COMPACT LIE GROUP ACTIONS ON CONTACT MANIFOLDS

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Zusammenfassung

Das Hauptergebnis dieser Arbeit besteht in der Klassifikation von SO(3)-Wirkungen auf 5-dimensionalen Kontaktmannigfaltigkeiten. Die Impulsabbildung ermöglicht die Reduktion eines solchen Raums um zwei Dimensionen. Diese Methode scheitert aber in den singulären Punkten, die man deshalb getrennt untersuchen muß. Für diese Punkte stellt man fest, daß alle möglichen Fälle durch 3 Modelle abgedeckt werden. Die ursprüngliche 5-dimensionale Mannigfaltigkeit kann man dadurch rekonstruieren, daß man den 3-dimensionalen Unterraum in verträglicher Weise auf die Menge der singulären Punkte klebt. Es ist bekannt, daß $\mathbb{S}^1$-Hauptfaserbündel über einer geschlossenen Fläche durch die Eulerzahl charakterisiert werden. In unserer Situation gibt es eine ähnliche Zahl, die die Verklebung der beiden oben genannten Mengen festlegt.

Abstract

The main result in this thesis is the classification of SO(3)-actions on contact 5-manifolds. Using properties of the moment map, one can reduce the manifold to a 3-dimensional contact manifold with an $\mathbb{S}^1$-action. This works everywhere outside of the singular orbits. For the singular orbits three models can be given that describe all possible cases. The 5-manifold is then obtained by gluing the singular set onto the 3-dimensional $\mathbb{S}^1$-manifold in a compatible way. As is well-known, $\mathbb{S}^1$-bundles over a closed surface are classified by an integer called the Euler number. A similar invariant can be recovered in our 3-dimensional setting. We call it the Dehn-Euler number.

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CHAPTER I

Introduction

The main objective of this is thesis is to explain the classification of 5-dimensional contact manifolds with SO(3)-symmetry.

Readers not familiar with the terms used above should think of contact manifolds as generalizations of energy hypersurfaces in a Hamiltonian mechanical system. I.e. imagine a system of N particles in the standard Euclidean space $\mathbb{R}^3$. The position of the j-th particle is given by the vector $\vec{q}_j$, its movement is described by the impulse (speed) $\vec{p}_j$. That means that the complete Hamiltonian system is described by a vector $(\vec{q}_1, \dots, \vec{q}_N; \vec{p}_1, \dots, \vec{p}_N) \in \mathbb{R}^{6N}$. In classical mechanics the energy of the system is given by a function

$$E(\vec{q}_1, \dots, \vec{q}_N; \vec{p}_1, \dots, \vec{p}_N) = \sum_{j=1}^{N} A_j \langle \vec{p}_j | \vec{p}_j \rangle + V(\vec{q}_1, \dots, \vec{q}_N) \ ,$$

where the first term (with A_j positive numbers) is called the kinetic energy, the second term, which describes the interaction of the particles with each other, is called the potential energy. The set of system configurations

$$M_\varepsilon := \{(\vec{q}_1, \dots, \vec{q}_N; \vec{p}_1, \dots, \vec{p}_N) | E(\vec{q}_1, \dots, \vec{p}_N) = \varepsilon\}$$

with the given energy ε is under reasonable assumptions a submanifold of dimension $6N - 1$ that carries a natural contact structure. At each point

$$(\vec{q}_1, \dots, \vec{q}_N; \vec{p}_1, \dots, \vec{p}_N) \in M_\varepsilon$$

there is a direction into which the system will move under time. The contact structure is the collection of planes normal to this direction at the points of M_ε. (For a definition of what a contact manifold really is, take a look at Chapter II.)

The symmetry group SO(3) is the set of rotations of the standard 3-dimensional Euclidean space.

The contact topology of 3-dimensional manifolds is a subject which has been studied for a long time, and with great success. Unfortunately, very little is known about higher dimensions. This thesis treats 5-dimensional manifolds, so we will now mainly focus on this dimension, and sketch some of the results known for this case.

1. Overview of 5-dimensional contact topology

1.1. Examples and existence results. Examples of 5-dimensional contact manifolds have been known for a long time. The unit cotangent bundle $\mathbb{S}(T^*M)$ of a 3-manifold M carries a natural contact structure (these examples describe mechanical systems like the one explained above). In particular, because all oriented 3-manifolds are parallelizable, for an orientable 3-manifold, we have

$$\mathbb{S}(T^*M) \cong M \times \mathbb{S}^2 \ .$$

Further examples are the Boothby-Wang manifolds ([**BW58**] or Section IV.1), which are $\mathbb{S}^1$-principal bundles over a suitable symplectic manifold.

Lutz and Meckert found a natural contact structure on all Brieskorn manifolds ([**LM76**] or Section IV.6.1), which are convex boundaries of Stein manifolds.

A more systematic approach was taken by Geiges in [**Gei91**], where he showed that any simply connected 5-manifold carries a contact structure in any homotopy class of hyperplane fields, provided some "obvious" topological conditions are met.

Some constructions exist to build new contact manifolds out of old ones, e.g. connected sum, Dehn twists (Appendix D) etc.

1.2. Invariants in 5-dimensional contact geometry.

The so-called classical contact invariants are topological ones. Any contact structure on a manifold M represents a hyperplane distribution, i.e. a codimension-1 subbundle of TM. Given two possibly non-equivalent contact structures on M, one can check if the corresponding subbundles are equivalent. This is done by comparing characteristic classes.

This method is relatively rough though Ustilovsky showed in [**Ust99**] using contact homology that the 5-sphere carries infinitely many non-equivalent contact structures that cannot be distinguished by the classical invariants.

In 3-dimensional contact topology the division into tight and overtwisted structures (see Section II.1) is one of the most fundamental discoveries in the field. No similar notion is known in higher dimensions.

2. Group symmetry in contact geometry

In Riemannian geometry having a metric that is symmetric under some transformation group is an exceptional situation. In fact, a generic metric does not have any symmetry at all, and even the standard sphere $\mathbb{S}^n$, which is the n-dimensional manifold with largest symmetry, has only an $\frac{n\,(n+1)}{2}$-dimensional symmetry group.

In contact topology the situation is completely different. Here any contact manifold has a symmetry group of infinite dimension. Hence one is interested in finding subgroups which are easy to handle, e.g. finite dimensional subgroups, or even better compact subgroups. Finding a compact symmetry group is a strong restriction on the smooth structure of the manifold. Any compact Lie group contains for example a circle group, but there are very few smooth manifolds admiting a circle action (in Chapter III you can find a classification of all 3-dimensional manifolds with an $\mathbb{S}^1$-action, but the general classification of 3-manifolds is still unknown to this date). It is also interesting to note that n-dimensional exotic spheres do not allow a smooth action of $\mathrm{SO}(n+1)$ (which implies that smooth actions are different from continous ones).

The most prominent results in contact group actions is probably the classification of $\mathbb{S}^1$-actions on contact 3-manifolds ([**Lut77**], [**KT91**]; Chapter IV), and the classification of toric contact manifolds (completed by Lerman in [**Ler03**]), i.e. the actions of an $(n+1)$-dimensional torus $\mathbb{T}^{n+1}$ on $(2n+1)$-dimensional contact manifolds.

For a symplectic manifold (M, ω) it is known that if $\pi_2(M)$ vanishes, there is no Hamiltonian action of a compact Lie group on M. Whether similar restrictions exist in contact geometry is not known to the author.

An indication that group actions can lead to interesting examples is given by the following: Considering $\mathbb{S}^1$-actions on 3-manifolds, Lutz showed for the first time that a manifold can carry non-equivalent contact structures (in fact he produced all contact structures on $\mathbb{S}^3$ in this way [**Lut77**] – that these were all, was shown later by Eliashberg [**Eli89**], [**Eli92**]).

3. The results of this thesis

The main result in this thesis is the classification of SO(3)-actions on contact 5-manifolds. Using properties of the moment map, one can reduce the manifold to a 3-dimensional contact manifold with an $\mathbb{S}^1$-action. This works everywhere outside of the singular orbits. For the singular orbits three models can be given that describe all possible cases. The 5-manifold is then obtained by gluing the singular set onto the 3-dimensional $\mathbb{S}^1$-manifold in a compatible way. As is well-known, $\mathbb{S}^1$-bundles over a closed surface are classified by an integer called the Euler number. A similar invariant can be recovered in our 3-dimensional setting. We call it the Dehn-Euler number.

Giroux proposed a method to produce new contact structures from a given one by applying a so-called Dehn twist: If one finds a closed chain of Legendrian spheres in a contact 5-manifold, its neighborhood is predetermined. One can cut out such a neighborhood, perform a Dehn twist (as defined by Seidel), and glue it back in. The smooth structure of the manifold is not changed, but the new manifold is often not contactomorphic to the initial one. For the contact SO(3)-manifolds, it can be shown that the integer described above is equal to the number of Dehn twists. Using this characterisation, it is for example easy to see that the Ustilovsky spheres can be obtained from the standard contact 5-sphere using the Dehn twist construction. One also obtains contact structures on $\mathbb{S}^5$ that are given by negative Dehn twists; Giroux has proposed negative Dehn twists as a generalization of the notion of overtwisted contact structures to higher dimensions.

CHAPTER II

Notation, definitions and preliminaries

In this chapter we will give basic definitions, and collect some necessary results without stating the proofs.

1. Contact manifolds

Definition. Let M be a $(2n+1)$-dimensional manifold with a hyperplane distribution ξ that is maximally non-integrable, i.e. if one represents ξ locally as the kernel of a smooth 1-form α (which is always possible), then $\alpha \wedge d\alpha^n$ does nowhere vanish. Such a ξ is called a **contact structure** on M.

The condition for a distribution χ which is the kernel of a 1-form β to be a foliation is $\beta \wedge d\beta \equiv 0$. The contact condition above is hence in a sense the exact opposite, and any submanifold N tangent to ξ on some open set $U \subset N$ can have at most dimension n.

Definition. A **contact form** α is a 1-form whose kernel is a contact structure. This is equivalent to requiring

$$\alpha \wedge d\alpha^n \neq 0 \ .$$

Remark II.1. Let ξ be a contact structure on M. There is a contact form α with $\ker \alpha = \xi$, if and only if the (real) line-bundle TM/ξ is trivial. Such a contact structure is called **coorientable**.

In this thesis all contact structures are assumed to be given by a contact form.

Remark II.2. Let α be a contact form, and f a nowhere vanishing smooth function. The contact form $f\,\alpha$ defines the same contact structure as ξ.

Example II.1. Let M be a closed manifold. The **canonical 1-form** λ_{can} on the contangent bundle is given at a point $\nu \in T_p^*M$ by

$$\lambda_{\text{can}} = \pi^*\nu \ ,$$

where $\pi : TM \to M$ is the bundle projection. The restriction of λ_{can} to the unit cotangent bundle $\mathbb{S}(T^*M)$ (with respect to any metric) is a contact form, and the differential $d\lambda_{\text{can}}$ is a symplectic form on the contangent bundle T^*M itself.

Definition. An submanifold N of a $(2n+1)$-dimensional contact manifold (M, ξ) is called **isotropic submanifold**, if N is tangent to ξ (i.e. $TN \subset \xi$). Such a manifold can have at most dimension n, and it that maximal case N is called a **Legendrian submanifold**.

Definition. Let (M, α) be a contact manifold. The **Reeb field** R of the contact form α is the unique vector field that satisfies

$$\alpha(R) \equiv 1 \quad \text{and} \quad \iota_R d\alpha \equiv 0 \ .$$

Two contact forms α_1, α_2 representing the same contact structure may have different Reeb fields.

In 3-dimensional contact topology the dichotomy between tight and overtwisted is one of the most fundamental notions.

Definition. Let (M, α) be a 3-dimensional closed contact manifold. It is called **overtwisted**, if there is an embedded 2-disc $\mathbb{D}^2$ with Legendrian boundary $\partial\mathbb{D}^2$

$$\iota : \mathbb{D}^2 \hookrightarrow M$$

such that $\iota^*\alpha$ vanishes only at the center of the disc (compare Figure 1). A non-overtwisted contact structure is called **tight**.

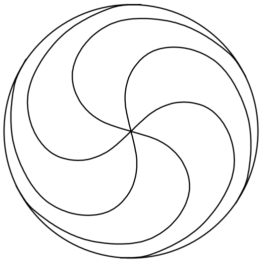


Figure 1. The induced foliation is asymptotic to the boundary

Often it is easier to find a disc $\mathbb{D}^2$ that is tangent to ξ along the whole boundary $\partial\mathbb{D}^2$ and at a single interior point. A proper overtwisted disc can be obtained from $\mathbb{D}^2$ by keeping $\mathbb{D}^2$ fixed along the boundary, while pushing the interior of $\mathbb{D}^2$ in the direction of the Reeb field (compare Figure 2).

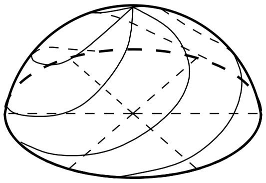


Figure 2. By pushing the singular disc a bit along the Reeb field, keeping the boundary fixed, we obtain a standard overtwisted disc.

Definition. Let (M, α) be a closed contact manifold. A symplectic manifold (W, ω) is called a **convex filling** of M, if M is the boundary of W, and if there is a vector field X defined in a neighborhood of M with the following properties

(i) X is an outward pointing vector field, transverse to $M = \partial W$, and $(\iota_X\omega)|_{TM} = \alpha$.
(ii) $\mathcal{L}_X\omega = \omega$.

Such a vector field is called a **Liouville vector field**. A contact manifold (M, α) is called **convex fillable**, if it allows a convex filling.

2. Group actions on contact manifolds

In this section let G be a compact Lie group acting on a manifold M, and let $\mathfrak{g}$ be the Lie-algebra of G. A very nice introduction to such actions can be found in [**Jän68**]. All actions are assumed to be **effective**, i.e. the map $G \to \mathrm{Diff}(M)$ is assumed to be injective.

Definition. The **stabilizer** (or isotropy group) $\mathrm{Stab}(p) \leq G$ is the closed subgroup that does not move $p \in M$, i.e.

$$\mathrm{Stab}(p) := \{g \in G |\, gp = p\} \ .$$

Sometimes we also write G_p instead of $\mathrm{Stab}(p)$. The **orbit** $\mathrm{Orb}(p)$ is the set

$$\mathrm{Orb}(p) := \{gp |\, g \in G\} \ .$$

Definition. One distinguishes the following types of orbits:

Principal orbits: An orbit $\mathrm{Orb}(p)$ is called principal, if there is no other point $q \in M$ such that $\mathrm{Stab}(q) \lneqq \mathrm{Stab}(p)$, i.e. the stabilizer is minimal. We denote the set of all principal orbits of M with $M_{(\mathrm{princ})}$.

Singular orbits: If the dimension of $\mathrm{Orb}(p)$ is smaller than the dimension of a principal orbit, then $\mathrm{Orb}(p)$ is called singular. We denote the set of all singular orbits with $M_{(\mathrm{sing})}$.

Regular orbits: Non-singular orbits are called regular, and we denote the set of all such orbits with $M_{(\mathrm{reg})}$.

Exceptional orbits: A regular, but non-principal orbit is called exceptional orbit, and we denote the set of all such orbits by $M_{(\mathrm{reg})}$.

Definition. The **infinitesimal generator** X_M of an element $X \in \mathfrak{g}$ is the vector field

$$X_M(p) := \left.\frac{d}{dt}\right|_{t=0} \exp(tX)\, p \ .$$

At any point $p \in M$, there exists a so-called **slice** S_p. This is a submanifold that is transverse to the orbit $\mathrm{Orb}(p)$, invariant under the action of $\mathrm{Stab}(p)$, and satisfies the condition that whenever $g \cdot q \in S_p$ (with $g \in G$ and $q \in S_p$), then $g \in \mathrm{Stab}(p)$.

Definition. A contact structure ξ is called G**-invariant**, if for every $g \in G$ and $p \in M$ the equation

$$g_* \xi_p = \xi_{gp}$$

holds. If ξ is given by a contact form α, this α need not be G-invariant, but one obtains an equivalent G-invariant contact form $\widetilde{\alpha}$ by averaging, i.e.

$$\widetilde{\alpha} := \int_G g^* \alpha \ .$$

A **contact** G**-manifold** (M, α) is a G-manifold with an invariant contact form α.

