Submanifolds with Parallel Focal Structure

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Deutsche Zusammenfassung

Sei $\varphi : M \to N$ eine isometrische Immersion einer vollständigen Mannigfaltigkeit M in eine vollständige Mannigfaltigkeit N. Diese Arbeit beschäftigt sich mit der Frage, wann es eine Zerlegung des umgebenden Raumes N durch parallele Untermannigfaltigkeiten und Fokalmannigfaltigkeiten von φ gibt. Um den Begriff der parallelen Untermannigfaltigkeit zu definieren, müssen wir zusätzlich fordern, dass M ein flaches Normalenbündel hat. Jeder Normalenvektor $v \in \nu M$ läßt sich zu einem parallelen Normalenfeld $\bar{v}$ auf der universellen Überlagerung $\tilde{M}$ von M fortsetzen. Wir fordern, dass die Abbildung $\exp^{\perp} \circ \bar{v}$ konstanten Rang für jedes $v \in \nu M$ hat; das Bild der Abbildung ist die parallele Mannigfaltigkeit bzw. die Fokalmannigfaltigkeit M_v. Dies sind die Minimalvoraussetzungen für unsere Fragestellung. Zusätzlich verlangen wir, dass $\exp^{\perp}(\nu_x M)$ eine totalgeodätische Untermannigfaltigkeit ist, ein sogenannter *Schnitt* durch x, und dass jeder reguläre Punkt der normalen Exponentialabbildung in genau einem Schnitt liegt. Wir sagen, dass eine isometrische Immersion mit den genannten Eigenschaften *parallele Fokalstruktur* hat. Nach [HLO] heißt $\mathcal{F} = \{M_v \mid v \in \nu M\}$ eine *globale Partition*, falls $\bigcup \mathcal{F} = N$ und falls $M_v \cap M_w \neq \emptyset \implies M_v = M_w$. Wir zeigen:

Satz *Eine eigentliche Immersion mit paralleler Fokalstruktur und endlicher normaler Holonomie induziert eine globale Partition $\mathcal{F}$. Diese Partition ist eine singuläre Riemannsche Blätterung im Sinn von [Mo].*

Umgekehrt hat ein reguläres Blatt einer singulären Riemannschen Blätterung mit Schnitten einer vollständigen Riemannschen Mannigfaltigkeit parallele Fokalstruktur nach [A]. Wir geben hierfür einen anderen Beweis.

Satz *Jede parallele Untermannigfaltigkeit einer eigentlichen Immersion mit paralleler Fokalstruktur und endlicher normaler Holonomie ist eine eingebettete und abgeschlossene Untermannigfaltigkeit mit paralleler Fokalstruktur und endlicher normaler Holonomie.*

Boualem definiert in [Bou] für eine singuläre Riemannsche Blätterung $\mathcal{F}$ mit Schnitten einer vollständigen Riemannschen Mannigfaltigkeit die Teilmenge

$$\hat{N} := \{T_p\Sigma \mid p \in N, \Sigma \text{ ist ein Schnitt von } \mathcal{F} \text{ durch } p\}$$

des Grassmannbündels $G_k(TN)$. Sei $\hat{\pi} : \hat{N} \to N$ die Einschränkung der kanonischen Projektion $G_k(TN) \to N$ auf $\hat{N}$. Er definiert eine differenzierbare Struktur auf $\hat{N}$ und zeigt, dass $\hat{\mathcal{F}} = \{\hat{\pi}^{-1}(M) \mid M \in \mathcal{F}\}$, die *Aufblasung* von $\mathcal{F}$, eine Riemannsche Blätterung von $\hat{N}$ für eine geeignete Metrik ist. Wie wir sehen werden, gelten seine Resultate auch für eine natürliche differenzierbare Struktur und eine natürliche Metrik. Mit Hilfe der Aufblasung und der Theorie der Riemannschen Blätterungen definieren wir für eine singuläre Riemannsche Blätterung mit Schnitten ein Pendant zur Weylgruppe einer polaren Wirkung. Abschließend zeigen wir:

Satz *Sei $\mathcal{F}$ eine singuläre Riemannsche Blätterung mit Schnitten in einem einfach zusammenhängenden symmetrischen Raum, deren reguläre Blätter eingebettet und abgeschlossen sind. Dann hat jedes reguläre Blatt triviale normale Holonomie.*

1 Introduction

One of the main topics of submanifold geometry is to study how the local invariants of a submanifold, like the first fundamental form, the shape operator and the normal connection, affect the global geometry. The functional dependence of the eigenvalues and their multiplicities of the shape operator to the focal distances and the focal multiplicities in space forms is a simple and well-known example for this. An *isoparametric submanifold* M is defined only by two conditions on its local invariants:

(1) νM is flat, and

(2) the eigenvalues of the shape operator A_ξ are constant for a locally defined parallel normal field ξ.

A rich theory has been developed for this class of submanifolds in $\mathbb{R}^n$ (see [PaTe]). Their structure is very similar to that of the regular orbits of an s-representation, which is the linear isotropy action of a symmetric space by definition; indeed, every homogeneous isoparametric submanifold of $\mathbb{R}^n$ is a regular orbit of an s-representation. An example for the special structure of an isoparametric submanifold M is, that it induces an orbit-like foliation of $\mathbb{R}^n$ by parallel submanifolds. The set of focal points in the affine plane $P = x + \nu_x M$ in $\mathbb{R}^n$ is a union of hyperplanes. The reflections of P in these hyperplanes generate a Coxeter group, in analogy to the Weyl group associated to a symmetric space. (For a detailed survey on isoparametric submanifolds and their relatives, see [Th2].)

Isoparametric submanifolds of S^n are also isoparametric submanifolds of $\mathbb{R}^{n+1}$; a theory for isoparametric submanifolds in the hyperbolic space H^n was developed in [Wu]. It is natural to ask whether one can obtain similar results as above for isoparametric submanifolds in a larger class of ambient spaces than space forms. It turns out that a definition of a submanifold like above that is only based on local invariants will not lead to similar results. Indeed, in a general ambient space there is no correspondence between the principal curvature and the focal distance. We can see this by perturbing the metric of the ambient space outside a tube of M which will affect the focal distances, but not the local invariants. Instead of demanding constant principal curvature, Terng and Thorbergsson considered in [TeTh] the focal structure. They found out that it carries the relevant information for a generalization of isoparametric submanifolds in a larger class of ambient spaces. They call a submanifold M of a simply connected, compact symmetric space N *equifocal*, if

(1) M has a globally flat normal bundle,

(2) the focal distances and multiplicities are constant along any parallel normal field, and in addition,

(3) $\exp(\nu_p M)$ is contained in a flat for every $p \in M$.

Later Ewert introduced the notion of a *submanifold with parallel focal structure* in [Ew], which is defined to satisfy (1), (2) and in addition the condition (3') that $\exp(\nu_x M)$ is a closed submanifold for every $x \in M$, called section, and meets all parallel submanifolds orthogonally. In our work we accept in the definition of a submanifold with parallel focal structure that νM is flat, possibly with non-trivial normal holonomy, but instead of (3'), we demand (3"), that the sections are totally geodesic submanifolds; we also change the first condition slightly (see section 3). Terng and Thorbergsson showed many properties similar to that for isoparametric submanifolds in $\mathbb{R}^n$ for an equifocal submanifold M in a simply connected, compact symmetric space N, among them the existence of an orbit-like foliation of N by parallel and focal submanifolds and the existence of a Weyl group acting on $\exp(\nu_p M)$, which turns out to be a flat submanifold. Their proofs rely heavily on the structure of symmetric spaces. Let $N = G/K$ for a symmetric pair (G, K) of compact type. Terng and Thorbergsson considered a submersion $\pi : V \to G$, where V is a certain Hilbert space. They derived geometric aspects on M from the analysis of the isoparametric submanifold $\pi^{-1}(M)$ in V. Later this technique was axiomatized in [HLO]. Of course, one cannot expect the existence of such a submersion for general N. Therefore it is desirable to implement appropriate conditions into the definition of a submanifold that will imply similar results as for equifocal submanifolds. We will focus on the following question: When is the set of all parallel and focal submanifolds of a given submanifold M in N a partition of N into submanifolds? More precisely, for $v \in \nu M$ we define

$$M_v = \left\{ \exp\left((\overset{1}{\underset{0}{\|}} c)v \right) \;\middle|\; c \text{ is a curve in } M \right\},$$

where $\| c$ denotes normal parallel translation along c. In [HLO] $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is called a *global foliation* if $\bigcup \mathcal{F} = N$ and $M_v \cap M_w \neq \emptyset$ implies $M_v = M_w$. We reformulate our question with these notions: Under which conditions on M is $\mathcal{F}$ a global foliation of N? Conditions (1) and (2) of a submanifold with parallel focal structure are necessary in order to guarantee that M_v is an immersed submanifold for every $v \in \nu M$. But these conditions are not sufficient. Let us consider two examples. First we take $N = S^2$ and M a parallel of the equator. Obviously M fulfills condition (1) and (2), and the parallel submanifolds of M are the other parallels of the equator, the focal submanifolds are the poles of S^2. Clearly M induces a partition by parallel and focal submanifolds. Next we consider the flat torus $N = T^2$ and a small distance circle M centered at a point p in N. Again M satisfies (1) and (2), but this time M does not induce a global foliation of N. For a unit vector $\xi \in \nu M$ we define a positive real $\sigma(\xi)$ as the maximal value t for which the geodesic $\gamma_\xi|[0, t]$ is the shortest connection between M and $\gamma_\xi(t)$. While σ is constant along a unit normal field of M in the first example, it is not in the second. This motivates us to introduce the *cut locus of a submanifold*, which is a generalization of the cut locus of a point (see 2.1). The constancy of σ along unit parallel normal fields is a necessary condition for M to induce a global foliation (see Proposition 3.12). Note that already Bolton has realized the relevance of the cut locus for transnormal partitions (see 2.2) with codimension 1 in [Bol]. In a way, he assumes the contrary point of view by considering the cut locus of one of the at most two singular leaves, while we consider the cut locus of a regular leaf in arbitrary codimension. We have seen above a necessary condition for M to induce a global foliation. Assuming (3"), the existence of sections, we found a necessary and sufficient condition:

Theorem A *Let $\varphi : M \to N$ be a proper immersion of a submanifold with parallel focal structure and finite normal holonomy. Then $\mathcal{F}$ is a global foliation of the ambient space N, if and only if there is only one section through each regular point of the normal exponential map of M. In this case $\mathcal{F}$ is a singular Riemannian foliation.*

Ewert states in [Ew] that the set of parallel submanifolds of a submanifold with parallel structure (according to his definition) is a foliation, which follows directly from condition (3'). Theorem A handles in addition the focal set. Moreover, it gives a link to the theory of singular Riemannian foliations ([Mo],[Bou],[A]), on which the theory of submanifolds with parallel focal structure now can capitalize. But the converse is true as well by a result of Marcos Alexandrino ([A]): A regular leaf of a singular Riemannian foliation admitting sections (for the definition see section 2.2) has parallel focal structure (we give a different proof in section 4.1). This means that these two theories are in general equivalent. From now on we will include the necessary and sufficient condition given in Theorem A into the definition of a submanifold with parallel focal structure.

Theorem B *Is M a closed and embedded submanifold with parallel focal structure and finite normal holonomy in a complete Riemannian manifold, so is every parallel submanifold.*

Ewert states this result for the case of trivial normal holonomy as Proposition 2.9 in [Ew]. His proof is not correct; we will explain his mistake later.

For a singular Riemannian foliation $\mathcal{F}$ admitting sections (for the definition see 2.2) of a Riemannian manifold N, Boualem defines the subset

$$\hat{N} := \{T_p\Sigma \mid p \in N, \Sigma \text{ is a section of } \mathcal{F} \text{ through } p\}$$

of the Grassmann bundle $G_k(TN)$ in [Bou]. Let $\hat{\pi} : \hat{N} \to N$ be the restriction of the canonical projection $G_k(TN) \to N$ to $\hat{N}$. He constructs a differentiable structure on $\hat{N}$ and shows that $\hat{\mathcal{F}} = \{\hat{\pi}^{-1}(M) \mid M \in \mathcal{F}\}$ is a regular Riemannian foliation of $\hat{N}$, the *blow-up* of $\mathcal{F}$, for some metric $\hat{g}$. We reprove his results with our theory and give the following extension:

Theorem C *The set $\hat{N}$ carries the unique differentiable structure with respect to which the inclusion into $G_k(TN)$ is an immersion. Let $\hat{g}$ be the pull-back of a natural metric on $G_k(TN)$ to $\hat{N}$. Then $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ is a bifoliation of $\hat{N}$ with respect to $\hat{g}$, where $\mathcal{F}$ is a Riemannian foliation and the orthogonal foliation $\hat{\mathcal{F}}^{\perp}$ is totally-geodesic.*

Using a result of Blumenthal and Hebda we can then describe the singular Riemannian foliation $\mathcal{F}$ from Theorem A by a map $\tilde{M} \times \tilde{\Sigma} \to N$, where $\tilde{M}$ respectively $\tilde{\Sigma}$ is the universal cover of a regular leaf M of $\mathcal{F}$ respectively of a section Σ. With this map we define the transversal holonomy group Γ acting by isometries on Σ. This is the analogue of the Weyl group for polar actions.

Theorem D *A singular Riemannian foliation admitting sections and whose leaves are properly embedded in a simply connected symmetric space has no exceptional leaves, i.e., every regular leaf has a trivial normal holonomy.*

Compare this to Lemma 1A.3 of [PoTh].

In section 2.1 we define and study the cut locus of a submanifold. In section 2.2 we introduce the necessary notions and tools of the theory of Riemannian foliations.

In section 3 we define the blow-up $\hat{\mathcal{F}}$ outgoing from a submanifold with parallel focal structure and prove Theorem C (3.7). We can then easily conclude Theorem A (3.10) and Theorem B (3.11).

In section 4.1 give an alternative proof of the converse of Theorem A (3.13). In section 4.2 we introduce the transversal holonomy group Γ, and we study the relation of the cut locus of M with the fundamental domains of Γ, and with the exceptional leaves. Then we will prove Theorem D (4.19).

I would like to thank my teacher Professor Thorbergsson for his long-term guidance and support of my work.

2 Preliminaries

2.1 Cut Locus of a Proper Immersion

Let N be a complete and connected Riemannian manifold and M a manifold. By $\varphi : M \to N$ we will always denote an isometric immersion. Let $\nu M = (\varphi_*(TM))^\perp$ and let $\iota : \nu M \to TN$ be the canonical immersion. We write $\eta := \exp^\perp = \exp \circ \iota : \nu M \to N$ and $\eta^r : B_r(\nu M) \to N$ for the restriction of η to the normal ball bundle of M of radius r and we write $\varphi_*^r : B_r(TM)$ for the restriction of the derivative of φ to the tangential ball bundle of M of radius r.

Lemma 2.1 *The following conditions are equivalent:*

(1) φ is proper.

(2) η^r is proper for all $r \geq 0$.

(3) φ_*^r is proper for all $r \geq 0$.

Proof The proof is clear. □

Definition A point $x \in M$ is called *tangential*, if for every $y \in M$ with the same image $\varphi_*(T_yM) = \varphi_*(T_xM)$, otherwise x is an *intersection point*.

Let φ be proper. Then the preimage of a point is compact and discrete and therefore finite. Let $x \in M$ a tangential point. We call x a *contact point* if there is a $y \in \varphi^{-1}(\varphi x)$ and a positive number ε such that $\varphi(B_{\varepsilon'}(x)) \neq \varphi(B_{\varepsilon'}(y))$ for any ε' with $0 < \varepsilon' < \varepsilon$.

By transversality the set of tangential points is dense in M. The next proposition shows that this set is also open in M, if M is embedded. This is not true for general immersions. Let us consider the immersion $\varphi : \mathbb{R} \to \mathbb{R}^2/_\sim, x \mapsto [(x, \sin x)]$, where $(x, y) \sim (x + 1, y)$. Since the periodicity of the sine-function is irrational, the set of intersection points of φ is dense in $\mathbb{R}$.

Proposition 2.2 *A proper immersion $\varphi : M \to N$ factorizes over an embedding if and only if it has neither intersection nor contact points.*

Proof We assume that φ has neither intersection nor contact points. Let p be an arbitrary point in $\varphi(M)$ and $\varphi^{-1}(p) = \{x_1, \ldots, x_l\}$. We choose $\varepsilon > 0$ such that $\varphi|B_\varepsilon(x_i)$ are diffeomorphisms onto their images for all i and that the $B_\varepsilon(x_i)$ are pairwise disjoint and that $\varphi(B_{\varepsilon'}(x_i))$ is the same for all i. We will now show that the function $k(x) := |\varphi^{-1}(\varphi x)|$ is locally constant. This will finish the proof. We see that $k(y) \geq k(x)$ for all $y \in B_\varepsilon(x)$. Assume there exists a sequence (y_n) converging to x with $k(y_n) > k(x)$ for all $n \in \mathbb{N}$. For every n we find a point z_n with $\varphi(z_n) = \varphi(y_n)$ and $(z_n) \notin B_\varepsilon(x_i)$ for all i. By properness (z_n) converges to a point $z \neq x_i, i = 1, \ldots, l$. So $k(z) > k(x)$ in spite of $\varphi(z) = \varphi(x)$. The converse is clear. □

Now we will introduce a generalization of the cut locus of a point that is defined for instance in [Kl]. Let $\varphi : M \to N$ be a proper immersion. Let γ_v denote the geodesic with initial vector $v \in TN$.

Definition We define $\sigma : \nu^1 M \to [0,\infty]$ by

$$\sigma(v) = \sup\{t \in \mathbb{R} \mid d(\gamma_{\iota(v)}(t), \varphi(M)) = t\}.$$

We call $\sigma(v)$ the *cut distance of* M *in direction* v. The *normal cut locus* $\mathcal{C}^{\nu M}$ of φ is defined by $\mathcal{C}^{\nu M} := \{\sigma(v)v \mid \sigma(v) < \infty, v \in \nu^1 M\}$ and the *cut locus* $\mathcal{C}_\varphi$ or $\mathcal{C}_{(M,N)}$ by $\exp^\perp \mathcal{C}^{\nu M}$.

Definition A vector $v \in \nu M$ is called a *normal* or *normal vector* and the geodesic $\gamma_{\iota(v)}$ a *normal geodesic*. If $\|v\| \le \sigma(v/\|v\|)$ for $v \in \nu M$, we call v *minimal* and $\gamma_v|[0,1]$, and any reparametrization of constant speed, a *minimal geodesic (segment)*. This terminology is justified by the fact that, in the set $\eta^{-1}(p)$, the minimal vectors have the least length. We call a normal vector v a *focal vector* and $\eta(v)$ a *focal point*, if v is singular with respect to η. We call a minimal vector v a *cut vector* and $\eta(v)$ a *cut point*, if there is a minimal $w \in \nu M$ with $\iota(w) \neq \iota(v)$ having the same endpoint as v. In this case $\|v\| = \sigma(v/\|v\|) = \sigma(w/\|w\|) = \|w\|$. If in addition there is no focal geodesic among the set of minimal geodesics to p, we call p of *pure cutting type*.

It is easy to see that the limit of a converging sequence of minimal vectors is minimal. In the sequel we will use the known fact that tv for $t > 1$ is not minimal if v is a focal vector or a cut vector. Also note that for every $p \in N$ there is a shortest curve from $\varphi(M)$ to p since $\varphi(M)$ is closed and N complete; this is a normal geodesic. This implies that η is surjective.

In contrast to the cut locus of a point the cut distance function is in general not continuous. The next lemma describes this situation.

Lemma 2.3 *The cut distance function σ is upper semi-continuous and it is discontinuous at a vector $v \in \nu^1 M$ if and only if $\sigma(v) > 0$ and $\liminf_{w\to v}\sigma(w) = 0$.*

Proof Note that a function is continuous at a point if and only if it is upper and lower semi-continuous at this point. We will prove the upper semi-continuity of σ, i.e. $\limsup_{w\to v}\sigma(w) \le \sigma(v)$ for every $v \in \nu^1 M$. Assume the existence of $v \in \nu^1 M$ such that $\sigma^* := \limsup_{w\to v}\sigma(w) > \sigma(v)$. Choose $a \in \mathbb{R}$ with $\sigma(v) < a < \sigma^*$. We find a sequence (v_n) in $\nu^1 M$ converging to v such that av_n is minimal for large n. Then av is also a minimal geodesic segment, which contradicts $\sigma(v) < a$.

Assume σ is not continuous at a vector $v \in \nu^1 M$. This implies that the lower semi-continuity at v fails. Thus $\sigma(v) > 0$ and $\sigma_* := \liminf_{Z\to v}\sigma(Z) < \sigma(v)$. We will show $\sigma_* = 0$. Let (v_n) be a sequence in $\nu^1 M$ converging to v with $\lim_{n\to\infty}\sigma(v_n) = \sigma_*$. We choose $a \in \mathbb{R}$ such that $\sigma_* < a < \sigma(v)$. Let (w_n) be a sequence in $\nu^1 M$ such that $\gamma_{\iota(w_n)}|[0,t_n]$ are minimal geodesic segments to $\eta(av_n)$ for some t_n. By the properness of $\eta^{\sigma(v)}$ we can assume that $t_n w_n$ converges to some $t_0 w$, where $w \in \nu^1 M$. It follows $\gamma_{\iota(w)}|[0,t_0]$ is also a minimal geodesic segment to $\eta(av)$, thus $\iota(w) = \iota(v)$ and $a = t_0$. Since av is not focal η is injective on a neighborhood of av. Thus $v \neq w$. Choose ε with $0 < \varepsilon < \sigma_*$ and a neighborhood U of εv not containing εw, such that $\eta|U$ is a diffeomorphism onto its image. Let $s_n\tilde{v}_n \in U$, where $s_n > 0$ and $\tilde{v}_n \in \nu^1 M$, be the unique vector in U such that $\gamma_{\iota(\tilde{v}_n)}|[0,s_n]$ is a minimal geodesic segment to

$\eta(\varepsilon w_n)$. It follows that $\tilde{v}_n$ converges to v. We have $\lim_{n\to\infty} \sigma(\tilde{v}_n) \le \varepsilon < \sigma_*$, which is a contradiction. □

In Corollary 2.6 we will give a geometric description of those points at which only discontinuity can occur.

