}_{2,1}(\tau) :=& \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(\boldsymbol{n})}\\ &+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2M_2\left(\sqrt{3};\sqrt{3v}(-2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(-n_1,n_2)},\\ \mathcal{E}_{2,2}(\tau) :=& \frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} e^{-\pi(3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)q^{-Q(\boldsymbol{n})}\\ &+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} e^{-\pi(-3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)q^{-Q(-n_1,n_2)},\end{aligned}$$

where $\widetilde{\eta}(\boldsymbol{\alpha}) := \eta(1-\alpha_1,\alpha_2)$. We then again use (IV.7.2), to split

$$\mathcal{E}_{2,1}(\tau) = \mathcal{E}_2^*(\tau)+H_2(\tau),$$

where

$$\mathcal{E}_2^*(\tau) := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2M_2^*\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(\boldsymbol{n})}$$

$$+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2 M_2^*\left(\sqrt{3};\sqrt{3v}(-2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(-n_1,n_2)},$$

$$H_2(\tau):=\frac{1}{2}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{m\in\beta+\mathbb{N}_0} mM\left(2\sqrt{v}m\right)q^{-m^2}.$$

Using that $\lim_{x\to 0^+} M^*(\pm x)=\mp 1$, where we let $M^*(x):=E(x)-\operatorname{sgn}^*(x)$, we split

$$\mathcal{E}_{2,2}(\tau)=\mathcal{E}_{2,2}^*(\tau)+H_3(\tau),$$

where

$$\mathcal{E}_{2,2}^*(\tau):=\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} e^{-\pi(3n_1+2n_2)^2 v}M^*\left(\sqrt{3v}n_1\right)q^{-Q(\boldsymbol{n})}$$

$$+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} e^{-\pi(-3n_1+2n_2)^2 v}M^*\left(-\sqrt{3v}n_1\right)q^{-Q(-n_1,n_2)},$$

$$H_3(\tau):=\frac{1}{4\pi\sqrt{v}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{m\in\beta+\mathbb{N}_0} e^{-4\pi m^2 v}q^{-m^2}.$$

We first investigate asymptotic properties of $\mathcal{E}_2^*$. Writing $\mathcal{G}_3(\boldsymbol{x}):=x_2\mathcal{F}_3(\boldsymbol{x})$ and $\widetilde{\mathcal{G}}_3(\boldsymbol{x}):=\mathcal{G}_3(-x_1,x_2)$, we have

$$\mathcal{E}_2^*\left(\frac{h}{k}+\frac{it}{2\pi}\right)=\frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{G}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)$$

$$+\frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(-\ell_1-\alpha_1,\ell_2+\alpha_2)}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{G}}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right).$$

The contribution from $\mathcal{G}_3$ to the Euler-Maclaurin main term is, as in Subsection IV.4.2,

$$\frac{\delta^2}{k^2p^2t^{\frac{3}{2}}}\mathcal{I}_{\mathcal{G}_3}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}=0.$$

In the same way we see that the contribution from $\widetilde{\mathcal{G}}_3$ to the main term vanishes.

The contribution to the second term in Euler-Maclaurin is, as in Subsection IV.4.2,

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge1}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!}$$

$$\times \int_0^\infty \left(\mathcal{G}_3^{(0,2n_2)}(x_1,0) + \widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0)\right) dx_1 \left(\frac{kp}{\delta}\right)^{2n_2-1} t^{n_2-1}.$$

We claim that

$$\int_0^\infty \left(\mathcal{G}_3^{(0,2n_2)}(x_1,0) + \widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0)\right) dx_1 = (-1)^{n_2+1} \int_0^\infty \mathcal{G}_1^{(0,2n_2)}(x_1,0) dx_1. \quad \text{(IV.7.7)}$$

Since we need to differentiate the x_2-factor exactly once, we have

$$\mathcal{G}_3^{(0,2n_2)}(x_1,0) + \widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0) = 2n_2 \left(\mathcal{F}_3^{(0,2n_2-1)}(x_1,0) + \widetilde{\mathcal{F}}_3^{(0,2n_1-1)}(x_1,0)\right),$$
$$\mathcal{G}_1^{(0,2n_2)}(x_1,0) = 2n_2 \mathcal{F}_1^{(0,2n_2-1)}(x_1,0).$$

The claim (IV.7.7) then follows from (IV.7.3). This gives the correspondence to (IV.4.4).

The third term in Euler-Maclaurin is, in the same way,

$$-2 \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \sum_{0\le \boldsymbol{\ell} \le \frac{kp}{\delta}-1} e^{-2\pi i \frac{h}{k} Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n_1\ge 0} \frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}$$
$$\times \int_0^\infty \left(\mathcal{G}_3^{(2n_1,0)}(0,x_2) - \widetilde{\mathcal{G}}_3^{(2n_1,0)}(0,x_2)\right) dx_2 \left(\frac{kp}{\delta}\right)^{2n_1-1} t^{n_1-1}.$$

To relate this to (IV.4.5) (skipping the $n_1 = 0$ term in both cases), we compute that

$$\mathcal{G}_3^{(2n_1,0)}(0,x_2) - \widetilde{\mathcal{G}}_3^{(2n_1,0)}(0,x_2) = x_2 \left(a_0^{(2n_1,0)}(0,x_2) - a_3^{(2n_1,0)}(0,x_2)\right).$$

Note that

$$a_0(\boldsymbol{x}) - a_3(\boldsymbol{x}) = -e^{Q(\boldsymbol{x})} M^* \left(\sqrt{\frac{1}{2\pi}}(3x_1+2x_2)\right). \quad \text{(IV.7.8)}$$

We next show that

$$\int_0^\infty x_2 \left(a_0^{(2n_1,0)}(0,x_2) - a_3^{(2n_1,0)}(0,x_2)\right) dx_2 \quad \text{(IV.7.9)}$$
$$= (-1)^{n_1+1} \int_0^\infty x_2 \left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} \left(e^{-x_2^2-3x_1x_2-3x_1^2}\right)\right]_{x_1=0} dx_2 + \frac{1}{\sqrt{2}} \left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} e^{\frac{3x_1^2}{4}}\right]_{x_1=0},$$

where the first terms on the right-hand side corresponds to (IV.4.5). We write it as

$$(-1)^{n_1+1} \left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} \left(e^{-\frac{3x_1^2}{4}} \int_0^\infty x_2 e^{-\left(x_2+\frac{3x_1}{2}\right)^2} dx_2\right)\right]_{x_1=0}.$$

Now we let

$$f(x_1)e^{\frac{3x_1^2}{4}}(-1)^{n_1+1} := \int_{\frac{3x_1}{2}}^{\infty}\left(x_2 - \frac{3x_1}{2}\right) e^{-x_2^2}dx_2 = \frac{1}{2}e^{-\frac{9x_1^2}{4}} - \frac{3x_1}{2}\frac{\sqrt{\pi}}{2}\left(1 - E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right),$$

using integration by parts. We then compute (using $n_1 > 0$)

$$f^{(2n_1)}(0) = \frac{(-1)^{n_1+1}}{2}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{-3x_1^2}\right]_{x_1=0} + \frac{(-1)^{n_1+1}3\sqrt{\pi}}{4}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1 e^{-\frac{3x_1^2}{4}}E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right]_{x_1=0}. \tag{IV.7.10}$$

For the left-hand side of (IV.7.9) we use (IV.7.8) and consider

$$-\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(\int_0^{\infty} x_2 M^*\left(\frac{2x_2+3x_1}{\sqrt{2\pi}}\right)e^{3x_1^2+3x_1x_2+x_2^2}dx_2\right)\right]_{x_1=0}.$$

Making the change of variables $u = \frac{2x_2+3x_1}{\sqrt{2\pi}}$, the integral before differentiation (including the minus sign) becomes

$$-\sqrt{\frac{\pi}{2}}e^{\frac{3x_1^2}{4}}\frac{1}{2}\int_{\frac{3x_1}{\sqrt{2\pi}}}^{\infty}\left(\sqrt{2\pi}u - 3x_1\right)M^*(u)e^{\frac{\pi u^2}{2}}du. \tag{IV.7.11}$$

Using integration by parts, the contribution from $\sqrt{2\pi}u$ equals

$$\frac{1}{2}e^{3x_1^2}\left(E\left(\frac{3x_1}{\sqrt{2\pi}}\right) - 1\right) + \frac{1}{\sqrt{2}}e^{\frac{3x_1^2}{4}}\left(1 - E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right).$$

Thus differentiating $2n_1$ times with respect to x_1 and then setting $x_1 = 0$ gives (using that $z \mapsto E(z)$ is odd)

$$\begin{aligned} -\frac{1}{2}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{3x_1^2}\right]_{x_1=0} &+ \frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0} \\ &= -\frac{1}{2}(-1)^{n_1}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{-3x_1^2}\right]_{x_1=0} + \frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}. \end{aligned}$$

The first term matches the first term in (IV.7.10), the second term is the second term on the right-hand side of (IV.7.9). For the second term in (IV.7.11), we split

$$\frac{3}{2}\sqrt{\frac{\pi}{2}}e^{\frac{3x_1^2}{4}}x_1\left(\int_0^{\infty}M^*(u)e^{\frac{\pi u^2}{2}}du - \int_0^{\frac{3x_1}{\sqrt{2\pi}}}E(u)e^{\frac{\pi u^2}{2}}du + \int_0^{\frac{3x_1}{\sqrt{2\pi}}}e^{\frac{\pi u^2}{2}}du\right).$$

Since we take an even number of derivatives only the last term survives, yielding the contribution

$$\frac{3}{2}\sqrt{\frac{\pi}{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1 e^{\frac{3x_1^2}{4}}\int_0^{\frac{3x_1}{\sqrt{2\pi}}} e^{\frac{\pi u^2}{2}}du\right)\right]_{x_1=0} = -(-1)^{n_1}\frac{3\sqrt{\pi}}{4}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1 e^{-\frac{3x_1^2}{4}}E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right]_{x_1=0}.$$

This is the second term in (IV.7.10), which implies (IV.7.9).

The left-over term from (IV.7.9) overall contributes as

$$-\sqrt{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge1}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-1}.$$

The final term in Euler-Maclaurin is

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}$$
$$\times\left(\mathcal{G}_3^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_3^{(n_1,n_2)}(\mathbf{0})\right)\left(\frac{kp}{\delta}\right)^{n_1+n_2}t^{\frac{n_1+n_2-1}{2}}.$$

Then

$$\mathcal{G}_3^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_3^{(n_1,n_2)}(\mathbf{0}) = i^{n_1+n_2-1}\mathcal{G}_1^{(n_1,n_2)}(\mathbf{0})$$

gives the relation to (IV.4.7).

We next consider H_2. We have, with $\mathcal{G}_4(x) := x\mathcal{F}_4(x)$,

$$H_2\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \frac{1}{2\sqrt{t}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}(r+\beta)^2}\sum_{m\in\frac{(r+\beta)\delta}{kp}+\mathbb{N}_0}\mathcal{G}_4\left(\frac{kp}{\delta}\sqrt{t}m\right).$$

The Euler-Maclaurin main term is

$$\frac{1}{2\sqrt{t}}\frac{\delta}{kp\sqrt{t}}\mathcal{I}_{\mathcal{G}_4}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}(r+\beta)^2} = \frac{\delta}{kpt}\mathcal{I}_{\mathcal{G}_4}\sum_{r \pmod{\frac{kp}{\delta}}} e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}.$$

The second term becomes

$$-\sum_{0\leq r\leq \frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\geq 0}\frac{B_{2m+2}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+2)!}\mathcal{G}_4^{(2m+1)}(0)\left(\frac{kp}{\delta}\right)^{2m+1}t^m.$$

Then

$$\mathcal{G}_4^{(2m+1)}(0)=(2m+1)\mathcal{F}_4^{(2m)}(0)=(2m+1)(-1)^{m+1}\mathcal{F}_2^{(2m)}(0)=(-1)^{m+1}\mathcal{G}_2^{(2m+1)}(0).$$

gives the relation to (IV.4.9).

Finally, we consider $\mathcal{E}_{2,2}$. We first study $\mathcal{E}_{2,2}^*$ and write

$$\begin{aligned}\mathcal{E}_{2,2}^*\left(\frac{h}{k}+\frac{it}{2\pi}\right)=&\frac{1}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{G}_5\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)\\&+\frac{1}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(-\ell_1-\alpha_1,\ell_2+\alpha_2)}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{G}}_5\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),\end{aligned}$$

where

$$\mathcal{G}_5(\boldsymbol{x}):=\frac{1}{2\sqrt{2}}e^{-\frac{3x_1^2}{2}-3x_1x_2-x_2^2}M^*\left(\sqrt{\frac{3}{2\pi}}x_1\right),\quad \widetilde{\mathcal{G}}_5(\boldsymbol{x}):=\mathcal{G}_5(-x_1,x_2).$$

As before the main term in Euler-Maclaurin vanishes. The second term equals

$$\begin{aligned}-\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\geq 0}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!}&\\ \times\int_0^\infty\left(\mathcal{G}_5^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_5^{(0,2n_2)}(x_1,0)\right)dx_1\left(\frac{kp}{\delta}\right)^{2n_2-1}t^{n_2-\frac{1}{2}}.&\end{aligned}$$

It is however not hard to see that

$$\mathcal{G}_5^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_5^{(0,2n_2)}(x_1,0)=0.$$

The third term in Euler-Maclaurin is

$$-\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\geq 0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}$$

$$\times \int_0^\infty \left(\mathcal{G}_5^{(2n_1,0)}(0,x_2) - \widetilde{\mathcal{G}}_5^{(2n_1,0)}(0,x_2)\right) dx_2 \left(\frac{kp}{\delta}\right)^{2n_1-1} t^{n_1-\frac{1}{2}}.$$

Now

$$\mathcal{G}_5^{(2n_1,0)}(0,x_2) - \widetilde{\mathcal{G}}_5^{(2n_1,0)}(0,x_2) = 2\mathcal{G}_{5,1}^{(2n_1,0)}(0,x_2),$$

where $\mathcal{G}_{5,1}(\boldsymbol{x}) := -\frac{1}{2\sqrt{2}} e^{-\frac{3x_1^2}{2}-3x_1x_2-x_2^2}$. We thus need to compute

$$2\int_0^\infty \mathcal{G}_{5,1}^{(2n_1,0)}(0,x_2)\, dx_2 = -\frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} e^{\frac{3x_1^2}{4}} \int_0^\infty e^{-\left(x_2+\frac{3}{2}x_1\right)^2} dx_2\right]_{x_1=0}$$
$$= -\frac{\sqrt{\pi}}{2\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(e^{\frac{3x_1^2}{4}}\left(1-E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right)\right]_{x_1=0} = -\frac{\sqrt{\pi}}{2\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} e^{\frac{3x_1^2}{4}}\right]_{x_1=0}.$$

This term then contributes as

$$\frac{1}{\sqrt{2}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}} e^{\frac{3x_1^2}{4}}\right]_{x_1=0}\left(\frac{kp}{\delta}\right)^{2n_1-1} t^{n_1-1}. \tag{IV.7.12}$$

The final term in the Euler-Maclaurin summation formula is

$$\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2 \pmod 2}}\frac{B_{n+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}$$
$$\times\left(\mathcal{G}_5^{(n_1,n_2)}(\mathbf{0}) + (-1)^{n_1+1}\widetilde{\mathcal{G}}_5^{(n_1,n_2)}(\mathbf{0})\right)\left(\frac{kp}{\delta}\sqrt{t}\right)^{n_1+n_2}.$$

It is easy to see that under the condition $n_1 \not\equiv n_2 \pmod 2$ we have

$$\mathcal{G}_5^{(n_1,n_2)}(\mathbf{0}) + (-1)^{n_1+1}\widetilde{\mathcal{G}}_5^{(n_1,n_2)}(\mathbf{0}) = 0.$$

Next, we consider

$$H_3\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \frac{1}{2\sqrt{2\pi t}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}(r+\beta)^2}\sum_{m\in\frac{\delta(r+\beta)}{kp}+\mathbb{N}_0}\mathcal{F}_2\left(\frac{kp}{\delta}\sqrt{t}m\right).$$

The Euler-Maclaurin main term is

$$\frac{\delta}{2kpt\sqrt{2\pi}}\mathcal{I}_{\mathcal{F}_2}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{r \pmod{\frac{kp}{\delta}}} e^{-2\pi i\frac{h}{k}(r+\beta)^2} = \frac{\delta}{kpt\sqrt{2\pi}}\mathcal{I}_{\mathcal{F}_2}\sum_{r \pmod{\frac{kp}{\delta}}} e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}.$$

The final term is

$$\frac{1}{\sqrt{2\pi t}} \sum_{0\le r\le \frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2} \sum_{m\ge 0} \frac{B_{2m+2}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+2)!} \mathcal{F}_2^{(2m+1)}(0) = 0$$

since $\mathcal{F}_2$ is an even function.

Collecting all growing terms gives

$$\frac{\delta}{kpt} \sum_{r \pmod{\frac{kp}{\delta}}} e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2} \left(\mathcal{I}_{\mathcal{G}_3(0,\cdot)-\widetilde{\mathcal{G}}_3(0,\cdot)} + \mathcal{I}_{\mathcal{G}_4} + \frac{1}{\sqrt{\pi}}\mathcal{I}_{\mathcal{G}_5(0,\cdot)-\widetilde{\mathcal{G}}_5(0,\cdot)} + \frac{\mathcal{I}_{\mathcal{F}_2}}{\sqrt{2\pi}}\right). \tag{IV.7.13}$$

We compute $\mathcal{I}_{\mathcal{F}_2} = \frac{\sqrt{\pi}}{2}$ and, using integration by parts,

$$\mathcal{I}_{\mathcal{G}_4} = \int_0^\infty xe^{x^2} M^*\left(\sqrt{\frac{2}{\pi}}x\right) dx = -\frac{M^*(0)}{2} - \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2} dx = \frac{1}{2} - \frac{1}{\sqrt{2}}$$

by conjugating (IV.4.8). Moreover, (IV.7.9) gives

$$\begin{aligned}
\mathcal{I}_{\mathcal{G}_3(0,\cdot)-\widetilde{\mathcal{G}}_3(0,\cdot)} &- \int_0^\infty x_2 e^{-x_2^2} dx_2 + \frac{1}{\sqrt{2}} = \frac{1}{2}\left[e^{-x_2^2}\right]_0^\infty + \frac{1}{\sqrt{2}} = -\frac{1}{2} + \frac{1}{\sqrt{2}},\\
\mathcal{I}_{\mathcal{G}_5(0,\cdot)-\widetilde{\mathcal{G}}_5(0,\cdot)} &= -\frac{\sqrt{\pi}}{2\sqrt{2}}.
\end{aligned}$$

Thus the term inside the paranthesis in (IV.7.13) vanishes.

We are left to show that the contributions from (IV.7.8) and (IV.7.12) vanish. For this it suffices to show that, for all $n \in \mathbb{N}$,

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \sum_{0\le \boldsymbol{\ell}\le \frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} B_{2n+1}\left(\frac{\delta\left(\ell_1+\alpha_1\right)}{kp}\right) = 0.$$

As in (IV.4.9) we get that this sum is zero for $\frac{p}{\delta} \notin \{1,2\}$. Next we consider $\frac{p}{\delta} = 1$. We first combine the first and third element in $\mathscr{S}^*$. Using (IV.4.10) and

$$B_{2m+1}(1-x) = -B_{2m+1}(x) \tag{IV.7.14}$$

gives that these cancel. Thus we need to show that

$$\sum_{0\le \ell<k} B_{2n+1}\left(\frac{\ell_1}{k}\right) e^{-2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)} = 0. \tag{IV.7.15}$$

We use (IV.4.11) and distinguish again whether k is even or odd. If k is odd we do the same change of variables and use (IV.7.14) to obtain that (IV.7.15) equals

$$B_{2n+1}(0) \sum_{\ell_2 \pmod k} e^{-2\pi i \frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0$$

since for $m \geq 3$ odd, $B_m(0) = 0$.