CHAPTER III

3-dimensional manifolds with $\mathbb{S}^1$-action

The aim of this chapter is to explain the classification of closed 3-dimensional $\mathbb{S}^1$-manifolds. The result was initially developed by [**Ray68**], but can be found in several other sources, as for example in [**Orl72**] or [**Aud04**] (the last reference is the most readable, but only treats the case of oriented manifolds). Note that we do not yet consider any contact structures on the manifolds in this chapter.

The main ideas for the classification are the following: The $\mathbb{S}^1$-manifold is almost everywhere a principal $\mathbb{S}^1$-bundle. Such a bundle would be classified by its base space, and a certain obstruction to finding a section. In our situation, the section has to be chosen with certain additional conditions to make it compatible with the non-principal orbits.

First we will describe the local features of an S^1-manifold.

1. The orbit types

The most important invariant of an orbit is the corresponding stabilizer. The only closed subgroups of the circle are $\{1\}$, $\mathbb{Z}_k$, and $\mathbb{S}^1$ itself. The principal orbits of an effective $\mathbb{S}^1$-action have trivial stabilizer, because principal stabilizers at different points are conjugate to each other, but since $\mathbb{S}^1$ is abelian, there would be a subgroup that acts trivially on the whole manifold.

1.1. Singular orbits. The only singular orbits are fixed points. We will denote the set of all fixed points of a manifold M by F.

With the help of the slice theorem, one sees that a neighborhood of $p \in F$ is determined by a faithful linear representation of $\mathbb{S}^1$ on T_pM. The only possible form is $T_pM \cong \mathbb{R} \oplus \mathbb{C}$ with action

$$e^{i\varphi}(t,z) = (t, e^{i\varphi}z) \ .$$

This means that the set F is composed by 1-dimensional submanifolds, and since M is closed, the components of F have to be diffeomorphic to $\mathbb{S}^1$. The neighborhood of a component of F is diffeomorphic to $\mathbb{R} \times \mathbb{C}/{\sim}$, where $(t,z) \sim (t+1, Az)$ with a linear map $A : \mathbb{C} \to \mathbb{C}$ that commutes with the $\mathbb{S}^1$-action. It is easy to check that $A \in \mathbb{C}^*$, but since $\mathbb{C}^*$ is connected, the model neighborhood can be represented as well by $\mathbb{S}^1 \times \mathbb{C}$ with the action $e^{i\varphi}(e^{it}, z) = (e^{it}, e^{i\varphi}z)$.

With this model it is easy to see that the projection $\pi : M \to M/\mathbb{S}^1$ to the orbit space can be described in a neighborhood of F by

$$\pi : \mathbb{S}^1 \times \mathbb{C} \to \mathbb{S}^1 \times [0,\infty),\ (e^{it}, re^{i\varphi}) \mapsto (e^{it}, r) \ .$$

Every section σ of the $\mathbb{S}^1$-action defined outside an open tubular neighborhood U_ε of F with radius ε is given by

$$\sigma : \mathbb{S}^1 \times [\varepsilon,\infty) \to \mathbb{S}^1 \times \mathbb{C},\ (e^{it}, r) \mapsto (e^{it}, re^{i\varphi(e^{it},r)}) \ ,$$

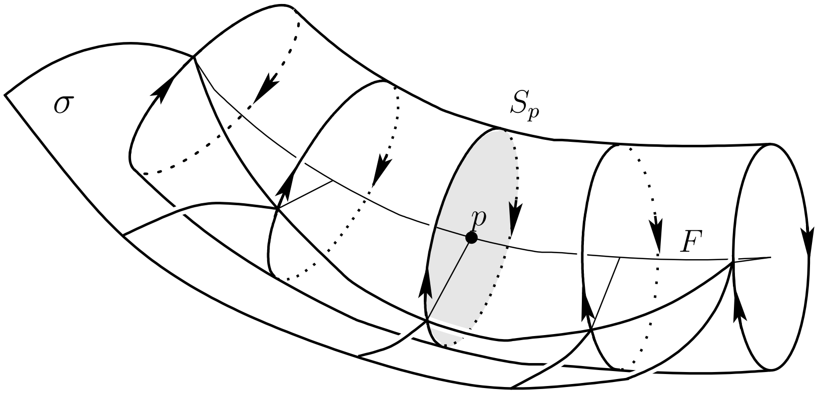


FIGURE 1. $\mathbb{S}^1$-action with fixed points

with a function $\varphi : \mathbb{S}^1 \times [\varepsilon, \infty) \to \mathbb{S}^1$. We can extend σ to the interior of U_ε by setting

$$\sigma(e^{it}, r) := (e^{it}, re^{i\varphi(e^{it},\varepsilon)})$$

that is, by connecting the section σ in ∂U_ε to the set of fixed points F with straight lines. The section constructed is only continuous, but it is possible to make σ smooth in a neighborhood of ∂U_ε.

1.2. Exceptional orbits. Denote the set of exceptional orbits by E. Exceptional orbits have stabilizer $\mathbb{Z}_k$. Their neighborhood is determined by the $\mathbb{Z}_k$-action on the 2-dimensional slice. The possible actions on the slice are given by rotations of the form

$$(\xi, z) \mapsto \xi^m \cdot z = e^{2\pi i m/k} z$$

for the generator $\xi = e^{2\pi i/k} \in \mathbb{Z}_k$, and m has to be an integer such that $\gcd(k, m) = 1$ (otherwise the action would not be effective). It is clear that m is only defined modulo k, but also the sign of m can change if we allow to invert the orientation of the slice.

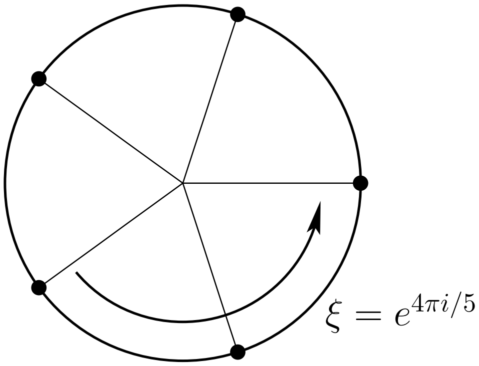


FIGURE 2. Slice of an exceptional orbit with $k = 5$ and $m = 2$ ($m = 3$ if the slice is given the opposite orientation)

If M is oriented, then one can fix m by the following argument: Orient the slice in such a way that its orientation, together with the direction of the $\mathbb{S}^1$-action give the orientation of

M. Then we can fix m in such a way that it lies between 1 and $k-1$. The numbers (k,m) are called the **oriented orbit invariants**.

Let U_ε be a tubular neighborhood of radius ε around an exceptional orbit. If M is non-orientable, then U_ε is *a priori* not oriented either, and the exceptional orbits with invariants (k,m) and $(k,k-m)$ are equivalent. The **unoriented orbit invariants** (k,m) are uniquely determined by requiring that $1 \leq m \leq k/2$. If the unoriented orbit invariants are not $(k,0)$ or $(k,k/2)$, then one can still give a canonical orientation to the neighborhood U_ε by choosing the orientation in such a way that the oriented and unoriented invariants coincide. For invariants $(k,0)$ or $(k,k/2)$, this does not distinguish the choices, because reverting the orientation gives $(k,k-0)=(k,k)\sim(k,0)$, and $(k,k-k/2)=(k,k/2)$ on U_ε. Fortunately because both numbers (k,m) are required to be coprime, the situation where one is not able to fix a preferred orientation for U_ε restricts to $(k,m)=(2,1)$.

Note that changing the direction of the $\mathbb{S}^1$-action has no effect on the invariants (k,m): For the unoriented invariants this is obvious. For the oriented ones, the orientation of the slice changes with that of the action, which both compensate each other.

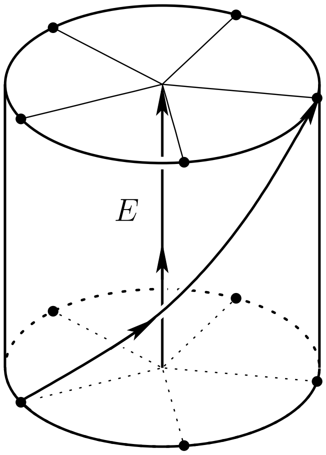


Figure 3. Neighborhood of an exceptional orbit with $k=5$ and $m=2$

The neighborhood of the exceptional orbit can be described by $\mathbb{S}^1\times\mathbb{D}^2_\varepsilon$ (where $\mathbb{D}^2_\varepsilon\subset\mathbb{C}$ is a disk of radius ε) with the action

$$e^{i\varphi}\cdot(e^{i\vartheta},z)=(e^{i(\vartheta+k\varphi)},e^{im\varphi}z)\ .$$

The next aim will be to find a section σ to the $\mathbb{S}^1$-action on the boundary of U_ε such that its homotopy class $[\sigma]$ is canonical (in the sense that it is uniquely determined by the pair of orbit invariants (k,m)).

The $\mathbb{S}^1$-action defines for every $q\in\partial U_\varepsilon$ the same class $[\mathrm{Orb}(q)]\in H_1(\partial U_\varepsilon,\mathbb{Z})$. A second class $[\mu]$ is given by the meridian, i.e. by the boundary μ of a slice. This class generates the kernel of the map $H_1(\partial U_\varepsilon)\to H_1(U_\varepsilon)$. If U_ε is oriented, then $[\mu]$ is uniquely determined, otherwise there is a choice of sign. Recall that U_ε can be canonically oriented, if M is oriented or if the unoriented orbit invariants are not $(2,1)$. Otherwise orient U_ε arbitrarily, but remember that the choice is not canonical.

A section σ intersects each orbit in a single point, hence the intersection number $\iota([\sigma],[\mathrm{Orb}(q)])$ is 1 (by choosing σ with the correct orientation; compare Figure 4). This does not fix the

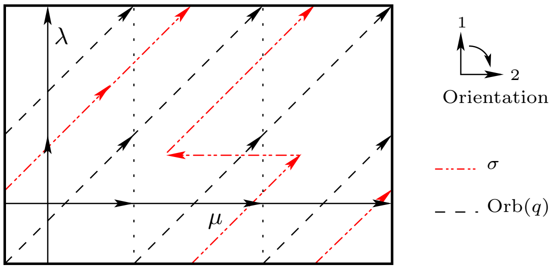


Figure 4. A section on ∂U_ε for an exceptional orbit with $k = 3$ and $m = 2$

class of σ, because any other class $[\sigma] + n[\mathrm{Orb}(q)]$ with $n \in \mathbb{Z}$ can also be represented by a section. The intersection number for this other section with the meridian would be

$$\iota([\sigma] + n[\mathrm{Orb}(p)], [\mu]) = \iota([\sigma], [\mu]) + nk \ ,$$

where k is the order of the stabilizer of the exceptional orbit. We can fix a standard class $[\sigma]$ by requiring that $\beta = \iota([\sigma], [\mu])$ has minimal positive value. Note that $m\beta \equiv 1 \mod k$, for the following reasons: The pair $\langle [\sigma], [\mathrm{Orb}(q)] \rangle$ is a basis of $H_1(\partial U_\varepsilon, \mathbb{Z})$, and we can choose a class $[\lambda]$ such that $\iota([\mathrm{Orb}(q)], [\lambda]) = -m$, and such that $\langle [\lambda], [\mu] \rangle$ is also a basis. With the relations

$$\iota([\mathrm{Orb}(q)], [\lambda]) = -m \qquad \iota([\mathrm{Orb}(q)], [\mu]) = k$$
$$\iota([\sigma], [\mathrm{Orb}(q)]) = 1 \qquad \iota([\sigma], [\mu]) = \beta \ ,$$

the second basis can be expressed by the first one in the form

$$\begin{aligned}[\lambda] &= m\,[\sigma] + C\,[\mathrm{Orb}(q)] \\ [\mu] &= -k\,[\sigma] + \beta\,[\mathrm{Orb}(q)] \ ,\end{aligned}$$

such that $m\beta = 1 - Ck$. Note that inverting the orientation of U_ε changes the orientation of ∂U_ε, μ and σ. Hence one gets $\iota([\sigma], [\mu]) = -\beta < 0$, and thus the canonical section σ' with respect to this orientation would be $\sigma' = \sigma + \mathrm{Orb}(q)$, and the new invariant β' would be $k - \beta$.

The only case where the orientation of U_ε was arbitrary was when M was non-orientable, and the unoriented orbit invariants were $(2, 1)$. In this situation the number β is 1, and for the opposite orientation of U_ε we also get $\beta = 2 - 1 = 1$.

If M is oriented or if the unoriented orbits invariants are not $(2, 1)$, then one can choose a unique canonical section in ∂U_ε. Otherwise, there are two possible choices σ_1 and σ_2, such that $[\sigma_2] = [\sigma_1] \pm [\mathrm{Orb}(q)]$.

Usually the **Seifert invariants** (α, β) are used to describe the exceptional orbits. In terms of (k, m) one can write $\alpha = k$ and $\beta m \equiv 1 \mod \alpha$ with $0 < \beta < \alpha$. It is easy to obtain the Seifert invariants from orbit invariants and vice versa.

1.3. Special exceptional orbits.

In the previous section the stabilizer of a point p was isomorphic to a finite group $\mathbb{Z}_k$, and it acted effectively on a 2-dimensional slice by rotations. If $k > 2$, rotations are indeed the only effective linear 2-dimensional actions of $\mathbb{Z}_k$, but if $\mathrm{Stab}(p) \cong \mathbb{Z}_2$, the slice representation can also be given by reflections. Such an action on a slice $\mathbb{D}^2_\varepsilon \subset \mathbb{C}$ can be written as

$$(\xi, z) = (\xi, x + iy) \mapsto \overline{z} = x - iy \ ,$$

where ξ is the generator of $\mathbb{Z}_2$. These actions gives rise to special exceptional orbits.

The neighborhood of a special exceptional orbit is diffeomorphic to

$$\mathbb{M}\ddot{\mathrm{o}}\mathrm{b} \times (-\varepsilon, \varepsilon) ,$$

where we describe the Möbius strip Möb by using the model $\mathbb{R} \times (-\delta, \delta)$ with the equivalence relation $(t, s) \sim (t + \pi, -s)$ and the $\mathbb{S}^1$-action $e^{i\varphi}(t, s) = (t + \varphi, s)$. In particular, any $\mathbb{S}^1$-manifold M with special exceptional orbits is non-orientable.

We will denote the set of all special exceptional orbits by SE. Each component of SE is an $\mathbb{S}^1$-bundle over a circle, i.e. a torus. The neighborhood of a component of SE is equivalent to $\mathbb{R} \times \mathbb{M}\ddot{\mathrm{o}}\mathrm{b}/\sim$, where $(u, p) \sim (u + 1, \Phi(p))$ with an $\mathbb{S}^1$-equivariant diffeomorphism $\Phi : \mathbb{M}\ddot{\mathrm{o}}\mathrm{b} \to \mathbb{M}\ddot{\mathrm{o}}\mathrm{b}$.

Let $\Phi : \mathbb{M}\ddot{\mathrm{o}}\mathrm{b} \to \mathbb{M}\ddot{\mathrm{o}}\mathrm{b}$ be an $\mathbb{S}^1$-equivariant diffeomorphism. Since $\mathbb{R} \times (-\delta, \delta)$ is contractible, any such map lifts to a diffeomorphism $\widetilde{\Phi} : \mathbb{R} \times (-\delta, \delta) \to \mathbb{R} \times (-\delta, \delta)$ which makes the diagram commutative

$$\begin{tikzcd}
\mathbb{R} \times (-\delta, \delta) \arrow[r, dotted, "\widetilde{\Phi}"] \arrow[d, "\pi"'] & \mathbb{R} \times (-\delta, \delta) \arrow[d, "\pi"] \\
\mathbb{M}\ddot{\mathrm{o}}\mathrm{b} \arrow[r, "\Phi"] & \mathbb{M}\ddot{\mathrm{o}}\mathrm{b}
\end{tikzcd}$$

To be compatible with the $\mathbb{S}^1$-action, $\widetilde{\Phi}$ has to be of the form $\widetilde{\Phi}(t, s) = (\Phi_1(s) + t, \Phi_2(s))$ with smooth maps $\Phi_1 : (-\delta, \delta) \to \mathbb{R}$, and $\Phi_2 : (-\delta, \delta) \to (-\delta, \delta)$. Since $\widetilde{\Phi}$ is a lift of Φ, also the equations $\Phi_1(-s) = \Phi_1(s)$ and $\Phi_2(-s) = -\Phi_2(s)$ hold. By assuming without loss of generality that $\Phi_1(0) = 0$ and $\Phi_2(s) > 0$ for $s > 0$, we obtain an isotopy $\widetilde{\Phi}_u : \mathbb{R} \times (-\delta, \delta) \to \mathbb{R} \times (-\delta, \delta)$, $(t, s) \mapsto (t + u\Phi(s), u\Phi_2(s) + (1 - u)\, s)$ that projects down to an isotopy between Φ and the identity on Möb which commutes with the $\mathbb{S}^1$-action. Thus we get that the neighborhood of a component of SE is equivalent to $\mathbb{S}^1 \times \mathbb{M}\ddot{\mathrm{o}}\mathrm{b}$.

