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Alexander Heinlein, Axel Klawonn, Jascha Knepper, Oliver Rheinbach, Olof B. Widlund

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ADAPTIVE GDSW COARSE SPACES OF REDUCED DIMENSION
FOR OVERLAPPING SCHWARZ METHODS

ALEXANDER HEINLEIN∗†, AXEL KLAWONN∗†, JASCHA KNEPPER∗, OLIVER
RHEINBACH‡, AND OLOF B. WIDLUND§

Abstract. A new reduced dimension adaptive GDSW (Generalized Dryja-Smith-Widlund) overlapping Schwarz method for linear second-order elliptic problems in three dimensions is introduced. It is robust with respect to large contrasts of the coefficients of the partial differential equations. The condition number bound of the new method is shown to be independent of the coefficient contrast and only dependent on a user-prescribed tolerance. The interface of the nonoverlapping domain decomposition is partitioned into nonoverlapping patches. The new coarse space is obtained by selecting a few eigenvectors of certain local eigenproblems which are defined on these patches. These eigenmodes are energy-minimally extended to the interior of the nonoverlapping subdomains and added to the coarse space. By using a new interface decomposition the reduced dimension adaptive GDSW overlapping Schwarz method usually has a smaller coarse space than existing GDSW and adaptive GDSW domain decomposition methods. A robust condition number estimate is proven for the new reduced dimension adaptive GDSW method which is also valid for existing adaptive GDSW methods. Numerical results for the equations of isotropic linear elasticity in three dimensions confirming the theoretical findings are presented.

Key words. domain decomposition, multiscale, GDSW, overlapping Schwarz, adaptive coarse spaces, reduced dimension

AMS subject classifications. 65F08,65F10,65N55,68W10

1. Introduction. Successful domain decomposition preconditioners for solving elliptic problems all require at least one global, coarse-level component in order to perform satisfactorily if the number of subdomains, into which the given domain has been decomposed, is relatively large. The design and analysis of these coarse components is central in most studies in this field given that they require global communication if the algorithms are implemented on distributed or parallel computing systems. In order to avoid creating a bottleneck, it is very important to keep the dimension of the related coarse space small.

In recent years, substantial progress has been possible by the development of algorithms which adaptively design the coarse space at a cost of solving local generalized eigenvalue problems. In this paper, we will focus on a particular family of domain decomposition algorithms, the two-level overlapping Schwarz methods, which use one coarse-level component in addition to local components each of which is defined on a subdomain which is part of an overlapping decomposition. We note that the use of adaptively designed coarse spaces has been very successful even with problems with very irregular coefficients; this is clearly demonstrated by examples in section 14 of this paper.

The robustness of many coarse spaces for arbitrary coefficient functions is obtained by using local generalized eigenvalue problems to adaptively enrich the coarse spaces with suitable basis functions; see, e.g., [14, 10, 41, 15, 20, 13]. These approaches differ, e.g., in the sizes of the eigenvalue problems, the coarse space dimensions, the class of problems considered, and their parallel efficiency. We also

∗Department of Mathematics and Computer Science, University of Cologne, Weyertal 86-90, 50931 Köln, Germany, \{alexander.heinlein, axel.klawonn, jascha.knepper\}@uni-koeln.de, http://www.numerik.uni-koeln.de
†Center for Data and Simulation Science, University of Cologne, 50931 Köln, Germany, http://www.cds.uni-koeln.de
‡Institut für Numerische Mathematik und Optimierung, Fakultät für Mathematik und Informatik, Technische Universität Freiberg, Akademiestr. 6, 09599 Freiberg, oliver.rheinbach@math.tu-freiberg.de, http://www.mathe.tu-freiberg.de/nmo/mitarbeiter/oliver-rheinbach
§Department of Mathematics, Courant Institute, 251 Mercer Street, New York, NY 10012, USA, widlund@cims.nyu.edu, https://cs.nyu.edu/faculty/widlund
mention success with adaptive coarse spaces for nonoverlapping domain decom-
position methods; see, e.g., [2, 34, 35, 42, 37, 31, 33, 38, 30, 32, 36].

Two-level overlapping Schwarz algorithms were first developed with coarse-
spaces based on a coarse triangulation of the domain and with subdomains obtained
by adding one or a few layers of fine elements to each coarse mesh element, see [43,
Chapter 3]. On the other hand, the iterative substructuring algorithms, developed
for decompositions of the domain into nonoverlapping subdomains, were immedi-
ately available for quite irregular subdomains such as those that can be obtained by
a mesh partitioner such as METIS [29]; see [43, Chapter 4, 5, and 6]. The iterative
substructuring algorithms have been very successful but they cannot be used unless
submatrices associated with the subdomains are available instead of just a fully
assembled stiffness matrix. This was a main reason why a new family of overlap-
ning Schwarz algorithms was developed, known as the GDSW methods (generalized
Dryja–Smith–Widlund), which borrow their coarse components from [43, Algorithm
5.16]. These ideas were first developed in [5, 6]. The elements of these coarse spaces
are defined by their values on the interface between the subdomains with values
in the interiors defined by energy-minimizing extensions. These algorithms were
further developed for almost incompressible elasticity in two papers [7, 8]; in the
second paper the dimension of the coarse spaces was considerably decreased; see
also [23, 16, 24, 25, 17, 22, 26] for further developments.

In this paper, we present an approach of constructing adaptive coarse spaces
for the two-level overlapping Schwarz method [40, 43] based on the adaptive GDSW
(AGDSW) coarse space of [21]. In particular, our focus is on one new coarse space –
the reduced dimension adaptive GDSW (RAGDSW) coarse space – and the reduc-
tion of the coarse space dimension. A proof of a condition number estimate, which
is independent of heterogeneities of the coefficient functions, is given in sections 10
and 11. We note that this proof is based on a more general decomposition of the
interface than the one in [21]; it applies to both, the original AGDSW and the new
RAGDSW coarse space. Supporting numerical results are presented in section 14.

In our adaptive algorithms, a user prescribed tolerance directly controls the
condition number of the preconditioned operator and, if this tolerance is chosen as
zero, adaptive GDSW is identical to GDSW and reduced dimension adaptive GDSW
is identical to reduced dimension GDSW, the latter being a variant of GDSW defined
on a specific interface partition of the domain decomposition; cf. section 8.

We note that our reduced dimension GDSW coarse space differs from the re-
duced dimension GDSW coarse spaces in [9]. However, they share the same core
idea: GDSW and AGDSW use basis functions associated with coarse nodes, edges,
and faces while the coarse spaces in [9], reduced dimension GDSW, and reduced
dimension adaptive GDSW use basis functions associated only with subdomain
vertices. Generally, this leads to a reduction in the coarse space dimension. See
also [8, 4, 27, 18] for reduced dimension GDSW coarse spaces.

We note that many other approaches to constructing coarse spaces exist. Some
borrow the idea from the multiscale finite element method (MsFEM) [28, 12] and
use basis functions of that type in the coarse space; c.f. [1, 3, 15, 20, 13]. However,
the coarse spaces in this paper are not based on MsFEM functions.

The outline of the paper is as follows: In section 2, we introduce our model
problem followed by the definition of the two-level additive overlapping Schwarz
methods in section 3. In the following five sections, we introduce four families of
GDSW algorithms. In section 9, we give a quite general description of adaptive
GDSW coarse spaces which covers both adaptive GDSW and reduced dimension
adaptive GDSW; see also section 12 for a variant which is computationally cheaper,
easier to implement and more efficient in a parallel implementation. In sections 10
and 11, we derive a condition number estimate for our new reduced dimension
adaptive GDSW preconditioner. In section 13, we address questions that may arise
about the implementation due to the encounter of singular matrices for certain extension operators described in section 9. Finally, in section 14, we present numerical results for a selection of coefficient functions.

For the reader’s convenience, an overview of some definitions is given in Table 1.

2. Linear elasticity. We will consider a variational formulation of the equations of compressible linear elasticity: Find $u \in (H^1_0(\Omega))^3$ such that

$$a_\Omega(u, v) = L(v) \quad \forall v \in (H^1_0(\Omega))^3,$$

where $\Omega \subset \mathbb{R}^3$ is a polyhedral domain and

$$a_\Omega(u, v) := \int_\Omega 2\mu(x) \left( \varepsilon(u(x)) : \varepsilon(v(x)) \right) dx + \int_\Omega \lambda(x) \left( \text{div}(u(x)) \text{div}(v(x)) \right) dx,$$

$$L(v) := \int_\Omega f(x) \cdot v(x) dx.$$

The Lamé parameters $0 < \lambda(x), \mu(x) : \mathbb{R}^3 \rightarrow \mathbb{R}$ are scalar coefficient functions, $f \in (L^2(\Omega))^3$, $\varepsilon(u) := \frac{1}{2} \left( \nabla u + \left( \nabla u \right)^T \right)$ and

$$A : B := \text{tr}(A^T B) = \sum_{i,j=1}^{d} A_{ij} B_{ij}$$

for any matrices $A, B \in \mathbb{R}^{3 \times 3}.$
We will consider problems with a highly heterogeneous Young modulus $E : \Omega \to \mathbb{R}$, $0 < E_{\min} \leq E(x) \leq E_{\max}$, and a positive Poisson ratio $\nu$, bounded away from above, by 1/2, and we define the Lamé parameters by

$$\lambda(x) := \frac{E(x)\nu}{(1+\nu)(1-2\nu)},$$

$$\mu(x) := \frac{E(x)}{2(1+\nu)}.$$ 

The algorithms described in this paper can also be applied to other linear, second-order elliptic problems including those in two dimensions.

Let $\tau_h := \tau_h(\Omega)$ be a finite element discretization of $\Omega$. We will use a conforming space $V^h(\Omega)$ of piecewise linear or trilinear finite elements on this mesh, and for simplicity assume that the Lamé parameters are constant on each element $T \in \tau_h$.

We will use the conjugate gradient method preconditioned by two-level overlapping Schwarz methods to solve the resulting linear system $Ku = b$.

For completeness, we note that the Dirichlet boundary condition has been incorporated into the global stiffness matrix by setting those rows and columns of $K$ to unit vectors that correspond to Dirichlet boundary nodes.

### 3. Two-level overlapping Schwarz methods.

We will now introduce the two-level Schwarz algorithms, mostly following [43, Chapter 2.2]. The different variants considered in this paper will differ in the coarse space chosen; the design of the coarse space is the main issue in this study and many other studies of algorithms of this kind. In the next five sections, we will introduce four different variants. In section 12, we also explore alternatives that decrease the costs of using the two algorithms which use adaptive choices of their coarse spaces.