If k is even, then we obtain

$$B_{2n+1}(0) \sum_{\ell_2 \pmod k} e^{-2\pi i \frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} + B_{2n+1}\left(\frac{1}{2}\right) \sum_{\ell_2 \pmod k} e^{-2\pi i \frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0$$

since for m odd $B_m(\frac{1}{2}) = 0$.

We next turn to the case $\frac{p}{\delta} = 2$. Then only the second element survives and we want

$$\sum_{0 \leq \boldsymbol{\ell} \leq 2k-1} B_{2n+1}(0) e^{-2\pi i \frac{h}{k}\left(\ell_1, \ell_2+1-\frac{1}{p}\right)} = 0. \tag{IV.7.16}$$

We obtain for the left-hand side of (IV.7.16)

$$\left(B_{2n+1}(0) + B_{2n+1}\left(\frac{1}{2}\right)\right) \sum_{\ell_2 \pmod k} e^{-2\pi i \frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0.$$

This finally proves the theorem.

□

IV.7.3 Proof of Theorem IV.1.1

We are now ready to prove a refined version of Theorem IV.1.1.

Theorem IV.7.3.(1) *The function $\widehat{F}_1 : \mathbb{Q} \to \mathbb{C}$ defined by $\widehat{F}_1(\frac{h}{k}) := F_1(e^{2\pi i \frac{ph}{k}})$ is a depth two quantum modular form of weight one for Γ_p with multiplier $(\frac{-3}{d})$.*

(2) *The function $\widehat{F}_2 : \mathbb{Q} \to \mathbb{C}$ defined by $\widehat{F}_2(\frac{h}{k}) := F_2(e^{2\pi i \frac{ph}{k}})$ is a depth two quantum modular form of weight two for Γ_p with multiplier $(\frac{3}{d})$.*

Proof. (1) We have, by Theorem IV.7.1,

$$\widehat{F}_1\left(\frac{h}{k}\right) = \lim_{t \to 0^+} F_1\left(e^{2\pi i \frac{ph}{k} - t}\right) = a_{hp_1, \frac{k}{p_2}}(0) = \lim_{t \to 0^+} \mathbb{E}_1\left(-\frac{h}{k} + \frac{it}{2\pi}\right),$$

where $p_1 := p/\gcd(k,p)$, $p_2 := \gcd(k,p)$. Proposition IV.5.2 then gives the claim.

(2) Theorem IV.7.2 gives

$$\widehat{F}_2\left(\frac{h}{k}\right) = \lim_{t\to 0^+} F_2\left(e^{2\pi i\frac{ph}{k}-t}\right) = b_{hp_1,\frac{k}{p_2}}(0) = \lim_{t\to 0^+} \mathbb{E}_2\left(-\frac{h}{k}+\frac{it}{2\pi}\right).$$

Proposition IV.5.3 then gives the claim. □

Remark 13. For odd d, we have that $(\frac{3}{d}) = (\frac{-3}{d}) = 1$ if and only if $d \equiv 1 \pmod{12}$ so that both F_1 and F_2 can be viewed as quantum modular forms with the trivial character under a suitable subgroup of Γ_p (e.g. the principal congruence subgroup $\Gamma(12p)$).

IV.8 Completed indefinite theta functions

In this section, we embed the double Eichler integrals in a modular context by viewing them as "purely non-holomorphic" parts of indefinite theta series.

IV.8.1 Weight one

The functions E_2 and M_2 were introduced in [1], where they played a crucial role in understanding modular indefinite theta functions of signature $(j,2)$ ($j \in \mathbb{N}_0$). We consider the quadratic form $Q_1(\boldsymbol{n}) := \frac{1}{2}\boldsymbol{n}^T A_1 \boldsymbol{n}$ and the bilinear form $B_1(\boldsymbol{n},\boldsymbol{m}) := \boldsymbol{n}^T A_1 \boldsymbol{m}$ given by $A_1 := \begin{pmatrix} 6&3&6&3\\3&2&3&2\\6&3&0&0\\3&2&0&0\end{pmatrix}$, and define $A_0 := \left(\begin{smallmatrix}6&3\\3&2\end{smallmatrix}\right)$, $P_0(\boldsymbol{n}) := M_2(\sqrt{3};\sqrt{3}\,(2n_1+n_2),n_2)$ and, for $\boldsymbol{n} \in \mathbb{R}^4$, set

$$\begin{aligned} P(\boldsymbol{n}) := {} & M_2\left(\sqrt{3};\sqrt{3}(2n_3+n_4),n_4\right) \\ & + (\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1))\,(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)) \\ & + (\operatorname{sgn}(n_4)+\operatorname{sgn}(n_2))\,M_1\left(\sqrt{3}(2n_3+n_4)\right) + (\operatorname{sgn}(n_3)+\operatorname{sgn}(n_1))\,M_1(3n_3+2n_4)\,. \end{aligned}$$

Note that, for $\boldsymbol{\alpha} \in \mathscr{S}^*$,

$$2\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) = \Theta_{-A_0,P_0,\boldsymbol{\alpha}}(\tau).$$

We view this function as "purely non-holomorphic" part of the indefinite theta function

$$\Theta_{A_1,P,\boldsymbol{a}}(\tau) = \sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4} P\left(\sqrt{v}\boldsymbol{n}\right) q^{Q_1(\boldsymbol{n})}, \tag{IV.8.1}$$

where $\boldsymbol{a} \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$ with $(a_3,a_4) = (\alpha_1,\alpha_2)$. One can either employ Section 4.3 of [1] or proceed directly (as we do here) to prove the following proposition.

Proposition IV.8.1. *Assume that $\boldsymbol{a} \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$ with $a_1, a_2 \notin \mathbb{Z}$.*

1. *We have*

$$\Theta_{A_1,P^-,\boldsymbol{a}}(\tau) = 2\mathcal{E}_{1,(a_3,a_4)}(\tau)\Theta_{A_0,1,(a_1-a_3,a_2-a_4)}(\tau),$$

   where

$$P^-(\boldsymbol{n}) := M_2\left(\sqrt{3}; \sqrt{3}\left(2n_3+n_4\right), n_4\right).$$

2. *The functions $\Theta_{A_1,P,\boldsymbol{a}}$ and $\Theta_{-A_0,P_0,(a_3,a_4)}$ converge absolutely and locally uniformly.*

3. *The function $\tau \mapsto \Theta_{A_1,P,a}(p\tau)$ transforms like a modular form of weight two for some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and some character.*

Remark 14. When considering indefinite theta functions of signature $(j, 2)$, one usually obtains four M_2-terms as the purely "non-holomorphic" part. The arguments of these four M_2-functions are dictated by the holomorphic part. The fact that $(1,0,0,0)^T$ and $(0,1,0,0)^T$ (which correspond to n_1 and n_2 occuring in P) have norm zero with respect to A_1^{-1} causes the "missing" M_2-terms to vanish. Therefore we refer to this situation as a *double null limit* (see [1]).

Proof of Proposition IV.8.1. (1) Shifting $(n_1, n_2, n_3, n_4) \mapsto (n_1 - n_3, n_2 - n_4, n_3, n_4)$ on the left hand side of the identity gives the claim.

(2) For $\Theta_{-A_0,P_0,(a_3,a_4)}$ we employ the asymptotic given in (IV.2.7), to obtain

$$\left|M_2\left(\sqrt{3}; \sqrt{3v}\left(2n_1+n_2\right), \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^T A_0 \boldsymbol{n}}\right| \leq \frac{e^{-\pi\left(3(2n_1+n_2)^2+n_2^2\right)v}}{\pi^2 n_1 n_2} e^{\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v}$$
$$\leq c_1 e^{-2\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v} e^{\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v} = c_1 e^{-\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v}$$

for some $c_1 \in \mathbb{R}^+$ and $(n_1, n_2) \in (a_3, a_4) + \mathbb{Z}^2$ with $n_1, n_2 \neq 0$. By plugging in the definition, one can show that for some $c_2 \in \mathbb{R}^+$ and $n = (0, n_2) \in (a_3, a_4) + \mathbb{Z}^2$

$$\left|M_2\left(\sqrt{3}; \sqrt{3v}n_2, \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^T A_0 \boldsymbol{n}}\right| \leq c_2 e^{-\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v}$$

(and similarly for the case $n_2 = 0$). Using that A_0 is positive definite, we obtain, for some $c_3 \in \mathbb{R}^+$

$$\sum_{\boldsymbol{n} \in (a_3,a_4)+\mathbb{Z}^2} \left|M_2\left(\sqrt{3}; \sqrt{3v}\left(2n_1+n_2\right), \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^T A_0 \boldsymbol{n}}\right| \leq c_3 \sum_{\boldsymbol{n} \in (a_3,a_4)+\mathbb{Z}^2} e^{-\pi \boldsymbol{n}^T A_0 \boldsymbol{n} v} < \infty,$$

implying the absolute and locally uniform convergence of $\Theta_{-A_0,P_0,(a_3,a_4)}$. Combining this with (1) and the convergence of the positive definite theta series $\Theta_{A_1,1,(a_1-a_3,a_2-a_4)}$, we obtain absolute and locally uniform convergence of the M_2-part of $\Theta_{A_1,P,\boldsymbol{a}}$.

For the part containing only sign-terms

$$\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4} \left(\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1)\right)\left(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)\right) q^{Q_1(\boldsymbol{n})}, \tag{IV.8.2}$$

we consider the determinant of $\Delta_{A_1}(\boldsymbol{n},\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4)$, where $\left(\Delta_M(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_5)\right)_{j,\ell} := \boldsymbol{v}_j^T M \boldsymbol{v}_\ell$ and

$$(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4) := \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\\ 2 & -3 & -1 & 0\\ -3 & 6 & 0 & -3 \end{pmatrix}.$$

We compute the determinant $\det(\Delta_{A_1}(\boldsymbol{n},\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4))$ via Laplace expansion to obtain

$$\begin{aligned} e^{-\pi v Q_1(\boldsymbol{n})} &\le e^{-\pi\left(\frac{15}{16}B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+\frac{2}{9}B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+B_1(\boldsymbol{b}_1,\boldsymbol{n})B_1(\boldsymbol{b}_3,\boldsymbol{n})+2B_1(\boldsymbol{b}_2,\boldsymbol{n})B_1(\boldsymbol{b}_4,\boldsymbol{n})\right)v}\\ &\le e^{-c_4\left(B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+|B_1(\boldsymbol{b}_3,\boldsymbol{n})|+|B_1(\boldsymbol{b}_4,\boldsymbol{n})|\right)v} \end{aligned}$$

with some $c_4\in\mathbb{R}^+$ for all $\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4$ which satisfy the condition

$$\left(\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1)\right)\left(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)\right)\neq 0.$$

Thus (IV.8.2) is dominated by

$$\begin{aligned} &\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4}\left|\left(\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1)\right)\left(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)\right)e^{-\pi Q_1(\boldsymbol{n})v}\right|\\ &\qquad\le 4\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4} e^{-c_4\left(B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+|B_1(\boldsymbol{b}_3,\boldsymbol{n})|+|B_1(\boldsymbol{b}_4,\boldsymbol{n})|\right)v}<\infty. \end{aligned}$$

To deal with the contribution of the third and fourth summand of P one combines the approaches of the two previous terms.

(3) We use Lemma IV.2.1 to rewrite P as a limit of E_2-functions, namely

$$P(\boldsymbol{n}) = \lim_{\varepsilon\to 0}\widehat{P}_\varepsilon(\boldsymbol{n}),$$

where

$$\begin{aligned} \widehat{P}_\varepsilon(\boldsymbol{n}) := \Bigg(& E_2\left(\frac{\varepsilon}{3};\sqrt{3}\left(2n_3+n_4\right),-\varepsilon\left(n_1+n_3+\frac{n_4}{\sqrt{3}}\right)+\frac{3n_2}{\varepsilon\left(2\sqrt{3}-3\right)}\right)\\ &+E_2\left(\frac{\varepsilon}{2};\left(3n_3+2n_4\right),\frac{3n_1}{\varepsilon\left(2\sqrt{3}-3\right)}-\varepsilon\left(n_2+\sqrt{3}n_3+n_4\right)\right) \end{aligned}$$

$$+ E_2\left(\sqrt{3};\sqrt{3}\left(2n_3+n_4\right),n_4\right)$$
$$+ E_2\left(-\sqrt{3};\frac{n_2}{2\varepsilon}-\frac{\varepsilon}{2}\left(n_2+2n_4\right),\frac{\sqrt{3}}{2\varepsilon}\left(2n_1+n_2\right)-\frac{\sqrt{3}}{2}\varepsilon\left(2n_1+n_2+4n_3+2n_4\right)\right)\Bigg).$$

One can then verify that each occuring term $E_2(\kappa; b^T n, c^T n)$ satisfies the Vignéras differential equation given in Theorem IV.2.3 with $\lambda = 0$ and $A = A_1$. A straightforward calculation shows that the Vignéras differential equation is satisfied for $\widehat{P}_\varepsilon$ with respect to A_1 if and only if it is satisfied for $\widehat{P}_{\varepsilon,p}(\boldsymbol{n}) := \widehat{P}_\varepsilon(\sqrt{p}\boldsymbol{n})$ with respect to pA_1. Furthermore, we have

$$\Theta_{A_1,P,\boldsymbol{a}}(p\tau) = \Theta_{pA_1,P_p,\boldsymbol{a}}(\tau) = \lim_{\varepsilon\to 0}\Theta_{pA_1,\widehat{P}_{\varepsilon,p},\boldsymbol{a}}(\tau)$$

where $P_p(\boldsymbol{n}) := P(\sqrt{p}\boldsymbol{n})$. We can apply Theorem IV.2.3 to obtain weight 2 modularity of $\Theta_{pA_1,\widehat{P}_{\varepsilon,p},\boldsymbol{a}}$ since $\boldsymbol{a} \in (pA_1)^{-1}\mathbb{Z}^4$. Now, taking the limit $\varepsilon \to 0$ proves the claim.

□

IV.8.2 Completion: weight two

Similarly as in the previous Section IV.8.1, the function $\mathbb{E}_2$ may be related to a modular object of weight three. This connection becomes evident when writing $\mathbb{E}_2$ as a Jacobi derivative as in Lemma IV.6.2. We leave the details to the reader.

IV.8.3 Lowering

The indefinite theta series considered in Subsection IV.8.1 are higher depth harmonic Maass forms following Zagier-Zwegers. Roughly speaking, by this we mean that applying the *Maass lowering operator* $L := -2iv^2\frac{\partial}{\partial\overline{\tau}}$ makes the function simpler. In particular, for the iterated Eichler integral, we have

$$L\left(I_{f_1,f_2}(\tau)\right) = 2^{k_1}v^{k_1}f_1\left(-\overline{\tau}\right)I_{f_2}(\tau).$$

Now $v^{k_1}f_1\left(-\overline{\tau}\right)$ is v^{k_1} times a conjugated modular form of weight k_1 (so transforming of weight $-k_1$) and I_{f_2}, defined in (IV.2.11), is the non-holomorphic part of an harmonic Maass form of weight $2-k_2$.

IV.9 Conclusion and further questions

We conclude here with several comments and research directions

(1) We plan to more systematically study higher depth quantum modular forms and to describe explicitly the quantum S-modular matrix of $F(q)$. This requires a modification of several arguments used here for $F_2(q)$ (note that we restricted ourselves to Γ_p out of necessity). This result would allow us to make a more precise connection between $W(p)_{A_2}$ and its irreducible modules. For one, we should be able to associate an S-matrix to the set of atypical irreducible $W(p)_{A_2}$-characters, in parallel to [8].

(2) Iterated (or multiple) Eichler integrals studied in Section 5 are of independent interest. As in other theories dealing with iterated integrals (e.g. non-commutative modular symbols, Chen's integrals and multiple zeta-values) shuffle relations are expected to play an important role. Another goal worth pursuing is to connect iterated Eichler integrals of half-integral weights to Manin's work [19].

(3) We plan to investigate the asymptotic of $F(q)$ in terms of finite q-series evaluated at root of unity. This requires certain hypergeometric type formulas for double rank two false theta functions.

(4) In recent work [4] we found a new expression for the error of modularity appearing in Propositions IV.5.2 and IV.5.3, at least if $M\tau = -\frac{1}{\tau}$. Our formulae involve what we end up calling, "double Mordell" integrals. In the rank one case this connection is well-understood [28, Theorem 1.16].

(5) Very recently, W. Yuasa [24] gave an explicit formula for the *tail* of $(2, 2p)$-torus links associated to the sequence of colored Jones polynomials: $J_{n\omega_j}(K, q)$, $n \in \mathbb{N}$, where ω_j, $j = 1, 2$ are the fundamental weights. We were able to identify the same tail as a summand of $F(q)$, up to the factor $1 - q$ (viz. extract the "diagonal" $m_1 = m_2$ in formula (IV.1.7)). This raises the following question: Is it true that $F(q)$ is the tail of $J_{n\rho}(K, q)$, $(n \in \mathbb{N})$ (here $\rho = \omega_1 + \omega_2$), up to a rational function of q? For related computations of tails colored with $\mathfrak{sl}_3$ representations see [13].

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Chapter V

Vector-valued higher depth quantum modular forms and higher Mordell integrals

This chapter is based on a manuscript published in *Journal of Mathematical Analysis and Applications* and is joint work with Prof. Dr. Kathrin Bringmann and Prof. Dr. Antun Milas [BKM2].

V.1 Introduction and statement of results

V.1.1 Mordell integrals and quantum modular forms

The Mordell integral is usually defined as a function of two variables

$$h(z) = h(z;\tau) := \int_{\mathbb{R}} \frac{\cosh(2\pi z w)}{\cosh(\pi w)} e^{\pi i \tau w^2} dw, \tag{V.1.1}$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the complex upper half-plane. Integrals of this form were studied by many mathematicians including Kronecker, Lerch, Ramanujan, Riemann, Siegel, and of course Mordell, who proved that a whole family of integrals reduces to (V.1.1). From these works it is also known that (V.1.1) occurs as the "error of modularity" of Lerch sums which have the shape ($q := e^{2\pi i\tau}$)

$$\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n z_1} q^{\frac{n^2+n}{2}}}{1 - e^{2\pi i z_2} q^n} \quad (z_1, z_2 \in \mathbb{C} \setminus \{0\}).$$

The Mordell integral plays an important role in the theory of mock modular forms as shown by Zwegers in his remarkable thesis [20]. Zwegers wrote the integrals in (V.1.1) as Eichler integrals. To be more precise, he showed that, for $a, b \in (-\frac{1}{2}, \frac{1}{2})$ we have

$$h(a\tau - b) = -e^{-2\pi i a\left(b+\frac{1}{2}\right)} q^{\frac{a^2}{2}} \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(w)}{\sqrt{-i(w+\tau)}} dw, \tag{V.1.2}$$

where, for $a, b \in \mathbb{R}$, $g_{a,b}$ is the weight $\frac{3}{2}$ unary theta function defined by

$$g_{a,b}(\tau) := \sum_{n \in a+\mathbb{Z}} n e^{2\pi i b n} q^{\frac{n^2}{2}}.$$

Zwegers then used (V.1.2) to find a completion of Lerch sums, by observing that the error of modularity $h(a\tau - b)$ also appears from integrals which have $-\overline{\tau}$ instead of 0 as the lower integration boundary.