The projection to the orbit space is given by

$$\pi : \mathbb{S}^1 \times \mathbb{M}\ddot{\mathrm{o}}\mathrm{b} \to \mathbb{S}^1 \times [0, \delta),\ (e^{i\vartheta}, (t, s)) \mapsto (e^{i\vartheta}, s) .$$

Any section given outside of an ε-neighborhood of SE can be extended to the interior by interpolation like it was done for fixed points.

2. Principal $\mathbb{S}^1$-bundles over surfaces

From the theory of classifying spaces, it is known that isomorphism classes of G-bundles over a manifold B are in one-to-one correspondence with the set $[B, BG]$ of homotopy classes of continuous maps from B to the classifying space BG.

In our case, we have that $B\mathbb{S}^1 \cong \mathbb{CP}^\infty$ is isomorphic to the Eilenberg-MacLane space $K(2, \mathbb{Z})$ and it follows that $[B, B\mathbb{S}^1] \cong H^2(B, \mathbb{Z})$. If B is an open surface, then the only principal $\mathbb{S}^1$-bundle over B is the trivial one. If B is closed, but non-orientable, then there are two non-isomorphic $\mathbb{S}^1$-bundles over B, and if B is closed and oriented, then there is a bijection between $\mathbb{Z}$ and the equivalence classes of $\mathbb{S}^1$-bundles over B.

For $\mathbb{S}^1$-bundles over surfaces, this classification result can be proved in a more intuitive way, which we will now sketch, because it helps to understand later the general $\mathbb{S}^1$-manifolds.

First note that a principal G-bundle P over B is trivial if and only if it has a global section σ. The trivialization is given by

$$G \times B \xrightarrow{\cong} P,\ (g,b) \mapsto \sigma(b) \cdot g\ .$$

Lemma III.1. *Let P be an $\mathbb{S}^1$-bundle over the closed 2-disk $\mathbb{D}^2$, and assume a continuous section σ is given over a closed proper subset A of the boundary $\partial\mathbb{D}^2$. Then one can extend σ to the whole disk.*

Proof. Of course, the lemma is a direct consequence of obstruction theory ([**Bre93**, Theorem VII.13.11]), but we want to give a more constructive proof.

Assume first that $\mathbb{D}^2$ is covered by a single bundle chart. Then $P \cong \mathbb{D}^2 \times \mathbb{S}^1$, and we can regard any section as a map $\mathbb{D}^2 \to \mathbb{S}^1$. The section $\sigma : A \subset \partial\mathbb{D}^2 \to \mathbb{S}^1$ can be extended to a map $\sigma : \partial\mathbb{D}^2 \to \mathbb{S}^1$, such that its degree is zero. For this note that $\partial\mathbb{D}^2 - A = \dot{\bigcup}_j I_j$, where each I_j is an open interval. Choose an arbitrary continuous map σ on I_j that is compatible with the boundary conditions on ∂I_j. Do this for all but one subset I_{j_0}. There, choose σ in such a way that it is not only compatible with the boundary conditions, but such that it rotates as often on I_{j_0} as it does on $\partial\mathbb{D}^2 - I_{j_0}$, but in opposite direction.

A map $f : \mathbb{S}^1 \to \mathbb{S}^1$ with $\deg f = 0$ is homotopic to a constant map. Hence one can define the global section by

$$\sigma : \mathbb{D}^2 \to \mathbb{S}^1, re^{i\varphi} \mapsto h_r(e^{i\varphi})\ ,$$

where h_r is a homotopy between the constant map h_0 and $\sigma = h_1$.

If $\mathbb{D}^2$ is not covered by a single chart, then subdivide the disk into four equal quarters $Q_1, \dots, Q_4$. Assume that they are arranged in clockwise direction and that Q_4 contains a part of $\partial\mathbb{D}^2 - A$. If each of the Q_j is contained in a bundle chart, it is easy to finish the proof. Extend σ from A over Q_1 (which is possible, because Q_1 is homeomorphic to a disk and σ is only predefined in a subset of ∂Q_1). Then construct σ on Q_2 such that it extends $\sigma|_A$ and $\sigma|_{\partial Q_1 \cap \partial Q_2}$. This is possible, because Q_2 has free boundary in $\partial Q_2 \cap \partial Q_3$. Repeat the analogous step for Q_3 and Q_4, by using that Q_3 has free boundary in $\partial Q_3 \cap \partial Q_4$ and Q_4 has free boundary in $\partial Q_4 \cap \partial\mathbb{D}^2$.

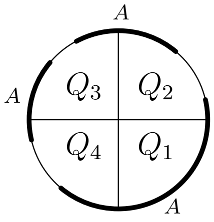


Figure 5. Q_4 has free boundary

If some of the Q_js are not covered by a chart proceed by induction: Subdivide Q_j further into $Q_{j1}, \dots Q_{j4}$, which can be arranged like above. By the Lemma of Lebesgue, after sufficiently many subdivision steps, each of the fragments is contained in a single chart. This finishes the proof. □

Corollary III.2. *Every $\mathbb{S}^1$-bundle over a closed 2-disk is trivial.*

Theorem III.3. *An $\mathbb{S}^1$-bundle over a compact surface B with boundary $\partial B \neq \emptyset$ is trivial.*

Proof. Every surface admits a triangulation. By spreading out the triangulation in a plane, one can represent B by a polytope $\widetilde{B}$ with edges $a_1, \ldots, a_n$ (compare Figure 6). The edges a_j of $\widetilde{B}$ either represent parts of the boundary of B or correspond to interior curves along which B was cut open. Edges created by cutting are identified pairwise, i.e. to each such edge a_j there corresponds an opposite edge a_k. If B is oriented, then the identification of the edges a_j and a_k reverses the orientation. If B is non-orientable, then there is at least a pair of edges $\{a_j, a_k\}$ that are identified with the same orientation. If a_j represents a boundary of B, it remains unpaired.

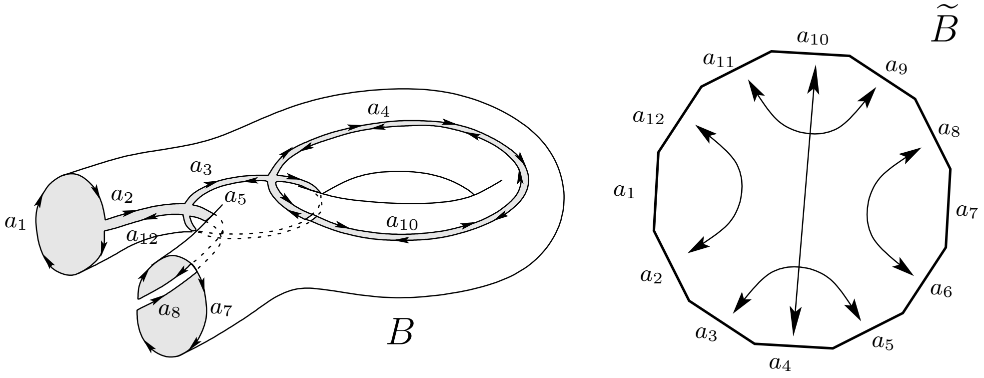


Figure 6. The edges a_1 and a_7 represent the boundary of the surface B. All other edges are identified in pairs as indicated by the arrows. Note that all identifications reverse the orientation, because B is an oriented surface.

Now we will define inductively a section over the edges a_j of $\widetilde{B}$. If a_j represents a boundary of B, or if the section over a_j has been previously defined, then skip a_j and go to the next a_{j+1}. If a_j corresponds to an interior curve of B, and if σ is not defined on a_j, then let a_k be the edge identified with a_j, and choose an arbitrary continuous section σ over a_j itself that is compatible with any possible previous definitions of σ on a_{j-1}. On a_k construct the section that is compatible with the identification between a_j and a_k.

Now we are in the situation that we can apply Lemma III.1 to find a section over the polytope, which induces a continuous section over B by our construction. □

The next aim will be to see how sections of an $\mathbb{S}^1$-bundle P can differ on the boundary ∂P. For this we will generalize the degree of a map.

Let B be a compact surface B with non-empty boundary, and let $P \cong \mathbb{S}^1 \times B$ be an $\mathbb{S}^1$-bundle over B. Let σ be a section of P. Its **degree on ∂B** will be defined like this: Choose an arbitrary trivialization of P. If B is oriented, then all of the components ∂B_j of the boundary ∂B receive a natural orientation, and by measuring $\sigma|_{\partial B_j}$ with respect to the trivialization of P, it can be considered as a map between oriented circles. Define

$$\deg(\sigma|_{\partial B}) := \sum_j \deg(\sigma|_{\partial B_j}) \ .$$

If B is non-orientable, then fix an arbitrary orientation for each component $\partial B_j \subset \partial B$. With these choices it is again possible to consider $\sigma|_{\partial B_j}$ as a map between oriented circles and use the above definition of the degree. The degree is not well-defined, because it can depend on the trivialization of P and the orientations chosen.

LEMMA III.4. *Let B be a compact surface B with non-empty boundary, and let $P \cong \mathbb{S}^1 \times B$ be an $\mathbb{S}^1$-bundle over B. If B is oriented, then for any section σ of P the equation* $\deg(\sigma|_{\partial B}) = 0$ *holds.*

If B is non-orientable, then $\deg(\sigma|_{\partial B}) \in 2\mathbb{Z}$. *If B is non-orientable, then for any even integer $n \in 2\mathbb{Z}$, there is a section σ in P, such that* $\deg(\sigma|_{\partial B}) = n$.

PROOF. By representing σ with respect to the trivialization, we can regard the section as a function $\sigma : B \to \mathbb{S}^1$. We have to show that $\deg(\sigma|_{\partial B}) = 0$, if B is orientable, and $\deg(\sigma|_{\partial B}) \in 2\mathbb{Z}$ otherwise.

First note that a map $f : \mathbb{D}^2 \to \mathbb{S}^1$ is always null-homotopic, and in particular $\deg(f|_{\partial \mathbb{D}^2}) = 0$. With a triangulation, we can represent B as a polytope $\widetilde{B}$, where certain edges of the boundary are identified as described in the proof of Theorem III.3. The total degree $\deg(\sigma|_{\partial \widetilde{B}})$ on the polytope vanishes. This number is obtained by adding two contributions: One comes from the edges a_j of $\widetilde{B}$ that correspond to interior curves in B, the other one comes from the edges that represent the boundary of B. This least part is identical to $\deg(\sigma|_{\partial B})$. If B is orientable, then the edges are identified with opposite orientations, i.e. the contribution of two identified edges cancel each other out. If B is non-orientable, then there is at last one pair of edges where the orientations of a_i and a_j agree. The contribution of these edges is then always an even number.

Given a section σ_1, we want to construct on a non-orientable surface B a section σ_2 such that the intersection number between $\sigma_1|_{\partial B}$ and $\sigma_2|_{\partial B}$ as curves on $\partial B \cong \mathbb{T}^2$ is $2k$. The intersection number is the difference of the degrees of both sections. Define σ_2 first only on the boundary of the polytope $\widetilde{B}$ by setting $\sigma_2|_{\partial \widetilde{B}} = \sigma_1|_{\partial \widetilde{B}}$. There are two edges a_{j_1} and a_{j_2} of $\widetilde{B}$ that are identified in B with equal orientation, because B is non-orientable. Change σ_2 on the free boundary of $\widetilde{B}$ by doing $2k$ positive turns with respect to σ_1, and on a_{j_1} by doing k negative turns. On a_{j_2} the section σ_2 has to be changed correspondingly, since a_{j_1} and a_{j_2} are identified. Now the total degree of σ_2 vanishes on $\partial \widetilde{B}$, and σ_2 can be extended in such a way to the interior of the polytope $\widetilde{B}$ that it induces the desired section on B. □

COROLLARY III.5. *Let P be an $\mathbb{S}^1$-bundle over a compact surface B with a single boundary component. Let σ_1 and σ_2 be two arbitrary sections over B. If B is orientable, then the restrictions $\sigma_1|_{\partial B}$ and $\sigma_2|_{\partial B}$ are homotopic. If B is non-orientable, then the restrictions $\sigma_1|_{\partial B}$ and $\sigma_2|_{\partial B}$ have even intersection number, and for every section σ_1 and every even integer $n \in 2\mathbb{Z}$, we can construct a section σ_2 such that the intersection number $\iota(\sigma_1|_{\partial B}, \sigma_2|_{\partial B}) = n$.*

Let P be an $\mathbb{S}^1$-bundle over a closed surface B without boundary. Choose a closed disk $D \subset B$ that lies inside a bundle chart, and denote the closure of the complement of D by $B^* := B - \operatorname{int} D$. We can decompose P into the two parts $P|_D$ and $P|_{B^*}$, which are the restriction of P to the corresponding subset of B. The intersection $P|_D \cap P|_{B^*}$ is an $\mathbb{S}^1$-invariant torus T. If B is oriented, then orient T as the boundary of $P|_{B^*}$, otherwise choose an arbitrary orientation. Since both $P|_D$ and $P|_{B^*}$ are $\mathbb{S}^1$-bundles over compact surfaces with boundary it is possible to find sections σ_D and σ_{B^*}. If B is orientable, let

$$e := \iota(\sigma_D|_T, \sigma_{B^*}|_T) \in \mathbb{Z}$$

be the intersection number of σ_D with σ_{B^*} inside T, where both sections carry the orientation inherited by B. If B is non-orientable, choose for the two sections an arbitrary orientation, and let

$$e := \Big(\iota(\sigma_D|_T, \sigma_{B^*}|_T) \mod 2\Big) \in \mathbb{Z}_2$$

be the intersection number in T modulo 2. We call e the **Euler invariant** of an $\mathbb{S}^1$-bundle.

LEMMA III.6. *The Euler invariant e of an $\mathbb{S}^1$-bundle is well-defined.*

PROOF. If B is oriented, then the section on $P|_D$ is well-defined up to homotopy. Consider now $P|_{B^*}$. By Corollary III.5, any two sections in $P|_{B^*}$ restricted to the boundary $P|_{\partial B^*} = T$ are homotopic, and hence e does not depend on the section chosen.

If B is non-orientable, then σ_D is well-defined up to homotopy and orientation. Any two sections in $P|_{B^*}$ have even intersection number on the boundary T. The number $\iota(\sigma_D|_T\,, \sigma_{B^*}|_T)$ is only well-defined up to sign and addition of even integers, but then e does not depend on the sections or any of the orientations chosen.

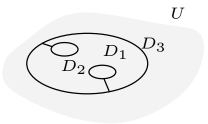


FIGURE 7.

To prove that e does not depend on the disk D in B, note that if D_1 and D_2 are two small closed disks in B that are sufficiently C^0-close, they are both contained in a third disk D_3 that lies inside a bundle chart U (like represented in Figure 7). A section over D_3 restricts to sections over D_1 and D_2. For the construction of the Euler invariant choose a section σ_j over $B - \operatorname{int} D_j$. These restrict to $B - \operatorname{int} D_3$, and we obtain a section suitable for the calculation of the Euler number e with respect to the disk D_3. Both e for D_j and D_3 are equal, because $D_3 - \operatorname{int} D_j$ is an annulus and by Lemma III.4 it follows that $\deg(\sigma_j|_{\partial D_3}) = \deg(\sigma_j|_{\partial D_j})$.

By the disk theorem, we can connect any two small disks on B by an isotopy, and by the argument above, e does not change along the path. Hence the Euler invariant does not depend on the disk. $\square$

LEMMA III.7. *Let M_1, M_2 be two 3-dimensional $\mathbb{S}^1$-manifolds, and let $V_1 \subset M_1$ and $V_2 \subset M_2$ be $\mathbb{S}^1$-invariant solid tori that contain only principal orbits. An $\mathbb{S}^1$-diffeomorphism*

$$\Phi : \, M_1 - \operatorname{int} V_1 \longrightarrow M_2 - \operatorname{int} V_2$$

extends to an $\mathbb{S}^1$-diffeomorphism $\widetilde{\Phi} : M_1 \to M_2$, if and only if the image $\Phi(\sigma_1|_{\partial V_1})$ of a section σ_1 in V_1 extends to a section σ_2 in V_2.