We partition the domain $\Omega$ into $N$ nonoverlapping subdomains $\Omega_i$ with a maximum diameter $H$, each a union of finite elements, and denote the corresponding interface by $\Gamma := \bigcup_{i \neq j} (\partial \Omega_i \cap \partial \Omega_j) \setminus \partial \Omega$. We extend each subdomain $\Omega_i$ by $k$ layers of finite elements to obtain an overlapping domain decomposition $\{\Omega'_i\}_{i=1}^N$ and introduce subspaces $V_i := V^h(\Omega'_i), i \in \{1, \ldots, N\}$, of finite element functions that vanish on $\partial \Omega'_i$ and in the complement of $\Omega'_i$.

Associated with each such subdomain is a restriction operator $R_i : V^h(\Omega) \to V_i$ and an extension operator $\tilde{R}^T_i : V_i \to V^h$. Furthermore, for any global coarse space $V_0 \subset V^h$, we define a linear interpolation operator $R_0 : V^h \to V_0$, where each of the columns of the matrix $R_0^T$ represents a coarse basis function defined on the fine mesh $\tau_h$.

We will use exact solvers for all the subspaces defined in terms of bilinear forms on $V_i, i \in \{0, 1, \ldots, N\}$, given by

$$\tilde{a}_i(u_i, v_i) = a_{\Omega_i}(R_i^T u_i, R_i^T v_i) \quad \forall u_i, v_i \in V_i;$$

cf. [43, Chapter 2.2]. The associated matrices are given by $K_i = R_iK_iR_i^T, i = 0, 1, \ldots, N$. The additive one-level Schwarz preconditioned operator is given by

$$P_{\text{OS-1}} = \sum_{i=1}^{N} R_i^T K_i^{-1} R_i,$$

and that of the additive two-level Schwarz operator by

$$P_{\text{OS-2}} = R_0^T K_0^{-1} R_0 + P_{\text{OS-1}}.$$ 

### 4. The GDSW preconditioner.

In what follows, $x^h$ will denote a finite element node. Those on the interface form the set $\Gamma^h := \{x^h \in \Gamma\}$. A key ingredient of each of our coarse spaces is a partition $\mathcal{P}$ of $\Gamma^h$ into disjoint interface components $\xi^h \subset \Gamma^h$, s.t.

$$\Gamma^h = \bigcup_{\xi^h \in \mathcal{P}} \xi^h.$$
Adaptive GDSW coarse spaces of reduced dimension for overlapping Schwarz methods

To simplify, we will omit the superscript \( h \) and write \( \xi \) instead of \( \xi^h \).

The GDSW, [5, 6], AGDSW, [19, 21], RGDSW, [9, 27] and section 6, and RAGDSW, section 7, preconditioners are two-level overlapping Schwarz methods, and their preconditioners can be written in matrix form as

\[
M^{-1} = \Phi \left( \Phi^T K \Phi \right)^{-1} \Phi^T + \sum_{i=1}^{N} R_i^T K_i^{-1} R_i.
\]

The basis functions of all our coarse spaces, i.e., the columns of \( \Phi \), are defined by an energy-minimal extension of the values \( \Phi_\Gamma \) on the interface \( \Gamma^h \) to the subdomains, i.e., by

\[
\Phi = \begin{bmatrix} \Phi_I \\ \Phi_\Gamma \end{bmatrix} = H_\Gamma \Phi_\Gamma, \quad H_\Gamma := \begin{bmatrix} -K_{II}^{-1} K_{II} \\ I_\Gamma \end{bmatrix}.
\]

Here \( I_\Gamma \) is the identity matrix on \( \Gamma^h \) and \( H_\Gamma \) is constructed from submatrices of the global stiffness matrix

\[
K := \begin{bmatrix} K_{II} & K_{II} \\ K_{I\Gamma} & K_{\Gamma\Gamma} \end{bmatrix},
\]

where \( I \) refers to the set of variables not associated with the interface. We note that \( I \) also contains boundary nodes of \( \Omega \). We note that \( K_{II} \) is block-diagonal and
that $K_{II} = K_{II}^T$ also can be written in block form as

$$
K_{II} = \begin{bmatrix}
K_{II}^{(1)} & \cdots & K_{II}^{(N)} \\
\cdots & \cdots & \cdots \\
K_{II}^{(N)} & \cdots & K_{II}^{(1)}
\end{bmatrix},
K_{II} = \begin{bmatrix}
K_{II}^{(1)} & \cdots & K_{II}^{(N)}
\end{bmatrix}.
$$

The superscripts of these matrices mark contributions from the subdomains $\Omega_i$ to the stiffness matrix $K$.

Given the sparsity of the stiffness matrix, reflecting the local coupling of the variables, all these matrix blocks are sparse and the coarse space basis functions are associated only with a few subdomains. In the original GDSW method for the scalar two-dimensional case, the columns of $\Phi_\Gamma$ are given by the characteristic functions of vertices and subdomain edges, i.e., the interface is partitioned as follows:

$$
\Gamma^h = (\bigcup_{v \in \mathcal{V}} v) \cup (\bigcup_{e \in \mathcal{E}} e),
$$

where $\mathcal{V}$ and $\mathcal{E}$ are the sets of subdomain vertices and edges, respectively, cf. Figure 1 (top-left) for the interface partition and (top-right) for two corresponding coarse functions. For the three dimensional case, the basis functions are defined analogously, using characteristic functions for interface vertices, edges, and faces.

In more general cases, the boundary values on $\Gamma$ span the restriction of the null space of $K^N$ to $\Gamma$, where $K^N$ is the stiffness matrix given by $a_\Omega(\cdot, \cdot)$ with a Neumann boundary condition on $\partial \Omega$. Thus, for linear elasticity in three dimensions, and any subdomain edge which is not straight, we obtain 6 functions: 3 translations and 3 rotations. We note that the restriction of the rigid body modes to a straight edge are linear dependent; see [7].

The matrix of the GDSW coarse operator can be computed either by forming the triple matrix product

$$
\Phi^T K \Phi
$$

or by exploiting the fact that

$$
\Phi^T K \Phi = \begin{bmatrix}
-K_{II}^{-1} K_{II} \Phi_\Gamma \\
\Phi_\Gamma
\end{bmatrix}^T \begin{bmatrix}
K_{II} & K_{II} \\
K_{II} & K_{II}\Gamma \Gamma \end{bmatrix} \begin{bmatrix}
-K_{II}^{-1} K_{II} \Phi_\Gamma \\
\Phi_\Gamma
\end{bmatrix}
$$

$$
= \Phi_\Gamma^T S_{II} \Phi_\Gamma,
$$

where $S_{II} = K_{II} - K_{II} K_{II}^{-1} K_{II}$ is the Schur complement obtained by eliminating the interior variables of all subdomains and those on the boundary of $\Omega$.

5. Standard adaptive GDSW coarse space. The standard adaptive GDSW method, the AGDSW method, uses the same interface partitioning $P$, based on subdomain vertices, edges, and faces, as the GDSW method. The coarse functions for the vertices are the same as for the GDSW variant but the columns of $\Phi$ corresponding to the edges and faces are not. Instead, we use a few of the eigenfunctions of local generalized eigenvalue problems of the form

$$
(5.1)
$$

$$
S_{\xi \xi} \tau_{*, \xi} = \lambda_{*, \xi} K_{\xi \xi}^{\Omega_\xi} \tau_{*, \xi},
$$

where $\xi$ corresponds to an edge or a face.

To define the Schur complement $S_{\xi \xi}$ and the matrix $K_{\xi \xi}^{\Omega_\xi}$, for any edge and face $\xi$, we will use the local stiffness matrix $K^{\Omega_\xi}$ on $\Omega_\xi$ with Neumann boundary conditions. Here $\Omega_\xi$ is the closure of the union of all subdomains which are adjacent to $\xi$ and $\Omega_\xi := \overline{\Omega_\xi \setminus \partial \Omega_\xi}$ its interior. The stiffness matrix $K^{\Omega_\xi}$ is defined by $a_{\Omega_\xi}(\cdot, \cdot)$ and can be assembled from the subdomain stiffness matrices of the subdomains adjacent to the edge or face.
We partition the degrees of freedom of \( \Omega_\xi \) into the set associated with \( \xi \) and the rest which forms a set \( R \) and write the stiffness matrix as

\[
K_{\Omega_\xi} = \begin{pmatrix}
K_{\Omega_\xi}^{\Omega_\xi} & K_{\Omega_\xi}^{\Omega_R}
K_{\Omega_R}^{\Omega_\xi} & K_{\Omega_R}^{\Omega_R}
\end{pmatrix},
\]

and can then define the Schur complement by

\[
S_{\xi\xi} = K_{\Omega_\xi}^{\Omega_\xi} - K_{\Omega_R}^{\Omega_\xi} \left( K_{\Omega_R}^{\Omega_R} \right)^+ K_{\Omega_\xi}^{\Omega_R},
\]

where \( \left( K_{\Omega_R}^{\Omega_R} \right)^+ \) is a pseudoinverse of \( K_{\Omega_R}^{\Omega_R} \); see Remark 9.1 and section 13.

We sort the eigenvalues of (5.1) in nondescending order; i.e., \( \lambda_1,\xi \leq \lambda_2,\xi \leq \ldots \leq \lambda_{m,\xi} \) where \( m \) is the number of unknowns of (5.1). We select all eigenvectors \( \tau_{*,\xi} \), with eigenvalues smaller or equal than a certain threshold, i.e., \( \lambda_{*,\xi} \leq \text{tol}_{\xi} \) and then define \( \tau_{*,\Gamma} \) as the extension by zero of \( \tau_{*,\xi} \) from \( \xi \) to \( \Gamma^h \). The coarse basis functions corresponding to \( \xi \) are then the extensions

\[
\nu_{*,\xi} := H_{\Gamma} \tau_{*,\Gamma}
\]

and the columns of \( \Phi \) are now given by the \( \nu_{*,\xi} \), selected, and the GDSW vertex functions.

Let \( \text{tol}_{\xi} \) and \( \text{tol}_{\mathcal{F}} \) be the smallest tolerance used for the subdomain edges and faces, respectively. The following condition number estimate for the preconditioned operator has been derived previously for scalar diffusion problems; see [21, Corollary 6.6]:

**Lemma 5.1.** The condition number of the AGDSW two-level Schwarz operator in three dimensions is bounded by

\[
\kappa(M_{\text{AGDSW}}^{-1}K) \leq \left( 20 + \frac{34(N^{\mathcal{F}})^2 \nu_{\text{max}}^e}{\text{tol}_{\mathcal{F}}} + \frac{68(N^{\mathcal{F}})^2}{\text{tol}_{\mathcal{F}}} \right) \left( \hat{N}_c + 1 \right).
\]

The constant \( \hat{N}_c \) is an upper bound of the number of overlapping subdomains that any point \( x^h \in \Omega \) can belong to. \( N^{\mathcal{E}} \) and \( N^{\mathcal{F}} \) are the maximum number of subdomain edges and faces, respectively, of any subdomain. \( \nu_{\text{max}}^e \) is the maximum number of subdomains that share a subdomain edge. All constants are independent of \( H, h, \) and the contrast of the coefficient function.

This kind of result also holds for linear elasticity; see Corollary 11.5 and section 11.