Starting with influential work of Zagier [18, 19], many authors studied related constructions with Eichler integrals from the perspective of quantum modular forms. In all of these examples the non-holomorphic part (or "companion") is given as

$$\int_{-\overline{\tau}}^{i\infty} \frac{f(w)}{(-i(\tau + w))^{\frac{3}{2}}} dw,$$

where f is a cuspidal theta function of weight $\frac{1}{2}$ or $\frac{3}{2}$.

The main motivation for this paper is to extend this well-known connection between Eichler and Mordell integrals to higher dimensions by using multiple integrals. We provide several explicit examples of this connection in the context of higher depth quantum modular forms introduced by the authors in [4] (see also [1]).

V.1.2 Vertex algebras and modular invariance of characters

Another, somewhat unrelated, motivation for this project comes from the study of characters in non-rational conformal field theories, where the modularity (or lack thereof) plays an important role.

There is already a growing body of research exploring the modularity of characters beyond the rational vertex operator algebras [6–9, 16]. One general feature of these irrational theories is that they admit typical modules (labelled by a continuous parameter) and atypical modules (parametrized by a discrete set which is mostly infinite). When it comes to modular transformation properties, the S-transformation (with $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$) of a character may produce both typical and atypical characters. So we expect that

$$\mathrm{ch}[M]\left(-\frac{1}{\tau}\right) = \int_{\Omega} S_{M,\nu} \mathrm{ch}[M_\nu](\tau) d\nu + \sum_{j \in \mathcal{D}} \alpha_{M,j} \mathrm{ch}[M_j](\tau), \qquad \text{(V.1.3)}$$

where $\mathrm{ch}[M_j]$ are atypical and $\mathrm{ch}[M_\nu]$ are typical characters. Note that the typical characters often have the form $\mathrm{ch}[M_\nu] = \frac{q^{\frac{\nu^2}{2}}}{\eta(\tau)^m}$, where $\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1}(1 - q^n)$ is Dedekind's η-function. Moreover Ω and $\mathcal{D}$ are domains parametrizing typicals and atypicals representations, respectively. The reader should exercise caution here – in some examples formulas like (V.1.3) only exist as formal distributions [8]. Also, as divergent integrals might appear, it is sometimes necessary to introduce additional variables (as in [6]) to ensure convergence.

This type of generalized modularity is known to hold for characters of certain representations of the affine Lie superalgebras $\widehat{\mathrm{sl}(n|1)}$ for $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras at admissible levels [13, 16]. In this work atypical characters transform as in (V.1.3) such that the integral part is a Mordell-type integral, which is essentially a consequence of Zwegers' thesis [20].

In this paper we take a slightly different point of view. As many important (algebraic, analytic and categorical) properties of rational vertex algebras are captured by the entries of the S-matrix (e.g. quantum dimensions, fusion rules), we expect that the full asymptotic expansion of characters and their quotients play a pivotal role for irrational theories. More precisely, we believe that these higher coefficients in the asymptotics determine the "fusion variety" via resummation and regularization. The latter was introduced by Creutzig and the third author [6]. As shown in [3], considerations of asymptotic expansion of characters naturally lead to quantum modular forms.

We now explain, with an example, how the concept of quantum modular forms can be used to obtain (V.1.3). For this we consider the $(1,p)$-singlet algebra and its characters. As explained in [6, 9], this vertex algebra admits typical and atypical representations. The characters of atypical representations $M_{r,s}$ are essentially false theta functions. To be more precise, for $1 \leq s \leq p-1$ and $r \in \mathbb{Z}$, we have

$$\mathrm{ch}[M_{r,s}](\tau) = \frac{1}{\eta(\tau)} \sum_{n \geq 0} \left(q^{\frac{1}{4p}(2pn-s-pr+2p)^2} - q^{\frac{1}{4p}(2pn+s-pr+2p)^2} \right).$$

Two of the authors have already proven in [3] that these characters are mixed quantum modular forms, in the sense that $\mathcal{M}_{r,s}(\tau) := \eta(\tau)\mathrm{ch}[M_{r,s}](\tau)$ is a weight $\frac{1}{2}$ quantum modular form whose companion (expressed as an Eichler integral) agrees with the original false theta function to all orders when expanding at roots of unity. This allows us to transfer modularity questions for characters to better behaved companions $\mathcal{M}^*_{r,s}$ as illustrated in the following

Example V.1.1. *For $1 \leq s \leq p-1$, we have*

$$\mathcal{M}^*_{1,s}(\tau) - \frac{1}{\sqrt{-i\tau}} \sqrt{\frac{2}{p}} \sum_{k=1}^{p-1} \sin\left(\frac{\pi k(p-s)}{p}\right) \mathcal{M}^*_{1,k}\left(-\frac{1}{\tau}\right) = i\sqrt{2p} \cdot r_{f_{p-s,p}}(\tau), \qquad \text{(V.1.4)}$$

where $r_{j_{j,p}}$ is the theta integral defined in (V.3.2) *for the theta function* (V.3.1)*. Note that $r_{f_{j,p}}$ also has the following representation as Mordell integral*

$$r_{f_{j,p}}(\tau) = -\int_{\mathbb{R}} \cot\left(\pi i w + \frac{\pi j}{2p}\right) e^{2\pi i p w^2 \tau} dw.$$

As typical characters take the form $\frac{e^{2\pi p i\tau x^2}}{\eta(\tau)}$*, the right-hand side of (V.1.4) can be viewed as the contribution from typical representations as in (V.1.3). For* $\mathrm{ch}[M_{r,s}]$ *with* $r \neq 1$ *a finite* q*-series has to be added to* $\mathrm{ch}[M_{1,s}]$ *so that the above formula looks slightly more complicated (cf.* [3,6]*).*

It is desirable to extend the modularity result in (V.1.4) to "higher rank" W-algebras, where false theta functions of higher rank appear as characters [5]. It was already observed earlier [7] that a regularization procedure can be used to derive a more complicated version of (V.1.3) involving iterated integrals. As the theory of higher depth quantum modular forms also involves multiple integrals [4], it is tempting to conjecture that these characters combine into vector-valued higher depth quantum modular forms. In this paper, we prove an analogue of (V.1.4) this for the simplest nontrivial example coming from an $\mathfrak{sl}_3$ false theta function $F(q)$ which was studied recently in [4].

V.1.3 Quantum invariants of knots and 3-manifolds

As discussed above quantum modular forms are connected to various aspects of number theory including Maass forms. But originally they appeared in the pioneering work of Zagier (and Zagier-Lawrence) on unified quantum invariants of certain 3-manifolds [18,19]. In a recent work of Gukov, Pei, Putrov, and Vafa [12], the authors proposed new quantum invariants of certain 3–manifolds expressed as holomorphic q-series with integral coefficients. These invariants are in many examples sums of ordinary quantum modular forms. It is expected that more general 3-manifolds as well as SU(3) unified WRT invariants exhibit a more complicated higher depth quantum modularity. Understanding their error of modularity certainly requires a solid understanding of higher Mordell integrals.

V.1.4 Statement of results

Define

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right).$$

In [4] the authors decomposed this function as $F(q) = \frac{2}{p}F_1(q^p) + 2F_2(q^p)$ with F_1 and F_2 defined in (V.4.1) and (V.4.2), respectively. The function F_1 and F_2 turn out to have generalized quantum modular properties. This connection goes asymptotically via two-dimensional Eichler integrals. For instance, we showed in [4] that F_1 asymptotically agrees with an integral of the shape

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f(w_1,w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

where $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$ (χ_j are certain multipliers and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$). The modular properties of these integrals follow from the modularity of f which in turn gives quantum modular properties of F_1. We call the resulting functions higher depth quantum modular forms. Roughly speaking, depth two quantum modular forms satisfy, in the simplest case, the modular transformation property with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R), \tag{V.1.5}$$

where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real-analytic functions defined on $R \subset \mathbb{R}$. Clearly, we can construct examples of depth two simply by multiplying two (depth one) quantum modular forms. In this paper, we prove a vector-valued version which refines (VI.1.2). Roughly speaking, $f(M\tau)$ in (VI.1.2) is replaced by $\sum_{1 \le \ell \le N} \chi_{j,\ell}(M) f_\ell(M\tau)$ (see Definition 2 for the notation).

Objects of similar nature - not invariant under the action of the relevant group but instead they satisfy "higher order" functional equations - have already appeared in the literature. *Higher-order* modular forms constitute a natural extension of the notion of classical modular form and can be constructed using iterated integrals [10, 11]; see also [15]. They also appear in connection to percolation theory in mathematical physics [14].

We prove.

Theorem V.1.2. *The function F_1 is a component of a vector-valued depth two quantum modular form of weight one. The function F_2 is a component of a vector-valued quantum modular form of depth two and weight two.*

We next consider higher-dimensional Mordell integrals. Set, for $\boldsymbol{\alpha} \in \mathbb{R}^2$,

$$H_{1,\boldsymbol{\alpha}}(\tau) := -\sqrt{3} \int_0^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha}; \boldsymbol{w}) + \theta_2(\boldsymbol{\alpha}; \boldsymbol{w})}{\sqrt{-i(w_1 + \tau)}\sqrt{-i(w_2 + \tau)}} dw_2 dw_1,$$

where the theta functions θ_1 and θ_2 are defined in (V.4.4) and (V.4.5), respectively and where throughout the paper we write two-dimensional vectors in bold letters and their components using subscript.

Remark 15. The function $H_{1,\boldsymbol{\alpha}}$ occurs (basically) as the holomorphic error of modularity (see Proposition V.5.4). The remaining piece is itself already an Eichler integral.

Setting

$$\mathcal{F}_\alpha(x) := \frac{\sinh(2\pi x)}{\cosh(2\pi x) - \cos(2\pi\alpha)}, \qquad \mathcal{G}_\alpha(x) := \frac{\sin(2\pi\alpha)}{\cosh(2\pi x) - \cos(2\pi\alpha)},$$

we define

$$g_{1,\boldsymbol{\alpha}}(\boldsymbol{w}) := \begin{cases} 2\mathcal{G}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2) - 2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2) & \text{if } \alpha_1, \alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_0(w_1)\mathcal{F}_{\alpha_2}(w_2) + \frac{2}{\pi w_1}\mathcal{F}_{\alpha_2}\left(w_2 + \frac{3w_1}{2}\right) & \text{if } \alpha_1 \in \mathbb{Z},\ \alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_0(w_2) + \frac{2}{\pi w_2}\mathcal{F}_{\alpha_1}\left(w_1 + \frac{w_2}{2}\right) & \text{if } \alpha_1 \notin \mathbb{Z},\ \alpha_2 \in \mathbb{Z}. \end{cases}$$

Theorem V.1.3. *If α_1, α_2 are not both in $\mathbb{Z}$, then we have, with $Q(\boldsymbol{w}) := 3w_1^2 + w_2^2 + 3w_1w_2$*

$$H_{1,\boldsymbol{\alpha}}(\tau) = \int_{\mathbb{R}^2} g_{1,\boldsymbol{\alpha}}(\boldsymbol{w}) e^{2\pi i\tau Q(\boldsymbol{w})} dw_1 dw_2.$$

In particular, if $\alpha_j \notin \mathbb{Z}$ for $j = 1, 2$, then we have

$$H_{1,\boldsymbol{\alpha}}(\tau) = \int_{\mathbb{R}^2} \cot\left(\pi i w_1 + \pi\alpha_1\right) \cot\left(\pi i w_2 + \pi\alpha_2\right) e^{2\pi i\tau Q(\boldsymbol{w})} dw_1 dw_2.$$

Remark 16. Note that there is a related statement if $\alpha_1, \alpha_2 \in \mathbb{Z}$; however for the purpose of this paper it is not required.

Similarly, set

$$\begin{aligned} H_{2,\boldsymbol{\alpha}}(\tau) :=& \frac{\sqrt{3}i}{2\pi} \int_0^{i\infty} \int_{w_1}^{i\infty} \frac{2\theta_3(\boldsymbol{\alpha}; \boldsymbol{w}) - \theta_4(\boldsymbol{\alpha}; \boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1 \\ &+ \frac{\sqrt{3}i}{2\pi} \int_0^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_5(\boldsymbol{\alpha}; \boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}} dw_2 dw_1, \end{aligned}$$

where θ_3, θ_4, and θ_5 are theta functions defined in (V.4.8), (V.4.9), and (V.4.10), respectively. The function $H_{2,\boldsymbol{\alpha}}$ occurs in Proposition V.5.4.

Define the function $g_{2,\alpha}$ as follows:

$$g_{2,\boldsymbol{\alpha}}(\boldsymbol{w}) := \begin{cases} -2iw_2\left(\mathcal{G}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2) + \mathcal{F}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2)\right) & \text{if } \alpha_1 \notin \mathbb{Z}, \\ -2i\left(\mathcal{F}_0(w_1)\mathcal{G}^*_{\alpha_2}(w_2) - \frac{1}{\pi w_1}\mathcal{G}^*_{\alpha_2}\left(w_2 + \frac{3w_1}{2}\right)\right) & \text{if } \alpha_1 \in \mathbb{Z}, \end{cases}$$

where $\mathcal{G}^*_\alpha(x) := x\mathcal{G}_\alpha(x)$.

Theorem V.1.4. *We have*

$$H_{2,\boldsymbol{\alpha}}(\tau) = \int_{\mathbb{R}^2} g_{2,\boldsymbol{\alpha}}(\boldsymbol{w}) e^{2\pi i\tau Q(\boldsymbol{w})} dw_1 dw_2.$$

V.1.5 Organization of the paper

The paper is organized as follows. In Section 2, we recall some basic facts on theta functions, certain (generalized) error functions, quantum modular forms, and higher-dimensional quantum modular forms. Section 3 describes the one-dimensional situation, and Section 4 records our previous results in the two-dimensional case. In Section 5 we develop general vector-valued transformations which we then use for our specific situation. In Section 6 we represent the two theta integrals $H_{1,\boldsymbol{\alpha}}$ and $H_{2,\boldsymbol{\alpha}}$ as double Mordell integrals.

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V.2 Preliminaries

V.2.1 Theta function transformation

Define, for $\nu \in \{0,1\}$, $h \in \mathbb{Z}$, $N, A \in \mathbb{N}$ with $A|N$ and $N|hA$, the theta function studied, for example, by Shimura [11]

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z} \\ m\equiv h \pmod{N}}} m^\nu q^{\frac{Am^2}{2N^2}}.$$

We have the transformation property

$$\Theta_\nu\left(A,h,N;\tau\right) = (-i)^\nu(-i\tau)^{-\frac{1}{2}-\nu}A^{-\frac{1}{2}} \sum_{\substack{k \pmod{N} \\ Ak\equiv 0 \pmod{N}}} e\left(\frac{Akh}{N^2}\right)\Theta_\nu\left(A,k,N;-\frac{1}{\tau}\right). \tag{V.2.1}$$

Also note that if $h_1 \equiv h_2 \pmod{N}$

$$\begin{aligned} \Theta_\nu(A,h_1,N;\tau) &= \Theta_\nu(A,h_2,N;\tau), \\ \Theta_\nu(A,-h,N;\tau) &= (-1)^\nu\Theta_\nu(A,h,N;\tau), \\ \Theta_\nu(A,N-h,2N;\tau) &= (-1)^\nu\Theta_\nu(A,N+h,2N;\tau). \end{aligned} \tag{V.2.2}$$

V.2.2 Special functions

Following [20], define for $u \in \mathbb{R}$

$$E(u) := 2\int_0^u e^{-\pi w^2}dw.$$

We have the representation

$$E(u) = \operatorname{sgn}(u) \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \pi u^2\right)\right),$$

where $\Gamma(\alpha, u) := \int_u^\infty e^{-w} w^{\alpha-1} dw$ is the *incomplete gamma function* and where for $u \in \mathbb{R}$, we let

$$\operatorname{sgn}(u) := \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0. \end{cases}$$

Moreover, for $u \neq 0$, set

$$M(u) := \frac{i}{\pi} \int_{\mathbb{R} - iu} \frac{e^{-\pi w^2 - 2\pi i u w}}{w} dw.$$

We have

$$M(u) = E(u) - \operatorname{sgn}(u).$$

We next turn to two-dimensional analogues, following [1], however using a slightly different notation. Setting $\boldsymbol{dw} := dw_1 dw_2$, define $E_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$E_2(\kappa; \boldsymbol{u}) := \int_{\mathbb{R}^2} \operatorname{sgn}(w_1) \operatorname{sgn}(w_2 + \kappa w_1) e^{-\pi\left((w_1-u_1)^2 + (w_2-u_2)^2\right)} \boldsymbol{dw}.$$

Moreover for $u_2, u_1 - \kappa u_2 \neq 0$:

$$M_2(\kappa; \boldsymbol{u}) := -\frac{1}{\pi^2} \int_{\mathbb{R} - iu_2} \int_{\mathbb{R} - iu_1} \frac{e^{-\pi w_1^2 - \pi w_2^2 - 2\pi i(u_1 w_1 + u_2 w_2)}}{w_2(w_1 - \kappa w_2)} \boldsymbol{dw}. \tag{V.2.3}$$

Then we have

$$\begin{aligned} M_2(\kappa; \boldsymbol{u}) &= E_2(\kappa; \boldsymbol{u}) - \operatorname{sgn}(u_2) M(u_1) \\ &\quad - \operatorname{sgn}(u_1 - \kappa u_2) M_1\left(\frac{u_2 + \kappa u_1}{\sqrt{1+\kappa^2}}\right) - \operatorname{sgn}(u_1) \operatorname{sgn}(u_2 + \kappa u_1). \end{aligned} \tag{V.2.4}$$

Note that (V.2.4) extends the definition of M_2 to $u_2 = 0$ or $u_1 = \kappa u_2$.

V.2.3 Vector-valued quantum modular forms

We next recall vector-valued quantum modular forms for the modular group.

Definition V.2.1. An N-tuple $\boldsymbol{f} = (f_1, \ldots, f_N)$ of functions $f_j : \mathbb{Q} \to \mathbb{C}$ for $1 \leq j \leq N$ is called a *vector-valued quantum modular form of weight* $k \in \frac{1}{2}\mathbb{Z}$*, multiplier* $\boldsymbol{\chi} = (\chi_{j,\ell})_{1\leq j,\ell\leq N}$, if for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the error of modularity

$$f_j(\tau) - (c\tau + d)^{-k} \sum_{1\leq \ell \leq N} \chi_{j,\ell}(M) f_\ell(M\tau) \tag{V.2.5}$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there. We denote the vector space of such forms by $\mathcal{Q}_k(\chi)$.

Remark 17. Since the matrices $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{Z})$, it is enough to check (V.2.5) for these matrices.

V.2.4 Higher depth vector-valued quantum modular forms

We next introduce vector-valued higher depth quantum modular forms. Note that higher depth quantum modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ were considered in [4].

Definition V.2.2. An N-tuple $\boldsymbol{f} = (f_1, \ldots, f_N)$ of functions $f_j : \mathbb{Q} \to \mathbb{C}$ with $1 \leq j \leq N$ is called a *vector-valued quantum modular form of depth* $P \in \mathbb{N}$*, weight* $k \in \frac{1}{2}\mathbb{Z}$*, multiplier* $\boldsymbol{\chi} = (\chi_{j,\ell})_{1\leq j,\ell\leq N}$, if for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\left(f_j(\tau) - (c\tau + d)^{-k} \sum_{1\leq \ell\leq N} \chi_{\ell,j}(M) f_\ell(M\tau) \right)_{1\leq j\leq N} \in \sum_m \mathcal{Q}^{P-1}_{\kappa_m}(\chi_m)\mathcal{O}(R),$$

where m runs through a finite set, $\kappa_m \in \frac{1}{2}\mathbb{Z}$, χ_m are rank N multipliers, $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$ which contains an open subset of $\mathbb{R}$, $\mathcal{Q}^1_k(\chi) := \mathcal{Q}_k(\chi)$, $\mathcal{Q}^0_k(\chi) := 1$, and $\mathcal{Q}^P_k(\chi)$ denotes the space of vector-valued forms of weight k, depth P, and multiplier $\boldsymbol{\chi}$.