PROOF. Fix a diffeomorphism $h : V_1/\mathbb{S}^1 \to V_2/\mathbb{S}^1$ such that $\pi_2(\Phi(p)) = h(\pi_1(p))$ for all points $p \in \partial V_1$.

$$\begin{tikzcd}
\partial V_1 \arrow[r] \arrow[d, "\Phi|_{\partial V_1}"'] & V_1 \arrow[r, "\pi_1"] \arrow[d, dotted, "\widetilde{\Phi}"'] & V_1/\mathbb{S}^1 \arrow[d, "h"] \\
\partial V_2 \arrow[r] & V_2 \arrow[r, "\pi_2"] & V_2/\mathbb{S}^1
\end{tikzcd}$$

If there is a section σ_2 in V_2 that extends $\Phi \circ \sigma_1|_{\partial V_1}$, then we can define $\widetilde{\Phi}$ by

$$\widetilde{\Phi} : \, M_1 \to M_2, \, p \mapsto \begin{cases} \Phi(p) & \text{if } p \in M_1 - \operatorname{int} V_1 \\ \sigma_2(h(\pi_1(p))) \cdot e^{i\varphi} & \text{if } p \in V_1, \text{ where } \varphi \text{ such that } \sigma_1(\pi_1(p)) \cdot e^{i\varphi} = p. \end{cases}$$

This map is an $\mathbb{S}^1$-homeomorphism. It is possible to smooth σ_2 in V_2 to make $\widetilde{\Phi}$ an $\mathbb{S}^1$-diffeomorphism.

Conversely, if $\widetilde{\Phi}$ is a continuation of Φ, then the map $h:\ V_1/\mathbb{S}^1 \to V_2/\mathbb{S}^1$ is induced by $\widetilde{\Phi}$. It is clear that $\widetilde{\Phi} \circ \sigma_1 \circ h^{-1}$ is a section in V_2 that extends $\Phi \circ \sigma_1|_{\partial V_1}$ to V_2. □

Theorem III.8. *An $\mathbb{S}^1$-bundle over a closed surface B is classified by its Euler invariant e.*

Proof. First note that it is possible to construct an $\mathbb{S}^1$-bundle P with any desired Euler invariant. Define $B^* := B - \operatorname{int} D$, where D is a small disk in B. The bundle over B^* is just $P_{B^*} := B^* \times \mathbb{S}^1$, and the bundle over D is $P_D := D \times \mathbb{S}^1$. The boundary of both bundles is a torus $\mathbb{S}^1 \times \mathbb{S}^1$, where the circle action is given by the natural action on the second factor. Glue P_{B^*} onto P_D via the $\mathbb{S}^1$-homeomorphism

$$(e^{i\varphi}, e^{i\vartheta}) \mapsto (e^{i\varphi}, e^{i(\vartheta + e\varphi)}) \ .$$

The section $p \mapsto (p, 1)$ in P_{B^*} is mapped to the curve $\{(e^{i\varphi}, e^{ie\varphi}) | \varphi \in [0, 2\pi)\}$ on the boundary of P_D, which intersects the trivial section e times.

Let P_1 and P_2 be $\mathbb{S}^1$-bundles over B both with Euler invariant e. The aim is to find a bundle isomorphism $\Phi:\ P_1 \to P_2$. Consider a small disk $D \subset B$ and denote the closure of the complement of D again by $B^* := B - \operatorname{int} D$. Choose a section σ_j in $P_j|_{B^*}$ and σ'_j in $P_j|_D$ for $j = 1, 2$. If B is non-orientable then take care to choose σ_2 in such a way that

$$\iota(\sigma_1|_{\partial D}\, , \sigma'_1\big|_{\partial D}) = \iota(\sigma_2|_{\partial D}\, , \sigma'_2\big|_{\partial D}) \ .$$

Note that this is possible by Corollary III.5 and Lemma III.6, since we can change the intersection number by any even integer.

Define Φ over B^* by

$$\Phi :\ P_1|_{B^*} \to P_2|_{B^*}\, ,\ p \mapsto \sigma_2(\pi(p)) \cdot e^{i\vartheta} \ ,$$

where ϑ is chosen is such a way that $p \cdot e^{-i\vartheta} = \sigma_1(\pi(p))$. By using Lemma III.7, we will now show that Φ extends to the whole $\mathbb{S}^1$-bundle P_1. The intersection number of $\sigma_j|_{\partial B^*}$ and $\sigma'_j\Big|_{\partial D}$ is equal for both $j = 1, 2$. It follows that $\Phi \circ \sigma'_1|_{\partial D}$ has the same intersection number with $\sigma_2|_{\partial D}$ as $\sigma'_2|_{\partial D}$, and then $\Phi \circ \sigma'_1|_{\partial D}$ is homotopic to it and extends to a section over D. □

3. The orbit space

Corollary III.9. *The orbit space $B := M/\mathbb{S}^1$ is a two-dimensional orbifold. The boundary of B is the projection of $F \cup SE$. The set $E/\mathbb{S}^1$ consists of discrete points in the interior of B.*

4. Equivalence between $\mathbb{S}^1$-manifolds

Let M be a 3-dimensional $\mathbb{S}^1$-manifold. Call M an **exceptional $\mathbb{S}^1$-manifold**, if M is non-orientable and has at least one exceptional orbit with unoriented orbit invariants $(2, 1)$.

The **Euler invariant** e of M will be defined in a similar way as was done above for $\mathbb{S}^1$-bundles: If the $\mathbb{S}^1$-action has fixed points or special exceptional orbits, or if M is an exceptional $\mathbb{S}^1$-manifold, then set the Euler invariant e of M to 0.

If M does not have

(i) fixed points

(ii) special exceptional orbits
(iii) or in case M is a non-exceptional $\mathbb{S}^1$-manifold

then choose a small disk $D \subset B$ covered only by free orbits, and let B^* be $B - \operatorname{int} D$. Choose a section σ_D in $M|_D$, and a section σ_{B^*} in the set of principal orbits of $M|_{B^*}$ that agrees with the standard sections (defined in Section 1.2) in the neighborhood of the exceptional orbits. If M is oriented, then there is a natural orientation on $M|_D$, and one can define a preferred orientation on σ_D and σ_{B^*}. These orientations induce an orientation on $M|_{\partial D}$, $\sigma_D|_{\partial D}$ and $\sigma_{B^*}|_{\partial D}$. If M is non-orientable, then choose an arbitrary orientation on each of $M|_{\partial D}$, $\sigma_D|_{\partial D}$ and $\sigma_{B^*}|_{\partial D}$. For M oriented, the Euler invariant is the intersection number between σ_D and σ_{B^*} in $M|_{\partial D}$. For M non-orientable, the Euler invariant $e \in \mathbb{Z}_2$ is the intersection number between σ_D and σ_{B^*} in $M|_{\partial D}$ modulo 2.

LEMMA III.10. *The Euler invariant e of a 3-dimensional $\mathbb{S}^1$-manifold is well-defined.*

PROOF. If M contains fixed points, special exceptional orbits or if M is an exceptional $\mathbb{S}^1$-manifold, then there is nothing to prove. Otherwise one needs to show that e does not depend on the section or on the disk $D \subset M/\mathbb{S}^1$, and if M is non-orientable on any of the orientations chosen. The proof is almost identical to the one of Lemma III.6.

The manifold $M|_{B^*}$ has only a single boundary component, but the sections over B^* are not defined in the exceptional orbits. Cut out small neighborhoods of these exceptional orbits, and apply Lemma III.4. If M is oriented, then any two sections on the boundary of $M|_{B^*}$ are homotopic, because the total degree on the boundary has to vanish, and both sections are equal on the neighborhood of the exceptional orbits. If M is non-orientable, then any two sections on the boundary of $M|_{B^*}$ have even intersection number, which gives no contribution to the Euler number $e \in \mathbb{Z}_2$. ☐

THEOREM III.11. *A 3-dimensional $\mathbb{S}^1$-manifold M is completely determined by the numbers*

$$(g, f, s, e, (k_1, m_1), \dots, (k_N, m_N)) \, ,$$

where g is the genus of the orbit space $M/\mathbb{S}^1$, the number of components in the fixed point set F is denoted by f, the number of components of special exceptional orbits SE is denoted by s, e is the Euler invariant, and the (k_j, m_j) are either the oriented or unoriented invariants of the exceptional orbits.

(i) *If M is oriented, then $s = 0$, the numbers (k_j, m_j) are the oriented orbit invariants, and e is an integer that has to vanish if $f > 0$.*
(ii) *If M is non-orientable, then (k_j, m_j) are the unoriented orbit invariants, and the Euler invariant e is an element in $\mathbb{Z}_2$ that is 0, if $f \neq 0$ or $s \neq 0$ or if M is an exceptional $\mathbb{S}^1$-manifold.*

Every combination of invariants described above, is realized by an $\mathbb{S}^1$-manifold.

PROOF. If M_1 and M_2 are $\mathbb{S}^1$-manifolds with identical invariants, we have to show that there is an $\mathbb{S}^1$-diffeomorphism $\Phi : M_1 \to M_2$.

We will first define Φ in a neighborhood of the exceptional orbits. As we explained in Section 1.2, the neighborhood of an exceptional orbit carries a preferred orientation unless M_1 is an exceptional $\mathbb{S}^1$-manifold. For the moment we will assume that we are not in this last situation. Then we can find an $\mathbb{S}^1$-diffeomorphism $\Phi_E : U_E \to M_2$ on a small neighborhood $U_E \subset M_1$ of the exceptional orbits E that respects the preferred orientations. The image of a standard section in each component of U_E is again a standard section.

There exists a diffeomorphism $h : M_1/\mathbb{S}^1 \to M_2/\mathbb{S}^1$ that extends the map induced by Φ_E on $U_E/\mathbb{S}^1$. We will denote both orbit spaces M_1/S^1 and $M_2/\mathbb{S}^1$ by B after identifying them with the diffeomorphism h.

It depends on the situation how we proceed: If $f \neq 0$ or $s \neq 0$, then there exists a section σ_1 in $M_1 - E$ that extends the canonical section on U_E. The same can be done in M_2, where we choose σ_2 to extend $\Phi \circ \sigma_1$ around the exceptional orbits. Define $\Phi : M_1 \to M_2$ by

$$\Phi(p) := \begin{cases} \Phi_E(p) & \text{if } p \in U_E \\ \sigma_2(\pi(p)) \cdot e^{i\varphi} & \text{otherwise,} \end{cases}$$

where $\pi : M_1 \to B$ denotes the projection onto the orbit space, and $\varphi \in \mathbb{S}^1$ is chosen such that $\sigma_1(\pi(p)) \cdot e^{i\varphi} = p$.

If $s = f = 0$, then choose a small disk $D \subset B$ suitable for the computation of the Euler invariant. The same strategy as above can be used for $M_1|_{B^*}$ and $M_2|_{B^*}$ with $B^* := B - D$ to construct an $\mathbb{S}^1$-diffeomorphism $\Phi : M_1|_{B^*} \to M_2|_{B^*}$. If M_1 is non-orientable, then one has to take care that the sections σ_1 and σ_2 have equal degree on the boundaries $M_1|_{\partial D}$ and $M_2|_{\partial D}$ with respect to sections over D. This allows us to apply Lemma III.7 to extend Φ to the whole of M_1.

If M_1 is non-orientable and contains an exceptional orbit $\mathrm{Orb}(p_0) \subset E$ with unoriented orbit invariants $(2, 1)$, then there are two possible choices for standard sections around this orbit $\mathrm{Orb}(p_0)$. The difference of the homotopy classes of these two choices correspond to the class generated by an orbit. This means that if we followed the steps used for the computation of the Euler invariant, depending on the choice for the standard section, e would be 1 or 0. The section that gives $e = 0$ is the one that will be used to do all the steps like in the construction above. In the end, one obtains an $\mathbb{S}^1$-diffeomorphism between M_1 and M_2.

To construct a manifold with a given set of invariant, start with a surface B^* with genus g and one more puncture than the number of components in F, SE, and E. Take $B^* \times \mathbb{S}^1$, and glue in the exceptional orbits by attaching the canonical sections around E to the trivial section of $B^* \times \mathbb{S}^1$, then attach the fixed points and special exceptional orbits. At the end, glue in a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ with the linear map described in the proof of Theorem III.8 to produce the desired Euler invariant e. □

5. Generalized connection 1-forms

In this chapter so far, all invariants necessary to classify 3-dimensional $\mathbb{S}^1$-manifolds were given. Unfortunately, depending on the form in which a certain manifold is given, it may be extremely hard to compute these numbers explicitly. In this section, we will describe an alternative method to find some of the invariants, which may or may not prove easier to apply for a given manifold. In any case, the theory developed here will be important for Chapter IV.

DEFINITION. Let M be an $\mathbb{S}^1$-manifold. Denote by Z_M the infinitesimal generator of the $\mathbb{S}^1$-action. A **generalized connection 1-form** A is a 1-form on M that satisfies the equations

$$\mathcal{L}_{Z_M} A = 0 \quad \text{and} \quad A(Z_M) \equiv 1 \,.$$

REMARK III.1. It is clear that generalized connection forms do not exist, when there are fixed points. But on any $\mathbb{S}^1$-manifold M with non-vanishing vector field Z_M, it is easy to construct a connection form A. Choose for example an $\mathbb{S}^1$-invariant metric g on M and define

$$A := \frac{1}{\|Z_M\|^2}\, g(Z_M, \cdot)\ .$$

Let M be an $\mathbb{S}^1$-manifold with connection form A. The 2-form dA is $\mathbb{S}^1$-invariant, and because of

$$\iota_{Z_M} dA = \mathcal{L}_{Z_M} A - d\big(A(Z_M)\big) = 0\ ,$$

it vanishes on orbits. Denote the set of principal orbits of M with M^*, and let $B^* := M^*/\mathbb{S}^1$ be the orbit space corresponding to M^*. From the equations above, it follows that dA induces a 2-form on B^*. (With orbifold theory, one can also define differential forms on the whole orbit space, but here we will avoid doing so.)

Definition. The **curvature form** F of a connection form A is the unique 2-form on B^* defined by the equation

$$dA = \pi^* F\ ,$$

with $\pi : M^* \to B^*$.

Lemma III.12. *Let A be a connection 1-form on M, and F its curvature form on B^*. Then*

$$\int_M A \wedge dA = 2\pi \int_{B^*} F\ .$$

Proof. Let $U \subset B^*$ be an open set, and $\Phi : U \times \mathbb{S}^1 \hookrightarrow M^*$ be a bundle chart with coordinates $(x, y, e^{i\varphi})$. The connection has the form $\Phi^* A = d\varphi + f(x,y)\, dx + g(x,y)\, dy$ on this chart, and $\Phi^* dA = (\partial_x g - \partial_y f)\, dx \wedge dy$. The curvature is the unique form on B^* such that $\pi^* F = dA$. Then we can write (with $\iota : U \to U \times \{1\}$)

$$\begin{aligned}\int_{U\times\mathbb{S}^1} \Phi^* A \wedge \Phi^* dA &= \int_{U\times\mathbb{S}^1} (\partial_x g - \partial_y f)\, d\varphi \wedge dx \wedge dy = 2\pi \int_U \Phi^* dA \\ &= 2\pi \int_U \iota^* \Phi^* \pi^* F = 2\pi \int_U (\pi \circ \Phi \circ \iota)^* F = 2\pi \int_U F\ ,\end{aligned}$$

because $\pi \circ \Phi \circ \iota : U \to U$ is the identity map on U. □

Theorem III.13. *Let M be a closed, oriented 3-dimensional $\mathbb{S}^1$-manifold determined by the invariants*

$$(g, f = 0, s = 0, e, (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)),$$

i.e. M does not have any fixed points, but it has N exceptional orbits with Seifert invariants (α_j, β_j) (remember that the Seifert invariants can be easily obtained from the orbit invariants), and the Euler number is e. Let A be a generalized connection 1-form on M. Then:

$$\int_M A \wedge dA = 4\pi^2 \left(e + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \right)\ .$$

Proof. The proof of this theorem will be postponed to Section IV.4. □

6. Examples

6.1. Brieskorn manifolds. The most important examples in this thesis are provided by **Brieskorn manifolds** (for a more detailed approach see [**Mil68**]). Let $a_0, \dots, a_n \in \mathbb{N}$, and let f be the polynomial

$$f : \mathbb{C}^{n+1} \to \mathbb{C}, \, (z_0, \dots, z_n) \mapsto z_0^{a_0} + \dots + z_n^{a_n} \; ,$$

The Brieskorn manifold $\Sigma(a_0, a_1, \dots, a_n) \subset \mathbb{C}^{n+1}$ is the intersection of the variety $V_f := f^{-1}(0)$ with a sphere $\mathbb{S}^{2n+1}$.