Proposition 2.4 *The cut locus only consists of focal and cut points.*

Proof We consider an $v \in \nu^1 M$ with $\sigma(v) < \infty$ such that $\sigma(v)v$ is not a focal normal. We have to show that $\eta(\sigma(v)v)$ is a cut point. We construct sequences $(\tilde{v}_n)$ in $\nu^1 M$ and $t_n > 0$ such that $t_n \tilde{v}_n$ is minimal with $\eta(t_n \tilde{v}_n) = \eta((\sigma(v) + 1/n)v)$. As η^r is proper for all $r \ge 0$ we can assume that $t_n \tilde{v}_n$ converges to, say $t_0 \tilde{v}$, where $\tilde{v} \in \nu^1 M$. Then $\sigma(v)v$ and $t_0 \tilde{v}$ are minimal and have the same endpoint. This implies $t_0 = \sigma(v)$. Since η is injective on a neighborhood of $\sigma(v)v$ we have $\iota(\tilde{v}) \ne \iota(v)$ and $\eta(\sigma(v)\tilde{v}) = \eta(\sigma(v)v)$. □

The next proposition characterizes intersection and contact points in terms of the cut locus.

Proposition 2.5 *A point $x \in M$ is an intersection point if and only if $\sigma(v) = 0$ for some $v \in \nu^1_x M$. An intersection point x is also characterized by the property that $\varphi(x)$ is in the cut locus; in this case $\varphi(x)$ is of pure cutting type. This implies that M has no self-intersections if and only if the cut locus has no common points with $\varphi(M)$. A point $x \in M$ is a contact point if and only if x is tangential and there is some $v \in \nu^1_x M$ with $\sigma(v) > 0$ and $\liminf_{w\to v} \sigma(w) = 0$.*

Proof Assume $0 \notin \sigma(\nu^1_x M)$. Take an arbitrary $v \in \nu^1_x M$ and choose t with $0 < t < \sigma(v)$. Then the geodesic $\gamma_{\iota(v)}|[0,t]$ is the unique shortest connection between $\eta(tv)$ and $\varphi(M)$. By the variation principle $\iota(v)$ is orthogonal to $\varphi_*(T_y M)$ for all $y \in \varphi^{-1}(\varphi x)$. Since v is arbitrary, x is a tangential point. Now we consider a tangential point $x_1 \in M$ with image p and $\varphi^{-1}(p) = \{x_1, \dots, x_l\}$. We take $\varepsilon > 0$ and a neighborhood $U_i = B_\varepsilon(x_i)$ of x_i for all i. If we assume $\sigma(v) = 0$ for some $v \in \nu^1_{x_1} M$ we find for each $n \in \mathbb{N}$ a point $y_n \in M$ for any i, with $d(\varphi(y_n), \gamma_{\iota(v)}(1/n)) = d(\varphi(M), \gamma_{\iota(v)}(1/n)) < 1/n$. It follows $d(\varphi(y_n), p) < 2/n$ and the sequence $(\varphi(y_n))$ converges to p. By eventually choosing a smaller ε we can assume that $y_n \notin U_i$ for all i using that x is tangential, $v \in (\varphi_*(T_{x_i} M))^\perp$ and the linear approximation of φ. Properness of φ implies the convergence of a subsequence of (y_n) to some point $y \ne x_i$, contradiction. The image of an intersection point is clearly of pure cutting type. If $\varphi(x)$ is in the cut locus for a point $x \in M$ then $0 \in \sigma(\nu^1_x M)$.

We are now going to show the last statement. Let $x \in M$ be a tangential point, such that there is some $v \in \nu^1_x M$ with $\sigma(v) > 0$ and a sequence (w_n) in $\nu^1 M$ converging to v with $\lim_{n\to\infty} \sigma(w_n) = 0$. Choose $\varepsilon > 0$ such that $\varphi|B_\varepsilon(x)$ is a diffeomorphism. For large n we find $y_n \in M, y_n \notin B_\varepsilon(x)$ with $d(\varphi(y_n), \varphi(\pi w_n) \le 2\sigma(w_n)$. So $\varphi(y_n)$ converges to $\varphi(x)$. By properness of φ there is a subsequence of (y_n) converging to some point $y \ne x$ with $\varphi(y) = \varphi(x)$. We deduce that the images of $B_{\varepsilon'}(x)$ and $B_{\varepsilon'}(y)$ do not coincide for any $\varepsilon' < \varepsilon$.
The converse follows from similar arguments. □

Corollary 2.6 *Discontinuity of the cut distance function can only occur over in-*

tersection or contact points. Let φ be a proper embedding. Then its cut distance function is continuous and its cut locus is closed.

Proof The first statement follows form 2.5. The second statement follows from the first together with Proposition 2.2. The proof of the last statement is analogous to that of Corollary 2.10, chapter 13, [dC]. $\square$

Proposition 2.7 *Let M be properly embedded in N. Then M is compact and its cut distance function is bounded from above if and only if N is compact.*

Proof Can be proven by similar arguments as employed in the proof of Corollary 2.10, chapter 13, [dC]. $\square$

Later we will frequently us the following notion. If $r = \inf\{\sigma(v) \mid v \in \nu^1 M\} > 0$ we call r *injectivity radius* of φ and $T_s = tube(M, s) = \exp B_s(\nu M)$ an *injectivity tube* of M with radius s for any s with $0 < s \leq r$. By definition for each $p \in T$ there is exactly one minimal normal v with endpoint p up to foot point. The map φ has neither intersection nor contact points and thus factorizes over an embedding. Therefore $\eta : B_s(\nu M) \to T_s$ is a covering, and it is a diffeomorphism, if φ is injective, i.e. an embedding.

Remark As in the case of the cut locus of a point, we can give a similar connection between the homology and the cut locus of a submanifold.

2.2 Foliations

Let $\mathcal{F}$ be a partition of a manifold N^{n+k} into connected, injectively immersed submanifolds with maximal dimension n. For a point $p \in N$ we denote the element of $\mathcal{F}$ containing p by M_p. We define $T\mathcal{F} = \bigcup_{p\in N} T_pM_p$. Let $\Xi(\mathcal{F})$ be the module of differentiable vector fields X tangential to $\mathcal{F}$, i.e., with $X_p \in T_pM_p$ for every $p \in N$. If the values of $\Xi(\mathcal{F})$ exhaust T_pM_p for every $p \in N$, we say that $\Xi(\mathcal{F})$ *acts transitively* (on $T\mathcal{F}$).

Definition A partition $\mathcal{F}$ as above is called a *singular foliation* of N of dimension n/codimension k, if $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$.

If all the elements of a singular foliation $\mathcal{F}$ have the same dimension n, then $\mathcal{F}$ is a foliation as we will see. First we give some other definitions. We call the elements of $\mathcal{F}$ *leaves*. A leaf is *regular* if it has dimension n otherwise *singular*. A point belonging to a regular leaf is *regular*, otherwise *singular*. The set N' of regular points in N is called the *regular stratum*. Let p be a point in N and M_p be a leaf of dimension q. By [St] there is a neighborhood U of p, ball neighborhoods B^q and B^{n+k-q} of the origin in $\mathbb{R}^q$ respectively $\mathbb{R}^{n+k-q}$ and a chart $\psi : U \to B^q \times B^{n+k-q}$ with $\psi(p) = (0,0)$ such that $\psi(M_p \cap U) = B^q \times A$ for a subset A of B^{n+k-q}. The intersection of M_p with the *slice* $\psi^{-1}(\{0\} \times B^{n+k-q})$ is transversal. Therefore it is a submanifold and as a consequence also A. We call ψ a *foliated chart* and U a *simple neighborhood* of p. For $q \in U$ we call the connected component $M_q \cap U$ containing q the *plaque* in U through $q \in U$. The property of a foliated chart shows that the function $\dim M_p$ in dependence of p is lower semi-continuous, hence the regular stratum N' is an open subset of N. We can see that that a singular foliation $\mathcal{F}$ restricted to the regular stratum N' is a foliation: Let $X, Y \in \Xi(\mathcal{F})$, let p be a regular point and $i : M_p \to N$ the inclusion map of the leaf through p. We denote by X' and Y' the i-related vector fields on M_p of X and Y. Then $[X,Y]_{(i(p))} = i_*[X',Y']_p \in T_pM_p$, so $([X,Y])|N' \in \Xi(\mathcal{F}|N')$. This shows that $\Xi(\mathcal{F})$ and $\Xi(\mathcal{F}|N')$ are Liealgebras acting transitively, thus the (differentiable) distribution $T(\mathcal{F}|N')$ is involutive and $\mathcal{F}|N'$ is a foliation by the Theorem of Frobenius. In particular, *a singular foliation having only leaves of the same dimension is a foliation.*

The morphisms between singular foliations are defined below.

Definition Let $\mathcal{F}_i$ be a singular foliation/partition of N_i for $i = 1,2$. A map $f : (N_1, \mathcal{F}_1) \to (N_2, \mathcal{F}_2)$ is a *foliated map* or *f-map*, if f maps each element of $\mathcal{F}_1$ onto an element of $\mathcal{F}_2$.

We now consider a foliation $\mathcal{F}$ of dimension n. The foliated chart ψ of a singular foliation specializes to a foliated chart $\psi : U \to B^n \times B^k$ in foliation theory: For $\psi = (x_1, \ldots, x_n, y_1, \ldots, y_k) = (x, y)$ the connected component of $\psi(M_q \cap U)$ containing $\psi(q)$ (the plaque) is equal to $B^n \times \{y(q)\}$ for all $q \in U$.

A vector field X of N is *foliated* if $[X,Y] \in \Xi(\mathcal{F})$ for all $Y \in \Xi(\mathcal{F})$. In other words, the vector space of foliated fields is the normalizer of $\Xi(\mathcal{F})$ in the Lie algebra $\Xi(N)$ of differentiable vector fields of N.

Lemma 2.8 *For $X \in \Xi(N)$ the following conditions are equivalent:*

(1) X is foliated.

(2) The local one parameter group $(\phi_t)_{-\varepsilon<t<\varepsilon}$ associated to X on a neighborhood of an arbitrary point of N leaves the distribution $\mathcal{D}$ invariant.

(3) In every simple open set U with local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_k)$ as above, the last k components of X only depend on the variables $y_1, \ldots, y_k$.

PROOF See [Mo], Proposition 2.2. □

Let U be simple. Then the quotient manifold $\bar{U} = U/(\mathcal{F}|U)$ carries a unique differentiable structure for which the projection $\pi : U \to \bar{U}$ becomes a submersion. The fibers of π are the plaques of U and the foliated fields are exactly the projectable fields on U. We see that the vector space of foliated vector fields on U modulo $\Xi(\mathcal{F}|U)$ is isomorphic to $\Xi(\bar{U})$ as vector spaces.

Now let (N, g) be a not necessarily complete Riemannian manifold and $\mathcal{F}$ be a foliation of N. Let $\mathcal{D}^\perp$ be the distribution orthogonal to $\mathcal{D}$. We call $\mathcal{D}^\perp$ horizontal and $\mathcal{D}$ vertical. We set $g_\perp$ to be the restriction of g to the horizontal distribution. Then g is called *bundle-like*, if $L_X g_\perp = 0$ for any vertical vector X. This is equivalent to the condition that $g(X, Y)$ is constant along the plaques of any simple set U for all horizontal foliated fields X and Y on U. If g is bundle-like we call $\mathcal{F}$ *Riemannian foliation*. A foliation $\mathcal{F}$ of (N, g) is obviously Riemannian if and only if we can cover N with simple sets U_i such that we can endow $\bar{U}_i = U/\mathcal{F}_{U_i}$ with a metric for which the canonical projection $\pi_i : U_i \to \bar{U}_i$ becomes a Riemannian submersion.

DEFINITION A partition $\mathcal{F}$ of a Riemannian manifold (N, g) into connected, injectively immersed submanifolds is called *transnormal* if for every $p \in N$ every geodesic in N starting orthogonally to T_pM_p intersects every element of $\mathcal{F}$ it meets orthogonally.

Proposition 2.9 *A foliation $\mathcal{F}$ is a Riemannian foliation if and only if it is transnormal.*

PROOF See [Rei]. □

This proposition justifies the following definition introduced in [Mo].

DEFINITION A transnormal singular foliation $\mathcal{F}$ of a Riemannian manifold N is called a *singular Riemannian foliation.*

$\mathcal{F}$ is *proper* if every leaf of $\mathcal{F}$ is properly embedded.

DEFINITION We say $\mathcal{F}$ *admits sections*, there is a complete, totally-geodesic submanifold Σ_p (called *section*) through every regular point $p \in N$ that meets every leaf and always orthogonally.

EXAMPLE The set of orbits of an isometric Lie group action on a Riemannian manifold N is a singular Riemannian foliation. The set of orbits of a polar action is a singular Riemannian foliation admitting sections.

Proposition 2.10 *Let $(\mathcal{F}, \mathcal{F}^\perp)$ be a bifoliation of a Riemannian manifold (N, g). Then $\mathcal{F}$ is a Riemannian foliation if and only if the leaves of $\mathcal{F}^\perp$ are totally geodesic.*

Proof This is the well-known duality between Riemannian and totally geodesic foliations: Following [BH3] we have for all vector fields V tangential to $\mathcal{F}$ (vertical) and H_1, H_2 tangential to $\mathcal{F}^\perp$ (horizontal):

$$\begin{aligned}
(L_V g)(H_1, H_2) &= Vg(H_1, H_2) - g([V, H_1], H_2) - g(H_1, [V, H_2]) \\
&= g(\nabla_V H_1 - [V, H_1], H_2) + g(H_1, \nabla_V H_2 - [V, H_2]) \\
&= g(\nabla_{H_1} V, H_2) + g(H_1, \nabla_{H_2} V) \\
&= -2g(V, \nabla_{H_1} H_2) \\
&= -2g(V, \alpha(H_1, H_2)),
\end{aligned}$$

where α is the second fundamental form of leaves of $\mathcal{F}^\perp$ and ∇ is the Levi-Civita connection of N. This means that g is bundle-like if and only if $\mathcal{F}^\perp$ is a totally geodesic foliation. □

Later we will need the following lemma.

Lemma 2.11 *Let $(\mathcal{F}, \mathcal{F}^\perp)$ be a bifoliation of (N, g), where $\mathcal{F}$ is Riemannian. Let U be simple. Then the Lie algebra of horizontal $\mathcal{F}$-foliated fields on U acts transitively on the normal space $\nu_p P$ for every $p \in U$, where P is the $\mathcal{F}$-plaque of $\mathcal{F}$ through p, and is isomorphic to the Lie algebra of vector fields on the quotient manifold $\bar{U}$ by projection π_*. The restriction of such a foliated field to an $\mathcal{F}$-plaque P is a parallel normal field of P. It is then obvious that a leaf of $\mathcal{F}$ must have a flat normal bundle.*

Proof The first part is clear. Now let P be an $\mathcal{F}$-plaque of U. We want to show that the restriction of a horizontal foliated field to P is a parallel normal field of P. Let V be a vertical and H_1, H_2 be horizontal foliated vector fields. Then

$$\begin{aligned}
g(\nabla_V H_1, H_2) &= g(\nabla_{H_1} V + [V, H_1], H_2) \\
&= g(\nabla_{H_1} V, H_2) \\
&= -g(V, \nabla_{H_1} H_2) \\
&= 0
\end{aligned}$$

since $[V, H_1]$ is vertical and $\nabla_{H_1} H_2$ is horizontal, because $\mathcal{F}^\perp$ is a totally geodesic foliation. Now take a point p in U and let $P \in \mathcal{F}|U$ be the plaque through p. The above calculation showed that a horizontal foliated vector field in U is a parallel normal field of P. Since the vector space of values of horizontal foliated vector fields in p spans $\nu_p P$, each parallel normal field along P is the restriction of a foliated horizontal vector field to P. Now it is clear that the normal bundle of a leaf of $\mathcal{F}$ is flat. □

Remark Along a curve in a leaf, normal parallel translation is the same as sliding along the leaves. For the definition of the latter, see [Mo].

Let $(\mathcal{F}, \mathcal{F}^\perp)$ be a Riemannian/totally-geodesic foliation of a Riemannian manifold (N, g) such that the leaves of $\mathcal{F}^\perp$ are complete with the induced metric. Note that if N is complete so are the leaves of $\mathcal{F}$ and $\mathcal{F}^\perp$ with respect to the induced metric. A curve in an element of $\mathcal{F}$ respectively $\mathcal{F}^\perp$ is called *vertical* respectively *horizontal.*

Lemma 2.12 (Blumenthal-Hebda) *Let $\tau : [0,1] \to N$ be a vertical and $\sigma : [0,1] \to N$ be a horizontal curve with $\tau(0) = \sigma(0)$. Then there is a unique map $H = H_{(\tau,\sigma)} : [0,1] \times [0,1] \to N$ with*

(1) $H(\,\cdot\,, 0) = \tau$,

(2) $H(0, \,\cdot\,) = \sigma$,

(3) $H(\,\cdot\,, t)$ *is vertical,*

(4) $H(s, \,\cdot\,)$ *is horizontal.*

Proof See [BH1], Corollary 2.7 which is based on Lemma 2.6. Note that for the proof of the latter one can drop completeness of N and assume completeness of the horizontal leaves instead. □

We write $T_\sigma\tau = H_{(\tau,\sigma)}(\,\cdot\,, 1)$ and $T_\tau\sigma = H_{(\tau,\sigma)}(1, \,\cdot\,)$. For a vertical curve τ in a leaf $M \in \mathcal{F}$ respectively a horizontal curve σ in a leaf $\Sigma \in \mathcal{F}^\perp$ we write $[\tau]$ respectively $[\sigma]$ for the equivalence class of curves under homotopy in M respectively Σ fixing endpoints. Then $[T_\sigma\tau]$ and $[T_\tau\sigma]$ only depend on $[\tau]$ and $[\sigma]$. A continuous map $H : [0,1]\times[0,1] \to N$, such that $H(\,\cdot\,, t)$ is vertical for any t and $H(s, \,\cdot\,)$ is horizontal for any s, is called *rectangle* with initial vertical/horizontal curve $H(\,\cdot\,, 0)/H(0, \,\cdot\,)$, terminal vertical/horizontal curve $H(\,\cdot\,, 1)/H(1, \,\cdot\,)$ and diagonal $t \mapsto H(t,t)$. The following lemma is easy to prove.

Lemma 2.13 *For any curve $\mu : [0,1] \to N$ we find a unique rectangle $H : [0,1] \times [0,1] \to N$ with diagonal μ.*

Let μ and H be as above. We write μ_v respectively μ_h for the initial vertical respectively horizontal curve of H and μ^v respectively μ^h for the terminal vertical respectively horizontal curve of H.

We recall that the universal cover $\tilde{M}$ of a manifold M is equal to the set of equivalence classes of curves starting from a fixed point x_0, where the equivalence is given by homotopy fixing endpoints; in some cases we write more precisely $\widetilde{(M, x_0)}$. The covering map $\tilde{M} \to M$ is given by $[\sigma] \mapsto \sigma(1)$. Let x_0 be arbitrary and let M respectively Σ be the element of $\mathcal{F}$ respectively $\mathcal{F}^\perp$ through x_0. Then

$$\begin{aligned} \tilde{M} &= \{[\tau] \mid \tau \text{ is vertical and } \tau(0) = x_0\} \\ \tilde{\Sigma} &= \{[\sigma] \mid \sigma \text{ is horizontal and } \sigma(0) = x_0\} \\ \tilde{N} &= \{[\mu] \mid \mu \text{ is a curve in } N \text{ and } \mu(0) = x_0\} \end{aligned}$$

Here $[\,]$ means in each case an equivalence class under a different homotopy.

Theorem 2.14 (Blumenthal-Hebda) *The map*

$$\begin{aligned} \Phi : \tilde{M} \times \tilde{\Sigma} &\cong \tilde{N} \\ ([\tau], [\sigma]) &\mapsto [\sigma T_\sigma \tau] \end{aligned}$$

is a diffeomorphism with $\Phi^{-1}([\mu]) = ([\mu_v], [\mu_h])$*. Both maps are foliated with respect to the product foliation on* $\tilde{M} \times \tilde{\Sigma}$ *and the lift of the bifoliation of* $\tilde{N}$ *via the universal covering map* $\tilde{N} \to N$*.*

Below we will work out the sketch of the proof given in [BH2] thereby introducing a little technique that will give us a new insight into the fundamental group of N. First we state the following result:

Theorem 2.15 (Blumenthal-Hebda) *The map*

$$\begin{aligned} \Psi : \tilde{M} \times \tilde{\Sigma} &\rightarrow N \\ ([\tau], [\sigma]) &\mapsto T_\tau \sigma(1) \end{aligned}$$

is the universal covering map, and it is foliated with respect to the product foliation of $\tilde{M} \times \tilde{\Sigma}$ *and the bifoliation* $(\mathcal{F}, \mathcal{F}^\perp)$*. Moreover, given* $[\tau] \in \tilde{M}$ *the map*

$$T_\tau : \widetilde{(\Sigma, x_0)} \rightarrow \widetilde{(\Sigma_x, x)}; [\sigma] \mapsto [T_\tau \sigma]$$

is an isometry, where $y = \tau(1)$*. In particular, the horizontal leaves have the same Riemannian universal cover. Similarly the vertical leaves have the same universal cover.*

Proof The first part follows from Theorem 2.14. For the second, see [BH2], Proposition 3.1. This proof makes only use of Lemma 2.12. Thus completeness of N can be replaced by completeness of the horizontal leaves. □

Now we will prove 2.14. The map $\Phi : \tilde{M} \times \tilde{\Sigma} \to \tilde{N}; ([\sigma], [\tau]) \mapsto [\sigma T_\sigma \tau]$ is clearly well-defined, because $T_\sigma \tau$ only depends on $[\sigma]$, and $[T_\sigma \tau]$ only on $[\sigma]$ and $[\tau]$. To prove that Φ^{-1} defined by $[\mu] \mapsto ([\mu_v], [\mu_h])$ is well-defined, let $[\mu_1] = [\mu_2] \in \tilde{N}$. Then $x_0 = \mu_1(0) = \mu_2(0)$ and $\mu_1(0) = \mu_2(0)$ which we denote by x. Let $G : [0,1] \times [0,1] \to N$ be a homotopy in N fixing endpoints with $G(\,\cdot\,, 0) = \mu_1$ and $G(\,\cdot\,, 1) = \mu_2$. We define $C : [0,1]^3 \to N$ such that $C(\,\cdot\,, \,\cdot\,, t)$ is the unique rectangle with diagonal $G(\,\cdot\,, t)$ for each t. We have $C(0, 0, \,\cdot\,) = c_{x_0}$ and $C(1, 1, \,\cdot\,) = c_x$, where c is a constant curve. Then $C(1, 0, \,\cdot\,)$ is a curve in $M \cap \Sigma_x$. Since $C(1, 0, \,\cdot\,)$ is continuous and $M \cap \Sigma_x$ is an at most countable set of points (M is second countable), it follows that $C(1, 0, \,\cdot\,)$ is the constant map. Similarly $C(0, 1, \,\cdot\,)$ is the constant map. Then

(1) $C(\,\cdot\,, 0, \,\cdot\,)$ is a homotopy in M between $(\mu_1)_v$ and $(\mu_2)_v$ fixing endpoints,

(2) $C(\,\cdot\,, 1, \,\cdot\,)$ is a homotopy in M_x between $(\mu_1)^v$ and $(\mu_2)^v$ fixing endpoints,

(3) $C(0, \,\cdot\,, \,\cdot\,)$ is a homotopy in Σ between $(\mu_1)_h$ and $(\mu_2)_h$ fixing endpoints,

(4) $C(1, \,\cdot\,, \,\cdot\,)$ is a homotopy in Σ_x between $(\mu_1)^h$ and $(\mu_2)^h$ fixing endpoints.