V.3 The one-dimensional case

Recall the classical false theta functions ($1 \leq j \leq p-1, p \geq 2$),

$$F_{j,p}(\tau) := \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2p}}} \mathrm{sgn}(n) q^{\frac{n^2}{4p}}.$$

The following theorem is shown in [3, 9] (note that here we renormalized in comparison to [4])

Theorem V.3.1. *The functions* $F_{j,p} : \mathbb{H} \to \mathbb{C}$ $(1 \leq j \leq p-1)$ *form a vector-valued quantum modular form.*

Proof. (Sketch) Define the *non-holomorphic Eichler integral*

$$F^*_{j,p}(\tau) := \frac{1}{\sqrt{\pi}} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2p}}} \operatorname{sgn}(n)\Gamma\left(\frac{1}{2}, \frac{\pi n^2 v}{p}\right) q^{-\frac{n^2}{4p}}.$$

Note that $F_{j,p}(it+\frac{h}{k})$ and $F^*_{j,p}(it-\frac{h}{k})$ agree asymptotically to infinite order. That is, if we write

$$F_{j,p}\left(it+\frac{h}{k}\right) \sim \sum_{m\geq 0} a_{h,k}(m)t^m \quad \left(t\to 0^+\right),$$

then

$$F^*_{j,p}\left(it-\frac{h}{k}\right) \sim \sum_{m\geq 0} a_{h,k}(m)(-t)^m \quad \left(t\to 0^+\right).$$

One may then show that

$$F^*_{j,p}(\tau) = -i\sqrt{2p}\cdot I_{f_{j,p}}(\tau),$$

where

$$f_{j,p}(z) := \frac{1}{2p} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2p}}} nq^{\frac{n^2}{4p}} \tag{V.3.1}$$

and for a holomorphic modular form from f of weight k, the *non-holomorphic Eichler integral* is

$$I_f(\tau) := \int_{-\overline{\tau}}^{i\infty} \frac{f(w)}{(-i(\tau+w))^{2-k}}dw.$$

Using (V.2.1), one can prove that

$$f_{j,p}(\tau) = \sqrt{\frac{2}{p}}(-i\tau)^{-\frac{3}{2}} \sum_{k=1}^{p-1} \sin\left(\frac{\pi kj}{p}\right) f_{k,p}\left(-\frac{1}{\tau}\right),$$

correcting a sign-error in [9]. From this one may conclude that

$$F^*_{j,p}(\tau) - \frac{1}{\sqrt{-i\tau}}\sqrt{\frac{2}{p}} \sum_{k=1}^{p-1} \sin\left(\frac{\pi kj}{p}\right) F^*_{k,p}\left(-\frac{1}{\tau}\right) = i\sqrt{2p}\cdot r_{f_{j,p}}(\tau),$$

where, for f a holomorphic modular form of weight k,

$$r_f(\tau) := \int_0^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}}dw. \tag{V.3.2}$$

The claim now follows since $r_{f_{j,p}}$ is real-analytic on $\mathbb{R}$. □

The next lemma writes the "error to modularity" as an Eichler integral. Following the approach of Zwegers [20] and using trigonometric identities, one finds the following.

Lemma V.3.2. *We have*

$$\begin{aligned}-i\sqrt{2p}\cdot r_{f_{j,p}}(\tau) &= \int_{\mathbb{R}} \cot\left(\pi i w + \frac{\pi j}{2p}\right) e^{2\pi i p\tau w^2} dw \\ &= \sin\left(\frac{\pi j}{p}\right)\frac{1}{2}\int_{\mathbb{R}} \frac{e^{2\pi i p\tau w^2}}{\sinh\left(\pi w + \frac{\pi i j}{2p}\right)\sinh\left(\pi w - \frac{\pi i j}{2p}\right)} dw.\end{aligned}$$

V.4 Previous results in the two-dimensional case

In this section, we recall the results from [4]. In that paper the following decomposition was shown

$$F(q) = \frac{2}{p}F_1\left(q^p\right) + 2F_2\left(q^p\right)$$

with

$$F_1(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})} + \frac{1}{2}\sum_{m\in\mathbb{Z}} \operatorname{sgn}\left(m+\frac{1}{p}\right) q^{\left(m+\frac{1}{p}\right)^2}, \tag{V.4.1}$$

where

$$\mathscr{S} := \left\{\left(1-\frac{1}{p},\frac{2}{p}\right), \left(\frac{1}{p},1-\frac{2}{p}\right), \left(1,\frac{1}{p}\right)\left(0,1-\frac{1}{p}\right), \left(\frac{1}{p},1-\frac{1}{p}\right), \left(1-\frac{1}{p},\frac{1}{p}\right)\right\},$$

and for $\boldsymbol{\alpha} \pmod{\mathbb{Z}^2}$, we set

$$\varepsilon(\boldsymbol{\alpha}) := \begin{cases} -2 & \text{if } \boldsymbol{\alpha} \in \left\{\left(1-\frac{1}{p},\frac{2}{p}\right), \left(\frac{1}{p},1-\frac{2}{p}\right)\right\}, \\ 1 & \text{otherwise.}\end{cases}$$

Moreover

$$F_2(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \eta(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2 q^{Q(\boldsymbol{n})} - \frac{1}{2}\sum_{m\in\mathbb{Z}} \left|m+\frac{1}{p}\right| q^{\left(m+\frac{1}{p}\right)^2}, \tag{V.4.2}$$

where for $\boldsymbol{\alpha} \pmod{\mathbb{Z}^2}$, we let

$$\eta(\boldsymbol{\alpha}) := \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in \left\{\left(1-\frac{1}{p},\frac{2}{p}\right), \left(0,1-\frac{1}{p}\right), \left(\frac{1}{p},1-\frac{1}{p}\right)\right\}, \\ -1 & \text{otherwise.}\end{cases}$$

In [4] the following theorem was shown.

Theorem V.4.1. *For $p \geq 2$, the functions F_1 and F_2 are quantum modular forms of depth two with quantum set $\mathbb{Q}$ and of weight one and weight two, respectively.*

Sketch of proof. Using the Euler-Maclaurin summation formula, it was shown in [4] that the higher rank false theta functions asymptotically equal double Eichler integrals. To be more precise, write

$$F_1\left(e^{2\pi i \frac{h}{k} - t}\right) \sim \sum_{m \geq 0} A_{h,k}(m) t^m \quad \left(t \to 0^+\right).$$

In [4], we proved that we have, for $h, k \in \mathbb{Z}$ with $k > 0$ and $\gcd(h,k) = 1$,

$$\mathbb{E}_1\left(\frac{it}{2\pi} - \frac{h}{k}\right) \sim \sum_{m \geq 0} A_{h,k}(m)(-t)^m \quad \left(t \to 0^+\right). \tag{V.4.3}$$

Here the double Eichler integral $\mathbb{E}_1$ is given as follows: Define for $\boldsymbol{\alpha} \in \mathscr{S}^* := \{(1 - \frac{1}{p}, \frac{2}{p}), (0, 1 - \frac{1}{p}), (\frac{1}{p}, 1 - \frac{1}{p})\}$

$$\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) := -\frac{\sqrt{3}}{4} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha}; \boldsymbol{w}) + \theta_2(\boldsymbol{\alpha}; \boldsymbol{w})}{\sqrt{-i(w_1 + \tau)}\sqrt{-i(w_2 + \tau)}} dw_2 dw_1$$

with

$$\theta_1(\boldsymbol{\alpha}; \boldsymbol{w}) := \sum_{n \in \alpha + \mathbb{Z}^2} (2n_1 + n_2) n_2 e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}}, \tag{V.4.4}$$

$$\theta_2(\boldsymbol{\alpha}; \boldsymbol{w}) := \sum_{\boldsymbol{n} \in \boldsymbol{\alpha} + \mathbb{Z}^2} (3n_1 + 2n_2) n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}. \tag{V.4.5}$$

Then set

$$\mathcal{E}_1(\tau) := \sum_{\boldsymbol{\alpha} \in \mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \mathcal{E}_{1,\boldsymbol{\alpha}}(p\tau), \qquad \mathbb{E}_1(\tau) := \mathcal{E}_1\left(\frac{\tau}{p}\right). \tag{V.4.6}$$

The double Eichler integral $\mathcal{E}_1$ satisfies modular transformation properties. To be more precise, we have, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_p$ (some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$),

$$\mathcal{E}_1(\tau) - \left(\frac{-3}{d}\right)(c\tau + d)^{-1} \mathcal{E}_1(M\tau) = \sum_{j=1}^{2} \left(r_{f_j, g_j, \frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j, \frac{d}{c}}(\tau)\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the extended Jacobi symbol, f_j, g_j are cusp forms of weight $\frac{3}{2}$ (with some multiplier), and for holomorphic modular forms f_1 and f_2 of weights κ_1 and κ_2, respectively, we set

$$r_{f_1,f_2,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}} dw_2 dw_1,$$

$$r_{f_1,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \frac{f_1(w)}{(-i(w+\tau))^{2-\kappa_1}} dw.$$

The situation is similar for F_2. To be more precise, writing

$$F_2\left(e^{2\pi i \frac{h}{k} - t}\right) \sim \sum_{m \geq 0} B_{h,k}(m) t^m \qquad \left(t \to 0^+\right),$$

we proved in [4] that we have, for $h, k \in \mathbb{Z}$ with $k > 0$ and $\gcd(h,k) = 1$,

$$\mathbb{E}_2\left(\frac{it}{2\pi} - \frac{h}{k}\right) \sim \sum_{m \geq 0} B_{h,k}(m)(-t)^m \qquad \left(t \to 0^+\right). \tag{V.4.7}$$

Here the Eichler integral $\mathbb{E}_2$ is given as follows: Define for $\boldsymbol{\alpha} \in \mathscr{S}^*$

$$\begin{aligned}\mathcal{E}_{2,\boldsymbol{\alpha}}(\tau) :=& \frac{\sqrt{3}}{8\pi} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{2\theta_3(\boldsymbol{\alpha}; \boldsymbol{w}) - \theta_4(\boldsymbol{\alpha}; \boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1 \\ &+ \frac{\sqrt{3}}{8\pi} \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_5(\boldsymbol{\alpha}; \boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}} dw_2 dw_1\end{aligned}$$

with

$$\theta_3(\boldsymbol{\alpha}; \boldsymbol{w}) := \sum_{n \in \boldsymbol{\alpha} + \mathbb{Z}^2} (2n_1+n_2) e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}}, \tag{V.4.8}$$

$$\theta_4(\boldsymbol{\alpha}; \boldsymbol{w}) := \sum_{\boldsymbol{n} \in \boldsymbol{\alpha} + \mathbb{Z}^2} (3n_1+2n_2) e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}, \tag{V.4.9}$$

$$\theta_5(\boldsymbol{\alpha}; \boldsymbol{w}) := \sum_{\boldsymbol{n} \in \boldsymbol{\alpha} + \mathbb{Z}^2} n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}. \tag{V.4.10}$$

We then set

$$\mathcal{E}_2(\tau) := \sum_{\boldsymbol{\alpha} \in \mathscr{S}^*} \mathcal{E}_{2,\boldsymbol{\alpha}}(p\tau), \qquad \mathbb{E}_2(\tau) := \mathcal{E}_2\left(\frac{\tau}{p}\right). \tag{V.4.11}$$

Again one can show transformations for $\mathcal{E}_2$. Namely for $M \in \Gamma_p$, one has

$$\mathcal{E}_2(\tau) - \left(\frac{3}{d}\right)(c\tau + d)^{-2}\mathcal{E}_2(M\tau) = \sum_{j=1}^{4} \left(r_{f_j,g_j,\frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j,\frac{d}{c}}(\tau)\right),$$

where f_j and g_j are holomorphic modular forms of weight $\frac{1}{2}$ or cusp forms of weight $\frac{3}{2}$, respectively. $\square$

V.5 Higher depth Vector-valued transformations

V.5.1 General double Eichler integrals

We first describe the general situation. For this assume that f_j, g_ℓ $(1 \leq j \leq N,\ 1 \leq \ell \leq M)$ are components of vector-valued modular forms and in particular transform as (with $\kappa_1, \kappa_2 \in \frac{1}{2} + \mathbb{N}_0$)

$$f_j\left(-\frac{1}{\tau}\right) = (-i\tau)^{\kappa_1} \sum_{1\leq k\leq N} \chi_{j,k} f_k(\tau), \quad g_\ell\left(-\frac{1}{\tau}\right) = (-i\tau)^{\kappa_2} \sum_{1\leq m\leq M} \psi_{\ell,m} g_m(\tau). \tag{V.5.1}$$

Following [4], define the *double Eichler integral*

$$I_{f_j,g_\ell}(\tau) := \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f_j(w_1) g_\ell(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}} dw_2 dw_1.$$

Remark 18. Related, but different, iterated integrals were studied by Manin in his work on non-commutative modular symbols [15].

We prove the following transformation.

Lemma V.5.1. *We have the following two transformations*

$$I_{f_j,g_\ell}(\tau) - I_{f_j|T,g_\ell|T}(\tau+1) = 0, \tag{V.5.2}$$

$$I_{f_j,g_\ell}(\tau) - (-i\tau)^{\kappa_1+\kappa_2-4} \sum_{\substack{1\leq k\leq N\\ 1\leq m\leq M}} \chi_{j,k}\psi_{\ell,m} I_{f_k,g_m}\left(-\frac{1}{\tau}\right) \tag{V.5.3}$$

$$= \int_0^{i\infty} \int_{w_1}^{i\infty} \frac{f_j(w_1) g_\ell(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}} dw_2 dw_1 + I_{f_j}(\tau) r_{g_\ell}(\tau) - r_{f_j}(\tau) r_{g_\ell}(\tau),$$

where $|_\kappa$ denotes the usual weight k slash operator.

Proof. The transformation (V.5.2) is clear. To show (V.5.3), we first compute, using (V.5.1),

$$(-i\tau)^{\kappa_1+\kappa_2-4}\sum_{\substack{1\le k\le N\\ 1\le m\le M}}\chi_{j,k}\psi_{\ell,m}I_{f_k,g_m}\left(-\frac{1}{\tau}\right)\\ =\int_{-\overline{\tau}}^{0}\int_{w_1}^{0}\frac{f_j(w_1)g_\ell(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}}dw_2dw_1.$$

Employing the splitting

$$\int_{-\overline{\tau}}^{0}\int_{w_1}^{0}=\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}+\int_{0}^{i\infty}\int_{0}^{i\infty}-\int_{0}^{i\infty}\int_{w_1}^{i\infty}-\int_{-\overline{\tau}}^{i\infty}\int_{0}^{i\infty}$$

then directly gives the claim. □

V.5.2 The function $\mathcal{E}_1$

We first rewrite $\mathcal{E}_1$. For this define, for $k_1,k_2\in\mathbb{Z}$ with $k_1\equiv k_2 \pmod 2$,

$$J_{\boldsymbol{k}}(\tau):=\sum_{\delta\in\{0,1\}}I_{(k_1+\delta p,k_2+3\delta p)}(\tau)\quad\text{with}\quad I_{\boldsymbol{k}}(\tau):=-\frac{\sqrt{3}}{4p}I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_1(6p,k_2,6p,\cdot)}(\tau),$$

$$r_{\boldsymbol{k}}(\tau):=\int_0^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(2p,k_1,2p;w_1)\Theta_1(6p,k_2,6p;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

We have the following transformation properties.

Proposition V.5.2. *We have, for $\ell_1\equiv\ell_2 \pmod 2$,*

$$J_{\boldsymbol{\ell}}(\tau)=-\frac{1}{\sqrt{3}p(-i\tau)}\sum_{\substack{k_1 \pmod p\\ k_2 \pmod{6p}\\ k_1\equiv k_2 \pmod 2}}\zeta_{2p}^{k_1\ell_1}\zeta_{6p}^{k_2\ell_2}J_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right)-\frac{\sqrt{3}}{4p}\sum_{\delta\in\{0,1\}}r_{(k_1+p\delta,k_2+3p\delta)}(\tau)\\ -\frac{\sqrt{3}}{4p}\sum_{\delta\in\{0,1\}}\left(I_{\Theta_1(2p,\ell_1+p\delta,2p;\,\cdot\,)}(\tau)-r_{\Theta_1(2p,\ell_1+p\delta,2p;\,\cdot\,)}(\tau)\right)r_{\Theta_1(6p,\ell_2+3p\delta,6p;\,\cdot\,)}(\tau),$$

where $\zeta_j:=e^{\frac{2\pi i}{j}}$.

Proof. Using (V.2.1) gives

$$\Theta_1\left(2p,a,2p;-\frac{1}{\tau}\right)=-i(-i\tau)^{\frac{3}{2}}(2p)^{-\frac{1}{2}}\sum_{k \pmod{2p}}\zeta_{2p}^{ka}\Theta_1(2p,k,2p;\tau),\tag{V.5.4}$$

$$\Theta_1\left(6p, a, 6p; -\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{3}{2}}(6p)^{-\frac{1}{2}} \sum_{k \pmod{6p}} \zeta_{6p}^{ka}\Theta_1(6p, k, 6p; \tau).$$

Thus by Lemma V.5.1, we obtain that $J_{\boldsymbol{\ell}}(\tau)$ equals

$$-\frac{1}{2\sqrt{3}p(-i\tau)} \sum_{\delta\in\{0,1\}} \sum_{\substack{k_1 \pmod{2p} \\ k_2 \pmod{6p}}} \zeta_{2p}^{k_1(\ell_1+p\delta)}\zeta_{6p}^{k_2(\ell_2+3p\delta)} I_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right) - \frac{\sqrt{3}}{4p} \sum_{\delta\in\{0,1\}} r_{(\ell_1+p\delta,\ell_2+3p\delta)}(\tau)$$
$$-\frac{\sqrt{3}}{4p} \sum_{\delta\in\{0,1\}} \left(I_{\Theta_1(2p,\ell_1+p\delta,2p;\cdot)}(\tau) - r_{\Theta_1(2p,\ell_1+p\delta,2p;\cdot)}(\tau)\right) r_{\Theta_1(6p,\ell_2+3p\delta,6p;\cdot)}(\tau).$$

To prove the proposition, we are left to simplify the first term. For this, we write

$$\sum_{\delta\in\{0,1\}} \sum_{\substack{k_1 \pmod{2p} \\ k_2 \pmod{6p}}} (-1)^{\delta(k_1+k_2)}\zeta_{2p}^{\ell_1 k_1}\zeta_{6p}^{\ell_2 k_2} I_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right) = 2 \sum_{\substack{k_1 \pmod{2p} \\ k_2 \pmod{6p} \\ k_1\equiv k_2 \pmod{2}}} \zeta_{2p}^{\ell_1 k_1}\zeta_{6p}^{\ell_2 k_2} I_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right).$$

Making the change of variables $k_1 \mapsto k_1 + p\delta$, $k_2 \mapsto k_2 + 3p\delta$ yields that this equals

$$2 \sum_{\substack{k_1 \pmod{p} \\ k_2 \pmod{6p} \\ k_1\equiv k_2 \pmod{2}}} \sum_{\delta\in\{0,1\}} \zeta_{2p}^{(k_1+p\delta)\ell_1}\zeta_{6p}^{(k_2+3p\delta)\ell_2} I_{(k_1+p\delta,k_2+3p\delta)}\left(-\frac{1}{\tau}\right) = 2 \sum_{\substack{k_1 \pmod{2p} \\ k_2 \pmod{6p} \\ k_1\equiv k_2 \pmod{2}}} \zeta_{2p}^{\ell_1 k_1}\zeta_{6p}^{\ell_2 k_2} J_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right).$$

□

To find transformation properties to use for $\mathcal{E}_1$, we write it as a J-function.

