Percolation for the Gaussian free field and random interlacements via the cable system

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Abstract

This thesis studies the phase transition associated to the percolation for two different models: level sets of the Gaussian free field and vacant set of random interlacements. The Gaussian free field is a classical model of statistical mechanics, and the study of percolation for its level sets has been initiated by Bricmont and Saleur in [56]. Random interlacements is a model recently introduced by Sznitman in [93], and the existence of an infinite component for its vacant set is linked to the existence of a giant component in the vacant set left by a random walk on a torus or a cylinder.

These two models have long-range correlations, and compared to the usual independent percolation problem, it is challenging to just prove that the phase transition is non-trivial, see [81] and [93]. We are interested in the existence of a coexistence phase, that is a phase on which the sets of open and closed vertices contain an infinite cluster at the same time. The underlying graph that we consider can be the integer lattice $\mathbb{Z}^d$, $d \geq 3$, or a more complicated graph, such as a Cayley or a fractal graph with some regularity conditions.

One of our main tools is the cable system, a continuous version of the graph, on which one can derive surprisingly explicit results for the percolation of the level sets of the Gaussian free field. This was first noticed by Lupu in [57] on $\mathbb{Z}^d$, $d \geq 3$. Deep results about the existence of a coexistence phase for the discrete Gaussian free field follow from this thorough understanding of the percolative properties on the cable system. A powerful isomorphism between the Gaussian free field and random interlacements, first introduced by Sznitman in [96], leads, in turn, to similar results for random interlacements.

In order to understand better the particularities of percolation for the Gaussian free field on the cable system of the integer lattice, we extend and find new results on the cable system of a very general class of graphs using three new independent techniques. For instance, there is no coexistence phase for the level sets of the Gaussian free field on the cable system, and the law of the capacity of a given cluster can be written explicitly.

Kurzzusammenfassung

Diese Doktorarbeit studiert die Phasenänderung bezüglich Perkolation für zwei verschiedene Modelle: Niveaumengen des Gauss'schen freien Feldes und vacant set von random interlacements. Das Gauss'sche freie Feld ist ein klassisches Modell in der statistischen Mechanik, und die Untersuchung von Perkolation seiner Niveaumengen wurde zuerst von Bricmont and Saleur in [56] vorgenommen. Random interlacements ist ein kürzlich von Sznitman in [93] eingeführtes Modell, und die Existenz einer unendlichen Komponente in seinem vacant set ist eng verknüpft mit der Existenz einer riesigen Komponente in dem vacant set, welches durch eine Irrfahrt auf einem Torus oder einem Zylinder hinterlassen wird.

Diese beiden Modelle besitzen eine weitreichende Korrelationsstruktur. Im Vergleich zur gewöhnlichen unabhängigen Perkolation ist es schon eine Herausforderung zu zeigen, dass der Phasenübergang nicht trivial ist, siehe [81] und [93]. Wir interessieren uns für die Existenz einer Koexistenzphase, das heißt eine Phase, auf welcher die Mengen geöffneter und geschlossener Knoten gleichzeitig eine unendliche Komponente enthalten. Der zugrundeliegende Graph kann das Gitter der ganzen Zahlen $\mathbb{Z}^d$, $d \geq 3$, oder ein komplizierterer Graph wie zum Beispiel ein Cayley- oder ein Fraktalgraph mit zusätzlichen Regularitätsbedingungen sein.

Eines unserer Hauptwerkzeuge ist das cable system, eine stetige Version des Graphen, auf welchem man erstaunlich explizite Resultate für die Perkolation der Niveaumengen des Gauss'schen freien Feldes gewinnen kann. Dies wurde zuerst von Lupu in [57] auf $\mathbb{Z}^d$, $d \geq 3$, bemerkt. Weitreichende Ergebnisse über die Existence einer Koexistenzphase für das Gauss'sche freie Feld resultieren aus dem tiefen Verständnis des cable systems. Ein starker Isomorphismus zwischen dem Gauss'schen freien Feld und random interlacements, welcher zuerst von Sznitman in [96] eingeführt wurde, erlaubt ähnliche Ergebnisse für random interlacements.

Um die Genauigkeit der Perkolation des Gauss'schen freien Feldes auf dem cable system des Gitters der ganzen Zahlen besser zu verstehen, erweitern wir bestehende und finden neue Ergebnisse auf dem cable system für eine sehr generelle Klasse von Graphen. Zum Beispiel gibt es keine Koexistenzphase für die Niveaumengen des Gauss'schen freien Feldes auf dem cable system und die Wahrscheinlichkeitsverteilung der Kapazität einer gegebenen Komponente kann explizit angegeben werden.

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Chapter I

Introduction

I.1 Definitions and known results

In this introduction, we define the objects of interest and present the results of this thesis in a rather informal way, which should be accessible to a general mathematical audience. This thesis investigates percolation for a few models with long-range correlations, namely random interlacements and the Gaussian free field, and gives new results about their phase transitions and related questions. Considering for instance the integer lattice $\mathbb{Z}^d$, $d \geq 2$, as a graph, and a random set $\mathcal{A} \subset \mathbb{Z}^d$ of "open" vertices, percolation theory studies the existence of infinite clusters in $\mathcal{A}$, that is the existence of connected and infinite components of open vertices. The law of the random set $\mathcal{A}$ usually depends on some parameter, and, depending on this parameter, $\mathcal{A}$ might contain an infinite cluster with probability zero or with positive probability, thus exhibiting a phenomenon of phase transition. Classical problems include physically relevant questions on the existence and uniqueness of the infinite cluster, the probability that two vertices are in the same cluster, and the typical size of a finite cluster.

Arguably the simplest model for percolation is that of Bernoulli, or independent, percolation. It has been introduced by Broadbent and Hammersley [17] at the end of the 1950s in their research on gas masks. Even though the model is easy to define, Bernoulli percolation has been the subject of intensive mathematical research during the last couple of decades, and deep results have been obtained. For each $p \in [0, 1]$ attach to each vertex $x \in \mathbb{Z}^d$, $d \geq 2$, (or sometimes edge e) a Bernoulli random variable $B_x^p \in \{0, 1\}$, such that, under some probability $\mathbb{P}^B$, the family of random variables $(B_x^p)_{x\in\mathbb{Z}^d}$ is independent and $\mathbb{P}^B(B_x^p = 1) = 1 - \mathbb{P}^B(B_x^p = 0) = p$. In other words, each vertex $x \in \mathbb{Z}^d$ is considered independently open, when $B_x^p = 1$, or closed, when $B_x^p = 0$, with respective probability p and $1 - p$. One then wants to investigate the existence of an infinite cluster in the set of open vertices $\mathcal{B}^p \stackrel{\text{def.}}{=} \{x \in \mathbb{Z}^d : B_x^p = 1\}$, that is the existence of a random infinite and connected set C^p such that $B_x^p = 1$ for all $x \in C^p$, and study the properties of this cluster. We then naturally define the following critical parameter

$$p_c \stackrel{\text{def.}}{=} \sup\{p \in [0,1] : \mathbb{P}^B(\mathcal{B}^p \text{ contains an infinite cluster}) = 0\}.$$

Note that p_c depends implicitly on the dimension d. By a simple coupling argument one can easily show that, $\mathbb{P}^B$-a.s, $\mathcal{B}^p$ does not contain any infinite cluster for $p < p_c$, and we call this the subcritical phase, and that $\mathcal{B}^p$ contains $\mathbb{P}^B$-a.s. at least one infinite cluster for $p > p_c$, and we call this the supercritical phase. Moreover in every dimension $d \geq 2$, the phase transition is non-trivial, that is $p_c \in (0,1)$, and so both phases exist. In dimension two, duality has proved to be an essential tool to obtain deep results about percolation, and several tools have been developed in this case, such as Russo-Seymour-Welsch theory, which provides the equality $p_c = \frac{1}{2}$ for bond percolation on $\mathbb{Z}^2$ [47, 52].

In dimension $d \geq 3$, the situation is more complicated. An interesting question is the existence of a coexistence phase, where it is possible to have both an infinite cluster of open and closed vertices at the same time, that is whether $\mathcal{B}^p$ and its complement can have an infinite cluster at the same time for some $p \in [0,1]$. Since $(\mathcal{B}^p)^c$ has the same law as $\mathcal{B}^{1-p}$, it is thus of interest to know whether percolation at $p = \frac{1}{2}$ occurs or not, which is not the case in dimension two. This was answered positively by Campanino and Russo in [19], where they prove that $p_c < \frac{1}{2}$ for all $d \geq 3$. In particular, for all $p \in (p_c, 1-p_c)(\neq \varnothing)$, both $(\mathcal{B}^p)^c$ and $\mathcal{B}^p$ contain an infinite component, which corresponds to the coexistence phase. We refer to the monographs [44] and [12] for an extensive presentation of the main results and ideas regarding Bernoulli percolation.

When it comes to percolation models with strong and long-range correlations, many of the techniques from Bernoulli percolation either have to be extended, or do not work anymore. Among them, the study of the percolation for the (massless) discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, was initiated at the end of 1980s by Bricmont, Lebowitz and Maes in [16]. It is defined as a centered Gaussian field $(\varphi_x)_{x \in \mathbb{Z}^d}$ on $\mathbb{Z}^d$ with covariance function

$$\mathbb{E}^G[\varphi_x \varphi_y] = g(x,y),$$

where $g(x,y)$ is the Green function for the simple random walk on $\mathbb{Z}^d$; i.e., it is the average number of time the random walk beginning in x hits the vertex y before blowing up. In other words, φ_x is a centered Gaussian variable for all $x \in \mathbb{Z}^d$, and the covariance of φ_x and φ_y is $g(x,y)$. Once one assumes $d \geq 3$, simple random walk on $\mathbb{Z}^d$ is transient, and so the Green function is finite, and the field is well-defined. Due to the slow decay properties of the Green function, this field has very strong correlations decaying as $|x-y|^{-d+2}$ at infinity, which makes it hard to study. On the other hand, it also exhibits features which help its investigation, such as the (spatial) Markov property: fixing the value of φ on a finite set K, the field $(\varphi_x)_{x\in K^c}$ is, after deterministic recentering, a Gaussian free field on K^c.

The naturally ensuing percolation model for the Gaussian free field is defined through its excursion / level sets, i.e., we are interested in the study of the excursion sets above level $h \in \mathbb{R}$, defined via

$$E^{\geq h} \stackrel{\text{def.}}{=} \{x \in \mathbb{Z}^d : \varphi_x \geq h\}.$$

We then define the critical parameter h_* for the percolation phase transition of the level sets of the Gaussian free field by

$$h_* \stackrel{\text{def.}}{=} \inf\{h \in \mathbb{R} : \mathbb{P}^G(\text{there exists an infinite cluster in } E^{\geq h}) = 0\}.$$

Note that $E^{\geq h}$ is decreasing in h, whereas $\mathcal{B}^p$ was increasing in p. Analogously to Bernoulli percolation we have that $\mathbb{P}^G$-a.s, $E^{\geq h}$ does not contain any infinite cluster for $h > h_*$, and that $E^{\geq h}$ contains $\mathbb{P}^G$-a.s. at least one infinite cluster for $h < h_*$. It is however less clear than for Bernoulli percolation if the phase transition is non-trivial, that is if $h_* \in (-\infty, \infty)$, and we will come back to this question later.

A more recent percolation problem with long-range correlations concerns random interlacements, a model introduced by Sznitman in [93] to study the vacant set left by a random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, and which has since experienced a lot of attention in mathematical research. We denote by $Z = (Z_n)_{n\geq 0}$ the simple random walk on $\mathbb{Z}^d$, starting at x under P^x, and for any doubly infinite trajectory $w : \mathbb{Z} \to \mathbb{Z}^d$, we call $(w(n))_{n\geq 0}$ the forwards part of w and $(w(-n))_{n\geq 0}$ the backwards part of w. The random interlacement process ω^u at level $u > 0$ under some probability $\mathbb{P}^I$ consists of an infinite number of independent doubly infinite trajectories modulo time shift, and the backwards and forwards part of each trajectory both behave like the random walk Z. In order to describe it further, let us define for all finite sets $A \subset \mathbb{Z}^d$ the equilibrium measure and capacity of A by

$$e_A(x) \stackrel{\text{def.}}{=} P^x(\widetilde{H}_A = \infty)\mathbb{1}_A(x) \text{ for all } x \in \mathbb{Z}^d \text{ and } \mathrm{cap}(A) \stackrel{\text{def.}}{=} \sum_{x\in A} e_A(x), \quad \text{(I.1.1)}$$

where $\widetilde{H}_A \stackrel{\text{def.}}{=} \inf\{n \geq 1,\ Z_n \in A\}$, with $\inf \varnothing = \infty$, is the first return time in A for the simple random walk Z. A possible interpretation for the capacity of A is to see it as the size of A viewed from the point of view of the random walk on

$\mathbb{Z}^d$ at infinity. It is increasing in A, that is $\mathrm{cap}(A) \leq \mathrm{cap}(B)$ if $A \subset B$, and the capacity of a ball of radius N is of order N^{d-2}.

Let $H_A \stackrel{\text{def.}}{=} \inf\{n \in \mathbb{Z} : w(n) \in A\}$ be the first time a doubly infinite trajectory w on G hits A. For each $x \in A$, the number of trajectories hitting A in x for the first time in the random interlacements process ω^u is a Poisson distributed random variable with parameter $ue_A(x)$, and each of these trajectories behaves like a simple random walk on $\mathbb{Z}^d$ after time H_A, and like a simple random walk conditioned on not coming back to A before time H_A. Denoting by $\mathcal{I}^u$ the random interlacement set, i.e. the set of vertices visited at least once by a trajectory in ω^u, we then have

$$\mathbb{P}^I(\mathcal{I}^u \cap A = \varnothing) = \exp(-u\mathrm{cap}(A)). \tag{I.1.2}$$

This equation actually entirely characterises the law of $\mathcal{I}^u$. The random interlacement set $\mathcal{I}^u$ consists of the trace on $\mathbb{Z}^d$ of doubly infinite random walks, and thus always contains at least one infinite cluster. However, an interesting percolation problems arises when considering its complement $\mathcal{V}^u \stackrel{\text{def.}}{=} (\mathcal{I}^u)^c$, the vacant set of random interlacements, which is a set decreasing in u, and which might contain an infinite cluster with positive probability for some values of the parameter $u > 0$. The usual percolation question arises: defining

$$u_* \stackrel{\text{def.}}{=} \inf\{u > 0 : \mathbb{P}^I(\text{there exists an infinite cluster in } \mathcal{V}^u) = 0\},$$

does $u_* \in (0, \infty)$? Similarly as for the Gaussian free field, one can prove that $\mathbb{P}^I$-a.s, $\mathcal{V}^u$ does not contain any infinite cluster for $u > u_*$, and that $\mathcal{V}^u$ contains $\mathbb{P}^I$-a.s. at least one infinite cluster for $u < u_*$.

Apart from the discrete graph $\mathbb{Z}^d$, this thesis studies extensively percolation problems on a continuous version of the graph $\mathbb{Z}^d$, the cable system, or metric graph, $\widetilde{\mathbb{Z}}^d$. The cable system $\widetilde{\mathbb{Z}}^d$ is obtained by replacing each edge e between two neighbors of $\mathbb{Z}^d$ by a continuous interval I_e, $e \in E$, isomorphic to $[0, \frac{1}{2}]$, and by glueing this intervals through their respective endpoints. One can endow $\widetilde{\mathbb{Z}}^d$ with a distance, which corresponds to the length of the shortest path between two points. The notion of random walk on $\mathbb{Z}^d$ can be generalized to a continuous time stochastic process $(X_t)_{t\geq 0}$ on $\widetilde{\mathbb{Z}}^d$, which is now also taking values inside the edges I_e, $e \in E$. This process X is continuous and has the Markov property, its restriction to $\mathbb{Z}^d$, which can be seen as a subset of $\widetilde{\mathbb{Z}}^d$, behaves like the simple random walk Z, and X behaves like a Brownian motion on each edge I_e, $e \in E$. The process X is sometimes called the Brownian motion on the cable system $\widetilde{\mathbb{Z}}^d$.

The definitions of the Gaussian free field and random interlacements were extended to the cable system by Lupu in [57]: there exists a continuous field

$(\widetilde{\varphi}_x)_{x\in\widetilde{\mathbb{Z}}^d}$ on $\widetilde{\mathbb{Z}}^d$, such that $\widetilde{\varphi}_x$ is a centered Gaussian variable for all $x \in \widetilde{\mathbb{Z}}^d$, and the covariance of $\widetilde{\varphi}_x$ and $\widetilde{\varphi}_y$ corresponds to the average total time spent in y by the diffusion X beginning in x. The restriction $(\widetilde{\varphi}_x)_{x\in\mathbb{Z}^d}$ of $\widetilde{\varphi}$ to $\mathbb{Z}^d$ is then a discrete Gaussian free field on $\mathbb{Z}^d$, as defined before. Similarly for each $u > 0$, there exists a random interlacement process $\widetilde{\omega}^u$ at level u, consisting of an infinite number of independent doubly infinite trajectories on $\widetilde{\mathbb{Z}}^d$, each with a forwards and a backwards part both behaving like the process X. Moreover, if $\widetilde{\mathcal{I}}^u$ denotes the set of points in $\widetilde{\mathbb{Z}}^d$ visited by at least one trajectory in the random interlacement process $\widetilde{\omega}^u$, then $\widetilde{\mathcal{I}}^u \cap \mathbb{Z}^d$ has the same law as the discrete random interlacement set $\mathcal{I}^u$. From now on, we will identify $(\widetilde{\varphi}_x)_{x\in\mathbb{Z}^d}$ and $(\varphi_x)_{x\in\mathbb{Z}^d}$, and see $\mathcal{I}^u$ as subset of $\widetilde{\mathcal{I}}^u$.

The percolation problem on the cable system corresponds to the existence of an unbounded connected set, or cluster, where a set $\widetilde{A} \subset \widetilde{\mathbb{Z}}^d$ is connected if and only if there exists a continuous path in $\widetilde{\mathbb{Z}}^d$ between every two points $x, y \in \widetilde{A}$. We will naturally study the percolation for the level sets of the Gaussian free field on the cable system, defined by $\widetilde{E}^{\geq h} \stackrel{\text{def.}}{=} \{x \in \widetilde{\mathbb{Z}}^d : \varphi_x \geq h\}$ for all $h \in \mathbb{R}$, and the associated critical parameter is

$$\widetilde{h}_* \stackrel{\text{def.}}{=} \inf\{h \in \mathbb{R}; \mathbb{P}^G(\widetilde{E}^{\geq h} \text{ contains an unbounded cluster}) = 0\}. \tag{I.1.3}$$

If $h < \widetilde{h}_*$, then $\widetilde{E}^{\geq h}$ contains $\mathbb{P}^G$-a.s. at least one unbounded cluster, and so $\widetilde{E}^{\geq h} \cap \mathbb{Z}^d = E^{\geq h}$ also contains an infinite cluster, that is $h < h_*$. We thus obtain that $\widetilde{h}_* \leq h_*$, and one might inquire whether this inequality is strict or not. In the case of random interlacements, it similarly holds that if $\widetilde{\mathcal{V}}^u \stackrel{\text{def.}}{=} (\widetilde{\mathcal{I}}^u)^c$ contains an infinite cluster, then $\mathcal{V}^u = \widetilde{\mathcal{V}}^u \cap \mathbb{Z}^d$ also contains an infinite cluster. However, looking at trajectories, since each discrete connected path $\pi \subset \mathcal{V}^u$ correspond to a continuous connected path $\widetilde{\pi} \subset \widetilde{\mathcal{V}}^u$ with $\widetilde{\pi} \cap \mathbb{Z}^d = \pi$, one can easily see that the reverse implication is true as well. The percolation of $\widetilde{\mathcal{V}}^u$ is thus equivalent to the percolation of $\mathcal{V}^u$, and we will not investigate it further.

These two models, the Gaussian free field and random interlacements, are actually linked by an isomorphism, both on discrete graphs and on their cable system. The study of such isomorphisms has been initiated by Dynkin [31], and takes its origin from ideas in physic from Symanzik [88]. The isomorphism between random interlacements and the Gaussian free field was first proved on discrete graphs by Sznitman in [96], and extended to the cable system $\widetilde{\mathbb{Z}}^d$ by Lupu in [57], and can be seen as an extension of the second Ray-Knight theorem for Markov processes, see [32]. If we call $(\widetilde{\ell}_{x,u})_{x\in\widetilde{\mathbb{Z}}^d}$ the local times of random interlacements on the cable system, that is $\widetilde{\ell}_{x,u}$ is the total time spent in x by all the trajectories in the random interlacement process $\widetilde{\omega}^u$, then:

$$\Big(\frac{1}{2}\widetilde{\varphi}_x^2 + \widetilde{\ell}_{x,u}\Big)_{x\in\widetilde{\mathbb{Z}}^d} \text{ has the same law as } \Big(\frac{1}{2}(\widetilde{\varphi}_x + \sqrt{2u})^2\Big)_{x\in\widetilde{\mathbb{Z}}^d}, \tag{I.1.4}$$

where the Gaussian free field and random interlacements on the left-hand side are taken independent. Let us explain a first consequence of this isomorphism, which highlights the importance of considering the cable system. Each $x \in \widetilde{\mathcal{I}}^u$ is visited by at least one trajectory in the random interlacement process, that is its local time $\widetilde{\ell}_{x,u}$ is strictly positive. Since $\widetilde{\mathcal{I}}^u$ consists of unbounded trajectories, it is clear that $\{x \in \widetilde{\mathbb{Z}}^d : \widetilde{\varphi}_x^2 + 2\widetilde{\ell}_{x,u} > 0\}$ contains an unbounded cluster. Therefore, by (I.1.4), $\{x \in \widetilde{\mathbb{Z}}^d : (\widetilde{\varphi}_x + \sqrt{2u})^2 > 0\}$ also contains an unbounded cluster, and by continuity of $\widetilde{\varphi}$ on $\widetilde{\mathbb{Z}}^d$, either $\{x \in \widetilde{\mathbb{Z}}^d : \widetilde{\varphi}_x > -\sqrt{2u}\}$ or $\{x \in \widetilde{\mathbb{Z}}^d : \widetilde{\varphi}_x < -\sqrt{2u}\}$ contains an unbounded connected cluster. By symmetry of the Gaussian free field, we obtain that, $\mathbb{P}^G$-a.s, either $\widetilde{E}^{\geq -\sqrt{2u}}$ or $\widetilde{E}^{\geq \sqrt{2u}}$ contains an unbounded cluster, and in both cases $\widetilde{h}_* \geq -\sqrt{2u}$. This holds for any $u \geq 0$, and in combination with the previously obtained inequality $\widetilde{h}_* \leq h_*$, we obtain

$$0 \leq \widetilde{h}_* \leq h_*. \tag{I.1.5}$$

One may naturally wonder which of these inequalities are actually strict, which is one of the main goals of this thesis. Note that the equality $h_* \geq 0$ was first proved in [16], using only the Markov property of the Gaussian free field. On $\mathbb{Z}^d$, $d \geq 3$, it was actually proved by Lupu in [57] that this simple observation describes the whole supercritical phase of the Gaussian free field on the cable system, that is $\widetilde{h}_* = 0$. Moreover, at level 0, the level sets $\widetilde{E}^{\geq 0}$ also contain $\mathbb{P}^G$-a.s. only bounded clusters, and by symmetry of the Gaussian free field, this implies that the sign clusters $\widetilde{E}^{|>0|} \stackrel{\text{def.}}{=} \{x \in \widetilde{\mathbb{Z}}^d : |\varphi_x| > 0\}$ contain $\mathbb{P}^G$-a.s. only bounded clusters. In particular, since $(\widetilde{E}^{\geq h})^c$ has the same law as $\widetilde{E}^{\geq -h}$, there is never an unbounded cluster for both $E^{\geq h}$ and its complement, and so no coexistence phase on the cable system, as opposed to discrete Bernoulli percolation. This result can be surprising since the critical parameter $\widetilde{h}_*$ is not only explicitly known, but it also does not dependent on the dimension, and we will generalize this fact to a much broader class of graphs later.

Let us now explain a second interesting direct consequence of the isomorphism (I.1.4) from [57]. Fixing some u such that $\sqrt{2u} < h_*$, by symmetry of the Gaussian free field the set $\{x \in \mathbb{Z}^d : \varphi_x \leq -\sqrt{2u}\}$ contains $\mathbb{P}^G$-a.s. an infinite cluster, that we denote by C_u. Since $\widetilde{h}_* = 0$, by symmetry of the Gaussian free field, we have that $\{x \in \widetilde{\mathbb{Z}}^d : \widetilde{\varphi}_x \leq -\sqrt{2u}\}$ contains $\mathbb{P}^G$-a.s. only bounded clusters, and so by continuity of the Gaussian free field on the cable system, the cluster of y in $\{x \in \widetilde{\mathbb{Z}}^d : (\widetilde{\varphi}_x + \sqrt{2u})^2 > 0\}$ is bounded for all y with $\varphi_y \leq -\sqrt{2u}$, and so also for all $y \in C_u$. Using (I.1.4), we obtain that there exists $\mathbb{P}^G$-a.s. an infinite cluster C_u' such that the cluster of y in $\{x \in \widetilde{\mathbb{Z}}^d : \widetilde{\varphi}_y^2 + 2\widetilde{\ell}_{y,u} > 0\}$ is bounded for all $y \in C_u'$. The random interlacement set only contains unbounded clusters on which its local times are strictly positive, and so we necessarily have

$y \in \mathcal{V}^u$ for all $y \in C'_u$, that is $\mathcal{V}^u$ contains $\mathbb{P}^G$-a.s. an infinite cluster. We thus obtain the following inequality

$$h_* \leq \sqrt{2u_*}.$$

These two applications of the isomorphism (I.1.4) show its importance, as well as the interest of considering the cable system, and we will present other applications in this thesis. In order to prove that the phase transition for the level sets of the Gaussian free field and for the vacant set of random interlacements is non-trivial, we still need to derive upper-bounds for h_* and u_*. This was done by Sznitman for random interlacements in its seminal paper [93] using a renormalization scheme, and combined with the previous inequalities we have

$$0 = \widetilde{h}_* \leq h_* \leq \sqrt{2u_*} < \infty \text{ on } \mathbb{Z}^d \text{ for all } d \geq 3. \tag{I.1.6}$$

The inequality $h_* < \infty$ was initially proved independently by Rodriguez and Sznitman in [81], without using the isomorphism (I.1.4) with random interlacements. The proofs of the inequalities $u_* < \infty$ and $h_* < \infty$ are more involved than the inequality $h_* \geq 0$, and both use crucially some decoupling inequalities, see [95, 68] for random interlacements and [67] for the Gaussian free field. Strict inequalities between the critical parameters in (I.1.6) are harder to obtain: the strict inequality $h_* > 0$ was proved in [81] when the dimension d is large enough, but this question is harder to investigate in lower dimension since the correlations are stronger, and we will come back to this soon. It is still an open question to know whether the inequality $h_* \leq \sqrt{2u_*}$ is strict or not on $\mathbb{Z}^d$, $d \geq 3$, but this has already been proved on a large class of trees, see [101, 1].

I.2 Existence of a coexistence phase

We now present the results about the positivity of the critical parameters h_*, for the discrete level sets of the Gaussian free field, and u_*, for the vacant set of random interlacements, which imply the existence of a coexistence phase for these models, as well as some additional results concerning the geometry for the level sets of the Gaussian free field. The complete statements and proofs are presented in Chapters II and III, which respectively correspond to the articles [25] and [26]. The first result, which is proved in Chapter II, is that

$$h_* > 0 \text{ on } \mathbb{Z}^d \text{ for all } d \geq 3. \tag{I.2.1}$$

This corresponds to the inequality $p_c < \frac{1}{2}$ for Bernoulli percolation from [19], and so the discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$ also exhibits a phenomenon of phase coexistence: for all $h \in (-h_*, h_*)(\neq \varnothing)$, both the discrete level sets of the Gaussian free field $E^{\geq h}$ and its complement have an infinite cluster at the same time. In particular, the discrete sign clusters $E^{|>0|}$, that is the union of the clusters above and below level 0, of the Gaussian free field percolate, whereas the continuous sign clusters $\widetilde{E}^{|>0|}$ of the Gaussian free field on the cable system don't. The inequality (I.2.1) was already expected to hold in dimension three since it has been supported by previous numerical evidence in [63]. It also generalizes the results from [81], where (I.2.1) is shown in high dimensions, and in fact in high dimensions one can even derive an asymptotic expression for h_*, see [29]. The proofs of these results in high dimension mostly rely on methods analogue to Bernoulli percolation, whereas our proof of (I.2.1) is new and specific to the Gaussian free field, since it relies on the isomorphism (I.1.4) with random interlacements, and takes advantage of the cable system.

The existence of a coexistence phase is expected to hold in dimension $d \geq 3$ for a variety of percolation models of random functions with long range correlation, and the result (I.2.1) for the level sets of the Gaussian free field can be seen as a milestone in the study of these models. An example is positively correlated discrete Gaussian fields on $\mathbb{Z}^d$, or continuous Gaussian fields on $\mathbb{R}^d$, with a covariance function decaying fast enough to infinity. In dimension two, several articles, see [64, 76, 9] for instance, have recently studied the phase transition for the level sets of smooth planar Gaussian fields and, using Russo-Seymour-Welsh theory, have proved in particular that the associated critical parameter is zero. One would then expect the critical parameter to increase with the dimension, and the result (I.2.1) would then hold on a large class of discrete or continuous Gaussian fields, but little is known rigorously about this subject for the moment. Another example on point is the nodal domain of monochromatic random wave, which should display a supercritical behavior in dimension $d \geq 3$, and we refer to [84] and the references therein for details.

Let us quickly comment on the proof of (I.2.1). As explained before, the random interlacement set on the cable system $\widetilde{\mathcal{I}}^u$ has only unbounded clusters, which correspond by the isomorphism (I.1.4) to unbounded clusters of $\{x \in \widetilde{G} : (\widetilde{\varphi}_x + \sqrt{2u})^2 > 0\}$. Since $\widetilde{h}_* = 0$, by symmetry of the Gaussian free field we have that $\{x \in \widetilde{G} : \widetilde{\varphi}_x < -\sqrt{2u}\}$ only contains bounded clusters, and so by continuity of the Gaussian free field on the cable system, $\widetilde{\mathcal{I}}^u$ corresponds in fact in (I.1.4) to unbounded clusters of $\widetilde{E}^{\geq -\sqrt{2u}}$.

Each unbounded cluster of $\widetilde{E}^{\geq -\sqrt{2u}}$ for the Gaussian free field on the cable system contains an infinite cluster $C_u \subset E^{\geq -\sqrt{2u}}$ for the discrete Gaussian free field such that, for all $x \in C_u$, there exists an edge $e = \{x, y\}$ with $y \in C_u$ and $\widetilde{\varphi} \geq -\sqrt{2u}$ on the whole edge I_e. Conditionally on φ_x and φ_y, one can show that the Gaussian free field $(\widetilde{\varphi}_z)_{z\in I_e}$ on the segment I_e is a Brownian bridge of length $\frac{1}{2}$ between φ_x and φ_y, which has a certain probability $p_e^u(\varphi_x,\varphi_y)$ to stay above level $-\sqrt{2u}$ on the whole edge I_e. For each $u>0$, each cluster of $\widetilde{\mathcal{I}}^u$ thus corresponds to an infinite cluster C_u in $E^{\geq -\sqrt{2u}}$, such that, conditionally on $(\varphi_x)_{x\in\mathbb{Z}^d}$, the Bernoulli percolation with parameter p^u on the edges of C_u contains an infinite cluster. If u is small enough, one can show that the probability that $\varphi_x \geq -\sqrt{2u}$ and at least one Bernoulli variable on the edges starting from x with parameter p^u is equal to 1 is strictly smaller than the probability that $\varphi_x \geq \sqrt{2u}$. Therefore, C_u corresponds to an infinite component of $E^{\geq\sqrt{2u}}$, that is $h_* \geq \sqrt{2u} > 0$.

In order to dominate the probability that $\varphi_x \geq -\sqrt{2u}$ plus a small Bernoulli noise p^u by the probability that $\varphi_x \geq \sqrt{2u}$, one needs in fact to assume that $|\varphi_x|$ is small enough. Adapting the stability of $\widetilde{\mathcal{I}}^u$ to small noise from [74], one can show via a renormalization scheme that an unbounded cluster $\widetilde{C}_u$ in $\widetilde{E}^{\geq -\sqrt{2u}}$ exists, such that $|\widetilde{\varphi}|$ is small enough on $\widetilde{C}_u$, and this concludes the proof. Most of the proof of the inequality (I.2.1) consists of proving the existence of such an unbounded cluster $\widetilde{C}_u$, but due to its technicality, we refer to Chapter II for details.

In Chapter III, we improve and generalize the inequality (I.2.1) in various ways. First we prove the inequality (I.2.1) on a large family of transient graphs, that we will assume to be unweighted in this introduction for simplicity. The notion of percolation, that is the existence of an infinite connected cluster, can naturally be extended from $\mathbb{Z}^d$ to any discrete graph G, as well as the definition of the Gaussian free field and random interlacement when the graph G is transient. In order to prove (I.2.1) on the transient graph G, let us add the following hypothesis:

$$\begin{gathered}\text{there exist parameters } \alpha \text{ and } \beta \text{ with } 2\leq\beta<\alpha \text{ such}\\ \text{that, } |B(x,L)| \asymp L^{\alpha} \text{ and } g(x,y)\asymp d(x,y)^{-(\alpha-\beta)}, \text{ for } x,y\in G.\end{gathered} \tag{I.2.2}$$

Here d is some distance function on the graph G, $|B(x,L)|$ denotes the volume of the ball centered in x and with radius L for this distance, and $g(x,y)$ is the Green function for the graph G. These conditions are actually related to some heat kernel bounds for the random walk on the graph G, see Chapter III for details, and have been studied extensively, see [42, 43] for instance. We will also need an isoperimetric inequality for the graph G, which, in essence, says that

$$\begin{gathered}\text{for each finite set } A\subset G, \text{ the boundary of } A \text{ contains an almost}\\ \text{connected path whose length is of the same order as the diameter of } A.\end{gathered} \tag{I.2.3}$$

Under these conditions, we prove that the inequalities (I.1.6) and (I.2.1) still hold, that is

$$0 = \widetilde{h}_* < h_* \leq \sqrt{2u_*} < \infty \text{ if (I.2.2) and (I.2.3) are verified.} \tag{I.2.4}$$

Examples of transient graphs verifying (I.2.2) and (I.2.3) include $\mathbb{Z}^d$, $d \geq 3$, but also Cayley graphs of suitably growing non-Abelian groups and various fractal graphs. As a direct consequence of (I.2.4), we obtain that $0 < u_* < \infty$, that is the phase transition for the vacant set of random interlacements in non-trivial. In particular for all $u \in (0, u_*)(\neq \varnothing)$, since the interlacement set $\mathcal{I}^u$ always contains infinite clusters, both $\mathcal{I}^u$ and its complement $\mathcal{V}^u$ contain infinite clusters, which corresponds to the existence of a coexistence phase for random interlacements.

The inequality $u_* > 0$ was first proved on $\mathbb{Z}^d$, $d \geq 7$, in [93], and for any dimension $d \geq 3$ in [87], see also [72] for a short proof of this result. It was then extended to any product graph G of the form $G = G' \times \mathbb{Z}$ in [95] under the condition (I.2.2) when $\alpha \geq 1+\beta$, and one can show that (I.2.3) is in fact verified on any such product graph. The result (I.2.4) thus generalize the inequality $u_* > 0$ to graphs with $\alpha < 1+\beta$, and a canonical example of such a graph is the "Toblerone bar" graph, that is the graph $G = G' \times \mathbb{Z}$ where G' is the Sierpinski gasket [50], and also to non-product graphs, such as the three-dimensional Sierpinksi carpet, see Section 2 of [6]. The inequality $u_* > 0$ can also hopefully lead to a better understanding of the trace of random walks on cylinders, see [91, 111, 90].

Improving the renormalization technique used to prove (I.2.1), we also obtain subsequent information on the geometry of the sign clusters for the GFF, which, under the conditions (I.2.2) and (I.2.3), can be summed up as follows:

$$\begin{array}{c}\text{there exist exactly two infinite sign clusters of } \varphi, \text{ one for} \\ \text{each sign, which consume `all the ambient space', up to} \\ \text{(stretched) exponentially small finite islands of +/- signs.}\end{array} \tag{I.2.5}$$

Adapting the proof of (I.2.1) to this larger class of graphs required us to overcome many challenges in order to obtain several intermediate results, by adapting some proofs from the case $\mathbb{Z}^d$, $d \geq 3$, and by finding some new arguments: a proof of (I.2.3) on product graphs, heat kernel estimates, strong connectivity of $\widetilde{\mathcal{I}}^u$, decoupling inequalities for both the Gaussian free field and random interlacements, and developing a stronger renormalization scheme to obtain (I.2.5). We refer to Chapter III for details. In order to adapt the proof of (I.1.6), the main challenges were to prove the equality $\widetilde{h}_* = 0$ and to obtain a stronger version of (I.1.4) including the sign of the processes, see [101], and we will now extend these results to a much larger class of graphs.

I.3 Percolation on the cable system

We now investigate percolation for the level sets of the Gaussian free field on the cable system of a general transient graph, which is the content of Chapter IV. The cable system $\widetilde{G}$ of a graph G is obtained similarly as before by replacing each edge e between two neighbors of G by a continuous interval I_e of length $\frac{1}{2}$. When G is transient, one can then also extend the definition of the Gaussian free field and random interlacements to the cable system $\widetilde{G}$, and we will keep the same notations as on $\mathbb{Z}^d$, $d \geq 3$. A natural questions is, to which transient graphs G does the equality $\widetilde{h}_* = 0$, see (I.1.3), verified on $\mathbb{Z}^d$ ([57]), can be extended?

As explained above (I.1.5), the inequality $\widetilde{h}_* \geq 0$ is a direct consequence of the isomorphism (I.1.4) between random interlacements and the Gaussian free field, which actually holds on any transient graph, see [96, 57]. However, the proof of the inequality $\widetilde{h}_* \leq 0$, that is $\widetilde{E}^{\geq 0}$ does not contain an unbounded cluster, from [57] can only be easily extended to amenable and transitive transient graphs. As mentioned in (I.2.4), it also holds on transient graphs such that (I.2.2) is verified, but also on a large class of trees, see [101, 1]. Let us now introduce a condition on our graph G, which is verified on all the previously mentioned examples:

$$\mathrm{cap}(A) = \infty \text{ for all infinite and connected sets } A \subset G, \tag{I.3.1}$$

where the capacity of a set was introduced in (I.1.1), and is extended to infinite sets by approximation. It can actually also be extended to any connected and closed set $\widetilde{A} \subset \widetilde{G}$, and the main result of Chapter IV can then be summed up as

$$\begin{gathered}\text{if } G \text{ is a transient graph such that (I.3.1) is fulfilled, then } \widetilde{h}_* = 0, \text{ and} \\ \text{for all } h \in \mathbb{R} \text{ and } x_0 \in \widetilde{G}, \text{ one can give explicitly the law of } \mathrm{cap}(\widetilde{E}^{\geq h}(x_0)).\end{gathered} \tag{I.3.2}$$

In the particular case $h = 0$, the law of the capacity of the sign clusters on the cable system can be described by the following Laplace transform

$$\mathbb{E}^G\left[\exp\left(-u\mathrm{cap}(\widetilde{E}^{\geq 0}(x_0))\right)\mathbb{1}_{\varphi_{x_0}\geq 0}\right] = \mathbb{P}^G(\varphi_{x_0} \geq \sqrt{2u}) \text{ for all } u \geq 0, \tag{I.3.3}$$

where $\widetilde{E}^{\geq 0}(x_0)$ is the set of $y \in \widetilde{G}$ connected to $x_0 \in \widetilde{\mathcal{G}}$ in $\widetilde{E}^{\geq 0}$. Note that this formula is explicit since φ_{x_0} is just a centered Gaussian variable with variance $g(x_0, x_0)$. One can actually prove the formula (I.3.3) on any graph such that $\widetilde{E}^{\geq 0}$ contains $\mathbb{P}^G$-a.s. only bounded clusters. The result (I.3.2) is thus a generalization of the equality $\widetilde{h}_* = 0$ to all previously known graphs, and one can actually prove that condition (I.3.1) also holds on any graph such that the Green function decays to zero at infinity, or also any transitive graph, and is therefore very general. We also provide in Chapter IV an example of a transient graph for which $\widetilde{h}_* = \infty$.

One might be surprised that the critical parameter $\widetilde{h}_*$ is almost independent of the choice of the graph and explicitly known, or that the law of the capacity of the level sets is also explicitly known, and actually only depends really little on the nature of the graph that we consider, see (I.3.3). For these questions, percolation for the Gaussian free field on the cable system is thus better understood than the, a priori simpler, Bernoulli percolation. In fact, one can prove the inequality $p_c < 1$ for Bernoulli percolation using the Gaussian free field on the cable system on a large class of graphs, see [30]. In order to hopefully reach better heuristics on this result, we give three new different proofs of (I.3.2).

The first proof involves a Russo's formula introduced in [79] for the Gaussian free field, see also [82] for the initial formula for Bernoulli percolation, which gives a formula for the derivative in h of the probability of events depending only on the level sets $\widetilde{E}^{\geq h}$. In some specific cases, this formula can be turned into an explicit differential equation, from which one can deduce (I.3.2). For the second proof, one explores the level sets of the Gaussian free field $\widetilde{E}^{\geq h}$ from the point of view of the equilibrium measure to obtain an "exploration martingale", similar to the one introduced in [24], and (I.3.2) then follows using some usual martingale theory.

The third proof involving random interlacements is more complicated, but also implies additional results, and leads to a deeper understanding of the relation between the level sets of the Gaussian free field on the cable system at different levels. The isomorphism (I.1.4) between random interlacements and the Gaussian free field can be extended to also include the sign of $\widetilde{\varphi}_x + \sqrt{2u}$ on the right-hand side, rather than its square, which is proved in [101] when $\widetilde{E}^{\geq 0}$ is $\mathbb{P}^G$-a.s. bounded and the Green function is bounded. In Chapter IV, we weaken these conditions under which this "signed" isomorphism holds to include any graph such that (I.3.3) holds, and let us now describe this result in details. We define for each $u > 0$ a process $(\sigma^u_x)_{x\in\widetilde{G}} \in \{-1,1\}^{\widetilde{G}}$, such that, conditionally on $(|\widetilde{\varphi}_x|)_{x\in\widetilde{G}}$ and the random interlacement process $\widetilde{\omega}_u$, σ^u is constant on each of the clusters of $\{x \in \widetilde{G} : \ 2\widetilde{\ell}_{x,u} + \widetilde{\varphi}_x^2 > 0\}$, $\sigma^u_x = 1$ for all $x \in \widetilde{\mathcal{I}}^u$, and the values of σ^u on each other clusters are independent and uniformly distributed. Then if either (I.3.3) holds or $\widetilde{E}^{\geq 0}$ is $\mathbb{P}^G$-a.s. bounded,

$$\Big(\sigma^u_x\sqrt{\widetilde{\varphi}_x^2 + 2\widetilde{\ell}_{x,u}}\Big)_{x\in\widetilde{G}} \text{ has the same law as } \big(\widetilde{\varphi}_x + \sqrt{2u}\big)_{x\in\widetilde{G}}. \tag{I.3.4}$$

In particular $\widetilde{E}^{\geq -\sqrt{2u}}$ has the same law as $\{x \in \widetilde{G} : \ \sigma^u_x = 1\}$, and, by symmetry of the Gaussian free field, $\widetilde{E}^{\geq \sqrt{2u}}$ has the same law as $\{x \in \widetilde{G} : \ \sigma^u_x = -1\}$. Using the definition of σ^u and (I.1.2), and approximating the graph $\widetilde{G}$ by finite graphs, one can prove that (I.3.2) and (I.3.3) hold. Note that this implies that (I.3.3) is actually equivalent to (I.3.4). The proof of (I.3.4) relies on an approximation of the Gaussian free field and random interlacements on $\widetilde{G}$ by Gaussian free fields and random interlacements on finite graphs, and takes advantage of an isomorphism similar to (I.3.4) between loop soups and the Gaussian free field, see [54, 57]. This proof also provides us with a version of (I.3.4) on the discrete graph G similar to the version of the second Ray Knight theorem from [58], where the signs σ^u are also only described in terms of the discrete processes $(\varphi_x)_{x\in G}$ and ω^u, and we refer to Chapter IV for details.

In addition to (I.3.2), the "signed" isomorphism (I.3.4) has several other applications. Concerning the discrete Gaussian free field, it has already been used, or more precisely its version from [101], to prove the strict inequality in (I.2.4) between h_* and $\sqrt{2u_*}$ on trees in [101, 1], and is expected to also help on several other graphs including $\mathbb{Z}^d$, $d \geq 3$. As explained in Chapter III, it is also useful to prove the statement (I.2.5) about the geometry of the discrete sign clusters. On the cable system, this isomorphism also implies that clusters of $\widetilde{E}^{\geq h}$ and $\widetilde{E}^{\geq -h}$ have the same law when they are bounded, and that the critical parameter $\widetilde{h}_*$ is either equal to zero, as under condition (I.3.1), or to infinity, that is $\widetilde{E}^{\geq h}$ then contains an unbounded cluster with positive probability for all $h \in \mathbb{R}$.

All the results of Chapter IV actually hold on any *weighted* transient graphs, that is on graphs on which the random walk on G crosses an edge after an exponential time whose parameter may depend on the choice of the edge. We also obtain some partial results when adding the possibility for the random walk on G to be stopped after a random time, whose law is described by a killing measure $\kappa : G \to [0, \infty]$, $\kappa \neq 0$, that is on *massive* graphs. When the random walk on G is in x, the probability that it is killed before jumping to a neighbor of x is then proportional to κ_x. There is no direct proof of the existence of random interlacements on massive graphs in the literature, and we give one in Chapter V, which also include the cable system. On massive graphs, trajectories in the discrete random interlacements process can have forwards parts, or backwards parts, which are finite when they are killed by the measure κ before escaping to infinity.

One can then naturally introduce the notions of killed random interlacements, which correspond to the doubly finite trajectories in the random interlacements process, and surviving random interlacements, which correspond to the doubly infinite trajectories in the random interlacements process. These definitions can be extended to the cable system, and other characterizations are available. For instance, killed random interlacements on the discrete graph G can be obtained by starting a Poissonian number of independent forwards trajectories in x for each $x \in G$, each trajectory behaving like a random walk on G until the first time it is killed.

Contrary to random interlacements on massless graphs, killed random interlacements consist of finite range trajectories, and therefore the killed random interlacements set do not always contain an infinite connected component. The existence of an infinite connected component depends both on the parameter u and on the choice of the killing measure κ, and let us denote by $u_*^{\mathcal{K},\mathcal{I}}(\kappa)$ the critical parameter associated with the percolation for the discrete killed random interlacements set on the graph G with killing measure κ. For $u > u_*^{\mathcal{I},\mathcal{K}}(\kappa)$, the killed random interlacements set contains an infinite connected component with positive probability, and with probability zero for $u < u_*^{\mathcal{I},\mathcal{K}}(\kappa)$. In Chapter V, we show that on $(0,\infty)$

$$\text{the function } a \mapsto a^2 u_*^{\mathcal{I},\mathcal{K}}(a\kappa) \text{ is increasing,} \tag{I.3.5}$$

where we write $a\kappa$ for the killing measure $(a\kappa_x)_{x\in G}$. In particular, if $u_*^{\mathcal{I},\mathcal{K}}(\kappa) > 0$, then $u_*^{\mathcal{I},\mathcal{K}}(a\kappa) > 0$ for all $a \geq 1$, and, if $u_*^{\mathcal{I},\mathcal{K}}(\kappa) < \infty$, then $u_*^{\mathcal{I},\mathcal{K}}(a\kappa) < \infty$ for all $a \leq 1$. One can also find results similar to (I.3.5) for the vacant set of killed random interlacement, instead of the killed random interlacements set, both on the discrete graph G and the cable system $\widetilde{G}$.

Another characterization of killed and surviving random interlacements can also be given using the notion of $\mathbf{h}$-transform $G_{\mathbf{h}}$ of a graph G, which can for instance correspond to a modification of the graph G such that the random walk on $G_{\mathbf{h}}$ is the random walk on G, conditionally on being killed by the measure κ in finite time. This characterization directly provides us with a version of the isomorphism (I.1.4), or even (I.3.4), but between the Gaussian free field and killed, or surviving, random interlacements. Similar isomorphisms also hold between the trajectories of random interlacements avoiding a given compact K, and the Gaussian free field conditioned on being equal to zero on K.

One can also define killed level sets of the Gaussian free field, which correspond to the level sets for the Gaussian free field associated to the random walk conditioned on being killed by the measure κ in finite time. The isomorphism between killed random interlacements and the Gaussian free field let us write negative killed level sets of the Gaussian free field as a coupling between killed random interlacements and sign clusters of the Gaussian free field. Using (I.3.5), one can prove an identity similar to (I.3.5), but for the critical parameter associated to the percolation of negative killed level sets of the Gaussian free field on the cable system. We also give an identity similar to (I.3.5) for positive killed level sets of the Gaussian free field on the cable system, and it would be interesting to have similar results for the discrete Gaussian free field, since it could provide us with a proof of the identity $h_* > 0$, see (I.2.1), on a larger class of graph.

I.4 Outlook

The Gaussian free field on the cable system is an interesting toy model to study percolation properties of strongly correlated random fields, since despite its apparent complexity, it is integrable enough to answer complex questions about its phase transition. It has also proved useful to study the more complex discrete Gaussian free field [101, 1, 25], or even other related percolation models [57, 59, 30], and it could thus be interesting to have an even better understanding of this model, and in particular its properties near criticality. More generally, percolation for the level sets of general smooth Gaussian fields in dimension $d \geq 3$, either on $\mathbb{R}^d$, $\mathbb{Z}^d$, or even the cable system $\widetilde{\mathbb{Z}}^d$, seems to be a more natural percolation problem than the Gaussian free field. It is however mathematically largely unexplored, especially in dimension $d \geq 3$, and it would be interesting to see how the previous results, in particular the existence of a coexistence phase, can be extended to this more general class of fields.

I.5 Organization of the thesis

Chapter II concerns the result (I.2.1) about the existence of a coexistence phase for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, which is then extended in Chapter III to the more general class of graphs verifying (I.2.2) and (I.2.3). On these graphs, the set of inequalities (I.2.4) and the strong percolation result (I.2.5) are also provided in Chapter III. The equality $\widetilde{h}_* = 0$ for the Gaussian free field on the cable system of transient graphs verifying (I.3.1) is proved in Chapter IV, as well as the formula (I.3.3) for the law of its level sets. Finally, random interlacements on massive graphs are studied in Chapter V, as well as killed and surviving random interlacements. Chapters II and III correspond respectively to the articles [25] and [26], Chapter IV to an article in preparation, in collaboration with Alexander Drewitz and Pierre-François Rodriguez, whereas Chapter V is additional material. Each chapter can be read independently of the others.

I.6 Notation

In order to improve readability, we gave a name to some of the most important conditions appearing in this thesis, and for the reader's orientation we indicate here the sections in which they can be found. Conditions (p_0), (V_α), (G_β) and (WSI) are introduced in Section III.2, conditions (Sign), (Law$_h$) and (Isom) in Section IV.1, and condition (Isom') is Section IV.3. Finally, let us mention that the notation in Chapters IV and V differs slightly from the the rest of the thesis, since we only study the cable system in these chapters, and we refer to the discussion at the end of Section IV.1 for details.

Chapter II

Percolation for the sign clusters on $\mathbb{Z}^d$, $d \geq 3$

II.1 Introduction

The present work studies the percolation phase transition of Gaussian free field level sets on $\mathbb{Z}^d$, $d \geqslant 3$, which provides a canonical example for a percolation model with strong, algebraically decaying correlations. It was first proved in [16] that the corresponding critical level $h_*(d)$, see (II.1.4) below for its definition, satisfies $h_*(d) \geqslant 0$ for every dimension $d \geqslant 3$ and that $h_*(3) < \infty$. It was later shown in [81] that $h_*(d)$ is finite in every dimension $d \geqslant 3$, and strictly positive when d is large, with leading asymptotics as $d \to \infty$ derived in [29]. We prove here that this parameter is actually strictly positive in all dimensions $d \geqslant 3$. This answers a question from [16], see also Remark 3.6 in [81], and fits with numerical evidence from [63], see Section 4.1.2 and Figure 4.1 therein. A corresponding classical result for Bernoulli site percolation, $p_c^{\text{site}}(\mathbb{Z}^d) < \frac{1}{2}$ for $d \geqslant 3$, has been known to hold for several decades already [19].

Our construction of infinite clusters (by which, adopting the usual terminology, we mean unbounded connected components) of excursion sets for the Gaussian free field crucially relies on another object, random interlacements. The model of random interlacements has originally been introduced in [93] to study certain geometric properties of random walk trajectories on large, asymptotically transient, finite graphs. The Dynkin-type isomorphism theorem relating interlacements and the Gaussian free field, see [96], has repeatedly proved a useful tool in their study, see [96], [98], [78], [57], [101] and [1]. In a broader scheme, the usefulness of similar random walk representations as a tool in field theory and statistical mechanics has been recognized for a long time, see [88], [18] and [31].

The cable system method introduced in [57] provides a continuous version of this isomorphism theorem, from which some links between the level sets of the Gaussian free field and the vacant sets $\mathcal{V}^u$, $u > 0$, of random interlacements can be derived. This method was used in [101] and [1] to find a suitable coupling between those two sets, and was applied in the case of transient trees. It was also proved in these papers that, under certain conditions on the geometry of the tree $\mathbb{T}$, the critical parameter $h_*(\mathbb{T})$ for level set percolation of the Gaussian free field on $\mathbb{T}$ is strictly positive. As will become apparent below, the isomorphism theorem on the cable system can be paired with renormalization techniques from random interlacements, and in particular from [74], which imply a certain robustness property of $\mathcal{I}^u = \mathbb{Z}^d \setminus \mathcal{V}^u$ with respect to small noise, to yield similar findings on $\mathbb{Z}^d$, for all $d \geqslant 3$.

Let us now describe the results in more details. For $d \geqslant 3$, we consider $\mathbb{Z}^d$ as a graph, with undirected edge set E, and take uniform weights equal to 1 on all edges in E, so that the sum of all weights around a vertex $x \in \mathbb{Z}^d$ is $2d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if and only if $\{x, y\} \in E$. Noting that $\mathbb{Z}^d$, $d \geqslant 3$, is transient for discrete time simple random walk, we define the symmetric Green function by

$$g(x,y) = \frac{1}{2d} E_x\Big[\int_0^\infty 1_{\{X_t = y\}}\,\mathrm{d}t\Big],\ x, y \in \mathbb{Z}^d, \tag{II.1.1}$$

where $(X_t)_{t\geq 0}$ denotes the canonical continuous time random walk on $\mathbb{Z}^d$, with constant jump rate 1, starting at x under P_x. We also set $g(x) = g(0,x)$, for $x \in \mathbb{Z}^d$. We define $\mathbb{P}^G$, a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$ endowed with its canonical σ-algebra generated by the coordinate maps Φ_x, $x \in \mathbb{Z}^d$, such that, under $\mathbb{P}^G$,

$$\begin{aligned}&(\Phi_x)_{x\in\mathbb{Z}^d} \text{ is a centered Gaussian field with}\\ &\text{covariance function } \mathbb{E}^G[\Phi_x \Phi_y] = g(x,y) \text{ for all } x, y \in \mathbb{Z}^d.\end{aligned} \tag{II.1.2}$$

(Any random field $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ with law $\mathbb{P}^G$ on $\mathbb{R}^{\mathbb{Z}^d}$ will henceforth be called a Gaussian free field on $\mathbb{Z}^d$). We are interested in level sets of Φ, and for every $h \in \mathbb{R}$, denote by $\{x \overset{\geqslant h}{\longleftrightarrow} \infty\}$ the event that $x \in \mathbb{Z}^d$ lies in an infinite connected component of

$$E^{\geqslant h} = \{y \in \mathbb{Z}^d;\, \Phi_y \geqslant h\}, \tag{II.1.3}$$

and by $\eta(h)$ its probability, which does not depend on the choice of x. The function $\eta(\cdot)$ is decreasing, and it is natural to ask whether it is strictly positive or not. This leads to the definition of the critical point

$$h_*(d) \overset{\text{def.}}{=} \inf\{h \in \mathbb{R};\, \eta(h) = 0\}. \tag{II.1.4}$$

By ergodicity, this definition corresponds to the phase transition for the existence of an infinite connected component in $E^{\geqslant h}$, see Lemma 1.5 in [81]. It is not a priori clear whether $|h_*| < \infty$ or not, and a summary of the *status quo* was given in the first paragraph. In summary, it is known that $h_*(d) \in [0, \infty)$ for all $d \geqslant 3$, and that $h_*(d) \sim (2g(0) \log d)^{1/2}$ as $d \to \infty$. Our main result is the following lower bound in all dimensions.

Theorem II.1.1.

$$h_*(d) > 0, \quad \textit{for all } d \geqslant 3. \tag{II.1.5}$$

Moreover, there exists $h_1 > 0$ such that for all $h \leqslant h_1$, there exists $L_0 > 0$ such that $E^{\geqslant h}$ contains an infinite cluster in the thick slab $\mathbb{Z}^2 \times [0, 2L_0)^{d-2}$.

In fact, one can replace $E^{\geqslant h}$ by $\{y \in \mathbb{Z}^d;\, K(h) \geqslant \Phi_y \geqslant h\} (\subset E^{\geqslant h})$ for sufficiently large $K(h)$ in the previous statement, see Remark II.5.2, 2) below. Note that the infinite cluster of $E^{\geqslant h}$ cannot be contained in $\mathbb{Z}^2 \times \{0\}^{d-2}$ for $0 \leqslant h < h_*(d)$, as explained in Remark 3.6.1 of [81]. As an immediate corollary of Theorem II.1.1, we note that there exists an open interval $I \subset \mathbb{R}$ containing the origin and such that, for all $h \in I$, the level set $E^{\geqslant h}$ and its complement $E^{<h} = \mathbb{Z}^d \setminus E^{\geqslant h}$ *both* percolate (with probability one). This follows readily from (II.1.5) and the fact that $E^{<h} \overset{\text{law}}{=} E^{\geqslant -h}$ for all $h \in \mathbb{R}$, by symmetry of Φ. In particular, choosing $h = 0$, this implies that

$$\Phi \text{ almost surely contains two infinite sign clusters (one for each sign).} \tag{II.1.6}$$

In Chapter III, we will extend the inequality (II.1.5) to other graphs than $\mathbb{Z}^d$, and also obtain more information on the geometry of the sign clusters, see (III.1.1) for instance. Put differently, Theorem II.1.1 asserts that the critical density $p_c^G(d) = \mathbb{P}^G[\Phi_0 \geqslant h_*(d)]$ satisfies $p_c^G(d) < \frac{1}{2}$, for all $d \geq 3$, thus mirroring the result $p_c^{\text{site}}(\mathbb{Z}^d) < \frac{1}{2}$, see [19], for independent Bernoulli site percolation on $\mathbb{Z}^d$, $d \geq 3$. However, the elegant geometric arguments developed therein to "interpolate" between two- and three-dimensional structures do not seem to transfer to the current situation: the correlations present a serious impediment. Moreover, there is no obvious monotonicity of $p_c^G(d)$ (or $h_*(d)$) as a function of d. One may also conjecture that $p_c^G(d) < p_c^{\text{site}}(\mathbb{Z}^d)$, the critical density for independent site percolation on the lattice, based on the reasonable intuition that positive correlations "help" in forming clusters of $E^{\geqslant h}$. We do not currently know a proof of this (nor of the more modest conjecture $p_c^G(d) \leq p_c^{\text{site}}(\mathbb{Z}^d)$).

A key tool in the proof of Theorem II.1.1 is a certain isomorphism, see Theorem II.2.2 below, which gives a link between random interlacements and the Gaussian free field. We now explain its benefits in some detail, and refer to Section II.2 for precise definitions. Suppose that ω denotes the interlacement point process defined in [93], with law $\mathbb{P}^I$, and let ω^u be the process consisting of the trajectories in the support of ω with label at most u. Somewhat informally, ω^u is a Poisson cloud of bi-infinite nearest neighbor trajectories modulo time-shift whose forward and backward parts escape all finite sets in finite time. One naturally associates to ω^u, see for instance (1.8) in [96], a field of occupation times $(\ell_{x,u})_{x\in\mathbb{Z}^d}$, where $\ell_{x,u} = \ell_{x,u}(\omega^u)$ collects the total amount of time spent at x by any of the trajectories in the support of ω^u. The interlacement set at level u is then defined as

$$\mathcal{I}^u = \{x \in \mathbb{Z}^d;\ \ell_{x,u} > 0\}. \tag{II.1.7}$$

It corresponds to the set of vertices visited by at least one trajectory in the support of ω^u. For any $u > 0$, the set $\mathcal{I}^u$ is almost surely unbounded and connected [93]. The following isomorphism was proved in Theorem 0.1 of [96], and has the same spirit as the generalized second Ray-Knight theorem, see for example [32], [62] or [98]:

$$\begin{aligned}&\Big(\ell_{x,u} + \frac{1}{2}\Phi_x^2\Big)_{x\in\mathbb{Z}^d} \text{ under } \mathbb{P}^I \otimes \mathbb{P}^G \text{ has the same law}\\ &\text{as } \Big(\frac{1}{2}(\Phi_x + \sqrt{2u})^2\Big)_{x\in\mathbb{Z}^d} \text{ under } \mathbb{P}^G.\end{aligned} \tag{II.1.8}$$

If one attaches to each edge e of $\mathbb{Z}^d$ a line segment I_e of length $\frac{1}{2}$, the resulting "graph" $\widetilde{\mathbb{Z}}^d$ is continuous and called the *cable system*, see Section II.2. On this cable system, one then defines probabilities $\widetilde{\mathbb{P}}^G$ and $\widetilde{\mathbb{P}}^I$ under which the fields $(\Phi_x)_{x\in\mathbb{Z}^d}$ and $(\ell_{x,u})_{x\in\mathbb{Z}^d}$ admit *continuous* extensions $\widetilde{\Phi} = (\widetilde{\Phi}_x)_{x\in\widetilde{\mathbb{Z}}^d}$ and $\widetilde{\ell} = (\widetilde{\ell}_{x,u})_{x\in\widetilde{\mathbb{Z}}^d}$, and the set $\widetilde{\mathcal{I}}^u = \{x \in \widetilde{\mathbb{Z}}^d;\ \widetilde{\ell}_{x,u} > 0\}$ is connected. It was proved in [57] that for each $u > 0$, a continuous version of the isomorphism (II.1.8) also holds on $\widetilde{\mathbb{Z}}^d$, see also (II.2.15) below, and in particular (somewhat inaccurately, but see (II.2.15), (II.2.16) below for precise statements) the sign of $\widetilde{\Phi}_x + \sqrt{2u}$ is constant as long as $\widetilde{\ell}_{x,u} > 0$, and thus by the continuity of $\widetilde{\Phi}$ and the connectivity of $\widetilde{\mathcal{I}}^u$, either $\widetilde{\Phi}_x > -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$ or $\widetilde{\Phi}_x < -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$. But $\widetilde{\mathcal{I}}^u$ is unbounded, hence, taking $h = \sqrt{2u}$, by symmetry of the Gaussian free field and ergodicity, $\widetilde{\mathbb{P}}^G$-a.s. the set

$$\{x \in \widetilde{\mathbb{Z}}^d;\ \widetilde{\Phi}_x \geqslant -h\} \text{ contains an unbounded cluster in the cable system } \widetilde{\mathbb{Z}}^d. \tag{II.1.9}$$

This result was already known to hold on $\mathbb{Z}^d$ without the isomorphism theorem [16], where it had been derived using a neat contour argument. It is interesting to note that, on the cable system, (II.1.9) is actually sharp, because $\widetilde{\mathbb{P}}^G$-a.s. the set

$$\{x \in \widetilde{\mathbb{Z}}^d;\ \widetilde{\Phi}_x \geqslant 0\} \text{ does not contain unbounded clusters in the cable system } \widetilde{\mathbb{Z}}^d, \tag{II.1.10}$$

see Proposition 5.5 in [57], which sharply contrasts with (II.1.6). We will extend the property (II.1.10) to a large class of graphs in Chapter IV. All in all, the infinite cluster in $E^{\geqslant 0}$ (part of $\mathbb{Z}^d$), which exists by Theorem II.1.1, "scatters" into finite pieces upon adding the field on the edges, but the infinite cluster of $E^{\geqslant -h}$ does not, for ever so small $h > 0$.

On our way towards proving Theorem II.1.1, we will first show that a truncated version of the level sets in (II.1.9) contains an unbounded cluster on $\widetilde{\mathbb{Z}}^d$. Indeed, it was proved in [74] that the intersection of the random interlacement set $\mathcal{I}^u$ with a Bernoulli percolation having large success parameter still contains an infinite cluster in $\mathbb{Z}^d$. By showing a similar stability result on the cable system, see Proposition II.4.1, and using the isomorphism theorem on the cable system, we will obtain, cf. Theorem II.3.1 below, that the truncated (continuous) level set

$$\{x \in \widetilde{\mathbb{Z}}^d;\, -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\} \tag{II.1.11}$$

contains an unbounded cluster on $\widetilde{\mathbb{Z}}^d$ for all $h > 0$ and large enough, but finite $K(h)$ (with $hK(h) \to 0$ as $h \searrow 0$). Once this has been proved, see Theorem II.3.1 for the precise technical statement, we no longer need to use random interlacements to prove Theorem II.1.1 (note however that the interlacements are crucial in generating a suitable percolating cluster to start with, i.e., one which is already reasonably "close" to being a sign cluster of the free field, see (II.1.11)).

We now describe the second part of the proof. By construction, one can view $\widetilde{\Phi}$, the Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$, as a Gaussian free field on $\mathbb{Z}^d$ with Brownian bridges of length $\frac{1}{2}$ attached on the edges, see (II.2.7) and thereafter. On an edge contained in the set of (II.1.11), those Brownian bridges never go below $-h$, which happens with low probability for small h. We are going to use this low probability to go from $-h \leqslant \widetilde{\Phi} \leqslant K(h)$ *on the edges* to $h \leqslant \widetilde{\Phi} \leqslant K(h)$ *on the endpoints* of theses edges and for small enough h, see in particular Lemma II.5.1, which will then imply that the set $\{x \in \mathbb{Z}^d;\, \widetilde{\Phi}_x \geqslant h\}$ has an infinite cluster on $\mathbb{Z}^d$, as asserted.

We now explain the organization of this chapter, and highlight its main contributions. In Section II.2, we recall the definitions of the Gaussian free field and random interlacements on the cable system, and the link between the two via the aforementioned isomorphism theorem. In Section II.3, we collect a few preparatory tools by showing some strong connectivity properties, a large deviation inequality as well as a version of the decoupling inequalities for random interlacements on the cable system. Most of these are well-known in spirit, but existing results do not entirely fit our needs.

The construction of the infinite cluster comes essentially in three steps, Proposition II.4.1, Theorem II.3.1, and Section II.5, which are the main reference points of this chapter. Proposition II.4.1 is a fairly generic result, which, roughly speaking, for *any* coupling of a continuous interlacement and a Gaussian free field, see (II.4.1), yields a percolating interlacement cluster, with good control on the free field part, and some room to play with along the edges. Its proof follows a standard static renormalization scheme from [74], [78], assembling the results of Section II.3. In Theorem II.3.1, we "translate" Proposition II.4.1, for a certain choice of the coupling, to show that suitably truncated level sets of the Gaussian free field on the cable system contain an unbounded connected component. The reference level for the excursion sets of Theorem II.3.1 is $-h$, for (small) positive h. Section II.5 contains the device to "flip the sign" and pass from $-h$ to h on the vertices, as indicated above. Together with Theorem II.3.1, this then yields a proof of Theorem II.1.1.

In the rest of this chapter, we denote by c and C positive constants that may change from place to place. Numbered constants such as C_0, c_0, C_1, C_1', ... are fixed until the end of the chapter. All constants are allowed to implicitly depend on the dimension d and a parameter $u_0 > 0$, which will first appear in Lemma II.3.2 and throughout the remaining sections.

II.2 Notation and useful facts about the cable system

In this section, we give a definition of the Gaussian free field and random interlacements on the cable system that will be useful later. We also discuss some aspects of the Markov property for the Gaussian free field and its consequences, and recall the isomorphism theorem which links random interlacements and the Gaussian free field.

For later convenience, we endow the graph $\mathbb{Z}^d$ with a distance function $d(\cdot,\cdot)$ which is half of the usual graph distance, i.e., half of the ℓ^1-distance $|\cdot|_1$ on $\mathbb{Z}^d$. Recall that we write $x \sim y$, for $x, y \in \mathbb{Z}^d$, if $|x-y|_1 = 1$. We define $V^0 = \{2x,\ x \sim 0\}$, so that, for all $x, y \in \mathbb{Z}^d$ with $x \sim y$, we can write $y = x + \frac{1}{2}v_{(x,y)}$ for a unique $v_{(x,y)} \in V^0$. Note that $d(x, x+\frac{1}{2}v) = \frac{1}{2}$, for all $x \in \mathbb{Z}^d$ and $v \in V^0$. We attach to each edge $e = \{x, y\}$ the following interval of length $\frac{1}{2}$:

$$I_e \stackrel{\text{def.}}{=} \Big\{x + tv_{(x,y)};\ t \in \Big(0, \frac{1}{2}\Big)\Big\} = \Big\{y + tv_{(y,x)};\ t \in \Big(0, \frac{1}{2}\Big)\Big\}, \qquad \text{(II.2.1)}$$

which is homeomorphic to an open interval of $\mathbb{R}$ of length $\frac{1}{2}$, and we write $\overline{I_e} = I_e \cup \{x, y\}$. The cable system $\widetilde{\mathbb{Z}}^d$ is then defined by glueing these intervals through their endpoints. $\widetilde{\mathbb{Z}}^d \backslash \mathbb{Z}^d$ is now the union of such I_e, one for every edge $e \in E$. We extend the definition of the distance d to $\widetilde{\mathbb{Z}}^d$ by setting $d(x+tv, x+t'v) = t - t'$ for all $x \in \mathbb{Z}^d$, $v \in V^0$ and $0 \leqslant t' \leqslant t \leqslant \frac{1}{2}$, and for all $x_1 \sim x_2$ and $y_1 \sim y_2$ in $\mathbb{Z}^d$, $z \in I_{\{x_1,x_2\}}$ and $z' \in I_{\{y_1,y_2\}}$,

$$d(z,z') = \min_{i,j\in\{1,2\}} \big\{ d(z,x_i) + d(x_i,y_j) + d(y_j,z') \big\}.$$

For all $e \in E$ and z_1, $z_2 \in \overline{I_e}$ we define $(z_1,z_2) \subset \widetilde{\mathbb{Z}}^d$ as the open interval in I_e between z_1 and z_2. We also define the distance between two subsets A_1 and A_2 of $\widetilde{\mathbb{Z}}^d$ by $d(A_1,A_2) = \inf_{x\in A_1, y\in A_2} d(x,y)$. For $R_1 < R_2$, we introduce the boxes $[R_1,R_2)^d = \{z \in \widetilde{\mathbb{Z}}^d;\ z \in \overline{I}_{\{x,y\}},\ \text{with } x_i, y_i \in [R_1,R_2) \text{ for all } i = 1,\dots,d\}$. The set $\mathbb{Z}^d$ will henceforth be considered as a subset of $\widetilde{\mathbb{Z}}^d$ and we will call vertices the elements of $\mathbb{Z}^d$.

One can define a continuous diffusion $\widetilde{X}$ on the cable system $\widetilde{\mathbb{Z}}^d$, via probabilities $\widetilde{P}_z$, $z \in \widetilde{\mathbb{Z}}^d$, with continuous local times with respect to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$. We now describe this construction from a simple random walk on $\mathbb{Z}^d$ with the help of the excursion process of Brownian motion as in Section 2 of [57], and refer to [33] or Section 2 of [36] for precise definitions. Let n be the intensity measure of Brownian excursions, see Chapter XII §2 in [75], and λ_+ be the Lebesgue measure on $[0,\infty)$. For all $x \in \mathbb{Z}^d$, we define under $\widetilde{P}_x$ a Poisson point process $\mathbf{e} = \sum_{n\in\mathbb{N}} \delta_{(e_n,t_n)}$ with intensity measure $n \otimes \lambda_+$, $(V_n)_{n\in\mathbb{N}}$ an i.i.d. sequence of uniform variables on V^0 independent of $\mathbf{e}$ (here and in the sequel $\mathbb{N} = \{0,1,2,\dots\}$), and $(Z_n)_{n\in\mathbb{N}}$ an independent simple random walk on $\mathbb{Z}^d$ with $Z_0 = x$. For any trajectory e in the space of excursions, let $R(e) = \inf\{t > 0 : \ e(t) = 0\}$ be the length of e, and we define for all $n \in \mathbb{N}$

$$\tau_n \equiv \tau_n(\mathbf{e}) := \sum_{\substack{p\in\mathbb{N} \\ t_p \leqslant t_n}} R(e_p), \qquad \tau_n^- \equiv \tau_n^-(\mathbf{e}) := \sum_{\substack{p\in\mathbb{N} \\ t_p < t_n}} R(e_p), \quad \text{if } \mathbf{e} = \sum_{n\in\mathbb{N}} \delta_{(e_n,t_n)},$$

and $T \in [0,\infty)$ such that there exists $N \in \mathbb{N}$ with $|e_N(T - \tau_N^-)| = \frac{1}{2}$ and for all $p \in \mathbb{N}$ such that $t_p < t_N$, $\sup_{s>0} e_p(s) < \frac{1}{2}$. In words, T is the first time that the graph obtained by concatenating the excursions in the support of $\mathbf{e}$ according to their label t_n reaches height $1/2$ in absolute value. For each $x \in \mathbb{Z}^d$, we then define under $\widetilde{P}_x$ for all $t < \tau_N^-$,

$$\widetilde{X}_t = x + |e_n(t - \tau_n^-)| V_n \text{ whenever } \tau_n^- \leqslant t \leqslant \tau_n.$$

and for all $t \in [\tau_N^-, T]$,

$$\widetilde{X}_t = x + |e_N(t - \tau_N^-)| v_{(Z_0,Z_1)}.$$

Note that, under $\widetilde{P}_x$, $(\widetilde{X}_t)_{t\leqslant T}$ is a continuous process on $\bigcup_{y\sim x}\overline{I}_{\{x,y\}}$, and that $\widetilde{X}_T = Z_1$, and we repeat this process after time T starting in Z_1 in such a way that, conditionally on $(\widetilde{X}_t)_{t\leqslant T}$, the law of $(\widetilde{X}_t)_{t\geqslant T}$ is $\widetilde{P}_x$-a.s. the same as the law of $(\widetilde{X}_t)_{t\geqslant 0}$ under $\widetilde{P}_{Z_1}$, and that the projection of the trajectory of $\widetilde{X}$ on $\mathbb{Z}^d$ is $(Z_n)_{n\in\mathbb{N}}$. On an edge, the process X behaves like a Brownian motion, see Chapter XII, Proposition 2.5 in [75] for a similar construction of the Brownian motion on $\mathbb{R}$ from the Poisson point process of excursions. Finally, for all $x \sim y \in \mathbb{Z}^d$ and $z \in I_{\{x,y\}}$, we construct $(\widetilde{X}_t)_{t\geqslant 0}$ under $\widetilde{P}_z$ as a Brownian motion beginning in z on $I_{\{x,y\}}$ until either x or y is reached, and then we continue with the previous construction beginning at this vertex.

Under $\widetilde{P}_x$ for $x \in \mathbb{Z}^d$, the local time in x of $\widetilde{X}$ at time T relative to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$ has the same law upon renormalization as the local time in 0 of a Brownian motion at the moment it leaves $(-\frac{1}{2},\frac{1}{2})$, and is thus an exponential variable, see for example Chapter VI, Proposition 4.6 in [75] for a similar result, with parameter 1, see Section 2 of [57] for details. For all $t \in [0,\infty]$, let us denote by $(L^y_t)_{y\in\widetilde{\mathbb{Z}}^d}$ the local times relative to the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$ of $\widetilde{X}$ at time t, see Section 2 in [57], then for all $x \in \mathbb{Z}^d$, $(L^y_\infty)_{y\in\mathbb{Z}^d}$ has the same law under $\widetilde{P}_x$ as the field of occupation times of the jump process X on $\mathbb{Z}^d$ under P_x (cf. below (II.1.1)). In particular, we can define for all $x, y \in \widetilde{\mathbb{Z}}^d$ the Green function

$$g(x,y) = \widetilde{E}_x[L^y_\infty], \tag{II.2.2}$$

and its restriction to $\mathbb{Z}^d$ is the same as the Green function on $\mathbb{Z}^d$ defined in (II.1.1), so the identical notation does not bear any risk of confusion.

We endow the canonical space $\Omega_0 := C(\widetilde{\mathbb{Z}}^d,\mathbb{R})$ of continuous real-valued functions on $\widetilde{\mathbb{Z}}^d$ with the canonical σ-algebra generated by the coordinate functions $\widetilde{\Phi}_x$, $x \in \widetilde{\mathbb{Z}}^d$, and let $\widetilde{\mathbb{P}}^G$ be the probability on Ω_0 such that, under $\widetilde{\mathbb{P}}^G$,

$$\begin{aligned}&(\widetilde{\Phi}_x)_{x\in\widetilde{\mathbb{Z}}^d} \text{ is a centered Gaussian field with}\\&\text{covariance function } \widetilde{\mathbb{E}}^G[\widetilde{\Phi}_x\widetilde{\Phi}_y] = g(x,y) \text{ for all } x,y\in\widetilde{\mathbb{Z}}^d,\end{aligned} \tag{II.2.3}$$

with $g(\cdot,\cdot)$ given by (II.2.2). With a slight abuse of notation, any random variable $\widetilde{\varphi} = (\widetilde{\varphi}_x)_{x\in\widetilde{\mathbb{Z}}^d}$ on $C(\widetilde{\mathbb{Z}}^d,\mathbb{R})$ with law $\widetilde{\mathbb{P}}^G$ under some $\widetilde{\mathbb{P}}$ will be called a Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$, and it is plain that the restriction of a Gaussian free field on the cable system to $\mathbb{Z}^d$ is a Gaussian free field on $\mathbb{Z}^d$, so we will often identify $\widetilde{\Phi}_x$ with Φ_x for $x \in \mathbb{Z}^d$.

Let us recall the simple Markov property for $\widetilde{\varphi}$. Let $K \subset \widetilde{\mathbb{Z}}^d$ be a compact subset with finitely many components, and let $U = \widetilde{\mathbb{Z}}^d \setminus K$. For all $x \in \widetilde{\mathbb{Z}}^d$, we define

$$\widetilde{\beta}^U_x = \widetilde{E}_x\left[\widetilde{\varphi}_{\widetilde{X}_{T_U}} 1_{\{T_U<\infty\}}\right] \quad \text{and} \quad \widetilde{\varphi}^U_x = \widetilde{\varphi}_x - \widetilde{\beta}^U_x, \tag{II.2.4}$$

where $T_U := \inf\{t \geqslant 0;\ \widetilde{X}_t \notin U\}$, with the convention $\inf \varnothing = \infty$, is the exit time from U of the diffusion $\widetilde{X}$ on $\widetilde{\mathbb{Z}}^d$. Moreover, for all $x, y \in \widetilde{\mathbb{Z}}^d$, we define similarly as in (II.2.2) the Green function $g_U(x,y) = \mathbb{E}_x[L^y_{T_U}]$ of the diffusion $\widetilde{X}$ under $\widetilde{P}_x$ killed when exiting U. Then,

$$\begin{aligned}&(\widetilde{\varphi}^U_x)_{x\in\widetilde{\mathbb{Z}}^d} \text{ is a centered Gaussian field with}\\ &\text{covariance function } g_U(x,y) \text{ for all } x,y \in \widetilde{\mathbb{Z}}^d.\end{aligned} \tag{II.2.5}$$

Furthermore, this field is continuous, vanishes on K and is independent of $\sigma(\widetilde{\varphi}_z,\ z \in K)$. A strong Markov property is also known to hold, but we will not need it here, see Section 1 of [101] for more details.

Following standard notation, we say that $(B_t)_{t\in[0,l]}$ is a *Brownian bridge of length $l > 0$ between x and y of a Brownian motion with variance σ^2 at time* 1 under a probability $\mathbb{P}^B$ if the process

$$W_t := B_t - \frac{t}{l}y - \Big(1 - \frac{t}{l}\Big)x, \quad t \in [0,l], \tag{II.2.6}$$

is a centered Gaussian field with covariance function

$$\mathbb{E}^B\left[W_{s_1}W_{s_2}\right] = \frac{\sigma^2 s_1(l-s_2)}{l} \text{ for all } s_1, s_2 \in [0,l] \text{ with } s_1 \leqslant s_2 \tag{II.2.7}$$

(the process $(W_{lt}/\sqrt{l\sigma^2})_{t\in[0,1]}$ is a standard Brownian bridge). Let $e \in E$, $z_1 \neq z_2 \in \overline{I_e}$, $v \in V^0$ and $t \in (0, \frac{1}{2}]$ such that $z_2 = z_1 + tv$, and let $s_1, s_2 \in [0,t]$ such that $s_1 \leqslant s_2$. Under $\widetilde{P}_{z_1+s_1v}$, until time $T_{(z_1,z_2)}$, the diffusion $\widetilde{X}$ behaves like a Brownian motion on I_e beginning at $z_1 + s_1 v$ until the hitting time of $(z_1,z_2)^c$. Using Chapter II.11 in [13] with $s(x) = x$, and noting that the function G_0 defined therein is $\frac{1}{2}g_{(z_1,z_2)}$, we have

$$g_{(z_1,z_2)}(z_1 + s_1 v, z_1 + s_2 v) = \frac{2s_1(t-s_2)}{t}.$$

The Markov property for the Gaussian free field implies that, under $\widetilde{\mathbb{P}}$ (under which $\widetilde{\varphi}$ is a Gaussian free field),

$$\Big(\widetilde{\varphi}_{z_1+sv} - \frac{t-s}{t}\widetilde{\varphi}_{z_1} - \frac{s}{t}\widetilde{\varphi}_{z_2}\Big)_{s\in[0,t]} \tag{II.2.8}$$

is a centered Gaussian field with covariance function $(g_{(z_1,z_2)}(z_1 + s_1 v, z_2 + s_2 v))_{s_1,s_2\in[0,t]}$, and is independent of $\sigma(\widetilde{\varphi}_z,\ z \in \widetilde{\mathbb{Z}}^d \setminus (z_1,z_2))$. Thus, it is a Brownian bridge of length t between 0 and 0 of a Brownian motion with variance 2 at time 1. In particular, knowing $\widetilde{\varphi} \upharpoonright \mathbb{Z}^d$, the Gaussian free field on the edges $((\widetilde{\varphi}_z)_{z\in I_e})_{e\in E}$ is an independent family of random processes such that, for each

$x \sim y \in \mathbb{Z}^d$, the process $(\widetilde{\varphi}_z)_{z\in I_{\{x,y\}}}$ has the same law as a Brownian bridge of length $\frac{1}{2}$ between $\widetilde{\varphi}_x$ and $\widetilde{\varphi}_y$ of a Brownian motion with variance 2 at time 1, as mentioned in Section 2 of [57] or in Section 2.2 of [58]. More precisely, let

$$B^e_t = \widetilde{\varphi}_{x_e+tv_{(x_e,y_e)}} - 2t\widetilde{\varphi}_{y_e} - (1-2t)\widetilde{\varphi}_{x_e}, \text{ for all } t \in [0,1/2] \text{ and } e \in E, \quad \text{(II.2.9)}$$

where we have given an (arbitrary) orientation (x_e, y_e) for each edge $e = \{x_e, y_e\} \in E$. Then, under $\widetilde{\mathbb{P}}$, $(B^e)_{e\in E}$ is a family of independent Brownian bridges of length $\frac{1}{2}$ between 0 and 0 of a Brownian motion with variance 2 at time 1. Note that this provides an explicit (and simple) construction of a Gaussian free field on the cable system starting from the Gaussian free field $(\varphi_x)_{x\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$: if one links independently each $x \sim y \in \mathbb{Z}^d$ via a Brownian bridge on $I_{\{x,y\}}$ of length $\frac{1}{2}$ between φ_x and φ_y of a Brownian motion with variance 2 at time 1, then the resulting process is a Gaussian free field on the cable system. In view of this construction, we will later need the following result on the probability that the maximum of a Brownian bridge exceeds some value M (see e.g. [13], Chapter IV.26).

Lemma II.2.1. *Let x, y be two real numbers, $M \geqslant \max(x, y)$ and, under $\mathbb{P}^B$, $(B_t)_{t\in[0,l]}$ a Brownian bridge of length l between x and y of a Brownian motion with variance σ^2 at time 1. One has*

$$\mathbb{P}^B\Big(\sup_{t\in[0,l]} B_t > M\Big) = \exp\Big(-\frac{2(M-x)(M-y)}{l\sigma^2}\Big). \quad \text{(II.2.10)}$$

Let us now turn to the definition of random interlacements on $\widetilde{\mathbb{Z}}^d$, as in [57] or [101]. The usual definition of random interlacements on $\mathbb{Z}^d$, see, for example, [93] or the monograph [27], can be adapted to define a Poisson point process $\widetilde{\omega}$ on $\widetilde{W}^* \times [0,\infty)$, where $\widetilde{W}^*$ is the space of doubly infinite trajectories on $\widetilde{\mathbb{Z}}^d$ modulo time-shift, endowed with its canonical σ-algebra, and where $[0,\infty)$ describes labels of the trajectories. Recall the law $(\widetilde{P}_z)_{z\in\widetilde{\mathbb{Z}}^d}$ of the diffusion $\widetilde{X}$ on the cable system, started at $z \in \widetilde{\mathbb{Z}}^d$. The intensity measure of $\widetilde{\omega}$ is characterized as follows: for some $N_1, N_2 \in \mathbb{Z}$ with $N_1 \leqslant N_2$, let $\widetilde{K} := [N_1, N_2]^d \cap \widetilde{\mathbb{Z}}^d$, let $K := \widetilde{K} \cap \mathbb{Z}^d$, let $\widetilde{\omega}^u$ be the point process which consists of the trajectories in $\widetilde{\omega}$ with label at most $u > 0$, and let $\widetilde{\omega}^u_{\widetilde{K}}$ be the point process comprising the forward trajectories of $\widetilde{\omega}^u$ hitting $\widetilde{K}$ and beginning at the first time $\widetilde{K}$ is reached. Then $\widetilde{\omega}^u_{\widetilde{K}}$ is a Poisson point process with intensity measure $u\widetilde{P}_{e_K} = u\sum_{x\in K} e_K(x)\widetilde{P}_x$, where e_K is the usual equilibrium measure of K on $\mathbb{Z}^d$, as mentioned in [101]. One can also construct the random interlacement process $\widetilde{\omega}^u$ at level $u > 0$ on the cable system from the corresponding interlacement process ω^u on $\mathbb{Z}^d$ by adding independent Brownian excursions on the edges for every trajectory in the support of ω^u in the same fashion as one can construct the diffusion $\widetilde{X}$ from a simple random walk on $\mathbb{Z}^d$.

One then defines

$$(\widetilde{\ell}_{z,u})_{z\in\widetilde{\mathbb{Z}}^d} \text{ the field of local times of random interlacements, for } u>0, \tag{II.2.11}$$

as the sum of the local times of each of the trajectories in the support of $\widetilde{\omega}^u$. The restriction of these local times to $\mathbb{Z}^d$ has the same law as the occupation times $(\ell_{x,u})_{x\in\mathbb{Z}^d}$ for random interlacements on $\mathbb{Z}^d$ alluded to in the introduction, cf. above (II.1.7). The random interlacement set is defined as

$$\widetilde{\mathcal{I}}^u = \{x \in \widetilde{\mathbb{Z}}^d;\, \widetilde{\ell}_{x,u} > 0\}, \tag{II.2.12}$$

which is an open connected subset of $\widetilde{\mathbb{Z}}^d$. Note that $\{x \in \mathbb{Z}^d;\, x \in \widetilde{\mathcal{I}}^u\}$ has the same law as $\mathcal{I}^u$, cf. (II.1.7).

We also recall the following formula for the Laplace transform of $(\ell_{x,u})_{x\in\mathbb{Z}^d}$, see for instance [96], (1.9)–(1.11) or Remark 2.4.4 in [97]: for all $V : \mathbb{Z}^d \to \mathbb{R}$ with finite support $K \subset \mathbb{Z}^d$ and satisfying

$$\|GV\|_\infty < 1, \text{ where } (GV)f(x) = \sum_{y\in\mathbb{Z}^d} g(x,y)V(y)f(y) \text{ for all } f \in \ell^\infty(\mathbb{Z}^d), \tag{II.2.13}$$

with $g(\cdot,\cdot)$ as in (II.1.1), and where $\|\cdot\|_\infty$ denotes the operator norm on $\ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d)$, one has

$$\begin{aligned}\widetilde{\mathbb{E}}^I\Big[\exp\Big\{\sum_{x\in\mathbb{Z}^d} V(x)\widetilde{\ell}_{x,u}\Big\}\Big] &= \exp\{u\langle V,(I-GV)^{-1}1\rangle_{L^2(\mathbb{Z}^d)}\} \\ &\Big(=\exp\Big\{u\sum_{x\in\mathbb{Z}^d} V(x)\sum_{n\geqslant 0}(GV)^n1(x)\Big\}\Big).\end{aligned} \tag{II.2.14}$$

Random interlacements are useful in the study of the Gaussian free field on the cable system $\widetilde{\mathbb{Z}}^d$ because of the existence of a Ray-Knight-type isomorphism theorem proved in Proposition 6.3 of [57], see also (1.30) in [101].

Theorem II.2.2. *For each $u>0$, there exists a coupling $\widetilde{\mathbb{P}}^u$ between two Gaussian free fields $\widetilde{\varphi}$ and $\widetilde{\gamma}$ and a random interlacement process $\widetilde{\omega}$ on the cable system $\widetilde{\mathbb{Z}}^d$ (i.e., under $\widetilde{\mathbb{P}}^u$, the law of $\widetilde{\varphi}$ and $\widetilde{\gamma}$ is $\widetilde{\mathbb{P}}^G$ each, and the law of $\widetilde{\omega}$ is the same as under $\widetilde{\mathbb{P}}^I$) such that $\widetilde{\gamma}$ and $\widetilde{\omega}$ are independent, and $\widetilde{\mathbb{P}}^u$-a.s.,*

$$\frac{1}{2}\left(\widetilde{\varphi}_x + \sqrt{2u}\right)^2 = \widetilde{\ell}_{x,u} + \frac{1}{2}\widetilde{\gamma}_x^2, \quad \text{for all } x \in \widetilde{\mathbb{Z}}^d, \tag{II.2.15}$$

where $(\widetilde{\ell}_{x,u})_{x\in\widetilde{\mathbb{Z}}^d}$ is the field of local times of the random interlacements process $\widetilde{\omega}$ at level u, cf. (II.2.11).

This coupling will be essential for the proof of Theorem II.1.1. In particular, we are going to use results from the theory of random interlacements, along with the coupling (II.2.15), to deduce certain properties of the level sets of the Gaussian free field. For now, let us note that $\widetilde{\mathbb{P}}^u$-a.s. on $\widetilde{\mathcal{I}}^u$, cf. (II.2.12), one has $|\widetilde{\varphi} + \sqrt{2u}| > 0$, where $\widetilde{\varphi}$ refers to the Gaussian free field from the coupling in Theorem II.2.2. Since $\widetilde{\mathcal{I}}^u$ is connected (and unbounded, by construction) and since $x \in \widetilde{\mathbb{Z}}^d \mapsto \widetilde{\varphi}_x$ is continuous, either $\widetilde{\varphi}_x > -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$, or $\widetilde{\varphi}_x < -\sqrt{2u}$ for all $x \in \widetilde{\mathcal{I}}^u$. But Proposition 5.5 in [57], cf. also (II.1.10) above, implies that the set $\{x \in \widetilde{\mathbb{Z}}^d;\, \widetilde{\varphi}_x < 0\}$, which contains $\{x \in \widetilde{\mathbb{Z}}^d;\, \widetilde{\varphi}_x < -\sqrt{2u}\}$, only has bounded components, hence

$$\widetilde{\mathbb{P}}^u - \text{a.s., } \forall x \in \widetilde{\mathcal{I}}^u,\ \widetilde{\varphi}_x > -\sqrt{2u}. \tag{II.2.16}$$

In particular, this means that the negative (upper) level sets percolate on $\widetilde{\mathbb{Z}}^d$, see (II.1.9).

II.3 Connectivity and a large deviation inequality for $\widetilde{\mathcal{I}}^u$

The following result, which is proved over the next two sections, is essentially a refinement of (II.2.16), which allows us to truncate $\widetilde{\Phi}$, cf. (II.2.3), at sufficiently large heights. This important technical step will be helpful in dealing with the fact that $\widetilde{\Phi}$ is a priori unbounded on sets of interest.

Theorem II.3.1. *For each $h_0 > 0$, there exist positive constants C_0 and c_0, only depending on d and h_0, with $C_0 h_0^{-c_0} \geqslant 1$ such that, for all $0 < h \leqslant h_0$, with*

$$K(h) = \sqrt{\log\left(\frac{C_0}{h^{c_0}}\right)}, \tag{II.3.1}$$

there exists $L_0 = L_0(h) > 0$ such that $\widetilde{\mathbb{P}}^G$-a.s. the set

$$\begin{aligned}\widetilde{A}_h(\widetilde{\Phi}) \overset{\text{def.}}{=} &\left\{x \in \widetilde{\mathbb{Z}}^d \setminus \mathbb{Z}^d;\, \widetilde{\Phi}_x \geqslant -h\right\} \\ &\cup \left\{x \in \mathbb{Z}^d;\, \forall\, v \in V^0,\, \forall t \in \left[0, \frac{1}{2}\right],\, |\widetilde{\Phi}_{x+tv}| \leqslant K(h)\right\}\end{aligned} \tag{II.3.2}$$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$.

Note that, since $\widetilde{\Phi}$ is continuous, asserting that $\widetilde{A}_h(\widetilde{\Phi})$ has an unbounded component is tantamount to saying that there exists an infinite path in the set $\{x \in \widetilde{\mathbb{Z}}^d;\, \widetilde{\Phi}_x \geqslant -h\}$, and that in addition, for every $y \in \widetilde{\mathbb{Z}}^d$ at distance less than $\frac{1}{2}$ from a vertex on this path, $|\widetilde{\Phi}_y| \leqslant K(h)$ holds. In particular, the set $\{x \in \widetilde{\mathbb{Z}}^d;\, -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$ also contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$, as stated in (II.1.11). In order to be able to prove Theorem II.1.1 with the help of Theorem II.3.1 in Section II.5, the key property of $K(h)$ in (II.3.1) is that

$$hK(h) \to 0, \text{ as } h \searrow 0, \tag{II.3.3}$$

see in particular the proof of Lemma II.5.1. The proof of Theorem II.3.1 will involve an application of the isomorphism (II.2.15), and therefore hinges on a corresponding statement "in the world of random interlacements," see Proposition II.4.1 at the beginning of the next section. The proof of the latter requires some preliminary results on the geometry of $\widetilde{\mathcal{I}}^u$, which we gather now. The dependence of these results on u needs to be precise enough to later deduce (II.3.3) when transferring Proposition II.4.1 back to the Gaussian free field.

In the remainder of this section, we consider, under $\widetilde{\mathbb{P}}^I$, and for each $u > 0$, the random interlacement set $\widetilde{\mathcal{I}}^u$ at level u on the cable system, see (II.2.12), and $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$ the field of local times of the underlying interlacement process $\widetilde{\omega}^u$, see (II.2.11). The following lemma asserts that $\widetilde{\mathcal{I}}^u$ is typically well-connected.

Lemma II.3.2. *Let $d \geqslant 3$, $\varepsilon \in (0,1)$ and $u_0 > 0$. There exist constants $c = c(d, \varepsilon, u_0)$ and $C = C(d, \varepsilon, u_0)$ such that for all $u \in (0, u_0]$ and $R \geqslant 1$,*

$$\widetilde{\mathbb{P}}^I\Bigg(\bigcap_{x,y \in \widetilde{\mathcal{I}}^u \cap [0,R)^d} \{x \leftrightarrow y \text{ in } \widetilde{\mathcal{I}}^u \cap [-\varepsilon R, (1+\varepsilon)R)^d\} \Bigg) \geqslant 1 - C \exp\left(-cR^{1/7} u\right), \tag{II.3.4}$$

where, for measurable $A \subset \widetilde{\mathbb{Z}}^d$, the event $\{x \leftrightarrow y \text{ in } \widetilde{\mathcal{I}}^u \cap A\}$ refers to the existence of a continuous path in the subset $\widetilde{\mathcal{I}}^u \cap A$ of the cable system connecting x and y.

This property is essentially known, see for instance Proposition 1 of [73] or Lemma 3.1 in [74]. However, we need to keep careful track of the dependence of error terms on the intensity u. For the reader's convenience, we have included a proof of Lemma II.3.2 in the Appendix.

Next, we will need to know how much time the trajectories of random interlacements typically spend in a large box with sufficiently high precision. This can be conveniently formulated in terms of a large deviation inequality for the local times.

Lemma II.3.3. *Let $d \geqslant 3$ and $\varepsilon \in (0,1)$. There exist constants $c = c(d, \varepsilon)$ and $C = C(d, \varepsilon)$ such that for all $u > 0$ and $R \geqslant 1$,*

$$\widetilde{\mathbb{P}}^I\Bigg(\Big| \frac{1}{R^d} \sum_{x \in [0,R)^d \cap \mathbb{Z}^d} \widetilde{\ell}_{x,u} - u \Big| > \varepsilon \cdot u \Bigg) \leqslant C \exp\left(-cR^{d-2} u\right). \tag{II.3.5}$$

Proof. Abbreviate $B_R = [0,R)^d \cap \mathbb{Z}^d$ and, for $\lambda > 0$, let us define $V(x) = (|B_R|)^{-1}\lambda 1_{\{x\in B_R\}}$. It follows, cf. (II.2.13) for notation, that there exists $K_1 < \infty$ such that for all $f \in \ell^\infty(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$,

$$|(GV)f(x)| = \lambda \Big| \sum_{y\in B_R} \frac{g(x,y)f(y)}{|B_R|} \Big| \le K_1 \lambda R^{2-d} \|f\|_{\ell^\infty(\mathbb{Z}^d)}, \tag{II.3.6}$$

using that $g(x,y) \le C'|x-y|^{2-d}$, for $x,y \in \mathbb{Z}^d$. Hence, for $\lambda = \lambda_0 R^{d-2}$, with $\lambda_0 < K_1^{-1}$, one obtains that $\|GV\|_\infty < 1$, for all $R \ge 1$. In view of (II.3.6), applying Markov's inequality and using (II.2.14) then yields, for all $\lambda_0 < K_1^{-1}$ and $R \ge 1$,

$$\begin{aligned} &\widetilde{\mathbb{P}}^I\Big(\frac{1}{R^d} \sum_{x\in[0,R)^d\cap\mathbb{Z}^d} \widetilde{\ell}_{x,u} > (1+\varepsilon)u \Big) \\ &\leqslant \exp\Big\{ -\lambda_0 R^{d-2} u \Big((1+\varepsilon) - \big(1 + \sum_{n\geqslant 1} (K_1\lambda_0)^n\big)\Big)\Big\}. \end{aligned}$$

The right-hand side is bounded from above by $C\exp\big(-cR^{d-2}u\big)$ upon choosing $\lambda_0(\varepsilon) < K_1^{-1}$ small enough such that $\sum_{n\geqslant 1}(K_1\lambda_0)^n \le \varepsilon/2$. In a similar fashion one bounds for V, λ as above,

$$\begin{aligned} &\widetilde{\mathbb{P}}^I\Big(\frac{1}{R^d} \sum_{x\in[0,R)^d\cap\mathbb{Z}^d} \widetilde{\ell}_{x,u} < (1-\varepsilon)u \Big) \\ &= \widetilde{\mathbb{P}}^I\Big(\exp\Big\{ -\sum_{x\in\mathbb{Z}^d} V(x)\widetilde{\ell}_{x,u} \Big\} > e^{-(1-\varepsilon)\lambda u} \Big) \\ &\leqslant \exp\Big\{ \lambda_0 R^{d-2} u \Big((1-\varepsilon) - \big(1 - \sum_{n\geqslant 1} (K_1\lambda_0)^n\big)\Big)\Big\}, \end{aligned}$$

from which (II.3.5) readily follows. □

As a direct application of Lemmas II.3.2 and II.3.3, we derive lower bounds for the probabilities of the following events.

Definition II.3.4. For all u, $u' > 0$, and integer $R \ge 1$, the events $E_R^{u,u'}$ and $F_R^{u,u'}$ are defined as follows:

(a) $E_R^{u,u'}$ occurs if and only if for each $e \in \{0,1\}^d$, the set $(eR + [0,R)^d) \cap \widetilde{\mathcal{I}}^u$ contains a connected component $\mathcal{A}_e$ such that

$$\sum_{y\in\mathcal{A}_e\cap\mathbb{Z}^d} \widetilde{\ell}_{y,u} > \frac{3}{4} u' R^d,$$

and such that the components $(\mathcal{A}_e)_{e\in\{0,1\}^d}$ are all connected in $\widetilde{\mathcal{I}}^u \cap [0,2R)^d$.

(b) $F_R^{u,u'}$ occurs if and only if for all $e \in \{0,1\}^d$,

$$\sum_{y \in (eR+[0,R)^d) \cap \mathbb{Z}^d} \widetilde{\ell}_{y,u} < \frac{5}{4} u' R^d.$$

Note that for fixed $u' > 0$, and positive integer R, the events $(E_R^{u,u'})_{u>0}$ are increasing, i.e., there exists a measurable and increasing function $f_R^{u'} : [0,\infty)^{\widetilde{\mathbb{Z}}^d} \to \{0,1\}$ such that $1_{E_R^{u,u'}} = f_R^{u'}(\widetilde{\ell}_{\cdot,u})$ for all $u > 0$, and that the events $(F_R^{u,u'})_{u>0}$ are decreasing, i.e., the events $((F_R^{u,u'})^c)_{u>0}$ are increasing. The following consequence of Lemmas II.3.2 and II.3.3 is tailored to our purposes in the next section.

Corollary II.3.5. *Let $d \geqslant 3$ and $u_0 > 0$. There exist $\delta \in (0,1)$, positive and finite constants $C = C(d,u_0)$ and $c = c(d,u_0)$ such that for all $u \in (0,u_0]$ and $R \geqslant 1$,*

$$\widetilde{\mathbb{P}}^I\big(E_R^{u(1-\delta),u}\big) \geqslant 1 - C\exp\big(-cR^{1/7}u\big) \tag{II.3.7}$$

and

$$\widetilde{\mathbb{P}}^I\big(F_R^{u(1+\delta),u}\big) \geqslant 1 - C\exp\big(-cR^{d-2}u\big)\,. \tag{II.3.8}$$

Proof. Let $\delta = \frac{1}{6}$. We begin with (II.3.8). In view of Definition II.3.4, it follows from Lemma II.3.3 applied with $u(1+\delta)$ instead of u and translation invariance that for all $e \in \{0,1\}^d$ and $u > 0$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}^I\left(\sum_{x \in (eR+[0,R)^d)\cap\mathbb{Z}^d} \widetilde{\ell}_{x,u(1+\delta)} < \frac{5}{4}R^d u\right) \\
&\geqslant \widetilde{\mathbb{P}}^I\left(\frac{1}{R^d}\sum_{x \in (eR+[0,R)^d)\cap\mathbb{Z}^d} \widetilde{\ell}_{x,u(1+\delta)} < \frac{15}{14}u(1+\delta)\right) \\
&\overset{\text{(II.3.5)}}{\geqslant} 1 - C\exp(-cR^{d-2}u),
\end{aligned}$$

which is (II.3.8).

In order to obtain (II.3.7), fix any two constants $\varepsilon = \varepsilon(d) \in (0,1)$ and $\mu = \mu(d) \in (0,1)$ in such a way that $(1-8\varepsilon)^d(1-\mu)(1-\delta) = \frac{3}{4}$. For all $e \in \{0,1\}^d$, we define the inner boxes $\mathtt{B}_e(\varepsilon) = eR + [2\lfloor \varepsilon R\rfloor, R - 2\lfloor \varepsilon R \rfloor)^d$. It is sufficient to prove (II.3.7) for R satisfying $\varepsilon R \geqslant 1$, which we now tacitly assume. We then have $|\mathtt{B}_e(\varepsilon) \cap \mathbb{Z}^d| \cdot (1-\mu)(1-\delta) \geqslant \frac{3}{4}R^d$, where $|A|$ denotes the cardinality of $A \subset \mathbb{Z}^d$. According to Lemma II.3.3,

$$\begin{aligned}
&\widetilde{\mathbb{P}}^I \left(\sum_{x \in \mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d} \widetilde{\ell}_{x,u(1-\delta)} > \frac{3}{4} R^d u \right) \\
&\geqslant \widetilde{\mathbb{P}}^I \left(\frac{1}{|\mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d|} \sum_{x \in \mathsf{B}_e(\varepsilon) \cap \mathbb{Z}^d} \widetilde{\ell}_{x,u(1-\delta)} > (1-\mu) u (1-\delta) \right) \\
&\geqslant 1 - C \exp\left(-c R^{d-2} u\right).
\end{aligned}$$

We now define $\mathcal{A}_e^1 = \mathsf{B}_e(\varepsilon) \cap \widetilde{\mathcal{I}}^{u(1-\delta)}$. According to Lemma II.3.2, for every $e \in \{0,1\}^d$, all the vertices of $\mathcal{A}_e^1$ are connected in $\widetilde{\mathcal{I}}^{u(1-\delta)} \cap (eR + [\lfloor \varepsilon R \rfloor, R - \lfloor \varepsilon R \rfloor)^d)$ with probability at least $1 - C\exp(-cR^{1/7}u)$, and on the corresponding event we define $\mathcal{A}_e \subset \widetilde{\mathcal{I}}^{u(1-\delta)} \cap (eR + [\lfloor \varepsilon R \rfloor, R - \lfloor \varepsilon R \rfloor)^d)$ such that $\mathcal{A}_e^1 \subset \mathcal{A}_e$ and $\mathcal{A}_e$ is connected.

Still according to Lemma II.3.2, all the $\mathcal{A}_e$ for $e \in \{0,1\}^d$ are connected with each other in $\widetilde{\mathcal{I}}^{u(1-\delta)} \cap [0, 2R)^d$ with probability at least $1 - C\exp(-cR^{1/7}u)$, which gives (II.3.7). □

Since the events $E_R^{u,u'}$ and $F_R^{u,u'}$ are defined in terms of local times and not in terms of the occupation field $\left(1_{\{x \in \widetilde{\mathcal{I}}^u\}}\right)_{x \in \mathbb{Z}^d}$, we now give a slightly different version of the decoupling inequality presented in [68] valid for the local times on the cable system. This inequality will later enable us to use $E_R^{u,u'}$ and $F_R^{u,u'}$ as seed events of a suitable multi-scale argument. In what follows, let $\widetilde{\mathbf{Q}}^u$, $u > 0$, be the law on $\overline{\Omega} = [0, \infty)^{\widetilde{\mathbb{Z}}^d}$ of the local times $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathbb{Z}}^d}$ of random interlacements on the cable system $\widetilde{\mathbb{Z}}^d$, and let $(p_x)_{x \in \widetilde{\mathbb{Z}}^d}$ denote the canonical coordinate functions on $\overline{\Omega}$, i.e., for all $f \in \overline{\Omega}$ and $x \in \widetilde{\mathbb{Z}}^d$, $p_x(f) = f(x)$.

Theorem II.3.6. *Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two measurable non-intersecting subsets of $\widetilde{\mathbb{Z}}^d$. Assume that $s := d(\widetilde{A}_1, \widetilde{A}_2) \geqslant 1$, and that the minimum r of the diameters of $\widetilde{A}_1$ and $\widetilde{A}_2$ is finite. Then there exist $\kappa_0(d)$ and $\kappa_1(d)$ such that for all $u > 0$ and $\varepsilon \in (0,1)$, for any functions $f_i : \overline{\Omega} \to [0,1]$ which are $\sigma(p_x,\ x \in \widetilde{A}_i)$ measurable for each $i \in \{1,2\}$, and which are both increasing or both decreasing,*

$$\widetilde{\mathbf{Q}}^u[f_1 f_2] \leqslant \widetilde{\mathbf{Q}}^{u(1\pm\varepsilon)}[f_1] \widetilde{\mathbf{Q}}^{u(1\pm\varepsilon)}[f_2] + \kappa_0 (r+s)^d \exp(-\kappa_1 \varepsilon^2 u s^{d-2}), \qquad \text{(II.3.9)}$$

where the plus sign corresponds to the case where the f_i's are increasing and the minus sign to the case where the f_i's are decreasing.

Proof. Let A_1 and A_2 be the smallest subsets of $\mathbb{Z}^d$ such that for all $i \in \{1,2\}$, and all $x \in \widetilde{A}_i$, there exist $y, z \in A_i$ such that $x \in \overline{I}_{\{y,z\}}$. Note that $\widetilde{A}_i \cap \mathbb{Z}^d \subset A_i$. Since $d(\widetilde{A}_1, \widetilde{A}_2) \geqslant 1$, the sets A_1 and A_2 are not intersecting (recall that the distance between two neighbors of $\mathbb{Z}^d$ is $\frac{1}{2}$). For two measures μ_1 and μ_2, we say that $\mu_1 \leqslant \mu_2$ if $\mu_2 - \mu_1$ is a non-negative measure. The proof of the main decoupling result, Theorem 2.1 in [68], see in particular Section 5 therein, implies that, for each $u > 0$, there exists a coupling $\mathbb{Q}_u^I$ between the random interlacement process ω on $\mathbb{Z}^d$ and two independent Poisson point processes ω_1 and ω_2 having the same law as ω, such that, for $B \subset \mathbb{Z}^d$, denoting by $(\omega^u)_{|B}$ the point process consisting of the restriction to B of the trajectories in ω^u which hit B,

$$\begin{aligned}&\mathbb{Q}_u^I\big[(\omega_i^{u(1-\varepsilon)})_{|A_i} \leqslant (\omega^u)_{|A_i} \leqslant (\omega_i^{u(1+\varepsilon)})_{|A_i},\ i=1,2\big]\\ &\geqslant 1-\kappa_0(r+s)^d\exp(-\kappa_1\varepsilon^2 us^{d-2}).\end{aligned} \tag{II.3.10}$$

For each $u > 0$ and $i \in \{1,2\}$, under an extended probability $\widetilde{\mathbb{Q}}_u^I$, one then constructs an interlacement process $\widetilde{\omega}_i^{u(1-\varepsilon)}$ at level $u(1-\varepsilon)$ on the cable system by adding independent Brownian excursions on the edges for every trajectory in the support of the random interlacement process $\omega_i^{u(1-\varepsilon)}$ on $\mathbb{Z}^d$, as in the construction of the diffusion $\widetilde{X}$, see the beginning of Section II.2.

We now construct a random interlacement process $\widetilde{\omega}^u$ at level u using $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and ω^u. Its trajectories are the trajectories of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ which have a projection on $\mathbb{Z}^d$ already contained in ω^u (i.e., all the trajectories of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ on the event in (II.3.10)) and the trajectories of ω^u which are not already in $\omega_i^{u(1-\varepsilon)}$, lifted to $\widetilde{\mathbb{Z}}^d$ using additional independent (of $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and ω^u) Brownian excursions on the edges. We repeat this construction to obtain a random interlacement process $\widetilde{\omega}_i^{u(1+\varepsilon)}$ at level $u(1+\varepsilon)$ in a similar way from $\widetilde{\omega}^u$ and $\omega_i^{u(1+\varepsilon)}$. Then, an analogue of (II.3.10) holds for these processes $\widetilde{\omega}^u$, $\widetilde{\omega}_i^{u(1-\varepsilon)}$ and $\widetilde{\omega}_i^{u(1+\varepsilon)}$ under $\widetilde{\mathbb{Q}}_u^I$. In particular, denoting by $\widetilde{\ell}_{x,u}$, $\widetilde{\ell}^i_{x,u(1-\varepsilon)}$ and $\widetilde{\ell}^i_{x,u(1+\varepsilon)}$ their respective local time fields on the cable system, see (II.2.11), it follows that

$$\widetilde{\mathbb{Q}}_u^I\big[\widetilde{\ell}^i_{x,u(1-\varepsilon)} \leqslant \widetilde{\ell}_{x,u} \leqslant \widetilde{\ell}^i_{x,u(1+\varepsilon)},\ x\in\widetilde{A}_i,\ i=1,2\big] \geqslant 1-\kappa_0(r+s)^d\exp(-\kappa_1\varepsilon^2 us^{d-2}) \tag{II.3.11}$$

The inequalities in (II.3.9) are a direct consequence of (II.3.11). □

II.4 Percolation for the truncated level set

In this section, we prove Theorem II.3.1: for each $h > 0$, there exists a finite constant $K(h)$ such that the level set of the Gaussian free field on the cable system truncated above level $-h$ and below level $K(h)$ contains an unbounded connected component. We will actually show a similar statement for random interlacements and use the coupling from Theorem II.2.2 to obtain Theorem II.3.1. The corresponding statement for random interlacement, see Proposition II.4.1, essentially asserts that one can intersect the continuous interlacement set $\widetilde{\mathcal{I}}^u$, the set $\{x \in \mathbb{Z}^d;\, |\varphi_x| < K\}$ and a Bernoulli family on the edges with parameter p and still retain an unbounded connected component in $\widetilde{\mathbb{Z}}^d$ for sufficiently large K and p close enough to 1. The proof of this statement bears similarities to the proof of Theorem 2.1 in [74], where it is shown that the intersection of $\mathcal{I}^u$ and a Bernoulli family with parameter p on $\mathbb{Z}^d$, not necessarily independent from $\mathcal{I}^u$, contains an infinite connected component in $\mathbb{Z}^d$ for large enough p.

Henceforth, for a given $p \in (0,1)$ (and $d \geqslant 3$), let $\widetilde{\mathbb{Q}}^p$ be any coupling between a Gaussian free field $\widetilde{\varphi}$, a random interlacement process $\widetilde{\omega}$ and a family of independent Bernoulli random variables on the edges $\mathcal{B}^p = (\theta^p_e)_{e\in E}$ with parameter p, i.e.,

$$\begin{gathered}\text{under } \widetilde{\mathbb{Q}}^p \text{, the law of } (\widetilde{\varphi}_x)_{x\in\widetilde{\mathbb{Z}}^d} \text{ is } \widetilde{\mathbb{P}}^G\text{, the process } \widetilde{\omega} \text{ has the same}\\ \text{law as under } \widetilde{\mathbb{P}}^I \text{ and } (\theta^p_e)_{e\in E} \text{ is an i.i.d. family of } \{0,1\}\text{-valued}\\ \text{random variables with } \widetilde{\mathbb{Q}}^p(\theta^p_e = 1) = p \text{ for each } e \in E.\end{gathered} \tag{II.4.1}$$

In particular, $\widetilde{\varphi}$, $\widetilde{\omega}$ and $\mathcal{B}^p$ need not be independent, and in fact, we will later use a coupling such that (II.2.16) holds. For any level $u > 0$, we define the random interlacement set $\widetilde{\mathcal{I}}^u$ as in (II.2.12) and the local times $(\widetilde{\ell}_{x,u})_{x\in\widetilde{\mathbb{Z}}^d}$ as in (II.2.11) in terms of $\widetilde{\omega}$. We further denote by φ the restriction of $\widetilde{\varphi}$ to $\mathbb{Z}^d$ and by $\mathcal{I}^u$ the restriction of $\widetilde{\mathcal{I}}^u$ to $\mathbb{Z}^d$.

Proposition II.4.1. ($d \geqslant 3$, $u_0 > 0$, (II.4.1))

There exist positive constants C_1, c_1, C_1' and c_1', only depending on d and u_0, satisfying $C_1 u_0^{-c_1} \geqslant 1$ and $C_1' u_0^{c_1'} < 1$, such that for all $u \in (0, u_0]$, with

$$\widetilde{K}(u) \stackrel{\text{def.}}{=} \sqrt{\log\left(\frac{C_1}{u^{c_1}}\right)} \quad \text{and} \quad p(u) \stackrel{\text{def.}}{=} 1 - C_1' u^{c_1'}, \tag{II.4.2}$$

there exists $L_0(u) > 0$ such that, if $p \in [p(u), 1]$, then $\widetilde{\mathbb{Q}}^p$-a.s. the set

$$\begin{aligned}\widetilde{A}'_{u,p} = {} & \Big(\widetilde{\mathcal{I}}^u \setminus \mathcal{I}^u\Big)\\ & \cup \Big\{x \in \mathcal{I}^u;\, |\varphi_x| \leqslant \widetilde{K}(u) \text{ and } \forall\, y \sim x,\, |\varphi_y| \leqslant \widetilde{K}(u) \text{ and } \theta^p_{\{x,y\}} = 1\Big\}\end{aligned} \tag{II.4.3}$$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0(u))^{d-2}$.

We now comment on (II.4.3). First, note that $\mathcal{I}^u \subset \mathbb{Z}^d$, so saying that $\widetilde{A}'_{u,p}$ contains an unbounded connected component implies that $\widetilde{A}'_{u,p} \cap \mathbb{Z}^d$ contains an infinite path such that all the edges of this path are in $\widetilde{\mathcal{I}}^u$, and for all vertices

x on this path and all $y \sim x$, $|\varphi_x| \leqslant \widetilde{K}(u)$ and $\theta^p_{\{x,y\}} = 1$. Proposition II.4.1 is true for any choice of coupling probability $\widetilde{\mathbb{Q}}^p$ satisfying (II.4.1). Once its proof is completed, we will choose the coupling introduced in (II.2.15). This will automatically enforce the lower bound $-h$ required for the proof of Theorem II.3.1, since $\widetilde{\mathcal{I}}^u \subset \{x \in \widetilde{\mathbb{Z}}^d;\, \widetilde{\varphi}_x > -\sqrt{2u}\}$, see (II.2.16). A good choice of $\theta^p_{\{x,y\}}$, cf. Lemma II.4.9 below, will then allow to control the height of the field along the edges. To this effect, (II.4.3) essentially guarantees that on $\widetilde{A}'_{u,p}$ we are dealing with Brownian bridges whose boundary values are uniformly bounded.

The proof of Proposition II.4.1 follows a strategy very similar to the proof of Theorem 2.1 in [74], but we need to pay diligent attention to the dependence on u in order to obtain the explicit bounds (II.4.2). We use a renormalization scheme akin to the one introduced in Section 4 of [74], which uses a sprinkling technique developed in [93] and later improved in [95] and [68]. For $n \geqslant 0$ and $L_0 \geq 1$, we define the geometrically increasing sequence

$$L_n = l_0^n L_0, \quad \text{where } l_0 = 4l(d) \text{ and } l(d) = 4(5 \cdot 4^d + 1) \tag{II.4.4}$$

and the coarse-grained lattice model

$$\mathbb{G}_0^{L_0} = L_0\mathbb{Z}^d \text{ and } \mathbb{G}_n^{L_0} = L_n\mathbb{Z}^d \subset \mathbb{G}_{n-1}^{L_0} \text{ for } n \geqslant 1.$$

Note that, albeit only implicitly, the sequence L_n depends on the choice of L_0, which is the only parameter in this scheme. For $x \in \mathbb{G}_n^{L_0}$, we further introduce the boxes

$$\Lambda_{x,n}^{L_0} = \mathbb{G}_{n-1}^{L_0} \cap (x + [0, L_n)^d), \tag{II.4.5}$$

and note that $\{\Lambda_{x,n}^{L_0};\, x \in \mathbb{G}_n^{L_0}\}$ forms a partition of $\mathbb{G}_{n-1}^{L_0}$. For a given collection of events indexed by $\mathbb{G}_0^{L_0}$, that we denote by $A = (A_x)_{x\in\mathbb{G}_0^{L_0}}$, we define recursively the events $G_{x,n}^{L_0}(A)$ such that $G_{x,0}^{L_0} = A_x$ for all $x \in \mathbb{G}_0^{L_0}$, and for all $n \geqslant 1$ and $x \in \mathbb{G}_n^{L_0}$,

$$G_{x,n}^{L_0}(A) = \bigcup_{\substack{x_1,x_2\in\Lambda_{x,n}^{L_0}\\ |x_1-x_2|_\infty \geqslant \frac{L_n}{l(d)}}} G_{x_1,n-1}^{L_0}(A) \cap G_{x_2,n-1}^{L_0}(A), \tag{II.4.6}$$

where $|\cdot|_\infty$ stands for the ℓ^∞-distance on $\mathbb{Z}^d$. For each $x \in \mathbb{Z}^d$, let T_x be the translation operator on the space of point measures on W^*, the space of doubly infinite trajectories on $\widetilde{\mathbb{Z}}^d$ modulo time-shift such that, if μ is such a measure, then $T_x(\mu)$ is the point measure where each trajectory in the support of μ has been translated by x. Moreover, in a slight abuse of notation, let τ_x be defined by

$$\widetilde{\omega} \circ \tau_x = T_x(\widetilde{\omega}). \tag{II.4.7}$$

We introduce a family of events on the space Ω_{coup} on which $\widetilde{\mathbb{Q}}^p$, cf. (II.4.1), is defined. We say that an event $A \in \sigma(\widetilde{\varphi}_z,\, z \in \widetilde{\mathbb{Z}}^d)$ is increasing if there exists an increasing and measurable function $f : \Omega_0 \to \{0,1\}$ such that, $\widetilde{\mathbb{Q}}^p$-a.s., $1_A = f(\widetilde{\varphi})$, and decreasing if A^c is increasing. Recall that the events $E_{L_0}^{u,u'}$ and $F_{L_0}^{u,u'}$ from Definition II.3.4 are respectively increasing and decreasing, and we will from now on tacitly consider them as subsets of Ω_{coup}.

Definition II.4.2. For each $u > 0$, integer $L_0 \geq 1$, $K > 0$ and $p \in [0,1]$ let

(a) $(\mathbf{E}_x^{L_0,u})_{x \in \mathbb{G}_0^{L_0}}$ be the family of increasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{E}_x^{L_0,u} = \tau_x^{-1}\left(E_{L_0}^{u,u}\right)$ occurs,

(b) $(\mathbf{F}_x^{L_0,u})_{x \in \mathbb{G}_0^{L_0}}$ be the family of decreasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{F}_x^{L_0,u} = \tau_x^{-1}\left(F_{L_0}^{u,u}\right)$ occurs,

(c) $(\mathbf{C}_x^{L_0,K})_{x \in \mathbb{G}_0^{L_0}}$ be the family of decreasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{C}_x^{L_0,K}$ occurs if and only if for all $y \in (x + [-1, 2L_0+1)^d) \cap \mathbb{Z}^d$, we have $\varphi_y \leqslant K$,

(d) $(\widehat{\mathbf{C}}_x^{L_0,K})_{x \in \mathbb{G}_0^{L_0}}$ be the family of increasing events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\widehat{\mathbf{C}}_x^{L_0,K}$ occurs if and only if for all $y \in (x + [-1, 2L_0+1)^d) \cap \mathbb{Z}^d$, we have $\varphi_y \geqslant -K$,

(e) $(\mathbf{D}_x^{L_0,p})_{x \in \mathbb{G}_0^{L_0}}$ be the family of events such that, for all $x \in \mathbb{G}_0^{L_0}$, the event $\mathbf{D}_x^{L_0,p}$ occurs if and only if for all $e \in (x + [-1, 2L_0+1)^d) \cap E$, we have $\theta_e^p = 1$.

A vertex $x \in \mathbb{G}_0^{L_0}$ is called a good (L_0, u, K, p) vertex if

$$\mathbf{C}_x^{L_0,K} \cap \widehat{\mathbf{C}}_x^{L_0,K} \cap \mathbf{D}_x^{L_0,p} \cap \mathbf{E}_x^{L_0,u} \cap \mathbf{F}_x^{L_0,u} \tag{II.4.8}$$

occurs, and otherwise a bad (L_0, u, K, p) vertex.

The reason for the choices in Definition II.3.4 and (II.4.8), with regards to Proposition II.4.1, comes in the following.

Lemma II.4.3. $(u > 0,\ L_0 \geqslant 1,\ p \in (0,1])$

If $(x_0, x_1, \dots)$ is an unbounded nearest neighbor path of good $(L_0, u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0}$, then the set $\bigcup_{i=0}^{\infty}(x_i + [0, 2L_0)^d)$ contains an unbounded connected path in $\widetilde{A}'_{u,p}$, cf. (II.4.3).

Proof. Let x, y be two good $(L_0, u, \widetilde{K}(u), p)$ vertices and neighbours in $\mathbb{G}_0^{L_0}$, and assume that there exists $e \in \{0,1\}^d$ such that $x + eL_0 = y$. Since $\mathbf{E}_x^{L_0,u}$ holds, there exist two random sets $\mathcal{A}_{x,0} \subset \widetilde{\mathcal{I}}^u \cap (x + [0, L_0)^d)$ and $\mathcal{A}_{x,e} \subset \widetilde{\mathcal{I}}^u \cap (x + eL_0 + [0, L_0)^d)$ which are connected in $\widetilde{\mathcal{I}}^u \cap (x + [0, 2L_0)^d)$, and such that the sum of the local times on the vertices of each of those two sets is larger than $\frac{3}{4}uL_0^d$. Moreover, since $\mathbf{F}_y^{L_0,u}$ occurs, the sum of the local times on the vertices of $\mathcal{A}_{x,e} \cup \mathcal{A}_{y,0}$ is smaller than $\frac{5}{4}uL_0^d$ because $\mathcal{A}_{x,e} \cup \mathcal{A}_{y,0} \subset \widetilde{\mathcal{I}}^u \cap (y + [0, L_0)^d)$. Hence, $\mathcal{A}_{x,e} \cap \mathcal{A}_{y,0} \neq \varnothing$, and this implies that $\mathcal{A}_{x,0}$ is connected to $\mathcal{A}_{y,0}$ in $\widetilde{\mathcal{I}}^u \cap (x + [0, 2L_0)^d)$.

Applying the above to each of the neighbors in our path $(x_0, x_1, \dots)$, we get that for all $i \in \mathbb{N}_0$, $\mathcal{A}_{x_i,0}$ is connected to $\mathcal{A}_{x_{i+1},0}$ in $\widetilde{\mathcal{I}}^u \cap ((x_i + [0, 2L_0)^d) \cup (x_{i+1} + [0, 2L_0)^d))$. Thus, one can find an unbounded connected path in $\widetilde{\mathcal{I}}^u \cap \bigcup_{i=0}^{\infty}(x_i + [0, 2L_0)^d)$, and this path is actually in $\widetilde{A}'_{u,p}$ since $|\varphi_x| \leqslant \widetilde{K}(u)$ and $\theta_e^p = 1$ for all $x, e \in \bigcup_{i=1}^{\infty}(x_i + [-1, 2L_0 + 1)^d)$ by Definition II.4.2, (c), (d), (e). □

To prove that an unbounded nearest neighbor path of good $(L_0, u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0}$ exists for a suitable choice of the parameters, we pair our good (seed) events with the renormalization scheme (II.4.6) to show that, if being a good seed is typical, i.e., if it occurs with probability sufficiently close to 1, then the probability of being good "at level n" cf. (II.4.6) and (II.4.21) below, is overwhelming. The respective bounds for all events of interest, cf. Definition II.4.2, can be found in Lemmas II.4.4, II.4.6 and II.4.7 below. We first consider the events $(\mathbf{E}_x^{L_0,u})_{x\in\mathbb{Z}^d}$ and $(\mathbf{F}_x^{L_0,u})_{x\in\mathbb{Z}^d}$, and take advantage of Corollary II.3.5 and Theorem II.3.6 to show the following.