Therefore $[(\mu_1)_v] = [(\mu_2)_v]$ in $\tilde{M}$ and $[(\mu_1)_h] = [(\mu_2)_h]$ in $\tilde{\Sigma}$, so Φ^{-1} is also well-defined.

We have

$$\Phi^{-1} \circ \Phi([\sigma], [\tau]) = \Phi^{-1}([\sigma T_\sigma \tau]) = ([\sigma c_{\sigma(1)}], [c_{x_0} \tau]) = ([\sigma], [\tau])$$

$$\Phi \circ \Phi^{-1}([\mu]) = \Phi([\mu_v], [\mu_h]) = [\mu_v \mu^h] = [\mu].$$

Thus Φ is a diffeomorphism. If $p : \tilde{N} \to N; [\mu] \mapsto \mu(1)$ denotes the universal covering map, we have $\pi_1(N, x_0) = p^{-1}(x_0)$.

Note that the action of $\pi_1(N, x_0)$ on $\tilde{N} \cong \tilde{M} \times \Sigma$ preserves the bifoliation. Consider $[\mu] \in \pi_1(N, x_0)$. There is a vertical curve σ and a horizontal curve τ with $\sigma(0) = \tau(1) = x_0$ and $\tau(0) = \sigma(1)$ such that $[\mu] = [\sigma\tau]$; we can choose $\sigma = \mu_v$ and $\tau = \mu^h$. By the previous discussion σ and τ are unique up to homotopy in the corresponding leaf fixing endpoints, i.e., up to $[\sigma]$ and $[\tau]$. The map $I : \pi_1(N, x_0) \to M \cap \Sigma; [\mu] \mapsto \mu_v(1)$ is well-defined and surjective. The action of $\pi_1(M, x_0) \times \pi_1(\Sigma, x_0)$ on $\pi_1(N, x_0)$ by $([\sigma], [\tau]) \cdot [\mu] := [\sigma\mu\tau^{-1}]$ is free: Assume $[\mu] \in \pi_1(N; x_0)$ is fixed by $([\sigma], [\tau]) \in \pi_1(M, x_0) \times \pi_1(\Sigma, x_0)$ then $[\mu] = [\sigma\mu\tau^{-1}] = [\sigma\mu_v\mu^h\tau^{-1}]$ and therefore, by applying Φ^{-1}, $[\mu_v] = [\sigma\mu_v]$ and $[\mu_h] = [\tau\mu_h]$, so $[\sigma]$ and $[\tau]$ are trivial. So the action is free. We observe that I maps all points of an orbit to a single value. Then we have

$$\pi_1(N, x_0) // (\pi_1(M, x_0) \times \pi_1(\Sigma, x_0)) \cong M \cap \Sigma.$$

This implies:

Proposition 2.16 *Let N be bifoliated as above. Then*

$$|\pi_1(N, x_0)| = |\pi_1(M, x_0)| \cdot |\pi_1(\Sigma, x_0)| \cdot |M \cap \Sigma|.$$

In the case of infinity, the interpretation of this equation is that the left value is infinity if and only if at least one of the factors on the right side is infinity.

3 Submanifolds with Parallel Focal Structure

Let M and N be complete and connected Riemannian manifolds and $\varphi : M \to N$ an isometric immersion. The aim of this section is to find minimal conditions for M to foliate the ambient space N with parallel submanifolds.

We take a look at the geometry of the tangent bundle and then at Jacobi fields. Let $v \in T_xN$. We define the *vertical space* $V_v^N = d\pi(v)^{-1}(0) \subset T_vTN$, where $\pi : TN \to N$ is the bundle projection. Since T_xN is a submanifold of TN, T_vT_xN is a subspace of T_vTN and coincides with V_v^N. The *horizontal space* H_v^N in T_vTN is defined to be the space of derivatives $\dot{w}(0)$ of parallel vector fields w with respect to the Levi-Civita connection ∇ along curves in N through x with $w(0) = v$. One can see $T_vTN = H_v^N \oplus V_v^N$. The linear map $d\pi(v)|H_v^N : H_v^N \to T_xN$ is an isomorphism. Let $\iota_v : T_vT_xN \to T_xN$ be the canonical identification. We define the *connection map* $K : TTN \to TN$ by $K(\xi) = \iota_v \circ \tau(\xi)$, where $\xi \in T_vTN$ and τ is the projection of T_vTN onto $V_v^N = T_vT_xN$ along H_v^N. The linear map $K|V_v^N : V_v^N \to T_xN$ is an isomorphism. Then $\pi_* \times K : T_vTN = H_v^M \oplus V_v^N \to T_xN \times T_xN$ is isomorphic. Summing up, the Levi-Civita connection ∇ gives us

$$T_vTN = H_v^M \oplus V_v^N \cong T_xN \times T_xN.$$

Pulling back the metric on N by $\pi_* \times K$ we obtain the so-called *Sasaki metric* on TN. For an element $\xi \in T_vTN$ we write $\xi = (\xi_h, \xi_v)$. If $\frac{dX}{dt}\big|_{t=0} = \xi$ for a vector field X along a curve c with $c(0) = x$ and $X(0) = v$ then $\xi_h = \dot{c}(0)$ and $\xi_v = (\nabla_{dt}X)(0)$.

We consider a submanifold M of N. For $v \in \nu_xM$ we have the decomposition

$$T_v\nu M = H_v^M \oplus V_v^M \cong T_xM \oplus \nu_xM$$

with respect to the Levi-Civita connection $\nabla^\perp$ on the normal bundle. Obviously $V^M \subset V^N$ but in general we do not have $H^M \subset H^N$. Indeed, an element $\xi = (\xi_h, \xi_v) \in T_v\nu M$ is equal to $(\xi_h, \xi_v - A_v\xi_h)$ as an element of T_vTN, where A is the shape operator of M.

We have the isomorphism between T_vTN and the vector space of Jacobi fields of N along γ_v mapping an element $\xi \in T_vTN$ to the Jacobi field J given by $(J(t), J'(t)) = \phi_*^t(\xi_h, \xi_v)$, where ϕ^t is the time t map of the geodesic flow $\phi : \mathbb{R} \times TN \to TN$. The inverse map is given by $J \mapsto (J(0), J'(0))$. The restriction of the first map to $T_v\nu M$ is an isomorphism onto the vector space $\mathcal{J}_M(v)$ of M-Jacobi fields along γ_v

$$T_v\nu M \to \mathcal{J}_M(v); \quad \xi \mapsto J_\xi \quad \text{with} \quad (J_\xi(t), J_\xi'(t)) = \phi_*^t(\xi_h, \xi_v - A_v\xi_h).$$

The inverse map is given by $J \mapsto (J(0), J'(0)^\perp)$. The decomposition $T_v\nu M = H_v^M \oplus V_v^M$ carries over to the decomposition of $\mathcal{J}_M(v)$ into a horizontal and a vertical subspace. We can describe a vertical/horizontal M-Jacobi field J with initial condition $\xi \in T_v\nu M$ by a variational vector field. Define $V(s,t) = \eta(tX(s))$, where X is a vector field along the constant curve $c \equiv x$ with $\frac{dX}{dt}|_{t=0} = \xi_v$ if J is vertical, and a parallel normal field along c in M with $\dot{c}(0) = \xi_h$ if J is horizontal. Then $J(t) = \partial_s V(0,t)$.

Definition Let $\varphi : M \to N$ be an immersion. For $x \in M$ we call an isometric immersion $i_x : \Sigma_x \to N$ with $(i_x)_*(T_x\Sigma_x) = \iota(\nu_x M)$ (or shorter Σ_x) a *section*, if it is totally geodesic in N and if Σ_x is complete. Since we want i_x to be unambiguous, we also demand $i_x(y) = i_x(z)$ whenever $(i_x)_*(T_y\Sigma_x) = (i_x)_*(T_z\Sigma_x)$. The immersion $\varphi : M \to N$ is said to *admit sections* if Σ_x is a section for every $x \in M$ and if there is exactly one section of φ through every regular point of the normal exponential map, i.e. if $p \in i_x(\Sigma_x) \cap i_y(\Sigma_y)$ is regular then $i_x \cong i_y \circ \alpha$ for some isometry $\alpha : \Sigma_x \to \Sigma_y$.

In order to avoid a cumbersome notation, we use Σ_x and the term section in two different ways. When it comes to point sets, for instance, if we write $p \in \Sigma_x$, we actually mean by Σ_x the image of the immersion i_x. If we talk about tangent vectors or curves of Σ_x, i.e., if the context is a topological or differentiable one, we are of course referring to the underlying manifold structure of the section. This distinction is particularly important here, since we allow Σ_x to have self-intersections.

Remark If φ admits sections then $\eta : \nu M \to N$ is surjective. Also note that there is a section through every point in N, if φ is proper.

Lemma 3.1 *Let γ be a geodesic in a section $\Sigma = \Sigma_x$ with $\gamma(0) = p = \varphi(x)$. Then any Jacobi field in N along γ can be decomposed into $J = J_1 + J_2$, where J_1 is a Jacobi field of Σ and J_2 is Jacobi field with $J_2(t) \in T_{\gamma(t)}\Sigma^\perp$ for every t. For an M-Jacobi field J this decomposition is exactly the one into vertical and horizontal M-Jacobi fields along γ. In particular we have $J(t) \perp T_{\gamma(t)}\Sigma$ for a horizontal M-Jacobi field J.*

Proof We write J_1 for the $T\Sigma$-part of J and J_2 for the orthogonal part. Since Σ is totally geodesic, the curvature operator $R_{\dot\gamma(t)}$ leaves $T_{\gamma(t)}\Sigma$ invariant and therefore, as a self-adjoint operator, also the orthogonal complement $T_{\gamma(t)}\Sigma^\perp$, so $R_{\dot\gamma(t)}J_1(t) \in T_{\gamma(t)}\Sigma$ and $R_{\dot\gamma(t)}J_2(t) \in T_{\gamma(t)}\Sigma^\perp$. On the other hand we have $J_1''(t) \in T_{\gamma(t)}\Sigma$, since Σ is totally geodesic, and $J_2''(t) \in T_{\gamma(t)}\Sigma^\perp$ for all t, because of $0 = \frac{d^2}{dt^2} g(J_2(t), X(t)) = g(J_2''(t), X(t))$ for any parallel field X of Σ along γ. The Jacobi identity for J gives

$$0 = R_{\dot\gamma(t)}J(t) + J''(t) = (R_{\dot\gamma(t)}J_1(t) + J_1''(t)) + (R_{\dot\gamma(t)}J_2(t) + J_2''(t)).$$

Since the term in the first bracket lies in $T_{\gamma(t)}\Sigma$ and the term in the second in $T_{\gamma(t)}\Sigma^\perp$, the vector fields J_1 and J_2 are also Jacobi fields. The second statement follows from the initial conditions $(J_i(0), J_i'(0)^\perp)$ of J_i for $i = 1, 2$. □

The kernel of $d\eta(v)$ consists of $(J(0), J'(0)^\perp)$, where J is an M-Jacobi field along γ_v with $J(1) = 0$. The decomposition $J = J_1 + J_2$ as in the lemma then implies that $\ker d\eta(v)$ is a direct sum of a horizontal and a vertical subspace of $T_v\nu M$ and that the kernel of $d\eta(v)$ only has a non-trivial vertical component if and only if $\eta(v)$ is a conjugate point of x along γ_v in Σ_x. Summing up, the decomposition of an M-Jacobi field J into $J = J_1 + J_2$ means that

$$(1) \qquad \begin{array}{rcl} d\exp^\perp(v) : H_v^M \oplus V_v^M & \to & T_{\eta(v)}\Sigma^\perp \oplus T_{\eta(v)}\Sigma \\ (J(0), J'(0)^\perp) & \mapsto & J_1(1) + J_2(1) \end{array}$$

splits as an orthogonal direct sum of linear maps $H_v^M \to T_{\eta(v)}\Sigma^\perp$ and $V_v^M \to T_{\eta(v)}\Sigma$.

Definition We call a focal normal v of *horizontal/vertical type* if $\ker d\eta(v)$ has a non-trivial horizontal/vertical component. If a normal vector v is not a focal normal of horizontal type we call v *f-regular*. A point $p \in N$ is called *f-regular* if there is an f-regular normal v such that $\eta(v) = p$. For a normal vector $v \in \nu_x M$ we call the dimension of the horizontal factor of $\ker d\eta(v)$ the *horizontal multiplicity* of v.

Compare these definitions with [Ew].

We assume that φ admits sections and that νM is flat. We define two distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on the set of f-regular points in N by $\mathcal{D}^\perp(p) = T_p\Sigma$, where Σ is a section through p; let $\mathcal{D}$ be the orthogonal distribution. The distribution $\mathcal{D}^\perp$ and therefore $\mathcal{D}$ are well-defined on the set of regular points, since M admits sections, but a priori not on the set of f-regular points. It is easy to see that both distributions are integrable on the regular set: Let p be a regular point and $v \in \nu M$ with $\eta(v) = p$. Recall that νM carries the horizontal foliation $\mathcal{P}$ given by normal parallelity, and the vertical foliation given by the fibers of the projection $\nu M \to M$. Now let U_v be an open neighborhood of $v \in \nu M$ such that $\eta|U_v : U_v \to V$ from U_v onto its image V is a diffeomorphism. The map $\eta|U_v$ maps vertical leaves diffeomorphically onto the connected components of the sections intersected with V. Because of (1), $d\eta$ maps the horizontal distribution on νM to $\mathcal{D}$, i.e. $d\eta(v)(T_x M) = \mathcal{D}(\eta(v))$. Since U_v is bifoliated and $\eta|U_v$ is a diffeomorphism, V is also bifoliated with respect to $\mathcal{D}$ and $\mathcal{D}^\perp$. We want to show that both distributions are also differentiable and well-defined on the set N_r of f-regular points in N. Integrability is clear.

Lemma 3.2 *There is exactly one section Σ through a given f-regular point p and $\eta^{-1}(p)$ only consists of f-regular vectors that are tangential to Σ. Moreover, N_r is open and dense in N and there is a unique differentiable extension of $\mathcal{D}^\perp$ on N_r.*

Proof Existence follows by surjectivity of η. We show uniqueness. Let $v_0 \in \nu_x M$ be an f-regular vector with $\eta(v_0) = p$. Then there is a simply connected neighborhood U of x in M such that $(\eta \circ v)|U : U \to P_{v_0} = \eta(U)$ is a diffeomorphism, where v is a parallel normal field on U with $v_x = v_0$. We define $T = tube(P_{v_0}, \varepsilon) = \{\exp(\xi) \mid \xi \in B_\varepsilon(\nu P_{v_0})\}$. By shrinking U we can assume that T is an injectivity tube around P_{v_0} for small $\varepsilon > 0$. Let $\rho : T \to P_{v_0}$ be the projection. We have $T_{\eta(v_z)}\Sigma_z \perp T_{\eta(v_z)}P_{v_0}$ for every $z \in U$ by (1). Therefore

- a slice of the tube T through $\eta(v_z) \in P_{v_0}$ coincides with the component of $\Sigma_z \cap T$ containing $\eta(v_z)$.

We can therefore extend $\mathcal{D}^\perp$ differentiably to T as the kernel of the differential of the submersion ρ. Since $\mathcal{D}^\perp$ is defined on the open and dense set of regular points of N,

- this extension is the unique differentiable extension of $\mathcal{D}^\perp$.

Let $w_0 \in \nu_y M$ be another f-regular vector with $\eta(w_0) \in T$. The same process as for v_0 gives us a simply connected neighborhood U' of y, a parallel normal field w extending w_0, P_{w_0} and its tube T' with the same properties. By eventually shrinking U' and the radius of T' we can assume $T' \subset T$. By the uniqueness of a differentiable extension of $\mathcal{D}^\perp$ we conclude that the slices of T' are equal to the slices of T intersected with the open set T'. In particular, if $\eta(w_0) = p$ this implies that v_0 and w_0 are tangent to the same section $\Sigma_x = \Sigma_y$. Again by (1) P_{w_0} intersects these slices orthogonally. Since w_0 is f-regular, $\eta \circ w$ has maximal rank on a neighborhood of y. We can assume this neighborhood to be U'. Then P_{w_0} intersects the slices transversally, i.e.

- $\rho \circ \eta \circ w : U' \to P_{v_0}$ is a diffeomorphism onto its image.

We have seen above that the f-regular vectors in $\eta^{-1}(p)$ are tangential to the same section. Now we are going to show that any $w_0 \in \nu M$ with $\eta(w_0) \in T$ is f-regular. Then $T \cap \Sigma$ is an open neighborhood of p in Σ only containing f-regular points. This implies that the set of f-regular points is open in N and that $\eta^{-1}(p)$ only consists of f-regular vectors. We remark that this even shows that the f-regular points in a section Σ are open in Σ (see Remark(3)). Assume there is a focal normal $w \in \nu_y M$ of horizontal type with $\eta(w_0) \in T$ and U' a neighborhood of w_0 in νM such that $\eta(U') \subset T$. We can locally define a parallel normal field w extending w_0. Then there is a simply connected neighborhood U of y in M and an $\varepsilon > 0$ such that the image of U under $(1+t)w$ lies in U' for all $t \in (0, \varepsilon)$ and such that w'_z is f-regular for every $z \in U$, where $w' = (1+\varepsilon)w$. The geodesic $\gamma_{w(z)}$ intersects $P_{(1+\varepsilon)w_0}$, the image of $\eta \circ w'$, orthogonally in $\gamma_{w(z)}(1+\varepsilon)$ for all $z \in U$ by (1) or the Gauss Lemma for the normal exponential map. Then the image of $\gamma_w|[1, 1+\varepsilon]$ lies in a slice of the tube T. Therefore

$$\rho \circ \eta \circ w = \rho \circ \eta \circ w'$$

on U. Since the right side is a diffeomorphism this implies that also $\eta \circ w$ has maximal rank, i.e. w_0 is f-regular. □

Remark

(1) The lemma says that the preimage of a focal point $\eta(v)$, where v is a focal normal of horizontal type, only consists of focal normals of horizontal type.

(2) The sections intersect the images of φ and $\eta \circ \bar{v}$ for f-regular v always orthogonally and transversally. This implies that φ and $\eta \circ \bar{v}$ factorize through injective immersions, the first even through an injective isometric immersion. A proper immersion φ factorizes finitely over an embedding (Proposition 2.2). We will furtheron assume that φ is injective and there is no loss of generality if we assume φ to be the inclusion map of M into N.

(3) By the Theorem of Sard the set of regular points of η is open and dense in N. Obviously the intersection of the set of regular points with Σ is open in Σ. But since a section Σ is a null-set in N, it is a priori not clear that the set of regular points in Σ is dense in Σ. The lemma can be applied to prove: *The subset of (almost) regular points in a section Σ is open and dense in Σ.* We will show that the complement C of the set of regular points in Σ is a null-set. C is the union of the set A of endpoints of focal normals of horizontal

type and the set B of f-regular focal points. We define $S := M \cap \Sigma$. Take an arbitrary point $x \in S$. Then A is contained in the null set of endpoints of singular vectors in $\nu_x M$ of η; if p is such an endpoint, there is a $v \in \nu_x M$ with $\eta(v) = p$ by completeness of Σ, and v is a focal normal of horizontal type by the lemma. B is the union of conjugate points of $\exp_x^\Sigma$ for every $x \in S$ (normal vectors $v \in \nu M$ such that $\eta(v)$ is f-regular are tangential to Σ by the lemma and therefore have their foot point contained in S). Since M intersects Σ orthogonally, the set S is at most countable. Thus B is a null set, too. This means that the set of regular points in Σ is open and dense in Σ. We remarked in the proof of the previous lemma that the subset of almost regular points in Σ is open in Σ. As the set of regular points is contained in the set of f-regular points, we have proved our claim.

Definition An immersion $\varphi : M \to N$ is said to have *parallel focal structure*, if

(1) νM is flat,

(2) $\dim(\ker d\eta(v) \cap H_v^M) = \dim \ker d(\eta \circ v)$ is constant for any local parallel normal field v, i.e. the horizontal focal data is invariant under normal parallel translation, and

(3) φ admits sections.

Note that this definition of a parallel focal structure differs from that in [Ew]; we do not demand the invariance of the vertical data. We will show in Proposition 3.12 that this second invariance is an implication.

Example Regular orbits of polar actions have parallel focal structure. Isoparametric submanifolds in $\mathbb{R}^{n+k}$ and equifocal submanifolds in simply connected, compact symmetric spaces obviously fulfill conditions (1) and (2) of a submanifold. The existence of sections is clear for the first class of submanifolds and a consequence for the second class. Theorem 3.10 will show, that they admit sections if and only if the set of parallel manifolds builds a foliation on the regular set, which is known for both classes.

Let $\bar{M}$ be the normal holonomy principal bundle over M equipped with the metric such that the projection $\bar{M} \to M$ becomes a Riemannian covering. Its normal bundle is globally flat and $\bar{M} \to M$ has the lowest degree among all coverings of M with this property. Each normal vector v of M canonically defines a global parallel normal field on $\bar{M}$, denoted by $\bar{v}$. We will denote the normal exponential map of $\bar{M}$ also by η.

Remark If φ is in addition proper and v is f-regular and has finite normal holonomy degree then $\eta_v = \eta \circ \bar{v} : \bar{M} \to N$ is also a proper immersion, since η^r is proper.

Definition Let $\varphi : M \to N$ have parallel focal structure. We call $\eta \circ \bar{v} : \bar{M} \to N$ a *focal submanifold* of M if $v \in \nu M$ is a focal normal of horizontal type, a *parallel submanifold*, if v is f-regular. In any case we denote the image by M_v.

Let v be a focal normal of horizontal type. Since the map $\eta \circ \bar{v}$ has constant rank, the set of connected components of preimages of $\eta \circ \bar{v}$ defines a foliation (the *focal foliation*) by the rank theorem which gives us simple sets for this foliation. The leaf through x is called the *focal leaf* $F_{\bar{v}_x}$ through x associated to v (or to $\bar{v}_x$).