Lemma V.5.3. *We have*

$$\mathcal{E}_1(\tau) = J_{(1,3)}(\tau). \tag{V.5.5}$$

Proof. As in the proof of Proposition 5.2 of [4] we see that

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha})\,\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) = \frac{1}{p^2}\sum_{\boldsymbol{A}\in\mathcal{A}} \varepsilon_1(\boldsymbol{A})\,\Theta_1\left(2p, A_1, 2p; \frac{3w_1}{p}\right)\Theta_1\left(2p, A_2, 2p; \frac{w_2}{p}\right)$$

with

$$\mathcal{A} :=\{(0,2), (p,p+2), (p-1,p-1), (-1,-1), (p+1,p-1), (1,-1)\},$$
$$\varepsilon_1(\boldsymbol{A}) :=\varepsilon\left(\frac{A_1-A_2}{2p}, \frac{A_2}{p}\right).$$

Using (V.2.2), it is not hard to prove that this sum vanishes.

Similarly

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon\left(\boldsymbol{\alpha}\right)\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) = \frac{1}{p^2}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2\left(\boldsymbol{B}\right)\Theta_1\left(2p,B_1,2p;\frac{w_1}{p}\right)\Theta_1\left(2p,B_2,2p;\frac{3w_2}{p}\right) \tag{V.5.6}$$

with

$$\begin{aligned}\mathcal{B} :=& \{(p+1,p-1),(1,-1),(p+2,p),(2,0),(1,1),(p+1,p+1)\},\\ \varepsilon_2(\boldsymbol{B}) :=& \varepsilon\left(\frac{B_2-3B_1}{2p},\frac{B_1}{p}\right).\end{aligned}$$

Using again (V.2.2) and $\Theta_1(2p,h,2p;3\tau)=\frac{1}{3}\Theta_1(6p,3h,6p;\tau)$, one obtains that (V.5.6) equals

$$\frac{1}{p^2}\sum_{\delta\in\{0,1\}}\Theta_1\left(2p,1+\delta p,2p;\frac{w_1}{p}\right)\Theta_1\left(6p,3+3\delta p,6p;\frac{w_2}{p}\right).$$

This yields the claim by (V.4.6). □

Proposition V.5.2 then implies the following transformation for $\mathcal{E}_1$.

Corollary V.5.4. *We have*

$$\begin{aligned}\mathcal{E}_1(\tau) =& -\frac{1}{\sqrt{3}p(-i\tau)}\sum_{\substack{k_1 \pmod p\\ k_2 \pmod{6p}\\ k_1\equiv k_2 \pmod 2}}\zeta_{2p}^{k_1+k_2}J_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right)+\frac{1}{4}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}(\tau)\\ &-\frac{\sqrt{3}}{4p}\sum_{\delta\in\{0,1\}}\left(I_{\Theta_1(2p,1+p\delta,2p;.)}(\tau)-r_{\Theta_1(2p,1+p\delta,2p;.)}(\tau)\right)r_{\Theta_1(6p,3+3p\delta,6p;.)}(\tau).\end{aligned}$$

Proof. We use Proposition V.5.2 with $\ell_1=1$ and $\ell_2=3$ and reversing the calculation used to show (V.5.5), we obtain that the second term equals $\frac{1}{4}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}(\tau)$. □

V.5.3 The function $\mathcal{E}_2$

We proceed in the same way as for $\mathcal{E}_1$. To rewrite $\mathcal{E}_2$, defined in (V.4.11), we set, for $k_1\equiv k_2 \pmod 2$,

$$\mathcal{K}_{\boldsymbol{k}}(\tau) := 2\mathcal{J}_{\boldsymbol{k}}(\tau)+\mathcal{J}_{\left(\frac{k_1+k_2}{2},\frac{k_2-3k_1}{2}\right)}(\tau),$$

where (note that we changed the normalization in comparison to [4])

$$\mathcal{J}_{\boldsymbol{k}}(\tau) := \sum_{\delta\in\{0,1\}} \mathcal{I}_{(k_1+p\delta,k_2+3p\delta)}(\tau), \quad \text{with} \quad \mathcal{I}_{\boldsymbol{k}}(\tau) := -\frac{\sqrt{3}}{8\pi} I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_0(6p,k_2,6p;\cdot)}(\tau).$$

Moreover set

$$R_{\boldsymbol{k}}(\tau) := \int_0^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(2p,k_1,2p;w_1)\Theta_0(6p,k_2,6p;w_2)}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1.$$

We have the following transformation law for the function $\mathcal{K}_\ell$.

Proposition V.5.5. *We have, for $\ell_1 \equiv \ell_2 \pmod 2$,*

$$\begin{aligned}\mathcal{K}_{\boldsymbol{\ell}}(\tau) = &\frac{i}{2\sqrt{3}p} \sum_{\substack{k_1 \pmod p\\ k_2 \pmod{6p}\\ k_1\equiv k_2 \pmod 2}} \zeta_{2p}^{k_1\ell_1}\zeta_{6p}^{k_2\ell_2}\mathcal{K}_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right)\\
&- \frac{\sqrt{3}}{8\pi}\sum_{\delta\in\{0,1\}}\left(2R_{(k_1+p\delta,k_2+3p\delta)}(\tau) + R_{\left(\frac{k_1+k_2}{2}+p\delta,\frac{k_2-3k_1}{2}+3p\delta\right)}(\tau)\right)\\
&- \frac{\sqrt{3}}{8\pi}\sum_{\delta\in\{0,1\}}\Bigg(2\left(I_{\Theta_1(2p,\ell_1+p\delta,2p;.)}(\tau) - r_{\Theta_1(2p,\ell_1+p\delta,2p;.)}(\tau)\right) r_{\Theta_0(6p,\ell_2+3p\delta,6p;.)}(\tau)\\
&+ \left(I_{\Theta_1\left(2p,\frac{\ell_1+\ell_2}{2}+p\delta,2p;.\right)}(\tau) - r_{\Theta_1\left(2p,\frac{\ell_1+\ell_2}{2}+p\delta,2p;.\right)}(\tau)\right) r_{\Theta_0\left(6p,\frac{\ell_2-3\ell_1}{2}+3p\delta,6p;.\right)}(\tau)\Bigg).\end{aligned}$$

Proof. Using (V.5.4) and

$$\Theta_0\left(6p,a,6p;-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\frac{1}{\sqrt{6p}}\sum_{k \pmod{6p}} \zeta_{6p}^{ka}\Theta_0(6p,k,6p;\tau),$$

Proposition V.5.7 gives that $\mathcal{K}_{\ell_1,\ell_2}(\tau)$ equals

$$\begin{aligned}&\sum_{\substack{k_1 \pmod{2p}\\ k_2 \pmod{6p}}}\left(2\zeta_{2p}^{k_1(\ell_1+p\delta)}\zeta_{6p}^{k_2(\ell_2+3p\delta)} + 2\zeta_{2p}^{k_1\left(\frac{\ell_1+\ell_2}{2}+p\delta\right)}\zeta_{6p}^{k_2\left(\frac{\ell_2-3\ell_1}{2}+3p\delta\right)}\right)\frac{i\mathcal{I}_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right)}{16p\pi(-i\tau)^2}\\
&- \frac{\sqrt{3}}{8\pi}\sum_{\delta\in\{0,1\}}\left(2R_{(\ell_1+p\delta,\ell_2+3p\delta)}(\tau) + R_{\left(\frac{\ell_1+\ell_2}{2}+p\delta,\frac{\ell_2-3\ell_1}{2}+3p\delta\right)}(\tau)\right)\end{aligned}$$

$$-\frac{\sqrt{3}}{8\pi}\sum_{\delta\in\{0,1\}}\left(2\left(I_{\Theta_1(2p,\ell_1+p\delta,2p;.)}(\tau)-r_{\Theta_1(2p,\ell_1+p\delta,2p;.)}(\tau)\right)r_{\Theta_0(6p,\ell_2+3p\delta,6p;\cdot)}(\tau)\right.$$
$$\left.+\left(I_{\Theta_1\left(2p,\frac{\ell_1+\ell_2}{2}+p\delta,2p;.\right)}(\tau)-r_{\Theta_1\left(2p,\frac{\ell_1+\ell_2}{2}+p\delta,2p;.\right)}(\tau)\right)r_{\Theta_0(6p,\ell_2+3p\delta,6p;\cdot)}(\tau)\right).$$

We are left to simplify the first term. As in the proof of Proposition 5.3 the sum on k_1, k_2 equals

$$2\sum_{\substack{k_1 \pmod p\\ k_2 \pmod{6p}\\ k_1\equiv k_2 \pmod 2}}\left(2\zeta_{2p}^{k_1+k_2}+\zeta_p^{k_1}\right)\mathcal{J}_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right).$$

In the contribution from the second term, we change k_1 into $\frac{k_1+k_2}{2}$ and k_2 into $\frac{k_2-3k_1}{2}$ giving the claim. □

We next write $\mathcal{E}_2$ in terms of the $\mathcal{K}$-functions.

Lemma V.5.6. *We have*

$$\mathcal{E}_2(\tau)=\mathcal{K}_{(1,3)}(\tau). \tag{V.5.7}$$

Proposition 5.5 yields the following transformation for $\mathcal{E}_2$.

Corollary V.5.7. *We have*

$$\mathcal{E}_2(\tau)=\frac{i}{8\pi p(-i\tau)^2}\sum_{\substack{k_1 \pmod p\\ k_2 \pmod{6p}\\ k_1\equiv k_2 \pmod 2}}\zeta_{2p}^{k_1+k_2}\mathcal{K}_{\boldsymbol{k}}\left(-\frac{1}{\tau}\right)+\frac{i}{4}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}H_{2,\boldsymbol{\alpha}}(\tau)$$
$$-\frac{\sqrt{3}}{8\pi}\sum_{\delta\in\{0,1\}}\left(2\left(I_{\Theta_1(2p,1+p\delta,2p;\cdot)}(\tau)-r_{\Theta_1(2p,1+p\delta,2p;\cdot)}(\tau)\right)r_{\Theta_0(6p,3+3p\delta,6p;\cdot)}(\tau)\right.$$
$$\left.-\left(I_{\Theta_1(2p,2+p\delta,2p;\cdot)}(\tau)-r_{\Theta_1(2p,1+p\delta,2p;\cdot)}(\tau)\right)r_{\Theta_0(6p,3p\delta,6p;\cdot)}(\tau)\right).$$

Proof. The claim follows from Proposition V.5.5. Reversing the calculations required for the proof of (V.5.7) yields that the second summand equals $\frac{i}{4}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}H_{2,\boldsymbol{\alpha}}(\tau)$. □

V.5.4 Proof Theorem V.1.2

We are now ready to prove a refined version of Theorem V.1.2.

Theorem V.5.8.(1) *The function $\widehat{F}_1:\mathbb{Q}\to\mathbb{C}$ defined by $\widehat{F}_1(\frac{h}{k}):=F_1(e^{2\pi i\frac{ph}{k}})$ is a component of a vector-valued quantum modular form of depth two and weight one.*

(2) *The function* $\widehat{F}_2 : \mathbb{Q} \to \mathbb{C}$ *defined by* $\widehat{F}_2(\frac{h}{k}) := F_2(e^{2\pi i \frac{ph}{k}})$ *is a component of a vector-valued quantum modular form of depth two and weight two.*

Proof. (1) We have, by (V.4.3),

$$\widehat{F}_1\left(\frac{h}{k}\right) = \lim_{t\to 0^+} F_1\left(e^{2\pi i\frac{ph}{k}-t}\right) = A_{hp_1,\frac{k}{p_2}}(0) = \lim_{t\to 0^+} \mathbb{E}_1\left(\frac{it}{2\pi} - \frac{h}{k}\right),$$

where $p_1 := p/\gcd(k,p)$, $p_2 := \gcd(k,p)$. Corollary V.5.4 and Proposition V.5.2 then give the claim.

(2) The relation (V.4.7) gives

$$\widehat{F}_2\left(\frac{h}{k}\right) = \lim_{t\to 0^+} F_2\left(e^{2\pi i\frac{ph}{k}-t}\right) = B_{hp_1,\frac{k}{p_2}}(0) = \lim_{t\to 0^+} \mathbb{E}_2\left(\frac{it}{2\pi} - \frac{h}{k}\right).$$

Corollary 5.4 and Proposition V.5.5 then yields the claim. □

V.6 Higher Mordell integrals

V.6.1 Proof of Theorem V.1.3

Proof of Theorem V.1.3. We first assume that $\alpha_j \notin \mathbb{Z}$. Via analytic continuation, it is enough to show the theorem for $\tau = iv$. We first claim that

$$H_{1,\boldsymbol{\alpha}}(iv) = 2\lim_{r\to\infty} \sum_{\substack{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2\\ |n_j-\alpha_j|\le r}} M_2\left(\sqrt{3};\sqrt{\frac{v}{2}}\left(\sqrt{3}\left(2n_1+n_2\right),n_2\right)\right)e^{2\pi Q(\boldsymbol{n})v}. \tag{V.6.1}$$

For this we write (which follows from shifting in (6.1) of [4] $w_j \mapsto 2iw_j - \overline{\tau}$)

$$\begin{aligned} &e^{4\pi Q(\boldsymbol{n})v} M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)\\ &\qquad = \sqrt{3}\left(2n_1+n_2\right)n_2 \int_0^\infty \frac{e^{-3\pi(2n_1+n_2)^2w_1}}{\sqrt{w_1+v}} \int_{w_1}^\infty \frac{e^{-\pi n_2^2 w_2}}{\sqrt{w_2+v}} dw_2 dw_1\\ &\qquad\quad + \sqrt{3}\left(3n_1+2n_2\right)n_1 \int_0^\infty \frac{e^{-\pi(3n_1+2n_2)^2w_1}}{\sqrt{w_1+v}} \int_{w_1}^\infty \frac{e^{-3\pi n_1^2 w_2}}{\sqrt{w_2+v}} dw_2 dw_1. \end{aligned}$$

Then we change $v \mapsto \frac{v}{2}$, sum over those $\boldsymbol{n} \in \boldsymbol{\alpha} + \mathbb{Z}^2$ satisfying $|n_j - \alpha_j| \le r$ and let $r \to \infty$. On the right-hand side we may use Lebesgue's dominated convergence theorem and can reorder the absolutely converging series inside the integral to obtain (V.6.1).

To finish the proof, we rewrite (V.2.3), to obtain (assuming $N_2, N_1 - \kappa N_2 \neq 0$)

$$M_2\left(\kappa; \sqrt{v}\boldsymbol{N}\right) = -\frac{1}{\pi^2} e^{-\pi v\left(N_1^2+N_2^2\right)} \int_{\mathbb{R}^2} \frac{e^{-\pi v w_1^2 - \pi v w_2^2}}{\left(w_2 - iN_2\right)\left(w_1 - \kappa w_2 - i\left(N_1 - \kappa N_2\right)\right)} \boldsymbol{dw}. \tag{V.6.2}$$

Thus in particular (for $n_1, n_2 \neq 0$)

$$\begin{aligned} &M_2\left(\sqrt{3}; \sqrt{\frac{v}{2}}\left(\sqrt{3}\left(2n_1+n_2\right), n_2\right)\right) \\ &\qquad = -\frac{1}{\pi^2} e^{-2\pi Q(\boldsymbol{n})v} \int_{\mathbb{R}^2} \frac{e^{-\frac{\pi v w_1^2}{2} - \frac{\pi v w_2^2}{2}}}{\left(w_2 - in_2\right)\left(w_1 - \sqrt{3}w_2 - 2\sqrt{3}in_1\right)} \boldsymbol{dw} \\ &\qquad = -\frac{1}{\pi^2} e^{-2\pi Q(\boldsymbol{n})v} \int_{\mathbb{R}^2} \frac{e^{-\frac{3\pi v\left(2w_1+w_2\right)^2}{2} - \frac{\pi v w_2^2}{2}}}{\left(w_2 - in_2\right)\left(w_1 - in_1\right)} \boldsymbol{dw}, \end{aligned}$$

making the change of variables $w_1 \mapsto 2\sqrt{3}w_1 + \sqrt{3}w_2$. This implies that

$$\begin{aligned} &\lim_{r\to\infty} \sum_{\substack{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2 \\ |n_j-\alpha_j|\le r}} M_2\left(\sqrt{3}; \sqrt{\frac{v}{2}}\left(\sqrt{3}\left(2n_1+n_2\right), n_2\right)\right) e^{2\pi Q(\boldsymbol{n})v} \\ &\qquad = -\frac{1}{\pi^2} \lim_{r\to\infty} \sum_{\substack{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2 \\ |n_j-\alpha_j|\le r}} \int_{\mathbb{R}^2} \frac{e^{-2\pi v Q(\boldsymbol{w})}}{\left(w_2 - in_2\right)\left(w_1 - in_1\right)} \boldsymbol{dw}. \end{aligned} \tag{V.6.3}$$

Using

$$\pi \cot(\pi x) = \lim_{r\to\infty} \sum_{k=-r}^{r} \frac{1}{x+k},$$

we obtain that the sum over the integrand (without the exponential factor) is

$$\begin{aligned} &-\lim_{r\to\infty} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^2 \\ |n_j|\le r}} \left(\frac{1}{iw_1+\alpha_1+n_1}\right)\left(\frac{1}{iw_2+\alpha_2+n_2}\right) \\ &\qquad = -\pi^2 \cot\left(\pi\left(iw_1+\alpha_1\right)\right) \cot\left(\pi\left(iw_2+\alpha_2\right)\right). \end{aligned}$$

Using again Lebesgue's theorem of dominated convergence, one can show that one can interchange the limit and the integration in (V.6.3) to obtain

$$H_{1,\boldsymbol{\alpha}}(\tau) = \int_{\mathbb{R}^2} \cot\left(\pi i w_1 + \pi\alpha_1\right) \cot\left(\pi i w_2 + \pi\alpha_2\right) e^{2\pi i \tau Q(\boldsymbol{w})} \boldsymbol{dw}.$$

Using

$$\cot(x+iy) = -\frac{\sin(2x)}{\cos(2x)-\cosh(2y)} + i\frac{\sinh(2y)}{\cos(2x)-\cosh(2y)}.$$

then yields,

$$H_{1,\boldsymbol{\alpha}}(\tau) = 2\int_{\mathbb{R}^2} \left(\mathcal{G}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2) - \mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2)\right) e^{2\pi i\tau Q(\boldsymbol{w})}\boldsymbol{dw}. \tag{V.6.4}$$

This gives the claim of Theorem V.1.3 in this case.