Lemma II.4.4. $(u_0 > 0)$

There exist $C_2 = C_2(d, u_0)$ and $C_2' = C_2'(d, u_0)$ such that for all $u \in (0, u_0]$ and $L_0 \geq 1$ with $L_0^{1/7}u \geqslant C_2$,

$$\widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0}\left((\mathbf{E}^{L_0,u})^c\right)\right] \leqslant 2^{-2^n}, \tag{II.4.9}$$

and for all $u \in (0, u_0]$ and $L_0 \geq 1$ with $L_0^{1/7}u \geqslant C_2'$,

$$\widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0}\left((\mathbf{F}^{L_0,u})^c\right)\right] \leqslant 2^{-2^n}. \tag{II.4.10}$$

Proof. We only prove (II.4.9). The proof of (II.4.10) is similar. Fix $\delta \in (0,1)$ as in Corollary II.3.5, and let $\delta' \in (0,1)$ be small enough such that

$$u_n := \frac{u(1-\delta)}{\prod_{k=1}^{n-1}\left(1 - \frac{\delta'}{2^k}\right)} < u, \ \forall n \geqslant 1 \tag{II.4.11}$$

(with $u_1 = u(1-\delta)$). For all $x \in \mathbb{G}_0^{L_0}$, let $\mathbf{E}_x^{L_0,u,u'} = \tau_x^{-1}(E_{L_0}^{u,u'})$, cf. (II.4.7) and Definition II.3.4, and note that $\mathbf{E}_x^{L_0,u,u} = \mathbf{E}_x^{L_0,u}$. For all $u \in (0, u_0]$, $L_0 \geqslant 1$, positive integer n, $i \in \{1,2\}$ and $x_i \in \mathbb{G}_n^{L_0}$ such that $|x_1 - x_2|_\infty \geqslant \frac{L_{n+1}}{l(d)} = 4L_n$, the events $G^{L_0}_{x_i,n-1}\left((\mathbf{E}^{L_0,u',u})^c\right)$ are $\sigma(\widetilde{\ell}_{z,u},\ z \in x_i + [0, L_n + L_0)^d)$ measurable for all $u' > 0$, and, defining $r_n := \frac{L_n+L_0}{2}$,

$$s_n := d\left(x_1 + [0,2r_n)^d, x_2 + [0,2r_n)^d]\right) \geqslant d(x_1,x_2) - r_n \geqslant L_n.$$

By Theorem II.3.6 applied with $\varepsilon = 1 - \frac{u_n}{u_{n+1}} = \delta' 2^{-n}$, and since the events $(\mathbf{E}^{L_0,u',u})^c$ are decreasing, there exist two constants C and c independent of u, n and L_0 such that

$$\begin{aligned}
&\widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x_1,n-1}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right) \cap G^{L_0}_{x_2,n-1}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right)\right] \\
&\quad \leqslant \widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x_1,n-1}\left((\mathbf{E}^{L_0,u_n,u})^c\right)\right]\widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x_2,n-1}\left((\mathbf{E}^{L_0,u_n,u})^c\right)\right] \\
&\quad + C(s_n + r_n)^d \exp\left(-cus_n^{d-2}4^{-n}\right).
\end{aligned} \tag{II.4.12}$$

We have chosen $l_0^{d-2} > 8$, see (II.4.4), whence for $L_0^{1/7}u \geqslant c(d,u_0)$,

$$l_0^{2d} C(s_n + L_n + L_0)^d \exp\left(-cus_n^{d-2}4^{-n}\right) \leqslant C' \exp\left(-cuL_0^{1/7}2^n\right) \leqslant \frac{1}{(4l_0^{2d})2^{2^{n+1}}}. \tag{II.4.13}$$

We now prove by induction over n that for all $x \in \mathbb{G}_n^{L_0}$, and all $u \in (0,u_0]$,

$$\widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x,n}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right)\right] \leqslant \frac{1}{(2l_0^{2d})2^{2^n}}, \quad \text{if } L_0^{1/7}u \geq c(d,u_0). \tag{II.4.14}$$

For $n = 0$, the bound on the right-hand side of (II.4.14) is purely numerical. Thus, it is clear from Corollary II.3.5, and since $G^{L_0}_{x,0}\left((\mathbf{E}^{L_0,u_1,u})^c\right) = (\mathbf{E}_x^{L_0,u_1,u})^c$, see above (II.4.6), that if one takes $L_0^{1/7}u$ large enough (only depending on u_0 and d), then (II.4.14) holds for $n = 0$ on account of (II.3.7). Suppose now it holds for $n - 1 \geq 0$. Then, according to (II.4.6)

$$\begin{aligned}
&\widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x,n}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right)\right] \\
&\leqslant \sum_{\substack{x_1,x_2 \in \Lambda^{L_0}_{x,n} \\ |x_1-x_2| \geqslant \frac{L_n}{l(d)}}} \widetilde{\mathbb{Q}}^p\left[G^{L_0}_{x_1,n-1}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right) \cap G^{L_0}_{x_2,n-1}\left((\mathbf{E}^{L_0,u_{n+1},u})^c\right)\right] \\
&\leqslant \frac{1}{(2l_0^{2d})2^{2^n}},
\end{aligned}$$

where the last equality follows from (II.4.12), (II.4.13), the induction hypothesis and $|\Lambda^{L_0}_{x,n}| \leqslant l_0^d$, and (II.4.14) follows. The claim (II.4.9) then follows from (II.4.11), (II.4.14) and the fact that the $\left(\mathbf{E}^{L_0,u',u}\right)^c$ are decreasing events. □

We now turn to the Gaussian free field part $\mathbf{C}^{L_0,K}$ and $\widehat{\mathbf{C}}^{L_0,K}$, see (c) and (d) in Definition II.4.2, of the good events in (II.4.8). Sprinkling techniques have been used successfully in investigating level set percolation of the Gaussian free field, see for example [81], [78] or [29], and also [80] with regard to non-Gaussian measures. These techniques imply similar results as for random interlacements, and this is mainly due to the fact that decoupling inequalities as (II.3.9) also hold for the Gaussian free field. With hopefully obvious notation, in writing $\Phi + c$ below, with Φ as in (II.1.2), we mean the field whose value is shifted by $c \in \mathbb{R}$ everywhere.

Theorem II.4.5 ([67, Corollary 1.3] and thereafter)**.** *Let A_1 and A_2 be two non intersecting subsets of $\mathbb{Z}^d$, define $s = d(A_1, A_2)$ and assume that the minimum r of their diameters is finite. Then, there exist positive constants $\kappa_0'(d)$ and $\kappa_1'(d)$ such that, for all $\varepsilon \in (0,1)$, and any two functions $f_i : \mathbb{R}^{\mathbb{Z}^d} \to [0,1]$ which are $\sigma(\Phi_x,\, x \in A_i)$ measurable for each $i \in \{1,2\}$, and either both increasing or both decreasing,*

$$\mathbb{E}^G[f_1(\Phi)f_2(\Phi)] \leqslant \mathbb{E}^G[f_1(\Phi \pm \varepsilon)]\mathbb{E}^G[f_2(\Phi \pm \varepsilon)] + \kappa_0'(r+s)^d \exp(-\kappa_1'\varepsilon^2 s^{d-2}), \tag{II.4.15}$$

where the plus sign corresponds to the case where the f_i's are increasing and the minus sign to the case where the f_i's are decreasing.

Theorem 1.2 in [67] gives a slightly better inequality, but (II.4.15) will be sufficient for our purposes, and readily yields the following analogue of Lemma II.4.4 for the events pertaining to the free field.

Lemma II.4.6. *There exist constants $C_3(d) > 1$ and $C_3'(d) > 0$ such that for all $L_0 \geqslant C_3$ and $K > 0$ with*

$$K \geqslant C_3' \sqrt{\log(L_0)}, \tag{II.4.16}$$

one has

$$\widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0}\left((\mathbf{C}^{L_0,K})^c \right) \right] \leqslant 2^{-2^n} \quad \text{and} \quad \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0}\left((\widehat{\mathbf{C}}^{L_0,K})^c \right) \right] \leqslant 2^{-2^n}. \tag{II.4.17}$$

Proof. One knows from (2.35) and (2.38) in [81] that if $K > C\sqrt{\log(L_0)}$ for some constant C large enough,

$$\begin{aligned} \widetilde{\mathbb{Q}}^p \left[\left(\widehat{\mathbf{C}}_0^{L_0,K} \right)^c \right] = \widetilde{\mathbb{Q}}^p \left[\left(\mathbf{C}_0^{L_0,K} \right)^c \right] &= \widetilde{\mathbb{Q}}^p \Big(\sup_{x \in [-1, 2L_0+1)^d} \varphi_x > K \Big) \\ &\leqslant e^{-\frac{\left(K - C\sqrt{\log(L_0)}\right)^2}{2g(0)}}. \end{aligned} \tag{II.4.18}$$

The claim (II.4.17) now follows by induction over n from (II.4.18) and Theorem II.4.5 in exactly the same way as Lemma II.4.4 was obtained from Corollary II.3.5 and Theorem II.3.6. $\square$

Finally, we collect a simple estimate for the Bernoulli part of our good events $\mathbf{D}^{L_0,p}$, see part (e) in Definition II.4.2.

Lemma II.4.7 ([74, Lemma 4.7]). *There exists $C_4 = C_4(d)$ such that for all $L_0 \geqslant 1$ and $p \in (0,1)$ satisfying*

$$p \geqslant \exp\left(-\frac{C_4}{L_0^d}\right), \tag{II.4.19}$$

one has

$$\widetilde{\mathbb{Q}}^p\left[G_{0,n}^{L_0}\left((\mathbf{D}^{L_0,p})^c\right)\right] \leqslant 2^{-2^n}. \tag{II.4.20}$$

The bounds of Lemmas II.4.4, II.4.6 and II.4.7 allow for a proof of Proposition II.4.1 by means of a standard duality argument. In view of Lemma II.4.3, this requires an estimate on the probability to see certain long (dual) paths. The relevant events, see (II.4.22), can be suitably expressed in terms of bad vertices at level n, as Lemma II.4.8 asserts.

Recall the definition of good (L_0, u, K, p) vertices in (II.4.8). For $n \geq 0$, we call $x \in \mathbb{G}_n^{L_0}$ a bad $n - (L_0, u, K, p)$ vertex if the event

$$G_{x,n}^{L_0}\big((\mathbf{C}^{L_0,K})^c\big) \cup G_{x,n}^{L_0}\big((\widehat{\mathbf{C}}^{L_0,K})^c\big) \cup G_{x,n}^{L_0}\big((\mathbf{D}^{L_0,p})^c\big) \cup G_{x,n}^{L_0}\big((\mathbf{E}^{L_0,u})^c\big) \cup G_{x,n}^{L_0}\big((\mathbf{F}^{L_0,u})^c\big) \tag{II.4.21}$$

occurs, and a good $n - (L_0, u, K, p)$ vertex otherwise. Note that a good $0 - (L_0, u, K, p)$ vertex is simply a good (L_0, u, K, p) vertex. We say that $(x_0, x_1, \dots, x_n, \dots)$ is a $*$-path in $\mathbb{G}_0^{L_0}$ if for all $i \in \{0, 1, \dots\}$, $x_i \in \mathbb{G}_0^{L_0}$ and $\|x_i - x_{i+1}\|_\infty = L_0$. For each $u, K > 0$, integer $L_0 \geq 1$, $p \in (0,1)$, $0 < M < N$ with M, N multiples of L_0, and $x \in \mathbb{G}_0^{L_0}$, let

$$\begin{aligned} H_M^N(x; L_0, u, K, p) = \big\{&(x + [-M, M]^d) \text{ is connected to } (x + \partial[-N, N]^d) \text{ by} \\ &\text{a } *\text{-path of bad } (L_0, u, K, p) \text{ vertices in } \mathbb{G}_0^{L_0}\big\}. \end{aligned} \tag{II.4.22}$$

Here, $\partial[-N, N]^d$ denotes the boundary of the set $[-N, N]^d$, which intersects $\mathbb{G}_0^{L_0}$ since N is a multiple of L_0. The following lemma asserts that $H_{L_n}^{2L_n}(x; L_0, u, K, p)$ can only happen if there is a bad $n - (L_0, u, K, p)$ vertex in the box of radius $2L_n$ around x.

Lemma II.4.8. *For all integers $n \geq 0$ and $L_0 \geq 1$, $u, K > 0$, $p \in (0,1)$, and $x \in \mathbb{G}_n^{L_0}$,*

$$H_{L_n}^{2L_n}(x; L_0, u, K, p) \subset \bigcup_{y \in \mathbb{G}_n^{L_0} \cap (x + [-2L_n, 2L_n)^d)} \{y \text{ is } n - (L_0, u, K, p) \text{ bad}\}. \tag{II.4.23}$$

Proof. This is a consequence of Lemma 4.4 of [78] (with $N \equiv 5$, $r \equiv l(d)$ and L_0, l_0 as in (II.4.4) above). We include the proof for the reader's convenience. We proceed by induction over n: it is clear that (II.4.23) is true for $n = 0$, and we assume that it holds for any choice of x up to level $n-1$. If $H_{L_n}^{2L_n}(x; L_0, u, K, p)$ occurs, there exists a $*$-path π of bad (L_0, u, K, p) vertices in $\mathbb{G}_0^{L_0}$ from $(x + [-L_n, L_n]^d)$ to $(x + \partial[-2L_n, 2L_n]^d)$. This path intersects the concentric ℓ^∞-spheres $(x + \partial[-L_n - 16iL_{n-1}, L_n + 16iL_{n-1}]^d)$ for all $i \in \{0, \dots, m-1\}$, where $m = 5 \cdot 4^d + 1$ (recall that $l_0 = 16m$). In view of (II.4.4), for all $i \in \{0, \dots, m-1\}$, one can thus find $y_i \in \mathbb{G}_{n-1}^{L_0} \cap (x + \partial[-L_n - 16iL_{n-1}, L_n + 16iL_{n-1}]^d)$ such that $\pi \cap (y_i + [-L_{n-1}, L_{n-1}]^d) \neq \varnothing$.

For each $i \in \{0, \dots, m-1\}$, we clearly have $(y_i + [-2L_{n-1}, 2L_{n-1}]^d) \subset (x + [-2L_n, 2L_n]^d)$, and so the connected $*$-path π in $\mathbb{G}_0^{L_0}$ connects $(y_i + [-L_{n-1}, L_{n-1}]^d)$ to $(y_i + \partial[-2L_{n-1}, 2L_{n-1}]^d)$, and thus the induction hypothesis implies that there exists $z_i \in (y_i + [-2L_{n-1}, 2L_{n-1})^d)$ which is $(n-1) - (L_0, u, K, p)$ bad, and in particular $z_i \in \mathbb{G}_{n-1}^{L_0}$. There are $m = 5 \cdot 4^d + 1$ such z_i, and since there are only 4^d elements in $\mathbb{G}_n^{L_0} \cap \left(x + [-2L_n, 2L_n)^d\right)$, one can find x_0 in this set such that $\Lambda_{x_0,n}^{L_0}$, cf. (II.4.5), contains at least 6 different z_i. By (II.4.21), one can thus find $k \neq j$ in $\{0, \dots, m-1\}$ and $A_0 \in \{(\mathbf{C}^{L_0,K})^c, (\widehat{\mathbf{C}}^{L_0,K})^c, (\mathbf{D}^{L_0,p})^c, (\mathbf{E}^{L_0,u})^c, (\mathbf{F}^{L_0,u})^c\}$ such that z_k and z_j are in $\Lambda_{x_0,n}^{L_0}$, and $G_{z_k,n-1}^{L_0}(A_0)$ and $G_{z_j,n-1}^{L_0}(A_0)$ both occur. Moreover,

$$\|z_k - z_j\|_\infty \geqslant \|y_k - y_j\|_\infty - 4L_{n-1} \geqslant 12L_{n-1} \geqslant L_n/l(d),$$

which, in view of (II.4.6), implies that $G_{x_0,n}^{L_0}(A_0)$ occurs, and thus x_0 is $n - (L_0, u, K, p)$ bad. □

By Lemmas II.4.4, II.4.6 and II.4.7, we know that for all $u \in (0, u_0]$, and for a suitable choice of the parameters L_0, K and p, the probability that a vertex is $n - (L_0, u, K, p)$ bad is very small. Lemma II.4.8 then yields that a $*$-path of (L_0, u, K, p) bad vertices in $\mathbb{G}_0^{L_0}$ exists with very small probability only, and on account of Lemma II.4.3, we can prove Proposition II.4.1 using a Peierls argument.

Proof of Proposition II.4.1. Choose a constant $C_5 = C_5(d, u_0)$ large enough such that, upon defining

$$L_0(u) = \lceil C_5/u^7 \rceil,$$

one has $L_0(u)^{1/7}u \geqslant \max(C_2, C_2')$, cf. Lemma II.4.4, and $L_0(u) \geqslant C_3$, cf. Lemma II.4.6, for all $u \in (0, u_0]$. One can now find constants c_1, C_1, c_1' and C_1' such that if (II.4.2) holds, then $\widetilde{K}(u) \geqslant C_3'\sqrt{\log(L_0(u))}$, cf. (II.4.16), and $p(u) \geqslant \exp(-\frac{C_4}{L_0(u)^d})$, cf. (II.4.19), for all $u \in (0, u_0]$. Let us now fix arbitrarily some $u \in (0, u_0]$ and $p \in [p(u), 1]$. Lemma II.4.8, Lemmas II.4.4 and II.4.6 and Lemma II.4.7 can now be applied with $L_0 = L_0(u)$, $K = \widetilde{K}(u)$ and p, to yield

$$\begin{aligned}
&\widetilde{\mathbb{Q}}^p \left(H_{L_n}^{2L_n}(0; L_0(u), u, \widetilde{K}(u), p) \right) \\
&\overset{\text{(II.4.23)}}{\leqslant} 4^d \widetilde{\mathbb{Q}}^p \big(0 \text{ is } n - (L_0(u), u, \widetilde{K}(u), p) \text{ bad} \big) \\
&\overset{\text{(II.4.21)}}{\leqslant} 4^d \Big\{ \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0(u)} \left(\left(\mathbf{C}^{L_0(u), \widetilde{K}(u)} \right)^c \right) \right] + \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0(u)} \left(\left(\widehat{\mathbf{C}}^{L_0(u), \widetilde{K}(u)} \right)^c \right) \right] \\
&\quad + \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0(u)} \left(\left(\mathbf{D}^{L_0(u), p} \right)^c \right) \right] + \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0(u)} \left(\left(\mathbf{E}^{L_0(u), u} \right)^c \right) \right] \\
&\quad + \widetilde{\mathbb{Q}}^p \left[G_{0,n}^{L_0(u)} \left(\left(\mathbf{F}^{L_0(u), u} \right)^c \right) \right] \Big\} \\
&\leqslant 5 \cdot 4^d \cdot 2^{-2^n},
\end{aligned}$$

using (II.4.9), (II.4.10), (II.4.17) and (II.4.20) in the last step. Since this bound holds for all $n \geqslant 0$, and we have, in view of (II.4.4), $H_0^N(0; L_0(u), u, \widetilde{K}(u), p) \subset H_{L_n}^{2L_n}(0; L_0(u), u, \widetilde{K}(u), p)$ for any $n \in \mathbb{N}$ such that $2L_n \leqslant N$, one can find constants $c, C > 0$ depending only on d, u and u_0 such that, for all integers N,

$$\widetilde{\mathbb{Q}}^p \left(H_0^N \big(0; L_0(u), u, \widetilde{K}(u), p\big) \right) \leqslant C \exp\left(-cN^c\right). \tag{II.4.24}$$

Given (II.4.24), the argument proceeds as follows. For any set $A \subset \mathbb{Z}^2 \times \{0\}^{d-2}$, define $(x_0, \dots, x_n, \dots)$ to be a nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ that connects A to ∞ if all the $x_i \in \mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ are good $(L_0(u), u, \widetilde{K}(u), p)$ vertices, $\|x_i - x_{i+1}\|_1 = L_0(u)$ for all $i \in \{0, 1, \dots\}$, $x_0 \in A$ and $\|x_i\|_\infty \to \infty$, as $i \to \infty$. Now, assume that there exists no unbounded nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$, and in particular that for all $M \in L_0(u) \cdot \mathbb{N} \equiv \mathbb{N}_u$ there is no nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices that connects $[M, M]^2 \times \{0\}^{d-2}$ to ∞. Then by planar duality, for all $M \in \mathbb{N}_u$, there exists a $*$-path π around $[-M, M]^2 \times \{0\}^{d-2}$ in $\mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ of bad $(L_0(u), u, \widetilde{K}(u), p)$ vertices. If $N \geqslant M$ denotes the smallest multiple of $L_0(u)$ such that $x_N \equiv (N, 0) \in \mathbb{N}_u \times \{0\}^{d-1}$ is in π, then $H_0^N(x_N; L_0(u), u, \widetilde{K}(u), p)$ occurs. Thus, the probability that there is no infinite nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ that connect $[-M, M]^2 \times \{0\}^{d-2}$ to ∞ is bounded by

$$\sum_{N \in \mathbb{N}_u :\, N \geqslant M} \widetilde{\mathbb{Q}}^p \left(H_0^N(0; L_0(u), u, \widetilde{K}(u), p \right) \leqslant \sum_{N \in \mathbb{N}_u :\, N \geqslant M} C \exp\left(-cN^c\right).$$

This is true for all $M \in \mathbb{N}_u$, hence the probability of having no unbounded nearest neighbor path of good $(L_0(u), u, \widetilde{K}(u), p)$ vertices in $\mathbb{G}_0^{L_0(u)} \cap (\mathbb{Z}^2 \times \{0\}^{d-2})$ is 0. Lemma II.4.3 then implies that the set $\widetilde{A}'_{u,p}$ percolates (almost surely), for any $u \in (0, u_0]$ and $p \in [p(u), 1]$, and the claim of Proposition II.4.1 follows. □

With Proposition II.4.1 at hand, it is possible to deduce Theorem II.3.1 for a good choice of coupling $\widetilde{\mathbb{Q}}^p$ in (II.4.1). The idea is to use (II.2.15), and to suitably couple the Bernoulli percolation $\mathcal{B}^p$ with $\{|\widetilde{\varphi}| \leqslant K(h)\}$ on the edges.

The key to the proof of Theorem II.3.1 is the following lemma, by which one can essentially couple a Bernoulli percolation $\{\mathcal{B}^p = 1\}$ on the edges with sufficiently large success parameter $p \geqslant 1 - C_1' u^{c_1'}$, cf. (II.4.2), with $\{|\widetilde{\varphi}| \leqslant K'\}$ on the edges for K' large enough.

Lemma II.4.9. *Let $\widetilde{\varphi}$ be a Gaussian free field on the cable system under $\widetilde{\mathbb{P}}$. For all $u_0 > 0$, there exist positive constants C_0 and c_0 such that, for all $u \in (0, u_0]$, with $\widetilde{K}(u)$ and $p(u)$ as defined in* (II.4.2)*, $h = \sqrt{2u}$ and $K(h)$ as defined in* (II.3.1)*, the following holds: under $\widetilde{\mathbb{P}}$, there exists a family of independent Bernoulli variables $\mathcal{B}^{\widetilde{p}(u)} = (\theta_e^{\widetilde{p}(u)})_{e\in E}$ with parameter $\widetilde{p}(u) \geqslant p(u)$, and the property that*

$$\begin{aligned} &\text{for all } e = \{x,y\} \in E, \text{ if } |\varphi_x| \leqslant \widetilde{K}(u) \text{ and } |\varphi_y| \leqslant \widetilde{K}(u), \\ &\text{then } \big\{\theta_e^{\widetilde{p}(u)} = 1 \Rightarrow \forall\, z \in \overline{I_e},\ |\widetilde{\varphi}_z| \leqslant K(h)\big\}. \end{aligned} \tag{II.4.25}$$

Proof. Let $u \in (0, u_0]$ and $h = \sqrt{2u}$. With C_1, c_1, C_1' and c_1' as given by Proposition II.4.1, fix constants C_0 and c_0 depending only on u_0 and d such that

$$\begin{aligned} K(h) &\overset{\text{(II.3.1)}}{=} \sqrt{\log\left(\frac{C_0}{(2u)^{c_0/2}}\right)} \geqslant \sqrt{\log\left(\frac{C_1}{u^{c_1}}\right)} + \sqrt{-\frac{1}{2}\log\left(\frac{C_1' u^{c_1'}}{2}\right)} \\ &\overset{\text{(II.4.2)}}{=} \widetilde{K}(u) + \sqrt{-\frac{1}{2}\log\left(\frac{1-p(u)}{2}\right)}. \end{aligned} \tag{II.4.26}$$

Let $(B^e)_{e\in E}$ be defined as in (II.2.9), and recall that $(B^e)_{e\in E}$ is an i.i.d. family of Brownian bridges with length $\frac{1}{2}$ of a Brownian motion with variance 2 at time 1. For all $e \in E$, define

$$\theta_e^{\widetilde{p}(u)} = \begin{cases} 1, & \text{if } |B_t^e| \leqslant K(h) - \widetilde{K}(u) \text{ for all } t \in \left[0, \frac{1}{2}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{II.4.27}$$

Then $\left(\theta_e^{\widetilde{p}(u)}\right)_{e\in E}$ is an i.i.d. family of Bernoulli variables with parameter

$$\widetilde{p}(u) \overset{\text{def.}}{=} \widetilde{\mathbb{P}}\left(\forall\, t \in [0,1/2],\ |B_t^e| \leqslant K(h) - \widetilde{K}(u)\right).$$

Moreover, by symmetry (the boundary values of B^e are both 0) and Lemma II.2.1

$$\begin{aligned} \widetilde{p}(u) &\geqslant 1 - 2\widetilde{\mathbb{P}}\left(\sup_{t\in[0,1/2]} B_t^e \geqslant K(h) - \widetilde{K}(u)\right) \\ &\overset{\text{(II.2.10)}}{\geqslant} 1 - 2\exp\left(-2(K(h) - \widetilde{K}(u))^2\right) \\ &\overset{\text{(II.4.26)}}{\geqslant} p(u), \end{aligned}$$

and, using (II.2.9) and (II.4.27), for all $e = \{x, y\} \in E$ such that $\varphi_x \leqslant \widetilde{K}(u)$ and $\varphi_y \leqslant \widetilde{K}(u)$,

$$\begin{aligned}\theta_e^{\widetilde{p}(u)} = 1 \quad &\Rightarrow \quad \forall t \in [0, 1/2],\ |\widetilde{\varphi}_{x+tv_{(x,y)}} - (1-2t)\varphi_x - 2t\varphi_y| \leqslant K(h) - \widetilde{K}(u) \\ &\Rightarrow \quad \forall z \in \overline{I_e},\ |\widetilde{\varphi}_z| \leqslant K(h),\end{aligned}$$

whence (II.4.25). □

Proof of Theorem II.3.1. Let $u_0 = \frac{h_0^2}{2}$, and, for any $h \in (0, h_0]$, define $u = \frac{h^2}{2}$. Let $\widetilde{\mathbb{P}}^u$ be the coupling from Theorem II.2.2, under which there exist a Gaussian free field $\widetilde{\varphi}$ and a random interlacement process $\widetilde{\omega}$ such that (II.2.16) holds. For this $\widetilde{\varphi}$, let $\mathcal{B}^{\widetilde{p}(u)} = \mathcal{B}^{\widetilde{p}(u)}(\widetilde{\varphi})$ be the family of independent Bernoulli variables under $\widetilde{\mathbb{P}}^u$ introduced in Lemma II.4.9. This yields a coupling $\widetilde{\mathbb{Q}}^{\widetilde{p}(u)}$ satisfying (II.4.1), with parameter $\widetilde{p}(u) \geqslant p(u)$. One can now apply Proposition II.4.1 to obtain that, $\widetilde{\mathbb{P}}^u$-a.s, the set $\widetilde{A}'_{u,\widetilde{p}(u)}$, cf. (II.4.3), contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, L_0(u))^{d-2}$, and thus (II.4.25) yields that $\widetilde{\mathbb{P}}^u$-a.s. the set

$$\big(\widetilde{\mathcal{I}}^u \cap (\widetilde{\mathbb{Z}}^d \setminus \mathbb{Z}^d)\big) \cup \big\{x \in \mathbb{Z}^d;\ \forall\, v \in V^0,\ \forall t \in [0, 1/2],\ |\widetilde{\varphi}_{x+tv}| \leqslant K(h)\big\} \quad \text{(II.4.28)}$$

contains an unbounded connected component in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, L_0(u))^{d-2}$. Now (II.2.16) implies that the set defined in (II.4.28) is included in $\widetilde{A}_h(\widetilde{\varphi})$, and Theorem II.3.1 follows. □

II.5 Percolation for positive level set

In this section, we prove our main result, Theorem II.1.1, with the help of Theorem II.3.1. We consider the Gaussian free field $\widetilde{\Phi}$ on $\widetilde{\mathbb{Z}}^d$ as defined in (II.2.3), and, with Theorem II.3.1 at hand, we will no longer need random interlacements nor the coupling (II.2.15) to prove Theorem II.1.1. A key ingredient is the following observation: we have shown, see (II.1.11) that the set $\{x \in \widetilde{\mathbb{Z}}^d;\ -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$ contains an unbounded connected component for large enough $K(h)$, cf. (II.3.1). Suppose that $x \in \mathbb{Z}^d$ is a vertex inside this unbounded component, and that I_e is attached to x (recall that $\Phi = \widetilde{\Phi} \restriction \mathbb{Z}^d$). Then, since $\widetilde{\Phi}$ behaves like a Brownian bridge on $\overline{I_e}$, see (II.2.9), the probability that $\widetilde{\Phi}_z \geqslant -h$ for all $z \in I_e$ becomes very small as $h \searrow 0$. In fact, since $hK(h) \to 0$ as $h \searrow 0$, if $e = \{x, y\}$, for sufficiently small $h > 0$, it is more costly to keep $\widetilde{\Phi} \geqslant -h$ along the entire cable $\overline{I_e}$, than to require $\Phi_x \geqslant h$ (*at* the vertex x only!), knowing that $-h \leqslant \Phi_x \leqslant K(h)$ and $|\Phi_y| \leqslant K(h)$, see Lemma II.5.1 for the corresponding statement. Accordingly, the probability that the set $\{x \in \widetilde{\mathbb{Z}}^d;\ -h \leqslant \widetilde{\Phi}_x \leqslant K(h)\}$

contains an unbounded connected component becomes smaller than the probability that the set $\{x \in \mathbb{Z}^d;\, h \leqslant \Phi_x \leqslant K(h)\}$ contains an infinite cluster (in $\mathbb{Z}^d$) as h goes to 0, which implies Theorem II.1.1.

Comparing the probability that $\Phi_x \geqslant h$ knowing that $-h \leqslant \Phi_x \leqslant K(h)$ with the probability that the Brownian bridge on I_e remains above level $-h$ in a uniform way requires some control on the Gaussian free field $\widetilde{\Phi}$ in the neighborhood of x, and for this purpose we are actually going to use Theorem II.3.1 and not only (II.1.11). We define, for $x \in \mathbb{Z}^d$ and $v \in V^0$, the subsets $U^{x,v}$ and U^x of $\widetilde{\mathbb{Z}}^d$ by

$$U^{x,v} = x + \Big[0, \frac{1}{4}v\Big) \text{ and } U^x = \bigcup_{v \in V^0} U^{x,v} = x + \bigcup_{v \in V^0} \Big[0, \frac{1}{4}v\Big). \tag{II.5.1}$$

We call $\mathcal{K}^x \equiv \partial U^x$ the boundary of U^x, which has exactly $2d$ elements, and define $\mathcal{K} = \bigcup_{x \in \mathbb{Z}^d} \mathcal{K}^x$. Henceforth, we set

$$h_0 = 1 \tag{II.5.2}$$

in all the previous definitions and results, and in particular in Theorem II.3.1 (this value is chosen arbitrarily in $(0, \infty)$). For any $h \in (0, 1]$, we define $K(h)$ as in (II.3.1) (with c_0, C_0 numerical constants depending only on d by the choice (II.5.2)). We further define two families of events $(E_h^{x,v})_{x \in \mathbb{Z}^d}$ and $(F_h^{x,v})_{x \in \mathbb{Z}^d, v \in V^0}$ (part of Ω_0, cf. above (II.2.3)) by

$$\begin{aligned} E_h^{x,v} &= \Big\{\widetilde{\Phi}_{x + \frac{1}{4}v} \geqslant -h\Big\} \cap \Big\{\forall y \in \mathcal{K}^x;\, |\widetilde{\Phi}_y| \leqslant K(h)\Big\} \\ \text{and } F_h^{x,v} &= \Big\{\forall\, z \in U^{x,v};\, \widetilde{\Phi}_z \geqslant -h\Big\}, \end{aligned} \tag{II.5.3}$$

as well as

$$E_h^x = E_h^{x,v} \text{ and } G_h^x = \bigcup_{v \in V^0} \big(E_h^{x,v} \cap F_h^{x,v}\big), \tag{II.5.4}$$

and the (random) subsets of $\mathbb{Z}^d$

$$E_h = \{x \in \mathbb{Z}^d;\, E_h^x \text{ occurs}\} \text{ and } G_h = \{x \in \mathbb{Z}^d;\, G_h^x \text{ occurs}\}. \tag{II.5.5}$$

For all $K \subset \widetilde{\mathbb{Z}}^d$, we denote by $\mathcal{A}_K$ the σ-algebra $\sigma\left(\widetilde{\varphi}_z,\ z \in K\right)$. We note that the sets U^x are disjoint when x varies, cf. (II.5.1) and (II.2.1), and that the events $E_h^{x,v}$ are $\mathcal{A}_{\mathcal{K}^x}$-measurable. Theorem II.3.1 implies that G_h contains an infinite connected component, and the goal is to go from this to the percolation of $E_h \cap \{x \in \mathbb{Z}^d;\, \Phi_x \geqslant h\}$. The following lemma makes the above observation, see the discussion at the beginning of this section, precise.

Lemma II.5.1. *There exists $h_1 \in (0,1]$ such that for all $h \in (0,h_1]$ and $x \in \mathbb{Z}^d$,*

$$\widetilde{\mathbb{P}}^G\left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \leqslant \widetilde{\mathbb{P}}^G\left(E_h^x \cap \{\Phi_x \geqslant h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right). \tag{II.5.6}$$

Proof. Let us fix some $x \in \mathbb{Z}^d$. It is sufficient to prove that there exists $h_1 \in (0,1]$ such that for all $h \in (0,h_1]$ and all $v \in V^0$,

$$1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G\left(F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \leqslant \frac{1}{2d} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G\left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right). \tag{II.5.7}$$

Indeed, if (II.5.7) holds, then

$$\begin{aligned}
\widetilde{\mathbb{P}}^G\left(G_h^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) &= \widetilde{\mathbb{P}}^G\left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) + \widetilde{\mathbb{P}}^G\left(G_h^x \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\overset{\text{(II.5.4)}}{\leqslant} \widetilde{\mathbb{P}}^G\left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\quad + \sum_{v \in V^0} \widetilde{\mathbb{P}}^G\left(E_h^{x,v} \cap F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\overset{\text{(II.5.7)}}{\leqslant} \widetilde{\mathbb{P}}^G\left(G_h^x \cap \{\Phi_x > 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\quad + \frac{1}{2d} \sum_{v \in V^0} \widetilde{\mathbb{P}}^G\left(E_h^{x,v} \cap \{h \leqslant \Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\leqslant \widetilde{\mathbb{P}}^G\left(E_h^x \cap \{\Phi_x \geqslant h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right),
\end{aligned}$$

noting that $G_h^x, E_h^{x,v} \subset E_h^x$ in the last inequality, and (II.5.6) follows. We now show (II.5.7). Let us fix some $v \in V^0$. We begin with the study of Φ_x, by decomposing it suitably. It follows from the Markov property, cf. (II.2.4), that $\widetilde{\Phi}_x^{U^x} = \Phi_x - \widetilde{\beta}_x^{U^x}$ is a centered Gaussian variable with variance $g_{U^x}(x,x)$. The value of the variance $g_{U^x}(x,x) \equiv \sigma_0^2$ does not depend on $x \in \mathbb{Z}^d$ (it actually follows from Section 2 of [57] that $\sigma_0^2 = \frac{1}{4d}$). Moreover, on the event $E_h^{x,v}$, it is clear that $|\widetilde{\beta}_x^{U^x}| \leqslant K(h)$. Thus, on the event $E_h^{x,v}$, since the harmonic average $\widetilde{\beta}_x^{U^x}$ is $\mathcal{A}_{\mathcal{K}^x}$-measurable, we obtain, for all $h > 0$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}^G\left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&= \widetilde{\mathbb{P}}^G\left(-h \leqslant \widetilde{\Phi}_x^{U^x} + \widetilde{\beta}_x^{U^x} \leqslant 2h \,\middle|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma_0^2}} \int_{-h}^{2h} \exp\left(-\frac{\left(y - \widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2}\right) \mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2}\right) \int_{-h}^{2h} \exp\left(-\frac{y^2}{2\sigma_0^2}\right) \exp\left(\frac{y\widetilde{\beta}_x^{U^x}}{\sigma_0^2}\right) \mathrm{d}y \\
&\leqslant \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2}\right) \times 3h \exp\left(\frac{2hK(h)}{\sigma_0^2}\right).
\end{aligned} \tag{II.5.8}$$

A similar calculation shows that on the event $E_h^{x,v}$, for $h > 0$,

$$\widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \geqslant \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\left(\widetilde{\beta}_x^{U^x}\right)^2}{2\sigma_0^2}\right) \times h \exp\left(-\frac{2h(K(h)+h)}{\sigma_0^2}\right). \tag{II.5.9}$$

Define

$$C_6(d) = \sup_{h\in(0,1]} \left\{3\exp\left(\frac{2hK(h)}{\sigma_0^2}\right) \times \exp\left(\frac{2h(K(h)+h)}{\sigma_0^2}\right)\right\},$$

and note that $C_6 < \infty$ since $hK(h) \to 0$ as $h \searrow 0$, cf. (II.3.1). Hence, by (II.5.8) and (II.5.9),

$$1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \leqslant C_6 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right), \text{ for } h \in (0,1]. \tag{II.5.10}$$

Let us now turn to the events $F_h^{x,v}$. It follows again from the Markov property for the Gaussian free field, see in particular the discussion below (II.2.8), that, knowing $\mathcal{A}_{\mathcal{K}^x\cup\{x\}}$, the process $(\widetilde{\Phi}_z)_{z\in U^{x,v}}$ is a Brownian bridge of length $\frac{1}{4}$ between Φ_x and $\widetilde{\Phi}_{x+\frac{1}{4}v}$ of a Brownian motion with variance 2 at time 1. Using Lemma II.2.1, one can then find $h_1 \in (0,1]$ such that, for all $h \in (0,h_1]$, on the $\mathcal{A}_{\mathcal{K}^x\cup\{x\}}$ measurable event $E_h^{x,v} \cap \{-h \leqslant \Phi_x \leqslant 2h\}$,

$$\begin{aligned}
\widetilde{\mathbb{P}}^G \left(F_h^{x,v} \,|\, \mathcal{A}_{\mathcal{K}^x\cup\{x\}}\right) &= \widetilde{\mathbb{P}}^G \left(\min_{z\in U^{x,v}} \widetilde{\Phi}_z \geqslant -h \,\middle|\, \mathcal{A}_{\mathcal{K}^x\cup\{x\}}\right) \\
&= 1 - \exp\left(-4(h+\Phi_x)(h+\widetilde{\Phi}_{x+\frac{1}{4}v})\right) \\
&\leqslant 1 - \exp\left(-12h(K(h)+h)\right) \\
&\leqslant \frac{1}{2dC_6}.
\end{aligned} \tag{II.5.11}$$

We now conclude using (II.5.10) and (II.5.11): for all $h \in (0,h_1]$,

$$\begin{aligned}
&1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(F_h^{x,v} \cap \{\Phi_x \leqslant 2h\} \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\overset{\text{(II.5.3)}}{=} \mathbb{E}\left[1_{E_h^{x,v}\cap\{-h\leqslant\Phi_x\leqslant 2h\}} \widetilde{\mathbb{P}}^G \left(F_h^{x,v} \,|\, \mathcal{A}_{\mathcal{K}^x\cup\{x\}}\right) \,\middle|\, \mathcal{A}_{\mathcal{K}^x}\right] \\
&\overset{\text{(II.5.11)}}{\leqslant} \frac{1}{2dC_6} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(-h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \\
&\overset{\text{(II.5.10)}}{\leqslant} \frac{1}{2d} 1_{E_h^{x,v}} \widetilde{\mathbb{P}}^G \left(h \leqslant \Phi_x \leqslant 2h \,|\, \mathcal{A}_{\mathcal{K}^x}\right),
\end{aligned}$$

which is (II.5.7). □

Lemma II.5.1 roughly asserts that it is more likely to have $\{\Phi_x \geqslant h\}$ than to have G_h^x (on E_h^x) for small $h > 0$ and we know by Theorem II.3.1 that G_h

has an infinite connected component in a thick slab. Using the Markov property for the Gaussian free field we will show that this implies that $E^{\geqslant h}$, see (II.1.3), percolates in a sufficiently thick slab for any such value of h, thus obtaining Theorem II.1.1.

Proof of Theorem II.1.1. Let us fix some $h \in (0, h_1]$, with h_1 as in Lemma II.5.1. The Markov property for the Gaussian free field, see (II.2.5), (II.5.1) and (II.2.1), implies that the family $(\widetilde{\Phi}^{U^x}_{\cdot})_{x\in\mathbb{Z}^d}$ is i.i.d. and independent of $\mathcal{A}_{\mathcal{K}}$, and that for all $x \in \mathbb{Z}^d$ and $z \in U^x$,

$$\widetilde{\Phi}_z = \widetilde{\beta}^{U^x}_z + \widetilde{\Phi}^{U^x}_z$$

where $\widetilde{\beta}^{U^x}$ is $\mathcal{A}_{\mathcal{K}^x}$-measurable for all $x \in \mathbb{Z}^d$. For each $x \in \mathbb{Z}^d$, there exists $f_x : C(\widetilde{\mathbb{Z}}^d, \mathbb{R}) \times \mathbb{R}^{\mathcal{K}^x} \to \{0,1\}$ and $g_x : C(\widetilde{\mathbb{Z}}^d, \mathbb{R}) \times \mathbb{R}^{\mathcal{K}^x} \to \{0,1\}$ such that

$$1_{G^x_h} = f_x(\widetilde{\Phi}^{U^x}, \widetilde{\Phi}_{|\mathcal{K}^x}) \text{ and } 1_{E^x_h \cap \{\Phi_x \geqslant h\}} = g_x(\widetilde{\Phi}^{U^x}, \widetilde{\Phi}_{|\mathcal{K}^x}). \tag{II.5.12}$$

For each measurable subset A of $\widetilde{\mathbb{Z}}^d$, let us denote by $\widetilde{\mathbb{P}}^{G,A}$ the law of $\widetilde{\Phi}_{|A}$ under $\widetilde{\mathbb{P}}^G$. Lemma II.5.1 now gives that for all $x \in \mathbb{Z}^d$ and for $\widetilde{\mathbb{P}}^{G,\mathcal{K}^x}$-a.s. all $\beta^x \in \mathbb{R}^{\mathcal{K}^x}$,

$$\widetilde{\mathbb{P}}^G\big(f_x(\widetilde{\Phi}^{U^x}, \beta^x) = 1\big) \leqslant \widetilde{\mathbb{P}}^G\big(g_x(\widetilde{\Phi}^{U^x}, \beta^x) = 1\big). \tag{II.5.13}$$

For each $x \in \mathbb{Z}^d$ and $\beta^x \in \mathbb{R}^{\mathcal{K}^x}$ such that (II.5.13) holds, abbreviating the left and right-hand sides of (II.5.13) by $p_f(\beta^x)$ and $p_g(\beta^x)$, respectively, so that $p_f(\beta^x) \leqslant p_g(\beta^x)$, we can now define a probability ν_{β^x} on $\{0,1\}^2$ such that, with π_1 and π_2 respectively denoting the projections onto the first and second coordinate of $\{0,1\}^2$, we have

$$\pi_1 \leqslant \pi_2, \quad \nu_{\beta^x}(\pi_1 = 1) = p_f(\beta^x) \quad \text{and} \quad \nu_{\beta^x}(\pi_2 = 1) = p_g(\beta^x). \tag{II.5.14}$$

The measure ν_{β^x} can for instance be constructed from a uniform random variable Y on $[0,1]$ as the law of $(1_{\{Y \leqslant p_f(\beta^x)\}}, 1_{\{Y \leqslant p_g(\beta^x)\}})$ on $\{0,1\}^2$. For each $\beta \in \mathbb{R}^{\mathcal{K}}$ and $x \in \mathbb{Z}^d$, let $\beta^x = (\beta_z)_{z\in\mathcal{K}^x} \in \mathbb{R}^{\mathcal{K}^x}$, and finally define, for $\widetilde{\mathbb{P}}^{G,\mathcal{K}}$-a.s. all $\beta \in \mathbb{R}^{\mathcal{K}}$ the following probabilities on $(\{0,1\}^{\mathbb{Z}^d})^2$

$$\nu_\beta = \bigotimes_{x\in\mathbb{Z}^d} \nu_{\beta^x} \qquad \text{and} \qquad \nu = \widetilde{\mathbb{E}}^G\left[\nu_{\widetilde{\Phi}_{|\mathcal{K}}}\right]. \tag{II.5.15}$$

Note that for all $B \subset \{0,1\}^2$, $\beta^x \mapsto \nu_{\beta^x}(B)$ is measurable, and thus ν is well-defined. Let π_1' and π_2' be the projections on the first and second coordinate of $(\{0,1\}^{\mathbb{Z}^d})^2$. Then $\pi_1' \leqslant \pi_2'$ ν-a.s by (II.5.14), (II.5.15), and on account of (II.5.12) π_1' has the same law under ν as $(1_{G^x_h})_{x\in\mathbb{Z}^d}$ under $\widetilde{\mathbb{P}}^G$ and π_2' has the same law under ν as $(1_{E^x_h\cap\{\Phi_x\geqslant h\}})_{x\in\mathbb{Z}^d}$ under $\widetilde{\mathbb{P}}^G$. Moreover, by Theorem II.3.1, there exists $L_0 = L_0(h) > 0$ such that the set $\widetilde{A}_h(\widetilde{\Phi})$, cf. (II.3.2), contains $\widetilde{\mathbb{P}}^G$-a.s.

an unbounded connected component M in the thick slab $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$. By definition, see (II.5.5), $M \cap \mathbb{Z}^d \subset G_h$, and thus $\{x \in \mathbb{Z}^d : \pi_1'(x) = 1\}$ contains ν-a.s. an unbounded component in $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$. Since $\pi_1' \leqslant \pi_2'$, this implies that $E_h \cap \{x \in \mathbb{Z}^d : \widetilde{\Phi}_x \geqslant h\}$ also contains an infinite connected component in $\widetilde{\mathbb{Z}}^2 \times [0, 2L_0)^{d-2}$, as desired. □

Remark II.5.2. 1) The result of [81] is actually slightly better than Theorem II.1.1 in high dimensions: if d is large enough, there exist $h_2 = h_2(d) > 0$ and $L_0 = L_0(d) \geqslant 1$ such that the level set $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h_2\}$ percolates in the slab $\mathbb{Z}^2 \times [0, 2L_0) \times \{0\}^{d-3}$. However, in all dimensions $d \geqslant 3$, the set $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$ never percolates for $h \geqslant 0$ in $\mathbb{Z}^2 \times \{0\}^{d-2}$, as explained in Remark 3.6.1 of [81].

2) It is possible to get a result similar to (II.1.11) for the positive level set of the Gaussian free field Φ on $\mathbb{Z}^d$ just constructed, thus obtaining the following strengthening of Theorem II.1.1. For all $h \leqslant h_1$, let $K(h)$ be as in (II.3.1) for $h_0 = 1$, then the set $\{x \in \mathbb{Z}^d; h \leqslant \Phi_x \leqslant K(h)\}$ contains a.s. an infinite connected component. Indeed, using an argument similar to that of Lemma II.5.1, one can prove that, conditionally on $\mathcal{A}_{\mathcal{K}^x}$, the probability of $G_h^x \cap \{\Phi_x \leqslant K(h)\}$ is smaller than the probability of $E_h^x \cap \{h \leqslant \Phi_x \leqslant K(h)\}$ and the result follows.

3) Theorem 2.2 in [74] can also easily be extended to the Gaussian free field: for each $h \leqslant h_1$, the set $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$ contains an almost surely transient component. Indeed, looking at the proof of Theorem 2.2 in [74], see also Theorem 1 in [73], we can use (II.4.24) instead of (5.1) in [74] to obtain that the set $\widetilde{A}_{u,p}'$ defined in (II.4.3) contains an unbounded connected and transient component for $p \in [p(u), 1]$. Using the same coupling as in Lemma II.4.9, we get that this is also true for the set $\widetilde{A}_h(\widetilde{\varphi})$ defined in (II.3.2), and the same proof as the proof of Theorem II.1.1 tells us that $\{x \in \mathbb{Z}^d; \Phi_x \geqslant h\}$ also contains an infinite connected and transient component for $h \leqslant h_1$.

4) Another parameter $\overline{h} \leqslant h_*$ has been introduced in [28], and a similar one has been used in [100]. This parameter describes a strong percolative regime for $E^{\geqslant h}$, when $h < \overline{h}$, i.e., all connected components of $E^{\geqslant h}$ in $[-R, R]^d$ with diameter at least $\frac{R}{10}$ are connected in $[-2R, 2R]^d$ with large enough probability when R goes to ∞. It has been proved that $\overline{h} > -\infty$ and it is believed that actually $\overline{h} = h_*$, but it is still unknown whether $\overline{h} \geqslant 0$ or not. Our methods may perhaps help in that regard.

II.A Appendix: Proof of Lemma II.3.2

The proof of Lemma II.3.2 is very close to the proof of Proposition 1 in [73], but we need to remove the dependence on u of the constants, and make the dependence on u of the error term explicit instead. We will henceforth refer to [73] whenever possible, and in particular, Lemmas 3 to 6 and 11 in [73] do not involve u at all, so we will use them without proof. Recall ω^u, the interlacement process at level u on $\mathbb{Z}^d$, and $\widetilde{\omega}^u$, the interlacement process at level u on the cable system, obtained from ω^u by adding independent Brownian excursions on the edges, as in the construction of the diffusion $\widetilde{X}$ from a simple random walk on $\mathbb{Z}^d$ in the beginning of section II.2. We denote by $\widehat{\mathcal{I}}^u$ the set of edges traversed by at least one of the trajectories in $\mathrm{supp}(\omega^u)$. Now observe that the event that every x and y in $\widetilde{\mathcal{I}}^u \cap [0,R)^d$ be connected in $\widetilde{\mathcal{I}}^u \cap [-\varepsilon R,(1+\varepsilon)R)^d$, which is the event of interest in (II.3.4), is more likely than every x and y in $\widehat{\mathcal{I}}^u \cap [-1,R+1)^d$ being connected in $\widehat{\mathcal{I}}^u \cap [-\varepsilon R,(1+\varepsilon)R)^d$. Thus, we only need to show the respective statement of Lemma II.3.2 for $\widehat{\mathcal{I}}^u$ instead of $\widetilde{\mathcal{I}}^u$, cf. Lemma II.A.5 below.

The idea of the proof is to show that there exists $C \geqslant 1$ such that for every integer $R \geqslant 1$ and every $x, y \in \mathcal{I}^u \cap [-R,R)^d$, the vertices x and y are connected through edges in $\widehat{\mathcal{I}}^u \cap [-CR,CR)^d$ with high enough probability. It is quite hard to directly link x and y, especially if R is large. Therefore, let us define, under some probability P, $\omega_{i,3}^{u/3}$ for $i \in \{1,2,3\}$, three independent Poisson point process with the same law as $\omega^{u/3}$ under $\mathbb{P}^I$, such that $\omega^u = \sum_{i=1}^3 \omega_{i,3}^{u/3}$. Let us call $\mathcal{I}_{i,3}^{u/3}$ the set of vertices visited by at least one of the trajectories from $\mathrm{supp}(\omega_{i,3}^{u/3})$, denote by $\widehat{\mathcal{I}}_{i,3}^{u/3}$ the set of edges traversed by at least one of the trajectories from $\mathrm{supp}(\omega_{i,3}^{u/3})$, and let $C_i^{u/3}(x,R)$ be the set of vertices connected to x by edges in $\widehat{\mathcal{I}}_{i,3}^{u/3} \cap [-R,R)^d$ for $i \in \{1,2,3\}$ and $x \in \mathbb{Z}^d$. We are going to prove that, if $x \in \mathcal{I}_{1,3}^{u/3}$ and $y \in \mathcal{I}_{2,3}^{u/3}$, then $C_1^{u/3}(x,R)$ and $C_2^{u/3}(y,R)$ are big enough, and that one can connect these two sets by edges in $\widehat{\mathcal{I}}_{3,3}^{u/3} \cap [-CR,CR)^d$ with high probability. In particular, this will imply that x and y are connected through edges in $\widehat{\mathcal{I}}^u \cap [-CR,CR)^d$ with high probability.

We first recall a property of the Poisson distribution (see for example (2.11) in [73]): let N be a random variable which has Poisson distribution with parameter λ, then there exist constants $c < 1$ and $C > 1$ independent of λ such that

$$\mathbb{P}\left(c\lambda \leqslant N \leqslant C\lambda\right) \geqslant 1 - C\exp\left(-c\lambda\right).$$

For $A \subset \mathbb{Z}^d$ finite and ω an interlacement process, we call for all $u > 0$ N_A^u the the number of trajectories in $\mathrm{supp}(\omega^u)$ which enter A, and write $Z_1, \dots, Z_{N_A^u}$ for the corresponding trajectories, parametrized such that $Z_i(0) \in A$ and $Z_i(-n) \notin A$

for all $n > 0$. Note that $Z_1, \dots, Z_{N_A^u}$ depend on ω, u and A even if this is only implicit in the notation. Then N_A^u is a Poisson variable with parameter $u\mathrm{cap}(A)$ and

$$\mathbb{P}^I\left(cu\mathrm{cap}(A) \leqslant N_A^u \leqslant Cu\mathrm{cap}(A)\right) \geqslant 1 - C\exp\left(-cu\mathrm{cap}(A)\right). \qquad \text{(II.A.1)}$$

Here, $\mathrm{cap}(A)$ is the capacity of the set A, i.e., the total mass of the equilibrium measure of A. The following standard bounds will soon prove to be useful: For any $A \subset [-R,R)^d \cap \mathbb{Z}^d$ and $R \geqslant 1$,

$$\mathrm{cap}(A) \leqslant \mathrm{cap}([-R,R)^d) \leqslant CR^{d-2} \text{ and } \mathrm{cap}([-R,R)^d) \geqslant cR^{d-2}. \qquad \text{(II.A.2)}$$

The next lemma gives a bound on the probability to connect the two sets $C_1^{u/3}(x,R)$ and $C_2^{u/3}(y,R)$ in $\widehat{\mathcal{I}}_3^{u/3} \cap [-CR,CR)^d$ in terms of capacity.

Lemma II.A.1. *There exist constants $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $R > 0$ and $u > 0$, for all subsets U and V of $[-R,R)^d$,*

$$\mathbb{P}^I\big(U \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} V\big) \geqslant 1 - C\exp\big(-cR^{2-d}u\mathrm{cap}(U)\mathrm{cap}(V)\big).$$

Proof. If there is a trajectory among $(Z_1, \dots, Z_{N_U^u})$, which hits V after 0 and before leaving $[-CR,CR)^d$, then U is connected to V through edges of $\widehat{\mathcal{I}}^u \cap [-CR,CR)^d$. We can use Lemma 11 in [73] to lower bound the probability of a trajectory to behave accordingly by $cR^{2-d}\mathrm{cap}(V)$, and thus we infer

$$\begin{aligned}\mathbb{P}^I\left(U \overset{\widehat{\mathcal{I}}^u \cap [-CR,CR)^d}{\longleftrightarrow} V\right) &\geqslant 1 - \mathbb{P}^I(N_U^u < cu\mathrm{cap}(U)) - \big(1 - cR^{2-d}\mathrm{cap}(V)\big)^{cu\mathrm{cap}(U)} \\ &\overset{(II.A.1),(II.A.2)}{\geqslant} 1 - C\exp\big(-cR^{2-d}u\mathrm{cap}(U)\mathrm{cap}(V)\big).\end{aligned}$$

□

We are now going to prove that $\mathrm{cap}\big(C_1^{u/3}(x,R)\big)$ and $\mathrm{cap}\big(C_2^{u/3}(y,R)\big)$ are large enough with high probability, and in particular that they grow faster in R than $R^{\frac{d-2}{2}}$. From now on we fix some $u_0 > 0$, and for all $u > 0$, $A \subset \mathbb{Z}^d$ finite and T a positive integer, we define the set $\Psi(u,A,T)$ by

$$\Psi(u,A,T) = A \cup \bigcup_{i=1}^{N_A^u} \{Z_i(n),\ 0 \leqslant n \leqslant T\}. \qquad \text{(II.A.3)}$$

Lemma II.A.2. *For all $\varepsilon \in (0,1)$, $k \geqslant 1$ and $\delta \geqslant \varepsilon$, there exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0,u_0]$, $A \subset \mathbb{Z}^d$ finite and T a positive integer,*

$$\begin{aligned}\mathbb{P}^I\Big(\mathrm{cap}\left(\Psi\left(u,A,T\right)\right) \geqslant c\,\min\Big(u\mathrm{cap}(A)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\Big)\Big) \\ \geqslant 1 - C\exp\big(-c\min\big(T^{\varepsilon/2}, u\mathrm{cap}(A)\big)\big),\end{aligned} \qquad \text{(II.A.4)}$$

and, if $A \subset B = [-kT^{\delta}, kT^{\delta})^d$,

$$\mathbb{P}^I\left(\Psi(u,A,T) \subset B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d\right) \geqslant 1 - C\exp\left(-cT^{\varepsilon}u\right). \qquad \text{(II.A.5)}$$

Proof. (II.A.4) is a simple consequence of Lemma 6 in [73] and (II.A.1). In order to prove (II.A.5), let us first define $h(T,\varepsilon)$, the probability that the simple random walk on $\mathbb{Z}^d$ beginning in 0 leaves $[-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d$ before time T. Hoeffding's inequality yields that $h(T,\varepsilon) \leqslant C\exp(-cT^{\varepsilon})$. Now, taking $B = [-kT^{\delta}, kT^{\delta})^d$ and assuming $A \subset B$, observing that, in order for $\Psi(u,A,T)$ not to be contained in $B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d$, at least one of the walks Z_i in (II.A.3) must reach distance $T^{\frac{1+\varepsilon}{2}}$ before time T, and noting that $N_A^u \leq N_B^u$, we get

$$\begin{aligned}
&\mathbb{P}^I\left(\Psi(u,A,T) \subset B + [-T^{\frac{1+\varepsilon}{2}}, T^{\frac{1+\varepsilon}{2}})^d\right) \\
&\geqslant 1 - \mathbb{P}^I\left(N_B^u \geqslant Cu\mathrm{cap}(B)\right) - Cu\mathrm{cap}(B)h(T,\varepsilon) \\
&\overset{(II.A.1),\,(II.A.2)}{\geqslant} 1 - C\exp\left(-cuT^{(d-2)\delta}\right) - CuT^{(d-2)\delta}\exp\left(-cT^{\varepsilon}\right) \\
&\geqslant 1 - C\exp\left(-cT^{\varepsilon}u\right),
\end{aligned}$$

which concludes the proof. □

We now iterate this process to find the desired bound on $\mathrm{cap}\left(C^u(x,R)\right)$. Consider, under some probability $\mathbb{Q}$, a sequence of independent random interlacement processes $(\omega_k)_{k\geqslant 1}$ which define an independent sequence $(\Psi_k)_{k\geqslant 2}$ such that for all $k \geqslant 2$, Ψ_k has the same law as Ψ (see (II.A.3) for notation), and let $\mathcal{I}_1^v$ be the random interlacement set associated with ω_1^v. For each $x \in \mathbb{Z}^d$, let Z^x be the trajectory with the smallest label v contained in ω_1 such that $Z^x(0) = x$, which exists since $x \in \mathcal{I}_1^v$ for $v \in (0,\infty)$ large enough. For all $x \in \mathbb{Z}^d$, $u > 0$ and T positive integer, we recursively define a sequence of subsets $(U_u^{(k)}(x,T))_{k\geqslant 1}$ of $\mathbb{Z}^d$ by

$$U_u^{(1)}(x,T) = \{Z^x(n),\ 0 \leqslant n \leqslant T\},$$

and, for all $k \geqslant 2$,

$$U_u^{(k)}(x,T) = \Psi_k\left(u, U_u^{(k-1)}(x,T), T\right).$$

Note that $U_u^{(k)}(x,T)$ depends on u only for $k \geqslant 2$. In the next lemma, we iterate the results of Lemma II.A.2 to find lower bounds on the capacity of $U_u^{(d-2)}(x,T)$ and upper bounds on the diameter of $U_u^{(d-2)}(x,T)$.

Lemma II.A.3. *For all* $\varepsilon \in (0, \frac{1}{3}]$, *there exist constants* $c > 0$ *and* $C < \infty$ *such that for every* $u \in (0,u_0]$, $x \in \mathbb{Z}^d$, *positive integer* T *and* $k \in \{1,\dots,d-2\}$,

$$\mathbb{Q}\left(\mathrm{cap}\left(U_u^{(k)}(x,T)\right) \geqslant u^{k-1}\left(cT^{\frac{1-\varepsilon}{2}}\right)^k\right) \geqslant 1 - C\exp\left(-cT^{\varepsilon/2}u\right) \qquad \text{(II.A.6)}$$

and

$$\mathbb{Q}\left(U_u^{(k)}(x,T) \subseteq x + [-kT^{\frac{1+\varepsilon}{2}}, kT^{\frac{1+\varepsilon}{2}})^d\right) \geqslant 1 - C\exp\left(-cT^{\varepsilon}u\right). \qquad \text{(II.A.7)}$$

Proof. Let us introduce the shorthand $U_u^{(k)} = U_u^{(k)}(x,T)$, and note that, albeit only implicitly, $U_u^{(k)}$ depends on x and T . We first prove (II.A.6) by induction on $k \in \{1,\dots,d-2\}$. For $k=1$, (II.A.6) follows directly from Lemma 6 in [73]. Let us assume that (II.A.6) holds true at level $k-1$ for some $k \in \{2,\dots,d-2\}$, and that c is small enough so that $u_0 c \leqslant 1$. The event in (II.A.6) is implied by the event

$$\begin{aligned}&\left\{\mathrm{cap}\left(U_u^{(k-1)}\right) \geqslant u^{k-2}\big(cT^{\frac{1-\varepsilon}{2}}\big)^{k-1}\right\}\\ &\cap\left\{\mathrm{cap}\left(U_u^{(k)}\right) \geqslant c\min\left(u\mathrm{cap}\left(U_u^{(k-1)}\right)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right\}\end{aligned}$$

since $k \leqslant d-2$. We only need to prove (II.A.6) if $cT^{\varepsilon/2}u \geqslant 1$, and then $cT^{\frac{1-\varepsilon}{2}}u \geqslant T^{\varepsilon/2}$ and (II.A.4) gives that, on the event $\mathrm{cap}\left(U_u^{(k-1)}\right) \geqslant u^{k-2}\left(cT^{\frac{1-\varepsilon}{2}}\right)^{k-1}$,

$$\begin{aligned}&\mathbb{Q}\left(\mathrm{cap}\left(U_u^{(k)}\right) \geqslant c\min\left(u\mathrm{cap}\left(U_u^{(k-1)}\right)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right) \,\Big|\, U_u^{(k-1)}\right)\\ &\qquad\geqslant 1 - C\exp\left(-c\min\left(T^{\varepsilon/2}, \left(cT^{\frac{1-\varepsilon}{2}}u\right)^{k-1}\right)\right) \geqslant 1 - C\exp\left(-cT^{\varepsilon/2}u\right),\end{aligned}$$

and (II.A.6) follows by the induction hypothesis. The proof of (II.A.7) is similar: one needs to use the fact that $h(T,\varepsilon) \leqslant C\exp(-cT^{\varepsilon})$ for $k=1$ (see the proof of Lemma II.A.2 for the definition of $h(T,\varepsilon)$), and then proceed by induction with (II.A.5) for $k \geqslant 2$. □

Corollary II.A.4. *For all $\varepsilon \in (0,\frac{1}{2}]$, there exist constants $c>0$ and $C<\infty$ such that for every $u \in (0,u_0]$, $x \in \mathbb{Z}^d$ and $R>0$,*

$$\mathbb{P}^I\left(x \in \mathcal{I}^u,\ \mathrm{cap}(C^u(x,R)) < cR^{(1-\varepsilon)(d-2)}u^{d-3}\right) \leqslant C\exp\left(-cR^{\varepsilon/2}u\right),$$

where $C^u(x,R)$ is the set of vertices connected to x by edges in $\widehat{\mathcal{I}}^u \cap [-R,R)^d$.

Proof. For all $v \in (0,\frac{u_0}{d-2}]$, let $\omega^{(d-2)v} := \sum_{i=1}^{d-2}\omega_i^v$, where $(\omega_i^v)_{i\geqslant 1}$ are the independent random interlacement processes at level v used in the definition of $\big(U_v^{(k)}(.\,,.)\big)_{k\geqslant 1}$, see above Lemma II.A.3, and let $\mathcal{I}^{(d-2)v}$ be the random interlacement set associated with $\omega^{(d-2)v}$. By definition, $\omega^{(d-2)v}$ has the same law under $\mathbb{Q}$ as a random interlacement process at level $(d-2)v$, and if $x \in \mathcal{I}_1^v$ then $U_v^{(k)}(x,T)$ is a connected subset of $\mathcal{I}^{(d-2)v}$ for all $k \in \{1,\dots,d-2\}$, $x \in \mathbb{Z}^d$ and positive integer T. In particular, if $x \in \mathcal{I}_1^v$ and if the event in (II.A.7) occurs with ε in that formula taking the value of some $\delta \in (0,\frac{1}{3})$, then

$$U_v^{(d-2)}(x,T) \subset C^{(d-2)v}\big(x,(d-2)T^{\frac{1+\delta}{2}}\big).$$

Using Lemma II.A.3 with $k = d-2$ we obtain, for all $\delta \in (0, \frac{1}{3})$, $v \in (0, u_0]$, $x \in \mathbb{Z}^d$ and positive integer T,

$$\mathbb{Q}\left(x \in \mathcal{I}_1^v,\ \operatorname{cap}\left(C^{(d-2)v}(x, (d-2)T^{\frac{1+\delta}{2}})\right) < cT^{\frac{(1-\delta)(d-2)}{2}} v^{d-3}\right)$$
$$\leqslant C \exp\left(-cT^{\delta/2} v\right).$$

The result follows by taking $v = \frac{u}{d-2}$, $T = \left\lfloor \left(\frac{R}{d-2}\right)^{2-\varepsilon} \right\rfloor$ and $\delta = \frac{\varepsilon}{2-\varepsilon}$. □

We now have all the tools required to connect $x, y \in \mathcal{I}^u \cap [-R, R)^d$ through edges in $\widehat{\mathcal{I}}^u \cap [-CR, CR)^d$, as mentioned at the beginning of the Appendix.

Lemma II.A.5. *There exist constants $c > 0$ and $C < \infty$ such that for every $u \in (0, u_0]$, $R > 0$ and $x, y \in [-R, R)^d$,*

$$\mathbb{P}^I\left(x, y \in \mathcal{I}^u, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} y\right\}^c\right) \leqslant C \exp\left(-cR^{1/6} u\right).$$

Proof. Using the notation introduced at the beginning of the Appendix, we have

$$\mathbb{P}^I\left(x, y \in \mathcal{I}^u, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} y\right\}^c\right)$$
$$\leqslant \sum_{i,j=1}^{3} P\left(x \in \mathcal{I}_{i,3}^{u/3}, y \in \mathcal{I}_{j,3}^{u/3}, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} y\right\}^c\right).$$

Let us now fix $i, j \in \{1, 2, 3\}$ and let $k \in \{1, 2, 3\}$ be different from i and j. We define the events

$$E_1 = \left\{\operatorname{cap}\left(C_i^{u/3}(x, R)\right) \geqslant CR^{\frac{2(d-2)}{3}} u^{d-3}\right\},$$
$$E_2 = \left\{\operatorname{cap}\left(C_j^{u/3}(y, R)\right) \geqslant CR^{\frac{2(d-2)}{3}} u^{d-3}\right\},$$

and note that $E_1 \subset \{x \in \mathcal{I}_{i,3}^{u/3}\}$ and $E_2 \subset \{y \in \mathcal{I}_{j,3}^{u/3}\}$. Thus,

$$P\left(x \in \mathcal{I}_{i,3}^{u/3}, y \in \mathcal{I}_{j,3}^{u/3}, \left\{x \overset{\widehat{\mathcal{I}}^u \cap [-CR, CR)^d}{\longleftrightarrow} y\right\}^c\right)$$
$$\leqslant P\left((E_1 \cap E_2) \setminus \left\{C_i^u(x, R) \overset{\widehat{\mathcal{I}}_{k,3}^{u/3} \cap [-CR, CR)^d}{\longleftrightarrow} C_j^u(y, R)\right\}\right)$$
$$+ P\left(\{x \in \mathcal{I}_{i,3}^{u/3}\} \setminus E_1\right) + P\left(\{y \in \mathcal{I}_{j,3}^{u/3}\} \setminus E_2\right). \tag{II.A.8}$$

For $uR^{1/6} \geqslant 1$, we can now use Lemma II.A.1 to bound the first summand of (II.A.8) as

$$\begin{aligned} &P\Big((E_1 \cap E_2) \setminus \Big\{C_i^u(x,R) \overset{\widehat{\mathcal{I}}_{k,3}^{u/3} \cap [-CR,CR)^d}{\longleftrightarrow} C_j^u(y,R)\Big\}\Big) \\ &\leqslant C \exp\left(-cR^{2-d}u \times R^{\frac{4(d-2)}{3}} u^{2(d-3)}\right) \\ &\leqslant C \exp\left(-cR^{1/6}u\right). \end{aligned}$$

The second summand of (II.A.8) can also be bounded using Corollary II.A.4 with $\varepsilon = \frac{1}{3}$, and the result follows. □

We now come to the

Proof of Lemma II.3.2. Lemma II.3.2 is a simple consequence of Lemma II.A.5. Indeed, let us define $R' = \lfloor \varepsilon R/2C \rfloor$ with C as in Lemma II.A.5, and we can assume without loss of generality that $\varepsilon R \geqslant 2C$. We define for each $x \in \mathbb{Z}^d$ the events

$$A_x^{(1)} = \left\{\mathcal{I}^u \cap \left(x + [-R', R')^d\right) \neq \varnothing\right\} \text{ and}$$
$$A_x^{(2)} = \bigcap_{x,y \in \mathcal{I}^u \cap (x+[-2R',2R')^d)} \left\{x \overset{\widehat{\mathcal{I}}^u \cap x + [-\varepsilon R, \varepsilon R)^d}{\longleftrightarrow} y\right\}.$$

Note that these events depend on our choice of u and R even if it does not appear in the notation. It follows from the definition of random interlacements, (II.A.2) and Lemma II.A.5 that

$$\mathbb{P}^I\left(A_x^{(1)}\right) \geqslant 1 - \exp\left(-cR^{d-2}u\right) \text{ and } \mathbb{P}^I\left(A_x^{(2)}\right) \geqslant 1 - CR^{2d} \exp\left(-cR^{1/6}u\right).$$

In particular, we get that

$$\mathbb{P}^I\Bigg(\bigcap_{x \in [0,R)^d \cap \mathbb{Z}^d} A_x^{(1)} \cap A_x^{(2)}\Bigg) \geqslant 1 - CR^{3d} \exp\left(-cR^{1/6}u\right) \geqslant 1 - C\exp\left(-cR^{1/7}u\right).$$

Let us call A the event on the left-hand side in the previous line, and suppose that A occurs. Then, for all $x, y \in \mathcal{I}^u \cap [0,R)^d$, one can find a path of nearest neighbors between x and y in $[0,R)^d$. Moreover, if x and x' are two neighbors in $[0,R)^d$, then $(x + [-R', R')^d) \cup (x' + [-R', R')^d) \subset x + [-2R', 2R')^d$, so every vertex in

$$\mathcal{I}^u \cap (x + [-R', R')^d) \text{ is connected to every vertex in } \mathcal{I}^u \cap (x' + [-R', R')^d) \tag{II.A.9}$$

by a path of edges in $\widehat{\mathcal{I}}^u \cap (x + [-\varepsilon R, \varepsilon R)^d) \subset \widehat{\mathcal{I}}^u \cap [-\varepsilon R, (1+\varepsilon)R)^d$, and the sets in (II.A.9) are not empty. This tells us that if A occurs, then every $x, y \in \mathcal{I}^u \cap [0,R)^d$ can be connected by edges in $\widehat{\mathcal{I}}^u \cap [-\varepsilon R, (1+\varepsilon)R)^d$, and thus A implies the event on the left-hand side of (II.3.2). □

Chapter III

Geometry of the sign clusters and random interlacements on transient graphs

III.1 Introduction

This chapter rigorously investigates the phenomenon of *phase coexistence* which is associated to the geometry of certain random fields in their supercritical phase, characterized by the presence of strong, slowly decaying correlations. Our aim is to prove the existence of such a regime, and to describe the random geometry arising from the competing influences between two supercritical phases. The leitmotiv of this work is to study the sign clusters of the Gaussian free field in "high dimensions" (transient for the random walk), which offer a framework that is analytically tractable and has a rich algebraic structure, but questions of this flavor have emerged in various contexts, involving fields with similar large-scale behavior. One such instance is the model of random interlacements, introduced in [93] and also studied in this chapter, which relates to the broad question of how random walks tend to create interfaces in high dimension, see e.g. [91], [92], and also [106], [20]. Another case in point (not studied in this chapter) is the nodal domain of a monochromatic random wave, e.g. a randomized Laplace eigenfunction on the n-sphere $\mathbb{S}^n$, at high frequency, which appears to display supercritical behavior when $n \geq 3$, see [84] and references therein.

As a snapshot of the first of our main results, Theorem III.1.1 below gives an essentially complete picture of the sign cluster geometry of the Gaussian free field Φ (see (III.1.5) for its definition) on a large class of transient graphs G. It can be informally summarized as follows. Under suitable assumptions on G, which hold e.g. when $G = \mathbb{Z}^d$, $d \geq 3$ –but see (III.1.4) below for further examples, which hopefully convey the breadth of our setup–,

$$\text{there exist exactly two infinite sign clusters of } \Phi, \text{ one for each sign, which "consume all the ambient space," up to (stretched) exponentially small finite islands of } +/- \text{ signs;} \tag{III.1.1}$$

see Theorem III.1.1 for the corresponding precise statement. In fact, we will show that this regime of phase coexistence persists for level sets above small enough height $h = \varepsilon > 0$. It is worth emphasizing that (III.1.1) really comprises two distinct features, namely (i) the presence of unbounded sign clusters, which is an *existence* result, and (ii) their ubiquity, which is *structural* and forces bounded connected components to be very small. Our results further indicate a certain universality of this phenomenon, as the class of transient graphs G for which we can establish (III.1.1) includes possibly fractal geometries, see the examples (III.1.4) below, where random walks typically experience slowdown due to the presence of "traps at every scale," see e.g. [6], [42], [43] and the monograph [4].

As it turns out, the phase coexistence regime for sign(Φ) described by (III.1.1) is also related to the existence of a supercritical phase for the vacant set of random interlacements; cf. [93] and below (III.1.15) for a precise definition. This is due to a certain algebraic relation linking Φ and the interlacements, see [96], [57], [101], whose origins can be traced back to early work in constructive field theory, see [88], and also [18], [31], and which will be a recurrent theme throughout this work. Interestingly, the arguments leading to the phase described in (III.1.1), paired with the symmetry of Φ, allow us to embed (in distribution) a large part of the interlacement set inside its complement, the vacant set, at small levels. As a consequence, we deduce the existence of a supercritical regime of the latter by appealing to the good connectivity properties of the former, for all graphs G belonging to our class. We will soon return to these matters and explain them in due detail. For the time being, we note that these insights yield the answer to an important open question from [95], see the final Remark 5.6(2) therein and our second main result, Theorem III.1.2 below.

We now describe our results more precisely, and refer to Section III.2 for the details of our setup. We consider an infinite, connected, locally finite graph G endowed with a positive and symmetric weight function λ on the edges. To the data (G, λ), we associate a canonical discrete-time random walk, which is the Markov chain with transition probabilities given by $p_{x,y} = \lambda_{x,y}/\lambda_x$, where $\lambda_x = \sum_{y \in G} \lambda_{x,y}$. It is characterized by the generator

$$Lf(x) = \frac{1}{\lambda_x} \sum_{y \in G} \lambda_{x,y}(f(y) - f(x)), \text{ for } x \in G, \tag{III.1.2}$$

for $f : G \to \mathbb{R}$ with finite support. We assume that the transition probabilities of this walk are uniformly bounded from below, see (p_0) in Section III.2, and writing $g(x,y)$, $x,y \in G$, for the corresponding Green density, see (III.2.3) below, that

$$\begin{aligned}&\text{there exist parameters } \alpha \text{ and } \beta \text{ with } 2 \le \beta < \alpha \\ &\text{such that, for some distance function } d(\cdot,\cdot) \text{ on } G, \\ &\lambda(B(x,L)) \asymp L^\alpha \text{ and } g(x,y) \asymp d(x,y)^{-(\alpha-\beta)}, \text{ for } x,y \in G,\end{aligned} \tag{III.1.3}$$

where $\asymp$ means that the quotient is uniformly bounded from above and below by positive constants, $B(x,L)$ is the ball of radius L in the metric $d(\cdot,\cdot)$ and $\lambda(A) = \sum_{x\in A} \lambda_x$ is the measure of $A \subset G$, see (V_α) and (G_β) in Section III.2 for the precise formulation of (III.1.3). The exponent β in (III.1.3) reflects the diffusive (when $\beta = 2$) or sub-diffusive (when $\beta > 2$) behavior of the walk on G, cf. Proposition III.3.3 below. Note that the condition on $g(\cdot,\cdot)$ in (III.1.3) implies in particular that G is transient for the walk. For more background on why condition (III.1.3) is natural, we refer to [42], [43] as well as Theorem III.2.2 and Remark III.3.4 below regarding its relation to heat kernel estimates. As will further become apparent in Section III.3, see in particular Proposition III.3.5 and Corollary III.3.9, choosing d to be the graph distance on G is not necessarily a canonical choice, for instance when G has a product structure.

Apart from (p_0), (V_α) and (G_β), we will often make one additional geometric assumption (WSI) on G, introduced in Section III.2. Roughly speaking, this hypothesis ensures a (weak) sectional isoperimetry of various large subsets of G, which allows for certain contour arguments. Rather than explaining this in more detail, we single out the following representative examples of graphs, which satisfy all four aforementioned assumptions (p_0), (V_α), (G_β) and (WSI), cf. Corollary III.3.9 below:

$$\begin{aligned}&G_1 = \mathbb{Z}^d, \text{ with } d \ge 3, \\ &G_2 = G' \times \mathbb{Z}, \text{ with } G' \text{ the discrete skeleton of the Sierpinski gasket}, \\ &G_3 = \text{the standard } d\text{-dimensional graphical Sierpinski carpet for } d \ge 3, \\ &G_4 = \begin{array}{l}\text{a Cayley graph of a finitely generated group } \Gamma = \langle S \rangle \text{ with } S = S^{-1} \\ \text{having polynomial volume growth of order } \alpha > 2\end{array}\end{aligned} \tag{III.1.4}$$

(see e.g. [6], pp.6–7 for definitions of G' and G_3, the latter corresponds to $V^{(d)}$ in the notation of [6]), all endowed with unit weights and a suitable distance function d (see Remark III.2.1 and Section III.3). The graph G_2 is a benchmark case for various aspects of [95], to which we will return in Theorem III.1.2 below. The case G_3 underlines the fact that even in the fractal context a product structure is not necessarily required.

The fact that (WSI) holds in cases G_2, G_3 and G_4 is not evident, and will follow by expanding on results of [108], see Section III.3. In the case of G_4, (WSI) crucially relies on Gromov's deep structural result [45]. The reader may choose to focus on (III.1.4), or even G_1, for the purpose of this introduction.

Our first main result deals with the Gaussian free field Φ on the weighted graph (G, λ). Its canonical law $\mathbb{P}^G$ is the unique probability measure on $\mathbb{R}^G$ such that $(\Phi_x)_{x\in G}$ is a mean zero Gaussian field with covariance function

$$\mathbb{E}^G[\Phi_x\Phi_y] = g(x,y), \text{ for any } x,y \in G. \tag{III.1.5}$$

On account of (III.1.3), Φ exhibits (strong) algebraically decaying correlations with respect to the distance d, captured by the exponent

$$\nu \overset{\text{def.}}{=} \alpha - \beta \ (> 0). \tag{III.1.6}$$

We study the geometry of Φ in terms of its level sets

$$E^{\geq h} \overset{\text{def.}}{=} \{y \in G;\ \Phi_x \geq h\},\ h \in \mathbb{R}. \tag{III.1.7}$$

The random set $E^{\geq h}$ decomposes into connected components, also referred to as *clusters*: two points belong to the same cluster of $E^{\geq h}$ if they can be joined by a path of edges whose endpoints all lie inside $E^{\geq h}$. Finite clusters are sometimes called *islands*.

As h varies, the onset of a supercritical phase in $E^{\geq h}$ is characterized by a critical parameter $h_* = h_*(G)$, which records the emergence of infinite clusters,

$$h_* \overset{\text{def.}}{=} \inf\big\{h \in \mathbb{R};\ \mathbb{P}^G\big(\text{there exists an infinite cluster in } E^{\geq h}\big) = 0\big\} \tag{III.1.8}$$

(with the convention $\inf\varnothing = \infty$). The existence of a nontrivial phase transition, i.e., the statement $-\infty < h_* < \infty$, was initially investigated in [16], and even in the case $G = G_1 = \mathbb{Z}^d$ with $d \geq 3$, has only been completely resolved recently in [81]. It was further shown in Corollary 2 of [16] that $h_* \geq 0$ on $\mathbb{Z}^d$, and this proof can actually be adapted to any locally finite transient weighted graph, see the Appendix of [1], or [57] for a different proof.

Of particular interest are the connected components of $E^{\geq 0}$. The symmetry of Φ implies that $E^{\geq 0}$ and its complement in G have the same distribution. The connected components of $E^{\geq 0}$ and its complement are referred to as the positive and negative sign clusters of Φ, respectively. It is an important problem to understand if these sign clusters fall into a *supercritical* regime (below h_*), and, if so, what the resulting sign cluster geometry of Φ looks like. In order to formulate our results precisely, we introduce a critical parameter $\bar{h}$ characterizing a regime of *local uniqueness* for $E^{\geqslant h}$, whose distinctive features (III.1.10) and (III.1.11) below reflect (i) and (ii) in the discussion following (III.1.1). Namely,

$$\bar{h} = \sup\{h \in \mathbb{R};\ \Phi \text{ strongly percolates above level } h' \text{ for all } h' < h\}, \quad \text{(III.1.9)}$$

where the Gaussian free field Φ is said to strongly percolate above level h if there exist constants $c(h) > 0$ and $C(h) < \infty$ such that for all $x \in G$ and $L \geqslant 1$,

$$\mathbb{P}^G\left(\begin{array}{c} E^{\geqslant h} \cap B(x,L) \text{ has no connected} \\ \text{component with diameter at least } \frac{L}{5} \end{array}\right) \leq Ce^{-L^c} \quad \text{(III.1.10)}$$

and

$$\mathbb{P}^G\left(\begin{array}{c} \text{there exist connected components of } E^{\geqslant h} \cap B(x,L) \\ \text{with diameter at least } \frac{L}{10} \text{ which are not connected} \\ \text{in } E^{\geqslant h} \cap B(x, 2C_{10}L) \end{array}\right) \leq Ce^{-L^c} \quad \text{(III.1.11)}$$

(the constant C_{10} is defined in (III.3.4) below). With the help of (III.1.10), (III.1.11) and a Borel-Cantelli argument, one can easily patch up large clusters in $E^{\geq h} \cap B(x, 2^k)$ for $k \geq 0$ when $h < \bar{h}$ to deduce that $\bar{h} \leq h_*$. One also readily argues that for all $h < \bar{h}$, there is a *unique* infinite cluster in $E^{\geq h}$, as explained in (III.2.12) below.

We will prove the following result, which makes (III.1.1) precise. For reference, conditions (p_0), (V_α), (G_β) and (WSI) appearing in (III.1.13) are defined in Section III.2. All but (p_0) depend on the choice of metric d on G. Following (III.1.3), in assuming that conditions (V_α), (G_β) and (WSI) are met in various statements below, we understand that

> (V_α), (G_β) and (WSI) hold with respect to *some* distance function $d(\cdot,\cdot)$ on G, for *some* values of α and β satisfying $\alpha > 2$ and $\beta \in [2, \alpha)$. (III.1.12)

Theorem III.1.1.

If (p_0), (V_α), (G_β) *and* (WSI) *hold, then* $\bar{h} > 0$. (III.1.13)

The proof of Theorem III.1.1 is given in Section III.9. For a list of pertinent examples, see (III.1.4) and Section III.3, notably Corollary III.3.9 below, which implies that all conditions appearing in (III.1.13) hold true for the graphs listed in (III.1.4), and in particular for $\mathbb{Z}^d$, $d \geq 3$. Some progress in the direction of Theorem III.1.1 was obtained in Chapter II, where it was shown that $h_*(\mathbb{Z}^d) >$

0. The sole existence of an infinite sign cluster without proof of (III.1.11) at small enough $h \geq 0$ can be obtained under slightly weaker assumptions, see condition ($\widetilde{\text{WSI}}$) in Remark III.8.5 and Theorem III.9.3 below. As an immediate consequence of (III.1.10), (III.1.11) and (III.1.13), we note that for all $h < \bar{h}$, and in particular when $h = 0$, denoting by $\mathscr{C}^h(x)$ the cluster of x in $E^{\geq h}$,

$$\mathbb{P}^G\big(L \leq \operatorname{diam}\big(\mathscr{C}^h(x)\big) < \infty\big) \leq Ce^{-L^c}. \tag{III.1.14}$$

The parameter $\bar{h}$, or a slight modification of it, see Remark III.9.4, 1) below, has already appeared when $G = \mathbb{Z}^d$ in [28], [100], [70], [83], [11] and [21] to test various geometric properties of the percolation cluster in $E^{\geqslant h}$ in the regime $h < \bar{h}$; note that $\bar{h} > -\infty$ is known to hold on $\mathbb{Z}^d$ as a consequence of Theorem 2.7 in [28], thus making these results not vacuously true, but little is known about $\bar{h}$ otherwise. These findings can now be combined with Theorem III.1.1. For instance, as a consequence of (III.1.13) and Theorem 1.1 in [70], when $G = \mathbb{Z}^d$, denoting by $\mathscr{C}_\infty^+$ the infinite +-sign cluster,

$\mathbb{P}^G$-a.s., conditionally on starting in $\mathscr{C}_\infty^+$, the random walk on $\mathscr{C}_\infty^+$
(see below (1.2) in [70] for its definition) converges weakly to a
non-degenerate Brownian motion under diffusive rescaling of space and time.
(III.1.15)

We refer to the above references for further results exhibiting, akin to (III.1.15), the "well-behavedness" of the phase $h < \bar{h}$, to which the sign clusters belong.

We now introduce and state our results regarding *random interlacements*, leading to Theorem III.1.2 below, and explain their significance. As alluded to above, cf. also the discussion following Theorem III.1.2 for further details, the interlacements, which constitute a Poisson cloud ω^u of bi-infinite random walk trajectories as in (III.1.2) modulo time-shift, were introduced on $\mathbb{Z}^d$ in [93], see also [103] and Section III.2, and naturally emerge due to their deep ties to Φ. The parameter $u > 0$ appears multiplicatively in the intensity measure of ω^u and hence governs how many trajectories enter the picture – the larger u, the more trajectories. The law of the interlacement process $(\omega^u)_{u>0}$ is denoted by $\mathbb{P}^I$ and the random set $\mathcal{I}^u \subset G$, the interlacement set at level u, is the subset of vertices of G visited by at least one trajectory in the support of ω^u. Its complement $\mathcal{V}^u = G \setminus \mathcal{I}^u$ is called the *vacant set (at level u).* The process ω^u is also related to the loop-soup construction of [53], if one "closes the bi-infinite trajectories at infinity," as in [98].

Originally, ω^u was introduced in order to investigate the local limit of the trace left by simple random walk on large, locally transient graphs $\{G_N;\, N \geq 1\}$ with $G_N \nearrow G$ as $N \to \infty$, when run up to suitable timescales of the form

$u\,t_N$ with $u > 0$ and $t_N = t_N(G_N)$, see [10], [89], [91], [92], [106], as well as [110] and [20]. The trajectories in the support of ω^u can roughly be thought of as corresponding to successive excursions of the random walk in suitably chosen sets, and the timescale t_N defines a Poissonian limiting regime for the occurrence of these excursions (note that this limit is hard to establish due to the long-range dependence between the excursions of the walk). Of particular interest in this context are the percolative properties of $\mathcal{V}^u$, as described by the critical parameter (note that $\mathcal{V}^u$ is decreasing in u)

$$u_* \stackrel{\text{def.}}{=} \inf\big\{u \geq 0;\, \mathbb{P}^I(\text{there exists an infinite connected component in } \mathcal{V}^u) = 0\big\}. \tag{III.1.16}$$

This corresponds to a drastic change in the behavior of the complement of the trace of the walk on G_N, as the parameter u appearing multiplicatively in front of t_N varies across u_*, provided this threshold is non-trivial; see for instance [106] for simulations when $G_N = (\mathbb{Z}/N\mathbb{Z})^d$ with $t_N = N^d$. The finiteness of u_*, i.e. the existence of a subcritical phase for $\mathcal{V}^u$, and even a phase of stretched exponential decay for the connectivity function of $\mathcal{V}^u$ at large values of u, can be obtained by adapting classical techniques, once certain decoupling inequalities are available. As a consequence of Theorem III.2.4 below, see Remark III.7.2, 1) and Corollary III.7.3, such a phase is exhibited for any graph G satisfying (p_0), (V_α) and (G_β) as in (III.1.12).

On the contrary, the existence of a supercritical phase is much less clear in general. It was proved in [95] that $u_* > 0$ for graphs of the type $G = G' \times \mathbb{Z}$, endowed with some distance d such that (III.1.3) holds, see (1.8) and (1.11) in [95]. However, in this source the condition $\nu \geq 1$ was required, cf. (III.1.6), excluding for instance the case $G = G_2$ in which $\nu = \frac{\log 9 - \log 5}{\log 4} < 1$, see [50] and [2]. As a consequence of the following result, we settle the question about positivity of u_* affirmatively under our assumptions. This solves a principal open problem from [95], see Remark 5.6(2) therein, and implies the existence of a phase transition for the percolation of the vacant set $\mathcal{V}^u$ of random interlacements on such graphs. We remind the reader of the convention (III.1.12) regarding conditions (V_α), (G_β) and (WSI), which is in force in the following:

Theorem III.1.2. *Suppose G satisfies (p_0), (V_α), (G_β) and (WSI). Then there exists $\widetilde{u} > 0$ and for every $u \in (0, \widetilde{u}]$, a probability space $(\Omega^u, \mathcal{F}^u, Q^u)$ governing three random subsets $\mathcal{I}$, $\mathcal{V}$ and $\mathcal{K}$ of G with the following properties:*

i) $\mathcal{I}$, resp. $\mathcal{V}$, have the law of $\mathcal{I}^u$, resp. $\mathcal{V}^u$, under $\mathbb{P}^I$.

ii) $\mathcal{K}$ is independent of $\mathcal{I}$. (III.1.17)

iii) Q^u-a.s., $\mathcal{I} \cap \mathcal{K}$ contains an infinite cluster, and $(\mathcal{I} \cap \mathcal{K}) \subset \mathcal{V}$.

A fortiori, $u_* \geq \widetilde{u}(>0)$.

Thus, our construction of an infinite cluster of $\mathcal{V}^u$ for small $u > 0$, and hence our resolution of the conjecture in [95], proceeds by stochastically embedding a large part of its complement, $\mathcal{I}^u \cap \mathcal{K}$ inside $\mathcal{V}^u$. The law of the set $\mathcal{K}$ can be given explicitly, see Remark III.9.4, 2), and $\mathcal{K}$ could also be chosen independent of $\mathcal{V}$ instead of $\mathcal{I}$, see Remark III.9.4, 3).

While we will in fact deal more generally with product graphs in Section III.3, let us elaborate shortly on the important case $G = G' \times \mathbb{Z}$ considered in [95]. In this setting, the conclusions of Theorem III.1.2 hold under the mere assumptions that (p_0) holds and G' satisfies the upper and lower heat kernel estimates (UHK(α,β)) and (LHK(α,β)), see Remark III.2.2, with respect to $d = d_{G'}$, the graph distance on G', for some $\alpha > 1$ and $\beta \in [2, 1+\alpha)$; for instance, if $G = G_2$ from (III.1.4), then $\alpha = \frac{\log 3}{\log 2}$ and $\beta = \frac{\log 5}{\log 2}$, see [7, 50]. This (and more) will follow from Proposition s III.3.5 and III.3.7 below; see also Remark III.3.10 for further examples. Incidentally, let us note that Theorem III.1.2 is also expected to provide further insights into the disconnection of cylinders $G_N \times \mathbb{Z}$ by a simple random walk trace, for G_N a large finite graph, for instance when G_N is a ball of radius N in the discrete skeleton of the Sierpinski gasket (corresponding to G_2 of (III.1.4)), cf. Remark 5.1 in [89].

Since Theorem III.1.2 builds on the arguments leading to Theorem III.1.1, we delay further remarks concerning (III.1.17) for a few lines, and first provide an overview of the proof of Theorem III.1.1.

As hinted at above, a key ingredient and the starting point of the proof of Theorem III.1.1 is a certain isomorphism theorem, see [96], [57], [101] and (III.5.2) and Corollary III.5.3 below, which links the free field Φ to the interlacement ω^u. The argument unfolds by first studying the random set $\mathcal{I}^u$, which has remarkable connectivity properties: even though its density tends to 0 as $u \downarrow 0$, $\mathcal{I}^u$ is an *unbounded connected* set for *every* $u > 0$. Much more is in fact true, see Section III.4, in particular Proposition III.4.1 below, the set $\mathcal{I}^u$ is actually *locally* well-connected. These features of $\mathcal{I}^u$, especially for u close to 0, will figure prominently in our construction of various large random sets, and ultimately serve as an indispensable tool to build percolating sign clusters. Indeed, as a consequence of the aforementioned correspondence between Φ and ω^u, see also (III.5.4) below, one can use $\mathcal{I}^u$ in a first step as a system of "highways" to produce connections inside $E^{\geq -h}$, for ever so small $h = \sqrt{2u} > 0$.

A substantial part of these connections persists to exist in $\widetilde{E}^{\geq -h}$ $(h > 0)$, the level sets of the free field $\widetilde{\varphi}$ on a continuous extension $\widetilde{G}$ of the graph, the associated *cable system.* This object, to which all above processes can naturally be extended, goes back at least to [8] and is obtained by replacing the edges between vertices by one-dimensional cables. This result, which quantifies and strengthens the early insight $h_*(\mathbb{Z}^d) \geq 0$ of [16] – deduced therein by a soft but indirect and general argument – is in fact sharp on the cables, see Theorem III.9.5 below. Importantly, the recent result of [101], which can be applied in our framework, see Corollary III.5.3, further allows to formulate a condition in terms of an (auxiliary) Gaussian free field $\widetilde{\gamma}$ appearing in the isomorphism and $\widetilde{\mathcal{I}}^u$, the continuous interlacement, for points in $\widetilde{E}^{\geq -h}$ to "rapidly" (i.e. at scale L_0 in the renormalization argument detailed in the next paragraph) connect to the interlacement $\widetilde{\mathcal{I}}^{u=h^2/2}$. Following ideas from Chapter II, we can then rely on a certain robustness property exhibited on the cables to pass from $\widetilde{E}^{\geq -h}$ to $E^{\geq +h}$ by means of a suitable coupling, which operates independently at any given vertex when certain favorable conditions are met. These conditions in turn become typical as $u \to 0^+$, see Lemma III.5.5 and Proposition III.5.6.

The previous observations can be combined into a set of *good features*, assembled in Definition III.7.4 below, which are both increasingly likely as $L_0 \to \infty$ and entirely local, in that all properties constituting a good vertex $x \in G$ are phrased in terms of the various fields inside balls of radius $\approx L_0$ in the distance d around x. This notion can then be used as the starting point of a renormalization argument, presented in Sections III.7 and III.8, to show that good regions form large connected connected components. Importantly, with a view towards (III.1.10) and (III.1.11), good regions need not only to *form* but do so *everywhere* inside of G. This comes under the proviso of (WSI) as a feature of the renormalization scheme, which ensures that subsets of G having large diameter are typically connected by paths of good vertices, see Lemmas III.8.6 and III.8.7 below. Using additional randomness, the connection by paths of good vertices is turned into a connection by paths in $E^{\geq h}$, and this completes the proof of Theorem III.1.1, see Section III.9.

A renormalization of the parameters involved in the scheme is necessary due to the presence of the strong correlations, and it relies on suitable decoupling inequalities, see Theorem III.2.4 below. At the level of generality considered here, namely assuming only (p_0), (V_α), (G_β), and particularly in the case of $\mathcal{I}^u$, see (III.2.21), these inequalities generalize results of [95] and are interesting in their own right. At the technical level, they are eventually obtained from the soft local time technique introduced in [68] and developed therein on $\mathbb{Z}^d$. The difficulty stems from having to control the resulting error term, which is key in obtaining (III.2.21). This control ultimately rests on chaining arguments and a suitable elliptic Harnack inequality, see in particular Lemmas III.6.5 and III.6.7, which provides good bounds if certain sets of interest do not get too close (note that, due to their Euclidean nature, the arguments leading to the precise controls of [68] valid even at short distances seem out of reach within the current setup). Fortunately, this is good enough for the purposes we have in mind.

The proof of Theorem III.1.2 then proceeds by using the results leading to Theorem III.1.1 and adding one more application of the coupling provided in Corollary III.5.3. Indeed, the above steps essentially allow to roughly translate the probabilities in (III.1.10) and (III.1.11) regarding $E^{\geq h}$, for $h > 0$ in terms of the interlacement $\mathcal{I}^u$, for $u = h^2/2$ and some "noise", see Lemma III.8.4 and (the proof of) Lemma III.8.7, but $E^{\geq h}$ is in turn naturally embedded into $\mathcal{V}^u$, see (III.5.4). Following how the percolative regime for $\mathcal{V}^u$ is obtained, one thus starts with its complement $\mathcal{I}^u$, first passes to Φ and proves the phase coexistence regime around $h = 0$ asserted in Theorem III.1.1, and then translates back to $\mathcal{V}^u$. The existence of the phase coexistence regime along with the symmetry of Φ is then ultimately responsible for producing the inclusion *iii*) in (III.1.17). The set $\mathcal{K}$ appearing there morally corresponds to all the undesired noise produced by bad regions in the argument leading to Theorem III.1.1. It would be interesting to devise a direct argument for $u_* > 0$ which by-passes the use of Φ. We are currently unable to do so, except when $\nu > 1$, in which case the reasoning of [95] can be adapted, see Remark III.7.2, 2). We refer to Remark III.9.4, 5)–8) for further open questions.

We now describe how this chapter is organized. Section III.2 introduces the precise framework, the processes of interest and, importantly, the conditions (p_0), (V_α), (G_β) and (WSI) appearing in our main results. We then collect some first consequences of this setup. The decoupling inequalities mentioned above are stated in Theorem III.2.4 at the end of that section.

Section III.3 has two main purposes. After gathering some preliminary tools from harmonic analysis (for L in (III.1.2)), which are used throughout, we first discuss in Proposition III.3.5 how (V_α), (G_β) are obtained for product graphs of the form $G = G' \times G''$, when the factors satisfy suitable heat kernel estimates. This has important applications, notably to the graph $G = G_2$ in (III.1.4), and requires that we work with general distances d in conditions (V_α), (G_β). For this reason, we have also included a proof of the classical (in case $d = d_G$, the graph distance) estimates of Proposition III.3.3 in the appendix. The second main result of Section III.3 is to deduce in Corollary III.3.9 that the relevant conditions (p_0), (V_α), (G_β) and (WSI) appearing in Theorems III.1.1 and III.1.2 apply in all cases of (III.1.4). In addition to Proposition III.3.5, this requires proving (WSI) and dealing with boundary connectivity properties of connected sets, which is the object of Proposition III.3.7.

Section III.4 collects the local connectivity properties of the continuous interlacement set $\widetilde{\mathcal{I}}^u$, see Proposition III.4.1 and Corollary III.4.2. The overall strategy is similar to what was done in [73] on $\mathbb{Z}^d$, see also Chapter II, to which we frequently refer. The proof of Proposition III.4.1 could be omitted on first reading.

Section III.5 is centered around the isomorphism on the cables. The main takeaway for later purposes is Corollary III.5.3, see also Remark III.5.4, which asserts that the coupling of Theorem 2.4 in [101] can be constructed in our framework. This requires that certain conditions be met, which are shown in Lemma III.5.1 and Proposition III.5.2. The latter also yields the desired inclusion (III.5.4). The generic absence of ergodicity makes the verification of these properties somewhat cumbersome. Lemma III.5.5 contains the adaptation of the sign-flipping argument from Chapter II, from which certain desirable couplings needed later on in the renormalization are derived in Proposition III.5.6. Section III.5 closes with a more detailed overview over the last four sections, leading to the proofs of our main results.

Section III.6 is devoted to the proof of Theorem III.2.4, which contains the decoupling inequalities. While the free field can readily be dispensed with by adapting results of [67], the interlacements are more difficult to deal with. We apply the soft local times technique from [68]. All the work lies in controlling a corresponding error term, see Lemma III.6.6. The regularity estimates for hitting probabilities needed in this context, see the proof of Lemma III.6.7, rely on Harnack's inequality, see Lemma III.6.5 for a tailored version.

Section III.7 introduces the renormalization scheme needed to put together the ingredients of the proof, which uses the decoupling inequalities of Theorem III.2.4. The important Definition III.7.4 of good vertices appears at the end of that section, and Lemma III.7.6 collects the features of good long paths, which are later relied upon. The good properties appearing in this context are expressed in terms of (an extension of) the coupling from Corollary III.5.3.

Section III.8 takes advantage of the renormalization scheme introduced in Section III.7 to create a giant and ubiquitous cluster of good vertices, and of random interlacements with suitable properties. Proposition III.8.3 first yields the desired estimate that long paths of bad vertices are very unlikely, for suitable choices of the parameters. Lemmas III.8.4 and III.8.7 provide precursor estimates to (III.1.10) and (III.1.11), which are naturally associated to our notion of goodness. In particular, Lemma III.8.7 directly implies that $\overline{h} \geq 0$ as a first step toward Theorem III.1.1, see Corollary III.8.8. An important technical step with regards to Lemma III.8.7 is Lemma III.8.6, which asserts that large sets in diameter are typically connected by a path of good vertices.

The pieces are put together in Section III.9, and the proofs of Theorems III.1.1 and III.1.2 appear towards the end of this last section. Proposition III.5.6

exhibits the coupling transforming (for instance) giant good regions from Lemma III.8.7 into giant subsets of $E^{\geq h}$, $h > 0$, see Lemma III.9.2, from which (III.1.10) and (III.1.11) are eventually inferred. Finally, Section III.9 also contains the simpler existence result, Theorem III.9.3, alluded to above, which can be obtained under a slightly weaker condition ($\widetilde{\text{WSI}}$), introduced in Remark III.8.5.

We conclude this introduction with our convention regarding constants. In the rest of this chapter, we denote by $c, c', \dots$ and $C, C', \dots$ positive constants changing from place to place. Numbered constants c_0, C_0, c_1, $C_1, \dots$ are fixed when they first appear and do so in increasing numerical order. All constants may depend implicitly "on the graph G" through conditions (p_0), (V_α) and (G_β) below, in particular they may depend on α and β. Their dependence on any other quantity will be made explicit.

For the reader's orientation, we emphasize that the conditions (p_0), (V_α), (G_β) and (WSI), which will be frequently referred to, are all introduced in Section III.2. We seize this opportunity to highlight the set of assumptions (III.3.1) on (G, λ) appearing at the beginning of Section III.3, which will be in force from then on until the end.

III.2 Basic setup and first properties

In this section, we introduce the precise framework alluded to in the introduction, formulate the assumptions appearing in Theorems III.1.1 and III.1.2, and collect some of the basic geometric features of our setup. We also recall the definitions and several useful facts concerning the two protagonists, random interlacements and the Gaussian free field on G, as well as their counterparts on the cable system. We then state in Theorem III.2.4 the relevant decoupling inequalities for both interlacements and the free field, which will be proved in Section III.6.

Let (G, E) be a countably infinite and connected graph with vertex set G and (unoriented) edge set $E \subset G \times G$. We will often tacitly identify the graph (G, E) with its vertex set G. We write $x \sim y$, or $y \sim x$, if $\{x, y\} \in E$, i.e., if x and y are connected by an edge in G. Such vertices x and y will be called *neighbors*. We also say that two edges in E are neighbors if they have a common vertex. A *path* is a sequence of neighboring vertices in G, finite or infinite. For $A \subset G$, we set $A^{\mathsf{c}} = G \setminus A$, we write $\partial A = \{y \in A;\, \exists\, z \in A^{\mathsf{c}},\, z \sim y\}$ for its inner boundary, and define the external boundary of A by

$$\partial_{ext} A \overset{\text{def.}}{=} \left\{ \begin{array}{c} y \in A^{\mathsf{c}};\ \exists \text{ an unbounded path in } A^{\mathsf{c}} \\ \text{beginning in } y \text{ and } \exists\, z \in A,\ z \sim y \end{array} \right\} \tag{III.2.1}$$

We write $x \leftrightarrow y$ in A (or $x \overset{A}{\longleftrightarrow} y$ in short) if there exists a nearest-neighbor path in A containing x and y, and we say that A is *connected* if $x \overset{A}{\longleftrightarrow} y$ for any $x, y \in A$. For all $A_1 \subset A_2 \subset G$, we write $A_1 \subset\subset A_2$ to express that A_1 is a finite subset of A_2. We endow G with a non-negative and symmetric weight function $\lambda = (\lambda_{x,y})_{x,y\in G}$, such that $\lambda_{x,y} \geq 0$ for all $x, y \in G$ and $\lambda_{x,y} > 0$ if and only if $\{x, y\} \in E$. We define the weight of a vertex $x \in G$ and of a set $A \subset G$ by $\lambda_x = \sum_{y\sim x} \lambda_{x,y}$ and $\lambda(A) = \sum_{x\in A} \lambda_x$. We often regard $\{\lambda_x : x \in G\}$ as a positive measure on G endowed with its power set σ-algebra in the sequel.

To the weighted graph (G, λ), we associate the discrete-time Markov chain with transition probabilities

$$p_{x,y} \overset{\text{def.}}{=} \frac{\lambda_{x,y}}{\lambda_x}, \quad \text{for } x, y \in G. \tag{III.2.2}$$

We write P_x, $x \in G$, for the canonical law of this chain started at x, and $Z = (Z_n)_{n\geq 0}$ for the corresponding canonical coordinates. For a finite measure μ on G, we also set

$$P_\mu \overset{\text{def.}}{=} \sum_{x\in G} \mu(x) P_x. \tag{III.2.3}$$

Our assumptions, see in particular (G_β) below, will ensure that Z is in fact transient. We assume that G has controlled weights, i.e., there exists a constant c_0 such that

$$p_{x,y} \geq c_0 \text{ for all } x \sim y \in G. \tag{p_0}$$

Note that (p_0) implies that each $x \in G$ has at most $\lfloor 1/c_0 \rfloor$ neighbors, so G has uniformly bounded degree.

We introduce the symmetric Green function associated to Z,

$$g(x, y) \overset{\text{def.}}{=} \frac{1}{\lambda_y} E_x\Big[\sum_{k=0}^{\infty} 1_{\{Z_k=y\}}\Big] \text{ for all } x, y \in G. \tag{III.2.4}$$

For $A \subset G$, we let $T_A \overset{\text{def.}}{=} \inf\{k \geq 0;\, Z_k \notin A\}$, the first exit time of A and $H_A \overset{\text{def.}}{=} T_{A^c} = \inf\{k \geq 0;\, Z_k \in A\}$ the first entrance time in A, and introduce the killed Green function

$$g_A(x, y) \overset{\text{def.}}{=} \frac{1}{\lambda_y} E_x\Big[\sum_{k=0}^{T_A} 1_{\{Z_k=y\}}\Big] \text{ for all } x, y \in A. \tag{III.2.5}$$

Applying the strong Markov property at time T_A for $A \subset\subset G$, we obtain the relation

$$E_x[g(Z_{T_A}, y)] + g_A(x, y) = g(x, y), \text{ for all } x, y \in A,. \tag{III.2.6}$$

Finally, the heat kernel of Z is defined as

$$p_n(x,y) = \lambda_y^{-1} P_x(Z_n = y) \text{ for all } x, y \in G \text{ and } n \in \mathbb{N}. \tag{III.2.7}$$

We further assume that G is endowed with a distance function d.

Remark III.2.1. A natural choice is $d = d_G$, the graph distance on G, but this does not always fit our needs. We will return to this point in the next section. Roughly speaking, some care is needed due to our interest in product graphs such as G_1 in (III.1.4), and more generally graphs of the type $G = G' \times \mathbb{Z}$ as in [95]. This is related to the way by which conditions (V_α) and (G_β) below propagate to a product graph, especially in cases where the factors have different diffusive scalings, see Proposition III.3.5 and in particular (III.3.22) below. □

We denote by $B(x,L)=\{y \in G : d(x,y) \leq L\}$ the closed ball of center x and radius L for the distance d and by $B_E(x,L)$ the set of edges for which both endpoints are in $B(x,L)$. For all $A \subset G$, we write $d(A,x) = \inf_{y\in A} d(y,x)$ for the distance between $A \subset G$ and $x \in G$, $B(A,L) \stackrel{\text{def.}}{=} \{y \in G : d(A,y) \leq L\}$ is the closed L-neighborhood of A and $\delta(A) \stackrel{\text{def.}}{=} \sup_{x,y\in A} d(x,y) \in [0,\infty]$ the diameter of A. Note that unless $d = d_G$, balls in the distance d are not necessarily connected in the sense defined below (III.2.1).

We now introduce two – natural, see Theorem III.2.2 below – assumptions on (G,λ), one geometric and the other analytic. We suppose that G has regular volume growth of degree α with respect to d, that is, there exists $\alpha > 2$ and constants $0 < c_1 \leq C_1 < \infty$ such that

$$c_1 L^\alpha \leq \lambda\big(B(x,L)\big) \leq C_1 L^\alpha, \text{ for all } x \in G \text{ and } L \geq 1. \tag{V_α}$$

We also assume that the Green function g has the following decay: there exist constants $0 < c_2 \leq C_2 < \infty$ such that, with α as in (V_α), for some $\beta \in [2,\alpha)$, g satisfies

$$\begin{aligned} &c_2 \leq g(x,x) \leq C_2 \text{ for all } x \in G \text{ and} \\ &c_2 d(x,y)^{-\nu} \leq g(x,y) \leq C_2 d(x,y)^{-\nu} \text{ for all } x \neq y \in G, \end{aligned} \tag{G_β}$$

where we recall that $\nu = \alpha - \beta$ from (III.1.6). The parameter $\beta \geq 2$ in (III.1.6) can be thought of as characterizing the order of the mean exit time from balls (of radius L), which grows like L^β as $L \to \infty$, see Lemma III.A.1.

Remark III.2.2 (Equivalence to heat kernel bounds). The above assumptions are very natural. Indeed, in case $d(\cdot,\cdot)$ is the graph distance – but see Remark III.2.1 above – the results of [42], see in particular Theorem 2.1 therein, assert that, assuming (p_0), the conditions (V_α) and (G_β) are equivalent to the following sub-Gaussian estimates on the heat kernel: for all $x, y \in G$ and $n \geq 0$,

$$p_n(x,y) \leq Cn^{-\frac{\alpha}{\beta}} \exp\left\{ -\left(\frac{d(x,y)^\beta}{Cn}\right)^{\frac{1}{\beta-1}} \right\} \tag{UHK(α,β)}$$

and, if $n \geq d_G(x,y)$,

$$p_n(x,y) + p_{n+1}(x,y) \geq cn^{-\frac{\alpha}{\beta}} \exp\left\{ -\left(\frac{d(x,y)^\beta}{cn}\right)^{\frac{1}{\beta-1}} \right\}. \tag{LHK(α,β)}$$

Many examples of graphs G for which (UHK(α,β)) and (LHK(α,β)) hold for the graph distance are given in [50], [5] and [46], and further characterizations of these estimates can be found in [43], [3], [7] and [4]. We will return to the consequences of (V_α), (G_β), and their relation to estimates of the above kind within our framework, i.e., for general distance function d, in Section III.3, cf. Proposition III.3.3 and Remark III.3.4 below. □

We now collect some simple geometric consequences of the above setup. We seize the opportunity to recall our convention regarding constants at the end of Section III.1.

Lemma III.2.3. *Assume (p_0), (V_α), and (G_β) to be fulfilled. Then:*

$$d(x,y) \leq C_3 d_G(x,y) \text{ for all } x,y \in G, \tag{III.2.8}$$

$$d(x,y) \geq c_3 \text{ for all } x \neq y \in G, \tag{III.2.9}$$

$$c_4 \leq \lambda_{x,y} \leq \lambda_x \leq C_4 \text{ for all } x \sim y \in G. \tag{III.2.10}$$

Proof. We first show (III.2.8). Using (p_0), (G_β), and the strong Markov property at time H_y, there exists $c > 0$ such that for all $x \sim y \in G$,

$$g(x,y) = P_x(H_y < \infty) g(y,y) \geq p_{x,y} g(y,y) \geq c_0 c_2,$$

where $p_{x,y}$ is the transition probability between x and y for the random walk Z, see (III.2.2). Thus, one can find C_3 such that

$$d(x,y) \overset{(G_\beta)}{\leq} \left(\frac{g(x,y)}{C_2}\right)^{-\nu} \leq C_3 \text{ for all } x \sim y \in G. \tag{III.2.11}$$

For arbitrary x and y in G, we then consider a geodesic for the graph distance between x and y, apply the triangle inequality (for d) and use (III.2.11) repeatedly to deduce (III.2.8). Similarly, for all $x \neq y \in G$,

$$d(x,y) \overset{(G_\beta)}{\geq} \left(\frac{g(x,y)}{c_2}\right)^{-\nu} \overset{(G_\beta)}{\geq} \left(\frac{C_2}{c_2}\right)^{-\nu} \overset{\text{def.}}{=} c_3.$$

We now turn to (III.2.10). For $x \sim y \in G$, we have $x \in B(x,1)$ and thus, by (V_α), $\lambda_{x,y} \leq \lambda_x \leq C_1 \stackrel{\text{def.}}{=} C_4$. Moreover, $g(x,x) \geq \lambda_x^{-1}$ by definition, and thus by (p_0) and (G_β),

$$\lambda_{x,y} \geq c_0 \lambda_x \geq \frac{c_0}{g(x,x)} \geq \frac{c_0}{C_2} \stackrel{\text{def.}}{=} c_4.$$

□

We now define the weak sectional isoperimetric condition alluded to in Section III.1. This is an additional condition on the geometry of G that will enter in Section III.8 to guarantee that certain "bad" regions are sizeable and thus costly in terms of probability, cf. the proofs of Lemma III.8.4 and Lemma III.8.6. We say that $(x_1, \dots, x_n)$ is an *R-path from x to $B(x,N)^{\mathsf{c}}$* if $x_1 = x$, $x_n \in B(x,N)^{\mathsf{c}}$, and $d(x_i, x_{i+1}) \leq R$ for all $i \in \{1, \dots, n-1\}$, with the additional convention that (x_1) is an R-path from x to $B(x,N)^{\mathsf{c}}$ if $N \leqslant R$. The weak sectional isoperimetric condition is a condition on the existence of long R-path in the boundary of sets, and similar conditions have already been used to study Bernoulli percolation, see [69]. More precisely, this weak sectional isoperimetric condition states that there exists $R_0 \geq 1$ and $c_5 \in (0,1]$ such that

$$\begin{aligned}&\text{for each finite connected subset } A \text{ of } G \text{ and all } x \in \partial_{ext} A,\\ &\text{there exists an } R_0\text{-path from } x \text{ to } B(x, c_5 \delta(A))^{\mathsf{c}} \text{ in } \partial_{ext} A.\end{aligned} \tag{WSI}$$

We now introduce the processes of interest. For each $x \in G$, we denote by Φ_x the coordinate map on $\mathbb{R}^G$ endowed with its canonical σ-algebra, $\Phi_x(\omega) = \omega_x$ for all $\omega \in \mathbb{R}^G$, and $\mathbb{P}^G$ is the probability measure defined in (III.1.5). Any process $(\varphi_x)_{x \in G}$ with law $\mathbb{P}^G$ will be called a *Gaussian free field on G*; see [86] as well as the references therein for a rigorous introduction to the relevance of this process. Recalling the definition of the level sets $E^{\geq h}$ of Φ in (III.1.7) and of the parameter $\bar{h}$ from (III.1.9), we now provide a simple argument that

$$\text{for each } h < \bar{h},\ \mathbb{P}^G\text{-a.s., } E^{\geq h} \text{ contains a unique infinite cluster.} \tag{III.2.12}$$

Indeed, if L is large enough, on the event $A_L^h = \{B(x, L/2)$ intersects at least two infinite clusters of $E^{\geq h}\}$, there is at least two clusters of $E^{\geq h} \cap B(x,L)$ with diameter at least $L/10$ which are not connected in G, and thus the event in (III.1.11) occurs. The events A_L^h are increasing toward $\{E^{\geq h}$ has at least two infinite clusters$\}$ as L goes to infinity, and thus by (III.1.11) $E^{\geq h}$ contains $\mathbb{P}^G$-a.s. at most one infinite cluster for all $h < \bar{h}$, and (III.2.12) follows since $\bar{h} \leqslant h_*$ as explained below (III.1.11).

On the other hand, random interlacements on a graph G as above are defined under a probability measure $\mathbb{P}^I$ as a Poisson point process ω on the product space of doubly infinite trajectories on G modulo time-shift, whose forward and backward parts escape all compact sets in finite time, times the label space $[0,\infty)$, see [103]. For $u>0$, we denote by ω^u the random interlacement process at level u, which consists of all the trajectories in ω with label at most u. By $\mathcal{I}^u$ we denote the random interlacement set associated to ω^u, which is the set of vertices visited by at least one trajectory in the support of ω^u, by $\mathcal{V}^u \stackrel{\text{def.}}{=} G\setminus\mathcal{I}^u$ the vacant set of random interlacements, and by $(\ell_{x,u})_{x\in G}$ the field of occupation times associated to ω^u, see (1.8) in [96], which collects the total time spent in each vertex of G by the trajectories in the support of ω^u. As stated in Corollary III.4.2 below, if (p_0), (V_α) and (G_β) hold,

$$\text{for all } u>0,\ \mathcal{I}^u \text{ is } \mathbb{P}^I\text{-a.s. an infinite connected subset of } G. \tag{III.2.13}$$

For vertex-transitive G, (III.2.13) is in fact a consequence of Theorem 3.3 of [105], since all graphs considered in the present chapter are amenable on account of (III.3.16) below as well as display (14) and thereafter in [105] (their spectral radius is equal to one).

Recall the definitions of the critical parameters h_* and u_* from (III.1.8) and (III.1.16), which describe the phase transition of $E^{\geq h}$, the level sets of Φ (as h varies), and that of $\mathcal{V}^u$ (as u varies). Note that (III.2.13) indicates a very different geometry of $\mathcal{I}^u$ and $\mathcal{V}^u$ as $u\to 0$ in comparison with independent Bernoulli percolation on G. Indeed, it is proved in [104] that for all the graphs from (III.1.4), both the set of open vertices and its complement undergo a non-trivial phase transition.

In order to derive an alternative representation of the critical parameters u_* and h_*, we recall that the FKG inequality was proved in Theorem 3.1 of [103] for random interlacements, and that it also holds for the Gaussian free field on G. Indeed, it is shown in [65] for any centered Gaussian field with non-negative covariance function on a finite space, and by conditioning on a finite set and using a martingale convergence theorem this result can be extended to an infinite space, see for instance the proof of Theorem 2.8 in [44]. As a consequence, for any $x\in G$, we have that

$$u_* = \inf\Big\{u\geq 0;\, \mathbb{P}^I\Big(\begin{array}{c}\text{the connected component of}\\ \mathcal{V}^u \text{ containing } x \text{ is infinite}\end{array}\Big)=0\Big\}, \tag{III.2.14}$$

and similarly for h_*.

The proofs of Theorems III.1.1 and III.1.2 involve a continuous version of the graph G, its cable system $\widetilde{G}$, and of the various processes associated to it. We attach to each edge $e=\{x,y\}$ of G a segment I_e of length $\rho_{x,y}=1/(2\lambda_{x,y})$, and $\widetilde{G}$ is obtained by glueing these intervals to G through their respective endpoints.

In other words, $\widetilde{G}$ is the metric graph where every edge e has been replaced by an interval of length ρ_e. We regard G as a subset of $\widetilde{G}$, and the elements of G will still be called vertices. One can define on $\widetilde{G}$ a continuous diffusion $\widetilde{X}$, via probabilities $\widetilde{P}_z$, $z \in \widetilde{G}$, such that for all $x \in G$, the projection on G of the trajectory of $\widetilde{X}$ under $\widetilde{P}_x$ has the same law as the discrete random walk Z on the weighted graph G under P_x. This diffusion can be defined from its Dirichlet form or directly constructed from the random walk Z by adding independent Brownian excursions on the edges beginning at a vertex. We refer to Section 2 of [57] or Section 2 of [36] for a precise definition and construction of the cable system $\widetilde{G}$ and the diffusion $\widetilde{X}$; see also Section II.2 for a detailed description in the case $G = \mathbb{Z}^d$. For all $x, y \in \widetilde{G}$ we denote by $\widetilde{g}(x, y)$, $x, y \in \widetilde{G}$, the Green function associated to $\widetilde{X}$, i.e., the density relative to the Lebesgue measure on $\widetilde{G}$ of the 0-potential of $\widetilde{X}$, which agrees with g on G, as well as $\widetilde{g}_U$ for $U \subset \widetilde{G}$ the Green function associated to the process $\widetilde{X}$ killed on exiting U.

We define for $\widetilde{A} \subset \widetilde{G}$ the set $\widetilde{A}^* \subset G$ as the smallest set such that $\widetilde{A}^* \supset G \cap \widetilde{A}$, and such that for all $z \in \widetilde{A} \setminus G$, there exist $x, y \in \widetilde{A}^*$ such that $z \in I_{\{x,y\}}$. For all $x \in G$ and $L > 0$, we write $\widetilde{B}(x, L)$ for the largest subset $\widetilde{B}$ of $\widetilde{G}$ such that $\widetilde{B}^* = B(x, L)$, and for all $\widetilde{A} \subset \widetilde{G}$ and $L > 0$, we let $\widetilde{B}(\widetilde{A}, L)$ denote the largest subset $\widetilde{B}$ of $\widetilde{G}$ such that $\widetilde{B}^* = B(\widetilde{A}^*, L)$. Moreover, for $\widetilde{A} \subset \widetilde{G}$, we write

$$z \overset{\sim}{\longleftrightarrow} z' \text{ in } \widetilde{A}, \tag{III.2.15}$$

if there exists a continuous path between z and z' in $\widetilde{A}$. We say that $\widetilde{A}$ is connected in $\widetilde{G}$ if $z \overset{\sim}{\longleftrightarrow} z'$ in $\widetilde{A}$ for all $z, z' \in \widetilde{A}$. Similarly, for $\widetilde{A}_1 \subset \widetilde{A}$ and $\widetilde{A}_2 \subset \widetilde{A}$, we write $\widetilde{A}_1 \overset{\sim}{\longleftrightarrow} \widetilde{A}_2$ in $\widetilde{A}$ if there exists a continuous path between $\widetilde{A}_1$ and $\widetilde{A}_2$ in $\widetilde{A}$.

The Gaussian free field naturally extends to the metric graph $\widetilde{G}$: Let $\widetilde{\Phi}_z$, $z \in \widetilde{G}$, be the coordinate functions on the space of continuous real-valued functions $C(\widetilde{G}, \mathbb{R})$, the latter endowed with the σ-algebra generated by the maps $\widetilde{\Phi}_z$, $z \in \widetilde{G}$. Let $\widetilde{\mathbb{P}}^G$ be the probability measure on $C(\widetilde{G}, \mathbb{R})$ such that, under $\widetilde{\mathbb{P}}^G$, $(\widetilde{\Phi}_z)_{z \in \widetilde{G}}$ is a centered Gaussian field with covariance function

$$\widetilde{\mathbb{E}}^G\big[\widetilde{\Phi}_{z_1}\widetilde{\Phi}_{z_2}\big] = \widetilde{g}(z_1, z_2) \text{ for all } z_1, z_2 \in \widetilde{G}. \tag{III.2.16}$$

The existence of such a continuous process was shown in [57]. Any random variable $\widetilde{\varphi}$ on $C(\widetilde{G}, \mathbb{R})$ with law $\widetilde{\mathbb{P}}^G$ will be called a Gaussian free field on $\widetilde{G}$. Moreover, if $\widetilde{\varphi}$ is a Gaussian free field on $\widetilde{G}$, then it is plain that $(\widetilde{\varphi}_x)_{x \in G}$ is a Gaussian free field on G. With a slight abuse of notation, we will henceforth write φ_x instead of $\widetilde{\varphi}_x$ when $x \in G$ for emphasis. We now recall the spatial Markov property for the Gaussian free field on $\widetilde{G}$, see Section 1 of [101]. Let $K \subset \widetilde{G}$ be a compact subset with finitely many connected components, and let

$U = \widetilde{G} \setminus K$ be its complement. We can decompose any Gaussian free field $\widetilde{\varphi}$ on $\widetilde{G}$ as

$$\widetilde{\varphi} = \widetilde{\varphi}^U + \widetilde{\beta}^U \text{ with } \widetilde{\beta}^U_z = \widetilde{E}_z\big[\widetilde{\varphi}_{\widetilde{X}_{T_U}} \mathbb{1}_{\{T_U<\infty\}}\big] \text{ for all } z \in \widetilde{G}, \qquad \text{(III.2.17)}$$

$\widetilde{\varphi}^U$ is a Gaussian free field independent of $\sigma(\widetilde{\varphi}_z,\, z \in K)$ and with covariance function $\widetilde{g}_U$, and in particular $\widetilde{\varphi}^U$ vanishes on K.

One can also adapt the usual definition of random interlacements on G, see [103], to the cable system $\widetilde{G}$ as in [57], [101] and Chapter II. For each $u > 0$, one thus introduces under a probability measure $\widetilde{\mathbb{P}}^I$ the random interlacement process $\widetilde{\omega}^u$ on $\widetilde{G}$ at level u, whose restriction to the trajectories hitting $K \subset\subset G$ can be described by a Poisson point process with intensity $u\widetilde{P}_{e_K}$ where e_K is the usual equilibrium measure of $K \subset\subset G$, see (III.3.6) below. One then defines a continuous field of local times $(\widetilde{\ell}_{z,u})_{z\in\widetilde{G}}$ relative to the Lebesgue measure on $\widetilde{G}$ associated to the random interlacement process on $\widetilde{G}$ at level u, i.e., $\widetilde{\ell}_{z,u}$ corresponds for all $z \in \widetilde{G}$ to the density with respect to the Lebesgue measure on $\widetilde{G}$ of the total time spent by the random interlacement process around z. For all $u > 0$, the restriction $(\widetilde{\ell}_{x,u})_{x\in G}$ of the local times to G coincides with the field of occupation times $(\ell_{x,u})_{x\in G}$ associated with the discrete random interlacement process ω^u defined above (III.2.13), and just like for the free field, we will write $\ell_{x,u}$ instead of $\widetilde{\ell}_{x,u}$ when $x \in G$. We also define for each measurable subset $\widetilde{B}$ of $\widetilde{G}$ and $u > 0$ the family

$$\widetilde{\ell}_{\widetilde{B},u} \stackrel{\text{def.}}{=} (\widetilde{\ell}_{z,u})_{z\in\widetilde{B}} \in C(\widetilde{B}, \mathbb{R}), \qquad \text{(III.2.18)}$$

and the random interlacement set at level u by

$$\widetilde{\mathcal{I}}^u = \{z \in \widetilde{G};\, \widetilde{\ell}_{z,u} > 0\}. \qquad \text{(III.2.19)}$$

The connectivity properties of $\widetilde{\mathcal{I}}^u$ will be studied in Section III.4. In particular, as stated in Corollary III.4.2, $\widetilde{\mathcal{I}}^u$ is $\widetilde{\mathbb{P}}^I$-a.s. an unbounded and connected subset of $\widetilde{G}$, and the same is true of $\mathcal{I}^u$ (as a subset of G). We will elaborate on an important link between the fields $\widetilde{\ell}_{\widetilde{G},u}$ and $\widetilde{\varphi}$ from (III.2.16) and (III.2.18) in Section III.5.

Finally, one of the main tools in the study of the percolative properties of the vacant set of random interlacements and of the level sets of the Gaussian free field, and the driving force behind the renormalization arguments of Section III.8 are a certain family of correlation inequalities on $\widetilde{G}$, which we now state. Their common feature is a small sprinkling for the parameters u and h, respectively, which partially compensates the absence of a BK-inequality (after van den Berg and Kesten, see for instance [44]) caused by the presence of long-range correlations in these models. The results below, in particular (III.2.21) below, are of independent interest. We recall the notation from the paragraph preceding (III.2.16) and (III.2.18) and use $C(A,\mathbb{R})$ to denote the space of continuous functions from A to the reals, where the topology on A is generally clear from the context.

Theorem III.2.4. *Suppose G is infinite, connected and (G,λ) such that (p_0), (V_α), (G_β) hold. Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two Borel-measurable subsets of $\widetilde{G}$, at least one of which is bounded. Let $s = d(\widetilde{A}_1^*, \widetilde{A}_2^*)$ and $r = \delta(\widetilde{A}_1^*) \wedge \delta(\widetilde{A}_2^*)$ (note that $r < \infty$). There exist C_6 and c_6 such that for all $\varepsilon \in (0,1)$, and all measurable functions $f_i : C(\widetilde{A}_i, \mathbb{R}) \to [0,1]$, $i = 1,2$, which are either both increasing or both decreasing, if $s > 0$,*

$$\begin{aligned}\widetilde{\mathbb{E}}^G & \left[f_1\big(\widetilde{\Phi}_{|\widetilde{A}_1}\big) f_2\big(\widetilde{\Phi}_{|\widetilde{A}_2}\big)\right] \\ & \leq \widetilde{\mathbb{E}}^G \left[f_1\big(\widetilde{\Phi}_{|\widetilde{A}_1} \pm \varepsilon\big)\right] \widetilde{\mathbb{E}}^G \left[f_2\big(\widetilde{\Phi}_{|\widetilde{A}_2} \pm \varepsilon\big)\right] + C_6 (r+s)^\alpha \exp\left\{-c_6 \varepsilon^2 s^\nu\right\},\end{aligned} \tag{III.2.20}$$

and there exist C_7, C_8 and c_8 such that for all $u > 0$, $\varepsilon \in (0,1)$ and f_i as above, if $s \geq C_7(r \vee 1)$,

$$\begin{aligned}\widetilde{\mathbb{E}}^I & \left[f_1\big(\widetilde{\ell}_{\widetilde{A}_1,u}\big) f_2\big(\widetilde{\ell}_{\widetilde{A}_2,u}\big)\right] \\ & \leq \widetilde{\mathbb{E}}^I \left[f_1\big(\widetilde{\ell}_{\widetilde{A}_1,u(1\pm\varepsilon)}\big)\right] \widetilde{\mathbb{E}}^I \left[f_2\big(\widetilde{\ell}_{\widetilde{A}_2,u(1\pm\varepsilon)}\big)\right] + C_8 (r+s)^\alpha \exp\left\{-c_8 \varepsilon^2 u s^\nu\right\},\end{aligned} \tag{III.2.21}$$

where the plus sign corresponds in both equations to the case where the functions f_i are increasing and the minus sign to the case where the functions f_i are decreasing.

The proof of Theorem III.2.4 is deferred to Section III.6. While (III.2.20) follows rather straightforwardly from the decoupling inequality from [67] for the Gaussian free field (see also Theorem III.6.2 for a strengthening of (III.2.20)), the proof of (III.2.21) is considerably more involved. It uses the soft local times technique introduced in [68] on $\mathbb{Z}^d$ for random interlacements, but a generalization to the present setup requires some effort (note also that for graphs of the type $G = G' \times \mathbb{Z}$, one could also use the inequalities of [95], which are proved by different means).