**Remark 5.2.** If \( \text{tol}_{\xi} = 0 \) for all \( \xi \in \mathcal{P} \), the AGDSW coarse space contains only the coarse functions of the GDSW coarse space. Thus, we obtain

\[
V_{\text{GDSW}} = V_{\text{AGDSW}}^0 \subset V_{\text{AGDSW}}^{\text{tol}(\mathcal{P})};
\]

cf. also Remark 7.1.

**6. A reduced dimension GDSW coarse space.** We will first give a simple description of an interface partition for a structured mesh and domain decomposition. This partition can also be used for the reduced dimension adaptive GDSW coarse spaces.

Our goal is to reduce the number of interface components. To this end, each vertex of the coarse mesh will be associated with an interface component \( \xi \) formed by parts of the edges and faces adjacent to the vertex. A disjoint partition is obtained by distributing parts of these faces and edges equally, or almost equally, between nearby vertices; see Figure 1 (bottom-left) for a two-dimensional representation.
The reduced dimension GDSW coarse space is then defined completely analogously to the GDSW coarse space. Thus the restriction of the null space elements to the interface components is first extended by zero to the rest of the interface nodes and then extended with minimal energy to the subdomain interiors to obtain the coarse functions; see Figure 1 (bottom-right) for one of the coarse functions for a two-dimensional diffusion problem.

We note that our RGDSW coarse space differs from those of [9] but that can be regarded as a variant of the coarse spaces introduced in that paper.

7. The reduced dimension adaptive GDSW coarse space. For the reduced adaptive GDSW coarse space, we need to partition each interface component \( \xi \), as those of the previous section, into subcomponents. For a structured mesh and domain decomposition, as in that section, we partition each \( \xi \) into subsets related to the subdomain vertices, edges, and faces. With \( \mathcal{V}, \mathcal{E}, \) and \( \mathcal{F} \) the sets of subdomain vertices, edges, and faces, respectively, we define subcomponents \( \xi \) of \( \xi \) such that

\[
\{ \xi \}_{i=1}^{n} = \{ \xi \cap c : c \in \mathcal{V} \cup \mathcal{E} \cup \mathcal{F} \land c \cap \xi \neq \emptyset \},
\]

where \( n_\xi \) is the number of subcomponents of \( \xi \); see Figure 2 (left) for a two-dimensional case. We next partition \( K_{\xi \xi}^{\Omega} \) with respect to the subsets \( \{ \xi_i \}_{i=1}^{n_\xi} \), into

\[
K_{\xi \xi}^{\Omega} = \left( K_{\xi_i \xi_j}^{\Omega} \right)_{i,j=1}^{n_\xi}
\]

and, as before, we define the Schur complement by

\[
S_{\xi \xi} := K_{\xi \xi}^{\Omega} - K_{\xi \xi}^{\Omega} \left( K_{R \xi R}^{\Omega} \right)^{+} K_{R \xi \xi}^{\Omega},
\]

where \( \left( K_{R \xi R}^{\Omega} \right)^{+} \) is a pseudoinverse of \( K_{R \xi R}^{\Omega} \), see Remark 9.1 and section 13. Furthermore, let

\[
\tilde{K}_{\xi \xi} := \text{blockdiag}(K_{\xi_i \xi_i}^{\Omega})
\]

and introduce a generalized eigenvalue problem, given in matrix form by

\[
S_{\xi \xi} \tau_{*, \xi} = \lambda_{*, \xi} \tilde{K}_{\xi \xi} \tau_{*, \xi}.
\]

As in section 5, the eigenvalues are sorted in a nondecreasing order and eigenvectors \( \tau_{*, \xi} \) corresponding to \( \lambda_{*, \xi} \leq \text{tol}_{\xi} \) are selected and then extended by zero to \( \Gamma^h \) as \( \tau_{*, \Gamma} \). The coarse basis functions, i.e., the columns of \( \Phi \), corresponding to \( \xi \) are the extensions \( v_{*, \xi} := H_{\Gamma} \tau_{*, \Gamma} \).
Remark 7.1. If $\text{tol}_\xi = 0$ for all $\xi \in \mathcal{P}$, the RAGDSW coarse space contains only the coarse functions associated with the null space of the Schur complement $S_{\xi \xi}$. The latter is identical to the null space of $K_{\xi \xi}^\Omega$ restricted to $\xi$. Thus, in this case, RAGDSW reduces to RGDSW, and we have

$$V_{\text{RGDSW}} = V_{\text{RAGDSW}}^0 \subset V_{\text{RAGDSW}}^{\text{tol}(\mathcal{P})}.$$

8. Interface partitioning for RAGDSW on unstructured meshes. For unstructured cases, we will define the partitioning $\mathcal{P}$ using nodal equivalence classes and begin with definitions of connected components of finite element nodes and of nodal equivalence classes. We note that equivalence classes have previously been used in [9] for similar purposes.

Two finite element nodes $x_1^h, x_2^h \in \Gamma^h$ are said to be adjacent, if there exists a finite element edge or face $z \subset \Gamma$ such that $x_1^h, x_2^h \in z$, the closure of $z$. A set of nodes $\gamma \subset \Gamma^h$ is said to form a connected component, if, for any two nodes $x_0^h, x_1^h \in \gamma$, there exists a path $(x_0^h, \ldots, x_1^h)$, $x_0^h, x_1^h \in \gamma$, of adjacent nodes.

For any node $x^h \in \Omega$, let

$$n(x^h) := \{ i \in \{1, 2, \ldots, N\} : x^h \in \Omega_i \}$$

be the set of indices of the subdomains which have $x^h$ in common. To partition a set of nodes $\gamma \subset \Gamma^h$, we define nodal equivalence classes (NECs) by the relation $x_1^h \sim x_2^h \iff n(x_1^h) = n(x_2^h)$, for any two nodes $x_1^h, x_2^h \in \gamma$. We further partition each NEC into its connected components based on the adjacency of nodes; cf. Figure 2 (right).

By $\mathcal{N}(x^h)$, we denote the NEC of a node $x^h \in \gamma$, i.e., $x^h \in \mathcal{N}(x^h)$. If $n(x_1^h) \subseteq n(x_2^h)$, then $\mathcal{N}(x_1^h)$ is said to be an ancestor of $\mathcal{N}(x_2^h)$ which in turn is a descendant of $\mathcal{N}(x_1^h)$. If a NEC does not have an ancestor, we call it a root.

We note that for $\gamma = \Gamma^h$ a root is a vertex (i.e., a coarse node) in the case of cuboid subdomains. However, often for unstructured domain decompositions obtained, e.g., by METIS [29], a root can be a coarse edge or coarse face as well; see further the discussion in [9]. We note that for special cases of structured domain decompositions, e.g., a beam built from a union of cubes, the same can occur.

We now give a general description of the interface partition for RAGDSW for an unstructured mesh and domain decomposition. We will define components $\xi$, s.t. each $\xi$ contains only one root and parts of its descendants. Furthermore, we will assure that the resulting interface partition $\mathcal{P}$ is nonoverlapping to obtain a partition $\mathcal{P}$ of connected disjoint components $\xi \in \mathcal{P}$ s.t.

$$\Gamma^h = \bigcup_{\xi \in \mathcal{P}} \xi.$$

Several specific constructions are possible. Relevant aspects are, e.g., obtaining components of similar size, nondegenerate components, and parallel efficiency of the construction.

For the results in this paper, we have constructed the interface partition in the following way: We initialize each component $\xi \in \mathcal{P}$ with the nodes of a root and add the remaining nodes in an iterative process.

Starting with the roots, we grow sets which will result in all the subsets $\xi \in \mathcal{P}$. In each step of an iteration, we add all nodes which are adjacent to elements of each of the current sets, which have not been previously assigned, and which are descendants of the root of the set. We repeat this process until all interface nodes have been assigned to a $\xi \in \mathcal{P}$. Figure 3 depicts sample partitions for two and three dimensions.

We note that for the unstructured meshes in section 14, the average number of degrees of freedom per eigenvalue problem is increased by roughly 50% and with
the maximum roughly doubled, compared to the face eigenvalue problems used in the standard AGDSW.

As before, we partition each interface component into its subcomponents. Let \( \mathcal{N}_{V_h} \) be the set of NECs of \( \Gamma^h \) and for \( \xi \in \mathcal{P} \) let

\[
\mathcal{N}_{\xi} := \{ \xi \cap c : c \in \mathcal{N}_{V_h} \land \xi \cap c \neq \emptyset \}.
\]

Let \( n_\xi := |\mathcal{N}_{\xi}| \) be the number of NECs of \( \xi \) and let \( \xi_i, i = 1, \ldots, n_\xi \), be the resulting decomposition of \( \xi \) into \( \{\xi_i\}_{i=1}^{n_\xi} = \mathcal{N}_{\xi} \). We then have \( \xi_i \cap \xi_j = \emptyset \ (i \neq j) \) and \( \xi = \bigcup_{i=1}^{n_\xi} \xi_i \).

**Remark 8.1.** If our problem satisfies a Neumann boundary condition on \( \partial \Omega_N \subset \partial \Omega \), in addition to a nonempty set \( \partial \Omega_D = \partial \Omega \setminus \partial \Omega_N \) with a Dirichlet boundary condition, then the construction of the RAGDSW coarse space and the proof of the condition number estimate in sections 10 and 11 will essentially be the same. The finite element nodes that lie on the Neumann boundary but not on the interface \( \Gamma = \bigcup_{i \neq j} (\partial \Omega_i \cap \partial \Omega_j) \setminus \partial \Omega_D \) are treated as interior nodes.

In the next section, we will first describe the adaptive GDSW coarse spaces in variational form. Thereafter, we will derive a condition number estimate for the preconditioned two-level additive Schwarz operator based on the coarse space introduced above. We note that the proof remains valid for quite general interface partitions \( \mathcal{P} \) and is not restricted to the one of RAGDSW.

**9. Variational description of adaptive GDSW-type coarse spaces.** For \( \xi \in \mathcal{P} \) the index set \( n^{\xi} \) contains the indices of all adjacent subdomains, i.e., the union of the index sets of all nodes \( x^h \in \xi \),

\begin{equation}
(9.1) \quad n^{\xi} = \bigcup_{x^h \in \xi} n(x^h).
\end{equation}

As in section 5, \( \overline{\Omega}_\xi \) is the closure of the union of adjacent subdomains, i.e., \( \overline{\Omega}_\xi = \bigcup_{\xi \in n^{\xi}} \overline{\Omega}_\xi \).

Let \( \overline{\mathcal{G}} \subset \overline{\Omega} \) be any union of sets \( s \in \{ \mathcal{T}_i \cap \mathcal{T}_j \neq \emptyset : \mathcal{T}_i, \mathcal{T}_j \in \tau_h \} \). By \( z_{\xi,\mathcal{G}}(\cdot) \), we
Adaptive GDSW coarse spaces of reduced dimension for overlapping Schwarz methods

Fig. 4. Graphical representation in two dimensions of the energy-minimal extension (9.3) from $\xi \in \mathcal{P}$ to $\overline{\Omega}_\xi$ (left) and sample energy-minimal extension for the diffusion equation (right) in which the RAGDSW interface component $\xi$ is highlighted in red and the remaining interface nodes in light gray.