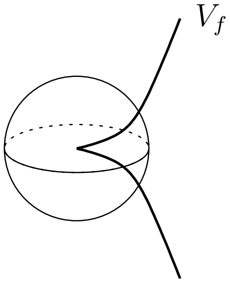


FIGURE 8. V_f is a variety with an isolated singularity at 0, but taking the intersection with $\mathbb{S}^{2n+1}$ gives a smooth manifold.

6.1.1. *The Milnor fibration.* There is a natural $\mathbb{R}$-action on each of these manifolds given by

$$\begin{aligned} \mathbb{R} \times \Sigma(a_0, \dots, a_n) &\to \Sigma(a_0, \dots, a_n) \\ \big(t, (z_0, \dots, z_n)\big) &\mapsto \big(e^{2\pi i t/a_0} z_0, \dots, e^{2\pi i t/a_n} z_n\big) \; . \end{aligned}$$

The orbits of this action give the **Milnor fibration**. The $\mathbb{R}$-action is never effective, but it induces an effective $\mathbb{S}^1$-action for $\mathbb{S}^1 \cong \mathbb{R}/c\mathbb{Z}$ with the least common multiple $c = \operatorname{lcm}(a_0, \dots, a_n)$. We will call this $\mathbb{S}^1$-action the **Milnor action** on a Brieskorn manifold.

From now on we will restrict to 3-dimensional Brieskorn manifolds $\Sigma(a_0, a_1, a_2)$. Assume that $t \in (0, c)$ leaves a point (z_0, z_1, z_2) fixed. Then the three equations

$$e^{2\pi i t/a_j} z_j = z_j$$

hold (with $j = 0, 1, 2$), i.e. either

$$z_j = 0 \quad \text{ or } \quad t = k_j a_j$$

with some $k_j \in \mathbb{N}$. It is not possible for two of the three components (z_0, z_1, z_2) to vanish at the same time (because the equations $z_j^{a_j} = 0$ and $|z_j|^2 = 1$ contradict each other). Assume first that none of the z_j vanishes. Then t is a multiple of all three a_j, and hence also of $c = \operatorname{lcm}(a_0, a_1, a_2)$, which is not possible by our assumption on t.

Therefore the only orbits which are not principal are given by $z_j = 0$ for exactly one $j = 0, 1, 2$. Assume that $z_0 = 0$. Then to satisfy

$$(0, e^{2\pi i t/a_1} z_1, e^{2\pi i t/a_n} z_n) = (0, z_1, z_2) \; ,$$

both t/a_1 and t/a_2 have to be integers, and hence t is a multiple of $\operatorname{lcm}(a_1, a_2)$. The stabilizer of $(0, a_1, a_2)$ is isomorphic to $\mathbb{Z}_k$ with $k = \operatorname{lcm}(a_0, a_1, a_2)/\operatorname{lcm}(a_1, a_2)$.

To compute the second invariant of the orbit $\operatorname{Orb}(0, z_1, z_2)$, note first that if $a_0 = 1$, then the point $(0, z_1, z_2)$ does not lie on an exceptional orbit, because the order of the stabilizer is $k = \operatorname{lcm}(1, a_1, a_2)/\operatorname{lcm}(a_1, a_2) = 1$. If $a_0 > 1$, then the complex plane $\{(z_0, 0, 0) | z_0 \in \mathbb{C}\}$ is a slice at $(0, z_1, z_2)$. The generator of $\operatorname{Stab}(0, z_1, z_2)$ is given by $\operatorname{lcm}(a_1, a_2)$. As explained in Section 1.2, the orbit invariant m can be read off from the equation

$$e^{2\pi i m/k} z_0 = e^{2\pi i \operatorname{lcm}(a_1,a_2)/a_0} z_0 \ .$$

It follows that $m = k \operatorname{lcm}(a_1, a_2)/a_0 = \operatorname{lcm}(a_0, a_1, a_2)/a_0$.

Lemma III.14. *The stabilizer of a point* $(0, z_1, z_2) \in \Sigma(a_0, a_1, a_2)$ *is isomorphic to* $\mathbb{Z}_k$ *with* $k = \operatorname{lcm}(a_0, a_1, a_2)/\operatorname{lcm}(a_1, a_2)$*, i.e. it lies only on an exceptional orbit if* $\operatorname{lcm}(a_0, a_1, a_2) \neq \operatorname{lcm}(a_1, a_2)$*. In that case the orbit invariants* (k, m) *are*

$$(k, m) = \left(\frac{\operatorname{lcm}(a_0, a_1, a_2)}{\operatorname{lcm}(a_1, a_2)}, \frac{\operatorname{lcm}(a_0, a_1, a_2)}{a_0}\right) \ .$$

Note that the set of points $\{(0, z_1, z_2) \in \Sigma(a_0, a_1, a_2)\}$ *is diffeomorphic to* $\Sigma(a_1, a_2)$.

6.1.2. *The Brieskorn manifolds* W_k^{2n-1}. In this section, we will follow the beautiful exposition in [**HM68**]. I would like to thank Otto van Koert for bringing these examples to my attention.

An interesting subfamily of Brieskorn manifolds are the ones of type $W_k^{2n-1} := \Sigma(k, 2, \dots, 2)$. (The upper index denotes the dimension of the manifold.) These spaces carry an $\mathrm{SO}(n)$-action that commutes with the Milnor action defined above. Set $z_j = x_j + i y_j$, and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{x} = (x_1, \dots, x_n)$, and $\mathbf{y} = (y_1, \dots, y_n)$. The $\mathrm{SO}(n)$-action on $\mathbb{C}^{n+1}$, given by $A \cdot (z_0, \mathbf{z}) = (z_0, A \cdot \mathbf{z})$ for a matrix $A \in \mathrm{SO}(n)$ (embed the orthogonal group in the standard way into $\mathrm{GL}(n, \mathbb{C})$) restricts to the manifold W_k^{2n-1}, because f can be written as

$$f(z_0, \mathbf{z}) = z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i \langle \mathbf{x} | \mathbf{y} \rangle \ .$$

The stabilizer of a point $(z_0, \mathbf{x} + i\mathbf{y})$ is given by the intersection $\operatorname{Stab}(\mathbf{x}) \cap \operatorname{Stab}(\mathbf{y})$. If follows that the stabilizer of $(z_0, \mathbf{x} + i\mathbf{y})$ with linearly dependent $\mathbf{x}$ and $\mathbf{y}$ is isomorphic to $\mathrm{SO}(n-1)$. The stabilizer of any other point is isomorphic to $\mathrm{SO}(n-2)$. Fixed points do not occur, because $(z_0, 0, \dots, 0)$ does not lie on W_k^{2n-1}.

To make computations easier, we define W_k^{2n-1} as the intersection of the variety V_f with the sphere of radius $\sqrt{2}$.

Lemma III.15. *The manifold* W_k^3 *is an* $\mathbb{S}^1$*-principal bundle over* $\mathbb{S}^2$ *with Euler number* $e = k$*. The orbit space of* W_k^{2n-1} *for* $2n - 1 \geq 5$ *is a closed disk.*

Proof. Note that the projection $\pi : W_k^{2n-1} \to \mathbb{C}$, $(z_0, \mathbf{z}) \mapsto z_0$ is compatible with the orbit structure of W_k^{2n-1}. The following computation (with $r_0 = |z_0|$) shows that $\pi(W_k^{2n-1})$ lies in a disk with radius 1:

$$\begin{aligned} f(z_0, \mathbf{z}) &= z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i \langle \mathbf{x} | \mathbf{y} \rangle = 0 \ , \\ \overline{f(z_0, \mathbf{z})} &= \bar{z}_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 - 2i \langle \mathbf{x} | \mathbf{y} \rangle = 0 \ , \\ \|(z_0, \mathbf{z})\|^2 &= r_0^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = 2 \ . \end{aligned}$$

By using the first two equations and the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} r_0^{2k} &= (\|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)^2 + 4 \langle \mathbf{x} | \mathbf{y} \rangle^2 = \|\mathbf{x}\|^4 - 2 \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 + \|\mathbf{y}\|^4 + 4 \langle \mathbf{x} | \mathbf{y} \rangle^2 \\ &\leq \|\mathbf{x}\|^4 + 2 \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 + \|\mathbf{y}\|^4 = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2 \ . \end{aligned}$$

Equality holds only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent (for $2n-1 \geq 5$ such a point $(z_0, \mathbf{z})$ lies on a singular orbit). With the sphere equation one gets

$$r_0^{2k} \leq (2 - r_0^2)^2 \ ,$$

and it is possible to take the square root on both sides, because $2 = \|(z_0, \mathbf{z})\|^2 \geq r_0^2$, and so

$$r_0^k + r_0^2 \leq 2 \ .$$

It follows that the image of π lies in a disk $\mathbb{D}^2$ of radius 1, and if $2n-1 \geq 5$, the set of singular orbits is equal to $\pi^{-1}(\partial\mathbb{D}^2)$.

Next we will show that $\pi : W_k^{2n-1} \to \mathbb{D}^2$ is surjective. Define

$$A(r) = \sqrt{2 - r^2 + \sqrt{(2-r^2)^2 - r^{2k}}} \ .$$

The map below is an embedding of the disk into W_k^{2n-1}

$$\sigma : \quad \mathbb{D}^2 \hookrightarrow W_k^{2n-1}, \quad z_0 \mapsto \left(z_0, \frac{i}{2A(|z_0|)}(A^2(|z_0|) + z_0^k), \frac{1}{2A(|z_0|)}(A^2(|z_0|) - z_0^k), 0, \dots, 0\right) \ ,$$

such that $\pi \circ \sigma = \mathrm{id}_{\mathbb{D}^2}$.

Each point $e^{i\varphi}$ in the boundary $\partial\mathbb{D}^2$ of the disk is covered by a single orbit. It is easy to see that

$$\pi^{-1}(e^{i\varphi}) = \left\{(e^{i\varphi}, ie^{ik\varphi/2}\mathbf{y}) \middle|\ \text{with } \|\mathbf{y}\| = 1\right\} \ ,$$

and all of these points lie on a single orbit, because $\mathrm{SO}(n)$ acts transitively on $\mathbb{S}^{n-1}$.

We want to show that the preimage of an interior point $z_0 \in \mathbb{D}^2_{<1}$ is composed by a single orbit, if $2n-1 \geq 5$, and composed of two orbits, if $2n-1 = 3$. Because the Milnor action commutes with the $\mathrm{SO}(n)$-action considered in this example, it is no restriction to the generality of the proof to assume that $z_0 = r_0$ is real. Any point $(r_0, \mathbf{z})$ can be rotated in a first step to a point with $\mathbf{x} = (x_1, 0, \dots, 0)$ such that $x_1 \geq 0$. If the dimension of W_k^{2n-1} is 5 or larger, then a second rotation allows to change $\mathbf{y}$ to the form $(y_1, y_2, 0, \dots, 0)$ with $y_2 \geq 0$. From $f(r_0, \mathbf{z}) = 0$, it follows $\langle \mathbf{x} | \mathbf{y} \rangle = 0$, such that $y_1 = 0$ (the case $x_1 = 0$ can be excluded, because then the orbit would lie in $\pi^{-1}(\partial\mathbb{D}^2)$. The orbit over r_0 can be represented by the point

$$(r_0, \sqrt{2 - r_0^2 - r_0^k}, i\sqrt{2 - r_0 + r_0^k}, 0, \dots, 0) \ .$$

For the 3-dimensional Brieskorn manifolds W_k^3, it is only possible to rotate every point to one of the form

$$(r_0, \sqrt{2 - r_0^2 - r_0^k}, \pm i\sqrt{2 - r_0 + r_0^k}) \ .$$

But depending on the sign of the last slot, the point lies on a different orbit.

It follows that the orbit space of W_k^{2n-1} with $2n-1 \geq 5$ is diffeomorphic to $\mathbb{D}^2$. The orbit space of W_k^3 is diffeomorphic to two copies of the disk that have been glued along the boundary. Hence the manifold W_k^3 is an $\mathbb{S}^1$-principal bundle over $\mathbb{S}^2$. Now we will show that the Euler number of such an $\mathbb{S}^1$-manifold W_k^3 is really k. For this, we will compute the intersection number between the two sections $\sigma_\pm$ on the common boundary,

$$\sigma_\pm : \mathbb{D}^2 \hookrightarrow W_k^3, \quad z_0 \mapsto \left(z_0, \frac{i}{2A}(A^2 + z_0^k), \pm\frac{1}{2A}(A^2 - z_0^k)\right) \ ,$$

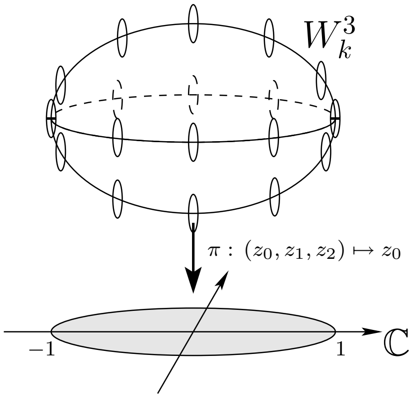


FIGURE 9. W_k^3 is obtained by taking two solid tori that are glued along the boundary. Each of the solid tori covers the disc.

with $A = \sqrt{2 - r_0^2 + \sqrt{(2 - r_0^2)^2 - r_0^{2k}}}$. The points where both sections intersect are the ones with $z_0 = e^{i\varphi_0}$, where $e^{ik\varphi_0} = 1$, i.e. there are k intersection points. $\square$

6.2. Lens spaces. A (3-dimensional) **lens space** $L(p, q)$ (with integers $1 \leq q < p$ and $\gcd(p, q) = 1$) is defined in the following way: Let $\xi = e^{2\pi i/p}$ be the generator of the cyclic group $\mathbb{Z}_p$. The action on the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ given by

$$\xi \cdot (z_1, z_2) = (\xi z_1, \xi^q z_2)\ .$$

is free, because

$$(\xi^n z_1, \xi^{nq} z_2) = (z_1, z_2)$$

can only hold if $z_1 = 0$ and $p|nq$, but since $\gcd(p, q) = 1$, it follows that $p|n$, and n has to be multiple of p. Hence the orbit space

$$L(p, q) := \mathbb{S}^3/\mathbb{Z}_p$$

is a smooth manifold. It is obvious that $\pi_1(L(p, q)) \cong \mathbb{Z}_p$. Lens spaces were the first examples of closed manifolds that are homotopy equivalent but not homeomorphic, e.g. $L(7, 1) \simeq L(7, 2)$, but $L(7, 1) \not\cong L(7, 2)$. More on this topic can be found in many books on algebraic topology (e.g. in [**Bre93**], [**Hat02**]). Here of course, we are interested in lens spaces as $\mathbb{S}^1$-manifolds.

The standard Hopf action of the circle on $\mathbb{S}^3$

$$\mathbb{S}^1 \times \mathbb{S}^3,\ (e^{i\varphi}, (z_1, z_2)) \mapsto (e^{i\varphi} z_1, e^{i\varphi} z_2)$$

commutes with the $\mathbb{Z}_p$-action defined above, and hence it induces a well-defined $\mathbb{S}^1$-action on $L(p, q)$. This $\mathbb{S}^1$-action is not effective in general. If there is an $n \in \{1, \dots, p-1\}$ such that $n(q-1)/p \in \mathbb{Z}$, then the two equations

$$e^{2\pi i t} z_1 = \xi^n z_1 \qquad \text{and} \qquad e^{2\pi i t} z_2 = \xi^{qn} z_2$$

are solved by $t = n/p$, i.e. $e^{2\pi i n/p}$ acts trivially on every point $[z_1, z_2] \in L(p, q)$. The kernel of the map $\mathbb{S}^1 \to \mathrm{Diff}(L(p, q))$ is isomorphic to $\mathbb{Z}_r$ with $r = \gcd(p, q-1)$. From now on divide the circle by $\mathbb{Z}_r$ to get an effective $\mathbb{S}^1$-action.