III.3 Preliminaries and examples

We now gather several aspects of potential theory for random walks on the weighted graphs introduced in the last section. These include estimates on killed

Green functions, see Lemma III.3.1 below, a resulting (elliptic) Harnack inequality, bounds on the capacities of various sets, see Lemma III.3.2, and on the heat kernel, see Proposition III.3.3, which will be used throughout. We then proceed to discuss product graphs in Proposition III.3.5 and, with a view towards (WSI), connectivity properties of external boundaries in Proposition III.3.7. These results are helpful in showing how the examples from (III.1.4), which constitute an important class, fit within the framework of the previous section. We conclude this section by deducing in Corollary III.3.9 that our main results, Theorems III.1.1 and III.1.2, apply in all cases of (III.1.4).

From now on,

$$\text{we assume that } (G,\lambda) \text{ is an infinite, connected, weighted graph endowed with a distance function } d \text{ that satisfies } (p_0),\ (V_\alpha) \text{ and } (G_\beta) \tag{III.3.1}$$

(see Section III.2). Throughout the remainder of this chapter, we *always* tacitly work under the assumptions (III.3.1). Any additional assumption will be mentioned explicitly.

The following lemma collects an estimate similar to (G_β) for the stopped Green function (III.2.5).

Lemma III.3.1. *There exists a constant $C_9 > 1$ such that, if $U_1 \subset U_2 \subset\subset G$ with $d(U_1, U_2^{\mathsf{c}}) \geq C_9(\delta(U_1) \vee 1)$, then*

$$\begin{aligned} \frac{c_2}{2} d(x,y)^{-\nu} \leq g_{U_2}(x,y) \leq C_2 d(x,y)^{-\nu} \text{ for all } x \neq y \in U_1, \text{ and} \\ \frac{c_2}{2} \leq g_{U_2}(x,x) \leq C_2 \text{ for all } x \in U_1. \end{aligned} \tag{III.3.2}$$

Proof. Let $U_1 \subset U_2 \subset\subset G$. The upper bound in (III.3.2) follows immediately from (G_β) since $g_{U_2}(x,y) \leq g(x,y)$ for all $x,y \in G$ by definition. For the lower bound, using (III.2.6) and (G_β), we obtain that for all $x \neq y \in U_1$,

$$g_{U_2}(x,y) \geq c_2 d(x,y)^{-\nu} - C_2 E_x\big[d(X_{T_{U_2}}, y)^{-\nu}\big] \geq c_2 d(x,y)^{-\nu} - C_2 d(U_1, U_2^{\mathsf{c}})^{-\nu}.$$

Thus, choosing C_9 large enough such that $\frac{c_2}{2} \geq \frac{C_2}{C_9^\nu}$, it follows that if $d(U_1, U_2^{\mathsf{c}}) \geq C_9\delta(U_1)\ (\geq C_9 d(x,y))$, then

$$g_{U_2}(x,y) \geq \frac{c_2}{2} d(x,y)^{-\nu} \text{ for all } x \neq y \in U_1.$$

The lower bound for $g_{U_2}(x,x)$, $x \in U_1$, is obtained similarly. □

Using Lemma A.2 in [94], which is an adaptation of Lemma 10.2 in [42], an important consequence of (III.3.2) is the elliptic Harnack inequality in (III.3.3) below. For this purpose, recall that a function f defined on $\overline{U_2} \stackrel{\text{def.}}{=} B_G(U_2, 1)$, the closed 1-neighborhood of U_2 for the graph distance, is called L-harmonic (or simply harmonic) in U_2 if $E_x[f(Z_1)] = f(x)$, or equivalently $Lf(x) = 0$ (see (III.1.2)), for all $x \in U_2$. The bounds of (III.3.2) imply that there exists a constant $c_9 \in (0, 1)$ such that for all $U_1 \subset U_2 \subset\subset G$ with $\delta(U_1) \geq 2C_3$ and $d(U_1, U_2^{\mathsf{c}}) \geq C_9(2\delta(U_1) \vee 1)$, and any non-negative function f on $\overline{U_2}$ which is harmonic in U_2,

$$\inf_{y \in U_1} f(y) \geq c_9 \sup_{y \in U_1} f(y). \tag{III.3.3}$$

Another important consequence of (III.3.2) is that the balls for the distance d are almost connected in the following sense:

$$\forall x \in G,\ R \geq 1 \text{ and } y, y' \in B(x, R),\ y \leftrightarrow y' \text{ in } B(x, C_{10}R), \text{ with } C_{10} = 2C_9 + 1. \tag{III.3.4}$$

Indeed, for all $U \subset\subset G$ and $y, y' \in G$, $y \stackrel{U}{\longleftrightarrow} y'$ is equivalent to $g_U(y, y') > 0$, and by definition,

$$d\big(B(x, R), B(x, C_{10}R)^{\mathsf{c}}\big) \geq 2C_9 R \geq C_9 \delta\big(B(x, R)\big). \tag{III.3.5}$$

As a consequence, (III.3.2) implies that $g_{B(x,C_{10}R)}(y, y') > 0$ for all $y, y' \in B(x, R)$.

We now recall some facts about the equilibrium measure and capacity of various sets. For $A \subset\subset U \subset G$, the equilibrium measure of A relative to U is defined as

$$e_{A,U}(x) \stackrel{\text{def.}}{=} \lambda_x P_x(\widetilde{H}_A > T_U) \mathbb{1}_A(x) \text{ for all } x \in G, \tag{III.3.6}$$

where $\widetilde{H}_A \stackrel{\text{def.}}{=} \inf\{n \geq 1,\ Z_n \in A_1\}$ is the first return time in A for the random walk on G, and the capacity of A relative to U as the total mass of the equilibrium measure,

$$\mathrm{cap}_U(A) \stackrel{\text{def.}}{=} \sum_{x \in A} e_{A,U}(x). \tag{III.3.7}$$

For all $A \subset\subset U \subset G$, the following last-exit decomposition relates the entrance time H_A of Z in A, the exit time T_U of U, the stopped Green function and the equilibrium measure:

$$P_x(H_A < T_U) = \sum_{y \in A} g_U(x, y) e_{A,U}(y) \text{ for all } x \in U. \tag{III.3.8}$$

For $A \subset\subset G$ and $x \in G$, we introduce the equilibrium measure, capacity and harmonic measure as

$$e_A(x) \stackrel{\text{def.}}{=} e_{A,G}(x), \qquad \mathrm{cap}(A) \stackrel{\text{def.}}{=} \mathrm{cap}_G(A) \qquad \text{and} \qquad \overline{e}_A(x) \stackrel{\text{def.}}{=} \frac{e_A(x)}{\mathrm{cap}(A)}, \tag{III.3.9}$$

respectively. The capacity is a central notion for random interlacements, since we have the following characterization for the random interlacement set $\mathcal{I}^u$

$$\mathbb{P}^I(\mathcal{I}^u \cap A = \varnothing) = \exp\{-u \cdot \mathrm{cap}(A)\} \text{ for all } A \subset\subset G; \tag{III.3.10}$$

see Remark 2.3 in [103]. With these definitions, it then follows using (III.3.8) and (III.2.8) that for all $R \geq C_3$ and $x_0 \in G$,

$$\begin{aligned} c_2 R^{-\nu} \mathrm{cap}\left(B(x_0,R)\right) \leq 1 = &\sum_{y \in \partial B(x_0,R)} g(x_0,y) e_{B(x_0,R)}(y) \\ &\leq C_2 (R - C_3)^{-\nu} \mathrm{cap}\left(B(x_0,R)\right), \end{aligned}$$

and hence there exist constants $0 < c_{11} \leq C_{11} < \infty$ only depending on G such that for all $R \geq 1$ and $x \in G$,

$$c_{11} R^{\nu} \leq \mathrm{cap}\left(B(x,R)\right) \leq C_{11} R^{\nu}. \tag{III.3.11}$$

A useful characterization of capacity in terms of a variational problem is given by

$$\mathrm{cap}(A) = \Big(\inf_{\mu} \sum_{x,y \in A} g(x,y)\mu(x)\mu(y)\Big)^{-1}, \quad \text{for } A \subset\subset G, \tag{III.3.12}$$

where the infimum is over probability measures μ on A, see e.g. Proposition 1.9 in [98] for the case of a finite graph with non-vanishing killing measure (the proof can be extended to the present setup). In particular, since every probability measure μ on A is also a probability measure on any set containing A, the capacity is increasing, so for $A, B \subset G$,

$$A \subset B \qquad \text{implies} \qquad \mathrm{cap}(A) \leqslant \mathrm{cap}(B). \tag{III.3.13}$$

Another consequence of the representation (III.3.12) is the following lower bound on the capacity of a set.

Lemma III.3.2. *There exists a constant c depending only on G such that for all $L \geqslant 1$ and $A \subset G$ connected with diameter at least L,*

$$\mathrm{cap}(A) \geqslant \begin{cases} cL, & \text{if } \nu > 1, \\ \frac{cL}{\log(L+1)}, & \text{if } \nu = 1, \\ cL^{\nu}, & \text{if } \nu < 1. \end{cases} \tag{III.3.14}$$

Moreover, if $A \subset G$ is infinite and connected, then for all $x_0 \in G$

$$\mathrm{cap}(A \cap B(x_0, L)) \to \infty \qquad \text{as } L \to \infty, \tag{III.3.15}$$

and thus $A \cap \mathcal{I}^u \neq \varnothing$ $\mathbb{P}^I$-a.s.

Proof. Let us fix some $L \geqslant 1$, A connected subset of G with diameter at least L, and $x_0 \in A$. We introduce $L' = \lfloor L/(2C_3) \rfloor$ and for each $k \in \{1, \dots, L'\}$ the set $A_k = A \cap (B(x_0, C_3 k) \setminus B(x_0, C_3(k-1)))$, which is non-empty by (III.2.8). Then

$$\sum_{p=1}^{k} \sup_{y \in A_p} g(x, y) \leqslant C_2 \Big(2 + C_3 \sum_{p=1}^{k-2} (k-1-p)^{-\nu} \Big) \leqslant C_2 \Big(2 + C_3 \sum_{p=1}^{L'} p^{-\nu} \Big),$$

Now let μ be the probability measure on A defined by $\mu(x) = (L'|A_k|)^{-1}$ if $x \in A_k$ for some $k \in \{1, \dots, L'\}$, and $\mu(x) = 0$ otherwise. For all $k \in \{1, \dots, L'\}$ and $x \in A_k$, we have by (G_β) that

$$\sum_{x,y \in A} g(x, y) \mu(x) \mu(y) \leqslant \frac{2C_2}{L'} \Big(2 + C_3 \sum_{p=1}^{L'} p^{-\nu} \Big).$$

Combining this bound with (III.3.12), the inequality (III.3.14) follows. If A is now an infinite and connected subset of G, then for each $x_0 \in G$ there exists $L_0 > 0$ such that for all $L \geqslant L_0$, the set $A \cap B_G(x_0, L/C_3)$ has diameter at least $\frac{L}{2C_3}$, and thus by (III.2.8) $A \cap B(x_0, L)$ contains at least a connected component of diameter $\frac{L}{2C_3}$, and (III.3.15) then follows directly from (III.3.14). Finally, by (III.3.10),

$$\begin{aligned} \mathbb{P}^I(A \cap \mathcal{I}^u = \varnothing) &\leqslant \mathbb{P}^I(A \cap \mathcal{I}^u \cap B(x_0, L) = \varnothing) \\ &\leqslant \exp\big\{ -u \cdot \mathrm{cap}(A \cap B(x_0, L)) \big\} \\ &\underset{L \to \infty}{\longrightarrow} 0. \end{aligned}$$

□

Next, we collect an upper bound on the heat kernel (III.2.7) and an estimate on the distribution of the exit time of a ball $T_{B(x,R)}$.

Proposition III.3.3.

i) There exists a constant C such that for all $x, y \in G$ and $n > 0$,

$$p_n(x, y) \leq C n^{-\frac{\alpha}{\beta}}. \tag{III.3.16}$$

ii) *There exist constants c and C such that for all $x \in G$, $R > 0$ and positive integer n,*

$$P_x\big(T_{B(x,R)} \le n\big) \le C \exp\Big\{ -\Big(\frac{cR^\beta}{n}\Big)^{\frac{1}{\beta-1}}\Big\}. \tag{III.3.17}$$

Proposition III.3.3 is essentially known, for instance if d is the graph distance d_G then these results (as well as (UHK(α,β)) and (LHK(α,β))) are proved in [42]. For a general distance d, some estimates similar to (III.3.16) and (III.3.17) (as well as (UHK(α,β)) and (LHK(α,β))) are also proved in [41] and [40] in the more general setting of metric spaces, and we could apply them to the variable rate continuous time Markov chain on G. However, there does not seem to be any proof in the literature that exactly fits our needs (general distance d, discrete time random walk Z), and so, for the reader's convenience, we have included a proof of Proposition III.3.3 in the Appendix.

Remark III.3.4. 1) With Proposition III.3.3 at our disposal, following up on Remark III.2.2, we briefly discuss the relation of the above assumptions (III.3.1) to heat kernel bounds within our setup. A consequence of (III.3.16) and (III.3.17) is that, under condition (p_0),

$$(V_\alpha) + (G_\beta) \Rightarrow (\mathrm{UHK}(\alpha,\beta)); \tag{III.3.18}$$

note that in contrast to the results of Remark 2.2, this holds true even when d is not the graph distance, where (UHK(α,β)) is defined in Remark III.2.2. Indeed, for $d = d_G$ this implication is part of Proposition 8.1 in [42], but the proof remains valid for any distance d. However the corresponding lower bound (LHK(α,β)) on the heat kernel does not always hold. To see this, take for example G a graph such that (p_0), (V_α) and (G_β) hold when d is the graph distance, and let $d' = d^{\frac{1}{\kappa}}$ for some $\kappa > 1$ (cf. Proposition III.3.5 and (III.3.22) below for a situation where this is relevant). Then for the graph G endowed with the distance d', the conditions (p_0), $(V_{\alpha'})$ and $(G_{\beta'})$ hold with $\alpha' = \alpha\kappa$ and $\beta' = \beta\kappa$. Moreover, using (UHK(α,β)) for the distance d, one obtains that $p_n(x,y) + p_{n+1}(x,y) \le 2Cn^{-\frac{\alpha'}{\beta'}} \exp\{-(\frac{d'(x,y)^{\beta'}}{Cn})^{\frac{1}{\beta-1}}\}$. Taking $n = d'(x,y)$ for instance, it follows that for any $c > 0$, since $\beta' > \beta$,

$$\begin{aligned}\big(p_n(x,y) + p_{n+1}(x,y)\big) n^{\frac{\alpha'}{\beta'}} \exp\Big\{\Big(\frac{d'(x,y)^{\beta'}}{cn}\Big)^{\frac{1}{\beta'-1}}\Big\} \\ \le 2C \exp\Big\{ -\Big(\frac{n^{\beta'-1}}{C}\Big)^{\frac{1}{\beta-1}} + \Big(\frac{n^{\beta'-1}}{c}\Big)^{\frac{1}{\beta'-1}}\Big\} \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

thus (LHK(α',β')) cannot hold for G endowed with the distance d'.

2) Even in cases where (LHK(α,β)) does not hold, it is still possible to obtain some slightly worse lower bounds for a general distance d. We will not need these results in the rest of the chapter, and therefore we only sketch the proofs. We introduce the following near-diagonal lower estimate

$$p_n(x,y)+p_{n+1}(x,y)\geqslant cn^{-\frac{\alpha}{\beta}} \quad \text{for all } x,y\in G \text{ and } n\geqslant cd(x,y)^{\beta}. \tag{NLHK(α,β)}$$

Let us assume that the condition (p_0) is fulfilled, we then have the following equivalences for all $\alpha>2$ and $\beta\in[2,\alpha)$

$$(V_\alpha)+(G_\beta)\Leftrightarrow(\text{UHK}(\alpha,\beta))+(\text{NLHK}(\alpha,\beta)). \tag{III.3.19}$$

The first implication follows from (13.3) in [42], whose proof remains valid for a general distance d, given (III.3.18), (III.3.16), (III.A.1) and (III.3.3), and the proof of its converse is exactly the same as the proof of Proposition 15.1 in [42] or Lemma 4.22 and Theorem 4.26 in [4]. Estimates similar to (UHK(α,β)) and (NLHK(α,β)) for the continuous time Markov chain on G with jump rates $(\lambda_x)_{x\in G}$ and transition probabilities $(p_{x,y})_{x,y\in G}$, see (III.2.2), are also equivalent to (III.3.19), see Theorem 3.14 in [41]. Let us now also assume that there exist constants $c>0$ and $\zeta\in[1,\beta)$ such that

$$\begin{aligned}&\text{for all } r>0,\ k\in\mathbb{N} \text{ and } x,y\in G \text{ such that } d(x,y)\leqslant ck^{\frac{1}{\zeta}}r, \text{ there exists}\\ &\text{a sequence } x_1=x,x_2,\dots,x_k=y \text{ with } d(x_{i-1},x_i)\leqslant r \text{ for all } i\in\{2,\dots,k\},\end{aligned} \tag{D_ζ}$$

then the conditions in (III.3.19) are also equivalent to (UHK(α,β)) plus the following lower estimate

$$p_n(x,y)+p_{n+1}(x,y)\geqslant cn^{-\frac{\alpha}{\beta}}\exp\Big\{-\Big(\frac{d(x,y)^{\beta}}{cn}\Big)^{\frac{\zeta}{\beta-\zeta}}\Big\} \quad \text{for all } n\geqslant d_G(x,y). \tag{LHK(α,β,ζ)}$$

Indeed, under condition (D_ζ), the proof that (III.3.19) implies (LHK(α,β,ζ)) is similar to the proof of Proposition 13.2 in [42] or Proposition 4.38 in [4], modulo some slight modifications when d is a general distance, and its converse is trivial. Note that if $d=d_G$, it is clear that (D_1) holds and that the lower estimate (LHK($\alpha,\beta,1$)) is the same as (LHK(α,β)), and thus we recover the results from Remark III.2.2. If $d'=d_G^{\frac{1}{\kappa}}$ for some $\kappa\geqslant1$ as in the counter-example of Remark III.3.4, 1), and (V_α) and (G_β) hold with the distance d_G, then (D_κ) hold for the distance d' and thus also (LHK(α',β',κ)) for the distance d', where $\beta'=\beta\kappa$ and $\alpha'=\alpha\kappa$, which is exactly the same as (LHK(α,β)) for the distance d_G. □

We now discuss product graphs. Let G_1 and G_2 be two graphs as in the previous section (countably infinite, connected and with bounded degree), endowed with weight functions λ^1 and λ^2. The graph $G = G_1 \times G_2$ is defined such that $x = (x_1, x_2) \sim y = (y_1, y_2)$ if and only there exists $i \neq j \in \{1, 2\}$ such that $x_i \sim y_i$ and $x_j = y_j$. One naturally associates with G the weight function λ such that for all $x = (x_1, x_2) \sim y = (y_1, y_2)$, one has

$$\lambda_{x,y} = \lambda^i_{x_i,y_i}, \text{ where } i \in \{1, 2\} \text{ is such that } x_i \neq y_i. \tag{III.3.20}$$

Proposition III.3.5. *Suppose that* (G_i, λ^i), $i = 1, 2$, *satisfy (UHK*(α_i, β_i)*)) and (LHK*(α_i, β_i)*)) with respect to the graph distance* d_{G_i}*, as well as* (p_0)*. Assume that*

$$\alpha_i \geq 1 \text{ and } 2 \leq \beta_i \leq 1 + \alpha_i, \text{for } i = 1, 2, \text{ and } \exists j \in \{1, 2\} \text{ s.t. } \alpha_j > 1 \text{ or } \beta_j > 2. \tag{III.3.21}$$

Then, if $\beta_1 \leq \beta_2$*, the graph* $G_1 \times G_2$ *endowed with the weights* (III.3.20) *satisfies* (V_α)*,* (G_β) *with*

$$\alpha = \alpha_1 \frac{\beta_2}{\beta_1} + \alpha_2, \ \beta = \beta_2 \text{ and } d(x, y) = \max\big(d_{G_1}(x_1, y_1)^{\frac{\beta_1}{\beta_2}}, d_{G_2}(x_2, y_2)\big). \tag{III.3.22}$$

Proof. We first argue that (G, λ) satisfies (V_α). By Remark III.2.2, (G_i, λ^i), $i = 1, 2$, satisfy (V_{α_i}). On account of (III.3.20), one readily infers that $\lambda(A \times B) = \lambda^1(A) \cdot |B| + |A| \cdot \lambda^2(B)$ for all $A \subset G_1$, $B \subset G_2$. Applying this to $A = B_{d_{G_1}}(x_1, R^{\beta_2/\beta_1})$, $B = B_{d_{G_2}}(x_2, R)$, observing that $B_d((x_1, x_2), R) = A \times B$ by definition of $d(\cdot, \cdot)$ and noting that $c_4|A| \leq \lambda^1(A) \leq C_4|A|$ (and similarly for B), see (III.2.10), it follows that uniformly in $(x_1, x_2) \in G$, $\lambda(B_d((x_1, x_2), R))$ is of order R^α with α given by (III.3.22), whence (V_α) is fulfilled.

It remains to show that (G_β) holds. Let $(\overline{X}^i_t)_{t\geq 0}$, $i = 1, 2$, denote the continuous time walk on G_i (resp. G) with jump rates $\lambda^i_x = \sum_{y: d_{G_i}(x,y)=1} \lambda^i_{x,y}$, and suppose $\overline{X}^1_\cdot$, $\overline{X}^2_\cdot$ are independent. Let $\overline{X}_\cdot$ be the corresponding walk on G (with jump rates λ_x, cf. (III.3.20)). Then $\overline{X}_\cdot$ has the same law as $(\overline{X}^1_\cdot, \overline{X}^2_\cdot)$ and in view of (III.2.4),

$$g(x, y) = \int_0^\infty P_x(\overline{X}_t = y)\,\mathrm{d}t = \int_0^\infty P_{x_1}(\overline{X}^1_t = y_1) P_{x_2}(\overline{X}^2_t = y_2)\,\mathrm{d}t, \tag{III.3.23}$$

with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We introduce for $i = 1, 2$, the additive functionals

$$A^i_t = \int_0^t \lambda_{\overline{X}^i_s}\,\mathrm{d}s, \quad \text{for } t \geq 0,\ i = 1, 2, \tag{III.3.24}$$

along with $\tau^i_t = \inf\{s \geq 0;\, A^i_s \geq t\}$ and the corresponding time-changed processes

$$(Y^i_t)_{t\geq 0} \overset{\text{def.}}{=} (\overline{X}^i_{\tau^i_t})_{t\geq 0}.$$

By the above assumptions, the discrete skeletons of $Y^i_\cdot$, $i = 1, 2$, satisfy the respective heat kernel bounds $\mathrm{HK}(\alpha_i, \beta_i)$ in the notation of [4], and thus by Theorem 5.25 in [4] (the process $Y^i_\cdot$ has unit jump rate), for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in G, abbreviating $d_i = d_{G_i}(x_i, y_i)$ and $d = d(x, y)$, so that $d^{\beta_2} = d_1^{\beta_1} \vee d_2^{\beta_2}$,

$$ct^{-\frac{\alpha_i}{\beta_i}} \exp\Big\{-\Big(\frac{d^{\beta_2}}{ct}\Big)^{\frac{1}{\beta_i-1}}\Big\} \leq P_{x_i}(Y^i_t = y_i) \leq c't^{-\frac{\alpha_i}{\beta_i}} \exp\Big\{-\Big(\frac{d_i^{\beta_i}}{c't}\Big)^{\frac{1}{\beta_i-1}}\Big\}, \tag{III.3.25}$$

where the lower bound holds for all $t \geq d_i \vee 1$ and the upper bound for all $t \geq d_i$. Going back to (III.3.23), noting that $\overline{X}^i_t = Y^i_{A^i_t}$ and that $c_4 t \leq A^i_t \leq C_4 t$ for all $t \geq 0$ by (III.2.10) and (III.3.24), and observe that

$$\inf_{i\in\{1,2\}} \sup_{t\leq C_4 c_4^{-1}(d_1\vee d_2)} P_{x_i}(Y^i_t = y_i) \leq Ce^{-c(d_1\vee d_2)}, \tag{III.3.26}$$

which follows for instance from Theorem 5.17 in [4]. We obtain for all x and y, with constants possibly depending on α_i and β_i, keeping in mind that $d^{\beta_2} = d_i^{\beta_i}$ for some i in the third line below,

$$\begin{aligned}
g(x,y) &\leq \int_0^\infty \sup_{c_4 t\leq s\leq C_4 t} \big\{P_{x_1}(Y^1_s = y_1)P_{x_2}(Y^2_s = y_2)\big\}\, \mathrm{d}t \\
&\overset{\text{(III.3.25),(III.3.26)}}{\leq} C(d_1 \vee d_2)e^{-c(d_1\vee d_2)} \\
&\qquad + C' \int_{c_4^{-1}(d_1\vee d_2)}^\infty t^{-(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2})} \exp\Big\{-\sum_{i=1}^2 \Big(\frac{d_i^{\beta_i}}{c''t}\Big)^{\frac{1}{\beta_i-1}}\Big\}\, \mathrm{d}t \\
&\overset{u=d^{-\beta_2}t}{\leq} Ce^{-cd} + C' \int_0^\infty d^{-(\frac{\beta_2\alpha_1}{\beta_1}+\alpha_2)} u^{-(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2})} \exp\Big\{-(c''u)^{-\frac{1}{\beta_i-1}}\Big\} d^{\beta_2}\, \mathrm{d}u \\
&\leq C'' d^{-(\alpha-\beta)},
\end{aligned} \tag{III.3.27}$$

recalling the definition of α and β from (III.3.22) in the last step; we also note that the integral over u in the last but one line is finite since $\alpha_i \geq 1$ and $\beta_i \leqslant 1+\alpha_i$, so that $\frac{\alpha_i}{\beta_i} \geq \frac{\alpha_i}{1+\alpha_i} \geq \frac{1}{2}$ with strict inequality for at least one of the i's due to (III.3.21), whence $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > 1$. In view of (III.1.6), (III.3.27) yields the desired upper bound. For the corresponding lower bound, one proceeds similarly, starting from (III.3.23), discarding the integral over $0 \leq t \leq c_4^{-1}(d_1 \vee d_2 \vee 1)$, and applying the lower bound from (III.3.25). Thus, (G_β) holds, which completes the proof. □

Remark III.3.6. 1) Proposition III.3.5 is sufficient for our purposes but one could also extend it to graphs (G_i, λ_i) which satisfy (p_0), $(\mathrm{UHK}(\alpha_i, \beta_i))$ and $(\mathrm{NLHK}(\alpha_i, \beta_i))$ under a general distance d_i for $i = 1, 2$.

2) Under the hypotheses of Proposition III.3.5, one can show that there exists constants $c > 0$ and $C < \infty$ such that for all $n \in \mathbb{N}$, $x_1 \in G_1$ and $x_2, y_2 \in G_2$, the upper bound $(\mathrm{UHK}(\alpha, \beta))$ and the lower bound $(\mathrm{LHK}(\alpha, \beta))$ for $p_n((x_1, x_2), (x_1, y_2))$ hold, and for all $n \in \mathbb{N}$, $x_1, y_1 \in G_1$ and $x_2 \in G_2$, the upper bound

$$p_n((x_1, x_2), (y_1, x_2)) \leqslant Cn^{-\frac{\alpha}{\beta}} \exp\Big\{ -\Big(\frac{d(x, y)^\beta}{Cn}\Big)^{\frac{1}{\beta_1 - 1}}\Big\}, \qquad \text{(III.3.28)}$$

and the similar lower bound $(\mathrm{LHK}(\alpha, \beta, \beta_2/\beta_1))$ for $p_n((x_1, x_2), (y_1, x_2))$ hold. In particular, the estimates $(\mathrm{UHK}(\alpha, \beta))$ and $(\mathrm{LHK}(\alpha, \beta, \beta_2/\beta_1))$ are the best estimates one can obtain for all $x, y \in G$. We only sketch the proofs since these results will not be needed in the rest of the chapter. Between vertices of the type $x = (x_1, x_2)$ and $y = (x_1, y_2)$, one can show that the condition (D_1) holds, and $(\mathrm{LHK}(\alpha, \beta))$=$(\mathrm{LHK}(\alpha, \beta, 1))$ is then proved as in Remark III.3.4, 2), and the upper bound $(\mathrm{UHK}(\alpha, \beta))$ is a consequence of (III.3.18). Between the vertices $x = (x_1, x_2)$ and $y = (y_1, x_2)$, one can prove a result similar to (III.A.1) but for the expected exit time of the cylinder $B'(x, R) = B_{G_1}(x_1, R^{\frac{\beta_2}{\beta_1}}) \times B_{G_2}(x_2, R^{\frac{\beta_2}{\beta_1}})$, and the proof of (III.3.28) is then similar to the proof of (III.3.18), and $(\mathrm{LHK}(\alpha, \beta, \beta_2/\beta_1))$ is proved in Remark III.3.4, 2) since $(D_{\frac{\beta_2}{\beta_1}})$ always holds on G. □

We now turn to the proof of (WSI) for product graphs and the standard d-dimensional Sierpinski carpet, $d \geqslant 3$. If $G = G_1 \times G_2$, we say that two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are $*$-neighbors if and only if both, the graph distance in G_1 between x_1 and y_1 and the graph distance in G_2 between x_2 and y_2, are at most 1. If G is the standard d-dimensional Sierpinski carpet, we say that $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in G are $*$-neighbors if and only if there exist $i, j \in \{1, \dots, d\}$ such that $|x_i - y_i| \leqslant 1$, $|x_j - y_j| \leqslant 1$, and $x_k = y_k$ for all $k \neq i, j$. Moreover, we say in both cases that $A \subset G$ is $*$-connected if every two vertices of A are connected by a path of $*$-neighbors vertices. We are going to prove that in these two examples, the external boundary of any finite and connected subset A of G is $*$-connected. In order to do this, we are first going to prove a property which generalizes Lemma 2 in [108], and then apply it to our graphs. In Proposition III.3.7, we say that C is a cycle of edges if it is a finite set of edges such that every vertex has even degree in C, that P is a path of edges between x and y in G if x and y are the only vertices with odd degree in P, and *we always understand the addition of sets of edges modulo* 2. We also define for all $x \in G$ the set $\partial_{ext}^x A = \{y \in \partial_{ext} A;\, y \overset{A^c}{\longleftrightarrow} x\}$.

Proposition III.3.7. *Let $\mathcal{C}$ be a set of cycles of edges such that for all finite sets of edges $\mathcal{S} \subset E$ and all cycles of edges Q,*

$$\text{there exists } \mathcal{C}_0 \subset \mathcal{C} \text{ with } \mathcal{S} \cap \Big(Q + \sum_{C \in \mathcal{C}_0} C\Big) = \varnothing. \tag{III.3.29}$$

Then for all finite and connected sets $A \subset G$ and for all $x \in A^c$, the set $\partial_{ext}^x A$ is connected in G^+, the graph with the same vertices as G and where $\{y, z\}$ is an edge of G^+ if and only if y and z are both traversed by some $C \in \mathcal{C}$.

In particular, if A is either a finite and connected subset of $G_1 \times G_2$ for two infinite and locally finite graphs G_1 and G_2, or of the standard d-dimensional Sierpinski carpet for $d \geqslant 3$, then $\partial_{ext} A$ is $*$-connected.

Proof. Let A be a finite and connected subset of G, and let us fix some $x_0 \in A$, $x_1 \in A^c$, and S_1 and S_2 two arbitrary non-empty disjoints subsets of G such that $\partial_{ext}^{x_1} A = S_1 \cup S_2$. Define $\mathcal{S}_i = \{(x, y) \in E;\, x \in A \text{ and } y \in S_i\}$ for each $i \in \{1, 2\}$. We will prove that there exists $C \in \mathcal{C}$ which contains at least one edge of $\mathcal{S}_1$ and one edge of $\mathcal{S}_2$; thus by contraposition $\partial_{ext}^{x_1} A$ will be connected in G^+ since S_1 and S_2 were chosen arbitrary. Since A is finite and connected and S_1 and S_2 are non-empty, there exist two paths P_1 and P_2 of edges between x_0 and x_1 such that $P_i \cap \mathcal{S}_i \neq \varnothing$ but $P_i \cap \mathcal{S}_{3-i} = \varnothing$ for all $i \in \{1, 2\}$, and then $Q = P_1 + P_2$ is a cycle of edges. By (III.3.29), there exists $\mathcal{C}_0 \subset \mathcal{C}$ such that

$$Q' = Q + \sum_{C \in \mathcal{C}_0} C$$

does not intersect $\mathcal{S}_2$. Let us define $\mathcal{C}_1 = \{C \in \mathcal{C}_0;\, C \cap \mathcal{S}_1 \neq \varnothing\}$ and $\mathcal{C}_2 = \mathcal{C}_0 \setminus \mathcal{C}_1$, then

$$P_2 + \sum_{C \in \mathcal{C}_2} C = Q' + P_1 + \sum_{C \in \mathcal{C}_1} C. \tag{III.3.30}$$

The left-hand side of (III.3.30) is a path of edges between x_0 and x_1 which does not intersect $\mathcal{S}_1$ by definition, and thus it intersects $\mathcal{S}_2$. Therefore, the right-hand side of (III.3.30) intersects $\mathcal{S}_2$ as well, i.e., there exists $C \in \mathcal{C}_1$ which intersects $\mathcal{S}_2$, and also $\mathcal{S}_1$ by definition.

We now prove that $\partial_{ext} A$ is $*$-connected when $G = G_1 \times G_2$, for G_1 and G_2 two infinite and locally finite graphs. We start with considering the case that G_2 is a tree, i.e., it does not contain any cycle. We define $\mathcal{C}$ by saying that $C \in \mathcal{C}$ if and only if it contains exactly every edge between (x_1, x_2), (x_1, y_2), (y_1, y_2) and (y_1, x_2) for some $x_1 \sim y_1 \in G_1$ and $x_2 \sim y_2 \in G_2$. Hence a set is connected in G^+ if and only if it is $*$-connected. Note that since G_1 and G_2 are infinite, $\partial_{ext}A = \partial_{ext}^x A$ for all $x \in A^{\mathsf{c}}$, and thus we only need to prove (III.3.29).

Let $\mathcal{S}$ be a finite set of edges and Q_0 be a cycle of edges. We fix a nearest-neighbor path of vertices $\pi = (y_0, y_1, \dots, y_p) \subset G_2^{p+1}$ such that all the vertices visited by the edges in Q_0 are contained in $G_1 \times \{\pi\}$, $y_p \notin \{y_0, \dots, y_{p-1}\}$, and $\mathcal{S} \cap (G_1 \times \{y_p\}) = \varnothing$. For all $n \in \{0, \dots, p-1\}$ and all edges $e = (e_1, y_n) \in E_1 \times \{y_n\}$, with E_1 denoting the edges of G_1, we define C_e^n as the unique cycle in $\mathcal{C}$ containing the edges e and (e_1, y_{n+1}). Next, we recursively define a sequence $(Q_n)_{n\in\{0,\dots,p\}}$ of sets of edges by

$$Q_{n+1} = Q_n + \sum_{e \in Q_n \cap (G_1 \times \{y_n\})} C_e^n \quad \text{for all } n \in \{0, \dots, p-1\}.$$

By construction, for all $n \in \{0, \dots, p-1\}$, Q_p does not contain any edge in $G_1 \times \{y_n\}$ and thus if e is an edge in Q_p of the form (e_1, y) for some $e_1 \in E_1$ and $y \in G_2$, then necessarily $y = y_p$. Since Q_p is a cycle of edges and since G_2 does not have any cycle, $Q_p \subset G_1 \times \{y_p\}$, and thus $Q_p \cap \mathcal{S} = \varnothing$, which gives us (III.3.29).

Let us now assume that G_2 contains exactly one cycle of edges, and let $\{x_2, y_2\}$ and $\{x_2, z_2\}$ be two different edges of this cycle. Let A be a finite and connected subset of G, then the exterior boundary of A in $G_1 \times (G_2 \setminus \{x_2, y_2\})$ and the exterior boundary of A in $G_1 \times (G_2 \setminus \{x_2, z_2\})$ are $*$-connected in G since $G_2 \setminus \{x_2, y_2\}$ and $G_2 \setminus \{x_2, z_2\}$ do not contain any cycle. First assume that there exists $x_1 \in G_1$ such that $(x_1, x_2) \in A$, $(x_1, y_2) \in \partial_{ext}A$ and $(x_1, z_2) \in \partial_{ext}A$, then (x_1, z_2) is $*$-connected in G to any vertex of the external boundary of A in $G_1 \times (G_2 \setminus \{x_2, y_2\})$ and (x_1, y_2) is $*$-connected in G to any vertex of the external boundary of A in $G_1 \times (G_2 \setminus \{x_2, z_2\})$, that is (x_1, y_2) and (x_1, z_2) are $*$-connected in G. The other cases are similar, and we obtain that the exterior boundary of A in G is $*$-connected. We can thus prove by induction on the number of cycles that if G_2 has a finite number of cycles of edges, then the external boundary of any finite and connected subset A of G is $*$-connected. Otherwise, let x and y be any two vertices in $\partial_{ext}A$, and let π^x be an infinite nearest-neighbor path in A^{c}, without loops, beginning in x, such that the projection of π^x on G_1 is a finite path on G_1, i.e. constant after some time, and π^y be a finite nearest-neighbor path in A^{c}, without loops, beginning in y and ending in π^x. Let G_2' be the graph with vertices the projection on G_2 of $A \cup \partial_{ext}A \cup \{\pi^x\} \cup \{\pi^y\}$, and with the same edges between two vertices of G_2' as in G_2. By definition G_2' is infinite and only contains a finite number of cycles of edges, so the exterior boundary of A in $G_1 \times G_2'$ is $*$-connected in $G_1 \times G_2'$, and thus x and y are $*$-connected in G.

Let us now take G to be the standard d-dimensional Sierpinski carpet, $d \geqslant 3$, that we consider as a subset of $\mathbb{N}^d$, and A a finite and connected subset of G. We define $\mathcal{C}$ as the set of cycles with exactly 4 edges, and then a set is connected in G^+ if and only if it is $*$-connected, thus we only need to prove (III.3.29). Let $\mathcal{S}$ be a finite set of edges, Q_0 be a cycle of edges, and $p \in \mathbb{N}$ such that $Q_0 \subset G \cap (\{0,\dots,p-1\} \times \mathbb{N}^{d-1})$ and $\mathcal{S} \cap (\{p\} \times \mathbb{N}^{d-1}) = \varnothing$. We also define $\mathcal{V}_n$ as the set of $d-1$-dimensional squares $V = \{n_2,\dots,n_2+m\} \times \dots \times \{n_d,\dots,n_d+m\}$ such that $\{n\} \times \overline{V} \subset G$ and $(\{n+1\} \times \overline{V}) \cap G = \{n+1\} \times (\overline{V} \setminus V)$, where $\overline{V} = \{n_2-1,\dots,n_2+m+1\} \times \dots \times \{n_d-1,\dots,n_d+m+1\}$. Let us now define recursively two sequences $(Q_n)_{n \in \{0,\dots,p\}}$ and $(R_n)_{n\in\{1,\dots,p\}}$ of cycles of edges such that $Q_n \subset \{n,\dots,p\} \times \mathbb{Z}^{d-1}$ for all $n \in \{0,\dots,p\}$. For each square $V \in \mathcal{V}_n$, all the vertices of $\{n\} \times V$ have an even degree in $Q_n \cap (\{n\} \times \overline{V})$ since $Q_n \cap (\{n-1\} \times V) = Q_n \cap (\{n+1\} \times V) = \varnothing$ and Q_n is a cycle of edges. Moreover, since $d \geqslant 3$, every cycle of edges in $\{n\} \times \overline{V}$ is a sum of cycles with exactly 4 edges in $\{n\} \times \overline{V}$, and thus one can find a set $\mathcal{C}_V \subset \mathcal{C}$ (with $\mathcal{C}_V = \varnothing$ if $(\{n\} \times V) \cap Q_n = \varnothing$) of cycle of edges included in $\{n\} \times \overline{V}$ such that

$$(\{n\} \times \overline{V}) \cap \Big(Q_n + \sum_{C \in \mathcal{C}_V} C\Big) \subset \{n\} \times (\overline{V} \setminus V)$$

We first define R_{n+1} by

$$R_{n+1} = Q_n + \sum_{V \in \mathcal{V}_n} \sum_{C \in \mathcal{C}_V} C.$$

By construction, every edge $e = (n, e_1) \in R_{n+1} \cap (\{n\} \times \mathbb{Z}^{d-1})$ is such that $(n+1, e_1) \in G$, and we then define C_e^n as the unique cycle in $\mathcal{C}$ containing the edges e and $(n+1, e_1)$, and we take

$$Q_{n+1} = R_{n+1} + \sum_{e \in R_{n+1} \cap (\{n\} \times \mathbb{Z}^{d-1})} C_e^n.$$

By construction, $Q_{n+1} \cap (\{0,\dots,n\} \times \mathbb{Z}^{d-1}) = \varnothing$ and since Q_{n+1} is a cycle of edges, we have $Q_{n+1} \subset \{n+1,\dots,p\} \times \mathbb{Z}^{d-1}$. Therefore, we have $Q_p \cap \mathcal{S} = \varnothing$ by our choice of p, which gives us (III.3.29). □

Remark III.3.8. 1) One can extend Proposition III.3.7 similarly to Theorem 3 in [108]. Let us assume that there exists $\mathcal{C}$ such that (III.3.29) hold, and that for each edge e of $E^+ \setminus E$, where E^+ is the set of edges of G^+, there exists a cycle O_e of edges of G^+ such that $O_e \setminus \{e\} \subset E$. Then for all finite set A connected in G^+ and for all $x \in A^{\mathsf{c}}$, the set $\partial_{ext}^x A$ is connected in G^{++}, the graph with the same vertices and edges as G^+ plus every edge of the type $\{x, y\}$ for x, y both crossed by O_e for some edge $e \in E^+ \setminus E$. Indeed let G_A^+ be the graph with the same vertices as G, and edge set E_A^+ which consists of E plus the edges in $E^+ \setminus E$ with both endpoints in A, and let $\mathcal{C}_A^+ = \mathcal{C} \cup \{O_e,\, e \text{ edge of } E_A^+ \setminus E\}$. For each cycle Q of edges in E_A^+ we then have that

$$Q + \sum_{e \in Q \setminus G} O_e$$

is a cycle of edges in E, and thus by (III.3.29) for G with the set of cycles of edges $\mathcal{C}$, one can easily show that (III.3.29) also hold for G_A^+ with the set of cycles of edges $\mathcal{C}_A^+$. Since A is connected in G_A^+, by Proposition III.3.7, $\partial_{ext}^x A$ is connected in G^{++}.

In particular, if G is either a product of infinite graphs $G_1 \times G_2$ or the d-dimensional Sierpinski carpet, $d \geqslant 3$, taking O_e such that $O_e \setminus \{e\}$ only contains two connected edges of E for each $e \in E^+ \setminus E$, we get that the external boundary of every finite and $*$-connected subset A of G is $*$-connected since $G^{++} = G^+$.

2) Proposition III.3.7 provides us with a stronger result than Lemma 2 in [108] even when $G = \mathbb{Z}^d$, $d \geqslant 3$. Indeed, $\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z}$ and thus the external boundary of every finite and connected (or even $*$-connected) subset of $\mathbb{Z}^d$ is $*$-connected in the sense of product graphs previously defined, i.e., it is connected in $\mathbb{Z}^d \cup \{\{(x,n),(y,n+1)\};\, n \in \mathbb{Z}, x \sim y \in \mathbb{Z}^{d-1}\}$.

3) An example of a graph G for which we cannot apply Proposition III.3.7, and in fact where we can find a finite and connected set whose boundary is not $*$-connected, and where (WSI) does not hold, but where (G_β) and (V_α) hold, is the Menger sponge. It is defined as the graph associated to the following generalized 3-dimensional Sierpinski carpet, see Section 2 of [6]: split $[0,1]^3$ into 27 cubes of size length 1/3, remove the central cube of each face as well as the central cube of $[0,1]^3$, and iterate this process for each remaining cube. It is easy to show that G endowed with the graph distance verifies (V_α) with $\alpha = \frac{\log(20)}{\log(3)}$, and (G_β) follows from Theorem 5.3 in [6] since the random walk on the Menger sponge is transient, see p.741 of [5]. One can then easily check that taking $A_n = (3^n/2, 5 \times 3^n/2)^3 \cap G$, where we see G as a subset of $\mathbb{R}^3$, then $\partial_{ext} A$ is not $*$-connected. In fact for each $x \in \partial_{ext} A_n$ and $p < n$, there is no 3^p path between x and $B(x, 2 \times 3^p)^{\mathsf{c}}$, and thus (WSI) does not hold. □

We can now conclude that our main results apply to the examples mentioned in the introduction.

Corollary III.3.9. *The graphs in* (III.1.4) *(endowed with unit weights) satisfy* (p_0), (V_α), (G_β), *for some* $\alpha > 2$, $\beta \in [2, \alpha)$ *and* (WSI), *with respect to a suit-*

able distance function $d(\cdot,\cdot)$. In particular, the conclusions of Theorems III.1.1 and III.1.2 hold for these graphs.

Proof. Condition (p_0) holds plainly in all cases since all graphs in (III.1.4) have unit weights and uniformly bounded degree. For G_1, we classically have $\alpha = d$, $\beta = 2$ and (WSI) follows e.g. from Proposition III.3.7 with $d = d_G$ (or even the ℓ^∞-norm) since $\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z}$. The case of G_2 is an application of Proposition s III.3.5 and III.3.7: it is known [7, 50] that G', the discrete skeleton of the Sierpinski gasket, satisfies (V_{α_2}) and (G_{β_2}) with $\alpha_2 = \frac{\log 3}{\log 2}$ and $\beta_2 = \frac{\log 5}{\log 2}$, whence (V_α), (G_β), hold for G_2 with respect to d in (III.3.22), for $\alpha = \frac{\log 45}{2\log 2}$ and $\beta = \frac{\log 5}{\log 2}$ as given by (III.3.22) with $\alpha_1 = 1$, $\beta_1 = 2$ (note that $\alpha_2 > 1$ so (III.3.21) holds), and it is easy to see that any $*$-connected path is also a 1-path for d in (III.3.22), hence (WSI) holds. Regarding G_3, the standard d dimensional graphical Sierpinski carpet endowed with the graph distance, with $d \geq 3$ (cf. p.6 of [6]), $\alpha = \log(3^d - 1)/\log(3)$ (with $d = d_G$) and (G_β) then follows from Theorem 5.3 in [6] since the random walk on G_3 is transient for $d \geq 3$, see p.741 of [5]. Moreover, (WSI) on G_3 follows from Proposition III.3.7 since any $*$-connected path in G_3 is also a 2-path.

Finally, G_4 endowed with the graph distance $d = d_{G_4}$ satisfies (V_α) for some $\alpha > 2$ by assumption and (G_β) holds with $\beta = 2$ by Theorem 5.1 in [48]. To see that (WSI) holds, we first observe that the group $\Gamma = \langle S \rangle$ which has G_4 as a Cayley graph is finitely presented. Indeed, by a classical theorem of Gromov [45], Γ is virtually nilpotent, i.e., it has a a normal subgroup H of finite index which is nilpotent. Furthermore, H is finitely generated (this is because Γ/H is finite, so writing gH, $g \in C$ with $|C| < \infty$ and $1 \in C$ for all the cosets, one readily sees that $H = \langle \{h \in H;\, h = g^{-1}sg' \text{ for some } g, g' \in C \text{ and some } s \in S\}\rangle$).

Since H is nilpotent and finitely generated, it is in fact finitely presented, see for instance 2.2.4 (and thereafter) and 5.2.18 in [77], and so is Γ/H, being finite. Together with the normality of H one straightforwardly deduces from this that Γ is finitely presented, see again 2.2.4 in [77]. As a consequence $\Gamma = \langle S|R \rangle$ for a suitable finite set of relators R. This yields a generating set of cycles for G_4 of maximal cycle length $t < \infty$, where t is the largest length of any relator in R, and Theorem 5.1 of [107] (alternatively, one could also apply Proposition III.3.7) readily yields that, for all $x \in \partial_{ext}A$, every two vertices of $\partial^x_{ext}A$ are linked via an R_0 path in $\partial^x_{ext}A$, with $R_0 = t/2$. Moreover, since G has sub-exponential growth, $\{\partial^x_{ext}A,\, x \in \partial_{ext}A\}$ contains at most two elements, see for instance Theorem 10.10 and 12.2, (g), in [112] and, since G does not have linear growth, in fact only 1, see for instance Lemma 5.4, (a), and Theorem 5.12 in [51]. We also prove this fact for any graph satisfying (III.3.1) in the course of proving Lemma III.6.5.

In order to prove (WSI), we thus only need to show that there exists $c > 0$ such that $\delta(\partial_{ext}A) \geqslant c\delta(A)$ for all finite and connected subgraphs A of G, and we are actually going to show this inequality in the general setting of vertex-transitive graphs G. Write $m \stackrel{\text{def.}}{=} \delta(\partial_{ext}A)$, let us fix some $x_0 \in \partial_{ext}A$, and let $\overline{B}(x,m) = \{y \in G;$ every unbounded path beginning in y intersects $B(x,m)\}$, for all $x \in G$. Let us assume that there exists $x_1 \in \overline{B}(x_0,m)$ such that $B(x_1,m) \cap B(x_0,m) = \varnothing$, and then we have $\overline{B}(x_1,m) \subset \overline{B}(x_0,m) \setminus B(x_0,m)$. Since G is vertex-transitive, there exists $x_2 \in \overline{B}(x_1,m)$ such that $B(x_2,m) \cap B(x_1,m) = \varnothing$. Moreover, by definition, $\overline{B}(x_2,m) \subset \overline{B}(x_1,m) \setminus B(x_1,m)$, and $x_1 \leftrightarrow x_2$ in $\overline{B}(x_1,m)$. Iterating this reasoning, we can thus construct recursively a sequence $(x_n)_{n\in\mathbb{N}}$ of vertices such that $\overline{B}(x_{n+1},m) \subset \overline{B}(x_n,m) \setminus B(x_n,m)$, and $x_n \leftrightarrow x_{n+1}$ in $\overline{B}(x_n,m)$ for all $n \in \mathbb{N}$. Therefore, there exists an unbounded path beginning in x_1 in $\overline{B}(x_0,m) \setminus B(x_0,m)$, which is a contradiction by definition of $\overline{B}(x_0,m)$. Hence, $\delta(\overline{B}(x_0,m)) \leqslant 4m$, and so $\delta(A) \leqslant 4\delta(\partial_{ext}A)$. □

Remark III.3.10. The conclusions of Theorems III.1.1 and III.1.2 do not only hold for G_2 in (III.1.4), but also for any product graphs $G_1 \times G_2$ under the same hypotheses as in Proposition III.3.5. Further interesting examples can be generated involving graphs G endowed with a distance $d \neq d_G$ which is not of the form of a product of graph distances as in (III.3.22). For instance, in Corollary 4.12 of [46], estimates similar to (UHK($\alpha', \alpha'+1$)) and (LHK($\alpha', \alpha'+1, \zeta$)) for some $\alpha' > 1$ and $\zeta \in [1, \alpha'+1)$ are proved for different recurrent fractal graphs G' when the distance d' on G' is the effective resistance as defined in (2.4) of [46]. By Lemma 3.2 in [46], ($V_{\alpha'}$) hold on G' endowed with the distance d', and thus one can then prove similarly as in the proof of Proposition III.3.5 that $G = G' \times \mathbb{Z}$ (or some other product with an infinite graph satisfying (UHK(α,β)) and (NLHK(α,β))) satisfy (V_α) and (G_β) with $\alpha = \frac{3\alpha'+1}{2}$ and $\beta = \alpha'+1$ for the distance

$$d((x',n),(y',m)) = d'(x',y') \vee |n-m|^{\frac{2}{\beta}} \quad \text{for all } x',y' \in G' \text{ and } n,m \in \mathbb{Z}.$$

Moreover, (WSI) is also verified on G by Proposition III.3.7, and thus the conclusions of Theorems III.1.1 and III.1.2 hold for G. It should be noted that d' is not always equivalent to the graph distance on G', see for instance the graph G' considered in Corollary 4.16 of [46]. This graph is also another example of a graph where (D_ζ) hold for some $\zeta > 1$ but not $\zeta = 1$, and where the estimates (UHK(α,β)) and (LHK(α,β,ζ)) are optimal at this level of generality. □

III.4 Strong connectivity of the interlacement set

We now prove a strong connectivity result for the random interlacement set on the cable system, Proposition III.4.1 below; see also Proposition 1 in [73] and Lemma II.3.2 for similar findings in the case $G = \mathbb{Z}^d$. We recall our standing assumption (III.3.1). The availability of controls on the heat kernel and exit times provided by Proposition III.3.3 will figure prominently in obtaining the desired estimates; see also Remark III.4.8 below. The connectivity result will play a crucial role in Section III.9, where $\widetilde{\mathcal{I}}^u$ will be used as a random network to construct certain continuous level-set paths for the free field. We recall the notation introduced in (III.2.15) and (III.3.4), and our standing assumptions (III.3.1).

Proposition III.4.1. *For each $u_0 > 0$, there exist constants $c_{12} > 0$, $c > 0$ and $C < \infty$ all depending on u_0 such that, for all $x_0 \in G$, $u \in (0, u_0]$ and $L \geq 1$,*

$$\widetilde{\mathbb{P}}^I\Big(\bigcap_{z,z' \in \widetilde{\mathcal{I}}^u \cap \widetilde{B}(x_0,L)} \big\{z \overset{\sim}{\longleftrightarrow} z' \text{ in } \widetilde{\mathcal{I}}^u \cap \widetilde{B}\,(x_0, 2C_{10}L)\,\big\}\Big) \geq 1 - C\exp\left\{-cL^{c_{12}}u\right\}. \tag{III.4.1}$$

The proof of Proposition III.4.1 requires some auxiliary lemmas and appears at the end of the section. In the rest of the chapter, we will not use directly Proposition III.4.1 because the event in (III.4.1) is neither increasing nor decreasing, see above (III.7.4), and therefore cannot be used in the decoupling inequalities, see Theorem III.2.4. We will however use two auxiliary results which together readily imply Proposition III.4.1, namely Lemma III.4.3 and Proposition III.4.7. Another interest of Proposition III.4.1 is the following corollary, which is a generalization of Corollary 2.3 of [93] from $\mathbb{Z}^d$ to G as in (III.3.1).

Corollary III.4.2. *Let $u > 0$. Then $\widetilde{\mathbb{P}}^I$-a.s., the subset $\widetilde{\mathcal{I}}^u$ of $\widetilde{G}$ is unbounded and connected. Analogously, $\mathbb{P}^I$-a.s., the subset $\mathcal{I}^u$ of G is infinite and connected.*

Proof of Corollary III.4.2. Fix any vertex $x_0 \in G$. Let A_L denote the event appearing on the left-hand side of (III.4.1), and $A'_L = \{\widetilde{\mathcal{I}}^u \cap \widetilde{B}\,(x_0, L) \neq \varnothing\}$. Note that $\{\widetilde{\mathcal{I}}^u$ is unbounded, connected$\} \supset (\bigcup_L A'_L) \cap \liminf_L A_L$. The events A'_L are increasing with $\lim_L \widetilde{\mathbb{P}}^I(A'_L) = 1$ by (III.3.11), and by (III.4.1) and a Borel-Cantelli argument, $\widetilde{\mathbb{P}}^I(\liminf_L A_L) = 1$. The same reasoning applies also to $\mathcal{I}^u$ (with (III.4.2) below in place of (III.4.1)). □

Let us denote for each $u > 0$ by $\widehat{\mathcal{I}}^u$ the set of edges of G traversed by at least one of the trajectories in the trace of the random interlacement process ω^u. From the construction of the random interlacement process on the cable system

$\widetilde{G}$ from the corresponding process on G by adding Brownian excursions on the edges, it follows that the inequality

$$\mathbb{P}^I\Big(\bigcap_{x,y\in\mathcal{I}^u\cap B(x_0,L)}\{x\overset{\wedge}{\longleftrightarrow} y\ \text{ in } \widehat{\mathcal{I}}^u\cap B_E\,(x_0,2C_{10}L)\,\}\Big)\geq 1-C(u_0)\exp(-L^{c(u_0)}u) \tag{III.4.2}$$

for all $u\leq u_0$, will entail (III.4.1), where for $x,y\in G$ and $A\subset E$, $\{x\overset{\wedge}{\longleftrightarrow} y \text{ in } A\}$ means that there exists a nearest neighbor path from x to y crossing only edges contained in A. We refer to the discussion at the beginning of the Appendix of Chapter II for a similar argument on why (III.4.2) implies (III.4.1). In order to prove (III.4.2), we will apply a strategy inspired by the proof of Proposition 1 in [73] for the case $G=\mathbb{Z}^d$.

For $U\subset\subset G$ let N_U^u be the number of trajectories in $\text{supp}(\omega^u)$ which enter U. By definition, N_U^u is a Poisson variable with parameter $u\text{cap}(U)$, and thus there exist constants $c,C\in(0,\infty)$ such that uniformly in $u\in(0,\infty)$,

$$\mathbb{P}^I\left(cu\cdot\text{cap}(U)\leq N_U^u\leq Cu\cdot\text{cap}(U)\right)\geq 1-C\exp\left\{-cu\cdot\text{cap}(U)\right\}, \tag{III.4.3}$$

cf. display (2.11) in [73]. We now state a lemma which gives an estimate in terms of capacity for the probability to link two subsets of $B(x,L)$ through edges in $\widehat{\mathcal{I}}^u\cap B(x,C_{10}L)$.

Lemma III.4.3. *There exist constants $c\in(0,1)$ and $C\in[1,\infty)$ such that for all $L\geq 1$, $u>0$ and all subsets U and V of $B(x,L)$,*

$$\mathbb{P}^I\big(U\overset{\wedge}{\longleftrightarrow} V\ \textit{in } \widehat{\mathcal{I}}^u\cap B_E(x,C_{10}L)\big)\geq 1-C\exp\big\{-cL^{-\nu}u\text{cap}(U)\text{cap}(V)\big\}, \tag{III.4.4}$$

with ν as in (III.1.6).

Proof. For U not to be connected to V through edges in $\widehat{\mathcal{I}}^u\cap B_E(x,C_{10}L)$, all of the N_U^u trajectories hitting U must not hit V after hitting U and before leaving $B(x,C_{10}L)$, so

$$\begin{aligned}&\mathbb{P}^I\Big(U\overset{\wedge}{\longleftrightarrow} V \text{ in } \widehat{\mathcal{I}}^u\cap B(x,C_{10}L)\Big)\\ &\quad\geq 1-\mathbb{P}^I(N_U^u<cu\text{cap}(U))-\big(P_{\overline{e}_U}(H_V>T_{B(x,C_{10}L)})\big)^{cu\text{cap}(U)}\end{aligned} \tag{III.4.5}$$

(recall (III.2.3) and (III.3.9) for notation). For all $y\in B(x,L)$, by (III.3.8), (III.3.5) and (III.3.2),

$$P_y(H_V>T_{B(x,C_{10}L)})\leq 1-\sum_{z\in B(x,L)}g_{B(x,C_{10}L)}(y,z)e_V(z)\leq 1-\frac{c_2}{2}(2L)^{-\nu}\text{cap}(V), \tag{III.4.6}$$

where we also used $e_V\leq e_{V,B(x,C_{10}L)}$ in the first inequality. Since $\text{cap}(V)\leq C_{11}L^\nu$ by (III.3.11), we can combine (III.4.5), (III.4.3) and (III.4.6) to get (III.4.4). □

For each $x \in G$ and $L \geqslant 1$, if $x \in \mathcal{I}^u$, we denote by $C^u(x, L)$ the set of vertices in G connected to x by a path of edges in $\widehat{\mathcal{I}}^u \cap B_E(x, L)$, and we take $C^u(x, L) = \varnothing$ otherwise. On our way to establishing (III.4.2) we introduce the following thinned processes. For each $i \in \{1, \dots, 3\}$, let $\omega_i^{u/3}$ be the Poisson point process which consists of those trajectories in ω^u which have label between $(i-1)u/3$ and $iu/3$. I.e., $\omega_i^{u/3}$, $i \in \{1, 2, 3\}$, have the same law as three independent random interlacement processes at level $u/3$ on G. For each $i \in \{1, \dots, 3\}$, let $\mathcal{I}_i^{u/3}$ and $\widehat{\mathcal{I}}_i^{u/3}$, respectively, be the set of vertices and edges, respectively, visited by at least one trajectory in $\mathrm{supp}(\omega_i^{u/3})$, and for each $x \in G$ and $L > 0$, let $C_i^{u/3}(x, L)$ be the set of vertices connected to x by a path of edges in $\widehat{\mathcal{I}}_i^{u/3} \cap B_E(x, L)$. Note that $\mathbb{P}^I$-a.s. we have $\mathcal{I}^u = \cup_{i=1}^3 \mathcal{I}_i^{u/3}$ and $\widehat{\mathcal{I}}^u = \cup_{i=1}^3 \widehat{\mathcal{I}}_i^{u/3}$. Now fix some $x_0 \in G$ and $L > 0$, and assume there exist $x, y \in \mathcal{I}^u \cap B(x_0, L)$ such that x is not connected to y through edges in $\widehat{\mathcal{I}}^u \cap B_E(x_0, C_{10}L)$. Let $i, j \in \{1, 2, 3\}$ be such that $x \in \mathcal{I}_i^{u/3}$ and $y \in \mathcal{I}_j^{u/3}$, and let $k = k(i, j) \in \{1, 2, 3\}$ be different from i and j. By definition, $C_i^{u/3}(x, L)$ is not connected to $C_j^{u/3}(y, L)$ through edges in $\widehat{\mathcal{I}}_k^{u/3} \cap B_E(x_0, 2C_{10}L)$, and so

$$\begin{aligned} &\mathbb{P}^I\left(x, y \in \mathcal{I}^u, \left\{x \overset{\wedge}{\longleftrightarrow} y \text{ in } \widehat{\mathcal{I}}^u \cap B_E(x_0, 2C_{10}L)\right\}^{\mathsf{c}}\right) \\ &\leq \sum_{i,j=1}^{3} \mathbb{P}^I\left(x \in \mathcal{I}_i^{u/3}, y \in \mathcal{I}_j^{u/3}, \left\{\begin{array}{c} C_i^{u/3}(x, L) \overset{\wedge}{\longleftrightarrow} C_j^{u/3}(y, L) \\ \text{in } \widehat{\mathcal{I}}_k^{u/3} \cap B_E(x_0, 2C_{10}L) \end{array}\right\}^{\mathsf{c}}\right). \end{aligned} \tag{III.4.7}$$

Since $\widehat{\mathcal{I}}_k^{u/3}$ is independent from $\widehat{\mathcal{I}}_i^{u/3}$ and $\widehat{\mathcal{I}}_j^{u/3}$ and $C_i^{u/3}(x, L) \subset B(x_0, 2L)$, we can use Lemma III.4.3 to upper bound the last probability in (III.4.7). In order to obtain (III.4.2), we now need a lower bound on the capacity of $C_i^{u/3}(x, L)$, and for this purpose we begin with a lower bound on the capacity of the range of N random walks. For each $N \in \mathbb{N}$ and $S_N = (x_1, \dots, x_N) \in G^N$ we define a sequence $(Z^i)_{i \in \{1, \dots, N\}}$ of independent random walks on G with fixed initial point $Z_0^i = x_i$ under some probability measure P^{S_N}, i.e., for each $i \in \{1, \dots, N\}$, Z^i has the same law under P^{S_N} as Z under P_{x_i}. For all positive integers M and N we define the trace $T(N, M)$ on G of the N first random walks up to time M by

$$T(N, M) \overset{\text{def.}}{=} \bigcup_{i=1}^{N} \bigcup_{p=0}^{M} \{Z_p^i\}.$$

For ease of notation, we also set

$$\gamma = \frac{\alpha}{\beta} > 1 \qquad \text{and} \qquad F_\gamma(M) = \begin{cases} M^{2-\gamma} & \text{if } \gamma < 2, \\ \log(M) & \text{if } \gamma = 2, \\ 1 & \text{otherwise,} \end{cases} \tag{III.4.8}$$

with α and β from (V_α) and (G_β). The function F_γ reflects the fact that the "size" of $\{Z_n;\, n \geq 0\}$ (as captured by β, see Lemma III.A.1) becomes increasingly small relative to the overall geometry of G (controlled by α) as γ grows. As a consequence, intersections between independent walks in $\mathcal{I}^u$ are harder to produce for larger γ. This is implicit in the estimates below.

Lemma III.4.4. *There exists $C < \infty$ such that for all $t > 0$, positive integers N and M, and starting points $S_N \in G^N$,*

$$P^{S_N}\left(\mathrm{cap}\big(T(N,M)\big) \leq t \min\left(\frac{NM}{F_\gamma(M)}, M^{\gamma-1}\right)\right) \leq Ct. \tag{III.4.9}$$

Proof. Consider positive integers N and M, and $S_N \in G^N$. By Markov's inequality,

$$\begin{aligned} &P^{S_N}\left(\mathrm{cap}\big(T(N,M)\big) \leq t \min\left(\frac{NM}{F_\gamma(M)}, M^{\gamma-1}\right)\right) \\ &\quad \leq t \min\left(\frac{NM}{F_\gamma(M)}, M^{\gamma-1}\right) E^{S_N}\left[\mathrm{cap}\big(T(N,M)\big)^{-1}\right]. \end{aligned} \tag{III.4.10}$$

Applying (III.3.12) with the probability $\mu = \frac{1}{(M-\lceil M/2\rceil+1)N} \sum_{i=1}^N \sum_{p=\lceil M/2\rceil}^M \delta_{Z_p^i}$, which has support in $T(N,M)$, yields

$$E^{S_N}\left[\mathrm{cap}(T(N,M))^{-1}\right] \leq E^{S_N}\Big[\frac{C}{N^2M^2} \sum_{i,j=1}^N \sum_{p,q=\lceil M/2\rceil}^M g\big(Z_p^i, Z_q^j\big)\Big]. \tag{III.4.11}$$

Moreover, using the heat kernel bound (III.3.16) and the Markov property at time p, we have uniformly in all $p \in \mathbb{N}$ and $x, y \in G$,

$$f_p^x(y) \stackrel{\text{def.}}{=} E_x\left[g(Z_p, y)\right] = \sum_{n=p}^\infty p_n(x,y) \leq C \sum_{n=p}^\infty n^{-\gamma} \leq Cp^{1-\gamma}, \tag{III.4.12}$$

and, thus, for $p < q$, with $\widetilde{P}_\cdot$ an independent copy of $P_\cdot$ governing the process $\widetilde{Z}$, using symmetry of $g(\cdot,\cdot)$,

$$E_{x_i}\left[g(Z_p^i, Z_q^i)\right] = E_{x_i}\left[\widetilde{E}_{Z_p^i}[g(\widetilde{Z}_0, \widetilde{Z}_{q-p})\right] = E_{x_i}\left[f_{q-p}^{Z_p^i}(Z_p^i)\right] \stackrel{\text{(III.4.12)}}{\leq} C(q-p)^{1-\gamma}, \tag{III.4.13}$$

and the same upper bound applies to $E_{x_i}\left[g(Z_q^i, Z_p^i)\right]$, again by symmetry of g. Considering the on-diagonal terms in the first sum on the right-hand side of

(III.4.11), we obtain

$$
\begin{aligned}
E^{S_N}\Bigg[\sum_{i=1}^{N}\sum_{p,q=\lceil M/2\rceil}^{M} g(Z_p^i,Z_q^i)\Bigg] &\le 2N \max_{i\in\{1,\dots,N\}} E^{S_N}\Bigg[\sum_{\substack{p,q=\lceil M/2\rceil\\ p\le q}}^{M} g(Z_p^i,Z_q^i)\Bigg] \\
&\overset{\text{(III.4.13)}}{\le} CNM\Big(1+\sum_{k=1}^{\lceil M/2\rceil} k^{1-\gamma}\Big) \\
&\overset{\text{(III.4.8)}}{\le} CNMF_\gamma(M).
\end{aligned}
\tag{III.4.14}
$$

For $i\neq j$ on the other hand, (III.4.12) implies

$$
\begin{aligned}
E^{S_N}\Bigg[\sum_{p,q=\lceil M/2\rceil}^{M} g\left(Z_p^i,Z_q^j\right)\Bigg] = \sum_{p,q=\lceil M/2\rceil}^{M} E^{S_N}\left[f_p^{x_i}\left(Z_q^j\right)\right] &\le CM\sum_{p=\lceil M/2\rceil}^{M} p^{1-\gamma} \\
&\le CM^{3-\gamma}.
\end{aligned}
$$

Combining this with (III.4.10), (III.4.11) and (III.4.14) yields (III.4.9). □

We now iterate the bound from Lemma III.4.4 over the different parts of the random walks $(Z^i)_{i\in\{1,\dots N\}}$ in order to improve it.

Lemma III.4.5. *For each $\varepsilon\in(0,1)$, there exist constants $c(\varepsilon)>0$ and $C(\varepsilon)\in[1,\infty)$ such that for all positive integers N and M, and $S_N\in G^N$,*

$$
P^{S_N}\left(\mathrm{cap}\big(T(N,M)\big)\le c\kappa\right)\le C\exp\{-cM^{\varepsilon}\},
\tag{III.4.15}
$$

where

$$
\kappa=\kappa(N,M,\gamma,\varepsilon)=\min\left(\frac{NM^{1-\varepsilon}}{F_\gamma(M^{1-\varepsilon})},M^{(\gamma-1)(1-\varepsilon)}\right).
\tag{III.4.16}
$$

Proof. For $\varepsilon\in(0,1)$, all positive integers N, M and k, we define

$$
T_k(N,M)=\bigcup_{i=1}^{N}\bigcup_{p=(k-1)M}^{kM-1}\{Z_p^i\}.
$$

By the Markov property and Lemma III.4.4, for all $t>0$, $\varepsilon\in(0,1)$ and $S_N\in G^N$, with $\mathcal{F}_k^{N,M}=\sigma(Z_p^i,1\le i\le N,1\le p\le(k-1)\lceil M^{1-\varepsilon}\rceil)$,

$$
P^{S_N}\left(\mathrm{cap}\big(T_k(N,\lceil M^{1-\varepsilon}\rceil)\big)\le t\kappa\,\Big|\,\mathcal{F}_k^{N,M}\right)\le Ct.
\tag{III.4.17}
$$

Moreover,

$$
\bigcup_{k=1}^{\lfloor M^{\varepsilon}/2\rfloor} T_k(N,\lceil M^{1-\varepsilon}\rceil)\subset T(N,M),
$$

whence $\mathrm{cap}\big(T(N,M)\big) \leq L$ implies $\mathrm{cap}\big(T_k(N,\lceil M^{1-\varepsilon}\rceil)\big) \leq L$ for all $1 \leq k \leq \lfloor M^\varepsilon/2\rfloor$ by the monotonicity property (III.3.13). Thus, applying the Markov property and using (III.4.17) inductively we obtain,

$$P^{S_N}\left(\mathrm{cap}\big(T(N,M)\big) \leq t\kappa\right) \leq (Ct)^{\lfloor M^\varepsilon/2\rfloor} \leq \exp\{-cM^\varepsilon\}$$

for all t small enough and $M \geq 2$. This yields (III.4.15). □

The next step is to transfer the bound in Lemma III.4.5 from the trace on G of N independent random walks to a subset of the random interlacement. For all $u > 0$ and $A \subset\subset G$, conditionally on the number N_A^u of trajectories in $\mathrm{supp}(\omega^u)$ which hit A, let $S_A^u \in G^{N_A^u}$ be the family of entrance points in A by trajectories in the support of the random interlacement process ω^u on G. With a slight abuse of notation, we identify $Z^1,\dots,Z^{N_A^u}$ under $P^{S_A^u}$ with the forward (seen from the first hitting time of A) parts of the trajectories in $\mathrm{supp}(\omega^u)$ which hit A under $\mathbb{P}^I(\cdot\,|\,S_A^u)$. We define $\Psi(u,A,M) = T(N_A^u,M)$ for all positive integers M.

Lemma III.4.6. *For each $u_0 > 0$ and $\varepsilon \in (0,1)$, there exist constants $c' = c'(\varepsilon) > 0$ independent of u_0, $c(u_0,\varepsilon) > 0$ and $C(u_0,\varepsilon) < \infty$ such that for all $u \in (0,u_0]$, $A \subset\subset G$, $x \in G$, and positive integers M, with $\widetilde{\kappa}_{u,A} \overset{def.}{=} \kappa(u\mathrm{cap}(A),M,\gamma,\varepsilon)$ (cf. (III.4.16)),*

$$\mathbb{P}^I\left(\mathrm{cap}\big(\Psi(u,A,M)\big) \leq c'\widetilde{\kappa}_{u,A}\right) \leq C\exp\big\{-c\,(u\mathrm{cap}(A)\wedge M^\varepsilon)\big\}, \qquad \text{(III.4.18)}$$

and for all positive integers k, if $A \subset B\big(x,kM^{\frac{1+\varepsilon}{\beta}}\big)$ (with β as in (G_β)),

$$\mathbb{P}^I\left(\Psi(u,A,M) \not\subset B\big(x,(k+1)M^{\frac{1+\varepsilon}{\beta}}\big)\right) \leq Ck^\nu \exp\big\{-cM^{\frac{\varepsilon(\nu\wedge 1)}{\beta}}u\big\}. \qquad \text{(III.4.19)}$$

Proof. Writing, with $N = \lceil cu\mathrm{cap}(A)\rceil$,

$$\begin{aligned}\mathbb{P}^I\left(\mathrm{cap}\big(\Psi(u,A,M)\big) \leq c'\widetilde{\kappa}_{u,A}\right) \leq &\mathbb{P}^I\left(N_A^u < N\right)\\ &+\sup_{S_N} P^{S_N}\left(\mathrm{cap}\big(T(N,M)\big) \leq c'\widetilde{\kappa}_{u,A}\right),\end{aligned}$$

the inequality (III.4.18) easily follows from the Poisson bound (III.4.3) and Lemma III.4.5. We turn to the proof of (III.4.19), and we fix $x \in G$, $\varepsilon \in (0,1\wedge(\gamma-1))$ as well as positive integers k and M. Let us write $A_k = B\big(x,kM^{\frac{1+\varepsilon}{\beta}}\big)$ to simplify notation. If $\Psi(u,A_k,M) \not\subset A_{k+1}$, then for at least one trajectory Z^i among the forward trajectories $Z^1,\dots,Z^{N_{A_k}^u}$ in $\mathrm{supp}(\omega^u)$ which hit A_k, the walk Z^i will leave $B\big(Z_0^i,M^{\frac{1+\varepsilon}{\beta}}\big)$ before time M, which is atypically short on account of Proposition III.3.3 ii). Therefore, since $N_A^u \leqslant N_{A_k}^u$,

$$\begin{aligned}&\mathbb{P}^I\left(\Psi(u,A,M) \not\subset A_{k+1}\right)\\ &\quad\leq \mathbb{P}^I\left(N_{A_k}^u \geq Cu\cdot\mathrm{cap}(A_k)\right) + Cu\cdot\mathrm{cap}(A_k)\sup_{y\in A_k} P_y\Big(T_{B(y,M^{(1+\varepsilon)/\beta})} \leq M\Big).\end{aligned}$$

Using (III.4.3), (III.3.11) and (III.3.17), we get

$$\begin{aligned}\mathbb{P}^I\big(\Psi(u,A,M)\not\subset A_{k+1}\big) \leq &C\exp\big\{-cuk^{\nu}M^{\frac{\nu(1+\varepsilon)}{\beta}}\big\}\\ &+Cuk^{\nu}M^{\frac{\nu(1+\varepsilon)}{\beta}}\exp\big\{-cM^{\frac{\varepsilon}{\beta-1}}\big\},\end{aligned}$$

and (III.4.19) follows. □

With Lemma III.4.6 at hand, we can finally produce the desired bound on the capacity of $\mathcal{C}^u(x,L)$ (see after Lemma III.4.3 for the definition).

Proposition III.4.7. *For each $u_0>0$ there exist $c_{13}>0$ and $C_{13}<\infty$ independent of u_0, $c=c(u_0)>0$ and $C=C(u_0)\in[1,\infty)$ such that for every $u\in(0,u_0]$, $x\in G$ and $L\geq 1$,*

$$\mathbb{P}^I\left(x\in\mathcal{I}^u,\ \mathrm{cap}(\mathcal{C}^u(x,L))\leq c_{13}L^{3\nu/4}u^{\lfloor\gamma-1\rfloor}\right)\leq C\exp\big\{-cuL^{C_{13}}\big\}. \quad \text{(III.4.20)}$$

Proof. We focus on the case $\gamma<2$. Let $u_0>0$, $x\in G$, and $u\in(0,u_0)$ as above and consider a positive integer M and $\delta\in(0,1)$ to be chosen suitably. Since $\gamma<2$, we have $F_\gamma(M)=M^{2-\gamma}$ by (III.4.8). Thus, by Lemma III.4.5,

$$\begin{aligned}&\mathbb{P}^I\left(x\in\mathcal{I}^u,\mathrm{cap}\big(\Psi(u,\{x\},M)\big)\leq c'M^{(1-\delta)(\gamma-1)}\right)\\ &\quad=\mathbb{E}^I\left[\mathbb{1}_{x\in\mathcal{I}^u}P_x\left(\mathrm{cap}\big(T(1,M)\big)\leq c'M^{(1-\delta)(\gamma-1)}\right)\right]\leq C\exp\{-cM^{\delta}\},\end{aligned}$$

and with (III.4.19),

$$\mathbb{P}^I\Big(\Psi(u,\{x\},M)\not\subset B\big(x,2M^{\frac{1+\delta}{\beta}}\big)\Big)\leq C\exp\big\{-cM^{\frac{\delta(\nu\wedge 1)}{\beta}}u\big\}.$$

Note that if $\Psi(u,\{x\},M)\subset B\big(x,2M^{\frac{1+\delta}{\beta}}\big)$, then $\Psi(u,\{x\},M)\subset\mathcal{C}^u\big(x,2M^{\frac{1+\delta}{\beta}}\big)$ by definition. Thus, combining the previous two estimates,

$$\mathbb{P}^I\Big(x\in\mathcal{I}^u,\ \mathrm{cap}\Big(\mathcal{C}^u\big(x,2M^{\frac{1+\delta}{\beta}}\big)\Big)\leq c'M^{(1-\delta)(\gamma-1)}\Big)\leq C\exp\big\{-cM^{\frac{\delta}{\beta-1}}u\big\}$$

and (III.4.20) follows by taking $M=\big\lfloor(L/2)^{\frac{7\beta}{8}}\big\rfloor$ and $\delta=\frac{1}{7}$ since $\beta(\gamma-1)=\nu$.

For $\gamma\geq 2$, stronger bounds are required to deduce (III.4.20) than the one provided by Lemma III.4.6. The idea is to apply recursively Lemma III.4.6 to a sequence of $\lfloor\gamma\rfloor$ independent random interlacement processes at level $u/\lfloor\gamma\rfloor$ as in Lemma 8, 9 and 10 of [73] or Lemma II.A.3 and Corollary II.A.4 for $G=\mathbb{Z}^d$. We refer the reader to these references for details. □

We conclude with the proof of Proposition III.4.1.

Proof of Proposition III.4.1. Fix some $u_0 > 0$. Recall the notation below Lemma III.4.3, and write for all $x_0 \in G$, $L \geq 1$, $u \in (0, u_0]$ and $x, y \in B(x_0, L)$,

$$E_1 = \Big\{ \mathrm{cap}\Big(C_i^{u/6}(x, L) \Big) \geq c_{13} L^{3\nu/4} u^{\lfloor \gamma - 1 \rfloor} \Big\},$$
$$E_2 = \Big\{ \mathrm{cap}\Big(C_j^{u/6}(y, L) \Big) \geq c_{13} L^{3\nu/4} u^{\lfloor \gamma - 1 \rfloor} \Big\}.$$

Noting that $E_1 \subset \{x \in \mathcal{I}_i^{u/3}\}$ and $E_2 \subset \{y \in \mathcal{I}_j^{u/3}\}$, the probability in the second line of (III.4.7) is upper bounded by

$$\begin{aligned} &\mathbb{P}^I \Big(E_1 \cap E_2 \setminus \Big\{ C_i^{u/3}(x, L) \overset{\wedge}{\longleftrightarrow} C_j^{u/3}(y, L) \text{ in } \widehat{\mathcal{I}}_k^{u/3} \cap B_E(x_0, 2C_{10}L) \Big\} \Big) \\ &\quad + \mathbb{P}^I \big(\{x \in \mathcal{I}_i^{u/3}\} \setminus E_1 \big) + \mathbb{P}^I \big(\{y \in \mathcal{I}_j^{u/3}\} \setminus E_2 \big). \end{aligned} \tag{III.4.21}$$

For the first term in (III.4.21), we fix the constant $c_{12} = c_{12}(\varepsilon) \in \big(0, C_{13}/2\big]$ small enough so that, using Lemma III.4.3 and the capacity estimates on the event $E_1 \cap E_2$, for all $x, y \in B(x_0, L)$, whenever $uL^{2c_{12}} \geq 1$,

$$\begin{aligned} &\mathbb{P}^I \Big(E_1 \cap E_2 \setminus \Big\{ C_i^{u/3}(x, L) \overset{\wedge}{\longleftrightarrow} C_j^{u/3}(y, L) \text{ in } \widehat{\mathcal{I}}_k^{u/3} \cap B_E(x_0, 2C_{10}L) \Big\} \Big) \\ &\leq C \exp\big\{ -cL^{-\nu} u \times L^{3\nu/2} u^{2\lfloor \gamma - 1 \rfloor} \big\} \leq C \exp\big\{ -cL^{2c_{12}} u \big\}. \end{aligned} \tag{III.4.22}$$

Note that when $uL^{2c_{12}} \leq 1$, it is easy to see that (III.4.22) still holds upon increasing the constant C. To bound the probabilities in the second line of (III.4.21), we apply Proposition III.4.7. Combining the resulting estimate with (III.4.7), (III.4.21), (III.4.22), we get for all $u \leq u_0$, $L \geq 1$ and $x, y \in B(x_0, L)$,

$$\mathbb{P}^I \Big(x, y \in \mathcal{I}^u, \Big\{ x \overset{\wedge}{\longleftrightarrow} y \text{ in } \widehat{\mathcal{I}}^u \cap B_E(x_0, 2C_{10}L) \Big\}^{\mathsf{c}} \Big) \leq C \exp\{-cL^{2c_{12}} u\},$$

and (III.4.2) follows from a union bound on $x, y \in B(x, L)$, (V_α) and (III.2.10). □

Remark III.4.8. The resulting connectivity estimate (III.4.1) is not optimal, see for instance (III.4.22). Notwithstanding, its salient feature for later purposes (see Section III.8) is that it imposes a polynomial condition on u and L of the type $u^a L^b \geq C$, for some $a, b > 0$, in order for the complement of the probability in (III.4.1) to fall below any given deterministic threshold (later denoted $c_{17} l_0^{-4\alpha}$, see Proposition III.7.1). □

III.5 Isomorphism, cable system and sign flipping

In the first part of this section we explore some connections between the interlacement $\widetilde{\mathcal{I}}^u$ and the (continuous) level sets

$$\widetilde{E}^{>h} \stackrel{\text{def.}}{=} \{z \in \widetilde{G};\, \widetilde{\Phi}_z > h\} \tag{III.5.1}$$

of the Gaussian free field on the cable system defined in (III.2.16). Among other things, we aim to eventually apply a recent strengthening of the Ray-Knight type isomorphism from [96], see Theorem 2.4 in [101] and Corollary III.5.3 below. This improvement will be crucial in our understanding that certain level sets tend to *locally* (i.e. at the smallest scale L_0 of our renormalization scheme – see Section III.7) connect to $\widetilde{\mathcal{I}}^u$ and that the latter can be used to build connections of desired type, but it requires that certain conditions be met within our framework (III.3.1). We will in fact prove that the critical parameter for the percolation of the (continuous) level sets (III.5.1) is zero, and that $\widetilde{E}^{>-h}$ contains $\widetilde{\mathbb{P}}^G$-a.s. a unique unbounded connected component for all $h > 0$. In the second part of this section, we use a "sign-flipping" device which we introduced in Chapter II, see Lemma III.5.5, but improve it in view of the isomorphism from Corollary III.5.3, which leads to certain desirable couplings gathered in Proposition III.5.6 as a first step in proving Theorem III.1.1 and III.1.2.

Our starting point is the following observation from [57], see also (1.27)–(1.30) in [96] (N.B.: (III.5.2) below is in fact true on any transient weighted graph (G, λ)). For each $u > 0$, there exists a coupling $\widetilde{\mathbb{P}}^u$ between two Gaussian free fields $\widetilde{\varphi}$ and $\widetilde{\gamma}$ on $\widetilde{G}$, and local times $\widetilde{\ell}_{\widetilde{G},u}$ of a random interlacement process on $\widetilde{G}$ at level u such that,

$$\begin{aligned}&\widetilde{\mathbb{P}}^u\text{-a.s., } \widetilde{\ell}_{\widetilde{G},u} \text{ and } \widetilde{\gamma} \text{ are independent and}\\ &\frac{1}{2}\big(\widetilde{\varphi}_z + \sqrt{2u}\big)^2 = \widetilde{\ell}_{z,u} + \frac{1}{2}\widetilde{\gamma}_z^2, \quad \text{for all } z \in \widetilde{G}.\end{aligned} \tag{III.5.2}$$

The isomorphism (III.5.2) has the following immediate consequence: $\widetilde{\mathbb{P}}^u$-a.s.,

$$\widetilde{\mathcal{I}}^u \subset \{z \in \widetilde{G};\, |\widetilde{\varphi}_z + \sqrt{2u}| > 0\}. \tag{III.5.3}$$

In particular, by continuity, $\widetilde{\mathcal{I}}^u$ is either included in $\{z \in \widetilde{G};\, \widetilde{\varphi}_z > -\sqrt{2u}\}$ or $\{z \in \widetilde{G};\, \widetilde{\varphi}_z < -\sqrt{2u}\}$. This result will be improved with the help of Corollary III.4.2 in Proposition III.5.2. We begin with the following lemma about the connected components of $\{z \in \widetilde{G};\, |\widetilde{\Phi}_z + h| > 0\}$.

Lemma III.5.1. *For each $h \neq 0$, $\widetilde{\mathbb{P}}^G$-a.s. the set*

$$\{z \in \widetilde{G};\, |\widetilde{\Phi}_z + h| > 0\}$$

contains a unique unbounded connected component.

Proof. By symmetry of $\widetilde{\Phi}$ it is sufficient to consider the case $h > 0$. For convenience, we write $h = \sqrt{2u}$ for suitable $u > 0$ and consider the field $\widetilde{\varphi}$ with law $\widetilde{\mathbb{P}}^G$ under $\widetilde{\mathbb{P}}^u$ instead of $\widetilde{\Phi}$. The existence of an unbounded connected component of $\{z \in \widetilde{G};\, |\widetilde{\varphi}_z + h| > 0\}$ follows from (III.5.3) in combination with Corollary III.4.2. Thus, it remains to show uniqueness. Assume on the contrary that the set $\{z \in \widetilde{G};\, |\widetilde{\varphi}_z + \sqrt{2u}| > 0\}$ contains at least two unbounded connected components. Then by connectivity of $\widetilde{\mathcal{I}}^u$, see Corollary III.4.2, and by the inclusion (III.5.3), at least one of these unbounded connected components does not intersect $\widetilde{\mathcal{I}}^u$. Call it $\mathcal{C}^u$. Since $\mathcal{C}^u \subset \widetilde{\mathcal{V}}^u$, the isomorphism (III.5.2) and continuity imply that $\mathcal{C}^u$ is an infinite cluster of $\{z \in \widetilde{G};\, |\widetilde{\gamma}_z| > 0\}$. But since $\widetilde{\gamma}$ and $\widetilde{\mathcal{I}}^u$ are independent, it follows from Lemma III.3.2 that $\widetilde{\mathbb{P}}^u$-a.s. all the unbounded connected components of $\{z \in G;\, |\widetilde{\gamma}_z| > 0\}$, and thus $\mathcal{C}^u$, intersect $\widetilde{\mathcal{I}}^u$, which is a contradiction. □

The uniqueness and existence of the unbounded component of $\{z \in \widetilde{G};\, |\widetilde{\Phi}_z + h| > 0\}$ for $h > 0$ ensured by Lemma III.5.1 implies that $\widetilde{\mathbb{P}}^G$-a.s. either $\widetilde{E}^{>-h}$ or $\widetilde{G} \setminus \widetilde{E}^{>-h}$ contains an unbounded connected component, and we are about to show that it is always $\widetilde{E}^{>-h}$. For graphs G having a suitable action by a group of translations (for instance graphs of the form $G = G' \times \mathbb{Z}$), this result is clear by ergodicity and symmetry of the Gaussian free field. Due to the lack of ergodicity, we use a different argument here. The measure $\widetilde{\mathbb{P}}^u$ refers to the coupling in (III.5.2).

Proposition III.5.2. *For all $h > 0$, $\widetilde{\mathbb{P}}^G$-a.s., the set $\widetilde{E}^{>h}$ only contains bounded connected components whereas the set $\widetilde{E}^{>-h}$ contains a unique unbounded connected component. Moreover, for all $u > 0$, $\widetilde{\mathbb{P}}^u$-a.s.,*

$$\widetilde{\mathcal{I}}^u \subset \{z \in \widetilde{G};\, \widetilde{\varphi}_z > -\sqrt{2u}\}. \tag{III.5.4}$$

Proof. We only need to show that for all $h > 0$

$$\widetilde{\mathbb{P}}^{u=\frac{h^2}{2}}\big(\{z \in \widetilde{G};\, \widetilde{\varphi}_z < -h\} \text{ contains an unbounded connected component}\big) = 0. \tag{III.5.5}$$

Indeed, if (III.5.5) holds then by symmetry $\widetilde{E}^{>h}$ only contains bounded connected components, by Lemma III.5.1 $\widetilde{E}^{>-h}$ contains $\widetilde{\mathbb{P}}^G$-a.s. a unique unbounded component and (III.5.4) follows from (III.5.3) and Corollary III.4.2.

Assume that (III.5.5) does not hold for some height $h > 0$, which is henceforth fixed, and set $u = \frac{h^2}{2}$. Let $\mathscr{C}^h \subset \widetilde{G}$ be the set of points belonging to the infinite connected component of $\{z \in \widetilde{G};\, \widetilde{\varphi}_z < -h\}$ whenever it exists ($\mathscr{C}^h = \varnothing$ if there is no such component). By a union bound there exists $x_0 \in G$ such that

$$\widetilde{\mathbb{P}}^u\big(x_0 \in \mathscr{C}^h\big) > 0. \tag{III.5.6}$$

For all $n \in \mathbb{N}$, we define the random variable

$$Y_n = \frac{|\mathcal{I}^u \cap B(x_0,n)|}{|B(x_0,n)|}, \quad (\text{where } u = h^2/2.) \tag{III.5.7}$$

All constants from here on until the end of this proof may depend implicitly on u (or h). By definition of random interlacements, $\mathbb{P}^I(x \in \mathcal{I}^u) = 1 - e^{-\frac{u}{g(x,x)}}$, whence for all $x \in G$, $c \leq \mathbb{P}^I(x \in \mathcal{I}^u) \leq C$ due to (G_β) and thus, in view of (III.5.7),

$$c \leq \widetilde{\mathbb{E}}^u[Y_n] = \frac{1}{|B(x_0,n)|} \sum_{x \in B(x_0,n)} \widetilde{\mathbb{P}}^u(x \in \mathcal{I}^u) \leq C. \tag{III.5.8}$$

Following the lines of the proof of (1.38) in [95] one finds with the help of (G_β) that there exists a constant C such that for all $x, x' \in G$,

$$\mathrm{Cov}_{\widetilde{\mathbb{P}}^u}\big(\mathbb{1}_{x \in \mathcal{I}^u}, \mathbb{1}_{x' \in \mathcal{I}^u}\big) = \mathrm{Cov}_{\mathbb{P}^I}\big(\mathbb{1}_{x \in \mathcal{V}^u}, \mathbb{1}_{x' \in \mathcal{V}^u}\big) \leq C g(x,x'). \tag{III.5.9}$$

Moreover, by (III.2.10) and Lemma III.A.1, there exists a constant $C < \infty$ such that for all $x \in G$ and $n \in \mathbb{N}$,

$$\sum_{y \in B(x,n)} g(x,y) \leq C n^\beta. \tag{III.5.10}$$

Combining (III.5.9), (III.5.10), (III.2.10) and (V_α) yields that for all $n \in \mathbb{N}$

$$\mathrm{Var}_{\widetilde{\mathbb{P}}^u}(Y_n) = \frac{1}{|B(x_0,n)|^2} \sum_{x,x' \in B(x_0,n)} \mathrm{Cov}_{\widetilde{\mathbb{P}}^u}\big(\mathbb{1}_{x \in \mathcal{I}^u}, \mathbb{1}_{x' \in \mathcal{I}^u}\big) \leq C n^{\beta - \alpha} = C n^{-\nu}. \tag{III.5.11}$$

With (III.5.8), (III.5.11) and Chebyshev's inequality, one then finds $N_0 > 0$ large enough such that for all $n \geq N_0$,

$$\widetilde{\mathbb{P}}^u\Big(Y_n \leq \frac{\widetilde{\mathbb{E}}^u[Y_n]}{2}\Big) \leq \frac{4\mathrm{Var}_{\widetilde{\mathbb{P}}^u}(Y_n)}{\widetilde{\mathbb{E}}^u[Y_n]^2} \leq C n^{-\nu} \leq \frac{\widetilde{\mathbb{P}}^u(x_0 \in \mathscr{C}^h)}{2}, \tag{III.5.12}$$

where the last step follows from the assumption (III.5.6). Using (III.5.12) and (III.5.8), we get that for all $n \geq N_0$,

$$\widetilde{\mathbb{E}}^u[Y_n \cdot \mathbb{1}_{x_0 \in \mathscr{C}^h}] \geq \frac{\widetilde{\mathbb{E}}^u[Y_n]}{2} \cdot \widetilde{\mathbb{P}}^u\Big(Y_n \geq \frac{\widetilde{\mathbb{E}}^u[Y_n]}{2}, x_0 \in \mathscr{C}^h\Big) \geq c\widetilde{\mathbb{P}}^u(x_0 \in \mathscr{C}^h). \tag{III.5.13}$$

If $x_0 \in \mathscr{C}^h$, then $\mathscr{C}^h$ is the unique connected component of $\{z \in \widetilde{G};\, |\widetilde{\varphi}_z + h| > 0\}$ by Lemma III.5.1, and thus by (III.5.3), (III.5.13), (V_α) and (III.2.10), for all $n \geq N_0$ the lower bound

$$\widetilde{\mathbb{E}}^u\left[\big|\mathscr{C}^h \cap B(x_0,n)\big| \cdot \mathbb{1}_{x_0 \in \mathscr{C}^h}\right] \geq \widetilde{\mathbb{E}}^u\Big[\big|\mathcal{I}^u \cap B(x_0,n)\big| \cdot \mathbb{1}_{x_0 \in \mathscr{C}^h}\Big] \geq c n^\alpha \widetilde{\mathbb{P}}^u(x_0 \in \mathscr{C}^h) \tag{III.5.14}$$

follows. On the other hand,

$$\widetilde{\mathbb{E}}^u\left[\left|\mathscr{C}^h\cap B(x_0,n)\right|\cdot \mathbb{1}_{x_0\in\mathscr{C}^h}\right]=\sum_{x\in B(x_0,n)}\widetilde{\mathbb{P}}^u(x\in\mathscr{C}^h,x_0\in\mathscr{C}^h), \tag{III.5.15}$$

and, according to Proposition 5.2 in [57], for all $x\in G$,

$$\begin{aligned}\widetilde{\mathbb{P}}^u(x\in\mathscr{C}^h,x_0\in\mathscr{C}^h)&\leq\widetilde{\mathbb{P}}^u(x\overset{\sim}{\longleftrightarrow}x_0\text{ in }\{z\in\widetilde{G};\,|\widetilde{\varphi}_z|>0\})\\&\leq\arcsin\left(\frac{g(x_0,x)}{\sqrt{g(x_0,x_0)g(x,x)}}\right)\overset{(G_\beta)}{\leq}Cg(x_0,x).\end{aligned} \tag{III.5.16}$$

Combining (III.5.15), (III.5.16) and (III.5.10) then yields the upper bound

$$\widetilde{\mathbb{E}}^u\left[\left|\mathscr{C}^h\cap B(x_0,n)\right|\cdot \mathbb{1}_{x_0\in\mathscr{C}^h}\right]\leq Cn^\beta. \tag{III.5.17}$$

Finally, by (III.5.14) and (III.5.17) one obtains, for all $n\geq N_0$, $\widetilde{\mathbb{P}}^u(x_0\in\mathscr{C}^h)\leq Cn^{\beta-\alpha}\leq Cn^{-\nu}$, which contradicts (III.5.6) as $n\to\infty$. □

Having shown Proposition III.5.2, taking complements in (III.5.4), we know that for all $u>0$,

$$\{z\in\widetilde{G};\,\widetilde{\varphi}_z<-\sqrt{2u}\}\subset\widetilde{\mathcal{V}}^u \tag{III.5.18}$$

(and in particular $h_*\leq\sqrt{2u_*}$) for all graphs G satisying our assumptions (III.3.1). Moreover, as will become clear in the proof of Corollary III.5.3 below, Proposition III.5.2 provides us with a very explicit way to construct a coupling $\widetilde{\mathbb{P}}^u$ as in (III.5.2) with the help of [101]. With a slight abuse of notation (which will soon be justified), for all $u>0$, we consider a (canonical) coupling $\widetilde{\mathbb{P}}^u$ between a Gaussian free field $\widetilde{\gamma}$ on $\widetilde{G}$ (with law $\widetilde{\mathbb{P}}^G$) and an independent family of local times $(\widetilde{\ell}_{z,u})_{z\in\widetilde{G}}$ continuous in $z\in\widetilde{G}$ of a random interlacement process with the same law as under $\widetilde{\mathbb{P}}^I$, cf. (III.2.18). Note that this defines the set $\widetilde{\mathcal{I}}^u$ by means of (III.2.19). We then define

$$\begin{aligned}&\mathcal{C}_u^\infty\text{ as the union of the connected components}\\&\quad\text{of }\{z\in\widetilde{G};\,2\widetilde{\ell}_{z,u}+\widetilde{\gamma}_z^2>0\}\text{ intersecting }\widetilde{\mathcal{I}}^u.\end{aligned} \tag{III.5.19}$$

The following is essentially an application of Theorem 2.4 in [101].

Corollary III.5.3. *The process* $(\widetilde{\varphi}_z)_{z\in\widetilde{G}}$ *defined by*

$$\widetilde{\varphi}_z=\begin{cases}-\sqrt{2u}+\widetilde{\gamma}_z & \text{if } z\notin\mathcal{C}_u^\infty,\\ -\sqrt{2u}+\sqrt{2\widetilde{\ell}_{z,u}+\widetilde{\gamma}_z^2} & \text{if } z\in\mathcal{C}_u^\infty.\end{cases} \tag{III.5.20}$$

for all $z\in\widetilde{G}$, *is a Gaussian free field, i.e., its law is* $\widetilde{\mathbb{P}}^G$, *and the joint field* $(\widetilde{\gamma}_\cdot,\widetilde{\ell}_{\cdot,u},\widetilde{\varphi}_\cdot)$ *thereby defined constitutes a coupling such that* (III.5.2) *holds. Moreover,* $\mathcal{C}_u^\infty$ *is the unique unbounded connected component of* $\{z\in\widetilde{G};\,\widetilde{\varphi}_z>-\sqrt{2u}\}$.

Proof. We aim at invoking Theorem 2.4 in [101] in order to deduce that the field $\widetilde{\varphi}$ defined in (III.5.20) is indeed a Gaussian free field. The conditions to apply this result are that

$$\widetilde{\mathbb{P}}^G\text{-a.s., } \{z \in \widetilde{G};\, |\widetilde{\Phi}_z| > 0\} \text{ only contains bounded connected components,} \tag{III.5.21}$$

and $g(x,x)$ is uniformly bounded. The latter is clear by (G_β), but it is not obvious that (III.5.21) holds. However, by direct inspection of the proof of Theorem 2.4 in [101], we see that (III.5.21) is only used to prove (1.33) and (2.48) in [101], and that it can be replaced by the following (weaker) conditions:

$$\text{for all } u > 0,\ \widetilde{\mathbb{P}}^u\text{-a.s., } \widetilde{\mathcal{I}}^u \subset \{z \in \widetilde{G};\, \widetilde{\varphi}_z > -\sqrt{2u}\} \text{ and} \tag{III.5.22}$$

$$\text{all the unbounded connected components of } \{z \in \widetilde{G};\, |\widetilde{\gamma}_z| > 0\} \text{ intersect } \widetilde{\mathcal{I}}^u, \tag{III.5.23}$$

and the proof of Theorem 2.4 in [101] continues to hold. For the class of graphs (III.3.1) considered here the condition (III.5.22) have been shown in (III.5.4) and the condition (III.5.23) follows from Lemma III.3.2 and the independence of $\widetilde{\gamma}$ and $\widetilde{\mathcal{I}}^u$. Thus, Theorem 2.4 in [101] applies and yields that $\widetilde{\varphi}$ defined in (III.5.20) has law $\widetilde{\mathbb{P}}^G$.

By (III.5.19), $\widetilde{\ell}_{z,u} = 0$ for $z \notin \mathcal{C}_u^\infty$ and it then follows plainly from (III.5.20) that (III.5.2) holds. Finally, the fact that $\mathcal{C}_u^\infty$ is the unique unbounded cluster of $\{z \in \widetilde{G};\, \widetilde{\varphi}_z > -\sqrt{2u}\}$ is a consequence of Proposition III.5.2 and the definitions of $\mathcal{C}_u^\infty$ and $\widetilde{\varphi}$, recalling that $\widetilde{\mathcal{I}}^u = \{z \in \widetilde{G};\, \widetilde{\ell}_{z,u} > 0\}$ is an unbounded connected set due to Corollary III.4.2 and (III.2.19). □

Remark III.5.4. 1) An interesting consequence of Corollary III.5.3 is that for all graphs satisfying our assumptions (III.3.1), the inclusion (III.5.18) can be strengthened to

$$\text{for all } A \subset (-\infty, 0),\ \{z \in \widetilde{G};\, \widetilde{\varphi}_z \in -\sqrt{2u} + A\} \subset \widetilde{\mathcal{V}}^u \cap \{z \in \widetilde{G};\, \widetilde{\gamma}_z \in A\}, \tag{III.5.24}$$

see Corollary 2.5 in [101].

2) For the remainder of this chapter, with a slight abuse of notation, we will *solely* refer to $\widetilde{\mathbb{P}}^u$ as the coupling between $(\widetilde{\gamma}_\cdot, \widetilde{\ell}_{\cdot,u}, \widetilde{\varphi}_\cdot)$ constructed around (III.5.19) and (III.5.20). Thus, the conclusions of Corollary III.5.3 hold, and in particular $\widetilde{\mathbb{P}}^u$ satisfies (III.5.2). □

3) In Chapter IV, we will extend the results from Proposition III.5.2 and Corollary III.5.3 to a way more general class of graph than the graphs satisfying (III.3.1) studied in this chapter, and will in fact show that the isomorphism

(III.5.20) is equivalent to the condition (III.5.21). In particular, if G is a graph satisfying (III.5.22) and (III.5.23), then (III.5.21) holds.

We now adapt a result from Section II.5 which roughly shows that, under $\widetilde{\mathbb{P}}^u$, for each $x \in G$ and with $u = h^2/2$ for a suitable $h > 0$, except on an event with small probability, a suitable conditional probability that $\widetilde{\varphi}_z \geq -h$ for all z on the first half of an edge starting in x is smaller than the respective conditional probability that $\varphi_x \geq h$ at the vertex x whenever h (or u) is small enough.

For each $x \sim y \in G$, we denote by $U^{x,y}$ the compact subset of $\widetilde{G}$ which consist of the points on the closed half of the edge $I_{\{x,y\}}$ beginning in x, and for $x \in G$ let $U^x = \bigcup_{y \sim x} U^{x,y}$ and $\mathcal{K}^x = \partial U^x$, i.e., $\mathcal{K}^x$ is the finite set of midpoints on any edge incident on x. For all $U \subset \widetilde{G}$, we denote by $\mathcal{A}_U$ the σ-algebra $\sigma(\widetilde{\varphi}_z,\, z \in U)$. For all $x \in G$, $u > 0$ and $K > 0$, we also define the events

$$\begin{aligned} R_u^x &= \big\{\exists\, y \in G;\, y \sim x \text{ and } \widetilde{\varphi}_z \geq -\sqrt{2u} \text{ for all } z \in U^{x,y}\big\}, \\ \overline{S}_K^x &= \big\{\widetilde{\varphi}_z \geq -K \text{ for all } z \in \mathcal{K}^x\big\}. \end{aligned} \tag{III.5.25}$$

For all $z \in \mathcal{K}^x$, let y_z be the unique $y \sim x$ such that $z \in U^{x,y}$. Recall that by the Markov property (III.2.17) of the free field, one can write, for all $x \in G$,

$$\varphi_x = \beta_x^{U^x} + \varphi_x^{U^x}, \text{ where } \beta_x^{U^x} = \sum_{z \in \mathcal{K}^x} P_x\big(\widetilde{X}_{T_{U^x}} = z\big)\widetilde{\varphi}_y = \frac{1}{\lambda_x} \sum_{z \in \mathcal{K}^x} \lambda_{x,y_z} \widetilde{\varphi}_z \tag{III.5.26}$$

is $\mathcal{A}_{\mathcal{K}^x}$ measurable and $\varphi_x^{U^x}$ is a centered Gaussian variable independent of $\mathcal{A}_{\mathcal{K}^x}$ and with variance $g_{U^x}(x,x) = \frac{2}{\sum_{y \sim x} (\rho_{x,y}/2)^{-1}} = \frac{1}{2\lambda_x}$, where we recall $\rho_{x,y} = 1/(2\lambda_{x,y})$ and refer to Section 2 of [57] for details on these calculations.

Lemma III.5.5. *There exists $c_{14} > 0$ such that for all $u > 0$, $x \in G$ and $K > \sqrt{2u}$ satisfying*

$$K\lambda_x\sqrt{2u} \leq c_{14}, \tag{III.5.27}$$

we have

$$\mathbb{1}_{\bar{S}_K^x} \widetilde{\mathbb{P}}^u\left(R_u^x \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \leq \frac{1}{2}\widetilde{\mathbb{P}}^u(\varphi_x \geq \sqrt{2u} \,|\, \mathcal{A}_{\mathcal{K}^x}) \quad \text{on } \{\beta_x^{U^x} \leq K\}, \tag{III.5.28}$$

and, denoting by F the cumulative distribution function of a standard normal variable,

$$\widetilde{\mathbb{P}}^u(\varphi_x \geq \sqrt{2u} \,|\, \mathcal{A}_{\mathcal{K}^x}) \geq F\big(\sqrt{2\lambda_x}(K - \sqrt{2u})\big) \quad \text{on } \{\beta_x^{U^x} > K\}. \tag{III.5.29}$$

Proof. We first consider the event $\{\beta_x^{U^x} \leq K\}$. For any $u > 0$ and $K > \sqrt{2u}$, on the event $\{\beta_x^{U^x} \leq K\} \cap \overline{S}_K^x$, we have $|\beta_x^{U^x}| \leq K$ by (III.5.25) and (III.5.26) and thus

$$\begin{aligned}&\widetilde{\mathbb{P}}^u\big(-\sqrt{2u} \leq \varphi_x \leq 2\sqrt{2u} \,\big|\, \mathcal{A}_{\mathcal{K}^x}\big)\\&= \sqrt{\frac{\lambda_x}{\pi}}\int_{-\sqrt{2u}}^{2\sqrt{2u}} \exp\big\{-\lambda_x(y-\beta_x^{U^x})^2\big\}\,\mathrm{d}y\\&\leq \sqrt{\frac{2u\lambda_x}{\pi}}\exp\big\{-\lambda_x(\beta_x^{U^x})^2\big\}\times 3\exp\big\{4\sqrt{2u}\lambda_x K\big\}.\end{aligned} \tag{III.5.30}$$

Similarly, still on the event $\{\beta_x^{U^x} \leq K\} \cap \overline{S}_K^x$,

$$\widetilde{\mathbb{P}}^u\big(\sqrt{2u} \leq \varphi_x \leq 2\sqrt{2u} \,\big|\, \mathcal{A}_{\mathcal{K}^x}\big) \geq \sqrt{\frac{2u\lambda_x}{\pi}}\exp\big\{-\lambda_x(\beta_x^{U^x})^2\big\}\exp\big\{-8\sqrt{2u}\lambda_x K\big\}. \tag{III.5.31}$$

For any $x \in G$ and $z \in \mathcal{K}^x$, by the Markov property (III.2.17), the law of the Gaussian free field $\widetilde{\varphi}$ on U^{x,y_z} conditionally on $\mathcal{A}_{\mathcal{K}^x\cup\{x\}}$ is that of a Brownian bridge of length $\rho_{x,y_z}/2 = (4\lambda_{x,y_z})^{-1}$ between φ_x and $\widetilde{\varphi}_z$ of a Brownian motion with variance 2 at time 1. Furthermore, still conditionally on $\mathcal{A}_{\mathcal{K}^x\cup\{x\}}$, these bridges form an independent family in $z \in \mathcal{K}^x$. Therefore, on the event $\{-\sqrt{2u} \leq \varphi_x \leq 2\sqrt{2u}\} \cap \{\beta_x^{U^x} \leq K\} \cap \overline{S}_K^x$, using an exact formula for the distribution of the maximum of a Brownian bridge, see for instance [13], Chapter IV.26, we obtain

$$\begin{aligned}\widetilde{\mathbb{P}}^u\big(R_u^x \,|\, \mathcal{A}_{\mathcal{K}^x\cup\{x\}}\big) &= 1 - \prod_{y\sim x}\widetilde{\mathbb{P}}^u\Big(\exists\, z \in U^{x,y};\, \widetilde{\varphi}_z < -\sqrt{2u}\,\Big|\, \mathcal{A}_{\mathcal{K}^x\cup\{x\}}\Big)\\&= 1 - \prod_{\substack{z\in\mathcal{K}^x\\ \widetilde{\varphi}_z \geq -\sqrt{2u}}} \exp\big\{-4\lambda_{x,y_z}(\widetilde{\varphi}_z + \sqrt{2u})(\varphi_x+\sqrt{2u})\big\}\\&\leq 1 - \exp\big\{-24\sqrt{2u}\lambda_x K\big\} \leq 24\sqrt{2u}\lambda_x K.\end{aligned} \tag{III.5.32}$$

Together, (III.5.30), (III.5.31) and (III.5.32) imply that for all $u > 0$ and $K > \sqrt{2u}$, on the event $\{\beta_x^{U^x} \leq K\} \cap \overline{S}_K^x$,

$$\frac{\widetilde{\mathbb{P}}^u\left(R_u^x \cap \{\varphi_x \leq 2\sqrt{2u}\} \,\big|\, \mathcal{A}_{\mathcal{K}^x}\right)}{\widetilde{\mathbb{P}}^u\left(\sqrt{2u} \leq \varphi_x \leq 2\sqrt{2u} \,\big|\, \mathcal{A}_{\mathcal{K}^x}\right)} \leq 72\sqrt{2u}\lambda_x K \exp\big\{12\sqrt{2u}\lambda_x K\big\}. \tag{III.5.33}$$

We now choose the constant c_{14} such that the right-hand side of (III.5.33) is smaller than 1/2 if $\sqrt{2u}\lambda_x K \leq c_{14}$, and (III.5.28) then readily follows from (III.5.33). The inequality (III.5.29) follows simply from (III.5.26): for all $u > 0$, $K > \sqrt{2u}$ and $x \in G$, on the event $\{\beta_x^{U^x} > K\}$,

$$\widetilde{\mathbb{P}}^u\big(\varphi_x \geq \sqrt{2u} \,|\, \mathcal{A}_{\mathcal{K}^x}\big) \geq \widetilde{\mathbb{P}}^u\big(\varphi_x^{U^x} \geq \sqrt{2u} - K \,|\, \mathcal{A}_{\mathcal{K}^x}\big) = F\big(\sqrt{2\lambda_x}(K-\sqrt{2u})\big).$$

This completes the proof of Lemma III.5.5. □

For all parameters $u > 0$ and $p \in (0,1)$, we consider a probability measure $\widetilde{\mathbb{Q}}^{u,p}$, extension of the coupling $\widetilde{\mathbb{P}}^u$ introduced above (III.5.19), see also Remark III.5.4, 2), governing the fields $((\widetilde{\gamma}_z)_{z\in\widetilde{G}}, (\widetilde{\ell}_{z,u})_{z\in\widetilde{G}}, (\mathcal{B}^p_x)_{x\in G})$ such that, under $\widetilde{\mathbb{Q}}^{u,p}$,

$$\begin{gathered}\text{the fields } \widetilde{\gamma}_\cdot,\ \widetilde{\ell}_{\cdot,u} \text{ are those from above (III.5.19) (and thus Corollary III.5.3}\\ \text{applies), } \mathcal{B}^p_x,\ x\in G \text{ are i.i.d. } \{0,1\}\text{-valued random variables with}\\ \widetilde{\mathbb{Q}}^{u,p}(\mathcal{B}^p_x = 1) = p, \text{ the three fields } \mathcal{B}^p_\cdot,\ \widetilde{\gamma}_\cdot,\ \widetilde{\ell}_{\cdot,u} \text{ are independent.}\end{gathered} \tag{III.5.34}$$

Let us introduce the following condition on $u > 0$, $K > \sqrt{2u}$ and $p \in (0,1)$

$$\frac{1}{2} \le p < \inf_{x\in G} F\big(\sqrt{2\lambda_x}(K - \sqrt{2u})\big). \tag{III.5.35}$$

Recalling the definition of the σ-algebra $\mathcal{A}_{\mathcal{K}^x}$, $x \in G$, we consider a family $(X^x_{u,K,p})_{x\in G} \in \{0,1\}^G$ of random variables defined with the same underlying probability $\widetilde{\mathbb{Q}}^{u,p}$ from (III.5.34) and the property that, for $K > \sqrt{2u}$ and all $x \in G$,

$$\mathbb{1}_{\beta^{U^x}_x \ge K} \cdot \widetilde{\mathbb{Q}}^{u,p}\left(X^x_{u,K,p} = 1 \,|\, \mathcal{A}_{\mathcal{K}^x}\right) \le p. \tag{III.5.36}$$

We will consider the following two natural choices for $X_{u,K,p}$, either

$$X^x_{u,K,p} = \mathcal{B}^p_x, \quad x \in G, \tag{III.5.37}$$

or

$$X^x_{u,K,p} = \mathbb{1}_{\{\varphi_x \le K\}}, \quad x \in G, \tag{III.5.38}$$

and we will allow for both. The reason for this twofold choice is explained below in Remark III.9.4, 2). In case (III.5.37), inequality (III.5.36) follows directly from the definition (III.5.34), whereas in the case (III.5.38) it is a consequence of the decomposition (III.5.26) and the fact that $p \ge 1/2 = \widetilde{\mathbb{Q}}^{u,p}(\varphi^{U_x}_x \le 0 \,|\, \mathcal{A}_{\mathcal{K}^x})$. We introduce the event

$$S^x_K \overset{\text{def.}}{=} \big\{\widetilde{\gamma}_y \ge -K + \sqrt{2u} \text{ for all } y \in \mathcal{K}^x\big\} \tag{III.5.39}$$

and the following random subsets of G, cf. (III.5.25) for the definitions of R^x_u and $\overline{S}^x_K$:

$$\begin{aligned} R_u &\overset{\text{def.}}{=} \{x \in G;\ R^x_u \text{ occurs}\},\\ S_K &\overset{\text{def.}}{=} \{x \in G;\ S^x_K \text{ occurs}\},\\ \overline{S}_K &\overset{\text{def.}}{=} \{x \in G;\ \overline{S}^x_K \text{ occurs}\},\\ X_{u,K,p} &\overset{\text{def.}}{=} \{x \in G;\ X^x_{u,K,p} = 1\}, \text{ and} \end{aligned} \tag{III.5.40}$$

By (III.5.20), under $\widetilde{\mathbb{Q}}^{u,p}$, if $\widetilde{\varphi}_z < -K$, then $\widetilde{\gamma}_z < -K + \sqrt{2u}$ for all $z \in \widetilde{G}$, and thus for all $x \in G$, in view of (III.5.25) and (III.5.39),

$$(\overline{S}_K^x)^c \subset (S_K^x)^c, \text{ and therefore } S_K \subset \overline{S}_K. \tag{III.5.41}$$

We now take advantage of Lemma III.5.5 to obtain the following coupling.

Proposition III.5.6. *For all $u > 0$, $K > \sqrt{2u}$ and $p \in (0,1)$ such that* (III.5.27) *and* (III.5.35) *hold true for all $x \in G$, with $(X^x_{u,K,p})_{x\in G}$ as in* (III.5.37) *or* (III.5.38)*, one can find an extension $(\Omega^{u,K,p}, \mathcal{F}^{u,K,p}, \mathcal{Q}^{u,K,p})$ of the probability space underlying $\widetilde{\mathbb{Q}}^{u,p}$ on which one can define for each $0 \leq v \leq u$ two random subsets $H = H_{u,v,K,p}$ and $\overline{E}^{\geq\sqrt{2v}}$ of G such that*

$$\overline{E}^{\geq\sqrt{2v}} \text{ has the same law under } \mathcal{Q}^{u,K,p} \text{ as } E^{\geq\sqrt{2v}} \text{ under } \mathbb{P}^G, \tag{III.5.42}$$

the family $\{x \in H\}_{x\in G}$ is i.i.d. and independent of $\widetilde{\mathcal{I}}^u$, $\widetilde{\gamma}$ and $(\mathcal{B}^p_x)_{x\in G}$, $\{x \in H\}$ is independent of $\{y \in \overline{E}^{\geq\sqrt{2v}}\}_{y\in G\setminus\{x\}}$, $\mathcal{Q}^{u,K,p}(x \in H) > 0$, and the following inclusion holds:

$$(R_u \cup H) \cap S_K \cap X_{u,K,p} \subset \overline{E}^{\geq\sqrt{2v}}. \tag{III.5.43}$$

Proof. For fixed values of u, K and p satisfying the above assumptions, we consider an extension $(\Omega^{u,K,p}, \mathcal{F}^{u,K,p}, \mathcal{Q}^{u,K,p})$ of the probability space underlying $\widetilde{\mathbb{Q}}^{u,p}$, on which we also have an i.i.d. family $(V_x)_{x\in G}$ of uniform random variable on $[0,1]$, independent of $\widetilde{\mathcal{I}}^u$, $\widetilde{\gamma}$ and $(\mathcal{B}^p_x)_{x\in G}$. For each $x \in G$ and $0 \leq v \leq u$, there exists a measurable function $f^x_{u,v} : \mathbb{R}^{\mathcal{K}^x} \to (-\infty, 1]$ such that, with $\mathcal{A}_{\mathcal{K}} = \sigma(\widetilde{\varphi}_x, x \in \mathcal{K})$,

$$f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x}) = \frac{\widetilde{\mathbb{Q}}^{u,p}(\varphi_x \geq \sqrt{2v} \,|\, \mathcal{A}_{\mathcal{K}}) - \widetilde{\mathbb{Q}}^{u,p}(R^x_u \cap \overline{S}^x_K \cap \{X^x_{u,K,p} = 1\} \,|\, \mathcal{A}_{\mathcal{K}})}{1 - \widetilde{\mathbb{Q}}^{u,p}(R^x_u \cap \overline{S}^x_K \cap \{X^x_{u,K,p} = 1\} \,|\, \mathcal{A}_{\mathcal{K}})} \tag{III.5.44}$$

(in particular the right-hand side depends on $\widetilde{\varphi}_{|\mathcal{K}}$ only through $\widetilde{\varphi}_{|\mathcal{K}^x}$). Moreover for each $x \in G$, by (III.5.26), (III.5.28), (III.5.29), (III.5.36) and since $v \leq u$, for all $\psi \in \mathbb{R}^{\mathcal{K}^x}$ with $\psi_y \geq -K$ for all $y \in \mathcal{K}^x$, we have

$$f^x_{u,v}(\psi) \geq \big(F\big(\sqrt{2\lambda_x}(K - \sqrt{2u})\big) - p\big) \wedge \frac{1}{2} F\big(-\sqrt{2\lambda_x}(K + \sqrt{2u})\big).$$

By (III.2.10) and (III.5.35), we thus have

$$f_{\min} \overset{\text{def.}}{=} \inf_{x\in G} \inf_{\substack{\psi\in\mathbb{R}^{\mathcal{K}^x}:\\ \psi_y\geq -K, y\in\mathcal{K}^x}} f^x_{u,v}(\psi) > 0. \tag{III.5.45}$$

For all $0 \leq v \leq u$, let

$$\overline{E}^{\geq\sqrt{2v}} \overset{\text{def.}}{=} \{x \in G;\, V_x \leq f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x})\} \cup \big(R_u \cap \overline{S}_K \cap X_{u,K,p}\big) \tag{III.5.46}$$

and

$$H \stackrel{\text{def.}}{=} \{x \in G;\, V_x \le f_{\min}\}. \tag{III.5.47}$$

It is clear that the family $\{x \in H\}_{x\in G}$ is i.i.d. and independent of $\widetilde{\mathcal{I}}^u$, $\widetilde{\gamma}$ and $(\mathcal{B}^p_x)_{x\in G}$, that $\{x \in H\}$ is independent of $\{y \in \overline{E}^{\ge\sqrt{2v}}\}_{y\in G\setminus\{x\}}$, and that $\widetilde{\mathbb{Q}}^{u,p}(x \in H) > 0$ due to (III.5.45). We proceed to verify (III.5.43) with $\overline{S}_K$ replacing S_K, which is sufficient due to (III.5.41). We have

$$\begin{aligned}
&\big(H \cap \overline{S}_K \cap X_{u,K,p}\big)\\
&\overset{\text{(III.5.25),(III.5.45)}}{\subset} \big(H \cap \overline{S}_K \cap \{x \in G:\, f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x}) \ge f_{\min}\}\big)\\
&\overset{\text{(III.5.47)}}{\subset} \{x \in G;\, V_x \le f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x})\}\\
&\overset{\text{(III.5.46)}}{\subset} \overline{E}^{\ge\sqrt{2v}},
\end{aligned}$$

from which (III.5.43) (with $\overline{S}_K$ in place of S_K) immediately follows, since $(R_u \cap \overline{S}_K \cap X_{u,K,p}) \subset \overline{E}^{\ge\sqrt{2v}}$.

It remains to check that (III.5.42) holds. By (III.5.46) and by definition of R^x_u, $\overline{S}^x_K$ and $X^x_{u,K,p}$ see (III.5.25) and (III.5.37) or (III.5.38), conditionally on $\mathcal{A}_{\mathcal{K}}$, the events $\{x \in \overline{E}^{\ge\sqrt{2v}}\}$, $x \in G$, are independent under $\mathcal{Q}^{u,K,p}$. Therefore, abbreviating $q = \mathcal{Q}^{u,K,p}\big(\, x \in \big(R_u \cap \overline{S}_K \cap X_{u,K,p}\big)\,\big|\mathcal{A}_{\mathcal{K}}\big)$, we have

$$\begin{aligned}
&\mathcal{Q}^{u,K,p}\big(\, x \in \overline{E}^{\ge\sqrt{2v}}\,\big|\mathcal{A}_{\mathcal{K}}\big)\\
&\overset{\text{(III.5.46)}}{=} q + \mathcal{Q}^{u,K,p}\big(\, V_x \le f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x}),\, x \in \big(R_u \cap \overline{S}_K \cap X_{u,K,p}\big)^{\mathsf{c}}\,\big|\mathcal{A}_{\mathcal{K}}\big)\\
&= \; q + f^x_{u,v}(\widetilde{\varphi}_{|\mathcal{K}^x})(1-q)\\
&\overset{\text{(III.5.44)}}{=} \widetilde{\mathbb{Q}}^{u,p}(\varphi_x \ge \sqrt{2v}\,|\,\mathcal{A}_{\mathcal{K}}).
\end{aligned} \tag{III.5.48}$$

Conditionally on $\mathcal{A}_{\mathcal{K}}$, the events $\{x \in \overline{E}^{\ge\sqrt{2v}}\}$, $x \in G$, respectively $\{\varphi_x \ge \sqrt{2v}\}$, $x \in G$, are independent and so by (III.5.48) $\overline{E}^{\ge\sqrt{2v}}$ and $\{x \in G : \varphi \ge -\sqrt{2u}\}$ have the same conditional law. Integrating, we obtain (III.5.42). □

Remark III.5.7. Lemma III.5.5 is stated in terms of the field $\widetilde{\varphi}$ under the measure $\widetilde{\mathbb{P}}^u$ with $u > 0$, or equivalently under the measure $\widetilde{\mathbb{Q}}^{u,p}$, to which it will eventually be applied. Nevertheless, let us note here that it could in fact be stated for the Gaussian free field $\widetilde{\Phi}$ under $\widetilde{\mathbb{P}}^G$ for any weighted graph (G,λ) since the assumptions (III.3.1) are not required for its proof. Proposition III.5.6 is valid on any transient weighted graph (G,λ) such that (III.2.10) and Corollary III.5.3 holds, i.e. on any graph such $g(x,x)$ is uniformly bounded and such that the conditions (III.2.10), (III.5.22) and (III.5.23) hold. In particular, the assumptions (III.3.1) are not necessarily required.

We close this section with an outlook of the remaining sections. Under $\mathcal{Q}^{u,K,p}$ from Proposition III.5.6 with $X_{u,K,p}$ from (III.5.37), we have that $S_K \cap X_{u,K,p}$ and $\mathcal{I}^u$ are independent, and by (III.5.4) that $\mathcal{I}^u \cap S_K \cap X_{u,K,p} \subset R_u \cap S_K \cap X_{u,K,p} \subset \overline{E}^{\geq \sqrt{2u}}$. Moreover by (III.5.42) and (III.5.24), we have that $\overline{E}^{\geq \sqrt{2u}}$ is stochastically dominated by $\mathcal{V}^u$. In order to prove Theorem III.1.2 (but not Theorem III.1.1), we thus only need to show that $\mathcal{I}^u \cap S_K \cap X_{u,K,p}$ percolates for a suitable choice of u, K and p with $K\lambda_x\sqrt{2u} \leq c_{14}$ and $p < F\big(\sqrt{2\lambda_x}(K - \sqrt{2u})\big)$ for all $x \in G$. A promising strategy to prove that the intersection of $\mathcal{I}^u$ and a large set percolates on G is to apply the decoupling inequalities of Theorem III.2.4 to a suitable renormalization scheme, similarly to [74] and Chapter II. This requires roughly the same amount of work as obtaining an estimate like (III.1.10) for small $h > 0$ (both are "existence"-type results), and they will follow as a by-product of the renormalization argument developed in the course of the next three sections. The actual renormalization scheme will be considerably more involved than the arguments presented in [74] and Chapter II in order to produce an estimate like (III.1.11) for small $h > 0$ and thereby allow us to deduce Theorem III.1.1.

III.6 Proof of decoupling inequalities

The coupling $\widetilde{\mathbb{Q}}^{u,p}$ of (III.5.34) will eventually feature within a certain renormalization scheme that will lead to the proof of our main results, Theorems III.1.1 and III.1.2. This is the content of Sections III.7 and III.8. The successful deployment of these multi-scale techniques hinges on the availability of suitable decoupling inequalities, which were stated in Theorem III.2.4 and which we now prove. In essence, both inequalities (III.2.20) (for the free field) and (III.2.21) (for interlacements) constituting Theorem III.2.4 will follow from two corresponding results in [67] and [68], see also (III.6.4) and (III.6.29) below (these results are stated in [67], [68], for $\mathbb{Z}^d$ but can be extended to $\widetilde{G}$, the cable system of any graph satisfying (III.3.1)), once certain error terms are shown to be suitably small. In the free field case, see Lemma III.6.4, the respective estimate is straightforward and we give the short argument, along with the proof of (III.2.20), first.

The issue of controlling the error term is considerably more delicate for the interlacement. The key control comes in Lemma III.6.6 below. Following arguments in [68], it essentially boils down to estimates on the second moment and on the tail of the so-called *soft local times* attached the relevant excursion process (for one random walk trajectory), see (III.6.25) below, which are given in Lemma III.6.7. For $G = \mathbb{Z}^d$, these bounds follow from the strong estimates of Proposition 6.1 in [68], but its proof is no longer valid at the level of generality considered here (the details of the argument are very Euclidean; see for instance Section 8 in [68]). We bypass this issue by presenting a way to obtain the desired bounds in Lemma III.6.7 and along with it, the decoupling inequality (III.2.21), without relying on (strong) estimates akin to Proposition 6.1 of [68]. This approach is shorter even when $G = \mathbb{Z}^d$ but comes at the price of requiring an additional assumption on the distance between the sets. An essential ingredient is a certain consequence of the Harnack inequality (III.3.3), see Lemma III.6.5 below.

The following lemma will be useful to find "approximate lattices" at all scales inside G. It will be applied in the context of certain chaining arguments below. These lattices will also be essential in setting up an appropriate renormalization scheme in Section III.7.

Lemma III.6.1. *Assume* (p_0), (V_α), *and* (G_β) *to be fulfilled. Then there exists a constant* C_{14} *such that for each* $L \geq 1$, *one can find a set of vertices* $\Lambda(L) \subset G$ *with*

$$\bigcup_{y \in \Lambda(L)} B(y, L) = G, \tag{III.6.1}$$

and for all $x \in G$ *and* $N \geq 1$,

$$|\Lambda(L) \cap B(x, LN)| \leq C_{14} N^\alpha. \tag{III.6.2}$$

Proof. For a given $L \geq 1$, let $\Lambda(L) \subset G$ have the following two properties: i) for all $y \neq y' \in \Lambda(L)$, $d(y, y') > L$, and ii) for all $x \in G$, there exists $y \in \Lambda(L)$ such that $d(x, y) \leq L$. Indeed, one can easily construct such a set $\Lambda(L) = \{y_0, y_1, \dots\}$, e.g. by labeling all the vertices in $G = \{x_0, x_1, \dots\}$ and then "exploring" G, starting at $y_0 = x_0 \in G$, then defining y_1 as the point with smallest label in the complement of $B(x_0, L)$, idem for y_2 in the complement of $B(y_0, L) \cup B(y_1, L)$, etc.