Denote an extension-by-zero operator from $\xi \subset G$ to $G$:

$$z_{\xi \rightarrow G} : X^h(\xi) \rightarrow \{ w|_{\overline{\Omega}} : w \in V^h(\Omega), w = 0 \text{ in } \overline{\Omega} \setminus \xi \}$$

$$v \mapsto z_{\xi \rightarrow G}(v) := \begin{cases} v(x^h) & \forall x^h \in \xi, \\ 0 & \forall x^h \in \overline{G} \setminus \xi. \end{cases}$$

Here, $X^h(\xi) := \{ v : \xi \rightarrow \mathbb{R}^3 \}$.

By $H_{\xi, \Omega(\xi)} \cdot \cdot \cdot$, we denote a possibly nonunique (cf. Remark 9.1) energy-minimal extension w.r.t. $a_{\Omega(\xi)} \cdot \cdot \cdot$ from $\xi$ to $\overline{\Omega}_\xi$: let $V_{0, \xi}^h(\Omega) := \{ w|_{\partial \Omega} : w \in V^h(\Omega), w(x^h) = 0 \ \forall x^h \in \xi \}$, then for $\tau_\xi \in X^h(\xi)$, an extension $v_\xi := H_{\xi, \Omega(\xi)}(\tau_\xi) \in V^h(\Omega)$ is given by a solution of

$$a_{\Omega(\xi)}(v_\xi, v) = 0 \ \forall v \in V_{0, \xi}^h(\Omega),$$

$$v_\xi(x^h) = \tau_\xi(x^h) \ \forall x^h \in \xi;$$

cf. Figure 4. We note that the extension is computed with a homogeneous Neumann boundary condition on $\partial \Omega_\xi$.

As in section 8, let $\{ \xi_i \}_{i=1}^n$ be the set of all NECs of a $\xi \in \mathcal{P}$. Then $\xi_i \cap \xi_j = \emptyset$ ($i \neq j$) and $\xi = \bigcup_{i=1}^n \xi_i$ holds. We define the symmetric, positive definite bilinear form

$$c_\xi(u, v) := \sum_{i=1}^{n_\xi} c_{\xi_i}(u, v) \ \forall u, v \in X^h(\xi),$$

with

$$c_{\xi_i}(u, v) := a_{\Omega(\xi_i)}(z_{\xi \rightarrow \Omega(\xi_i)}(u), z_{\xi \rightarrow \Omega(\xi_i)}(v)) \ \forall u, v \in X^h(\xi).$$

The corresponding norm is defined by

$$\| u \|_{c_\xi}^2 := c_\xi(u, u) \ \forall u \in X^h(\xi).$$

We define the following generalized eigenvalue problem on $\xi \in \mathcal{P}$: Find $\tau_{\ast, \xi} \in X^h(\xi)$ such that

$$a_{\Omega(\xi)}(H_{\xi, \Omega(\xi)}(\tau_{\ast, \xi}), H_{\xi, \Omega(\xi)}(\theta)) = \lambda_{\ast, \xi} c_\xi(\tau_{\ast, \xi}, \theta) \ \forall \theta \in X^h(\xi).$$
The eigenvalues are again sorted in non-descending order; i.e., $\lambda_{1,\xi} \leq \lambda_{2,\xi} \leq \ldots \leq \lambda_{m,\xi}$ and the eigenmodes accordingly, where $m = \dim (X^h (\xi))$. Furthermore, let the eigenmodes $\tau_{\ast,\xi}$ satisfy $c_\xi (\tau_{\ast,\xi}, \tau_{\ast,\xi}) = \delta_{kj}$, where $\delta_{kj}$ is the Kronecker delta symbol. We select all eigenmodes $\tau_{\ast,\xi}$ where the eigenvalues are below a certain threshold, i.e., $\lambda_{\ast,\xi} \leq tol$. Then, the coarse basis functions corresponding to $\xi$ are the extensions

\begin{equation}
(9.8)
v_{\ast,\xi} := \mathcal{H}_{\Gamma^0}\left( \tau_\ast \right) \in V^h_0 (\Omega), \quad \tau_\ast := z_{\ast,\xi} (\tau_{\ast,\xi}),
\end{equation}

of the selected $\tau_{\ast,\xi}$, where $v_{\ast,\xi} = \mathcal{H}_{\Gamma^0} (\tau_\ast)$ is given by the solution $v_{\ast,\xi} \in V^h_0 (\Omega)$ that satisfies

\begin{equation}
(9.9)
\begin{align*}
a_{\Omega^k} (v_{\ast,\xi}, w) &= 0 \quad \forall w \in V^h_0 (\Omega_l), l = 1, \ldots, N, \\
v_{\ast,\xi} (x^h) &= \tau_\ast (x^h) \quad \forall x^h \in \Gamma^h.
\end{align*}
\end{equation}

We note that, contrary to (9.7), $v_{\ast,\xi}$ vanishes on $\partial \Omega_\xi$ since $\tau_\ast = z_{\ast,\xi} (\tau_{\ast,\xi})$ and since $v_{\ast,\xi} = \mathcal{H}_{\Gamma^0} (\tau_\ast) \in V^h_0 (\Omega)$. Therefore, (9.9) has a unique solution.

For a general interface partition $\mathcal{P}$, we define the adaptive GDSW coarse space as

\begin{equation}
(9.10)
V_P := \bigoplus_{\xi \in \mathcal{P}} \text{span} \{ v_{k,\xi} : \lambda_{k,\xi} \leq tol \}.
\end{equation}

The standard AGDSW coarse space (see [21]) is based on the partition

\begin{equation}
\mathcal{P} := \mathcal{F} \cup \mathcal{E} \cup \mathcal{V}.
\end{equation}

Since vertices, edges, and faces are NECs, we then have

\begin{equation}
c_\xi (u, v) = a_{\Omega^k} (z_{\xi,\Omega^k} (u), z_{\xi,\Omega^k} (v))
\end{equation}

if $\xi$ is a vertex, an edge, or a face.

Remark 9.1. For the diffusion case the energy-minimal extension defined by (9.3) has a unique solution. If an interface component $\xi$ is a straight edge or a vertex then 1 or 3 rotations, respectively, are in the null space of the extension operator for linear elasticity. However, as all solutions of (9.3) have the same energy, the choice of the particular solution does not influence the solution of the generalized eigenvalue problem (9.7): let $v_{\ast,\xi} = \mathcal{H}_{\xi,\Omega^k} (\tau_{\ast,\xi})$ be a solution of (9.3). Then all solutions are given by $v_{\ast,\xi} + r$, where $r \in \text{range} (\mathcal{H}_{\xi,\Omega^k} (0))$; for linear elasticity $r$ is a rigid body mode. Since $r \in V^h_0 (\Omega_\xi)$, we have $a_{\Omega^k} (r, \mathcal{H}_{\xi,\Omega^k} (\theta)) = 0$ by the definition of $\mathcal{H}_{\xi,\Omega^k} (\theta)$. Therefore, $v_{\ast,\xi} + r$ solves (9.3) and

\begin{equation}
a_{\Omega^k} (v_{\ast,\xi} + r, \mathcal{H}_{\xi,\Omega^k} (\theta)) = a_{\Omega^k} (v_{\ast,\xi}, \mathcal{H}_{\xi,\Omega^k} (\theta)) \quad \forall \theta \in X^h (\xi).
\end{equation}

As a consequence, any operator defined by (9.3) yields the same generalized eigenvalue problem (9.7). In section 13, we will provide some remarks on how to find the solution of (9.3) when it is not unique.

Remark 9.2. We note that the left hand side of eigenvalue problem (9.7) is singular and its kernel contains the constant functions for the scalar diffusion case and the rigid body modes for linear elasticity. Therefore, the null space has a dimension of 1 for the scalar diffusion problem and at least 3 for linear elasticity. For a vertex (i.e., $\xi \in \mathcal{V}$) the problem has only one (scalar diffusion) and three (linear elasticity) degrees of freedom. Thus, in the latter case, the solution is given by the vertex basis functions of the GDSW coarse space, i.e., the three translations in case of linear elasticity; cf. [21] and [7].
10. Spectral projections. We will now consider the projections

\[ \Pi_P w := \sum_{\xi \in P} \Pi_\xi w, \quad \Pi_\xi w := \sum_{\lambda_{k, \xi} \leq \text{tol}_\xi} c_\xi(w, v_{k, \xi}) v_{k, \xi} \]

onto the space \( V_P \). Here, \( v_{k, \xi} \) are the energy-minimal extensions of the eigenfunctions determined by (9.8) and \( \lambda_{k, \xi} \) the corresponding eigenvalues from (9.7). For \( \xi \in \mathcal{P} \), let \( d_\xi : X^h(\xi) \times X^h(\xi) \to \mathbb{R} \) be the symmetric, positive semidefinite bilinear form

\[ d_\xi(\cdot, \cdot) := a_{\Omega_\xi}(H_{\xi\Omega_\xi}(\cdot), H_{\xi\Omega_\xi}(\cdot)). \]

For any union \( B \subset \Omega \) of finite elements \( T \in \tau_h \), let

\[ |v|_{a(B)} := \sqrt{a_{\Omega}(v, v)} \quad \forall v \in V^h(\Omega). \]

We find that

\[ |v|^2_{d_\xi} := d_\xi(v, v) = |H_{\xi\Omega_\xi}(v)|^2_{a(\Omega_\xi)} \leq |v|^2_{a(\Omega_\xi)} \quad \forall v \in V^h(\Omega), \]

due to the energy-minimal property of the extension operator.

Using standard arguments of spectral theory, we obtain two important properties of the projection \( \Pi_\xi \), required for the proof of the condition number estimate in section 11; cf., e.g., [21, Lemma 5.3] and [20, Lemma 4.1].

**Lemma 10.1.** Let the eigenpairs \( \{(\tau_k, \xi, \lambda_k, \xi)\}_{k=1}^{\dim\{X^h(\xi)\}} \) from (9.7) be chosen such that \( c_\xi(\tau_k, \xi, \tau_j, \xi) = \delta_{kj} \) and such that the eigenpairs are sorted in nondescending order w.r.t. the eigenvalues. Then the operator \( \Pi_\xi \) defines a projection which is orthogonal with respect to the bilinear form \( d_\xi(\cdot, \cdot) \) and therefore

\[ |u|_{d_\xi}^2 = |\Pi_\xi u|_{d_\xi}^2 + |u - \Pi_\xi u|_{d_\xi}^2, \quad \forall u \in X^h(\xi). \]

In addition, we have, from spectral theory,

\[ \|u - \Pi_\xi u\|_{c_\xi}^2 \leq \frac{1}{\text{tol}_\xi} |u - \Pi_\xi u|_{d_\xi}^2. \]

The following lemma follows directly from Lemma 10.1; cf. [21, Lemma 2].