We next turn to the case that $\alpha_j \in \mathbb{Z}$ for exactly one $j \in \{1,2\}$. We only consider the case $\alpha_1 \in \mathbb{Z}$, since the case $\alpha_2 \in \mathbb{Z}$ goes analogously. Since the integrand in $H_{1,\boldsymbol{\alpha}}$ is invariant under $\alpha_j \mapsto \alpha_j + 1$, we may assume that $\alpha_1 = 0$. One directly sees from (V.6.4) that in this case

$$H_{1,(0,\alpha_2)}(\tau) = -2\lim_{\alpha_1\to 0}\int_{\mathbb{R}^2} \mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2)e^{2\pi i\tau Q(\boldsymbol{w})}\boldsymbol{dw}.$$

Using that $\mathcal{F}_0(-w_1) = -\mathcal{F}_0(w_1)$, we obtain

$$H_{1,(0,\alpha_2)}(\tau) = -\int_{\mathbb{R}^2} \mathcal{F}_0(w_1)\mathcal{F}_{\alpha_2}(w_2)e^{2\pi i\tau\left(3w_1^2+w_2^2\right)}\sum_{\pm} \pm e^{\pm 6\pi i\tau w_1w_2}\boldsymbol{dw}. \tag{V.6.5}$$

Now write

$$\mathcal{F}_0(w_1) = \left(\mathcal{F}_0(w_1) - \frac{1}{\pi w_1}\right) + \frac{1}{\pi w_1}.$$

The contribution of the first term to the integral now exists and gives, changing $w_1 \mapsto -w_1$ for the minus sign

$$-2\int_{\mathbb{R}^2}\left(\mathcal{F}_0(w_1) - \frac{1}{\pi w_1}\right)\mathcal{F}_{\alpha_2}(w_2)e^{2\pi i\tau Q(\boldsymbol{w})}\boldsymbol{dw}.$$

For the second term we write

$$\mathcal{F}_{\alpha_2}(w_2) = \left(\mathcal{F}_{\alpha_2}(w_2) - \mathcal{F}_{\alpha_2}\left(w_2 \pm \frac{3w_1}{2}\right)\right) + F_{\alpha_2}\left(w_2 \pm \frac{3w_1}{2}\right). \tag{V.6.6}$$

The first term in (V.6.6) contributes to (V.6.5), changing $w_1 \mapsto -w_1$ for the minus sign

$$-\frac{2}{\pi}\int_{\mathbb{R}^2} w_1^{-1}\left(\mathcal{F}_{\alpha_2}(w_2) - \mathcal{F}_{\alpha_2}\left(w_2 + \frac{3w_1}{2}\right)\right)e^{2\pi i\tau Q(\boldsymbol{w})}\boldsymbol{dw}.$$

For the final term in (V.6.6) we use that $3w_1^2+w_2^2\pm 3w_1w_2=(w_2\pm\frac{3w_1}{2})^2+\frac{3}{4}w_1^2$, to obtain

$$-\frac{1}{\pi}\int_{\mathbb{R}}\frac{e^{\frac{3\pi i\tau w_1^2}{2}}}{w_1}\int_{\mathbb{R}}\sum_{\pm}\pm\mathcal{F}_{\alpha_2}\left(w_2\pm\frac{3w_1}{2}\right)e^{2\pi i\tau\left(w_2\pm\frac{3w_1}{2}\right)^2}dw_2dw_1.$$

The inner integral on w_2 now vanishes, which may be seen by changing in the integral on w_2 for the minus sign $w_2\mapsto w_2+3w_1$. Combining, the theorem statement follows. □

V.6.2 Proof of Theorem V.1.4

Proof of Theorem V.1.4. From (6.3) and (6.4) of [4], one obtains that

$$\begin{aligned}
&\frac{1}{2\pi i}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi in_2z}\right)\right]_{z=0}e^{4\pi vQ(\boldsymbol{n})}\\
&=-\frac{\sqrt{3}}{2\pi}(2n_1+n_2)\int_0^\infty\frac{e^{-\frac{3\pi}{2}(2n_1+n_2)^2w_1}}{\sqrt{w_1+2v}}\int_{w_1}^\infty\frac{e^{-\frac{\pi}{2}n_2^2w_2}}{(w_2+2v)^{\frac{3}{2}}}dw_2dw_1\\
&\quad+\frac{\sqrt{3}}{4\pi}(3n_1+2n_2)\int_0^\infty\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{\sqrt{w_1+2v}}\int_{w_1}^\infty\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{(w_2+2v)^{\frac{3}{2}}}dw_2dw_1\\
&\quad-\frac{\sqrt{3}}{4\pi}n_1\int_0^\infty\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{(w_1+2v)^{\frac{3}{2}}}\int_{w_1}^\infty\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{\sqrt{w_2+2v}}dw_2dw_1.
\end{aligned}$$

Then we sum over $\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2$ satisfying $|n_j-\alpha_j|\le r$ and let $r\to\infty$. On the right hand side we use Lebesgue's dominated convergence theorem and can reorder the absolutely converging series inside the integral to obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{r\to\infty}\sum_{\substack{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2\\|n_j-\alpha_j|\le r}}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi in_2z}\right)\right]_{z=0}\\
&\qquad\times e^{4\pi vQ(\boldsymbol{n})}=\frac{1}{2i}H_{2,\boldsymbol{\alpha}}(2iv).
\end{aligned}$$

We now use (V.6.2) and change variables $w_1\mapsto 2\sqrt{3}w_1+\sqrt{3}w_2$, to obtain

$$\begin{aligned}
&M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi in_2z}\\
&=-\frac{1}{\pi^2}e^{-\pi v\left(3(2n_1+n_2)^2+\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)^2\right)+2\pi in_2z}
\end{aligned}$$

$$\times \int_{\mathbb{R}^2} \frac{e^{-\pi v w_1^2 - \pi v w_2^2}}{\left(w_2 - i\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right)\left(w_1 - \sqrt{3}w_2 - i\left(2\sqrt{3}n_1 + 2\sqrt{3}\frac{\mathrm{Im}(z)}{v}\right)\right)} \boldsymbol{dw}$$

$$= -\frac{1}{\pi^2} e^{-\pi v\left(3(2n_1+n_2)^2 + \left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)^2\right) + 2\pi i n_2 z}$$

$$\times \int_{\mathbb{R}^2} \frac{e^{-4\pi v Q(\boldsymbol{w})}}{\left(w_2 - i\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right)\left(w_1 - i\left(n_1 + \frac{\mathrm{Im}(z)}{v}\right)\right)} \boldsymbol{dw}.$$

Thus

$$\left[\frac{\partial}{\partial z} M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right]_{z=0} e^{4\pi v Q(\boldsymbol{n})}$$

$$= \frac{2}{\pi} \int_{\mathbb{R}^2} \frac{w_2 e^{-4\pi v Q(\boldsymbol{w})}}{(w_2 - i n_2)(w_1 - i n_1)} \boldsymbol{dw}.$$

Exactly as in the proof of Theorem V.1.3, we then obtain

$$\lim_{r\to\infty} \sum_{\substack{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2 \\ |n_j-\alpha_j|\le r}} \left[\frac{\partial}{\partial z} M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right]_{z=0} e^{4\pi v Q(\boldsymbol{n})}$$

$$= \frac{\pi}{\sqrt{3}} \int_{\mathbb{R}^2} w_2 e^{-4\pi v Q(\boldsymbol{w})} \cot(\pi i w_2 + \pi\alpha_2) \cot(\pi i w_1 + \pi\alpha_1) \boldsymbol{dw}.$$

Observing that on the right hand side the integral over the real part vanishes, gives

$$H_{2,\boldsymbol{\alpha}}(\tau) = -2i \int_{\mathbb{R}^2} w_2 \left(\mathcal{G}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2) + \mathcal{F}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2)\right) e^{2\pi i\tau Q(\boldsymbol{w})} \boldsymbol{dw}.$$

The case $\alpha_1 \notin \mathbb{Z}$ follows directly.

For $\alpha_1 \in \mathbb{Z}$, we obtain

$$H_{2,\boldsymbol{\alpha}}(\tau) = -2i \int_{\mathbb{R}^2} \mathcal{F}_0(w_1)\mathcal{G}^*_{\alpha_2}(w_2) e^{2\pi i\tau Q(\boldsymbol{w})} \boldsymbol{dw}.$$

Now the claim follows as in the proof of Theorem V.1.3. □

V.7 Future work

Here we discuss a few future directions. We also announce a result that will appear in full detail in our forthcoming work [5].

V.7.1 Further examples of rank two false theta functions

In addition to the function F studied in [4], there are additional rank two false theta functions studied by the first and third author in [5]. To be more precise, define, for $1 \leq s_1, s_2 \leq p$

$$\mathbb{F}_{s_1,s_2}(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(\left(m_1-\frac{s_1}{p}\right)^2+\left(m_2-\frac{s_2}{p}\right)^2+\left(m_1-\frac{s_1}{p}\right)\left(m_2-\frac{s_2}{p}\right)\right)}$$
$$\times\left(1-q^{m_1s_1}-q^{m_2s_2}+q^{m_1s_1+(m_1+m_2)s_2}+q^{m_2s_2+(m_1+m_2)s_1}-q^{(m_1+m_2)(s_1+s_2)}\right).$$

We will show in [5] that these series are also higher depth quantum modular forms with quantum set $\mathbb{Q}$. We believe that these series decompose into two vector-valued higher depth quantum modular forms of weight one and two.

V.7.2 Example: two-dimensional vector-valued quantum modular forms of depth two

The previous two-parametric family of rank two false theta functions $\mathbb{F}_{s_1,s_2}(q)$ takes a particularly nice shape for $p = 2$. In this case it can be shown that the only contribution comes from the weight one component and that only two false theta functions contribute. Their companions are double Eichler integrals with a basis

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\eta(w_1)^3\eta(3w_2)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$
$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\eta(w_1)^3\eta\left(\frac{w_2}{3}\right)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

This gives a two-dimensional vector-valued quantum modular form of depth two and weight one.

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Chapter VI

Some examples of higher depth vector-valued quantum modular forms

This chapter is based on a manuscript accepted for publication in the conference proceedings of the conference "Number Theory: Arithmetic, Diophantine and Transcendence" at the Indian Institute of Technology in Ropar celebrating the 130th birth anniversary of S. Ramanujan and is joint work with Prof. Dr. Kathrin Bringmann and Prof. Dr. Antun Milas [BKM3].

VI.1 Introduction and statement of results

For $p \in \mathbb{N}$, define the following $\mathfrak{sl}_3$ false theta function

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right).$$

This function was introduced in [3] as the numerator of the character of a certain W-algebra associated to $\mathfrak{sl}_3$. A more direct connection between the series and Lie theory can be readily seen from its coefficient $\min(m_1, m_2)$ - the value of Kostant's partition function of $\mathfrak{sl}_3$.

In [4] we decomposed F as

$$F(q) = \frac{2}{p} F_1\left(q^p\right) + 2F_2\left(q^p\right), \tag{VI.1.1}$$

where F_1 and F_2 are generalizations of quantum modular forms. Roughly speaking Zagier [12] defined *quantum modular forms* to be function $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subset \mathbb{Q}$) such that the "obstruction to modularity"

$$f(\tau) - (c\tau+d)^{-k} f(M\tau) \qquad M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$$

is “nice”. One can show quantum modular properties of the F_j by using two-dimensional Eichler integrals. For instance, as $\tau \to \frac{h}{k} \in \mathbb{Q}$, F_1 agrees with an integral of the shape ($q := e^{2\pi i\tau}$)

$$\int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f(\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1,$$

where $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$ (χ_j are certain multipliers and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$). Throughout we write vectors in bold letters and their components with subscripts. The modular properties of the integral in (VI.1.1) follow from the modularity of f which in turn gives quantum modular properties of F_1. We call the resulting functions higher depth quantum modular forms. Roughly speaking, *depth two quantum modular forms* satisfy, in the simplest case, the modular transformation property with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R), \tag{VI.1.2}$$

where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real-analytic functions defined on $R \subset \mathbb{R}$. In [5], we proved that F_1 and F_2 are components of vector-valued quantum modular forms of depth two, generalizing (VI.1.2).

A natural question that arises is what the other components of the vector-valued forms are as q-series. To investigate this, we define, for $1 \leq s_1, s_2 \leq p \in \mathbb{N}$,

$$\begin{aligned}\mathbb{F}_{\boldsymbol{s}}(q) := &\sum_{\substack{m_1,m_2 \geq 1 \\ m_1 \equiv m_2 \pmod 3}} \min(m_1, m_2) q^{\frac{p}{3}\left(\left(m_1 - \frac{s_1}{p}\right)^2 + \left(m_2 - \frac{s_2}{p}\right)^2 + \left(m_1 - \frac{s_1}{p}\right)\left(m_2 - \frac{s_2}{p}\right)\right)} \\ &\times \left(1 - q^{m_1 s_1} - q^{m_2 s_2} + q^{m_1 s_1 + (m_1+m_2)s_2} + q^{m_2 s_2 + (m_1+m_2)s_1} - q^{(m_1+m_2)(s_1+s_2)}\right).\end{aligned}$$

Note that $\mathbb{F}_{(1,1)}(q) = F(q)$. As discussed in [3] these series are in fact parametrized by dominant integral weights $(s_1 - 1)\omega_1 + (s_2 - 1)\omega_2$ for $\mathfrak{sl}_3$, where ω_j are fundamental weights (dual to simple roots α_1 and α_2).

We may decompose $\mathbb{F}_{\boldsymbol{s}}$ as in (VI.1.1) (see Lemma VI.2.1). The corresponding functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$ are again generalized quantum modular forms. More precisely, we have.

Theorem VI.1.1. *The functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$ are depth two quantum modular forms (with respect to some subgroup) of weights one and two, respectively.*

To prove Theorem VI.1.1, we show that $\mathbb{F}_{1,\boldsymbol{s}}(\tau)$ asymptotically agrees to infinite order with a certain Eichler integral $\mathcal{E}_{1,\boldsymbol{s}}(\frac{\tau}{p})$ defined in (VI.2.1). Similarly, $\mathbb{F}_{2,\boldsymbol{s}}(\tau)$ asymptotically agrees with an Eichler integral $\mathcal{E}_{2,\boldsymbol{s}}(\frac{\tau}{p})$ given in (VI.2.2).

We next restrict to the special case $p = 2$. It turns out (see Lemma VI.2.2) that for $p = 2$ all $\mathbb{F}_{2,\boldsymbol{s}}$ vanish. Thus we only need to consider $\mathbb{F}_{1,\boldsymbol{s}}$.

Theorem VI.1.2. *For $p = 2$, the space spanned by $\mathcal{E}_{1,(1,1)}$ and $\mathcal{E}_{1,(1,2)}$ is essentially invariant under modular transformations. By this we mean that the only terms appearing in the modular transformations which do not lie in the space are simpler (see* (VI.2.6) *and* (VI.2.7) *for the case of inversion).*

Motivated by representation theory of the W-algebra $W^0(p)_{A_2}$ studied in [3, 8], we raise the following.

Conjecture. *After multiplication with η^2, the characters of $W^0(p)_{A_2}$ given in* [3, Section 5] *(which also includes the series $\mathbb{F}_{\boldsymbol{s}}$) combine into a vector-valued quantum modular form of depth two.*

The second goal of this paper is to determine the asymptotic behavior of $\mathcal{E}_{1,\boldsymbol{s}}(it)$ as $t \to 0^+$. It is well-known that asymptotic behaviors of vector-valued modular forms (as $t \to 0^+$) can be computed by applying the S-transformation $\tau \mapsto -\frac{1}{\tau}$, and then analyzing the dominating term. This method is widely used for studying quantum dimensions of modules of vertex algebras (and affine Lie algebras) as their characters often transform invariantly under $\mathrm{SL}_2(\mathbb{Z})$. In this paper we work with functions (coming also from characters) that transform with higher depth error terms so their asymptotics are more interesting and harder to analyze. We show that asymptotic behavior of double Eichler integrals can be also analyzed by using a similar approach. We do this directly from the integral representation of the error function. In the body of the paper, we show that it is enough to study

$$\mathbb{E}_{1,(1,1)}(\tau) := 4I_{(1,3)}(\tau) \quad \text{and} \quad \mathbb{E}_{1,(1,2)}(\tau) := 2I_{(1,1)}(\tau) + 2I_{(1,5)}(\tau), \tag{VI.1.3}$$

where the theta integrals $I_{\boldsymbol{k}}$ are defined in (VI.2.3). We prove the following.

Theorem VI.1.3. *We have, as $t \to 0^+$,*

$$\mathbb{E}_{1,(1,1)}(it) \sim \frac{1}{4}, \qquad \mathbb{E}_{1,(1,2)}(it) \sim \frac{3}{4}.$$

Note that the asymptotics in Theorem VI.1.3 agree with the answer which one obtains from [5] by using two-dimensional false theta functions.

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VI.2 Proof of Theorem VI.1.1 and Theorem VI.1.2

To prove Theorem VI.1.1 and Theorem VI.1.2, we let

$$\mathbb{F}_{1,\boldsymbol{s}}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}} \varepsilon_{\boldsymbol{s}}(\alpha) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} q^{pQ(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where $Q(x_1, x_2) := 3x_1^2 + 3x_1x_2 + x_2^2$ and where

$$\begin{aligned}\mathscr{S}_{\boldsymbol{s}} := \Bigg\{ &\left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right),\\ &\left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right),\\ &\left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right),\\ &\left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right), \left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right), \left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right)\Bigg\},\end{aligned}$$

$$\varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha}) := \begin{cases} s_2 & \text{if } \boldsymbol{\alpha} \in \Big\{ \left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right),\\ & \qquad \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right) \Big\},\\ s_1 & \text{if } \boldsymbol{\alpha} \in \Big\{ \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right),\\ & \qquad \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right) \Big\},\\ -(s_1+s_2) & \text{if } \boldsymbol{\alpha} \in \Big\{ \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right),\\ & \qquad \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right) \Big\} \end{cases}$$

and

$$\mathbb{F}_{2,\boldsymbol{s}}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}} \eta_{\boldsymbol{s}}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where

$$\eta_{\boldsymbol{s}}(\boldsymbol{\alpha}) := \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in \Big\{ \left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \\ & \quad \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right) \Big\}, \\ -1 & \text{if } \boldsymbol{\alpha} \in \Big\{ \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right), \\ & \quad \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right), \left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right), \left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right) \Big\}. \end{cases}$$

Remark 19. We have

$$\mathbb{F}_{(p,p)}(q) = 1.$$

Thus we may throughout assume that $\boldsymbol{s} \neq (p,p)$.

Similarly as in the case $\boldsymbol{s} = (1,1)$, a lengthy calculation gives.

Lemma VI.2.1. *We have*

$$\mathbb{F}_{\boldsymbol{s}}(q) = \frac{1}{p}\mathbb{F}_{1,\boldsymbol{s}}\left(q^p\right) + \mathbb{F}_{2,\boldsymbol{s}}\left(q^p\right).$$

The following theorem states quantum modular properties of the functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$, using the method of [4]. Let

$$\mathcal{E}_{1,\boldsymbol{s}}(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha})\mathcal{E}_{1,\boldsymbol{\alpha}}(p\tau), \tag{VI.2.1}$$

where

$$\mathscr{S}_{\boldsymbol{s}}^* := \Bigg\{ \left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \\ \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right) \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right) \Bigg\}.$$

Moreover, the Eichler integrals $\mathcal{E}_{1,\boldsymbol{\alpha}}$ are given as

$$\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) := -\frac{\sqrt{3}}{4}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$

with

$$\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (2n_1+n_2)n_2 e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1+\frac{\pi i n_2^2 w_2}{2}},$$

$$\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (3n_1+2n_2)n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.$$

Finally let

$$\mathcal{E}_{2,\boldsymbol{s}}(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \mathcal{E}_{2,\boldsymbol{\alpha}}(p\tau). \tag{VI.2.2}$$

Here

$$\begin{aligned}\mathcal{E}_{2,\boldsymbol{\alpha}}(\tau) :=& \frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{2\theta_3(\boldsymbol{\alpha};\boldsymbol{w})-\theta_4(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1\\ &+ \frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_5(\boldsymbol{\alpha};\boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}} dw_2 dw_1\end{aligned}$$

with

$$\begin{aligned}\theta_3(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (2n_1+n_2) e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}},\\ \theta_4(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} (3n_1+2n_2) e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}},\\ \theta_5(\boldsymbol{\alpha};\boldsymbol{w}) &:= \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.\end{aligned}$$

Furthermore define, for $\nu\in\{0,1\}$, $h\in\mathbb{Z}$, $N, A\in\mathbb{N}$ with $A|N$ and $N|hA$, the theta function studied, for example, by Shimura [11]

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z}\\ m\equiv h \pmod{N}}} m^\nu q^{\frac{Am^2}{2N^2}}.$$

We are now ready to prove Theorem VI.1.1.