Proposition 3.3 *The set M_v is the image of an immersed submanifold for any $v \in \nu M$.*

PROOF See [Ew], Proposition 2.7. The statement is clear by definition for f-regular v. Let v be a focal normal of horizontal type. The focal foliation $\mathcal{G}$ given by $\eta \circ \bar{v}$ is regular in the sense of [Pa] because of the rank theorem. Then Theorem VIII of [Pa] implies that the quotient $\bar{M}/\mathcal{G}$ is a differentiable manifold such that the quotient map is a submersion. Note that it is not necessarily Hausdorff or second countable. The map $\eta \circ \bar{v} : \bar{M} \to M_v$ now induces an immersion $M/\mathcal{G} \to M_v$. □

A priori parallel or focal manifolds can have intersections with themselves or with other parallel or focal submanifolds. According to [HLO], we say that M gives rise to a *global foliation* or $\mathcal{F} = \{M_v \mid v \in \nu_p M\}$ of N, if $\bigcup \mathcal{F} = N$, and $M_v \cap M_w \neq \emptyset$ implies $M_v = M_w$. The aim of this section is to show that the parallel foliation induced by M is a global foliation of N. We will see that the normal exponential map $\exp^\perp$ of M becomes a foliated map, mapping the horizontal foliation on νM to the parallel foliation $\mathcal{F}$ on N; moreover $\exp^\perp$ respects the vertical foliation, mapping the vertical spaces $\nu_x M$ onto the totally geodesic "leaves", the sections.

Our aim is to show that there is a bifoliation $(N_r, \mathcal{F}, \mathcal{F}^\perp)$, where $\mathcal{F}$ is a regular Riemannian foliation of parallel submanifolds of M and $\mathcal{F}^\perp$ a foliation with totally geodesic submanifolds, the sections restricted to N_r. Properness of φ is not assumed.

Proposition 3.4 *The two distributions from above give rise to a bifoliation $(\mathcal{F}, \mathcal{F}^\perp)$ on N_r, where $\mathcal{F}$ is a Riemannian foliation and $\mathcal{F}^\perp$ a totally geodesic foliation. The leaves of $\mathcal{F}$ respectively of $\mathcal{F}^\perp$ are the parallel submanifolds respectively the connected components of the sections restricted to N_r. Therefore the parallel submanifolds have a flat normal bundle.*

PROOF By Lemma 3.2, $\mathcal{D}^\perp$ is differentiable. Thus $\mathcal{D}$ as the orthogonal distribution is differentiable, too. Obviously both distributions are integrable. $\mathcal{F}^\perp$ restricted to N_r is a totally geodesic foliation. Then Proposition 2.10 implies that $\mathcal{F}$ restricted to N_r is a Riemannian foliation.

Now we want to show that the parallel submanifolds M_v for f-regular v are exactly the leaves of $\mathcal{F}$. We first prove that M is a leaf, where we consider M to be included in N (see Remark(2)). Choose a point p in M. Since the bundle TM is equal to the distribution $\mathcal{D}$ of $\mathcal{F}$ restricted to M, we have $M \subset M_p$, where M_p is the leaf of $\mathcal{F}$ through p. We endow M_p with the induced metric. As M is a connected, open and complete subset of M_p it follows $M = M_p$: Let q be a border point of M in M_p, so $q \in N \backslash M$. Let B be an injectivity ball of M_p around q. Then there is a point q' of M in B. Let $\gamma : [0,1] \to B$ be a geodesic of N from q to q'. We have $w = -\dot{\gamma}(1) \in T_{q'}M$ since q' has an open neighborhood in M. As M is complete, the geodesic γ^{-1} from q' to q with initial vector w is a geodesic of M, so $q = \gamma^{-1}(1)$ lies in M, contradiction.

Now let $v \in \nu_p M$ be an f-regular vector and $q = \eta(v)$. M_p and M_q are the leaves through p respectively q. We want to show $M_v = M_q$. The regular leaves have a flat normal bundle by Lemma 2.11. Let $w = \phi^1(v)$. Let L_v and L_w be the horizontal leaves of νM_p respectively νM_q through v respectively w. We are going to show that the restriction of the time 1 map ϕ^1 of the geodesic flow of N to L_v, $\phi^1|L_v : L_v \to L_w$, is a diffeomorphism. It suffices to show $\phi^1(L_v) = L_w$. Since $M = M_p$ and since $\eta \circ \bar{v}$ has constant rank, we have

$$d(\pi \circ \phi^1) = \mathcal{D} \circ (\pi \circ \phi^1)$$

on L_v by (1), where $\mathcal{D}$ is the distribution of $\mathcal{F}$ and $\pi : TN \to N$ the foot point map. Thus $\pi \circ \phi^1(L_v) \subset M_q$, because $\mathcal{F}$ is a foliation. We claim that $\phi^1(L_v)$ is horizontal in νM_q. It suffices to verify this locally. Let $x \in M$ be arbitrary and $v' \in L_v$ with foot point x. There is a neighborhood U of x in Σ_x and a vector field X on U with $X(x) = v'$ and such that $\pi \circ \phi^1 \circ X$ is a diffeomorphism to a neighborhood V of $y := \eta(v')$ in Σ_x. We extend X to a simple neighborhood U' of x with $U' \cap \Sigma_x = U$ such that the restriction of X to the plaques in U' are parallel normal fields. Then X is a foliated field on U' by Lemma 2.11. As above we see that $\pi \circ \phi^1 \circ X : U' \to V'$ is a diffeomorphism onto its image V', mapping plaques to plaques. Then the unique vector field X' on V' with $X' \circ \pi \circ \phi^1 = \phi^1 \circ X$ is foliated by Lemma 2.8. Thus the restriction of X' to a plaque P' of V' is a normal parallel field of P' by Lemma 2.11. In particular, this proves our claim that $\phi^1(L_v)$ is horizontal in νM_q and $\phi^1(L_v) \subset L_w$. By the same argument as above we have $\phi^1(-L_w) \subset L_{-v} = -L_v$. Since $\phi^{-1}(w') = -\phi^1(-w')$, this implies $\phi^1(L_v) = L_w$ and $\phi^1|L_v : L_v \to L_w$ is a diffeomorphism. Since $M = M_p$ we have $M_v = \eta \circ \bar{v}(\bar{M}) = \pi \circ \phi^1(L_v) = \pi(L_w) = M_q$.

Lemma 2.11 implies that every parallel submanifold has a flat normal bundle. □

In Proposition 2.2 in [HLO] it is shown that each parallel normal field of M is transported to a parallel normal field of M_v by the parallel transport in the sections.

Let $\varphi : M \to N$ be an immersion with parallel focal structure. The map φ factorizes over an injective isometric immersion, so we can assume $M \subset N$ and that φ is the inclusion. Then we have a bifoliation on N_r given by parallel submanifolds and restrictions of sections to N_r. The last proposition says that each parallel submanifold M_v has a flat normal bundle. Any parallel manifold has the same set of sections as M; in fact, if $p = \eta(v_x)$ is an arbitrary point in M_v, then $\exp(\nu_p M_v) = \Sigma_x$. By a similar argument as in Lemma 3.2, we can show that each parallel submanifold has the same set of f-regular points in N, namely N_r, and therefore admits sections. This means that in order to show that M_v has parallel focal structure it remains to prove property (2). We will see this in Theorem 3.11.

Let η' be the normal exponential map of a parallel manifold M' and L be a horizontal leaf of $\nu M'$ containing an f-regular vector. Then $\eta'(L)$ lies in a parallel manifold M'' by (1). A similar argument as in Proposition 3.4 shows that $\eta'|L : L \to M''$ is a covering. Note that $\eta'|L$ has (constant) maximal rank. In other words $\eta'^{-1}(N_r)$ is saturated with horizontal leaves and $\eta' : \eta'^{-1}(N_r) : \eta'^{-1}(N_r) \to N_r$ is foliated with respect to the horizontal foliation of $\eta'^{-1}(N_r)$ and the foliation of N_r by parallel manifolds. Note that in order to prove that M' has parallel focal structure, it now suffices to prove that η' has maximal rank when restricted to a horizontal leaf in $\nu M'$ through a focal normal of horizontal type. We will handle this problem by considering a map, where these focal points of horizontal type are dissolved.

Our main goal in this section is to show first that $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a global foliation and then a singular Riemannian foliation. We will associate to $(N, \mathcal{F})$ a certain foliated manifold $(\hat{N}, \hat{\mathcal{F}})$. An analysis of this foliation will yield the results. Boualem defines this Riemannian foliation $\hat{\mathcal{F}}$ in [Bou] from a singular Riemannian foliation $\mathcal{F}$. Thus we cannot use his construction. Instead we build up $\hat{\mathcal{F}}$ with the normal exponential map.

For an f-regular point $x \in N$ let $\eta_x : \nu M_x \to N$ be the normal exponential map of the leaf M_x. We define

$$\hat{\eta}_x : \nu M_x \to G_k(TN); v \mapsto T_{\eta_x(v)}\Sigma_{\pi(v)}.$$

Let

$$\hat{N} = \{T_q\Sigma \mid \Sigma \text{ is a section}, q \in \Sigma\}.$$

and let $\hat{\pi} : \hat{N} \to N$ be the footpoint map of $G_k(TN)$ restricted to $\hat{N}$. Then we have $\hat{N} = \hat{\eta}_x(\nu M_x)$ for any f-regular point $x \in N$ since the set of sections of two different parallel manifolds coincide. Our next aim is to give a bifoliated manifold structure to $\hat{N}$. The idea is to model $\hat{N}$ on the normal bundles of the parallel submanifolds. The normal bundle νM has two natural, complemetary foliations, one given by the flat horizontal structure, the other by the fibers of the projection $\nu M \to M$.

Let $p \in N$ be arbitrary. We fix $r > 0$ and take $\varepsilon' > 0$ to be smaller than the injectivity radius of any point $q \in \bar{B}_r(p)$ in N. There is an f-regular point x and a vector $v \in \nu_x M_x$ with $\eta_x(v) = p$ that is not a focal normal of vertical type. One can see that $d\hat{\eta}_x(w)|\mathcal{H}^M_w$ is injective for any $w \in \nu M_x$. Therefore $\hat{\eta}_x$ has maximal rank on a neighborhood of v, even if v is a focal normal of horizontal type (this is what we meant before by dissolving focal points). This means there is a neighborhood U of v in νM_x such that $\hat{\eta}_x|U : U \to G_k(TN)$ is an embedding into $G_k(TN)$ and such that the footpoint set V of $\hat{V} := \hat{\eta}_x(U)$ is contained in $\bar{B}_{\varepsilon'}(p)$. We take a ball neighborhood P of x in M_x and a neighborhood U_0 of v in $\nu_x M_x$ such that $\phi : P \times U_0 \to U; (y, w) \to w_y$ is an injective immersion into U, where w_y is the normal parallel displacement of w to y. We reduce U to the image of ϕ so that ϕ becomes a diffeomorphism onto U. We choose an f-regular point p' in $B^{\Sigma_x}_{\varepsilon'}(p)$, such that $p \in B^{\Sigma_x}_{\varepsilon}(p')$ for some ε with $0 < \varepsilon < \varepsilon'$. The map $\eta_x|\phi(\{y\} \times U_0)$ is a diffeomorphism onto its image V_y for any $y \in L$ by choice of U (note that U does not contain any focal normals of vertical type). We shrink U_0 such that this map is a diffeomorphism onto $V_x = B^{\Sigma_x}_{\varepsilon}(p')$ for $y = x$.

Lemma 3.5 *The map $\alpha_y : V_x \to V_y; \eta_x(v_x) \mapsto \eta_x(v_y)$ is an isometry, where $v_x \in U_0$ and v_y is the normal parallel displacement of v_x to $y \in P$.*

Proof The set $V_r = V \cap N_r$ is open and dense in V and $U_r = \eta_x^{-1}(V_r)$, saturated by leaves of the shape $P \times \{w\}, w \in U_0$, is open and dense in U. We consider the diffeomorphism $\eta : U_r \to V_r$. A horizontal foliated field on U_r maps to a foliated field von V_r that is a parallel normal field when restricted to the plaques of parallel manifolds in V_r by Lemma 2.11. Moreover, any such parallel normal field along a regular plaque is given this way. If $w \in U_0$ is f-regular, $P_w = \eta(\phi(P \times \{w\}))$ and X is a parallel normal field on P_w, then $\|(\alpha_y)_* X(\eta(w))\| = \|X(\alpha_y(\eta(w)))\|$. It follows that $\alpha_y : V_x \cap N_r \to V_y$ is a local isometry. As $V_x \cap N_r$ is open and dense in V_x, $\alpha_y : V_x \to V_y$ is an isometry. □

There is exactly one $v' \in U_0$ with $\eta(v') = p'$. Let $P' := \eta_x(\phi(P \times \{v'\}))$. We define the diffeomorphism $h : P \to P'; y \mapsto \eta_p(\phi(y, v'))$. Similarly as for U we have a natural diffeomorphism $\phi' : P' \times U_0' \to B_\varepsilon(\nu P')$. The map $\eta_{p'} : B_\varepsilon(\nu_q P') \to B_\varepsilon^{\Sigma_q}(q)$ is a diffeomorphism for any $q \in P'$. Obviously $B_\varepsilon^{\Sigma_{h(y)}}(h(y)) = V_y$ for any $y \in P$. Then $\hat{\eta}_{p'} \circ (\phi'(\{h(y)\} \times U_0'))$ is equal to the transversal plaque $\hat{V}_y$ for any $y \in P$ by choice of ε $(*)$. Moreover, the map $k : U_0 \to U_0'$ defined by $k(w) = (\eta_{p'}|\phi'(\{p'\} \times U_0'))^{-1} \circ (\eta_x|\phi(x, w))$ is diffeomorphism. Now let $w \in U_0$ be an arbitrary f-regular vector and $u = k(w) \in U_0'$. We extend w and u to parallel normal fields on P respectively P'. The images of $\eta_x \circ w$ and $\eta_{p'} \circ u$ lie in the same plaque in V. As $\hat{\pi}$ is injective over N_r, the image of $\hat{\eta}_{p'} \circ u$ lies in the plaque $\hat{\eta}_x(\phi(P \times \{w\}))$ in $\hat{V}$. Together with $(*)$ we have $\hat{\eta} \circ w = \hat{\eta}_{p'} \circ u \circ h$ on P. By continuity we have

$$\hat{\eta} \circ \phi(y, w) = \hat{\eta}_{p'} \circ \phi(h(y), k(w))$$

for any $y \in P$ and $w \in U_0$.

So far we have the following. Given any k-plane $\xi \in \hat{N}$, any normal vector v of a parallel manifold M_x (where x is the footpoint of v) that is not a focal normal of vertical type, defines as above a neighborhood $\hat{V}$ of ξ. A chart is given by $\hat{\eta}_x : U \to \hat{V}$. The discussion above implies that any two chart domains V intersect in open subsets of each other. So the union of topologies on the various neighborhoods V forms a basis for the topology on $\hat{N}$, and $\hat{N}$ is a topological manifold. In addition we see that the change of coordinates (h, k) is differentiable, so $\hat{N}$ carries a differentiable structure. Since $\hat{\eta}_x$ is also differentiable as a map into $G_k(TN)$, the differentiable structure is the unique one for which the inclusion $\hat{N} \to G_k(TN)$ is an immersion. Moreover, the chart $\hat{\eta}_x : U \to \hat{V}$ induces two foliations on $\hat{V}$ that are complementary to each other. The leaves of the first are given by $\hat{\eta}_x(\phi(P \times \{*\}))$, the second by $\hat{\eta}_x(\phi(\{*\} \times U_0))$. A look at the change of coordinates (h, k) reveals that these local foliations coincide on intersections. This gives us a (vertical) foliation $\hat{\mathcal{F}}$ and a complementary (horizontal) foliation $\hat{\mathcal{F}}^\perp$ on $\hat{N}$.

Proposition 3.6 *$\hat{N}$ carries a natural differentiable structure, for which the inclusion into $G_k(TN)$ is an immersion. Moreover $\hat{N}$ has a natural bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$.*

Since we have not yet defined a metric on $\hat{N}$, the denotation of $\hat{\mathcal{F}}^\perp$ has to be justified. The Grassmann bundle carries a canonical metric (see appendix) for which the projection $G_k(TN) \to N$ is a Riemannian submersion. The horizontal distribution of this bundle is given as follows. Let $\xi \in G_k(TN)$ be a k-plane through a point

$p \in N$ spanned by an orthonormal k-frame $(v_1, \ldots, v_k)$. Then the horizontal lift $\tilde{c}$ of a curve c in N with $c(0) = p$ to ξ is given by

$$\tilde{c}(t) = \operatorname{span}\Big\{ (\overset{t}{\underset{0}{\|}} c) v_1, \ldots, (\overset{t}{\underset{0}{\|}} c) v_k \Big\}.$$

In particular, the tangent bundle $T\Sigma$ of a totally geodesic submanifold Σ of N is horizontal with respect to $G_k(TN) \to N$. We denote the pullback of this metric under ι by $\hat{g}$.

Proposition 3.7 *The foliation $\hat{\mathcal{F}}^\perp$ is orthogonal to $\hat{\mathcal{F}}$ and we have*

$$\hat{\mathcal{F}}^\perp = \{T\Sigma \mid \Sigma \textit{ is a section of } M\}.$$

In particular $\hat{\mathcal{F}}^\perp$ has complete totally geodesic leaves. Therefore $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ is a Riemannian/totally-geodesic bifoliation of $(\hat{N}, \hat{g})$.

Boualem says in [Bou] that this is true for some metric on $\hat{N}$. We prove it for the natural metric $\hat{g}$.

Proof For a k-plane $\xi \in \hat{N}$ through a point p in N there is a section Σ such that $T_p\Sigma = \xi$. Let $\hat{\eta}_x : U \to \hat{V}$ be a chart with $\xi \in \hat{V}$. Then there is a $u \in U$ with $\hat{\eta}_x(u) = \xi$ and V_y is an open neighborhood of ξ in Σ_y. Thus the distribution $T\Sigma$ is open in the leaf $L_\xi \in \hat{\mathcal{F}}^\perp$ through ξ. By the definition of $\hat{g}$, the submanifold $T\Sigma$ is horizontal for $G_k(TN) \to N$. Since the horizontal lift of a geodesic along the Riemannian submersion $G_k(TN) \to N$ is a geodesic, $T\Sigma$ is a complete, totally geodesic submanifold of $G_k(TN)$ and of $\hat{N}$. Then $T\Sigma = L_\xi$ since $T\Sigma$ is open in L_ξ, connected and complete.

We consider a chart $\hat{\eta}_x : U \to \hat{V}$. For $v \in U$ with footpoint x and a horizontal vector $X \in T_vU$ and a vertical vector $Y \in T_vU$. We have

$$\hat{g}\big(d\hat{\eta}(v)X, d\hat{\eta}(v)Y\big) = g\big(d\eta(v)X, d\eta(v)Y\big) = 0.$$

The first equality is valid because $d\hat{\eta}(v)Y \in T_{\hat{\eta}(v)}T\Sigma$ is horizontal for $\pi : G_k(TN) \to N$ and π is a Riemannian submersion. The second equality follows from $d\eta(v)Y \in T_{\eta(v)}\Sigma_x$ and $d\eta(v)X \perp T_{\eta(v)}\Sigma_x$ by (1). This implies that $\hat{\mathcal{F}}^\perp$ is the orthogonal foliation to $\hat{\mathcal{F}}$ with respect to $\hat{g}$. $\square$

We define $\hat{M}_x = \hat{\pi}^{-1}(M_x)$ for f-regular $x \in N$.

Lemma 3.8 *Let x be an f-regular point in N. Then the leaves of $\hat{\mathcal{F}}$ are the parallel submanifolds of $\hat{M}_x$ which have the shape $\hat{\eta}_x \circ \bar{v}(\bar{M}_x)$ and the map $\hat{\eta}_x : \nu M_x \to \hat{N}$ is foliated with respect to the natural bifoliation on νM_x and $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$.*

Proof For any f-regular x the restriction of $\hat{\eta}_x$ to the set of f-regular vectors in νM_x respects the foliations by definition, i.e., it maps leaves into leaves. By continuity $\hat{\eta}_x : \nu M_x \to \hat{N}$ respects the foliations on the whole domain. At the end of the proof we see that this map is an f-map, which means that it sends leaves *onto* leaves.

Let x be f-regular. Then there is exactly one section through x, so $\hat{\pi}^{-1}(x) = \{T_x\Sigma_x\}$. Let L be the leaf of $\hat{\mathcal{F}}$ through V, where $V = T_x\Sigma_x$. We want to show that the map

$\hat{\pi} : L \to M_x$ is a diffeomorphism and $L = \hat{M}_x$. The set Z of points y in L such that $\hat{\pi}(y) \in N_r$ is clearly open. Let $y \in L\backslash Z$ be arbitrary, i.e., there is a focal normal $v \in \nu M$ of horizontal type with $\eta(v) = \hat{\pi}(y)$. We find a neighborhood W of v in the horizontal leaf of νM through v such that $\hat{\eta}|W$ is a diffeomorphism onto its image. Then $\eta(W)$ is an open neighborhood of y in L. Then $\hat{\pi}(\hat{\eta}(W)) = \eta(W)$ has no intersection with N_r. Thus Z closed in L and $Z = L$ by connectivity. Therefore $\hat{\pi}|L$ is injective. Now $\eta_{x_i} = \hat{\pi} \circ \hat{\eta}_{x_i}$ implies $\hat{\pi}(L) \subset M_x$. Let $s : M_x \to \hat{N}, y \mapsto T_y\Sigma_y$. Then $s(M_x) \subset L$ because $s(y) = \hat{\eta}_x(0_y)$ and 0_y is f-regular for η_x. Therefore $\hat{\pi} : L \to M_x$ is a diffeomorphism with inverse map s and $L = \hat{M}_x$. Using that the horizontal leaves are complete we can argue as in the proof of Proposition 3.4 to show that the leaves of $\hat{\mathcal{F}}$ are exactly the parallel submanifolds of $\hat{M}_x$.

Now let x be an f-regular point. If we identify νM_x and $\nu \hat{M}_x$ then $\hat{\eta}_x : \nu M_x \to \hat{N}$ is the normal exponential map of $\hat{M}_x$ in $\hat{N}$. Since we know that the image of a horizontal leaf under $\hat{\eta}_x$ is a parallel manifold of $\hat{M}_x$, thus a leaf of $\mathcal{F}$, we conclude that $\hat{\eta}_x$ is an f-map for $\hat{\mathcal{F}}$. It is an f-map for $\mathcal{F}^\perp$ since the horizontal leaves of $\hat{N}$ are complete. □

Note that $\hat{\pi}|\hat{\pi}^{-1}(N_r) : \hat{\pi}^{-1}(N_r) \to N_r$ is an f-isomorphism. If we already knew that $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a global foliation we would have that $\hat{\pi} : (\hat{N}, \hat{\mathcal{F}}) \to (N, \mathcal{F})$ is foliated.