**Lemma 10.2.** For \( \xi \in \mathcal{P} \) and \( u \in V^h(\Omega) \) it holds that

\[ \|u - \Pi_\xi u\|_{c_\xi}^2 \leq \frac{1}{\text{tol}_\xi} \sum_{k \in n(\xi)} |u|_{a(\Omega_k)}^2. \]

**Proof.** We have

\[ \|u - \Pi_\xi u\|_{c_\xi}^2 \overset{\text{Lemma 10.1}}{\leq} \frac{1}{\text{tol}_\xi} |u - \Pi_\xi u|_{d_\xi}^2 \leq \frac{1}{\text{tol}_\xi} |u|_{d_\xi}^2 \]

\[ \overset{(10.4)}{\leq} \frac{1}{\text{tol}_\xi} |u|_{a(\Omega_\xi)}^2 = \frac{1}{\text{tol}_\xi} \sum_{k \in n(\xi)} |u|_{a(\Omega_k)}^2. \]
11. Convergence analysis. To prove a condition number estimate, we will prove the existence of a stable decomposition; cf. [43, Chapter 2]. We therefore define the coarse interpolation \( I_0 := \Pi_P \) as the projection onto the coarse space \( V_0 := V_P \); cf. (9.10) and (10.1). Thus the coarse component of the stable decomposition is defined as

\[
u_0 := I_0 u := \Pi_P u.\]

**Lemma 11.1.** For \( \xi \in \mathcal{P} \) and \( u \in V^h(\Omega) \), we have

\[
\| u - u_0 \|_{c_\xi}^2 = c_\xi (u - u_0, u - u_0) \leq \frac{1}{\text{tol}_\xi} \sum_{k \in n\xi} |u|^2_{a(\Omega_k)}.
\]

**Proof.** We have

\[
\| u - u_0 \|_{c_\xi}^2 = \sum_{i=1}^{n\xi} |z_{\xi_i \Omega_{\xi_i}}(u - \Pi_{\mathcal{P}} u)|^2_{a(\Omega_{\xi_i})}
\]

\[
= \sum_{i=1}^{n\xi} |z_{\xi_i \Omega_{\xi_i}}(u - \Pi_{\mathcal{P}} u)|^2_{a(\Omega_{\xi_i})}
\]

\[
= \| u - \Pi_{\mathcal{P}} u \|_{c_\xi}^2 \leq \frac{1}{\text{tol}_\xi} \sum_{k \in n\xi} |u|^2_{a(\Omega_k)}.
\]

\[
\text{Lemma 11.2.}\]

Next, we derive an estimate for the energy of the coarse component.

**Lemma 11.2.** It holds that

\[
|u_0|^2_{a(\Omega)} \leq 2 |u|^2_{a(\Omega)} + \frac{2C_{\tau}}{\text{tol}_\mathcal{P}} \sum_{\xi \in \mathcal{P}} \sum_{k \in n\xi} |u|^2_{a(\Omega_k)} \leq 2 \left( 1 + \frac{C_{\tau} N^\xi}{\text{tol}_\mathcal{P}} \right) |u|^2_{a(\Omega)},
\]

where \( C_{\tau} \) is the maximum number of vertices of any element \( T \in \tau_h(\Omega) \), and

\[
(11.1) \quad N^\xi := \max_{1 \leq i \leq N} |\mathcal{P}(\Omega_i)|, \quad \mathcal{P}(\Omega_i) := \{ \xi \in \mathcal{P} : \xi \cap \Omega_i \neq \emptyset \}
\]

is the maximum number of interface components \( \xi \in \mathcal{P} \) of any subdomain, and

\[
\text{tol}_\mathcal{P} := \min_{\xi \in \mathcal{P}} \text{tol}_\xi.
\]

**Proof.** We can use the fact that \( u_0 \) is energy-minimal w.r.t. \( |\cdot|_{a,\Omega_i} \) for each subdomain \( \Omega_i \), i.e., \( u_0 = H_{\tau \Omega_i}(u_0) \), and obtain

\[
|u_0|^2_{a(\Omega)} \leq 2 |H_{\tau \Omega_i}(u)|^2_{a(\Omega)} + 2 |H_{\tau \Omega_i}(u - u_0)|^2_{a(\Omega)}
\]

\[
\leq 2 |u|^2_{a(\Omega)} + 2|z_{\tau \Omega_i}(u - u_0)|^2_{a(\Omega)}.
\]

Let

\[
N_{\text{ec,} \mathcal{P}} := \bigcup_{\xi \in \mathcal{P}} \{ \xi_i, i = 1, \ldots, n_\xi \}
\]

be the set of interface components of the \( \xi \in \mathcal{P} \) partitioned into their nodal equivalence classes \( \xi_i, i = 1, \ldots, n_\xi \). Then, \( \xi_i \cap \xi_j = \emptyset \) for \( i \neq j \), and \( \bigcup_{\xi_i \in N_{\text{ec,} \mathcal{P}}} \xi_i = \Gamma^h \), and

\[
|z_{\tau \Omega}(u - u_0)|^2_{a(\Omega)} = \sum_{\xi_i \in N_{\text{ec,} \mathcal{P}}} z_{\xi_i \Omega_i}(u - u_0)|^2_{a(\Omega)}
\]

\[
= \sum_{T \in \tau_h(\Omega)} \sum_{\xi_i \in N_{\text{ec,} \mathcal{P}}} z_{\xi_i \Omega_i}(u - u_0)|^2_{a(T)}.
\]
There can be at most $C_\tau$ NECs $\xi_i$ that are nonzero in any element $T$. Thus, we have using the Cauchy–Schwarz inequality

$$\sum_{T \in \mathcal{T}_h(\Omega)} \sum_{\xi_i \in \mathcal{N}_{ec,p}} z_{\xi_i,\Omega}(u - u_0)^2_{a(T)} \leq \sum_{T \in \mathcal{T}_h(\Omega)} C_\tau \sum_{\xi_i \in \mathcal{N}_{ec,p}} |z_{\xi_i,\Omega}(u - u_0)|^2_{a(T)}$$

$$= C_\tau \sum_{\xi_i \in \mathcal{N}_{ec,p}} |z_{\xi_i,\Omega}(u - u_0)|^2_{a(\Omega_{\xi_i})}$$

$$= C_\tau \sum_{\xi_i \in \mathcal{P}} \|u - u_0\|^2_{\xi_i}$$

$$\leq \frac{C_\tau}{tol_P} \sum_{\xi_i \in \mathcal{P}} \sum_{k \in \mathcal{N}_\xi} |u|^2_{a(\Omega_k)}$$

where in the last step we have used Lemma 11.1. Thus,

$$|u_0|^2_{a(\Omega)} \leq 2|u|^2_{a(\Omega)} + 2\frac{C_\tau}{tol_P} \sum_{\xi_i \in \mathcal{P}} \sum_{k \in \mathcal{N}_\xi} |u|^2_{a(\Omega_k)} \leq 2 \left(1 + \frac{C_\tau N_\xi}{tol_P} \right)|u|^2_{a(\Omega)}.$$
Since $0 \leq \Psi \leq 1$ is constant on a NEC $\xi_i \in \mathcal{N}_{ec}(\Omega_l)$, we have

$$
\sum_{\xi_i \in \mathcal{N}_{ec}(\Omega_l)} \left| z_{\xi_i \cdot \Omega_l} \left( \Psi \cdot (u - u_0) \right) \right|^2_{a(\Omega_{\xi_i})} = \sum_{\xi_i \in \mathcal{N}_{ec}(\Omega_l)} \left( \Psi|_{\xi_i} \right)^2 \left| z_{\xi_i \cdot \Omega_l} (u - u_0) \right|^2_{a(\Omega_{\xi_i})}
$$

$$
\leq \sum_{\xi_i \in \mathcal{N}_{ec}(\Omega_l)} \left| z_{\xi_i \cdot \Omega_l} (u - u_0) \right|^2_{a(\Omega_{\xi_i})}
$$

$$
\leq \sum_{\xi_i \in \mathcal{P}(\Omega_l)} \sum_{i=1}^{n_{\xi_i}} \left| z_{\xi_i \cdot \Omega_l} (u - u_0) \right|^2_{a(\Omega_{\xi_i})}
$$

$$
= \sum_{\xi_i \in \mathcal{P}(\Omega_l)} c_{\xi} \left( u - u_0, u - u_0 \right).
$$

Using Lemma 11.1, we obtain

$$
C_{\tau} \sum_{\xi_i \in \mathcal{P}(\Omega_l)} c_{\xi} \left( u - u_0, u - u_0 \right) \leq \frac{C_{\tau}}{tol_p} \sum_{\xi_i \in \mathcal{P}(\Omega_l)} \sum_{k \in \ell^i} |u|^2_{a(\Omega_{\xi_k})}.
$$

Thus, in total, we have

$$
\left| I^h (\Psi \cdot (u - u_0)) \right|^2_{a(\Omega)} \leq \frac{C_{\tau}}{tol_p} \sum_{\xi_i \in \mathcal{P}(\Omega_l)} \sum_{k \in \ell^i} |u|^2_{a(\Omega_{\xi_k})}.
$$

Now, we are able to prove the existence of a stable decomposition.

**Theorem 11.4 (Stable Decomposition).** For each $u \in V^h(\Omega)$, there exists a decomposition $u = \sum_{i=0}^{N} R_i^u u_i$, $u_i \in V_i = V^h(\Omega'_i)$, where $\Omega'_0 := \Omega$, such that

$$
\sum_{i=0}^{N} |u_i|^2_{a(\Omega'_i)} \leq C_0^2 |u|^2_{a(\Omega)},
$$

where $C_0^2 = \left( 14 + (12 N^\xi + C) \frac{C_{\tau}}{tol_p} \right)$ and

$$
\mathcal{C} := \mathcal{C}(\left\{ \Omega_i \right\}_{i=1}^{N}, \mathcal{P}) := \max_{1 \leq i \leq N} \sum_{j=1}^{N} |\{ \xi \in \mathcal{P} : i, j \in \ell^i \}|.
$$

(11.3) $\mathcal{C}$ is a measure for the $\mathcal{P}$-connectivity of the domain decomposition: Two subdomains $i, j$ are connected, if they touch the same interface component $\xi \in \mathcal{P}$, i.e., if $i, j \in \ell^i$.