Proof of Theorem VI.1.1 (Sketch). We start with $\mathbb{F}_{1,\boldsymbol{s}}$. Write

$$\mathbb{F}_{1,\boldsymbol{s}}\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} A_{\boldsymbol{s},h,k}(m) t^m \quad \left(t\to 0^+\right).$$

Using the Euler-Maclaurin summation formula (in the shape stated in (28) of [4]) one can prove, following the proof of Theorem 7.1 of [4], that

$$\mathbb{E}_{1,\boldsymbol{s}}\left(\frac{it}{2\pi}-\frac{h}{k}\right) \sim \sum_{m\geq 0} A_{\boldsymbol{s},h,k}(m)(-t)^m \quad \left(t\to 0^+\right).$$

Here

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} M_2\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right) q^{-Q(\boldsymbol{n})},$$

where $\boldsymbol{w}\in\mathbb{R}^2$ and $\kappa\in\mathbb{R}$ with $w_2,\ w_1-\kappa w_2\neq 0$, we set

$$M_2(\kappa;\boldsymbol{w}) := -\frac{1}{\pi^2}\int_{\mathbb{R}^2-i\boldsymbol{w}} \frac{e^{-\pi t_1^2-\pi t_2^2-2\pi i(t_1w_1+t_2w_2)}}{t_2(t_2-\kappa t_1)}\, dt_1dt_2.$$

In particular, $\mathbb{E}_{1,s}$ agrees with $\mathbb{F}_{1,s}$ on $\mathbb{Q}$. Proceeding as in the proof of Lemma 6.1 of [4] one can then show that

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) = \mathcal{E}_{1,\boldsymbol{s}}\left(\frac{\tau}{p}\right).$$

To determine the transformation behaviour, we rewrite the theta functions in $\mathcal{E}_{1,\boldsymbol{s}}$ in terms of Shimura theta functions to obtain, as in the proof of Proposition 5.2 of [4]

$$3p\mathcal{E}_{1,\boldsymbol{s}}\left(\frac{\tau}{p}\right) = (2s_1+s_2)J_{(s_2,s_2+2s_1)}(\tau)+(2s_2+s_1)J_{(s_1,s_1+2s_2)}(\tau)+(s_2-s_1)J_{(s_1+s_2,s_1-s_2)}(\tau),$$

where

$$J_{\boldsymbol{k}}(\tau) := \sum_{\delta\in\{0,1\}} I_{(k_1+\delta p,k_2+3\delta p)}(\tau) \quad \text{with} \quad I_{\boldsymbol{k}}(\tau) := -\frac{\sqrt{3}}{4p} I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_1(6p,k_2,6p,\cdot)}(\tau). \tag{VI.2.3}$$

Here, for modular forms f and g of weights κ_1 and κ_2, respectively,

$$I_{f,g}(\tau) := \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f(w_1)g(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}} dw_2dw_1.$$

Now the transformation properties follow as in the proof of Proposition 5.2 of [5].

For the function $\mathbb{F}_{2,\boldsymbol{s}}$, we proceed in the same way. Writing

$$\mathbb{F}_{2,\boldsymbol{s}}\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} B_{\boldsymbol{s},h,k}(m)t^m \quad \left(t\to 0^+\right)$$

we may show in a similar manner as in the proof of Theorem 7.2 of [4], using the Euler-Maclaurin summation formula, that

$$\mathbb{E}_{2,\boldsymbol{s}}\left(\frac{it}{2\pi}-\frac{h}{k}\right) \sim \sum_{m\geq 0} B_{\boldsymbol{s},h,k}(m)(-t)^m.$$

Here

$$\mathbb{E}_2(\tau) = \mathbb{E}_{2,\boldsymbol{s}}(\tau) := \frac{1}{4\pi i} \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} \times \left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}\left(n_2 - \frac{2\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right)\right]_{z=0} q^{-Q(\boldsymbol{n})}.$$

Following the proof of Lemma 6.2 of [4], one may then prove that

$$\mathbb{E}_{2,\boldsymbol{s}}(\tau) = \mathcal{E}_{2,\boldsymbol{s}}\left(\frac{\tau}{p}\right).$$

To finish the proof one may show that, proceeding as in the proof of Proposition 5.2 of [4].

$$\mathbb{E}_{2,\boldsymbol{s}}(\tau) = \frac{2}{p}\left(-\mathcal{J}_{(s_1+s_2,s_1-s_2)}(\tau) + \mathcal{J}_{(s_2,2s_1+s_2)}(\tau) + \mathcal{J}_{(s_1,2s_2+s_1)}(\tau)\right),$$

where

$$\mathcal{J}_{\boldsymbol{k}}(\tau) := \sum_{\delta\in\{0,1\}} \mathcal{I}_{(k_1+p\delta,k_2+3p\delta)}(\tau), \quad \text{with} \quad \mathcal{I}_{\boldsymbol{k}}(\tau) := -\frac{\sqrt{3}}{8\pi} I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_0(6p,k_2,6p;\cdot)}(\tau).$$

Again the transformation properties follow as in the proof of Proposition 5.5 of [5]. □

We now restrict to $p = 2$. The following lemma shows the vanishing of $\mathbb{F}_{2,\boldsymbol{s}}$ in this case.

Lemma VI.2.2. *For $p = 2$, the functions $\mathbb{F}_{2,\boldsymbol{s}}$ and $\mathbb{E}_{2,\boldsymbol{s}}$ vanish identically.*

Proof. We start by proving that $\mathbb{F}_{2,\boldsymbol{s}} = 0$. It is enough to consider $\boldsymbol{s} \in \{(1,1),(1,2)\}$. The claim for $\boldsymbol{s} = (1,1)$ follows directly by plugging in the definition of $\mathbb{F}_{2,(1,1)}$ and canceling terms.

We next consider $\mathbb{F}_{2,(1,2)}$. By definition

$$\mathbb{F}_{2,(1,2)}(q) = \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,2)}} \eta_{(1,2)}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where

$$\eta_{(1,2)}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in \left\{\left(\frac{1}{6},0\right), \left(\frac{5}{6},\frac{1}{2}\right), \left(\frac{2}{3},\frac{1}{2}\right), \left(\frac{5}{6},0\right), \left(\frac{1}{6},\frac{3}{2}\right), \left(\frac{1}{3},\frac{3}{2}\right)\right\}, \\ -1 & \text{if } \boldsymbol{\alpha} \in \left\{\left(\frac{2}{3},-\frac{1}{2}\right), \left(\frac{5}{6},-\frac{1}{2}\right), \left(\frac{1}{6},1\right), \left(\frac{1}{3},\frac{1}{2}\right), \left(\frac{1}{6},\frac{1}{2}\right), \left(\frac{5}{6},1\right)\right\}. \end{cases}$$

Note that

$$\begin{aligned}H_{\boldsymbol{\alpha}}(q) &:= \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})} - \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2-1)\, q^{Q(\boldsymbol{n}+(\alpha_1,\alpha_2-1))}\\ &= (1-\alpha_2)\, q^{\frac{1}{4}(\alpha_2-1)^2} \sum_{n\in\alpha_1+\frac{\alpha_2-1}{2}+\mathbb{N}_0} q^{3n^2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{F}_{2,(1,2)}(q) &= -H_{\left(\frac{1}{6},1\right)}(q) + H_{\left(\frac{5}{6},\frac{1}{2}\right)}(q) + H_{\left(\frac{2}{3},\frac{1}{2}\right)}(q) - H_{\left(\frac{5}{6},1\right)}(q) + H_{\left(\frac{1}{6},\frac{3}{2}\right)}(q) + H_{\left(\frac{1}{3},\frac{3}{2}\right)}(q)\\ &= \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{7}{12}+\mathbb{N}_0} q^{3n^2} + \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{5}{12}+\mathbb{N}_0} q^{3n^2} - \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{5}{12}+\mathbb{N}_0} -\frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{7}{12}+\mathbb{N}_0} q^{3n^2} = 0.\end{aligned}$$

To see that $\mathbb{E}_{2,\mathbf{s}} = 0$, it is sufficient to prove

$$-\mathcal{J}_{(s_1+s_2,s_1-s_2)} + \mathcal{J}_{(s_2,2s_1+s_2)} + \mathcal{J}_{(s_1,2s_2+s_1)} = 0,$$

which is a straightforward computation with theta series. □

We are now ready to prove Theorem VI.1.2.

Sketch of proof of Theorem VI.1.2. We write

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) = -\frac{\sqrt{3}}{2}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1.$$

We next show the identities in (VI.1.3). We start with $\boldsymbol{s} = (1,1)$. We use the theta relation

$$\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,1)}^*} \varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right) = \frac{1}{2}\Theta_1(4,1,4;w_1)\Theta_1(12,3,12;w_2). \tag{VI.2.4}$$

Equation (VI.2.4) yields

$$\mathbb{E}_{1,(1,1)}(\tau) = -\frac{\sqrt{3}}{2}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(4,1,4;w_1)\Theta_1(12,3,12;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1 = 4I_{(1,3)}(\tau),$$

which is the first identity in (VI.1.3).

We next consider $\mathbb{E}_{1,(1,2)}$ and use that

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,2)}^*} \varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right) \tag{VI.2.5}$$

$$= \frac{1}{2}\Theta_1(4,1,4;w_1)\left(\Theta_1(12,1,12;w_2) + \Theta_1(12,5,12;w_2)\right).$$

Thus

$$\begin{aligned}\mathbb{E}_{1,(1,2)}(\tau) &= -\frac{\sqrt{3}}{4}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(4,1,4;w_1)\left(\Theta_1(12,1,12;w_2)+\Theta_1(12,5,12;w_2)\right)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1\\ &= 2(I_{(1,1)}(\tau)+I_{(1,5)}(\tau)),\end{aligned}$$

which is the second identity in (VI.1.3).

We next use Lemma 5.1 of [5], to obtain

$$I_{\boldsymbol{k}}(\tau) = (-i\tau)^{-1}\frac{1}{\sqrt{3}}\sum_{k=1}^{5}\sin\left(\frac{\pi k k_2}{6}\right)I_{(k_1,k)}\left(-\frac{1}{\tau}\right)+\mathbb{A}_{\boldsymbol{k}}(\tau),$$

where $\mathbb{A}_{\boldsymbol{k}}$ contributes the simpler terms mentioned in Theorem VI.1.2, and is explicitly given by

$$\begin{aligned}\mathbb{A}_{\boldsymbol{k}}(\tau) := &-\frac{\sqrt{3}}{8}\int_0^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(4,k_1,4;w_1)\Theta_1(12,k_2,12;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1\\ &-\frac{\sqrt{3}}{8}I_{\Theta_1(4,k_1,4;\cdot)}(\tau)r_{\Theta_1(12,k_2,12;\cdot)}(\tau)+\frac{\sqrt{3}}{8}r_{\Theta_1(4,k_1,4;\cdot)}(\tau)r_{\Theta_1(12,k_2,12;\cdot)}(\tau),\end{aligned}$$

where, for f a holomorphic modular form of weight k,

$$r_f(\tau) := \int_0^{i\infty} f(w)(-i(w+\tau))^{k-2}dw.$$

In particular

$$\begin{aligned}\mathbb{E}_{1,(1,1)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)-\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)+4\mathbb{A}_{(1,3)}(\tau),\\ \mathbb{E}_{1,(1,2)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)+\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)\right)+2\mathbb{A}_{(1,1)}(\tau)+2\mathbb{A}_{(1,5)}(\tau).\end{aligned}$$

Inverting and reordering gives

$$\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)=-\frac{i\tau}{\sqrt{3}}\big(2\mathbb{E}_{1,(1,2)}(\tau)-\mathbb{E}_{1,(1,1)}(\tau)\big)-\frac{4i\tau}{\sqrt{3}}\big(\mathbb{A}_{(1,3)}(\tau)-\mathbb{A}_{(1,1)}(\tau)-\mathbb{A}_{(1,5)}(\tau)\big), \tag{VI.2.6}$$

$$\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)=-\frac{i\tau}{\sqrt{3}}\big(\mathbb{E}_{1,(1,2)}(\tau)+\mathbb{E}_{1,(1,1)}(\tau)\big)+\frac{2i\tau}{\sqrt{3}}\big(\mathbb{A}_{(1,1)}(\tau)+\mathbb{A}_{(1,5)}(\tau)+2\mathbb{A}_{(1,3)}(\tau)\big). \tag{VI.2.7}$$

The claim follows using that

$$\mathbb{E}_{1,(1,1)}(\tau+1) = -\mathbb{E}_{1,(1,1)}(\tau), \quad \mathbb{E}_{1,(1,2)}(\tau+1) = e^{-\frac{\pi i}{6}}\mathbb{E}_{1,(1,2)}(\tau).$$

□

VI.3 The asymptotic behavior of $H_{1,\boldsymbol{\alpha}}$

To prove Theorem VI.1.3 we need to compute

$$H_{\boldsymbol{\alpha}} := \lim_{t\to 0^+} \frac{H_{1,\boldsymbol{\alpha}}\left(\frac{i}{t}\right)}{t},$$

where, for $\boldsymbol{\alpha} \in \mathbb{R}^2$,

$$H_{1,\boldsymbol{\alpha}}(\tau) := -\sqrt{3}\int_0^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) + \theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1.$$

Proposition VI.3.1. *Assume that α_1, α_2 are not both in $\mathbb{Z}$. We have*

$$H_{\boldsymbol{\alpha}} = \begin{cases} \frac{2}{\sqrt{3}}\frac{\sin(2\pi\alpha_1)\sin(2\pi\alpha_2)}{(1-\cos(2\pi\alpha_1))(1-\cos(2\pi\alpha_2))} & \textit{if } \alpha_1, \alpha_2 \notin \mathbb{Z}, \\ \frac{2\sqrt{3}}{1-\cos(2\pi\alpha_2)} & \textit{if } \alpha_1 \in \mathbb{Z}, \alpha_2 \notin \mathbb{Z}, \\ \frac{2}{(1-\cos(2\pi\alpha_1))\sqrt{3}} & \textit{if } \alpha_1 \notin \mathbb{Z}, \alpha_2 \in \mathbb{Z}. \end{cases}$$

Proof. We first rewrite $H_{1,\boldsymbol{\alpha}}(\tau)$. By Theorem 1.2 of [5], we have

$$H_{1,\boldsymbol{\alpha}}(\tau) = \int_{\mathbb{R}^2} g_{1,\boldsymbol{\alpha}}(\boldsymbol{w}) e^{2\pi i \tau Q(\boldsymbol{w})} dw_1 dw_2.$$

Here we define

$$g_{1,\boldsymbol{\alpha}}(\boldsymbol{w}) := \begin{cases} 2\mathcal{G}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2) - 2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2) & \text{if } \alpha_1, \alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_0(w_1)\mathcal{F}_{\alpha_2}(w_2) + \frac{2}{\pi w_1}\mathcal{F}_{\alpha_2}\left(w_2 + \frac{3w_1}{2}\right) & \text{if } \alpha_1 \in \mathbb{Z},\ \alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_0(w_2) + \frac{2}{\pi w_2}\mathcal{F}_{\alpha_1}\left(w_1 + \frac{w_2}{2}\right) & \text{if } \alpha_1 \notin \mathbb{Z},\ \alpha_2 \in \mathbb{Z}, \end{cases}$$

setting

$$\mathcal{F}_\alpha(x) := \frac{\sinh(2\pi x)}{\cosh(2\pi x) - \cos(2\pi\alpha)}, \qquad \mathcal{G}_\alpha(x) := \frac{\sin(2\pi\alpha)}{\cosh(2\pi x) - \cos(2\pi\alpha)}.$$

Applying the two-dimensional saddle point method gives that

$$H_{\boldsymbol{\alpha}} = \frac{g_{1,\boldsymbol{\alpha}}(0,0)}{\sqrt{3}}.$$

Explicitly computing $g_{1,\boldsymbol{\alpha}}(0,0)$ yields the claim of Proposition VI.3.1.

□

VI.4 Proof of Theorem VI.1.3.

Inverting (VI.2.6) and (VI.2.7) gives

$$\begin{aligned}
\mathbb{E}_{1,(1,1)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) - \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)\\
&\qquad + \frac{4}{\sqrt{3}(-i\tau)}\left(\mathbb{A}_{(1,3)}\left(-\frac{1}{\tau}\right) - \mathbb{A}_{(1,1)}\left(-\frac{1}{\tau}\right) - \mathbb{A}_{(1,5)}\left(-\frac{1}{\tau}\right)\right),\\
\mathbb{E}_{1,(1,2)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) + \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)\\
&\qquad - \frac{2}{\sqrt{3}(-i\tau)}\left(\mathbb{A}_{(1,1)}\left(-\frac{1}{\tau}\right) + \mathbb{A}_{(1,5)}\left(-\frac{1}{\tau}\right) + 2\mathbb{A}_{(1,3)}\left(-\frac{1}{\tau}\right)\right).
\end{aligned}$$

We next rewrite the first summand of $\mathbb{A}_{(1,j)}$, denoting it by $\mathbb{B}_{(1,j)}$. For this, we again use the theta relations (VI.2.4) and (VI.2.5). This yields

$$\mathbb{B}_{(1,3)}(\tau) = \frac{1}{16}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}(2\tau), \quad \mathbb{B}_{(1,1)}(\tau) + \mathbb{B}_{(1,5)}(\tau) = \frac{1}{8}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}(2\tau).$$

Thus

$$\begin{aligned}
\mathbb{E}_{1,(1,1)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) - \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)\\
&\quad + \frac{1}{2\sqrt{3}(-i\tau)}\left(\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right) - \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right)\right)\\
&\quad - \frac{1}{2(-i\tau)}\left(I_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right)\right)\\
&\qquad \times\left(r_{\Theta_1(12,3,12)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(12,1,12)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(12,5,12)}\left(-\frac{1}{\tau}\right)\right),\\
\mathbb{E}_{1,(1,2)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) + \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)\\
&\quad - \frac{1}{4\sqrt{3}(-i\tau)}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right) + \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha})H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right)\right)\\
&\quad + \frac{1}{4(-i\tau)}\left(I_{\Theta_1(4,1,4,1)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right)\right.
\end{aligned}$$

$$\times \left(2r_{\Theta_1(12,3,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right)+r_{\Theta_1(12,1,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right)+r_{\Theta_1(12,5,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right)\right)\Bigg).$$

Letting $\tau = it \to 0$ yields

$$\mathbb{E}_{1,(1,1)}(it) \sim \frac{1}{8\sqrt{3}}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} - 2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}}\right)+\frac{1}{2}(h_3-h_1-h_5), \quad \text{(VI.4.1)}$$

$$\mathbb{E}_{1,(1,2)}(it) \sim -\frac{1}{8\sqrt{3}}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}}\right)-\frac{1}{4}(2h_3+h_1+h_5), \quad \text{(VI.4.2)}$$

where

$$h_j := \lim_{t\to 0}\frac{1}{t}r_{\Theta_1(4,1,4;\,\cdot\,)}\left(\frac{i}{t}\right)r_{\Theta_1(12,j,12;\,\cdot\,)}\left(\frac{i}{t}\right).$$

We have

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{\boldsymbol{s}}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = s_2H_{\left(\frac{s_2-s_1}{6},1-\frac{s_2}{2}\right)}+s_1H_{\left(1-\frac{s_2-s_1}{6},1-\frac{s_1}{2}\right)}+s_1H_{\left(\frac{2s_1+s_2}{6},1-\frac{s_1}{2}\right)}$$
$$+s_2H_{\left(\frac{2s_2+s_1}{6},1-\frac{s_2}{2}\right)}-(s_1+s_2)H_{\left(1-\frac{s_1+2s_2}{6},\frac{s_1+s_2}{2}\right)}-(s_1+s_2)H_{\left(1-\frac{s_2+2s_1}{6},\frac{s_1+s_2}{2}\right)}.$$

In particular, using Proposition 1.1, we evaluate

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = \frac{2}{\sqrt{3}}, \qquad \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = \frac{16}{\sqrt{3}}. \quad \text{(VI.4.3)}$$

To compute $\lim_{t\to 0} t^{-\frac{1}{2}}r_{\Theta_1(N,a,N;\,\cdot\,)}(\frac{i}{t})$ we employ Lemma 3.2 of [5] to obtain

$$r_{\Theta_1(N,a,N;\,\cdot\,)}\left(\frac{i}{t}\right) = \frac{i\sqrt{N}}{2}\sin\left(\frac{2\pi a}{N}\right)\int_{\mathbb{R}}\frac{e^{-\frac{\pi N}{t}x^2}}{\sinh\left(\pi x+\frac{\pi i a}{N}\right)\sinh\left(\pi x-\frac{\pi i a}{N}\right)}dx.$$

The saddle point method then yields that

$$r_{\Theta_1(N,a,N;\,\cdot\,)}\left(\frac{i}{t}\right) = i\sqrt{t}\cot\left(\frac{\pi a}{N}\right).$$

Thus

$$h_j = \cot\left(\frac{\pi j}{12}\right).$$

In particular

$$h_1 = -\cot\left(\frac{\pi}{12}\right), \qquad h_3 = -1, \qquad h_5 = -\cot\left(\frac{5\pi}{12}\right).$$

Plugging this and (VI.4.3) into (VI.4.1) and (VI.4.2) gives the claim.