Later we prove that $\hat{M}_x$ is connected also for a focal point x of horizontal type. This will show that $\mathcal{F}$ is a global foliation.

Up to now we have not assumed properness of φ. Now let $\varphi : M \to N$ be a proper immersion with parallel focal structure and finite normal holonomy. We know that the parallel submanifolds are injectively immersed and orthogonal to the sections in each point of intersection. So far this is not clear for the focal submanifolds. Please note that the fact $d(\eta \circ \bar{v})(T_x\bar{M}) \perp T_{\eta(v)}\Sigma_x$ for $v \in \nu_x\bar{M}$ (see (1)) does not imply that M_v is orthogonal to Σ_x in $\eta(v)$. So far we do not know that $d(\eta \circ \bar{v})(T_x\bar{M}) = d(\eta \circ \bar{v})(T_y\bar{M})$ for an arbitrary $y \in \bar{M}$ with $(\eta \circ \bar{v})(y) = (\eta \circ \bar{v})(x)$. We will be able to show this if $\bar{v}(y)$ is tangential to the same section as $\bar{v}(x)$ is. The next lemma will enable us to reduce our problem to this case and we will finally show that the focal submanifolds are embedded in Theorem 3.10. We need some preparations. Let $v \in \nu_x\bar{M}$ be a focal normal of horizontal type and $p = \eta(v)$. Let $F = F_{\bar{v}_x}$ be the focal leaf associated to v containing x. Define $F_v^1 = \bar{v}(F)$ and $V = (d(\eta \circ \bar{v})(x)(T_x\bar{M}))^\perp$. The rank theorem states that we can write $\eta \circ \bar{v} : \bar{M} \to N$ locally in coordinates as $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-\mu(v)}, 0, \ldots, 0)$, where $\mu(v)$ is the horizontal multiplicity of v. This implies that for arbitrary $y_1, y_2 \in F$ we find neighborhoods U_1, U_2 such that images of $(\eta \circ \bar{v})|U_1$ and $(\eta \circ \bar{v})|U_2$ coincide. In particular, we have $(d(\eta \circ \bar{v})(y)(T_y\bar{M}))^\perp = V$ for every $y \in F$. By (1) $T_p\Sigma_y \subset V$ for every $y \in F$.

Lemma 3.9 *Let $\varphi : M \to N$ be a proper immersion with parallel focal structure and finite normal holonomy. Let $v \in \nu_x M$ be a focal normal of horizontal type, $p = \eta(v)$. Then the set of sections Σ containing p is $J = \{\Sigma_y \mid y \in F\}$, where F is the focal leaf associated to v through x. Moreover, we have $\bigcup_{\Sigma \in J} T_p\Sigma = V$.*

PROOF We first prove the second statement. The inclusion from left to right was already shown before the lemma. Now let $w \in V$ be arbitrary. $F_v^1 = \bar{v}(F)$ is compact since η^r is proper and φ has finite normal holonomy. ϕ^1 maps F_v^1 diffeomorphically onto a compact submanifold F' of V. Therefore we find a shortest ray γ in V from F' to w. Then γ is orthogonal to F' in some point $v' := \phi^1(u)$, where $u \in F_v^1$ with foot point $y \in F$. As we will soon see $T_{v'}T_p\Sigma_y = \nu_{v'}F'$ in V (we have $T_p\Sigma_y \subset V$), which implies that γ and therefore w lies in $T_p\Sigma_y$. We want to show $T_{v'}T_p\Sigma_y = \nu_{v'}F'$. First we prove $T_{v'}T_p\Sigma_y \subset \nu_{v'}F'$. We have

$$T_{v'}F' = \{(0, J'_\xi(1)) \mid \xi \in T_uF_v^1\},$$

because $T_{v'}F'$ consists of elements $d\phi^1(u)\xi = (J_\xi(1), J'_\xi(1)) = (0, J'_\xi(1))$ for $\xi \in T_uF_v^1$. This implies $T_{v'}F' \subset T_{v'}T_pN = V_{v'}^N$. Since ξ is horizontal, $J'_\xi(t)$ is orthogonal to $T_{\gamma_u(t)}\Sigma_y$ by (1). So $T_{v'}T_p\Sigma_y \perp T_{v'}F'$ also in $T_{v'}T_pN = V_{v'}^N$ for the Sasaki metric, hence $T_{v'}T_p\Sigma_y \subset \nu_{v'}F'$, where we consider F' as a submanifold of V. Since $\dim F' = \dim F = \mu(v)$ and $\dim V = \mu(v) + k$, where $\mu(v)$ is the horizontal multiplicity of v and k the codimension of M, we have $T_{v'}T_p\Sigma_y = \nu_{v'}F'$ by equality of dimensions.

We will now prove the first statement of the lemma. Assume Σ is an arbitrary section containing p. Then by completeness of Σ there is a $u \in \nu M$ tangential to Σ with $\eta(v) = p$. Then $\phi^1(u) \in V$. We have seen above that $\phi^1(u)$ is tangential to a section Σ of J. Then the same is true for u. By Lemma 3.2 there is only one section containing the geodesic γ_u. Thus $\Sigma \in J$. □

Compare the following statement with the weaker result of Corollary 2.14 in [Ew]. That corollary is based on Lemma 2.13, [Ew] which is not proved correctly (see the first sentence of the proof).

Theorem 3.10 *If $\varphi : M \to N$ is a proper immersion with parallel focal structure and finite normal holonomy, then $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a transnormal global foliation and the leaves of $\mathcal{F}$ are closed, embedded and orthogonal to each section they meet.*

PROOF Assume $\eta(v) = \eta(w) =: p$. We have to show $M_v = M_w$. If p is f-regular, then v and w are tangential to the same section by Lemma 3.2, so $\hat{\eta}(v) = \hat{\eta}(w)$. By Proposition 3.8 it follows $\hat{\eta} \circ \bar{v}(\bar{M}) = \hat{\eta} \circ \bar{w}(\bar{M})$ and therefore $M_v = M_w$ because $\hat{\pi} \circ \hat{\eta} = \eta$. Now let p be a focal point of horizontal type and let Σ be the section to which v is tangential. The focal leaf F associated to w induces the set of sections J_w as above. Because of $\Sigma \in J_w$, there is a normal parallel translation $w' \in F_w^1$ of w that is tangential to Σ. Because of $\hat{\eta}(v) = \hat{\eta}(w')$, we conclude $M_v = M_{w'} = M_w$ as above, so $\mathcal{F}$ is a global foliation.

We already know that the parallel submanifolds are closed and embedded. We will use a similar argument as above to show that the same is true for the focal submanifolds. Let $v \in \nu_x\bar{M}$ be a focal normal of horizontal type with endpoint p. Assume there is a $y \in \bar{M}$ such that $\eta(\bar{v}_y) = p$. As above we find a point y' in the focal leaf associated to v through y such that $\bar{v}_{y'}$ is tangential to the same section as $\bar{v}_x$, so $\hat{\eta}(\bar{v}_x) = \hat{\eta}(\bar{v}_{y'})$. Since $\hat{\eta}$ is foliated and $d\hat{\eta}(w)|H_w^M$ has rank n for any $w \in \nu M$ there are neighborhoods U_1 and U_2 of x and y' in $\bar{M}$ such that $(\eta\circ\bar{v})|U_1$ and $(\eta\circ\bar{v})|U_2$

are diffeomorphisms onto the same image. As a consequence of the rank theorem applied to $\eta \circ \bar{v} : \bar{M} \to N$ and the compactness of focal leaves we can replace U_1 and U_2 by neighborhoods of x and y' that are saturated with focal leaves associated to v such that $(\eta \circ \bar{v})|U_i$ is a submersion onto its image in M_v. Since y was arbitrary, this means that M_v has neither intersection points nor contact points. As $\eta \circ \bar{v} : \bar{M} \to N$ is proper, M_v is closed and embedded by Proposition 2.2. Now νM_p is well-defined for $p \in M_v$. Lemma 3.9 now states that $\nu_p M_v$ is the union of all $T_p\Sigma$, where Σ is a section through p. This implies that also a focal submanifold intersects each section it meets orthogonally. Moreover, this shows that $\mathcal{F}$ is transnormal: If γ is a geodesic with initial vector $v \in \nu_p M_p$, then v is tangential to a section Σ through p and γ is contained in Σ. Therefore γ is orthogonal to every leaf of $\mathcal{F}$ it meets. □

The starting point of our work was the question, under which conditions a submanifold M in N induces a global foliation through parallel submanifolds. In order to define parallel submanifolds we have to demand flatness of νM and that the maps $\eta \circ \bar{v}$ have constant rank. These are conditions (1) and (2) of a submanifold with parallel focal structure. The existence of sections is a common condition in related theories, like in the theory of polar actions for instance, which is one part of condition (3). The theorem now states that M with the above properties induces a global foliation if and only if M admits sections. (Necessity is clear. Otherwise there is a regular point p and two sections Σ_1 and Σ_2 with $T_p\Sigma_1 \neq T_p\Sigma_2$. Then there are two parallel manifolds M_{v_i} with $T_pM_{v_i} \perp T_p\Sigma_i$ $(i = 1, 2)$, thus $T_pM_{v_1} \neq T_pM_{v_2}$, contradicting that M induces a global foliation.)

We call the elements of $\mathcal{F}$ *leaves*. A leaf is called *regular* if its dimension is maximal in $\mathcal{F}$, otherwise *singular*. A regular leaf with non-trivial normal holonomy is called *exceptional*.

A point in N is f-regular if and only if it is contained in a regular leaf of $\mathcal{F}$. This justifies the denotation: the "f" in f-regular stands for *foliation*.

Remark Let M be a regular leaf and let $\mathcal{P}$ be the horizontal foliation on νM. Then $\eta : (\nu M, \mathcal{P}) \to (N, \mathcal{F})$ is foliated.

In Theorem 4.3 we will show that $\mathcal{F}$ is a singular Riemannian foliation admitting sections. Then we can apply the Slice Theorem of Alexandrino, Theorem 4.4, and derive Corollary 4.5, which states that there is a neighborhood of a given leaf M_v containing no leaf of lower dimension. But this result can also be obtained easily by using the lower semi-continuity of the rank of $\tilde{\eta} : \bar{M} \times \nu_x M \to N; (y, w) \mapsto \eta(\bar{w}(y))$. For this purpose we have to assume that v is not a focal normal of vertical type, otherwise we replace M appropriately.

Theorem 3.11 *If φ is a proper immersion with parallel focal structure and finite normal holonomy, then also every parallel manifold M_v is embedded, has parallel focal structure and finite normal holonomy.*

Moreover, a focal point of horizontal type of M is also a focal point with the same horizontal multiplicity of any other parallel submanifold of M and vice versa. In other words, each parallel submanifold has the same focal submanifolds.

Proof We have already seen that a parallel submanifold is embedded and has a flat normal bundle. Let $M_v, v \in \nu_x M$ be a parallel submanifold and $p = \eta(v)$. Let M_u be a focal submanifold of M and let L be the leaf of $\hat{\mathcal{F}}$ over M_u, i.e., $L = \hat{\pi}^{-1}(M_u)$. We recall that $d\hat{\eta}(u')|\mathcal{H}^M_{u'} : \mathcal{H}^M_{u'} \to T_{\hat{\eta}(u')}L$ is an isomorphism for any u' parallel to u in νM. Since M has parallel focal structure, $\mu(u') = \operatorname{rank} d\eta(u')|H^M_{u'}$, the horizontal multiplicity, is constant for all u' which are parallel to u in νM. Now $\eta = \hat{\pi} \circ \hat{\eta}$ implies that $c := \operatorname{rank} d\hat{\pi}|T_V L, V \in L$ is constant. Let $w \in \nu M_p$ be an arbitrary vector with endpoint in M_u. Again by $\eta_p = \hat{\pi} \circ \hat{\eta}_p$ we can now conclude that the horizontal multiplicity $d\eta_p(w')|\mathcal{H}^{M_v}_{w'}$ is equal to c for any w' parallel to w in νM_p. In particular, f-regular points of M_v and of M coincide. Since M_v has the same sections as M, there is exactly one section of M_v through a given f-regular point of M_v, i.e. parallel submanifolds also admit sections. Thus M_v has parallel focal structure.

It remains to show that every parallel submanifold of M has finite normal holonomy. Let M_x be the parallel submanifold through a point x, $\Sigma = \Sigma_x$ and Γ_x the normal holonomy group of M_x in x, acting on $\nu_x M_x = T_x\Sigma$. Since M_x has parallel focal structure, the focal points of horizontal type of M_x are bounded away from 0 by a number $\varepsilon' > 0$. Let ε be the minimum of ε' and the injectivity radius of Σ in x. Then Γ_x acts on $B^\Sigma_\varepsilon(x)$, such that both actions of Γ_x, restricted to the balls of radius ε, are equivariant with respect to $\exp^\Sigma_x$. The orbit $\Gamma_x(q)$ of an arbitrary point $q \in B^\Sigma_\varepsilon(x)$ is contained in $M_q \cap \Sigma$, where M_q is the parallel submanifold through q. Since M_q is closed and embedded, $M_q \cap \Sigma$ is closed and discrete, so $\Gamma_x(q)$ is finite. Therefore each orbit of the action of Γ_x on $\nu_x M_x$ is finite. As this action is linear and effective, Γ_x is finite and M_x has finite normal holonomy. □

Ewert states this result in Proposition 2.9 in [Ew], but his proof is not correct. In the fourth last line of p. 20 he writes that $V_*\partial_t(1, \cdot, t)$ is a parallel normal field along the focal submanifold through $V(1, 0, t)$. This is not true. Indeed, he refers to Proposition 2.4, [Ew], which is not correct if M_z is a focal submanifold; take $x := z \circ c$ for instance.

The theorem shows that every vector in $\eta^{-1}(p)$ has the same horizontal multiplicity. Since a normal vector is f-regular if and only if its horizontal mulitplicity is zero, this is a generalization of Lemma 3.2.

Proposition 3.12 *Let M be a closed and embedded submanifold with parallel focal structure and finite normal holonomy. If $v \in \nu M$ is a multiplicity k focal normal of vertical type so are its normal parallel translations. In other words the vertical focal data is also invariant under normal parallel translation. If v is a cut normal, so are its normal parallel translations. In particular, the cut distance function is constant along the parallel normal fields.*

The proposition can be easily proved with Theorem 4.7, whose assumption is that $\mathcal{F}$ is a singular Riemannian foliation admitting sections, but which is also valid in our situation. We will introduce the theorem in the context of singular Riemannian foliations, section 4 in order not to disturb the current development in this section. Therefore we postpone the proof until the next section.

This proposition says that the cut locus of M is already determined by its intersection with a section and that we can easily distinguish a submanifold with parallel focal structure from other submanifolds by its cut locus. Our next aim is to show the following theorem.

Theorem 3.13 *If $\varphi : M \to N$ is a proper immersion with parallel focal structure and finite normal holonomy, then $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a singular Riemannian foliation of N that admits sections.*

We need some preparations.

Lemma 3.14 *Let $v \in \nu_x \bar{M}$ be a focal normal of horizontal type that is not of vertical type. Then there is a neighborhood U of x that is saturated by focal leaves of v, an open relatively compact neighborhood P of $\eta(v) \in M_v$, $\varepsilon > 0$, a neighborhood V of $\bar{v}_x \in \nu_x \bar{M}$ such that*

(1) $\eta \circ \bar{v} : U \to P$ is a surjective fibration whose fibers are the focal leaves. This gives a local trivialization $U \cong F_v \times P$.

(2) $\tilde{\eta} : U \times V \to T(P, \varepsilon); (y, \bar{w}_x) \mapsto \eta(\bar{w}_y)$ is surjective.

(3) $F_{\bar{w}_y} \subset F_{\bar{v}_y}$ for any $(y, \bar{w}_x) \in U \times V$. That means that the focal foliation given by $\eta \circ \bar{w}$ is finer than the focal foliation given by $\eta \circ \bar{v}$.

(4) Each section through a point $q \in T$ also contains the unique point p' in P that is in the same slice as q, i.e., $J_q \subset J_{p'}$.

(5) Let $p \in P$ and S_p be the slice in T through p. Then $S_q \subset S_p$ for any $q \in S_p$.

Proof Let $v \in \nu_x \bar{M}$ be a focal normal of horizontal type. The focal foliation given by the submersion $\eta \circ \bar{v}$ is regular in the sense of [Pa] and the leaves are compact. This implies that $\eta \circ \bar{v}$ is locally a fibration by Corollary 2 of Theorem X of [Pa], i.e., there exist local trivializations of $\eta \circ \bar{v}$. Therefore we find a saturated neighborhood U of the focal leaf F_v through x such that $(\rho, \eta \circ \bar{v}) : U \to F_v \times P$ is a diffeomorphism for some projection $\rho : U \to F_v$ and $P = \eta \circ \bar{v}(U)$.

Now let $v \in \nu_x \bar{M}$ be not a focal normal of vertical type. We can choose U such that P is relatively compact. There is a number $\varepsilon > 0$ such that $T(P, \varepsilon)$ is an injectivity tube of P with radius ε. By Proposition 3.12 every normal parallel translation of v is not a focal normal of vertical type either. Then there is a neighborhood V of v in $\nu_x \bar{M}$ such that $\tilde{\eta}(y, \cdot\,) : V \to \Sigma_y$ is a diffeomorphism onto its image for every $y \in U$, where $\tilde{\eta} : U \times V \to N; (y, \bar{w}_x) \mapsto \eta(\bar{w}_y)$. We can assume $\tilde{\eta}(y, \cdot\,) : V \to B_\varepsilon^{\Sigma_y}(p)$ is a diffeomorphism, eventually shrinking ε and V. By Lemma 3.5 and the remark following it $\alpha_y : B_\varepsilon^{\Sigma_x}(p) \to B_\varepsilon^{\Sigma_y}(\eta(\bar{v}_y))$ is an isometry. As $\tilde{\eta}(y, \cdot\,) = \alpha_y \circ \tilde{\eta}(x, \cdot\,)$ we have $\tilde{\eta}(\{y\} \times V) = B_\varepsilon^{\Sigma_y}(\eta(\bar{v}_y))$. Then $\tilde{\eta} : U \times V \to T$ is surjective because the slice S_q of P in T through $q \in P$ is equal to

$$S_q = \bigcup \{B_\varepsilon^{\Sigma_y}(q) \mid y \text{ is in the focal leaf associated to } v \text{ through } y\}$$

for any $q \in P$ by Lemma 3.9.

Let $y \in U$ and $u \in V$ be arbitrary. Let F' be the focal leaf associated to u through y. Let $q = \eta(\bar{u}_y)$ and $p' = \eta(\bar{v}_y) \in P$. We want to show that F' is contained in the focal leaf F associated to v through y. This is clear if u is f-regular. We assume that u is a focal normal of horizontal type. The section $\Sigma := \Sigma_y$ contains p' and q. There is a vector $w \in T_{p'}\Sigma \subset \nu_{p'} M_v$ of length smaller than ε with endpoint q. For $z \in U$ we define $w_z = d\alpha_z(p')w$, where $\alpha_z : B_\varepsilon^{\Sigma_y}(p') \to B_\varepsilon^{\Sigma_z}(\eta(\bar{v}_z))$ as above but with central point p' instead of p. The endpoint $\alpha_z(q)$ of w_z is still in $T(P, \varepsilon)$ because $\|w_z\| = \|w\| < \varepsilon$ for all $z \in U$. For all $z \in F' \subset U$ we have $q = \eta(\bar{u}_z) = \alpha_z(q)$, thus $w_z = w$ since w is unique among the vectors of νP of length smaller than ε with endpoint $q \in T$. Therefore $\eta(\bar{v}_z) = \alpha_z(p') = p'$ for all $z \in F'$, so $F' \subset F_{\bar{v}_y}$. (In other words, the foliation of focal leaves given by $\eta \circ \bar{u}$ is finer than the foliation of focal leaves given by $\eta \circ \bar{v}$.) By Lemma 3.9 we obtain that the set $J_{p'}$ of sections through p' contains the set J_q of sections through q. Therefore

$$S_q \subset \bigcup_{\Sigma \in J_q} (\Sigma \cap T)_q \subset \bigcup_{\Sigma \in J_{p'}} (\Sigma \cap T)_q = S_{p'},$$

where $(\Sigma \cap T)_q$ denotes the connected component of $\Sigma \cap T$ containing q. $\square$

Proof of Theorem 3.13. By Theorem 3.11, it remains to show that $\Xi(\mathcal{F})$ acts transitively at a given point p. This is clear for f-regular $p \in N$, since the set of f-regular points N_r is foliated by Proposition 3.4. Therefore we assume that p is a focal point of horizontal type of M and $v \in \nu_x \bar{M}$ with $\eta(v) = p$. We assume that v is not a focal normal of vertical type, otherwise we replace M by a parallel manifold. Now we use the same objects as in the previous lemma. We want to define a distribution $\mathcal{D}'$ of dimension $\dim P$ on T such that $\mathcal{D}'(q) \subset T_q M_q$. Let $q \in T$ be arbitrary. Let S_q be a slice of M_q through q. Then there is a uniqe point $p' \in P$ such that the slice $S_{p'}$ of T through p' contains q. We define $\mathcal{D}'(q) = T_q S_{p'}^{\perp}$. Since the distribution tangential to the slices is differentiable so is $\mathcal{D}'$. Since $S_q \subset S_{p'}$ we have $\mathcal{D}'(q) \subset T_q M_q$. Thus, for any $p \in P$ and $X_0 \in T_p P$ there is a vector field X of $\mathcal{D}'$ in T extending X_0. If $f : N \to \mathbb{R}$ is a bump function with support in U and $f(p) = 1$ then $fX \in \Xi(\mathcal{F})$ with $(fX)_p = X_0$. Since p and X_0 were arbitrary, $\Xi(\mathcal{F})$ acts transitively. $\square$

We can now exploit the theory of singular Riemannian foliations for submanifolds with parallel focal structure. Implications will be given in the next section. The converse was proven in [A]. We give a different proof in the next section.

4 Singular Riemannian Foliations

4.1 Parallel Focal Structure of Regular Leaves

Let $\mathcal{F}$ be a singular Riemannian foliation admitting sections of a complete Riemannian manifold N. Let M be a regular leaf and $\eta : \nu M \to N$ be its normal exponential map. We have seen in Lemma 2.11 that νM is flat. Therefore νM is endowed with a natural foliation of horizontal leaves.