By ii), for each $x \in G$, there exists $y \in \Lambda(L)$ such that $d(x, y) \leq L$, and so in particular $\bigcup_{y \in \Lambda(L)} B(y, L) = G$. Moreover, for all $x \in G$ and $N \geq 1$,

$$\bigcup_{y \in \Lambda(L) \cap B(x, NL)} B\Big(y, \frac{L}{2}\Big) \subset B(x, L(N+1)),$$

and the balls $B\big(y, \frac{L}{2}\big)$, $y \in \Lambda(L)$, are disjoint by i). Combining this with (V_α) we infer that for $L \geq 2$,

$$|\Lambda(L) \cap B(x, NL)| \leq \frac{C_1 (L(N+1))^\alpha}{c_1 (L/2)^\alpha} \leq \frac{4^\alpha C_1}{c_1} N^\alpha,$$

and the proof of (III.6.2) for $1 \leq L < 2$ is trivial by (V_α) and (III.2.10) (choose $\Lambda(L) = G$). □

We start with some preparation towards (III.2.20). Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two disjoints measurable subsets of $\widetilde{G}$ such that $\widetilde{A}_1$ is compact with finitely many connected components, and let $\widetilde{U}_1 = \widetilde{A}_1^{\mathsf{c}}$. We recall the definition of the harmonic extension $\widetilde{\beta}^{\widetilde{U}_1}$ of the Gaussian free field $\widetilde{\Phi}$ from (III.2.17), and for each $\varepsilon > 0$ define the event

$$H_\varepsilon = \Big\{ \sup_{z\in\widetilde{A}_2} |\widetilde{\beta}_z^{\widetilde{U}_1}| \leq \frac{\varepsilon}{2} \Big\}. \tag{III.6.3}$$

The following result is stated on $\mathbb{Z}^d$ in [67] but its proof is actually valid on $\widetilde{G}$, for any G as in (III.3.1), using the Markov property of the free field on $\widetilde{G}$, cf. (III.2.17), instead of the Markov property on $\mathbb{Z}^d$.

Theorem III.6.2 ([67, Theorem 1.2])**.** *Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two disjoints measurable subsets of $\widetilde{G}$ such that $\widetilde{A}_1$ is compact with finitely many connected components, and let $f_2 : C(\widetilde{A}_2,\mathbb{R}) \to [0,1]$ be a measurable and increasing or decreasing function. Then for all $\varepsilon > 0$, $\widetilde{\mathbb{P}}^G$-a.s.,*

$$\begin{aligned}&\Big\{\widetilde{\mathbb{E}}^G \Big[f_2(\widetilde{\Phi}_{|\widetilde{A}_2} - \sigma\varepsilon)\Big] - \widetilde{\mathbb{P}}^G\left(H_\varepsilon^{\mathsf{c}}\right)\Big\}\mathbb{1}_{H_\varepsilon}\\ &\quad\leq \widetilde{\mathbb{E}}^G\Big[f_2(\widetilde{\Phi}_{|\widetilde{A}_2}) \,\big|\, \widetilde{\varphi}_{|\widetilde{A}_1}\Big]\mathbb{1}_{H_\varepsilon} \leq \Big\{\widetilde{\mathbb{E}}^G\Big[f_2(\widetilde{\Phi}_{|\widetilde{A}_2} + \sigma\varepsilon)\Big] + \widetilde{\mathbb{P}}^G\left(H_\varepsilon^{\mathsf{c}}\right)\Big\}\mathbb{1}_{H_\varepsilon}\end{aligned} \tag{III.6.4}$$

where $\sigma = 1$ if f_2 is increasing and $\sigma = -1$ if f_2 is decreasing.

Remark III.6.3. We note in passing that conditions (p_0), (V_α) and (G_β) are not even necessary here: Theorem III.6.2 holds on any locally finite, transient, connected weighted graph (G, λ). □

Assume now that $\widetilde{A}_1$ is no longer compact, but only bounded (and measurable) and let $\widetilde{A}_1'$ be the largest subset $\widetilde{B}$ of $\widetilde{G}$ such that $\widetilde{B}^* = \widetilde{A}_1^*$ (see before display (III.2.15) for a definition of $\widetilde{B}^*$), i.e., $\widetilde{A}_1'$ is the closure of the set where one adds to $\widetilde{A}_1$ all the edges I_e such that $\widetilde{A}_1 \cap I_e \neq \varnothing$, and $\widetilde{A}_1'^* = \widetilde{A}_1^* \subset G$ is the "print" of $\widetilde{A}_1'$ in G. Note that every continuous path started in $\widetilde{G}\setminus\widetilde{A}_1'$ and entering $\widetilde{A}_1'$ will do so by traversing one of the vertices in $\widetilde{A}_1^*$. The set $\widetilde{A}_1'$ is a compact subset of $\widetilde{G}$ with finitely many connected components. We can thus define H_ε' as in (III.6.3) but with $\widetilde{U}_1' \overset{\text{def.}}{=} (\widetilde{A}_1')^{\mathsf{c}}$ in place of $\widetilde{U}_1$, for any bounded measurable set $\widetilde{A}_1 \subset \widetilde{G}$. The inequality (III.2.20) will readily follow from Theorem III.6.2 once we have the following lemma, which is similar to Proposition 1.4 in [67].

Lemma III.6.4. *Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two Borel-measurable subsets of $\widetilde{G}$, $s = d(\widetilde{A}_1^*, \widetilde{A}_2^*)$ and $r = \delta(\widetilde{A}_1^*)$. Assume that $s > 0$ and $r < \infty$. There exist constants $c_6 > 0$ and $C_6 < \infty$ such that for all such $\widetilde{A}_1$, $\widetilde{A}_2$ and all $\varepsilon > 0$,*

$$\widetilde{\mathbb{P}}^G\left(H_\varepsilon'^{\mathsf{c}}\right) \leq \frac{C_6}{2}(r+s)^\alpha \exp\left\{-c_6\varepsilon^2 s^\nu\right\}. \tag{III.6.5}$$

Proof. Let $K = \partial B(\widetilde{A}_1^*, s)$. By assumption, every connected path on $\widetilde{G}$ from $\widetilde{A}_2$ to $\widetilde{A}_1$ must enter K prior to $\widetilde{A}_1^*$. By the strong Markov property of $\widetilde{X}$, we have $\widetilde{\beta}_z^{\widetilde{U}'_1} = \sum_{x\in K} \widetilde{P}_z(H_K < \infty, \widetilde{X}_{H_K} = x)\widetilde{\beta}_x^{\widetilde{U}'_1}$ for all $z \in \widetilde{A}_2$ and therefore, in view of (III.6.3), we obtain the bound

$$\widetilde{\mathbb{P}}^G\left(H_\varepsilon'^{\mathsf{c}}\right) \leq \widetilde{\mathbb{P}}^G\left(\sup_{x\in K} |\widetilde{\beta}_x^{\widetilde{U}'_1}| > \frac{\varepsilon}{2}\right) = \mathbb{P}^G\left(\sup_{x\in K} |\beta_x^{\widetilde{A}_1^*}| > \frac{\varepsilon}{2}\right), \tag{III.6.6}$$

with $\beta_x^{\widetilde{A}_1^*} = E_x\big[\Phi_{Z_{H_{\widetilde{A}_1^*}}} \mathbb{1}_{H_{\widetilde{A}_1^*}<\infty}\big]$. Here, the equality follows from the fact that under $\widetilde{P}_x$ for $x \in K$, $\widetilde{X}_{T_{\widetilde{U}'_1}} = \widetilde{X}_{H_{\widetilde{A}'_1}}$ is always on $\widetilde{A}_1^*$ (cf. the discussion below Remark III.6.3), that the law of $\widetilde{\Phi}_{|G}$ under $\widetilde{\mathbb{P}}^G$ is $\mathbb{P}^G$, and that the law of $\widetilde{X}_{|G}$ under $\widetilde{P}_x$ is P_x for each $x \in G$. Following the proof of Proposition 1.4 in [67] (see the computation of $\mathrm{Var}(h_x)$ therein), if $s > 2C_3$, then for each $x \in K$, $\beta_x^{\widetilde{A}_1^*}$ is a centered Gaussian variable with variance upper bounded by

$$\sup_{y\in\widetilde{A}_1^*} g(x,y) \overset{(G_\beta)}{\leq} C_2 \sup_{y\in\widetilde{A}_1^*} d(x,y)^{-\nu} \overset{\text{(III.2.8)}}{\leq} C_2(s - C_3)^{-\nu} \leq Cs^{-\nu}, \tag{III.6.7}$$

noting that $d(K, \widetilde{A}_1^*) \geq s - C_3$ by (III.2.8). By possibly adjusting the constant C, we see that (III.6.7) continues to hold if $s \leq 2C_3$, for then $s^{-\nu} \geq c$ and $\sup_{x\in K, y\in\widetilde{A}_1^*} g(x,y) \leq \sup_{x\in G} g(x,x) \leq C_2$ by (G_β) and using that $g(x,y) = P_x(H_y < \infty)g(y,y) \leq g(y,y)$. By a union bound, using (V_α) and (III.2.10), we finally get with (III.6.7) and (III.6.6),

$$\widetilde{\mathbb{P}}^G\left(H_\varepsilon'^{\mathsf{c}}\right) \leq 2C_1(r+s)^\alpha \exp\big\{-cs^\nu\varepsilon^2\big\},$$

for all $s > 0$ and $r < \infty$, which completes the proof. □

Proof of (III.2.20). We may assume without loss of generality that $\widetilde{A}_1$ is bounded and $r = \delta(\widetilde{A}_1)$. Applying Theorem III.6.2 with $\widetilde{A}'_1$ and $\widetilde{A}_2$, multiplying the upper bound in (III.6.4) by $f_1(\widetilde{\varphi}_{|\widetilde{A}_1})$ for some monotone function $f_1 : C(\widetilde{A}_1, \mathbb{R}) \to [0,1]$ and integrating yields

$$\widetilde{\mathbb{E}}^G\left[f_1\big(\widetilde{\Phi}_{|\widetilde{A}_1}\big)f_2\big(\widetilde{\Phi}_{|\widetilde{A}_2}\big)\right] \leq \widetilde{\mathbb{E}}^G\left[f_1\big(\widetilde{\Phi}_{|\widetilde{A}_1} \pm \varepsilon\big)\right]\widetilde{\mathbb{E}}^G\left[f_2\big(\widetilde{\Phi}_{|\widetilde{A}_2} \pm \varepsilon\big)\right] + 2\widetilde{\mathbb{P}}^G\left(H_\varepsilon'^{\mathsf{c}}\right). \tag{III.6.8}$$

The inequality (III.2.20) then follows from (III.6.8) and (III.6.5). □

We now turn to (III.2.21), the decoupling inequality for random interlacements. We will eventually use the soft local times technique which has been introduced in [68] to prove a similar (stronger) inequality on $\mathbb{Z}^d$, for $d \geq 3$. In anticipation of arising difficulties when estimating the error term which naturally appears within this method, we first show a certain Harnack-type inequality, see (III.6.11) below, which will be our main tool to deal with this issue. Let

$$K \geq 5 \vee (2C_3)^2 \tag{III.6.9}$$

be a parameter to be fixed later (the choice of K will correspond to the constant C_7 appearing above (III.2.21), see (III.6.36) below). We consider $\widetilde{A}_1$ and $\widetilde{A}_2$ two measurable subsets of $\widetilde{G}$ and we assume that the diameter r of $\widetilde{A}_1^*$ is finite and smaller than the diameter of $\widetilde{A}_2^*$ (recall the definition of $\widetilde{A}^* \subset G$ for $\widetilde{A} \subset \widetilde{G}$ from Section III.2), and that $s = d(\widetilde{A}_1^*, \widetilde{A}_2^*) \geq K(r \vee 1)$ and $s > 0$. We then define

$$A_1 = \widetilde{A}_1^*,\ A_2 = B\Big(A_1, \frac{s}{2}\Big)^{\mathsf{c}} \text{ and } V = \partial B\Big(A_1, \frac{s}{\sqrt{K}}\Big). \tag{III.6.10}$$

These assumptions imply that $s \geq K \overset{\text{(III.6.9)}}{\geq} 2C_3\sqrt{K}$, so that by (III.2.8), the sets A_1, A_2 and V are disjoints subsets of G, $A_2 \supset \widetilde{A}_2^*$ and any nearest neighbor path from A_1 to A_2 crosses V. The following lemma will follow from (III.3.3) and a chaining argument.

Lemma III.6.5. *For all $K \geq c$, there exists $C_{15} = C_{15}(K) \geq 1$ such that for any A_1, A_2, V as above, $B \in \{A_1, A_2, A_1 \cup A_2\}$, v a non-negative function on G, L-harmonic on B^{c},*

$$\sup_{y\in V} v(y) \leq C_{15} \inf_{y \in V} v(y). \tag{III.6.11}$$

Proof. Set $\varepsilon(K) = \frac{1}{\sqrt{K}}$ and

$$U_0 = B\big(A_1, \varepsilon^2(2C_9+1)s\big),\ U_1 = B\big(A_1, \varepsilon s\big),\ U_2 = B\big(A_2, \varepsilon^2(2C_9+1)s\big)^{\mathsf{c}},$$

and V' the largest component of $V (= \partial U_1)$ which is connected in $U_0^{\mathsf{c}} \cap U_2$, where C_9 corresponds to the constant in the elliptic Harnack inequality, see above (III.3.3) and Lemma III.3.1. We first prove that if $K \geq c$ (so that ε is small enough) then $V' = V$, i.e., V is connected in $U_0^{\mathsf{c}} \cap U_2$. We first assume that $K \geq c$ so that $U_0 \subset U_1 \subset U_2$. If $V' \neq V$, then there exist $y, y' \in V$ such that y is not connected to y' in $U_0^{\mathsf{c}} \cap U_2$, and in particular using the strong Markov property of Z at time H_{U_0},

$$P_y(H_{y'} < T_{U_2}) \leq P_y(H_{U_0} < T_{U_2}) \sup_{x \in U_0} P_x(H_{y'} < T_{U_2}). \tag{III.6.12}$$

Recall the relative equilibrium measure $e_{U_0,U_2}(\cdot)$ and capacity $\mathrm{cap}_{U_2}(U_0)$ from (III.3.6) and (III.3.7). Using that $s \geq Kr$, it follows that for $K \geq c'$, $d(U_1, U_2^{\mathsf{c}}) \geq C_9\delta(U_1)$ so that, by (III.3.2) and (III.3.8), one obtains for all $x \in A_1 \subset U_0$,

$$\begin{aligned} 1 = \sum_{x' \in U_0} g_{U_2}(x,x') e_{U_0,U_2}(x') &\geq \frac{c_2}{2}\big(2r + \varepsilon^2(2C_9+1)s\big)^{-\nu} \mathrm{cap}_{U_2}(U_0) \\ &\overset{r \leq \varepsilon^2 s}{\geq} \frac{c_2}{2}\big(\varepsilon^2(2C_9+3)s\big)^{-\nu} \mathrm{cap}_{U_2}(U_0). \end{aligned} \tag{III.6.13}$$

We further assume that $K \geq c$ and ε is small enough so that $d(U_0, V) \geq \frac{\varepsilon s}{4}$, and then, using again (III.3.2) and (III.3.8), for all $y \in V$,

$$P_y(H_{U_0} < T_{U_2}) = \sum_{x \in U_0} g_{U_2}(y, x) e_{U_0,U_2}(x) \leq C_2 d(U_0, V)^{-\nu} \mathrm{cap}_{U_2}(U_0) \overset{\text{(III.6.13)}}{\leq} C \times \varepsilon^{\nu}. \tag{III.6.14}$$

We stress that C is uniform in K (and ε) in (III.6.14). On the other hand, applying the strong Markov property at time $H_{y'}$ and (III.3.2) we find for all $x \in U_1$

$$\frac{c_2}{2C_2} d(x, y')^{-\nu} \leq P_x(H_{y'} < T_{U_2}) = \frac{g_{U_2}(x, y')}{g_{U_2}(y', y')} \leq \frac{2C_2}{c_2} d(x, y')^{-\nu}. \tag{III.6.15}$$

Combining (III.6.12) with (III.6.14) and (III.6.15) (recall that $U_0 \subset U_1$ and $y \in U_1$) we get, for $K \geq c$

$$d(y, y')^{-\nu} \leq C \times \varepsilon^{\nu} \times \left(\frac{\varepsilon s}{4}\right)^{-\nu} \leq C' \times s^{-\nu} \tag{III.6.16}$$

(with constants C and C' uniform in K and ε). But since $y, y' \in V$,

$$d(y, y') \leq 2\,(r + \varepsilon s) \leq 4\varepsilon s. \tag{III.6.17}$$

Clearly, upon choosing K large enough, as $\varepsilon(K) \to 0$ as $K \to \infty$, (III.6.16) and (III.6.17) lead to a contradiction. Thus V' is connected in $U_0^{\mathsf{c}} \cap U_2$.

For all $x \in B(A_1, 2\varepsilon^2 C_9 s)^{\mathsf{c}} \cap B(A_2, 2\varepsilon^2 C_9 s)^{\mathsf{c}}$, v is harmonic on $B(x, 2\varepsilon^2 C_9 s)$ by assumption and thus (III.3.3) gives

$$\inf_{z \in B(x, \varepsilon^2 s)} v(z) \geq c_9 \sup_{z \in B(x, \varepsilon^2 s)} v(z). \tag{III.6.18}$$

By connectivity of V' in $U_0^{\mathsf{c}} \cap U_2$ and (III.6.1), for all $y, y' \in V$, one can find $N \in \mathbb{N}$, a sequence $z_0, \ldots, z_N$ in $\Lambda\big(\varepsilon^2 s/2\big) \cap B(A_1, 2\varepsilon^2 C_9 s)^{\mathsf{c}} \cap B(A_2, 2\varepsilon^2 C_9 s)^{\mathsf{c}}$, with $\Lambda\big(\varepsilon^2 s/2\big)$ as in Lemma III.6.1, such that $z_i \neq z_j$ for $i \neq j$, $y \in B(z_0, \varepsilon^2 s)$, $y' \in B(z_N, \varepsilon^2 s)$ and for all $i \in \{1, \ldots, N\}$, there exists $y_i \in B(z_{i-1}, \varepsilon^2 s) \cap B(z_i, \varepsilon^2 s)$. Note that with the help of (III.6.2), we can choose N uniformly in s and $y, y' \in V$ (but still as a function of K). We then apply (III.6.18) recursively on each of the balls $B(z_i, \varepsilon^2 s)$, $i \in \{0, \ldots, N\}$, to find

$$v(y) \geq c_9^{N+1} v(y'),$$

and (III.6.11) follows. □

We now recall some facts about soft local times from [68]. We continue with the setup of (III.6.10) and introduce the excursion process between $B \in \{A_1, A_2, A_1 \cup A_2\}$ and V for the Markov chain $Z_\cdot$ on G as follows. Let $\theta_n : G^{\mathbb{N}} \to G^{\mathbb{N}}$ denote the canonical time shifts on $G^{\mathbb{N}}$, that is for all $n, p \in \mathbb{N}$ and $\omega \in G^{\mathbb{N}}$, $(\theta_n(\omega))_p = \omega_{n+p}$. The successive return times to B and V are recursively defined by $D_0 = 0$ and for all $k \geq 1$,

$$R_k = H_B \circ \theta_{D_{k-1}} + D_{k-1} \qquad\qquad D_k = H_V \circ \theta_{R_k} + R_k, \tag{III.6.19}$$

where H_B is the first hitting of B by $Z_\cdot$, cf. below (III.2.4). Let $N^B = \inf\{k \geq 0 : R_k = \infty\}$, and note that $N^B < \infty$ a.s. since $Z_\cdot$ is transient. For $k \in \{1, \dots, N^B - 1\}$, a trajectory $\Sigma_k \stackrel{\text{def.}}{=} (Z_n)_{n \in \{R_k, \dots, D_k\}}$ is called an excursion between B and V. It takes values in Ξ_B, the set of trajectories starting in ∂B and either ending the first time V is hit or never visiting V. We add a cemetery point Δ to Ξ_B and, with a slight abuse of notation, introduce a new point Δ' in G such that for any random variable $H \in \mathbb{N} \cup \{\infty\}$, $Z_H = \Delta'$ if $H = \infty$. For each $x \in \partial B$, let $\Xi_B(x)$ be the set of trajectories in $\Xi_B \setminus \{\Delta\}$ starting in x. Set $\Xi_B(\Delta') = \{\Delta\}$ and for all $\sigma \in \Xi_B$, let $\sigma^e \in V$ be the last point visited by σ if σ is a finite trajectory of $\Xi_B \setminus \Delta$, and $\sigma^e = \Delta'$ otherwise. Upon defining $\Sigma_k = \Delta$ for $k \geq N^B$, the sequence $(\Sigma_k)_{k \geq 1}$ can be viewed as a Markov process on Ξ_B, called the excursion process between B and V.

We now sample the Markov chain $(\Sigma_k)_{k \geq 1}$ using a Poisson point process as described in Section 4 of [68]. Let μ_B be the measure on Ξ_B given by

$$\mu_B(S) = \sum_{x \in \partial B} P_x(\Sigma_1 \in S) + \delta_\Delta(S) \tag{III.6.20}$$

for all S in the σ-algebra generated by the canonical coordinates, where δ_Δ denotes a Dirac mass at Δ, and let $p_B : \Xi_B \times \Xi_B \to [0, \infty)$ be defined (see also (5.18) of [68]) by

$$p_B(\sigma, \sigma') = P_{\sigma^e}(H_B = x) \text{ for all } \sigma \in \Xi_B \text{ and } \sigma' \in \Xi_B(x),\ x \in \partial B \cup \{\Delta'\}, \tag{III.6.21}$$

with the convention $P_{\Delta'}(H_B = \Delta') = 1$. Let η be a Poisson random measure on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with intensity $\mu_B \otimes \lambda$, where λ is the Lebesgue measure on $[0, \infty)$. Let σ_0 be a random variable on Ξ_B independent of η such that

$$\mathbb{P}(\sigma_0^e = y) = \overline{e}_V(y) \text{ for all } y \in V$$

(see (III.3.9) for notation). Moreover, set $\Gamma_0 : \Xi_B \to \mathbb{R}_+$ with $\Gamma_0(\sigma) = 0$ for all $\sigma \in \Xi_B$. We now define recursively the random variables ξ_n, σ_n, v_n and Γ_n: for all $n \geq 1$, (σ_n, v_n) is the $\mathbb{P}$-a.s. unique point in $\Xi_B \times [0, \infty)$ such that

$$\xi_n \stackrel{\text{def.}}{=} \inf_{(\sigma, v)} \frac{v - \Gamma_{n-1}(\sigma)}{p_B(\sigma_{n-1}, \sigma)} \tag{III.6.22}$$

is reached in (σ_n, v_n), where the infimum is taken among all the possible pairs (σ, v) in $\mathrm{supp}(\eta) \setminus \{(\sigma_1, v_1), \dots, (\sigma_{n-1}, v_{n-1})\}$, and define

$$\Gamma_n(\sigma) = \Gamma_{n-1}(\sigma) + \xi_n p_B(\sigma_{n-1}, \sigma) \text{ for all } \sigma \in \Xi_B. \tag{III.6.23}$$

Note that, for all $n \geq 1$ and $(\sigma, v) \in \mathrm{supp}(\eta)$, as follows from (III.6.22) and (III.6.23), $\mathbb{P}$-a.s,

$$v \leq \Gamma_n(\sigma) \implies (\sigma, v) \in \{(\sigma_1, v_1), \dots, (\sigma_n, v_n)\}. \tag{III.6.24}$$

According to Proposition 4.3 in [68], $(\sigma_n)_{n\geq 1}$ has the same law under $\mathbb{P}$ as $(\Sigma_n)_{n\geq 1}$ under $P_{\bar{e}_V}$ (recall the notation from (III.2.3)). By definition, see (III.6.21), for all $\sigma, \sigma' \in \Xi_B$, $p_B(\sigma, \sigma')$ only depend on the last vertex visited by σ and on the first vertex visited by σ' and thus, on account of (III.6.23), for all $x \in \partial B \cup \{\Delta'\}$ and $\sigma, \sigma' \in \Xi_B(x)$, $\Gamma_n(\sigma) = \Gamma_n(\sigma')$. In particular, we can define the *soft local time* up to time $T^B \overset{\text{def.}}{=} \inf\{n;\, \sigma_n = \Delta\}$ of the excursion process between B and V by

$$F_1^B(x) = \Gamma_{T^B}(\sigma_x) \text{ for all } x \in \partial B \cup \{\Delta'\}, \tag{III.6.25}$$

where σ_x is any trajectory in $\Xi_B(x)$. By definition, see (III.6.23), we can also write

$$F_1^B(x) = \sum_{k=1}^{T^B} \xi_k p_B(\sigma_{k-1}, \sigma_x), \quad \text{for all } x \in \partial B \cup \{\Delta'\}. \tag{III.6.26}$$

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is suitably enlarged as to carry a family $F = \{F_k^B;\, k = 1, 2, \dots\}$ of i.i.d. random variables with the same law as F_1^B, and, for each $u > 0$, a random variable Θ_u^V with law Poisson$(u \cdot \mathrm{cap}(V))$ independent of F. The variables F_k^B, $1 \leq k \leq \Theta_u^V$ correspond to the soft local times attached to each of the trajectories in the support of ω^u, the interlacement point process, which visit the set V (by (III.6.10) these are the trajectories causing correlations between $\widetilde{\ell}_{\widetilde{A}_1,u}$ and $\widetilde{\ell}_{\widetilde{A}_2,u}$). For all $u > 0$ and $x \in \partial B$, we then set

$$G_u^B(x) = \sum_{k=1}^{\Theta_u^V} F_k^B(x), \tag{III.6.27}$$

which has the same law as the accumulated soft local time of the excursion process between B and V up to level u defined in (5.22) of [68] (note that Section 5 in [68] can be adapted, mutatis mutandis, to any transient graph).

The proof of Proposition 5.3 in [68] then asserts that there exists a coupling $\mathbb{Q}$ between three random interlacements processes ω, ω_1 and ω_2 such that ω_1 and

ω_2 are independent and, for all $u > 0$ and $\varepsilon \in (0,1)$,

$$\mathbb{Q}\Big[(\omega_i^{u(1-\varepsilon)})_{|A_i} \leq (\omega^u)_{|A_i} \leq (\omega_i^{u(1+\varepsilon)})_{|A_i},\ i=1,2\Big] \geq 1 - \sum_{\substack{(v,B)=(u(1\pm\varepsilon),A_1),\\ (u(1\pm\varepsilon),A_2),(u,A_1\cup A_2)}} \sum_{x\in\partial B} \mathbb{P}\Big(\big|G_v^B(x) - \mathbb{E}[G_v^B(x)]\big| \geq \frac{\varepsilon}{3}\mathbb{E}[G_v^B(x)]\Big), \tag{III.6.28}$$

where $(\omega^u)_{|A_i}$ is the point process consisting of the restriction to A_i of the trajectories in ω^u hitting A_i and we write $\mu \leq \nu$ if and only if $\nu - \mu$ is a non-negative measure. Adding independent Brownian excursions on the cable system $\widetilde{G}$ as in the proof of Theorem II.3.6, one then easily infers that (III.6.28) can be extended to the local times on the cable system, and thus, in the framework of (III.6.10), since $A_1 = \widetilde{A}_1^*$ and $\widetilde{A}_2^* \subset A_2$, that there exists a coupling $\widetilde{\mathbb{Q}}$ such that

$$\widetilde{\mathbb{Q}}\Big[\widetilde{\ell}^{\,i}_{x,u(1-\varepsilon)} \leq \widetilde{\ell}_{x,u} \leq \widetilde{\ell}^{\,i}_{x,u(1+\varepsilon)},\ x \in \widetilde{A}_i,\ i=1,2\Big] \geq 1 - \sum_{\substack{(v,B)=(u(1\pm\varepsilon),A_1),\\ (u(1\pm\varepsilon),A_2),(u,A_1\cup A_2)}} \sum_{x\in\partial B} \mathbb{P}\Big(\big|G_v^B(x) - \mathbb{E}[G_v^B(x)]\big| \geq \frac{\varepsilon}{3}\mathbb{E}[G_v^B(x)]\Big), \tag{III.6.29}$$

where $(\widetilde{\ell}_{x,u})_{x\in\widetilde{G}}$, $(\widetilde{\ell}^{\,1}_{x,u})_{x\in\widetilde{G}}$ and $(\widetilde{\ell}^{\,2}_{x,u})_{x\in\widetilde{G}}$ have the law under $\widetilde{\mathbb{Q}}$ of local times of random interlacements on the cable system $\widetilde{G}$, cf. around (III.2.18), with $\widetilde{\ell}^{\,1}$ independent from $\widetilde{\ell}^{\,2}$. The decoupling inequality (III.2.21) will follow at once from (III.6.29), see the end of this section, once the following large deviation inequality on the error term is shown. We continue with the setup leading to (III.6.10). Recall the multiplicative parameter K in (III.6.9) controlling the distance $d(\widetilde{A}_1^*, \widetilde{A}_2^*)$.

Lemma III.6.6. *There exists $K_0 \geq 5 \vee (2C_3)^2$ such that for all $u > 0$, $\varepsilon \in (0,1)$ and $B \in \{A_1, A_2, A_1 \cup A_2\}$ as in* (III.6.10) *with $K \geq K_0$ and $x \in \partial B$,*

$$\mathbb{P}\Big(\big|G_u^B(x) - \mathbb{E}[G_u^B(x)]\big| \geq \frac{\varepsilon}{3}\mathbb{E}[G_u^B(x)]\Big) \leq C(K)\exp\big\{-c(K)\varepsilon^2 u s^{\nu}\big\}.$$

In order to prove Lemma III.6.6, cf. (III.6.27), we need some estimates on the law of $F_1^B(x)$, which deals with one excursion process between B and V. Let us define

$$\pi^B(y,x) = E_y\Big[\sum_{k=1}^{N^B-1} \delta_{Z_{R_k},x}\Big], \quad \text{for } x \in B \text{ and } y \in V, \tag{III.6.30}$$

the average number of times an excursion starts in x for the excursion process beginning in y (here, $\delta_{x,y}=1$ if $x=y$ and 0 otherwise; recall N^B from below (III.6.19)). It follows from (5.24) in [68] that

$$\pi^B(x) \stackrel{\text{def.}}{=} \mathbb{E}[F_1^B(x)] = \sum_{y\in V} \overline{e}_V(y)\pi^B(y,x). \tag{III.6.31}$$

The following estimates will be useful to prove Lemma III.6.6.

Lemma III.6.7. *For $K\geq K_0$, there exist $c_{16}(K)>0$ and $C_{16}(K)<\infty$ such that, for all $B\in\{A_1,A_2,A_1\cup A_2\}$ as in* (III.6.10)*, all $x\in\partial B$ and $v\in(0,\infty)$,*

(i) $\mathbb{E}\left[F_1^B(x)^2\right]\leq 4C_{15}\pi^B(x)^2$,

(ii) $\mathbb{P}\left(F_1^B(x)\geq \pi^B(x)v\right)\leq C_{16}\exp\{-c_{16}v\}$.

Proof. We tacitly assume throughout the proof that $K\geq c$ so that Lemma III.6.5 applies. Theorem 4.8 in [68] asserts that for all $x\in B$

$$\mathbb{E}\left[F_1^B(x)^2\right]\leq 4\pi^B(x)\sup_{y'\in V}\pi^B(y',x).$$

The function $y'\mapsto \pi^B(y',x)$ is L-harmonic on B^{c}, and (i) follows from (III.6.31) and Lemma III.6.5. We now turn to the proof of (ii). Using (III.6.26) and (III.6.21), we have for all $x\in\partial B$ and $x'\in\partial B\cup\{\Delta'\}$, $\mathbb{P}$-a.s.,

$$\begin{aligned} F_1^B(x') = \sum_{k=1}^{T^B}\xi_k P_{\sigma^e_{k-1}}(Z_{H_B}=x') &\geq \inf_{y'\in V}\left\{\frac{P_{y'}(Z_{H_B}=x')}{P_{y'}(Z_{H_B}=x)}\right\}\sum_{k=1}^{T^B}\xi_k P_{\sigma^e_{k-1}}(Z_{H_B}=x)\\ &\geq \frac{1}{C_{15}}\frac{\inf_{y'\in V}P_{y'}(Z_{H_B}=x')}{\inf_{y'\in V}P_{y'}(Z_{H_B}=x)}F_1^B(x), \end{aligned} \tag{III.6.32}$$

where we used the fact that $y\mapsto P_y(Z_{H_B}=x)$ is harmonic on B^{c} and Lemma III.6.5 in the last inequality. Slight care is needed above if $\sigma^e_{T^B-1}=\Delta'$, in which case $P_{\sigma^e_{T^B-1}}(Z_{H_B}=x')\geq P_{\sigma^e_{T^B-1}}(Z_{H_B}=x)=0$ for all $x\in\partial B$ and $x'\in\partial B\cup\{\Delta'\}$ so that (III.6.32) continues to hold. With (III.6.32), we obtain for all $x\in\partial B$ and $v\in(0,\infty)$,

$$\begin{aligned} &\mathbb{P}\left(F_1^B(x)\geq \pi^B(x)v\right)\\ &\leq \mathbb{P}\left(\forall\, x'\in\partial B\cup\{\Delta'\}:\ F_1^B(x')\geq \frac{1}{C_{15}}\frac{\inf_{y'\in V}P_{y'}(Z_{H_B}=x')}{\inf_{y'\in V}P_{y'}(Z_{H_B}=x)}\pi^B(x)v\right)\\ &\leq \mathbb{P}\left(\forall\, x'\in\partial B\cup\{\Delta'\}:\ F_1^B(x')\geq \frac{1}{C_{15}}\inf_{y'\in V}P_{y'}(Z_{H_B}=x')v\right), \end{aligned} \tag{III.6.33}$$

since $\pi^B(x) \geq \inf_{y'\in V} P_{y'}(Z_{H_B} = x)$ by (III.6.30) and (III.6.31). By (III.6.24) and (III.6.25), if $F_1^B(x') \geq u$ for some $u > 0$ and $x' \in \partial B \cup \{\Delta'\}$, then for every $\sigma \in \Xi_B(x')$ and $v' \in [0,u]$ such that $(\sigma, v') \in \text{supp}(\eta)$, $(\sigma, v') \in \{(\sigma_1, v_1), \dots, (\sigma_{T^B}, v_{T^B})\}$, and thus by (III.6.33), for all $x \in \partial B$ and $v \in (0,\infty)$,

$$\begin{aligned}
&\mathbb{P}\left(F_1^B(x) \geq \pi^B(x)v\right) \\
&\leq \mathbb{P}\Big[\eta\Big(\bigcup_{x'\in\partial B\cup\{\Delta'\}} \{\Xi_B(x')\} \times \Big[0, \frac{1}{C_{15}} \inf_{y'\in V} P_{y'}(Z_{H_B} = x')v\Big]\Big) \leq T^B\Big] \\
&\leq a_1 + a_2,
\end{aligned}$$

where

$$a_1 = \mathbb{P}\Big[\eta\Big(\bigcup_{x'\in\partial B\cup\{\Delta'\}} \{\Xi_B(x')\} \times \Big[0, \frac{1}{C_{15}} \inf_{y'\in V} P_{y'}(Z_{H_B} = x')v\Big]\Big) \leq \frac{v}{2C_{15}^2}\Big], \tag{III.6.34}$$

$$a_2 = \mathbb{P}\left(T^B \geq \frac{v}{2C_{15}^2}\right). \tag{III.6.35}$$

We bound a_1 and a_2 separately. For all $x' \in \partial B \cup \{\Delta'\}$, $\mu_B(\Xi_B(x')) = 1$, see (III.6.20), so the parameter of the Poisson variable in (III.6.34) is

$$\frac{1}{C_{15}} \sum_{x'\in\partial B\cup\{\Delta'\}} \inf_{y'\in V} P_{y'}(Z_{H_B} = x')v \geq \frac{v}{C_{15}^2}$$

by Lemma III.6.5, and thus a_1 in (III.6.34) is bounded by $C(K)\exp\{-c'(K)v\}$ by a standard concentration estimate for the Poisson distribution (recall that $C_{15} = C_{15}(K)$). We now seek an upper bound for a_2. Assume for now that $B = A_1$, whence $\{\Sigma_1 = \Delta\} = \{H_{A_1} = \infty\}$ P_y-a.s. for all $y \in V$, and thus $T^B(= \inf\{n;\, \Sigma_n = \Delta\})$ is dominated by a geometric random variable with parameter $\inf_{y\in V} P_y(H_{A_1} = \infty) = 1 - \sup_{y\in V} P_y(H_{A_1} < \infty)$. By (III.3.8) and (III.6.10), for all $y \in V$,

$$\begin{aligned}
P_y(H_{A_1} < \infty) = \sum_{x\in A_1} g(y,x)e_{A_1}(x) &\overset{(G_\beta)}{\leq} C_2\Big(\frac{s}{\sqrt{K}} - C_3\Big)^{-\nu} \text{cap}(A_1) \\
&\overset{\text{(III.3.11)}}{\leq} 2^\nu C_2 C_{11} K^{-\nu/2},
\end{aligned} \tag{III.6.36}$$

for all $y \in V$, where we used $s \geq (2C_3\sqrt{K}) \vee (Kr)$ in the last inequality (this is guaranteed, cf. around (III.6.10)). By choosing K_0 large enough, we can ensure that the last constant in (III.6.36) is, say, at most $1/2$ for all $K \geq K_0$, so that T^B is dominated by a geometric random variable with positive parameter and then

a_2 in (III.6.35) is bounded by $C(K)\exp\{-c(K)v\}$, for all $K \geq 0$ and $v \in (0,\infty)$. The proof is essentially the same if $B = A_2$ or $B = A_1 \cup A_2$; the only point that requires slight care is that $T^B \geq 2$ on account of (III.6.10), and thus we use instead that $T^B - 1$ is bounded by a suitable geometric random variable. □

With Lemma III.6.7 at hand, we are now able to prove Lemma III.6.6 using arguments similar to those appearing in the proof of Theorem 2.1 in [68].

Proof of Lemma III.6.6. By (III.6.27), (III.6.31) and Markov's inequality, we can write for all $a > 0$, $x \in \partial B$ and $\varepsilon \in (0,1)$, recalling that Θ_u^V and the family F are independent,

$$\begin{aligned}\mathbb{P}\Big(G_u^B(x) \geq (1+\frac{\varepsilon}{3})\mathbb{E}[G_u^B(x)]\Big) \\ &\leq \mathbb{E}\left[\Big(\mathbb{E}\big[\exp\big\{aF_1^B(x)\big\}\big]\Big)^{\Theta_u^V}\right]\exp\Big\{-a\big(1+\frac{\varepsilon}{3}\big)u\mathrm{cap}(V)\pi^B(x)\Big\} \\ &\leq \exp\Big\{u\mathrm{cap}(V)\Big(\mathbb{E}\big[\exp\big\{aF_1^B(x)\big\}\big] - 1 - a\big(1+\frac{\varepsilon}{3}\big)\pi^B(x)\Big)\Big\}. \end{aligned} \tag{III.6.37}$$

We now bound $\mathbb{E}\big[\exp\big\{aF_1^B(x)\big\}\big]$ for small enough a. If $t \in [0,1]$, $e^t \leq 1+t+t^2$, so by (i) of Lemma III.6.7, for $K \geq K_0$, $x \in \partial B$ and $a > 0$,

$$\mathbb{E}\Big[\exp\big\{aF_1^B(x)\big\}\mathbb{1}_{\{F_1^B(x)\leq a^{-1}\}}\Big] \leq 1 + a\pi^B(x) + 4a^2 C_{15}\pi^B(x)^2 \tag{III.6.38}$$

(recall for purposes to follow that C_{15} and also C_{16}, c_{16} all depend on K). Moreover, by (ii) of Lemma III.6.7, for all $K \geq K_0$, $x \in \partial B$ and $a \in \big(0, \frac{c_{16}}{2\pi^B(x)}\big]$,

$$\begin{aligned}\mathbb{E}\Big[\exp\big\{aF_1^B(x)\big\}\mathbb{1}_{\{F_1^B(x)> a^{-1}\}}\Big] &\leq a\int_{a^{-1}}^{\infty} e^{at}\mathbb{P}(F_1^B(x) > t)\,\mathrm{d}t + e\mathbb{P}(F_1^B(x) > a^{-1}) \\ &\leq a\pi^B(x)C_{16}\int_{(a\pi^B(x))^{-1}}^{\infty} e^{(a\pi^B(x)-c_{16})t}\,\mathrm{d}t + e\times C_{16}e^{-\frac{c_{16}}{a\pi_B(x)}} \\ &\leq C_{16}(1+e)e^{-\frac{c_{16}}{2a\pi^B(x)}} \leq C_{16}(1+e)\left(\frac{2a\pi^B(x)}{c_{16}}\right)^2, \end{aligned} \tag{III.6.39}$$

where we took advantage of the inequality $e^{-x} < \frac{1}{x^2}$ for $x > 0$ in the last step. Thus, combining (III.6.37), (III.6.38) and (III.6.39) with the choice $a = \frac{c(K)\varepsilon}{\pi^B(x)}$ for a small enough constant $c(K) > 0$, we have for all $x \in \partial B$ and $\varepsilon \in (0,1)$ and $K \geq K_0$,

$$\begin{aligned}\mathbb{P}\Big(G_u^B(x) \geq (1+\frac{\varepsilon}{3})\mathbb{E}[G_u^B(x)]\Big) &\leq \exp\big\{-c'(K)u\varepsilon^2\mathrm{cap}(V)\big\} \\ &\overset{\text{(III.3.11)}}{\leq} \exp\big\{-c''(K)u\varepsilon^2 s^{\nu}\big\}. \end{aligned}$$

In a similar way, one can bound $\mathbb{P}(G_u^B(x) \leq (1-\frac{\varepsilon}{3})\mathbb{E}[G_u^B(x)])$ from above. Indeed, using instead that for all $t > 0$, $e^{-t} \leq 1 - t + t^2$, and so by (i) of Lemma III.6.7, one obtains for $a > 0$, $x \in \partial B$ and $K \geq K_0$,

$$\mathbb{E}\left[\exp\left\{-aF_1^B(x)\right\}\right] \leq 1 - a\pi^B(x) + 4a^2 C_{15}\pi^B(x)^2.$$

This completes the proof. □

We can now conclude.

Proof of (III.2.21). Consider $\widetilde{A}_1$ and $\widetilde{A}_2$ as in the statement of Theorem III.2.4 and set $C_7 = K_0$ with K_0 as appearing in Lemma III.6.6. This fits within the framework described above (III.6.10) with $K = K_0$, whence (III.6.29) and Lemma III.6.6 apply. Thus, (III.2.21) follows upon using (V_α), (III.2.10) and (III.6.10) to bound $|\partial B|$ for any $B \in \{A_1, A_2, A_1 \cup A_2\}$. □

III.7 General renormalization scheme

We now set up the framework for the multi-scale analysis that will lead to the proof of Theorems III.1.1 and III.1.2 in Section III.9. This will bring together the coupling $\widetilde{\mathbb{P}}^u$ from Section III.5, see Corollary III.5.3 and Remark III.5.4, 2), and the decoupling inequalities of Theorem III.2.4, which have been proved in Section III.6 and which will be used to propagate certain estimates from one scale to the next, see Proposition III.7.1 below, much in the spirit of [93] and [95]. Crucially, this renormalization scheme will be applied to a carefully chosen set of "good" local features indexed by points on the approximate lattice $\Lambda(L_0)$ (cf. Lemma III.6.1) at the lowest scale L_0, see Definition III.7.4, which involve the fields $(\widetilde{\gamma}_\cdot, \widetilde{\ell}_{\cdot,u}, \mathcal{B}^p_\cdot)$ from the coupling $\widetilde{\mathbb{Q}}^{u,p}$, see (III.5.34). Importantly, good regions will allow for good local control on the set $\mathcal{C}_u^\infty$ which is defining for $\widetilde{\varphi}_\cdot$, see (III.5.20), and in particular of the $\widetilde{\gamma}_\cdot$-sign clusters in the vicinity to the interlacement, cf. (III.5.19). This will for instance be key in obtaining the desired ubiquity of the two infinite sign clusters in (III.1.13), see also (III.1.10) and (III.1.11).

Following ideas of [93], improved in [95], [68] for random interlacements and extended in [81], [67] to the Gaussian free field, we first introduce an adequate renormalization scheme. As before, G is any graph satisfying the assumptions (III.3.1). We introduce a triple $\mathcal{L} = (L_0, \bar{l}, l_0)$ of parameters

$$L_0 \geq C_3,\ \bar{l} \geq 2 \quad \text{and} \quad l_0 \geq 8^{1/\nu} \vee C_{14}^{-\frac{1}{2\alpha}} \vee (8 + 4C_7)\bar{l} \qquad \text{(III.7.1)}$$

(cf. (III.2.8) for the definition of C_3, before (III.2.21) for C_7, (III.6.2) for C_{14}, and recall ν from (III.1.6)), and define

$$L_n = l_0^n L_0 \quad \text{and} \quad \Lambda_n^{\mathcal{L}} = \Lambda(L_n) \text{ for all } n \in \{0,1,2,\dots\}. \tag{III.7.2}$$

Here, $\Lambda(L)$, $L \geq 1$ is any henceforth fixed sequence of subsets of G as given by Lemma III.6.1. For any family $B = \{B_x : x \in \Lambda_0^{\mathcal{L}}\}$ of events defined on a common probability space, we introduce the events $G_{x,n}^{\mathcal{L}}(B)$ for all $x \in \Lambda_n^{\mathcal{L}}$ recursively in n by setting

$$\begin{aligned} G_{x,0}^{\mathcal{L}}(B) &= B_x \qquad \text{for all } x \in \Lambda_0^{\mathcal{L}}, \text{ and} \\ G_{x,n}^{\mathcal{L}}(B) &= \bigcup_{\substack{y,y' \in \Lambda_{n-1}^{\mathcal{L}} \cap B(x,\bar{l}L_n) \\ d(y,y') \geq L_n}} G_{y,n-1}^{\mathcal{L}}(B) \cap G_{y',n-1}^{\mathcal{L}}(B) \quad \text{for all } n \geq 1 \text{ and } x \in \Lambda_n^{\mathcal{L}}. \end{aligned} \tag{III.7.3}$$

We recall here that the distance d in (III.7.3) and entering the definition of balls is the one from (III.3.1) (consistent with the regularity assumptions (V_α) and (G_β)) and thus in general *not* the graph distance, cf. Remark III.3.4. Note that since $L_0 \geqslant C_3$ and $l_0 \geqslant 2\bar{l} \geqslant 4$, see (III.7.1), then by (III.2.8), (III.6.1) and (III.7.2) the union in (III.7.3) is not empty. For $\widetilde{A}$ any measurable subset of $\widetilde{G}$ and B a measurable subset of $C(\widetilde{A},\mathbb{R})$, we say that B is *increasing* if for all $f \in B$ and $f' \in C(\widetilde{A},\mathbb{R})$ with $f \leq f'$, $f' \in B$, and B is *decreasing* if B^{c} is increasing. For $h \in \mathbb{R}$ and $u > 0$, we define the events

$$B^{G,h} = \{\widetilde{\Phi}_{|\widetilde{A}} + h \in B\} \quad \text{and} \quad B^{I,u} = \{\widetilde{\ell}_{\widetilde{A},u} \in B\}, \tag{III.7.4}$$

and we add the convention $B^{I,u} = \varnothing$ for $u \leq 0$. If B is increasing then (III.7.4) implies that $B^{G,h} \subset B^{G,h'}$ for $h < h'$ and $B^{I,u} \subset B^{I,u'}$ for $u < u'$.

Proposition III.7.1. *For all graphs G satisfying* (III.3.1)*, there exist $c_{17} > 0$ and $C_{17} \geq 1$ such that for all all L_0, $\bar{l}$ and l_0 as in* (III.7.1)*, all $\varepsilon > 0$ and $h \in \mathbb{R}$ (resp. $u > 0$) with*

$$\frac{\varepsilon^2(\sqrt{l_0}L_0)^\nu}{\log(L_0+1)} \geq C_{17} \qquad \Big(\textit{resp. } \frac{u\varepsilon^2(\sqrt{l_0}L_0)^\nu}{\log(L_0+1)} \geq C_{17}\Big), \tag{III.7.5}$$

and all families $B = \{B_x : x \in \Lambda_0^{\mathcal{L}}\}$ such that the sets B_x, $x \in \Lambda_0^{\mathcal{L}}$, are either all increasing or all decreasing measurable subsets of $C(\widetilde{B}(x,\bar{l}L_0),\mathbb{R})$ satisfying

$$\widetilde{\mathbb{P}}^G(B_x^{G,h}) \leq \frac{c_{17}}{l_0^{4\alpha}} \qquad \Big(\textit{resp. } \widetilde{\mathbb{P}}^I(B_x^{I,u}) \leq \frac{c_{17}}{l_0^{4\alpha}}\Big) \qquad \textit{for all } x \in \Lambda_0^{\mathcal{L}}, \tag{III.7.6}$$

one has for all $n \in \{0,1,2,\dots\}$ and $x \in \Lambda_n^{\mathcal{L}}$,

$$\widetilde{\mathbb{P}}^G\big(G_{x,n}^{\mathcal{L}}(B^{G,h\pm\varepsilon})\big) \leq 2^{-2^n} \qquad \Big(\textit{resp. } \widetilde{\mathbb{P}}^I\big(G_{x,n}^{\mathcal{L}}(B^{I,u(1\pm\varepsilon)})\big) \leq 2^{-2^n}\Big), \tag{III.7.7}$$

where the plus sign corresponds to the case where the sets B_x are all decreasing and the minus sign to the case where the sets B_x are all increasing.

Proof. We give the proof for the Gaussian free field in the case of decreasing events. The proof for increasing events and/or random interlacements is similar and relies in the latter case on (III.2.21) rather than (III.2.20), which will be used below. Thus, fix some $\varepsilon > 0$, $h \in \mathbb{R}$, $\bar{l}$ and l_0 as in (III.7.1), and assume $B = \{B_x : x \in \Lambda_0^{\mathcal{L}}\}$ is such that B_x is a decreasing subset of $C(\widetilde{B}(x, \bar{l}L_0), \mathbb{R})$ satisfying (III.7.6), for all $x \in \Lambda_0^{\mathcal{L}}$. The sequence $(h_n)_{n\geq 0}$ is defined by $h_0 = h$ and for all $n \geq 1$, $h_n = h + \sum_{k=1}^{n} \frac{\varepsilon \wedge 1}{2^k}$, whence $h_n \leq h + \varepsilon$ for all n.

We now argue that there exists a constant C_{17} such that, if the first inequality in (III.7.5) holds, then for all $n \in \{0, 1, 2, \dots\}$,

$$\widetilde{\mathbb{P}}^G\big(G_{x,n}^{\mathcal{L}}(B^{G,h_n})\big) \leq \frac{2^{-2^n}}{2C_{14}^2 l_0^{4\alpha}} \text{ for all } x \in \Lambda_n^{\mathcal{L}}, \tag{III.7.8}$$

with α as in (V_α) and C_{14} defined by (III.6.2). It is then clear that (III.7.7) follows from (III.7.8) since $l_0 \geqslant C_{14}^{-\frac{1}{2\alpha}}$ and the sets B_x, $x \in \Lambda_0^{\mathcal{L}}$, are decreasing. We prove (III.7.8) by induction on n: for $n = 0$, (III.7.8) is just (III.7.6) upon choosing

$$c_{17} \overset{\text{def.}}{=} \frac{1}{4C_{14}^2}.$$

Assume that (III.7.8) holds at level $n-1$ for some $n \geq 1$. Note that by (III.7.3), (III.7.1), for all $h' > 0$ and $x \in \Lambda_{n-1}^{\mathcal{L}}$, $G_{x,n-1}^{\mathcal{L}}(B^{G,h'}) \in \sigma\big(\widetilde{\Phi}_x,\, x \in \widetilde{B}(x, 2\bar{l}L_{n-1})\big)$. Let $r_n = 2\bar{l}L_{n-1}$. Then, for all $x \in \Lambda_n^{\mathcal{L}}$ and $y, y' \in \Lambda_{n-1}^{\mathcal{L}} \cap B(x, \bar{l}L_n)$ such that $d(y, y') \geq L_n$ (as appearing in the union in (III.7.3)),

$$\bar{l}L_n \geq d\big(B(y, r_n), B(y', r_n)\big) \geq \big(l_0 - 4\bar{l}\big)\, L_{n-1} \overset{\text{(III.7.1)}}{\geq} \frac{l_0}{2} L_{n-1} \overset{\text{(III.7.1)}}{\geq} C_7 r_n \overset{\text{def.}}{=} s_n.$$

Using (III.6.2), (III.7.3), (III.7.2), a union bound and the decoupling inequality (III.2.20), we get

$$\begin{aligned}
&\widetilde{\mathbb{P}}^G\big(G_{x,n}^{\mathcal{L}}(B^{G,h_n})\big) \\
&\leq \big(C_{14} l_0^{2\alpha}\big)^2 \bigg[\Big(\sup_y \widetilde{\mathbb{P}}^G\big(G_{y,n-1}^{\mathcal{L}}(B^{G,h_{n-1}})\big)\Big)^2 + C_6 L_{n+1}^{\alpha} \exp\Big(-c_6 \frac{\varepsilon^2}{2^{2n}} s_n^{\nu}\Big)\bigg],
\end{aligned}$$

where the supremum is over all $y \in \Lambda_{n-1}^{\mathcal{L}} \cap B(x, \bar{l}L_n)$. Then (III.7.8) follows by the induction hypothesis upon choosing C_{17} large enough such that for all $\bar{l}$ and l_0 as in (III.7.1), $\varepsilon \in (0, 1)$ and $L_0 \geqslant 1$ such that the first inequality in (III.7.5) holds, as well as all $n \geq 1$,

$$C_6 C_{14}^{2\alpha} l_0^{(5+n)\alpha} L_0^{\alpha} \exp\Big(-c_6 \frac{\varepsilon^2}{2^{2n}} s_n^{\nu}\Big) \leq \frac{2^{-2^n}}{4C_{14}^2 l_0^{4\alpha}},$$

which is possible since $\varepsilon^2 s_n^\nu \geqslant \varepsilon^2(L_0 l_0^n/2)^\nu \geqslant C_{17}\log(L_0+1)(\sqrt{l_0}l_0^{n-1}/2)^\nu$ and $l_0^\nu \geq 8$. □

Remark III.7.2. 1) (Existence of a subcritical regime) As a first consequence of the scheme put forth in (III.7.1)–(III.7.4) and noteworthily under the mere assumptions (III.3.1), Proposition III.7.1 can be readily applied to a suitable family of events $B = \{B_x : x \in \Lambda_0^{\mathcal{L}}\}$ and of parameters $\mathcal{L}$ in (III.7.1) to obtain (stretched) exponential controls on the connectivity function above large levels. This complements results in [95]. The argument is classical, see e.g. [95], so we collect this result and simply sketch its proof. Let

$$h_{**} \stackrel{\text{def.}}{=} \inf\big\{h \in \mathbb{R};\ \liminf_{L\to\infty}\sup_{x\in G}\mathbb{P}^G\big(B(x,L) \stackrel{E^{\geq h}}{\longleftrightarrow} \partial B(x,2L)\big) = 0\big\}, \quad \text{(III.7.9)}$$

where the event under the probability refers to the existence of a nearest neighbor path of vertices from the ball $B(x,L)$ to the boundary of the ball $\partial B(x,2L)$ in $E^{\geq h}$. The parameter u_{**} is defined similarly, but with the infimum ranging over $u \geq 0$ in (III.7.9) and the probability under consideration replaced by $\mathbb{P}^I\big(B(x,L) \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B(x,2L)\big)$. By definition, $h_* \leq h_{**}$ and $u_* \leq u_{**}$, cf. (III.1.8) and (III.1.16).

Corollary III.7.3. *For G satisfying* (III.3.1)*, there exists $c_{18} > 0$ such that*

$$h_{**} = \inf\big\{h \in \mathbb{R};\ \liminf_{L\to\infty}\sup_{x\in G}\mathbb{P}^G\big(B(x,L) \stackrel{E^{\geq h}}{\longleftrightarrow} \partial B(x,2L)\big) < c_{18}\big\} < \infty \quad \text{(III.7.10)}$$

and

$$u_{**} = \inf\big\{u \geq 0;\ \liminf_{L\to\infty}\sup_{x\in G}\mathbb{P}^I\big(B(x,L) \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B(x,2L)\big) < c_{18}\big\} < \infty. \quad \text{(III.7.11)}$$

Moreover, for all $h > h_{**}$ and $u > u_{**}$, there exist constants $c > 0$ and $C < \infty$ depending on u and h such that for all $x \in G$ and $L \geq 1$,

$$\mathbb{P}^G\big(x \stackrel{E^{\geq h}}{\longleftrightarrow} \partial B(x,L)\big) \leq Ce^{-L^c} \text{ and } \mathbb{P}^I\big(x \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B(x,L)\big) \leq Ce^{-L^c}. \quad \text{(III.7.12)}$$

We now outline the proof, and focus on (III.7.11). One chooses $\bar{l} = 4$ and $l_0 = 8^{1/\nu} \vee C_{14}^{-\frac{1}{2\alpha}} \vee (8+4C_7)\bar{l}$ in (III.7.1), takes $\varepsilon = 1$ and fixes some L_0 large enough so that the second condition in (III.7.5) holds for all $u \geq 1$. It is then clear from (V_α), (G_β) and (III.2.10) that one can find $u \geq 1$ large enough such that $\mathbb{P}^I\big(B(x,2L_0) \stackrel{\mathcal{V}^u}{\longleftrightarrow} \partial B(x,4L_0)\big) \leq \sum_{y\in B(x,2L_0)} e^{-\frac{u}{g(y,y)}} \leq c_{18} \stackrel{\text{def.}}{=} c_{17} l_0^{-4\alpha}$, for all $x \in G$, and where we used (III.3.10) and a union bound to infer the first inequality. Having fixed such u, one first shows that $u_{**} \leqslant 2u$ and hence u_{**} is finite as asserted by applying Proposition III.7.1 as follows: for $x \in G$, one considers

$$B_x = \big\{f \in C(\widetilde{B}(x,4L_0),\mathbb{R}) : \, B(x,2L_0) \overset{\{x\in G;\, f(x)\leq 0\}}{\longleftrightarrow} \partial B(x,4L_0)\big\},$$

which are decreasing measurable subsets of $C(\widetilde{B}(x,4L_0),\mathbb{R})$, and one proves by induction over n with the help of (III.6.1) that for all $n \in \{0,1,2,\dots\}$ and $x \in G$,

$$\big(\{0 \overset{\mathcal{V}^{u(1+\varepsilon)}}{\longleftrightarrow} \partial B(x,4L_n)\} \subset \big) \;\; \{B(x,2L_n) \overset{\mathcal{V}^{u(1+\varepsilon)}}{\longleftrightarrow} \partial B(x,4L_n)\} \subset G^{\mathcal{L}}_{x,n}(B^{I,u(1+\varepsilon)}) \tag{III.7.13}$$

(for now $\varepsilon = 1$ but this is true for any $\varepsilon, u > 0$). By these choices, Proposition III.7.1 applies, yielding for all $n \geq 0$ that $\widetilde{\mathbb{P}}^I\big(G^{\mathcal{L}}_{x,n}(B^{I,2u})\big) \leq 2^{-2^n} \leq C\exp\{-L_n^c\}$, and in particular, $\lim_n \widetilde{\mathbb{P}}^I\big(B(x,2L_n) \overset{\mathcal{V}^{2u}}{\longleftrightarrow} \partial B(x,4L_n)\}\big) = 0$, as desired.

To prove the equality in (III.7.11), one repeats the above argument but with different choices of u, L_0 and ε. Namely, one considers any $u > 0$ for which

$$\liminf_{L_0\to\infty} \sup_{x\in G} \mathbb{P}^I\big(B(x,2L_0) \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B(x,4L_0)\big) < c_{18}. \tag{III.7.14}$$

It suffices to show that $u(1+\varepsilon) \geq u_{**}$, for then by letting $\varepsilon \downarrow 0$, it follows that u_{**} is smaller or equal than the infimum in (III.7.11), and the reverse inequality is obvious, as follows from (III.7.9). With u and ε fixed, one selects $L_0 \geq 1$ large enough so as to ensure (III.7.5), and such that the probabilities in (III.7.14) are smaller than c_{18}. Proposition III.7.1 then implies as explained above that $\lim_n \widetilde{\mathbb{P}}^I\big(B(x,2L_n) \overset{\mathcal{V}^{u(1+\varepsilon)}}{\longleftrightarrow} \partial B(x,4L_n)\}\big) = 0$ and $L \mapsto \mathbb{P}^I\big(x \overset{\mathcal{V}^{u(1+\varepsilon)}}{\longleftrightarrow} \partial B(x,L)\big)$ has stretched exponential decay in L for all $x \in G$, thus yielding that $u(1+\varepsilon) \geq u_{**}$ and the interlacement part of (III.7.12) as a by-product. The proof of (III.7.10) and the free field part of (III.7.12) follow similar lines. □

2) (Existence of a supercritical regime for $\nu > 1$) Another simple consequence of Proposition 7.1 is that if G is a graph satisfying (III.3.1) with $\nu > 1$ which contains a subgraph isomorphic to $\mathbb{N}^2$, then, identifying with a slight abuse of notation this subgraph with $\mathbb{N}^2$, there exists $u > 0$ such that $\mathbb{P}^I$-a.s.,

$$\mathcal{V}^u \cap \mathbb{N}^2 \text{ contains an infinite connected component,} \tag{III.7.15}$$

and in particular $u_* > 0$. In the proof of Theorem III.1.2, we only show that under the same assumptions there exists $u > 0$ and $L > 0$ such that $\mathcal{V}^u \cap$

$B(\mathbb{N}^2, L)$ contains an infinite connected component, see Theorem III.9.3 and Remark III.9.4, 5). Thus, (III.7.15) provides us with a stronger, and easier to prove, result for random interlacements. Examples of graphs for which we can prove (III.7.15) are product graphs $G = G_1 \times G_2$ as in Proposition III.3.5 with $\nu = \alpha - \beta > 1$ since if P_1 and P_2 are two semi-infinite geodesics of G_1 and G_2, which exist by Theorem 3.1 in [109], then $P_1 \times P_2$ is a subgraph of G isomorphic to $\mathbb{N}^2$. Also, finitely generated Cayley graphs verifying (V_α) for some $\alpha > 3$ which are not almost isomorphic to $\mathbb{Z}$, see Theorem 7.18 in [61], are covered by this setting.

Let us now sketch the proof of (III.7.15). Using the result from Exercise 1.16 in [27], which is given for $\mathbb{Z}^d$ but immediately transfers to our setting, we have for all positive integer L, M and N, since $\nu > 1$,

$$\begin{aligned}\mathrm{cap}([M, M+L] \times \{N\}) &\leq \frac{L+1}{\inf_{k \in [M,M+L]} \sum_{p=M}^{M+L} g((k,N),(p,N))}\\ &\overset{(G_\beta)}{\leq} \frac{L+1}{C_2 C_3^{-\nu} \sum_{p=1}^{L} p^{-\nu}} \leq CL.\end{aligned}$$

Here, we used that $d((k,N),(p,N)) \leqslant C_3 d_G((k,N),(p,N)) \leqslant C_3|k-p|$ in the second inequality, see (III.2.8), and we also have a similar bound on the capacity of $\{M\} \times [N, N+L]$. For all positive integer L and all $x \in \{L+1, L+2, \dots\}^2$, we write $\mathcal{S}(x,L) = x + \mathbb{N}^2 \cap \partial_{\mathbb{N}^2}[-L,L]^2$, where $\partial_{\mathbb{N}^2} A$ is the boundary of A as a subset of $\mathbb{N}^2$, and we thus get by a union bound

$$\mathrm{cap}\left(\mathcal{S}(x,L)\right) \leq CL. \tag{III.7.16}$$

Fix $\bar{l} = 4$ and $l_0 = 8^{1/\nu} \vee C_{14}^{-\frac{1}{2\alpha}} \vee (8 + 4C_7)\bar{l}$ in (III.7.1), take $\varepsilon = 1/2$, and let C_{18} be such that for all $u > 0$ and $L_0 \geqslant C_3$ with $uL_0 \leqslant C_{18}$, and all $x \in \{4L_0+1, 4L_0+2, \dots\}^2$,

$$\mathbb{P}^I\left(\mathcal{S}(x, 2L_0) \overset{*\text{-}\mathcal{I}^u \cap \mathbb{N}^2}{\longleftrightarrow} \mathcal{S}(x, 4L_0)\right) \overset{\text{(III.7.16)}}{\leqslant} 1 - \exp\{-2CuL_0\} \leqslant \frac{c_{17}}{l_0^2},$$

where $A \overset{*\text{-}B}{\longleftrightarrow} C$ means that there exists a $*$-path in $B \subset \mathbb{N}^2$, as defined above Proposition III.3.7, beginning in A and ending in C. Since $\nu > 1$ one can find L_0 large enough so that (III.7.5) hold when $u = C_{18}L_0^{-1}$, and, applying Proposition III.7.1 and using a property similar to (III.7.13) for $*$-paths of $\mathcal{I}^u$, we get that $L \mapsto \sup_x \mathbb{P}^I(x \overset{*\text{-}\mathcal{I}^{u/2} \cap \mathbb{N}^2}{\longleftrightarrow} \mathcal{S}(x,L))$ has stretched exponential decay, with the supremum ranging over all $x \in \{L+1, L+2, \dots\}^2$. If $\mathcal{V}^u \cap \mathbb{N}^2$ has no infinite connected component, then for any positive integer L the sphere $\partial_{\mathbb{N}^2}[0,L]^2$ is not connected to ∞ in $\mathcal{V}^u \cap \mathbb{N}^2$. Thus, by planar duality, see for instance Proposition III.3.7, there exists $L' \geqslant L-1$ and $x \in \{L'+1, L'+2\} \times \{L'+1\}$ which is connected to $\mathcal{S}(x, L')$ by a $*$-path in $\mathcal{I}^u \cap \mathbb{N}^2$, which happens with probability 0.

In order to prove $u_* > 0$ for $\nu = 1$ by the same method, one would need to remove the polynomial term $(r_n + s_n)^\alpha$ in the decoupling inequality (III.2.21), and it seems plausible that one could do that for a large class of graphs (including $\mathbb{Z}^3$), using arguments similar to [23] or [102]. This is proved in the case $G = G' \times \mathbb{Z}$ in [95]. However, this method does not seem to work in the case $\nu < 1$. A (simpler) proof of $u_* > 0$ is given for $G = \mathbb{Z}^d$ in [72] without using decoupling inequalities, but it seems that one cannot adapt simply its proof to more general graphs if $\nu < 1$. Therefore, the result $u_* > 0$ from Theorem III.1.2 is particularly interesting when $\nu < 1$. □

We now introduce the families of events of the form (III.7.4) to which Proposition III.7.1 will eventually be applied. The reason for the following choices will become apparent in the next section. The strategy developed in Chapter II to prove $h_* > 0$ on $\mathbb{Z}^d$, $d \geq 3$, serves as a starting point in the current setting, but the desired ubiquity result (III.1.13) requires a considerably finer analysis, which is more involved, see also Remark III.7.5 below. All our events will be defined under the probability $\widetilde{\mathbb{Q}}^{u,p}$ from (III.5.34), under which the Gaussian free field $\widetilde{\varphi}_\cdot$ on $\widetilde{G}$ is defined in terms of $(\widetilde{\gamma}_\cdot, \widetilde{\ell}_{\cdot,u})$ by means of (III.5.20).

We now come to the central definition of good vertices. As usual, we denote by $(\ell_{x,u})_{x\in G} = (\widetilde{\ell}_{x,u})_{x\in G}$, $\mathcal{I}^u = \widetilde{\mathcal{I}}^u \cap G$, $\gamma = (\widetilde{\gamma}_x)_{x\in G}$ and $\varphi = (\widetilde{\varphi}_x)_{x\in G}$ the projections of $\widetilde{\ell}$, $\widetilde{\mathcal{I}}^u$, $\widetilde{\gamma}$ and $\widetilde{\varphi}$ on the graph G. For all $u > 0$, these fields have the same law as the occupation time field of random interlacements at level u, a random interlacement set at level u and two Gaussian free fields on G, respectively. We recall the definition of the constants C_{10} from (III.3.4), C_3 from (III.2.8), and c_{13} from Proposition III.4.7, the definition of $\mathcal{B}_y^p$ from (III.5.34), the definition of $\widehat{\mathcal{I}}^u$ from above (III.4.2), and that $C^u(x, L)$ is the set of vertices in G connected to x by a path of edges in $\widehat{\mathcal{I}}^u \cap B_E(x, L)$, see below Lemma III.4.3.

Definition III.7.4 (Good vertex). For $u > 0$, $L_0 \geq 1$, $K > 0$, $p \in (0,1)$, $x \in G$, the event

(i) $C_x^{L_0,K}$ occurs if and only if $\widetilde{\gamma}_z \geq -K/2$ for all $z \in \widetilde{G}$ such that $z \in \widetilde{B}(x, 3C_{10}(L_0 + C_3) + 2L_0 + C_3)$,

(ii) $D_x^{L_0,u}$ occurs if and only if $\mathcal{I}^{u/4} \cap B(x, L_0) \neq \varnothing$,

(iii) $\widehat{D}_x^{L_0,u}$ occurs if and only if $\mathrm{cap}\big(C^{u/2}(y, 2(L_0+C_3))\big) \geq c_{13}(L_0+C_3)^{\frac{3\nu}{4}}(\frac{u}{8})^{\lfloor \gamma - 1 \rfloor}$ for all $y \in \mathcal{I}^{u/4} \cap B(x, L_0 + C_3)$,

(iv) $\overline{D}_x^{L_0,u}$ occurs if and only if

$$\bigcap_{\substack{y,y' \in \mathcal{I}^{u/2} \cap B(x, L_0+C_3); \\ \mathrm{cap}\left(C^{u/2}(y, 2(L_0+C_3))\right) \geq c_{13}(L_0+C_3)^{3\nu/4}(u/8)^{\lfloor \gamma-1 \rfloor}, \\ \mathrm{cap}\left(C^{u/2}(y', 2(L_0+C_3))\right) \geq c_{13}(L_0+C_3)^{3\nu/4}(u/8)^{\lfloor \gamma-1 \rfloor}}} \big\{ y \overset{\wedge}{\longleftrightarrow} y' \text{ in } \widehat{\mathcal{I}}^u \cap B_E(x, 3C_{10}(L_0+C_3)) \big\}, \tag{III.7.17}$$

(v) $E_x^{L_0,u}$ occurs if and only if every component of $\{y \in G;\, \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0/2)$ with diameter at least $L_0/4$ is connected to $\mathcal{I}^{u/4}$ in $\{y \in G;\, \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0)$,

(vi) $F_x^{L_0,p}$ occurs if and only if $\mathcal{B}_y^p = 1$ for all $y \in B(x, 3C_{10}(L_0 + C_3) + 2L_0)$.

Moreover, a vertex $x \in G$ is said to be (L_0, u, K, p)-*good* if the event

$$C_x^{L_0,K} \cap D_x^{L_0,u} \cap \widehat{D}_x^{L_0,u} \cap \overline{D}_x^{L_0,u} \cap E_x^{L_0,u} \cap F_x^{L_0,p} \tag{III.7.18}$$

occurs, and (L_0, u, K, p)-*bad* otherwise.

Remark III.7.5. The above definition of good vertices differs in a number of ways from a corresponding notion introduced in Chapter II (cf. Definition II.4.2 therein) by the authors. This is due to the refined understanding of the isomorphism (III.5.2) stemming from (III.5.19) and (III.5.20). Notably, property (i) above is new in dealing directly with $\widetilde{\gamma}_\cdot$ (rather than $\widetilde{\varphi}_\cdot$). Observe that (v) involves both the field $\widetilde{\varphi}$ and the random interlacements set $\widetilde{\mathcal{I}}^u$ simultaneously, coupled as in (III.5.20). It will lead to a direct proof of the inequality $\overline{h} \geq 0$, see Corollary III.8.8, without using our sign-flipping method, Proposition III.5.6. Properties (ii), (iii) and (iv) can be viewed as a more transparent substitute for the events involved in Lemma II.3.3 and Definition II.3.4 (see also (4.1) in [74]), and have the advantage of preserving the local uniqueness of interlacements, at the cost of introducing a sprinkling between $u/4$ and u. It would be possible to find sharp estimates on the 'size' of the interlacement in a ball similar to Lemma II.3.3 on the class of graphs considered here, but such bounds are in fact unnecessary once we have Lemma III.4.3 and Proposition III.4.7. □

We conclude this section by collecting the following result, which will be crucially used in the next section. It sheds some light on why good vertices may be useful.

Lemma III.7.6. *For all $u > 0$, $L_0 \geq 1$, $K > 0$, $p \in (0, 1)$ and any connected set $A \subset G$ such that each $x \in A$ is an (L_0, u, K, p)-good vertex, there exists a* *connected set* $\widetilde{A}$ *such that*

$$\varnothing \neq \mathcal{I}^{u/4} \cap B(x, L_0) \subset \widetilde{A} \text{ for all } x \in A,\ \widetilde{A} \subset \widetilde{\mathcal{I}}^u \cap \widetilde{B}(A, 3C_{10}(L_0 + C_3)), \tag{III.7.19}$$

as well as

$$\begin{aligned}&\text{for all } x \in A,\ \widetilde{A} \cap B(x, L_0) \neq \varnothing \text{ and every connected component}\\ &\quad \text{of } \{y \in G;\ \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0/2) \text{ with diameter at least}\\ &\quad L_0/4 \text{ is connected to } \widetilde{A} \text{ in } \{y \in G;\ \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0).\end{aligned} \tag{III.7.20}$$

and

$$\widetilde{\gamma}_z \geq -K/2 \text{ for all } z \in \widetilde{B}(\widetilde{A}, 2L_0 + C_3) \text{ and } \mathcal{B}^p_y = 1 \text{ for all } y \in B(\widetilde{A} \cap G, 2L_0). \tag{III.7.21}$$

Proof. For all $x_1 \sim x_2 \in A$, by (ii) of Definition III.7.4, there exists $y_i \in \mathcal{I}^{u/4} \cap \widetilde{B}(x_i, L_0)$ for each i. By (III.2.8), $d(x_1, y_2) < L_0 + C_3$ and by (iii) of Definition III.7.4 $\mathrm{cap}\big(C^{u/2}(y_i, L_0 + C_3)\big) \geq c_{13}(L_0 + C_3)^{3\nu/4}(u/8)^{\lfloor \gamma - 1 \rfloor}$ for each $i \in \{1, 2\}$. Therefore, by (III.7.17), $y_1 \overset{\wedge}{\longleftrightarrow} y_2$ in $\widehat{\mathcal{I}}^u \cap B_E(x_1, 3C_{10}(L_0 + C_3))$, and since each edge traversed by a trajectory of the random interlacement process is included in $\widetilde{\mathcal{I}}^u$, we also have that $y_1 \overset{\sim}{\longleftrightarrow} y_2$ in $\widetilde{\mathcal{I}}^u \cap \widetilde{B}(x_1, 3C_{10}(L_0 + C_3))$. We now define $\widetilde{A}$ as the union of the connected paths in $\widetilde{\mathcal{I}}^u \cap \widetilde{B}(x, 3C_{10}(L_0 + C_3))$ between y and y' for all $x \in A$ and $y, y' \in B(x, L_0 + C_3) \cap \mathcal{I}^{u/4}$, which is thus connected and it is clear that (III.7.19) holds.

For all $x \in A$, we clearly have $\widetilde{A} \cap B(x, L_0) \neq \varnothing$ by (III.7.19). Moreover, we have by (v) of Definition III.7.4 that every connected component of $\{y \in G;\ \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0/2)$ with diameter at least $L_0/2$ is connected to $\mathcal{I}^{u/4}$ in $\{y \in G;\ \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0)$, and thus is also connected to $\widetilde{A}$ in $\{y \in G;\ \varphi_y \geq -\sqrt{2u}\} \cap B(x, L_0)$, and we obtain (III.7.20). One infers from (i) and (vi) of Definition III.7.4 that (III.7.21) also hold. □

III.8 Construction of a giant cluster

We are now going to use the general renormalization scheme from Proposition III.7.1 to find a giant, or ubiquitous, cluster of (L_0, u, K, p)-good vertices, as defined in Definition III.7.4, or of $\widetilde{\mathcal{I}}^u$ with suitable properties. This comes in several steps. The first one is reached in Proposition III.8.3 below and yields under the mere assumptions (III.3.1) that long good (R-)paths, cf. Definition III.7.4, are very likely for suitable choices of the parameters. The second step is to prove the existence of a suitable infinite cluster $\widetilde{A}$ of $\widetilde{\mathcal{I}}^u$ and is presented in Lemma III.8.4, and the third step is to prove that this cluster is ubiquitous, see

Lemma III.8.7. This giant cluster $\widetilde{A}$ of $\widetilde{\mathcal{I}}^u$ verifies (III.7.21) and is the neighborhood of a cluster A of good vertices, for which (III.7.20) hold. It can be seen as precursor of the giant cluster of $E^{\geq h}$, $h > 0$, that we will construct in Section III.9, which will lead to (III.1.10) and (III.1.11) (for small $h > 0$). In a sense, the resulting estimates (III.8.15) and (III.8.23) provide a rough translation of the events appearing in (III.1.10) and (III.1.11) to the world of interlacements, and deliver directly (III.1.10) and (III.1.11) for any $h < 0$, see Corollary III.8.8. Apart from the quantitative bounds leading to Proposition III.8.3, these two estimates crucially rely on the additional geometric information provided by (WSI), on all aspects of Definition III.7.4 and on certain features of the renormalization scheme, in particular with regards to the desired ubiquity, gathered in Lemma III.8.6 below.

We continue in the framework of the previous section and recall in particular the scheme (III.7.1)–(III.7.3), the measure $\widetilde{\mathbb{Q}}^{u,p}$ from (III.5.34) and Definition III.7.4. We also keep our standing (but often implicit) assumption that G satisfies (III.3.1) and mention any other condition, such as (WSI), explicitly. Henceforth, we set

$$\bar{l} = 22c_{19}C_{10}, \quad l_0 = 8^{1/\nu} \vee C_{14}^{-\frac{1}{2\alpha}} \vee (8+4C_7)\bar{l}, \tag{III.8.1}$$

where

$$c_{19} \overset{\text{def.}}{=} 7(1+7c_5^{-1}) \text{ if G satisfies (WSI) and } c_{19} \overset{\text{def.}}{=} 7 \text{ otherwise.} \tag{III.8.2}$$

Note that $\bar{l}$ and l_0 satisfy the conditions appearing in (III.7.1). For all $L_0 \geq C_3$, we write $\mathcal{L}_0 = (L_0, \bar{l}, l_0)$ rather than $\mathcal{L}$ to insist on the choice (III.8.1). Thus $L_0 \geq C_3$ remains a free parameter at this point. We now define bad vertices at all scales L_n, $n \geq 0$, cf. (III.7.2). For all $L_0 \geq C_3$, $x \in \Lambda(L_0) = \Lambda_0^{\mathcal{L}_0}$, $u > 0$, $K > 0$ and $p \in (0,1)$, we introduce

$$\mathbf{C}_x^{L_0,K} = \bigcap_{y \in B(x, 20c_{19}C_{10}L_0)} C_y^{L_0,K}, \tag{III.8.3}$$

and similarly $\mathbf{D}_x^{L_0,u}$, $\widehat{\mathbf{D}}_x^{L_0,u}$, $\overline{\mathbf{D}}_x^{L_0,u}$ $\mathbf{E}_x^{L_0,u}$ and $\mathbf{F}_x^{L_0,p}$ by replacing $C_y^{L_0,K}$ with the relevant events $D_y^{L_0,u}$, $E_y^{L_0,u}$ and $F_y^{L_0,p}$ in Definition III.7.4, (ii)–(iv). The families $(\mathbf{C}^{L_0,K})^{\mathsf{c}} = \{(\mathbf{C}_x^{L_0,K})^{\mathsf{c}} : x \in \Lambda_0^{\mathcal{L}_0}\}$ and $(\mathbf{D}^{L_0,u})^{\mathsf{c}}$, $(\widehat{\mathbf{D}}_x^{L_0,u}))^{\mathsf{c}}$, $(\overline{\mathbf{D}}_x^{L_0,u}))^{\mathsf{c}}$ $(\mathbf{E}^{L_0,u})^{\mathsf{c}}$ and $(\mathbf{F}^{L_0,p})^{\mathsf{c}}$ are defined correspondingly. For $n \geq 0$ and $x \in \Lambda_n^{\mathcal{L}_0}$ (cf. (III.7.2)), we then say that the vertex x is $n-(L_0,u,K,p)$ *bad* if (recall (III.7.3))

$$\begin{aligned} &G_{x,n}^{\mathcal{L}_0}\big((\mathbf{C}^{L_0,K})^{\mathsf{c}}\big) \cup G_{x,n}^{\mathcal{L}_0}\big((\mathbf{D}^{L_0,K})^{\mathsf{c}}\big) \cup G_{x,n}^{\mathcal{L}_0}\big((\mathbf{E}^{L_0,u})^{\mathsf{c}}\big) \\ &\cup G_{x,n}^{\mathcal{L}_0}\big((\widehat{\mathbf{E}}^{L_0,u})^{\mathsf{c}}\big) \cup G_{x,n}^{\mathcal{L}_0}\big((\overline{\mathbf{E}}^{L_0,u})^{\mathsf{c}}\big) \cup G_{x,n}^{\mathcal{L}_0}\big((\mathbf{F}^{L_0,p})^{\mathsf{c}}\big) \end{aligned} \tag{III.8.4}$$

occurs (under $\widetilde{\mathbb{Q}}^{u,p}$), and x is $n-(L_0,u,K,p)$ *good* otherwise. In view of (III.7.18) and the first line of (III.7.3), an (L_0,u,K,p)-bad vertex in $\Lambda_0^{\mathcal{L}_0}$ is always a $0-(L_0,u,K,p)$ bad vertex, but not vice versa. A key to Proposition III.8.3, see (III.8.14) below, is to prove that the probability of having an $n-(L_0,u,K,p)$ bad vertex decays rapidly in n for a suitable range of parameters (L_0,u,K,p). This relies on individual bounds for each of the events in (III.8.4), which are the objects of Lemmas III.8.1 and III.8.2 as well as (III.8.10) below. Due to the presence of long-range correlations, the decoupling estimates from Proposition III.7.1 will be crucially needed.

Lemma III.8.1. *There exist constants $C_{19}<\infty$ and $C'_{19}<\infty$ such that for all $L_0\geq C_{19}$, $K\geq C'_{19}\sqrt{\log(L_0)}$, $n\in\{0,1,2,\dots\}$ and $x\in\Lambda_n^{\mathcal{L}_0}$, and all $u>0$, $p\in(0,1)$,*

$$\widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\mathbf{C}^{L_0,K})^{\mathsf{c}}\big)\big)\leq 2^{-2^n}. \tag{III.8.5}$$

Proof. In view of (III.8.3), Definition III.7.4 (i), and (III.8.1), if $L_0\geq C_3$, the event $(\mathbf{C}_x^{L_0,K})^{\mathsf{c}}$ is measurable with respect to the σ-algebra generated by $\widetilde{\gamma}_{|\widetilde{B}(x,\bar{l}L_0)}$, and $(\mathbf{C}_x^{L_0,K})^{\mathsf{c}}$ is of the form $\{\widetilde{\gamma}_{|\widetilde{B}(x,\bar{l}L_0)}+K\in B_x\}$, cf. (III.7.4), for a suitable decreasing subset B_x of $C(\widetilde{B}(x,\bar{l}L_0),\mathbb{R})$. With this observation, and since $\widetilde{\gamma}$ has the same law under $\widetilde{\mathbb{Q}}^{u,p}$ as $\widetilde{\Phi}$ under $\widetilde{\mathbb{P}}^G$, in order to show (III.8.5), it is enough by Proposition III.7.1 to prove that there exists C'_{19} such that

$$\text{for all } L_0\geq C_{19},\ K\geq C'_{19}\sqrt{\log(L_0)}-1 \text{ and } x\in\Lambda_0^{\mathcal{L}_0}:\ \widetilde{\mathbb{Q}}^{u,p}\left((\mathbf{C}_x^{L_0,K})^{\mathsf{c}}\right)<\frac{c_{17}}{l_0^{4\alpha}}, \tag{III.8.6}$$

where $C_{19}\geq C_3\vee 2$ is chosen so that the first inequality in (III.7.5) holds for all $L_0\geq C_{19}$, with l_0 as in (III.8.1) and $\varepsilon=1$. Conditionally on the field $\gamma=\widetilde{\gamma}_{|G}$, and for each edge $e=\{y,y'\}$, the process $(\widetilde{\gamma}_{y+te})_{t\in[0,\rho_{y,y'}]}$ on I_e has the same law as a Brownian bridge of length $\rho_{y,y'}=1/(2\lambda_{y,y'})$ (the length of I_e, cf. below (III.2.14)) between γ_y and $\gamma_{y'}$ of a Brownian motion with variance 2 at time 1, as defined in Section II.2. This fact has already appeared in the literature, see Section 2 of [57], Section 1 of [60] or Section 2 of [58] for example. We refer to Section II.2 for a proof of this result when $G=\mathbb{Z}^d$, which can be easily adapted to a general graph satisfying (III.3.1). Let us denote by $(W_t^{y,y'})_{t\in[0,\rho_{y,y'}]}$ defined as $W_t^{y,y'}=\widetilde{\gamma}_{y+te}-2\lambda_{y,y'}t\widetilde{\gamma}_{y'}-(1-2\lambda_{y,y'}t)\widetilde{\gamma}_y$ the Brownian bridge of length $\rho_{y,y'}$ between 0 and 0 of a Brownian motion with variance 2 at time 1 associated with

$(\widetilde{\gamma}_{y+te})_{t\in[0,\rho_{y,y'}]}$. For all $L \geq 1$, $K > 0$ and $x \in G$, we thus have

$$\widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{z\in\widetilde{B}(x,L)} \widetilde{\gamma}_z \geq \frac{K}{2}\Big) \leq \widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{y\in B(x,L)} \gamma_y \geq \frac{K}{4}\Big) + \sum_{\{y,y'\}\in B_E(x,L)} \widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{t\in[0,\rho_{y,y'}]} W_t^{y,y'} \geq \frac{K}{4}\Big). \tag{III.8.7}$$

We consider both terms in (III.8.7) separately. For all $y \in B(x,L)$, γ_y is a centered Gaussian variable with variance $g(y,y)$, thus by (V_α) and (G_β)

$$\widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{y\in B(x,L)} \gamma_y \geq \frac{K}{4}\Big) \leq \sum_{y\in B(x,L)} C\sqrt{\frac{g(y,y)}{K^2}} \exp\Big\{-\frac{K^2}{32g(y,y)}\Big\} \leq \frac{CL^\alpha}{K}\exp\{-cK^2\}.$$

The law of the maximum of a Brownian bridge is well-known, see for instance [13], Chapter IV.26, and so for all $y \sim y'$ in G, by (III.2.10),

$$\widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{t\in[0,\rho_{y,y'}]} W_t^{y,y'} \geq \frac{K}{4}\Big) = \exp\Big\{-\frac{K^2}{16\rho_{y,y'}}\Big\} \leq \exp\{-cK^2\},$$

where to obtain the inequality we took advantage of the fact that $\frac{1}{\rho_{y,y'}} = 2\lambda_{y,y'} \geq c$, cf. (III.2.10). Therefore, returning to (III.8.7), using (V_α), (III.2.10) and the fact that G has uniformly bounded degree, we obtain that for all $L \geq 1$ and $K \geq 1$, $\widetilde{\mathbb{Q}}^{u,p}(\sup_{z\in\widetilde{B}(x,L)} \widetilde{\gamma}_z \geq K) \leq CL^\alpha \exp\{-cK^2\}$. Choosing $L = \bar{l}L_0$ and using the symmetry of $\widetilde{\gamma}_\cdot$, we can finally bound for all $L_0 \geq C_{19}$ and $K \geq 1$,

$$\widetilde{\mathbb{Q}}^{u,p}\big((\mathbf{C}_x^{L_0,K})^{\mathsf{c}}\big) \leq \widetilde{\mathbb{Q}}^{u,p}\Big(\sup_{z\in\widetilde{B}(x,\bar{l}L_0)} \widetilde{\gamma}_z \geq \frac{K}{2}\Big) \leq CL_0^\alpha \exp\{-cK^2\},$$

from which (III.8.6) readily follows for a suitable choice of C'_{19}. □

The next lemma deals with the events involving the families $\mathbf{D}_x^{L_0,u}$, $\widehat{\mathbf{D}}_x^{L_0,u}$, $\overline{\mathbf{D}}_x^{L_0,u}$ and $\mathbf{E}_x^{L_0,u}$ in (III.8.4), which all involve the interlacement parameter $u > 0$. In the former case, this will bring into play the connectivity estimates from Section III.4 in order to initiate the decoupling.

Lemma III.8.2. *For all $u_0 > 0$, there exist constants c_{20} and C_{20} depending on u_0 such that for all $u \in (0,u_0)$, $L_0 \geq C_3$ with $L_0 u^{c_{20}} \geq C_{20}$, $n \in \{0,1,2,\dots\}$, $x \in \Lambda_n^{\mathcal{L}_0}$, and $p \in (0,1)$,*

$$\widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\mathbf{D}^{L_0,u})^{\mathsf{c}}\big)\big) \leq 2^{-2^n}, \quad \widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\widehat{\mathbf{D}}^{L_0,u})^{\mathsf{c}}\big)\big) \leq 2^{-2^n},$$
$$\widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\overline{\mathbf{D}}^{L_0,u})^{\mathsf{c}}\big)\big) \leq 2^{-2^n} \text{ and } \widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\mathbf{E}^{L_0,u})^{\mathsf{c}}\big)\big) \leq 2^{-2^n}. \tag{III.8.8}$$

Proof. We start with the estimate involving the family $(\mathbf{D}^{L_0,u})^{\mathsf{c}}$. By (III.3.10) and (III.3.11) we have

$$\widetilde{\mathbb{Q}}^{u,p}\left((D_x^{L_0,u/2})^{\mathsf{c}}\right) \leq \exp(-c_{11}(u/8)L_0^{\nu})$$

By (III.8.3) and a union bound, this readily implies that both (III.7.5), for l_0 as in (III.8.1) and $\varepsilon = 1$, and $\widetilde{\mathbb{Q}}^{u,p}\big((\mathbf{D}_x^{L_0,u/2})^{\mathsf{c}}\big) \leq c_{17}l_0^{-4\alpha}$ hold for all $u \in (0,u_0)$ and $L_0 \geq C_3 \vee Cu^{-c}$ (and all $x \in \Lambda_0^{\mathcal{L}_0}$). For all $L_0 \geq C_3$, $v > 0$ and $x \in G$ the events $(\mathbf{D}_x^{L_0,u})^{\mathsf{c}}$ are measurable with respect to the σ-algebra generated by $\widetilde{\ell}_{\widetilde{B}(x,\bar{l}L_0),u}$ and decreasing in u. Therefore, Proposition III.7.1 with $\varepsilon = 1$ applies and (III.7.7) yields the first part of (III.8.8).

Let us now turn to the events $(\widehat{\mathbf{D}}^{L_0,u})^{\mathsf{c}}$. For all $L_0 > 0$, $v \geq u/8$ and $x \in G$, we say that the event $\widehat{D}_x^{L_0,v,u}$ occurs if and only if $\mathrm{cap}\big(C^{u/4+v}(y,2(L_0+C_3))\big) \geq c_{13}(L_0+C_3)^{3\nu/4}(u/8)^{\lfloor\gamma-1\rfloor}$ for all $y \in \mathcal{I}^{u/4} \cap B(x,L_0+C_3)$, and we define $\widehat{\mathbf{D}}_x^{L_0,v,u}$ similarly as in (III.8.3), replacing $C_y^{L_0,u}$ by $\widehat{D}_y^{L_0,v,u}$. Consider a fixed value of $u_0 > 0$. Note that the law of $\widehat{\mathcal{I}}^{u/4+v} \setminus \widehat{\mathcal{I}}^{u/4}$ conditionally on $\widehat{\mathcal{I}}^{u/4}$ is the same as the law of $\widehat{\mathcal{I}}^{v}$. By (III.2.8) the set $C^{u/4}(y,L_0+C_3)$ has diameter at least L_0 for all $y \in \mathcal{I}^{u/4}$, and thus by (III.3.10) and (III.3.14), we have for all $v \geq u/8$ and $y \in \mathcal{I}^{u/4} \cap B(x,L_0+C_3)$

$$\widetilde{\mathbb{Q}}^{u,p}\big((\widehat{\mathcal{I}}^{u/4+v} \setminus \widehat{\mathcal{I}}^{u/4}) \cap C^{u/4}(y,L_0+C_3) = \varnothing \,|\, \widehat{\mathcal{I}}^{u/4}\big) \leq \exp\big(-cuL_0^{\frac{\nu}{2}\wedge 1}\big).$$

Moreover, if on the other hand $(\widehat{\mathcal{I}}^{u/4+v} \setminus \widehat{\mathcal{I}}^{u/4}) \cap C^{u/4}(y,L_0+C_3) \neq \varnothing$ for some $y \in \mathcal{I}^{u/4} \cap B(x,L_0+C_3)$, then $C^{u/4+v}(y,2(L_0+C_3))$ contains the cluster of edges in $B(y',L_0+C_3)$ traversed by at least one of the trajectories of $\widehat{\mathcal{I}}^{u/4+v} \setminus \widehat{\mathcal{I}}^{u/4}$ for some $y' \in (\mathcal{I}^{u/4+v} \setminus \mathcal{I}^{u/4}) \cap B(x,2(L_0+C_3))$. By Proposition III.4.7 applied to $\widehat{\mathcal{I}}^{u/4+v} \setminus \widehat{\mathcal{I}}^{u/4}$, (V_α) and a union bound, we thus have for all $u < u_0$ and $v \in [u/8,u_0]$

$$\widetilde{\mathbb{Q}}^{u,p}\left((\widehat{D}_x^{L_0,v,u})^{\mathsf{c}} \,\middle|\, \widehat{\mathcal{I}}^{u/4}\right) \leq C(u_0)(L_0+C_3)^{\alpha}\Big(\exp\big(-c(u_0)u(L_0+C_3)^{C_{13}}\big) + \exp\big(-cuL_0^{\frac{\nu}{2}\wedge 1}\big)\Big).$$

Moreover, conditionally on $\widehat{\mathcal{I}}^{u/4}$, the events $(\widehat{\mathbf{D}}_x^{L_0,v,u})^{\mathsf{c}}$ are decreasing in v, i.e., there exists a decreasing subset B_x of $C(\widetilde{B}(x,\bar{l}L_0),\mathbb{R})$ (depending on L_0 and $\widehat{\mathcal{I}}^{u/4}$) such that $(\widehat{\mathbf{D}}_x^{L_0,v,u})^{\mathsf{c}}$ has the same law as $B_x^{I,v}$ for all $u > 0$ and $v \geq u/8$, see (III.7.4). By a union bound, we have that $\widetilde{\mathbb{Q}}^{u,p}\big((\widehat{\mathbf{D}}_x^{L_0,u/8,u})^{\mathsf{c}}\big) \leq c_{17}l_0^{-4\alpha}$ and the second part of (III.7.5) with l_0 as in (III.8.1) and $\varepsilon = 1$ simultaneously hold for all $u \in (0,u_0)$, and $L_0 \geq C_3 \vee C(u_0)u^{-c(u_0)}$, and by another application of Proposition III.7.1 with $\varepsilon = 1$ we obtain that for all $u \in (0,u_0)$

$$\widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\widehat{\mathbf{D}}^{L_0,u/4,u})^{\mathsf{c}}\big) \,|\, \widehat{\mathcal{I}}^{u/4}\big) \leq 2^{-2^n}.$$

Since $\widehat{\mathbf{D}}^{L_0,u/4,u}=\widehat{\mathbf{D}}^{L_0,u}$, we obtain directly the second part of (III.8.8) by integrating over $\widehat{\mathcal{I}}^v$.

We now consider the events $(\overline{\mathbf{D}}^{L_0,u})^{\mathsf{c}}$. For all $L_0>0$, $u>0$, $v>0$ and $x\in G$, we say that the event $\overline{D}_x^{L_0,v,u}$ occurs if and only if

$$\bigcap_{\substack{y,y'\in\mathcal{I}^{u/2}\cap B(x,L_0+C_3);\\ \mathrm{cap}\left(C^{u/2}(y,2(L_0+C_3))\right)\geq c_{13}(L_0+C_3)^{3\nu/4}(u/8)^{\lfloor\gamma-1\rfloor},\\ \mathrm{cap}\left(C^{u/2}(y',2(L_0+C_3))\right)\geq c_{13}(L_0+C_3)^{3\nu/4}(u/8)^{\lfloor\gamma-1\rfloor}}}\big\{y\stackrel{\wedge}{\longleftrightarrow}y' \text{ in } \widehat{\mathcal{I}}^{u/2+v}\cap B_E(x,3C_{10}(L_0+C_3))\big\},$$

where $C^{u/2}(z,L_0)$ is defined below Lemma III.4.3, and we define $\overline{\mathbf{D}}_x^{L_0,v,u}$ similarly as in (III.8.3), replacing $C_y^{L_0,u}$ by $\overline{D}_y^{L_0,v,u}$. Note that $C^{u/2}(y,2(L_0+C_3))\subset B(x,3(L_0+C_3))$ for all $y\in B(x,L_0+C_3)$. By (V_α), Lemma III.4.3 and a union bound, we have for all $u\in(0,u_0)$, $v\in[u/4,u/2]$, $x\in G$ and $L_0\geq C_3$,

$$\widetilde{\mathbb{Q}}^{u,p}\left((\overline{D}_x^{L_0,v,u})^{\mathsf{c}}\,|\,\widehat{\mathcal{I}}^{u/2}\right)\leq C(L_0+C_3)^\alpha\exp\big(-cu^{2\lfloor\gamma-1\rfloor+1}(L_0+C_3)^{\nu/2}\big).$$

Conditionally on $\widehat{\mathcal{I}}^{u/2}$, the events $(\overline{\mathbf{D}}_x^{L_0,v,u})^{\mathsf{c}}$ are decreasing in v, and similarly as before we can apply Proposition III.7.1 with $\varepsilon=1$ to obtain the third bound of (III.8.8) for all $u\in(u,u_0)$ and $L_0\geq C_3\vee C(u_0)u^{-c(u_0)}$ since $\overline{\mathbf{D}}_x^{L_0,u/2,u}=\overline{\mathbf{D}}_x^{L_0,u}$.

Regarding $(\mathbf{E}^{L_0,u})^{\mathsf{c}}$, under $\widetilde{\mathbb{Q}}^{u,p}$, note that by (III.5.20), the clusters of $\{y\in G;\,\varphi_y>-\sqrt{2u}\}$ are the same as the clusters of $\{y\in G;\,y\in\mathcal{C}_u^\infty$ or $\gamma_y>0\}$. Therefore if the cluster $\mathcal{U}_x$ of x in $\{y\in G;\,\varphi_y>-\sqrt{2u}\}\cap B(x,L_0/2)$ has diameter at least $L_0/4$ and is not connected to $\mathcal{I}^{u/4}$ in $\{y\in G;\,\varphi_y>-\sqrt{2u}\}\cap B(x,L_0)$, then either $\mathcal{U}_x$ is a cluster of $\{y\in G;\,y\in\mathcal{C}_u^\infty\setminus\mathcal{C}_{u/4}^\infty$ or $\gamma_y>0\}\cap B(x,L_0/2)$ of diameter at least $L_0/4$, or $\mathcal{U}_x$ contains a vertex y in $\mathcal{C}_{u/4}^\infty\cap B(x,L_0/2)$ not connected to $\mathcal{I}^{u/4}$ in $\{y\in G;\,\varphi_y>-\sqrt{2u}\}\cap B(x,L_0)$, and then by (III.5.19) and (III.5.20), y is in a connected component of $\{z\in\widetilde{G};\,|\widetilde{\gamma}_z|>0\}\cap\widetilde{B}(x,L_0)$ of diameter $\geq L_0/4$ not intersecting $\mathcal{I}^{u/4}$. Therefore, defining the event

$$E_x^{L_0,v,u}=\left\{\begin{array}{c}\text{all the connected components of}\\ \{y\in G;\,y\in\mathcal{C}_u^\infty\setminus\mathcal{C}_{u/4}^\infty \text{ or } \gamma_y>0\}\cap B(x,L_0/2)\\ \text{or of } \{z\in\widetilde{G};\,|\widetilde{\gamma}_z|>0\}\cap\widetilde{B}(x,L_0)\\ \text{with diameter } \geq L_0/4 \text{ intersect } \mathcal{I}^v\end{array}\right\}$$

for all $v\leq u/4$, we have $E_x^{L_0,v,u}\subset E_x^{L_0,u}$ by Definition III.7.4 (v). We also define $\mathbf{E}_x^{L_0,v,u}$ similarly as in (III.8.3), replacing $C_y^{L_0,u}$ by $E_y^{L_0,v,u}$. Let $\widetilde{\mathcal{I}}_2^{3u/4}=\widetilde{\mathcal{I}}^u\setminus\widetilde{\mathcal{I}}^{u/4}$, then $\mathcal{C}_u^\infty\setminus\mathcal{C}_{u/4}^\infty$ is $\widetilde{\mathcal{I}}_2^{3u/4}$ measurable. Moreover $\widetilde{\gamma}$ is independent from the random interlacement set $\mathcal{I}^{u/4}$, see (III.5.34), $\widetilde{\mathcal{I}}_2^{3u/4}$ is also independent from $\mathcal{I}^{u/4}$, and there are at most $2|B(x,L_0)|$ connected components of either $(\{y\in\mathcal{C}_u^\infty\setminus\mathcal{C}_{u/4}^\infty\}\cup\{y\in G;\,\gamma_y>0\})\cap B(x,L_0/2)$ or $\{z\in\widetilde{G};\,|\widetilde{\gamma}_z|>0\}\cap\widetilde{B}(x,L_0)$ with diameter at least $\frac{L_0}{4}$. Thus, by (V_α), Lemma III.3.2, and (III.3.10), $\widetilde{\mathbb{Q}}^{u,p}$-a.s., for all $u > 0$, $v \in [u/8, u/4]$ and $p \in (0,1)$,

$$\widetilde{\mathbb{Q}}^{u,p}\Big((E_x^{L_0,v,u})^{\mathsf{c}} \,\Big|\, \widetilde{\gamma}, \widetilde{\mathcal{I}}_2^{3u/4}\Big) \le 2C_1 L_0^{\alpha} \exp\big\{-cuL_0^{\frac{\nu}{2}\wedge 1}\big\}. \tag{III.8.9}$$

The fourth bound in (III.8.8) is then obtained by virtue of another application of Proposition III.7.1 under the conditional measure $\widetilde{\mathbb{Q}}^{u,p}(\cdot \,|\, \widetilde{\gamma}, \widetilde{\mathcal{I}}_2^{3u/4})$, using (III.8.9) and a union bound to deduce that $\widetilde{\mathbb{Q}}^{p}((\mathbf{E}_x^{L_0,u/8,u})^{\mathsf{c}} \,|\, \widetilde{\gamma}, \widetilde{\mathcal{I}}_2^{3u/4}) \le c_{17} l_0^{-4\alpha}$; the second part of (III.7.5) with l_0 as in (III.8.1) and $\varepsilon = 1$ simultaneously holds true whenever $L_0 u^c \ge C'$. Noting that, for all $v \le u/4$, conditionally on $\widetilde{\gamma}$ and $\widetilde{\mathcal{I}}_2^{3u/4}$, $(E_x^{L_0,v,u})^{\mathsf{c}}$ is a decreasing $\sigma(\widetilde{\ell}_{B(x,\bar{l}L_0),v})$-measurable event in v, Proposition III.7.1 yields an upper bound similar to (III.8.8) but for $G_{x,n}^{\mathcal{L}_0}\big((\mathbf{E}_x^{L_0,u/4,u})^{\mathsf{c}}\big)$ under $\widetilde{\mathbb{Q}}^{u,p}(\cdot \,|\, \widetilde{\gamma}, \widetilde{\mathcal{I}}_2^{3u/4})$. The desired bound (III.8.8) then follows by integrating over $\widetilde{\gamma}$ and $\widetilde{\mathcal{I}}_2^{3u/4}$ since $G_{x,n}^{\mathcal{L}_0}\big((\mathbf{E}_x^{L_0,u})^{\mathsf{c}}\big) \subset G_{x,n}^{\mathcal{L}_0}\big((\mathbf{E}_x^{L_0,u/4,u})^{\mathsf{c}}\big)$. □

Finally for the events involving the family $(\mathbf{F}^{L_0,p})^{\mathsf{c}}$ in (III.8.4), by a similar reasoning as in Lemma 4.7 of [74] and using (V_α), there exists a constant C_{21} such that for all $p \in (0,1)$ such that $p \ge \exp\{-C_{21} L_0^{-\alpha}\}$, all $u > 0$, $n \ge 0$ and $x \in \Lambda_n^{\mathcal{L}_0}$,

$$\widetilde{\mathbb{Q}}^{u,p}\big(G_{x,n}^{\mathcal{L}_0}\big((\mathbf{F}^{L_0,p})^{\mathsf{c}}\big)\big) \le 2^{-2^n}. \tag{III.8.10}$$

For all $u_0 > 0$ and $R \ge 1$ we define

$$L_0(u) = R \vee C_3 \vee C_{19} \vee C_{20} u^{-c_{20}}, \tag{III.8.11}$$

where we keep the dependence of various constants and of $L_0(u)$ on u_0 and R implicit. Furthermore, we choose constants C_{22} and c_{22} such that $\sqrt{\log(C_{22}u^{-c_{22}})} \ge C_{19}' \sqrt{\log(l_0 L_0(u))}$ for all $u \in (0, u_0)$, and constants C_{23} and c_{23} such that $1 - C_{23} u^{c_{23}} \ge \exp\big\{-C_{21}(l_0 L_0(u))^{-\alpha}\big\}$ for all $u \in (0, u_0)$, which can both be achieved on account of (III.8.11). Then, by (III.8.4), Lemmas III.8.1 and III.8.2 and (III.8.10), for all $n \in \mathbb{N}$ and $u \in (0, u_0)$

$$\left.\begin{array}{c} L_0 \in [L_0(u), l_0 L_0(u)], \\ K \ge \sqrt{\log(C_{22}u^{-c_{22}})} \\ \text{and } p \ge 1 - C_{23}u^{c_{23}} \end{array}\right\} \quad \text{imply} \quad \widetilde{\mathbb{Q}}^{u,p}(x \text{ is } n-(L_0,u,K,p) \text{ bad}) \le 6 \times 2^{-2^n}. \tag{III.8.12}$$

Relying on (III.8.12), we now deduce a strong bound on the probability to see long R-paths of (L_0, u, K, p)-bad vertices (see above (WSI) for a definition of R-paths). We emphasize that the following result holds for all graphs satisfying (III.3.1). In particular, (WSI) is not required for (III.8.13) below to hold.

Proposition III.8.3. *For G satisfying* (III.3.1) *and each $u_0 > 0$, there exist constants $c(u_0), C(u_0) \in (0,\infty)$ such that for all $R \geq 1$, $x \in G$, $u \in (0,u_0)$, $K > 0$ with $K \geq \sqrt{\log(C_{22}u^{-c_{22}})}$, $p \in (0,1)$ with $p \geq 1 - C_{23}u^{c_{23}}$, and $N > 0$,*

$$\widetilde{\mathbb{Q}}^{u,p}\Big(\begin{array}{c} \textit{there exists an } R\textit{-path of } (L_0,u,K,p) \\ \textit{-bad vertices from } x \textit{ to } B(x,N)^{\mathsf{c}} \end{array} \Big) \leq C(u_0) \exp\left\{ -(N/L_0(u))^{c(u_0)} \right\}. \tag{III.8.13}$$

Proof. We will show by induction that for all $n \in \{0,1,2,\dots\}$, $L_0 \geq R \vee C_3$, and $x \in \Lambda_n^{\mathcal{L}_0}$,

$$\left\{ \begin{array}{l} \text{there exists an } R\text{-path of } (L_0,u,K,p)\text{-bad} \\ \text{vertices from } B(x,L_n) \text{ to } B(x,\bar{l}L_n)^{\mathsf{c}} \end{array} \right\} \subset \{x \text{ is } n-(L_0,u,K,p) \text{ bad}\}. \tag{III.8.14}$$

If (III.8.14) holds, then Proposition III.8.3 directly follows from (III.8.11) and (III.8.12) by taking $n \in \mathbb{N}$ and $L_0 \in [L_0(u), l_0L_0(u))$ such that $\bar{l}l_0^nL_0 = N$. Let us fix some $L_0 \geq R \vee C_3$. For $n = 0$, if there exists a bad vertex in $B(x,L_0)$, then, see below (III.8.4), x is $0-(L_0,u,K,p)$ bad. Suppose now that (III.8.14) holds at level $n-1$ for all $x \in \Lambda_{n-1}^{\mathcal{L}_0}$ for some $n \geq 1$. Then, since $L_0 \geq R \vee C_3$ and $\bar{l} \geq 22$, if there exists an R-path π of (L_0,u,K,p)-bad vertices from $B(x,L_n)$ to $B(x,\bar{l}L_n)^{\mathsf{c}}$, one can find for each $k \in \{1,\dots,7\}$ a vertex

$$y_k \in \pi \cap \big(B(x,3kL_n) \setminus B(x,(3k-1)L_n)\big).$$

Using (III.6.1), one then picks for each $k \in \{1,\dots,7\}$ a vertex $z_k \in \Lambda_{n-1}^{\mathcal{L}_0}$ such that $y_k \in B(z_k,L_{n-1})$. One then easily checks that with the choice of $\bar{l}$ and l_0 in (III.8.1), for all $k \neq k'$ in $\{1,\dots,7\}$, $d(z_k,z_{k'}) \geq L_n$, and $B(z_k,\bar{l}L_{n-1}) \subset B(x,\bar{l}L_n) \setminus B(x,L_n)$. In particular, for each $k \in \{1,\dots,7\}$, π yields an R-path of (L_0,u,K,p)-bad vertices from $B(z_k,L_{n-1})$ to $B(z_k,\bar{l}L_{n-1})^{\mathsf{c}}$, and the induction hypothesis implies that z_k is $(n-1)-(L_0,u,K,p)$ bad. Among these seven $(n-1)-(L_0,u,K,p)$ bad vertices, there exist $i \neq j \in \{1,\dots,7\}$ and $A \in \{(\mathbf{C}^{L_0,K})^{\mathsf{c}}, (\mathbf{D}^{L_0,K})^{\mathsf{c}}, (\widehat{\mathbf{D}}^{L_0,K})^{\mathsf{c}}, (\overline{\mathbf{D}}^{L_0,K})^{\mathsf{c}}, (\mathbf{E}^{L_0,u})^{\mathsf{c}}, (\mathbf{F}^{L_0,p})^{\mathsf{c}}\}$ such that $G_{z_i,n-1}^{\mathcal{L}_0}(A)$ and $G_{z_j,n-1}^{\mathcal{L}_0}(A)$ both occur, whence z_i and z_j appear in the union for $G_{x,n}^{\mathcal{L}_0}(A)$, see (III.7.3). By definition (III.8.4), x is $n-(L_0,u,K,p)$ bad and (III.8.14) follows. □

Using the additional condition (WSI), Proposition III.8.3 together with Lemma III.7.6 can be used to show the existence of a certain set $\widetilde{A}$, see Lemma III.8.4 below, from which the prevalence of the infinite cluster of $E^{\geq h}$, $h > 0$ small, will eventually be deduced. The bound obtained in (III.8.15) will later lead to (III.1.10).

Lemma III.8.4. *Assume G satisfies* (WSI) *(in addition to* (III.3.1)*), and let $R = R_0$ as in* (WSI)*. Furthermore, let $u_0 > 0$, $u \in (0, u_0)$, $K > 0$ with $K \geq \sqrt{\log(C_{22}u^{-c_{22}})}$, and $p \in (0,1)$ with $p \geq 1 - C_{23}u^{c_{23}}$. Then $\widetilde{\mathbb{Q}}^{u,p}$-a.s. there exists $L_0 \geq 1$ and a connected and unbounded set $\widetilde{A}^u_\infty \subset \widetilde{\mathcal{I}}^u$ such that* (III.7.21) *holds and there exist constants $c > 0$ and $C < \infty$ depending on u and u_0 such that for all $x_0 \in G$ and $L > 0$,*

$$\widetilde{\mathbb{Q}}^{u,p}\big(\widetilde{A}^u_\infty \cap \widetilde{B}(x_0, L) = \varnothing\big) \leq C \exp\{-L^c\}. \tag{III.8.15}$$

Proof. Fix a vertex $x_0 \in G$. By (WSI), there exists $R_0 \geq 1$ such that, for all finite connected subsets A of G with $x_0 \in A$ and $\delta(A) \geq C_3$, noting that $d(x, x_0) \leq \delta(A) + C_3 \leq 2\delta(A)$ for all $x \in \partial_{ext}A$ by (III.2.8),

$$\text{for all } x \in \partial_{ext}A,\ \exists \text{ an } R_0\text{-path from } x \text{ to } B(x, c_6 d(x, x_0)/2)^{\mathsf{c}} \text{ in } \partial_{ext}A. \tag{III.8.16}$$

It is then enough to prove that for all $u \in (0, u_0)$, $K \geq \sqrt{\log(C_{22}u^{-c_{22}})}$ and $p \geq 1 - C_{23}u^{c_{23}}$, the probability under $\widetilde{\mathbb{Q}}^{u,p}$ of the event

$$\left\{ \begin{array}{c} \text{there does not exist an unbounded nearest neighbor path in } G \\ \text{of } (L_0, u, K, p)\text{-good vertices starting in } B(x_0, L) \end{array} \right\} \tag{III.8.17}$$

has stretched-exponential decay in L for some $L_0 \geq 1$ (with constants depending on u and u_0). Indeed, if (III.8.17) does not occur, then by Lemma III.7.6 there exists an unbounded connected component $\widetilde{A}^u_\infty \subset \widetilde{\mathcal{I}}^u$ intersecting $\widetilde{B}(x_0, L)$ such that (III.7.21) holds; therefore, the bound (III.8.15) follows.

Thus, in order to establish the desired decay, assume that (III.8.17) occurs for some $u \in (0, u_0)$, $K \geq \sqrt{\log(C_{22}u^{-c_{22}})}$, $p \geq 1 - C_{23}u^{c_{23}}$, a positive integer L and L_0 as in (III.8.11). We may assume that $L \geq C_3$. We now use Proposition III.8.3 and a contour argument involving (III.8.16) to bound its probability. Note that the assumptions of Proposition III.8.3 on the set of parameters (L_0, u, K, p) are met for all $u \in (0, u_0)$ by our choice of constants. Define

$$A_L = B(x_0, L) \cup \big\{x \in G;\, x \leftrightarrow B(x_0, L) \text{ in the set of } (L_0, u, K, p)\text{-good vertices}\big\},$$

which is the set of vertices in G either in, or connected to $B(x_0, L)$ by a nearest neighbor path of (L_0, u, K, p)-good vertices in G. Since (III.8.17) occurs, A_L is finite. It is also connected, and $\delta(A_L) \geq C_3$. Hence, since every vertex in $\partial_{ext}A_L$ is (L_0, u, K, p)-bad, by (III.8.16) there exists $x \in \partial_{ext}A_L$ and an R_0-path of (L_0, u, K, p)-bad vertices from x to $B(x, c_6 d(x, x_0)/2)^{\mathsf{c}}$. Let $N = \lfloor d(x, x_0) \rfloor$, then $N \geq L$, and thus by a union bound the probability that the event (III.8.17) occurs is smaller than

$$\sum_{N=L}^{\infty} \sum_{x \in B(x_0, N+1)} \widetilde{\mathbb{Q}}^{u,p} \left(\begin{array}{c} \text{there exists an } R_0\text{-path of } (L_0, u, K, p) \\ \text{-bad vertices from } x \text{ to } B(x, cN)^{\mathsf{c}} \end{array} \right),$$

which has stretched-exponential decay in L by (V_α), (III.2.10) and Proposition III.8.3. □

Remark III.8.5. One can replace (WSI) by the following (weaker) condition $(\widetilde{\text{WSI}})$ and still retain a statement similar to Lemma III.8.4. This is of interest in order to determine how little space (in G) one can afford to use in order for various sets, in particular $\mathcal{V}^u$ at small $u > 0$ in Theorem III.1.2, to retain an unbounded component; see Theorem III.9.3 and Remark III.9.4, 5) below. We first introduce $(\widetilde{\text{WSI}})$. Suppose that there exists an infinite connected subgraph G_p of G, $\zeta > 0$, $R_0 \geq 1$, a vertex $x_0 \in G_p$ and $c_{24} > 0$ such that

$$\begin{aligned} &\text{for all finite connected } A \subset G_p \text{ with } x_0 \in A, \text{ there exists } x \in (\partial_{ext} A) \cap G_p \\ &\text{and an } R_0\text{-path from } x \text{ to } B(x, c_{24} d(x,x_0)^\zeta)^{\mathsf{c}} \text{ in } (\partial_{ext} A) \cap G_p, \end{aligned} \tag{$\widetilde{\text{WSI}}$}$$

i.e., all the vertices of this path are in $(\partial_{ext} A) \cap G_p$. It is easy to see that (WSI) implies $(\widetilde{\text{WSI}})$ with $\zeta = 1$. Suppose now that instead of (WSI), condition $(\widetilde{\text{WSI}})$ hold for some subgraph G_p of G. Then the conclusions of Lemma III.8.4 leading to (III.7.19) still hold and the set $\widetilde{A}^u_\infty$ thereby constructed satisfies $\widetilde{A}^u_\infty \subset \widetilde{B}(G_p, 3C_{10}(L_0(u) + C_3))$. To see this, one replaces (III.8.16) by the following consequence of $(\widetilde{\text{WSI}})$: there exists $R_0 \geq 1$, $x_0 \in G_p$ and $c > 0$ such that for all finite connected subsets A of G_p with $x_0 \in A$,

$$\exists\, x \in (\partial_{ext} A) \cap G_p \text{ and a } R_0\text{-path from } x \text{ to } B(x, cd(x,x_0)^\zeta)^{\mathsf{c}} \text{ in } (\partial_{ext} A) \cap G_p. \tag{III.8.16'}$$

One then argues as above, with small modifications due to (III.8.16'), whence, in particular, the set A_L needs to be replaced by $A_L(G_p) \stackrel{\text{def.}}{=} \big(B(x_0, L) \cap G_p\big) \cup \{x \in G_p;\, x \leftrightarrow B(x_0, L) \cap G_p$ in the set of (L_0, u, K, p)-good vertices in $G_p\}$, so that $A_L = A_L(G)$. □

The bound (III.8.15) will be useful to prove that (III.1.10) holds, and we seek a similar result which roughly translates (III.1.11) to the world of random interlacements. This appears in Lemma III.8.7 below. Its proof rests on the following technical result, which is a feature of the renormalization scheme.

Lemma III.8.6. *Assume G satisfies* (WSI)*, and recall the definition of c_{19} from* (III.8.2)*. For any $L_0 \geq C_3$, $K > 0$, $u > 0$ and $n \in \{0, 1, 2, \dots\}$, if there exists a vertex $x \in \Lambda_n^{\mathcal{L}_0}$ which is $n - (L_0, u, K, p)$ good, then every two connected components of $B(x, 20c_{19}L_n)$ with diameter at least $c_{19}L_n$ are connected via a path of (L_0, u, K, p)-good vertices in $B(x, 30c_{19}C_{10}L_n)$.*

Proof. We use induction on n. For $n = 0$, if x is $0 - (L_0, u, K, p)$ good, then in view of (III.8.3), (III.8.4) and Definition III.7.4, every path in $B(x, 20c_{19}L_0)$

is a path of (L_0,u,K,p)-good vertices and all the vertices in $B(x,20c_{19}C_{10}L_0)$ are (L_0,u,K,p)-good, so the result follows directly from (III.3.4). Let us now assume that the conclusion of the lemma holds at level $n-1$ for some $n\geq 1$ and let

$$x \text{ be an } n-(L_0,u,K,p) \text{ good vertex.} \tag{III.8.18}$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be any two connected components of $B(x,20c_{19}L_n)$ with diameter at least $c_{19}L_n$. We are first going to show that

$$\begin{aligned}&\mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ are linked via } (n-1)-(L_0,u,K,p)\\ &\text{-good vertices in } B(x,22c_{19}C_{10}L_n),\end{aligned} \tag{III.8.19}$$

by which we mean that there exists a subset S of $\Lambda_{n-1}^{\mathcal{L}_0}\cap B(x,22c_{19}C_{10}L_n)$ containing only $(n-1)-(L_0,u,K,p)$ good vertices and such that $\bigcup_{y\in S}B(y,L_{n-1})$ contains a connected component intersecting both $\mathcal{U}_1$ and $\mathcal{U}_2$. To see that (III.8.19) holds, for each $i\in\{1,2\}$ choose seven connected subsets $(\mathcal{U}_i^k)_{k\in\{1,\dots,7\}}$ of $\mathcal{U}_i$ such that for all $k\neq k'\in\{1,\dots,7\}$,

$$d(\mathcal{U}_i^k,\mathcal{U}_i^{k'})\geq L_n+2L_{n-1} \quad \text{and} \quad \delta(\mathcal{U}_i^k)\geq 7L_nc_5^{-1};$$

such a choice is possible since $L_0\geq C_3$, $l_0\geq\bar{l}\geq 22$ and $c_{19}=7(1+7c_5^{-1})$. If for each $k\in\{1,\dots,7\}$ there exists an $(n-1)-(L_0,u,K,p)$ bad vertex $y_k\in\Lambda_{n-1}^{\mathcal{L}_0}$ such that $B(y_k,L_{n-1})\cap\mathcal{U}_i^k\neq\varnothing$, then there are at least seven $(n-1)-(L_0,u,K,p)$ bad vertices in $B(x,20c_{19}L_n+L_{n-1})\subset B(x,\bar{l}L_n)$ with mutual distance at least L_n, which contradicts (III.8.18) by (III.8.4) and the definition of the renormalization scheme, see (III.7.3). For each $i\in\{1,2\}$ we can thus find k_i such that each $y\in\Lambda_{n-1}^{\mathcal{L}_0}$ with $B(y,L_{n-1})\cap\mathcal{U}_i^{k_i}\neq\varnothing$ is $(n-1)-(L_0,u,K,p)$ good. Recalling that $\mathcal{U}_i^{k_i}$ is connected, we can define for each $i\in\{1,2\}$ the set $\mathrm{comp}_{n-1}(\mathcal{U}_i^{k_i})\subset G$ as the connected component in

$$\bigcup_{\substack{y\in\Lambda_{n-1}^{\mathcal{L}_0}\cap B(x,22c_{19}C_{10}L_n),\\ y\text{ is }(n-1)-(L_0,u,K,p)\text{ good}}} B(y,L_{n-1}) \tag{III.8.20}$$

containing $\mathcal{U}_i^{k_i}$.

The claim (III.8.19) amounts to showing that $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})=\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$. Suppose on the contrary that $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ and $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ are not equal. By (III.3.4), there is a nearest neighbor path $(x_1,\dots,x_p)$ in $B(x,20c_{19}C_{10}L_n)$ connecting $\mathcal{U}_1^{k_1}$ and $\mathcal{U}_2^{k_2}$. Recalling the notion of external boundary from (III.2.1), since $x_1\in\mathcal{U}_1^{k_1}$, either there exists $m\in\{1,\dots,p\}$ such that $x_m\in\partial_{ext}\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, or every unbounded nearest neighbor path beginning in x_p intersects $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, and likewise for $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$. If every unbounded path beginning in x_p hits $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ and every unbounded path beginning in x_1 hits $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$, then by connectivity every unbounded path beginning in $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ hits $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ and every unbounded path beginning in $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ hits $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, which is impossible since $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \neq \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ (indeed, unless $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) = \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$, these conditions would require any such path to 'oscillate' between $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ and $\mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$ infinitely often and thus it remains bounded). Therefore, we may assume that $\partial_{ext}\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) \cap B(x, 20c_{19}C_{10}L_n) \neq \varnothing$ (otherwise exchange the roles of $\mathcal{U}_1$ and $\mathcal{U}_2$), and by (WSI), there exists an R_0-path in $\partial_{ext}\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$ of diameter between $7L_n$ and $8L_n$ beginning in $B(x, 20c_{19}C_{10}L_n)$. By definition of $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1})$, see (III.8.20), every vertex of this R_0-path is contained in $B(y, L_{n-1})$ for some $(n-1)-(L_0,u,K,p)$ bad vertex y in $\Lambda_{n-1}^{\mathcal{L}_0} \cap B(x, (20c_{19}C_{10} + 8 + l_0^{-1})L_n) \subset B(x, 22c_{19}L_n)$, and, since $L_0 \geq C_3$ and $l_0 \geq \bar{l} \geq 22$, there are at least 7 $(n-1)-(L_0,u,K,p)$ bad vertices in $B(x, 22c_{19}C_{10}L_n) = B(x, \bar{l}L_n)$ with mutual distance at least L_n. By (III.7.3) and (III.8.4), x is $n-(L_0,u,K,p)$ bad, which is a contradiction.

Therefore, we have $\mathrm{comp}_{n-1}(\mathcal{U}_1^{k_1}) = \mathrm{comp}_{n-1}(\mathcal{U}_2^{k_2})$, i.e., (III.8.19) holds. Thus, by (III.6.1) there exists $y_0 \in \mathcal{U}_1^{k_1}$, $y_{m+1} \in \mathcal{U}_2^{k_2}$ and a sequence of vertices $y_1, \ldots, y_m \in \Lambda_{n-1}^{\mathcal{L}_0} \cap B(x, 22c_{19}C_{10}L_n)$ of good $(n-1)-(L_0,u,K,p)$ vertices such that

$$5c_{19}L_{n-1} \leq d(y_{j-1}, y_j) \leq 6c_{19}L_{n-1} \forall\, j \in \{1, \ldots, m\} \text{ and } d(y_m, y_{m+1}) \leq 6c_{19}L_{n-1}. \tag{III.8.21}$$

We now construct the desired nearest neighbor path of (L_0,u,K,p)-good vertices connecting $\mathcal{U}_1$ and $\mathcal{U}_2$. To this end, we fix a nearest neighbor path π_0 in $\mathcal{U}_1^{k_1}$ beginning in y_0, a nearest neighbor path π_{m+1} in $\mathcal{U}_2^{k_2}$ beginning in y_{m+1}, and, for each $j \in \{1, \ldots, m\}$ a nearest neighbor path π_j beginning in y_j such that for all $j \in \{0, \ldots, m+1\}$, $c_{19}L_{n-1} \leq \delta(\pi_j) \leq 2c_{19}L_{n-1}$, which is always possible since $7l_0c_5^{-1} \geq c_{19}$, see (III.8.1). Note that, using (III.8.21),

$$\pi_0, \pi_1 \subset B(y_1, 20c_{19}L_{n-1}) \text{ and } d(\pi_0, \pi_1) \geq c_{19}L_{n-1}. \tag{III.8.22}$$

In view of (III.8.22), applying the induction hypothesis to π_0 and π_1, we can construct a nearest neighbor path $\bar{\pi}_1$ of (L_0,u,K,p)-good vertices in $B(y_1, 30c_{19}C_{10}L_{n-1}) \subset B(x, 30c_{19}C_{10}L_n)$ with diameter at least $c_{19}L_{n-1}$ connecting π_0 and π_1. Moreover, we can further extract from $\bar{\pi}_1$ a nearest neighbor path $\bar{\pi}_1'$ included in $B(y_1, 2c_{19}L_{n-1})$ and with diameter at least $c_{19}L_{n-1}$, and so we have $\bar{\pi}_1' \subset B(y_2, 20c_{19}L_{n-1})$ and $d(\bar{\pi}_1', \pi_2) \geq c_{19}L_{n-1}$. By the induction hypothesis, we can thus find a nearest neighbor path $\bar{\pi}_2$ of (L_0,u,K,p)-good vertices in $B(y_2, 30c_{19}C_{10}L_{n-1}) \subset B(x, 30c_{19}C_{10}L_n)$ with diameter at least $c_{19}L_{n-1}$ between $\bar{\pi}_1$ and π_2. Iterating this construction, we find a sequence of $(\bar{\pi}_j)_{j\in\{1,\ldots,m+1\}}$ of nearest neighbors paths of (L_0,u,K,p)-good vertices in $B(x,30c_{19}C_{10}L_n)$ such that $\pi_0\cap\bar{\pi}_1\neq\varnothing$, $\bar{\pi}_j\cap\bar{\pi}_{j+1}\neq\varnothing$ for all $j\in\{1,\dots,m\}$ and $\bar{\pi}_{m+1}\cap\pi_{m+1}\neq\varnothing$. Concatenating the paths $\bar{\pi}_0,\dots,\bar{\pi}_{m+1}$ provides a path of (L_0,u,K,p)-good vertices in $B(x,30c_{19}C_{10}L_n)$ connecting $\mathcal{U}_1$ and $\mathcal{U}_2$, as desired. □

Using Lemma III.7.6 and the quantitative bounds derived earlier in this section, we infer from Lemma III.8.6 the following estimate tailored to our later purposes. Let us define

$$E_{\varphi}^{\geq-\sqrt{2u}}=\{y\in G;\,\varphi_y>-\sqrt{2u}\}.$$

Lemma III.8.7. *Assume G satisfies* (WSI) *(in addition to* (III.3.1)*), and take $R=R_0$ from* (WSI)*. Then for all $u_0>0$, $u\in(0,u_0)$, $x\in G$, $K>0$ with $K\geq\sqrt{\log(C_{22}u^{-c_{22}})}$, $p\in(0,1)$ with $p\geq1-C_{23}u^{c_{23}}$ and $L>0$, there exists $L_0=L_0(L)\in\big[L_0(u),l_0L_0(u)\big)$, $C<\infty$ and $c>0$ depending on u and u_0 such that*

$$\widetilde{\mathbb{Q}}^{u,p}\left(\mathcal{E}^u_{x,L}\right)\geq1-C(u,u_0)\exp\{-L^c(u,u_0)\},$$

where $\mathcal{E}^u_{x,L}$ is the event

$$\left\{\begin{array}{c}\exists\text{ a connected set }A^u_{x,L}\subset B(x,2C_{10}L)\text{ which intersects every cluster}\\ \text{of }B(x,L)\text{ with diameter}\geq\sqrt{L},\text{ and a connected set }\widetilde{A}^u_{x,L}\subset\widetilde{\mathcal{I}}^u\cap\\ \widetilde{B}(x,2C_{10}L)\text{ verifying (III.7.21), such that }B(y,L_0)\cap\widetilde{A}^u_{x,L}\neq\varnothing\\ \text{for all }y\in A^u_{x,L}\text{ and every cluster of }E_{\varphi}^{\geq-\sqrt{2u}}\cap B(x,L)\text{ with}\\ \text{diameter}\geq L/10\text{ is connected to }\widetilde{A}^u_{x,L}\cap G\text{ in }E_{\varphi}^{\geq-\sqrt{2u}}\cap B(x,2L)\end{array}\right\}.\tag{III.8.23}$$

Proof. As a direct consequence of Lemma III.8.6 and (III.8.12) with $R=R_0$ from (WSI), we obtain that for all $u_0>0$, $u\in(0,u_0]$, $K\geq\sqrt{\log(C_{22}u^{-c_{22}})}$, $p\geq1-C_{23}u^{c_{23}}$ $n\in\mathbb{N}$, $x\in\Lambda_n^{\mathcal{L}_0}$, and $L_0\in[L_0(u),l_0L_0(u)]$, see (III.8.11),

$$\widetilde{\mathbb{Q}}^{u,p}\left(\begin{array}{c}\text{there exist connected components of }B(x,20c_{19}L_n)\\ \text{with diameter }\geq c_{19}L_n\text{ which are not connected by}\\ \text{a path of }(L_0,u,K,p)\text{-good vertices in }B(x,30c_{19}C_{10}L_n)\end{array}\right)\leq6\times2^{-2^n}.\tag{III.8.24}$$

Therefore, for all L large enough, taking $L_0=L_0(L)\in\big[L_0(u),l_0L_0(u)\big)$ and $n\in\mathbb{N}$ such that $L=20c_{19}l_0^nL_0$, we have

$$\begin{aligned}\widetilde{\mathbb{Q}}^{u,p}\left(\begin{array}{c}\text{there exist connected components of }B(x,L)\\ \text{with diameter }\geq\frac{L}{10}\text{ which are not connected by a}\\ \text{path of }(L_0,u,K,p)\text{-good vertices in }B(x,2C_{10}L)\end{array}\right)&\leq6\times2^{-2^n}\\ &\leq C\exp\{-L^c\},\end{aligned}\tag{III.8.25}$$

for some constants $C = C(u, u_0)$ and $c = c(u, u_0)$. Let us call $\overline{\mathcal{E}}^u_{x,L}$ the complement of the event on the left-hand side of (III.8.25). On the event $\overline{\mathcal{E}}^u_{x,L}$, there exists a connected set $\mathcal{A}^u_{x,L} \subset B(x, 2C_{10}L)$ of (L_0, u, K, p)-good vertices which intersects every connected component of $B(x, L)$ with diameter $\geq \frac{L}{10}$. One can construct such a set by starting with a path π of (L_0, u, K, p)-good vertices in $B(x, L)$ with diameter $\geq \frac{L}{10}$, and taking $\mathcal{A}^u_{x,L}$ as the union of all the paths of (L_0, u, K, p)-good vertices between π and every other connected component of $B(x, L)$ with diameter $\geq \frac{L}{10}$.