**Proof.** On the overlapping decomposition $\{\hat{\Omega}_i\}_{i=1}^{N}$ of width $h$, we consider the local components $u_i := I^h (\theta_i \cdot (u - u_0))$ with the partition of unity $\left\{ \theta_i \right\}_{i=1}^{N}$, $\theta_i : x^h \in \hat{\Omega} \rightarrow \mathbb{R}$, where

$$
\theta_i(x^h) := \begin{cases} 
\frac{1}{|n(x^h)|} & \text{if } x^h \in \hat{\Omega}_i, \\
0 & \text{elsewhere},
\end{cases}
$$

where $x^h$ is a finite element node and $|n(x^h)|$ is the number of subdomains the node $x^h$ is contained in.

We note that, in general, $\{\hat{\Omega}_i\}_{i=1}^{N}$ differs from the decomposition $\{\Omega'_i\}_{i=1}^{N}$ used in the first level of the preconditioner, in which an overlap with one or more layers of finite elements is used. The decomposition $\{\hat{\Omega}_i\}_{i=1}^{N}$ is only used in the proof.
and, since \( \bar{\Omega}_i \subset \Omega_i' \), we have \( u_i \in V_i \). Thus, no restriction is placed on the size of the overlap of \( \{ \Omega_i' \}_{i=1}^N \). The condition number estimate in Corollary 11.5 does not reflect the fact that the rate of convergence of the algorithm often improves when the overlap is increased.

We define the cutoff function \( \theta : \{ x^h \in \overline{\Omega} \} \rightarrow [0, 1] \) s.t.

\[
\theta(x^h) := 1 - \frac{1}{|n(x^h)|} \quad \text{for any node } x^h \in \overline{\Omega}.
\]

Then, we have

\[
|u_i|^2_{a(\Omega')} = |u_i|^2_{a(\bar{\Omega}_i)} = |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i)}^2
\]

\[
= |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i)}^2 + |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i \setminus \Omega_i)}^2
\]

\[
\leq 2|I^h((1 - \theta_i)(u - u_0))|_{a(\bar{\Omega}_i)}^2 + 2|u - u_0|_{a(\bar{\Omega}_i)}^2 + |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i \setminus \Omega_i)}^2
\]

\[
\leq 2|I^h(\theta(u - u_0))|_{a(\bar{\Omega}_i)}^2 + 4|u^2|_{a(\bar{\Omega}_i)} + 4|u_0|_{a(\bar{\Omega}_i)}^2 + |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i \setminus \Omega_i)}^2.
\]

As \( \theta \) is only nonzero on \( \Gamma^h \), it follows from Lemma 11.3 that

\[
\sum_{i=1}^N 2|I^h(\theta(u - u_0))|_{a(\bar{\Omega}_i)}^2 = 2|I^h(\theta(u - u_0))|_{a(\Omega)}^2
\]

\[
\leq 2 \frac{C_r}{tol_p} \sum_{\xi \in \mathcal{P}} \sum_{k \in n^h} |u_i|^2_{a(\bar{\Omega}_k)}
\]

\[
(11.4)
\]

(11.4)

Similarly, we have

\[
\sum_{i=1}^N |I^h(\theta_i(u - u_0))|_{a(\bar{\Omega}_i \setminus \Omega_i)}^2 \leq \frac{C_r}{tol_p} \sum_{i=1}^N \sum_{\xi \in \mathcal{P}(\bar{\Omega}_i)} \sum_{k \in n^h} |u_i|^2_{a(\bar{\Omega}_k)} \leq \frac{C_r}{tol_p} |u|^2_{a(\Omega)}.
\]

Thus, using (11.4), (11.5), and Lemma 11.2, we obtain

\[
\sum_{i=0}^N |u_i|^2_{a(\Omega')} = |u_0|^2_{a(\Omega)} + \sum_{i=1}^N |u_i|^2_{a(\bar{\Omega}_i)}
\]

\[
\leq 5|u_0|^2_{a(\Omega)} + 4|u|_{a(\Omega)}^2 + 2 \frac{C_r N^\xi}{tol_p} |u|^2_{a(\Omega)} + \frac{C_r C}{tol_p} |u|^2_{a(\Omega)}
\]

\[
\leq 5 \cdot 2 \left( 1 + \frac{C_r N^\xi}{tol_p} \right) |u|^2_{a(\Omega)} + \left( 4 + (2N^\xi + C) \frac{C_r}{tol_p} \right) |u|^2_{a(\Omega)}
\]

\[
= \left( 14 + (12N^\xi + C) \frac{C_r}{tol_p} \right) |u|^2_{a(\Omega)}.
\]

From Theorem 11.4, we directly obtain a condition number estimate for the preconditioned system.

**Corollary 11.5.** The condition number of the RAGDSW two-level Schwarz operator in three dimensions is bounded by

\[
\kappa \left( M_{\text{RAGDSW}}^{-1} K \right) \leq \left( 14 + (12N^\xi + C) \frac{C_r}{tol_p} \right) \left( \bar{N} + 1 \right),
\]

where \( \bar{N} \) is an upper bound for the number of overlapping subdomains \( \{ \Omega_i' \}_{i=1}^N \). Any point \( x^h \in \Omega \) can belong to. All constants are independent of \( H, h, \) and the contrast of Young’s modulus \( E \).
Proof. Since we use exact local solvers, we directly obtain
\[
\kappa \left( M_{\text{RAGDSW}}^{-1} \right) \leq C_0^2 \left( \tilde{N}_c + 1 \right),
\]
where \( C_0^2 \) is the constant of the stable decomposition; cf. [43, Lemma 3.11] and the follow-up discussion and the proof of [11, Theorem 4.1]. We obtain the final estimate using Theorem 11.4.

12. A variant using local Neumann problems. We will now describe a technique that can significantly speed up the algorithm in a parallel setting and greatly facilitate its implementation.

We first consider the case of an interface component which is a coarse face \( f \). The energy-minimal extension used in the generalized eigenvalue problem (9.7) is only weakly coupled between the two subdomains via the nodes adjacent to the face, i.e. \((\Gamma^h \cap \Omega_i \cap \Omega_j) \setminus f\) contains relatively few nodes on certain coarse edges and at certain coarse nodes. Instead of computing this coupled extension \( \mathcal{H}_{f \cap \Omega_j} (\cdot) \) from the face \( f \) to the two adjacent subdomains as in (9.3), we can compute the extensions to each subdomain \( \Omega_i, \Omega_j \) separately. We expect that little information will be lost. We find that
\[
a_{\Omega_i} \left( \mathcal{H}_{\xi \cap \Omega_i} (\theta), \mathcal{H}_{\xi \cap \Omega_i} (\theta) \right) \geq \sum_{k \in n^i} a_{\Omega_k} \left( \mathcal{H}_{\xi \cap \Omega_k} (\theta), \mathcal{H}_{\xi \cap \Omega_k} (\theta) \right),
\]
for \( \theta \in X^h (\xi) \). Since the subdomains are only weakly coupled via these adjacent nodes of the face, we expect only a small change if we replace the left hand side of (9.7) using this alternative extension and that the dimension of the coarse space will increase only slightly.

The same technique can be applied to arbitrary interface components \( \xi \in \mathcal{P} \). We might expect that the coupling will be stronger between subdomains for smaller interface components but our numerical results in section 14 suggest that the increase in the coarse space dimension is moderate in all cases considered.

We indicate that this technique is employed by adding a trailing \( S \) to the coarse space name: \( V_{\text{RAGDSW}} - S \) and \( V_{\text{RAGDSW}} - S \). Using this modification yields the same condition number bound as in Corollary 11.5, since the modified \( d_\xi (\cdot, \cdot), d_\xi^S (\cdot, \cdot), \) satisfies the same inequality as in (10.4):
\[
|v|_{d_\xi^2}^2 := d_\xi^2 (v, v) := \sum_{k \in n^i} |\mathcal{H}_{\xi \cap \Omega_k} (v)|_{a_{\Omega_k}}^2 \leq \sum_{k \in n^i} |v|_{a_{\Omega_k}}^2 = |v|_{a_{\Omega}}^2 \quad \forall v \in V^h (\Omega).
\]

Let the local (nonoverlapping) stiffness matrices with a Neumann boundary for the corresponding bilinear forms \( a_{\Omega_k} (\cdot, \cdot) \) be given by \( K_{\Omega_k} \). For each \( \xi \in \mathcal{P} \), we partition the degrees of freedom of \( \Omega_i \) into those in \( \xi \cap \Omega_i \) and the remaining ones, \( R \). We have
\[
K_{\Omega_i} = \begin{pmatrix}
K_{\Omega_i}^{\Omega_i} & K_{\Omega_i}^{\Omega_k} \\
K_{\Omega_k}^{\Omega_i} & K_{\Omega_k}^{\Omega_k}
\end{pmatrix},
\]
Let \( R_{\xi \cap \Omega_k}^T \) map the degrees of freedom of \( \xi \cap \Omega_k \) to \( \xi \). We define
\[
S^{\xi S}_{\xi \xi} := \sum_{k \in n^i} R_{\xi \cap \Omega_k}^T S^k_{\xi \xi},
\]
with the Schur complements
\[
S^k_{\xi \xi} := K_{\xi \xi}^{\Omega_k} - K_{\xi \xi}^{\Omega_k} \left( K_{\xi \xi}^{\Omega_k} \right)^+ K_{\xi \xi}^{\Omega_k}, \quad k \in n^\xi,
\]
where \((K_{Rk})^+\) is a pseudoinverse of \(K_{Rk}\), cf. Remark 9.1 and section 13. Using the definition of \(\tilde{K}_{\xi}\) from (7.2), we obtain the modified generalized eigenvalue problem given in matrix form by

\[
S_{\xi}^S \tau_{*, \xi} = \lambda_{*, \xi} \tilde{K}_{\xi} \tau_{*, \xi}.
\]

13. Remarks on the computation of the energy-minimal extension.

For an interface component \(\xi \in \mathcal{P}\), the energy-minimal extension (9.3) satisfies a homogeneous Neumann boundary condition on \(\partial \Omega_k \setminus \xi\). Therefore, for linear elasticity, if \(\xi\) consists only of a single node or if it is given by a straight edge, then all three rotations or the rotation around the edge are in the null space of the problem; cf. Remark 9.1. Thus, in such cases, the operator \(\mathcal{H}_{\xi, \Omega_k}(\cdot)\) defined by (9.3) is symmetric and only positive semidefinite.

We also note that if the variant described in section 12 is used, the extension operators are even more likely to be only positive semidefinite, since the extension is defined on the sets \(\xi \cap \Omega_k, k \in n^\xi\).

In an implementation, we have several options. Theoretically, we could compute a full pseudoinverse, however, this is very expensive in terms of processor time and memory. As an algebraic alternative, a pivoted factorization can be computed such that the diagonal is rank revealing. Alternatively, we can add a small regularization term \(\varepsilon \mathcal{R}\) to obtain a symmetric, positive definite problem; e.g., \(\varepsilon \mathcal{R} = 10^{-13} \text{K}_{\text{diag}}\), where \(\text{K}_{\text{diag}}\) is the diagonal of the respective matrix.