Lemma 4.1 *A point of N is $\mathcal{F}$-regular if and only if it is f-regular. In particular the subset of $\mathcal{F}$-regular points in a section is open and dense.*

Proof Let p be $\mathcal{F}$-regular and $v \in \eta^{-1}(p)$ with footpoint y. We have to show that v is f-regular. Let P_p be a relativey compact of p in M_p, the leaf of $\mathcal{F}$ through p, let T be an injectivity tube of P_p and $\rho : T \to P_p$ be the orthogonal projection which is the projection along the sections. We will now use an argument similiar to one in Lemma 3.2. We extend v to normal parallel field on a simply connected neighborhood of y in M_y. By the Morse Index Theorem $\eta \circ ((1+t)v)$ has maximal rank on a small neighborhood U of y for small t, therefore $\rho \circ \eta \circ ((1+t)v)$ is a submersion onto its image in P_p by (1). We have

$$\rho \circ \eta \circ ((1+t)v) = \rho \circ \eta \circ v,$$

so $\eta \circ v$ has maximal rank on U, so v is f-regular.

Now let p be f-regular, i.e. there is a vector $v \in \nu M$ with endpoint p. We denote the footpoint of v by x. For a small simply connected relatively compact neighborhood U of x in M, $\eta \circ v'|U$ is a diffeomorphism onto its image P_v where v' is the parallel normal field on U extending v. Then there is an injectivity tube T' of P_v with radius ε'. The tube T' is open in N and is foliated by its slices. The slice through $\eta(v'_y)$ is the connected component in T' of the section Σ_y containing y for $y \in U$. Now assume that p is singular with respect to $\mathcal{F}$. Let T be an injectivity tube of a small open subset P_p of the singular leaf M_p containing p with radius $\varepsilon < \varepsilon'$ and $T \subset T'$. Note that T is a distinguished neighborhood of P_p in the sense of Molino (see [Mo]). Since the set of $\mathcal{F}$-regular points is open and dense in N by [Mo] there is an $\mathcal{F}$-regular point q' in T. The plaque $P_{q'}$ of the regular leaf $M_{q'}$ in T' intersects the slices of T' transversally and orthogonally. Indeed the slices are exactly the connected components of the sections of $P_{q'}$ in T'. Since T is open in N there is a slice S of P_p of T whose subset R of $\mathcal{F}$-regular points is non-empty; otherwise we would obtain a contradiction to the density of $\mathcal{F}$-regular points in N. Since R is open in S and the dimension k of a section is smaller than the dimension of S there are at least two vectors $w_1, w_2 \in \nu_p P_p = T_p S$ with $\exp(w_i) \in R$ such that $T_p\Sigma_1 \neq T_p\Sigma_2$, where Σ_i is the unique section to which w_i is tangential (namely $\Sigma_i = \exp(\nu M_{\eta(w_i)})$). But this is a contradiction to the fact that T' is foliated by the restriction of the sections of $P_{q'}$ to T'. □

Remark The set of singular points on a bounded segment of a geodesic γ_v for a vector $v \in \nu M$ is finite by the lemma and the Morse Index Theorem.

The lemma implies that a regular leaf M admits sections in the sense of section 4.

Proposition 4.2 *The map $\eta : \nu M \to N$ is foliated and the restriction of η to a horizontal leaf in νM has constant rank.*

Proof Let $v \in \nu M$ with endpoint p and footpoint x. We define

$$Z = \{w \in L_v \mid \eta(w) \in M_p\},$$

where L_v is the horizontal leaf of νM through v. We want to show that Z is open and closed in L_v and therefore equal to L_v by connectivity. Let $w \in Z$ with footpoint y and $q = \eta(w)$. Let P_q be a relatively compact open neighborhood of q in M_q and let T be a distinguished neighborhood of P_q. Since $\Xi(\mathcal{F})$ acts transitively we can assume that each plaque in T intersects each slice of T and always transversally. Thus the restriction of the projection $\rho : T \to P_q$ to an arbitrary plaque in T is a surjective submersion. We choose a positive number $t < 1$ such that tv is f-regular and $\gamma_w|[t, 1]$ lies in T. We see that γ_w intersects P_q orthogonally for $t = 1$ since $\mathcal{F}$ is transnormal and $\gamma_w|[t, 1]$ lies in the slice of T through q. The leaf M_{tv} is regular by the previous lemma. Let $L' = L_{(1-t)\phi^t(v)}$ be the horizontal leaf in νM_{tv} containing $(1-t)\phi^t(v)$. Observe that the map $\alpha : L_v \to L'; \xi \mapsto (1-t)\phi^t(\xi)$ is a diffeomorphism and that $\eta_x|L_v = (\eta_{\eta(tv)} \circ \alpha)|L_v$. This means that we can replace M by M_{tv} for our considerations and assume that $\gamma_w|[0, 1]$ is contained in a slice of T, so in particular $y \in T$. Let P_y be the connected component of M_y in T containing y. We define the function $r : T \backslash P_q \to \mathbb{R}$ measuring the distance to P_q and let $X = -\operatorname{grad} r$ be the negative of the radial vector field. Then $w = \|w\| X_y$. Note that $X|P_y$ is a normal vector field of P_y. The flow of X is a family of homotheties in T centered at P_q which respects the singular Riemannian foliation by the Homothety Lemma (see [Mo]). Lemma 2.8 now implies that X is a foliated vector field on a neighborhood of P_y in T. Thus $X|P_y$ is a normal parallel field of P_y by Lemma 2.11 and the image of P_y under X is an open subset of the horizontal leaf L_v in νM_x containing w. We want to show that $(\eta \circ (\|w\| X))|P_y = \rho|P_y$ which implies that Z is open in L_v. But this follows from the observation that $\phi_X(t, z) = \gamma_{X_z}(t)$ for $t \in [0, \|w\|)$ and $z \in P_x$ where ϕ_X is the flow of X; note that $\|w\|$ is the distance of P_y and P_q. We remark that this implies that $\eta|L_v$ has constant rank and its image is open in M_p. Now let $w \notin Z$ with footpoint y and endpoint q. By assumption $q \notin M_p$. As above we show that an open neighborhood of w in L_v is mapped to M_q which is disjoint to M_p by definition of $\mathcal{F}$. Therefore the complement of Z is also open. Thus $\eta(L_v) \subset M_p$.

We will now show $\eta(L_v) = M_p$. We have seen above that $\eta(L_v)$ is open in M_p. It suffices to show that $\eta(L_v)$ is also closed in M_p. Let q be an arbitrary point on the boundary of $\eta(L_v)$ in M_p. We have to show $q \in \eta(L_v)$. There is an injectivity tube T of some open neighborhood P_q of q in M_q. As $\Xi(\mathcal{F})$ acts transitively, we can assume that any plaque in T meets any slice of P_q, and always transversally. Now there is a $w \in L_v$ such that $\eta(w) \in P_q$. As above we can assume that the footpoint y of w is contained in T. Then we define $X = -\operatorname{grad} r$ on $T \backslash P_q$ and we have $w = \|w\| X_y$. The endpoint of $\|w\| X_{y'}$ for $y' \in P_y$ is the unique point in the intersection of P_q and the slice of P_q containing y'. Since $P_{y'}$ meets any slice of P_q, in particular the slice through q, we have $q \in \eta(L_v)$ and $\eta(L_v) = M_p$. □

As a direct consequence we obtain the following theorem of Marcos Alexandrino.

Theorem 4.3 (Alexandrino) *A regular leaf of a singular Riemannian foliation admitting sections of a complete Riemannian manifold has parallel focal structure.*

REMARK We need the following discussion for Theorem 4.19. Let p be a singular, P a relatively compact neighborhood of p in M_p, T an injecitivity tube of P and S the slice through p. Let M be a regular leaf that intersects S in a point x. Then there is a geodesic $\gamma_v : [0, 1] \to N$ in S from x to p. Let $\rho : \bar{M} \to M$ and $\rho' : \nu\bar{M} \to \nu M$ be the canonical projections. We choose a point $\bar{x}$ with $\rho(\bar{x}) = x$. Let F be the focal leaf in $\bar{M}$ associated to $\bar{v}(\bar{x})$. Then $F^1 = \bar{v}(F)$ is the connected component of $(\eta \circ \rho')^{-1}(p)$ containing $\bar{v}(\bar{x})$. It is clear that the connected component A of $M \cap S$ through x contains $\rho(F)$. From the proof above we know that $X = -\|v\| \operatorname{grad} r$ is a parallel normal field when restricted to $M \cap T$. We see that $\rho' : F^1 \to X(M \cap S)$ and $\rho : F \to M \cap S$ are coverings. We can push down the focal leaf F to a submanifold in M and the focal parallel normal field on F to one on that submanifold. We have done this for M intersecting S. But this is true for any regular leaf M with $M_v = M_p$ using the technique introduced in the proof of the Proposition 4.2.

The following is a slice theorem for singular Riemannian foliations admitting sections.

Theorem 4.4 (Alexandrino) *Let $\mathcal{F}$ be a singular Riemannian foliation admitting sections of a complete Riemannian manifold N. Let $p \in N$, $B_\varepsilon^\perp(0_p)$ be the ball of 0_p in $\nu_p M_p$ for a small radius ε and $S_p = \exp^\perp(B_\varepsilon^\perp(0_p))$. Then the restriction $\mathcal{F}|S_p$ is a singular Riemannian foliation admitting sections that is isomorphic to an isoparametric partition $\mathcal{F}'$ of $\mathbb{R}^m$, where m is the codimension of M_p. This isomorphism is given by $\exp^\perp : B_\varepsilon^\perp(0_p) \to S_p$, and it maps flat sections of $\mathcal{F}'$ to sections of $\mathcal{F}$ restricted to S_p.*

An isoparametric family of submanifolds of $\mathbb{R}^m$ is given as the level sets of a transnormal map. Therefore the isoparametric family in $\mathbb{R}^m$ and $\mathcal{F}|S_q$ are proper singular Riemannian foliations, i.e., its leaves are closed and embedded.

Corollary 4.5 *Let $\mathcal{F}$ be as above and let M be a leaf of $\mathcal{F}$. Then there is a neighborhood of M that contains no leaf of lower dimension than* $\dim M$*.*

PROOF The proof is clear. □

We already know this result. It follows from the existence of foliated charts for singular foliations, see 2.2.

4.2 Transversal Holonomy

Let $(N, \mathcal{F})$ be as in the previous subsection. Then by Theorem 4.3 and section 3

$$\hat{N} := \{T_p\Sigma \mid p \in N, \Sigma \text{ is a section of } \mathcal{F} \text{ through } p\}$$

carries the unique differentiable structure for which the inclusion $\hat{N} \to G_k(TN)$ is an immersion (see Proposition 3.6). Moreover, $\hat{N}$, endowed with the pull-back metric, carries a Riemannian/totally-geodesic bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$, where

$$\hat{\mathcal{F}}^\perp = \{T\Sigma \mid \Sigma \text{ is a section of } \mathcal{F}\}.$$

The footpoint map $\hat{\pi} : (\hat{N}, \hat{\mathcal{F}}) \to (N, \mathcal{F})$ is foliated and it maps a horizontal leaf $T\Sigma$ isometrically onto the section Σ. Thus we know that the image of a leaf of $\hat{\mathcal{F}}$ is a leaf of $\hat{\pi}$. We want to see that $\hat{\pi}^{-1}(M)$ is a leaf, where $M \in \mathcal{F}$. This is clear for regular M. Let M be singular and $p \in M$. By defintion $\hat{\pi}^{-1}(p)$ is the set of sections through p. It suffices to show that this set is contained in one leaf of $\hat{\mathcal{F}}$. Let S_p be a slice through p. The corresponding isoparametric partition of $\nu_p M_p$ given by Theorem 4.4 has closed and embedded regular leaves with parallel focal structure and finite normal holonomy. Now Proposition 3.9 describes the set of sections through p as the image of a focal leaf associated to v under $\hat{\eta} \circ \bar{v}$ for some $v \in \nu M$, so $\hat{\pi}^{-1}(p)$ is contained in one leaf. This means $\hat{M}_p := \hat{\pi}^{-1}(M_p)$ is a leaf. Therefore

$$\hat{\mathcal{F}} = \{\hat{\pi}^{-1}(M) \mid M \in \mathcal{F}\}.$$

For a curve $\tau : [0,1] \to N$ in a regular leaf of $\mathcal{F}$ and a curve $\sigma : [0,1] \to N$ in a section, both starting in an $\mathcal{F}$-regular point, we define $\hat{\tau}(t) := T_{\tau(t)}\Sigma_{\tau(t)}$ and $\hat{\sigma}(t) := T_{\sigma(t)}\Sigma_{\sigma(0)}$. Obviously $\hat{\pi} \circ \hat{\tau} = \tau$ and $\hat{\pi} \circ \hat{\sigma} = \sigma$.

Lemma 4.6 *Let x_0 be $\mathcal{F}$-regular, let $\tau : [0,1] \to N$ be a curve in M_{x_0} and $\sigma : [0,1] \to N$ be a curve in Σ_{x_0} with $\tau(0) = \sigma(0) = x_0$. Then there is a unique map $H = H_{(\tau,\sigma)} : [0,1] \times [0,1] \to N$ with*

(1) $H(\,\cdot\,, 0) = \tau$,

(2) $H(0, \,\cdot\,) = \sigma$,

(3) $H(\,\cdot\,, t)$ is contained in a leaf of $\mathcal{F}$,

(4) $H(s, \,\cdot\,)$ is contained in a section.

Moreover we have $H_{(\tau,\sigma)} = \hat{\pi} \circ \hat{H}_{(\hat{\tau},\hat{\sigma})}$, where $\hat{H}_{(\hat{\tau},\hat{\sigma})}$ is the homotopy given in Lemma 2.12 for $(\hat{N}; \hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$.

Proof Existence follows by $H_{(\tau,\sigma)} = \hat{\pi} \circ H_{(\hat{\tau},\hat{\sigma})}$, where $H_{(\hat{\tau},\hat{\sigma})}$ is the homotopy defined in Lemma 2.12 for the bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ of $\hat{N}$. We want to show uniqueness of $H_{(\tau,\sigma)}$. Let H be an arbitrary homotopy with the four properties in the lemma. We define $\hat{H}(s,t) := T_{H(s,t)}\Sigma_{\tau(s)}$. Obviously $\hat{H}(s, \,\cdot\,)$ lies in the horizontal leaf $T\Sigma_{\tau(s)}$.

The curve $H(\,\cdot\,,t)=\hat\pi\circ\hat H(\,\cdot\,,t)$ lies in the leaf $M_{\sigma(t)}$ by assumption. By the discussion at the beginning of this subsection, $\hat\pi^{-1}(M_{\sigma(t)})$ is a leaf of $\hat{\mathcal{F}}$. Therefore $\hat H(\,\cdot\,,t)$ is contained in a vertical leaf. By Lemma 2.12 we have $\hat H=\hat H_{(\hat\tau,\hat\sigma)}$ and therefore $H=\hat\pi\circ H_{(\hat\tau,\hat\sigma)}$. □

Let x_0 be $\mathcal{F}$-regular, $M=M_{x_0}$ and $\Sigma=\Sigma_{x_0}$. In the sequel we think of the universal cover $\tilde M$ respectively $\tilde\Sigma$ as the set of equivalence classes of vertical respectively horizontal curves starting from x_0, where the equivalence is given by homotopy fixing endpoints. We define

$$\begin{array}{rcl}\psi:\tilde M\times\tilde\Sigma & \to & N\\ ([\tau],[\sigma]) & \mapsto & H_{(\tau,\sigma)}(1,1)\end{array}$$

and

$$\begin{array}{rcl}\Psi:\tilde M\times\tilde\Sigma & \to & \hat N\\ ([\tau],[\sigma]) & \mapsto & T_{\psi(\tau,\sigma)}\Sigma_{\tau(1)}=\hat H_{(\hat\tau,\hat\sigma)}(1,1).\end{array}$$

Obviously

$$\hat\pi\circ\Psi=\psi.$$

We could have defined ψ by the above formula. The reason that we did not is that we wanted to emphasize that the definition of ψ only depends on $\mathcal{F}$, namely on the property of Lemma 4.6, and not primarily on its blow-up. For the proof of this property we have used the blow-up nevertheless.

Theorem 4.7 *The map Ψ is the universal covering map, and it is foliated with respect to the product foliation of $\tilde M\times\tilde\Sigma$ and to $(\hat N;\hat{\mathcal{F}},\hat{\mathcal{F}}^\perp)$. The footpoint map $\hat\pi:(\hat N,\hat{\mathcal{F}})\to(N,\mathcal{F})$ is foliated and it maps a horizontal leaf $T\Sigma$ isometrically onto the section Σ. The map ψ is foliated with respect to the vertical foliation on $\tilde M\times\tilde\Sigma$ and $(N,\mathcal{F})$. Given $[\tau]\in\tilde M$ the map $\psi:\{[\tau]\}\times\tilde\Sigma\to\Sigma_{\tau(1)}$ is a Riemannian covering and*

$$T_\tau:\widetilde{(\Sigma,x_0)}\to\widetilde{(\Sigma,\tau(1))};[\sigma]\mapsto H_{(\tau,\sigma)}(1,\,\cdot\,)$$

is an isometry. In particular, the sections have the same Riemannian universal cover. Similarly the regular leaves of $\mathcal{F}$ have the same universal cover.

Proof Identifying M respectively Σ with $\hat M$ respectively $\hat\Sigma$ by $\hat\pi$ and then their universal covers with each other, the map Ψ is a foliated universal covering by Theorem 2.15. The results about $\hat\pi$ are clear and have been stated before. Then $\psi=\hat\pi\circ\Psi$ is foliated. The map T_τ is the isometry T_τ in Theorem 2.15 up to isometric identification of $\tilde\Sigma_x$ and $\widetilde{T\Sigma_x}$ for $x=\tau(0),\tau(1)$. Then the remaining statements follow. □

Remark The map ψ completely describes the singular Riemannian foliation $\mathcal{F}$ of N. The singular values of ψ are exactly the singularities of $\mathcal{F}$. It is a covering when restricted to the regular set.

This theorem describes a topological difference between a singular Riemannian foliation admitting sections and a polar action, namely the normal holonomy of a section. While the sections of a polar action are isometric to each other, the sections of a singular Riemannian foliation only have the same Riemannian universal cover. We want to explain this in more detail. We can define a local isometry along a vertical curve τ starting in Σ similarly as in Verweisl 3.5. It is important to know that in general such a map cannot be extended to an isometry that is defined on all of Σ. For instance consider the Klein bottle $N = [0,1]^2/\sim$, where we identify the two vertical edges in opposite direction and the horizontal ones in common direction. The two partitions, the one into vertical, the other into horizontal lines, build a bifoliation, so in particular a singular Riemannian foliation admitting sections. Take M to be a vertical line and Σ to be the exceptional horizontal line. Let τ be a curve in M from a point in Σ to a point that is not in Σ. Obviously we cannot extend a local isometry defined as above to a map defined on Σ that respects the foliation. But we can develop these maps on the Riemannian universal cover $\tilde{\Sigma}$ and this is $T_{[\tau]} : \widetilde{(\Sigma, \tau(0))} \to \widetilde{(\Sigma, \tau(1))}$. The set of the above local isometries along vertical curves τ that start and end in Σ becomes a pseudogroup of local isometries on Σ while the set of its developments $T_{[\tau]}$ becomes a group acting on $\tilde{\Sigma}$ that we will later denote by $\tilde{\Gamma}$. It is more convenient to work with this group than with the corresponding pseudogroup. On the other hand we have to handle additional elements, namely the deck transformations of $\pi_\Sigma : \tilde{\Sigma} \to \Sigma$, which are contained in $\tilde{\Gamma}$, but do not contribute to the geometry of $\mathcal{F}$. Therefore we will divide them out and obtain a group Γ acting on Σ, that completely describes the holonomy of $\mathcal{F}$.

For a vertical/horizontal curve c and a horizontal/vertical curve d starting in the same point we denote by $T_c d$ the terminal horizontal/vertical edge of the homotopy $H_{(c,d)}$. Let $[c]$ and $[d]$ be the equivalence classes under homotopy in the corresponding leaf or section fixing endpoints. Then $T_c d$ depends only on $[c]$ and $[T_c d]$ only on $[c]$ and $[d]$; we write $T_{[c]}[d] := [T_c d]$.

Lemma 4.8

$$T_{c_1 c_2} = T_{c_2} \circ T_{c_1} \qquad \textit{and} \qquad T_c(d_1 d_2) = T_c d_1 \cdot T_{T_{d_1} c} d_2.$$

Proof The proof is clear. □

Let $M \cap \Sigma = \{x_i\}_{i \in I}$. We define

$$\tilde{\Lambda} = \left\{ [\tau] T_{[\sigma]} : \tilde{M} \to \tilde{M} \;\middle|\; \begin{array}{c} \tau \text{ is vertical}, \sigma \text{ is horizontal} \\ \tau(0) = \sigma(0) = x_0, \sigma(1) = \tau(1) \end{array} \right\}$$

and

$$\tilde{\Gamma} = \left\{ [\sigma] T_{[\tau]} : \tilde{\Sigma} \to \tilde{\Sigma} \;\middle|\; \begin{array}{c} \tau \text{ is vertical}, \sigma \text{ is horizontal} \\ \tau(0) = \sigma(0) = x_0, \sigma(1) = \tau(1) \end{array} \right\}.$$

Lemma 4.9 *$\tilde{\Gamma}$ is a subgroup of $I(\tilde{\Sigma})$ and $\tilde{\Lambda}$ a subgroup of Diff$(\tilde{M})$.*

Proof We only prove that $\tilde{\Gamma}$ is a subgroup of $I(\tilde{\Sigma})$, the proof for $\tilde{\Lambda}$ is similar. Let $[\sigma] T_{[\tau]} \in \tilde{\Gamma}$. Since $T_{[\tau]} : \widetilde{(\Sigma, x_0)} \to \widetilde{(\Sigma, \tau(1))}$ is an isometry and left multiplication with $[\sigma]$ is an isometry from $\widetilde{(\Sigma, \tau(1))}$ to $\widetilde{(\Sigma, x_0)}$, the given element is clearly an isometry of $\widetilde{(\Sigma, x_0)}$. We want to determine its inverse. Let $\tau' = T_{\sigma^{-1}}\tau^{-1}$ and $\sigma' = (T_{\tau'}\sigma)^{-1}$. Then $T_{[\sigma]}[\tau'] = (T_{[\sigma]} \circ T_{[\sigma]^{-1}})[\tau]^{-1} = [\tau]^{-1}$. Now

$$[\sigma']T_{[\tau']}([\sigma]T_{[\tau]}) = [\sigma'] \underbrace{T_{[\tau']}[\sigma]}_{[\sigma']^{-1}} \cdot T_{\underbrace{T_{[\sigma]}[\tau']}_{[\tau]^{-1}}} \circ T_{[\tau]} = [c_{x_0}],$$

where c_{x_0} is the constant curve with image x_0. Thus any element of $\tilde{\Gamma}$ has a left inverse. Now we want to show that the product of two elements $[\sigma]T_{[\tau]}$ and $[\sigma']T_{[\tau']}$ of $\tilde{\Gamma}$ lies in $\tilde{\Gamma}$.

$$[\sigma]T_{[\tau]}([\sigma']T_{[\tau']}) = [\sigma]T_{[\tau]}[\sigma'] \cdot T_{T_{[\sigma']}[\tau]} \circ T_{[\tau']} = ([\sigma]T_{[\tau]}[\sigma'])T_{[\tau']T_{[\sigma']}[\tau]}$$

lies in $\tilde{\Gamma}$, because $(\sigma T_\tau \sigma')(1) = (T_\tau \sigma')(1) = (T_{\sigma'}\tau)(1) = (\tau' T_{\sigma'}\tau)(1)$. Since $\tilde{\Gamma} \subset I(\tilde{\Sigma})$ it follows that $\tilde{\Gamma}$ is a group. □

The next lemma shows how $\tilde{\Gamma}$ depends on the choice of the base point x_0.