The only two exceptional orbits E_1 and E_2 are given by the circles

$$E_1 := \{[z_1, 0] | \, z_1 \in \mathbb{C}\} \quad \text{and} \quad E_2 := \{[0, z_2] | \, z_2 \in \mathbb{C}\}$$

in $L(p,q)$. The element ξ generates the stabilizer of both orbits, and the stabilizer is isomorphic to $\mathbb{Z}_k$ with $k = p/\gcd(p, q-1)$ (remember that the circle acting on $L(p,q)$ had to be reduced to obtain an effective action). Note that the greatest common divisor of p and 0 is $\gcd(p, 0) = p$, such that for $q = 1$, $\mathbb{Z}_p$ acts as the obvious restriction of the Hopf action, and there are no exceptional orbits. That means that all $L(p,1)$ are principal $\mathbb{S}^1$-bundles.

To compute the second orbit invariant m, we will do most of the necessary computations on the 3-sphere, and later apply the equivalence relations. Note that the slice at $[1,0] \in E_1$ and $[0,1] \in E_2$ can be written as

$$S_{E_1} := \{(0, z_2) | \, z_2 \in \mathbb{C}\} \quad \text{and} \quad S_{E_2} := \{(z_1, 0) | \, z_1 \in \mathbb{C}\} \ .$$

The slices lift to $\mathbb{S}^3$, and the action of the generator ξ of the stabilizer on $(0, z_2) \in T_{(1,0)}\mathbb{S}^3$ and on $(z_1, 0) \in T_{(0,1)}\mathbb{S}^3$ gives

$$\xi(0, z_2) = (0, \xi z_2) \in T_{(\xi,0)}\mathbb{S}^3 \quad \text{and} \quad \xi(z_1, 0) = (\xi z_1, 0) \in T_{(0,\xi)}\mathbb{S}^3 \ .$$

Projecting to the lens space $L(p,q)$ gives the following equivalence relations: $T_{(1,0)}\mathbb{S}^3 \ni (0, z_2) \sim (0, \xi^q z_2) \in T_{(\xi,0)}\mathbb{S}^3$, and $T_{(0,1)}\mathbb{S}^3 \ni (z_1, 0) \sim (\xi z_1, 0) \in T_{(0,\xi^q)}\mathbb{S}^3$. For the action of the generator ξ, this means

$$\xi(0, z_2) = (0, \xi^{1-q} z_2) \in T_{[1,0]}L(p,q) \quad \text{and} \quad \xi(z_1, 0) = (\xi^{1-a} z_1, 0) \in T_{[0,1]}L(p,q) \ ,$$

where $a, b \in \mathbb{Z}$ are chosen in such a way that $aq + bp = 1$. The orbit invariants (k, m) are defined by the representation of ξ on the slice

$$\xi z = e^{2\pi i m/k} z \ ,$$

and accordingly the orbit invariants of E_1 are

$$(k_1, m_1) = \left(\frac{p}{\gcd(p, q-1)}, \frac{c_1}{\gcd(p, q-1)} \right) \ ,$$

and the ones of E_2 are

$$(k_2, m_2) = \left(\frac{p}{\gcd(p, q-1)}, \frac{c_2}{\gcd(p, q-1)} \right) \ ,$$

where c_1 is the smallest positive number that can be obtained from $(1-q)$ by adding multiples of p, and c_2 is the smallest positive number that can be obtained in the same way from $(1-a)$.

The orbit space of $L(p,q)$ is isomorphic to the double quotient of $\mathbb{S}^3$ first by $\mathbb{Z}_p$ and then by $\mathbb{S}^1$, but since both actions commute, we have

$$L(p,q)/\mathbb{S}^1 \cong \mathbb{CP}^1/\mathbb{Z}_p \ .$$

The exceptional orbits in $\mathbb{CP}^1$ correspond to the points $[1:0]$ and $[0:1]$. Hence the orbit space of $L(p,q)$ with the exceptional orbits removed is equal to the quotient of the punctured plane $\mathbb{C}^*$ by $\mathbb{Z}_p$, which is still diffeomorphic to $\mathbb{C}^*$. The total orbit space $L(p,q)/\mathbb{S}^1$ is homeomorphic to $\mathbb{S}^2$.

To obtain the Euler invariant, we will make use of Theorem III.13. The 1-form

$$\alpha = x_1 \, dy_1 - y_1 \, dx_1 + x_2 \, dy_2 - y_2 \, dx_2$$

is invariant under both the Hopf- and the $\mathbb{Z}_p$-action, and thus projects down to an $\mathbb{S}^1$-invariant form $\widetilde{\alpha}$ on $L(p,q)$. The integral of $\alpha \wedge d\alpha$ over $\mathbb{S}^3$ gives

$$\int_{\mathbb{S}^3} \alpha \wedge d\alpha = \int_{\mathbb{B}^4} d\alpha \wedge d\alpha = 8 \int_{\mathbb{B}^4} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = 4\pi^2 \ ,$$

but the 3-sphere $\mathbb{S}^3$ is a p-fold cover of $L(p,q)$. Therefore the 3-form $\widetilde{\alpha} \wedge d\widetilde{\alpha}$ evaluates to $4\pi^2/p$ on the lens space. Also, in general the 1-form $\widetilde{\alpha}$ has to be rescaled by $\gcd(p, q-1)$ to correct for the non-effectiveness of the standard Hopf-action. Finally we obtain for a connection 1-form A on $L(p,q)$ that

$$\int_{L(p,q)} A \wedge dA = \frac{4\pi^2 \gcd(p, q-1)^2}{p} \ .$$

According to Theorem III.13, the Euler number e of the lens space is given by

$$e = \frac{\gcd(p, q-1)}{p} \Big(\gcd(p, q-1) - \beta_1 - \beta_2\Big) \ ,$$

where the Seifert invariant β_j is the smallest positive number such that $\beta_j c_j / \gcd(p, q-1) \equiv 1 \mod (p/\gcd(p, q-1))$. All of the invariants given so far can easily be computed for any given $L(p,q)$, but I have not been able to find a nice closed formula.

If $q = 1$, then there are no exceptional orbits, and one obtains that the Euler number is

$$e = \frac{\gcd(p, 0)^2}{p} = p \ ,$$

and hence $L(p,1)$ is the principal $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ with $e = p$, and

$$L(p,1) \cong W_p^3 \ .$$

CHAPTER IV

Contact $\mathbb{S}^1$-manifolds

In this chapter we will give the classification of 3-dimensional contact $\mathbb{S}^1$-manifolds (Theorem IV.16). Several people have contributed to this result. Probably the first to consider $\mathbb{S}^1$-invariant contact structures were Boothby and Wang ([**BW58**]). They constructed contact structures on manifolds (of any dimension) with a free $\mathbb{S}^1$-action, where all the orbits are transverse to the contact structure. In [**Lut77**], Lutz extended the result to 3-dimensional $\mathbb{S}^1$-bundles with an invariant contact structure allowing Legendrian orbits. He was able to show that these contact structures sometimes lie in different homotopy classes of plane fields, providing different contact structures on the same manifold. Finally, Kamishima and Tsuboi gave a full classification of 3-dimensional contact $\mathbb{S}^1$-manifolds in [**KT91**]. Unfortunately, the proof in [**BW58**] is wrong, and the one in [**KT91**] explains in great detail the easy parts, but skips completely the more difficult arguments. In this chapter I hope to fill the missing gaps in the literature.

REMARK IV.1. In this section we will only consider contact structures induced by a global contact form α. Such a contact structure defines an orientation on the manifold in question, because it is not possible to change the sign of $\alpha \wedge d\alpha$ by changing the sign of α. On an oriented manifold M a contact structure ξ is called **positive**, if the orientation given by ξ coincides with the one of M.

Note that many invariants of an $\mathbb{S}^1$-manifold M described in Chapter III depended on the orientation of M. All such invariants below will be computed with respect to the orientation induced by the contact structure. This subtle point may seem unnecessary, but it is quite important as can be seen in Example 6.1.

1. Contact $\mathbb{S}^1$-bundles

It has been known for a long time that $\mathbb{S}^1$-principal bundles over certain symplectic manifolds carry a natural contact structure.

THEOREM IV.1 (Boothby-Wang). *Let (M, ω) be an* ***integral symplectic manifold****, i.e. a symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$. The $\mathbb{S}^1$-bundle (P, M, π) over M with Euler class $[\omega]$ has a connection α that represents an $\mathbb{S}^1$-invariant contact form.*

DEFINITION. The manifold (P, α) in the theorem above is called the **Boothby-Wang fibration** over (M, ω).

REMARK IV.2. The proof below assumes for higher dimension that the reader is familiar with the relation between the classification of $\mathbb{S}^1$-bundles and the curvature form is known (see [**Wel80**]). For dimension 3 these results can be found in Section III.2 and III.5.

PROOF OF THEOREM IV.1. Let β be an arbitrary connection on P, i.e. an $\mathbb{S}^1$-invariant 1-form with $\beta(Z_P) = 1$, where Z_P is the infinitesimal generator of the $\mathbb{S}^1$-action.

The curvature of β is a 2-form ω' on M such that $d\beta = \pi^*\omega'$. The curvature represents the Euler class, thus $[\omega'] = [\omega]$, and one finds a 1-form γ on M such that $d\gamma = \omega' - \omega$.

Define $\alpha = \beta - \pi^*\gamma$. This is also a connection, because $\alpha(Z_P) = \beta(Z_P) = 1$ and $\mathcal{L}_{Z_P}\alpha = 0$, and it is a contact form, because $d\alpha = d\beta - \pi^*(\omega' - \omega) = \pi^*\omega$, and $\alpha\wedge(d\alpha)^n = \alpha\wedge\pi^*\omega^n \neq 0$. $\square$

Let X be a nowhere vanishing vector field on an $(n+1)$-dimensional manifold M. According to the flow box theorem (e.g. [**PdM82**, Theorem 2.1.1]), there exists around any point $p \in M$ a chart

$$(-\varepsilon, \varepsilon)^{n+1} \subset \mathbb{R}^{n+1} \longrightarrow U \subset M$$

with coordinates $(x_0, x_1, \ldots, x_n)$ such that the vector field X is given by $\frac{\partial}{\partial x_0}$. A chart U around $p \in M$ is called a **regular chart for** X, if it is of the form above, and if the intersection of any trajectory of X with U is either empty or a single line

$$\{(t, x_1, \ldots, x_n) | x_j \text{ fixed and } t \in (-\varepsilon, \varepsilon)\} .$$

A contact form α on M is called **regular**, if every point $p \in M$ is contained in a regular chart for the Reeb field X_{Reeb}.

LEMMA IV.2. *Let X be a vector field on a closed manifold M such that there is a regular chart around every point $p \in M$. Every trajectory of X is a closed loop, and the function $\lambda : M \to \mathbb{R}^+$, which assigns to every p the period of the flow Φ^X, i.e.*

$$\lambda(p) := \min\bigl\{t \in (0, \infty) |\, \Phi_t^X(p) = p\bigr\}$$

is smooth.

PROOF. First we will show that every trajectory is closed, i.e. a circle. If the flow line through $p \in M$ is not a circle, it cannot be a closed subset either. Then there is a point q lying in the closure of the orbit through p, but not on the integral curve itself. There is a regular chart with coordinates $(x_0, x_1, \ldots, x_n)$ around q such that the field is of the form $\frac{\partial}{\partial x_0}$. By our assumption the trajectory through p enters at most once a small neighborhood of q, but at the same time q has to lie in the closure of this trajectory. Hence q lies on the trajectory, which is a contradiction, and every trajectory is closed.

It follows that the function λ is defined, but a priori it does not need to be continuous, and thus is is not obvious that λ is bounded from below by a positive number. Because M is compact, it can be covered with finitely many regular charts, where the smallest one is a cube say $(-\varepsilon_0, \varepsilon_0)^{n+1}$. We get

$$\lambda \geq 2\varepsilon_0 .$$

Now we will show that λ is a smooth function. Let U be a regular chart around p_0 such that p_0 corresponds to the point $(0, \ldots, 0)$ in coordinates, and let $t_0 = \lambda(p_0)$. The time-t_0-flow $T := \Phi_{t_0}^X : M \to M$ of the field X is a diffeomorphism that leaves p_0 fixed. Because the chart U is regular, the map T has on an open subset of U (compare Figure 1) the form

$$T(x_0, x_1, \ldots, x_n) = (\widetilde{x}_0, x_1, \ldots, x_n) ,$$

and because T is a diffeomorphism, $\widetilde{x}_0 = \widetilde{x}_0(x_0, \ldots, x_n)$ is a smooth function. Using that $(x_0, x_1, \ldots, x_n) = \Phi_{x_0}^X(0, x_1, \ldots, x_n)$, and that T commutes with the flow, it follows that

$$\widetilde{x}_0(x_0, x_1, \ldots, x_n) = \widetilde{x}_0(0, x_1, \ldots, x_n) + x_0 .$$

This allows us to define a smooth function

$$F : U \longrightarrow \mathbb{R}, (x_0, x_1, \ldots, x_n) \longmapsto t_0 - \widetilde{x}_0(0, x_1, \ldots, x_{2n}) ,$$

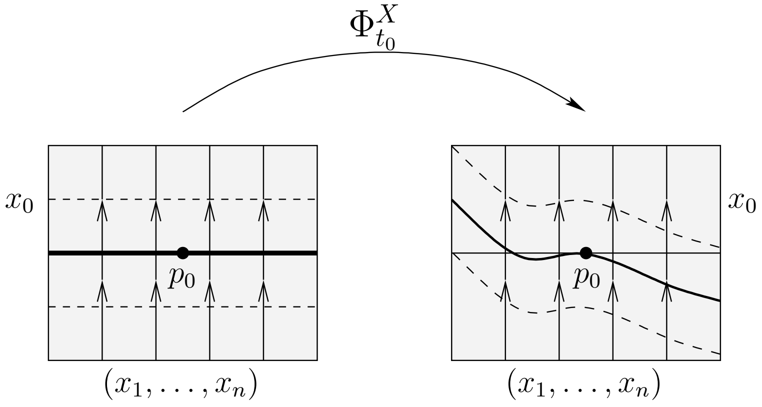


FIGURE 1. $T = \Phi^X_{t_0}$ is a diffeomorphism with fixed point p_0

which associates to points in U a time, where the flow Φ^X returns:

$$\begin{aligned}\Phi^X_{F(x_0,x_1,\dots,x_n)}(x_0, x_1, \dots, x_n) &= \Phi^X_{-\widetilde{x}_0(0,x_1,\dots,x_n)} \circ \Phi^X_{t_0}(x_0, x_1, \dots, x_n)\\ &= \Phi^X_{-\widetilde{x}_0(0,x_1,\dots,x_n)}\big(x_0 + \widetilde{x}_0(0, x_1, \dots, x_n), x_1, \dots, x_n\big)\\ &= (x_0, x_1, \dots, x_{2n})\ .\end{aligned}$$

Note that F does not depend on the x_0-coordinate. If we are able to show that λ coincides around p_0 with F, we have shown that λ is smooth. In fact, it is enough to show that $|\lambda - F| < \varepsilon$, because inside a regular chart $(-\varepsilon, \varepsilon)^{n+1}$ the period of the field X cannot be smaller than 2ε.

We will now prove that λ is continuous. Then it follows that F and λ are arbitrarily close around p_0 and hence equal. Let p_n be a sequence of points in U such that $p_n \to p_0$. We have to show that $\lim_{n\to\infty} \lambda(p_n) = \lambda(p_0)$.

First note that $F \geq \lambda$, because λ is the smallest positive return time for the flow, and then the inequality

$$\lambda(p_0) = t_0 = F(p_0) = \limsup F(p_n) \geq \limsup \lambda(p_n)$$

holds.

If $\liminf \lambda(p_n) < \lambda(p_0)$, then there is a subsequence p_k such that $\lambda(p_k) \to \lambda_0 < \lambda(p_0)$. Note that $\Phi^X_{\lambda_0}(p_0) = p_0$, because on one hand we get

$$\lim_{k\to\infty} \Phi^X_{\lambda(p_k)}(p_k) = \lim_{k\to\infty} p_k = p_0\ ,$$

but since the flow map $\Phi^X : \mathbb{R} \times M \to M$ is continuous, we also have $\Phi^X_{\lambda(p_k)}(p_k) \to \Phi^X_{\lambda_0}(p_0)$. From the equation $\Phi^X_{\lambda_0}(p_0) = p_0$ it either follows that $\lambda_0 = \lambda(p_0)$, which contradicts the assumption above, or $\lambda_0 = 0$, but this is not possible because we showed that $\lambda \geq 2\varepsilon_0$.