We have also considered two further, geometric approaches. One approach is to remove the null space by a projection. For this, we need to determine a basis of the null space, i.e., compute the rotations which requires geometric information. This approach has another downside, if we want to use a direct solver on the resulting system, since transforming the system is quite expensive and the transformed system is generally more dense.

A second geometric approach is less algebraic and eliminates a subset of the degrees of freedom of the matrix \(H\) corresponding to \(\mathcal{H}_{\xi, \Omega_k}(\cdot)\) at the expense of solving a small Schur complement system using a pseudoinverse. At best, this amounts to prescribing a zero Dirichlet boundary condition on some additional degrees of freedom. We partition the matrix \(\tilde{H}\) w.r.t. \(\xi\) and the remaining degrees of freedom \(R\). To evaluate \(\mathcal{H}_{\xi, \Omega_k}(\cdot)\) requires the application of \(H_{1, R}\). However, if \(\xi\) is a straight edge or a vertex, the submatrix \(H_{RR}\) has a null space of 1 or 3 rotations.

In general, we pick as least as many degrees of freedom \(\tilde{D} \subset \tilde{R}\) as the dimension of the null space of \(H_{RR}\). Let the remaining degrees of freedom be denoted by \(\tilde{R} \subset R\). The matrix \(H_{RR}\) is partitioned by \(\tilde{R}\) and \(\tilde{D}\) s.t.

\[
H_{RR} = \begin{pmatrix}
H_{R, \tilde{R}} & H_{R, \tilde{D}} \\
H_{\tilde{D}, R} & H_{\tilde{D}, \tilde{D}}
\end{pmatrix}.
\]

The variables \(\tilde{R}\) are then eliminated to obtain a Schur complement system

\[
\begin{pmatrix}
H_{R, \tilde{R}} & H_{R, \tilde{D}} \\
0 & S_{\tilde{D}, \tilde{D}}
\end{pmatrix}, \quad S_{\tilde{D}, \tilde{D}} = H_{\tilde{D}, \tilde{D}} - H_{\tilde{D}, \tilde{R}} H_{\tilde{R}, \tilde{R}}^{-1} H_{\tilde{R}, \tilde{D}}.
\]

If \(\tilde{D}\) was chosen properly, the submatrix \(H_{R, \tilde{R}}\) is invertible. For example, if \(\xi\) is a straight edge and \(\tilde{D}\) corresponds to a node which does not lie on the same straight as the edge (note that three degrees of freedom are associated with each node), then \(H_{R, \tilde{R}}\) is invertible. In that case, the Schur complement is well defined and has a null space of the same dimension as \(H_{RR}\). Thus, we can solve the corresponding
system using a pseudoinverse. This is much cheaper than using a pseudoinverse on $K_{RR}$, since $S_{\tilde{D}, \tilde{D}}$ is of a much smaller dimension than $K_{RR}$.

If we select the degrees of freedom in $\tilde{D}$ carefully, the Schur complement will be identically zero, i.e., evaluating $H_{\xi_{\Omega}}(\cdot)$ is no more expensive than solving a linear system with $K_{\tilde{R}, \tilde{R}}$ and the cost will be comparable to that of a case with an invertible $K_{RR}$.

14. Numerical results. In this section, we present numerical results to compare the nonadaptive coarse spaces GDSW and RGDSW, the adaptive coarse spaces AGDSW (section 5) and RAGDSW (section 8), and their S-variants AGDSW–S and RAGDSW–S; cf. section 12.

We show numerical results for a discretization of problem (2.1) with a Poisson ratio $\nu = 0.4$, the right hand side $f \equiv (1, 1, 1)^T$, and several coefficient functions given by different choices of the Young modulus function $E(\cdot)$. The smallest Young modulus $E_{\text{min}} := \min_{x \in \Omega} E(x)$ is always set to 1 and the maximum $E_{\text{max}} := \max_{x \in \Omega} E(x)$ is specified in the respective figure and table caption. Except for the test case of Figure 6 and Table 3, the computational domain is the unit cube with a zero Dirichlet condition prescribed on all its boundary.

We use piecewise linear basis functions on tetrahedra and we solve the resulting linear system with the preconditioned conjugate gradient (PCG) method and a relative stopping criterion of $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-8}$, where $r^{(0)}$ and $r^{(k)}$ are the initial and the $k$th unpreconditioned residuals. The reported condition numbers are the estimates obtained after the last iteration of the PCG method using the Lanczos method [39, Chapter 6.7.3]. We partition the domain into subdomains using METIS [29]. In all experiments, we use an overlap of two layers of finite elements; see section 3 for the definition of the overlap.

The coefficient function of the first test problem is depicted in Figure 5; the corresponding results are given in Table 2. Experiments with both nonadaptive coarse spaces GDSW and RGDSW failed to converge in 2000 iterations, clearly showing that adaptivity is required to obtain a robust preconditioner. By using the adaptive coarse spaces, we obtain acceptable condition numbers and iteration counts. The results show a significant reduction in the coarse space dimension for the RAGDSW variant compared to AGDSW. For example ($tol = 0.05$), the dimension

Fig. 5. Cross section (left) of a domain decomposition of a cube and a discontinuous coefficient function $E$ with beams of large coefficients (light blue) crossing the domain. The beams of large coefficients do not touch the domain boundary. The light blue color corresponds to a coefficient of $E_{\text{max}} = 10^6$ and the remainder is set to $E_{\text{min}} = 1.0$. Number of subdomains: 125; number of nodes: 132,651 (degrees of freedom: 397,953); average degrees of freedom per overlapping subdomain: 6,198; overlap: two layers of finite elements. Structured tetrahedral mesh; unstructured domain decomposition (METIS). For the corresponding results, see Table 2. Taken from [21, Figure 8].
### Table 2

Results for the coefficient function in Figure 5: iteration counts, condition numbers, and resulting coarse space dimension for different coarse spaces. Number of subdomains: 125; degrees of freedom: 397 953; overlap: two layers of finite elements; maximum coefficient $E_{\text{max}} = 10^9$; relative stopping criterion $\|\cdot\|_2/\|\cdot\|_2 < 10^{-8}$. Structured tetrahedral mesh; unstructured domain decomposition (METIS).

<table>
<thead>
<tr>
<th>$V_0$</th>
<th>$\text{tol}$</th>
<th>$\text{it}$</th>
<th>$\kappa$</th>
<th>$\text{dim} V_0$</th>
<th>$(V/P, \xi, \zeta)$</th>
<th>$(V_{AGDSW})$</th>
<th>$(V_{RAGDSW})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{GDSW}$</td>
<td>$-$</td>
<td>$&gt;$2000</td>
<td>$3.1\cdot10^9$</td>
<td>9996</td>
<td>$(1 707, 4 618, 3 367)$</td>
<td>2.51%</td>
<td>$\text{AGDSW}$</td>
</tr>
<tr>
<td>$\text{RAGDSW}$</td>
<td>$-$</td>
<td>$&gt;$2000</td>
<td>$3.9\cdot10^9$</td>
<td>3358</td>
<td>$(3 358, 0, 0)$</td>
<td>0.84%</td>
<td>$\text{AGDSW}$</td>
</tr>
<tr>
<td>$\text{AGDSW-S}$</td>
<td>0.100</td>
<td>71</td>
<td>41.1</td>
<td>14 439</td>
<td>$(1 707, 4 943, 7 789)$</td>
<td>3.63%</td>
<td>$\text{RAGDSW}$</td>
</tr>
<tr>
<td>$\text{RAGDSW-S}$</td>
<td>0.050</td>
<td>90</td>
<td>59.5</td>
<td>13 945</td>
<td>$(1 707, 4 915, 7 323)$</td>
<td>3.50%</td>
<td>$\text{AGDSW}$</td>
</tr>
<tr>
<td>$\text{AGDSW-S}$</td>
<td>0.010</td>
<td>132</td>
<td>161.1</td>
<td>13 763</td>
<td>$(1 707, 4 912, 7 144)$</td>
<td>3.46%</td>
<td>$\text{AGDSW}$</td>
</tr>
<tr>
<td>$\text{RAGDSW-S}$</td>
<td>0.001</td>
<td>327</td>
<td>971.8</td>
<td>13 721</td>
<td>$(1 707, 4 907, 7 107)$</td>
<td>3.45%</td>
<td>$\text{AGDSW}$</td>
</tr>
</tbody>
</table>

![Discontinuous coefficient function](image1.png)

**Fig. 6.** (left) Discontinuous coefficient function $E$ with coefficient layers of $E = 10^6$ in light gray and an inclusion at the top right with $E = 10^9$ in dark grey. The remainder of the coefficient in white is set to $E_{\text{min}} = 1.0$. (center) Boundary partition for Dirichlet (blue) and Neumann (orange) boundary. (right) Domain decomposition of 50 subdomains. Number of nodes: 56 053 (degrees of freedom: 168 159); average degrees of freedom per overlapping subdomain: 5 632.2; overlap: two layers of finite elements. Unstructured tetrahedral mesh; unstructured domain decomposition (METIS). For the corresponding results, see Table 3. Taken from [21, Figure 9].

712 of $V_{\text{AGDSW-S}}$ is reduced by 43.6% by using $V_{\text{RAGDSW-S}}$. And even while GDSW does not converge in 2000 iterations, its coarse space is 26.5% larger than that of RAGDSW–S ($\text{tol} = 0.05$).

715 For the next example, we consider a problem, for which we impose a Neumann boundary condition on most of the domain boundary; see Figure 6. The results in Table 3 show an even larger reduction in the coarse space dimension from AGDSW to RAGDSW compared to the previous case. We obtain a reduction of 69.4% ($\text{tol} = 0.05$). The reason for this is the larger number of interface components: Since the AGDSW space contains the GDSW space and the RAGDSW space contains the RGDSW space, a significant part of the coarse space reduction can be attributed to the smaller dimension of RGDSW compared to GDSW. This highlights the core idea behind the reduced dimension GDSW spaces in [9]; the explanation is supported by the fact that the dimension of $V_{\text{RAGDSW}}$ is fairly close to that of $V_{\text{RGDSW}}$. Therefore, since the coefficient function contains only relatively few connected large coefficient components, only a few additional coarse basis functions are required.
Table 3

Results for the coefficient function in Figure 6: iteration counts, condition numbers, and resulting coarse space dimension for different coarse spaces. Number of subdomains: 50; degrees of freedom: 168 159; overlap: two layers of finite elements; maximum coefficient $E_{\text{max}} = 10^6$; relative stopping criterion $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-8}$. Unstructured tetrahedral mesh; unstructured domain decomposition (METIS).