By Lemma III.7.6, for L large enough, this implies the existence of a connected set $\widetilde{\mathcal{A}}^u_{x,L} \subset \widetilde{B}(x, 2C_{10}L + 3C_{10}(L_0 + C_3)) \subset \widetilde{B}(x, 3C_{10}L)$ such that (III.7.19), (III.7.20) and (III.7.21) hold when replacing A by $\mathcal{A}^u_{x,L}$ and $\widetilde{A}$ by $\widetilde{\mathcal{A}}^u_{x,L}$. Moreover, if $\mathcal{V}$ is a cluster of $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, L)$ with diameter at least $L/10$, then there exists $z \in \mathcal{V} \cap \mathcal{A}^u_{x,L}$, and thus $\mathcal{V}$ contains a cluster of $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(z, L_0/2)$ with diameter at least $L_0/4$. By (III.7.20) we obtain that $\mathcal{V}$ is connected to $\widetilde{A}^u_{x,L}$ in $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(z, L_0) \subset B(x, 3C_{10}L)$.

If $\overline{\mathcal{E}}^u_{y,L}$ and $\overline{\mathcal{E}}^u_{y',L}$ happen for y and y' in G with $y \sim y'$, then $\delta(\mathcal{A}^u_{y,L} \cap B(y', L)) \geq \frac{L}{10}$, and so there exists $z \in \mathcal{A}^u_{y,L} \cap \mathcal{A}^u_{y',L}$. By (III.7.19), $\varnothing \neq B(z, L_0) \cap \mathcal{I}^{u/4} \subset \widetilde{\mathcal{A}}^u_{y,L} \cap \widetilde{\mathcal{A}}^u_{y',L}$. If $\overline{\mathcal{E}}^u_{y,10\sqrt{L}}$ happens for all $y \in B(x, C_{10}L)$, let us define $B_L \subset B(x, C_{10}L)$ a connected set containing $B(x, L)$, which exists by (III.3.4), and

$$A^u_{x,L} = \bigcup_{y \in B_L} \mathcal{A}^u_{y,10\sqrt{L}} \qquad \text{and} \qquad \widetilde{A}^u_{x,L} = \bigcup_{y \in B_L} \widetilde{\mathcal{A}}^u_{y,10\sqrt{L}}.$$

Then $A^u_{x,L}$ is a connected subset of $B(x, C_{10}(L + 20\sqrt{L})) \subset B(x, 2C_{10}L)$ and $\widetilde{A}^u_{x,L}$ is a connected subset of $B(x, C_{10}(L + 30\sqrt{L})) \subset B(x, 2C_{10}L)$ for L large enough. Changing L_0 into $L_0(10\sqrt{L})$, we clearly have that (III.7.21) still holds, that $B(y, L_0) \cap \widetilde{A}^u_{x,L} \neq \varnothing$ for all $y \in A^u_{x,L}$, that every cluster of $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, L)$ with diameter at least $\sqrt{L} \leq L/10$ is connected to $\widetilde{A}_{x,L}$ in $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, L + 30C_{10}\sqrt{L}) \subset E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, 2L)$, and that $A_{x,L}$ intersects every connected component of $B(x, L)$ with diameter at least $\sqrt{L}$. Therefore by (V_α) and (III.8.25), we have

$$\widetilde{\mathbb{Q}}^{u,p}\left(\mathcal{E}^u_{x,L}\right) \geq \widetilde{\mathbb{Q}}^{u,p}\left(\bigcap_{y \in B(x, C_{10}L)} \overline{\mathcal{E}}^u_{y,10\sqrt{L}}\right) \geq 1 - CL^{\alpha} \exp\left\{-(10\sqrt{L})^c\right\}.$$

□

Under $\mathcal{E}^u_{x,L}$, we have constructed by (III.5.4) a giant cluster $\widetilde{A}^u_{x,L} \cap G$ intersecting $B(x, L/2)$, with $\widetilde{A}^u_{x,L} \cap G \subset E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, 2C_{10}L)$ and such that $\widetilde{A}^u_{x,L} \cap G$ is connected in $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, 2L)$ to every cluster of $E_{\varphi}^{\geq -\sqrt{2u}} \cap B(x, L)$ with diameter at least $L/10$. We readily obtain by (III.8.23) that:

Corollary III.8.8. *For all $h < 0$, there exists constants $c(h) > 0$ and $C(h) < \infty$ such that* (III.1.10) *and* (III.1.11) *hold, and thus $\overline{h} \geq 0$.*

Remark III.8.9. As the perceptive reader will have already noticed, one does not need to use our "sign flipping" result, Proposition III.5.6, to prove $\overline{h} \geq 0$. One also does not need local uniqueness for random interlacements on the cable system, see Proposition III.4.1, but only on the discrete graph. We need to use percolation results for random interlacements on the cable system and Proposition III.5.6 only to prove $\overline{h} > 0$, which is the content of the next section. This is similar to the case of h_* on $\mathbb{Z}^d$, $d \geq 3$, where one can prove $h_* \geq 0$ without using Proposition III.5.6, see for instance [16] or (III.5.4), but an equivalent of Proposition III.5.6 is used to prove $h_* > 0$, see Lemma II.5.1.

III.9 Denouement

We proceed to the proof of our main results, Theorems III.1.1 and III.1.2. In Lemma III.9.2, we first use Proposition III.5.6 to translate the result of Lemma III.8.7, which is stated in terms of $\mathcal{I}^u$ and $E_{\varphi}^{\geq -\sqrt{2u}}$, to a similar result in terms of $\overline{E}^{\geq \sqrt{2v}}$ and $E_{\varphi}^{\geq -\sqrt{2u}}$, $0 \leq v < u$, which correspond to level sets of a Gaussian free field, see (III.5.42). This gives us directly, with overwhelming probability, that a giant cluster of $\overline{E}^{\geq \sqrt{2u}}$ intersecting every large connected component of $E_{\varphi}^{\geq -\sqrt{2u}}$ exists, see Lemma III.9.2. The sets $H_{u,v,K,p}$ from Proposition III.5.6 provide us with additional randomness, and we will take advantage of it to connect the giant cluster of $\overline{E}^{\geq \sqrt{2u}}$ not only to every large connected component of $E_{\varphi}^{\geq -\sqrt{2u}}$, but also to every large connected component of $\overline{E}^{\geq \sqrt{2u}}$, and this delivers Theorem III.1.1. We then use the couplings from (III.5.24) and Proposition III.5.6 as well as Lemma III.8.4 to also obtain Theorem III.1.2. As a by-product of our methods, Theorem III.9.3 asserts the existence of infinite sign clusters (in slabs) without any statements regarding their local structural properties under the slightly weaker assumption $(\widetilde{\text{WSI}})$, introduced in Remark III.8.5 above. We then conclude with some final remarks.

Let us first choose the parameters $u > 0$, $K < \infty$, $p \in (0,1)$ in such a way that the conclusions of Proposition III.5.6 and Lemmas III.8.4 and III.8.7 simultaneously hold. Recall that $\lambda_x \leq C_4$ for all $x \in G$, see (III.2.10). We now specify the range of values of $u > 0$ and $p \in (0,1)$ for which we will consider. Fix an arbitrary reference level $u_0 > 0$, say $u_0 = 1$, and choose $u_1 \in (0, u_0)$ such that, for all $0 < u \le u_1$,

$$\begin{aligned} &\exists\, K \ge 2\sqrt{2u} \text{ with } \sqrt{\log(C_{22}u^{-c_{22}})} \vee \frac{c_{14}}{2\sqrt{2u}\,C_4} \le K \le \frac{c_{14}}{\sqrt{2u}\,C_4},\\ &\text{and } \exists\, p \in \Big[\frac{1}{2},1\Big) \text{ such that } 1 - C_{23}u^{c_{23}} \le p \le F\Big(\frac{\sqrt{c_4}c_{14}}{4\sqrt{u}\,C_4}\Big), \end{aligned} \tag{III.9.1}$$

where we recall that F denotes the cumulative distribution function of a standard normal distribution. Also, note that u_1 with the desired properties exists by considering the limit as $u \downarrow 0$ and using the standard bound $F(x) \ge 1 - \frac{1}{\sqrt{2\pi}x}\exp\{-\frac{x^2}{2}\}$ for all $x > 0$ in the second line. For a given $u \in (0, u_1]$, we then select any specific value of $K = K(u)$ and $p = p(u)$ satisfying the constraints in (III.9.1), and henceforth refer to these values when writing K and p, and in particular we take the probability $\widetilde{\mathbb{Q}}^{u,p}$, cf. (III.5.34), and $\mathcal{Q}^{u,K,p}$, cf. Proposition III.5.4, for this particular value of K and p. Then K satisfies the constraint in (III.5.27) and p satisfies the constraint in (III.5.35) on account of (III.9.1) and (III.2.10). Therefore Proposition III.5.6 applies for $u \in (0, u_1]$. Noting that $K/2 \le K - \sqrt{2u}$, recalling S_K from (III.5.39) and (III.5.40), taking $X_{u,K,p}$ as in (III.5.37) and (III.5.40) and using (III.2.8), we have for all set $\widetilde{A}$ such that (III.7.21) holds that $B(\widetilde{A} \cap G, L_0) \subset S_K \cap X_{u,K,p}$. Moreover, recalling R_u from (III.5.25) and (III.5.40), and using (III.5.4), we have that $\mathcal{I}^u \subset R_u$. We thus obtain by (III.5.43) that for all $u \in (0, u_1]$ and $v \le u$, under $\mathcal{Q}^{u,K,p}$,

$$\begin{aligned} &\text{if } \widetilde{A} \subset \widetilde{\mathcal{I}}^u \text{ is a connected set such that (III.7.21) holds for}\\ &\text{some } L_0 \ge 1, \text{ then } \widetilde{A} \cap G \subset \mathcal{I}^u \cap S_K \cap X_{u,K,p} \subset \overline{E}^{\ge\sqrt{2v}}\\ &\text{and } B(\widetilde{A} \cap G, L_0) \cap H_{u,v,K,p} \subset \overline{E}^{\ge\sqrt{2v}}. \end{aligned} \tag{III.9.2}$$

Definition III.9.1. For all $x \in G$, $L > 0$, $L_0 = L_0(L)$ as in Lemma III.8.7, $u \in (0, u_1)$ and $0 \le v < u$ let us define the event $\overline{\mathcal{E}}^{u,v}_{x,L}$ as the event that

i) there exists a $\sigma\big(\widetilde{\mathcal{I}}^u, \widetilde{\gamma}, (\mathcal{B}^p_x)_{x\in G}\big)$-measurable and connected set $A^{u,v}_{x,L} \subset B(x, 2C_{10}L)$ such that $A^{u,v}_{x,L}$ intersects every connected component of $B(x, L)$ with diameter at least $\sqrt{L}$,

ii) there exists a connected set $\mathcal{C}^{u,v}_{x,L} \subset \overline{E}^{\ge\sqrt{2v}} \cap B(x, 2C_{10}L)$ such that $B(\mathcal{C}^{u,v}_{x,L}, L_0) \cap H_{u,v,K,p} \subset \overline{E}^{\ge\sqrt{2v}}$,

iii) for all $y \in A^{u,v}_{x,L}$, $B(y, L_0) \cap \mathcal{C}^{u,v}_{x,L} \neq \varnothing$.

Applying (III.9.2) to the set $\widetilde{A}^u_{x,L}$ from (III.8.23) and taking $A^{u,v}_{x,L} = A^u_{x,L}$ and $\mathcal{C}^{u,v}_{x,L} = \widetilde{A}^u_{x,L} \cap G$, it is clear that $\mathcal{E}^u_{x,L} \subset \overline{\mathcal{E}}^{u,v}_{x,L}$, see (III.8.23) for the definition of $\mathcal{E}^u_{x,L}$. Moreover, it is clear that Lemma III.8.7 holds for any $0 < u \le u_1$, and K and p as in (III.9.1), and we obtain:

Lemma III.9.2. *For all $x \in G$, $L > 0$, $L_0 = L_0(L)$ as in Lemma III.8.7, $u \in (0, u_1)$, K and p as in* (III.9.1)*, and $0 \leq v < u$, there exist constants $C < \infty$ and $c > 0$ depending on u such that*

$$\mathcal{Q}^{u,K,p}\left(\overline{\mathcal{E}}^{u}_{x,L}\right) \geq 1 - C\exp\{-L^c\},$$

Under $\overline{\mathcal{E}}^{u,v}_{x,L}$, we have thus constructed a giant component $\mathcal{C}^{u,v}_{x,L} \subset \overline{E}^{\geq\sqrt{2u}} \cap B(x, 2C_{10}L)$ such that, by i), any cluster of $\overline{E}^{\geq\sqrt{2v}} \cap B(x,L)$ with diameter at least $\sqrt{L}$ intersect the set $A^{u,v}_{x,L}$, and, by iii), it also intersects $B(y,L_0)$ for some $y \in \mathcal{C}^{u,v}_{x,L}$. Therefore, any cluster of $\overline{E}^{\geq\sqrt{2v}} \cap B(x,L)$ with diameter at least $L/10$ is connected to $B(y,L_0)$ for many vertices $y \in \mathcal{C}^{u,v}_{x,L}$, and if $B(y,L_0) \subset H_{u,v,K,p}$ for one of these y, by ii), this cluster would be connected to the giant component $\mathcal{C}^{u,v}_{x,L}$ in $\overline{E}^{\geq\sqrt{2v}} \cap B(y,L_0)$. We use this remark and the independence of $H_{u,v,K,p}$ from $A^{u,v}_{x,L}$ to deduce Theorem III.1.1 from (III.5.42) and Lemma III.9.2.

Proof of Theorem III.1.1. We first show that for all $h \leq h_1 = \sqrt{2u_1}$, (III.1.10) holds. On the event $\overline{\mathcal{E}}^{u,u}_{x,L}$, with $u = h^2/2$, we have that $\mathcal{C}^{u,u}_{x,L}$ is a connected component of $\overline{E}^{\geq\sqrt{2u}} \cap B(x, 2C_{10}L)$ such that $d(\mathcal{C}^{u,u}_{x,L}, B(x, L/2)) < L_0$, and thus $\mathcal{C}^{u,u}_{x,L}$ intersects $B(x, L/2 + L_0)$. In particular $\mathcal{C}^{u,u}_{x,L} \cap B(x,L)$ has diameter at least $L/5$, and we can conclude by (III.5.42) and Lemma III.9.2.

Let us now prove that (III.1.11) holds for all $h \leq h_1$. By Corollary III.8.8, it is enough to prove that (III.1.11) holds for all $0 \leq h \leq h_1$, and let us fix $u = u_1$ and $v = h^2/2$. We will simply denote by H the event $H_{u,v,K,p}$ from Proposition III.5.6. Let us define for all $x \in G$, L large enough, L_0 as in Lemma III.8.7, $k \in \{2, \dots, \lfloor \frac{\sqrt{L}}{20} \rfloor\}$ and $y \in B(x,L)$

$$\widehat{\mathcal{E}}^{y,k}_{x,L} = \overline{\mathcal{E}}^{u,v}_{x,L} \cap \left\{ \begin{array}{c} \text{the cluster of } y \text{ in } \overline{E}^{\geq\sqrt{2v}} \cap B(y, 2k\sqrt{L}) \cap B(x,L) \\ \text{intersects } \partial B(y, 2k\sqrt{L}) \text{ but does not intersect } \mathcal{C}^{u,v}_{x,L} \end{array} \right\}.$$

Let also $\mathcal{Z}^{y,k}_{x,L} = A^{u,v}_{x,L} \cap B(y, 2k\sqrt{L} - L_0 - C_3) \cap B(x,L) \setminus B(y, 2(k-1)\sqrt{L} + L_0)$, and Z_k be the smallest $z \in \mathcal{Z}^{y,k}_{x,L}$ (in some deterministic fixed order on the vertices of G) such that

$$y \longleftrightarrow \partial_{ext} B(z, L_0) \text{ in } \overline{E}^{\geq\sqrt{2v}} \cap B(y, 2k\sqrt{L}) \cap B(x,L) \setminus \bigcup_{z' \in \mathcal{Z}^{y,k}_{x,L}} B(z', L_0). \quad \text{(III.9.3)}$$

We fix arbitrarily $Z_k = y$ if (III.9.3) never happens. By (III.2.8), if $\widehat{\mathcal{E}}^{y,k}_{x,L}$ happens and L is large enough, since the set of vertices in $B(y, 2k\sqrt{L} - L_0 - C_3) \cap B(x,L) \setminus B(y, 2(k-1)\sqrt{L} + L_0)$ connected to y in $\overline{E}^{\geq\sqrt{2v}} \cap B(y, 2k\sqrt{L}) \cap B(x,L)$ contains a connected component with diameter $\geq 2\sqrt{L} - 2L_0 - 3C_3 \geq \sqrt{L}$, by i) of Definition III.9.1 it must intersect some $z\in A_{x,L}^{u,v}$, and so $Z_k\neq y$. Since under $\widehat{\mathcal{E}}_{x,L}^{y,k}$ the cluster of y in $\overline{E}^{\geq\sqrt{2v}}\cap B(y,2k\sqrt{L})\cap B(x,L)$ does not intersect $\mathcal{C}_{x,L}^{u,v}$, we obtain by ii) and iii) of Definition III.9.1 that $H^c\cap B(Z_k,L_0)\neq\varnothing$. Therefore

$$\widehat{\mathcal{E}}_{x,L}^{y,k}\subset\{Z_k\neq y,\ H^c\cap B(Z_k,L_0)\neq\varnothing\}. \tag{III.9.4}$$

Since $A_{x,L}^{u,v}$ is $\sigma(\widetilde{\mathcal{I}}^u,\widetilde{\gamma},(\mathcal{B}_x^p)_{x\in G})$ measurable, we have that the events $\{Z_k=z\}$ are $\mathcal{F}_z$ measurable for all $z\in B(y,2(k-1)\sqrt{L}+L_0)^c$, where

$$\mathcal{F}_z=\sigma\big(\widetilde{\mathcal{I}}^u,\widetilde{\gamma},(\mathcal{B}_x^p)_{x\in G},\{x'\in\overline{E}^{\geq\sqrt{2v}}\}_{x'\in B(z,L_0)^c}\big).$$

Moreover by Lemma III.5.6 the event $\{x'\in H\}$ is independent of $\mathcal{F}_z$ for all $z\in G$ and $x'\in B(z,L_0)$ and so, under $\mathcal{Q}^{u,K,p}(\cdot\,|\,\mathcal{F}_z)$, $\{x'\in H\}_{x'\in B(z,L_0)}$ is an i.i.d. sequence of events with common probability $\mathcal{Q}^{u,K,p}(x\in H)>0$. Since for all $k\in\mathbb{N}$ we have $\widehat{\mathcal{E}}_{x,L}^{y,k}\subset\widehat{\mathcal{E}}_{x,L}^{y,k-1}$ and $\widehat{\mathcal{E}}_{x,L}^{y,k-1}$ is $\mathcal{F}_z$ measurable for all $z\in B(y,2(k-1)\sqrt{L}+L_0)^c$, with the convention $\widehat{\mathcal{E}}_{x,L}^{y,0}=\overline{\mathcal{E}}_{x,L}^{u,v}$, we obtain by (V_α) and (III.9.4) that

$$\begin{aligned}
&\mathcal{Q}^{u,K,p}(\widehat{\mathcal{E}}_{x,L}^{y,k})\\
&\leq\sum_{z\in B(y,2(k-1)\sqrt{L}+L_0)^c}\mathbb{E}_{\mathcal{Q}^{u,K,p}}\Big[\mathbb{1}_{\widehat{\mathcal{E}}_{x,L}^{y,k-1}\cap\{Z_k=z\}}\mathcal{Q}^{u,K,p}(H^c\cap B(z,L_0)\neq\varnothing\,|\,\mathcal{F}_z)\Big]\\
&\leq\mathcal{Q}^{u,K,p}(\widehat{\mathcal{E}}_{x,L}^{y,k-1})\left(1-\mathcal{Q}^{u,K,p}(x\in H^c)^{C_1L_0^\alpha}\right),
\end{aligned}$$

Iterating, we obtain that there exists constants $c=c(u,v)>0$ and $C=c(u,v)<\infty$ such that for all $k\in\{2,\dots,\lfloor\frac{\sqrt{L}}{20}\rfloor\}$,

$$\mathcal{Q}^{u,K,p}(\widehat{\mathcal{E}}_{x,L}^{y,k})\leq C\exp(-ck). \tag{III.9.5}$$

By ii) of Definition III.9.1, we have moreover under $\overline{\mathcal{E}}_{x,L}^{u,v}$ that $\mathcal{C}_{x,L}^{u,v}\subset\overline{E}^{\geq\sqrt{2v}}\cap B(x,2C_{10}L)$ and is connected. Now the event in (III.1.11) for $h=\sqrt{2v}$ and $\overline{E}^{\geq\sqrt{2v}}$ instead of $E^{\geq h}$ implies that either $\overline{\mathcal{E}}_{x,L}^{u,v}$ does not happen, or it happens and there exists $y\in B(x,L)$ such that the component of y in $\overline{E}^{\geq\sqrt{2v}}\cap B(x,L)$ has diameter at least $L/10$ and is not connected to $\mathcal{C}_{x,L}^{u,v}$ in $\overline{E}^{\geq\sqrt{2v}}\cap B(x,2C_{10}L)$, and then there exists $y\in B(x,L)$ such that $\widehat{\mathcal{E}}_{x,L}^{y,\lfloor\frac{\sqrt{L}}{20}\rfloor}$ happens. By (III.5.42) $\overline{E}^{\geq\sqrt{2v}}$ has the same law under $\mathcal{Q}^{u,K,p}$ as $E^{\geq h}$ under $\mathbb{P}^G$, and thus by (V_α), Lemma III.9.2 and (III.9.5), we obtain that the probability in (III.1.11) is smaller than

$$\widetilde{\mathbb{Q}}^{u,p}(\overline{\mathcal{E}}_{x,L}^{u,v})+\mathcal{Q}^{u,K,p}\Big(\bigcup_{y\in B(x,L)}\widehat{\mathcal{E}}_{x,L}^{y,\lfloor\frac{\sqrt{L}}{20}\rfloor}\Big)\leq C\exp(-L^c)+CL^\alpha\exp(-c\sqrt{L}).$$

□

We now continue with the proof of Theorem III.1.2.

Proof of Theorem III.1.2. We continue with the setup of (III.9.1), and fix some $u \leq \widetilde{u} \stackrel{\text{def.}}{=} u_1$. We now define the probability ν_1 on $(\{0,1\}^G)^2 \times (\{0,1\}^G)$ as the (joint) law of

$$\Big(\big(\mathbb{1}_{\{x\in\mathcal{I}^u\}}, \mathbb{1}_{\{S_K^x\cap\{X^x_{u,K,p}=1\}\}}\big)_{x\in G}, \big(\mathbb{1}_{\{x\in\overline{E}^{\geq\sqrt{2u}}\}}\big)_{x\in G}\Big)$$

under $\mathcal{Q}^{u,K,p}$, and the probability ν_2 on $(\{0,1\}^G) \times (\{0,1\}^G)^2$ as the law of

$$\Big(\big(\mathbb{1}_{\{-\varphi_x\geq\sqrt{2u}\}}\big)_{x\in G}, \big(\mathbb{1}_{\{x\in\mathcal{V}^u\}}, \mathbb{1}_{\{-\gamma_x\geq 0\}}\big)_{x\in G}\Big)$$

under $\widetilde{\mathbb{Q}}^{u,p}$. We concatenate these probabilities by defining the probability Q^u on the product space $(\{0,1\}^G)^2\times(\{0,1\}^G)\times(\{0,1\}^G)^2$ such that for all measurable sets $A_1 \subset (\{0,1\}^G)^2$, $A_2 \subset \{0,1\}^G$ and $A_3 \subset (\{0,1\}^G)^2$

$$Q^u\big(A_1 \times A_2 \times A_3\big) = \mathbb{E}_{\nu_1}\Big[\mathbb{1}_{\{\eta_1^1\in A_1,\eta_2^1\in A_2\}}\nu_2\big(\eta_2^2 \in A_3 \,\big|\, \eta_1^2 = \eta_2^1\big)\Big],$$

where we wrote the coordinates under ν_i as (η_1^i, η_2^i) for all $i \in \{1,2\}$, and furthermore $\nu_2(\eta_2^2 \in \cdot \,|\, \eta_1^2 = \cdot)$ is a regular conditional probability distribution on $\{0,1\}^G$ for η_2^2 given $\sigma(\eta_1^2)$. One then defines the three random sets from the statement of the theorem under $\mathcal{Q}^u$ as follows: the sets $\mathcal{I}$ and $\mathcal{K}$ are defined by the marginals of η_1^1 and the set $\mathcal{V}$ as the first marginal of η_2^2. With this choices, part *i*) and *ii*) of (III.1.17) are clear by definition, noting that $\mathcal{I}^u$ and $S_K \cap X_{u,K,p}$ with $X_{u,K,p}$ coming from (III.5.37) are independent under $\widetilde{\mathbb{Q}}^{u,p}$, which follows from (III.5.34) on account of (III.5.39). Since $\mathcal{I}^u \cap S_K \cap X_{u,K,p} \subset \overline{E}^{\geq\sqrt{2u}}$ by (III.5.4), (III.5.25) and (III.5.43), $\overline{E}^{\geq\sqrt{2u}}$ has the same law as $\{x \in G; -\varphi_x \geq \sqrt{2u}\}$ by (III.5.42) and symmetry of φ, and $\{x \in G; -\varphi_x \geq \sqrt{2u}\} \subset \mathcal{V}^u$ by (III.5.24), one can easily check that the inclusion $\mathcal{I} \cap \mathcal{K} \subset \mathcal{V}$ holds under Q^u. Finally, $\mathcal{I}^u \cap S_K \cap X_{u,K,p}$ contains $\widetilde{\mathbb{Q}}^{u,p}$-a.s. an infinite cluster by Lemma III.8.4 and (III.9.2), and thus $\mathcal{I} \cap \mathcal{K}$ under Q^u too. This completes the proof. □

As the perceptive reader will have noticed, the inclusion in part *iii*) of Theorem III.1.2 can be somewhat strengthened to a statement of the form $(\mathcal{I} \cap \mathcal{K}) \subset (\mathcal{V} \cap \mathcal{K}')$ with $\mathcal{K}'$ independent of $\mathcal{V}$ and with the same law as $\{x \in G;\, \Phi_x > 0\}$ under $\mathbb{P}^G$ by taking into account the effect of $\widetilde{\gamma}$ in (III.5.24), cf. (III.5.34) regarding the asserted independence.

The sole existence of an infinite cluster without the local connectivity picture entailed in (III.1.11) can be obtained under the slightly weaker geometric assumption $(\widetilde{\text{WSI}})$ from Remark III.8.5. We record this in the following

Theorem III.9.3. *Under the assumptions* (III.3.1) *and* ($\widetilde{\text{WSI}}$) *on G, there exists $h_1 > 0$ such that for all $h \leq h_1$,* (III.1.10) *holds for some $x \in G$ and there exists a.s. an infinite connected component in $E^{\geq h} \cap B(G_p, CL_0(h^2/2))$ and in $\mathcal{V}^{h^2/2} \cap B(G_p, CL_0(h^2/2))$ with $L_0(\cdot)$ given by* (III.8.11)*. In particular $h_* > 0$ and $u_* > 0$.*

Proof. One adapts the argument leading to (III.1.10) in the proof of Theorem III.1.1, replacing the use of Lemma III.8.7 by Lemma III.8.4, or more precisely by the corresponding result obtained under the weaker assumption ($\widetilde{\text{WSI}}$) described in Remark III.8.5. We omit further details. □

We conclude with several comments.

Remark III.9.4. 1) In [28], on $\mathbb{Z}^d$, $d \geq 3$, a slightly different parameter $\bar{h}_1$ is introduced since only a super-polynomial decay in L is required in the conditions corresponding to (III.1.10) and (III.1.11), and in [100] yet another parameter $\bar{h}_2$ is introduced by allowing the addition of a small sprinkling parameter h' to connect together the large paths of $E^{\geq h}$. However, it is clear that $\bar{h} \leq \bar{h}_1 \leq \bar{h}_2$, and so the parameters $\bar{h}_1$ and $\bar{h}_2$ are also positive as a consequence of Theorem III.1.1.

2) Looking at the proof of Theorem 1.2, one sees that for u small enough, the set $\mathcal{K}$ can be taken with the same law under Q^u as $S_K \cap X_{u,K,p}$ under $\widetilde{\mathbb{Q}}^{u,p}$, for some $K > 0$ and $p \in (0,1)$ as in (III.9.1), where S_K is defined in (III.5.39) and (III.5.40), and $X_{u,K,p}$ in (III.5.37) and (III.5.40). Changing the event $C_x^{L_0,p}$ in Definition III.7.4 by the increasing event $\overline{C}_x^{L_0,p}$ which occurs if and only if for all $z \in \widetilde{B}(x, 2C_{10}(L_0 + C_3) + C_3)$, $\widetilde{\varphi}_z \geq -K$, and the event $F_x^{L_0,p}$ by the decreasing event $\overline{F}_x^{L_0,p}$ which occurs if and only if for all $z \in \widetilde{B}(x, 2C_{10}(L_0 + C_3) + C_3)$, $\widetilde{\varphi}_z \leq K$, one can show as in Lemma III.8.4 that there exists a connected and unbounded set $\widetilde{A} \subset \widetilde{G}$ such that

$$\widetilde{A} \subset \widetilde{\mathcal{I}}^u, \text{ and } |\widetilde{\varphi}_z| \leq K \text{ for all } z \in \widetilde{B}(\widetilde{A}, 2L_0 + C_3).$$

Therefore, adapting the proof of Theorem III.1.2, one can take $\mathcal{K}$ with the same law under Q^u as $\overline{S}_K \cap X_{u,K,p}$ under $\widetilde{\mathbb{Q}}^{u,p}$, for some $K > 0$ and $p \in (0,1)$ as in (III.9.1), where $\overline{S}_K$ is defined in (III.5.25) and (III.5.40), and $X_{u,K,p}$ in (III.5.38) and (III.5.40), or with the same law as $\{x \in G;\, |\widetilde{\varphi}_z| \leq K$ for all $z \in U^x\}$, and i) and iii) in (III.1.17) still hold. This choice for $\mathcal{K}$ has a simple expression and would be enough for the purpose of proving $\bar{h} > 0$ and $u_* > 0$, but has the disadvantage of not being independent from $\mathcal{I}$. Independence, however, is expected to be useful for future applications.

3) Taking complements in the inclusion $\mathcal{I}\cap\mathcal{K}\subset\mathcal{V}$, see Theorem III.1.2, and intersecting with $\mathcal{K}$, we obtain that $\mathcal{V}^c\cap\mathcal{K}\subset\mathcal{I}^c$. Taking $\mathcal{I}'=\mathcal{V}^c$ and $\mathcal{V}'=\mathcal{I}^c$, we obtain the inclusion $\mathcal{I}'\cap\mathcal{K}\subset\mathcal{V}'$, and $\mathcal{K}$ is independent of $\mathcal{I}$, and thus of $\mathcal{V}'$. Therefore, we could have chosen $\mathcal{K}$ independent of $\mathcal{V}$ in ii) of (III.1.17) instead of $\mathcal{K}$ independent of $\mathcal{I}$.

4) Using a similar reasoning as the one leading to Corollary III.8.8, one can prove strong percolation, as in (III.1.9), for the level sets $\widetilde{E}^{>h}$, see (III.5.1), for all $h<0$, in the sense that (III.1.10) and (III.1.11) hold but for the level sets $\widetilde{E}^{>h}$ of the Gaussian free field on the cable system $\widetilde{G}$ instead of the graph G. Moreover, the critical parameter $\widetilde{h}_*$ for percolation of the continuous level sets $\widetilde{E}^{>h}$ is exactly equal to 0 by Proposition III.5.2, and thus the strongly percolative phase consists of the *entire* supercritical phase for the Gaussian free field on the cable system, i.e. if one introduces $\widetilde{\overline{h}}$ as in (III.1.9), but putting "tildes everywhere" in (III.1.10) and (III.1.11), one arrives at the following

Theorem III.9.5. *If G satisfies* (III.3.1) *and* (WSI)*, then* $\widetilde{\overline{h}}=\widetilde{h}_*=0$.

This result can also be proved without condition (WSI). Indeed, by (III.5.4), (III.3.11) and the definition of random interlacements, the probability that $\widetilde{E}^{>-\sqrt{2u}}$ does not contain a connected component of diameter at least $L/10$ has stretched exponential decay in L for any $u>0$. Moreover, by Corollary III.5.3, any connected component of $\{z\in\widetilde{G};\,\widetilde{\varphi}_z>-\sqrt{2u}\}\cap B(x,L)$ either intersects $\widetilde{\mathcal{I}}^u$ or is a connected component of $\{z\in\widetilde{G};\,\widetilde{\gamma}_z>0\}$ not intersecting $\widetilde{\mathcal{I}}^u$. Since $\widetilde{\mathcal{I}}^u$ and $\widetilde{\gamma}$ are independent under $\widetilde{\mathbb{Q}}^{u,p}$, the probability that $\widetilde{\mathcal{I}}^u$ does not intersect a component of $\{z\in\widetilde{G};\,\widetilde{\gamma}_z>0\}$ with diameter at least $L/10$ has stretched exponential decay by Lemma III.3.2 and (III.3.10). Therefore, with high enough probability, any connected component of $\{z\in\widetilde{G};\,\widetilde{\varphi}_z>-\sqrt{2u}\}\cap B(x,L)$ with diameter at least $L/10$ intersects $\widetilde{\mathcal{I}}^u$, and strong connectivity of $\widetilde{E}^{-\sqrt{2u}}$ then readily follows from Proposition III.4.1.

5) Looking at Theorem III.9.3, we have in fact proved that if $(\widetilde{\text{WSI}})$ holds for some subgraph in G_p of G, then there exists $0<h_1\leq h_*$ such that for all $h<h_1$, there exists $L>0$ with

$$\mathbb{P}^G\big(\text{there exists an infinite connected components in } E^{\geq h}\cap B(G_p,L)\big)=1.$$

It then follows by (III.5.18), that the same is true for $\mathcal{V}^u$ i.e., there exists $0<u_1\leq u_*$ such that for all $u<u_1$, and some $L>0$,

$$\mathbb{P}^I\big(\text{there exists an infinite connected components in } \mathcal{V}^u\cap B(G_p,L)\big)=1.$$

If $G = G_1 \times G_2$, we may choose $G_p = P_1 \times P_2$ a half-plane, where P_1 and P_2 are two semi-infinite geodesic in G_1 and G_2. Hence, we obtain that $E^{\geq h}$ and $\mathcal{V}^u$ percolate in thick planes $B(G_p, L)$ for $h > 0$ and $u > 0$ small enough. If $\nu > 1$, then $\mathcal{V}^u$ actually percolates in the plane G_p for u small enough, see Remark III.7.2, 2), and in Theorem 5.1 of [95], it is shown that this is also true if $\nu = 1$ and $G_1 = \mathbb{Z}$. It is still unclear, and an interesting open question, whether this holds true for $\nu < 1$ or not.

6) The existence of a non-trivial supercritical phase for Bernoulli percolation (and other models) is proved in [104] if G satisfies the volume upper bound of (V_α) and a local isoperimetric inequality. The proof involves events similar to those considered in (III.1.11), and it is possible that our condition (WSI) could be replaced by this local isoperimetric inequality, which would for example cover the case of the Menger sponge, see Remark III.3.8, 3). However, one would then need to take a super-geometric scale in our renormalization scheme (III.7.2), and then lose the stretched exponential decay in (III.1.10) and (III.1.11).

7) One may also inquire whether a phase coexistence regime for percolation of $\{|\varphi| > h\}$ and $\{|\varphi| < h\}$ exists, or similarly for the level sets of local times $\{x \in G;\, \ell_{x,u} > \alpha\}$ of random interlacements, with $u > 0$, $\alpha \geq 0$, considered in [78]. For instance, regarding the latter, is it possible for all $\alpha > 0$ to find $u \geq 0$ such that percolation for the local times at level u above and below α occur simultaneously?

8) Finally, it would be desirable to have a conceptual understanding of the mechanism that lurks behind the percolation above small enough levels $h \geq 0$ for the discrete level sets $E^{\geq h}$ (as opposed to their continuous counterparts $\widetilde{E}^{\geq h}$, cf. 4) above). Our current techniques are based on stochastic comparison, see Lemma III.5.5 and Proposition III.5.6, but the induced couplings suggest that one should be able to exhibit these features as a property of $\widetilde{\varphi}$ *itself*, without resorting to additional randomness.

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III.A Appendix: Proof of Proposition III.3.3

Proposition III.3.3 is proved in [42] when d is the graph distance, and we are going to adapt its proof for a general distance d. Let us begin with the

Proof of Proposition III.3.3, i). Using (G_β) and (V_α), we have for all $x \in G$ and $t \le C_2$,

$$\lambda(\{y \in G: \ g(x,y) > t\}) \overset{(G_\beta)}{\le} \lambda(\{y \in G: \ C_2 d(x,y)^{-\nu} > t) \le \lambda\Big(B\big(x, \big(\frac{t}{C_2}\big)^{-\frac{1}{\nu}}\big)\Big) \overset{(V_\alpha)}{\le} Ct^{-\frac{\alpha/\beta}{\alpha/\beta - 1}}$$

Moreover, by (G_β), $\lambda(\{y \in G: \ g(x,y) > t\}) = 0$ for all $x \in G$ and $t > C_2$, and (III.3.16) follows directly from Proposition 5.1 in [42]. □

In order to prove Proposition III.3.3, *ii*) we first need the following bounds on the expected time at which the random walk Z on G leaves a ball.

Lemma III.A.1. *There exist constants* $0 < c_{25} \le C_{25} < \infty$ *only depending on* G *such that for all* $x \in G$ *and* $R \ge 1$,

$$c_{25} R^\beta \le E_x[T_{B(x,R)}] = \sum_{y \in B(x,R)} \lambda_y g_{B(x,R)}(x,y) \le \sum_{y \in B(x,R)} \lambda_y g(x,y) \le C_{25} R^\beta \tag{III.A.1}$$

Proof. Let us fix some $x \in G$ and $R \ge 1$. The equality in (III.A.1) is true by definition of the stopped Green function (III.2.5). Partitioning $B(x,R) \setminus B(x,1)$ into $B_k = B(x, 2^{-k}R) \setminus B(x, 2^{-k-1}R)$ for $k \in \{0, \dots, \lfloor \log_2 R \rfloor\}$, we have

$$\sum_{y \in B(x,R) \setminus B(x,1)} \lambda_y g(x,y) \le \sum_{k=0}^{\lfloor \log_2 R \rfloor} \sum_{y \in B_k} \lambda_y g(x,y) \overset{(G_\beta)}{\le} C_2 \sum_{k=0}^{\lfloor \log_2 R \rfloor} \lambda(B_k) \big(2^{-k-1}R\big)^{-\nu} \overset{(V_\alpha)}{\le} CR^{\alpha - \nu} \sum_{k=0}^{\infty} 2^{-k(\alpha-\nu)},$$

and the upper bound in (III.A.1) follows since $\alpha - \nu = \beta > 0$ and

$$\sum_{y \in B(x,1)} \lambda_y g_{B(x,R)}(x,y) \le C_1 C_2.$$

For the lower bound, we can assume w.l.o.g. that R is large, and we write

$$\sum_{y \in B(x,R)} \lambda_y g_{B(x,R)}(x,y) \ge \sum_{y \in B(x, \frac{R}{1+2C_9})} \lambda_y g_{B(x,R)}(x,y) \overset{\text{(III.3.2)}}{\ge} \frac{C_2}{2} \sum_{y \in B(x, \frac{R}{1+2C_9}) \setminus \{x\}} \lambda_y d(x,y)^{-\nu} \overset{(V_\alpha)}{\ge} cR^{\alpha-\nu}.$$

□

We now follow the proof of Proposition 4.33 in [4]. One can see the bounds in Lemma III.A.1 on the expected exit time of a ball give us the following lemma as a first step in the proof of Proposition III.3.3, *ii*).

Lemma III.A.2. *There exist constants* $C_{26} > 0$ *and* $c_{26} > 0$ *only depending on* G *such that for all* $x \in G$ *and* $R > 0$,

$$P_x\big(T_{B(x,R)} > C_{26}R^\beta\big) \geq c_{26}$$

Proof. Take $C_{26} = (c_{25} \wedge 1)/4$. Let us fix $x \in G$ and $R > 0$, and we can assume w.l.o.g. that $C_{26}R^\beta \geq 1/2$ (and then $R \geq 1$). We first need to remark that, by Lemma III.A.1, for all $y \in B(x, R)$,

$$E_y\left[T_{B(x,R)}\right] \leq E_y\left[T_{B(y,2R)}\right] \leq C_{25}(2R)^\beta.$$

Let us write $n = \left\lceil C_{26}R^\beta \right\rceil$. An application of the Markov property of Z at time n gives us

$$E_x\big[T_{B(x,R)}\mathbb{1}_{T_{B(x,R)}>n}\big] = E_x\big[E_{X_n}[T_{B(x,R)}]\mathbb{1}_{T_{B(x,R)}>n}\big] \leq C_{25}(2R)^\beta P_x(T_{B(x,R)} > n). \tag{III.A.2}$$

On the other hand, by Lemma III.A.1,

$$E_x\big[T_{B(x,R)}\mathbb{1}_{T_{B(x,R)}>n}\big] \geq c_{25}R^\beta - n \geq C_{26}R^\beta, \tag{III.A.3}$$

and combining (III.A.2) and (III.A.3) let us conclude. □

It is interesting to note that Lemma III.A.2 is analogue to Proposition III.3.3, *ii*) for $n = \left\lfloor C_{26}R^\beta \right\rfloor$, and we are going to use it iteratively with the help of (III.2.8) to finish the proof of Proposition III.3.3.

Proof of Proposition III.3.3, ii). Let us fix $x \in G$, $r > 0$ and a positive integer m. We define recursively the sequence of stopping time S_p, $p \in \mathbb{N}$ by

$$S_0 = x, \qquad \text{and for all } p \geq 1,\ S_p = T_{B(X_{S_{p-1}},r)}.$$

For all $p \in \mathbb{N}$, $d(Z_{S_{p-1}}, Z_{S_p-1}) \leq r$ and by (III.2.8), $d(Z_{S_{p-1}}, Z_{S_p}) \leq r + C_3$. In particular, $d(x, Z_k) \leq (r+C_3)m$ for all $0 \leq k \leq S_m$ and thus $S_m \leq T_{B(x,(r+C_3)m)}$. Let us define

$$\xi_p = \mathbb{1}_{S_p - S_{p-1} \geq C_{26}r^\beta} \qquad \text{and} \qquad N = \sum_{p=0}^{m} \xi_p.$$

By definition, $T_{B(x,(r+C_3)m)} \geq S_m \geq C_{26}r^\beta N$. Moreover, by the strong Markov property and Lemma III.A.2, $E_x[\xi_p \,|\, \mathcal{F}_{S_{p-1}}] \geq c_{26}$, where $\mathcal{F}_i = \sigma(Z_0, \ldots, Z_i)$ for all $i \geq 0$. Using a martingale inequality, Lemma A.8 in [4], we thus get

$$P_x\Big(T_{B(x,(r+C_3)m)} < \frac{C_{26}c_{26}}{2}r^\beta m\Big) \leq P_x\Big(N < \frac{mc_{26}}{2}\Big) \leq \exp\{-cm\}. \tag{III.A.4}$$

Let us now fix a constant c_{27} small enough so that, if $C_3^{-1}R \leq n \leq c_{27}R^{\beta}$, then

$$m \stackrel{\text{def.}}{=} \left\lceil \left(\frac{c_{27}R^{\beta}}{n}\right)^{\frac{1}{\beta-1}} \right\rceil \leq 2\left(\frac{c_{27}R^{\beta}}{n}\right)^{\frac{1}{\beta-1}}, \quad r \stackrel{\text{def.}}{=} \frac{R}{m} - C_3 \geq \frac{1}{4}\left(\frac{n}{c_{27}R}\right)^{\frac{1}{\beta-1}},$$

and

$$\frac{C_{26}c_{26}}{2} r^{\beta} m \geq \frac{C_{26}c_{26}}{2 \times 4^{\beta}} \times \frac{n}{c_{27}} \geq n,$$

and (III.3.17) (with $C = 1$) then readily follows from (III.A.4) as long as $C_3^{-1}R < n < c_{27}R^{\beta}$. Finally, if $n < C_3^{-1}R$, then by (III.2.8) $B_G(x,n) \subset B(x,R)$ and the left-hand side of (III.3.17) is always 0, and it is easy to find a constant C large enough so that the right-hand side of (III.3.17) is always larger than 1 whenever $n > c_{27}R^{\beta}$. □

Chapter IV

Percolation for the Gaussian free field on the cable system of transient graphs

IV.1 Introduction

This chapter studies the percolative properties of the level sets of the Gaussian free field on the cable system, or metric graph, a continuous version of the Gaussian free field on a discrete weighted graph investigated in [57]. Percolation for the level sets of the discrete Gaussian free field was first studied in [16] on $\mathbb{Z}^d$, $d \geq 3$, more than three decades ago. Its investigation has recently resurged by the introduction of new ideas from related topics, such as random interlacements introduced [93] or the cable system, see e.g. [81], [57] and [24]. Regarding the discrete lattice $\mathbb{Z}^d$, in [81] the non-triviality of the phase transition has been established for the level sets of the Gaussian free field. Considering continuous level sets of the Gaussian free field on the cable system, it was shown in [57] that the phase transition was not only non-trivial, but actually happens at level 0. That is, the critical parameter $\widetilde{h}_*$, see (IV.1.4) below, for the percolation of the Gaussian free field on the cable system of $\mathbb{Z}^d$ is equal to 0, which philosophically corresponds to $p_c = \frac{1}{2}$ in the language of Bernoulli percolation. This is different from the situation for the discrete Gaussian free field, where the phase transition actually happens at a strictly positive level, see Chapter II. The equality $\widetilde{h}_* = 0$ was proved on several other transient graphs, see [101], [1] or Chapter III for instance, and we generalize all these results here by showing that $\widetilde{h}_* = 0$ on massless graphs under the weak condition (Cap) that the capacity of a connected and infinite set is infinite, see also (IV.4.1).

Moreover, under condition (Cap), we also derive an explicit formula for the law of the capacity of the component of some point x_0 in the level sets above any level $h \in \mathbb{R}$, see (Law$_h$), and this law surprisingly only depends on the choice of the graph via the Green function at x_0. We are going to give three different proofs of these results, and one of them involves a strong version of the isomorphism (Isom) between the Gaussian free field and the model of random interlacements, introduced in [93]. This isomorphism has several consequences, such as (Law$_h$), but also implies the dichotomy $\widetilde{h}_* \in \{0, \infty\}$ on massless graphs. We also give an example of a graph for which $\widetilde{h}_* = \infty$.

Let us now explain the settings and results in details. We consider a weighted graph $\mathcal{G} = (V, \lambda, \kappa)$, where V is a finite or countably infinite set, $\lambda_{x,y} \in [0, \infty)$, $x, y \in V$, are non-negative weights satisfying $\lambda_{x,y} = \lambda_{y,x} \geq 0$ and $\lambda_{x,x} = 0$ for all $x, y \in V$, and $\kappa_x \in [0, \infty]$, $x \in V$, is a killing measure, possibly infinite. We always assume that the induced graph with vertex set $G = \{x \in V : \kappa_x < \infty\}$ and edge set $E = \{\{x, y\} : x \in G,\, y \in V,\, \lambda_{x,y} > 0\}$ is connected and locally finite. Unless explicitly mentioned otherwise, we henceforth assume without loss of generality that $G = V$, that is $\kappa_x < \infty$ for all $x \in G$, and refer to (IV.2.5) et (IV.2.6) below regarding how to reduce to this case. We write $x \sim y$ when $\{x, y\} \in E$, and we define

$$\lambda_x = \kappa_x + \sum_{y \in G} \lambda_{x,y},\ \rho_x = \frac{1}{2\kappa_x} \text{ for } x \in G \text{ and } \rho_{x,y} = \frac{1}{2\lambda_{x,y}} \text{ for } x \sim y \in G \tag{IV.1.1}$$

(with $\rho_x = \infty$ when $\kappa_x = 0$).

One naturally associates to $\mathcal{G}$ a continuous version $\widetilde{\mathcal{G}}$, the corresponding cable system or metric graph, which will be our main object of interest. The cable system $\widetilde{\mathcal{G}}$ is obtained by replacing each edge $e = \{x, y\} \in E$ by an open interval I_e of length $\rho_{x,y}$, glued to G through its endpoints x and y. In order to take into account the effect of the killing measure κ, which was supposed to be equal to 0 in the previous chapters, one additionally attaches to each vertex $x \in G$ an interval I_x isometric to $[0, \rho_x)$, glued to x through 0.

One then defines (e.g. in terms of its associated Dirichlet form, see (IV.2.1) and (IV.2.2) for details) a diffusion process $(X_t)_{t\geq 0}$ on $\widetilde{\mathcal{G}} \cup \{\Delta\}$, where Δ denotes an (absorbing) cemetery state, which can be seen as Brownian motion on the cable system. The process X induces a pure jump process $Z = (Z_t)_{t\geq 0}$ on $G \cup \{\Delta\}$, its *print on* $\mathcal{G}$, see Section IV.2 for its precise definition, which has the law of the continuous-time Markov chain that jumps from $x \in G$ to $y \in G$ at rate $\lambda_{x,y}$ and is killed at rate κ_x. We write P_x for the canonical law of X with starting point $x \in \widetilde{\mathcal{G}}$, and occasionally $P_x^{\widetilde{\mathcal{G}}}$ in place of P_x to stress the dependence on the datum $\widetilde{\mathcal{G}}$.

Our findings deal with the graph $\mathcal{G}$ and its associated metric graph $\widetilde{\mathcal{G}}$, when $\mathcal{G}$ is transient, that is when the Markov chain Z is transient, which we assume tacitly from now on. In particular, the graph $\mathcal{G}$ may be finite when $\kappa \not\equiv 0$. We then define the Gaussian free field on $\widetilde{\mathcal{G}}$, whose canonical law $\mathbb{P}^G$, defined on the space $C(\widetilde{\mathcal{G}}, \mathbb{R})$, endowed with coordinate maps φ_x, $x \in \widetilde{\mathcal{G}}$, and the σ-algebra they generate, is such that

$$\text{under } \mathbb{P}^G,\ (\varphi_x)_{x\in\widetilde{\mathcal{G}}} \text{ is a centered Gaussian field with covariance function } g(\cdot,\cdot), \tag{IV.1.2}$$

where $g(\cdot,\cdot)$ is the Green density of $X.$ with respect to m, the Lebesgue measure on $\widetilde{\mathcal{G}}$, see (IV.2.12). The restriction of this process to G has the same law as the usual Gaussian free field on $\mathcal{G}$ associated to the discrete Markov chain Z.

We now describe our main results. We are interested in the geometry of the level sets $E^{\geq h} \stackrel{\text{def.}}{=} \{y \in \widetilde{\mathcal{G}} : \varphi_y \geq h\}$ of φ, for $h \in \mathbb{R}$, and endow $\widetilde{\mathcal{G}}$ with the (geodesic) distance $d_{\widetilde{\mathcal{G}}}(\cdot,\cdot)$ such that all intervals I_e, $e \in E$, and I_x, when $\rho_x < \infty$, have length one (rather than ρ_e, resp. ρ_x), Albeit not imprescindible, we assume for convenience that $d_{\widetilde{\mathcal{G}}}$ also assigns length one to I_x when $\rho_x = \infty$ (by means of some strictly increasing bijection $[0,1] \to [0,\infty)$). We introduce the clusters of $E^{\geq h}$,

$$E^{\geq h}(x_0) \stackrel{\text{def.}}{=} \big\{y \in \widetilde{\mathcal{G}} : x_0 \leftrightarrow y \text{ in } E^{\geq h}\big\}, \text{ for } x_0 \in \widetilde{\mathcal{G}},\ h \in \mathbb{R}; \tag{IV.1.3}$$

here, for a measurable $A \subset \widetilde{\mathcal{G}}$ and $x, y \in \widetilde{\mathcal{G}}$, we write $\{x \leftrightarrow y \text{ in } A\}$ if there exists a (continuous) path from x to y in A, and we say that A is connected in $\widetilde{\mathcal{G}}$ if $z \leftrightarrow z'$ in A for all $z, z' \in A$.

Our principal focus is on the percolative properties of the set $E^{\geq h}$ (with respect to $d_{\widetilde{\mathcal{G}}}$) and we introduce the corresponding critical parameter

$$\widetilde{h}_* = \inf\big\{h \in \mathbb{R};\ \mathbb{P}^G(E^{\geq h} \text{ contains an unbounded connected component}) = 0\big\} \tag{IV.1.4}$$

(with the convention $\inf \varnothing = \infty$). The definition (IV.1.4) of $\widetilde{h}_*$ also corresponds to the usual definition of the critical parameter for percolation on an infinite discrete graph $\mathcal{G}$, more precisely it is the critical parameter associated with the percolation of $E^{\geq h}(x_0) \cap G$, $x_0 \in \widetilde{\mathcal{G}}$. In particular, the critical parameter for the percolation of the discrete Gaussian free field on the graph $\mathcal{G}$ is larger than $\widetilde{h}_*$. Other definitions of the critical parameter for the percolation of the Gaussian free field on the cable system are possible and will be useful for us, see (IV.3.1) and (IV.3.2).

We now briefly introduce the process of random interlacements on $\widetilde{\mathcal{G}}$, see [93] and [103], which will play a prominent role in this context, due to recent isomorphisms, see [57], [101] and (Isom) below, relating it to the Gaussian free field in a very explicit fashion. Under a suitable probability $\mathbb{P}^I$, for each $u > 0$, random interlacements at level u on the cable system constitute a Poisson point process ω_u with intensity $u\nu_{\widetilde{\mathcal{G}}}$, where $\nu_{\widetilde{\mathcal{G}}}$ is a measure on doubly non-compact trajectories modulo time-shift (when $\kappa \not\equiv 0$, these trajectories can be killed by the measure κ before escaping to infinity, i.e. they can exit $\widetilde{\mathcal{G}}$ via I_x for some $x \in G$ with $\kappa_x > 0$; see (IV.2.36) and (IV.2.37) for the precise definition of $\nu_{\widetilde{\mathcal{G}}}$). We denote by $(\ell_{x,u})_{x\in\widetilde{\mathcal{G}}}$ the continuous field of local times associated with ω_u, i.e. the sum of the local times relative to the Lebesgue measure on $\widetilde{\mathcal{G}}$ of all the trajectories in ω_u, and by $\mathcal{I}^u \subset \widetilde{\mathcal{G}}$ the interlacement set, defined as the open set of points with positive local times. Without any further assumptions on $\mathcal{G}$, one knows that for all $u > 0$,

$$\begin{aligned}\Big(\ell_{x,u} + \frac{1}{2}\varphi_x^2\Big)_{x\in\widetilde{\mathcal{G}}} &\text{ has the same law under } \mathbb{P}^G \otimes \mathbb{P}^I \\ \text{as } \Big(\frac{1}{2}(\varphi_x + \sqrt{2u})^2\Big)_{x\in\widetilde{\mathcal{G}}} &\text{ under } \mathbb{P}^G\end{aligned} \tag{IV.1.5}$$

(this was first derived for the square of the processes on discrete graphs with $\kappa \equiv 0$ in [96], as a consequence of the second Ray-Knight theorem for Markov processes, cf. [32]; this identity can actually be easily extended in a variety of ways, e.g. to any discrete transient graph, without the condition $\kappa \equiv 0$. It was also generalized to the cable system with $\mathbb{Z}^d$, $d \geq 3$ as underlying graph in Proposition 6.3 of [57], see also (1.27)–(1.30) in [101] for general graphs). We now introduce an additional structural property of $\widetilde{\mathcal{G}}$, namely

$$\mathrm{cap}(A) = \infty \text{ for all } (d_{\widetilde{\mathcal{G}}}\text{-})\text{unbounded, closed, connected sets } A \subset \widetilde{\mathcal{G}} \tag{Cap}$$

(see (IV.2.20) and (IV.2.27) for the definition of the capacity $\mathrm{cap}(\cdot)$ in this context). One can for instance show that (Cap) is verified whenever the Green function g decays to 0 at infinity, see Lemma IV.4.1 below. In particular, (Cap) holds on any vertex-transitive graph. As will turn out, cf. (IV.3.1) and (IV.3.2), the "magnitude" of clusters in $E^{\geq h}$ can be measured in several natural ways, and condition (Cap) reflects such a choice, based on capacity as a measure of size.

Our main result investigates the connection between the percolation phase transition for the level sets of the Gaussian free field on $\widetilde{\mathcal{G}}$, the value of the associated critical parameter $\widetilde{h}_*$, see (IV.1.4), and the following properties of $E^{\geq h}$, $h \geq 0$:

$$\begin{aligned}&\mathbb{P}^G\text{-a.s. the clusters of } E^{\geq 0} \text{ only contain} \\ &\text{bounded connected components;}\end{aligned} \tag{Sign}$$

$$\begin{aligned}\mathbb{E}^G\big[\exp\big(-u\,\mathrm{cap}\big(E^{\geq h}(x_0)\big)\big)\mathbb{1}_{\varphi_{x_0}\geq h}\big] &= \mathbb{P}^G\big(\varphi_{x_0} \geq \sqrt{2u+h^2}\big) \\ &\text{for all } u \geq 0 \text{ and } x_0 \in \widetilde{\mathcal{G}};\end{aligned} \tag{Law$_h$}$$

$$\begin{aligned}&\big(\varphi_x 1_{x\notin\mathcal{C}_u^\infty} + \sqrt{\varphi_x^2 + 2\ell_{x,u}}1_{x\in\mathcal{C}_u^\infty}\big)_{x\in\widetilde{\mathcal{G}}} \text{ has the same law}\\ &\text{under } \mathbb{P}^I\otimes\mathbb{P}^G \text{ as } \big(\varphi_x + \sqrt{2u}\big)_{x\in\widetilde{\mathcal{G}}} \text{ under } \mathbb{P}^G, \text{ for all } u\geq 0,\end{aligned} \tag{Isom}$$

where $\mathcal{C}_u^\infty$ denotes the closure of the union of the connected component of the sign clusters $\{x\in\widetilde{\mathcal{G}}:|\varphi_x|>0\}$ intersecting the interlacement set $\mathcal{I}^u$. These conditions will be duly discussed below. For the time being, we just note that the identity (Isom), derived in [101] under certain assumptions on $\widetilde{\mathcal{G}}$, among which (Sign), considerably strengthens (IV.1.5). Moreover, the random variable $\mathrm{cap}(E^{\geq h}(x_0))$, $h\geq 0$, with moment-generating function given by (Law$_h$), can be equivalently described in terms of its density ρ_h, which is explicit, see (IV.3.6) and Lemma IV.4.5 below.

We now present our main result, which is a combination of various findings presented in more details in Section IV.3.

Theorem IV.1.1. *Let $\mathcal{G}=(V,\lambda,\kappa)$ be a transient weighted graph as above. Then:*

1) *$\mathbb{P}^G$-a.s., the random variable $\mathrm{cap}(E^{\geq 0}(x_0))$ is finite for all $x_0\in\widetilde{\mathcal{G}}$. In particular, the condition* (Cap) *implies* (Sign) *(see Theorem IV.3.1 and Corollary IV.3.2 for details).*

2) *If $\kappa\equiv 0$, then $\widetilde{h}_*\geq 0$ (see Corollary IV.3.2 for details).*

3) *The following implications hold true:*

$$\widetilde{h}_*\leq 0 \overset{\text{Cor. IV.3.7}}{\underset{a)}{\Longleftrightarrow}} \text{(Sign)} \overset{\text{Thm. IV.3.3}}{\underset{b)}{\Longrightarrow}} (\text{Law}_h)_{h\geq 0}$$

and

$$(\text{Law}_0) \overset{\text{Thm. IV.3.4}}{\underset{c)}{(\Leftarrow)\!\!\Longrightarrow}} \text{(Isom)} \overset{\text{Prop. IV.4.7}}{\underset{d)}{(\Leftarrow)\!\!\Longrightarrow}} (\text{Law}_h)_{h\geq 0}.$$

In particular, if $\mathcal{G}$ is a transient weighted graph such that $\kappa\equiv 0$ and (Cap) *is fulfilled, then $\widetilde{h}_*=0$ and the law of $\mathrm{cap}(E^{\geq h}(x_0))$ is characterized by* (Law$_h$)*, for $h\geq 0$ (equivalently,* (Isom) *holds).*

Let us explain and comment the results in Theorem IV.1.1 in details. In Theorem IV.1.1,1), one can see the inequality $\mathrm{cap}(E^{\geq 0}(x_0))<\infty$ as an indication that the sign clusters of the Gaussian free field on the cable system do not percolate, at least in terms of capacity, see Theorem IV.3.1 for details. Condition (Cap) makes the previous intuition correct, since it directly implies that closed connected sets have finite capacity if and only if they are bounded, and so (Sign) holds, that is $\widetilde{h} \leq 0$, see (IV.1.4). The inequality $\widetilde{h}_* \leq 0$ had previously already been proved on a certain number of graphs with $\kappa \equiv 0$:

- $\mathbb{Z}^d$, $d \geq 3$, with unit weights, see Theorem 1 and Proposition 5.5 in [57]. This proof could actually easily be extended to any vertex-transitive and amenable graph, and these graphs verify (Cap), see Lemma IV.4.1.

- The $(d+1)$-regular tree, $d \geq 2$, with unit weights, see Proposition 4.1 in [101]. It is easy to prove that these graphs verify (Cap), since one can find a uniform upper bound for the equilibrium measure of a set.

- Any tree T with unit weights such that $\{x \in T : R_x^\infty > A\}$ only has bounded components for some $A > 0$, where R_x^∞ is the effective resistance between x and infinity for the descendants of x, see Proposition 2.2 in [1]. These graphs verify (Cap), see Lemma IV.4.2.

- Any transient graph with bounded weights, such that (V_α) holds, that is the volume of a ball increases polynomially fast, and (G_β) holds, that is the Green function decreases polynomially fast, see Proposition III.5.2. These graphs verify (Cap), see Lemma III.3.2.

Therefore, Theorem IV.1.1,1) is a generalization of all these previously known result, and also include many other graphs, such as any vertex-transitive graphs, see Lemma IV.4.1. Note that without assuming that (Cap) is fulfilled, it is possible to find a graph $\mathcal{G}$ such that $\widetilde{h}_* > 0$, and we give an example of such a graph in Section IV.9. One can also easily find an example of a graph such that (Sign) is verified, but not (Cap), see Remark IV.9.2,4)

Theorem IV.1.1,2) asserts that, in the case of massless graphs $\kappa \equiv 0$, $E^{\geq h}$ contains an unbounded and connected set with positive probability for all $h < 0$. This was first proved on $\mathbb{Z}^d$, $d \geq 3$, for the discrete Gaussian free field in [16], but when $\kappa \equiv 0$ one can adapt their proof to the Gaussian free field on the cable system on any transient graph, and obtain that $E^{\geq h}$ contains an unbounded and connected set with positive probability for all $h < 0$, see also the Appendix of [1]. We also give new proofs of this result. If $\kappa \not\equiv 0$, then it is possible to have $\widetilde{h}_* < 0$. For instance, if $\mathcal{G}$ is a finite transient graph, then it only contains bounded sets, and so $\widetilde{h}_* = -\infty$.

Let us now comment on the various implications in Theorem IV.1.1,3). The equivalence a) asserts that if $\widetilde{h}_* = 0$, then the level sets of the Gaussian free field never percolate at the critical point $h = 0$, even if (Cap) is not verified. The implication b) asserts that, if the sign clusters of the Gaussian free field are bounded, for instance under condition (Cap), then there is an explicit formula for the law of the positive level sets of the Gaussian free field on the cable system given by $(\mathrm{Law}_h)_{h\geq 0}$. One can use this formula to directly obtain bounds on the probability that $E^{\geq h}(x_0)$ exits a large ball, as well as on the critical window for h, see (IV.3.9) or (IV.3.11), which generalize the results from Theorems 3 to 6 in [24], from $\mathbb{Z}^d$, $d \geq 3$, to any transient weighted graph fulfilling (Sign). In Theorem IV.3.3, we also give an explicit formula for the law of the negative level sets of the Gaussian free field under condition (Cap). An explicit formula for the probability that $x \longleftrightarrow y$ in $E^{\geq 0}$ has also already been obtained in Proposition 5.2 of [57], and was the key ingredient for all the previous proofs of the inequality $\widetilde{h}_* \leq 0$.

The direct implication in c) asserts that it is enough to know that the law of the capacity of the sign clusters is given by (Law_0) to obtain the strong version of the isomorphism (Isom). In particular, if (Sign) is verified, then (Isom) holds, which is a generalization of the version of the isomorphism from [101], where an additional assumption on the boundedness on the Green function is required. Moreover, there are graphs verifying (Law_0), and thus (Isom), but not (Sign), see Remark IV.9.2,3). According to the direct implication in d), the isomorphism (Isom) implies that for all $h \geq 0$, the law of the capacity of the level sets is given by (Law_h). This follows directly, see Lemma IV.4.7, from a slightly different version of the isomorphism that we denote by (Isom'), and which is stated in Section IV.3. It includes the law of the signs of φ on the left-hand side of (Isom), which are given in Lemma 3.2 of [57]. We also give a version of the isomorphism (Isom') for the discrete graph $\mathcal{G}$ in Theorem IV.3.4, similar to the version of the second Ray-Knight theorem from Theorem 8 in [58]. Finally, since $(\mathrm{Law}_h)_{h\geq 0}$ trivially implies (Law_0), the direct implications in c) and d) are actually equivalences.

We now present the ideas behind the proofs of the various results in Theorem IV.1.1. It is surprising that, under the weak condition $\kappa \equiv 0$ and (Cap), the critical parameter $\widetilde{h}_*$ and the law of the capacity of the positive level sets $\mathrm{cap}(E^{\geq h}(x_0))$ are explicitly known, and are almost independent of the nature of the graph. In order to gain a better understanding of this result, we present in Sections IV.5, IV.6 and IV.7 three independent proofs of 1), 2) and 3,b) in Theorem IV.1.1. The first two proofs both involve the average value M_K of the Gaussian free field weighted by the equilibrium measure of K, where K is a compact of $\widetilde{\mathcal{G}}$, that is

$$M_K = \sum_{x\in\partial K} e_K(x)\varphi_x. \tag{IV.1.6}$$

One can show using the strong Markov property for the Gaussian free field, see (IV.2.31), that if $\mathcal{K}$ is a random compact of $\widetilde{\mathcal{G}}$, which essentially only depends on the Gaussian free field around this compact, then the law of M_K, conditionally on knowing φ on $\mathcal{K}$, is Gaussian with mean $M_{\mathcal{K}}$ and variance $\mathrm{cap}(K) - \mathrm{cap}(\mathcal{K})$; we refer to Lemma IV.2.3 for a precise statement.

The first proof involves Russo's formula for the Gaussian free field on the cable system, see Proposition IV.5.1, which was first introduced on discrete graphs in Proposition 3.2 of [79], and bears some similarities to Russo's formula for Bernoulli percolation, see [82]. This formula relates the derivative in h of the probability of an event depending only on the level-sets $E^{\geq h} \cap K$, for some compact K, to the average value of M_K when this event happens. In order to obtain Theorem IV.3.1, one consider events of the type $\{\mathrm{cap}(E^{\geq h}(x_0) \cap K) \geq t\}$ for some $t > \mathrm{cap}(\{x_0\})$. Conditionally on $\{\mathrm{cap}(E^{\geq h}(x_0) \cap K) \geq t\}$, one can then use the previously mentioned Markov property to replace the average value of M_K by $M_{E^{\geq h}(x_0) \cap K}$ in Russo's formula. Moreover, $\varphi \geq h$ on $E^{\geq h}(x_0) \cap K$, and so $M_{E^{\geq h}(x_0) \cap K} \geq ht$, and we thus obtain a differential inequality for the probability of the event $\{\mathrm{cap}(E^{\geq h}(x_0) \cap K) \geq t\}$. Solving this differential inequality let us compare this probability of this event at level h and level 0, see (IV.5.5), and taking $K \nearrow \widetilde{\mathcal{G}}$ and $t \to \infty$ we obtain Theorem IV.3.1. The proof of Theorem IV.3.3 is similar, but considering events of the type "$\{\mathrm{cap}(E^{\geq h}(x_0)) \approx t, E^{\geq h}(x_0) \subset K\}$" instead, and using that, if $E^{\geq h}(x_0) \subset K$, then $\varphi = h$ on the boundary of $E^{\geq h}(x_0)$ by continuity of the Gaussian free field, to obtain a differential equality instead of a differential inequality. This implies in particular that a version of (IV.3.7) still holds on *any* transient graph, that is $\mathrm{cap}(E^{\geq h}(x_0))$ have the same law for positive and negative h when $E^{\geq h}(x_0)$ is compact, see (IV.5.9).

The second proof involves an exploration martingale, similar to the one introduced in [24] on $\mathbb{Z}^d$, $d \geq 3$, see also Lemma 4.2 in [101] for a similar martingale on the $(d+1)$-regular tree, $d \geq 2$. To explore $E^{\geq h}(x_0)$ for some fixed vertex $x_0 \in \widetilde{\mathcal{G}}$, define $\mathcal{K}_t^{(h)}$ the set of points in $E^{\geq h}(x_0)$ at (chemical) distance at most t from x_0 and $\mathcal{M}_t^{(h)} = M_{\mathcal{K}_t^{(h)}}$, the average value of the Gaussian free field weighted by the equilibrium measure of the set of points explored at time $t \geq 0$. One can then use the previously mentioned Markov property to prove that $(\mathcal{M}_t^{(h)})_{t \geq 0}$ is a continuous martingale, and that its quadratic variation is $\mathrm{cap}(\mathcal{K}_t^{(h)})$, see Lemma IV.6.1. In particular, by usual martingale theory, up to time modification by $(\mathrm{cap}(\mathcal{K}_t^{(h)}))_{t \geq 0}$, $(\mathcal{M}_t^{(h)})_{t \geq 0}$ is a Brownian motion, see Lemma IV.6.2. Moreover, $\varphi \geq h$ on the boundary of $\mathcal{K}_t^{(h)}$, and so $\mathcal{M}_t^{(h)} \geq h\mathrm{cap}(\mathcal{K}_t^{(h)})$, and, if the exploration is stopped at time T, then by continuity of the Gaussian free field $\varphi = h$ on the boundary of $\mathcal{K}_T^{(h)}$ and $\mathcal{K}_T^{(h)} = E^{\geq h}(x_0)$, and so $\mathcal{M}_T^{(h)} = h\mathrm{cap}(\mathcal{K}_T^{(h)})$ and $\mathrm{cap}(\mathcal{K}_T^{(h)}) = \mathrm{cap}(E^{\geq h}(x_0))$. After time change by $\mathrm{cap}(\mathcal{K}_t^{(h)})$, one can thus link the probability that $\mathrm{cap}(E^{\geq h}(x_0)) > s$ to the probability that a Brownian motion starting at a positive value and with drift $-ht$ has not reached 0 at time s, see Proposition IV.6.3, and using usual formulas for this distribution, see (IV.6.5), Theorems IV.3.1 and IV.3.3 follow.

The last proof is more involved, but let us obtain all the results from Theorem IV.3.1 at once. We are mainly going to prove the strong isomorphism (Isom') between random interlacements and the Gaussian free field on an adequate class of graphs, and the identity $(\mathrm{Law}_h)_{h\geq 0}$ then follows at once, that is the direct implication in Theorem IV.1.1,3,d) holds. When $\mathcal{G}$ is a finite transient graph, the isomorphism (Isom) is a direct consequence of the isomorphism between loop soups, see [54] and [57], and the Gaussian free field that we recall in Theorem IV.7.1, and we refer to Lemma IV.7.2, proved in the Appendix, for details. Once (Isom), and thus $(\mathrm{Law}_h)_{h\geq 0}$, have been proved on finite transient graphs, we are going to approximate the Gaussian free field on any infinite transient graph $\mathcal{G}$ by the Gaussian free field on a sequence of finite transient graph $\mathcal{G}_n$ increasing to $\mathcal{G}$, see Lemma IV.7.4, to obtain our third proof of 1), 2) and 3,b) in Theorem IV.1.1, see the end of Section IV.7 for details.

It is interesting to extend (Isom) to infinite transient graphs, which is done in Section IV.8, since it also provides us with the equivalences a), c) and d) in Theorem IV.1.1,3). Moreover, the isomorphism (Isom), as stated in Theorem 2.4 in [101] under stronger conditions, has already been useful in [101] and [1] to compare the critical parameter for the percolation of random interlacements and the Gaussian free field on discrete trees, and in Chapter III to prove strong percolation for the level sets of the discrete Gaussian free field at a positive level on a large class graphs, for instance $\mathbb{Z}^d$, $d \geq 3$, or various fractal graphs. It is not always easy to check that the conditions of Theorem 2.4 in [101] are exactly verified, see the proof of Corollary III.5.3 which sparked our interest, and it thus can be interesting to replace them by the stronger condition (Law_0). Moreover, if $\mathcal{G}$ is a graph such that (Isom) holds, this implies a stronger statement than (IV.3.7): conditionally on being compact, the level sets of the Gaussian free field above level h and $-h$ have not only the same law for their capacity, but in fact the same law, see Proposition IV.4.7.

In order to prove (Isom) on infinite transient graphs, we approximate random interlacements on infinite graphs by random interlacements on finite graphs, see Lemma IV.8.2, and using as well the previously mentioned approximation for the Gaussian free field, we obtain (Isom) if (Sign) or (Law_0) is fulfilled, see Lemma IV.8.3, that is Theorem IV.1.1,3,c) holds. Moreover, our proof of (Isom) by approximation by finite graphs, instead of the Markov property as in [101], let us also derive immediately a signed version of the isomorphism for random interlacements on discrete graphs, (IV.3.15), from the equivalent discrete isomorphism for the loop soup, (IV.7.3).

Finally, the isomorphism (Isom) has another interesting consequence, stated in Corollary IV.3.6: if (Sign) does not hold, then $\widetilde{h}_* = \infty$. In particular, if $\mathcal{G}$ is a graph such that $\widetilde{h}_* \leq 0$, then $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded for all $h > 0$, and thus (Law$_h$) holds for all $h > 0$, see Theorem IV.3.3. Taking the limit as $h \searrow 0$, one can then prove that (Law$_0$), and thus (Isom), hold. Since $\widetilde{h}_* \neq \infty$, this means that (Sign) must hold, and thus we obtain Theorem IV.1.1,3,a).

We now explain how the chapter is organized. Section IV.2 recalls the main objects of interest, the diffusion X, the Gaussian free field, and random interlacements on the cable system, as well as their properties. For later use, the notion of equilibrium measure and capacity are also extended to the cable system, see Lemma IV.2.1, (IV.2.18) and (IV.2.20).

Section IV.3 collects and explains in details our results, which together imply Theorem IV.1.1, and that we will prove throughout the rest of the chapter.

Section IV.4 gathers various interesting results. It first gives some simple consequences and equivalent formulations of our main conditions (Cap) and $\kappa \equiv 0$, and presents several example of graphs on which they are verified. Then, the law of $\mathrm{cap}(E^{\geq h}(x_0))$, as given by its Laplace transform in (Law$_h$), is further described by computing its density and asymptotics for its cumulative distribution function. Finally, simple consequences of the isomorphism (Isom) are derived at the end of the section, such as the the identity (Law$_h$).

Section IV.5, IV.6 and IV.7 are devoted to the three proofs of 1), 2) and 3,b) in Theorem IV.1.1, or more precisely Theorems IV.3.1 and IV.3.3. Section IV.5 contains the proof using Russo's formula, Proposition IV.5.1. An interesting porism, Corollary IV.5.3, is a relationship between the law of $\mathrm{cap}(E^{\geq h}(x_0))$ when $E^{\geq h}(x_0)$ is compact, and the law of $\mathrm{cap}(E^{\geq 0}(x_0))$ when $E^{\geq 0}(x_0)$ is compact, for all $h \in \mathbb{R}$ and on any transient graph, which generalize the result from (IV.3.7).

Section IV.6 is centered around the proof using an exploration martingale (IV.6.1). The main interest of this proof is a condition equivalent to (Law$_0$), even when (Sign) does not hold, in terms of the limit of the exploration martingale, see Remark IV.6.4.

The proof in Section IV.7 consists in proving the isomorphism (Isom) on finite graphs, see Lemma IV.7.2, as well as the approximation of the Gaussian free field on a graph $\mathcal{G}$, by Gaussian free fields on a sequence of graphs increasing to $\mathcal{G}$, Lemma IV.7.4.

Section IV.8 is devoted to the proof of the isomorphism between random interlacements and the Gaussian free field (Isom) under the condition (Law$_0$), and to its consequences, Corollaries IV.3.6 and IV.3.7. An important role is played by the approximation of random interlacements on a graph $\mathcal{G}$, by random interlacements on a sequence of graphs increasing to $\mathcal{G}$, see Lemmas IV.8.1 and IV.8.2. Some concluding remarks and open questions are gathered at the end of the section.

Section IV.9 gives an example of a graph for which $\widetilde{h}_* = \infty$, as opposed to Theorem IV.1.1. This graph is constructed as a d-regular tree, but with unbounded weights, or equivalently with unbounded length of the edges for its cable system.

For the reader's orientation, we note that the conditions (Sign), (Law$_h$) and (Isom) are introduced above Theorem IV.1.1, and that the condition (Isom') is introduced above Theorem IV.3.4.

The notations in this chapter, as well as in Chapter V, differ slightly from the notation in Chapters I, II III. First we denote by $\mathcal{G}$ the graph and by G its vertex set, whereas G was denoting both in previous chapters. The reason for this choice is that we are going to consider in this chapter different graphs with the same vertex set, see (IV.2.7) for instance, and it is thus now important to distinguish them. In the previous chapters, we denoted by $\widetilde{G}$ the cable system associated to G, and we will consequently denote it by $\widetilde{\mathcal{G}}$ from now on. Note that one additionally attaches to the graph $\widetilde{\mathcal{G}}$ an interval I_x for all $x \in G$, and we will denote by $\widetilde{\mathcal{G}}^E$ the graph obtained by glueing together only the intervals I_e for $e \in E$, which corresponds to the definition of the cable system from the previous chapters. When $\kappa \equiv 0$, which we always assumed until this chapter, adding the intervals I_x, $x \in G$, with infinite length plays essentially no role, see (IV.2.11) and Lemma IV.4.3.

Moreover, we will often add $\widetilde{\mathcal{G}}$ as a subscript or superscript to the notation when we want to precise the graph that we consider for our cable system, and $\mathcal{G}$ when we want to precise that we only consider the discrete graph. For instance the diffusion X on $\widetilde{\mathcal{G}}$ is defined under $P^x_{\widetilde{\mathcal{G}}}$, whereas the jump process Z on G is defined under $P^x_{\mathcal{G}}$. Finally, since we almost always only consider the cable system from now on, we removed many tilde to avoid cumbersome notation, for instance on the Gaussian free field φ, the level sets $E^{\geq h}$, the local times of random interlacements $\ell_{x,u}$, the random interlacements $\mathcal{I}^u$, or on the probabilities $\mathbb{P}^G$, $\mathbb{P}^I$ and P_x on the cable system, which were denoted by $\widetilde{\varphi}$, $\widetilde{E}^{\geq h}$, $\widetilde{\ell}_{x,u}$, $\widetilde{\mathcal{I}}^u$, $\widetilde{E}^{\geq h}$, $\widetilde{\mathbb{P}}^G$, $\widetilde{\mathbb{P}}^I$ and $\widetilde{P}_x$ in the previous chapters.

IV.2 Definition and useful results

Let us consider a weighted graph as described above (IV.1.1). For all $x_0 \in G$, we define a Markov jump process on $G \cup \{\Delta\}$, where Δ is a cemetery state, as follows: under the probabilities $P^{\mathcal{G}}_{x_0}$ the process $Z = (Z_t)_{t \geq 0}$ is started in x_0. If at x at a certain time, Z jumps to y with rate $\lambda_{x,y}$ and is killed with rate κ_x. The process Z is defined up to time ζ, see (IV.2.3), such that $\zeta < \infty$ if this process is either killed by κ at time ζ or blows up in finite time ζ. We always assume that Z is transient, and we denote by $(\widehat{Z}_n)_{n \in \mathbb{N}}$ the sequence of sites visited by the process Z, and the former is usually referred to as the discrete time skeleton of Z.

Let us now define the cable system $\widetilde{\mathcal{G}}$, a continuous version of the graph $\mathcal{G}$, and the diffusion associated to it. We first assume that $\kappa_x < \infty$ for all $x \in V$, the general case being addressed below (IV.2.4). To each edge $e = \{x, y\}$ of $\mathcal{G}$, an open interval I_e, isometric to $[0, \rho_{x,y}]$, see (IV.1.1), is attached; furthermore, to each vertex x of G, an open interval I_x of length $\rho_x (= \frac{1}{2\kappa_x})$ isometric to $[0, \rho_x)$ (possibly infinite) is attached. The cable system $\widetilde{\mathcal{G}}$ is then obtained by glueing together the intervals I_e, $e \in E$, to $\mathcal{G}$ through their respective endpoints, and glueing the starting point of I_x, $x \in G$, to x. In other words, $\widetilde{\mathcal{G}}$ is the metric graph where every edge e has been replaced by an interval of length ρ_e, and where we have added an interval of length ρ_x, possibly infinite, starting at every vertex $x \in G$. Then G can be interpreted as a subset of $\widetilde{\mathcal{G}}$. The elements of G will still be called vertices and the intervals I_e, $e \in E$ and I_x, $x \in G$, are referred to as the edges of $\widetilde{\mathcal{G}}$. The canonical distance on each I_e, $e \in E$, and I_x, $x \in G$, is denoted by $D_{\widetilde{\mathcal{G}}}$. Note that $D_{\widetilde{\mathcal{G}}}(x, y)$ is not defined if x and y are not on the same edge. For any edge $e = \{x, y\} \in E$ and any $t \in [0, \rho_{x,y}]$, we denote by $x + t \cdot I_e = y + (\rho_{x,y} - t) \cdot I_e$ the point of I_e at distance t from x, and for any vertex $x \in G$ and any $t \in [0, \rho_x)$, by $x + t \cdot I_x$ the point of I_x at distance t from x.

We also define a distance $d_{\widetilde{\mathcal{G}}}$ on $\widetilde{\mathcal{G}}$, such that $d_{\widetilde{\mathcal{G}}}(x, y)$, $x, y \in \widetilde{\mathcal{G}}$ is the minimal length of a continuous path between x and y, when changing the length of each I_e, $e \in E \cup G$ from ρ_e to 1. In particular the restriction of $d_{\widetilde{\mathcal{G}}}$ to the graph $\mathcal{G}$ is just the graph distance $d_{\mathcal{G}}$ on $\mathcal{G}$. One can see $\widetilde{\mathcal{G}}$ as a metric space for the distance $d_{\widetilde{\mathcal{G}}}$, and for a subset A of $\widetilde{\mathcal{G}}$ we define ∂A as the boundary of A in $\widetilde{\mathcal{G}}$ for this distance. For simplicity, we will say that a set K is a compact of $\widetilde{\mathcal{G}}$, if it is compact for the distance $d_{\widetilde{\mathcal{G}}}$ and has finitely many connected component, or equivalently if K is a finite unions of compacts of I_e, $e \in E \cup G$, with finitely many components. A connected set K is then compact if and only if it is closed, bounded and $K \cap I_x$ is a connected compact of I_x for all $x \in G$. Note that I_x

itself is not compact.

IV.2.1 The canonical diffusion on the cable system

We define the set of forward trajectories $W^+_{\widetilde{\mathcal{G}}}$ as the set of functions $w^+ : [0,\infty) \to \widetilde{\mathcal{G}} \cup \{\Delta\}$, for which there exists $\widetilde{\zeta} \in [0,\infty]$ such that $w^+_{|[0,\widetilde{\zeta})} \in C([0,\widetilde{\zeta}),\widetilde{\mathcal{G}})$ and, when $\widetilde{\zeta} < \infty$, $w^+(t) = \Delta$ for all $t \geq \widetilde{\zeta}$. For each $t \geq 0$ we denote by X_t the projection at time t, i.e. $X_t(w^+) = w^+(t)$ for all $w^+ \in W^+_{\widetilde{\mathcal{G}}}$, and by $\mathcal{W}^+_{\widetilde{\mathcal{G}}}$ the σ-algebra on $W^+_{\widetilde{\mathcal{G}}}$ generated by X_t, $t \geq 0$. The Lebesgue measure on $\widetilde{\mathcal{G}}$, which can informally be interpreted as the sum of the Lebesgue measures on each I_e, $e \in E$, and I_x, $x \in G$, is denoted by m, with the normalization $m(I_e) = \rho_e$ and $m(I_x) = \rho_x$. Let us now define a diffusion on $\widetilde{\mathcal{G}}$, which we will characterize through its associated Dirichlet form. In order to introduce the latter, we define for $f : \widetilde{\mathcal{G}} \to \mathbb{R}$ measurable

$$(f,f)_m = \sum_{e\in E\cup G} \int_{I_e} f^2 \,\mathrm{d}m_{|I_e}, \tag{IV.2.1}$$

$L^2(\widetilde{\mathcal{G}},m) := \{f : \widetilde{\mathcal{G}} \to \mathbb{R} \text{ measurable}; (f,f)_m < \infty\}$ and $(f,g)_m$ the associated quadratic form on $L^2(\widetilde{\mathcal{G}},m)$. Let $C_0(\widetilde{\mathcal{G}})$ be the closure for the $\|\cdot\|_\infty$ norm of the set of continuous function with compact support on $\widetilde{\mathcal{G}}$ and let $D(\widetilde{\mathcal{G}},m) \subset L^2(\widetilde{\mathcal{G}},m)$ be the space of functions $f \in C_0(\widetilde{\mathcal{G}})$ such that $f_{|I_e} \in W^{1,2}(I_e, m_{|I_e})$ for all $e \in E \cup G$ and

$$\sum_{e\in E\cup G} \|f_{|I_e}\|^2_{W^{1,2}(I_e,m_{|I_e})} < \infty,$$

where $W^{1,2}(I_e, m_{|I_e})$ denotes the Sobolev space on I_e. We now define the Dirichlet form on $L^2(\widetilde{\mathcal{G}},m)$

$$\mathcal{E}_{\widetilde{\mathcal{G}}}(f,g) \overset{\text{def.}}{=} \frac{1}{2}(f',g')_m \text{ for all } f,g \in D(\widetilde{\mathcal{G}},m). \tag{IV.2.2}$$

By Theorem 7.2.2. in [37], one can associate for each $x \in \widetilde{\mathcal{G}}$ an m-symmetric diffusion starting in x with state space $\widetilde{\mathcal{G}}$ to the Dirichlet form $\mathcal{E}_{\widetilde{\mathcal{G}}}$, and we denote by $P^{\widetilde{\mathcal{G}}}_x$ its law on $(W^+_{\widetilde{\mathcal{G}}}, \mathcal{W}^+_{\widetilde{\mathcal{G}}})$. We then let

$$\widetilde{\zeta} = \inf\{t \geq 0 : X_t = \Delta\} \text{ and } \zeta = \inf\{t \geq 0 : Z_t = \Delta\}. \tag{IV.2.3}$$

Informally $\widetilde{\zeta}$ is either ∞, or the first time X blows up (i.e. escapes all bounded sets) or gets killed (i.e. exits $\widetilde{\mathcal{G}}$ through some I_x with $\kappa_x > 0$). We refer to Section 5 of [14], Section 2 of [36] and Section 2 of [57] for more details about the metric graph $\widetilde{\mathcal{G}}$ and its associated diffusion X, the Brownian motion on $\widetilde{\mathcal{G}}$. Note that in these references the cable system is defined without the cables I_x, $x \in G$, but in fact the two definitions are essentially equivalent, see the discussion below (IV.2.11). Informally, one can obtain a diffusion with law $P_x^{\widetilde{\mathcal{G}}}$ as follows: first run a Brownian motion starting at x on I_e, with $x \in I_e$, $e \in E \cup G$, until a vertex y is reached, then choose uniformly at random an edge, or a vertex, $e \in E \cup G$ which contains y, or is equal to y, run a Brownian excursion on I_e until a vertex is reached, and iterate this process until either the process blows up (i.e., it escapes all bounded sets) or the open end of the interval I_x is reached for some $x \in G$, and in the latter case the process is killed at that time. We refer to Section II.2 for a more accurate description of this construction on $\mathbb{Z}^d$, $d \geq 3$. We also define for any non-negative measure μ on $\widetilde{\mathcal{G}}$ with countable support $\mathrm{supp}(\mu)$ the probabilities

$$P_\mu^{\widetilde{\mathcal{G}}} \overset{\text{def.}}{=} \sum_{x \in \mathrm{supp}(\mu)} \mu_x P_x^{\widetilde{\mathcal{G}}}. \tag{IV.2.4}$$

Let us now extend our definition of the cable system $\widetilde{\mathcal{G}}$ and the diffusion $P_x^{\widetilde{\mathcal{G}}}$ to any graph $\mathcal{G}$ with $\kappa \in [0, \infty]$. Let

$$E_\kappa = \big\{\{x, y\} : x, y \in V, \lambda_{x,y} > 0, \kappa_x = \infty \text{ and } \kappa_y < \infty\big\}. \tag{IV.2.5}$$

Let $\mathcal{G}^{(\infty)}$ be the graph with vertex set $G \cup E_\kappa$, edges between each $x \sim y \in G$, as well as edges between x and e for all $e \in E_\kappa$ and $x \in G \cap e$. The symmetric weights and killing measure on $\mathcal{G}^{(\infty)}$ are given for all $x, y \in G \cup E_\kappa$ by

$$\lambda_{x,y}^{(\infty)} = \begin{cases} \lambda_{x,y} & \text{if } x, y \in G, \\ 2\lambda_{x,z} & \text{if } y = e = \{x, z\} \in E_\kappa, \\ 0 & \text{otherwise,} \end{cases}$$
$$\text{and} \quad \kappa_x^{(\infty)} = \begin{cases} \kappa_x & \text{if } x \in G, \\ 2\lambda_{y,z} & \text{if } x = e = \{y, z\} \in E_\kappa. \end{cases} \tag{IV.2.6}$$

We then have that $\mathcal{G}^{(\infty)}$ is a graph with $\kappa^{(\infty)} < \infty$, and we can define $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}^{(\infty)}$ and $P_x^{\widetilde{\mathcal{G}}} := P_x^{\widetilde{\mathcal{G}}^{(\infty)}}$ for all $x \in \widetilde{\mathcal{G}}$. Using properties of exponential and geometric random variables, it is easy to prove that for all $x \in G$, the restriction of Z to G under $P_x^{\mathcal{G}^{(\infty)}}$ has the same law as Z under $P_x^{\mathcal{G}}$. To simplify notation, we will identify $\mathcal{G}$ and $\mathcal{G}^{(\infty)}$ in the rest of the chapter; in particular, unless explicitly mentioned otherwise, one can assume without loss of generality that $\kappa_x < \infty$ for all $x \in V$.

In the following, it will be useful to compare the diffusion X on $\widetilde{\mathcal{G}}$ for different values of the killing measure κ. For any killing measure κ' on V, we define

$\mathcal{G}_{\kappa'} = (G, \lambda, \kappa')$, and we view

$$\widetilde{\mathcal{G}}_{\kappa'} \subset \widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_{\kappa} \quad \text{if } \kappa' \geq \kappa, \tag{IV.2.7}$$

where $\kappa' \geq \kappa$ means $\kappa'_x \geq \kappa_x$ for all $x \in V$. We set, under $P_x^{\widetilde{\mathcal{G}}}$, $x \in \widetilde{\mathcal{G}}_{\kappa'} (\subset \widetilde{\mathcal{G}})$,

$$\begin{cases} X_t^{\kappa'}, \text{ if } t < \widetilde{\zeta}^{\kappa'} \\ \Delta, \text{ if } t \geq \widetilde{\zeta}^{\kappa'} \end{cases} \quad \text{where } \widetilde{\zeta}^{\kappa'} = \inf\{t \geq 0 : X_t \notin \widetilde{\mathcal{G}}_{\kappa'}\}, \tag{IV.2.8}$$

By Theorem 4.4.2. in [37], the Dirichlet form associated to $X_t^{\kappa'}$ is $\mathcal{E}_{\widetilde{\mathcal{G}}_{\kappa'}}$, and so

$$\text{the law of } X_t^{\kappa'} \text{ under } P_x^{\widetilde{\mathcal{G}}_{\kappa'}} \text{ is } P_x^{\widetilde{\mathcal{G}}} \text{ for all } x \in \widetilde{\mathcal{G}}_{\kappa'}. \tag{IV.2.9}$$

One can show, analogously to Section 2 of [57], that the process X under $P_x^{\widetilde{\mathcal{G}}}$ has a space-time continuous family of local times $(\ell_y(t))_{y \in \widetilde{\mathcal{G}}, t \geq 0}$. Therefore, using that $P_x^{\widetilde{\mathcal{G}}}$ lives on the canonical space $(W_{\widetilde{\mathcal{G}}}^+, \mathcal{W}_{\widetilde{\mathcal{G}}}^+)$, for all sets $F \subset E \cup G$ of the form

$$F = \bigcup_{e \in F_1} I_e \cup \bigcup_{x \in F_2} \{x\},$$

where $F_1 \subset E \cup G$ and $F_2 \subset G$ arbitrary, we can define the time change

$$\tau_t^F \overset{\text{def.}}{=} \inf\Big\{s > 0 : \int_0^s 1_{X_u \in \bigcup_{e \in F_1} I_e} \, \mathrm{d}u + \sum_{y \in F_2} \ell_y(s) > t\Big\} \text{ for all } t \geq 0.$$

Here, we use the convention $\inf \varnothing = \widetilde{\zeta}_\kappa$ and denote the print of X on F by $X^F = (X_{\tau_t^F})_{t \geq 0}$, which corresponds to a time changed process with respect to a PCAF, see (A.2.36) and below in [37] for example. As a first application of this definition, it follows from Theorem 6.2.1. in [37] that for all $x \in G$

$$\text{the print } X^G \text{ of } X \text{ on } \mathcal{G} \text{ has the same law under } P_x^{\widetilde{\mathcal{G}}} \text{ as } Z \text{ under } P_x^{\mathcal{G}}, \tag{IV.2.10}$$

and that the local times $(\ell_y(\widetilde{\zeta}))_{y \in G}$ of X after being killed have the same law under $P_x^{\widetilde{\mathcal{G}}}$ as the occupation times of the Markov jump process Z after being killed under $P_x^{\mathcal{G}}$, see for instance (1.97) and (2.80) in [98].

We define $\widetilde{\mathcal{G}}^E$ as the closed subset of $\widetilde{\mathcal{G}}$ which consist of the union of the intervals I_e, $e \in E$, and $X^{\widetilde{\mathcal{G}}^E}$, the print on $\widetilde{\mathcal{G}}^E$ of X, that is the cable system that we considered in the previous chapters when κ was equal to zero. One can prove by Theorem 6.2.1. in [37] that the Dirichlet form on $L_E^2(\widetilde{\mathcal{G}}^E, m_{|\widetilde{\mathcal{G}}^E}) := \{f \in L^2(\widetilde{\mathcal{G}}^E, m_{|\widetilde{\mathcal{G}}^E}) : \sum_{x \in G} \kappa_x f(x)^2 < \infty\}$ associated to the print $X^{\widetilde{\mathcal{G}}^E}$ of X on $\widetilde{\mathcal{G}}$ is

$$\begin{aligned} &\mathcal{E}_{\widetilde{\mathcal{G}}^E}(f,g) \overset{\text{def.}}{=} \frac{1}{2}(f',g')_{m_{|\widetilde{\mathcal{G}}^E}} + \sum_{x \in G} \kappa_x f(x) g(x) \\ &\text{for all } f,g \in D(\widetilde{\mathcal{G}}^E, m_{|\widetilde{\mathcal{G}}^E}) \cap L_E^2(\widetilde{\mathcal{G}}^E, m_{|\widetilde{\mathcal{G}}^E}). \end{aligned} \tag{IV.2.11}$$

If $\kappa \equiv 0$, the process $X^{\widetilde{\mathcal{G}}^E}$ thus corresponds to the usual diffusion on the cable system $\widetilde{\mathcal{G}}^E$. If $\kappa \geq 0$, it follows from Theorems 6.1.1. and A.2.11. in [37] that $X^{\widetilde{\mathcal{G}}^E}$ has the same law under $P_x^{\widetilde{\mathcal{G}}}$ as the diffusion $X^{\widetilde{\mathcal{G}}_0^E}$ under $P_x^{\widetilde{\mathcal{G}}_0}$ killed at time

$$\widetilde{\zeta}_\kappa^E = \inf\Big\{t < \widetilde{\zeta}_0 : \sum_{x\in G} \ell_x(t)\kappa_x \geq \xi\Big\},$$

where ξ is an independent exponential variable with parameter 1 and with the convention $\inf \varnothing = \widetilde{\zeta}_0$. This process has been studied in Section 2 of [57]: $X^{\widetilde{\mathcal{G}}^E}$ under $P^{\widetilde{\mathcal{G}}}$ also has print Z on G, and so the local times $(\ell_y(t))_{y\in\widetilde{\mathcal{G}}^E, t\geq 0}$ have the same law under $P_x^{\widetilde{\mathcal{G}}}$ as the local times of the process $X^{\widetilde{\mathcal{G}}^E}$ under $P_x^{\widetilde{\mathcal{G}}}$, that is the local times of the process introduced in Section 2 of [57].

We define the Green function on an open set $U \subset \widetilde{\mathcal{G}}$ by

$$g_U(x,y) = E_x^{\widetilde{\mathcal{G}}}[\ell_y(T_U)] \text{ for all } x,y \in \widetilde{\mathcal{G}}, \tag{IV.2.12}$$

where

$$T_U = \inf\{t \geq 0 : X_t \notin U\}$$

is the first exit time of U, with the convention $\inf \varnothing = \widetilde{\zeta}$, and $g_{\widetilde{\mathcal{G}}}$ is then the usual Green function on $\widetilde{\mathcal{G}}$. This definition of the Green function agrees with the usual definition of the Green function on $\mathcal{G}$ associated with the Markov process Z. Using the Markov property for X at time T_U, we have for all $U \subset \widetilde{\mathcal{G}}$

$$g_U(x,y) = g_{\widetilde{\mathcal{G}}}(x,y) - E_x^{\widetilde{\mathcal{G}}}\big[g_{\widetilde{\mathcal{G}}}(X_{T_U},y)\mathbb{1}_{T_U<\widetilde{\zeta}}\big]. \tag{IV.2.13}$$

IV.2.2 Equilibrium measure and capacity on the cable system

We now introduce the notions of equilibrium measure and capacity on the cable system $\widetilde{\mathcal{G}}$ as generalizations of the respective standard versions of these notions on transient graphs, see (IV.2.18), (IV.2.20) and (IV.2.27), and give several identities between the diffusion X and the equilibrium measure. For all finite $A \subset G$ the equilibrium measure and capacity of A are defined by

$$e_{A,\mathcal{G}}(x) \stackrel{\text{def.}}{=} \lambda_x P_x^{\mathcal{G}}(\widetilde{H}_A = \infty)\mathbb{1}_A(x) \text{ for all } x \in G, \quad \text{and} \quad \mathrm{cap}(A) = \sum_{x\in A} e_{A,\mathcal{G}}(x); \tag{IV.2.14}$$

here, $\widetilde{H}_A \stackrel{\text{def.}}{=} \inf\{n \geq 1,\ \widehat{Z}_n \in A\}$, with $\inf\varnothing = \infty$, is the first return time to A for the discrete time random walk $\widehat{Z}$ on $\mathcal{G}$. We now define a graph $\mathcal{G}^A$ for each finite $A \subset \widetilde{\mathcal{G}}$, which has almost the same cable system as $\mathcal{G}$, but contains A in its vertex set, and such that the diffusions X on $\widetilde{\mathcal{G}}_A$ and $\widetilde{\mathcal{G}}$ are essentially the same. For instance, $\mathcal{G}^{\{x\}}$ corresponds to the graph where we added x as a vertex between y and z when $x \in I_e$, $e = \{y, z\}$, or to the graph where we added x as a vertex with an edge between x and y and an adapted killing mass at x when $x \in I_y$, $y \in G$.

Lemma IV.2.1. *For all finite set $A \subset \widetilde{\mathcal{G}}$, there exists a unique graph $\mathcal{G}^A = (V^A, \lambda^A, \kappa^A)$ with vertex set $G^A = A \cup G$, such that*

- *in a slight abuse of notation, $\widetilde{\mathcal{G}}$ is a subset of $\widetilde{\mathcal{G}}^A$;*
- $$\begin{gathered}\text{for all } x \in G^A, \text{ the law of the print } (X_{\tau_t^{G^A}})_{t\geq 0} \\ \text{of } X \text{ on } G^A \text{ under } P_x^{\widetilde{\mathcal{G}}} \text{ is } P_x^{\mathcal{G}^A};\end{gathered} \tag{IV.2.15}$$
- *for all finite sets $K \subset G$,*
$$e_{K,\mathcal{G}^A}(x) = e_{K,\mathcal{G}}(x) \text{ for all } x \in G. \tag{IV.2.16}$$

Proof. We first consider the case where $A = \{y\}$ for some $y \in \widetilde{\mathcal{G}}$. If $y \in I_{\{x_0,x_1\}} \setminus \{x_0, x_1\}$ for some $x_0, x_1 \in G$ with $x_0 \sim x_1$, we introduce the graph $\mathcal{G}^{\{y\}}$ as the graph obtained by adding a vertex y to V, as well as replacing the edge between x_0 and x_1 by two edges between x_0 and y and between y and x_1. We thus take $V^{\{y\}} = V \cup \{y\}$, and the symmetric weights and killing measure on $\mathcal{G}^{\{y\}}$ are defined by

$$\lambda_{x,x'}^{\{y\}} = \begin{cases} \lambda_{x,x'} & \text{if } x, x' \notin \{x_0, x_1, y\}, \\ 0 & \text{if } \{x, x'\} = \{x_0, x_1\}, \\ \frac{1}{2D_{\widetilde{\mathcal{G}}}(x_i,y)} & \text{if } \{x, x'\} = \{x_i, y\} \end{cases} \text{ and } \kappa_x^{\{y\}} = \begin{cases} \kappa_x & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

By (IV.1.1) we have

$$\rho_{x_0,y}^{\{y\}} + \rho_{y,x_1}^{\{y\}} = D_{\widetilde{\mathcal{G}}}(x_0, y) + D_{\widetilde{\mathcal{G}}}(y, x_1) = D_{\widetilde{\mathcal{G}}}(x_0, x_1) = \rho_{x_0,x_1}.$$

One can then identify $\widetilde{\mathcal{G}}^{\{y\}} \setminus I_y$ with $\widetilde{\mathcal{G}}$, and by a similar reasoning as in (IV.2.11), for all $x \in \widetilde{\mathcal{G}}$, the law of the print of X on $\widetilde{\mathcal{G}}$ under $P_x^{\widetilde{\mathcal{G}}^{\{y\}}}$ is $P_x^{\widetilde{\mathcal{G}}}$. By (IV.2.10) applied to the graph $\mathcal{G}^{\{y\}}$, we thus obtain that (IV.2.15) holds for $A = \{y\}$. Moreover if K is a finite subset of G containing x_0, then we have

$$\begin{aligned} e_{K,\mathcal{G}^{\{y\}}}(x_0) &= \lambda_{x_0}^{\{y\}}\big(P_{x_0}^{\mathcal{G}^{\{y\}}}(\widehat{Z}_1 \neq y, \widetilde{H}_K = \infty) + P_{x_0}^{\mathcal{G}^{\{y\}}}(\widehat{Z}_1 = y, \widehat{Z}_2 = x_1, \widetilde{H}_K = \infty)\big) \\ &= \lambda_{x_0} P_{x_0}^{\mathcal{G}}(\widehat{Z}_1 \neq x_1, \widetilde{H}_K = \infty) + \frac{\lambda_{x_0,y}^{\{y\}}\lambda_{y,x_1}^{\{y\}}}{\lambda_{x_0,y}^{\{y\}} + \lambda_{y,x_1}^{\{y\}}} P_{x_1}^{\mathcal{G}}(\widetilde{H}_K = \infty) \\ &= \lambda_{x_0} P_{x_0}^{\mathcal{G}}(\widehat{Z}_1 \neq x_1, \widetilde{H}_K = \infty) + \lambda_{x_0,x_1} P_{x_1}^{\mathcal{G}}(\widetilde{H}_K = \infty) \\ &= e_{K,\mathcal{G}}(x_0). \end{aligned}$$

One can easily similarly prove that if K is a finite subset of G, then (IV.2.16) holds for $A=\{y\}$. If now $y\in I_{x_0}\setminus\{x\}$ for some $x_0\in G$, the graph $\mathcal{G}^{\{y\}}$ is the graph obtained by adding a vertex y to the graph $\mathcal{G}$ and an edge between x_0 and y. We thus take $V^{\{y\}}=V\cup\{y\}$, and the symmetric weights and killing measure on $\mathcal{G}^{\{y\}}$ are defined by

$$\lambda^{\{y\}}_{x,x'}=\begin{cases}\lambda_{x,x'} & \text{if } x\neq y \text{ and } x'\neq y,\\ \frac{1}{2D_{\widetilde{\mathcal{G}}}(x_0,y)} & \text{if } \{x,x'\}=\{x_0,y\},\end{cases}\quad \kappa^{\{y\}}_x=\begin{cases}\kappa_x & \text{if } x\notin\{y,x_0\},\\ 0 & \text{if } x=x_0,\\ \frac{\kappa_{x_0}}{1-2\kappa_{x_0}D_{\widetilde{\mathcal{G}}}(x_0,y)} & \text{if } x=y.\end{cases} \tag{IV.2.17}$$

One can show similarly as before that one can identify $\widetilde{\mathcal{G}}^{\{y\}}$ with $\widetilde{\mathcal{G}}$, and that (IV.2.15) and (IV.2.16) hold for $A=\{y\}$. If $y\in G$, we simply define $\mathcal{G}^{\{y\}}=\mathcal{G}$. One can easily conclude by induction on the number of vertices in A, noting that (IV.2.15) uniquely define the weights and killing measure of $\mathcal{G}^A$. □

When K is a compact subset of $\widetilde{\mathcal{G}}$, then since the number of connected component of K is finite by assumption, ∂K is finite. Thus, we can define the equilibrium measure of K in $\widetilde{\mathcal{G}}$ by

$$e_{K,\widetilde{\mathcal{G}}}(x)\stackrel{\text{def.}}{=}e_{\partial K,\mathcal{G}^{\partial K}}(x) \text{ for all } x\in\partial K \text{ and } e_{K,\widetilde{\mathcal{G}}}(x)\stackrel{\text{def.}}{=}0 \text{ for all } x\in(\partial K)^{\mathrm{c}}. \tag{IV.2.18}$$

For all $K\subset\widetilde{\mathcal{G}}$ compact and $A\subset\widetilde{\mathcal{G}}$ finite such that $\partial K\subset A$, one can see $\mathcal{G}^A$ as $(\mathcal{G}^{\partial K})^A$, and so applying (IV.2.16) to the graph $\mathcal{G}^{\partial K}$ we have

$$e_{\partial K,\mathcal{G}^A}(x)=e_{K,\widetilde{\mathcal{G}}}(x) \text{ for all } x\in A. \tag{IV.2.19}$$

Note that (IV.2.19) is trivial if $x\in A\setminus\partial K$ since both side of the equation are 0. We now define the capacity of a compact K with finitely many connected components on $\widetilde{\mathcal{G}}$ as the total mass of the equilibrium measure

$$\mathrm{cap}_{\widetilde{\mathcal{G}}}(K)\stackrel{\text{def.}}{=}\sum_{x\in\partial K}e_{K,\widetilde{\mathcal{G}}}(x), \tag{IV.2.20}$$

which coincides with the definition of the capacity from (IV.2.14) if $K\subset G$. We can also define $\mathrm{cap}_{\widetilde{\mathcal{G}}^E}(K):=\mathrm{cap}_{\widetilde{\mathcal{G}}}(K)$ for all compacts $K\subset\widetilde{\mathcal{G}}^E$, see above (IV.2.11). This definition clearly corresponds to the natural notion of capacity on $\widetilde{\mathcal{G}}^E$ that one could also construct directly similarly as in (IV.2.18) and (IV.2.20). When there is no ambiguity, in order to simplify notation we will write

$$P_x \text{ for } P_x^{\widetilde{\mathcal{G}}},\quad g(x,y) \text{ for } g_{\widetilde{\mathcal{G}}}(x,y),\quad e_K \text{ for } e_{K,\widetilde{\mathcal{G}}},\quad \text{and}\quad \mathrm{cap}(K) \text{ for } \mathrm{cap}_{\widetilde{\mathcal{G}}}(K).$$

Using (IV.2.15), (IV.2.18), and (IV.2.19), we can extend most of the results on equilibrium measures from the discrete case to the equilibrium measure on $\widetilde{\mathcal{G}}$. By

(1.57) in [98], one can easily show the following identity between the entrance time H_K of X in K, the stopped Green function and the equilibrium measure:

$$P_x(H_K < \widetilde{\zeta}) = \sum_{y \in \partial K} g(x,y) e_K(y). \tag{IV.2.21}$$

A useful characterization of the capacity in terms of a variational problem is

$$\mathrm{cap}(K) = \Big(\inf_{\mu} \sum_{x,y \in \partial K} g(x,y)\mu(x)\mu(y)\Big)^{-1} \tag{IV.2.22}$$

where the infimum is over all probability measures μ on ∂K, see for e.g. Proposition 1.9 in [98]. When $K \subset K'$ are two compacts of $\widetilde{\mathcal{G}}$, one has

$$P_{e_{K'}}(X_{H_K} = x, H_K < \widetilde{\zeta}) = e_K(x) \text{ for all } x \in \widetilde{\mathcal{G}}, \tag{IV.2.23}$$

which is usually referred to as "sweeping identity," see for e.g. (1.59) in [98]. In particular, summing (IV.2.23) on $x \in \partial K$, we infer the monotonicity property

$$\mathrm{cap}(K) \leq \mathrm{cap}(K'); \tag{IV.2.24}$$

note in the above references that while [98] deals with discrete graphs, the transfer of the respective results to the cable graph setting is immediate, also in the references below.

For any function $f : \widetilde{\mathcal{G}} \to \mathbb{R}$ and compact $K \subset \widetilde{\mathcal{G}}$ of $\widetilde{\mathcal{G}}$, we define the harmonic extension η_K^f of f on K by

$$\eta_K^f(x) = \sum_{y \in \partial K} P_x(X_{H_K} = y, H_K < \widetilde{\zeta}) f(y) \text{ for all } x \in \widetilde{\mathcal{G}}. \tag{IV.2.25}$$

Moreover, we say that an increasing sequence of compacts $(K_n)_{n\in\mathbb{N}}$ increases to a compact K if K is the closure of the union of K_n, $n \in \mathbb{N}$, and that a decreasing sequence of compacts $(K_n)_{n\in\mathbb{N}}$ decreases to a compact K if K is the intersection of K_n, $n \in \mathbb{N}$.

Lemma IV.2.2. *Let $f : \widetilde{\mathcal{G}} \to \mathbb{R}$ be a continuous function, and let $(K_n)_{n\in\mathbb{N}}$ and K be compacts of $\widetilde{\mathcal{G}}$ such that K_n increases to K or K_n decreases to K. Then for all $x \in \widetilde{\mathcal{G}}$*

$$\eta_{K_n}^f(x) \underset{n\to\infty}{\longrightarrow} \eta_K^f(x).$$

Proof. For all $y \in \partial K$, let $A_n^y = \{z \in \partial K_n : d_{\widetilde{\mathcal{G}}}(z,y) \leq d_{\widetilde{\mathcal{G}}}(z,y') \text{ for all } y' \in \partial K\}$. Then $\max_{z \in A_n^y} d_{\widetilde{\mathcal{G}}}(z,y) \underset{n\to\infty}{\longrightarrow} 0$ for all $y \in \partial K$, and there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $(A_n^y)_{y\in\partial K}$ is a partition of ∂K_n. By (IV.2.25), we have for all $x \in \widetilde{\mathcal{G}}$ and $n \geq N$ that

$$\begin{aligned}&\eta_K^f(x) - \eta_{K_n}^f(x)\\&= \sum_{y\in\partial K}\Big(P_x(X_{H_K} = y, H_K < \widetilde{\zeta})f(y) - \sum_{z\in A_n^y} P_x(X_{H_{K_n}} = z, H_{K_n} < \widetilde{\zeta})f(z)\Big).\end{aligned}$$

For any $\varepsilon > 0$ one can find $N' \geq N$ such that for all $n \geq N'$, $y \in \partial K$ and $z \in A_n^y$ we have $|f(y) - f(z)| \leq \varepsilon$. Therefore for all $x \in \widetilde{\mathcal{G}}$ and $n \geq N'$

$$\begin{aligned}&|\eta_K^f(x) - \eta_{K_n}^f(x)|\\&\leq \varepsilon + \sum_{y\in\partial K} f(y)\big|P_x(X_{H_K} = y, H_K < \widetilde{\zeta}) - P_x(X_{H_{K_n}} \in A_n^y, H_{K_n} < \widetilde{\zeta})\big|.\end{aligned}$$

Since for all $x \in \widetilde{\mathcal{G}}$ and $y \in \partial K$ we have

$$\begin{aligned}&\big|P_x(X_{H_K} = y, H_K < \widetilde{\zeta}) - P_x(X_{H_{K_n}} \in A_n^y, H_{K_n} < \widetilde{\zeta})\big|\\&\leq P_x(X_{H_K} = y, X_{H_{K_n}} \notin A_n^y, H_K < \widetilde{\zeta}, H_{K_n} < \widetilde{\zeta}) + P_x(H_K < \widetilde{\zeta}, H_{K_n} = \widetilde{\zeta})\\&+ P_x(X_{H_K} \neq y, X_{H_{K_n}} \in A_n^y, H_K < \widetilde{\zeta}, H_{K_n} < \widetilde{\zeta}) + P_x(H_K = \widetilde{\zeta}, H_{K_n} < \widetilde{\zeta})\\&\underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

we can conclude. □

Another interesting consequence of (IV.2.23) is then that, if $(K_n)_{n\in\mathbb{N}}$ and K are compacts of $\widetilde{\mathcal{G}}$ such that K_n increases to K or K_n decreases to K, by Lemma IV.2.2 with $f = 1$ we have

$$\lim_{n\to\infty} \mathrm{cap}(K_n) = \mathrm{cap}(K). \tag{IV.2.26}$$

Therefore, we can extend the definition of the capacity to any closed set $A \subset \widetilde{\mathcal{G}}$ with finitely many components by taking

$$\mathrm{cap}(A) = \lim_{n\to\infty} \mathrm{cap}(A \cap K_n), \tag{IV.2.27}$$

where $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compacts of $\widetilde{\mathcal{G}}$ converging to $\widetilde{\mathcal{G}}$. This limit exists, does not depend on the choice of the sequence $(K_n)_{n\in\mathbb{N}}$ by (IV.2.24), and is consistent with our previous definition of the capacity for compacts by (IV.2.26).

We now define for any compact K of $\widetilde{\mathcal{G}}$ the last exit time L_K of K by $L_K = \sup\{t \geq 0 : X_t \in K\}$, with the convention $\sup\varnothing = -\infty$. For all $x \in \partial K$ with $P_x^{\widetilde{\mathcal{G}}}(X_{L_K} = x) > 0$ we define

$$P_x^{K,\widetilde{\mathcal{G}}} \text{ as the law of } (X_{t+L_K})_{t\geq 0} \text{ under } P_x^{\widetilde{\mathcal{G}}}(\cdot \,|\, X_{L_K} = x). \tag{IV.2.28}$$

For all $x \in \partial K$ and $y \in \widetilde{\mathcal{G}}$, by (1.56) in [98] applied to the graph $\mathcal{G}^{\partial K \cup \{y\}}$ and (IV.2.15), we have

$$P_y^{\widetilde{\mathcal{G}}}(X_{L_K} = x) = g(y,x)e_K(x). \tag{IV.2.29}$$

In particular for each $x \in \partial K$, $P_x^{\widetilde{\mathcal{G}}}(X_{L_K} = x) > 0$ is in fact equivalent to $P_x^{\mathcal{G}^{\partial K}}(\widetilde{H}_K = \infty) > 0$, and the law of the print of X on $\mathcal{G}^{\partial K}$ under $P_x^{K,\widetilde{\mathcal{G}}}$ is then $P_x^{\mathcal{G}^{\partial K}}(\cdot \mid \widetilde{H}_K = \infty)$. In fact, one can show that the law of $(X_{t+L_K})_{t\geq 0}$ under $P_y^{\widetilde{\mathcal{G}}}(\cdot \mid L_K, X_{L_K} = x)$ is $P_x^{K,\widetilde{\mathcal{G}}}$ if $L_K \geq 0$ for all $y \in \widetilde{\mathcal{G}}$, and that $(X_t)_{t>0}$ is a strong Markov process, see Theorem 2.12 in [38].

IV.2.3 Gaussian free field

Let us now define recall some properties of the Gaussian free field $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ on the cable system $\widetilde{\mathcal{G}}$ of a transient graph $\mathcal{G}$, as defined in (IV.1.2). The process $(\varphi_x)_{x\in\widetilde{\mathcal{G}}^E}$ is a Gaussian free field on $\widetilde{\mathcal{G}}^E$ and has been studied in [57], and $(\varphi_x)_{x\in G}$ then has the same law under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ as the discrete Gaussian free field on the graph $\mathcal{G}$. We will write $\mathbb{P}^G_{\mathcal{G}}$ instead of $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ when we want to stress that we only consider the discrete Gaussian free field $(\varphi_x)_{x\in\mathcal{G}}$, and $\mathbb{P}^G$ instead of $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ when there is no ambiguity about the graph $\mathcal{G}$ that we consider. One of the most important properties of the Gaussian free field is the strong spatial Markov property, which we now shortly recall, see Section 1 of [101] for details. For any open set $O \subset \widetilde{\mathcal{G}}$, let us define the σ-algebra $\mathcal{A}_O = \sigma(\varphi_x,\, x \in O)$, and for any compact $K \subset \widetilde{\mathcal{G}}$ we define

$$\mathcal{A}_K^+ = \bigcap_{\varepsilon>0} \mathcal{A}_{K^\varepsilon},$$

where K^ε is the open ε ball around K for the distance $d_{\widetilde{\mathcal{G}}}$. We say that $\mathcal{K}$ is a compatible random compact of $\widetilde{\mathcal{G}}$ if $\mathcal{K}$ is always a compact of $\widetilde{\mathcal{G}}$ and $\{\mathcal{K} \subset O\} \in \mathcal{A}_0$ for any open set $O \subset \widetilde{\mathcal{G}}$. We then define

$$\mathcal{A}_{\mathcal{K}}^+ = \big\{A \in \mathcal{A}_{\widetilde{\mathcal{G}}} : A \cap \{\mathcal{K} \subset K\} \in \mathcal{A}_K^+ \text{ for all compacts } K \subset \widetilde{\mathcal{G}} \text{ which is the closure of its interior}\big\}. \tag{IV.2.30}$$

Now the Markov property states that for any compatible random compact $\mathcal{K}$,

$$\text{conditionally on } \mathcal{A}_{\mathcal{K}}^+,\ (\varphi_x)_{x\in\widetilde{\mathcal{G}}} \text{ is a Gaussian field with mean } \eta_{\mathcal{K}}^{\varphi} \text{ and covariance } g_{\mathcal{K}^c}, \tag{IV.2.31}$$

where $\eta_{\mathcal{K}}^{\varphi}$ was defined in (IV.2.25) and $g_{\mathcal{K}^c}$ in (IV.2.12). An application of the Markov property is that, conditionally on $(\varphi_x)_{x\in G}$, if $e = \{y,z\} \in E$, one can describe the law of $(\varphi_x)_{x\in I_e}$, as a Brownian bridge of length ρ_e between φ_y and φ_z of a Brownian motion with variance 2 at time 1, and these Brownian bridges are independent. Similarly, conditionally on $(\varphi_x)_{x\in G}$, one can describe the law of $(\varphi_x)_{x\in I_y}$, as a Brownian bridge of length ρ_y between φ_x and 0 of a Brownian motion with variance 2 at time 1 if $\kappa_y > 0$, and as a Brownian motion starting in φ_y with variance 2 at time 1 if $\kappa_y = 0$, and all these Brownian bridges and Brownian motions are independent. We refer to Section II.2 for a proof of this result on $\mathbb{Z}^d$, $d \geq 3$, which can easily be adapted to any transient graph. In particular, we have that

$$\begin{gathered}\text{conditionally on } (\varphi_x)_{x\in G}, \text{ the random variables } (\varphi_x)_{x\in I_e}, e \in E\cup G,\\ \text{are independent, and for all } e\in E\cup G, (\varphi_x)_{x\in I_e} \text{ only depends on } \varphi_{|e},\end{gathered} \tag{IV.2.32}$$

where $\varphi_{|e} = (\varphi_x, \varphi_y)$ if $e = \{x, y\} \in E$ and $\varphi_{|e} = \varphi_x$ if $e = x \in G$. Moreover, using the exact formula for the distribution of the maximum of a Brownian bridge, see for instance [13], Chapter IV.26, we have for all $e \in E \cup G$ and $h \geq 0$

$$\mathbb{P}^G(|\varphi_z - h| > 0 \text{ for all } z \in I_e \,|\, \varphi_{|e}) = \big(1 - p_e^{\mathcal{G}}(\varphi - h)\big)\mathbb{1}_{e\in E}, \tag{IV.2.33}$$

where for all $e = \{x, y\} \in E$ and $f : G \to \mathbb{R}$

$$p_e^{\mathcal{G}}(f) \stackrel{\text{def.}}{=} p_e^{u,\mathcal{G}}(f,0) = \begin{cases}\exp\big(-2\lambda_{x,y}f(x)f(y)\big) & \text{if } f(x)f(y)\geq 0,\\ 1 & \text{otherwise,}\end{cases} \tag{IV.2.34}$$

and $p_e^{u,\mathcal{G}}(f, 0)$ is defined in (IV.3.13), and is independent of the choice of $u > 0$. Let us now give another consequence of the Markov property (IV.2.31), which will later be essential to the proof of Theorems IV.3.1 and IV.3.3 both via Russo's formula in Section IV.5 and via the exploration martingale in Section IV.6. Recall the definition of M_K from (IV.1.6).