This gives

$$\lambda(p_0) \geq \limsup \lambda(p_k) \geq \liminf \lambda(p_k) \geq \lambda(p_0)\ ,$$

and the function λ is continuous in p_0. □

THEOREM IV.3 (Boothby-Wang). *Let P be a manifold with a regular contact form α. Then α can be rescaled by a constant, such that the Reeb flow induces a free $\mathbb{S}^1$-action on P and the orbit space is a symplectic manifold.*

PROOF. Apply Lemma IV.2 to the Reeb field X_{Reeb} to see that all Reeb orbits are closed and to obtain the function λ. We will show that λ is constant in this situation.

The vector field $X(p) := \lambda(p) \cdot X_{\text{Reeb}}(p)$ is smooth and has closed orbits with return-time 1. The flow of X gives P the structure of an $\mathbb{S}^1$-bundle. There are bundle charts of the form $U \times \mathbb{S}^1 = \{(x_1, \dots, x_{2n}, e^{i\varphi})\}$ around an arbitrary point. The contact form α can be written in this chart as

$$\alpha = f(p)\,d\varphi + \sum_{j=1}^{2n} g_j(p)\,dx_j\ .$$

Because $X_{\text{Reeb}} = \frac{1}{\lambda}\,\partial_\varphi$ is the Reeb field, it follows that $f(p) = \lambda(p)$, and using Cartan formula

$$0 = \mathcal{L}_{X_{\text{Reeb}}}\alpha = \frac{1}{\lambda}\sum_{j=1}^{2n}\left(\frac{\partial g_j}{\partial\varphi}\,dx_j - \frac{\partial\lambda}{\partial x_j}\,dx_j\right)\ ,$$

and as a consequence $\partial_{x_j}\lambda = \partial_\varphi g_j$. The function λ does not change along the φ-direction, hence $\partial_\varphi^2 g_j = \partial_{x_j}\partial_\varphi\lambda = 0$, and so

$$g_j = \frac{\partial\lambda}{\partial x_j}\cdot\varphi + c(x_1, \dots, x_{2n})\ ,$$

but this means that $\frac{\partial\lambda}{\partial x_j} = 0$, because g_j has to be 2π-periodic in φ. The function λ is constant.

Divide α by λ. It is clear that P is an $\mathbb{S}^1$-bundle over its orbit space M with the action induced by the Reeb flow. The contact form is a connection for this bundle and its curvature F is a 2-form on M such that $d\alpha = \pi^*F$. It is well-known that the curvature F represents an integral cohomology class, and it is also clear that F is non-degenerate, because $\alpha \wedge d\alpha^n = \alpha \wedge \varphi^* F^n \neq 0$. ☐

2. Local behavior of the contact structure

In Section III.1, all local invariants of a 3-dimensional $\mathbb{S}^1$-manifold (without any contact structure) were given. In this section, the aim will be to specify all possible behaviors of invariant contact structures in the neighborhood of such orbits.

The result is that principal orbits can be either Legendrian or transverse to the contact structure, and orbits with non-trivial stabilizer allow at most one contact structure up to $\mathbb{S}^1$-contactomorphisms.

DEFINITION. Two G-invariant contact forms α_1 and α_2 on a G-manifold M are called **locally G-equivalent around a submanifold** $N \hookrightarrow M$, if there is a G-diffeomorphism $\Phi : M \to M$ with arbitrarily small support around N such that $\Phi^*\alpha_1$ and α_2 represent the same contact structure on a small neighborhood of N.

2.1. Fixed points.

Recall that the set F of fixed points is a disjoint union of circles.

LEMMA IV.4. *Let M be a closed oriented 3-dimensional $\mathbb{S}^1$-manifold. Any two positive $\mathbb{S}^1$-invariant contact forms are locally $\mathbb{S}^1$-equivalent around the set of fixed points F.*

PROOF. A neighborhood U of a component of F is $\mathbb{S}^1$-diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2_\varepsilon$ with the action $e^{i\varphi}(e^{it}, z) = (e^{it}, e^{i\varphi}z)$. The 1-form

$$\alpha_0 := dt + \frac{1}{2}\Big(x\,dy - y\,dx\Big)$$

is an $\mathbb{S}^1$-invariant contact form on U, where we used $z = x + iy$.

Assume now another positive invariant contact form $\alpha_1 = f\,dt + g\,dx + h\,dy$ is given on U with functions $f, g, h : U \to \mathbb{R}$.

The tangent space T_pM at a fixed point $p = (e^{it_0}, 0) \in F$ splits as $\mathbb{S}^1$-module into $\langle \partial_t \rangle \oplus \langle \partial_x, \partial_y \rangle$, but also into $\varepsilon^1 \oplus \xi_p$, where ε^1 is the line generated by the Reeb field Y of α_1, and $\xi_p = \ker \alpha_1$, because it has been shown in the proof of Lemma A.2 in Appendix A that Y remains invariant under the $\mathbb{S}^1$-action. If follows that $\varepsilon^1 = \langle \partial_t \rangle$ and $\xi_p = \langle \partial_x, \partial_y \rangle$, and then $g(e^{i\varphi}, 0) = h(e^{i\varphi}, 0) = 0$ and

$$\alpha_1 = f(e^{it}, 0)\, dt \text{ on } F,$$

with $f(e^{i\varphi}, 0) \neq 0$. We can divide α_1 by the function f (possibly only on a smaller neighborhood of F) to obtain an equivalent contact form $\tilde{\alpha}_1 = dt + g\,dx + h\,dy$, with new functions $g, h : U \to \mathbb{R}$.

The Reeb field $Y = \partial_t$ lies in the kernel of $d\tilde{\alpha}_1$, hence we obtain that $d\tilde{\alpha}_1 = (\partial_x h - \partial_y g)\; dx \wedge dy$ with $\partial_x h - \partial_y g > 0$ on F, because $\tilde{\alpha}_1$ is a positive contact form. The linear interpolation

$$\tilde{\alpha}_s := (1 - s)\,\alpha_0 + s\,\tilde{\alpha}_1$$

with $s \in [0, 1]$ consists in a neighbhorhood of F, of positive invariant contact forms, because on F the contact condition is

$$\tilde{\alpha}_s \wedge d\tilde{\alpha}_s := (1 - s + s\;(\partial_x h - \partial_y g))\, dt \wedge dx \wedge dy > 0\;.$$

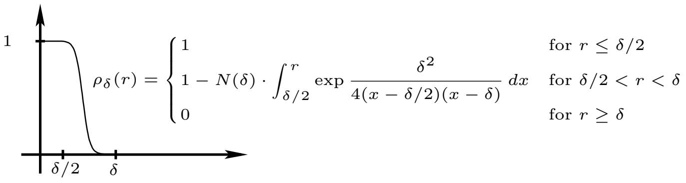
This allows to apply Lemma A.2: The vector field X_s is defined by the equations

$$\iota_{X_s} \tilde{\alpha}_s = 0 \quad \text{ and } \quad \iota_{X_s} d\tilde{\alpha}_s = r_s\,\tilde{\alpha}_s - \dot{\tilde{\alpha}}_s\;,$$

where $r_s = \dot{\tilde{\alpha}}_s(Y_s)$ with Y_s the Reeb field of the form $\tilde{\alpha}_s$. On F the equations reduce to

$$\iota_{X_s} \tilde{\alpha}_s = 0 \quad \text{ and } \quad \iota_{X_s} d\tilde{\alpha}_s = 0\;,$$

because $\dot{\tilde{\alpha}}_s = 0$, and hence the vector field X_s vanishes on the fixed point set.



$$\rho_\delta(r) = \begin{cases} 1 & \text{for } r \leq \delta/2 \\ 1 - N(\delta) \cdot \int_{\delta/2}^{r} \exp \frac{\delta^2}{4(x - \delta/2)(x - \delta)}\, dx & \text{for } \delta/2 < r < \delta \\ 0 & \text{for } r \geq \delta \end{cases}$$

FIGURE 2. ρ_δ is a cut-off function, with $N(\delta)$ the reciprocal value of $\int_{\delta/2}^{\delta} \exp \frac{\delta^2}{4(x-\delta/2)(x-\delta)}\, dx$

There is a small neighborhood $\widetilde{U}$ of F, where the flow $\Phi_s^{X_s}$ is defined for all $s \in [0, 1]$. To finish the proof choose a cut-off function ρ_δ that is equal to 1 on the set $[0, \delta/2)$ and whose support lies in the interval $[0, \delta)$ (for example the choice depicted in Figure 2 would do), and consider the time-one-flow of the vector field $\widetilde{X}_s(e^{it}, z) = \rho_\delta(|z|) \cdot X_s(e^{it}, z)$ with support in $\widetilde{U}$. The map $\Phi_1^{\widetilde{X}_s}$ gives the desired equivalence. $\square$

2.2. Exceptional orbits. The exceptional orbits of $\mathbb{S}^1$-actions have been described in Section III.1.2.

LEMMA IV.5. *Let M be an oriented 3-manifold with an $\mathbb{S}^1$-action. Any two positive $\mathbb{S}^1$-invariant contact forms on M are locally $\mathbb{S}^1$-equivalent around the set of exceptional orbits E.*

PROOF. A neighborhood U of an exceptional orbit with orbit invariants (k,m) is $\mathbb{S}^1$-diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2_\varepsilon$ with the action

$$e^{i\varphi} \cdot (e^{it}, z) = (e^{i(t+k\varphi)}, e^{im\varphi} z) \ .$$

The 1-form

$$\alpha_0 := dt + \frac{1}{2}\Big(x\, dy - y\, dx\Big)$$

is an $\mathbb{S}^1$-invariant contact form in the neighborhood of the exceptional orbit.

The procedure to show that any other $\mathbb{S}^1$-invariant contact structure is locally equivalent to the one given by α_0, is almost equal to the one given in the proof of Lemma IV.4.

Assume α_1 is another $\mathbb{S}^1$-invariant contact form in the neighborhood U. The tangent space T_pM splits as $\mathbb{Z}_k$-module canonically into $\varepsilon^1 \oplus \xi_p$, where p is a point on the exceptional orbit, and $\operatorname{Stab}(p) \cong \mathbb{Z}_k$. The line ε^1 is generated by the Reeb field of α_1. It follows that the Reeb field is parallel to the exceptional orbit. Dividing by the function $\alpha_1(Z_M)$, we obtain a contact form $\tilde\alpha_1$ with $\alpha_0 = \tilde\alpha_1$ and $d\tilde\alpha_1 = f\, d\alpha_0$ on E with $f > 0$. The proof is completed by the same arguments as those of Lemma IV.4. ☐

2.3. Special exceptional orbits. We are only considering contact manifolds (M,α) with a contact form. In particular M is naturally oriented, but the existence of special exceptional orbits implies that the manifold is non-orientable (see III.1.3). Hence 3-dimensional contact $\mathbb{S}^1$-manifolds do not have any special exceptional orbits.

2.4. Legendrian orbits. A 1-dimensional $\mathbb{S}^1$-orbit $\operatorname{Orb}(p)$ is called **Legendrian**, if it is everywhere tangent to the contact structure. This is equivalent to requiring that the generator $Z_M(p)$ of the action does not vanish at p, and that $\alpha_p(Z_M) = 0$.

LEMMA IV.6. *Let M be a closed, oriented 3-manifold with an $\mathbb{S}^1$-action. The set of Legendrian orbits Σ is a submanifold, whose components are embedded tori. All Legendrian orbits have trivial stabilizer.*

PROOF. Let α be an invariant contact form. Define a function $H : M \to \mathbb{R}$ which is constant along the orbits by

$$H(p) := \alpha_p(Z_M) \ .$$

The set Σ can be written as $\Sigma = H^{-1}(0) - F$ (with F the set of fixed points). Since F is the union of a finite number of embedded circles, and their neighborhood is of the form described in the proof of Lemma IV.4, it follows that the subsets F and Σ do not touch each other.

To see that the set Σ is a closed submanifold, it is enough to prove that dH does not vanish on Σ. This is shown by the following calculation using Cartan formula

$$0 \ = \ \mathcal{L}_{Z_M}\alpha = \iota_{Z_M} d\alpha + dH \ .$$

If dH vanishes as some point p of Σ, then p is a fixed point, because $\alpha_p(Z_M) = 0$ and $\iota_{Z_M} d\alpha_p = 0$, and so $Z_M(p) = 0$.

The Legendrian orbits have trivial stabilizer, because on the exceptional orbits the Reeb field R is parallel to the field Z_M and thus Z_M does not lie in $\xi = \ker \alpha$. By continuity it follows that exceptional orbits have a small neighborhood that does not contain any Legendrian orbit.

To see that the components of Σ are tori, note that the orbit space of a small neighborhood of Σ is a smooth surface, and that H induces a smooth function H^* on this surface. The zero set of H^* is composed of circles and thus the set of Legendrian orbits Σ is a torus (because the $\mathbb{S}^1$-principal bundle over a circle is trivial). □

COROLLARY IV.7. *The zero-set of the function* $H : M \to \mathbb{R}$, $p \mapsto \alpha_p(X_M)$ *projects to a collection of circles* $L_1 \cup \ldots \cup L_N$ *in the orbit space* $B = M/\mathbb{S}^1$. *The complement of* $L_1 \cup \ldots \cup L_N$ *decomposes into* $B_+ := \{p \in B | H(\pi^{-1}(p)) > 0\}$ *and* $B_- := \{p \in B | H(\pi^{-1}(p)) < 0\}$. *The orbit space* B *is partioned by the circles* L_j *into* B_+ *and* B_-.

LEMMA IV.8. *Let* M *be an oriented* 3*-manifold with an* $\mathbb{S}^1$*-action. Any two positive* $\mathbb{S}^1$*-invariant contact forms on* M *with identical set of Legendrian orbits* Σ *are locally* $\mathbb{S}^1$*-equivalent around* Σ.

PROOF. This has been proved in [**Lut77**], but for completeness, we rewrite the proof more explicitly: There is a neighborhood of the torus that looks like $U := \mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$, where the circle acts on the first component:

$$\left(e^{i\varphi'}, (e^{i\varphi}, e^{i\vartheta}, s)\right) \mapsto \left(e^{i(\varphi+\varphi')}, e^{i\vartheta}, s\right) .$$

We will show that any positive $\mathbb{S}^1$-invariant contact form on U with Legendrian orbits in Σ is locally $\mathbb{S}^1$-equivalent to

$$\alpha_0 = s\, d\varphi + d\vartheta .$$

Let $\alpha_1 = f(\vartheta, s)\, d\varphi + g(\vartheta, s)\, d\vartheta + h(\vartheta, s)\, ds$ be a second positive invariant contact form on U with the same Legendrian orbits, i.e. with $f(\vartheta, 0) = 0$ and in particular $\partial_\vartheta f(\vartheta, 0) = 0$. The contact condition on Σ is $\alpha_1 \wedge d\alpha_1 = g\, \partial_s f\, d\vartheta \wedge ds \wedge d\varphi \neq 0$, and hence $g(\vartheta, 0) \neq 0$. By possibly restricting to a smaller neighborhood and after dividing by g, we can assume $\alpha_1 = f(\vartheta, s)\, d\varphi + d\vartheta + h(\vartheta, s)\, ds$ (with new functions f, h). Let $\alpha_t := \alpha_0 + t\,(\alpha_1 - \alpha_0)$ be the linear interpolation of the two forms. All 1-forms in this family are positive invariant contact forms in a neighborhood of Σ. This can be checked by computing $\alpha_t \wedge d\alpha_t$ only on Σ:

$$\begin{aligned} \alpha_t \wedge d\alpha_t &= (d\vartheta + th\, ds) \wedge d\alpha_t \\ &= (1-t)\, \alpha_0 \wedge d\alpha_0 + t\, \alpha_1 \wedge d\alpha_1 = \omega_0 + t(\omega_1 - \omega_0) , \end{aligned}$$

where $\omega_i := \alpha_i \wedge d\alpha_i$. Both volume forms are by assumption positive, and their convex span does not vanish anywhere.

If the time-dependent vector field X_t defined in Lemma A.2 has a global flow Φ_t, then the equivariant Gray stability shows that the two contact forms α_0 and α_1 are equivalent. The vector field X_t is given by the equations $0 = \iota_{X_t}\alpha_t$ and $\iota_{X_t} d\alpha_t = r_t\, \alpha_t - \dot\alpha_t$, where $r_t = \dot\alpha_t(Y_t)$ with Y_t the Reeb field of the form α_t.