VI.5 Simplification for $p = 2$

We first recall the one-dimensional situation for $p = 2$. There is a unique false theta function

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{2\left(n+\frac{1}{4}\right)^2},$$

whose corresponding Eichler integral is (see [3])

$$F^*_{1,2}(\tau) := -2i\int_{-\overline{\tau}}^{i\infty} \frac{\Theta_1(4,1,4;w)}{\sqrt{-i(w+\tau)}}dw.$$

Noting that

$$\Theta_1(4,1,4;\tau) = \eta(\tau)^3, \tag{VI.5.1}$$

this integral transforms as a scalar-valued quantum modular form of weight $\frac{1}{2}$.

In the two-dimensional case, a similar "higher depth" picture emerges. Observing (VI.5.1) and

$$\Theta_1(12,3,12;\tau) = 3\eta(3\tau)^3, \quad \Theta_1(12,1,12;\tau) + \Theta_1(12,5,12;\tau) = 3\eta(3\tau)^3 + \eta\left(\frac{\tau}{3}\right)^3$$

we obtain that the space spanned by $\mathbb{E}_{1,(1,1)}(\tau)$ and $\mathbb{E}_{1,(1,2)}(\tau)$ is also spanned by

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\eta(w_1)^3\eta(3w_2)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1, \tag{VI.5.2}$$

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\eta(w_1)^3\eta\left(\frac{w_2}{3}\right)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

The next result can be found in [10, Corollary 6.6] (it can be also derived by using representation theory of $\widehat{\mathfrak{sl}}_3$ as discussed in [2]).

Proposition VI.5.1. *We have*

$$\eta(\tau)\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn} = 3\eta(3\tau)^3 + \eta\left(\frac{\tau}{3}\right)^3, \quad \eta(\tau)q^{\frac{1}{3}}\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn+n} = 3\eta(3\tau)^3.$$

According to [9], $\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn}$ and $q^{\frac{1}{3}} \sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn+n}$ are numerators of two characters of irreducible highest weight $\widehat{\mathfrak{sl}}_3$-modules of level one. Therefore modular properties of the double Eichler integrals in (VI.5.2), modulo correction factors, are identical to modular transformation properties of the span of characters of the level one simple affine vertex algebra of $\widehat{\mathfrak{sl}}_3$. It would be interesting to understand a possible connection from a purely representation theoretic perspective. This is closely related to a conjecture of Creutzig and the third author [8] pertaining to quantum modular properties of characters of $W^0(p)_{A_2}$, representations of affine Lie algebras, and representations of quantum groups at a root of unity (see also [1, 6, 7] for other appearances of this and related vertex algebras).

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Chapter VII

Summary and Discussion

In this chapter, the results of this thesis are summarized and further related research opportunities discussed.

VII.1 Indefinite theta functions arising in Gromov-Witten theory of elliptic orbifolds

In Chapter II (combined with [BRZ1]) we showed that the coefficients of the open Gromov-Witten potential $W_q(X, Y, Z)$ of $\mathbb{P}^1_{(2,3,6)}$ are essentially modular forms or mock modular forms of depth up to 3. To accomplish that, we used the strategies and tools devised in [Zw] and [ABMP] and built on them to describe the modularity properties of an indefinite theta function of signature $(1, 3)$. Chapter III we gave additional details for the proof of Proposition II.5.2.

The generating functions appearing in this Gromov-Witten potential come up by enumerating holomorphic discs on elliptic curves bounded by a set of Lagrangians, and can be expressed combinatorially as a counting function of hexagons whose edges lie on a certain set of lines, as shown by Cho, Hong, Kim, and Lau in [CHKL]. Similar counting functions for other planar polygons appear in homological mirror symmetry and are frequently related to theta functions [P1, P2, P3, P4, BHLW]. While the generating functions studied here correspond to using fixed Lagrangians, it is also possible to allow deformations of the Lagrangians as described in [P2]. It is expected that the corresponding counting functions of planar convex polygons (which have up to 6 vertices) will produce definite and indefinite theta functions with discontinuities (in the elliptic variable) along hyperplanes, with the values of the jumps corresponding to the theta functions for polygons with fewer vertices. This behavior has been verified for polygons of up to 5 vertices [BKZ], but homological mirror symmetry suggests that more relations between the generating functions should hold.

In regards to understanding indefinite theta functions in their own right, Nazaroglu [N] showed how to complete indefinite theta functions of arbitrary signature, but the proof of convergence relies on strict restrictions on the cone. It would be interesting to optimize his proof and push for less restrictive conditions on the cones. The cones considered in [ABMP] and [N] are called cubical cones since their cross-sections are (affine transforms of) hypercubes.

In contrast, other authors considered simplicial cones whose cross-sections are (affine transforms of) simplices and used abstract geometric objects in their proofs. Kudla showed that indefinite theta functions can be viewed as integrals of Kudla-Millson theta series [Ku], and extended this project with Funke to obtain general statements on modular completions of indefinite theta functions with simplicial cones, expressing the conditions on the cone as some related geometric objects being in "good position" [FK].

At the same time, Westerholt-Raum used a more abstract geometric setting to obtain results on the completions of indefinite theta function [WR]. However, some special cases with "degeneracies" such as those in Chapter II do not seem to be covered by these general statements.

While a lot of insight into the modular properties of indefinite theta functions has been gained by now, another possible research topic are the differential properties of the resulting completions. While some iterative structure in the appearing terms is evident, it should be possible to find a suitable analogue of the connection between mock modular forms and Maass forms for the mock modular forms of higher depth.

Once a proper structure is defined, one should study the dimensions of the spaces of fixed weight and (some kind of) depth, whether they are generated by indefinite theta functions, and what other kind of elements can be found.

VII.2 Higher depth quantum modular forms, multiple Eichler integrals, and $\mathfrak{sl}_3$ false theta functions

In Chapter IV we considered generating functions appearing as the character of the 0 weight space of the Lie algebra representation corresponding to the vertex algebra $W^0(p)_{A_2}$. We showed that the higher rank false theta functions appearing for the simple Lie algebra $\mathfrak{sl}_3$ is the sum of two depth two quantum modular forms and that a companion on the lower half plane is closely related to the "purely non-holomorphic part" of an indefinite theta function of signature $(2,2)$.

It would be interesting to generalize these results by considering the characters of 0 weight spaces of the Lie algebra representation corresponding to the lattices A_d, $d \geq 3$ (which were given in [BM1]). One could try to compute explicit expressions for the appearing false theta functions and find suitable companions on the lower half plane to determine the modularity properties of these functions. This is interesting from a representation theoretic standpoint because the vector-valued transformation under

$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ may produce typical and atypical characters in an identity of the shape

$$\mathrm{ch}[M]\left(-\frac{1}{\tau}\right) = \int_\Omega S_{M,\nu}\mathrm{ch}[M_\nu](\tau)d\nu + \sum_{j\in\mathcal{D}} \alpha_{M,j}\mathrm{ch}[M_j](\tau),$$

where the $\mathrm{ch}[M_j]$ are atypical characters and $\mathrm{ch}[M_\nu]$ are typical characters and Ω and $\mathcal{D}$ are domains parametrizing those types of characters, respectively [STT, KW].

Additionally, the S-transformation captures many important (algebraic, analytic, and categorical) properties of rational vertex algebras such as quantum dimensions and fusion rules and this could advance the understanding of modularity properties of characters beyond the rational vertex operator algebras.

Closer inspection of the generating functions appearing for $d \geq 3$ revealed some combinatorial problems related to the appearing representations that would have to be solved. Additionally, for a generalization to $d \geq 3$, the proof equating the asymptotic expansions has to be modified since growing terms appear that do not cancel completely without further conditions (which they did for $d = 2$). This could also be interesting independently of solving the combinatorial problem by considering suitable linear combinations of partial theta functions for arbitrary positive definite quadratic forms of rank d, which would give examples of quantum modular forms of depth d. Males performed this analysis for the case of $d = 2$ in [Mal1].

Problem VII.2.1. *Generalize the results of* [BKM1] *to the false theta functions of rank at least* 3 *produced by characters of W-algebras described in* [BM2]. *Further find companions for suitable linear combinations of partial theta functions for arbitrary positive definite quadratic forms of rank at least* 3.

Another possible connection is with colored Jones polynomials $J_{n\omega_j}(K, q)$, where ω_j, $j = 1, 2$ are the fundamental weights. Yuasa gave an explicit formula for the tail of $(2, 2p)$-torus links belonging to the sequence $(J_{n\omega_j}(K, q))_{n\in\mathbb{N}}$ [Yu]. Up to a factor $1 - q$, the same tail appeared as a summand of $F(q)$. Therefore one could try to show that $F(q)$ is the tail of $J_{n\rho}(K, q)$ up to a rational function in q. Related computations of tails colored with $\mathfrak{sl}_3$ representations can be found in [GV].

VII.3 Vector-valued higher depth quantum modular forms and higher Mordell integrals

In Chapter V we analyzed further aspects of the functions introduced in Chapter IV. Specifically, we showed that the quantum modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ appearing there also satisfy vector-valued higher depth quantum modularity with respect to the full modular group. Furthermore, we investigated the iterated Eichler integrals appearing in the companion and found a two-dimensional analogue of an identity between Eichler integrals and Mordell integrals shown by Zwegers [Zw].

By carefully inspecting the second identity given in Theorem V.1.3, one can guess such an identity for arbitrary positive definite quadratic forms of rank 2, and it suggests a generalization to positive definite quadratic forms of arbitrary rank. To that end, note that the theta functions in the Eichler integral come from the two ways of diagonalizing the quadratic form by completing the square. A more general analysis for quadratic forms of rank 2 using the same approach was performed by Males in the unpublished note [Mal2], but the arbitrary rank case would be an interesting continuation of this research that could shed light on higher dimensional analogues. The Eichler integral should have a term for each possible way of diagonalizing the quadratic form by iteratively completing the square, so we raise the following conjecture.

Conjecture VII.3.1. *For $\alpha \in (\mathbb{R} \setminus \mathbb{Z})^m$ and a positive definite quadratic form Q of rank m we have*

$$\int_{\mathbb{R}^m} e^{2\pi i\tau Q(w)} \prod_{j=1}^{m} \cot\left(\pi i w_j + \pi\alpha_j\right) dw = \int_0^{i\infty} \int_{w_1}^{i\infty} \cdots \int_{w_{m-1}}^{i\infty} \frac{\sum_k \theta_k\left(\alpha; w\right)}{\prod_{j=1}^{m} \sqrt{-i\left(w_j+\tau\right)}} dw_m \ldots dw_1,$$

where the sum on the right hand side runs over the $m!$ diagonalizations of Q obtained by iteratively completing the square, and $\theta_k\left(\alpha; w\right)$ is a theta function corresponding to the k-th diagonalization.

Chapters IV, V, and VI contain multiple non-holomorphic iterated Eichler integrals of specific theta functions and their companions on the upper half-plane. It would be interesting to study them in their own right and to answer questions such as the following.

Problem VII.3.2. *Can one find interesting companions on the upper complex half-plane (in the sense of agreeing asymptotic expansions at the rational points) for non-holomorphic Eichler integrals of a larger family of theta functions or other modular forms (instead of just the few examples appearing in Chapters IV, V, and VI)?*

Furthermore, one can draw inspiration from work by Brown [Bro], where he constructed a class of real-analytic modular forms as linear combinations of regularized iterated holomorphic Eichler integrals of Eisenstein series (see also [Ma]).

Problem VII.3.3. *Can one find linear combinations of holomorphic and non-holomorphic iterated Eichler integrals that are real-analytic modular forms?*

The family of rank 2 partial theta functions appearing in Chapters IV, V, and VI are Fourier coefficients of the Jacobi form of weight 0 and matrix index $-\frac{1}{2}\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ given by

$$\frac{\vartheta\left(z_1;2\tau\right)\vartheta\left(z_2;2\tau\right)\vartheta\left(z_1+z_2;2\tau\right)}{\vartheta\left(z_1;\tau\right)\vartheta\left(z_2;\tau\right)\vartheta\left(z_1+z_2;\tau\right)}.$$

Similar phenomena have been studied previously for Jacobi forms of index $m \in \mathbb{Z}$ [BCR, BRZ2, EZ, Ol, Zw]. For example, Eichler and Zagier showed that the Fourier coefficients of holomorphic Jacobi forms of index $m \in \mathbb{Z}$ give rise to vector valued modular forms via their theta decomposition [EZ]. More recently, Bringmann, Rolen, and Zwegers showed that the Fourier coefficients of negative index $m \in -\mathbb{N}$ meromorphic Jacobi forms are built from partial theta functions [BRZ2, Theorem 1.4]. It would be interesting to show similar properties for the coefficients of negative definite matrix index Jacobi forms in general.

Problem VII.3.4. *Can the coefficients of negative definite matrix index Jacobi forms be expressed in terms of partial theta functions? Are they quantum modular forms of higher depth?*

VII.4 Some examples of higher depth vector-valued quantum modular forms

Chapter V determined higher depth vector-valued quantum modular transformation behavior, but left open whether the other components are naturally connected to the q-series $F(q)$ by a corresponding family of q-series. In Chapter VI we proved that the family $(\mathbb{F}_{\boldsymbol{s}}(q))_{1\leq s_1,s_2\leq p}$ (that contains $F(q)=\mathbb{F}_{(1,1)}(q)$) consists of depth 2 vector-valued quantum modular forms, but only showed for $p=2$ that they form a closed space under modular transformation. Notably, the other components $\mathbb{F}_{\boldsymbol{s}}$ are connected from a representation theoretic viewpoint, so from the example of $p=2$ it is reasonable to expect them to combine naturally in general.

Therefore, a straightforward continuation would be to generalize our result to arbitrary $p \geq 2$ as follows.

Conjecture VII.4.1. *The functions $\mathbb{F}_{\boldsymbol{s}}$ combine into a vector-valued quantum modular form possibly after inclusion of additional characters (or functions).*

Similarly, one could generalize the study of the asymptotic behavior that we did for $p=2$ to general $p \geq 2$.

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Declaration

I hereby declare that the article *Indefinite theta functions arising in Gromov-Witten theory of elliptic orbifolds* [BKR] was jointly written with Prof. Dr. Kathrin Bringmann and Prof. Dr. Larry Rolen and my share of the work amounted to 33%. The articles *Higher depth quantum modular forms, multiple Eichler integrals, and* $\mathfrak{sl}_3$ *false theta functions* [BKM1], *Vector-valued higher depth quantum modular forms and higher Mordell integrals* [BKM2], and *Some examples of higher depth vector-valued quantum modular forms* [BKM3] were jointly written with Prof Dr. Kathrin Bringmann and Prof. Dr. Antun Milas and my share of the work amounted to 33% for each of these articles.

Köln, 24. September 2019

Jonas Kaszián

Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Kathrin Bringmann betreut worden.

1. K. Bringmann, J. Kaszian, and L. Rolen, *Indefinite theta functions arising in Gromov-Witten theory of elliptic orbifolds*, Camb. J. Math. **6** (2018), 25-57.

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3. K. Bringmann, J. Kaszian, and A. Milas, *Vector-valued higher depth quantum modular forms and higher Mordell integrals*, J. Math. Anal. Appl., doi:10.1016/j.jmaa.2019.123397.

4. K. Bringmann, J. Kaszian, and A. Milas, *Some examples of higher depth vector-valued quantum modular forms*, conference proceedings of "Number Theory: Arithmetic, Diophantine and Transcendence" at the IIT Ropar celebrating the 130th birth anniversary of S. Ramanujan.

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Lebenslauf

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Bildung

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Berufserfahrung

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Seminarvorträge

2019	Emory Algebra Seminar, Atlanta, USA
2017	51. ABKLS Seminar an der Universität zu Köln, Deutschland
2016	Dublin Mathematics Colloquium, Geometry Seminar an der School of Mathematics, Trinity College Dublin, Ireland

Sommerschulen

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2016	“Characters of Representations and Modular Forms” am Max-Planck-Institut für Mathematik, Bonn, Deutschland
2014	“Building Bridges: 2nd EU/US Summer School + Workshop on Automorphic Forms and Related Topics” an der Universität von Bristol, United Kingdom

Lehre

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Zweitbetreuungen von Abschlussarbeiten

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Winter 2016/17	Alicia Buhk, Bachelorarbeit über “Beweis von Eisensteinreihenidentitäten mit elementaren Mitteln”

Veröffentlichungen

eingereicht	K. Bringmann, J. Kaszián and J. Zhou, *Generating functions of planar polygons from homological mirror symmetry of elliptic curves*, arXiv:1904.05058.
eingereicht	K. Bringmann, J. Kaszián, A. Milas and S. Zwegers, *Rank two false theta functions and Jacobi forms of negative matrix index*, arXiv:1902.10554.
angenommen	K. Bringmann, J. Kaszián and A. Milas, *Some examples of higher depth vector-valued quantum modular forms*, conference proceedings of the conference "Number Theory: Arithmetic, Diophantine and Transcendence" for Ramanujan's 130th birthday at the IIT Ropar, India.
2019	K. Bringmann, J. Kaszián and A. Milas, *Vector-valued higher depth quantum modular forms and higher Mordell integrals*, Journal of Mathematical Analysis and Applications, doi:10.1016/j.jmaa.2019.123397.
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2018	K. Bringmann, J. Kaszián and L. Rolen, *Indefnite theta functions arising in Gromov-Witten Theory of elliptic orbifolds*, Cambridge Journal of Mathematics **6** (2018), 25-57.
2014	J. Kaszián, P. Moree and I. Shparlinski, *Period Structure of the Exponential Pseudorandom Number Generator*, Applied Algebra and Number Theory, Cambridge University Press, doi:10.1017/CBO9781139696456.