Lemma IV.2.3. *For any compact K of $\widetilde{\mathcal{G}}$ and compatible random compact $\mathcal{K}$ such that $\mathcal{K} \subset K$ $\mathbb{P}^G$-a.s,*

$$\begin{gathered}\textit{conditionally on } \mathcal{A}^+_{\mathcal{K}},\ M_K \textit{ is Gaussian}\\ \textit{with mean } M_{\mathcal{K}} \textit{ and variance } \operatorname{cap}(K) - \operatorname{cap}(\mathcal{K}).\end{gathered}$$

Proof. By (IV.2.31) and (IV.1.6), we have that, conditionally on $\mathcal{A}^+_{\mathcal{K}}$, M_K is Gaussian

$$\text{with mean } \sum_{x\in\partial K} e_K(x)\eta^{\varphi}_{\mathcal{K}}(x) \text{ and variance } \sum_{x,y\in\partial K} e_K(x)e_K(y)g_{\mathcal{K}^c}(x,y).$$

By (IV.2.23) and (IV.2.25) we have, $\mathbb{P}^G$-a.s,

$$\begin{aligned}\sum_{x\in\partial K} e_K(x)\eta_{\mathcal{K}}^{\varphi}(x) &= \sum_{y\in\partial\mathcal{K}}\varphi_y\sum_{x\in\partial K} e_K(x)P_x(X_{H_{\mathcal{K}}}=y,\, H_{\mathcal{K}}<\widetilde{\zeta})\\ &= \sum_{y\in\partial\mathcal{K}} e_{\mathcal{K}}(y)\varphi_y\\ &= M_{\mathcal{K}},\end{aligned}$$

and so the conditional mean of M_K is $M_{\mathcal{K}}$. Moreover, for any compact $K'\subset K$ of $\widetilde{\mathcal{G}}$, we have by (IV.2.21) and (IV.2.23) that

$$\begin{aligned}&\sum_{x,y\in\partial K} e_K(x)e_K(y)E^x\big[g(y,X_{H_{K'}})\mathbb{1}_{H_{K'}<\widetilde{\zeta}}\big]\\ &= \sum_{x\in\partial K} e_K(x)E^x\Big[\mathbb{1}_{H_{K'}<\widetilde{\zeta}}\sum_{y\in\partial K} e_K(y)g(y,X_{H_{K'}})\Big]\\ &= \sum_{x\in\partial K} e_K(x)P_x(H_{K'}<\widetilde{\zeta}) = \mathrm{cap}(K').\end{aligned}$$

Using (IV.2.13), and noting that $T_{\mathcal{K}^c}=H_{\mathcal{K}}$, we have for all $x,y\in K$

$$\begin{aligned}g_{\mathcal{K}^c}(x,y) &= g(x,y) - E^x\big[g(y,X_{H_{\mathcal{K}}})\mathbb{1}_{H_{\mathcal{K}}}<\widetilde{\zeta}\big]\\ &= E^x\big[g(y,X_{H_K})\mathbb{1}_{H_K}<\widetilde{\zeta}\big] - E^x\big[g(y,X_{H_{\mathcal{K}}})\mathbb{1}_{H_{\mathcal{K}}}<\widetilde{\zeta}\big],\end{aligned}$$

and therefore the conditional variance of M_K is $\mathrm{cap}(K)-\mathrm{cap}(\mathcal{K})$. □

One can also describe the law of the restriction of the Gaussian free field on any transient graph $\mathcal{G}$ to a connected compact K of $\widetilde{\mathcal{G}}$ by a Gaussian free field on a finite graph. Indeed, if $\partial K\subset G$, following Proposition 1.11 in [98], one can define a graph $\mathcal{G}_*^K$ with vertex set $G_*^K := \{x\in G:\,\exists\, e\in E, x\in e, K\cap I_e\neq\varnothing\}$, such that the restriction of the weights to $E\cap K^2$ is still $\lambda_{x,y}$ for all $\{x,y\}\in E\cap K^2$, the killing measure is $e_{G_*^K}(x)$ for all $x\in G_*^K$, and $g_{\mathcal{G}_*^K}(x,y)=g_{\mathcal{G}}(x,y)$ for all $x,y\in G_*^K$. We can then also see K as a subset of $\widetilde{\mathcal{G}}_*^K$, and for all $x,y\in K$ with $x\neq y$, considering the graph $\mathcal{G}^{\{x,y\}}$, it is easy to see that $g_{\widetilde{\mathcal{G}}_*^K}(x,y)=g_{\widetilde{\mathcal{G}}}(x,y)$ for all $x,y\in K$. Therefore, considering the graph $\mathcal{G}^{\partial K}$, for all connected compacts K of $\widetilde{\mathcal{G}}$, there exists a graph $\mathcal{G}_*^K$ with vertex set $G_*^K\cup\partial K$, killing measure $e_K(x)$ for all $x\in G_*^K\cup\partial K$, and such that, using (IV.1.2),

$$(\varphi_x)_{x\in K} \text{ has the same law under } \mathbb{P}^G_{\widetilde{\mathcal{G}}} \text{ and } \mathbb{P}^G_{\widetilde{\mathcal{G}}_*^K}. \tag{IV.2.35}$$

IV.2.4 Random interlacements

Let us now define our second object of interest, random interlacements on the cable system $\widetilde{\mathcal{G}}$, similarly as in [57] or [101]. We define the set of doubly infinite trajectories $W_{\widetilde{\mathcal{G}}}$ as the set of functions $w:\mathbb{R}\to\widetilde{\mathcal{G}}\cup\Delta$, for which there exist $-\infty\leq\widetilde{\zeta}^-<\widetilde{\zeta}^+\leq\infty$ such that $w_{|(\widetilde{\zeta}^-,\widetilde{\zeta}^+)}\in C((\widetilde{\zeta}^-,\widetilde{\zeta}^+),\widetilde{\mathcal{G}})$ and $w(t)=\Delta$ for all $t\notin(\widetilde{\zeta}^-,\widetilde{\zeta}^+)$. For each $w\in W_{\widetilde{\mathcal{G}}}$, we also define $p^*_{\widetilde{\mathcal{G}}}(w)$ as the equivalence class of w modulo time-shift; here, w and w' are equal modulo time-shift if there exists $t_0\in\mathbb{R}$ such that $w(t+t_0)=w(t)$ for all $t\in\mathbb{R}$, and $W^*_{\widetilde{\mathcal{G}}}=\{p^*_{\widetilde{\mathcal{G}}}(w),\,w\in W_{\widetilde{\mathcal{G}}}\}$. We define $\mathcal{W}_{\widetilde{\mathcal{G}}}$ the σ-algebra on $W_{\widetilde{\mathcal{G}}}$ generated by the coordinate functions, and $\mathcal{W}^*_{\widetilde{\mathcal{G}}}=\{A\subset W^*_{\widetilde{\mathcal{G}}}:(p^*_{\widetilde{\mathcal{G}}})^{-1}(A)\in\mathcal{W}_{\widetilde{\mathcal{G}}}\}$. For each compact K of $\widetilde{\mathcal{G}}$, we denote by $W^0_{K,\widetilde{\mathcal{G}}}$ the set of $w\in W_{\widetilde{\mathcal{G}}}$ with $H_K(w)=0$, where $H_K(w)=\inf\{t\in\mathbb{R}:w(t)\in K\}$, with the convention $\inf\varnothing=\widetilde{\zeta}^+$. By $W^*_{K,\widetilde{\mathcal{G}}}$ we denote the set of $w^*\in W^*_{\widetilde{\mathcal{G}}}$ such that $(p^*_{\widetilde{\mathcal{G}}})^{-1}(\{w^*\})\cap W^0_{K,\widetilde{\mathcal{G}}}\neq\varnothing$. For $w\in W_{\widetilde{\mathcal{G}}}$, we define the forward part of w as $(w(t))_{t\geq0}$ and the backward part of w as $(w(-t))_{t\geq0}$, which both are elements of $W^+_{\widetilde{\mathcal{G}}}$, see above (IV.2.1). For $w^*\in W^*_{K,\widetilde{\mathcal{G}}}$ we define the forward and backward part of w^* on hitting K, respectively, as the forward and the backward part of w, respectively, where w is the only trajectory in $(p^*_{\widetilde{\mathcal{G}}})^{-1}(\{w^*\})\cap W^0_{K,\widetilde{\mathcal{G}}}$. For a set $B\in\mathcal{W}_{\widetilde{\mathcal{G}}}$ we write

$$B^+\stackrel{\text{def.}}{=}\{(w(t))_{t\geq0}:\,w\in B\}\text{ and }B^-\stackrel{\text{def.}}{=}\{(w(-t))_{t\geq0}:\,w\in B\},$$

and it is clear that $B^+,B^-\in\mathcal{W}^+_{\widetilde{\mathcal{G}}}$. The set of $B\in\mathcal{W}_{\widetilde{\mathcal{G}}}$, $B\subset W^0_{K,\widetilde{\mathcal{G}}}$, such that B is equal to the set of $w\in W^0_{K,\widetilde{\mathcal{G}}}$ whose forward part is in B^+ and whose backward part is in B^-, is denoted by $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}}$. We then observe that $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}}$ and $\{A\in\mathcal{W}_{\widetilde{\mathcal{G}}}:\,W^0_{K,\widetilde{\mathcal{G}}}\cap A=\varnothing\}$ generate $\mathcal{W}_{\widetilde{\mathcal{G}}}$. Recalling the notational convention of (IV.2.28), we define a measure $Q_{K,\widetilde{\mathcal{G}}}$ on $\mathcal{W}_{\widetilde{\mathcal{G}}}$, whose restriction to $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}}$ is given by

$$Q_{K,\widetilde{\mathcal{G}}}=\sum_{x\in\partial K}e_K(x)P^{\widetilde{\mathcal{G}}}_x(\cdot^+)P^{K,\widetilde{\mathcal{G}}}_x(\cdot^-),\tag{IV.2.36}$$

and such that $Q_{K,\widetilde{\mathcal{G}}}(A)=0$ for all $A\in\mathcal{W}_{\widetilde{\mathcal{G}}}$ with $A\cap W^0_{K,\widetilde{\mathcal{G}}}=\varnothing$. Note that $P^{K,\widetilde{\mathcal{G}}}_x(\cdot^-)$ is well-defined whenever $P^{\widetilde{\mathcal{G}}}_x(X_{L_K}=x)>0$, that is $e_K(x)>0$ by (IV.2.29), and so the sum in (IV.2.36) is well-defined. There exists a unique measure $\nu_{\widetilde{\mathcal{G}}}$ on $W^*_{\widetilde{\mathcal{G}}}$, which is the intensity measure underlying random interlacements on $\widetilde{\mathcal{G}}$, such that for all compacts $K\subset\widetilde{\mathcal{G}}$,

$$\nu_{\widetilde{\mathcal{G}}}(A)=Q_{K,\widetilde{\mathcal{G}}}\big((p^*_{\widetilde{\mathcal{G}}})^{-1}(A)\big)\text{ for all }A\in\mathcal{W}^*_{\widetilde{\mathcal{G}}},A\subset W^*_{K,\widetilde{\mathcal{G}}}.\tag{IV.2.37}$$

We will not give a proof of the existence of the measure $\nu_{\widetilde{\mathcal{G}}}$; instead, we refer to [103] for a proof of the existence of such a measure on the discrete graph $\mathcal{G}$ when $\kappa\equiv0$, and to [57] for a proof of the existence of such a measure on the cable system associated to $\mathbb{Z}^d$, $d\geq3$. One can easily adapt these proofs to obtain a measure $\nu_{\widetilde{\mathcal{G}}}$ such that (IV.2.37) hold for all compacts K of $\widetilde{\mathcal{G}}$ with $\partial K\subset G$, even when $\kappa \not\equiv 0$. Now considering any compact K of $\widetilde{\mathcal{G}}$, one can thus construct a measure $\nu_{\widetilde{\mathcal{G}}^{\partial K}}$ such that (IV.2.37) holds for $\nu_{\widetilde{\mathcal{G}}^{\partial K}}$ and K, and using the fact that $P_x^{\widetilde{\mathcal{G}}}$ is the law of the print of X on $\widetilde{\mathcal{G}}$ under $P_x^{\widetilde{\mathcal{G}}^{\partial K}}$, one can easily prove that $\nu_{\widetilde{\mathcal{G}}}$ is the print on $\widetilde{\mathcal{G}}$ of $\nu_{\widetilde{\mathcal{G}}^{\partial K}}$, and so that (IV.2.37) also holds for $\nu_{\widetilde{\mathcal{G}}}$ and K. We refer to Section V.2 for a complete proof of the existence of the measure $\nu_{\widetilde{\mathcal{G}}}$, and an in depth study of the associated interlacements process.

The random interlacement process ω under some probability $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$ is a Poisson point process on $W^*_{\widetilde{\mathcal{G}}} \times (0,\infty)$ with intensity measure $\nu_{\widetilde{\mathcal{G}}} \otimes \lambda$, where λ is the Lebesgue measure on $(0,\infty)$. When $\kappa \not\equiv 0$, these trajectories can be killed before blowing up; in our setup this is realized by the trajectory exiting $\widetilde{\mathcal{G}}$ via I_x for some $x \in G$ with $\kappa_x > 0$, both for their forwards and backwards part. We also denote by ω^u the point process which consist of the trajectories in ω with label less than u, by $(\ell_{x,u})_{x\in\widetilde{\mathcal{G}}}$ the continuous field of local times relative to m on $\widetilde{\mathcal{G}}$ of ω_u and by $\mathcal{I}^u = \{x \in \widetilde{\mathcal{G}} : \ \ell_{x,u} > 0\}$ the interlacement set at level u, and one can easily show that it is characterized by the following identity: for any closed set $A \subset \widetilde{\mathcal{G}}$ with finitely many components, possibly non-compact,

$$\mathbb{P}^I_{\widetilde{\mathcal{G}}}(\mathcal{I}^u \cap A = \varnothing) = \exp\left(-u \operatorname{cap}(A)\right). \tag{IV.2.38}$$

The print $\omega_u^{\mathcal{G}}$ of ω_u on G has the same law under $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$, or equivalently $\mathbb{P}^I_{\mathcal{G}}$, as the usual discrete random interlacement process, see [103] in the case $\kappa \equiv 0$. If $\kappa \not\equiv 0$, a trajectory in $\omega_u^{\mathcal{G}}$ can start or end at a fixed point $x \in G$, and in this case we say that this trajectory is *killed* at x. We also define $\mathcal{I}^u_E \subset E \cup G$ to be the set of edges crossed by at least one single trajectory in $\omega_u^{\mathcal{G}}$, union with the set of vertices at which a trajectory in $\omega_u^{\mathcal{G}}$ is killed, and we shall write $\mathbb{P}^I_{\mathcal{G}}$ instead of $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$ in case we want to emphasize that we only consider $\omega_u^{\mathcal{G}}$. In the case $\lambda_{x,y} = \frac{T}{T+1}$ for all $x, y \in E$ and $\kappa_x = \frac{\deg(x)}{T+1}$ for all $x \in G$, $T > 0$, the discrete random interlacement process $\omega_u^{\mathcal{G}}$ corresponds to the model of finitary random interlacements studied in [15], and we refer to Proposition 4.1 in [15] for a proof of the correspondence.

Moreover, we can describe ω_u as follows: for any compact K of $\widetilde{\mathcal{G}}$, the law of the forward trajectories in ω_u on hitting K is a Poisson point process with intensity $uP^{\widetilde{\mathcal{G}}}_{e_K}$, and so it can be constructed from a Poisson point process with intensity $uP^{\mathcal{G}^{\partial K}}_{e_K}$ by adding Brownian excursions on the edges. Thus ω_u can be constructed from $\omega_u^{\mathcal{G}}$ by adding independent Brownian excursion on the edges, and we refer to [57] for more details on this construction. In particular

$$\begin{gathered}\text{conditionally on } \omega_u^{\mathcal{G}}, \text{ the random variables } (\ell_{x,u})_{x\in I_e}, e \in E \cup G,\\ \text{are independent, and for all } e \in E \cup G, (\ell_{x,u})_{x\in I_e} \text{ only depends on } \omega_{e,u}^{\mathcal{G}},\end{gathered} \tag{IV.2.39}$$

where $\omega_{e,u}^{\mathcal{G}}$ is the set of trajectories in $\omega_u^{\mathcal{G}}$ hitting e. When there is no ambiguity, we will simply write $\mathbb{P}^I$ for $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$, and ν for $\nu_{\widetilde{\mathcal{G}}}$.

IV.3 Statement of the Results

In this section, we explain and state our results, which we will prove in the rest of the chapter. Put together, these results imply in particular Theorem IV.1.1, but we are going to give more details here. Theorem IV.3.1, together with its consequence, Corollary IV.3.2, correspond to 1) and 2) in Theorem IV.1.1. In Theorem IV.3.3, we give the law of the capacity for all $h \in \mathbb{R}$ under condition (Cap), and not only for $h \geq 0$ as in Theorem IV.1.1. Once the law of the capacity is known, one can derive some bounds on the critical window as $h \to 0$, see (IV.3.9), and we give an example of this in (IV.3.11) under the condition (IV.3.10). We then study the isomorphism (Isom) between random interlacements and the Gaussian free field, and Theorem IV.3.4 gathers results corresponding to the equivalences 3),c) and 3),d) in Theorem IV.1.1. We also give another formulation of this isomorphism on the cable system, see (Isom'), as well as a version on the discrete graph, see (IV.3.15). Finally, we gather some interesting consequences of the isomorphism (Isom') in Proposition IV.3.5 and Corollaries IV.3.6 and IV.3.7.

In order to get a better understanding on why the conditions $\kappa \equiv 0$ and (Cap) are introduced in Theorem IV.1.1, we now introduce some useful additional critical parameters. Recall the definition of a compact set introduced at the beginning of Section IV.2, and that a set is compact if and only if it is bounded and its intersection with I_x is a compact of I_x (identified with a semi-open interval of length ρ_x) for all $x \in G$. Our second critical parameter (after $\widetilde{h}_*$, see (IV.1.4)) is then defined as

$$h_*^{\text{com}} = \inf\left\{h \in \mathbb{R};\, \mathbb{P}^G\left(\begin{array}{c} E^{\geq h} \text{ contains a non-compact} \\ \text{connected component}\end{array}\right) = 0\right\}. \tag{IV.3.1}$$

Every compact set is bounded, and so we always have $h_*^{\text{com}} \geq \widetilde{h}_*$. A third critical parameter, involving the capacity of $E^{\geq h}$, is

$$h_*^{\text{cap}} = \inf\left\{h \in \mathbb{R};\, \mathbb{P}^G\left(\begin{array}{c} E^{\geq h} \text{ contains a connected} \\ \text{component with infinite capacity}\end{array}\right) = 0\right\}. \tag{IV.3.2}$$

Every compact set has finite capacity, and so $h_*^{\text{com}} \geq h_*^{\text{cap}}$, and we therefore have that

$$\text{on any transient graph, } h_*^{\text{com}} \geq h_*^{\text{cap}} \text{ and } h_*^{\text{com}} \geq \widetilde{h}_*. \tag{IV.3.3}$$

On any graph such that $\kappa \equiv 0$ or (Cap) is verified, the situation is simpler. On the one hand, if $\kappa \equiv 0$, since I_x is an interval of infinite length for each $x \in \mathcal{G}$, one can easily prove that $E^{\geq h} \cap I_x$ is a compact of I_x for all $h \in \mathbb{R}$, and so $E^{\geq h}$ is compact if and only if it is bounded, see Proposition IV.4.4. By (IV.1.4), (IV.3.1) and (IV.3.3), we thus obtain that

$$\text{if } \mathcal{G} \text{ is a transient graph with } \kappa \equiv 0 \text{ then, } h_*^{\mathrm{com}} = \widetilde{h}_* \geq h_*^{\mathrm{cap}}. \tag{IV.3.4}$$

We refer to Proposition IV.9.1 for an example of a graph for which the inequality in (IV.3.4) is strict. On the other hand, if condition (Cap) is fulfilled, then every connected closed set with finite capacity is bounded, and so $h_*^{\mathrm{cap}} \geq \widetilde{h}_*$ by (IV.1.4) and (IV.3.2). In fact, one can also show that for all $x_0 \in \widetilde{\mathcal{G}}$, if $\mathrm{cap}(E^{\geq h}(x_0)) < \infty$, then $E^{\geq h}(x_0)$ is also compact, and so $h_*^{\mathrm{cap}} \geq h_*^{\mathrm{com}}$, see Proposition IV.4.4. By (IV.3.3), we thus obtain that

$$\text{if } \mathcal{G} \text{ is a transient graph verifying (Cap) then, } h_*^{\mathrm{com}} = h_*^{\mathrm{cap}} \geq \widetilde{h}_*. \tag{IV.3.5}$$

In particular, if $\mathcal{G}$ satisfies (Cap) and $\kappa \equiv 0$ as in Theorem IV.1.1, then from (IV.3.5) and (IV.3.4) it is clear that the three critical parameter h_*^{com}, $\widetilde{h}_*$ and h_*^{cap} coincide; hence, in this case, for proving that they are equal to zero, it is now enough to show that one is non-negative and another one is non-positive.

Theorem IV.3.1. *Assume $\mathcal{G}$ is transient. For all $x_0 \in \mathcal{G}$ and $h \geq 0$ the random variable* $\mathrm{cap}(E^{\geq h}(x_0))$ *is $\mathbb{P}^G$-a.s. finite, and for all $h < 0$ the level sets $E^{\geq h}(x_0)$ of x_0 is non-compact with positive probability. In particular, we have $h_*^{\mathrm{cap}} \leq 0$ and $h_*^{\mathrm{com}} \geq 0$.*

Using (IV.3.4) and (IV.3.5), as well as Proposition IV.4.4, we directly obtain the following corollary.

Corollary IV.3.2. *Assume $\mathcal{G}$ is transient. If $\mathcal{G}$ satisfies* (Cap)*, then* (Sign) *holds; in this case, in particular, $h_*^{\mathrm{com}} = h_*^{\mathrm{cap}} = 0$. If $\kappa \equiv 0$, then for all $h < 0$ the level sets $E^{\geq h}(x_0)$ of x_0 is unbounded with positive probability; in this case, in particular, $h_*^{\mathrm{com}} = \widetilde{h}_* \geq 0$. Therefore, if $\mathcal{G}$ satisfies* (Cap) *and $\kappa \equiv 0$, then $\widetilde{h}_* = h_*^{\mathrm{com}} = h_*^{\mathrm{cap}} = 0$.*

We refer to Proposition IV.9.1 for an example of a graph for which (Cap) is not satisfied, and $h_*^{\mathrm{com}} = \widetilde{h}_* = \infty$, and to Remark IV.9.2,4) for an example of a graph for which $h_*^{\mathrm{com}} = h_*^{\mathrm{cap}} = 0$, but the condition (Cap) is not fulfilled. An interesting direct consequence of Corollary IV.3.2 is that if $\mathcal{G}$ satisfies (Cap), then by symmetry $\{x \in \widetilde{\mathcal{G}} : |\varphi_x| > 0\}$ only contains compact connected components, and so the loop soup on $\widetilde{\mathcal{G}}$ at level $\frac{1}{2}$ also only contains compact connected component on which its local time are positive, see Theorem 1 in [57]. A fortiori, the discrete loop soup on G at level $\frac{1}{2}$ then only consists of finite clusters.

Let us now consider the case where $\mathcal{G}$ is a finite transient graph as a first example of graphs on which some critical parameters are equal to 0, in order to give an intuition about the results from Theorem IV.3.1 and Corollary IV.3.2. For all $h < 0$ and $x \in G$ such that $\kappa_x > 0$, since φ on I_x conditionally on φ_x has the same law as a Brownian bridge of length $\rho_x < \infty$ between φ_x and 0 of a Brownian motion with variance 2 at time 1, see the discussion below (IV.2.31), we have that $\mathbb{P}^G(\varphi_y \geq h \text{ for all } y \in I_x) > 0$, and since I_x is non-compact, we obtain $h_*^{\mathrm{com}} \geq 0$. Now similarly if $h \geq 0$, then $\mathbb{P}^G(\varphi_y \geq h \text{ for all } y \in I_x) = 0$ for all $x \in G$, and since G is finite, we obtain that $h_*^{\mathrm{com}} \leq 0$. Since (Cap) is trivially verified on finite graphs, we thus have by (IV.3.5) that $h_*^{\mathrm{com}} = h_*^{\mathrm{cap}} = 0$. Note however that $\widetilde{h}_* = -\infty$ since there is no unbounded sets on finite graphs, and so the inequality in (IV.3.5) can be strict.

We are now interested in the law of the level sets, under the condition that the level sets $E^{\geq h}$, $h \geq 0$, of the Gaussian free field only contain compact connected components. Note that by Corollary IV.3.2, on any graph $\mathcal{G}$ such that the condition (Cap) is satisfied, condition (Sign) is also satisfied, and so $E^{\geq h}$ contain only compact connected components for all $h \geq 0$.

Theorem IV.3.3. *Assume $\mathcal{G}$ is transient. For all $x_0 \in \mathcal{G}$ and $h \geq 0$ such that $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded, on the event $\{\varphi_{x_0} \geq h\}$ (in order to ensure non-triviality), the random variable* $\mathrm{cap}(E^{\geq h}(x_0))$ *has moment generating function given by* (Law$_h$) *and density given by*

$$\rho_h(t) = \frac{1}{2\pi t\sqrt{g(x_0,x_0)(t-g(x_0,x_0)^{-1})}} \exp\Big(-\frac{h^2 t}{2}\Big)\mathbb{1}_{t\geq g(x_0,x_0)^{-1}}. \tag{IV.3.6}$$

If $\mathcal{G}$ satisfies (Cap)*, then for each $h \geq 0$,* (Law$_h$) *holds, the random variable*

$$\mathrm{cap}(E^{\geq -h}(x_0))\mathbb{1}_{\mathrm{cap}(E^{\geq -h}(x_0))\in(0,\infty)} \text{ has the same law as } \mathrm{cap}(E^{\geq h}(x_0))\mathbb{1}_{\varphi_{x_0}\geq h}, \tag{IV.3.7}$$

and so

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq -h}(x_0)) = \infty) = \mathbb{P}^G(\varphi_{x_0} \in (-h,h)). \tag{IV.3.8}$$

Three independent proofs of Theorems IV.3.1 and IV.3.3 are given at the end of Sections IV.5, IV.6 and IV.7. In the case $\kappa \equiv 0$ one can replace in Theorems IV.3.1 and IV.3.3 the cable system $\widetilde{\mathcal{G}}$ by the cable system $\widetilde{\mathcal{G}}^E$, which corresponds to removing the edges I_x, $x \in G$, from $\widetilde{\mathcal{G}}$, see Lemma IV.4.3, and is the usual definition of the cable system, see [57]. One can deduce from Theorem IV.3.3 some bounds on the critical window as $h \to 0$, which are similar to the ones obtained in [24] on $\mathbb{Z}^d$, $d \geq 3$. Indeed, for all $n \in \mathbb{N}$ and $h \in \mathbb{R}$, due to the monotonicity of capacity (cf. (IV.2.24)) one has that

$$\begin{aligned}\mathbb{P}^G\big(\mathrm{cap}(E^{\geq h}(x_0)) \geq \mathrm{cap}(B(x_0,n)\big) &\leq \mathbb{P}^G(x_0 \longleftrightarrow B(x_0,n)^c \text{ in } E^{\geq h}) \\ &\leq \mathbb{P}^G\big(\mathrm{cap}(E^{\geq h}(x_0)) \geq \inf_{x_0\in A\subset G,\delta(A)\geq n} \mathrm{cap}(A)\big),\end{aligned} \tag{IV.3.9}$$

where the infimum is over connected sets A, $\delta(A)$ is the diameter of A and $B(x_0,n)$ is the ball of radius n for the distance $d_{\widetilde{\mathcal{G}}}$. One can then exploit the findings of Theorem IV.3.3 in order to derive asymptotics for $\mathbb{P}^G(\mathrm{cap}(E^{\geq h_n}(x_0)) \geq \cdot)$ as $h_n \to 0$, and we gather these results in Lemma IV.4.6. A particularly interesting example is when $\mathcal{G}$ is a transient graph such that for all $x_0 \in G$,

$$\mathrm{cap}(B(x_0,n)) \underset{n\to\infty}{\longrightarrow} \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\mathrm{cap}(B(x_0,n))}{\inf_{x_0\in A\subset G,\delta(A)\geq n}\mathrm{cap}(A)} < \infty. \tag{IV.3.10}$$

Note that (IV.3.10) directly implies that (Cap) is fulfilled, and so (Sign) holds true as well. The bound (IV.3.10) is fulfilled on any graph such that $\kappa \equiv 0$, the volume of $B(x_0,n)$ increases as n^α for some $\alpha > 2$, and the Green function $g(x,y)$ decreases as $|x-y|^{-\nu}$ for some $\nu \in (0,1)$, see (III.3.11) and (III.3.14), and an example of such a graph is $\mathcal{G} = \mathcal{G}' \times \mathbb{Z}$, where $\mathcal{G}'$ is the Sierpinski gasket, see [50]. As a direct consequence of (IV.3.9) and the asymptotics from Lemma IV.4.6, we then obtain that, under (IV.3.10), for any sequence $h_n \geq 0$,

$$\frac{\mathbb{P}^G(0 \longleftrightarrow B(x_0,n)^c \text{ in } E^{\geq h_n})}{\mathbb{P}^G(0 \longleftrightarrow B(x_0,n)^c \text{ in } E^{\geq 0})} \underset{n\to\infty}{\longrightarrow} 0 \text{ if and only if } h_n\sqrt{\mathrm{cap}(B(x_0,n)} \underset{n\to\infty}{\longrightarrow} \infty. \tag{IV.3.11}$$

One could also find some results on the critical window for $h_n \leq 0$ using (IV.3.9) and Lemma IV.4.6.

We now turn to our results about the isomorphism between random interlacements and the Gaussian free field (Isom), and its link with the condition (Law$_h$). We first present another formulation of the isomorphism (Isom), which will be useful later. It includes the law of the sign of φ on the left-hand sign of (Isom), which was first given in Lemma 3.2 in [57], and the simple proof of the equivalence between (Isom') and (Isom) is part of the proof of Theorem IV.3.4 given at the end of Section IV.8.

On some extension $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}$ of $\mathbb{P}^G_{\widetilde{\mathcal{G}}} \otimes \mathbb{P}^I_{\widetilde{\mathcal{G}}}$, that we simply denote by $\widetilde{\mathbb{P}}^I$ when there is no ambiguity, let us define for each $u > 0$ an additional process $(\sigma^u_x)_{x\in\widetilde{\mathcal{G}}} \in \{-1,1\}^{\widetilde{\mathcal{G}}}$, such that, conditionally on $(|\varphi_x|)_{x\in\widetilde{\mathcal{G}}}$ and ω_u, σ^u is constant on each of the cluster of $\{x \in \widetilde{\mathcal{G}} : \ 2\ell_{x,u} + \varphi_x^2 > 0\}$, $\sigma^u_x = 1$ for all $x \in \mathcal{I}^u$, and the values of σ^u on each other cluster are independent and uniformly distributed.

If $2\ell_{x,u}+\varphi_x^2=0$, the value of σ_x^u will not play any role in what follows, and one can fix it arbitrarily. Recalling the definition of $\mathcal{C}_u^\infty$ from below (Isom), it is clear that the clusters of $\{x\in\widetilde{\mathcal{G}}:\ 2\ell_{x,u}+\varphi_x^2>0\}$ are the union of the clusters of the interior of $\mathcal{C}_u^\infty$ and the clusters of $\{x\in\widetilde{\mathcal{G}}:|\varphi_x|>0\}\cap(\mathcal{C}_u^\infty)^c$, and so one can equivalently define σ^u as follows: $\sigma_x^u=1$ for all $x\in\mathcal{C}_u^\infty$, σ^u is constant on each of the cluster of $\{x\in\widetilde{\mathcal{G}}:\ |\varphi_x|>0\}\cap(\mathcal{C}_u^\infty)^c$, and its values on each cluster are independent and uniformly distributed. Let us now introduce another isomorphism between random interlacements and the Gaussian free field

$$\begin{aligned}&\big(\sigma_x^u\sqrt{2\ell_{x,u}+\varphi_x^2}\big)_{x\in\widetilde{\mathcal{G}}}\text{ has the same law under }\widetilde{\mathbb{P}}^I\\&\text{as }\big(\varphi_x+\sqrt{2u}\big)_{x\in\widetilde{\mathcal{G}}}\text{ under }\mathbb{P}^G\text{ for all }u>0.\end{aligned}\tag{Isom'}$$

If (Isom') holds, it follows by symmetry of the Gaussian free field that for all $u>0$, $E^{\geq\sqrt{2u}}$ has the same law under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ as $\{x\in\widetilde{\mathcal{G}}:\sigma_x^u=-1\}$ under $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}$ and $E^{\geq-\sqrt{2u}}$ has the same law under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ as $\{x\in\widetilde{\mathcal{G}}:\ \sigma_x^u=1\}$ under $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}$. Moreover, by definition of σ^u, the expectation of σ_x^u is the probability that $x\in\mathcal{C}_u^\infty$, that is the probability that the cluster of x in $\{x\in\widetilde{\mathcal{G}}:|\varphi_x|>0\}$ intersects $\mathcal{I}^u$. Using (IV.2.38), one can then directly prove that (Isom') implies (Law$_0$), see Proposition IV.4.7 for details. In the next theorem, that we prove at the end of Section IV.8, we show that (Isom'), or equivalently (Isom), is actually equivalent to (Law$_0$), or even $(\text{Law}_h)_{h>0}$, and we also give a formulation of (Isom') for the discrete graph $\mathcal{G}$.

Theorem IV.3.4. *Assume that $\mathcal{G}$ is transient. Then*

$$(\text{Law}_0)\iff(\text{Law}_h)_{h>0}\iff(\text{Isom})\iff(\text{Isom'}).\tag{IV.3.12}$$

Moreover, if one of the previous condition is fulfilled, on some extension $\widetilde{\mathbb{P}}^I_{\mathcal{G}}$ of $\mathbb{P}^G_{\mathcal{G}}\otimes\mathbb{P}^I_{\mathcal{G}}$, let us define for each $u>0$ a random set $\widehat{\mathcal{E}}_u\subset E\cup G$ such that, conditionally on $(\varphi_x)_{x\in G}$ and $\omega_u^{\mathcal{G}}$, $\widehat{\mathcal{E}}_u$ contains each edge and vertex in $\mathcal{I}_E^u$, and each additional edge and vertex $e\in E\cup G$ conditionally independently with probability $1-p_e^{u,\mathcal{G}}(\varphi,\ell_{\cdot,u})$, where $p_e^{u,\mathcal{G}}:\mathbb{R}^G\times[0,\infty)^G\to[0,1]$ is defined for each $e=\{x,y\}\in E$ by

$$p_e^{u,\mathcal{G}}(f,g)=\exp\Big(-\lambda_{x,y}\big(f(x)f(y)+\sqrt{(f(x)^2+2g(x))(f(y)^2+2g(y))}\big)\Big),\tag{IV.3.13}$$

and for each $x\in G$ by

$$p_x^{u,\mathcal{G}}(f,g)=\exp\Big(-\kappa_x\sqrt{2u(f(x)^2+2g(x))}\Big).\tag{IV.3.14}$$

Then $\widehat{\mathcal{E}}_u$ has the same law under $\widetilde{\mathbb{P}}^I_{\mathcal{G}}$ as $\mathcal{E}_u:=\{e\in E\cup G:\ 2\ell_{x,u}+\varphi_x^2>0\text{ for all }x\in I_e\}$ under $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}$. In particular, if we define a process $(\widehat{\sigma}_x^u)_{x\in G}\in$

$\{-1,1\}^G$, *such that, conditionally on* $(\varphi_x)_{x\in G}$, $\omega_u^{\mathcal{G}}$ *and* $\widehat{\mathcal{E}}_u$, $\widehat{\sigma}^u$ *is constant on each of the clusters induced by* $\widehat{\mathcal{E}}_u$, $\widehat{\sigma}_x^u=1$ *for all* $x\in\widehat{\mathcal{E}}_u\cap G$, *and the values of* $\widehat{\sigma}^u$ *on each other cluster are independent and uniformly distributed, then*

$$\big(\widehat{\sigma}_x^u\sqrt{2\ell_{x,u}+\varphi_x^2}\big)_{x\in G} \text{ has the same law under } \widetilde{\mathbb{P}}_{\mathcal{G}}^I \text{ as } \big(\varphi_x+\sqrt{2u}\big)_{x\in G} \text{ under } \mathbb{P}_{\mathcal{G}}^G. \tag{IV.3.15}$$

The main interest of the isomorphism (Isom') comes in the following proposition, whose simple proof is given in Section IV.4.3. It shows that (IV.3.7) also holds for the law of the compact clusters of the level sets, and not only for the law of their capacity.

Proposition IV.3.5. *Let* $\mathcal{G}$ *be a graph such that* (Isom') *holds. Then the compact clusters of* $E^{\geq -h}$ *have the same law as the compact clusters of* $E^{\geq h}$.

Let us finally give some additional consequences of Theorem IV.3.4. We denote by 0 the constant killing measure equal to 0, and we define

$$\mathbf{h}_0(x)\stackrel{\text{def.}}{=}P_x^{\widetilde{\mathcal{G}}_0}(\widetilde{\zeta}_\kappa<\widetilde{\zeta}_0) \text{ for all } x\in G, \tag{IV.3.16}$$

in the notation of (IV.2.7) and (IV.2.8). In other words, $\mathbf{h}_0(x)$ is the probability that the diffusion X starting in x is killed before blowing up, or equivalently the probability that the discrete Markov chain Z on G starting in x is killed by the measure κ before blowing up, and we say that $\mathbf{h}_0<1$ if $\mathbf{h}_0(x)<1$ for all $x\in G$. Under condition (Law$_0$), another interesting consequence of Theorem IV.3.4 is that one can replace the condition $\kappa\equiv 0$ by $\mathbf{h}_0<1$ in Corollary IV.3.2. It also provides us the following dichotomy for the value of $\widetilde{h}_*$.

Corollary IV.3.6. *Assume* $\mathcal{G}$ *is transient and satisfies* (Law$_0$). *Then either the sign clusters* $E^{\geq 0}$ *of the Gaussian free field on* $\widetilde{\mathcal{G}}$ *only contain compact connected components* $\mathbb{P}^G$*-a.s, or for all* $h\in\mathbb{R}$ *the level sets* $E^{\geq h}$ *contain at least one unbounded connected component with* $\mathbb{P}^G$*-positive probability. If moreover* $\mathbf{h}_0<1$, *then for all* $x_0\in\widetilde{\mathcal{G}}$ *and* $h<0$ *the level set* $E^{\geq h}(x_0)$ *of* x_0 *is unbounded with positive probability, and in particular* $\widetilde{h}_*=h_*^{\mathrm{com}}\in\{0,\infty\}$.

The proof of Corollary IV.3.6 is done at the end of Section IV.8. We refer to Remark IV.9.2,3) for an example of a graph satisfying (Law$_0$) and $\mathbf{h}_0<1$, but for which $\widetilde{h}_*=h_*^{\mathrm{com}}=\infty$, that is the dichotomy from Corollary IV.3.6 is non-trivial. Note that however we still have $h_*^{\mathrm{cap}}\leq 0$ by Theorem IV.3.1. In view of Corollary IV.3.6, an interesting open question is then whether a transient graph with $\widetilde{h}_*\in(0,\infty)$, or $h_*^{\mathrm{com}}\in(0,\infty)$, exists or not. An interesting consequence of Corollary IV.3.6 is that if $\widetilde{h}_*=0$, then the level sets of the Gaussian free field do no percolate at the critical point $h=0$, and we refer to the end of Section IV.8 for a proof.

Corollary IV.3.7. *If G is a transient graph such that $\widetilde{h}_* \leq 0$, then $E^{\geq 0}$ contains only bounded connected components.*

IV.4 Preliminaries

IV.4.1 The conditions (Cap) and $\kappa \equiv 0$.

In this subsection, we prove another characterization of (Cap) in terms of the discrete graph $\mathcal{G}$, see (IV.4.1), and give several example of graphs verifying (Cap): graphs with a Green function decreasing to zero at infinity, and in particular transitive graphs, see Lemma IV.4.1, or the trees studied in [1], see Lemma IV.4.2. We then show that, if $\kappa \equiv 0$, the law of the capacity of the level sets of the Gaussian free field on $\widetilde{\mathcal{G}}$ from (Law_h) can be equivalently stated directly on the graph $\widetilde{\mathcal{G}}^E$, see Lemma IV.4.3. We finally explain in Proposition IV.4.4 under which conditions compactness and boundedness are equivalent for the level sets of the Gaussian free field, which directly imply the equalities in (IV.3.4) and (IV.3.5), and Corollary IV.3.2 follows then directly from Theorem IV.3.1.

The condition (Cap) plays an essential role in the proof of $\widetilde{h}_* = 0$, and we now give an equivalent condition in terms of the cable system $\widetilde{\mathcal{G}}$, which shows in combination with Theorem IV.3.1 that $E^{\geq 0}$ is bounded on any graph satisfying (IV.4.1). We also give a condition which implies (Cap), but is stated only in terms of the Green function on $\mathcal{G}$, and thus can be easier to verify. It implies for instance that any vertex-transitive graph verifies (Cap), and so that Corollary IV.3.2 generalizes the results from Proposition 5.5 in [57].

Lemma IV.4.1. *1. Condition* (Cap) *holds true if and only if*

$$\mathrm{cap}(A) = \infty \text{ for all infinite and connected sets } A \subset G. \tag{IV.4.1}$$

2. If

$$\begin{array}{c}\text{there exists } g_0 < \infty \text{ such that } \{x \in G : g(x,x) > g_0\} \\ \text{has no unbounded connected component}\end{array} \tag{IV.4.2}$$

and if

$$\text{for any sequence } (x_k), (y_k) \in G^{\mathbb{N}} \text{ with } d_{\mathcal{G}}(x_k, y_k) \underset{k\to\infty}{\longrightarrow} \infty : \ g(x_k, y_k) \underset{k\to\infty}{\longrightarrow} 0, \tag{IV.4.3}$$

then condition (Cap) *is verified for $\mathcal{G}$. In particular, if $\mathcal{G}$ is vertex-transitive, then condition* (Cap) *is verified.*

Proof. (a) Let us first assume that (Cap) holds true for the graph $\mathcal{G}$, then for all infinite and connected $A \subset G$, writing $\widetilde{A}$ for the union of the I_e for all edges $e \in E$ between two vertices of A, we have by (IV.2.18) and (IV.2.27)

$$\mathrm{cap}(A) = \mathrm{cap}(\widetilde{A}) = \infty,$$

since $\widetilde{A}$ is an unbounded and connected set of $\widetilde{\mathcal{G}}$, and so (IV.4.1) is satisfied. Assume now that $\mathcal{G}$ is a graph such that (IV.4.1) is verified, and let $\widetilde{A}$ be a connected and unbounded subset of $\widetilde{\mathcal{G}}$. Then $\widetilde{A}$ contains an infinite and connected set $A \subset \mathcal{G}$, and so by (IV.2.24) and (IV.4.1) $\mathrm{cap}(\widetilde{A}) \geq \mathrm{cap}(A) = \infty$, that is (Cap) holds.

(b) Let us now assume that $\mathcal{G}$ is a graph such that (IV.4.2) and (IV.4.3) are satisfied. We can assume that G is an infinite graph, otherwise condition (IV.4.1), and thus (Cap), is trivially satisfied. Let A be an infinite and connected subset of $\mathcal{G}$, which contains an infinite path $\pi = (x_0, x_1, \dots)$ such that $x_{i-1} \sim x_i$ for all $i \in \mathbb{N}$, and $x_i \in G$. Let us define recursively $a_0 = 0$ and

$$a_n = \inf\big\{i > a_{n-1} : d_{\mathcal{G}}(x_i, x_{a_j}) \geq n \text{ for all } j \leq n-1 \text{ and } g(x_i, x_i) \leq g_0\big\}.$$

The existence of a_n is guaranteed by (IV.4.2). Let us fix some $\varepsilon > 0$, and let $A = \{x_{a_0}, x_{a_1}, \dots\}$ and $K_{\varepsilon,x} = \{y \in A \setminus \{x\} : g(x,y) \geq \varepsilon\}$ for all $x \in A$. For all $n \in \mathbb{N}$ such that $K_{\varepsilon,x_{a_n}} \neq \varnothing$, there exists $y_n \in A$, $y_n \neq x_{a_n}$, such that $g(x_{a_n}, y_n) \geq \varepsilon$, and by definition of the sequence $(a_k)_{k\geq 0}$, we then have $d_{\mathcal{G}}(x_{a_n}, y_n) \geq n$. By (IV.4.3), this is only possible for finitely many n, that is there exists $N \in \mathbb{N}$ such that $K_{\varepsilon,x_{a_n}} = \varnothing$ for all $n \geq N$. Let us define $B_n = \{x_{a_N}, \dots, x_{a_{N+n-1}}\}$ for all $n \in \mathbb{N}$, we then have that $g(x,y) \leq \varepsilon$ and $g(x,x) \leq g_0$ for all $x \neq y \in B_n$ and $n \in \mathbb{N}$. Therefore, we have by (IV.2.22) that

$$\mathrm{cap}(B_n) \geq \Big(\frac{1}{n^2} \sum_{x,y \in B_n} g(x,y)\Big)^{-1} \geq \Big(\frac{g_0}{n} + \varepsilon\Big)^{-1}.$$

Using (IV.2.24) and (IV.2.27), we obtain that $\mathrm{cap}(A) = \infty$, that is (IV.4.1), and thus (Cap), holds.

Let us now assume that $\mathcal{G}$ is vertex-transitive. Then $g(x,x) = g_0$ is constant, and so (IV.4.2) holds, and by transitivity, (IV.4.3) is equivalent for any $x \in G$ to the following condition:

$$\text{for any sequence } (y_k) \in G^{\mathbb{N}} \text{ with } d_{\mathcal{G}}(x, y_k) \underset{k\to\infty}{\longrightarrow} \infty, \text{ we have } g(x, y_k) \underset{k\to\infty}{\longrightarrow} 0.$$

Let us assume that this does not hold, that is there exists $\varepsilon > 0$, $x \in G$, and a sequence $(y_k) \in G^{\mathbb{N}}$ with $d_{\mathcal{G}}(x, y_k) \underset{k\to\infty}{\longrightarrow} \infty$ and $g(x, y_k) \geq \varepsilon$ for all $k \in \mathbb{N}$. Since for all $y \in G$ $g(x,y) = P_x(H_y < \zeta)g(y,y)$ and g is symmetric, we then have for all $k \in \mathbb{N}$ that

$$P_x(H_{y_k} < \zeta) \geq g_0^{-1}\varepsilon \quad \text{and} \quad P_{y_k}(H_x < \zeta) \geq g_0^{-1}\varepsilon$$

For each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $d_{\mathcal{G}}(x, y_{k_n}) \geq n$, and then by the strong Markov property

$$P_x(\exists\, t \geq T_{B(x,n)} : Z_t = x) \geq P_x(H_{y_{k_n}} < \zeta, \exists\, t \geq H_{y_{k_n}} : Z_t = x) \geq g_0^{-2}\varepsilon^2.$$

Since $T_{B(x,n)}$ increases to ζ, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$P_x(\exists\, t \in [T_{B(x,a_n)}, T_{B(x,a_{n+1})}) : Z_t = x) \geq \frac{\varepsilon^2}{2g_0^2}.$$

Now

$$g(x,x) = E_x\Big[\int_0^\infty \mathbb{1}_{Z_t=x}\,\mathrm{d}t\Big] \geq \sum_{n\in\mathbb{N}} E_x\Big[\int_{T_{B(x,a_n)}}^{T_{B(x,a_{n+1})}} \mathbb{1}_{Z_t=x}\,\mathrm{d}t\Big] \geq \sum_{n\in\mathbb{N}} \frac{\varepsilon^2}{2g_0^2\lambda_x} = \infty,$$

where we used in the last inequality the fact that Z_t stays a time $\mathcal{E}(\lambda_x)$ in x whenever it is hit, and this is a contradiction. Therefore (IV.4.3) holds, and so also (Cap). □

In Proposition 2.2 of [1], it is proved that the sign clusters $E^{\geq 0}$ of the Gaussian free field on the cable system on $\widetilde{\mathbb{T}}$ only contains compact connected components $\mathbb{P}^G$-a.s, when $\mathbb{T}$ is a tree with unit weights and zero killing measure, and such that $\{x \in \mathbb{T} : R_x^\infty > A\}$ only has bounded components for some $A > 0$. Here R_x^∞ denotes the effective resistance between x and infinity for $\mathbb{T}_x$ (the tree consisting only of x and its descendants); in other words, $\frac{1}{\lambda_x R_x^\infty}$ is the probability that the Markov chain Z on $\mathbb{T}$ starting in x first visit a child of x, and then never comes back to x. We are now going to prove that (Cap) is always fulfilled on such trees; a fortiori, as a consequence of the previous discussion, Corollary IV.3.2 generalizes Proposition 2.2 of [1].

Lemma IV.4.2. *If $\mathbb{T}$ is a transient tree with zero killing measure and unit weights such that $\{x \in \mathbb{T} : R_x^\infty > A\}$ only has bounded connected components for some $A > 0$, then* (Cap) *is verified.*

Proof. By Lemma IV.4.1 and (IV.2.24), it is enough to prove that $\mathrm{cap}(B) = \infty$ for all infinite sets B of the form $B = \{x_0, x_1, \dots\}$, where x_i has degree at least 3 and is some descendant of x_{i-1} for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, $\{x \in \mathbb{T}_{x_i} \setminus B : R_x^\infty > A\}$ is finite, and so there exists a cut-set C_i between x_i and infinity in $\mathbb{T}_{x_i} \setminus B$, such that $R_y^\infty \leq A$ for all $y \in C_i$. Adding a vertex to $\mathbb{T}$ in the middle of each I_e,

$e \in E$, one can assume without loss of generality that each $y \in C_i$ has degree two, see the discussion below (2.18) in [1] for details. Taking $B_n = \{x_0, \dots, x_n\}$, we have for all $n \in \mathbb{N}$ and $i \in \{1, \dots, n-1\}$ that

$$\begin{aligned} e_{B_n}(x_i) &= \lambda_{x_i} P^{\mathbb{T}}_{x_i}(X_n \in \mathbb{T}_{x_i} \setminus B \text{ for all } n \in \mathbb{N}) \\ &\geq \lambda_{x_i} \sum_{y \in C_i} P^{\mathbb{T}}_{x_i}(X_{H_{C_i}} = y, H_{C_i} < \widetilde{H}_{x_i}) \frac{1}{\lambda_y R^{\infty}_y} \geq \frac{\lambda_{x_i}}{2A} P^{\mathbb{T}}_{x_i}(H_{C_i} < \widetilde{H}_{x_i}). \end{aligned}$$

Since $\mathbb{T}$ is transient and the random walk on $\mathbb{Z}$ is recurrent, it is easy to see that B is visited infinitely often with probability 0. Therefore for each $i \in \mathbb{N}$, under $P^{\mathbb{T}}_{x_i}$, if $Z_n \in \mathbb{T}_{x_i}$ for all $n \in \mathbb{N}$, then there exists $p \geq i$ such that $H_{C_p} < \widetilde{H}_{x_i}$, and so

$$\frac{1}{R^{\infty}_{x_i}} \leq \lambda_{x_i} P^{\mathbb{T}}_{x_i}(\exists\, p \geq i, H_{C_p} < \widetilde{H}_{x_i}) \leq \sum_{p \geq i} \lambda_{x_p} P^{\mathbb{T}}_{x_p}(H_{C_p} < \widetilde{H}_{x_i}),$$

where in the last inequality we used $\lambda_{x_i} P^{\mathbb{T}}_{x_i}(H_{x_p} < \widetilde{H}_{x_i}) = \lambda_{x_p} P^{\mathbb{T}}_{x_p}(H_{x_i} < \widetilde{H}_{x_p}) \leq \lambda_{x_p}$. Since for each $x \in \mathbb{T}$ with degree 2, using (1.11) in [1], we have $R^{\infty}_x = R^{\infty}_y + 1 \geq R^{\infty}_y$ when y is the first descendant of x, we have that $\frac{1}{R^{\infty}_{x_i}} \geq A^{-1}$ infinitely often, and so, using (IV.2.20) and (IV.2.27),

$$\operatorname{cap}(B) = \lim_{n \to \infty} \sum_{i \in \{0, \dots, n\}} e_{B_n}(x_i) \geq \frac{1}{2A} \sum_{i \in \mathbb{N}} \lambda_{x_i} P^{\mathbb{T}}_{x_i}(H_{C_i} < \widetilde{H}_{x_i}) = \infty.$$

□

We now turn to the proof that Corollary IV.3.2 follows from Theorem IV.3.1, and we begin with an auxiliary result about the capacity of I_x, $x \in G$, and the capacity on $\widetilde{\mathcal{G}}^E$, defined below (IV.2.27). It implies that, if $\kappa \equiv 0$, the law of the capacity of the level sets of the Gaussian free field on $\widetilde{\mathcal{G}}$ obtained in Theorem IV.3.3 can be equivalently stated directly on the graph $\widetilde{\mathcal{G}}^E$, for which the Gaussian free field was defined in the previous chapters. Recall that $\operatorname{cap}_{\widetilde{\mathcal{G}}^E}$ was defined below (IV.2.20).

Lemma IV.4.3. *For all $x \in G$, we have the following dichotomy*

$$\text{if } \kappa_x > 0, \text{ then } \operatorname{cap}(I_x) = \infty, \text{ and if } \kappa_x = 0, \text{ then } \operatorname{cap}(I_x) = \operatorname{cap}(\{x\}). \tag{IV.4.4}$$

Moreover, if $\kappa \equiv 0$, then for all connected and closed sets $A \subset \widetilde{\mathcal{G}}$ such that $A \cap \widetilde{\mathcal{G}}^E \neq \varnothing$, we have $\operatorname{cap}_{\widetilde{\mathcal{G}}}(A) = \operatorname{cap}_{\widetilde{\mathcal{G}}^E}(A \cap \widetilde{\mathcal{G}}^E)$.

Proof. Let us first prove (IV.4.4). If $\kappa_x > 0$, then for all $t \in (0, \rho_x)$, writing $y_t = x + (\rho_x - t) \cdot I_x$, we have by (IV.2.17) that $\kappa_{y_t}^{\{y_t\}} = \frac{1}{2t}$, and so $\lambda_{y_t}^{\{y_t\}} \geq \frac{1}{2t}$. Let $I_x^t = x + [x, y_t] \cdot I_x$, then by (IV.2.17)

$$P_{y_t}^{\mathcal{G}^{\{y_t\}}}(\widetilde{H}_{I_x^t} = \infty) = \frac{\kappa_{y_t}^{\{y_t\}}}{\lambda_{x,y_t}^{\{y_t\}} + \kappa_{y_t}^{\{y_t\}}} = 1 - \frac{t}{\rho_x},$$

and so by (IV.2.18) we have $\mathrm{cap}(I_x^t) \geq e_{I_x^t}(y_t) \geq \frac{1}{2t} - \frac{1}{2\rho_x}$, and by (IV.2.27) we obtain $\mathrm{cap}(I_x) = \infty$. If $\kappa_x = 0$, then keeping the same notation we have for all $t \in (0, \infty)$ $P_{y_t}^{\mathcal{G}^{\{y_t\}}}(\widetilde{H}_{I_x^t} = \infty) = 0$ since X behave like a Brownian motion on I_x, and thus will always come back in I_x^t in finite time, and $P_x^{\mathcal{G}^{\{y_t\}}}(\widetilde{H}_{I_x^t} = \infty) = P_x^{\mathcal{G}^{\{y_t\}}}(\widetilde{H}_{\{x\}} = \infty)$. Therefore by (IV.2.18), we have $\mathrm{cap}(I_x^t) = e_{I_x^t}(x) + 0 = e_{\{x\}}(x) = \mathrm{cap}(\{x\})$, and by (IV.2.27) we obtain $\mathrm{cap}(I_x) = \mathrm{cap}(\{x\})$.

Let us now assume that $\kappa \equiv 0$, and let $K \subset \widetilde{\mathcal{G}}$ be a connected and compact set such that $K \cap \widetilde{\mathcal{G}}^E \neq \varnothing$. Then for all $x \in \partial(K \cap \widetilde{\mathcal{G}}^E)$, we have that

$$e_{K \cap \widetilde{\mathcal{G}}^E}(x) = \lambda_x^{\partial K} P_x^{\mathcal{G}^{\partial K}}(\widetilde{H}_{K \cap \widetilde{\mathcal{G}}^E} = \infty) = \lambda_x^{\partial K} P_x^{\mathcal{G}^{\partial K}}(\widetilde{H}_K = \infty) = e_K(x),$$

and $e_K(y) = 0$ if $y \in \partial K \setminus \partial(K \cap \widetilde{\mathcal{G}}^E)$, and we can conclude by (IV.2.27). □

We now state a general lemma about the level sets $E^{\geq h}$ of the Gaussian free field, from which the equalities in (IV.3.4) and (IV.3.5) follow directly. Thus, Corollary IV.3.2 will be entailed as well once we will have proved Theorem IV.3.1.

Proposition IV.4.4. *Fix $h \in \mathbb{R}$ and $x_0 \in \widetilde{\mathcal{G}}$ arbitrarily. $\mathbb{P}^G$-a.s, if either $h \geq 0$, $\mathrm{cap}(E^{\geq h}(x_0)) < \infty$ or $\kappa \equiv 0$ on G, then the level set $E^{\geq h}(x_0)$ of x_0 is compact if and only if it is bounded.*

Proof. Observe that by definition, a connected set K is compact if and only if it is a closed and bounded subset of $\widetilde{\mathcal{G}}$ such that $I_x \cap K$ is a connected compact subset of I_x for all $x \in G$. Therefore, if the level set $E^{\geq h}(x_0)$ of x_0 is compact, then it is bounded. Hence, we only have to show the remaining implication, and we assume from now on that $E^{\geq h}(x_0)$ is bounded. First note that, as explained below (IV.2.31), if $\kappa_x = 0$, since φ on I_x conditioned on φ_x has the same law as a Brownian motion starting in φ_x with variance 2 at time 1, we have that $I_x \cap E^{\geq h}(x_0)$ is $\mathbb{P}^G$-a.s. a connected compact of I_x. Therefore $E^{\geq h}(x_0)$ is compact if $\kappa \equiv 0$. If $\kappa_x > 0$ we have by (IV.4.4) applied to the graph $\mathcal{G}^{\{x+t \cdot I_x\}}$ that $\mathrm{cap}([t, \rho_x) \cdot I_x) = \infty$. If $\mathrm{cap}(E^{\geq h}(x_0)) < \infty$, by (IV.2.24) we obtain $[t, \rho_x) \cdot I_x \not\subset E^{\geq h}(x_0)$, that is $I_x \cap E^{\geq h}(x_0)$ is a connected compact of I_x, and so $E^{\geq h}(x_0)$ is compact. Finally, if $\kappa_x > 0$ and $h \geq 0$, as explained below (IV.2.31), since φ on I_x conditioned on φ_x has the same law as a Brownian bridge of finite length between φ_x and 0 of a Brownian motion with variance 2 at time 1, $I_x \cap E^{\geq h}(x_0)$ is a connected compact of I_x, and so $E^{\geq h}(x_0)$ is compact. □

IV.4.2 Description of the law of $\mathrm{cap}(E^{\geq h}(x_0))$

In this subsection, we will study properties of the law of the capacity of the level sets of the Gaussian free field, when their the Laplace transform is given in (Law$_h$) (see above Theorem IV.1.1). The first lemma implies that $\mathrm{cap}(E^{\geq h}(x_0))$ satisfies (Law$_h$) if and only if its density (on the event $\{E^{\geq h}(x_0)\neq\varnothing\}$) is given by $\rho_h(\cdot)$, see (IV.3.6), for any $h\geq 0$, as mentioned in Theorem IV.3.3.

Lemma IV.4.5. *For all $u\geq 0$ and $h\in\mathbb{R}$,*

$$\int_{g(x_0,x_0)^{-1}}^{\infty}\rho_h(t)\exp(-ut)\,\mathrm{d}t=\mathbb{P}^G\big(\varphi_{x_0}\geq\sqrt{2u+h^2}\big),\tag{IV.4.5}$$

where ρ_h is defined as in (IV.3.6)*.*

Proof. Taking $v=u+h^2/2$ and $a=g(x_0,x_0)^{-1}$, it is enough to show that

$$\int_a^{\infty}\frac{1}{t\sqrt{2\pi(t-a)}}\exp(-vt)\,\mathrm{d}t=\int_{\sqrt{2v}}^{\infty}\exp\Big(-\frac{at^2}{2}\Big)\,\mathrm{d}t \text{ for all } v,a\geq 0.\tag{IV.4.6}$$

For $v=0$ we have, taking $s=\sqrt{t-a}$,

$$\int_a^{\infty}\frac{1}{t\sqrt{2\pi(t-a)}}\,\mathrm{d}t=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{s^2+a}\,\mathrm{d}s=\sqrt{\frac{2}{a\pi}}\Big[\arctan\Big(\frac{s}{\sqrt{a}}\Big)\Big]_0^{\infty}=\sqrt{\frac{\pi}{2a}},$$

and so (IV.4.6) holds for $v=0$. Moreover by dominated convergence, the left-hand side of (IV.4.6) viewed as a function of $v\geq 0$ is C^1 and its derivative is given by

$$\begin{aligned}-\int_a^{\infty}\frac{1}{\sqrt{2\pi(t-a)}}\exp(-vt)\,\mathrm{d}t&=-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp\big(-v(a+s^2)\big)\,\mathrm{d}s\\&=-\frac{1}{\sqrt{2v}}\exp(-va),\end{aligned}$$

and so is equal to the derivative with respect to v of the term on the right-hand side of (IV.4.6). This yields (IV.4.6), and thus (IV.4.5). □

We now give some asymptotics on the tail of $\mathbb{P}^G(\mathrm{cap}(E^{\geq h_N}(x_0)\geq\cdot)$, for certain sequences $h_N\underset{N\to\infty}{\longrightarrow}0$, from which the result (IV.3.11) on the critical window as $h\searrow 0$ under condition (IV.3.10) follows directly in view of (IV.3.9). One could also use these asymptotics to obtain bounds on the critical window, even when the condition (IV.3.10) does not hold or when $h\nearrow 0$, similarly as in [24].

Lemma IV.4.6. *Assume $\mathcal{G}$ satisfies* $(\mathrm{Law}_h)_{h\geq 0}$. *If $h_N \geq 0$ is a sequence such that $h_N\sqrt{N} \underset{N\to\infty}{\longrightarrow} h_\infty \in [0,\infty]$, then for all $x_0 \in \widetilde{\mathcal{G}}$,*

$$\sqrt{N}\mathbb{P}^G\big(\mathrm{cap}(E^{\geq h_N}(x_0)) \geq N\big) \underset{N\to\infty}{\longrightarrow} \frac{1}{\pi\sqrt{g(x_0,x_0)}} \exp\Big(-\frac{h_\infty^2}{2}\Big). \tag{IV.4.7}$$

Assume that $\mathcal{G}$ also satisfies (IV.3.7) *and* (IV.3.8). *If $h_N > 0$ is a sequence such that $h_N\sqrt{N} \underset{N\to\infty}{\longrightarrow} h_\infty \in [0,\infty]$, then for all $x_0 \in \widetilde{\mathcal{G}}$,*

$$\sqrt{N}\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h_N}(x_0)) \geq N\big) \underset{N\to\infty}{\longrightarrow} \frac{1}{\pi\sqrt{g(x_0,x_0)}} \exp\Big(-\frac{h_\infty^2}{2}\Big) + h_\infty\sqrt{\frac{2}{\pi g(x_0,x_0)}}, \tag{IV.4.8}$$

and if $h_N\sqrt{N} \underset{N\to\infty}{\longrightarrow} \infty$, then for all $x_0 \in \widetilde{\mathcal{G}}$,

$$h_N^{-1}\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h_N}(x_0)) \geq N\big) \underset{N\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi g(x_0,x_0)}}. \tag{IV.4.9}$$

Proof. Let use first assume that $(\mathrm{Law}_h)_{h\geq 0}$ is fulfilled, then ρ_h, $h \geq 0$, see (IV.3.6), is the density of $\mathrm{cap}(E^{\geq h}(x_0))$ by Lemma IV.4.5. Let $h_N \geq 0$ be a sequence such that $h_N\sqrt{N} \underset{N\to\infty}{\longrightarrow} h_\infty \in [0,\infty)$, then, as $N \to \infty$,

$$\begin{aligned}
&\mathbb{P}^G\big(\mathrm{cap}(E^{\geq h_N}(x_0) \geq N\big) \\
&= \int_N^\infty \rho_{h_N}(t)\,\mathrm{d}t \\
&= \int_1^\infty \frac{1}{2\pi t\sqrt{g(x_0,x_0)(tN - g(x_0,x_0)^{-1})}} \exp\Big(-\frac{Nh_N^2 t}{2}\Big)\,\mathrm{d}t \\
&\sim \exp\Big(-\frac{h_\infty^2}{2}\Big)\int_1^\infty \frac{1}{2\pi t\sqrt{g(x_0,x_0)tN}}\,\mathrm{d}t \\
&= \frac{1}{\pi\sqrt{g(x_0,x_0)N}} \exp\Big(-\frac{h_\infty^2}{2}\Big),
\end{aligned}$$

and so we obtain (IV.4.7). The proof is similar if $h_\infty = \infty$. Let us now assume that (IV.3.7) and (IV.3.8) are fulfilled. Then, since φ_{x_0} is $\mathcal{N}(0, g(x_0,x_0))$-distributed, we have that if $h_N > 0$, $h_N \underset{N\to\infty}{\longrightarrow} 0$, then, as $N \to \infty$,

$$\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h_N}(x_0) = \infty\big) = \mathbb{P}^G(\varphi_{x_0} \in (-h_N, h_N)) \sim h_N\sqrt{\frac{2}{\pi g(x_0,x_0)}}.$$

One can then directly obtain (IV.4.8) and (IV.4.9) from (IV.3.7) and (IV.4.7). □

IV.4.3 Consequences of the isomorphism

In this subsection, we prove that the signed isomorphism (Isom') between random interlacements and the Gaussian free field implies that the law of the compact clusters of the Gaussian free field at positive and negative levels are the same, see Proposition IV.3.5, and that the law of the capacity of the level sets is given by $(\mathrm{Law}_h)_{h\geq 0}$ and (IV.3.7), and we will use this fact several times in Sections IV.7 and IV.8.

Proof of Proposition IV.3.5. If (Isom') holds, then the compact clusters of $E^{\geq -\sqrt{2u}}$ have the same law as the closure of the compact clusters of $\{x \in \widetilde{\mathcal{G}} : \sigma_x^u = 1\}$. Each cluster of $\mathcal{I}^u$ is non-compact, and so by definition of σ^u, the compact clusters of $E^{\geq -\sqrt{2u}}$ have the same law as the compact clusters of $E^{|\geq 0|}$ not intersected by $\mathcal{I}^u$ with $\sigma^u = 1$ on these clusters, and so the same law as the compact clusters of $E^{|\geq 0|}$ not intersected by $\mathcal{I}^u$ with $\sigma^u = -1$, that is all the compact clusters of $E^{|\geq 0|}$ with $\sigma^u = -1$. By (Isom'), they also have the same law as the compact clusters of $\{x \in \widetilde{\mathcal{G}} : \varphi_x \leq -\sqrt{2u}\}$, and by symmetry of the Gaussian free field, the same law as the compact clusters of $E^{\geq \sqrt{2u}}$. □

Proposition IV.4.7. *Let $\mathcal{G}$ be a graph such that* (Isom') *is verified for all $u > 0$, then* $(\mathrm{Law}_h)_{h\geq 0}$ *holds on $\mathcal{G}$. Moreover, if* (Cap) *is also fulfilled, then* (IV.3.7) *also holds on $\mathcal{G}$..*

Proof. We first consider the case $h = 0$, and for all $x \in \widetilde{\mathcal{G}}$ let us denote by $E^{|>0|}(x) = \{y \in \widetilde{\mathcal{G}} : y \longleftrightarrow x \text{ in } \{z \in \widetilde{\mathcal{G}} : |\varphi_z| > 0\}\}$, and by $\overline{E^{|>0|}(x)}$ its closure. Note that if $E^{|>0|}(x) \cap \mathcal{I}^u = \varnothing$, then the cluster of x in $\{y \in \widetilde{\mathcal{G}} : 2\ell_{y,u} + \varphi_y^2 > 0\}$ is equal to $E^{|>0|}(x)$, and so $\sigma_x^u = \pm 1$ with probability $\frac{1}{2}$, and if $E^{|>0|}(x) \cap \mathcal{I}^u \neq \varnothing$, then $x \longleftrightarrow \mathcal{I}^u$ in $\{y \in \widetilde{\mathcal{G}} : 2\ell_{y,u} + \varphi_y^2 > 0\}$, and so $\sigma_x^u = 1$. By (Isom'), (IV.2.38) and symmetry of the Gaussian free field, we thus have for all $u > 0$ and $x \in \widetilde{\mathcal{G}}$

$$\begin{aligned} 2\mathbb{P}^G(\varphi_x \geq \sqrt{2u}) &= 1 - \mathbb{E}^G\big[\mathrm{sign}(\varphi_x + \sqrt{2u})\big] \\ &= 1 - \widetilde{\mathbb{E}}^I[\sigma_x^u] \\ &= 1 - \widetilde{\mathbb{P}}^I\big(E^{|>0|}(x) \cap \mathcal{I}^u \neq \varnothing\big) \\ &= \mathbb{E}^G\left[\exp\left(-u\mathrm{cap}\big(\overline{E^{|>0|}(x)}\big)\right)\right], \end{aligned}$$

where we used that $\mathcal{I}^u$ is open in the last equality to replace $E^{|>0|}(x)$ by $\overline{E^{|>0|}(x)}$. By the Markov property (IV.2.31), for all compacts K of $\widetilde{\mathcal{G}}$, conditionally on $\overline{E^{|>0|}(x)} \cap K$, $(\varphi_y)_{y \in (\overline{E^{|>0|}(x)} \cap K)^c}$ is a Gaussian free field on $\widetilde{\mathcal{G}} \setminus (\overline{E^{|>0|}(x)} \cap K)$, and thus φ behave like Brownian bridges on the boundary of $\overline{E^{|>0|}(x)}$. Therefore, any point y in the boundary of $\overline{E^{|>0|}(x)}$ can be approximated by a sequence

$y_n \in (\overline{E^{|>0|}(x)})^c$, $n \in \mathbb{N}$, such that $\varphi_{y_n} < 0$, and if $\varphi_x > 0$ we obtain that $\overline{E^{|>0|}(x)} = E^{\geq 0}(x)$ $\mathbb{P}^G$-a.s. By symmetry of the Gaussian free field, we thus have

$$\mathbb{E}^G\left[\exp\left(-u\mathrm{cap}\left(E^{\geq 0}(x)\right)\right)\mathbb{1}_{\varphi_x\geq 0}\right] = \frac{1}{2}\mathbb{E}^G\left[\exp\left(-u\mathrm{cap}\left(\overline{E^{|>0|}(x)}\right)\right)\right] \\ = \mathbb{P}^G(\varphi_x \geq \sqrt{2u}).$$

Let us now consider some $h > 0$, and let $u_0 = h^2/2$. By symmetry of the Gaussian free field and (Isom'), we have that $E^{\geq h}(x)$ has the same law under $\mathbb{P}^G$ as the closure of the connected component of x in $\{y \in \widetilde{\mathcal{G}} : \sigma_y^{u_0} = -1\}$ under $\widetilde{\mathbb{P}}^I$, that is the same law as $\overline{E^{|>0|}(x)}$ if $\mathcal{I}^{u_0} \cap \overline{E^{|>0|}(x)} = \varnothing$ and $\sigma_x = -1$, and $\varnothing$ otherwise. Therefore by (IV.2.38) we have for all $u > 0$

$$\begin{aligned}
&\mathbb{E}^G\left[\exp\left(-u\mathrm{cap}(E^{\geq h}(x)\right)\mathbb{1}_{\varphi_{x_0}\geq h}\right] \\
&= \widetilde{\mathbb{E}}^I\left[\mathbb{1}_{\mathcal{I}^{u_0}\cap\overline{E^{|>0|}(x)}=\varnothing,\sigma_x^{u_0}=-1}\exp\left(-u\mathrm{cap}(\overline{E^{|>0|}(x)}\right)\right] \\
&= \frac{1}{2}\mathbb{E}^G\left[\exp\left(-(u+u_0)\mathrm{cap}(\overline{E^{|>0|}(x)}\right)\right] \\
&= \mathbb{P}^G\left(\varphi_x \geq \sqrt{2u+h^2}\right).
\end{aligned}$$

Now assume that (Cap) holds on $\mathcal{G}$, and let us fix some $h > 0$. Then the clusters of $E^{\geq -h}$ with finite capacity are the bounded clusters of $E^{\geq -h}$, and so by Proposition IV.4.4 the compact clusters of $E^{\geq -h}$. By Proposition IV.3.5, they thus have the same law as the compact clusters of $E^{\geq h}$, that is all the clusters of $E^{\geq h}$, and so (IV.3.7) follows. □

IV.5 Proof using Russo's formula

For any compact K of $\widetilde{\mathcal{G}}$, and event $A \in \{0,1\}^{\otimes K}$, we write

$$A_K^{(h)} = \{E^{\geq h} \cap K \in A\}. \tag{IV.5.1}$$

Proposition IV.5.1. *For all compact K of $\widetilde{\mathcal{G}}$ and event $A_K \in \{0,1\}^{\otimes K}$,*

$$-\frac{\mathrm{d}\mathbb{P}^G(A_K^{(a)})}{\mathrm{d}a} = \mathbb{E}^G[M_K\mathbb{1}_{A_K^{(a)}}] \text{ for all } a \in \mathbb{R}. \tag{IV.5.2}$$

Proof. Fix a compact K of $\widetilde{\mathcal{G}}$. For arbitrary $a \in \mathbb{R}$, introducing the function

$$\mathbf{h}_a(x) = -a\mathbf{h}(x), \quad \mathbf{h}(x) \stackrel{\text{def.}}{=} P_x^{\widetilde{\mathcal{G}}}[H_K < \widetilde{\zeta}], \quad x \in \widetilde{\mathcal{G}},$$

one has in particular $\mathbf{h}_a|_K = -a$ and therefore, in view of (IV.5.1),

$$\mathbb{P}^G(A_K^{(a)}(\varphi)) = \mathbb{P}^G(A_K^{(0)}(\varphi + \mathbf{h}_a)). \tag{IV.5.3}$$

By (IV.2.21), we moreover have for all $x \in \widetilde{\mathcal{G}}$

$$\mathbf{h}_a(x) = -a \sum_{y \in \partial K} e_K(y) g(y,x) = \mathbb{E}^G\big[\varphi_x\big(-aM_K\big)\big].$$

By (IV.2.20) with $\mathcal{K} = \varnothing$ and the Cameron-Martin theorem, see for instance [49], Theorems 14.1 and 14.13, one knows that $\widetilde{\mathbb{P}}_a^G$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_a^G}{\mathrm{d}\mathbb{P}^G} = \exp\Big(-aM_K - \frac{a^2 \mathrm{cap}(K)}{2}\Big) \tag{IV.5.4}$$

is a probability measure and that φ under $\widetilde{\mathbb{P}}_a^G$ has the law of $\varphi + \mathbf{h}_a$ under $\mathbb{P}^G$. In view of (IV.5.3) and by dominated convergence, this implies that

$$\begin{aligned} -\frac{\mathrm{d}\mathbb{P}^G(A_K^{(a)})}{\mathrm{d}a} &= -\tfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{\mathbb{P}}_a^G(A_K^{(0)}(\varphi)) \\ &\overset{\text{(IV.5.4)}}{=} \mathbb{E}^G\Big[1_{A_K^{(0)}(\varphi)}\Big(-\tfrac{\mathrm{d}}{\mathrm{d}a}\exp\Big(-aM_K - \tfrac{a^2\mathrm{cap}(K)}{2}\Big)\Big)\Big] \\ &= \widetilde{\mathbb{E}}_a^G\Big[1_{A_K^{(0)}(\varphi)}\big(M_K + a\mathrm{cap}(K)\big)\Big] \\ &\overset{\text{(IV.5.4)}}{=} \mathbb{E}^G\Big[1_{A_K^{(a)}(\varphi)} M_K\Big]. \end{aligned}$$

□

Corollary IV.5.2. *For all $x_0 \in \widetilde{\mathcal{G}}$, compact K of $\widetilde{\mathcal{G}}$ such that $x_0 \in K$, $h \geq 0$ and $\mathrm{cap}(\{x_0\}) \leq s < t \leq \infty$, we have*

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq h}(x_0) \cap K) \in (s,t]) \leq \mathbb{P}^G(\mathrm{cap}(E^{\geq 0}(x_0) \cap K) \in (s,t]) \exp\Big(-\frac{h^2 s}{2}\Big). \tag{IV.5.5}$$

Proof. The bound (IV.5.5) will follow by integrating a suitable differential inequality, see (IV.5.8) below. Let $C_{s,t}^h = \{\mathrm{cap}(E^{\geq h}(x_0) \cap K) \in (s,t]\}$ and

$$\mathcal{K}^h \overset{\text{def.}}{=} E^{\geq h}(x_0) \cap K. \tag{IV.5.6}$$

Note that $C_{s,t}^h$ is of the form (IV.5.1) (with $A = \{\omega \in \{0,1\}^K : \mathrm{cap}(C_\omega(x_0) \cap K) \in (s,t]\}$, where $C_\omega(x_0)$ refers to the cluster of x_0), whence Proposition IV.5.1 applies to this event. Moreover, the set $\mathcal{K}^h$ in (IV.5.6) is a compatible random subset of $\widetilde{\mathcal{G}}$ and $\mathcal{K}^h \subset K$. Lastly, on account of (IV.2.30), the event $C_{s,t}^h$ is $\mathcal{A}_{\mathcal{K}^h}^+$-measurable. Together these observations imply that

$$-\frac{\mathrm{d}\mathbb{P}^G(C_{s,t}^h)}{\mathrm{d}h} = \mathbb{E}^G\big[\mathbb{E}^G[M_K \,|\, \mathcal{A}_{\mathcal{K}^h}^+]\mathbb{1}_{C_{s,t}^h}\big], \quad h \in \mathbb{R}. \tag{IV.5.7}$$

Applying Lemma IV.2.3, noting that $\varphi \geq h$ on $\partial\mathcal{K}^h$, we deduce that for $h \geq 0$,

$$\begin{aligned}\mathbb{E}^G[M_K \mid \mathcal{A}^+_{\mathcal{K}^h}]\mathbb{1}_{C^h_{s,t}} = M_{\mathcal{K}^h}\mathbb{1}_{C^h_{s,t}} &= \sum_{x\in\partial\mathcal{K}^h} e_{\mathcal{K}^h}(x)\varphi_x\mathbb{1}_{C^h_{s,t}} \\ &\geq h\mathrm{cap}(\mathcal{K}^h)\mathbb{1}_{C^h_{s,t}} \geq hs\mathbb{1}_{C^h_{s,t}}.\end{aligned}$$

Substituting this into (IV.5.7) yields

$$\frac{\mathrm{d}\log\mathbb{P}^G(C^h_{s,t})}{\mathrm{d}h} \leq -hs, \quad h \geq 0. \tag{IV.5.8}$$

Integrating (IV.5.8) then gives (IV.5.5). □

In case $E^{\geq h}(x_0)$ is compact, we can be more precise than Corollary IV.5.2. For all $x_0 \in \widetilde{\mathcal{G}}$, closed connected set F of $\widetilde{\mathcal{G}}$ such that $x_0 \in F$ and $h \in \mathbb{R}$, let μ^F_h be the law of $\mathrm{cap}(E^{\geq h}(x_0))\mathbb{1}_{E^{\geq h}(x_0)\subset F, E^{\geq h}(x_0) \text{ compact}, \varphi_{x_0}\geq h}$ under $\mathbb{P}^G$.

Corollary IV.5.3. *For all $x_0 \in \widetilde{\mathcal{G}}$, closed connected set F of $\widetilde{\mathcal{G}}$ such that $x_0 \in F$, and $h, h' \in \mathbb{R}$*

$$\frac{\mathrm{d}\mu^F_h}{\mathrm{d}\mu^F_{h'}}(t) = \exp\Big(-\frac{(h^2-(h')^2)t}{2}\Big), \quad t \in (\mathrm{cap}(x_0), \infty) \tag{IV.5.9}$$

(here with a slight abuse of notation we identify μ^F_h with its restriction to $(\mathrm{cap}(x_0), \infty)$).

Proof. Assume first that F is compact. Let X^F_h have law μ^F_h and note that $X^F_h \overset{\text{law}}{=} \mathrm{cap}(E^{\geq h}(x_0))\mathbb{1}_{E^{\geq h}(x_0)\subset F}$. For $h \in \mathbb{R}$, $0 < \varepsilon < \mathrm{cap}(x_0)\,(> 0)$ and $t > \mathrm{cap}(x_0)$, consider the event

$$A^h_\varepsilon(t) \overset{\text{def.}}{=} \{t-\varepsilon < X^F_h \leq t\} = \{E^{\geq h}(x_0) \subset F,\, \mathrm{cap}(E^{\geq h}(x_0) \cap F) \in (t-\varepsilon, t]\} \tag{IV.5.10}$$

and write

$$\mathbb{P}^G(A^h_\varepsilon(t)) = \exp\left\{\int_{h'}^h \frac{\mathrm{d}\log\mathbb{P}^G(A^a_\varepsilon(t))}{\mathrm{d}a}\,\mathrm{d}a\right\} \times \mathbb{P}^G(A^{h'}_\varepsilon(t)). \tag{IV.5.11}$$

Since the event in (IV.5.10) is of the form (IV.5.1), the logarithmic derivative can be computed by means of Proposition IV.5.1, yielding (with $K = F$)

$$\frac{\mathrm{d}\log\mathbb{P}^G(A^a_\varepsilon(t))}{\mathrm{d}a} = \frac{1}{\mathbb{P}^G(A^a_\varepsilon(t))}\frac{\mathrm{d}\mathbb{P}^G(A^a_\varepsilon(t))}{\mathrm{d}a} \overset{\text{(IV.5.2)}}{=} \frac{-1}{\mathbb{P}^G(A^a_\varepsilon(t))}\mathbb{E}^G\left[M_F 1_{A^a_\varepsilon(t)}\right]. \tag{IV.5.12}$$

As $A^a_\varepsilon(t) \in \mathcal{A}^+_{\mathcal{K}^a}$ with $\mathcal{K}^a = E^{\geq a}(x_0) \cap F$, using that $\mathcal{K}^a 1_{A^a_\varepsilon(t)} = E^{\geq h}(x_0) 1_{A^a_\varepsilon(t)}$ and that, still on the event $A^a_\varepsilon(t)$, one has $\varphi_x = a$ for all $x \in \partial\mathcal{K}^a$, a similar computation as below (IV.5.7) involving Lemma IV.2.3 gives

$$\frac{\mathbb{E}^G\left[M_F 1_{A^a_\varepsilon(t)}\right]}{\mathbb{P}^G(A^a_\varepsilon(t))} = a \cdot \frac{\mathbb{E}^G\left[\mathrm{cap}(E^{\geq h}(x_0)) 1_{A^a_\varepsilon(t)}\right]}{\mathbb{P}^G(A^a_\varepsilon(t))} \overset{\text{(IV.5.10)}}{\in} (a(t-\varepsilon), at]. \quad \text{(IV.5.13)}$$

Substituting (IV.5.12) and (IV.5.13) into (IV.5.11) then yields

$$e^{-t\frac{(h^2-(h')^2)}{2}} \leq \frac{\frac{1}{\varepsilon}\mathbb{P}^G(A^h_\varepsilon(t))}{\frac{1}{\varepsilon}\mathbb{P}^G(A^h_{h'}(t))} \leq e^{-(t-\varepsilon)\frac{(h^2-(h')^2)}{2}}, \quad \text{(IV.5.14)}$$

from which the claim follows by letting $\varepsilon \to 0$. To obtain (IV.5.9) in case F is non-compact, one writes F as the increasing limit of a sequence $\{F_n\}$ of compact sets, to which (IV.5.14) applies and gives uniform bounds (in n). One then first takes the monotone limit of the two probabilities as $n \to \infty$ using (IV.2.27), and lets then $\varepsilon \to 0$. □

We now proceed to do the

First proof of Theorem IV.3.1. Recall the definition of $\mathcal{K}^h$ for the quantity defined in (IV.5.6). Noting that $\mathcal{K}^h \nearrow E^{\geqslant h}(x_0)$ as $K \nearrow \widetilde{\mathcal{G}}$ and therefore $\mathrm{cap}(\mathcal{K}^h) \nearrow \mathrm{cap}(E^{\geqslant h}(x_0))$ by (IV.2.24) and (IV.2.27), one has, for all $s \geq \mathrm{cap}(\{x_0\})$ and $h \geq 0$,

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq h}(x_0)) > s) = \lim_{K \nearrow \widetilde{\mathcal{G}}} \lim_{t \to \infty} \mathbb{P}^G(\mathrm{cap}(\mathcal{K}^h) \in (s,t]) \overset{\text{(IV.5.5)}}{\leq} e^{-\frac{h^2 s}{2}}. \quad \text{(IV.5.15)}$$

Letting $s \to \infty$ in (IV.5.15) yields

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq h}(x_0)) < \infty) = 1, \text{ for all } h > 0, \quad \text{(IV.5.16)}$$

and therefore $h_*^{\mathrm{cap}} \leq 0$ in view of (IV.3.2). In order to deduce (IV.5.16) for $h = 0$ one considers instead (IV.5.5) with $s = \mathrm{cap}(\{x_0\})$, noting that $\mathrm{cap}(E^{\geq h}(x_0)) > \mathrm{cap}(x_0)$ is $\mathbb{P}^G$-a.s. equivalent to $\varphi_{x_0} > h$, to obtain for all $h > 0$ and $t \geq 0$

$$\mathbb{P}^G\big(\varphi_{x_0} > h, \mathrm{cap}(\mathcal{K}^h) \leq t\big) \leq \mathbb{P}^G\big(\varphi_{x_0} > 0, \mathrm{cap}(\mathcal{K}^0) \leq t\big) e^{-\frac{h^2 \mathrm{cap}(\{x_0\})}{2}}. \quad \text{(IV.5.17)}$$

Letting $K \nearrow \widetilde{\mathcal{G}}$, $t \to \infty$ and $h \searrow 0$ in (IV.5.17), the claim follows again by monotone convergence as the limit on the left-hand side equals $\frac{1}{2}$ by (IV.5.16).

It remains to argue that $h_*^{\mathrm{com}} \geq 0$, i.e. that for all $h > 0$ the level set $E^{\geq -h}(x_0)$ of x_0 is non-compact with positive probability. Assuming on the contrary that

$E^{\geq -h}$ is $\mathbb{P}^G$-a.s. compact for some $h > 0$, one deduces, noting that any compact set has finite capacity, that

$$\mu_s^{\widetilde{\mathcal{G}}}((\mathrm{cap}(x_0), \infty)) = \mathbb{P}^G(\varphi_{x_0} > s, E^{\geq s}(x_0) \text{ is compact}) = \mathbb{P}^G(\varphi_{x_0} > s), \ s \geq -h. \tag{IV.5.18}$$

But (IV.5.18) and (IV.5.9) applied with $h' = -h$ imply

$$\mathbb{P}^G(\varphi_{x_0} > -h) = \mathbb{P}^G(\varphi_{x_0} > h),$$

a contradiction. This completes the proof. □

We can now do the

First proof of Theorem IV.3.3. Let us first fix some $h \geq 0$ such that $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded. Note that by Proposition IV.4.4, for any $x_0 \in \widetilde{\mathcal{G}}$ and $u \geq 0$, $E^{\geqslant h+\sqrt{2u}}(x_0)$ is $\mathbb{P}^G$-a.s. compact, hence

$$\mu_{h+\sqrt{2u}}^{\widetilde{\mathcal{G}}} \text{ is the law of } \mathrm{cap}(E^{\geq h+\sqrt{2u}}(x_0))\mathbb{1}_{\varphi_{x_0} \geq h+\sqrt{2u}} \text{ for all } u \geq 0. \tag{IV.5.19}$$

In particular, for any $u > 0$

$$\begin{aligned}
\mathbb{P}^G(\varphi_{x_0} \geq h + \sqrt{2u}) &= \mathbb{P}^G\big(\mathrm{cap}(E^{\geq h+\sqrt{2u}}(x_0)) > \mathrm{cap}(x_0)\big) \\
&\overset{\text{(IV.5.19)}}{=} \mu_{h+\sqrt{2u}}^{\widetilde{\mathcal{G}}}\big((\mathrm{cap}(x_0), \infty)\big) \\
&\overset{\text{(IV.5.9)}}{=} \int_{\mathrm{cap}(x_0)}^{\infty} e^{ut} \mathrm{d}\mu_h^F(t) \\
&\overset{\text{(IV.5.19)}}{=} \mathbb{E}^G\big[\exp\big(-u\mathrm{cap}(E^{\geq h}(x_0))\big)\mathbb{1}_{\{\varphi_{x_0} \geq h\}}\big],
\end{aligned}$$

as desired. The fact that the density is ρ_h, as given by (IV.3.6), then follows from Lemma IV.4.5.

Assume now that (Cap) holds. This implies that for any $h \in \mathbb{R}$, $E^{\geq h}(x_0)$ being compact is equivalent to $\mathrm{cap}(E^{\geq h}(x_0))$ being finite, whence for all $h \geq 0$, $\mu_{-h}^{\widetilde{\mathcal{G}}}$ is the law of $\mathrm{cap}(E^{\geq -h}(x_0))1_{\mathrm{cap}(E^{\geq -h}(x_0)) < \infty, \varphi_{x_0} \geq -h}$. Moreover $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded for all $h \geq 0$ by Corollary IV.3.2, and so (Law$_h$) and (IV.5.19) hold for all $h \geq 0$. From this and (IV.5.19), the asserted equality in law in (IV.3.7) follows immediately from the equality

$$\mu_{-h}^{\widetilde{\mathcal{G}}}|_{(0,\infty)} = \mu_h^{\widetilde{\mathcal{G}}}|_{(0,\infty)}, \text{ for all } h > 0$$

(by which we mean the equality of the restriction of the respective measures to the space $((0, \infty), \mathcal{B}(0, \infty)))$, itself an immediate consequence of (IV.5.9). Therefore by (Law$_h$) for $u = 0$ we have

$$\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h}(x_0)) \in (\mathrm{cap}(\{x_0\}), \infty)\big) = \mathbb{P}^G(\varphi_{x_0} \geq h).$$

Since $\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h}(x_0)) \leq \mathrm{cap}(\{x_0\})\big) = \mathbb{P}^G(\varphi_{x_0} \leq -h)$, we obtain (IV.3.8). □

Remark IV.5.4. 1) Using (IV.5.9) with $F=\widetilde{\mathcal{G}}$ we get

$$\mathbb{E}^G\big[\exp\big(-u\mathrm{cap}(E^{\geq 0}(x_0))\big)\mathbb{1}_{\varphi_{x_0}\geq h, E^{\geq 0}(x_0)\text{ compact}}\big]=\mathbb{P}^G\left(\begin{array}{c}\varphi_{x_0}\geq\sqrt{2u},\\ E^{\geq\sqrt{2u}}\text{ compact}\end{array}\right).$$

In particular by Proposition IV.4.4, (Law_0) is equivalent to

$$\mathbb{E}^G\big[\exp\big(-u\mathrm{cap}(E^{\geq 0}(x_0))\big)\mathbb{1}_{\varphi_{x_0}\geq h, E^{\geq 0}(x_0)\text{ unbounded}}\big]=\mathbb{P}^G(E^{\geq\sqrt{2u}}\text{ unbounded}). \tag{IV.5.20}$$

2) One can easily deduce from (IV.5.20) and Theorem IV.3.1 that if $E^{\geq 0}$ is unbounded with positive probability, then $E^{\geq\sqrt{2u}}$ is unbounded with positive probability for all $u\geq 0$, that is $\widetilde{h}_*=\infty$. Moreover, by Theorem IV.3.1, when $\kappa\equiv 0$, if $E^{\geq 0}$ is compact, then $h_*^{\mathrm{com}}=0$. By Proposition IV.4.4 and (IV.3.4), we thus get that $h_*^{\mathrm{com}}=\widetilde{h}_*\in\{0,\infty\}$ when $\kappa\equiv 0$. One can thus obtain partial results in Corollary IV.3.6, replacing the condition $\mathbf{h}_0<1$ by the condition $\kappa\equiv 0$, without using random interlacements. One can then deduce Corollary IV.3.7 from Corollary IV.3.6, see the end of Section IV.8 for details.

IV.6 Proof using exploration martingales

In this section, we are going to prove Theorems IV.3.1 and IV.3.3 using an exploration martingale, similar to the one introduced in [24] on $\mathbb{Z}^d$, $d\geq 3$, or in Lemma 4.2 of [101] on the $(d+1)$-regular tree, $d\geq 2$. For a set $F\subset\widetilde{\mathcal{G}}$, we define the distance d_F^∞ on $\widetilde{\mathcal{G}}$, such that $d_F^\infty(x,y)$, $x,y\in\widetilde{\mathcal{G}}$, is the minimal length of a continuous path in F between x and y, when changing the length of each I_e, $e\in E$ from ρ_e to 1, and of each I_x, $x\in G$, from ρ_x to ∞, by means of some strictly increasing bijection $[0,\rho_x)\to[0,\infty)$ when $\kappa_x>0$. We take the convention $d_F^\infty(x,x)=0$ for all $x\in\widetilde{\mathcal{G}}$, and $d_F^\infty(x,y)=\infty$ if $x\neq y$ and either $x\notin F$ or $y\notin F$. Let us fix some $x_0\in\widetilde{\mathcal{G}}$, and let B_t be the closed ball of radius t around x_0 for the distance $d_{\widetilde{\mathcal{G}}}^\infty$. Note that B_t is compact for all $t\geq 0$, hence our choice of the distance d^∞ here. Under $\mathbb{P}^G$, we then define

$$\mathcal{K}_t^{(h)}=\{x\in E^{\geq h}: d_{E^{\geq h}}^\infty(x,x_0)\leq t\}\text{ and }\mathcal{M}_t^{(h)}=M_{\mathcal{K}_t^{(h)}}, \tag{IV.6.1}$$

see (IV.1.6) for the definition of M_K. Note that $\mathcal{K}_t^{(h)}$ is an increasing sequence of connected compact with $\mathcal{K}_t^{(h)}\subset B_t$, and that $\mathcal{K}_0^{(h)}=\{x_0\}$ is deterministic. Moreover, looking at geodesics one can easily prove that for all open sets $\varnothing\neq O\subset\widetilde{\mathcal{G}}$, taking a sequence K_p, $p\in\mathbb{N}$, of compacts increasing to O, and all $t\geq 0$

we have

$$\{\mathcal{K}_t^{(h)} \subset O\} = \bigcup_{p\in\mathbb{N}} \big\{\{x \in O :\, d^{\infty}_{E^{\geq h}\cap O}(x, x_0) \leq t\} \subset K_p\big\},$$

and so $\mathcal{K}_t^{(h)}$ is a compatible random compact of $\widetilde{\mathcal{G}}$, see above (IV.2.30). By Lemma IV.2.3, we thus have

$$\mathcal{M}_s^{(h)} = \mathbb{E}[M_{B_t} \,|\, \mathcal{F}_s] \text{ for all } 0 \leq s \leq t, \text{ with } \mathcal{F}_s^{(h)} = \mathcal{A}^+_{\mathcal{K}_s^{(h)}} \text{ for all } s \geq 0. \quad \text{(IV.6.2)}$$

Lemma IV.6.1. *For all $h \in \mathbb{R}$, the process $(\mathcal{M}_t^{(h)})_{t\geq 0}$ is a continuous martingale with respect to $(\mathcal{F}_t^{(h)})_{t\geq 0}$ with continuous quadratic variation given by*

$$\langle \mathcal{M}^{(h)} \rangle_t = \mathrm{cap}(\mathcal{K}_t^{(h)}) - \mathrm{cap}(\{x_0\}).$$

Proof. Using (IV.2.30), one can easily prove that $(\mathcal{F}_t^{(h)})_{t\geq 0}$ is a filtration, and so $(\mathcal{M}_t^{(h)})_{t\geq 0}$ is a $(\mathcal{F}_t^{(h)})_{t\geq 0}$ martingale, since for all $t \geq 0$, by (IV.6.2), $(\mathcal{M}_s^{(h)})_{s\leq t}$ is a Doob martingale. Let us fix some $t_0 > 0$. By (IV.2.31) and (IV.6.2), we can write for all $0 \leq t < t_0$

$$\mathcal{M}_t^{(h)} = \mathbb{E}[M_{B_{t_0}} \,|\, \mathcal{F}_t^{(h)}] = \sum_{x\in\partial B_{t_0}} e_{B_{t_0}}(x)\mathbb{E}^G[\varphi_x \,|\, \mathcal{A}^+_{\mathcal{K}_t^{(h)}}] = \sum_{x\in\partial B_{t_0}} e_{B_{t_0}}(x)\eta^{\varphi}_{\mathcal{K}_t^{(h)}}(x).$$

Since for each $0 \leq t < t_0$, $(\mathcal{K}_s^{(h)})_{t<s<t_0}$ decreases to $\mathcal{K}_t^{(h)}$ and for each $0 < t < t_0$, $(\mathcal{K}_s^{(h)})_{0\leq s<t}$ increases to $\mathcal{K}_t^{(h)}$, we have by Lemma IV.2.2 that $(\mathcal{M}_t^{(h)})_{0\leq t<t_0}$ is continuous, and thus $(\mathcal{M}_t^{(h)})_{t\geq 0}$ is continuous. We now compute the quadratic variation of $(\mathcal{M}_t^{(h)})_{t\geq 0}$. By Lemma IV.2.3 we have for all $s \leq t$

$$\begin{aligned}
&\mathbb{E}^G[(\mathcal{M}_t^{(h)})^2 - \mathrm{cap}(\mathcal{K}_t^{(h)}) \,|\, \mathcal{F}_s^{(h)}] \\
&= \mathbb{E}^G[\mathbb{E}^G[M_{B_t} \,|\, \mathcal{F}_t^{(h)}]^2 \,|\, \mathcal{F}_s^{(h)}] - \mathbb{E}^G[\mathrm{cap}(\mathcal{K}_t^{(h)}) \,|\, \mathcal{F}_s^{(h)}] \\
&= \mathrm{Var}(M_{B_t} \,|\, \mathcal{F}_s^{(h)}) + \mathbb{E}^G[M_{B_t} \,|\, \mathcal{F}_s^{(h)}]^2 - \mathbb{E}^G[\mathrm{Var}(M_{B_t} \,|\, \mathcal{F}_t^{(h)}) + \mathrm{cap}(\mathcal{K}_t^{(h)}) \,|\, \mathcal{F}_s^{(h)}] \\
&= \mathrm{cap}(B_t) - \mathrm{cap}(\mathcal{K}_s^{(h)}) + (\mathcal{M}_s^{(h)})^2 - \mathbb{E}^G[\mathrm{cap}(B_t) \,|\, \mathcal{F}_s^{(h)}] \\
&= (\mathcal{M}_s^{(h)})^2 - \mathrm{cap}(\mathcal{K}_s^{(h)}).
\end{aligned}$$

By (IV.2.24) and (IV.2.26), the function $t \mapsto \mathrm{cap}(\mathcal{K}_t^{(h)})$ is continuous and increasing, and so we can conclude since $\mathcal{K}_0^{(h)} = \{x_0\}$. □

As a consequence of Lemma IV.6.1, as well as Proposition 1.26, Chapter IV, and Theorem 1.7, Chapter V, in [75], we obtain the following.

Lemma IV.6.2. *Let $(\beta_t)_{t\geq 0}$ be a standard Brownian motion under some probability space $(\Omega^\beta, \mathcal{F}^\beta, \mathbb{P}^\beta)$, and $T_t = \inf\{s \geq 0 :\, \mathrm{cap}(\mathcal{K}_s^{(h)}) > t\}$. If $\mathrm{cap}(E^{\geq h}(x_0)) <$*

∞, *then there exists a random variable* $\mathcal{M}_\infty^{(h)}$ *such that* $\mathcal{M}_t^{(h)} \underset{t\to\infty}{\longrightarrow} \mathcal{M}_\infty^{(h)}$. *Moreover, under* $\mathbb{P}^\beta \otimes \mathbb{P}^G$, *the process* $(W_t^{(h)})_{t\geq 0}$, *defined by*

$$W^{(h)}_{t-\mathrm{cap}(\{x_0\})} = \begin{cases} \mathcal{M}^{(h)}_{T_t} - \mathrm{cap}(\{x_0\})\varphi_{x_0} & \text{if } t \in \big[\mathrm{cap}(\{x_0\}), \mathrm{cap}(E^{\geq h}(x_0))\big) \\ \mathcal{M}^{(h)}_\infty - \mathrm{cap}(\{x_0\})\varphi_{x_0} + \beta_{t-\mathrm{cap}(E^{\geq h}(x_0))} & \text{if } t \geq \mathrm{cap}(E^{\geq h}(x_0)), \end{cases}$$

is a standard Brownian motion.

Moreover, one can easily see that the results of Lemmas IV.6.1 and IV.6.2 still hold under $\mathbb{P}^G(\cdot \,|\, \varphi_{x_0})$. Therefore the law of $(W_t^{(h)})_{t\geq 0}$ under $\mathbb{P}^G(\cdot \,|\, \varphi_{x_0})$ does not depend on φ_{x_0}, and so

$$(W_t^{(h)})_{t\geq 0} \text{ is independent of } \varphi_{x_0}. \tag{IV.6.3}$$

We now state the main result of this section

Proposition IV.6.3. *Let*

$$g(h,a,M) = \mathbb{P}^\beta\big(\beta_t > ht + a \text{ for all } t \in [0,M)\big) \text{ for all } h,a \in \mathbb{R} \text{ and } M > 0. \tag{IV.6.4}$$

For all $h \in \mathbb{R}$, *abbreviating* $X_h = \mathrm{cap}(E^{\geq h}(x_0)) - \mathrm{cap}(\{x_0\})$, *for all* $M > 0$ *we have*

$$\begin{aligned} \mathbb{P}^G\big(X_h \geq M\big) &= \mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h} g(h, \mathrm{cap}(\{x_0\})(h-\varphi_{x_0}), M)\big] \\ &\quad - \mathbb{E}^G\big[\mathbb{1}_{X_h \in (0,M)} g(h, h\mathrm{cap}(E^{\geq h}(x_0)) - \mathcal{M}^{(h)}_\infty, M - X_h)\big]. \end{aligned}$$

Proof. Note that $\mathbb{P}^G$-a.s., $\{\varphi_{x_0} \geq h\} = \{\varphi_{x_0} > h\} = \{X_h > 0\}$, and so by Lemma IV.6.2 we have for all $M > 0$, that $\mathbb{P}^G$-a.s,

$$\begin{aligned} &\big\{\varphi_{x_0} \geq h, W^{(h)}_{t-\mathrm{cap}(\{x_0\})} > ht - \mathrm{cap}(\{x_0\})\varphi_{x_0} \, \forall\, t \in [\mathrm{cap}(\{x_0\}), M + \mathrm{cap}(\{x_0\}))\big)\big\} \\ &= \{X_h \geq M, \mathcal{M}^{(h)}_{T_t} > ht \text{ for all } t \in [\mathrm{cap}(\{x_0\}), M + \mathrm{cap}(\{x_0\})))\big\} \\ &\cup \left\{ \begin{matrix} X_h \in (0,M), \mathcal{M}^{(h)}_{T_t} > ht \text{ for all } t \in \big[\mathrm{cap}(\{x_0\}), \mathrm{cap}(E^{\geq h}(x_0))\big), \\ \beta_t > h(t + \mathrm{cap}(E^{\geq h}(x_0))) - \mathcal{M}^{(h)}_\infty \text{ for all } t \in \big[0, M - X_h\big) \end{matrix} \right\}. \end{aligned}$$

Moreover, if $\mathrm{cap}(E^{\geq h}(x_0)) > \mathrm{cap}(\{x_0\})$, for all $t \in \big[\mathrm{cap}(\{x_0\}), \mathrm{cap}(E^{\geq h}(x_0))\big)$, we have by continuity of $(\mathrm{cap}(\mathcal{K}_s^{(h)}))_{s\geq 0}$, that $T_t < \infty$, $\mathrm{cap}(\mathcal{K}^{(h)}_{T_t}) = t < \mathrm{cap}(\mathcal{K}^{(h)}_\infty) = \mathrm{cap}(E^{\geq h}(x_0))$. Noting that if $e_{\mathcal{K}_t^{(h)}}(x) = 0$ for some $x \in \partial\mathcal{K}_t^{(h)}$, then $e_{\mathcal{K}_s^{(h)}}(y) = 0$ for all $s \geq t$ and $y \in \partial\mathcal{K}_s^{(h)}$ such that $y \longleftrightarrow x$ in $(\mathcal{K}_t^{(h)})^c$, we have that there exists $x \in \partial\mathcal{K}^{(h)}_{T_t}$ with $e_{\mathcal{K}_t^{(h)}}(x) \neq 0$ such that $\varphi_x > h$, that is by (IV.6.1)

$$\Big\{X_h > 0,\, t \in \big[\mathrm{cap}(\{x_0\}), \mathrm{cap}(E^{\geq h}(x_0))\big)\Big\} \implies \mathcal{M}^{(h)}_{T_t} > h\mathrm{cap}(\mathcal{K}_t^{(h)}) = ht.$$

We thus obtain

$$\begin{aligned}&\big\{\varphi_{x_0}\geq h, W^{(h)}_{t-\mathrm{cap}(\{x_0\})}>ht-\mathrm{cap}(\{x_0\})\varphi_{x_0}\,\forall\,t\in[\mathrm{cap}(\{x_0\}),M+\mathrm{cap}(\{x_0\})))\big\}\\&=\{X_h\geq M\}\\&\cup\big\{X_h\in(0,M),\beta_t>h(t+\mathrm{cap}(E^{\geq h}(x_0)))-\mathcal{M}^{(h)}_\infty\text{ for all }t\in\big[0,M-X_h)\big)\big\}.\end{aligned}$$

Moreover by Lemma IV.6.2 and (IV.6.3)

$$\begin{aligned}&\mathbb{P}^G\left(\begin{array}{c}\varphi_{x_0}\geq h, W^{(h)}_{t-\mathrm{cap}(\{x_0\})}>ht-\mathrm{cap}(\{x_0\})\varphi_{x_0}\\ \text{for all }t\in[\mathrm{cap}(\{x_0\}),M+\mathrm{cap}(\{x_0\}))\end{array}\right)\\&=\mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h}g(h,\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}),M)\big],\end{aligned}$$

and so we can conclude. □

Theorem IV.3.1 follows directly from Proposition IV.6.3.

Second proof of Theorem IV.3.1. Note that a Brownian motion with drift $-h$ never hits a fixed negative level a with probability 0 if $h\geq 0$, and with strictly positive probability if $h<0$, see for instance equation 2.0.2 (1), in Part II of [13], and so by dominated convergence

$$\lim_{M\to\infty}\mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h}g(h,\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}),M)\big]=\begin{cases}0 & \text{if } h\geq 0,\\ >0 & \text{if } h<0.\end{cases}$$

Let us fix some $h\geq 0$, then by Proposition IV.6.3 we have that

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq h}(x_0))=\infty)\leq\lim_{M\to\infty}\mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h}g(h,\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}),M)\big]=0.$$

Let us now fix some $h<0$, and let us assume that $E^{\geq h}(x_0)$ is $\mathbb{P}^G$-a.s. compact, then there exists $\mathbb{P}^G$-a.s. $t_0<\infty$ such that $d^\infty_{E^{\geq h}}(x,x_0)\leq t_0$ for all $x\in E^{\geq h}(x_0)$, and then for all $t\geq t_0$ we have $\mathcal{K}^{(h)}_t=E^{\geq h}(x_0)$. Moreover since by continuity $\varphi_x=h$ for all $x\in\partial E^{\geq h}(x_0)$, we get by (IV.1.6) and (IV.6.1) that $\mathcal{M}^{(h)}_t=h\mathrm{cap}(E^{\geq h}(x_0))$ for all $t\geq t_0$, and thus $\mathcal{M}^{(h)}_\infty=h\mathrm{cap}(E^{\geq h}(x_0))$ $\mathbb{P}^G$-a.s. Since $\beta_0=0$ $\mathbb{P}^\beta$-a.s, we have $g(h,0,M)=0$ for all $h\in\mathbb{R}$ and $M>0$, and therefore by Proposition IV.6.3 if $h<0$ is such that $E^{\geq h}(x_0)$ is $\mathbb{P}^G$-a.s. compact, then

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq h}(x_0))=\infty)=\lim_{M\to\infty}\mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h}g(h,\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}),M)\big]>0.$$

This is a contradiction since the capacity of any compact is finite. □

Moreover, it follows from equation 2.0.2, Part II in [13] that the function g from (IV.6.4) is derivable with respect to M and that

$$\frac{\mathrm{d}g(h,a,M)}{\mathrm{d}M}=\frac{a}{\sqrt{2\pi}M^{3/2}}\exp\Big(-\frac{(hM+a)^2}{2M}\Big)\text{ for all } h\in\mathbb{R}, a\leq 0 \text{ and } M>0. \tag{IV.6.5}$$

Let us define

$$F(h,M)=\mathbb{E}^G\big[\mathbb{1}_{\varphi_{x_0}\geq h}g(h,\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}),M)\big]\text{ for all } h\in\mathbb{R}\text{ and } M>0.$$

By (IV.6.5) and dominated convergence, we have that

$$\begin{aligned}
&\frac{\mathrm{d}F(h,M)}{\mathrm{d}M}\\
&=\mathbb{E}^G\Big[\mathbb{1}_{\varphi_{x_0}\geq h}\frac{\mathrm{cap}(\{x_0\})(h-\varphi_{x_0})}{\sqrt{2\pi}M^{3/2}}\exp\Big(-\frac{(hM+\mathrm{cap}(\{x_0\})(h-\varphi_{x_0}))^2}{2M}\Big)\Big]\\
&=\int_{-\infty}^0\frac{\mathrm{cap}(\{x_0\})y}{2\pi\sqrt{g(x_0,x_0)}M^{3/2}}\exp\Big(-\frac{(h-y)^2}{2g(x_0,x_0)}-\frac{(hM+\mathrm{cap}(\{x_0\})y)^2}{2M}\Big)\,\mathrm{d}y\\
&=\frac{\exp\big(-(h^2/2)(M+\mathrm{cap}(\{x_0\}))\big)}{2\pi(g(x_0,x_0)M)^{3/2}}\int_{-\infty}^0 y\exp\Big(-\frac{y^2(1+Mg(x_0,x_0))}{2Mg(x_0,x_0)^2}\Big)\,\mathrm{d}y\\
&=-\frac{1}{2\pi\sqrt{g(x_0,x_0)M}(g(x_0,x_0)^{-1}+M)}\exp\big(-(h^2/2)(M+g(x_0,x_0)^{-1})\big),
\end{aligned}$$

where we used the fact that φ_{x_0} is a centered Gaussian variable with variance $g(x_0,x_0)$ and that $\mathrm{cap}(\{x_0\})=g(x_0,x_0)^{-1}$. By (IV.3.6), we obtain that

$$\frac{\mathrm{d}F(h,M-g(x_0,x_0)^{-1})}{\mathrm{d}M}=-\rho_h(M)\text{ for all } M>g(x_0,x_0)^{-1}\text{ and } h\in\mathbb{R}. \tag{IV.6.6}$$

In view of Proposition IV.6.3, this result provides us with the desired explicit formula for the law of $\mathrm{cap}(E^{\geq h}(x_0))$, Theorem IV.3.3.

Second proof of Theorem IV.3.3. Let us fix some $h\geq 0$ such that $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded, and $M>0$. By Proposition IV.4.4, we have that $E^{\geq h}(x_0)$ is compact, and then $\mathcal{K}_t^{(h)}=E^{\geq h}(x_0)$ for t large enough and $\mathcal{M}_\infty^{(h)}=h\mathrm{cap}(E^{\geq h}(x_0))$. By Proposition IV.6.3, since $g(h,0,M')=0$ for all $M'>0$, we have for all $M>\mathrm{cap}(\{x_0\})$

$$\mathbb{P}^G\big(\mathrm{cap}(E^{\geq h}(x_0))\geq M\big)=F(h,M-\mathrm{cap}(\{x_0\})=F(h,M-g(x_0,x_0)^{-1}). \tag{IV.6.7}$$

Therefore by (IV.6.6), under $\{\varphi_{x_0}\geq h\}=\{\mathrm{cap}(E^{\geq h}(x_0))\geq\mathrm{cap}(\{x_0\})\}$ the density of $\mathrm{cap}(E^{\geq h}(x_0))$ is ρ_h, see (IV.3.6). In view of Lemma IV.4.5, this also implies (Law_h).

Let us now assume that (Cap) is verified, and let us fix some $h \geq 0$. By Corollary IV.3.2, $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded, and so (Law$_h$) and (IV.6.7) hold. Under condition (Cap), by Proposition IV.4.4, we have that if $\mathrm{cap}(E^{\geq -h}(x_0)) < \infty$, then $E^{\geq -h}(x_0)$ is compact. Therefore, $\mathcal{M}_t^{(-h)}$ is constant and equal to $-h\mathrm{cap}(E^{\geq -h}(x_0))$ for all t large enough, and so $\mathcal{M}_\infty^{(-h)} = -h\mathrm{cap}(E^{\geq -h}(x_0))$. By Proposition IV.6.3, we thus have

$$\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h}(x_0)) \in [M, \infty)\big) = F(-h, M - g(x_0, x_0)^{-1}).$$

Since by (IV.6.6) and (IV.3.6) for all $M > 0$

$$\frac{\mathrm{d}F(-h, M - g(x_0, x_0)^{-1})}{\mathrm{d}M} = -\rho_{-h}(M) = -\rho_h(M) = \frac{\mathrm{d}F(h, M - g(x_0, x_0)^{-1})}{\mathrm{d}M},$$

we obtain (IV.3.7). The identity (IV.3.8) follows readily from (IV.3.7). □

Remark IV.6.4. Following the proof of Theorem IV.3.3, one can easily see by Proposition IV.6.3 and (IV.6.6) that if $\mathcal{G}$ is a graph such that $\mathcal{M}_\infty^{(0)} = 0$ $\mathbb{P}^G$-a.s, then (Law$_0$) holds. This is obviously the case when condition (Sign) is verified as in Theorem IV.3.3, but one could also prove that $\mathcal{M}_\infty^{(0)} = 0$ even without that condition, see Remark IV.9.2,3). This is actually an equivalence: if $\mathcal{M}_\infty^{(0)} > 0$ with positive probability, then by Proposition IV.6.3 and (IV.6.6), it is clear that (Law$_0$) does not hold.

IV.7 Proof using random interlacements

In this section, we are going to prove Theorems IV.3.1 and IV.3.3 using the isomorphisms between random interlacements, or loop soups, and the Gaussian free field. We first recall the isomorphism theorems between loop soups and the Gaussian free field from [57], and deduce from it a version of Theorem IV.3.4, see Lemma IV.7.2, on finite graphs. Then approximating the Gaussian free field on any transient graph by Gaussian free field on infinite graphs, see Lemma IV.7.4, we can prove Theorems IV.3.1 and IV.3.3 with the help of Proposition IV.4.7.

Following [35], one can define a measure on loops $\mu^L_{\widetilde{\mathcal{G}}}$ associated with $P_x^{\widetilde{\mathcal{G}}}$, $x \in \widetilde{\mathcal{G}}$, and, under some probability measure $\mathbb{P}^L_{\widetilde{\mathcal{G}}}$, we define for all $\alpha > 0$ the loop soup $\mathcal{L}_\alpha$ with parameter α as a Poisson point process in the space of loops on $\widetilde{\mathcal{G}}$ with intensity $\alpha\mu^L_{\widetilde{\mathcal{G}}}$. We denote by $(L_x^{(\alpha)})_{x\in\widetilde{\mathcal{G}}}$ its field of local times relative to m on $\widetilde{\mathcal{G}}$, which can be taken continuous, see Lemma 2.2 in [57]. Moreover we denote by $\mathcal{L}_{\mathcal{G},\alpha}$ the Poisson point process which consist of the print on G of each loop in $\mathcal{L}_\alpha$, which has the same law as the loop soup associated with $P_x^{\mathcal{G}}$, see Section 2 of [57] or Section 7.3 of [35] for details, and we will write $\mathbb{P}^L_{\mathcal{G}}$ instead of $\mathbb{P}^L_{\widetilde{\mathcal{G}}}$ when we want to stress that we only consider the discrete loops $\mathcal{L}_{\mathcal{G},\alpha}$. An important property of the loop soup $\mathcal{L}_\alpha$ is the restriction property, see Section 6 of [35]: for all connected and open subsets A of $\widetilde{\mathcal{G}}$, if we denote by $\mathcal{L}^A_\alpha$ the set of trajectories in $\mathcal{L}^A_\alpha$ entirely included in A, then

$$\mathcal{L}^A_\alpha \text{ has the same law under } \mathbb{P}^L_{\widetilde{\mathcal{G}}} \text{ as } \mathcal{L}_\alpha \text{ under } \mathbb{P}^L_{\widetilde{\mathcal{G}}^A_\infty}, \tag{IV.7.1}$$

where $\mathcal{G}^A_\infty$ is a graph with the same vertices, edges and weights as $\mathcal{G}^{\partial A}$, but with killing measure equal to κ on $(G\cap A)\setminus\partial A$ and infinity on $\partial A\cup(G\cap A^c)$, that is for all $x\in A$, the diffusion X under $P_x^{\widetilde{\mathcal{G}}^A_\infty}$ has the same law as the diffusion X killed on exiting A under $P^{\widetilde{\mathcal{G}}}$.

When $\alpha=\frac{1}{2}$, loop soups and the Gaussian free field are linked via an isomorphism, first derived for the square of the Gaussian free field and discrete graphs by Le Jan, see Theorem 2 of [54], and extended to include the sign of the Gaussian free field both on $\widetilde{\mathcal{G}}^E$ and G by Lupu, see [57]. To simplify notation, we define $L_x=L_x^{(\frac{1}{2})}$ the local time of the loop soup with parameter $\frac{1}{2}$.

Theorem IV.7.1 ([57])**.** *On some extension $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}}$ of $\mathbb{P}^L_{\widetilde{\mathcal{G}}}$, let us define an additional process $(\sigma_x)_{x\in\widetilde{\mathcal{G}}}\in\{-1,1\}^{\widetilde{\mathcal{G}}}$, such that, conditionally on $\mathcal{L}_{\frac{1}{2}}$, σ is constant on each cluster of $\{x\in\widetilde{\mathcal{G}}:L_x>0\}$ and its values on each cluster are independent and uniformly distributed. Then*

$$\text{under } \widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}} \text{ the law of } \big(\sigma_x\sqrt{2L_x}\big)_{x\in\widetilde{\mathcal{G}}} \text{ is } \mathbb{P}^G_{\widetilde{\mathcal{G}}}. \tag{IV.7.2}$$

Moreover, on some extension $\widetilde{\mathbb{P}}^L_{\mathcal{G}}$ of $\mathbb{P}^L_{\mathcal{G}}$, let us define a random set $\widehat{\mathcal{E}}\subset E$ such that, conditionally on $\mathcal{L}_{\mathcal{G},\frac{1}{2}}$, $\widehat{\mathcal{E}}$ contains each edge crossed by a loop in $\mathcal{L}_{\mathcal{G},\frac{1}{2}}$, and each additional edge $e\in E$ conditionally independently with probability $1-p_e^{\mathcal{G}}(\sqrt{L})$, where $p_e^{\mathcal{G}}$ is defined in (IV.2.34)*. Then $\widehat{\mathcal{E}}$ has the same law under $\widetilde{\mathbb{P}}^L_{\mathcal{G}}$ as $\mathcal{E}:=\{e\in E: L_x>0$ for all $x\in I_e\}$ under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}}$. In particular, if we define a process $(\widehat{\sigma}^u_x)_{x\in G}\in\{-1,1\}^{\mathcal{G}}$, such that, conditionally on $\mathcal{L}_{\mathcal{G},\frac{1}{2}}$ and $\widehat{\mathcal{E}}$, $\widehat{\sigma}$ is constant on each of the clusters induced by $\widehat{\mathcal{E}}$ and its values on each cluster are independent and uniformly distributed, then*

$$\big(\widehat{\sigma}_x\sqrt{2L_x}\big)_{x\in G} \text{ has the same law under } \widetilde{\mathbb{P}}^L_{\mathcal{G}} \text{ as } (\varphi_x)_{x\in G} \text{ under } \mathbb{P}^G_{\mathcal{G}}. \tag{IV.7.3}$$

The equality between the squares of the processes in (IV.7.2) follows from Theorem 3.1 in [35] and the law of σ on $\widetilde{\mathcal{G}}$ follows from a version of Lemma 3.2 in [57] on $\widetilde{\mathcal{G}}$ instead of $\widetilde{\mathcal{G}}^E$. Moreover Corollary 3.6 in [57] provides us with the law of $\{e\in\mathcal{E}\}$ conditionally on $\mathcal{L}_{\mathcal{G},\frac{1}{2}}$, and one can then directly derive (IV.7.3), see Theorem 1.bis in [57]. Note that the process σ on $\widetilde{\mathcal{G}}$ can be explicitly constructed.

Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in $\widetilde{\mathcal{G}}$ and $(\sigma'_n)_{n\in\mathbb{N}}\in\{-1,1\}^{\mathbb{N}}$ be a sequence of independent and uniformly distributed random variables under some probability $\mathbb{Q}$. Under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}}=\mathbb{P}^L_{\widetilde{\mathcal{G}}}\otimes\mathbb{Q}$, we define $m(x)$ as the smallest $n\in\mathbb{N}$ such that x_n and x are in the same cluster of $\{x\in\widetilde{\mathcal{G}}:\ L_x>0\}$, and since $(x_n)_{n\in\mathbb{N}}$ is dense and $(L_y)_{y\in\widetilde{\mathcal{G}}}$ is continuous, we have that $m(x)<\infty$ when $L_x>0$. We then define $\sigma_x=\sigma'_{m(x)}$ if $L_x>0$, and $\sigma_x=1$ otherwise, which has the desired properties. In the isomorphism between random interlacements and the Gaussian free field, Theorem IV.3.4, one could also construct explicitly the law of the signs σ^u by a similar procedure.

For each $x_0\in\widetilde{\mathcal{G}}$, one can use the following decomposition for the loop soup $\mathcal{L}_{\frac12}=\mathcal{L}_{\frac12}^{\{x_0\}^c}+\overline{\mathcal{L}}_{\frac12}^{\{x_0\}}$, into the loops $\mathcal{L}_{\frac12}^{\{x_0\}^c}$ which never hit x_0, and the loops $\overline{\mathcal{L}}_{\frac12}^{\{x_0\}}$ which hit x_0 at least once, and these two processes are independent. In Theorem 2 of [58], see also section 2 of [55], this decomposition is used to deduce the second Ray-Knight theorem from Theorem IV.7.1, which is an equivalent of Theorem IV.3.4, but replacing random interlacements by the diffusion X killed at time $\tau_u^{x_0}$, the first time $\ell_{x_0}(t)$ reaches u, and φ by the Gaussian free field conditioned on being equal to 0 in x_0. More precisely, they use that by (IV.7.1), $\mathcal{L}_{\frac12}^{\{x_0\}^c}$ is just a loop soup on $\widetilde{\mathcal{G}}_\infty^{\{x_0\}^c}$, and so its local times is the square of the Gaussian free field on $\widetilde{\mathcal{G}}_\infty^{\{x_0\}^c}$ by (IV.7.2), and that the concatenation of the loops in $\overline{\mathcal{L}}_{\frac12}^{\{x_0\}}$ has the same law under $\mathbb{P}^L$ as $(X_t)_{t<\tau_u^{x_0}}$ under P_{x_0}. In fact, as noted in [54], $\overline{\mathcal{L}}_{\frac12}^{\{x_0\}}$ is also linked to random interlacements on finite graphs, and very similarly as in the proof of Theorem 2 of [58], we can obtain the isomorphism between random interlacements and the Gaussian free field, Theorem IV.3.4, on finite graphs. For completeness, we have included the proof in the case of random interlacements in the Appendix.

Lemma IV.7.2. *If $\mathcal{G}$ is a transient graph such that G is finite, then* (Isom') *holds. Moreover, conditionally on $\omega_u^{\mathcal{G}}$ and $(\varphi_x)_{x\in G}$, the family $\{e\in\mathcal{E}_u\}$, $e\in E\cup G$, in independent, and for all $e\in E\cup G$*

$$\widetilde{\mathbb{P}}^I(e\in\mathcal{E}_u\,|\,\omega_u^{\mathcal{G}},\varphi)=\mathbb{1}_{e\in\mathcal{I}_E^u}\vee(1-p_e^{u,\mathcal{G}}(\varphi,\ell_{\cdot,u})). \tag{IV.7.4}$$

Note that the probability $1-p_e^{u,\mathcal{G}}(\varphi,\ell_{\cdot,u})$ in (IV.7.4) corresponds in [58], after replacing random interlacements by the diffusion X, to the probability of the combination of $\mathcal{O}(\varphi)$ and the additional edges opened independently of $(X_t)_{t<\tau_u^{x_0}}$.

Remark IV.7.3. 1) In the Appendix, we prove Lemma IV.7.2 using the isomorphism between loop soups and the Gaussian free field, Theorem IV.7.1. Similarly as in Theorem 2.4 of [101], one could in fact use the Markov property

(IV.2.31) to prove that (Isom) holds on any finite transient graph, or even on any transient graph with bounded Green function such that (Sign) holds. However, this proof does not directly provide us with the discrete isomorphism described by (IV.7.4). Moreover, it uses the "squared version" of the isomorphism (IV.1.5), which can also be seen as a consequence of the "squared version" of Theorem IV.7.1, thus making our proof conceptually shorter.

2) Similarly as in Theorem 2 of [58], one could also use Theorem IV.7.1 to deduce a theorem between random interlacements and the Gaussian free field even if G is infinite. More precisely, if $\mathcal{G}$ is a graph such that $|\{x \in G : \kappa_x > 0\}| < \infty$, then one can prove an isomorphism similar to Theorem IV.3.4, but replacing random interlacements on $\widetilde{\mathcal{G}}$ by killed random interlacements on $\widetilde{\mathcal{G}}$, that is all the trajectories in the random interlacement process whose forward and backward parts are both killed before escaping all bounded sets, and replacing $\varphi + \sqrt{2u}$ by $\varphi + \sqrt{2u}\mathbf{h}_0$, see (IV.3.16). One could then try to extend this theorem to any graph $\mathcal{G}$ with $\kappa \not\equiv 0$ by using a similar strategy as in Section IV.8, but we will not need this fact here.

We are now going to approximate the Gaussian free field on any transient graph $\mathcal{G}$ by Gaussian free fields on finite graphs. We say that a sequence of graphs $\mathcal{G}_n$ increases to $\mathcal{G}$ if $\mathcal{G}_n = \mathcal{G}_{\kappa^{(n)}}$ for some sequence $\kappa^{(n)} \subset [0,\infty]^{\mathcal{G}}$ of killing measures such that $\kappa_x^{(n)}$ decreases to κ_x for all $x \in G$. Note that we can see $\widetilde{\mathcal{G}}_n$ as a subset of $\widetilde{\mathcal{G}}$, that $\widetilde{\mathcal{G}}_n$ increases to $\widetilde{\mathcal{G}}$, and that for each compact K of $\widetilde{\mathcal{G}}$ we have $K \subset \widetilde{\mathcal{G}}_n$ for n large enough.

Lemma IV.7.4. *Let $\mathcal{G}$ be a transient graph, and let $\mathcal{G}_n$, $n \in \mathbb{N}$, be a sequence of transient graphs increasing to $\mathcal{G}$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which one can define processes $(\varphi_x^{(n)})_{x\in\widetilde{\mathcal{G}}_n}$, $n \in \mathbb{N}$, and $(\varphi_x^{(\infty)})_{x\in\widetilde{\mathcal{G}}}$, with the following properties: for each $n \in \mathbb{N}\cup\{\infty\}$, taking $\mathcal{G}_\infty = \mathcal{G}$, the process $(\varphi_x^{(n)})_{x\in\widetilde{\mathcal{G}}_n}$ has law $\mathbb{P}^G_{\widetilde{\mathcal{G}}_n}$. Moreover, $\mathbb{P}$-a.s, for each compact $K \subset \widetilde{\mathcal{G}}$, $\varphi_x^{(n)} = \varphi_x^{(\infty)}$ for all $x \in K$ and n large enough, and, for all $x_0 \in \widetilde{\mathcal{G}}$ and $h \in \mathbb{R}$, defining $E_n^{\geq h}(x_0)$ as in* (IV.1.3) *but for $\varphi^{(n)}$, $n \in \mathbb{N} \cup \{\infty\}$, we have*

$$\liminf_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq h}(x_0)\big) \geq \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq h}(x_0)\big). \tag{IV.7.5}$$

Proof. Let $\mathcal{L}_{\frac{1}{2}}^{(\infty)}$ be a process under some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same law as $\mathcal{L}_{\frac{1}{2}}$ under $\mathbb{P}^L_{\widetilde{\mathcal{G}}}$. For each $n \in \mathbb{N}$ we define $(L_x^{(n)})_{x\in\widetilde{\mathcal{G}}_n}$ as the total local times of the loops in $\mathcal{L}_{\frac{1}{2}}^{(\infty)}$ which are entirely contained in $\widetilde{\mathcal{G}}_n \subset \widetilde{\mathcal{G}}$. One can clearly identify $\widetilde{\mathcal{G}}_n$ with $\widetilde{\mathcal{G}}_\infty^{\widetilde{\mathcal{G}}_n}$, and by (IV.7.1), the law of $(L_x^{(n)})_{x\in\widetilde{\mathcal{G}}_n}$ is the same as the law of $(L_x)_{x\in\widetilde{\mathcal{G}}}$ under $\mathbb{P}^L_{\widetilde{\mathcal{G}}_n}$. Moreover for each $x \in \widetilde{\mathcal{G}}$, the sequence $L_x^{(n)}$,

$n\in\mathbb{N}$, is increasing, and we denote by $L_x^{(\infty)}$ its limit. Since each loop of $\mathcal{L}_{\frac{1}{2}}^{(\infty)}$ is compact, it is contained in $\widetilde{\mathcal{G}}_n$ for n large enough, and so $(L_x^{(\infty)})_{x\in\widetilde{\mathcal{G}}}$ correspond to the total local times of the loops in $\mathcal{L}_{\frac{1}{2}}^{(\infty)}$ which are entirely contained in $\widetilde{\mathcal{G}}$.

For each $n\in\mathbb{N}$, let $(\mathcal{A}_p^{(n)})_{p\in\mathbb{N}}$ be some enumeration of the clusters $\{x\in\widetilde{\mathcal{G}}:L_x^{(n)}>0\}$, and let $(\sigma_p)_{p\in\mathbb{N}}\in\{-1,1\}^{\mathbb{N}}$ be an independent sequence of uniformly distributed random variables. For each $n\in\mathbb{N}$ and $x\in\widetilde{\mathcal{G}}_n$ we define $E_n^{\mathcal{L}}(x)=\{y\in\widetilde{\mathcal{G}}_n:x\leftrightarrow y\text{ in }\{z\in\widetilde{\mathcal{G}}_n:L_z^{(n)}>0\}\}$, and if $L_x^{(n)}\neq 0$, we define $k_n(x)\in\{1,\dots,n\}$ such that $E_n^{\mathcal{L}}(x)\cap\widetilde{\mathcal{G}}_{k_n(x)}\neq\varnothing$ and $E_n^{\mathcal{L}}(x)\cap\widetilde{\mathcal{G}}_{k_n(x)-1}=\varnothing$, with the convention $\widetilde{\mathcal{G}}_0=\varnothing$. We also define $p_n(x)=\inf\{p\in\mathbb{N}:\mathcal{A}_p^{k_n(x)}\subset E_n^{\mathcal{L}}(x)\}$, with the convention $\inf\varnothing=+\infty$. Note that since $L_x^{(n)}$, $n\in\mathbb{N}$, is increasing for all $x\in\widetilde{\mathcal{G}}$ and $k_n(x)\leq n$, we have that $p_n(x)<\infty$. For each $n\in\mathbb{N}$ and $x\in\widetilde{\mathcal{G}}_n$, we also define $\sigma_x^{(n)}=\sigma_{p_n(x)}$ if $L_x^{(n)}>0$ and $\sigma_x^{(n)}=1$ otherwise, as well as $\varphi_x^{(n)}=\sigma_x^{(n)}\sqrt{2L_x^{(n)}}$. By Theorem IV.7.1, $(\varphi_x^{(n)})_{x\in\widetilde{\mathcal{G}}}$ has law $\mathbb{P}^G_{\widetilde{\mathcal{G}}_n}$. Moreover for each $x\in\widetilde{\mathcal{G}}$ with $L_x^{(\infty)}>0$, we have that $x\in\widetilde{\mathcal{G}}_n$ with $L_x^{(n)}>0$ for n large enough, and that $k_n(x)$ is constant for n large enough since $E_n^{\mathcal{L}}(x)$ increases to $E_\infty^{\mathcal{L}}(x)$. Therefore the sequence $p_n(x)$, $n\in\mathbb{N}$, is decreasing for n large enough, and we denote by $p_\infty(x)$ its limit. Note that we then have $p_n(x)=p_\infty(x)$ for n large enough. We define $\sigma_x^{(\infty)}=\sigma_{p_\infty(x)}$ if $L_x^{(\infty)}>0$ and $\sigma_x^{(\infty)}=1$ otherwise, and $\varphi_x^{(\infty)}=\sigma_x^{(\infty)}\sqrt{2L_x^{(\infty)}}$. We then have $\varphi_x^{(n)}\underset{n\to\infty}{\longrightarrow}\varphi_x^{(\infty)}$ for all $x\in\widetilde{\mathcal{G}}$ and since $g_{\widetilde{\mathcal{G}}_n}(x,y)\underset{n\to\infty}{\longrightarrow}g_{\widetilde{\mathcal{G}}}(x,y)$ for all $x,y\in\widetilde{\mathcal{G}}$, we have that $(\varphi_x^{(\infty)})_{x\in\widetilde{\mathcal{G}}}$ has law $\mathbb{P}^G$.

For each connected compact K of $\widetilde{\mathcal{G}}$, there exists $N\in\mathbb{N}$, such that for all $n\geq N$, $K\subset\widetilde{\mathcal{G}}_n$ and no trajectory in $\mathcal{L}_{\frac{1}{2}}^{(\infty)}$ hitting K hits $\widetilde{\mathcal{G}}\setminus\widetilde{\mathcal{G}}_n$, and then for all $n\geq N$, $L_x^{(n)}=L_x^{(\infty)}$ for all $x\in K$ and the clusters of $\{x\in\widetilde{\mathcal{G}}:L_x^{(n)}>0\}$ entirely included in K are equal to the clusters of $\{x\in\widetilde{\mathcal{G}}:L_x^{(\infty)}>0\}$ entirely included in K. Therefore, $\sigma_x^{(n)}=\sigma_x^{(\infty)}$ for all these clusters and $n\geq N$. Since ∂K is finite, we also have $\sigma_x^{(n)}=\sigma_x^{(\infty)}$ for all $x\in\partial K$ and n large enough. We thus obtain that $\varphi_x^{(n)}=\varphi_x^{(\infty)}$ for all $x\in K$ and n large enough.