Lemma 4.10 *Let $\tilde{\Gamma}$ respectively $\tilde{\Gamma}'$ be defined as above with respect to the base point x respectively y, where x and y are $\mathcal{F}$-regular points in the same section Σ. Then $\tilde{\Gamma} = [\gamma]\tilde{\Gamma}'[\gamma]^{-1}$, where γ is an arbitrary horizontal curve from x to y.*

Proof Let $[\sigma]T_{[\tau]} \in \tilde{\Gamma}$. Then $[\gamma^{-1}\sigma T_\tau \gamma]T_{[T_\gamma \tau]} \in \tilde{\Gamma}'$. Now we have for any horizontal curve δ with $\delta(0) = x_0$

$$\begin{aligned}
& [\gamma] \cdot [\gamma^{-1}\sigma T_\tau \gamma]T_{[T_\gamma \tau]}([\gamma^{-1}][\delta]) \\
= \; & [\sigma]T_{[\tau]}[\gamma]T_{[T_\gamma \tau]}([\gamma^{-1}][\delta]) \\
= \; & [\sigma]T_{[\tau]}[\gamma]T_{[T_\gamma \tau]}([\gamma]^{-1})T_{[T_{\gamma^{-1}}T_\gamma \tau]}[\delta] \\
= \; & [\sigma](T_{[\tau]}[\gamma]T_{[T_\gamma \tau]}([\gamma]^{-1}))T_{[\tau]}[\delta] \\
= \; & [\sigma]T_{[\tau]}[\delta],
\end{aligned}$$

because $[T_{T_\gamma \tau}\gamma^{-1}] = [T_\tau \gamma]^{-1}$ by Lemma 4.8. □

There is a natural injective representation $\rho_\Sigma : \pi_1(\Sigma, x_0) \to \tilde{\Gamma}; [\sigma] \mapsto [\sigma]$. This means that $\tilde{\Gamma}$ contains the deck transformations of $\pi_\Sigma : \tilde{\Sigma} \to \Sigma; [\sigma] \mapsto \sigma(1)$. It is easy to see that $\tilde{\Gamma}$ normalizes $\pi_1(\Sigma, x_0)$. We call the group

$$\Gamma = \tilde{\Gamma}/\pi_1(\Sigma, x_0)$$

the *transversal holonomy group* of Σ. It is a subgroup of $I(\Sigma)$. The transversal holonomy group generalizes the Weyl group of an s-representation (or polar action) and the fundamental domains of Γ generalize the Weyl chambers. Note that Γ is independent of x_0. But it depends on the choice of the section Σ, unlike $\tilde{\Gamma}$. Also let $\Lambda = \tilde{\Lambda}/\pi_1(M, x_0)$.

Moreover there is a representation $\rho_M : \pi_1(M, x_0) \to \tilde{\Gamma}; [\tau] \mapsto T_{[\tau]}$ that is in general not injective. Let K_{x_0} be the kernel of ρ_M and $H_{x_0} = \pi_1(M, x_0)/K_{x_0}$. Since the action of $\pi_1(M, x_0)$ on $\tilde{\Sigma}$ by ρ_M is isometric, it is already determined by its infinitesimal (orthogonal) action on $T_{x_0}\tilde{\Sigma} = \nu_{x_0}M$, which is

$$\begin{aligned}
\pi_1(M, x_0) \times \nu_{x_0}M & \to \nu_{x_0}M \\
([\alpha], v) & \mapsto (\overset{1}{\underset{0}{\|}}\alpha)v.
\end{aligned}$$

This implies that H_{x_0} is isomorphic to the normal holonomy group of M. Thus we can write $\bar{M} = \tilde{M}/K_{x_0}$ for the normal holonomy principal bundle $\bar{M}$, and H_{x_0} is the group of deck transformations of $\tilde{M} \to \bar{M}$.

We define $[\sigma_0] = [c_{x_0}]$ and $[\tau_0] = [c_{x_0}]$. For each $i \in I, i \neq 0$, we choose a horizontal curve $[\sigma_i]$ and a vertical curve $[\tau_i]$ from x_0 to x_i. We can write any element $[\sigma]T_{[\tau]} \in \tilde{\Gamma}$ as

$$\alpha(i, g, h) := g[\sigma_i]T_{h[\tau_i]},$$

where $g \in \pi_1(\Sigma, x_0)$ and $h \in \pi_1(M, x_0)$.

Lemma 4.11 $\alpha(i, g, h) = \alpha(j, g', h') \iff i = j, g = g'$ *and* $h^{-1}h' \in K_{x_0}$.

Proof ($\Leftarrow$) follows from $T_{hk[\tau_i]} = T_{[\tau_i]} \circ T_k \circ T_h = T_{[\tau_i]} \circ T_h = T_{h[\tau_i]}$ for $k \in K_{x_0}$, since $T_k = \mathrm{id}_{\tilde{\Sigma}}$. For ($\Rightarrow$) apply $[c_{x_0}]$ to both sides. We see that $g[\sigma_i] = \alpha(i, g, h)[c_{x_0}] = \alpha(j, g', h')[c_{x_0}] = g'[\sigma_j]$. The endpoints x_i and x_j are equal, so $i = j$. Thus $g = g'$. It follows $T_h = T_{h'}$, or $T_{h^{-1}h'} = \mathrm{id}_{\tilde{\Sigma}}$. Observe that $h^{-1}h' \in \pi_1(M, x_0)$. Now we have $h^{-1}h' \in K_{x_0}$. □

We remark $\rho_\Sigma(g) = \alpha(0, g, [c_{x_0}])$ for any $g \in \pi_1(\Sigma, x_0)$ and $\rho_M(h) = \alpha(0, [c_{x_0}], h)$ for any $h \in \pi_1(M, x_0)$.

Lemma 4.12 *The action of* Γ *respects the foliation. The set of leaves of* $\mathcal{F}$ *is* $\Sigma/\Gamma = \tilde{\Sigma}/\tilde{\Gamma}$. *The set of sections is* $\tilde{M}/\tilde{\Lambda}$.

Proof For this proof it is advisable to recall the precise definition of the section $i : \Sigma \to N$ and to distinguish between the manifold Σ and the image $\Sigma' = i(\Sigma)$ since Σ can have self-intersections. We can take Σ to be the appropriate leaf of $\mathcal{F}^\perp$ in $\hat{N}$ and i to be the projection $\hat{\pi}|\hat{\Sigma} : \hat{\Sigma} \to \hat{\pi}(\Sigma) = \Sigma'$. The set of leaves is $N/\mathcal{F}$. We have seen at the beginning of this subsection, that $\hat{\pi}$ defines a bijection between the set of leaves of $\mathcal{F}$ and that of $\hat{\mathcal{F}}$. Therefore $N/\mathcal{F} = \hat{N}/\hat{\mathcal{F}}$. For the bifoliated manifold $\hat{N}$ we can easily prove $\tilde{\Gamma}[\sigma] = \pi_\Sigma^{-1}(\hat{M}_{\sigma(1)} \cap \Sigma)$ for any $[\sigma] \in \tilde{\Sigma}$ and $\Gamma(x) = \hat{M}_x \cap \Sigma$ for any $x \in \Sigma$. Thus $\hat{N}/\hat{\mathcal{F}} = \Sigma/\Gamma = \tilde{\Sigma}/\tilde{\Gamma}$. □

The description of a $\tilde{\Gamma}$-orbit in the proof implies in particular that each element of $\tilde{\Gamma}$ permutes the set $\{g[\sigma_i] \mid i \in I, g \in \pi_1(\Sigma, x_0)\}$. In other words, this defines a representation of $\tilde{\Gamma}$ as a permutation group. This representation is faithful if M has trivial normal holonomy, because of $K_{x_0} = 1$ and Lemma 4.11. Now let $\mathcal{F}$ be a proper singular Riemannian foliation admitting sections. Then each regular leaf M has parallel focal structure and finite normal holonomy. The set $\{x_i\}$ is disrete and closed. We call

$$\mathcal{D}_{x_i} = \{q \in \Sigma \mid d(x_i, q) < d(x_j, q) \text{ for all } j \neq i\}$$

a *Dirichlet region* of the set $\{x_i\}$, where d is the distance function in Σ. These sets are open and disjoint and we have $\bigcup_i \overline{\mathcal{D}}_{x_i} = \Sigma$. The set $\mathcal{D}_{x_i}$ is star-shaped and therefore 1-connected; thus the universal covering $\pi_\Sigma : \tilde{\Sigma} \to \Sigma$ is trivial over $\mathcal{D}_{x_i}$ and we denote the connected component of $\pi_\Sigma^{-1}(\tilde{\mathcal{D}}_{x_i})$ containing $g[\sigma_i], g \in \pi_1(\Sigma, x_0)$ by $\tilde{\mathcal{D}}_{g[\sigma_i]}$. Then $\{\tilde{\mathcal{D}}_{g[\sigma_i]} \mid g \in \pi_1(\Sigma, x_0), i \in I\}$ is the set of Dirichlet regions for $\pi_\Sigma^{-1}(M \cap \Sigma)$.

Proposition 4.13 *Let $\mathcal{F}$ be a proper singular Riemannian foliation admitting sections. Then the action of Γ on Σ is properly discontinuous. It acts transitively on the set of Dirichlet regions $\{\mathcal{D}_{x_i}\}_{i\in I}$, and simply transitive, if M has trivial normal holonomy. The same holds for $\tilde{\Gamma}, \tilde{\Sigma}$ and $\{\tilde{\mathcal{D}}_{g[\sigma_i]} \mid g \in \pi_1(\Sigma, x_0), i \in I\}$. The set of leaves Σ/Γ is an orbifold.*

Proof The action of $\tilde{\Gamma}$ on $\tilde{\Sigma}$ is isometric and has discrete orbits, thus it is properly discontinous, i.e., for any compact subset K of $\tilde{\Sigma}$ the intersection $\phi(K)\cap K$ is nonempty for only a finite number of $\phi \in \tilde{\Gamma}$. This implies that the set of leaves is an orbifold. The rest follows from Lemma 4.11. □

Remark As we can see later, the singular leaves of $\mathcal{F}$ lift to exceptional leaves. Therefore the nonregular points of the orbifold Σ/Γ correspond exactly to leaves of $\mathcal{F}$ that are either exceptional or singular.

Lemma 4.14 *The isotropy group $\tilde{\Gamma}_{[c_{x_0}]} = \rho_M(\pi_1(M, x_0)) \cong H_{x_0}$ is characterized in $\tilde{\Gamma}$ by mapping $\tilde{\mathcal{D}}_{[c_{x_0}]}$ onto itself. Consequently $\tilde{\Gamma}_{[\sigma]} \subset \tilde{\Gamma}_{[c_{x_0}]}$ for any $[\sigma] \in \tilde{\mathcal{D}}_{[c_{x_0}]}$. An anlogous property holds for Γ.*

Proof Clear. □

$H_{x_0} \cong \Gamma_{x_0}$ means that the normal holonomy of a leaf is just the isotropy group of the larger action Γ.

Before we come to applications, we want to show a relation between $\pi_1(\hat{N}, x_0), \tilde{\Gamma}$ and $\tilde{\Lambda}$. We identify the actions of $\tilde{\Gamma}$ and $\tilde{\Lambda}$ with the corresponding actions for $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$. For $[\mu] \in \pi_1(\hat{N}, x_0)$ we define $\tilde{\gamma}_{[\mu]} := [(\mu_h)]T_{[(\mu^v)^{-1}]} \in \tilde{\Gamma}$ and $\tilde{\lambda}_{[\mu]} := [(\mu_v)]T_{[(\mu^h)^{-1}]} \in \tilde{\Lambda}$ (for the notation see 2.2). One can show that $\tilde{\gamma} : \pi_1(\hat{N}, x_0) \to \tilde{\Gamma}; [\mu] \mapsto \tilde{\gamma}_{[\mu]}$ and $\tilde{\lambda} : \pi_1(\hat{N}, x_0) \to \tilde{\Lambda}; [\mu] \mapsto \tilde{\lambda}_{[\mu]}$ are homomorphisms. $\pi_1(\hat{N}, x_0)$ acts naturally from the left on the universal cover of $\hat{N}$. We want to transfer this action to $\tilde{M} \times \tilde{\Sigma}$ via the f-isomorphism Φ. Let $[\mu] \in \pi_1(\hat{N}, x_0)$ and $\Phi([\tau], [\sigma]) = [\nu]$. Then

$$\Phi^{-1}([\mu][\nu]) = ([(\mu_h)]T_{[(\mu^v)^{-1}]}[\tau], [\mu_v]T_{[(\mu^h)^{-1}]}[\sigma]) = (\tilde{\gamma}_{[\mu]}([\tau]), \tilde{\lambda}_{[\mu]}([\sigma])).$$

This shows that the action of $\pi_1(\hat{N}, x_0)$ respects the product foliation on $\tilde{M} \times \tilde{\Sigma}$, so $\pi_1(\hat{N}, x_0)$ is a subgroup of $\mathrm{Diff}(\tilde{M}) \times \mathrm{I}(\tilde{\Sigma})$. The projection of $\pi_1(\hat{N}, x_0)$ on the first component is $\tilde{\Gamma}$, the one on the second is $\tilde{\Lambda}$. The projection homomorphisms are $\tilde{\gamma}$ and $\tilde{\lambda}$. This describes a new view on the transversal holonomy group.

We will now give an application for the action of Γ. Reinhart showed in [Rei] that the nearby leaves of a leaf M in a Riemannian foliation are coverings of M. The next proposition describes the maximal neighborhood for which this is true. Compare with the proof in [Rei].

Proposition 4.15 *Let M be a regular leaf of a proper singular Riemannian foliation $\mathcal{F}$ admitting sections and let $x_0 \in M$ be arbitrary. Then any regular leaf M' through $\mathcal{D}_{x_0}$ covers M and the degree is equal to the holonomy orbit $\tilde{\Gamma}_{[c_{x_0}]}[\gamma]$ (or $\Gamma_{x_0}(\gamma(1))$), where γ is a shortest geodesic in Σ from x_0 to a point in M'.*

Proof Let $y_0 \in \mathcal{D}_{x_0} \cap M'$ and let γ_0 be a shortest geodesic from x_0 to y_0 which is contained in $\mathcal{D}_{x_0}$. Moreover let $Y := \{[\gamma_j]\}_{j\in J} := \tilde{\Gamma}_{[c_{x_0}]}[\gamma_0]$. We define an action

$h \cdot ([\tau],[\gamma_j]) = (h[\tau], T_{[h^{-1}]}[\gamma_j])$ of $\pi_1(M,x_0)$ on $\tilde{M} \times Y$. Note that this group acts from the left by Lemma 4.8, and the action is free and properly discontinuous. Let $\tilde{M} \times_{\pi_1(M,x_0)} Y := (\tilde{M} \times Y)/\pi_1(M,x_0)$ We want to show that

$$\begin{array}{rcl} \tilde{M} \times_{\pi_1(M,x_0)} Y & \to & M' \\ ([\tau],[\gamma_j]) & \mapsto & (T_\tau \gamma_j)(1) \end{array}$$

is a diffeomorphism. The map is clearly surjective. We show that it is well-defined. Let $h \in \pi_1(M,x_0)$ and $([\tau],[\gamma_j]) \in \tilde{M} \times Y$. Then

$$(T_{T_{h^{-1}}\gamma_j} h\tau)(1) = (T_{T_{h^{-1}}\gamma_j} h \cdot T_{T_h T_{h^{-1}}\gamma_j} \tau)(1) = (T_{\gamma_j}\tau)(1),$$

so the map is well-defined. We prove injectivity. Let $(T_{\gamma_j}\tau)(1) = (T_{\gamma_k}\tau')(1)$ for vertical curves τ, τ' starting at x_0. Then $\tau(1) = \tau'(1)$, so there is exactly one $h \in \pi_1(M,x_0)$ such that $h[\tau] = [\tau']$. We claim $\gamma_k = T_{h^{-1}}\gamma_j$. We have $(T_\tau \gamma_j)(1) = (T_{\gamma_j}\tau)(1) = (T_{\gamma_k}\tau')(1) = (T_{\tau'}\gamma_k)(1)$. Thus $(T_\tau\gamma_j)(1) = (T_{h\tau}\gamma_k)(1) = (T_\tau(T_h\gamma_k))(1)$. Applying $T_{\tau^{-1}}$ shows $\gamma_j(1) = (T_h\gamma_k)(1)$, i.e., $\pi_\Sigma([\gamma_j]) = \pi_\Sigma(T_h[\gamma_k])$. Since $[\gamma_j]$ and $T_h[\gamma_k]$ lie in $\tilde{\mathcal{D}}_{[c_{x_0}]}$ this implies $[\gamma_j] = T_h[\gamma_k]$ and we proved our claim. Now the above map is a diffeomorphism. Thus $M' = \tilde{M} \times_{\pi_1(M,x_0)} Y$ covers M with typical fiber Y. □

Remark It is clear that $Y = (\pi_\Sigma|\tilde{\mathcal{D}}_{[c_{x_0}]})^{-1}(M' \cap \mathcal{D}_{x_0})$. The preimage of x_0 under the covering $M' \to M$ is $M' \cap \mathcal{D}_{x_0}$. Moreover we have $T_{[\gamma_j]}K_{[\gamma_j]} = \pi_1(M',y)$, where $K_{[\gamma_j]}$ is the subgroup of $\pi_1(M,x_0)$ of elements k such that $\rho_M(k)$ fixes $[\gamma_j]$.

Proof of Proposition 3.12 The first statement, that we have already proved, follows also directly from the fact that T_τ is an isometry. So the normal parallel translation of a focal normal is a focal normal with the same horizontal and vertical multiplicity. We will now prove the second statement. Let $v \in \nu M$ with footpoint x be arbitrary. We claim $\sigma(v'/\|v'\|) \le \sigma(v/\|v\|)$ where v' is an arbitrary normal parallel translation of v and σ is the cut distance function. This is clear if v is a focal normal by the first statement. We assume that v is a regular cut vector. Then there is a point $y \in M$ in the same section as x and let w be minimal with $\eta(w) = \eta(v)$. We define $\gamma = \gamma_v\gamma_w^{-1}$.

$$\begin{array}{ccccc} \widetilde{(M,x)} & & \xrightarrow{T_{[\gamma]}} & & \widetilde{(M,y)} \\ \Big\downarrow & \searrow & & \swarrow & \Big\downarrow \\ & & N & & \\ \Big\downarrow & \nearrow & & \nwarrow & \Big\downarrow \\ M & & \longrightarrow & & M \end{array}$$

The arrow $(\tilde{M},x) \to N$ respectively $(\tilde{M},y) \to N$ is the map $[\tau] \mapsto (T_\tau\gamma_v)(1)$ respectively $[\tau] \mapsto (T_\tau\gamma_w)(1)$. Then the upper triangle commutes because

$$T_{[\tau]}[\gamma_v] = T_{[\tau]}[\gamma\gamma_w] = (T_{[\tau]}[\gamma])T_{T_{[\gamma]}[\tau]}[\gamma_w].$$

The two vertical arrows are the natural projections $p_1 : (\tilde{M}, x) \to M/K_x = \bar{M}$ and $p_2 : (\tilde{M}, y) \to M/K_y = \bar{M}$. We can push down the map $p_2 \circ T_{[\gamma]}$ along p_1 to the lower horizontal arrow, which we denote by α, because

$$p_2(T_{[\gamma]}(k[\tau])) = p_2(T_{[\gamma]}k \cdot T_{[\gamma]}[\tau]) = p_2(T_{[\gamma]}[\tau])$$

for any $k \in \pi_1(M, x)$ since $T_{[\gamma]}k \in K_y$, so the rectangle commutes. The left diagonal arrow $\bar{M} \to N$ is $\eta \circ \bar{v}$, the right diagonal arrow is $\eta \circ \bar{w}$. The left and the right triangles commute. By the commutativity of the three triangles and the rectangle

$$(\eta \circ \bar{v}) \circ p_1 = (\eta \circ \bar{w}) \circ (p_2 \circ T_{[\gamma]}) = (\eta \circ \bar{w}) \circ (\alpha \circ p_1).$$

Consequently $\eta \circ \bar{v} = (\eta \circ \bar{w}) \circ \alpha$ (that means also the lower triangle commutes). This proves our claim. By the same reason there cannot be a normal parallel translation v' of v with $\sigma(v'/\|v'\|) < \sigma(v/\|v\|)$. It follows that the cut distance function of M is constant under normal parallel translation and that the normal parallel translation of a cut vector is a cut vector. □

The next proposition shows the relation between $\{\mathcal{D}_{x_i}\}$ and the cut locus of a regular leaf, which has parallel focal structure as we know. It also shows that a leaf M' through $\mathcal{D}_{x_0}$ is regular if M has a globally flat normal bundle.