<table>
<thead>
<tr>
<th>$V_0$</th>
<th>$\text{tol}$</th>
<th>$\text{it.}$</th>
<th>$\kappa$</th>
<th>$\dim V_0$</th>
<th>$\dim V_0/\text{dof}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{GDSW}}$</td>
<td>-</td>
<td>1349</td>
<td>1.0·10'</td>
<td>2319 (291, 1058, 15028)</td>
<td>1.38%</td>
</tr>
<tr>
<td>$V_{\text{RGDSW}}$</td>
<td>-</td>
<td>1549</td>
<td>1.0·10'</td>
<td>572 (572, 0, 0)</td>
<td>0.34%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.100</td>
<td>60</td>
<td>20.2</td>
<td>2732 (291, 1058, 1383)</td>
<td>1.62%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.050</td>
<td>69</td>
<td>28.1</td>
<td>2631 (291, 1058, 1282)</td>
<td>1.56%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.010</td>
<td>71</td>
<td>28.2</td>
<td>2626 (291, 1058, 1277)</td>
<td>1.56%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.001</td>
<td>152</td>
<td>1162.2</td>
<td>2613 (291, 1052, 1270)</td>
<td>1.55%</td>
</tr>
<tr>
<td>$V_{\text{RGDSW}}$</td>
<td>0.100</td>
<td>58</td>
<td>18.9</td>
<td>2741 (291, 1059, 1391)</td>
<td>1.63%</td>
</tr>
<tr>
<td>$V_{\text{RGDSW}}$</td>
<td>0.050</td>
<td>69</td>
<td>28.1</td>
<td>2631 (291, 1058, 1282)</td>
<td>1.56%</td>
</tr>
<tr>
<td>$V_{\text{RGDSW}}$</td>
<td>0.010</td>
<td>72</td>
<td>28.2</td>
<td>2626 (291, 1058, 1277)</td>
<td>1.56%</td>
</tr>
<tr>
<td>$V_{\text{RGDSW}}$</td>
<td>0.001</td>
<td>142</td>
<td>733.7</td>
<td>2614 (291, 1053, 1270)</td>
<td>1.55%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.100</td>
<td>68</td>
<td>27.1</td>
<td>988 (988, 0, 0)</td>
<td>0.59%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.050</td>
<td>85</td>
<td>43.8</td>
<td>804 (804, 0, 0)</td>
<td>0.48%</td>
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<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.010</td>
<td>100</td>
<td>88.5</td>
<td>781 (781, 0, 0)</td>
<td>0.46%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.001</td>
<td>183</td>
<td>769.1</td>
<td>774 (774, 0, 0)</td>
<td>0.46%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.100</td>
<td>60</td>
<td>20.7</td>
<td>1152 (1152, 0, 0)</td>
<td>0.69%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.050</td>
<td>78</td>
<td>35.2</td>
<td>868 (868, 0, 0)</td>
<td>0.52%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.010</td>
<td>100</td>
<td>87.6</td>
<td>790 (790, 0, 0)</td>
<td>0.47%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.001</td>
<td>115</td>
<td>141.1</td>
<td>786 (786, 0, 0)</td>
<td>0.47%</td>
</tr>
</tbody>
</table>

We conclude with averaged results for 100 random coefficient functions showing

Fig. 7. Partial visualization of an unstructured tetrahedral mesh consisting of several disconnected components of foam-like structures. On the corresponding mesh of a cube, foam corresponds to a large coefficient of $E_{\text{max}} = 10^6$ with $E_{\text{min}} = 1.0$ elsewhere. The large coefficient does not touch the domain boundary. Number of subdomains: 100; number of nodes: 588 958 (degrees of freedom: 1766 874); average degrees of freedom per overlapping subdomain: 19 969; overlap: two layers of finite elements. Unstructured tetrahedral mesh; unstructured domain decomposition (METIS). For the corresponding results, see Table 4. Taken from [21, Figure 10].

We consider another realistic geometry in Figure 7 with a foamlike structure. We note that the foam is not a single connected structure but consists of several smaller disconnected foamlike structures. The results in Table 4 are similar to the previous ones. By using RAGDSW–S, we obtain a coarse space reduction of 49.9% compared to AGDSW–S ($\text{tol} = 0.05$). However, here, the dimension of $V_{\text{RAGDSW}}$ is more than double that of $V_{\text{RGDSW}}$ indicating that $V_{\text{RAGDSW}}$ is adaptively enriched with quite a few additional basis functions compared to $V_{\text{RGDSW}}$.

We conclude with averaged results for 100 random coefficient functions showing
Table 4

Results for the coefficient function in Figure 7: iteration counts, condition numbers, and resulting coarse space dimension for different coarse spaces. Number of subdomains: 100; degrees of freedom: 1 760 874; overlap: two layers of finite elements; maximum coefficient $E_{\text{max}} = 10^6$; relative stopping criterion $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-8}$. Unstructured tetrahedral mesh; unstructured domain decomposition (METIS).

<table>
<thead>
<tr>
<th>$V_0$</th>
<th>tol</th>
<th>it.</th>
<th>$\kappa$</th>
<th>dim $V_0$ (V/f, $\ell$, $\ell'$)</th>
<th>dim $V_0$/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{GDSW}}$</td>
<td>$-$</td>
<td>1.865</td>
<td>1.1-10$^6$</td>
<td>8 311 (1 167, 4 108, 3 036)</td>
<td>0.47%</td>
</tr>
<tr>
<td>$V_{\text{RGSW}}$</td>
<td>$-$</td>
<td>1.613</td>
<td>9.3-10$^5$</td>
<td>2 313 (2 313, 0, 0)</td>
<td>0.13%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.10</td>
<td>52</td>
<td>21.4</td>
<td>12 367 (1 167, 4 358, 6 842)</td>
<td>0.70%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.05</td>
<td>68</td>
<td>43.8</td>
<td>10 940 (1 167, 4 351, 5 422)</td>
<td>0.62%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.01</td>
<td>167</td>
<td>333.4</td>
<td>10 304 (1 167, 4 324, 4 813)</td>
<td>0.58%</td>
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<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.10</td>
<td>50</td>
<td>18.7</td>
<td>12 539 (1 167, 4 389, 6 983)</td>
<td>0.71%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.05</td>
<td>63</td>
<td>32.2</td>
<td>11 005 (1 167, 4 362, 5 476)</td>
<td>0.62%</td>
</tr>
<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.01</td>
<td>147</td>
<td>158.1</td>
<td>10 320 (1 167, 4 338, 4 815)</td>
<td>0.58%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.10</td>
<td>54</td>
<td>22.9</td>
<td>6 641 (6 641, 0, 0)</td>
<td>0.38%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.05</td>
<td>80</td>
<td>45.2</td>
<td>4 868 (4 868, 0, 0)</td>
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</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
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<td>280.2</td>
<td>4 019 (4 019, 0, 0)</td>
<td>0.23%</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW-S}}$</td>
<td>0.10</td>
<td>50</td>
<td>18.4</td>
<td>7 833 (7 833, 0, 0)</td>
<td>0.44%</td>
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<td>$V_{\text{RAGDSW-S}}$</td>
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<td>69</td>
<td>46.1</td>
<td>5 519 (5 519, 0, 0)</td>
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<tr>
<td>$V_{\text{RAGDSW-S}}$</td>
<td>0.01</td>
<td>151</td>
<td>202.6</td>
<td>4 152 (4 152, 0, 0)</td>
<td>0.23%</td>
</tr>
</tbody>
</table>

Table 5

Averaged results for 100 random coefficient functions (average large coefficient density: 11.08%): tolerance for the selection of the eigenfunctions, iteration counts, condition numbers, and resulting coarse space dimension for different coarse spaces; maximum in brackets. Number of subdomains: 512; number of nodes: 4 551 522 (degrees of freedom: 1 357 566); average degrees of freedom per overlapping subdomains: 5 906.4; overlap: two layers of finite elements; maximum coefficient $E_{\text{max}} = 10^6$; relative stopping criterion $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-8}$. Unstructured tetrahedral mesh; unstructured domain decomposition (METIS). $V_{\text{GDSW}}$ and $V_{\text{RGSW}}$ never converged in 2000 iterations.

<table>
<thead>
<tr>
<th>$V_0$</th>
<th>tol</th>
<th>it.</th>
<th>$\kappa$</th>
<th>dim $V_0$</th>
<th>Random coefficient function $E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{GDSW}}$</td>
<td>$-$</td>
<td>2 000 (-)</td>
<td>2.1-10$^7$</td>
<td>49 862 (49 862)</td>
<td>$3.7%$ (3.7%)</td>
</tr>
<tr>
<td>$V_{\text{RGSW}}$</td>
<td>$-$</td>
<td>2 000 (-)</td>
<td>2.4-10$^7$</td>
<td>17 778 (17 778)</td>
<td>1.3% (1.3%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.10</td>
<td>81.8 (93)</td>
<td>56.2 (80.7)</td>
<td>69 006 (69 892)</td>
<td>5.1% (5.1%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.05</td>
<td>106.3 (118)</td>
<td>92.1 (145.2)</td>
<td>66 482 (67 273)</td>
<td>4.9% (5.0%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW}}$</td>
<td>0.01</td>
<td>180.8 (228)</td>
<td>293.3 (662.9)</td>
<td>64 508 (65 235)</td>
<td>4.8% (4.8%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.10</td>
<td>76.4 (84)</td>
<td>44.1 (54.2)</td>
<td>70 570 (71 632)</td>
<td>5.2% (5.3%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.05</td>
<td>99.3 (112)</td>
<td>77.9 (110.7)</td>
<td>67 445 (68 360)</td>
<td>5.0% (5.0%)</td>
</tr>
<tr>
<td>$V_{\text{AGDSW-S}}$</td>
<td>0.01</td>
<td>168.1 (195)</td>
<td>247.5 (448.4)</td>
<td>65 212 (66 046)</td>
<td>4.8% (4.9%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.10</td>
<td>89.5 (100)</td>
<td>60.9 (82.2)</td>
<td>39 081 (39 780)</td>
<td>2.9% (2.9%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.05</td>
<td>115.1 (129)</td>
<td>104.8 (152.5)</td>
<td>35 961 (36 649)</td>
<td>2.6% (2.7%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW}}$</td>
<td>0.01</td>
<td>200.3 (232)</td>
<td>342.8 (525.6)</td>
<td>33 370 (34 058)</td>
<td>2.5% (2.5%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW-S}}$</td>
<td>0.10</td>
<td>74.9 (88)</td>
<td>42.8 (59.6)</td>
<td>44 045 (44 167)</td>
<td>3.2% (3.3%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW-S}}$</td>
<td>0.05</td>
<td>97.1 (112)</td>
<td>72.9 (103.5)</td>
<td>39 076 (39 730)</td>
<td>2.9% (2.9%)</td>
</tr>
<tr>
<td>$V_{\text{RAGDSW-S}}$</td>
<td>0.01</td>
<td>167.8 (199)</td>
<td>244.7 (469.9)</td>
<td>35 399 (36 137)</td>
<td>2.6% (2.7%)</td>
</tr>
</tbody>
</table>

the robustness of the methods; cf. Table 5. Despite comparable number of iterations and condition numbers, the coarse space dimensions of RAGDSW(-S) are smaller by a factor of 1.6 compared to those of AGDSW(-S) (at an equal tolerance).

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