For each connected compact K of $\widetilde{\mathcal{G}}$ and $n\in\mathbb{N}$ such that $K\subset\widetilde{\mathcal{G}}_n$, since $P_x^{\mathcal{G}}(\widetilde{H}_K=\infty)\leq P_x^{\mathcal{G}_n}(\widetilde{H}_K=\infty)$ for all $x\in\partial K$, we have $\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(K)\geq\mathrm{cap}_{\widetilde{\mathcal{G}}}(K)$ by (IV.2.14), (IV.2.18) and (IV.2.20). For each connected compact $K\subset\widetilde{\mathcal{G}}$ containing x_0, defining $E_n^{\geq h}(x_0,K)=\{x\in\widetilde{\mathcal{G}}_n\cap K:x_0\longleftrightarrow x\text{ in }E_n^{\geq h}\cap K\}$ for all $n\in\mathbb{N}\cup\{\infty\}$, we have that $\varphi^{(n)}=\varphi^{(\infty)}$ on K for n large enough, and so $E_n^{\geq h}(x_0,K)=E_\infty^{\geq h}(x_0,K)$ for n large enough. Therefore, by (IV.2.24) and

(IV.2.26)

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq h}(x_0)\big) &\geq \liminf_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq h}(x_0,K)\big)\\ &\geq \liminf_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_n^{\geq h}(x_0,K)\big)\\ &= \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq h}(x_0,K)\big).\end{aligned}$$

Taking a sequence of compacts $(K_n)_{n\in\mathbb{N}}$ increasing to $\widetilde{\mathcal{G}}$, $E_\infty^{\geq h}(x_0,K_n)$ is a sequence of compacts increasing to $E_\infty^{\geq h}(x_0)$, and we obtain (IV.7.5) by (IV.2.27). □

We are now ready to do the proof of Theorem IV.3.1, as a consequence of (IV.1.5) and Lemmas IV.7.2 and IV.7.4.

Third proof of Theorem IV.3.1. Let us fix some $x_0 \in G$ and for a sequence U_n, $n \in \mathbb{N}$, of finite connected subsets of G, increasing to G, let us define the killing measure $\kappa_x^{(n)} = \kappa_x$ if $x \in U_n$ and $\kappa_x^{(n)} = \infty$ otherwise, and let $\mathcal{G}_n = \mathcal{G}_{\kappa^{(n)}}$. Then the sequence of graphs $\mathcal{G}_n$, $n \in \mathbb{N}$, increases to $\mathcal{G}$, and G_n is finite for each $n \in \mathbb{N}$. Considering the sequence $(\varphi_x^{(n)})_{x\in\widetilde{\mathcal{G}}_n}$ from Lemma IV.7.4, we have by Lemma IV.7.2 and (Law$_0$) that for all $n \in \mathbb{N}$

$$\mathbb{E}\left[\exp\left(-u\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(E_n^{\geq 0}(x_0))\right)\mathbb{1}_{\varphi_{x_0}^{(n)}\geq 0}\right] = \mathbb{P}(\varphi_{x_0}^{(n)} \geq \sqrt{2u}) \text{ for all } u > 0. \quad \text{(IV.7.6)}$$

Taking the limit as $n \to \infty$, by dominated convergence and (IV.7.5), we thus have

$$\mathbb{E}^G\left[\exp\left(-u\mathrm{cap}_{\widetilde{\mathcal{G}}}(E^{\geq 0}(x_0))\right)\mathbb{1}_{\varphi_{x_0}\geq 0}\right] \geq \mathbb{P}^G(\varphi_{x_0} \geq \sqrt{2u}) \text{ for all } u > 0.$$

Taking the limit as $u \to 0$, we obtain by dominated convergence

$$\mathbb{P}^G(\mathrm{cap}(E^{\geq 0}(x_0)) < \infty, \varphi_{x_0} \geq 0) \geq \frac{1}{2}.$$

Since $E^{\geq 0}(x_0) = \varnothing$ when $\varphi_{x_0} < 0$ and $\mathbb{P}^G(\varphi_{x_0} < 0) = \frac{1}{2}$, we obtain that $\mathrm{cap}(E^{\geq 0}(x_0))$ is $\mathbb{P}^G$-a.s. finite.

Let us now fix some $h < 0$, and let $u = h^2/2$. By (IV.2.38), $\mathcal{I}^u = \{x \in \widetilde{\mathcal{G}} : \ell_{x,u} > 0\}$ intersects $\{x_0\}$ with positive probability, and all components of $\mathcal{I}^u$ are non-compact by definition. Therefore by (IV.1.5) the component of x_0 in $\{x \in \widetilde{\mathcal{G}} : (\varphi_x + h)^2 > 0\}$ is non-compact with positive probability, that is by continuity and symmetry of the Gaussian free field, either $E^{\geq h}(x_0)$ or $E^{\geq -h}(x_0)$ is non-compact with positive probability. Since $E^{\geq -h}(x_0) \subset E^{\geq h}(x_0)$, we obtain that $E^{\geq h}(x_0)$ is non-compact with positive probability. □

Third proof of Theorem IV.3.3. Let $\mathcal{G}$ be a transient graph, and let $\mathcal{G}_n$, $n \in \mathbb{N}$, be a sequence of finite graphs increasing to $\mathcal{G}$ with G_n finite for all $n \in \mathbb{N}$, as in the third proof of Theorem IV.3.1. Let us fix some $x_0 \in \widetilde{\mathcal{G}}$, and assume that $E_\infty^{\geq h}(x_0)$ is compact. By Lemma IV.8.2, we have that $\varphi^{(n)} = \varphi^{(\infty)}$ on a neighborhood of $E_\infty^{\geq h}(x_0)$ for all n large enough. Moreover, one can easily prove by (IV.2.14) that the equilibrium measure of any compact set K on $\widetilde{\mathcal{G}}_n$ converge to the equilibrium measure of K on $\widetilde{\mathcal{G}}$, and thus by (IV.2.20) we obtain that for all $h \in \mathbb{R}$, if $E_\infty^{\geq h}(x_0)$ is compact

$$\lim_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq h}(x_0)\big) = \lim_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_\infty^{\geq h}(x_0)\big) = \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq 0}(x_0)\big).$$

Since G_n is finite for each $n \in \mathbb{N}$, we have by Lemma IV.7.2 that (Isom') holds on $\mathcal{G}_n$, and so by Proposition IV.4.7 we have

$$\mathbb{E}\left[\exp\left(-u\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(E_n^{\geq h}(x_0))\right) \mathbb{1}_{\varphi_{x_0}^{(n)} \geq h}\right] = \mathbb{P}(\varphi_{x_0}^{(n)} \geq \sqrt{2u+h^2}) \text{ for all } u > 0.$$

Now assume that (Sign) is fulfilled on $\mathcal{G}$ and fix some $h \geq 0$, then $E_\infty^{\geq h}(x_0)$ is $\mathbb{P}$-a.s. compact, and then taking the limit as $n \to \infty$, we obtain by dominated convergence that (Law$_h$) holds on $\mathcal{G}$, and thus (IV.3.6) also holds on $\mathcal{G}$ by Lemma IV.4.5.

If (IV.3.8) is fulfilled on $\mathcal{G}$, then either $\mathrm{cap}(E_\infty^{\geq h}(x_0)) < \infty$, and then $E_\infty^{\geq h}(x_0)$ is bounded, and thus compact by Proposition IV.4.4, or $\mathrm{cap}(E_\infty^{\geq h}(x_0)) = \infty$, and so, using (IV.7.5), we have that for all $h < 0$, $\mathbb{P}$-a.s,

$$\lim_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq h}(x_0)\big) = \lim_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_\infty^{\geq h}(x_0)\big) = \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq 0}(x_0)\big).$$

For each $n \in \mathbb{N}$, (Cap) holds trivially on $\mathcal{G}_n$, and so by Proposition IV.4.7 we have for all $h > 0$

$$\mathbb{E}\left[\exp\left(-u\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(E_n^{\geq -h}(x_0))\right) \mathbb{1}_{\varphi_{x_0}^{(n)} \geq h}\right] = \mathbb{P}(\varphi_{x_0}^{(n)} \geq \sqrt{2u+h^2}) \text{ for all } u > 0.$$

Taking the limit as $n \to \infty$, we obtain by dominated convergence that (IV.3.7) holds on $\mathcal{G}$. Therefore by (Law$_h$) for $u = 0$ we have

$$\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h}(x_0)) \in (\mathrm{cap}(\{x_0\}), \infty)\big) = \mathbb{P}^G(\varphi_{x_0} \geq h).$$

Since $\mathbb{P}^G\big(\mathrm{cap}(E^{\geq -h}(x_0)) \leq \mathrm{cap}(\{x_0\})\big) = \mathbb{P}^G(\varphi_{x_0} \leq -h)$, we obtain (IV.3.8). □

Remark IV.7.5. 1) With the help of Lemma IV.7.4, we directly obtain a way to prove (Law$_0$). Indeed if $\mathcal{G}$ is a graph such that there exists a sequence $\mathcal{G}_n$ of graphs increasing to $\mathcal{G}$, and such that (Law$_0$) hold on $\mathcal{G}_n$ for all $n \in \mathbb{N}$ and, $\mathbb{P}$-a.s,

$$\limsup_{n\to\infty} \mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq 0}(x_0)\big) \leq \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq 0}(x_0)\big) \text{ for all } x_0 \in \widetilde{\mathcal{G}}, \tag{IV.7.7}$$

then, using (IV.7.5), we thus have $\mathrm{cap}_{\widetilde{\mathcal{G}}_n}\big(E_n^{\geq 0}(x_0)\big) \underset{n\to\infty}{\longrightarrow} \mathrm{cap}_{\widetilde{\mathcal{G}}}\big(E_\infty^{\geq 0}(x_0)\big)$, and taking the limit in (IV.7.6), we obtain that (Law_0) holds.

2) By Proposition IV.4.7 and Lemma IV.7.2, we have that if $\mathcal{G}$ is finite graph, then the compact clusters of $E^{\geq -h}$ and $E^{\geq h}$ have the same law, and so for all compacts K of $\widetilde{\mathcal{G}}$, the clusters of $E^{\geq -h}$ and $E^{\geq h}$ included in K have the same law. Let us now consider a general transient graph $\mathcal{G}$, then the graph $\mathcal{G}_*^K$, defined above (IV.2.35), is finite, and so we directly obtain that the clusters of $E^{\geq -h}$ and $E^{\geq h}$ included in K also have the same law for the graph $\mathcal{G}$. Approximating any connected and closed set F by an increasing sequence of compacts, this is thus a generalization of Corollary IV.5.3.

IV.8 Proof of the signed isomorphism with random interlacements

In this section we prove the isomorphism between random interlacements and the Gaussian free field, Theorem IV.3.4. We first compare random interlacements on $\mathcal{G}$ with random interlacements on $\mathcal{G}_{\kappa'}$ for some $\kappa' \geq \kappa$ in Lemma IV.8.1, and use this comparison to approximate random interlacements on any transient graphs by random interlacements on finite graphs in Lemma IV.8.2. Using the approximation of the Gaussian free field on transient graphs by Gaussian free fields on finite graphs from Lemma IV.7.4, and that Theorem IV.3.4 holds on finite graph, see Lemma IV.7.2, we can prove Theorem IV.3.4, see Lemma IV.8.3. Finally at the end of the section, we prove Proposition IV.3.5 and deduce from Theorem IV.3.4, that Corollaries IV.3.6 and IV.3.7 also hold.

We are now going to approximate random interlacements on any transient graph $\mathcal{G}$ by random interlacements on a sequence of finite graphs $\mathcal{G}_n$ increasing to $\mathcal{G}$. We are first going to compare random interlacements on $\mathcal{G}$ with random interlacements on $\mathcal{G}_{\kappa'}$, for some killing measure $\kappa' \geq \kappa$. Note that we can see $\widetilde{\mathcal{G}}_{\kappa'}$ as a subset of $\widetilde{\mathcal{G}}$, and for all compacts K of $\widetilde{\mathcal{G}}_{\kappa'}$ and $w \in W^0_{K,\widetilde{\mathcal{G}}}$, we define we define the killing times $\widetilde{\zeta}^+_{\kappa',K}(w)$ and $\widetilde{\zeta}^-_{\kappa',K}(w)$ by

$$\widetilde{\zeta}^+_{\kappa',K}(w) \stackrel{\text{def.}}{=} \inf\big\{t \in [0,\widetilde{\zeta}^+) : w(t) \notin \widetilde{\mathcal{G}}_{\kappa'}\big\} \text{ and}$$
$$\widetilde{\zeta}^-_{\kappa',K}(w) \stackrel{\text{def.}}{=} \sup\big\{t \in (\widetilde{\zeta}^-,0] : w(t) \notin \widetilde{\mathcal{G}}_{\kappa'}\big\},$$

with the convention $\inf\varnothing = \widetilde{\zeta}^+(w)$ and $\sup\varnothing = \widetilde{\zeta}^-(w)$. We also define $\pi_{\widetilde{\mathcal{G}},\kappa',K}$:

$W^0_{K,\widetilde{\mathcal{G}}} \to W^0_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ by

$$\pi_{\widetilde{\mathcal{G}},\kappa',K}(w) = \begin{cases} w(t) & \text{if } t \in (\widetilde{\zeta}^-_{\kappa',K}(w), \widetilde{\zeta}^+_{\kappa',K}(w)), \\ \Delta & \text{otherwise,} \end{cases}$$

as well as $\pi^*_{\widetilde{\mathcal{G}},\kappa',K} : W^*_{K,\widetilde{\mathcal{G}}} \to W^*_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ as the unique function such that $p^*_{\widetilde{\mathcal{G}}_{\kappa'}} \circ \pi_{\widetilde{\mathcal{G}},\kappa',K} = \pi^*_{\widetilde{\mathcal{G}},\kappa',K} \circ p^*_{\widetilde{\mathcal{G}}}$. In other words $\pi^*_{\widetilde{\mathcal{G}},\kappa',K}(w^*)$ is the doubly infinite trajectory modulo time-shift on $\widetilde{\mathcal{G}}_{\kappa'}$, whose forward and backward parts on hitting K are the forward and backward parts on hitting K of w^*, both stopped on exiting $\widetilde{\mathcal{G}}_{\kappa'}$.

Lemma IV.8.1. *Let $\mathcal{G}$ be a graph with killing measure κ, let κ' be another killing measure such that $\kappa' \geq \kappa$, and let K and K' be compacts of $\widetilde{\mathcal{G}}_{\kappa'}$ with $K' \subset K$. There exists a measure $\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}$ on $W^*_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ such that*

$$(\nu_{\widetilde{\mathcal{G}}})_{|W^*_{K,\widetilde{\mathcal{G}}} \setminus W^*_{K',\widetilde{\mathcal{G}}}} \circ \big(\pi^*_{\widetilde{\mathcal{G}},\kappa',K}\big)^{-1} + \mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'} = (\nu_{\widetilde{\mathcal{G}}_{\kappa'}})_{|W^*_{K,\widetilde{\mathcal{G}}_{\kappa'}} \setminus W^*_{K',\widetilde{\mathcal{G}}_{\kappa'}}}. \tag{IV.8.1}$$

Moreover we have

$$\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}(W^*_{K,\widetilde{\mathcal{G}}_{\kappa'}}) = \mathrm{cap}_{\widetilde{\mathcal{G}}_{\kappa'}}(K) - \mathrm{cap}_{\widetilde{\mathcal{G}}_{\kappa'}}(K') - \mathrm{cap}_{\widetilde{\mathcal{G}}}(K) + \mathrm{cap}_{\widetilde{\mathcal{G}}}(K'). \tag{IV.8.2}$$

Proof. Considering the graph $\mathcal{G}^{\partial K \cup \partial K'}$, see Lemma IV.2.1, we can assume without loss of generality that $\partial K \subset G$ and $\partial K' \subset G$. Considering the graph $\mathcal{G}^A$, where $A \subset \widetilde{\mathcal{G}}_{\kappa'}$ is a set containing exactly 1 vertex on I_x for all $x \in K \cap G$, we can also assume without loss of generality that $\kappa_x = \kappa'_x = 0$ for all $x \in K \cap G$. Let us recall the notation $X^{\kappa'}$ and $\widetilde{\zeta}_{\kappa'}$ from (IV.2.8), and note that for all $w \in W^0_{K,\widetilde{\mathcal{G}}}$, the forward part of $\pi_{\widetilde{\mathcal{G}},\kappa',K}(w)$ on hitting K is $X^{\kappa'}(w^+)$, where w^+ is the forward part of w. We define the signed measure $\widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}$ on $\mathcal{W}_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ which is given on $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ by

$$\begin{aligned} \widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'} = \sum_{x \in \partial K} \Big(& e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x) P^{\widetilde{\mathcal{G}}_{\kappa'}}_x \big(\cdot^+, H_{K'} = \widetilde{\zeta}_{\kappa'}\big) P^{K,\widetilde{\mathcal{G}}_{\kappa'}}_x(\cdot^-) \\ & - e_{K,\widetilde{\mathcal{G}}}(x) P^{\widetilde{\mathcal{G}}}_x \big(X^{\kappa'} \in \cdot^+, H_{K'} = \widetilde{\zeta}_\kappa\big) P^{K,\widetilde{\mathcal{G}}}_x \big(X^{\kappa'} \in \cdot^-\big)\Big), \end{aligned}$$

and such that $\widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}(A) = 0$ for all $A \in \mathcal{W}_{K,\widetilde{\mathcal{G}}_{\kappa'}}$ with $A \cap W^0_{K,\widetilde{\mathcal{G}}_{\kappa'}} = \varnothing$. We also define $\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'} = \widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'} \circ (p^*_{\widetilde{\mathcal{G}}_{\kappa'}})^{-1}$, and by (IV.2.36) and (IV.2.37), it is clear that (IV.8.1) holds. Let us denote by $(\widehat{X}^\kappa_n)_{n \in \mathbb{N}}$ the discrete Markov chain which jumps to a new vertex of G every time X^κ hit this new vertex, that is for each $x \in G$, the Markov chain $\widehat{X}^\kappa$ has the same law under $P^{\widetilde{\mathcal{G}}}_x$ as $\widehat{Z}$ under $P^{\mathcal{G}}_x$. Let us denote by $\widehat{L}^\kappa_K = \sup\{n \in \mathbb{N} : \widehat{X}^\kappa \in K\}$ the last exit time of K for $\widehat{X}^\kappa$, and

$L_K^\kappa = \sup\{t \geq 0 : X^\kappa \in K\}$ the last exit time of K for X, with the convention $\sup \varnothing = \infty$, and then $\{X_{L_K^\kappa}^\kappa = x\} = \{\widehat{X}_{\widehat{L}_K^\kappa}^\kappa = x\}$. By definition of $P_x^{K,\widetilde{\mathcal{G}}}$ and (IV.2.29) we have for all $x \in \partial K$ with $e_{K,\widetilde{\mathcal{G}}}(x) > 0$

$$\begin{aligned}
e_{K,\widetilde{\mathcal{G}}}(x) P_x^{K,\widetilde{\mathcal{G}}}\big(X^{\kappa'} \in \cdot\big) &= e_{K,\widetilde{\mathcal{G}}}(x) P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^\kappa} \in \cdot \,|\, X_{L_K^\kappa}^\kappa = x\big) \\
&= \frac{1}{g_{\widetilde{\mathcal{G}}}(x,x)} P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^\kappa} \in \cdot, \widehat{X}_{\widehat{L}_K^\kappa}^\kappa = x\big) \\
&= \frac{1}{g_{\widetilde{\mathcal{G}}}(x,x)} \sum_{n \geq 0} P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^\kappa} \in \cdot, \widehat{X}_n^\kappa = x, \widehat{L}_K^\kappa = n\big) \\
&= \lambda_x P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^\kappa} \in \cdot, \widehat{L}_K^\kappa = 0\big),
\end{aligned}$$

where we used in the fourth inequality the strong Markov property and the identity

$$g_{\widetilde{\mathcal{G}}}(x,x) = \frac{1}{\lambda_x} \sum_{n \geq 0} P_x(\widehat{X}_n^\kappa = x).$$

Similarly we have, using similar notations, by (IV.2.9),

$$\begin{aligned}
e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x) P_x^{K,\widetilde{\mathcal{G}}_{\kappa'}}(\,\cdot\,) &= \lambda_x' P_x^{\widetilde{\mathcal{G}}_{\kappa'}}\big((X_t^{\kappa'})_{t > L_K^{\kappa'}} \in \cdot, \widehat{L}_K^{\kappa'} = 0\big) \\
&= \lambda_x' P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^{\kappa'}} \in \cdot, \widehat{L}_K^{\kappa'} = 0\big).
\end{aligned}$$

On the event $\widehat{L}_K^\kappa = 0$, since $\kappa_x = \kappa_x'$ for all $x \in K$, we have $L_K^{\kappa'} = L_K^\kappa$. Therefore, by a similar argument as before, for all $x \in \partial K$ with $e_{K,\widetilde{\mathcal{G}}}(x) > 0$,

$$\begin{aligned}
&\frac{\lambda_x}{\lambda_x'} e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x) P_x^{K,\widetilde{\mathcal{G}}_{\kappa'}}(\,\cdot\,) - e_{K,\widetilde{\mathcal{G}}}(x) P_x^{K,\widetilde{\mathcal{G}}}\big(X^{\kappa'} \in \cdot\big) \\
&= \lambda_x P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^{\kappa'}} \in \cdot, \widehat{L}_K^{\kappa'} = 0 < \widehat{L}_K^\kappa\big) \\
&= \frac{\lambda_x}{\lambda_x'} e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x) P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t > L_K^{\kappa'}}, L_K^{\kappa'} < L_K^\kappa \,|\, X_{L_K^{\kappa'}} = x\big).
\end{aligned}$$

Note that if $e_{K,\widetilde{\mathcal{G}}}(x) = 0$ and $e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x) > 0$, then $L_K^{\kappa'} < \infty = L_K^\kappa$ $P_x^{\widetilde{\mathcal{G}}}$-a.s, and so the previous equality still holds. Moreover, using (IV.2.9), we have for all $x \in \partial K$

$$P_x^{\widetilde{\mathcal{G}}_{\kappa'}}\big(\cdot, H_{K'} = \widetilde{\zeta}_{\kappa'}\big) - P_x^{\widetilde{\mathcal{G}}}\big(X^{\kappa'} \in \cdot, H_{K'} = \widetilde{\zeta}_\kappa\big) = P_x^{\widetilde{\mathcal{G}}}\big(X^{\kappa'} \in \cdot, \widetilde{\zeta}_\kappa > H_{K'} > \widetilde{\zeta}_{\kappa'}\big).$$

We obtain that, on $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}_{\kappa'}}$,

$$\begin{aligned}
&\widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}\\
&=\sum_{x\in\partial K}\Big(\frac{\lambda'_x-\lambda_x}{\lambda'_x}e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x)P_x^{\widetilde{\mathcal{G}}_{\kappa'}}\big(\cdot^+,H_{K'}=\widetilde{\zeta}_{\kappa'}\big)P_x^{K,\widetilde{\mathcal{G}}_{\kappa'}}\big(\cdot^-\big)\\
&\quad+\frac{\lambda_x}{\lambda'_x}e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x)P_x^{\widetilde{\mathcal{G}}_{\kappa'}}\big(\cdot^+,H_{K'}=\widetilde{\zeta}_{\kappa'}\big)P_x^{\widetilde{\mathcal{G}}}\big((X_t^{\kappa'})_{t>L_K^{\kappa'}}\in\cdot^-,L_K^{\kappa'}<L_K^{\kappa}\,|\,X_{L_K^{\kappa'}}=x\big)\\
&\quad+e_{K,\widetilde{\mathcal{G}}}(x)P_x^{\widetilde{\mathcal{G}}}\big(X^{\kappa'}\in\cdot^+,\widetilde{\zeta}_{\kappa}>H_{K'}>\widetilde{\zeta}_{\kappa'}\big)P_x^{K,\widetilde{\mathcal{G}}}\big(X^{\kappa'}\in\cdot^-\big)\Big),\qquad\text{(IV.8.3)}
\end{aligned}$$

and so $\widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}$ is positive on $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}_{\kappa'}}$. By the monotone class theorem it is a positive measure on $\mathcal{W}_{K,\widetilde{\mathcal{G}}_{\kappa'}}$, and so $\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}$ is also a positive measure. Moreover we have by (IV.2.20) and (IV.2.23)

$$\begin{aligned}
\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}(W^*_{K,\widetilde{\mathcal{G}}_{\kappa'}})&=\widetilde{\mu}^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}(W^0_{K,\widetilde{\mathcal{G}}_{\kappa'}})\\
&=\sum_{x\in\partial K}\Big(e_{K,\widetilde{\mathcal{G}}_{\kappa'}}(x)P_x^{\widetilde{\mathcal{G}}_{\kappa'}}(H_{K'}=\widetilde{\zeta}_{\kappa'})-e_{K,\widetilde{\mathcal{G}}}(x)P_x^{\widetilde{\mathcal{G}}}(H_{K'}=\widetilde{\zeta}_{\kappa})\Big)\\
&=\mathrm{cap}_{\widetilde{\mathcal{G}}_{\kappa'}}(K)-\mathrm{cap}_{\widetilde{\mathcal{G}}_{\kappa'}}(K')-\mathrm{cap}_{\widetilde{\mathcal{G}}}(K)+\mathrm{cap}_{\widetilde{\mathcal{G}}}(K').
\end{aligned}$$

□

The difference between the trajectories under $\nu_{\widetilde{\mathcal{G}}}$ and $\nu_{\widetilde{\mathcal{G}}_{\kappa'}}$ hitting K but not K', when $K'\subset K$ compact $\widetilde{\mathcal{G}}_{\kappa'}$, comes into three parts: first the difference between the weights $\lambda'_x\geq\lambda_x$, then it is more likely for the forward trajectories to not hit K' before time $\widetilde{\zeta}_{\kappa'}$ than before time $\widetilde{\zeta}_{\kappa}$, and finally it is more likely for the backwards trajectories to not come back in K before time $\widetilde{\zeta}_{\kappa'}$ than before time $\widetilde{\zeta}_{\kappa}$. These three differences are all contained in the measure $\mu^{K,K'}_{\widetilde{\mathcal{G}},\kappa'}$ from (IV.8.1), see (IV.8.3). Taking a sequence $(K_p)_{p\in\mathbb{N}}$ of compacts increasing to $\widetilde{\mathcal{G}}_{\kappa'}$, one can then use Lemma IV.8.1 to construct a random interlacement process on $\widetilde{\mathcal{G}}_{\kappa'}$ from the random interlacement process ω on $\widetilde{\mathcal{G}}_{\kappa}$: take the image through $\pi^*_{\widetilde{\mathcal{G}},\kappa',K_p}$ of each trajectory in ω hitting K_p but not K_{p-1} for all $p\in\mathbb{N}$, with $K_0=\varnothing$, and add Poisson point processes with intensity $\mu^{K_p,K_{p-1}}_{\widetilde{\mathcal{G}},\kappa'}\otimes\lambda$ for all $p\in\mathbb{N}$. Using this construction and the estimate (IV.8.2), we can now approximate random interlacements on $\mathcal{G}$ by random interlacements on $\mathcal{G}_n$, where $\mathcal{G}_n$ is a sequence of finite graphs increasing to $\mathcal{G}$.

Lemma IV.8.2. *Let $\mathcal{G}$ be a transient graph, and let $\mathcal{G}_n$, $n\in\mathbb{N}$, be a sequence of transient graphs increasing to $\mathcal{G}$. There exists a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ on which one can define a sequence of processes $\omega^{(n)}$, $n\in\mathbb{N}$, and $\omega^{(\infty)}$ with the following properties: for each $n\in\mathbb{N}\cup\{\infty\}$, taking $\mathcal{G}^{(\infty)}=\mathcal{G}$, the process $\omega^{(n)}$*

has the same law as ω under $\mathbb{P}^I_{\widetilde{\mathcal{G}}_n}$. Moreover, there exists an increasing sequence $(a_n)_{n\in\mathbb{N}}$ such that for each $u>0$ and compact K of $\widetilde{\mathcal{G}}$, $\mathbb{P}'$-a.s, there exists $N\in\mathbb{N}$ such that for all $n\geq N$ the restriction to K of the set of trajectories hitting K is the same for $\omega_u^{(a_n)}$ and $\omega_u^{(\infty)}$.

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be such that K_n is a compact of $\widetilde{\mathcal{G}}_n$ for each $n\in\mathbb{N}$, and K_n, $n\in\mathbb{N}$, increases to $\widetilde{\mathcal{G}}$. Let $\omega^{(\infty)}$ be a Poisson point process under $(\Omega',\mathcal{F}',\mathbb{P}')$ with the same law as the random interlacement process ω under $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$. For each $n\in\mathbb{N}$ and $k\in\{1,\dots,n\}$, we define $\omega_1^{(k,n)}$ as the Poisson point process which consist of the image through $\pi^*_{\widetilde{\mathcal{G}},\kappa^{(n)},K_k}$ of all the trajectories in $\omega_u^{(\infty)}$ which hit K_k but not K_{k-1}, with the notation $K_0=\varnothing$, which is a Poisson point process with intensity $(\nu_{\widetilde{\mathcal{G}}})_{|W^*_{K_k,\widetilde{\mathcal{G}}}\setminus W^*_{K_{k-1},\widetilde{\mathcal{G}}}}\circ\big(\pi^*_{\widetilde{\mathcal{G}},\kappa^{(n)},K_k}\big)^{-1}$. We also define $\omega_2^{(k,n)}$ as an independent Poisson point process with intensity $\mu^{K_k,K_{k-1}}_{\widetilde{\mathcal{G}},\kappa^{(n)}}\otimes\lambda$ and $\omega_3^{(n)}$ as an independent Poisson point process with intensity $(\nu_{\widetilde{\mathcal{G}}_n})_{|(W^*_{K_n,\widetilde{\mathcal{G}}_n})^c}\otimes\lambda$. Defining for each $n\in\mathbb{N}$

$$\omega^{(n)}=\omega_3^{(n)}+\sum_{k=1}^{n}\Big(\omega_1^{(k,n)}+\omega_2^{(k,n)}\Big),$$

we have by Lemma IV.8.1 that $\omega^{(n)}$ has the same law as as ω under $\mathbb{P}^I_{\widetilde{\mathcal{G}}_n}$. Let us now fix some $u>0$ and $p\in\mathbb{N}$. By definition, no trajectories of $\omega_1^{(k,n)}$, $\omega_2^{(k,n)}$ and $\omega_3^{(n)}$ hits K_p if $p<k\leq n$. Moreover there is a only a finite number of trajectories in $\omega_u^{(\infty)}$ hitting K_p, and so for each $k\in\{1,\dots,p\}$, we have that the restriction to K_p of all the trajectories of $\omega_1^{(k,n)}$ at level u hitting K_p is constant for all n large enough. By (IV.8.2), for each $n\geq p$, the number of trajectories in $\sum_{k=1}^p\omega_2^{(k,n)}$ at level u is a Poisson random variable with parameter $u(\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(K_p)-\mathrm{cap}_{\widetilde{\mathcal{G}}}(K_p))$, and one can easily prove by (IV.2.14), (IV.2.18) and (IV.2.20) a since K_p compact that $\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(K_p)-\mathrm{cap}_{\widetilde{\mathcal{G}}}(K_p)\underset{n\to\infty}{\longrightarrow}0$. Using Borel-Cantelli Lemma, one can find a sequence $(a_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}'$-a.s, $\sum_{k=1}^p\omega_2^{(k,a_n)}$ contains no trajectory at level u for all $u>0$ and n large enough, and by a diagonal argument, one can take $(a_n)_{n\in\mathbb{N}}$ independent of the choice of p. Therefore for all compacts K of $\widetilde{\mathcal{G}}$, there exist $p\in\mathbb{N}$ such that $K\subset K_p$, and $\mathbb{P}'$-a.s, the restriction to K_p of all the trajectories of $\omega_u^{(a_n)}$ hitting K_p is constant for all n large enough, and we can conclude. □

From Lemmas IV.7.4 and IV.8.2, we obtain a way to approximate the Gaussian free field and random interlacements on a graph $\widetilde{\mathcal{G}}$ by Gaussian free fields and random interlacements on a sequence of graphs finite graphs increasing to $\mathcal{G}$. With the help of Lemma IV.7.2, we obtain the following Lemma, from which Theorem IV.3.4 readily follows.

Lemma IV.8.3. *If either* (Sign) *or* (Law$_0$) *is fulfilled, then* (Isom') *and* (IV.7.4) *hold on* $\mathcal{G}$.

Proof. Let $\mathcal{G}_n$, $n \in \mathbb{N}$ be a sequence of finite graphs increasing to $\mathcal{G}$, and consider the space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$ from Lemmas IV.7.4 and IV.8.2. Up to considering a subsequence of $\mathcal{G}_n$, $n \in \mathbb{N}$, we can assume that $a_n = n$ in Lemma IV.8.2. Note that G_n is finite for each $n \in \mathbb{N}$, and so one can use Lemma IV.7.2 for $\mathcal{G}_n$. For each $n \in \mathbb{N} \cup \{\infty\}$, let $(\ell^{(n)}_{x,u})_{x \in \widetilde{\mathcal{G}}_n}$ be the total local times of the trajectories of $\omega^{(n)}_u$, $\mathcal{I}^u_n = \{x \in \widetilde{\mathcal{G}}_n : \ell^{(n)}_{x,u} > 0\}$ and $E^{|\geq 0|}_n(x) = \{y \in \widetilde{\mathcal{G}}_n : x \longleftrightarrow y \text{ in } \{z \in \widetilde{\mathcal{G}}_n : \varphi^{(n)}_z \mathrm{sign}(\varphi^{(n)}_x) \geq 0\}\}$ for all $x \in \widetilde{\mathcal{G}}_n$. Let us prove that there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that, $\mathbb{P} \otimes \mathbb{P}'$-a.s. for all $x \in \widetilde{\mathcal{G}}$

$$\begin{aligned}\big\{\mathcal{I}^u_\infty \cap E^{|\geq 0|}_\infty(x) \neq \varnothing\big\} &= \liminf_{n\to\infty} \big\{\mathcal{I}^u_{b_n} \cap E^{|\geq 0|}_{b_n}(x) \neq \varnothing\big\} \\ &= \limsup_{n\to\infty} \big\{\mathcal{I}^u_{b_n} \cap E^{|\geq 0|}_{b_n}(x) \neq \varnothing\big\}.\end{aligned} \tag{IV.8.4}$$

If $y \in \mathcal{I}^u_\infty \cap E^{|\geq 0|}_\infty(x)$, then $y \in \mathcal{I}^u_n$ for n large enough and there is a path $\pi \subset \widetilde{\mathcal{G}}$ between x and y in $\{z \in \widetilde{\mathcal{G}} : \varphi^{(\infty)}_z \mathrm{sign}(\varphi^{(\infty)}_x) \geq 0\}$. Since π is compact, by Lemma IV.8.2, $\varphi^{(n)} = \varphi^{(\infty)}$ on π for n large enough, and thus π is also a path between x and y in $\{z \in \widetilde{\mathcal{G}} : \varphi^{(n)}_z \mathrm{sign}(\varphi^{(n)}_x) \geq 0\}$, and so $y \in \mathcal{I}^u_n \cap E^{|\geq 0|}_n(x)$ for n large enough, that is

$$\begin{aligned}\big\{\mathcal{I}^u_\infty \cap E^{|\geq 0|}_\infty(x) \neq \varnothing\big\} &\subset \liminf_{n\to\infty} \big\{\mathcal{I}^u_n \cap E^{|\geq 0|}_n(x) \neq \varnothing\big\} \\ &\Big(\subset \limsup_{n\to\infty} \big\{\mathcal{I}^u_n \cap E^{|\geq 0|}_n(x) \neq \varnothing\big\}\Big).\end{aligned} \tag{IV.8.5}$$

We now prove the other inclusions in (IV.8.4), and first assume that (Sign) is fulfilled. Let us fix some $x \in \widetilde{\mathcal{G}}$ such that $\mathcal{I}^u_n \cap E^{|\geq 0|}_n(x) \neq \varnothing$ infinitely often. By Lemma IV.8.2, since $E^{|\geq 0|}_\infty(x)$ is compact, we have that $\varphi^{(n)}$ and $\mathcal{I}^u_n$ are constant for n large enough on an open neighborhood of $E^{|\geq 0|}_\infty(x)$, and then $E^{|\geq 0|}_n(x) \cap \mathcal{I}^u_n = E^{|\geq 0|}_\infty(x) \cap \mathcal{I}^u_\infty$ for n large enough. Therefore, infinitely often, $\mathcal{I}^u_\infty \cap E^{|\geq 0|}_\infty(x) = \mathcal{I}^u_n \cap E^{|\geq 0|}_n(x) \neq \varnothing$, and combining with (IV.8.5), we obtain (IV.8.4) for $b_n = n$.

Let us now assume that (Law$_0$) is fulfilled for $\mathcal{G}$. For all $n \in \mathbb{N} \cup \{\infty\}$, by (IV.2.38)

$$(\mathbb{P} \otimes \mathbb{P}')\big(\mathcal{I}^u_n \cap E^{|\geq 0|}_n(x) \neq \varnothing\big) = 1 - \mathbb{E}\big[\exp(-u\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(E^{|\geq 0|}_n(x)))\big].$$

Since G_n is finite for each $n \in \mathbb{N}$, we have by Lemma IV.7.2 and Proposition IV.4.7 that (Law$_0$) holds on $\widetilde{\mathcal{G}}_n$, and therefore, denoting by Φ the distribution function of a $\mathcal{N}(0,1)$-distributed random variable, we have by (IV.2.38) and symmetry of the Gaussian free field

$$\begin{aligned}(\mathbb{P}\otimes\mathbb{P}')\big(\mathcal{I}_n^u\cap E_n^{|\geq 0|}(x)\neq\varnothing\big)&=1-\mathbb{E}\big[\exp(-u\mathrm{cap}_{\widetilde{\mathcal{G}}_n}(E_n^{|\geq 0|}(x)))\big]\\&=1-2\mathbb{P}^G_{\widetilde{\mathcal{G}}_n}(\varphi_x\geq\sqrt{2u})\\&=2\Phi\big(\sqrt{2u}(g_{\widetilde{\mathcal{G}}_n}(x,x))^{-1/2}\big)-1\\&\underset{n\to\infty}{\longrightarrow}2\Phi\big(\sqrt{2u}(g_{\widetilde{\mathcal{G}}}(x,x))^{-1/2}\big)-1\\&=(\mathbb{P}\otimes\mathbb{P}')\big(\mathcal{I}_\infty^u\cap E_\infty^{|\geq 0|}(x)\neq\varnothing\big),\end{aligned}$$

where we used (Law_0) for the graph $\mathcal{G}$ in the last equality. Therefore, using (IV.8.5), there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$

$$\sum_{n\in\mathbb{N}}\mathbb{P}\otimes\mathbb{P}'\Big(\big\{\mathcal{I}_{b_n}^u\cap E_{b_n}^{|\geq 0|}(x)\neq\varnothing\big\}\setminus\big\{\mathcal{I}_\infty^u\cap E_\infty^{|\geq 0|}(x)\neq\varnothing\big\}\Big)<\infty.$$

By Borel-Cantelli Lemma, we thus obtain that $\mathbb{P}\otimes\mathbb{P}'$-a.s.

$$\limsup_{n\to\infty}\big\{\mathcal{I}_{b_n}^u\cap E_{b_n}^{|\geq 0|}(x)\neq\varnothing\big\}=\big\{\mathcal{I}_\infty^u\cap E_\infty^{|\geq 0|}(x)\neq\varnothing\big\}.$$

Using a diagonal argument and separability of $\widetilde{\mathcal{G}}$, we can actually choose the sequence $(b_n)_{n\in\mathbb{N}}$ uniformly in $x\in\widetilde{\mathcal{G}}$. Combining with (IV.8.5), we obtain (IV.8.4).

Up to taking a subsequence of $\mathcal{G}_n$, $n\in\mathbb{N}$, we can from now assume that $b_n=n$ in (IV.8.4). For each $n\in\mathbb{N}\cup\{\infty\}$ and $x\in\widetilde{\mathcal{G}}_n$, we define $\sigma_x^{u,n}=1$ if $E_n^{|\geq 0|}(x)\cap\mathcal{I}_n^u\neq\varnothing$ or $\varphi_x^{(n)}=0$, and $\sigma_x^{u,n}=\mathrm{sign}(\varphi_x^{(n)})$ otherwise. Then by Theorem IV.7.1, for each $n\in\mathbb{N}\cup\{\infty\}$, the law of $(\ell_{x,u}^{(n)},|\varphi_x^{(n)}|,\sigma_x^{u,n})_{x\in\widetilde{\mathcal{G}}_n}$ under $\mathbb{P}\otimes\mathbb{P}'$ is the same as the law of $(\ell_{x,u},|\varphi_x|,\sigma_x^u)_{x\in\widetilde{\mathcal{G}}_n}$ under $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}_n}$, for a certain choice of σ^u, and thus by Lemma IV.7.2 we have for all $n\in\mathbb{N}$

$$\Big(\sigma_x^{u,n}\sqrt{2\ell_{x,u}^{(n)}+(\varphi_x^{(n)})^2)}\Big)_{x\in\widetilde{\mathcal{G}}_{\kappa(n)}}\text{ has the same law as }\big(\varphi_x^{(n)}+\sqrt{2u}\big)_{x\in\widetilde{\mathcal{G}}_{\kappa(n)}}.\tag{IV.8.6}$$

Let $x\in\widetilde{\mathcal{G}}$. If $\mathcal{I}_\infty^u\cap E_\infty^{|\geq 0|}(x)\neq\varnothing$, then by (IV.8.4), we have $\sigma_x^{u,n}=1$ for all n large enough, and thus $\sigma_x^{u,n}\underset{n\to\infty}{\longrightarrow}\sigma_x^{u,\infty}$. If $\varphi_x^{(\infty)}=0$, then $\varphi_x^{(n)}=0$ for n large enough, and thus $\sigma_x^{u,n}\underset{n\to\infty}{\longrightarrow}\sigma_x^{u,\infty}$. If $\varphi_x^{(\infty)}\neq 0$ and $\mathcal{I}_\infty^u\cap E_\infty^{|\geq 0|}(x)=\varnothing$, then by (IV.8.4), $\mathcal{I}_n^u\cap E_n^{|\geq 0|}(x)=\varnothing$ for n large enough, and thus $\sigma_x^{u,n}=\mathbb{1}_{\varphi_x^{(n)}\geq 0}\underset{n\to\infty}{\longrightarrow}\mathbb{1}_{\varphi_x^{(\infty)}\geq 0}=\sigma_x^{u,\infty}$. Therefore, $\mathbb{P}\otimes\mathbb{P}'$ a.s,

$$\sigma_x^{u,n}\sqrt{2\ell_{x,u}^{(n)}+(\varphi_x^{(n)})^2}\underset{n\to\infty}{\longrightarrow}\sigma_x^{u,\infty}\sqrt{2\ell_{x,u}^{(\infty)}+(\varphi_x^{(\infty)})^2}\text{ for all }x\in\widetilde{\mathcal{G}},$$

and $\varphi_x^{(n)}+\sqrt{2u}\underset{n\to\infty}{\longrightarrow}\varphi_x^{(\infty)}+\sqrt{2u}$. Using (IV.8.6), we obtain that (Isom') holds for $\mathcal{G}$.

Let us now define for all $n \in \mathbb{N} \cup \{\infty\}$ the random set of edges and vertices $\mathcal{E}_u^{(n)} = \{e \in E \cup G : 2\ell_{x,u}^{(n)} + (\varphi_x^{(n)})^2 > 0 \text{ for all } x \in I_e\}$. By (IV.7.4), we have for all $n \in \mathbb{N}$ that

$$(\mathbb{P} \otimes \mathbb{P}')\big(e \in \mathcal{E}_u^{(n)} \,|\, \omega_u^{\mathcal{G},(n)}, \varphi^{(n)}\big) = \mathbb{1}_{e \in \mathcal{I}_{E,n}^u} \vee p_e^{\mathcal{G}_n}(\varphi^{(n)}, \ell_{\cdot,u}^{(n)}),$$

where $\mathcal{I}_{E,n}^u$ is the set of edges crossed by the trace $\omega_u^{\mathcal{G},(n)}$ of $\omega_u^{(n)}$ on $\mathcal{G}$, and of vertices on which a trajectory of $\omega_u^{\mathcal{G},(n)}$ is killed. Moreover, using (IV.2.32) and (IV.2.39), we have that for each $n \in \mathbb{N} \cup \{\infty\}$, conditionally on $(\varphi_x^{(n)})_{x \in G}$ and $\omega_u^{\mathcal{G},(n)}$, the family $\{e \in \mathcal{E}_u^{(n)}\}$, $e \in E \cup G$, is independent, and for all $e \in E \cup G$,

$$(\mathbb{P} \otimes \mathbb{P}')\big(e \in \mathcal{E}_u^{(n)} \,|\, \omega_u^{\mathcal{G},(n)}, \varphi^{(n)}\big) = (\mathbb{P} \otimes \mathbb{P}')\big(e \in \mathcal{E}_u^{(n)} \,|\, \omega_{e,u}^{\mathcal{G},(n)}, (\varphi^{(n)})_{|e}\big).$$

Note that $(\mathbb{P} \otimes \mathbb{P}')$-a.s. for each $e \in E \cup G$, $(\varphi^{(n)})_{|e} = (\varphi^{(\infty)})_{|e}$, $\omega_{e,u}^{\mathcal{G},(n)} = \omega_{e,u}^{\mathcal{G},(\infty)}$, $\mathcal{I}_{E,n}^u = \mathcal{I}_{E,\infty}^u$, and by (IV.3.13) and (IV.3.14), $p_e^{\mathcal{G}_n}(\varphi^{(n)}, \ell_{\cdot,u}^{(n)}) = p_e^{u,\mathcal{G}}(\varphi^{(\infty)}, \ell_{\cdot,u}^{(\infty)})$ for all n large enough, and so

$$(\mathbb{P} \otimes \mathbb{P}')\big(e \in \mathcal{E}_u^{(\infty)} \,|\, \omega_u^{\mathcal{G},(\infty)}, \varphi^{(\infty)}\big) = \mathbb{1}_{e \in \mathcal{I}_{E,\infty}^u} \vee p_e^{u,\mathcal{G}}(\varphi^{(\infty)}, \ell_{\cdot,u}^{(\infty)}),$$

which is equivalent to (IV.7.4) for the graph $\mathcal{G}$. □

Let us now quickly explain how one can deduce Theorem IV.3.4 and Corollaries IV.3.6 and IV.3.7 from Lemma IV.8.3, and prove Proposition IV.3.5.

Proof of Theorem IV.3.4. We start with the proof of (IV.3.12). If (Isom') holds, then $(\text{Law}_h)_{h>0}$ also holds by Proposition IV.4.7. If $(\text{Law}_h)_{h>0}$ holds, then (Law_0) also holds by taking the limit as $h \searrow 0$ in (Law_h) and using (IV.2.24). If (Law_0) holds, then (Isom') also holds by Lemma IV.8.3. Therefore, we only need to prove that the two versions (Isom) and (Isom') of the isomorphism between random interlacements and the Gaussian free field are equivalent. Since for all $x \notin \mathcal{C}_u^\infty$, $\varphi_x = \text{sign}(\varphi_x)\sqrt{\varphi_x^2 + 2\ell_{x,u}}$, and by Theorem IV.7.1, the law of $(\text{sign}(\varphi_x)\mathbb{1}_{x \notin \mathcal{C}_u^\infty} + \mathbb{1}_{x \in \mathcal{C}_u^\infty})_{x \in \widetilde{\mathcal{G}}}$ under $(\mathbb{P}_{\widetilde{\mathcal{G}}}^I \otimes \mathbb{P}_{\widetilde{\mathcal{G}}}^G)(\cdot \,|\, |\varphi|, \omega^u)$ is the same as the law of σ^u under $\widetilde{\mathbb{P}}_{\widetilde{\mathcal{G}}}^I(\cdot \,|\, |\varphi|, \omega^u)$, this is clear.

Let us now assume that one of the conditions in (IV.3.12) hold. Then by Lemma IV.8.3, we have that (Isom') and (IV.7.4) hold. Moreover, the family $\{e \in \mathcal{E}_u\}$, $e \in E \cup G$, is independent by (IV.2.32) and (IV.2.39), and, by (IV.7.4) it is thus clear that $(\mathcal{E}_u, (\sigma_x^u)_{x \in G}, (\varphi_x)_{x \in G}, \omega_u^{\mathcal{G}})$ has the same law under $\widetilde{\mathbb{P}}_{\widetilde{\mathcal{G}}}^I$ as $(\widehat{\mathcal{E}}_u, \widehat{\sigma}^u, \varphi, \omega_u^{\mathcal{G}})$ under $\widetilde{\mathbb{P}}_{\mathcal{G}}^I$. The equality (IV.3.15) then follows directly from (Isom') since by (IV.2.38) and (IV.4.4), $\mathbb{P}^I(\mathcal{I}^u \cap I_x \neq \varnothing) = 1$ for all $x \in G$ with $\kappa_x > 0$. □

Proof of Corollary IV.3.6. Let $\mathcal{G}$ be a graph such that (Law$_0$) is fulfilled, then (Isom') holds by Lemma IV.8.3. Let us assume that $E^{\geq 0}$ contains at least one non-compact component with positive probability, then there exists $x_0 \in \widetilde{\mathcal{G}}$ such that $E^{\geq 0}(x_0)$ is non-compact with positive probability. Moreover by Theorem IV.3.1, we have $\mathrm{cap}(E^{\geq 0}(x_0)) < \infty$ $\mathbb{P}^G$-a.s, and so by Proposition IV.4.4, $E^{\geq 0}(x_0)$ is also unbounded with positive probability. By (IV.2.38) we have that, for all $u > 0$, with $\widetilde{\mathbb{P}}^I$ positive probability, $E^{\geq 0}(x_0)$ is unbounded and $E^{\geq 0}(x_0) \cap \mathcal{I}^u = \varnothing$, and therefore the component of x_0 in $\{x \in \widetilde{\mathcal{G}} : \sigma_x^u = -1\}$ is unbounded with positive probability. By (Isom') and symmetry of the Gaussian free field, we obtain that for all $u > 0$ $E^{\geq \sqrt{2u}}(x_0)$ is unbounded wit positive probability. Therefore if $h_*^{\mathrm{com}} > 0$, $E^{\geq 0}$ contains a non-compact component with positive probability, and so $E^{\geq h}$ contains an unbounded component for all $h > 0$, that is $\widetilde{h}_* = \infty$.

If moreover $\mathbf{h}_0 < 1$, then with positive probability $\mathcal{I}^u$ contains a trajectory which is not killed before blowing up, and thus unbounded, and by (Isom') we then have that $E^{\geq -\sqrt{2u}}$ is also unbounded with positive probability, that is $\widetilde{h}_* \geq 0$. Therefore by (IV.3.3), we have $h_*^{\mathrm{com}} \geq \widetilde{h}_* \geq 0$. Since $\widetilde{h}_* = \infty$ if $h_*^{\mathrm{com}} > 0$, we thus obtain $\widetilde{h}_* = h_*^{\mathrm{com}} \in \{0, \infty\}$. □

Proof of Corollary IV.3.7. Let us assume that $\widetilde{h}_* \leq 0$, then $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded for all $h > 0$. By Theorem IV.3.3, we thus have that (Law$_h$) holds for all $h > 0$, and so (Law$_0$) also holds by (IV.3.12). Since $E^{\geq h}$ is $\mathbb{P}^G$-a.s. bounded for all $h > 0$, we thus obtain by Corollary IV.3.6 that $E^{\geq 0}$ is $\mathbb{P}^G$-a.s. bounded. □

Remark IV.8.4. 1) From Proposition IV.4.7 and Lemma IV.8.3, one could immediately prove again Theorem IV.3.3.

2) In view of Remark IV.7.5,1), if (Law$_0$) and (IV.7.7) hold for some sequence $\mathcal{G}_n$ of graphs increasing to $\mathcal{G}$, then (Law$_0$) holds on $\mathcal{G}$, and thus also (Isom') and (Law$_h$). Similarly, by Remark IV.6.4, if $\mathcal{M}_\infty^{(0)} = 0$ $\mathbb{P}^G$-a.s, then (Law$_0$), (Isom') and (Law$_h$) hold on $\mathcal{G}$.

3) As explained in Corollary IV.3.6, if (Isom') and $\mathbf{h}_0 < 1$ are fulfilled, then $\widetilde{h}_* \in \{0, \infty\}$. If $\mathbf{h}_0 = 1$ it is however possible that (Isom') is verified but $\widetilde{h}_* = -\infty$, for instance on finite graphs since finite graphs are bounded but trivially verify (Cap). It is thus an interesting open question to know whether the equality $\widetilde{h}_* = h_*^{\mathrm{com}} = 0$ can still hold on some transient graphs satisfying (Isom'), or equivalently (Law$_0$), and $\mathbf{h}_0 = 1$.

4) Another interesting open question is whether a transient graph $\mathcal{G}$ exists such that (Law$_0$) does not hold. In view of Corollary IV.3.6, one could also ask if

a transient graph $\mathcal{G}$ exists, such that $\mathbf{h}_0 < 1$ is fulfilled, but $\widetilde{h}_* \in (0,\infty)$ or $h_*^{\mathrm{com}} \in (0,\infty)$, and then (Law$_0$) would not hold. On such a graph, we would still have by Theorem IV.3.3 that (Law$_h$) holds for all $h > h_*^{\mathrm{com}}$.

IV.9 An example of a graph with infinite critical parameter

In this section, we are going to provide an example of a graph for which the critical parameters $\widetilde{h}_*$ and h_*^{com} are strictly positive, and in fact infinite, thus providing a counterexample to Corollary IV.3.2 if we do not assume (Cap) to hold. For any $\alpha \in (0,1)$ and $d \in \mathbb{N}$, $d \geq 2$, we define $\mathbb{T}_d^\alpha$ the $(d+1)$-regular tree, such that, denoting by T_n the set of vertices in $\mathbb{T}_d^\alpha$ at generation n,

$$\lambda_{x,y}^{(\alpha)} = \alpha^n \text{ if } x \in T_n \text{ and } y \in T_{n+1},$$

and 0 otherwise. We moreover take $\kappa^{(\alpha)} = 0$ if $\alpha > \frac{1}{d}$ and $\kappa^{(\alpha)} = \mathbb{1}_0$ otherwise, where 0 is the root of the tree. Since for $x \in T_n$ and $\alpha > \frac{1}{d}$,

$$P_x^{\mathbb{T}_d^\alpha}(\widehat{Z}_1 \in T_{n+1}) = d\frac{\alpha^n}{\alpha^{n-1} + d\alpha^n} = \frac{d\alpha}{1 + d\alpha} > \frac{1}{2},$$

we have that $\mathbb{T}_d^\alpha$ is a transient graph for all $\alpha \in (0,1)$ and $d \in \mathbb{N}$, $d \geq 2$.

Proposition IV.9.1. *There exists a constant $C_0 < \infty$, such that for any $\alpha \in (0,1)$ and $d \in \mathbb{N}$, $d \geq 2$, with $d\big(1 - \exp\big(-\frac{\sqrt{\alpha}}{d\alpha+1}\big)\big) > C_0$, $E_{\mathbb{T}_d^\alpha}^{\geq h}$ contains $\mathbb{P}_{\widetilde{\mathbb{T}}_d^\alpha}^G$-a.s. an unbounded connected component for all $h \in \mathbb{R}$, and so $\widetilde{h}_* = h_*^{\mathrm{com}} = \infty$ and $h_*^{\mathrm{cap}} \leq 0$.*

Proof. Using the Markov property (IV.2.31), one can construct a Gaussian free field on $(\widetilde{\mathbb{T}}_d^\alpha)^E$ recursively on the generation T_n. Indeed let Y_x, $x \in \mathbb{T}_d^\alpha$, be a family of i.i.d. random variables with distribution $\mathcal{N}(0,1)$, and let $\psi_0 = Y_0\sqrt{g_{\mathbb{T}_d^\alpha}(0,0)}$. Recursively on n, we then define

$$\psi_x \stackrel{\text{def.}}{=} \psi_{x^-}\mathbb{P}_{\mathbb{T}_d^\alpha}^x(H_{\{x^-\}} < \infty) + Y_x\sqrt{g_{T_n^c}(x,x)} \text{ for all } x \in T_{n+1},$$

where x^- is the first ancestor of x. One can then easily prove by (IV.2.31) that $(\psi_x)_{x\in\mathbb{T}_d^\alpha}$ has the same law as $(\varphi_x)_{x\in\mathbb{T}_d^\alpha}$ under $\mathbb{P}_{\mathbb{T}_d^\alpha}^G$. Moreover, let B^e, $e \in E \cup G$, be a family of independent process, such that for each edge $e = \{x,y\} \in E$ between $x \in T_n$ and $y \in T_{n+1}$, B^e is a Brownian bridge of length $\frac{1}{2\alpha^n}$ between 0 and 0 of a Brownian motion with variance 2 at time 1, and let

$$\psi_{x+t\cdot I_e} = 2\alpha^n t\psi_y + (1 - 2\alpha^n t)\psi_x + B_t \text{ for all } t \in \big[0, \frac{1}{2\alpha^n}\big].$$

Then $(\psi_x)_{x\in(\widetilde{\mathbb{T}}_d^\alpha)^E}$ has the same law as $(\varphi_x)_{x\in(\widetilde{\mathbb{T}}_d^\alpha)^E}$ under $\mathbb{P}^G_{\widetilde{\mathbb{T}}_d^\alpha}$, see Section II.2 for a proof of a similar fact on $\mathbb{Z}^d$, $d\geq 3$. One could also easily extend this construction to $\widetilde{\mathbb{T}}_d^\alpha$ by similarly adding on I_0 an independent Brownian bridge of length $1/2$ between φ_0 and 0 with variance 2 at time 1 if $\alpha\leq\frac{1}{d}$, as well as an independent Brownian motion starting in φ_x and with variance 2 at time 1 for every other vertices x.

Now for each $x\in T_{n+1}$, we have that,

$$\begin{aligned}\mathbb{P}\big(\psi_x\geq(\lambda_x^{(\alpha)})^{-1/2}\,\big|\,\psi_{x^-}\big)\mathbb{1}_{\psi_{x^-}\geq(\lambda_{x^-}^{(\alpha)})^{-1/2}}&\geq\mathbb{P}\big(Y_x\geq(\lambda_x^{(\alpha)}g_{T_n^c}(x,x))^{-1/2}\big)\\&\geq\mathbb{P}(Y_0\geq 1),\end{aligned}$$

since x spends at least a time $\mathcal{E}(\lambda_x^{(\alpha)})$ in x before hitting T_n under $P_x^{\mathbb{T}_d^\alpha}$, and so $g_{T_n^c(x,x)}\lambda_x^{(\alpha)}\geq 1$. Moreover using (IV.2.33), we have for all $x\in T_{n+1}$, writing $e=\{x,x^-\}$, on the event $\psi_x\geq(\lambda_x^{(\alpha)})^{-1/2}$, $\psi_{x^-}\geq(\lambda_{x^-}^{(\alpha)})^{-1/2}$,

$$\begin{aligned}&\mathbb{P}(\psi_y\geq(\lambda_x^{(\alpha)})^{-\frac14}\forall\,y\in I_e\,|\,\psi_x,\psi_{x^-})\\&=1-\exp\big(-2\alpha^n(\psi_x-(\lambda_x^{(\alpha)})^{-\frac14})(\psi_{x^-}-(\lambda_x^{(\alpha)})^{-\frac14})\big)\\&\geq 1-\exp\big(-\alpha^n(\lambda_x^{(\alpha)})^{-\frac12}(\lambda_{x^-}^{(\alpha)})^{-\frac12}\big)\\&\geq 1-\exp\Big(-\frac{\sqrt{\alpha}}{d\alpha+1}\Big),\end{aligned}$$

for all n large enough. Therefore for all n large enough and each $y\in T_{n+1}$, the cluster of y in $\{x\in(\widetilde{\mathbb{T}}_d^\alpha)^E:\psi_x\geq(\lambda_x^{(\alpha)})^{-\frac14}\}$ contains with positive probability an independent Galton-Watson tree, with average number of children

$$d\Big(1-\exp\Big(-\frac{\sqrt{\alpha}}{d\alpha+1}\Big)\Big)\mathbb{P}(Y_0\geq 1).$$

Taking $C_0>\mathbb{P}(Y_0\geq 1)^{-1}$, letting n go to infinity, we thus have that $\{x\in\widetilde{\mathbb{T}}_d^\alpha:\psi_x\geq(\lambda_x^{(\alpha)})^{-\frac14}\}$ contains $\mathbb{P}$-a.s. an unbounded component if $d\big(1-\exp\big(-\frac{\sqrt{\alpha}}{d\alpha+1}\big)\big)>C_0$, and since $\lambda_x^{(\alpha)}\to 0$, $|x|\to\infty$, we obtain $\widetilde{h}_*=\infty$. By (IV.3.4), we thus have $h_*^{\mathrm{com}}=\infty$, and the inequality $h_*^{\mathrm{cap}}\leq 0$ follows directly from Theorem IV.3.1. □

Remark IV.9.2. 1) In both cases $\alpha>\frac1d$ and $\alpha\leq\frac1d$, It is possible to find $\alpha\in(0,1)$ and $d\in\mathbb{N}$, $d\geq 2$ such that $d\big(1-\exp\big(-\frac{\sqrt{\alpha}}{d\alpha+1}\big)\big)>C_0$. For instance, one can take $\alpha=\frac{a}{d}$ for some constant $a>0$, and choose $d_{\widetilde{\mathcal{G}}}$ large enough.

2) One can easily derive from Proposition IV.9.1 an example of a graph $\mathcal{G}$ with $\lambda_{x,y}=1$ for all $x,y\in G$ and $\kappa_x=0$ for all $x\in G$, which is the usual setting of graphs without weights, on which $\widetilde{h}_*=h_*^{\mathrm{com}}=\infty$. Fix some $d\in 2\mathbb{N}$, $d\geq 2$ such that $d\big(1-\exp\big(-\frac{\sqrt{2/d}}{3}\big)\big)>C_0$, and then by Proposition IV.9.1

we have that $E^{\geq h}$ contains $\mathbb{P}^G_{\widetilde{\mathbb{T}}_d^{2/d}}$-a.s. an unbounded connected component for all $h\in\mathbb{R}$. Now let U be the set which contains $x^-+(k/2)\cdot I_{\{x^-,x\}}$ for all $k\in\{1,\dots,(d/2)^{n-1}-1\}$, $x\in T_n$ and $n\in\mathbb{N}$, and let $\mathcal{G}=(\mathbb{T}_d^{2/d})^U$, the graph which corresponds to the tree $\mathbb{T}_d^{2/d}$ plus $2^{n-1}-1$ equidistant vertices on $I_{\{x^-,x\}}$ for each $x\in T_n$ and $n\in\mathbb{N}$. Then one can identify $\widetilde{\mathcal{G}}^E$ with $(\widetilde{\mathbb{T}}_d^{2/d})^E$, and so $E^{\geq h}$ contains also $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$-a.s. an unbounded connected component for all $h\in\mathbb{R}$, and $\lambda\equiv1$ and $\kappa\equiv0$ on $\mathcal{G}$.

3) If $\alpha\leq\frac{1}{d}$ and $d\big(1-\exp\big(-\frac{\sqrt{\alpha}}{d\alpha+1}\big)\big)>C_0$, we have $P_x^{\mathbb{T}_d^\alpha}(H_{T_{n-1}}<\zeta)=1$ for all $x\in T_n$, since the generation of Z has the same law as a random walk on $\mathbb{N}$, with a negative drift and a killing at 0. Therefore for each compact K of $\widetilde{\mathbb{T}}_d^\alpha$, if $x\in K$ is such that $x^-\in K$, we have $e_K(x)=0$, and so $\{x\in\mathcal{K}_t^{(0)}: e_{\mathcal{K}_t^{(0)}}(x)\neq0\}$ is constant and $\mathcal{M}_t^{(0)}=0$ for all t large enough, where $\mathcal{M}^{(0)}$ is the martingale from (IV.6.1). By Remark IV.6.4, $\mathbb{T}_d^\alpha$ is thus an example of a graph on which (Law$_0$) holds but $\widetilde{h}_*=h_*^{\text{com}}=\infty$, and so both cases in Corollary IV.3.6 are possible. Note that one could also construct such a graph on which $\kappa\equiv0$, and thus $\mathbf{h}_0<1$ is verified, by replacing the killing at 0 on $\mathbb{T}_d^\alpha$ by a copy of $\mathbb{Z}^3$ attached to 0.

4) One can also easily construct an example of a graph $\mathcal{G}$ not fulfilling condition (Cap), but for which we still have $\widetilde{h}_*=h_*^{\text{cap}}=h_*^{\text{com}}=0$. Consider $\mathcal{G}$ to be the graph $\mathbb{Z}^3$ plus a copy of $\mathbb{N}$ attached to the origin, with unit weights and zero killing measure. The cable system $\widetilde{\mathcal{G}}$ can then be identified with $\widetilde{\mathbb{Z}}^3$, the cable system of $\mathbb{Z}^3$, by identifying $I_0\subset\widetilde{\mathbb{Z}}^3$ with the cables corresponding to the edge of $\mathbb{N}$ in $\widetilde{\mathcal{G}}$. Since condition (Cap) is clearly verified on $\mathbb{Z}^3$, using Lemma IV.4.1 for instance, we obtain $\widetilde{h}_*=h_*^{\text{cap}}=h_*^{\text{com}}=0$ by Corollary IV.3.2. Now for each $n\in\mathbb{N}$, the equilibrium measure of $\{0,\dots,n\}$ is only supported on 2 points, and so the capacity of $\mathbb{N}$ is at most 2 by (IV.2.27), that is $\mathcal{G}$ does not fulfills (Cap).

IV.A Appendix: Proof of Lemma IV.7.2

In this Appendix we are going to prove that the coupling between loop soups and the Gaussian free field, Theorem IV.7.1, implies the coupling between random interlacements and the Gaussian free field on finite graphs, Lemma IV.7.2, following similar ideas to the proof of Theorem 2 in [58]. From now on, we will always assume that $\mathcal{G}$ is a transient graph such that G is finite and $\kappa_x\in[0,\infty)$ for all $x\in V$, which we can assume without loss of generality by considering the graph $\mathcal{G}^{(\infty)}$, see below (IV.2.4). Let us define

$$U_\kappa \stackrel{\text{def.}}{=} \{x \in G : \kappa_x > 0\},$$

and let $\mathcal{G}^*$ be the graph with vertex set G, plus an additional vertex x_*. The symmetric weights on $\mathcal{G}^*$ are

$$\lambda^*_{x,y} = \begin{cases} \lambda_{x,y} & \text{when } x, y \in G \\ \kappa_x & \text{when } x \in U_\kappa \text{ and } y = x_* \\ 0 & \text{when } x \notin U_\kappa \text{ and } y = x_*, \end{cases}$$

and the killing measure $\kappa^* = \mathbb{1}_{x_*}$. We write $G^* = G \cup \{x_*\}$ and $E^* = \{\{x, y\} \in G^* : \lambda^*_{x,y} > 0\}$ for the vertex and edge set of $\mathcal{G}^*$. Note that each edge I_e of $\widetilde{\mathcal{G}}^*$, $e \in E^*$, can be identified with some edge I_e of $\widetilde{\mathcal{G}}$, $e \in E \cup U_\kappa$, and one can then identify the cable system $\widetilde{\mathcal{G}}^* \setminus \{I_{x_*} \cup \bigcup_{x \in U_\kappa} I_x\}$ with $\widetilde{\mathcal{G}}$ and then by (IV.2.9), one can show that for all $x \in \widetilde{\mathcal{G}}$ the law of the print of X on $\widetilde{\mathcal{G}}$ killed on hitting x_* under $P_x^{\widetilde{\mathcal{G}}^*}$ is $P_x^{\widetilde{\mathcal{G}}}$. Recall the decomposition of the loop soup $\mathcal{L}_{\frac{1}{2}} = \mathcal{L}_{\frac{1}{2}}^{\{x_*\}^c} + \overline{\mathcal{L}}_{\frac{1}{2}}^{\{x_*\}}$ on $\mathcal{G}^*$ defined above Theorem IV.7.2, and define the local times $(\overline{L}_x^{x_*})_{x \in \widetilde{\mathcal{G}}^*}$ of $\overline{\mathcal{L}}_{\frac{1}{2}}^{\{x_*\}}$ under $\mathbb{P}^L_{\widetilde{\mathcal{G}}^*}$, and $\overline{\mathcal{L}}_{\mathcal{G}^*, \frac{1}{2}}^{\{x_*\}}$ as the print of $\overline{\mathcal{L}}_{\frac{1}{2}}^{\{x_*\}}$ on $\mathcal{G}^*$. Each loop in $\overline{\mathcal{L}}_{\mathcal{G}^*, \frac{1}{2}}^{\{x_*\}}$ can be decomposed into its excursions outside x_*, that is a trajectory entirely contained in G, starting and ending in U_κ, and the process $\overline{\mathcal{L}}_{\mathcal{G}^*, \frac{1}{2}}^{e,\{x_*\}}$ of excursions is then defined as the point process consisting of all the excursions outside x_* for all the loops in $\overline{\mathcal{L}}_{\mathcal{G}^*, \frac{1}{2}}^{\{x_*\}}$. We can now compare the Gaussian free field on $\widetilde{\mathcal{G}}^*$ with the Gaussian free field on $\widetilde{\mathcal{G}}$, and the loops $\overline{\mathcal{L}}_{\frac{1}{2}}^{\{x_*\}}$ hitting x_* on $\widetilde{\mathcal{G}}^*$ with random interlacements on $\widetilde{\mathcal{G}}$.

Proposition IV.A.1. *Let $\mathcal{G}$ be a transient graph such that G is finite. For any $u > 0$,*

$$(\varphi_x)_{x \in \widetilde{\mathcal{G}}} \text{ has the same law under } \mathbb{P}^G_{\widetilde{\mathcal{G}}^*}(\cdot \,|\, \varphi_{x_*} = \sqrt{2u}) \text{ as } (\varphi_x + \sqrt{2u})_{x \in \widetilde{\mathcal{G}}} \text{ under } \mathbb{P}^G_{\widetilde{\mathcal{G}}}, \tag{IV.A.1}$$

and

$$\overline{\mathcal{L}}_{\mathcal{G}^*, \frac{1}{2}}^{e,\{x_*\}} \text{ has the same law under } \mathbb{P}^L_{\mathcal{G}^*}(\cdot \,|\, L_{x_*} = u) \text{ as } \omega_u^{\mathcal{G}} \text{ under } \mathbb{P}^I_{\mathcal{G}}. \tag{IV.A.2}$$

In particular,

$$(\overline{L}_x^{x_*})_{x \in \widetilde{\mathcal{G}}} \text{ has the same law under } \mathbb{P}^L_{\widetilde{\mathcal{G}}^*}(\cdot \,|\, L_{x_*} = u) \text{ as } (\ell_{x,u})_{x \in \widetilde{\mathcal{G}}} \text{ under } \mathbb{P}^I_{\widetilde{\mathcal{G}}}. \tag{IV.A.3}$$

Proof. We begin with (IV.A.1). By the Markov property applied to the graph $\mathcal{G}^*$, see (IV.2.31), conditionally on $\mathcal{A}^+_{\{x_*\}}$, $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ is a Gaussian field with mean $\eta^{\varphi}_{\{x_*\}}=\varphi_{x_*}$ and variance $g_{\{x_*\}^c}=g_{\widetilde{\mathcal{G}}}$, and thus $(\varphi_x-\varphi_{x_*})_{x\in\widetilde{\mathcal{G}}}$ has the same law under $\mathbb{P}^G_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,\mathcal{A}^+_{\{x_*\}})$ as φ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$, and (IV.A.1) follows.

Let us now prove (IV.A.2). Following Section 7 of [54], see also Proposition 3.7 in [58], conditionally on $L_{x_*}=\overline{L}^{x_*}_{x_*}=u$, the excursions outside x_* in $\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G}^*,\frac12}$ have the same law as the excursions of the Markov jump process Z outside x_* stopped when reaching local time u at x_* under $P^{\mathcal{G}^*}_{x_*}$, which can be described as follows: first stay an exponential time with parameter λ_{x_*} in x_*, then jump to an $x\in U_\kappa$ with probability $\frac{\kappa_x}{\lambda_{x_*}}$ and follow on $\mathcal{G}$ a process with the same law as Z under $P^{\mathcal{G}}_x$. Once this process is killed, jump back in x_* and iterate this process until reaching local time u in x_*. By a property of exponential variables, the number of time this process is iterated is a Poisson variable with parameter $u\lambda_{x_*}$, and thus $\overline{\mathcal{L}}^{e,\{x_*\}}_{\mathcal{G}^*,\frac12}$ is a Poisson point process with intensity

$$u\sum_{x\in U_\kappa}\kappa_x P^{\mathcal{G}}_x.$$

Note that, under $P^{\mathcal{G}}_x$, we have $\widetilde{H}_G=\infty$ if and only if $x\in U_\kappa$ and the discrete skeleton $\widehat{Z}$ of Z is killed at time 1, and thus $e_G(x)=\kappa_x$ for all $x\in U_\kappa$ and $e_G(x)=0$ otherwise. Therefore by (IV.2.36) and (IV.2.37) with $K=G$, conditionally on $L_{x_*}=u$, $\overline{\mathcal{L}}^{e,\{x_*\}}_{\mathcal{G}^*,\frac12}$ is a Poisson point process with intensity $u\nu_{\mathcal{G}}$, where $\nu_{\mathcal{G}}$ is the intensity measure of discrete random interlacements on $\mathcal{G}$, and we obtain (IV.A.2). This implies in particular that $(\overline{L}^{x_*}_x)_{x\in G}$ has the same law under $\mathbb{P}^L_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,L_{x_*}=u)$ as $(\ell_{x,u})_{x\in G}$ under $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$, and thus (IV.A.3) follows by considering the graph $\mathcal{G}^A$ for any finite subset A of $\widetilde{\mathcal{G}}$, see Lemma IV.2.1. □

Using Theorem IV.7.1 for the graph $\mathcal{G}^*$, and decomposing $\mathcal{L}_{\frac12}$ on $\widetilde{\mathcal{G}}^*$ into $\mathcal{L}^{\{x_*\}^c}_{\frac12}$ and $\overline{\mathcal{L}}^{\{x_*\}}_{\frac12}$, we are now ready to prove Lemma IV.7.2.

Proof of Lemma IV.7.2. Let us define $(L^{x_*}_x)_{x\in\widetilde{\mathcal{G}}^*}$ the total local times of the loops in $\mathcal{L}^{\{x_*\}^c}_{\frac12}$ under $\mathbb{P}^L_{\widetilde{\mathcal{G}}^*}$. By (IV.7.1), $(L^{x_*}_x)_{x\in\widetilde{\mathcal{G}}}$ has the same law as the restriction to $\widetilde{G}$ of the local time of a loop soup on $\widetilde{\mathcal{G}}^{\{x_*\}^c}_\infty$, and thus the same law as the local time of a loop soup on $\widetilde{\mathcal{G}}$. By (IV.7.2), $(L^{x_*}_x)_{x\in\widetilde{\mathcal{G}}}$ has thus the same law under $\mathbb{P}^L_{\widetilde{\mathcal{G}}^*}$, or also $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,\sigma_{x_*}=1,\overline{L}^{x_*}_{x_*}=u)$, as $\frac12\varphi^2$ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$. Moreover, under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,\sigma_{x_*}=1,\overline{L}^{x_*}_{x_*}=u)$, using the equality $L_x=L^{x_*}_x+\overline{L}^{x_*}_x$ for all $x\in\widetilde{\mathcal{G}}$, the law of $(\sigma_x)_{x\in\widetilde{\mathcal{G}}}$ can be described as follows: conditionally on $(L^{x_*}_x)_{x\in\widetilde{\mathcal{G}}}$ and $(\overline{L}^{x_*}_x)_{x\in\widetilde{\mathcal{G}}}$, σ is constant on each cluster of $\{x\in\widetilde{\mathcal{G}}:\overline{L}^{x_*}_x+L^{x_*}_x>0\}$, with $\sigma_x=1$ for all $x\in\widetilde{\mathcal{G}}$ such that $\overline{L}^{x_*}_x>0$, and the values of σ on each other cluster are independent and uniformly distributed. Using (IV.A.3) we thus have that, under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,\sigma_{x_*}=1,\overline{L}^{x_*}_{x_*}=u)$,

$$(\sigma_x\sqrt{2L_x})_{x\in\widetilde{\mathcal{G}}}\text{ has the same law as }(\sigma^u_x\sqrt{2\ell_{x,u}+\varphi_x^2})_{x\in\widetilde{\mathcal{G}}}\text{ under }\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}.$$

According to (IV.7.2), the law of $(\sigma_x\sqrt{2L_x})_{x\in\widetilde{\mathcal{G}}}$ under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}}(\cdot\,|\,\sigma_{x_*}=1,L_{x_*}=u)$ is the same as the law of $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,\varphi_{x_*}=\sqrt{2u})$, and thus by (IV.A.1) the same as the law of $(\varphi_x+\sqrt{2u})_{x\in\widetilde{\mathcal{G}}}$ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$, and we obtain (Isom').

By (IV.2.32) and (IV.2.39), it is clear that, conditionally on $\omega^{\mathcal{G}}_u$ and $(\varphi_x)_{x\in G}$, the family $\{e\in\mathcal{E}_u\}$, $e\in E\cup G$, is independent. We define $\mathcal{E}^{x_*}=\{e\in E^*: L^{x_*}_x>0\text{ for all }x\in I_e\}$, and conditionally on $\mathcal{L}^{\{x_*\}^c}_{\frac12}$, let σ^{x_*} be an independent additional process, such that σ^{x_*} is constant on each cluster of $\{x\in\widetilde{\mathcal{G}}^*:L^{x_*}>0\}$ and its values on each cluster are independent and uniformly distributed. Under $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,L_{x_*}=u)$, by (IV.7.1) and (IV.7.2), $(\sigma^{x_*}_x\sqrt{2L^{x_*}_x})_{x\in\widetilde{\mathcal{G}}}$ has then the same law as φ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$, by (IV.A.2) $\overline{\mathcal{L}}^{e,\{x_*\}}_{\mathcal{G}^*,\frac12}$ has the same law as $\omega^{\mathcal{G}}_u$ under $\mathbb{P}^I_{\mathcal{G}}$, and by (IV.A.3), $\mathcal{E}$ has the same law as $\mathcal{E}_u\setminus\{I_x,x\in U^c_\kappa\}$ under $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}$. Let us also write $\mathcal{I}^{\mathcal{L}}_E\subset E\cup G$ for the set of edges of $\mathcal{G}$ which are entirely crossed by a trajectory in $\mathcal{L}^{e,\{x_*\}}_{\mathcal{G}^*,\frac12}$ and of vertices in G killed by a trajectory in $\mathcal{L}^{e,\{x_*\}}_{\mathcal{G}^*,\frac12}$, which has the same law under $\widetilde{\mathbb{P}}_{\widetilde{\mathcal{G}}^*}(\cdot\,|\,L_{x_*}=u)$ as $\mathcal{I}^u_E$ under $\mathbb{P}^I_{\mathcal{G}}$. For each $e\in E^*$, the event $\{e\notin\mathcal{E}^{x_*}\}$ is independent of $\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12}$, and, conditionally on $\{e\notin\mathcal{E}^{x_*}\}$, $\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12}$ and $L^{x_*}_{|G}=(L^{x_*}_x)_{x\in G}$, the event $\{e\notin\mathcal{E}\}$ is independent of $\sigma^{x_*}_{|G}=(\sigma^{x_*}_x)_{x\in G}$. Therefore, since $\{e\notin\mathcal{E}^{x_*}\}\subset\{e\notin\mathcal{E}\}$, we obtain

$$\begin{aligned}&\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}\,|\,\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12},L^{x_*}_{|G},\sigma^{x_*}_{|G}\big)\\&=\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}^{x_*}\,|\,L^{x_*}_{|G},\sigma^{x_*}_{|G}\big)\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}\,|\,\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12},L^{x_*}_{|G},e\notin\mathcal{E}^{x_*}\big).\end{aligned}$$

Now, since $(\sigma^{x_*}_x\sqrt{2L^{x_*}_x},\{e\notin\mathcal{E}^{x_*}\})$ has the same law as $((\varphi_x)_{x\in\mathcal{G}},\{\forall y\in I_e:|\varphi_y|>0\}^c)$ under $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$, it follows from (IV.2.33) that for all $e\in E^*$,

$$\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}^{x_*}\,|\,(\sigma^{x_*}_x\sqrt{2L^{x_*}_x})_{x\in G},\big)=p^{\mathcal{G}}_e(\sigma^{x_*}\sqrt{2L^{x_*}})\mathbb{1}_{e\in E}+\mathbb{1}_{e\notin E},\qquad\text{(IV.A.4)}$$

where we identified e with the corresponding edge or vertex of $E\cup G$. Now since $\{e\notin\mathcal{E}^{x_*}\}$ is independent of $\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12}$, we have by Theorem IV.7.1 that for all edges $e\in E$

$$\begin{aligned}\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}\,|\,\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12},L^{x_*}_{|G},e\notin\mathcal{E}^{x_*}\big)&=\frac{\widetilde{\mathbb{E}}^L_{\widetilde{\mathcal{G}}^*}\big[\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}\,|\,\mathcal{L}_{\mathcal{G},\frac12}\big)\,|\,\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac12},L^{x_*}_{|G}\big]}{\widetilde{\mathbb{E}}^L_{\widetilde{\mathcal{G}}^*}\big[\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e\notin\mathcal{E}^{x_*}\,|\,\mathcal{L}^{\{x_*\}^c}_{\mathcal{G},\frac12}\big)\,|\,L^{x_*}_{|G}\big]}\\&=\frac{p^{\mathcal{G}^*}_e\big(\sqrt{L^{x_*}+\overline{L}^{x_*}}\big)}{p^{\mathcal{G}}_e(\sqrt{L^{x_*}})}\mathbb{1}_{e\notin\mathcal{I}^{\mathcal{L}}_E},\end{aligned}$$

Combining with (IV.A.4), we thus obtain that for all edges $e \in E$,

$$\begin{aligned}\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e \notin \mathcal{E} \,|\, \overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac{1}{2}}, L^{x_*}_{|G}, \sigma^{x_*}_{|G}\big) &= \frac{p_e^{\mathcal{G}^*}\big(\sqrt{L^{x_*}+\overline{L}^{x_*}}\big) p_e^{\mathcal{G}}\big(\sigma^{x_*}\sqrt{2L^{x_*}}\big)}{p_e^{\mathcal{G}}\big(\sqrt{L^{x_*}}\big)} \mathbb{1}_{e\notin \mathcal{I}_E^{\mathcal{L}}} \\ &= p_e^{u,\mathcal{G}}\big(\sigma^{x_*}\sqrt{2L^{x_*}}, \overline{L}^{x_*}\big)\mathbb{1}_{e\notin \mathcal{I}_E^{\mathcal{L}}},\end{aligned}$$

where we used (IV.3.13) and (IV.2.34) in the last equality. Now if $e \in E^* \setminus E$, then one can identify e with some $x \in U_\kappa$, and by (IV.A.4), we have $e \notin \mathcal{E}^{x_*}$ $\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}$-a.s, and so e is crossed by a loop in $\mathcal{L}_{\mathcal{G},\frac{1}{2}}$ if and only if e is crossed by a loop in $\overline{\mathcal{L}}^{\{x_*\}}_{\mathcal{G},\frac{1}{2}}$, that is $e \in \mathcal{I}_E^{\mathcal{L}}$. Therefore by Theorem IV.7.1,

$$\begin{aligned}\widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e \notin \mathcal{E} \,|\, \overline{\mathcal{L}}^{e,\{x_*\}}_{\mathcal{G}^*,\frac{1}{2}}, L^{x_*}_{|G}, \sigma^{x_*}_{|G}, \overline{L}^{x_*}_{x_*} = u\big) &= \widetilde{\mathbb{P}}^L_{\widetilde{\mathcal{G}}^*}\big(e \notin \mathcal{E} \,|\, \overline{\mathcal{L}}^{e,\{x_*\}}_{\mathcal{G}^*,\frac{1}{2}}, L^{x_*}_{|G}, \overline{L}^{x_*}_{x_*} = u\big) \\ &= p_e^{\mathcal{G}^*}\big(\sqrt{L^{x_*}+\overline{L}^{x_*}}\big)\mathbb{1}_{e\notin \mathcal{I}_E^{\mathcal{L}}, \overline{L}^{x_*}_{x_*}=u} \\ &= p_x^{u,\mathcal{G}}\big(\sigma^{x_*}\sqrt{2L^{x_*}}, \overline{L}^{x_*}\big)\mathbb{1}_{e\notin \mathcal{I}_E^{\mathcal{L}}},\end{aligned}$$

where we used (IV.3.13) and (IV.2.34) in the last equality. Finally, if $x \in G \setminus U_\kappa$, then $\kappa_x = 0$, $x \notin \mathcal{I}_E^u$, and $\widetilde{\mathbb{P}}^I_{\widetilde{\mathcal{G}}}(x \notin \mathcal{E}_u \,|\, \omega_u^{\mathcal{G}}, (\varphi_x)_{x\in G}) = 1 = p_x^{u,\mathcal{G}}\big(\widehat{\sigma}'\sqrt{2L^{x_*}}, \overline{L}^{x_*}\big)$, and we can conclude.

□

Chapter V

Random interlacements on massive graphs

V.1 Introduction

Random interlacements is a model introduced on $\mathbb{Z}^d$, $d \geq 3$, in [93], to study disconnection of cylinders by a random walk, and which was extended to any transient weighted graphs in [103]. In this chapter, we are going to be interested in random interlacements on *massive* transient weighted graphs, that is a graph $\mathcal{G}$ on which the random walk can be killed in a finite random time by a killing measure κ. In Chapter IV, we already took advantage of this definition of random interlacements on massive graphs, see for instance Lemmas IV.7.2, IV.8.1 or IV.8.2, and we are going to investigate the particularities of this model in depth here. Random interlacements on a massive graph consist of a Poisson point process of trajectories modulo time-shift, and, when fixing arbitrarily a time zero for each trajectory, the corresponding forwards trajectory behave like a random walk on $\mathcal{G}$ and can thus either be finite, when they are killed in finite time by the killing measure κ, or infinite, and similarly for the corresponding backwards trajectories. In particular, if the probability that the random walk on $\mathcal{G}$ is killed in finite time by the killing measure κ is equal to one, then random interlacements consists of doubly finite trajectories modulo time-shift, with a starting and ending point.

Since the trajectories of random interlacements on massive graphs can be finite, it is not anymore true that the random interlacement set always contain an infinite connected component, and it will in fact often exhibit a phenomenon of phase transition. We are interested in understanding how does changing the killing measure κ, i.e. changing the speed at which the trajectories are killed, affects this phase transition, and in particular under which conditions does the phase transition stays non-trivial when changing the killing measure. Similar questions are raised for level sets of the Gaussian free field, and the results are gathered in Theorems V.1.1 and V.1.2.

The setting for this chapter is similar to the setting for Chapter IV, and we recall it briefly. We consider a graph $\mathcal{G} = (G, \lambda, \kappa)$, where G is a countable set of vertices, $\lambda = (\lambda_{x,y})_{x,y\in G} \in [0,\infty)^{G\times G}$ are called weights, and $\kappa = (\kappa_x)_{x\in G} \in [0,\infty)^G$ is called the killing measure. We always assume that the associated graph with vertex set G and edge set $E = \{\{x,y\} \in G^2 : \lambda_{x,y} > 0\}$ is connected, locally finite, and that the Markov jump process Z on this massive weighted graph is transient, where Z starts in z under $P_z^{\mathcal{G}}$, $z \in G$, and jumps from x to y at exponential speed with parameter $\lambda_{x,y}$ and is killed at x at exponential speed with parameter κ_x. We denote by $\lambda_x = \sum_{y\in G} \lambda_{x,y} + \kappa_x$ the total weight of the vertex $x \in G$.

Our study of random interlacements on massive graphs also includes the cable system $\widetilde{\mathcal{G}}$ of $\mathcal{G}$, which is defined by glueing together segments I_e with length $\rho_e = 1/(2\lambda_{x,y})$, $e = \{x,y\} \in E$, through their endpoints, and glueing the endpoint of half-open intervals I_x with length $\rho_x = 1/(2\kappa_x)$ to x, $x \in G$. One can endow $\widetilde{\mathcal{G}}$ with a distance $d_{\widetilde{\mathcal{G}}}$, such that $d_{\widetilde{\mathcal{G}}}(x,y)$ is the length of the shortest path between x and y when replacing the length of I_e by 1 for each $e \in E \cup G$, through some given increasing bijection $[0,\infty) \to [0,1)$ for I_x when $\kappa_x = 0$. The associated metric space $\widetilde{\mathcal{G}}$ is a Polish space, and a connected set K is compact for this topology if and only $K \cap G$ is finite and $K \cap I_e$ is a connected compact of I_e for all $e \in E \cup G$. For simplicity, we say that K is a compact of $\widetilde{\mathcal{G}}$ if it is compact for the distance $d_{\widetilde{\mathcal{G}}}$ and has finitely many components. One can then define a diffusion X on $\widetilde{\mathcal{G}}$, starting at x under $P_x^{\widetilde{\mathcal{G}}}$, $x \in \widetilde{\mathcal{G}}$, through its associated Dirichlet form, see (V.2.1). It behaves locally like a Brownian motion on each I_e, $e \in G \cup E$, and its print on G behaves like the Markov jump process Z.

The diffusion X stays in $\widetilde{\mathcal{G}}$ until a time $\widetilde{\zeta} \in [0,\infty]$, after which it remains in some cemetery state Δ, and, as $t \nearrow \widetilde{\zeta}$, either X_t reaches the open end of the cable I_x for some $x \in G$, and we say that X has been killed, or X_t exits every bounded and connected sets, and we say that X blows up. We define $\mathbf{h}_0$ as the probability to be killed before blowing up and $\mathbf{h}_1$ as the probability to blow up before being killed: for all $x \in \widetilde{\mathcal{G}}$,

$$\mathbf{h}_0(x) \stackrel{\text{def.}}{=} P_x^{\widetilde{\mathcal{G}}}\big((X_t)_{t\geq 0} \text{ is killed before blowing up}\big) \text{ and } \mathbf{h}_1(x) \stackrel{\text{def.}}{=} 1 - \mathbf{h}_0(x). \tag{V.1.1}$$

We define similarly the time ζ at which the jump process Z on G is either killed by the killing measure κ or exits every finite set, that is blow up, and $\mathbf{h}_0(x)$ is also equal to the probability under $P_x^{\mathcal{G}}$ that Z is killed before blowing up for all $x \in G$. One can easily check that this definition of $\mathbf{h}_0$ is equivalent to the one given in (IV.3.16).

We show in Proposition V.2.2 that a measure $\nu_{\widetilde{\mathcal{G}}}$ on the set $W^*_{\widetilde{\mathcal{G}}}$ of doubly non-compact trajectories modulo time-shift on $\widetilde{\mathcal{G}}$ exists, which correspond to the usual measure underlying random interlacements on the cable system, see (V.2.11) and (V.2.12). One can then classically define under some probability $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$ the random interlacement process ω as a Poisson point process on $W^*_{\widetilde{\mathcal{G}}} \times (0,\infty)$ with intensity $\nu_{\widetilde{\mathcal{G}}} \otimes \lambda$, where λ is the Lebesgue measure on $(0,\infty)$. It consists of an infinite number of independent doubly non-compact trajectories modulo time-shift, each with a forwards and backwards part behaving like the diffusion X. Random interlacements on the discrete graph $\mathcal{G}$ then correspond to the print of ω on G.

When $\kappa \not\equiv 0$, there are four possible types of trajectories in ω: they are either killed, that is both their forwards and backwards part are killed before blowing up, or surviving, that is both their forwards and backwards parts blow up before being killed, or backwards-killed, that is their backwards parts are killed before blowing up but their forwards parts blow up before being killed, or forwards-killed, that is their backwards parts blow up before being killed, but their forwards parts are killed before blowing up. We call killed random interlacements, surviving random interlacements, backwards-killed random interlacements and forwards-killed interlacements the point processes consisting of the corresponding trajectories in ω, and we denote them respectively by $\omega^{\mathcal{K}}$, $\omega^{\mathcal{S}}$, $\omega^{\mathcal{KS}}$ and $\omega^{\mathcal{SK}}$, see Definition V.3.1 for details on notation.

For random interlacements on transient massless graphs, the random interlacement set $\mathcal{I}^u$ consists of unbounded trajectories, and thus always contains an unbounded connected component, and this is also true on massive graph for surviving, backwards-killed or forwards-killed random interlacements whenever they are not empty. However, the killed random interlacement set $\mathcal{I}^u_{\mathcal{K}}$, i.e. the open set of points in $\widetilde{\mathcal{G}}$ reached by a trajectory in the killed random interlacement process $\omega^{\mathcal{K}}$ with label at most u, does not automatically contain an unbounded connected component, since it consists only of bounded trajectories. We thus naturally define the following critical parameter associated to the percolation of $\mathcal{I}^u_{\mathcal{K}}$

$$\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}) \stackrel{\text{def.}}{=} \sup\big\{u \geq 0 : \mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}(\mathcal{I}^u_{\mathcal{K}} \text{ contains an unbounded cluster}) = 0\big\}.$$

It depends on the choice of the graph $\mathcal{G}$ whether the phase-transition for the percolation of $\mathcal{I}^u_{\mathcal{K}}$ is non-trivial, that is $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}) \in (0,\infty)$, or not, see Remark V.5.3,5. In particular, it also depends on the choice of the killing measure κ, and we will now compare the values of $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}$ when changing the value of κ.

For any killing measures $\kappa' \in [0,\infty)^G$, we define the graph $\mathcal{G}_{\kappa'} = (G,V,\kappa')$, that is $\mathcal{G}_{\kappa'}$ is the same graph as $\mathcal{G}$ but with killing measure κ' instead of κ, and we define $\mathbf{h}_0^{\kappa'}$ as in (V.1.1) but for the graph $\mathcal{G}_{\kappa'}$. Our first result compare the critical parameter for the percolation of killed random interlacements on $\mathcal{G} = \mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$ when $\kappa' \leq \kappa$.

Theorem V.1.1. *Let $\mathcal{G} = \mathcal{G}_\kappa$ be a transient graph with $\kappa \not\equiv 0$. For all killing measures $\kappa' \neq 0$ with $\kappa' \leq \kappa$ let*

$$c(\kappa, \kappa') \stackrel{\text{def.}}{=} \Big(\sup_{x\in G} \frac{\kappa_x}{\kappa'_x}\Big)\Big(\sup_{x\in G} \frac{\kappa_x}{\kappa'_x + (\kappa_x - \kappa'_x)\mathbf{h}_0^{\kappa'}(x)}\Big). \tag{V.1.2}$$

If $\sup_{x\in G} \frac{\kappa_x}{\kappa'_x} < \infty$, then

$$\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_{\kappa'}) \leq \widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_\kappa) c(\kappa, \kappa').$$

Note that $\sup_{x\in G} \frac{\kappa_x}{\kappa'_x} \leq c(\kappa, \kappa') \leq (\sup_{x\in G} \frac{\kappa_x}{\kappa'_x})^2$, and so $c(\kappa, \kappa') < \infty$ if and only if $\sup_{x\in G} \frac{\kappa_x}{\kappa'_x} < \infty$. An easy consequence of Theorem V.1.1 is that if $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_\kappa) > 0$, then for all constants $C \geq 1$ we have $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_{C\kappa}) > 0$, and if $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_\kappa) < \infty$, then for all constants $0 < c \leq 1$ we have $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_{c\kappa}) < \infty$. Similar results could also be obtained when considering percolation for the vacant set $\mathcal{V}_{\mathcal{K}}^u := \mathcal{I}_{\mathcal{K}}^u$ of killed random interlacements or percolation for the discrete set $\mathcal{I}_{\mathcal{K}}^u \cap G$ of random interlacements, see Remark V.5.3,4). If $\mathbf{h}_0^{\kappa} = \mathbf{h}_0^{\kappa'} = 1$, then all the trajectories in the random interlacement process are doubly-killed, and so killed random interlacements are equal to random interlacements, and Theorem V.1.1 let us compare critical parameter for the percolation of the random interlacement set at different values of κ.

Random interlacements are linked through isomorphisms theorems to the Gaussian free field, see [96], [57] and [101], and this let us derive a result similar to Theorem V.1.1 for the Gaussian free field on the cable system, that we now describe. We denote by $\varphi = (\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ the Gaussian free field on $\widetilde{\mathcal{G}}$ under some probability $\mathbb{P}_{\widetilde{\mathcal{G}}}^G$, that is the canonical centered Gaussian free field with covariance function the Green function $(g(x,y))_{x,y\in\widetilde{\mathcal{G}}}$. We are going to study percolation for the killed level sets of the Gaussian free field, defined for all $h \in \mathbb{R}$ and $x_0 \in \widetilde{\mathcal{G}}$ by

$$E_{\mathcal{K}}^{\geq h} \stackrel{\text{def.}}{=} \{x \in \widetilde{\mathcal{G}} : \varphi_x \geq h \times \mathbf{h}_0(x)\} \text{ and } E_{\mathcal{K}}^{\geq h}(x_0) \stackrel{\text{def.}}{=} \big\{x \in \widetilde{\mathcal{G}} : x \longleftrightarrow x_0 \text{ in } E_{\mathcal{K}}^{\geq h}\big\}.$$

One can see $E_{\mathcal{K}}^{\geq h}$ as the level sets for the Gaussian free field associated with the diffusion X conditioned on being killed before blowing up, see Proposition V.4.2 for details. We define similarly the surviving level sets of the Gaussian free field $E_{\mathcal{S}}^{\geq h}$ by replacing $\mathbf{h}_0$ by $\mathbf{h}_1$. When $h = 0$, the level sets $E_{\mathcal{K}}^{\geq 0} = E_{\mathcal{S}}^{\geq 0}$, and also coincide with the usual level sets of the Gaussian free field, and we will often simply denote them by $E^{\geq 0}$. Let $\widetilde{h}^{\mathcal{K}}$ be the critical parameter associated with the percolation of the killed level sets of the Gaussian free field, that is

$$\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}) \overset{\text{def.}}{=} \inf\big\{h \in \mathbb{R} : \mathbb{P}^G_{\widetilde{\mathcal{G}}}(E^{\geq h}_{\mathcal{K}} \text{ contains an unbounded cluster}) = 0\big\}. \quad \text{(V.1.3)}$$

Let us also recall the following condition on the graph $\mathcal{G}$ from Chapter IV

$$E^{\geq 0} \text{ contains } \mathbb{P}^G_{\widetilde{\mathcal{G}}}\text{-a.s. only bounded connected components.} \quad \text{(Sign)}$$

In Corollary IV.3.2, we proved that (Sign) is verified on any graph such that (Cap) hold, that is the capacity of any unbounded set is infinite, and is thus verified on a very large class of graphs, but there are also examples of graphs on which (Sign) does not hold, see Proposition IV.9.1. In both cases, let us now present the analogue of Theorem V.1.1 for the Gaussian free field.

Theorem V.1.2. *Let $\mathcal{G} = \mathcal{G}_\kappa$ be a transient graph with $\kappa \not\equiv 0$. For all killing measures $\kappa' \neq 0$ with $\kappa' \leq \kappa$ we have*

i) if (Sign) *does not hold for $\mathcal{G}_\kappa$, then* (Sign) *does not hold for $\mathcal{G}_{\kappa'}$ and*

$$\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) \geq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa) \geq 0,$$

ii) if (Sign) *holds for $\mathcal{G}_{\kappa'}$ and $\sup_{x\in G} \frac{\kappa_x}{\kappa'_x} < \infty$, then*

$$0 \geq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) \geq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa)\sqrt{c(\kappa,\kappa')},$$

iii) if (Sign) *holds for $\mathcal{G}_\kappa$ but does not hold for $\mathcal{G}_{\kappa'}$, then*

$$\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) \geq 0 \geq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa).$$

Similarly as for random interlacements, we have that if $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa) < 0$, then for all constants $C \geq 1$ we have $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{C\kappa}) < 0$, and if $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa) > -\infty$, then for all constants $0 < c \leq 1$ we have $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{c\kappa}) > -\infty$. Moreover, if $\mathbf{h}_0^{\kappa'} = \mathbf{h}_0^{\kappa} = 1$, then Theorem V.1.2 let us compare usual level sets of the Gaussian free field on the cable system on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$. Note that iii) of Theorem V.5.5 is a trivial consequence of the definitions (V.1.3) and (Sign), and we only include it to list all the possible cases. A result similar to i) of Theorem V.1.2 holds for surviving level sets of the Gaussian free field, see Remark V.5.6,2), and this is the only relevant case since the critical parameter associated to the percolation of the surviving level sets of the Gaussian free field is always non-negative, see Remark V.4.7,4). It would be interesting to prove an equivalent of Theorem V.5.5 for the Gaussian free field on the discrete graph $\mathcal{G}$, which could imply percolation for the discrete sign clusters of the Gaussian free field, as studied in Chapters II and III, see Remark V.5.6,2) for details.

Let us now describe the proofs of Theorem V.1.1 and V.1.2, as well as various intermediate results, which are also interesting in their own right. The intensity measure underlying killed, surviving, backwards-killed or forwards-killed interlacements can be described directly, in a similar way as the intensity measure $\nu_{\widetilde{\mathcal{G}}}$ underlying random interlacements, see for instance (V.3.2) and (V.3.3) in the case of killed random interlacements. In particular, one can define the notion of killed capacity $\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(K)$ of a compact K of $\widetilde{\mathcal{G}}$, see (V.3.1) and (V.3.4), and the number of trajectories hitting K in the killed random interlacement process at level u is then $\mathrm{Poi}(u\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(K))$-distributed, similarly as for random interlacements.

There is however a simpler description of killed random interlacements: for each $x \in G$, start in x a $\mathrm{Poi}(u\mathbf{h}_0(x)\kappa_x)$-distributed number of independent trajectories, each distributed like the Markov jump process Z conditioned on being killed before blowing up. Then the point process consisting of these trajectories modulo time-shift has the same law as the discrete killed random interlacement process $\omega_u^{\mathcal{K},\mathcal{G}}$ at level u, that is

$$\begin{gathered}\omega_u^{\mathcal{K},\mathcal{G}} \text{ has the same law under } \mathbb{P}^{KI}_{\widetilde{\mathcal{G}}} \text{ as a Poisson point process with}\\ \text{intensity } u\widetilde{\nu}^{\mathcal{K}}_{\mathcal{G}} \text{ modulo time-shift, with } \widetilde{\nu}^{\mathcal{K}}_{\mathcal{G}} := \sum_{x\in G}\kappa_x\mathbf{h}_0(x)P_x^{\mathcal{G}}\big(\,\cdot\,\,|\,W_{\mathcal{G}}^{\mathcal{K},+}\big).\end{gathered} \tag{V.1.4}$$

We also extend this description of killed random interlacements to the cable system, and give similar descriptions for backwards-killed and forwards-killed random interlacements, see Proposition V.3.3.

Using the description (V.1.4) of killed random interlacements, one can show that for all $\kappa' \leq \kappa$ and $u' \geq uc(\kappa,\kappa')$, see (V.1.2), and $x, y \in G$, the number of trajectories starting in x and killed in y in the killed random interlacement process at level u on $\mathcal{G}_\kappa$ is smaller than the number of trajectories starting in x and killed by κ in y for the killed random interlacement process at level u' on $\mathcal{G}_{\kappa'}$, see Lemma V.5.1. One can do a similar reasoning on the cable system, which let us find a coupling of the local times of killed random interlacements on $\widetilde{\mathcal{G}}_\kappa$ at level u and on $\widetilde{\mathcal{G}}_{\kappa'}$ at level u', see Proposition V.5.2, and Theorem V.1.1 follows directly from this coupling.

Another possible description of killed, or surviving, random interlacements is through the Doob $\mathbf{h}$-transform. For any harmonic functions $\mathbf{h}$ on $\widetilde{\mathcal{G}}$, we define a graph $\mathcal{G}_{\mathbf{h}}$, the $\mathbf{h}$-transform of the graph $\mathcal{G}$, such that, after time change, see (V.4.2) and (V.4.3), the diffusion X on $\widetilde{\mathcal{G}}_{\mathbf{h}}$ corresponds to the usual $\mathbf{h}$-transform of the diffusion X on $\widetilde{\mathcal{G}}$, see for example Chapter 11 in [22], and we refer to

Lemma V.4.1 for a more precise statement. In particular, if $\mathbf{h} = \mathbf{h}_0$, then after time change, the law of X under $P_x^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}$ is the same as the law of X under $P_x^{\widetilde{\mathcal{G}}}$, conditioned on being killed before blowing up, see (V.4.2). Moreover,

$$\begin{gathered}\text{the random interlacement process } \omega \text{ on } \widetilde{\mathcal{G}}_{\mathbf{h}_0} \text{ has the same law} \\ \text{after time change as the killed random interlacement process } \omega^{\mathcal{K}} \text{ on } \widetilde{\mathcal{G}},\end{gathered}$$

see (V.4.5) for details. Similar statements hold when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$ and killed random interlacements by surviving random interlacements.

It is easy to also compare the Gaussian free field on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ and $\widetilde{\mathcal{G}}$, see (V.4.6), and using the description of killed random interlacements as random interlacements on the $\mathbf{h}_0$-transform graph $\mathcal{G}_{\mathbf{h}_0}$, we can adapt the results from Chapter IV about the Gaussian free field and random interlacements to results about the Gaussian free field and killed random interlacements. We thus obtain the law of the killed capacity of the sign clusters of the Gaussian free field on the cable system, see (V.4.14), and an isomorphism between killed random interlacements and the Gaussian free field, Theorem V.4.6, similar to Theorem IV.3.4. Similar considerations are also available when $\mathbf{h} = \mathbf{h}_1$ for surviving random interlacements.

The isomorphism between killed random interlacements and the Gaussian free field (V.4.15) provides us with the following description of negative killed level sets of the Gaussian free field on the cable system: for all $u > 0$, $E_{\mathcal{K}}^{\geq -\sqrt{2u}}$ has the same law as the union of $E_{\mathcal{K}}^{\geq 0}$ and the clusters of $(E_{\mathcal{K}}^{\geq 0})^c$ intersecting $\mathcal{I}_{\mathcal{K}}^u$. Following ideas from [57], one can easily find a coupling of non-negative killed level sets of the Gaussian free field on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$ when $\kappa' \leq \kappa$, see Lemma V.5.4, and easily deduce i) of Theorem V.1.2. Combining these observations with the previously mentioned coupling of the killed random interlacements $\mathcal{I}_{\mathcal{K}}^u$ on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$, we thus also obtain a coupling of negative killed level sets of the Gaussian free field on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$, see Proposition V.5.5, from which ii) of Theorem V.1.2 follows readily.

Let us finally give an interesting consequence of the isomorphism between surviving random interlacements and the Gaussian free field, which let us find an isomorphism between the trajectories in the surviving random interlacement process avoiding a compact K of $\widetilde{\mathcal{G}}$, and the Gaussian free field conditioned on being equal to 0 on K, and is proved at the end of Section V.4. We define $H_K = \inf\{t \geq 0 : X_t \in K\}$, with $\inf \varnothing = \widetilde{\zeta}$, $\mathbf{h}_K(x) = P_x^{\widetilde{\mathcal{G}}}(H_K = \widetilde{\zeta}, (X_t)_{t\geq 0}\text{blows up before being killed})$ for all $x \in \widetilde{\mathcal{G}}$, $\ell_{\cdot,u}^{\mathcal{S},K^c}$ the total local times of the trajectories in the surviving random interlacement process $\omega_u^{\mathcal{S}}$ at level u never hitting K and $\mathcal{I}_{\mathcal{S},K^c}^u = \{x \in \widetilde{\mathcal{G}} : \ell_{x,u}^{\mathcal{S},K^c} > 0\}$.

Theorem V.1.3. *Assume that $\mathcal{G}$ is a transient graph such that condition* (Cap) *is fulfilled, and let K be a compact of $\widetilde{\mathcal{G}}$. On some extension $\widetilde{\mathbb{P}}^{KI}_{K,\widetilde{\mathcal{G}}}$ of $\mathbb{P}^G_{\widetilde{\mathcal{G}}}(\cdot\,|\,\varphi_{|K}=0)\otimes\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}$, let us define for each $u>0$ an additional process $(\sigma_x^{K,u})_{x\in\widetilde{\mathcal{G}}}\in\{-1,1\}^{\widetilde{\mathcal{G}}}$, such that, conditionally on $(|\varphi_x|)_{x\in\widetilde{\mathcal{G}}}$ and ω_u, $\sigma^{u,K}$ is constant on each of the cluster of $\{x\in\widetilde{\mathcal{G}}:\ 2\ell_{x,u}^{\mathcal{S},K^c}+\varphi_x^2>0\}$, $\sigma_x^{u,K}=1$ for all $x\in\mathcal{I}^u_{\mathcal{S},K^c}$, and the values of $\sigma^{u,K}$ on each other cluster are independent and uniformly distributed. Then*

$$\begin{aligned}\Big(\sigma_x^{K,u}\sqrt{2\ell_{x,u}^{\mathcal{S},K^c}+\varphi_x^2}\Big)_{x\in\widetilde{\mathcal{G}}}&\text{ has the same law under }\widetilde{\mathbb{P}}^{KI}_{K,\widetilde{\mathcal{G}}}\\ \text{as }\big(\varphi_x+\sqrt{2u}\mathbf{h}_K(x)\big)_{x\in\widetilde{\mathcal{G}}}&\text{ under }\mathbb{P}^G_{\widetilde{\mathcal{G}}}(\cdot\,|\,\varphi_{|K}=0).\end{aligned} \tag{V.1.5}$$

We now describe how this chapter is organized. Section V.2 introduces various definitions and notations, which are useful to obtain a last exit decomposition (V.2.10) for the canonical diffusion X on the cable system $\widetilde{\mathcal{G}}$, and then to prove the existence of random interlacements on the cable system of massive weighted transient graphs, see Proposition V.2.2.

Section V.3 is devoted to the various definitions or characterizations of killed, surviving, backwards-killed and forwards-killed random interlacements. They are first introduced as mappings of the random interlacement process ω, see Definition V.3.1, then directly constructed from the law of X conditioned on being killed before blowing up, or blowing up before being killed, see (V.3.2) and (V.3.3) for instance, and finally a more direct description of killed, backwards-killed and forwards-killed random interlacements is given in Proposition V.3.3.

Section V.4 is centered around the notion of Doob $\mathbf{h}$-transform and its various consequences. For any harmonic functions $\mathbf{h}$, the $\mathbf{h}$-transform $\mathcal{G}_{\mathbf{h}}$ of a graph $\mathcal{G}$ is introduced, so that the diffusion X on $\widetilde{\mathcal{G}}_{\mathbf{h}}$ is related to the usual $\mathbf{h}$-transform of the diffusion X on $\widetilde{\mathcal{G}}$, see Lemma V.4.1. If $\mathbf{h}=\mathbf{h}_0$, see (V.1.1), we can then relate the diffusion X, random interlacements or the Gaussian free field on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ to the diffusion X, conditioned on being killed before blowing up, killed interlacements or the Gaussian free field on $\widetilde{\mathcal{G}}$, and similarly when $\mathbf{h}=\mathbf{h}_1$, see Proposition V.4.2. These relations are turned into correspondences for local times in Corollary V.4.5, which let us use the results from Chapter IV to obtain the law for the killed, or surviving, capacity of the sign clusters of the Gaussian free field, see (V.4.14), and a signed isomorphism between killed, or surviving, random interlacements and the Gaussian free field, see Theorem V.4.6. Finally, the proof of Theorem V.1.3 is given at the end of the section.

Section V.5 combines the previous results to give couplings of killed random interlacement sets, or killed level sets of the Gaussian free field at positive or negative levels, on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$ when $\kappa'\leq\kappa$, see Proposition V.5.2, Lemma V.5.4 and Proposition V.5.5, from which Theorems V.1.1 and V.1.2 follow readily.

Throughout the chapter, we will often remove the subscript or superscript $\widetilde{\mathcal{G}}$ from the notation when there is no ambiguity about the choice of the graph $\mathcal{G}$.

V.2 Definition of (massive) interlacements

In this section, we explain how to extend the definition of random interlacements to the cable system of massive weighted graphs. We first recall the definition of the canonical diffusion X on $\widetilde{\mathcal{G}}$, and present its last exit decomposition in (V.2.10), that is a decomposition of the law of X before and after the time L_K at which X leaves the compact K of $\widetilde{\mathcal{G}}$ forever. We then use this last exit decomposition to prove the existence of the measure $\nu_{\widetilde{\mathcal{G}}}$ underlying random interlacements, see Proposition V.2.2.

Let m be the Lebesgue measure on $\widetilde{\mathcal{G}}$, that is the sum of the Lebesgue measure on each I_e $e \in E \cup G$, W^+, be the set of continuous functions from $[0,\infty)$ to $\widetilde{\mathcal{G}} \cup \Delta$, where Δ is some cemetery point, X_t be the projection function at time t for all $t \geq 0$, and $\mathcal{W}^+$ the algebra generated by X_t, $t \geq 0$. We simply write W^+ and $\mathcal{W}^+$ when there is no ambiguity about the choice of the graph $\mathcal{G}$. For all measures $\widetilde{m}$ on $\widetilde{\mathcal{G}}$, that is $\widetilde{m}_{|I_e}$ is a measure on $(I_e, \mathcal{B}(I_e))$ for all $e \in E \cup G$, and measurable function $f : \widetilde{\mathcal{G}} \to \mathbb{R}$, we define

$$(f,f)_{\widetilde{m}} \stackrel{\text{def.}}{=} \sum_{e \in E \cup G} \int_{I_e} f^2 \, \mathrm{d}\widetilde{m}_{|I_e},$$

$L^2(\widetilde{\mathcal{G}}, \widetilde{m}) = \{f : (f,f)_{\widetilde{m}} < \infty\}$, and $(f,g)_{\widetilde{m}}$ the associated Dirichlet form on $L^2(\widetilde{\mathcal{G}}, \widetilde{m})$. Let also $D(\widetilde{\mathcal{G}}, \widetilde{m}) \subset L^2(\widetilde{\mathcal{G}}, \widetilde{m})$ be the space of function $f \in C_0(\widetilde{\mathcal{G}})$ such that $f_{|I_e} \in W^{1,2}(I_e, \widetilde{m}_{|I_e})$ for all $e \in E \cup G$ and

$$\sum_{e \in E \cup G} \|f_{|I_e}\|^2_{W^{1,2}(I_e, \widetilde{m}_{|I_e})} < \infty.$$

The canonical Brownian motion on $\widetilde{\mathcal{G}}$ is then defined by taking probabilities $P_x^{\widetilde{\mathcal{G}}}$, or simply P_x when there is no ambiguity about the choice of the graph $\mathcal{G}$, $x \in \widetilde{\mathcal{G}}$, under which the process X is an m-symmetric diffusion on $\widetilde{\mathcal{G}}$ with associated Dirichlet form on $L^2(\widetilde{\mathcal{G}}, m)$

$$\mathcal{E}_{\widetilde{\mathcal{G}}}(f,g) \stackrel{\text{def.}}{=} \frac{1}{2}(f',g')_m \text{ for all } f,g \in D(\widetilde{\mathcal{G}}, m). \tag{V.2.1}$$

These definitions could also be extended to any killing measure $\kappa \in [0,\infty]$ by replacing $\mathcal{G}$ by the graph $\mathcal{G}^{(\infty)}$, which is the graph with finite killing measure, obtained by keeping only the vertices x with $\kappa_x < \infty$, and adding a vertex between each x and y such that $\kappa_x < \infty$, $\kappa_y = \infty$ and $\lambda_{x,y} > 0$, and such that the restriction of the random walk on $\mathcal{G}^{(\infty)}$ to G is a random walk on $\mathcal{G}$. We refer to Section IV.2 for more details and properties of the cable system $\widetilde{\mathcal{G}}$ and its associated diffusion X.

In order to study random interlacements on the cable system, we are first interested in describing a decomposition of $(X_t)_{t\geq 0}$ before and after the last time L_K at which X exits a compact K of $\widetilde{\mathcal{G}}$, which will be given in (V.2.10). Using Theorems 4.1.2 and 4.2.4 in [37], the 0-potential of X has a density that we denote by $(g(x,y))_{x,y\in\widetilde{\mathcal{G}}}$, the Green function on $\widetilde{\mathcal{G}}$, and one can associate to the diffusion X a family of probability densities $(p_t(x,y))_{t>0}$, $x,y\in\widetilde{\mathcal{G}}$, such that

$$P_x(X_t\in \mathrm{d}y)=p_t(x,y)m(\mathrm{d}y) \quad \text{and} \quad g(x,y)=\int_0^\infty p_t(x,y)\,\mathrm{d}t. \tag{V.2.2}$$

One can show that this definition of the Green function corresponds to the definition given in (IV.2.12), using for instance Theorem 3.6.5 in [62]. Let us now recall some useful results from Section 2 of [34] about the existence of Markovian bridges, that we apply to our m-symmetric diffusion X. Under P_x, the process $(p_{t-s}(X_s,y))_{s\in[0,t)}$ is a martingale, and thus we can define

$$P_{x,y,t}(A)\overset{\text{def.}}{=}\frac{E_x[p_{t-s}(X_s,y)\mathbb{1}_A]}{p_t(x,y)} \text{ for all } A\in\mathcal{F}_s:=\sigma(X_u,u\leq s) \text{ and } 0\leq s<t,$$

and this definition is consistent. One can extend the definition of $P_{x,y,t}$ to a probability measure on $\mathcal{F}_t$, which informally corresponds to the law of a bridge of length t between x and y for X. Applying the optional stopping theorem to the martingale $(p_{t-s}(X_s,y))_{s\in[0,t)}$, see for instance Theorem 3.2 in Chapter II of [75], we have that for all $t>0$ and stopping time T

$$E_x[p_{t-T}(X_T,y)\mathbb{1}_{A,T<t}]=P_{x,y,t}(A,T<t)p_t(x,y) \text{ for all } A\in\mathcal{F}_T, \tag{V.2.3}$$

where $\mathcal{F}_T=\{F\in\mathcal{F}_t : F\cap\{T\leq s\}\in\mathcal{F}_s \text{ for all } s<t\}$ is the filtration associated with T. Moreover by m-symmetry of X, we have for all $t>0$ and $x,y\in\widetilde{\mathcal{G}}$ that

$$(X_{t-s})_{s\in[0,t]} \text{ has the same law under } P_{x,y,t} \text{ as } (X_s)_{s\in[0,t]} \text{ under } P_{y,x,t}. \tag{V.2.4}$$

Using (V.2.4), one can derive a decomposition for stopping time on the reversed time scale: for all random times τ such that $\{\tau\geq t\}$ is in $\sigma(X_{t+u},u\geq 0)$,

$$(X_s)_{s\in[0,\tau]} \text{ has the same law under } P_x(\cdot\,|\,\mathcal{G}_\tau) \text{ as } (X_s)_{s\in[0,\tau]} \text{ under } P_{x,X_\tau,\tau}, \tag{V.2.5}$$

where $\mathcal{G}_\tau=\sigma(\tau,X_{\tau+u},u\geq 0)$. We now define for any compacts K of $\widetilde{\mathcal{G}}$ the last exit time L_K of K by $L_K=\sup\{t\geq 0: X_t\in K\}$, with $\sup\varnothing=-\infty$, and, for all $x\in\partial K$ with $P_x(X_{L_K}=x)>0$ we define $P_x^{K,\widetilde{\mathcal{G}}}$ as the law of $(X_{t+L_K})_{t\geq 0}$ under

$P_x(\cdot\,|\,X_{L_K}=x)$, and we simply write P_x^K when there is no ambiguity about the choice of the graph $\mathcal{G}$. Using results for general Hunt processes, see either Theorem 8 in [66], Proposition 5.9 in [39] or Theorem 2.12 in [38], under P_x, on the event $L_K\geq 0$, $(X_{s+L_K})_{s>0}$ is a Markov process depending on the past only through X_{L_K}, and so we have for all $x\in\widetilde{\mathcal{G}}$,

$$(X_{s+L_K})_{s\geq 0} \text{ has the same law under } P_x(\cdot\,|\,L_K,X_{L_K}) \text{ as } (X_s)_{s\geq 0} \text{ under } P^K_{X_{L_K}}. \tag{V.2.6}$$

Combining (V.2.5) and (V.2.6), one can thus describe the law of $(X_t)_{t\geq 0}$ both before and after the last visit L_K of K. Let us now describe the law of L_K and X_{L_K}. We define a measure the equilibrium measure and the capacity of a set $K\subset G$ by

$$e_{K,\widetilde{\mathcal{G}}}(x)\stackrel{\text{def.}}{=}\lambda_x P_x^{\mathcal{G}}(\widetilde{H}_K=\zeta) \text{ for all } x\in G \text{ and } \mathrm{cap}_{\widetilde{\mathcal{G}}}(K)\stackrel{\text{def.}}{=}\sum_{x\in K}e_{K,\widetilde{\mathcal{G}}}(x), \tag{V.2.7}$$

where $\widetilde{H}_K$ is the first time the Markov jump process Z on G return in K after its first jump time, which is equal to ζ if Z never comes back in K. We simple write $e_K(x)$ and $\mathrm{cap}(K)$ when there is no ambiguity about the choice of the graph $\mathcal{G}$. One can extend these definitions to any compacts K of $\widetilde{\mathcal{G}}$, see (IV.2.18) and (IV.2.20). Using (IV.2.29), we have that

$$P_y(X_{L_K}=x)=g(y,x)e_K(x) \text{ for all } x,y\in\widetilde{\mathcal{G}}. \tag{V.2.8}$$

This leads to the following description of the law of L_K and X_{L_K}.

Lemma V.2.1. *For all compacts K of $\widetilde{\mathcal{G}}$ and $x,y\in\widetilde{\mathcal{G}}$, we have*

$$P_x(L_K\in \mathrm{d}t, X_{L_K}=y)=p_t(x,y)e_K(y)\mathrm{d}t. \tag{V.2.9}$$

Proof. For all $t>0$, we have by the Markov property at time t and (V.2.8)

$$P_x(L_K\geq t,X_{L_K}=y)=E_x\big[P_{X_t}(X_{L_K}=y)\big]=E_x\big[g(X_t,y)\big]e_K(y).$$

Using (V.2.3), we moreover have $E_x[p_{s-t}(X_t,y)]=p_s(x,y)$ for all $s>t$, and so by (V.2.2)

$$E_x\big[g(X_t,y)\big]=\int_t^\infty E_x\big[p_{s-t}(X_t,y)\big]\,\mathrm{d}s=\int_t^\infty p_s(x,y)\,\mathrm{d}s,$$

and we can conclude. □

We are now ready to give the last exit decomposition of $(X_t)_{t\geq 0}$ before and after time L_K. We denote by $W^{+,f}$ the set of continuous trajectories in $\widetilde{\mathcal{G}}$ with finite length, that is of continuous functions from $[0,t]$ to $\widetilde{\mathcal{G}}$ for some $t>0$. Let $\pi : W^+ \times (0,\infty) \to W^{+,f}$ the application $\pi(w,t) = w_{|[0,t]}$, $\pi_t = \pi(\cdot,t)$ for all $t>0$, and $\mathcal{W}^{+,f}$ the smallest σ-algebra such that π is measurable with respect to $\mathcal{W}^+ \otimes \mathcal{B}((0,\infty))$. For all $A_1 \in \mathcal{W}^{+,f}$ and $A_2 \in \mathcal{W}^+$, using (V.2.5) with $\tau = L_K$, we have that for all $x \in \widetilde{\mathcal{G}}$ and $y \in \partial K$

$$\begin{aligned} &P_x\big((X_t)_{t\in[0,L_K]} \in A_1, (X_{t+L_K})_{t\geq 0} \in A_2, X_{L_K} = y\big) \\ &= E_x\Big[\mathbb{1}_{(X_{t+L_K})_{t\geq 0}\in A_2, X_{L_K}=y}(P_{x,y,L_K} \circ \pi_{L_K}^{-1})(A_1)\Big] \\ &= P_y^K(A_2)E_x\Big[\mathbb{1}_{X_{L_K}=y}(P_{x,y,L_K} \circ \pi_{L_K}^{-1})(A_1)\Big], \end{aligned}$$

where we used (V.2.6) in the last equality. By (V.2.9), we moreover have that

$$E_x\Big[\mathbb{1}_{X_{L_K}=y}(P_{x,y,L_K} \circ \pi_{L_K}^{-1})(A_1)\Big] = e_K(y)\int_0^\infty (P_{x,y,s} \circ \pi_s^{-1})(A_1)p_s(x,y)\,\mathrm{d}s.$$

Summing over y in ∂K, we thus obtain the following last exit-decomposition for all compacts K of $\widetilde{\mathcal{G}}$, $x \in \widetilde{\mathcal{G}}$, $A_1 \in \mathcal{W}^{+,f}$ and $A_2 \in \mathcal{W}^+$

$$\begin{aligned} &P_x\big((X_t)_{t\in[0,L_K]} \in A_1, (X_{t+L_K})_{t\geq 0} \in A_2, L_K \geq 0\big) \\ &= \sum_{y\in\partial K} e_K(y)P_y^K(A_2)\int_0^\infty P_{x,y,s}(\pi_s^{-1}(A_1))p_s(x,y)\,\mathrm{d}s. \end{aligned} \tag{V.2.10}$$

The last exit decomposition (V.2.10) will now let us define random interlacements on the cable system of any massive transient graph. The random interlacement measure was first defined on $\mathbb{Z}^d$, $d\geq 3$, in [93], and then on any discrete transient graph with $\kappa \equiv 0$ in [103]. It was then extended to the cable system of $\mathbb{Z}^d$ in [57] using the fact that one can obtain the diffusion X by adding Brownian excursions on the edges to a discrete random walk on $\mathbb{Z}^d$, and this proof can easily be extended to any transient graph on which discrete random interlacements exist. A continuous analogue of random interlacements, Brownian interlacements, has also been defined on $\mathbb{R}^d$, $d \geq 3$, in [99]. We seize the opportunity here to give a direct construction of random interlacements on the cable system without using Brownian excursions, which also include the case $\kappa \not\equiv 0$.

Let us first recall some definitions from Chapter IV. The set of doubly non-compact trajectories $W_{\widetilde{\mathcal{G}}}$ is the set of continuous functions from $\mathbb{R}$ to $\widetilde{\mathcal{G}} \cup \Delta$, which take values in $\widetilde{\mathcal{G}}$ between times $\widetilde{\zeta}^- \in [-\infty,\infty)$ and $\widetilde{\zeta}^+ \in (-\infty,\infty]$, and is equal to Δ on $(\widetilde{\zeta}^-,\widetilde{\zeta}^+)^c$. We denote by $p^*_{\widetilde{\mathcal{G}}}(w)$ the equivalence class of w modulo time-shift for each $w \in W_{\widetilde{\mathcal{G}}}$, and $W^*_{\widetilde{\mathcal{G}}} = \{p^*_{\widetilde{\mathcal{G}}}(w),\, w \in W_{\widetilde{\mathcal{G}}}\}$. We define $\mathcal{W}_{\widetilde{\mathcal{G}}}$ the σ-Algebra on $W_{\widetilde{\mathcal{G}}}$ generated by the coordinate functions, and $\mathcal{W}^*_{\widetilde{\mathcal{G}}} = \{A \subset W^*_{\widetilde{\mathcal{G}}} :$

$(p^*_{\widetilde{\mathcal{G}}})^{-1}(A) \in \mathcal{W}_{\widetilde{\mathcal{G}}}\}$. For each compact K of $\widetilde{\mathcal{G}}$, we denote by $W^0_{K,\widetilde{\mathcal{G}}}$ the set of trajectories in $W_{\widetilde{\mathcal{G}}}$ hitting K for the first time at time 0, and by $W^*_{K,\widetilde{\mathcal{G}}}$ the set of trajectories modulo time-shift in $W^*_{\widetilde{\mathcal{G}}}$ hitting K. The forwards part of a trajectory $w \in W_{\widetilde{\mathcal{G}}}$ is $(w(t))_{t\geq 0}$ and its backwards part $(w(-t))_{t\geq 0}$, and we denote by $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}}$ the set of events $B \in \mathcal{W}_{\widetilde{\mathcal{G}}}$, $B \subset W^0_{K,\widetilde{\mathcal{G}}}$, which can be uniquely decomposed into an event $B^+ \in W^+_{\widetilde{\mathcal{G}}}$ concerning the forwards part of the trajectories and an event $B^- \in W^+_{\widetilde{\mathcal{G}}}$ concerning the backwards part of the trajectories. We define a measure $Q_{K,\widetilde{\mathcal{G}}}$ on $\mathcal{W}_{\widetilde{\mathcal{G}}}$, whose restriction to $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}}$ is given by

$$Q_{K,\widetilde{\mathcal{G}}} \stackrel{\text{def.}}{=} \sum_{x\in\partial K} e_{K,\widetilde{\mathcal{G}}}(x) P^{\widetilde{\mathcal{G}}}_x(\cdot^+) P^{K,\widetilde{\mathcal{G}}}_x(\cdot^-), \tag{V.2.11}$$

and such that $Q_{K,\widetilde{\mathcal{G}}}(A) = 0$ for all $A \in \mathcal{W}_{\widetilde{\mathcal{G}}}$ with $A \cap W^0_{K,\widetilde{\mathcal{G}}} = \varnothing$. As usual, we will simply remove the subscript $\widetilde{\mathcal{G}}$ to all the notation introduced in the previous paragraph when there is no ambiguity about the choice of the graph $\mathcal{G}$.