Proposition 4.16 *$\bigcup_{i \in I} \partial\mathcal{D}_{x_i} \subset \mathcal{C}_{(M,N)}$ and $\mathcal{C}_{(M,N)} \cap \mathcal{D}_{x_i}$ are points of the cut locus of $\exp^{\Sigma}_{x_i}$. If M has a globally flat normal bundle, then the focal points of horizontal type are contained in $\bigcup_{i \in I} \partial\mathcal{D}_{x_i}$.*

Proof Take a point $p \in \partial\mathcal{D}_{x_i}$. If p is a focal point of horizontal type, then each minimal normal vector with endpoint p is a focal normal of horizontal type by Lemma 3.2. Then $p \in \mathcal{C}_{(M,N)}$. Now assume $p \in \partial\mathcal{D}_{x_i}$ is f-regular. Then there exists a $j \neq i$ such that $d(p, x_i) = d(p, x_j)$. Thus there are vectors $v \in T_{x_i}\Sigma$ and $w \in T_{x_j}\Sigma$ of length $d(p, x_i)$ with endpoint p. These vectors are contained in νM. Observe that any other normal vector u with endpoint p is tangential to Σ in a point $x_k \in M$ and $\|u\| \geq d(p, x_k) \geq d(p, x_i)$. Thus v and w are minimal for η and p is a cut point of M.

Take $p \in \mathcal{C}_{(M,N)} \cap \mathcal{D}_{x_i}$. First we assume that p is a focal point of horizontal type. Let $v \in \nu M$ be minimal with $\eta(v) = p$. Using Lemma 3.9 we can assume that v is tangential to Σ. Then $v \in \nu_{x_i}M$ by minimality. We now construct F, F^1_v, F' as in the proof of Lemma 3.9. We have seen that F' is a compact submanifold of $\nu_p M_v$ intersecting $T_p\Sigma$ in $\phi^1(v)$. Moving $T_p\Sigma$ homotopically beyond F' in $\nu_p M_v$ shows that the intersection index of $T_p\Sigma$ with F' modulo 2 is zero. Thus there is another point w' in the intersection of $T_p\Sigma$ and F'. Then $w := \phi^{-1}w' \in F^1_v$ is tangential to Σ in a point $x_j \in M \cap \Sigma$ and we have $\eta(w) = p$. Since $\|w\| = \|v\| = d(x_i, p) < d(x_k, p)$ for every $k \neq i$, it follows $j = i$. Thus M has non-trivial normal holonomy, which proves the third statement. As v is minimal and $\|w\| = \|v\|$, $p = \eta(v) = \eta(w)$ is a cut point for η and $\exp^{\Sigma}_{x_i}$.

Now assume that $p \in \mathcal{C}_{(M,N)} \cap \mathcal{D}_{x_i}$ is an f-regular point. Let $v \in \nu M$ be minimal with $\eta(v) = p$. Lemma 3.2 implies that v is tangential to Σ and the foot point is

x_i by minimality. We know that p is a focal point (in this case it can only be of vertical type) or a cut point by Proposition 2.4. Assume p is a cut point. Then there is another minimal normal $w \in \nu_{x_j} M = T_{x_j}\Sigma$ with $\eta(w) = p$ for some j. Since v is minimal for η, it follows $j = i$ and that v is minimal for $\exp^{\Sigma}_{x_i} : T_{x_i}\Sigma \to \Sigma$, so p is also cut point of $\exp^{\Sigma}_{x_i}$. Now assume that p is not a cut point. Then v is the unique minimal vector in νM with endpoint p and v is a focal normal. Since p is f-regular, v is singular and minimal for $\exp^{\Sigma}_{x_i}$. □

In the proof we have seen that F' intersects the given section at least twice. Together with Lemma 3.8 this implies that the lift of a singular leaf is exceptional (see the remark after Proposition 4.13).

Example We will give an example such that a Dirichlet region $\mathcal{D}_{x_i}$ contains a focal point of horizontal type. Consider the image M of a geodesic in $P^2\mathbb{R}$. This is a submanifold with parallel focal structure with normal holonomy group $\mathbb{Z}_2$. M intersects $\Sigma = P^1\mathbb{R} = S^1$ in exactly one point. The Dirichlet region is equal to Σ and contains the unique focal point of M.

Now we express Proposition 4.15 as a corollary in terms of the cut locus.

Corollary 4.17 *Let M be a closed and embedded submanifold with parallel focal structure and finite normal holonomy. Then the parallel submanifolds that are not contained in the cut locus of M are coverings of M.*

The following result is a another corollary of Proposition 4.15.

Corollary 4.18 *The regular leaves with a globally flat normal bundle are diffeomorphic to each other. They cover any other regular leaf, the exceptional leaves. These exceptional leaves, if they exist, are contained in the cut locus of any regular leaf with a globally flat normal bundle. The union of regular leaves with a globally flat normal bundle is open and dense in N.*

Proof The action of Γ on Σ is isometric and properly discontinuous. It therefore has a set of fundamental domains on which Γ acts simply transitive. For any point p in a fundamental domain we have therefore $H_p \cong \Gamma_p = 1$. Since the union of the set of fundamental domains is open and dense in Σ, so is its intersection with the $\mathcal{F}$-regular points, and the last statement of the corollary follows. Alternatively we can see this by recalling that the subset of reguar points of an orbifold, in this case Σ/Γ, is open and dense.

Let $M = M_{x_0}$ be a regular leaf with a globally flat normal bundle, i.e. $\Gamma_{x_0} = 1$. Then any regular leaf through a point $y \in \mathcal{D}_{x_0}$ is diffeomorphic to M by Proposition 4.15 and has a globally flat normal bundle because $\Gamma_y \subset \Gamma_{x_0} = 1$ by Lemma 4.14. Now let M' be a regular leaf that does not intersect $\mathcal{D}_{x_0}$. Then it intersects $\partial\mathcal{D}_{x_0}$. We know that it is covered by a nearby leaf M'', which intersects $\mathcal{D}_{x_0}$ and is thereby diffeomorphic to M. So M covers M'. Now we assume that M' has a globally flat normal bundle. Then it is diffeomorphic to M'' and therefore to M. □

Remark Let G be a Riemannian transformation group of (N, g) and let S be a slice through a point $x \in N$ of an orbit Gx. It is known that $G_y \subset G_x$ for every $y \in S$. If Gx is an orbit of maximal dimension, this means that the orbit type of

Gx is smaller or equal to that of nearby orbits. This corresponds in our theory to $\Gamma_y \subset \Gamma_x$ and that M_y is covering of M_x.

Consider the trace of the cut locus of M in a section Σ. How does this trace differ from that of the cut locus of a nearby parallel submanifold? Roughly speaking, the focal points of the cut locus are fixed. But the cut points can move; this can occur for instance, if the transversal holonomy group Γ contains translations. The above corollary says that certain cut points, namely the points of exceptional leaves intersected with Σ, are fixed as well.

Our next aim is to show that there are no exceptional leaves if N is a simply connected symmetric space. For a point $p \in N$ we define $\mathcal{P} = \mathcal{P}(N, \varphi \times p)$ as the set of pairs (x, γ), where $x \in M$ and $\gamma : [0,1] \to N$ is a H^1-curve in N with $\gamma(0) = \varphi(x)$ and $\gamma(1) = p$. We write $\mathcal{P}(N, M \times p)$ for the path space if $\varphi : M \to N$ is the inclusion map. It is known that $\mathcal{P}$ is a Hilbert manifold. The smooth function

$$\begin{aligned} E_p : \mathcal{P} &\to \mathbb{R} \\ (x, \gamma) &\mapsto \int_0^1 \|\dot{\gamma}(t)\|^2 \, dt \end{aligned}$$

is called the *energy functional* (associated to p). The map E_p is a Morse function, i.e., it has only non-degenerate critical points, if and only if p is not a focal point of φ. We assume that p is not a focal point, i.e., p is regular for the normal exponential map of M. The energy functional is bounded below by zero and it is known that it satisfies the Palais-Smale condition. For $s \in \mathbb{R}$ we write $\mathcal{P}^s = E_p^{-1}\{[-\infty, s]\}$ and $\mathcal{P}^{s-} = E_p^{-1}\{[-\infty, s)\}$. Let $\mathbb{F}$ be a field and s be a regular value of E_p. The Morse inequalities state $b_k(\mathcal{P}^s, \mathbb{F}) \leq \mu_k(E_p|\mathcal{P}^s)$, where $b_k(\mathcal{P}^s, \mathbb{F})$ is the k-th Betti number of $\mathcal{P}^s$ with respect to $\mathbb{F}$ and $\mu_k(E_p|\mathcal{P}^s)$ is the number of critical points of index k of E_p below s. We call φ *taut* with respect to $\mathbb{F}$, if, for every regular point $p \in N$ and every regular value s, $E_p|\mathcal{P}^s$ is *perfect* with respect to $\mathbb{F}$, i.e., if $b_k(\mathcal{P}^s, \mathbb{F}) = \mu_k(E_p|\mathcal{P}^s)$ for all k. If $b_i(\mathcal{P}^s, \mathbb{F}) = \mu_i(E_p|\mathcal{P}^s)$ for all i with $i \leq k$, all regular points p and regular values s, we say that φ is k-taut with respect to $\mathbb{F}$.

The following paragraph is a brief summary of [PaTe] about critical points of linking type with slight changes in the definitions. Let κ be a critical level of E_p. There is a real number $\varepsilon > 0$ such that κ is the only critical level of E_p in $[\kappa-\varepsilon, \kappa+\varepsilon]$. One result of Morse theory is, that by properly attaching a k-cell e^k for each critical point of index k on level κ to $\mathcal{P}^{\kappa-\varepsilon}$ in $\mathcal{P}^{\kappa+\varepsilon}$, we obtain a deformation retract of $\mathcal{P}^{\kappa+\varepsilon}$. Each k-cell e_i^k attached gives a generator $[e_i^k]$ of $H_k(\mathcal{P}^\kappa, \mathcal{P}^{\kappa-}) = H_k(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon}) = \oplus_{i=1}^{r_k} \mathbb{F}[e_i^k]$, where r_k is the number of all critical points of index k on level κ. We consider the following long exact homology sequence of the pair $(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$ with respect to $\mathbb{F}$:

$$\to H_{k+1}(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon}) \xrightarrow{\partial_{k+1}} H_k(\mathcal{P}^{\kappa-\varepsilon}) \xrightarrow{i_k} H_k(\mathcal{P}^{\kappa+\varepsilon}) \xrightarrow{j_k} H_k(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon}) \to$$

We say that a critical point of index k on level κ is of *linking type* (with respect to $\mathbb{F}$), if each generator $[e_i^k]$ of $H_k(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$ is in the image of j_k, or equivalently, if $\partial_k([e_i^k]) = 0$. If every critical point of index k on level κ is of linking type, we call any k-cycle z of $\mathcal{P}^{\kappa+\varepsilon}$ with $j_k([z]) = [e_i^k]$ for some i a *linking cycle*. If all critical points of index l for $l \leq k+1$ on level κ are of linking type, then we can read off the exact sequence

$$0 \to H_l(\mathcal{P}^{\kappa-\varepsilon}) \xrightarrow{i_l} H_l(\mathcal{P}^{\kappa+\varepsilon}) \xrightarrow{j_l} \bigoplus_{i=1}^{r_l} \mathbb{F}[e_i^l] \to 0$$

for all $l \leq k$ from the long exact homology sequence above. This sequence clearly splits, so $H_l(\mathcal{P}^{\kappa+\varepsilon}) = H_l(\mathcal{P}^{\kappa-\varepsilon}) \oplus (\oplus_{i=1}^{r_l} \mathbb{F}[e_i^l])$ for $l \leq k$. Now we assume that all critical points of index l for $l \leq k+1$ are of linking type. Let s be a regular value of E_p. By induction it follows that for each $l \leq k$ we have

$$H_l(\mathcal{P}^s) = \mathbb{F}^r,$$

where r is the number of all critical points of index l of level smaller than s. In particular, φ is k-taut.

Thorbergsson proved in [Th1] that a compact proper Dupin hypersurface in S^n is taut with respect to $\mathbb{Z}_2$ by constructing concrete linking cycles. Ewert states that a proper immersion $\varphi : M \to N$ with parallel focal structure and a globally flat normal bundle is taut ([Ew], Theorem 2.19). But his proof contains a gap as Thorbergsson pointed out to me. If N contains no conjugate points, his construction of linking cycles is correct. In this case he constructs like Thorbergsson a variation of a given normal geodesic $\gamma : [0, l] \to N$ of length l to a regular point p. Then there are $t_i \in \mathbb{R}, i = 1, \dots k$ with $0 < t_1 < \dots < t_k < l$, sucht that $\gamma|[0, t_i]$ is a focal geodesic with multiplicity μ_i and any focal geodesic segment on γ is covered this way. The desired variation $\lambda : K \to \mathcal{P}$ is defined on an iterated fiber bundle K, one iteration for every focal point $\gamma(t_i)$ (increasing the dimension by μ_i). Each $\lambda(x)$ is a broken geodesic of length l such that for a vertex $\lambda(x)(t)$ we have $t = t_i$ for some i. In the case that N has conjugate points, Ewert adds an iteration for any conjugate point/focal point of vertical type on γ. But then a focal point of horizontal type can cross a focal point of vertical type in this variation λ. Thus K is more complicated than an iterated fiber bundle and it is not clear if K is a manifold at all. We restate Ewert's Theorem 2.19 below as Theorem 4.20.

For our considerations understanding 0-tautness is enough. The construction of linking cycles is then much simpler. In the proof below, we follow Ewert's construction of these cycles. Please note the following: Ewert states that a proper immersion $\varphi : M \to N$ with parallel focal structure and a globally flat normal bundle in a simply connected symmetric space is 0-taut, thus embedded. This is not true. Consider an example of $\varphi : M \to N$, where M is not simply connected, meeting the requirements of the statement. Then there is a covering map $p : M' \to M$ of finite degree and $\varphi \circ p$ is not an embedding, contradiction. Indeed, the cycles constructed in [Ew], Lemma 2.10, are only linking cycles for embeddings. Therefore the line of argumentation must be reversed. First we show that φ factorizes finitely over an embedding φ_0; then we show that φ_0 is 0-taut with the technique of linking cycles. Moreover, we extend his result to non-trivial normal holonomy and relate it to the cut locus and to exceptional leaves.

Theorem 4.19 *Let $\varphi : M \to N$ be a proper immersion with parallel focal structure with finite normal holonomy into a simply connected symmetric space N. Then φ factorizes finitely over a 0-taut (with respect to $\mathbb{F}$) embedding that has a globally flat normal bundle. There are no exceptional parallel submanifolds and the cut locus of M only consists of focal points.*

It is well-known that a closed hypersurface M of a simply connected manifold N is orientable and thus has a globally flat normal bundle. If the codimension is greater than one we have to argue differently.

Proof of Theorem 4.19 By remark (3) following Lemma 3.2, φ factorizes finitely over an embedding. So we can assume that φ is this embedding. Let p be a regular point in N with respect to the normal exponential map of M. Let $\gamma \in \mathcal{P}$ be an arbitrary critical point of index 1 of E_p, i.e., γ is a normal geodesic of index 1, with $E_p(\gamma) = \kappa$. Let e^1 be the corresponding 1-cell in $\mathcal{P}^{\kappa+\varepsilon}$ attached to $\mathcal{P}^{\kappa-\varepsilon}$. Let $v \in \nu_x M, \varphi(x) = \gamma(0)$ with $\gamma_v = \gamma$, and let $t_0 v$, $0 < t_0 < 1$ be the focal normal with multiplicity 1. First we assume that v is a focal normal of horizontal type. Let F be the footpoint set of the associated focal leaf. By the remark after Theorem 4.3 this set is a submanifold on which we can extend v to a parallel normal field, also denoted by v. Since F is 1-dimensional and compact we have $F \cong S^1$. We construct a variation $\lambda : F \to \mathcal{P}$ of γ by

$$\lambda(y)(t) := \begin{cases} \eta(tv_y) & \text{if } t \in [0, t_0] \\ \gamma(t) & \text{if } t \in [t_0, 1] \end{cases}$$

This smooth map is injective and Ewert deforms it under the negative gradient flow of E_p to a map $\lambda' : F \to \mathcal{P}$ that has a unique non-degenerate maximum in x. If we denote the generator of $H_1(F) = H_1(S^1)$ by z', then $z := \lambda_*(z') = \lambda'_*(z') \in H_1(\mathcal{P}^{\kappa+\varepsilon})$ is a so-called *Bott-Samelson cycle*, a special kind of linking cycle (see [PaTe]), with $j_1(z) = [e^1]$. Now we assume that v is a focal normal of vertical type, i.e., $\gamma(t_0)$ is conjugate to x along γ in Σ_x with multiplicity 1. Since Σ_x is a symmetric space as a totally geodesic submanifold of N, an S^1-action fixing x and $\gamma(t_0)$ applied to $\gamma|[0, t_0]$ gives an S^1-familiy of geodesics from $\gamma(0)$ to $\gamma(t_0)$. We extend this variation as above to a map $\lambda : S^1 \to \mathcal{P}$ and Ewert proves that also $z := \lambda_*(z') \in H_1(\mathcal{P}^{\kappa+\varepsilon})$, where z' is the generator of $H_1(S^1)$, is a linking cycle with $j_1(z) = [e^1]$. Thus every critical point of index 1 is of linking type, which implies that φ is 0-taut. As N is simply connected, $b_0(\mathcal{P}, \mathbb{F}) = 1$ by the homotopy sequence for the fibration $\mathcal{P} \to M; (x, \gamma) \mapsto x$, and E_p has only one local minimum. If there is a cut point that is not a focal point we would have at least two local minima by the definition of a cut point, contradicting 0-tautness. Assume now that there is an exceptional parallel manifold M' of M. We choose $\varepsilon > 0$ smaller than the injectivity radius of M', $p \in M'$ and $v \in \nu_p M'$ with non-trivial holonomy degree and $\|v\| < \varepsilon$. Let $w \neq v$ in $\nu_p M'$ be a normal parallel translation of v. The endpoints of v and w are contained in $M'_v \cap \mathcal{D}_p$, where $\mathcal{D}_p$ is a Dirichlet region for $M'_v \cap \Sigma_p$. We know from our previous observations that any two elements of $M'_v \cap \mathcal{D}_p$ have the same distance to p in Σ_p. Hence the geodesics $\gamma_v|[0, 1]$ and $\gamma_w|[0, 1]$, if parameterized in reverse direction, are minimal for M'_v. By the choice of ε they have index 0. Therefore there are at least two minima of $E_p : \mathcal{P}(N, M'_v \times p) \to \mathbb{R}$. Since also M'_v is 0-taut, this is a contradiction. In particular, M has a globally flat normal bundle. □

Remark The assumption that N is a symmetric space was necessary to construct cycles if the focal point on γ is a conjugate point. Therefore, if Σ is a symmetric space or if it has no conjugate points, e.g., if M is equifocal, we can drop the symmetry of N.

Theorem 4.19 has the following converse. If M is a closed and embedded submanifold with parallel focal structure such that its cut locus only consists of focal points of horizontal type, then $b_0(\mathcal{P}, \mathbb{F}) = 1$ for any regular point $p \in N$ and M (but not necessarily every parallel submanifold) is 0-taut with respect to $\mathbb{F}$: Let p be an arbitrary regular point. Since p is not in the cut locus, there is a unique minimal normal geodesic from M to p. Any other normal geodesic from M to p thus has to pass the cut locus and is therefore not of index 0 (note that here we use Remark(1) after Lemma 3.2). So $\mu_0(E_p) = 1$ and therefore, by the first Morse inequality, $b_0(\mathcal{P}, \mathbb{F}) = \mu_0(E_p) = 1$ (if $\mathcal{P} \neq \emptyset$).

Theorem 4.20 (Ewert) *A proper immersion $\varphi : M \to N$ with parallel focal structure in a symmetric space of noncompact type factorizes finitely over a taut embedding.*

5 Appendix

We want to introduce a natural metric on the Grassmann bundle $G_k(TN)$. Let $O(TN) \to N$ be the principal bundle of all orthonormal m-frames on TN, where $m = \dim N$. An orthonormal m-frame z over T_pN can be seen as a linear isometry $z : \mathbb{R}^m \to T_pN$, the identification being $z \mapsto (ze_1, \dots, ze_m)$, where $\{e_i\}$ is the canonical base of $\mathbb{R}^m$. Then the orthogonal group $O(m)$ acts naturally on $O(TN)$ from the right. For any X in the Lie algebra $L(O(n))$ of $O(n)$ we define the fundamental vector field $\tilde{X}$ on $O(TN)$ by

$$\tilde{X}_z = \frac{d}{dt}\bigg|_{t=0} z \cdot \exp(tX),$$

for $z \in O(TN)$. Let $L(O(n))$ be endowed with the Ad-invariant scalar product that is unique up to a factor. Let $z \in O(TN)$ and let $F_z = z \cdot O(n)$ be the fiber of $O(TN) \to N$ through z. We endow T_zF_z with the scalar product such that $\bar{z} : L(O(n)) \to T_zF_z; X \mapsto \tilde{X}_z$ becomes a linear isometry. Because of $R_{g*}\tilde{X}_z = (\widetilde{\mathrm{Ad}_{g^{-1}} X})_{zg}$ we obtain an $O(n)$-invariant fiber metric on $O(TN) \to N$. The horizontal distribution on $O(TN) \to N$ induced by the connection on N is invariant under $O(n)$. Therefore $O(TN)$ carries an $O(n)$-invariant metric such that the bundle projection $O(TN) \to N$ is a Riemannian submersion.

The Grassmannian $G_k(\mathbb{R}^m)$ carries an $O(n)$-invariant metric that is unique up to a constant factor. With the given metrics, $O(n)$ acts isometrically on $O(TN) \times G_k(\mathbb{R}^m)$ from the left by $g \cdot (z, V) = (zg^{-1}, gV)$. We write $O(TN) \times_{O(n)} G_k(\mathbb{R}^m) = (O(TN) \times G_k(\mathbb{R}^m))/O(n)$ for the quotient. This fiber bundle over N is the associated bundle to $O(TN)$ with typical fiber $G_k(\mathbb{R}^m)$, which is the Grassmann bundle $G_k(TN)$. Since $O(n)$ acts by isometries on $O(TN) \times G_k(\mathbb{R}^k)$, we can endow $G_k(TN)$ with the unique metric g' such that $O(TN) \times G_k(\mathbb{R}^m) \to G_k(TN)$ is a Riemannian submersion. Also note that, since the horizontal distribution on $O(TN) \times G_k(\mathbb{R}^m) \to N$ is preserved under $O(n)$, the projection of this distribution gives a horizonal distribution on $\pi' : G_k(TN) \to N$. Let $V \in G_k(TN)$ be a k-plane over a point $p \in N$ spanned by an orthonormal k-frame $(v_1, \dots, v_k)$. Then the horizontal lift $\tilde{c}$ of a curve c in N with $c(0) = p$ to V is given by

$$\tilde{c}(t) = \mathrm{span}\left\{ (\overset{t}{\underset{0}{\|}} c) v_1, \dots, (\overset{t}{\underset{0}{\|}} c) v_k \right\}.$$

In particular, the tangent bundle $T\Sigma$ of a totally geodesic submanifold Σ of N is horizontal.

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