Proposition V.2.2. *There exists a unique measure $\nu_{\widetilde{\mathcal{G}}}$ on $W^*_{\widetilde{\mathcal{G}}}$ such that for all compacts K of $\widetilde{\mathcal{G}}$*

$$\nu_{\widetilde{\mathcal{G}}}(A) = Q_{K,\widetilde{\mathcal{G}}}\big((p^*_{\widetilde{\mathcal{G}}})^{-1}(A)\big) \text{ for all } A \in \mathcal{W}^*_{\widetilde{\mathcal{G}}}, A \subset W^*_{K,\widetilde{\mathcal{G}}}. \tag{V.2.12}$$

Proof. The uniqueness is clear since W^*_K increases to W^* as K increases to $\widetilde{\mathcal{G}}$. Let us now fix some compacts K and K' of $\widetilde{\mathcal{G}}$ with $K \subset K'$. For all $A \in \mathcal{W}^0_K$, we denote by $A' = \{(w(t + H_{K'}))_{t\in\mathbb{R}},\, w \in A\}$. In order to prove (V.2.12), it is enough to prove that for all $A \in \mathcal{W}^0_K$ such that $A' \in \mathcal{W}^0_{K'}$

$$Q_K(A) = Q_{K'}(A'). \tag{V.2.13}$$

Indeed one can then define $\mathbb{1}_{W^*_K}\nu = Q_K \circ (p^*)^{-1}$ for all compacts K of $\widetilde{\mathcal{G}}$, and this definition is consistent by (V.2.13), and we can conclude by taking a sequence of compacts increasing to $\widetilde{\mathcal{G}}$. Using (V.2.11) and (V.2.8) we have

$$Q_K(A) = \sum_{x\in\partial K} \frac{1}{g(x,x)} P_x(A^+) P_x\big((X_{t+L_K})_{t\geq 0} \in A^-, X_{L_K} = x\big).$$

Taking $A^\pm = \{(w(t)_{t\in[0,H_K]} : \omega \in A'\}$, one can easily check that $(X_{t+L_K})_{t\geq 0} \in A^-$ if and only if $(X_{t+L_{K'}})_{t\geq 0} \in (A')^-$ and $(X_{-t+L_{K'}})_{t\in[0,L_{K'}-L_K]} \in A^\pm$. Therefore using (V.2.10) for K' and (V.2.4), we obtain that for all $x \in \partial K$

$$\begin{aligned}
&P_x\big((X_{t+L_K})_{t\geq 0} \in A^-, X_{L_K} = x\big)\\
&= \sum_{y\in\partial K'} e_{K'}(y) P^{K'}_y\big((A')^-\big) \int_0^\infty P_{x,y,s}\big((X_{s-t})_{t\in[0,s-L_K]} \in A^\pm, X_{L_K} = x\big) p_s(x,y)\,\mathrm{d}s\\
&= \sum_{y\in\partial K'} e_{K'}(y) P^{K'}_y\big((A')^-\big) \int_0^\infty P_{y,x,s}\big((X_t)_{t\in[0,H_K]} \in A^\pm, X_{H_K} = x\big) p_s(y,x)\,\mathrm{d}s.
\end{aligned}$$

Moreover by (V.2.3), we can write

$$\begin{aligned}&\int_0^\infty P_{y,x,s}\big((X_t)_{t\in[0,H_K]}\in A^\pm, X_{H_K}=x\big)p_s(y,x)\,\mathrm{d}s\\&=\int_0^\infty E_y\big[p_{s-H_K}(x,x)\mathbb{1}_{(X_t)_{t\in[0,H_K]}\in A^\pm,X_{H_K}=x}\big]\,\mathrm{d}s\\&=E_y\Big[\mathbb{1}_{(X_t)_{t\in[0,H_K]}\in A^\pm,X_{H_K}=x}\int_{H_K}^\infty p_{s-H_K}(x,x)\,\mathrm{d}s\Big]\\&=g(x,x)P_y\big((X_t)_{t\in[0,H_K]}\in A^\pm,X_{H_K}=x\big),\end{aligned}$$

where we used (V.2.2) in the last equality. Combining the previous equations, we thus obtain by the strong Markov property at time H_K that

$$\begin{aligned}Q_K(A)&=\sum_{x\in\partial K,y\in\partial K'}e_{K'}(y)P_x(A^+)P_y\big((X_t)_{t\in[0,H_K]}\in A^\pm,X_{H_K}=x\big)P_y^{K'}\big((A')^-\big)\\&=\sum_{x\in\partial K,y\in\partial K'}e_{K'}(y)P_y\big((A')^+,X_{H_K}=x\big)P_y^{K'}\big((A')^-\big)\\&=Q_{K'}(A'),\end{aligned}$$

where we used in the second equality the fact that $(X_t)_{t\geq0}\in(A')^+$ if and only if $(X_t)_{t\in[0,H_K]}\in A^\pm$ and $(X_{t+H_K})_{t\geq0}\in A^+$, and we can conclude. □

The measure $\nu_{\widetilde{\mathcal{G}}}$ from Proposition V.2.2 is the intensity measure underlying random interlacements on $\widetilde{\mathcal{G}}$, and as usual we then define ω the random interlacement process under some probability $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$ as a Poisson point process with intensity measure $\nu_{\widetilde{\mathcal{G}}}\otimes\lambda$, where λ is the Lebesgue measure on $(0,\infty)$. We simply write ν and $\mathbb{P}^I$ when there is no ambiguity about the choice of the graph $\mathcal{G}$. We also denote by ω^u the point process, which consist of the trajectories in ω with label less than u, by $(\ell_{x,u})_{x\in\widetilde{\mathcal{G}}}$ the continuous field of local times with respect to m on $\widetilde{\mathcal{G}}$ of ω_u and by $\mathcal{I}^u=\{x\in\widetilde{\mathcal{G}}:\ \ell_{x,u}>0\}$ the interlacement set at level u.

Remark V.2.3. Similarly as in (1.40) of [93] or (2.16) of [99], it is easy to show that random interlacements on the cable system are invariant under time reversal. Indeed for all connected compacts K of $\widetilde{\mathcal{G}}$ we have by (V.2.11), (V.2.10) and (V.2.5) that for all $A',A''\in W^+$ and $A'\in W^{+,f}$,

$$\begin{aligned}&Q_K\big((X_{-t})_{t\geq0}\in A,(X_t)_{t\in[0,L_K]}\in A',(X_t)_{t\geq L_K}\in A''\big)\\&=\sum_{x,y\in\partial K}e_K(x)e_K(y)P_y^K(A'')P_x^K(A)\int_0^\infty P_{x,y,s}(A')p_s(x,y)\,\mathrm{d}s\\&=Q_K\big((X_{-t})_{t\geq0}\in A'',(X_{L_K-t})_{t\in[0,L_K]}\in A',(X_t)_{t\geq L_K}\in A\big).\end{aligned}$$

Denoting $\check{\nu}$ the image of ν under time reversal, taking a sequence of compacts increasing to $\widetilde{\mathcal{G}}$, we thus directly obtain by (V.2.12) that

$$\nu=\check{\nu}.\tag{V.2.14}$$

V.3 Killed and surviving random interlacements

In this section, we introduce the notion of killed, or surviving, random interlacements, corresponding to the trajectories in the random interlacement process ω which are doubly killed before blowing up, or blows up doubly before being killed. We also present a direct construction of killed and surviving random interlacements, see (V.3.2) and (V.3.3), and prove another characterization of killed random interlacements when replacing doubly non-compact trajectories by forwards trajectories starting at a given vertex, see Proposition V.3.3 and (V.1.4). Similar results are also presented for backwards-killed and forwards-killed random interlacements.

For $w^* \in W^*_{\widetilde{\mathcal{G}}}$, we say that the forwards part of w^* has been killed before blowing up if there exists $w \in (p^*_{\widetilde{\mathcal{G}}})^{-1}(w^*)$ such that $w(0) \in \widetilde{\mathcal{G}}$ and the trajectory $(w(t))_{t\geq 0}$ has been killed before blowing up, using similar terminology as above (V.1.1), and similarly for the backwards part of w^* by considering $(w(-t))_{t\geq 0}$ instead. Let $W^{\mathcal{K},*}_{\widetilde{\mathcal{G}}}$ be the set of doubly non-compact trajectories modulo time-shift whose forwards and backwards parts have been killed before blowing up, $W^{\mathcal{S},*}_{\widetilde{\mathcal{G}}}$ be the set of doubly non-compact trajectories modulo time-shift whose forwards and backwards parts blow up before being killed, $W^{\mathcal{KS},*}_{\widetilde{\mathcal{G}}}$ be the set of doubly non-compact trajectories modulo time-shift whose backwards parts have been killed before blowing up and forwards parts blow up before being killed, and $W^{\mathcal{SK},*}_{\widetilde{\mathcal{G}}}$ be the set of doubly non-compact trajectories modulo time-shift whose forwards parts have been killed before blowing up and backwards parts blow up before being killed. We also define similarly the subsets $W^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$, $W^{\mathcal{S}}_{\widetilde{\mathcal{G}}}$, $W^{\mathcal{KS}}_{\widetilde{\mathcal{G}}}$ and $W^{\mathcal{SK}}_{\widetilde{\mathcal{G}}}$ of the set $W_{\widetilde{\mathcal{G}}}$ of doubly non-compact trajectories, $W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}$ the set of forwards trajectories in $W^+_{\widetilde{\mathcal{G}}}$ which are killed before blowing up, and $W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}$ the set of forwards trajectories in $W^+_{\widetilde{\mathcal{G}}}$ which blow up before being killed, and we denote by $\mathcal{W}^{\cdot}_{\widetilde{\mathcal{G}}}$ all the associated σ-algebras, generated by the coordinate functions.

Definition V.3.1. Let $\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}} := (\nu_{\widetilde{\mathcal{G}}})_{|W^{\mathcal{K},*}_{\widetilde{\mathcal{G}}}}$ the measure underlying killed random interlacements, $\nu^{\mathcal{S}}_{\widetilde{\mathcal{G}}} := (\nu_{\widetilde{\mathcal{G}}})_{|W^{\mathcal{S},*}_{\widetilde{\mathcal{G}}}}$ the measure underlying surviving random interlacements, $\nu^{\mathcal{KS}}_{\widetilde{\mathcal{G}}} := (\nu_{\widetilde{\mathcal{G}}})_{|W^{\mathcal{KS},*}_{\widetilde{\mathcal{G}}}}$ the measure underlying backwards-killed random interlacements and $\nu^{\mathcal{SK}}_{\widetilde{\mathcal{G}}} := (\nu_{\widetilde{\mathcal{G}}})_{|W^{\mathcal{SK},*}_{\widetilde{\mathcal{G}}}}$ the measure underlying forwards-killed random interlacements. We also define the killed random interlacement process $\omega^{\mathcal{K}}$ under some probability space $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}$, as a Poisson point process with intensity $\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}} \otimes \lambda$, where λ is the Lebesgue measure on $\mathbb{R}_+$, $\omega^{\mathcal{K}}_u$ the Poisson point process which consist of the trajectories in $\omega^{\mathcal{K}}$ with label at most $u > 0$, $(\ell^{\mathcal{K}}_{x,u})_{x\in\widetilde{\mathcal{G}}}$ the field of total local times with respect to m of all the trajectories in $\omega^{\mathcal{K}}_u$ and

$\mathcal{I}^u_{\mathcal{K}} := \{x \in \widetilde{\mathcal{G}} : \ell^{\mathcal{K}}_{x,u} > 0\}$ the killed random interlacement set on $\widetilde{\mathcal{G}}$. We use similar notations for surviving random interlacements under $\mathbb{P}^{SI}_{\widetilde{\mathcal{G}}}$, backwards-killed random interlacements under $\mathbb{P}^{KSI}_{\widetilde{\mathcal{G}}}$, and forwards-killed random interlacements under $\mathbb{P}^{SKI}_{\widetilde{\mathcal{G}}}$.

Note that by definition, we have that

$$\begin{gathered}\omega \text{ has the same law under } \mathbb{P}^I_{\widetilde{\mathcal{G}}} \text{ as}\\ \omega^{\mathcal{K}} + \omega^{\mathcal{S}} + \omega^{\mathcal{KS}} + \omega^{\mathcal{SK}} \text{ under } \mathbb{P}^{KI}_{\widetilde{\mathcal{G}}} \otimes \mathbb{P}^{SI}_{\widetilde{\mathcal{G}}} \otimes \mathbb{P}^{KSI}_{\widetilde{\mathcal{G}}} \otimes \mathbb{P}^{SKI}_{\widetilde{\mathcal{G}}},\end{gathered}$$

and that $\omega^{\mathcal{K}} = \omega^{\mathcal{KS}} = \omega^{\mathcal{SK}} = 0$ if $\mathbf{h}_0 \equiv 0$, i.e. $\kappa \equiv 0$, and $\omega^{\mathcal{S}} = \omega^{\mathcal{KS}} = \omega^{\mathcal{SK}} = 0$ if $\mathbf{h}_0 \equiv 1$, where $\mathbf{h}_0(x) = P^{\widetilde{\mathcal{G}}}_x(W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}})$ was defined in (V.1.1). If $\kappa \not\equiv 0$, one can directly describe the law of killed random interlacements as follows. We say that the Markov jump process Z is killed before blowing up if Z is killed by the killing measure κ at time $\zeta < \infty$, and we denote by $W^{\mathcal{K},+}_{\mathcal{G}}$ the corresponding event. For any finite sets $K \subset G$, let us define similarly as in (V.2.7) the killed equilibrium measure by

$$e^{\mathcal{K}}_{K,\mathcal{G}}(x) \stackrel{\text{def.}}{=} \lambda_x \mathbf{h}_0(x) P^{\mathcal{G}}_x\big(\widetilde{H}_K = \zeta, W^{\mathcal{K},+}_{\mathcal{G}}\big) \text{ for all } x \in \partial K. \tag{V.3.1}$$

This definition of the killed equilibrium measure can be extended to an equilibrium measure $e^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ on the cable system $\widetilde{\mathcal{G}}$ for any compacts K of $\widetilde{\mathcal{G}}$ by considering the graph $G^{\partial K}$ with $\partial K \subset G$, similarly as for the equilibrium measure on $\widetilde{\mathcal{G}}$, see Lemma IV.2.1 and (IV.2.18). Let now $Q^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ be the probability measure on $\mathcal{W}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$, whose restriction to $\mathcal{W}^0_{K,\widetilde{\mathcal{G}}} \cap \mathcal{W}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$ is given by

$$Q^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}} \stackrel{\text{def.}}{=} \sum_{x \in \partial K} e^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}(x) P^{\widetilde{\mathcal{G}}}_x(\cdot^+ \,|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}) P^{K,\widetilde{\mathcal{G}}}_x(\cdot^- \,|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}), \tag{V.3.2}$$

and such that $Q^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}(A) = 0$ for all $A \in W^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$ with $A \cap W^0_{K,\widetilde{\mathcal{G}}} \cap W^{\mathcal{K}}_{\widetilde{\mathcal{G}}} = \varnothing$. Since $P^{K,\widetilde{\mathcal{G}}}_x(W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}) = P^{\mathcal{G}}_x(W^{\mathcal{K},+}_{\mathcal{G}} \,|\, \widetilde{H}_K = \zeta)$ for all $K \subset G$ and $x \in K$ with $P^{\mathcal{G}}_x(\widetilde{H}_K = \zeta) > 0$, one can easily show by (V.2.12) and Definition V.3.1 that for all compacts K of $\widetilde{\mathcal{G}}$

$$\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(A) = Q^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}\big((p^*_{\widetilde{\mathcal{G}}})^{-1}(A)\big) \text{ for all } A \in \mathcal{W}^*_{\widetilde{\mathcal{G}}}, A \subset W^{\mathcal{K},*}_{K,\widetilde{\mathcal{G}}}, \tag{V.3.3}$$

with $W^{\mathcal{K},*}_{K,\widetilde{\mathcal{G}}} = W^*_{K,\widetilde{\mathcal{G}}} \cap W^{\mathcal{K},*}_{\widetilde{\mathcal{G}}}$. In other words, the restriction of $\omega^{\mathcal{K}}_u$ to the trajectories hitting a compact K of $\widetilde{\mathcal{G}}$ can be described as follows: for each $x \in \partial K$, there are $\mathrm{Poi}(ue^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}(x))$ independent trajectories hitting K for the first time in x, and each of these trajectories has a forwards part which behave like a Brownian motion on $\widetilde{\mathcal{G}}$ conditioned on being killed before blowing up, and a backwards part which behave like a Brownian motion on $\widetilde{\mathcal{G}}$ conditioned on never coming back in K and being killed before blowing up. We also define the killed capacity of a compact K of $\widetilde{\mathcal{G}}$ by

$$\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(K) \stackrel{\mathrm{def.}}{=} \sum_{x\in\partial K} e^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}(x), \tag{V.3.4}$$

and the killed interlacement set is then characterized by the following identity

$$\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}(\mathcal{I}^u_{\mathcal{K}}\cap K=\varnothing)=\exp\Big(-u\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(K)\Big). \tag{V.3.5}$$

One can also give similar definitions and results for surviving random interlacements by replacing $\mathbf{h}_0$ by $\mathbf{h}_1$ and $W^{\mathcal{K},\cdot}_{\widetilde{\mathcal{G}}}$ by $W^{\mathcal{S},\cdot}_{\widetilde{\mathcal{G}}}$, or for backwards-killed and forwards-killed random interlacements. Note that since random interlacements are invariant under time reversal, see (V.2.14), killed and surviving random interlacements are also invariant under time reversal, and the time reversal of backwards-killed interlacements is forwards-killed interlacements. In particular, the law of the number of trajectories hitting a given set is the same for backwards-killed and forwards-killed interlacements, that is backwards-killed and forwards-killed capacity are equal: for all finite $K\subset G$,

$$\begin{aligned}\mathrm{cap}^{\mathcal{KS}}_{\widetilde{\mathcal{G}}}(K) &\stackrel{\mathrm{def.}}{=} \sum_{x\in K}\lambda_x\mathbf{h}_1(x)P^{\mathcal{G}}_x\big(\widetilde{H}_K=\zeta,W^{\mathcal{K},+}_{\mathcal{G}}\big)\\ &=\sum_{x\in K}\lambda_x\mathbf{h}_0(x)P^{\mathcal{G}}_x\big(\widetilde{H}_K=\zeta,W^{\mathcal{S},+}_{\mathcal{G}}\big) \stackrel{\mathrm{def.}}{=} \mathrm{cap}^{\mathcal{SK}}_{\widetilde{\mathcal{G}}}(K).\end{aligned} \tag{V.3.6}$$

This equality can easily be generalized to any compacts K of $\widetilde{\mathcal{G}}$ by considering the graph $\mathcal{G}^{\partial K}$ from Lemma IV.2.1, and one could also prove it directly using the last exit decomposition (V.2.10). As usual, we will simply remove the subscript $\widetilde{\mathcal{G}}$ to all the previous definitions when there is no ambiguity about the graph the choice of the graph $\mathcal{G}$.

Let $\widetilde{\mathcal{G}}^E$ be the subset of $\widetilde{\mathcal{G}}$ consisting of only the edges I_e for $e\in E$, that is removing the edge I_x starting from x for all $x\in G$. There is an easier way to directly describe the restriction to $\widetilde{\mathcal{G}}^E$ of killed, backwards-killed, or forwards-killed random interlacements, instead of describing them through their restriction to compacts as in (V.3.3). Before giving this description, let us begin with an intermediate lemma. In essence, it states that, starting with the " killed equilibrium measure of G", that is $\kappa\mathbf{h}_0$, the probability to hit U in y and then to be killed before blowing up is the killed equilibrium measure of U in y, which bears some similarity with (IV.2.23).

Lemma V.3.2. *For all $y\in G$ and all finite $U\subset G$ we have*

$$\sum_{x\in G}\kappa_xP^{\mathcal{G}}_x(H_U<\zeta,Z_{H_U}=y)=\lambda_yP^{\mathcal{G}}_y(\widetilde{H}_U=\zeta,W^{\mathcal{K},+}_{\mathcal{G}})\mathbb{1}_{y\in U}, \tag{V.3.7}$$

and

$$\sum_{x\in G}\kappa_x P_x^{\mathcal{G}}(H_U<\zeta,W_{\mathcal{G}}^{\mathcal{K},+})=\mathrm{cap}_{\widetilde{\mathcal{G}}}^{\mathcal{K}}(U). \tag{V.3.8}$$

Proof. Let us denote by $(\widehat{Z}_n)_{n\in\mathbb{N}_0}$ the discrete skeleton of the jump process Z on $\mathcal{G}$, which is equal to Δ after being killed. Let us fix some sequence U_n, $n\in\mathbb{N}$, of finite subsets of G increasing to G, and let

$$\widehat{L}_{U_n}\stackrel{\text{def.}}{=}\inf\{k\geq 0:\ \widehat{Z}_k\in U_n,\widehat{Z}_{k+1}\neq\Delta,\widehat{Z}_p\notin U_n\text{ for all }p\geq n+1\},$$

with the convention $\inf\varnothing=\infty$. By the Markov property, we have for all $x\in U_n$

$$\begin{aligned}P_y^{\mathcal{G}}(\widehat{Z}_{\widehat{L}_{U_n}}=x,\widehat{L}_{U_n}<\infty)&=\sum_{k\geq 0}P_y^{\mathcal{G}}(\widehat{Z}_k=x,\widehat{Z}_{k+1}\neq\Delta,\widehat{Z}_p\notin U_n\text{ for all }p\geq n+1)\\&=\sum_{k\geq 0}P_y^{\mathcal{G}}(\widehat{Z}_k=x)P_x^{\mathcal{G}}(\widehat{Z}_1\neq\Delta,\widehat{Z}_p\notin U_n\text{ for all }p\geq 1)\\&=g(y,x)(e_{U_n}(x)-\kappa_x).\end{aligned}$$

Therefore we obtain by (IV.2.21), that for all $n\in\mathbb{N}$ large enough such that $y\in U_n$

$$\begin{aligned}\sum_{x\in U_n}\kappa_x g(x,y)&=\sum_{x\in U_n}g(y,x)e_{U_n}(x)-\sum_{x\in U_n}P_y^{\mathcal{G}}(\widehat{Z}_{\widehat{L}_{U_n}}=x,\widehat{L}_{U_n}<\infty)\\&=1-P_y^{\mathcal{G}}(\widehat{L}_{U_n}<\infty).\end{aligned}$$

Note that $\widehat{L}_{U_n}<\infty$ for all n large enough if and only if the trajectory blows up before being killed, and thus

$$\sum_{x\in G}\kappa_x g(x,y)=\lim_{n\to\infty}1-P_y^{\mathcal{G}}(\widehat{L}_{U_n}<\infty)=1-P_y^{\mathcal{G}}(W_{\mathcal{G}}^{\mathcal{S},+})=\mathbf{h}_0(y). \tag{V.3.9}$$

Now we have for all $z\in G$ that

$$\sum_{y\in G}g(z,y)\sum_{x\in G}\kappa_x P_x^{\mathcal{G}}(H_U<\zeta,Z_{H_U}=y)=\sum_{x\in G}\kappa_x E_x^{\mathcal{G}}[g(Z_{H_U},z)\mathbb{1}_{H_U<\zeta}].$$

Moreover by Hunt's switching identity, see for instance (1.50) in [98], we have $E_x^{\mathcal{G}}[g(Z_{H_U},z)\mathbb{1}_{H_U<\zeta}]=E_z^{\mathcal{G}}[g(Z_{H_U},x)\mathbb{1}_{H_U<\zeta}]$, and so using (V.3.9) we obtain

$$\begin{aligned}\sum_{y\in G}g(z,y)\sum_{x\in G}\kappa_x P_x^{\mathcal{G}}(H_U<\zeta,Z_{H_U}=y)&=E_z^{\mathcal{G}}\left[\sum_{x\in G}\kappa_x g(Z_{H_U},x)\mathbb{1}_{H_U<\zeta}\right]\\&=E_z^{\mathcal{G}}\left[\mathbf{h}_0(Z_{H_U})\mathbb{1}_{H_U<\zeta}\right]\\&=P_z^{\mathcal{G}}(H_U<\zeta,W_{\mathcal{G}}^{\mathcal{K},+}),\end{aligned}$$

where we used the strong Markov property at time H_U in the last equality. Let us now define

$$L_U \overset{\text{def.}}{=} \inf\{k \geq 0 : \widehat{Z}_k \in U, \widehat{Z}_p \notin U \text{ for all } p \geq k+1\},$$

with the convention $\inf \varnothing = \infty$. By simple Markov property, we have for all $z \in G$

$$\begin{aligned}
\sum_{y\in G} g(z,y)\lambda_y P_y^{\mathcal{G}}(\widetilde{H}_U = \zeta, W_{\mathcal{G}}^{\mathcal{K},+})\mathbb{1}_{y\in U} &= \sum_{y\in U}\sum_{k\geq 0} P_z^{\mathcal{G}}(\widehat{Z}_k = y) P_y^{\mathcal{G}}(\widetilde{H}_U = \zeta, W_{\mathcal{G}}^{\mathcal{K},+}) \\
&= \sum_{y\in U}\sum_{k\geq 0} P_z^{\mathcal{G}}(\widehat{Z}_k = y, L_U = k, W_{\mathcal{G}}^{\mathcal{K},+}) \\
&= P_z^{\mathcal{G}}(L_U < \infty, W_{\mathcal{G}}^{\mathcal{K},+}).
\end{aligned}$$

Noting that $\{L_U < \infty\} = \{H_U < \zeta\}$, and that the operator $Gf(x) := \sum_{x\in G} g(x,y)f(x)$ is invertible, see for instance (1.37) in [98], we obtain (V.3.7). The equality (V.3.8) follows directly since by the strong Markov property, (V.3.1) and (V.3.4) we have

$$\begin{aligned}
\sum_{x\in G} \kappa_x P_x^{\mathcal{G}}(H_U < \zeta, W_{\mathcal{G}}^{\mathcal{K},+}) &= \sum_{y\in U}\sum_{x\in G} \kappa_x P_x^{\mathcal{G}}(H_U < \zeta, Z_{H_U} = y, W_{\mathcal{G}}^{\mathcal{K},+}) \\
&= \sum_{y\in U} \mathbf{h}_0(y) \sum_{x\in G} \kappa_x P_x^{\mathcal{G}}(H_U < \zeta, Z_{H_U} = y) \\
&= \sum_{y\in U} \lambda_y \mathbf{h}_0(y) P_y^{\mathcal{G}}(\widetilde{H}_U = \zeta, W_{\mathcal{G}}^{\mathcal{K},+}) \\
&= \mathrm{cap}^{\mathcal{K}}(U).
\end{aligned}$$

□

Lemma V.3.2 let us derive another description of killed, backwards-killed and forwards-killed random interlacements. For all $x \in G$ with $\kappa_x > 0$, we denote by $P_x^{I_x^c,\widetilde{\mathcal{G}}}$ the law of $(X_{t+L_{I_x^c}})_{t\geq 0}$ under $P_x(\cdot \,|\, X_{L_{I_x^c}} = x, L_{I_x^c} < \widetilde{\zeta})$, which has the same law as a $\mathrm{BES}^3(0)$ process on I_x starting in x and stopped when reaching the open end of I_x, see for instance Theorem 4.5, Chapter XII in [75]. Similarly as above (V.2.11), we define $W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$ as the set of trajectories in W hitting $\widetilde{\mathcal{G}}^E$ at time 0 for the first time, the set $\mathcal{W}^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$ of $B \in \mathcal{W}$, $B \subset W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$, which can be uniquely decomposed into an event B^+ concerning the forwards part of the trajectories and an event B^- concerning the backwards part of the trajectories, and the set $W^{\mathcal{K},*}_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}} = p^*(W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}} \cap W^{\mathcal{K}})$. We then have that $\mathcal{W}^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$ and $\{A \in \mathcal{W} : A \cap W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}} = \varnothing\}$ generate $\mathcal{W}$.

Proposition V.3.3. *Let* $\widetilde{\nu}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$, $\widetilde{\nu}^{\mathcal{KS}}_{\widetilde{\mathcal{G}}}$ *and* $\widetilde{\nu}^{\mathcal{SK}}_{\widetilde{\mathcal{G}}}$ *be the probabilities on* $(W_{\widetilde{\mathcal{G}}},\mathcal{W}_{\widetilde{\mathcal{G}}})$ *given on* $\mathcal{W}^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$ *by*

$$\widetilde{\nu}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}\stackrel{\text{def.}}{=}\sum_{x\in G}\kappa_x\mathbf{h}_0(x)P_x^{\widetilde{\mathcal{G}}}\big(\cdot^+_{\widetilde{\mathcal{G}}^E}\,|\,\mathcal{W}^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}\big)P_x^{I_x^c,\widetilde{\mathcal{G}}}\big(\cdot^-_{\widetilde{\mathcal{G}}^E}\big),$$
$$\widetilde{\nu}^{\mathcal{KS}}_{\widetilde{\mathcal{G}}}\stackrel{\text{def.}}{=}\sum_{x\in G}\kappa_x\mathbf{h}_1(x)P_x^{\widetilde{\mathcal{G}}}\big(\cdot^+_{\widetilde{\mathcal{G}}^E}\,|\,\mathcal{W}^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}\big)P_x^{I_x^c,\widetilde{\mathcal{G}}}\big(\cdot^-_{\widetilde{\mathcal{G}}^E}\big),$$
$$\widetilde{\nu}^{\mathcal{SK}}_{\widetilde{\mathcal{G}}}\stackrel{\text{def.}}{=}\sum_{x\in G}\kappa_x\mathbf{h}_1(x)P_x^{I_x^c,\widetilde{\mathcal{G}}}\big(\cdot^+_{\widetilde{\mathcal{G}}^E}\big)P_x^{\widetilde{\mathcal{G}}}\big(\cdot^-_{\widetilde{\mathcal{G}}^E}\,|\,\mathcal{W}^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}\big),$$

and such that $\widetilde{\nu}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}=\widetilde{\nu}^{\mathcal{KS}}_{\widetilde{\mathcal{G}}}=\widetilde{\nu}^{\mathcal{SK}}_{\widetilde{\mathcal{G}}}=0$ *for all* $A\in\mathcal{W}_{\widetilde{\mathcal{G}}}$ *with* $A\cap W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}=\varnothing$. *Then*

$$\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(A)=\widetilde{\nu}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}\big((p^*_{\widetilde{\mathcal{G}}})^{-1}(A)\big)\ \text{for all } A\in\mathcal{W}^*_{\widetilde{\mathcal{G}}}, A\subset W^{\mathcal{K},*}_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}},\tag{V.3.10}$$

and similarly for backwards-killed and forwards-killed random interlacements.

Proof. Let us first consider killed random interlacements. Let K be a compact of $\widetilde{\mathcal{G}}$ such that $K\subset\widetilde{\mathcal{G}}^E$ and $\partial K\subset G$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compacts increasing to $\widetilde{\mathcal{G}}^E$ such that for all $n\in\mathbb{N}$ $K\subset K_n$ and $\partial K_n\subset G$. If $A\in\mathcal{W}$ with $A\subset W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}:=(p^*)^{-1}(W^*_{K,\widetilde{\mathcal{G}}})\cap W^{\mathcal{K}}$, we write $A_{\widetilde{\mathcal{G}}_E}=\{(w(t+H_{\widetilde{\mathcal{G}}^E}))_{t\in\mathbb{R}}:\,w\in A\}$. For all $n\in\mathbb{N}$ and $A\in\mathcal{W}^0_{K_n,\widetilde{\mathcal{G}}}$ such that $A\subset W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ and $A_{\widetilde{\mathcal{G}}^E}\in\mathcal{W}^0_{\widetilde{\mathcal{G}}_E,\widetilde{\mathcal{G}}}$, we have by (V.3.2)

$$\begin{aligned}Q^{\mathcal{K}}_{K_n}(A)-\widetilde{\nu}^{\mathcal{K}}(A_{\widetilde{\mathcal{G}}^E})=&\sum_{x\in\partial K_n}e^{\mathcal{K}}_{K_n}(x)P_x(A^+\,|\,W^{\mathcal{K},+})P_x^{K_n}(A^-\,|\,W^{\mathcal{K},+})\\&-\sum_{x\in G}\kappa_x\mathbf{h}_0(x)P_x(A^+_{\widetilde{\mathcal{G}}^E}\,|\,W^{\mathcal{K},+})P_x^{I_x^c}(A^-_{\widetilde{\mathcal{G}}^E}).\end{aligned}$$

Let us define $\partial_{ext}K_n=\{x\in\partial K_n:\,\exists\,y\in G\cap K_n^c,\,y\sim x\}$ for each $n\in\mathbb{N}$, as well as a measure μ_n on $\mathcal{W}^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}:=\{A\in\mathcal{W}^{\mathcal{K}}:\,A\subset W_{K,\widetilde{\mathcal{G}}}\}$, which is given for all $A\in\mathcal{W}^0_{K_n,\widetilde{\mathcal{G}}}$ such that $A\subset W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ and $A_{\widetilde{\mathcal{G}}^E}\in\mathcal{W}^0_{\widetilde{\mathcal{G}}_E,\widetilde{\mathcal{G}}}$ by

$$\begin{aligned}\mu_n(A)\stackrel{\text{def.}}{=}&\sum_{x\in\partial_{ext}K_n}e^{\mathcal{K}}_{K_n}(x)P_x(A^+\,|\,W^{\mathcal{K},+})P_x^{K_n}(A^-\,|\,W^{\mathcal{K},+})\\&+\sum_{x\in K_n^c\cup\partial_{ext}K_n}\kappa_x\mathbf{h}_0(x)P_x(A^+_{\widetilde{\mathcal{G}}^E}\,|\,W^{\mathcal{K},+})P_x^{I_x^c}(A^-_{\widetilde{\mathcal{G}}^E}),\end{aligned}$$

and $\mu_n(A)=0$ if $A\cap W^0_{K_n,\widetilde{\mathcal{G}}}=\varnothing$. Such sets A generate $\mathcal{W}^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$, and so this is enough to define μ_n. For all $x\in\partial K_n\setminus\partial_{ext}K_n$, we have

$$e^{\mathcal{K}}_{K_n}(x)=\lambda_x\mathbf{h}_0(x)P_x^{\mathcal{G}}(\widetilde{H}_{K_n}=\zeta,W^{\mathcal{K},+}_{\mathcal{G}})=\lambda_x\mathbf{h}_0(x)P_x^{\mathcal{G}}(\widehat{Z}_1=\Delta)=\kappa_x\mathbf{h}_0(x),$$

$A_{\widetilde{\mathcal{G}}^E}=A$ and $P_x^{K_n}(\cdot\,|\,W^{\mathcal{K},+})=P_x^{I_x^c}$ if $\kappa_x>0$. Therefore we obtain that

$$|Q^{\mathcal{K}}_{K_n}(A)-\widetilde{\nu}^{\mathcal{K}}(A_{\widetilde{\mathcal{G}}^E})|\leq \mu_n(A), \tag{V.3.11}$$

for all $A\in\mathcal{W}^0_{K_n,\widetilde{\mathcal{G}}}$ such that $A\subset W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ and $A_{\widetilde{\mathcal{G}}^E}\in\mathcal{W}^0_{\widetilde{\mathcal{G}}_E,\widetilde{\mathcal{G}}}$. Moreover, we have by (V.3.2), (V.3.3) and (V.3.8)

$$\begin{aligned}Q^{\mathcal{K}}_{K_n}(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}})=Q^{\mathcal{K}}_{K}(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}})&=\mathrm{cap}^{\mathcal{K}}(\partial K)\\&=\sum_{x\in G}\kappa_xP_x(H_K<\widetilde{\zeta},W^{\mathcal{K},+})\\&=\widetilde{\nu}^{\mathcal{K}}\big(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}\big),\end{aligned} \tag{V.3.12}$$

and so (V.3.11) holds for for all $A\in\mathcal{W}^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ with $A\subset W^0_{K_n,\widetilde{\mathcal{G}}}$ by the π-lambda theorem, and we obtain that

$$|Q^{\mathcal{K}}_{K_n}(A)-\widetilde{\nu}^{\mathcal{K}}(A_{\widetilde{\mathcal{G}}^E})|\leq \mu_n(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}})\text{ for all }A\in\mathcal{W}^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}\text{ with }A\subset W^0_{K_n,\widetilde{\mathcal{G}}}. \tag{V.3.13}$$

Since $(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}\cap W^0_{K_n,\widetilde{\mathcal{G}}})^+=\{X_0\in K_n,H_K<\widetilde{\zeta}\}\cap W^{\mathcal{K},+}$, $(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}\cap W^0_{K_n,\widetilde{\mathcal{G}}})^-=\{X_0\in K_n\}\cap W^{\mathcal{K},+}$, and similarly when considering $W^0_{\widetilde{\mathcal{G}}^E,\widetilde{\mathcal{G}}}$, we have

$$\begin{aligned}\mu_n(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}})=&\sum_{x\in\partial_{ext}K_n}e^{\mathcal{K}}_{K_n}(x)P^{\mathcal{G}}_x(H_K<\zeta\,|\,W^{\mathcal{K},+}_{\mathcal{G}})\\&+\sum_{x\in K_n^c\cup\partial_{ext}K_n}\kappa_xP_x(H_K<\widetilde{\zeta},W^{\mathcal{K},+}).\end{aligned}$$

Using again the equality $e^{\mathcal{K}}_{K_n}(x)=\kappa_x\mathbf{h}_0(x)$ for all $x\in\partial K_n\setminus\partial_{ext}K_n$, we have by (V.3.2) and (V.3.12)

$$\begin{aligned}&\sum_{x\in\partial_{ext}K_n}e^{\mathcal{K}}_{K_n}(x)P^{\mathcal{G}}_x(H_K<\zeta\,|\,W^{\mathcal{K},+}_{\mathcal{G}})\\&=Q^{\mathcal{K}}_{K_n}(W^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}})-\sum_{x\in\partial K_n\setminus\partial_{ext}K_n}\kappa_xP^{\mathcal{G}}_x(H_K<\zeta,W^{\mathcal{K},+}_{\mathcal{G}})\\&=\sum_{x\in K_n^c\cup\partial_{ext}K_n}\kappa_xP^{\mathcal{G}}_x(H_K<\zeta,W^{\mathcal{K},+}_{\mathcal{G}}).\end{aligned}$$

We obtain for all $A\in\mathcal{W}^{\mathcal{K}}_{K,\widetilde{\mathcal{G}}}$ with $A\subset W^0_{K_n,\widetilde{\mathcal{G}}}$, that by (V.3.13) and (V.3.8)

$$|Q^{\mathcal{K}}_{K_n}(A)-\widetilde{\nu}^{\mathcal{K}}(A_{\widetilde{\mathcal{G}}^E})|\leq 2\sum_{x\in K_n^c\cup\partial_{ext}K_n}\kappa_xP_x(H_K<\widetilde{\zeta},W^{\mathcal{K},+})\underset{n\to\infty}{\longrightarrow}0.$$

Using (V.3.3), we thus have that (V.3.10) hold for all $A\in\mathcal{W}_{\widetilde{\mathcal{G}}^*}$ such that $A\subset W^{\mathcal{K},*}_{K,\widetilde{\mathcal{G}}}$. Since this is true for any compacts $K\subset\widetilde{\mathcal{G}}^E$ with $\partial K\subset G$, we obtain (V.3.10) by the π-lambda theorem. For backwards-killed random interlacements, one can easily obtain from (V.3.7) an equality similar to (V.3.8) but for $\mathrm{cap}^{\mathcal{K}\mathcal{S}}(U)$ by replacing $W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}$ by $W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}$, and the rest of the proof is similar. One can then also conclude for forwards-killed random interlacements using that by (V.2.14) the forwards-killed random interlacement measure is the image under time reversal of the backwards-killed random interlacement measure. □

Proposition V.3.3 provides us with an interesting description of killed interlacements. Indeed, if one is only interested in the killed interlacement process $\omega_u^{\widetilde{\mathcal{G}}^E,\mathcal{K}}$ on $\widetilde{\mathcal{G}}^E$, i.e. the print on $\widetilde{\mathcal{G}}^E$ of each forwards part on hitting $\widetilde{\mathcal{G}}^E$ of the trajectories in the killed interlacement process $\omega_u^{\mathcal{K}}$, then its law can be described as follows: for each $x \in G$, take a Poisson number of trajectories with parameter $u\kappa_x\mathbf{h}_0(x)$, each independent and with law $P_x^{\widetilde{\mathcal{G}}^E}(\cdot \,|\, W^{\mathcal{K},+})$, then the point process which consist of all these trajectories modulo time-shift has the same law as $\omega_u^{\widetilde{\mathcal{G}}^E,\mathcal{K}}$ under $\mathbb{P}^{KI}$. The description (V.1.4) of the discrete killed random interlacement process $\omega_u^{\mathcal{K},\mathcal{G}}$, the print of $\omega_u^{\mathcal{K}}$ on $\mathcal{G}$ follows also directly from Proposition V.3.3. The restriction of $\mathcal{I}^u_{\mathcal{K}}$ to G has thus the same law as the set of vertices reached by a Poisson point process of trajectories with intensity $u\widetilde{\nu}^{\mathcal{K}}_{\mathcal{G}}$, which could be directly proved by (V.3.5) and (V.3.8). Similar descriptions can also be obtained for backwards-killed and forwards-killed random interlacements. Note that finitary interlacements, as introduced in [15], are a special case of killed random interlacements, and (V.1.4) can be seen as generalization of Proposition 4.1 in [15].

Remark V.3.4. 1) One could also prove (V.3.10) similarly as (V.2.13), but replacing Q_K by $Q_K^{\mathcal{K}}$ and $Q_{K'}$ by $\widetilde{\nu}^{\mathcal{K}}$, and using that the killed equilibrium measure of G, defined similarly as in (V.3.1) for $K = G$, is equal to $\kappa\mathbf{h}_0$. One would also need to extend the decomposition (V.2.10) to include the case $K = G$, and then the general strategy is very similar to the proof of Proposition V.2.2. We chose to present another proof here, which consists of taking a sequence of compacts K_n, $n \in \mathbb{N}$, increasing to $\widetilde{\mathcal{G}}^E$, and show that $Q^{\mathcal{K}}_{K_n}$ increases to $\widetilde{\nu}^{\mathcal{K}}$.

2) If one applies (V.3.10) to a new graph $\mathcal{G}'$ which is like $\mathcal{G}$, plus an additional vertex x' on each I_x, $x \in G$, then (V.3.10) describes the law of $\nu^{\mathcal{K}}$ on $(\widetilde{\mathcal{G}}')^E$, that is on $\widetilde{\mathcal{G}}^E$ and on $[x, x'](\subset I_x)$, $x \in G$. We can approximate the whole cable system $\widetilde{\mathcal{G}}$ in that way by letting $[x, x']$ increase to I_x for all $x \in G$, and thus (V.3.10) is enough to obtain the complete law of $\nu^{\mathcal{K}}$. One cannot however find a direct description similar to (V.3.10) for the complete law of $\nu^{\mathcal{K}}$ since for all $x \in G$ with $\kappa_x > 0$, $\nu^{\mathcal{K}}(W_{I_x^*}) = \mathrm{cap}^{\mathcal{K}}(I_x) = \infty$, by a similar argument as in (IV.4.4), and so there is an infinite number of trajectories in

the killed random interlacement process starting at the open end of I_x.

3) When $\mathbf{h}_0 \equiv 1$, for instance on finite transient graphs or on graphs with bounded degree, constant weights and constant killing measure, killed random interlacements and random interlacements coincide. Therefore, (V.3.10) then provides us with a description of the restriction of random interlacements to $\widetilde{\mathcal{G}}^E$ and (V.1.4) with a description of discrete random interlacements.

V.4 Doob h-transform

In this section, we introduce the notion of the Doob $\mathbf{h}$ transform $\mathcal{G}_{\mathbf{h}}$ of a graph $\mathcal{G}$, when $\mathbf{h} : \widetilde{\mathcal{G}} \to (0,\infty)$ is an harmonic function, so that the diffusion X on the cable system $\widetilde{\mathcal{G}}_{\mathbf{h}}$ of $\mathcal{G}_{\mathbf{h}}$ is related to the $\mathbf{h}$-transform of the diffusion X on $\widetilde{\mathcal{G}}$, see Lemma V.4.1. In particular, if $\mathbf{h} = \mathbf{h}_0$, then the diffusion X on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ is related to the diffusion X conditioned on being killed before blowing up, and if $\mathbf{h} = \mathbf{h}_1$, conditioned on blowing before being killed, see (V.4.4). One can then also relate the law of random interlacements on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ to killed random interlacements on $\widetilde{\mathcal{G}}$, and the Gaussian free field on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ to the Gaussian free field on $\widetilde{\mathcal{G}}$, and similarly when $\mathbf{h} = \mathbf{h}_1$ for surviving random interlacements, see (V.4.5) and (V.4.6). Similar relations also hold for the field of local times associated to X, or to random interlacements, see Corollary V.4.5. Therefore, one can use the results from Chapter IV about the Gaussian free field and local times of random interlacements on $\mathcal{G}_{\mathbf{h}_0}$, to obtain similar results about the Gaussian free field and local times of killed random interlacements on $\mathcal{G}$, or surviving random interlacements when $\mathbf{h} = \mathbf{h}_1$, see (V.4.14) and Theorem V.4.6. Finally, we use these results for surviving random interlacements on a suitable graph to obtain the isomorphism between random interlacements not hitting K and the Gaussian free field, Theorem V.1.3.

We first define the Doob $\mathbf{h}$-transform, or $\mathbf{h}$-transform, of the graph $\mathcal{G}$, using similar ideas as in the proof of Proposition 4.6. in [58]. For all $e = \{x, y\} \in E$ and $t \in [0, \rho_e]$ we denote by $x + t \cdot I_e$ the point of I_e at distance t from x, that is $x = x + 0 \cdot I_e = y + \rho_e \cdot I_e$, and similarly for all $x \in G$ and $t \in [0, \rho_x)$, we denote by $x + t \cdot I_x$ the point of I_x at distance t from x. We say that a function $\mathbf{h} : \widetilde{\mathcal{G}} \to (0,\infty)$ is harmonic on $\widetilde{\mathcal{G}}$, when $\mathbf{h}(x+t\cdot I_e) = t\rho_e^{-1}\mathbf{h}(y) + (1-\rho_e^{-1}t)\mathbf{h}(x)$ for all $e = \{x, y\} \in E$ and $t \in (0, \rho_e)$, $\mathbf{h}(\partial I_x) := \lim_{t\to\rho_x} \mathbf{h}(x + t \cdot I_x)$ exists, is finite, and $\mathbf{h}(x+t\cdot I_x) = t\rho_x^{-1}\mathbf{h}(\partial I_x)+(1-\rho_x^{-1}t)\mathbf{h}(x)$ for all $x \in G$ and $t \in [0, \rho_x)$, and

$$\sum_{y\sim x} \lambda_{x,y}\mathbf{h}(y) + \kappa_x\mathbf{h}(\partial I_x) = \lambda_x\mathbf{h}(x) \text{ for all } x \in G. \tag{V.4.1}$$

When $\mathbf{h}$ is an harmonic function on $\widetilde{\mathcal{G}}$, let us denote by $\mathcal{G}_{\mathbf{h}}$ the graph with the same vertex set $G_{\mathbf{h}} = G$, with weights $\lambda^{(\mathbf{h})}_{x,y} = \mathbf{h}(x)\mathbf{h}(y)\lambda_{x,y}$, $x, y \in G$, and with killing measure $\kappa^{(\mathbf{h})}_x = \kappa_x \mathbf{h}(x)\mathbf{h}(\partial I_x)$, $x \in G$. We say that $\mathcal{G}_{\mathbf{h}}$ is the $\mathbf{h}$-transform of the graph $\mathcal{G}$, and we will often write $\mathbf{x}$ for the vertex of $G_{\mathbf{h}}$ corresponding to $x \in G$, and $\mathbf{e}$ for the edge of $\mathcal{G}_{\mathbf{h}}$ corresponding to $e \in E$. By (V.4.1), the total weight of a vertex $\mathbf{x} \in G_{\mathbf{h}}$ is then $\lambda^{(\mathbf{h})}_{\mathbf{x}} = \mathbf{h}(x)^2\lambda_x$. We also define a function $\psi_{\mathbf{h}} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}_{\mathbf{h}}$ such that if $e \in E \cup G$ and $x \in \partial I_e$, then

$$\psi_{\mathbf{h}}(x + t \cdot I_e) \stackrel{\text{def.}}{=} \mathbf{x} + \frac{t}{\mathbf{h}(x)\mathbf{h}(x + t \cdot I_e)} \cdot I_{\mathbf{e}} \text{ for all } t \in [0, \rho_e),$$

and we will take the notation $\psi_{\mathbf{h}}(\Delta) = \Delta$. One can easily check that this definition does not depend on the choice of the endpoint x or y of I_e when $e = \{x, y\} \in E$, that $\psi_{\mathbf{h}}$ is bijective, and that $\psi_{\mathbf{h}}(x) = \mathbf{x}$ for all $x \in G$. For any forwards trajectories $w^+ \in W^+_{\widetilde{\mathcal{G}}_{\mathbf{h}}}$ on $\widetilde{\mathcal{G}}_{\mathbf{h}}$, we define the time change

$$\theta^{w^+}_{\mathbf{h}}(t) \stackrel{\text{def.}}{=} \inf\Big\{s \geq 0 : \int_0^s \mathbf{h}\big(\psi_{\mathbf{h}}^{-1}(w^+(u))\big)^4 \,\mathrm{d}u > t\Big\}, \tag{V.4.2}$$

with the conventions $\mathbf{h}(\Delta) = 0$ and $\inf \varnothing = \widetilde{\zeta}$, and

$$(\xi_{\mathbf{h}}(w^+))(t) \stackrel{\text{def.}}{=} \psi_{\mathbf{h}}^{-1}\big(w^+(\theta^{w^+}_{\mathbf{h}}(t))\big) \text{ for all } t \in [0, \infty). \tag{V.4.3}$$

The process $\xi_{\mathbf{h}}(X)$ is thus a stochastic process on $\widetilde{\mathcal{G}}$ under $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\psi_{\mathbf{h}}(x)}$, $x \in \widetilde{\mathcal{G}}$, and let us now prove that it corresponds to the $\mathbf{h}$-transform of X. We recall the definition of the Dirichlet form $\mathcal{E}_{\widetilde{\mathcal{G}}}$ and of the domain $D(\widetilde{\mathcal{G}}, \widetilde{m})$ for any measures $\widetilde{m}$ on $\widetilde{\mathcal{G}}$ from (V.2.1) and above.

Lemma V.4.1. *If $\mathbf{h}$ is an harmonic function on $\widetilde{\mathcal{G}}$, then $\xi_{\mathbf{h}}(X)$ is an $(\mathbf{h}^2 \cdot m)$ symmetric diffusion on $\widetilde{\mathcal{G}}$ under $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\psi_{\mathbf{h}}(x)}$, $x \in \widetilde{\mathcal{G}}$, with associated Dirichlet form $\mathcal{E}_{\widetilde{\mathcal{G}}}(f\mathbf{h}, g\mathbf{h})$ on $L^2(\widetilde{\mathcal{G}}, \mathbf{h}^2 \cdot m)$ with domain $D(\widetilde{\mathcal{G}}, \mathbf{h}^2 \cdot m)$.*

Proof. Let $m_{\mathbf{h}}$ be the Lebesgue measure on $\mathcal{G}_{\mathbf{h}}$. The Dirichlet form associated to the $((\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \cdot m_{\mathbf{h}})$-symmetric diffusion $(X_{\theta^X_{\mathbf{h}}(t)})_{t < (\theta^X_{\mathbf{h}})^{-1}(\widetilde{\zeta})}$ under $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_x$, $x \in \widetilde{\mathcal{G}}_{\mathbf{h}}$ is $\mathcal{E}_{\widetilde{\mathcal{G}}_{\mathbf{h}}}(f, g)$ on $L^2(\widetilde{\mathcal{G}}_{\mathbf{h}}, (\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \cdot m_{\mathbf{h}})$ with domain $\{f \in L^2(\widetilde{\mathcal{G}}_{\mathbf{h}}, (\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \cdot m_{\mathbf{h}}) \cap \mathcal{C}_0(\widetilde{\mathcal{G}}_{\mathbf{h}}) : f' \in L^2(\widetilde{\mathcal{G}}_{\mathbf{h}}, m_{\mathbf{h}})\}$, where $m_{\mathbf{h}}$ is the Lebesgue measure on $\widetilde{\mathcal{G}}_{\mathbf{h}}$, see Theorem 6.2.1 in [37]. Let $m'_{\mathbf{h}} = ((\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \cdot m_{\mathbf{h}}) \circ \psi_{\mathbf{h}}$. Following Section 13 of [85], one can prove that the Dirichlet form associated to the $m'_{\mathbf{h}}$-symmetric diffusion $\xi_{\mathbf{h}}(X)$ under $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\psi_{\mathbf{h}}(x)}$, $x \in \widetilde{\mathcal{G}}$, is $\mathcal{E}_{\widetilde{\mathcal{G}}_{\mathbf{h}}}(f \circ \psi_{\mathbf{h}}^{-1}, g \circ \psi_{\mathbf{h}}^{-1})$ on $L^2(\widetilde{\mathcal{G}}, m'_{\mathbf{h}})$ with domain $\{f \in L^2(\widetilde{\mathcal{G}}, m'_{\mathbf{h}}) \cap \mathcal{C}_0(\widetilde{\mathcal{G}}) : (f \circ \psi_{\mathbf{h}}^{-1})' \in L^2(\widetilde{\mathcal{G}}_{\mathbf{h}}, m_{\mathbf{h}})\}$. Let $e \in E \cup G$, then $(\psi_{\mathbf{h}})'(x) = \mathbf{h}(x)^{-2}$ for all $x \in I_e$, and $(\psi_{\mathbf{h}}^{-1})'(x) = (\mathbf{h} \circ \psi_{\mathbf{h}}^{-1}(x))^2$

for all $x \in I_{\mathbf{e}}$, and so we have by substitution for all $e \in E \cup G$ for any Borel sets $A \subset I_e$

$$\begin{aligned} m'_{\mathbf{h}}(A) = ((\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \cdot m_{\mathbf{h}})(\psi_{\mathbf{h}}(A)) &= \int_{\psi_{\mathbf{h}}(A)} (\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \, \mathrm{d}m_{\mathbf{h}} \\ &= \int_A \mathbf{h}^2 \, \mathrm{d}m = (\mathbf{h}^2 \cdot m)(A), \end{aligned}$$

and so $m'_{\mathbf{h}} = \mathbf{h}^2 \cdot m$. Moreover for any functions f, g in $D(\widetilde{\mathcal{G}}, \mathbf{h}^2 \cdot m)$ and $e \in E \cup G$ we have

$$\begin{aligned} \int_{\psi_{\mathbf{h}}(I_e)} (f \circ \psi_{\mathbf{h}}^{-1})'(g \circ \psi_{\mathbf{h}}^{-1})' \, \mathrm{d}m_{\mathbf{h}} &= \int_{\psi_{\mathbf{h}}(I_e)} (f' \circ \psi_{\mathbf{h}}^{-1})(g' \circ \psi_{\mathbf{h}}^{-1})(\mathbf{h} \circ \psi_{\mathbf{h}}^{-1})^4 \, \mathrm{d}m_{\mathbf{h}} \\ &= \int_{I_e} f'g'\mathbf{h}^2 \, \mathrm{d}m. \end{aligned}$$

Integrating by parts and noting that $\mathbf{h}' = \mathbf{h}'_e$ is constant on I_e we have

$$\begin{aligned} \int_{I_e} f'g'\mathbf{h}^2 \, \mathrm{d}m &= \int_{I_e} (f\mathbf{h})'(g\mathbf{h})' \, \mathrm{d}m - \mathbf{h}'_e \int_{I_e} (fg)'\mathbf{h} \, \mathrm{d}m - (\mathbf{h}'_e)^2 \int_{I_e} fg \, \mathrm{d}m \\ &= \int_{I_e} (f\mathbf{h})'(g\mathbf{h})' \, \mathrm{d}m - \mathbf{h}'_e [fg\mathbf{h}]_{I_e}. \end{aligned}$$

Moreover, if $e = \{x, y\} \in E$, then $\mathbf{h}'_e = \rho_e^{-1}(\mathbf{h}(y) - \mathbf{h}(x)) = 2\lambda_{x,y}(\mathbf{h}(y) - \mathbf{h}(x))$, and so we have

$$\begin{aligned} \sum_{e \in E} \mathbf{h}'_e [fg\mathbf{h}]_{I_e} &= 2 \sum_{e = \{x,y\} \in E} \lambda_{x,y} \big(f(y)g(y)\mathbf{h}(y) - f(x)g(x)\mathbf{h}(x) \big) \big(\mathbf{h}(y) - \mathbf{h}(x) \big) \\ &= 2 \sum_{x,y \in G} \lambda_{x,y} f(x)g(x)\mathbf{h}(x) \big(\mathbf{h}(x) - \mathbf{h}(y) \big), \end{aligned}$$

and if $e = x \in G$, then $\mathbf{h}'_e = \rho_x^{-1}(\mathbf{h}(\partial x) - \mathbf{h}(x)) = 2\kappa_x(\mathbf{h}(\partial x) - \mathbf{h}(x))$, and so

$$\sum_{x \in G} \mathbf{h}'_x [fg\mathbf{h}]_{I_x} = 2 \sum_{x \in G} \kappa_x f(x)g(x)\mathbf{h}(x) \big(\mathbf{h}(x) - \mathbf{h}(\partial I_x) \big).$$

Therefore we obtain by (V.2.1) that the process $\xi_{\mathbf{h}}(X)$ under $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\psi_{\mathbf{h}}(x)}$ is a $(\mathbf{h}^2 \cdot m)$-symmetric diffusion, and its associated Dirichlet form is

$$\mathcal{E}_{\widetilde{\mathcal{G}}}(f\mathbf{h}, g\mathbf{h}) + \sum_{x \in G} f(x)g(x)\mathbf{h}(x) \Big(\kappa_x \big(\mathbf{h}(\partial I_x) - \mathbf{h}(x) \big) + \sum_{y \in G} \lambda_{x,y} \big(\mathbf{h}(y) - \mathbf{h}(x) \big) \Big)$$

on $L^2(\widetilde{\mathcal{G}}, \mathbf{h}^2 \cdot m)$ with domain $D(\widetilde{\mathcal{G}}, \mathbf{h}^2 \cdot m)$. We can conclude by (V.4.1). □

Lemma V.4.1 implies that $\xi_{\mathbf{h}}(X)$ corresponds to the $\mathbf{h}$-transform of X, see for instance Chapter 11 of [22], and when $\mathbf{h}=\mathbf{h}_0$, see (V.1.1), one can then classically relate the law of the diffusion X on $\widetilde{\mathcal{G}}$ conditioned on being killed before blowing up with the $\mathbf{h}_0$ transform of X, see Theorem 11.26 in [22]. Therefore, the law of X on $\widetilde{\mathcal{G}}$ conditioned on being killed before blowing up can be related to the diffusion X on the $\mathbf{h}_0$-transform $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ of $\widetilde{\mathcal{G}}$, and since the proof of this result is short, we include it below for completeness. Similarly, the law of X on $\widetilde{\mathcal{G}}$ conditioned on blowing up before being killed can be related to the law of X on $\widetilde{\mathcal{G}}_{\mathbf{h}_1}$, where $\mathbf{h}_1:=1-\mathbf{h}_0$ is the probability that X blows up before being killed.

As a consequence, one can also relate killed random interlacements on $\widetilde{\mathcal{G}}$ with random interlacements on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$, and surviving random interlacements on $\widetilde{\mathcal{G}}$ with random interlacements on $\widetilde{\mathcal{G}}_{\mathbf{h}_1}$, as well as the corresponding Gaussian free fields. If $w^*\in W^*_{\widetilde{\mathcal{G}}_{\mathbf{h}}}$, we denote by $\xi^*_{\mathbf{h}}(w^*)$ the trajectory in $W^*_{\widetilde{\mathcal{G}}}$ which corresponds to taking the image modulo time-shift of a trajectory with backwards part $\xi_{\mathbf{h}}((w(-t))_{t\geq0})$ and forwards part $\xi_{\mathbf{h}}((w(t)_{t\geq0})$, for some $w\in(p^*_{\widetilde{\mathcal{G}}_{\mathbf{h}}})^{-1}(w^*)$, and one can easily check that this definition does not depend on the choice of w.

Proposition V.4.2. *If $\mathcal{G}$ is a graph with $\mathbf{h}_0\neq0$, then the function $\mathbf{h}_0$ is harmonic on $\widetilde{\mathcal{G}}$. Moreover, for all $x\in\widetilde{\mathcal{G}}$, the diffusion*

$$\xi_{\mathbf{h}_0}(X) \text{ has the same law under } P^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}_{\psi_{\mathbf{h}_0}(x)} \text{ as } X \text{ under } P^{\widetilde{\mathcal{G}}}_x(\cdot\,|\,\mathcal{W}^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}), \tag{V.4.4}$$

the random interlacement process

$$\omega\circ(\xi^*_{\mathbf{h}_0})^{-1} \text{ has the same law under } \mathbb{P}^I_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}} \text{ as } \omega^{\mathcal{K}} \text{ under } \mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}, \tag{V.4.5}$$

and the Gaussian free field

$$\big(\mathbf{h}_0(x)\varphi_{\psi_{\mathbf{h}_0}(x)}\big)_{x\in\widetilde{\mathcal{G}}} \text{ has the same law under } \mathbb{P}^G_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}} \text{ as } (\varphi_x)_{x\in\widetilde{\mathcal{G}}} \text{ under } \mathbb{P}^G_{\widetilde{\mathcal{G}}}. \tag{V.4.6}$$

Similar results hold when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$, $\mathcal{W}^{\mathcal{K},+}_{\widetilde{\mathcal{G}}}$ by $\mathcal{W}^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}$, and killed random interlacements by surviving random interlacements.

Proof. We only do the proof for $\mathbf{h}_0$, the proof for $\mathbf{h}_1$ is similar. It is clear that $\mathbf{h}_0(\partial I_x)=1$ if $\kappa_x\neq0$ and that $\mathbf{h}_0(\partial I_x)=\mathbf{h}_0(x)$ if $\kappa_x=0$. If $e=\{x,y\}\in E$ and $t\in[0,\rho_e]$, then the probability beginning in $x+t\cdot e$ that X hits y before x is $t\rho_e^{-1}$, and by the Markov property $\mathbf{h}_0$ is harmonic on I_e. If $x\in G$ with $\kappa_x\neq0$ and $t\in[0,\rho_x)$, then the probability beginning in $x+t\cdot e$ that X hits x before being killed is $1-\rho_x^{-1}t$, and thus by the Markov property $\mathbf{h}_0$ is harmonic on I_x. If $x\in G$ with $\kappa_x=0$, then $\mathbf{h}_0$ is constant equal to $\mathbf{h}_0(x)$ on I_x, and thus harmonic on I_x. Moreover for all $x \in G$ using the Markov property at the first time another vertex of $\mathcal{G}$ is hit, we have

$$\lambda_x \mathbf{h}_0(x) = \sum_{y\sim x} \lambda_{x,y}\mathbf{h}_0(x) + \kappa_x = \sum_{y\sim x}\lambda_{x,y}\mathbf{h}_0(x) + \kappa_x \mathbf{h}_0(\partial I_x).$$

The function $\mathbf{h}_0$ is thus harmonic on $\widetilde{\mathcal{G}}$. For all $x \in \widetilde{\mathcal{G}}$, $t \in [0,\infty)$ and $f \in L^2(\widetilde{\mathcal{G}}, \mathbf{h}_0^2 \cdot m)$ we have by the Markov property at time t

$$E_x^{\widetilde{\mathcal{G}}}\big[f(X_t)\,|\,W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+}\big] = \frac{1}{\mathbf{h}_0(x)} E_x^{\widetilde{\mathcal{G}}}\big[f(X_t)\mathbb{1}_{W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+}}\big] = \frac{1}{\mathbf{h}_0(x)} E_x^{\widetilde{\mathcal{G}}}\left[f(X_t)\mathbf{h}_0(X_t)\right].$$

Let P_t be the semi-group associated with X under $P_x^{\widetilde{\mathcal{G}}}$, then $\frac{1}{\mathbf{h}_0}P_t(\mathbf{h}_0 f)$ is the semi-group associated with the $(\mathbf{h}_0^2 \cdot m)$-symmetric diffusion X under $P_x^{\widetilde{\mathcal{G}}}(\cdot\,|\,\mathcal{W}_{\widetilde{\mathcal{G}}}^{\mathcal{K},+})$, and thus its associated Dirichlet form on $L^2(\widetilde{\mathcal{G}}, \mathbf{h}_0^2\cdot m)$ is

$$\lim_{t\to 0}\frac{1}{t}\big(f - \frac{1}{\mathbf{h}_0}P_t(f\mathbf{h}_0), g\big)_{\mathbf{h}_0^2\cdot m} = \lim_{t\to 0}\frac{1}{t}(f\mathbf{h}_0 - P_t(f\mathbf{h}_0), g\mathbf{h}_0)_m = \mathcal{E}_{\widetilde{\mathcal{G}}}(f\mathbf{h}_0, g\mathbf{h}_0),$$

with domain $D(\widetilde{\mathcal{G}}, \mathbf{h}_0^2\cdot m)$, and we obtain (V.4.4) by Lemma V.4.1.

We now turn to the proof of the identity (V.4.12) for random interlacements. By (V.2.12) and (V.3.3), it is enough to prove that

$$Q_{\psi_{\mathbf{h}_0}(K),\widetilde{\mathcal{G}}_{\mathbf{h}_0}} \circ (\xi_{\mathbf{h}_0}^* \circ p_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}^*)^{-1} = Q_{K,\widetilde{\mathcal{G}}}^{\mathcal{K}} \circ (p_{\widetilde{\mathcal{G}}}^*)^{-1} \tag{V.4.7}$$

for all compacts K of $\widetilde{\mathcal{G}}$, see (V.2.11) and (V.3.2). Considering the graph $\mathcal{G}^{\partial K}$ from Lemma IV.2.1, we can assume without loss of generality that $\partial K \subset G$. By (V.2.7) and (V.3.1) we have for all $x \in \partial K$ that

$$\begin{aligned} e_{\psi_{\mathbf{h}_0}(K),\widetilde{\mathcal{G}}_{\mathbf{h}_0}}(\psi_{\mathbf{h}_0}(x)) &= \lambda_{\mathbf{x}}^{(\mathbf{h}_0)} P_{\psi_{\mathbf{h}_0}(x)}^{\mathcal{G}_{\mathbf{h}_0}}(\widetilde{H}_{\psi_{\mathbf{h}_0}(K)} = \zeta) \\ &= \lambda_x \mathbf{h}_0(x)^2 P_x^{\mathcal{G}}(\widetilde{H}_K = \zeta\,|\,W_{\mathcal{G}}^{\mathcal{K},+}) \\ &= e_{K,\widetilde{\mathcal{G}}}^{\mathcal{K}}(x), \end{aligned} \tag{V.4.8}$$

where we used (V.4.4) in the second equality, and the fact that Z has the same law as the print of X on G. Moreover by (V.4.4), one can easily prove that $P_{\psi_{\mathbf{h}_0}(x)}^{\psi_{\mathbf{h}_0}(K),\widetilde{\mathcal{G}}_{\mathbf{h}_0}}(\xi_{\mathbf{h}_0}(X)\in\cdot) = P_x^{K,\widetilde{\mathcal{G}}}(\cdot\,|\,W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+})$, and we obtain (V.4.7), and thus (V.4.5).

Let us now prove (V.4.6). By the Markov property, we have that for all $t\geq 0$ and $x,y\in\widetilde{\mathcal{G}}$

$$P_x^{\widetilde{\mathcal{G}}}(X_t \in \mathrm{d}y\,|\,W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+}) = \frac{1}{\mathbf{h}_0(x)}p_t(x,y)\mathbf{h}_0(y)m(\mathrm{d}y),$$

and so by (V.2.2) we obtain that, with respect to m,

$$\text{the Green function associated to } X \text{ under } P_{\cdot}^{\widetilde{\mathcal{G}}}(\cdot \,|\, W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+}) \text{ is } \Big(\frac{\mathbf{h}_0(y)g_{\widetilde{\mathcal{G}}}(x,y)}{\mathbf{h}_0(x)}\Big)_{x,y\in\widetilde{\mathcal{G}}}. \tag{V.4.9}$$

Let us denote by $(\widehat{X}_n)_{n\in\mathbb{N}}$ the discrete time Markov chain on G, which corresponds to the discrete skeleton of the print of X on G. In other words, if $\widehat{X}_n = x \in G$, then $\widehat{X}_{n+1}$ jumps to $y \in G$ with probability $\frac{\lambda_{x,y}}{\lambda_x}$ and is killed in Δ with probability $\frac{\kappa_x}{\lambda_x}$. It easily follows from (V.4.3) and (V.4.4) that $\psi_{\mathbf{h}_0}^{-1}(\widehat{X})$ has the same under $P_{\psi_{\mathbf{h}_0}(x)}^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}$ as $\widehat{X}$ under $P_x^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}(\cdot \,|\, W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+})$ for all $x \in G$. Therefore we have for all $x, y \in G$

$$\begin{aligned}
g_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}(\psi_{\mathbf{h}_0}(x), \psi_{\mathbf{h}_0}(y)) &= \frac{1}{\lambda^{(\mathbf{h}_0)}_{\psi_{\mathbf{h}_0}(y)}} E_{\psi_{\mathbf{h}_0}(x)}^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}\Big[\sum_{k=0}^{\infty} \mathbb{1}_{\widehat{X}_k=\psi_{\mathbf{h}_0}(y)}\Big] \\
&= \frac{1}{\mathbf{h}_0(y)^2} \times \frac{1}{\lambda_y} E_x^{\widetilde{\mathcal{G}}}\Big[\sum_{k=0}^{\infty} \mathbb{1}_{\widehat{X}_k=y} \,\Big|\, W_{\widetilde{\mathcal{G}}}^{K,+}\Big] \\
&= \frac{g_{\widetilde{\mathcal{G}}}(x,y)}{\mathbf{h}_0(x)\mathbf{h}_0(y)},
\end{aligned}$$

where we used (V.4.9) in the last equality. This relation can be extended to any $x, y \in \widetilde{\mathcal{G}}$ by considering the graph $\mathcal{G}^{\{x,y\}}$ from Lemma IV.2.1, and therefore the two processes considered in (V.4.6) are centered Gaussian fields with covariance function $g_{\widetilde{\mathcal{G}}}$, and they thus have the same law. □

Remark V.4.3. Let us describe the analogue of Proposition V.4.2 but for the discrete graph $\mathcal{G}$. We define for all continuous time trajectories $\overline{w}^+$ on $G_{\mathbf{h}}$

$$\begin{aligned}
\overline{\theta}_{\mathbf{h}}^{\overline{w}^+}(t) &\stackrel{\text{def.}}{=} \inf\Big\{s \geq 0 : \int_0^s \mathbf{h}\big(\psi_{\mathbf{h}}^{-1}(\overline{w}^+(u))\big)^2 \, \mathrm{d}u > t\Big\} \\
&= \inf\Big\{s \geq 0 : \sum_{x\in G} \ell_{\mathbf{x}}(s)\mathbf{h}(x)^2 > t\Big\},
\end{aligned}$$

with the conventions $\mathbf{h}(\Delta) = 0$ and $\inf \varnothing = \zeta$, and

$$(\overline{\xi}_{\mathbf{h}}(\overline{w}^+))(t) \stackrel{\text{def.}}{=} \psi_{\mathbf{h}}^{-1}\big(\overline{w}^+(\overline{\theta}_{\mathbf{h}}^{\overline{w}^+}(t))\big) \text{ for all } t \in [0, \infty).$$

Then the results from Proposition V.4.2 still hold when replacing $\xi_{\mathbf{h}}$ by $\overline{\xi}_{\mathbf{h}}$, the diffusion X by the jump process Z, the random interlacement process ω by the discrete random interlacement process $\omega^{\mathcal{G}}$, and the Gaussian free field $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ on $\widetilde{\mathcal{G}}$ by the Gaussian free field $(\varphi_x)_{x\in G}$ on $\mathcal{G}$. One can deduce this statement from Proposition V.4.2 by using the fact that Z and $\omega^{\mathcal{G}}$ are the prints of X and ω on G, or prove it directly, see the proof of Proposition 4.6 in [58] for a proof of a similar statement.

In view of (V.4.5) and (V.4.6), one can transform any results about random interlacements or Gaussian free field on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ into results about killed random interlacements or Gaussian free field on $\widetilde{\mathcal{G}}$. We want to apply this strategy to the results of Chapter IV, which involve the field of local times associated to random interlacements, and in order to do that, let us first compute the local times of $\xi_{\mathbf{h}}$.

Lemma V.4.4. *Let* $\mathbf{h}$ *be an harmonic function on* $\widetilde{\mathcal{G}}$. *Under* $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\psi_{\mathbf{h}}(\cdot)}$, *with respect to the measure* m,

$$\text{the field of local times associated to } \xi_{\mathbf{h}}(X) \text{ is } \big(\mathbf{h}(x)^2\ell_{\psi_{\mathbf{h}}(x)}(\theta^X_{\mathbf{h}}(t))\big)_{t\geq 0, x\in\widetilde{\mathcal{G}}}. \tag{V.4.10}$$

Proof. Following Section 2 of [57], we have for all $x\in\widetilde{\mathcal{G}}$ and $t\geq 0$ that $P^{\widetilde{\mathcal{G}}_{\mathbf{h}}}_{\cdot}$-a.s.

$$\ell_{\psi_{\mathbf{h}}(x)}(t)=\lim_{\varepsilon\to 0}\frac{1}{m_{\mathbf{h}}(\psi_{\mathbf{h}}(B(x,\varepsilon)))}\int_0^t \mathbb{1}_{X_u\in\psi_{\mathbf{h}}(B(x,\varepsilon))}\,\mathrm{d}u,$$

where $B(x,\varepsilon)=\{x+t\cdot I_e\in\widetilde{\mathcal{G}}: t\in[0,\varepsilon] \text{ and } e\in E\cup G \text{ with } x\in I_e\}$. Taking $u=\theta^X_{\mathbf{h}}(s)$, we have

$$\int_0^t \mathbb{1}_{(\xi_{\mathbf{h}}(X))_s\in B(x,\varepsilon)}\,\mathrm{d}s=\int_0^{\theta^X_{\mathbf{h}}(t)}\mathbb{1}_{X_u\in\psi_{\mathbf{h}}(B(x,\varepsilon))}\mathbf{h}\big(\psi_{\mathbf{h}}^{-1}(X_u)\big)^4\,\mathrm{d}u,$$

and since

$$\frac{m_{\mathbf{h}}\big(\psi_{\mathbf{h}}(B(x,\varepsilon))\big)}{m\big(B(x,\varepsilon)\big)}\underset{\varepsilon\to 0}{\longrightarrow}\frac{1}{\mathbf{h}(x)^2},$$

we obtain that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{m(B(x,\varepsilon))}\int_0^t \mathbb{1}_{(\xi_{\mathbf{h}}(X))_s\in B(x,\varepsilon)}\,\mathrm{d}s\\&=\lim_{\varepsilon\to 0}\frac{1}{\mathbf{h}(x)^2 m_{\mathbf{h}}\big(\psi_{\mathbf{h}}(B(x,\varepsilon))\big)}\int_0^{\theta^X_{\mathbf{h}}(t)}\mathbb{1}_{X_u\in\psi_{\mathbf{h}}(B(x,\varepsilon))}\mathbf{h}\big(\psi_{\mathbf{h}}^{-1}(X_u)\big)^4\,\mathrm{d}u\\&=\mathbf{h}(x)^2\ell_{\psi_{\mathbf{h}}(x)}(\theta^X_{\mathbf{h}}(t)).\end{aligned}$$

This corresponds to the field of local times associated to the process $\xi_{\mathbf{h}}(X)$ in x at time t, see for instance Theorem 3.6.3 in [62]. □

Combining (V.4.4) and (V.4.10) let us compare the local times of the diffusion X conditioned on being killed before blowing up on $\widetilde{\mathcal{G}}$ with the local times of X on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$, see (V.4.11). Using (V.4.5) and (V.4.10), one can also compare the local times of killed random interlacements on $\widetilde{\mathcal{G}}$ with the local times of random interlacements on $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$, see (V.4.12). Let us now gather these results, as well as the corresponding statements for $\mathbf{h}_1$.

Corollary V.4.5. *For each $y \in \widetilde{\mathcal{G}}$, if $\mathbf{h}_0 \neq 0$, the process*

$$\begin{aligned}(\ell_x(t))_{t\geq 0, x\in\widetilde{\mathcal{G}}} \text{ has the same law under } P_y^{\widetilde{\mathcal{G}}}(\cdot \,|\, W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+})\\ \text{as } \big(\mathbf{h}_0(x)^2\ell_{\psi_{\mathbf{h}_0}(x)}(\theta_{\mathbf{h}_0}^X(t))\big)_{t\geq 0, x\in\widetilde{\mathcal{G}}} \text{ under } P_{\psi_{\mathbf{h}_0}(y)}^{\widetilde{\mathcal{G}}_{\mathbf{h}_0}},\end{aligned} \tag{V.4.11}$$

and the process

$$(\ell_{x,u}^{\mathcal{K}})_{x\in\widetilde{\mathcal{G}}} \text{ has the same law under } \mathbb{P}_{\widetilde{\mathcal{G}}}^{KI} \text{ as } (\mathbf{h}_0(x)^2\ell_{\psi_{\mathbf{h}_0}(x),u})_{x\in\widetilde{\mathcal{G}}} \text{ under } \mathbb{P}_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}^{I}. \tag{V.4.12}$$

Similar results hold when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$, $W_{\widetilde{\mathcal{G}}}^{\mathcal{K},+}$ by $W_{\widetilde{\mathcal{G}}}^{\mathcal{S},+}$, and killed random interlacements by surviving random interlacements.

We are now ready to take advantage of the results from Chapter IV about the Gaussian free field and random interlacements on $\mathcal{G}_{\mathbf{h}}$. The first interesting result is Theorem IV.3.3, which gives the law of the capacity of the level sets of the Gaussian free field on the cable system. We first need to relate the capacity of a set in $\widetilde{\mathcal{G}}_{\mathbf{h}_0}$ to the killed capacity of a set in $\widetilde{\mathcal{G}}$. Considering the graph $\mathcal{G}^{\partial K}$ from Lemma IV.2.1, one can easily extend the equality (V.4.8) to any compacts K of $\widetilde{\mathcal{G}}$, and thus by (V.2.7) and (V.3.4)

$$\mathrm{cap}_{\widetilde{\mathcal{G}}_{\mathbf{h}_0}}\big(\psi_{\mathbf{h}_0}(K)\big) = \mathrm{cap}_{\widetilde{\mathcal{G}}}^{\mathcal{K}}(K) \text{ for all compacts } K \text{ of } \widetilde{\mathcal{G}}. \tag{V.4.13}$$

A similar equality also holds when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$ and killed capacity by surviving capacity. It then follows directly from (Law$_h$), (V.4.6) and (V.4.13), that, if (Sign) holds, then for all $h \geq 0$, $u \geq 0$ and $x_0 \in \widetilde{\mathcal{G}}$

$$\mathbb{E}_{\widetilde{\mathcal{G}}}^{G}\Big[\exp\Big(-u\mathrm{cap}_{\widetilde{\mathcal{G}}}^{\mathcal{K}}\big(E_{\mathcal{K}}^{\geq h}(x_0)\big)\Big)\mathbb{1}_{\varphi_{x_0}\geq h\times\mathbf{h}_0(x_0)}\Big] = \mathbb{P}_{\widetilde{\mathcal{G}}}^{G}(\varphi_{x_0} \geq \mathbf{h}_0(x_0)\sqrt{2u+h^2}), \tag{V.4.14}$$

and a similar identity holds when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$, killed capacity by surviving capacity and killed level sets by surviving level sets. In particular for $h = 0$, one has an explicit formula for the capacity, the killed capacity, and the surviving capacity of the sign clusters of the Gaussian free field on the cable system. When $h < 0$, one could also derive identities similar to (IV.3.7) and (IV.3.8) for the law of the killed capacity of the level sets $E_{\mathcal{K}}^{\geq h}$, and the law of the surviving capacity of the level sets $E_{\mathcal{S}}^{\geq h}$.

One can similarly deduce from the isomorphism between random interlacements and the Gaussian free field, Theorem IV.3.4, an isomorphism between killed interlacements, or surviving interlacements, and the Gaussian free field. Let us first introduce some notation: we denote by $\omega_u^{\mathcal{K},\mathcal{G}}$ the print of ω_u on G, which corresponds to a killed random interlacement process on the discrete graph $\mathcal{G}$, and by $\mathcal{I}_{E,\mathcal{K}}^u \subset E \cup G$ the set of edges crossed by at least one trajectory of $\omega_u^{\mathcal{K},\mathcal{G}}$, and of vertices on which a trajectory of $\omega_u^{\mathcal{K},\mathcal{G}}$ is killed, either for its forwards or its backwards part. We will write $\mathbb{P}^I_{\mathcal{G}}$ instead of $\mathbb{P}^I_{\widetilde{\mathcal{G}}}$ when we want to stress that we only consider $\omega_u^{\mathcal{K},\mathcal{G}}$, and $\mathbb{P}^G_{\mathcal{G}}$ instead of $\mathbb{P}^G_{\widetilde{\mathcal{G}}}$ when we want to stress that we only consider $(\varphi_x)_{x\in G}$, which has the same law as a discrete Gaussian free field on $\mathcal{G}$.

Theorem V.4.6. *Assume that $\mathcal{G}$ is transient and $\mathbf{h}_0\neq 0$. On some extension $\widetilde{\mathbb{P}}^{KI}_{\widetilde{\mathcal{G}}}$ of $\mathbb{P}^G_{\widetilde{\mathcal{G}}}\otimes\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}$, let us define for each $u>0$ an additional process $(\sigma_x^{\mathcal{K},u})_{x\in\widetilde{\mathcal{G}}}\in\{-1,1\}^{\widetilde{\mathcal{G}}}$, such that, conditionally on $(|\varphi_x|)_{x\in\widetilde{\mathcal{G}}}$ and $\omega_u^{\mathcal{K}}$, $\sigma^{u,\mathcal{K}}$ is constant on each of the cluster of $\{x\in\widetilde{\mathcal{G}}:\ 2\ell_{x,u}^{\mathcal{K}}+\varphi_x^2>0\}$, $\sigma_x^{u,\mathcal{K}}=1$ for all $x\in\mathcal{I}_{\mathcal{K}}^u$, and the values of $\sigma^{u,\mathcal{K}}$ on each other cluster are independent and uniformly distributed. Then* (V.4.14) *holds for $h=0$ if and only if for all $u>0$*

$$\Big(\sigma_x^{\mathcal{K},u}\sqrt{2\ell_{x,u}^{\mathcal{K}}+\varphi_x^2}\Big)_{x\in\widetilde{\mathcal{G}}}\text{ has the same law under }\widetilde{\mathbb{P}}^{KI}_{\widetilde{\mathcal{G}}}\text{ as }\big(\varphi_x+\sqrt{2u}\,\mathbf{h}_0(x)\big)_{x\in\widetilde{\mathcal{G}}}\text{ under }\mathbb{P}^G_{\widetilde{\mathcal{G}}}.\qquad\text{(V.4.15)}$$

Moreover, if (V.4.14) *holds for $h=0$, let us define for each $u>0$ a random set $\widehat{\mathcal{E}}_u^{\mathcal{K}}\subset E\cup G$ such that, conditionally on $(\varphi_x)_{x\in G}$ and $\omega_u^{\mathcal{K},\mathcal{G}}$, $\widehat{\mathcal{E}}_u^{\mathcal{K}}$ contains each edge and vertex in $\mathcal{I}^u_{E,\mathcal{K}}$, and each additional edge and vertex $e\in E\cup G$ conditionally independently with probability $1-p_e^{u,\mathcal{G}}(\varphi,\ell^{\mathcal{K}}_{\cdot,u})$, where $p_e^{u,\mathcal{G}}$ is defined in* (IV.3.13) *and* (IV.3.14). *Then $\widehat{\mathcal{E}}_u^{\mathcal{K}}$ has the same law under $\widetilde{\mathbb{P}}^{KI}_{\mathcal{G}}$ as $\mathcal{E}_u^{\mathcal{K}}:=\{e\in E\cup G:\ 2\ell_{x,u}^{\mathcal{K}}+\varphi_x^2>0\text{ for all }x\in I_e\}$ under $\widetilde{\mathbb{P}}^{KI}_{\widetilde{\mathcal{G}}}$. In particular, if we define a process $(\widehat{\sigma}_x^{\mathcal{K},u})_{x\in G}\in\{-1,1\}^G$, such that, conditionally on $(\varphi_x)_{x\in G}$, $\omega_u^{\mathcal{K},\mathcal{G}}$ and $\widehat{\mathcal{E}}_u^{\mathcal{K}}$, $\widehat{\sigma}^{\mathcal{K},u}$ is constant on each of the clusters induced by $\widehat{\mathcal{E}}_u^{\mathcal{K}}$, $\widehat{\sigma}_x^{\mathcal{K},u}=1$ for all $x\in\widehat{\mathcal{E}}_u^{\mathcal{K}}\cap G$, and the values of $\widehat{\sigma}^{\mathcal{K},u}$ on each other cluster are independent and uniformly distributed, then*

$$\Big(\widehat{\sigma}_x^{\mathcal{K},u}\sqrt{2\ell_{x,u}^{\mathcal{K}}+\varphi_x^2}\Big)_{x\in G}\text{ has the same law under }\widetilde{\mathbb{P}}^{KI}_{\mathcal{G}}\text{ as }\big(\varphi_x+\sqrt{2u}\,\mathbf{h}_0(x)\big)_{x\in G}\text{ under }\mathbb{P}^G_{\mathcal{G}}.\qquad\text{(V.4.16)}$$

Similar results hold when replacing $\mathbf{h}_0$ by $\mathbf{h}_1$, killed random interlacements by surviving random interlacements and $1-p_e^{u,\mathcal{G}}$ by $\mathbb{1}_{e\in E}(1-p_e^{u,\mathcal{G}})$.

Remark V.4.7. 1) When $\kappa\not\equiv 0$ and $\{x\in G:\ \kappa_x>0\}$ is finite, (V.4.16) can be seen as a reformulation of Theorem 8 in [58]. Indeed, one can then define the graph $\mathcal{G}^*$ which corresponds to $\mathcal{G}$, but replacing the open end of each I_x, $x\in G$ with $\kappa_x>0$, by a common vertex x_*, and using (V.4.5), one can show that the law of the excursions on G of $(X_t)_{t<\tau_u^{x_*}}$ under $P^{\mathcal{G}^*}_{x_*}(\cdot\,|\,\tau_u^{x_*}<\zeta)$ is the same as the law of $\omega_u^{\mathcal{K},\mathcal{G}}$ under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}$, where $\tau_u^{x_*}=\inf\{s>0:\ \ell_{x_*}(s)>u\}$, see (IV.A.2) for a proof of a similar statement. One can then easily find an equivalence between (V.4.16) and Theorem 8 in [58], and we refer to the proof of Lemma IV.7.2 in the Appendix of Chapter IV for details. Moreover, it is easy to see that a version of Theorem 8 in [58] on the cable system, as given in their proof, holds on any transient graph, and by a similar equivalence as before we obtain that (V.4.15) holds on any transient graph such that $\kappa \not\equiv 0$ and $\{x \in \mathcal{G} : \kappa_x > 0\}$ is finite, and thus (V.4.14) as well.

2) One can prove (V.4.15) or (V.4.16) directly, without using Theorem IV.3.4. Indeed, let K_n, $n \in \mathbb{N}$, be a sequence of finite subsets of G increasing to G, $\kappa^{(n)} = \kappa \mathbb{1}_{K_n}$, and $\mathcal{G}_n$ be the same graph as $\mathcal{G}$, but with killing measure $\kappa^{(n)}$ instead of κ. Since $\{x \in G : \kappa^{(n)}_x > 0\}$ is finite, as explained before, one can use Theorem 8 in [58] to obtain Theorem V.4.6 on $\mathcal{G}_n$ for all $n \in \mathbb{N}$. Using the description of killed random interlacements from (V.3.10) and Remark V.3.4,2), one can compare for each $n \in \mathbb{N}$ the killed interlacements measures $\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}_n}$ and $\nu^{\mathcal{K}}_{\widetilde{\mathcal{G}}}$ on the whole cable system, instead of their restriction to compacts as in Lemma IV.8.1. Proceeding similarly as in the proof of Lemma IV.8.3, one can then approximate killed random interlacements on $\widetilde{\mathcal{G}}$ by killed random interlacements on the sequence $\widetilde{\mathcal{G}}_n$, decreasing to $\widetilde{\mathcal{G}}$, to obtain (V.4.15) for $\widetilde{\mathcal{G}}$ if (V.4.14) holds for $h = 0$. It seems more complicated to find a direct proof for surviving random interlacements.

3) Following the proof of Proposition IV.4.7, one can easily prove that (V.4.15) implies (V.4.14) for all $h \geq 0$, and in particular, if the law of the killed capacity of the level sets of the Gaussian free field is given by (V.4.14) for $h = 0$, then the law of the killed level sets of the Gaussian free field for all $h \geq 0$ is also given by (V.4.14).

4) Since $\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(A) \leq \mathrm{cap}_{\widetilde{\mathcal{G}}}(A)$ and $\mathrm{cap}^{\mathcal{S}}_{\widetilde{\mathcal{G}}}(A) \leq \mathrm{cap}_{\widetilde{\mathcal{G}}}(A)$ for all connected and closed sets $A \subset \widetilde{\mathcal{G}}$, we have by Theorem IV.3.1 that $\mathrm{cap}^{\mathcal{K}}_{\widetilde{\mathcal{G}}}(E^{\geq 0}(x_0)) < \infty$ and $\mathrm{cap}^{\mathcal{S}}_{\widetilde{\mathcal{G}}}(E^{\geq 0}(x_0)) < \infty$. Moreover, by Corollary IV.3.2, if condition (Cap) is fulfilled, then $E^{\geq h}_{\mathcal{K}}(x_0)$ and $E^{\geq h}_{\mathcal{S}}(x_0)$ contains $\mathbb{P}^G$-a.s. only compact connected components for all $h \geq 0$, and so (V.4.14) and (V.4.15) hold for both killed and surviving random interlacements. When $h < 0$, the situation is less clear. Using Theorem IV.3.1 for the graph $\mathcal{G}_{\mathbf{h}_1}$, we have that $E^{\geq h}_{\mathcal{S}}(x_0)$ is unbounded with positive probability if $\mathbf{h}_1 \neq 0$. We however expect that $E^{\geq h}_{\mathcal{K}}(x_0)$ stays compact for some $h < 0$ on a large class of graphs with $\mathbf{h}_0 \neq 0$, that is $\widetilde{h}^{\mathcal{K}}_*(\mathcal{G}) < 0$.

Using Theorem V.4.6 for surviving random interlacements on a suitable graph, let us finally prove the isomorphism between the trajectories in the random interlacement process ω^u avoiding a compact K of $\widetilde{\mathcal{G}}$ and the Gaussian free field conditioned on being equal to 0 on K, Theorem V.1.3.

Proof of Theorem V.1.3. We write $\mathcal{G}^{K^c}_{\infty}$ for the same graph as $\mathcal{G}^{\partial K}$, as defined in Lemma IV.2.1, but with killing measure equal to κ on $G\cap K^c$, and infinity on $(G\cap K)\cup\partial K$. We refer to the discussion below (IV.2.4) for an explanation of why all our results still hold when allowing infinite killing measure, and, up to considering each connected component of K^c individually, we will assume that K^c is connected. In other words, $\mathcal{G}^{K^c}_{\infty}$ is the graph such that $\widetilde{\mathcal{G}}^{K^c}_{\infty}$ is obtained by "removing" K from $\widetilde{\mathcal{G}}$, and then the law of $(X_t)_{t<H_K}$ under $P_x^{\widetilde{\mathcal{G}}}$ is $P_x^{\widetilde{\mathcal{G}}^{K^c}_{\infty}}$ for all $x\in K^c$, see Theorem 4.4.2 in [37], and we will often identify K^c and $\widetilde{\mathcal{G}}^{K^c}_{\infty}$. Moreover, using the Markov property for the Gaussian free field, see (IV.2.31), one can easily see that

$$(\varphi_x)_{x\in K^c} \text{ has the same law under } \mathbb{P}^G_{\widetilde{\mathcal{G}}}(\cdot\,|\,\varphi_{|K}=0) \text{ as } (\varphi_x)_{x\in\widetilde{\mathcal{G}}^{K^c}_{\infty}} \text{ under } \mathbb{P}^G_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}. \tag{V.4.17}$$

Using the last exit decomposition (V.2.10) and that the event $W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}$ for a trajectory in $W^{+}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}$ corresponds to the event $\{H_K=\widetilde{\zeta}\}\cap W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}$, for a trajectory in $W^{+}_{\widetilde{\mathcal{G}}}$, one can also easily show that for all compacts K' of $\widetilde{\mathcal{G}}^{K^c}_{\infty}$

$$e_{K',\widetilde{\mathcal{G}}^{K^c}_{\infty}}(x)P_x^{K',\widetilde{\mathcal{G}}^{K^c}_{\infty}}\big(\cdot,W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}\big)=e_{K',\widetilde{\mathcal{G}}}(x)P_x^{K',\widetilde{\mathcal{G}}}(\cdot,H_K=\widetilde{\zeta},W^{\mathcal{S},+}_{\widetilde{\mathcal{G}}}) \text{ for all } x\in\widetilde{\mathcal{G}}^{K^c}_{\infty}.$$

Therefore, by (V.2.11), (V.2.12) and Definition V.3.1, we obtain that $\nu^{\mathcal{S}}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}=\nu^{\mathcal{S}}_{\widetilde{\mathcal{G}}}((W^*_{K,\widetilde{\mathcal{G}}})^c,\cdot)$, where we identify with a slight abuse of notation the trajectories in $W^*_{\widetilde{\mathcal{G}}}$ not hitting K with trajectories in $W^*_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}$. In particular we obtain that

$$(\ell^{\mathcal{S},K^c}_{x,u})_{x\in K^c} \text{ has the same law under } \mathbb{P}^{SI}_{\widetilde{\mathcal{G}}} \text{ as } (\ell^{\mathcal{S}}_{x,u})_{x\in\widetilde{\mathcal{G}}^{K^c}_{\infty}} \text{ under } \mathbb{P}^{SI}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}. \tag{V.4.18}$$

Moreover, the function $\mathbf{h}_K$ on K^c corresponds to the function $\mathbf{h}_1$ on $\widetilde{\mathcal{G}}^{K^c}_{\infty}$, and since K is compact and $\mathcal{G}$ is transient, we have $\mathbf{h}_K\neq 0$. Noting that $\mathrm{cap}_{\widetilde{\mathcal{G}}^{K^c}_{\infty}}(F)\geq\mathrm{cap}_{\widetilde{\mathcal{G}}}(F)$ for all $F\subset G\cap K^c$, we also have that if condition (Cap) holds for $\mathcal{G}$, then it also holds for $\mathcal{G}^{K^c}_{\infty}$. By Remark V.4.7,4), we thus have that (V.4.15) holds for surviving random interlacements on $\widetilde{\mathcal{G}}^{K^c}_{\infty}$, and thus by (V.4.17) and (V.4.18), we obtain (V.1.5). □

Remark V.4.8. One could also derive a version of (V.1.5) for the discrete graph $\mathcal{G}$ similar to (V.4.16), or a formula for the law of the capacity, for the diffusion X on $\widetilde{\mathcal{G}}$ not hitting K, of the level sets of the Gaussian free field conditioned on being equal to 0 on K, similar to (V.4.14).

V.5 Coupling for different killing measures

In this section we prove Theorems V.1.1 and V.1.2. We first use Proposition V.3.3 to obtain a coupling of the local times of killed random interlacements under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$ for different values of the killing measure κ, see Proposition V.5.2, from which Theorem V.1.1 follows readily. Following ideas from [57], we also present a coupling of positive level sets $E^{\geq h}_{\widetilde{\mathcal{K}}}$ of the Gaussian free field under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$, $h \geq 0$, for different values of the killing measure κ, see Lemma V.5.4. Combining the two previous results with the isomorphism between killed random interlacements and the Gaussian free field, Theorem V.4.6, we finally obtain a coupling of the negative level sets $E^{\geq h}_{\widetilde{\mathcal{K}}}$ of the Gaussian free field under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$, $h \leq 0$, for different values of κ, see Proposition V.5.5, and deduce Theorem V.1.2.

Let us begin with an auxiliary lemma, which will be useful in the proof of Theorem V.1.1. Recall that for any killing measures κ' we defined $\mathcal{G}_{\kappa'} = (G, \lambda, \kappa')$, that we see $\widetilde{\mathcal{G}}_\kappa$ as a subset of $\widetilde{\mathcal{G}}_{\kappa'}$ if $\kappa' \leq \kappa$, and the definition of $c(\kappa, \kappa')$ from (V.1.2). We also write $W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y) \subset W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}$ for the event that a trajectory on $\widetilde{\mathcal{G}}_\kappa$ is killed on I_y, $y \in G$ and $H_{\widetilde{\mathcal{G}}^c_\kappa}$ the first time the diffusion X on $\widetilde{\mathcal{G}}_{\kappa'}$ leaves $\widetilde{\mathcal{G}}_\kappa$.

Lemma V.5.1. *For any $\kappa' \leq \kappa$, $x, y \in G$ and $A \in \mathcal{W}^+_{\widetilde{\mathcal{G}}_\kappa}$ we have*

$$\frac{\kappa_x \mathbf{h}^\kappa_0(x) P^{\widetilde{\mathcal{G}}_\kappa}_x\big(A \cap W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y) \,\big|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}\big)}{\kappa'_x \mathbf{h}^{\kappa'}_0(x) P^{\widetilde{\mathcal{G}}_{\kappa'}}_x\big((X_t)_{t<H_{\widetilde{\mathcal{G}}^c_\kappa}} \in A \cap W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y) \,\big|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_{\kappa'}}\big)} = \frac{\kappa_x \kappa_y}{\kappa'_x(\kappa'_y + (\kappa_y - \kappa'_y)\mathbf{h}^{\kappa'}_0(y))}$$
$$\leq c(\kappa, \kappa').$$

Proof. Let $z_y \in \widetilde{\mathcal{G}}_{\kappa'}$ be the only point in $I_y \cap \partial \widetilde{\mathcal{G}}_\kappa$ if $\kappa'_y < \kappa_y$, and $z_y = \Delta$ otherwise. By (IV.2.9) and the strong Markov property at time $H_{\widetilde{\mathcal{G}}^c_\kappa}$, we have

$$P^{\widetilde{\mathcal{G}}_{\kappa'}}_x\big((X_t)_{t<H_{\widetilde{\mathcal{G}}^c_\kappa}} \in A \cap W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y), W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_{\kappa'}}\big) = P^{\widetilde{\mathcal{G}}_\kappa}_x\big(A \cap W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y)\big)\mathbf{h}^{\kappa'}_0(z_y),$$

with the convention $\mathbf{h}^{\kappa'}_0(\Delta) = 1$. Since X behave like a Brownian motion on I_y until the first time it hits y, we have using general results about Brownian motion, see for instance equation 3.0.4 (b), in Part II of [13]

$$P^{\widetilde{\mathcal{G}}_{\kappa'}}_{z_y}(H_y = \widetilde{\zeta}) = \frac{1/(2\kappa_y)}{1/(2\kappa'_y)} = \frac{\kappa'_y}{\kappa_y},$$

and so

$$\mathbf{h}^{\kappa'}_0(z_y) = \frac{\kappa'_y}{\kappa_y} + \big(1 - \frac{\kappa'_y}{\kappa_y}\big)\mathbf{h}^{\kappa'}_0(y) = \frac{\kappa'_y + (\kappa_y - \kappa'_y)\mathbf{h}^{\kappa'}_0(y)}{\kappa_y},$$

and the result then follows from the definition of $c(\kappa, \kappa')$, see (V.1.2). □

Applying Proposition V.3.3 and Lemma V.5.1, we find a coupling of local times of killed random interlacements on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$, $\kappa' \leq \kappa$. The restriction $\widetilde{\mathcal{G}}_{\kappa'}^E$ of $\widetilde{\mathcal{G}}_{\kappa'}$ to edges, as defined above Lemma V.3.2, is independent of the choice of κ', and we will simply denote it by $\widetilde{\mathcal{G}}^E$.

Proposition V.5.2. *For any $\kappa' \leq \kappa$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which one can define for all $u > 0$ families $(\ell_{x,u}^{\mathcal{K},\kappa})_{x\in\widetilde{\mathcal{G}}^E} \in [0,\infty)^{\widetilde{\mathcal{G}}^E}$ and $(\ell_{x,u}^{\mathcal{K},\kappa'})_{x\in\widetilde{\mathcal{G}}^E} \in [0,\infty)^{\widetilde{\mathcal{G}}^E}$ with the same law as the restriction of the local times $(\ell_{x,u}^{\mathcal{K}})_{x\in\widetilde{\mathcal{G}}^E}$ of killed random interlacements to $\widetilde{\mathcal{G}}^E$, respectively under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$ and $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}$, and such that if $u' \geq uc(\kappa,\kappa')$, then $\ell_{x,u}^{\mathcal{K},\kappa} \leq \ell_{x,u'}^{\mathcal{K},\kappa'}$ for all $x \in \widetilde{\mathcal{G}}^E$.*

Proof. On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us define for each $x \in G$ a Poisson point process ω^x on $W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_{\kappa'}} \times (0,\infty)$ with intensity $P_x^{\widetilde{\mathcal{G}}_{\kappa'}}(\cdot \,|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_{\kappa'}}) \otimes \lambda$, where λ is the Lebesgue measure on $(0,\infty)$. For all $x \in G$ and $u > 0$, let $\omega_u^{x,\kappa'}$ be the point process on $W^+_{\widetilde{\mathcal{G}}_{\kappa'}}$ consisting of the trajectories of ω^x with label at most $u\kappa'_x \mathbf{h}_0^{\kappa'}(x)$. For all $x, y \in G$ let $\omega_u^{x,y,\kappa}$ be the point process on $W^+_{\widetilde{\mathcal{G}}_\kappa}$ obtained by only keeping the trajectories of ω^x with label at most

$$u\kappa'_x \mathbf{h}_0^{\kappa'}(x) \frac{\kappa_x \kappa_y}{\kappa'_x(\kappa'_y + (\kappa_y - \kappa'_y)\mathbf{h}_0^{\kappa'}(y))},$$

which leave $\widetilde{\mathcal{G}}_\kappa$ on I_y for the first time, and stopping them after the first leaving time $H_{\widetilde{\mathcal{G}}_\kappa^c}$ of $\widetilde{\mathcal{G}}_\kappa$. It follows from Lemma V.5.1 that $\omega_u^{x,y,\kappa}$ is a Poisson point process with intensity $u\kappa_x \mathbf{h}_0^\kappa(x) P_x^{\widetilde{\mathcal{G}}_\kappa}\big(\cdot, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}(y) \,|\, W^{\mathcal{K},+}_{\widetilde{\mathcal{G}}_\kappa}\big)$. For each $x, y \in G$, let $(\ell_{z,u}^{x,\kappa'})_{z\in\widetilde{\mathcal{G}}^E}$ be the restriction to $\widetilde{\mathcal{G}}^E$ of the local times of all the trajectories in $\omega_u^{x,\kappa'}$ and $(\ell_{z,u}^{x,y,\kappa})_{z\in\widetilde{\mathcal{G}}^E}$ be the restriction to $\widetilde{\mathcal{G}}^E$ of the local times of all the trajectories in $\omega_u^{x,y,\kappa}$. Defining for all $z \in \widetilde{\mathcal{G}}^E$ and $u > 0$

$$\ell_{z,u}^{\mathcal{K},\kappa'} = \sum_{x\in G} \ell_{z,u}^{x,\kappa'} \quad \text{and} \quad \ell_{z,u}^{\mathcal{K},\kappa} = \sum_{x,y\in G} \ell_{z,u}^{x,y,\kappa},$$

it follows from Proposition V.3.3 that $(\ell_{z,u}^{\mathcal{K},\kappa'})_{z\in\widetilde{\mathcal{G}}^E}$ and $(\ell_{z,u}^{\mathcal{K},\kappa})_{z\in\widetilde{\mathcal{G}}^E}$ have the same law as $(\ell_{x,u}^{\mathcal{K}})_{x\in\widetilde{\mathcal{G}}^E}$ respectively under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}$ and $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$. By Lemma V.5.1, if $u' \geq uc(\kappa,\kappa')$, then for all $x, y \in G$ any trajectory in $\omega_u^{x,y,\kappa}$ corresponds to a trajectory in $\omega_{u'}^{x,\kappa'}$ stopped when exiting $\widetilde{\mathcal{G}}_\kappa$ on I_y, and thus $\ell_{z,u}^{\mathcal{K},\kappa} \leq \ell_{z,u'}^{\mathcal{K},\kappa'}$ for all $z \in \widetilde{\mathcal{G}}^E$. □

Noting that finding an unbounded connected component in $\mathcal{I}_{\mathcal{K}}^u$ is equivalent to finding an unbounded connected component in $\mathcal{I}_{\mathcal{K}}^u \cap \widetilde{\mathcal{G}}^E$, Theorem V.1.1 is a direct consequence of Proposition V.5.2

Proof of Theorem V.1.1. Let us define for all $u>0$ the random sets $\mathcal{I}^u_{\mathcal{K},\kappa}=\{x\in\widetilde{\mathcal{G}}^E:\ell^{\mathcal{K},\kappa}_{x,u}>0\}$ and $\mathcal{I}^u_{\mathcal{K},\kappa'}=\{x\in\widetilde{\mathcal{G}}^E:\ell^{\mathcal{K},\kappa'}_{x,u}>0\}$, which have the same law as the killed random interlacement set $\mathcal{I}^u_{\mathcal{K}}\cap\widetilde{\mathcal{G}}^E$ respectively under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$ and $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}$. For all $u>u_*^{\mathcal{K},\mathcal{I}}(\widetilde{\mathcal{G}}_\kappa)$, we have that $\mathcal{I}^u_{\mathcal{K}}$ contains $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$-a.s. an unbounded connected component, and so $\mathcal{I}^u_{\mathcal{K},\kappa}$ contains $\mathbb{P}$-a.s. an unbounded connected component. Using Proposition V.5.2, taking $u'=uc(\kappa,\kappa')$, we have $\mathcal{I}^u_{\mathcal{K},\kappa}\subset\mathcal{I}^{u'}_{\mathcal{K},\kappa'}$, and so $\mathcal{I}^{u'}_{\mathcal{K},\kappa'}$ contains $\mathbb{P}$-a.s. an unbounded connected component, that is $u'\geq u_*^{\mathcal{K},\mathcal{I}}(\widetilde{\mathcal{G}}_{\kappa'})$. □

Remark V.5.3. 1) One could also derive from Proposition V.3.3 results similar to Proposition V.5.2 and Theorem V.1.1 for backward-killed and forwards-killed random interlacements, but this is less interesting since backwards-killed and forward-killed interlacements are either empty or always contain an unbounded connected component, that is their phase transition is always trivial.

2) One cannot extend Proposition V.5.2 in order to find families $(\ell^{\mathcal{K},\kappa}_{x,u})_{x\in\widetilde{\mathcal{G}}_\kappa}$ and $(\ell^{\mathcal{K},\kappa'}_{x,u})_{x\in\widetilde{\mathcal{G}}_{\kappa'}}$ with the same law as the local times of killed random interlacements respectively under $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$ and $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}$, and such that for a right choice of $u,u'>0$, $\ell^{\mathcal{K},\kappa}_{x,u}\leq\ell^{\mathcal{K},\kappa'}_{x,u'}$ for all $x\in\widetilde{\mathcal{G}}_\kappa$. Indeed let us fix some $x\in G$ with $\kappa'_x\neq\kappa_x$, and let $y\in I_x\cap\partial\widetilde{\mathcal{G}}_\kappa$. Then $\lim_{z\to y}\ell^{\mathcal{K}}_{z,u}=u$ $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_\kappa}$-a.s, which follows from instance from (V.4.15), whereas $\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}(\ell^{\mathcal{K}}_{y,u'}=0)=\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}_{\kappa'}}(y\notin\mathcal{I}^{u'})>0$ by (V.3.5), which would be a contradiction.

3) Since $c(\kappa,a\kappa)\leq a^{-2}$ for all constants $a\in(0,1]$, it follows from ii) and iii) of Theorem V.5.5 that if (Sign) holds for $\mathcal{G}_\kappa$, then $a^2\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_{a\kappa})$ is increasing in $a\in(0,\infty)$. In particular, if $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_\kappa)>0$, then $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_{a\kappa})>0$ for all $a\geq1$. It is an interesting open question whether this also holds when $\sup_{x\in G}\frac{\kappa_x}{\kappa'_x}=\infty$, or more generally if $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}_\kappa)$ is actually increasing in κ.

4) Taking complements in Proposition V.5.2, one can derive a result analogue to Theorem V.1.1 but for the vacant set of killed random interlacements $\mathcal{V}^u_{\mathcal{K}}:=(\mathcal{I}^u_{\mathcal{K}})^c$. More precisely let us denote by $\widetilde{u}_*^{\mathcal{K},\mathcal{V}}$ the critical parameter associated with the percolation of the vacant set of killed random interlacements on the cable system, that is

$$\widetilde{u}_*^{\mathcal{K},\mathcal{V}}(\mathcal{G})\stackrel{\text{def.}}{=}\inf\{u\geq0:\mathbb{P}^{KI}_{\widetilde{\mathcal{G}}}(\mathcal{V}^u_{\mathcal{K}}\text{ contains an unbounded cluster })=0\}.\tag{V.5.1}$$

For all killing measures κ' with $\kappa\geq\kappa'$, we then have

$$\widetilde{u}_*^{\mathcal{K},\mathcal{V}}(\mathcal{G}_{\kappa'})\leq\widetilde{u}_*^{\mathcal{K},\mathcal{V}}(\mathcal{G}_\kappa)c(\kappa,\kappa').$$

Similar results can also be obtained when considering percolation for the discrete killed random interlacement set $\mathcal{I}^u_{\mathcal{K}} \cap G$, or the discrete vacant set of killed random interlacements, $\mathcal{V}^u_{\mathcal{K}} \cap G$.

5) The phase transition for the percolation of $\mathcal{I}^u_{\mathcal{K}}$ can either be trivial or not depending on the choice of the graph $\mathcal{G}$, as we now explain. First consider the case where $\mathcal{G}$ is a finite graph with $\kappa > 0$, then $\widetilde{\mathcal{G}}$ is bounded, and so it is clear that $\widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}) = \infty$, that is the phase transition is trivial. Let us now consider for some $T > 0$ and $d \geq 3$ the graph $\mathcal{G}^T = (\mathbb{Z}^d, \lambda^T, \kappa^T)$, $d \geq 3$, where $\lambda^T_{x,y} = \frac{T}{T+1}$ and $\kappa^T_x = \frac{2d}{T+1}$. Then by Proposition V.3.3, or Corollary 4.2 in [15], $\mathcal{I}^u \cap \mathbb{Z}^d = \mathcal{I}^u_{\mathcal{K}} \cap \mathbb{Z}^d$ has the same law as the finitary random interlacement set introduced in [15]. One can prove similarly as in Theorem 2 of [71] that there exists T small such that $\mathcal{I}^1_{\mathcal{K}}$ is $\mathbb{P}^{\mathcal{K}I}_{\widetilde{\mathcal{G}}^T}$-a.s. bounded. Moreover for each edge $e = \{x, y\}$, the number of trajectories starting in x and crossing first e in a Poisson point process with intensity $u\widetilde{\nu}^{\mathcal{K}}_{\mathcal{G}}$, see (V.1.4), has law

$$\mathrm{Poi}\Big(u\kappa^T_x \frac{\lambda^T_{x,y}}{\lambda^T_x}\Big) = \mathrm{Poi}\Big(\frac{uT}{(T+1)^2}\Big).$$

Therefore for any u large enough so that $1 - \exp(-uT/(T+1)^2) > p_c$, where p_c is the critical parameter for Bernoulli bond percolation on $\mathbb{Z}^d$, there is an infinite connected component of edges crossed by the discrete killed random interlacement process $\omega_u^{\mathcal{K},\mathcal{G}^T}$, and thus $\mathcal{I}^u_{\mathcal{K}}$ contains an infinite connected component. We obtain that $0 < \widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}^T) < \infty$ when T is small enough, and thus the phase transition is non-trivial. We expect that this result could actually be extended to any $T > 0$.

We now turn to the proof of the inequalities between the critical parameters for the level sets of the Gaussian free field for different values of the killing measure, Theorem V.1.2. We first present a coupling of positive killed level sets of the Gaussian free field under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$ for different values of κ, which is a direct consequence of the isomorphism between loop soups and the Gaussian free field from [57], and will easily imply i) of Theorem V.1.2. The proof is similar to the proof of Lemma IV.7.4, but we still include it for completeness.

Lemma V.5.4. *For any $\kappa' \leq \kappa$, $\kappa' \neq 0$, and $x_0 \in \widetilde{\mathcal{G}}_\kappa$, there exist a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which one can define random fields $(\varphi_x^{(\kappa)})_{x \in \widetilde{\mathcal{G}}_\kappa}$ and $(\varphi_x^{(\kappa')})_{x \in \widetilde{\mathcal{G}}_{\kappa'}}$ with respective laws $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$ and $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$, and such that $|\varphi_x^{(\kappa)}| \leq |\varphi_x^{(\kappa')}|$ for all $x \in \widetilde{\mathcal{G}}_\kappa$ and $sign(\varphi_{x_0}^{(\kappa)}) = sign(\varphi_{x_0}^{(\kappa')})$. In particular, for all $0 \leq h' \leq h$, denoting by $E^{\geq h}_{\mathcal{K},\kappa}(x_0)$ the cluster of x_0 in $\{x \in \widetilde{\mathcal{G}}^E : \varphi_x^{(\kappa)} \geq h \times \mathbf{h}_0^\kappa(x)\}$ and $E^{\geq h'}_{\mathcal{K},\kappa'}(x_0)$ the cluster of x_0 in $\{x \in \widetilde{\mathcal{G}}^E : \varphi_x^{(\kappa')} \geq h' \times \mathbf{h}_0^{\kappa'}(x)\}$, we have $E^{\geq h}_{\mathcal{K},\kappa}(x_0) \subset E^{\geq h'}_{\mathcal{K},\kappa'}(x_0)$.*

Proof. Let $(\Omega',\mathcal{F}',\mathbb{P}')$ be some probability space on which one can define a loop soup $\mathcal{L}'_{\frac12}$ with parameter $\frac12$ associated with the diffusion X under $P_x^{\widetilde{\mathcal{G}}_{\kappa'}}$, $x\in\widetilde{\mathcal{G}}_{\kappa'}$, as defined in Section 3 of [35]. Let $\mathcal{L}_{\frac12}$ be the point process which consists of the loops in $\mathcal{L}'_{\frac12}$ entirely included in $\widetilde{\mathcal{G}}_{\kappa}(\subset\widetilde{\mathcal{G}}_{\kappa'})$, which has the same law by Theorem 6.1 in [35] as the loop soup with parameter $\frac12$ associated with the diffusion X under $P_x^{\widetilde{\mathcal{G}}_{\kappa}}$, $x\in\widetilde{\mathcal{G}}_{\kappa}$. We define $(L_x)_{x\in\widetilde{\mathcal{G}}_{\kappa}}$ and $(L'_x)_{x\in\widetilde{\mathcal{G}}_{\kappa'}}$ the continuous fields of local times respectively associated with $\mathcal{L}_{\frac12}$ and $\mathcal{L}'_{\frac12}$, which exist by Lemma 2.2 in [57]. Let finally $\sigma\in\{-1,1\}^{\widetilde{\mathcal{G}}_{\kappa}}$ and $\sigma'\in\{-1,1\}^{\widetilde{\mathcal{G}}_{\kappa'}}$ be two additional processes such that, conditionally on $\mathcal{L}'_{\frac12}$, σ is constant on each cluster of $\{x\in\widetilde{\mathcal{G}}_{\kappa}: L_x>0\}$, and its values on each cluster is independent and uniformly distributed, σ' is constant on each cluster of $\{x\in\widetilde{\mathcal{G}}_{\kappa'}: L'_x>0\}$, and its values on each cluster is independent and uniformly distributed, and coupled so that $\sigma_{x_0}=\sigma'_{x_0}$. Then by Theorem 3.1 in [35] and Lemma 3.2 in [57], we have that

$$\varphi^{(\kappa)}\stackrel{\text{def.}}{=}(\sigma_x\sqrt{2L_x})_{x\in\widetilde{\mathcal{G}}_{\kappa}}\text{ has law }\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa}}\text{ and }\varphi^{(\kappa')}\stackrel{\text{def.}}{=}(\sigma'_x\sqrt{2L'_x})_{x\in\widetilde{\mathcal{G}}_{\kappa'}}\text{ has law }\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}.$$

Since $\sigma_{x_0}=\sigma'_{x_0}$, $L_x\leq L'_x$ and $\mathbf{h}_0^{\kappa'}(x)\leq\mathbf{h}_0^{\kappa}(x)$ for all $x\in\widetilde{\mathcal{G}}_{\kappa}$, we can easily conclude. □

Using Theorem V.4.6, one can relate negative killed level sets of the Gaussian free field to sign clusters of the Gaussian free field and killed random interlacement set, and with the help of Proposition V.5.2 and Lemma V.5.4, one can also find a coupling of negative level sets of the Gaussian free field on the cable system under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa}}$ for different values of κ, which will easily imply ii) of Theorem V.1.2.

Proposition V.5.5. *Let $\kappa'\leq\kappa$, $\kappa'\neq0$, and assume that $\mathcal{G}_{\kappa}$ and $\mathcal{G}_{\kappa'}$ both satisfies* (Sign). *For any $x_0\in\widetilde{\mathcal{G}}^E$, there exist a probability space $(\Omega'',\mathcal{F}'',\mathbb{P}'')$ on which one can define for all $h\leq0$ random sets $E^{\geq h}_{\mathcal{K},\kappa'}(x_0)\subset\widetilde{\mathcal{G}}^E$ and $E^{\geq h}_{\mathcal{K},\kappa'}(x_0)\subset\widetilde{\mathcal{G}}^E$ with the same law as the level set $E^{\geq h}_{\mathcal{K}}(x_0)\cap\widetilde{\mathcal{G}}^E$ respectively under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa}}$ and $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$, and such that if $h,h'\leq0$ with $h'\leq h\sqrt{c(\kappa,\kappa')}$, then $E^{\geq h}_{\mathcal{K},\kappa}(x_0)\subset E^{\geq h'}_{\mathcal{K},\kappa'}(x_0)$ if $E^{\geq h}_{\mathcal{K},\kappa}(x_0)$ is unbounded.*

Proof. Let $(\Omega'',\mathcal{F}'',\mathbb{P}'')=(\Omega\times\Omega',\mathcal{F}\otimes\mathcal{F}',\mathbb{P}\otimes\mathbb{P}')$ be the product of the probability spaces from Proposition V.5.2 and Lemma V.5.4, and $(\ell^{\mathcal{K},\kappa}_{x,u})_{x\in\widetilde{\mathcal{G}}_{\kappa}}$ and $(\ell^{\mathcal{K},\kappa'}_{x,u})_{x\in\widetilde{\mathcal{G}}_{\kappa'}}$ some extensions of $(\ell^{\mathcal{K},\kappa}_{x,u})_{x\in\widetilde{\mathcal{G}}^E}$ and $(\ell^{\mathcal{K},\kappa'}_{x,u})_{x\in\widetilde{\mathcal{G}}^E}$ with the same law as $\ell^{\mathcal{K}}_{\cdot,u}$ respectively under $\mathbb{P}^I_{\widetilde{\mathcal{G}}_{\kappa}}$ and $\mathbb{P}^I_{\widetilde{\mathcal{G}}_{\kappa'}}$. Let us define for all $u>0$ the random sets $\mathcal{I}^u_{\mathcal{K},\kappa}=\{x\in\widetilde{\mathcal{G}}_{\kappa}:\ell^{\mathcal{K},\kappa}_{x,u}>0\}$ and $\mathcal{I}^u_{\mathcal{K},\kappa'}=\{x\in\widetilde{\mathcal{G}}_{\kappa'}:\ell^{\mathcal{K},\kappa'}_{x,u}>0\}$, and for each $x\in\widetilde{\mathcal{G}}_{\kappa}$, $E^{|>0|}_{\mathcal{K},\kappa}(x)$ the cluster of x in $\{y\in\widetilde{\mathcal{G}}_{\kappa}:|\varphi^{(\kappa)}_y|>0\}$ and $E^{|>0|}_{\mathcal{K},\kappa'}(x)$ the cluster of x in $\{y \in \widetilde{\mathcal{G}}_{\kappa'} : |\varphi_y^{(\kappa')}| > 0\}$. We also define for all $u > 0$ and $x \in \widetilde{\mathcal{G}}_\kappa$

$$\psi_x^{(\kappa,u)} \stackrel{\text{def.}}{=} \begin{cases} -\sqrt{2u}\mathbf{h}_0^\kappa(x) + \sqrt{2\ell_{x,u}^{\mathcal{K},\kappa} + (\varphi_x^{(\kappa)})^2} & \text{if } \mathcal{I}_{\mathcal{K},\kappa}^u \cap E_{\mathcal{K},\kappa}^{|>0|}(x) \neq \varnothing \\ -\sqrt{2u}\mathbf{h}_0^\kappa(x) + \varphi_x^{(\kappa)} & \text{otherwise,} \end{cases}$$

and $\psi_x^{(\kappa',u)}$ for all $x \in \widetilde{\mathcal{G}}_{\kappa'}$ similarly by replacing κ by κ'. Since (Sign) holds for both $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$, we also have that (V.4.14), and thus (V.4.15), hold as well. One can thus easily check that $\psi^{(\kappa,u)}$ and $\psi^{(\kappa',u)}$ have the same law as φ respectively under $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$ and $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$, similarly as in (Isom) in Chapter IV. For all $u \geq 0$, we then define $E_{\mathcal{K},\kappa}^{\geq -\sqrt{2u}}(x_0)$ as the closure of the cluster of x_0 in $\{x \in \widetilde{\mathcal{G}}^E : \psi_x^{(\kappa,u)} > -\sqrt{2u}\mathbf{h}_0^\kappa(x)\}$ and $E_{\mathcal{K},\kappa'}^{\geq -\sqrt{2u}}(x_0)$ as the closure of the cluster of x_0 in $\{x \in \widetilde{\mathcal{G}}^E : \psi_x^{(\kappa',u)} > -\sqrt{2u}\mathbf{h}_0^{\kappa'}(x)\}$, with the convention $\psi^{(\kappa,0)} = \varphi^{(\kappa)}$ and $\psi^{(\kappa',0)} = \varphi^{(\kappa')}$.

Let us now fix $u, u' > 0$ with $u' \geq uc(\kappa,\kappa')$ and $x \longleftrightarrow x_0$ in $\{y \in \widetilde{\mathcal{G}}^E : \psi_y^{(\kappa,u)} > -\sqrt{2u}\}$, and assume that $E_{\mathcal{K},\kappa}^{\geq -\sqrt{2u}}(x_0)$ is unbounded. We then necessarily have $\mathcal{I}_{\mathcal{K},\kappa}^u \cap E_{\mathcal{K},\kappa}^{|>0|}(x_0) \cap \widetilde{\mathcal{G}}^E \neq \varnothing$, since otherwise $E_{\mathcal{K},\kappa}^{\geq -\sqrt{2u}}(x_0)$ would be either empty or the closure of the cluster of x_0 in $\{x \in \widetilde{\mathcal{G}}^E : |\varphi_x| > 0\}$, and thus bounded by (Sign). We then have that $x \longleftrightarrow x_0$ in $\{y \in \widetilde{\mathcal{G}}^E : 2\ell_{y,u}^{\mathcal{K},\kappa} + (\varphi_y^{(\kappa)})^2 > 0\}$, thus $x \longleftrightarrow x_0$ in $\{y \in \widetilde{\mathcal{G}}^E : 2\ell_{y,u'}^{\mathcal{K},\kappa'} + (\varphi_y^{(\kappa')})^2 > 0\}$ and $\mathcal{I}_{\mathcal{K},\kappa'}^{u'} \cap E_{\mathcal{K},\kappa'}^{|>0|}(x_0) \neq \varnothing$, by Proposition V.5.2 and Lemma V.5.4, and so $x \in E_{\mathcal{K},\kappa'}^{\geq -\sqrt{2u'}}(x_0)$. Taking closure, we thus obtain $E_{\mathcal{K},\kappa}^{\geq -\sqrt{2u}}(x_0) \subset E_{\mathcal{K},\kappa'}^{\geq -\sqrt{2u'}}(x_0)$, and we can easily conclude. □

Noting that, $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$-a.s, $E_{\mathcal{K}}^{\geq h}$ contains an unbounded cluster if and only if $E_{\mathcal{K}}^{\geq h} \cap \widetilde{\mathcal{G}}^E$ also contains an unbounded cluster, Theorem V.1.2 is a simple consequence of the couplings from Lemma V.5.4 and Proposition V.5.5.

Proof of Theorem V.1.2. Let us first assume that (Sign) does not hold for $\mathcal{G}_\kappa$. Then for all $x_0 \in \widetilde{\mathcal{G}}^E$, $E_{\mathcal{K}}^{\geq 0}(x_0)$ is unbounded with $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$ positive probability, and by Lemma V.5.4 with $h = h' = 0$, $E_{\mathcal{K}}^{\geq 0}(x_0)$ is also unbounded with $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$ positive probability, that is (Sign) does not hold for $\mathcal{G}_{\kappa'}$, $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa) \geq 0$ and $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) \geq 0$. For all $h > \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'})$ and $x_0 \in \widetilde{\mathcal{G}}^E$, we have that $E_{\mathcal{K}}^{\geq h}(x_0)$ is $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$-a.s. bounded, and by Lemma V.5.4 with $h = h'$, $E_{\mathcal{K}}^{\geq h}(x_0)$ is also $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$-a.s. bounded, that is $h > \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa)$, and we obtain i) of Theorem V.1.2.

Let us now assume that (Sign) holds for $\mathcal{G}_{\kappa'}$ and $\sup_{x \in G} \frac{\kappa_x}{\kappa'_x} < \infty$. Then (Sign) also holds for $\mathcal{G}_\kappa$ by i) of Theorem V.1.2, and so (V.4.14) holds for $h = 0$, and thus also (V.4.15), for both $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$. Moreover $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) \leq 0$ and $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa) \leq 0$, and for all $h < \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa)$ and $x_0 \in \widetilde{\mathcal{G}}^E$, we have that $E_{\mathcal{K}}^{\geq h}(x_0)$ is unbounded with $\mathbb{P}^G_{\widetilde{\mathcal{G}}_\kappa}$ positive probability. Taking $h' = h\sqrt{c(\kappa,\kappa')}$, we have by Proposition V.5.5 that

$E_{\mathcal{K}}^{\geq h'}(x_0)$ is unbounded with $\mathbb{P}^G_{\widetilde{\mathcal{G}}_{\kappa'}}$ positive probability, that is $h' < \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{\kappa'})$, and we obtain ii) of Theorem V.1.2. Finally, iii) of Theorem V.1.2 follows directly from the definitions (V.1.3) and (Sign). □

Remark V.5.6. 1) Since $c(\kappa, a\kappa) \leq a^{-2}$ for all constants $a \in (0,1]$, it follows from ii) and iii) of Theorem V.5.5 that if (Sign) holds for $\mathcal{G}_\kappa$, then $a\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{a\kappa}) \geq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa)$ for all $a \in (0,1]$, and from i) and iii) of Theorem V.5.5 that if (Sign) does not hold for $\mathcal{G}_\kappa$, then $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{a\kappa}) \leq \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_\kappa)$ for all $a \in [1,\infty)$. It is an interesting open question to prove that either $a \mapsto a\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{a\kappa})$ or $a \mapsto \widetilde{h}_*^{\mathcal{K}}(\mathcal{G}_{a\kappa})$ is actually decreasing on $(0,\infty)$, or even as functions of κ.

2) One can adapt the proof of Lemma V.5.4 to find a coupling of positive surviving level sets of the Gaussian free field on $\mathcal{G}_\kappa$ and $\mathcal{G}_{\kappa'}$, and obtain a result similar to i) of Theorem V.1.2 for positive surviving level sets of the Gaussian free field, with an extra term since $\mathbf{h}_1^\kappa \leq \mathbf{h}_1^{\kappa'}$, or even positive usual level sets of the Gaussian free field. It would be interesting to prove an equivalent of Theorem V.5.5 for killed level sets of the Gaussian free field $(\varphi_x)_{x\in G}$ on the discrete graph $\mathcal{G}$, for instance by proving an equivalent of Lemma V.5.4 for the discrete Gaussian free field. Indeed, it is reasonable to think that percolation for the sign clusters of the Gaussian free field converges to Bernoulli percolation for $p = \frac{1}{2}$ as $\kappa \to \infty$, and a statement similar to i) in Theorem V.1.2 would imply that sign clusters of the discrete Gaussian free field percolate whenever $p_c < \frac{1}{2}$, which is so far only known on a smaller class of graph studied in Chapter III.

3) Proceeding similarly as in the proof of Theorem IV.3.6, one can easily prove that if (V.4.14) holds and (Sign) does not hold, then $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}) = \infty$. We currently do not know any examples of a graph under which (V.4.14) does not hold, and thus all the critical parameters appearing in i) of Theorem V.1.2 might always be infinite, and the statement then only says that if $h_*^{\mathcal{K}}(\mathcal{G}_\kappa) = \infty$, then $h_*^{\mathcal{K}}(\mathcal{G}_{\kappa'}) = \infty$ for all $\kappa' \leq \kappa$. However, when (Sign) does hold, we expect that there are many examples of graphs with $\widetilde{h}_*^{\mathcal{K}}(\mathcal{G}) \in (-\infty, 0)$, for instance on the graphs $\mathcal{G}^T$ considered in Remark V.5.3,5) on which we already know that $0 < \widetilde{u}_*^{\mathcal{K},\mathcal{I}}(\mathcal{G}^T) < \infty$ for T small enough.

4) One can easily see that if (V.4.15) holds, then for all $u > 0$, $\mathcal{I}_{\mathcal{K}}^u$ is stochastically dominated by $E_{\mathcal{K}}^{\geq -\sqrt{2u}}$ and $E_{\mathcal{K}}^{\geq \sqrt{2u}}$ by $\mathcal{V}_{\mathcal{K}}^u$, similarly as in Theorem 3 of [57], and thus

$$-\sqrt{2\widetilde{u}_*^{\mathcal{K},\mathcal{I}}} \leq \widetilde{h}_*^{\mathcal{K}} \leq \sqrt{2\widetilde{u}_*^{\mathcal{K},\mathcal{V}}}, \tag{V.5.2}$$

where $\widetilde{u}_*^{\mathcal{K},\mathcal{V}}$ is the critical parameter corresponding to the percolation of $\mathcal{V}_{\mathcal{K}}^u$ as defined in (V.5.1). The inequalities in (V.5.2) also hold on the discrete graph G, and it would be interesting to know which of these inequalities are strict, similarly as in Theorem 3.4 in [1] for instance.

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Alexander Drewitz betreut worden.

In Cologne, July 17, 2020

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Curriculum Vitae

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Research Interests

title	Percolation with long-range correlation
description	Percolation properties of models with long-range correlations, including the Gaussian free field and random interlacements

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Publications

[25] The sign clusters of the massless Gaussian free field percolate on $\mathbb{Z}^d$, $d \geq 3$ (and more)

[26] Geometry of Gaussian free field sign clusters and random interlacements