Both the Reeb field Y_t and the vector field X_t are tangent to Σ, because

$$d\alpha_t = (1 - t + t\, \partial_s f)\, ds \wedge d\varphi + t\, \partial_\vartheta h\, d\vartheta \wedge ds \neq 0$$

on $TM|_\Sigma$. If Y_t had a ∂_s-component, Y_t would not lie in the kernel of $d\alpha_t$. The second defining equation for X_t simplifies on Σ to $\iota_{X_t} d\alpha_t = -h\, ds$, and in particular it follows that X_t does not have a ∂_s-component either.

As a consequence, the flow of X_t is defined on the closed submanifold Σ up to time 1, and by continuity also on a small neighborhood $\widetilde{U}$ around Σ. To finish the proof choose again a cut-off function $\rho_\delta : \mathbb{R} \to [0,1]$ like the one in Figure 2, and consider the vector field $\rho_\delta(|s|) \cdot X_t$. Its time-1 flow is the map that gives the desired equivalence. $\square$

3. Uniqueness of Contact Structures

In Theorem III.11 the classification of closed 3-dimensional $\mathbb{S}^1$-manifolds was given. Two contact $\mathbb{S}^1$-manifolds (M_1, α_1) and (M_2, α_2) can only be equivalent if they are $\mathbb{S}^1$-diffeomorphic, i.e. if all the invariants given in the theorem agree. In this case, we can identify M_1 and M_2 and speak instead of a single $\mathbb{S}^1$-manifold M with two different contact structures α_1 and α_2.

Lemma IV.9. *Let M be a 3-dimensional $\mathbb{S}^1$-manifold with two positive $\mathbb{S}^1$-invariant contact forms α_0 and α_1. Assume that the set Σ of Legendrian orbits for both forms is the same collection of tori. Then there is an $\mathbb{S}^1$-diffeomorphism $\Phi : M \to M$ such that $\Phi^*\alpha_1$ and α_0 represent the same contact structure on M.*

Proof. If necessary multiply $\alpha_1(Z_M)$ by -1 to make sure that it has the same sign as $\alpha_0(Z_M)$. This assures that the sign of $\alpha_0(Z_M)$ and $\alpha_1(Z_M)$ agrees everywhere. By Lemma IV.4, Lemma IV.5 and Lemma IV.8 we find an $\mathbb{S}^1$-diffeomorphism φ that is the identity outside a small neighborhood of $E \cup F \cup \Sigma$, and such that in a smaller neighborhood the 1-forms $\varphi^*\alpha_1$ and α_0 represent the same contact structure. We can thus assume without loss of generality that $\alpha_0 = \alpha_1$ on a small neighborhood U of $E \cup F \cup \Sigma$.

Now divide the form α_0 by the smooth $\mathbb{S}^1$-invariant function $\alpha_0(Z_M)$, and α_1 by $\alpha_1(Z_M)$ on $M - F \cup \Sigma$. The neighborhood of a principal orbit is $\mathbb{S}^1$-diffeomorphic to the set $\mathbb{S}^1 \times \mathbb{D}^2$ with coordinates (φ, x, y) and the natural $\mathbb{S}^1$-action on the first factor. The scaled contact forms α_0 and α_1 are in these charts of the form

$$\alpha_j = d\varphi + f_j(x,y)\,dx + g_j(x,y)\,dy\ .$$

All of the 1-forms in the linear interpolation $\alpha_t = \alpha_0 + t\,(\alpha_1 - \alpha_0)$ are positive $\mathbb{S}^1$-invariant contact forms, because α_t is given by

$$\alpha_t = d\varphi + f_t(x,y)\,dx + g_t(x,y)\,dy\ ,$$

with $f_t(x,y) := (1-t)\,f_0(x,y) + t\,f_1(x,y)$ and $g_t(x,y) := (1-t)\,g_0(x,y) + t\,g_1(x,y)$. Then the 3-form

$$\begin{aligned}\alpha_t \wedge d\alpha_t &= \left(\partial_x g_t(x,y) - \partial_y f_t(x,y)\right)\, d\varphi \wedge dx \wedge dy \\ &= (1-t)\,\alpha_0 \wedge d\alpha_0 + t\,\alpha_1 \wedge d\alpha_1\end{aligned}$$

is positive by the assumption that α_0 and α_1 are positive contact forms.

The flow of the vector field X_t defined in Lemma A.2 exists, because α_t is constant in a small neighborhood U of $E \cup F \cup \Sigma$, and X_t vanishes on U, so that the flow cannot "escape". This shows that both contact forms are equivalent. $\square$

Remark IV.3. The lemma can easily be generalized to a compact manifold with non-empty boundary if the orbits in ∂M are Legendrian for both contact forms.

Let (M, α) be a contact $\mathbb{S}^1$-manifold. If M has Legendrian orbits, then construct a graph Γ_M in the following way: To every component $M_j \subset M - \Sigma$ associate a vertex V_j, and attach to V_j the sign of $\alpha(Z_{M_j})$, the number of fixed point components, the invariants $(\alpha_{j,1}, \beta_{j,1}), \dots, (\alpha_{j,N_j}, \beta_{j,N_j})$ of the exceptional orbits lying in this component, and the genus of the orbit space of M_j. Connect two vertices V_j and V_k by an edge only if the corresponding components in M touch each other. The edge is labeled by the number of components in Σ

between M_j and M_k. Note that V_j can only be connected to V_k if they carry different signs (as a special case it is not possible to connect V_j to itself).

We call two such graphs Γ_1 and Γ_2 isomorphic, if there exists a bijective map Φ from the vertices $\{V_{1,1},\dots,V_{1,N}\}$ of Γ_1 to the ones of Γ_2 that respects all numbers associated to the vertices, and such that the edges and their labels are conserved. The sign of the vertices must either be equal for all pairs $V_{1,j}$ and $\Phi(V_{1,j})$ or always opposite.

Lemma IV.10. *Two 3-dimensional contact $\mathbb{S}^1$-manifolds (M_1,α_1) and (M_2,α_2) with Legendrian orbits are equivalent, if and only if the associated graphs Γ_{M_1} and Γ_{M_2} are isomorphic.*

Proof. If the two manifolds are equivalent, it is obvious that the graphs are isomorphic. The opposite implication of the proof is based on Baer's theorem [**Bae28**], the classification of 3-dimensional $\mathbb{S}^1$-manifolds, and Lemma IV.9. $\square$

4. Existence of a contact structure

The first results in this section will be a generalization of Theorem IV.1, where it was shown that a non-trivial principal $\mathbb{S}^1$-bundle allows a connection 1-form α that defines an invariant contact structure. For the definition of generalized connection forms, we refer to Section III.5. Below we will also give the proof of Theorem III.13, which was postponed in Chapter III. Probably the proof is a direct consequence of the theory of characteristic classes on orbifolds, but here we will only use "smooth" techniques.

Lemma IV.11. *Let M be a closed oriented 3-dimensional $\mathbb{S}^1$-manifold without fixed points. Any generic connection form A on M is a contact form in the neighborhood of the exceptional orbits.*

Proof. We can perturb any A in a neighborhood U_E of the exceptional orbits E to make it of contact type on U_E. If A is already of contact type, leave it unchanged, otherwise define

$$\widetilde{A} := \frac{1}{f}\,\left(A + \varepsilon\rho\,\alpha_K\right)\ ,$$

where α_K is an invariant contact form on U_E with $\alpha_K(Z_M) = 1$ (take the one defined in the proof of Lemma IV.5, and rescale it), ρ is an $\mathbb{S}^1$-invariant cut-off function around E with support in U_E. The function f is given by $f := A(Z_M) + \varepsilon\rho\,\alpha_K(Z_M) = 1 + \varepsilon\rho$, and $\varepsilon > 0$ is an arbitrarily small number.

Consider the 3-form

$$\begin{aligned}\widetilde{A}\wedge d\widetilde{A} &= \frac{1}{f^2}\,(A + \varepsilon\rho\,\alpha_K)\wedge d(A + \varepsilon\rho\,\alpha_K)\\ &= \frac{1}{f^2}\,\left(A\wedge dA + \varepsilon\rho\,\alpha_K\wedge dA + \varepsilon^2\rho^2\,\alpha_K\wedge d\alpha_K + \varepsilon\rho\,A\wedge d\alpha_K + \varepsilon\,A\wedge d\rho\wedge\alpha_K\right)\,.\end{aligned}$$

Note that $\widetilde{A}\wedge d\widetilde{A}$ gets arbitrarily close to $A\wedge dA$ when ε decreases. On the exceptional fiber the 3-form reduces to

$$\widetilde{A}\wedge d\widetilde{A} = \frac{\varepsilon}{(1+\varepsilon)^2}\,\left(\alpha_K\wedge dA + \varepsilon\,\alpha_K\wedge d\alpha_K + A\wedge d\alpha_K\right)\,.$$

This form vanishes on E at most for a single ε, but by making ε smaller, we can always assume that $\widetilde{A}$ is of contact type on E, and thus on a small neighborhood of E. $\square$

THEOREM III.13. Let M be a closed oriented 3-dimensional $\mathbb{S}^1$-manifold determined by the invariants

$$(g, f=0, s=0, e, (\alpha_1,\beta_1),\dots,(\alpha_N,\beta_N)),$$

i.e. M does not have any fixed points, but N exceptional orbits with Seifert invariants (α_j,β_j), and the Euler number is e. Let A be a generalized connection 1-form on M, and F the corresponding curvature form on B^*. Then

$$\int_{B^*} F = 2\pi \left(e + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \right) .$$

PROOF. By the lemma above, we can perturb A to make it of contact type around the exceptional orbits E. Since the difference between the integral of the original form and the perturbed form can be made arbitrarily small, it is no restriction to assume that A itself is of contact type around E.

According to Section III.4, the Euler number is computed by taking a small $\mathbb{S}^1$-invariant solid torus V in M containing only free orbits. Define $M^* := M - E$, $B^* := M^*/\mathbb{S}^1$ and $D := V/\mathbb{S}^1$, and choose a section $\sigma_1 : D \hookrightarrow V$ and another section $\sigma_2 : B^* - D \hookrightarrow M^* - V$ that agrees with the canonical sections around the exceptional orbits $E_1,\dots,E_N$. The Euler number is equal to the intersection number between the two sections on ∂D.

The curvature form F is exact over the domain of each of the sections σ_j, because from the equations $\pi\circ\sigma_j = \mathrm{id}$, we can deduce

$$F = \mathrm{id}^* F = (\pi\circ\sigma_j)^* F = \sigma_j^*\pi^* F = \sigma_j^* dA = d(\sigma_j^* A) .$$

By splitting B^* into D and $B^* - D$, the integral can be written as

$$\int_{B^*} F = \int_{B^*-D} F + \int_D F = \int_D d(\sigma_1^* A) + \int_{B^*-D} d(\sigma_2^* A) .$$

Now we would wish to apply Stoke's Theorem. Though $B^* - D$ has open ends at the exceptional orbits, it is easy to see that one can cut off small punctured disks C_j around E_j in B^*, and not change the integral by much. The reason is that the integral over this small disk is equal to the integral of $(2\pi)^{-1} A\wedge dA$ over the corresponding neighborhood of an exceptional orbit (compare Lemma III.12). But $A\wedge dA$ is bounded in M, and so the integral of this 3-form over C_j goes to zero if the size of the disk decreases. Thus we can write

$$\int_{B^*} F = \int_{\partial D} (\sigma_1^* A - \sigma_2^* A) + \sum_{j=1}^{N} \int_{\partial C_j} \sigma_2^* A + \varepsilon , \tag{1}$$

where each of the ∂C_j is the outer boundary of the punctured disk C_j, and ε is a rest term that becomes arbitrarily small as the radius of C_j goes to zero.

To evaluate the contribution of exceptional orbits in Equation (1), consider such an orbit E_j with orbit invariants (k,m). Note that by assumption, α is of contact type around E_j. We can apply Lemma IV.5 to find a neighborhood U of E_j that is $\mathbb{S}^1$-contactomorphic to $\mathbb{S}^1\times\mathbb{D}^2_\varepsilon$ with the action

$$e^{i\varphi}\cdot(e^{it}, z) = (e^{i(t+k\varphi)}, e^{im\varphi} z)$$

and with contact form

$$A = \frac{2}{2k - m|z|^2} \left(dt + \frac{1}{2}\Big(x\,dy - y\,dx \Big) \right) .$$

Note that a contactomorphism could change the scaling of the 1-form, but since A is a connection 1-form, the correct scaling is the one given above. The canonical section σ in such a neighborhood can be written as a map

$$\sigma : \mathbb{D}^2_\varepsilon - \{0\} \to U,\, re^{i\varphi} \mapsto (e^{i\lambda_1\varphi},\, re^{i\lambda_2\varphi}) \;,$$

with suitable integers λ_1 and λ_2. In Section III.1.2, we expressed the homology classes of the meridian μ and the longitude λ as linear combinations of the canonical section σ and an orbit. It is easy to invert the corresponding matrix, and one obtains

$$[\sigma] = \beta\,[\lambda] - (1 - m\beta)/k\,[\mu] \;,$$

where $C = (1 - m\beta)/k$ by the requirement that the determinant of the matrix should be one, hence $\lambda_1 = k$, and $\lambda_2 = -(1 - \beta m)/k$.

With the pull-back

$$\sigma^* A = \frac{2}{2k - mr^2}\,(\lambda_1 + \frac{\lambda_2 r^2}{2})\,d\varphi \;,$$

one can integrate

$$\int_{\mathbb{S}^1_r} \sigma^* A = \frac{4\pi}{2k - mr^2}\,(\lambda_1 + \frac{\lambda_2 r^2}{2}) \;,$$

over the circle of radius r, and one sees that the contribution around the exceptional orbit E_j goes to $2\pi\lambda_1/k$ as we send the radius of the punctured disk C_j to 0.

Remember that the orbit invariants (k, m) and the Seifert invariants $(\alpha.\beta)$ are related by $k = \alpha$, and β is the smallest positive number such that $m\beta \equiv 1 \mod k$. Then Equation (1) simplifies with the arguments given so far to

$$\int_{B^*} F = \int_{\partial D} (\sigma_1^*\alpha - \sigma_2^*\alpha) + 2\pi \sum_{j=1}^N \frac{\beta_j}{\alpha_j} \;.$$

To compute the first term, note that $\sigma_2(p)$ can be described as $\sigma_1(p) \cdot e^{i\gamma(p)}$ on ∂D. One can easily check in bundle coordinates that $\sigma_2^*\alpha = \sigma_1^*\alpha + d\gamma$. Hence, one gets that $\sigma_1^*\alpha - \sigma_2^*\alpha = -d\gamma$, and integration of this form gives $-2\pi \deg \gamma$, which is equal to $2\pi e$, because γ counts how often σ_2 rotates in comparison to σ_1 in the fiber direction, which is just the Euler number.

Finally we get the formula

$$\int_{B^*} F = 2\pi \left(e + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} \right) \;. \quad \square$$

With this result, we can reproduce Theorem IV.1 in the presence of exceptional orbits.

Corollary IV.12. *Let M be a closed oriented 3-dimensional $\mathbb{S}^1$-manifold with N exceptional orbits that have Seifert invariants (α_j, β_j) $j = 1, \ldots, N$, but assume there are no fixed points in M. There is an $\mathbb{S}^1$-invariant contact structure α on M without Legendrian orbits, if and only if the Euler number e is not equal to*

$$e_0 := -\sum_{j=1}^N \frac{\beta_j}{\alpha_j} \;.$$

*We call the number $e + e_0$ the **orbifold Euler number**.*

Definition. We call an $\mathbb{S}^1$-manifold manifold M with the contact structure α given in the corollary above a (3-dimensional) **generalized Boothby-Wang fibration**.

Proof. Notice that any contact form without Legendrian orbits can be rescaled such that it becomes a generalized connection form. If $e = e_0$, then by the theorem above

$$\int_M \alpha \wedge d\alpha = 0 \; ,$$

which is a contradiction to the contact condition.

To prove the opposite implication, use an arbitrary generalized connection A on M that is a positive contact form around the exceptional orbits (possible by Lemma IV.11). The differential dA is equal to the pull-back of the curvature 2-form F on B^*. According to Theorem III.13, the integral of F over B^* is given by the formula $2\pi\,(e + \sum \beta_j/\alpha_j)$. Choose a volume form Ω on B^* that agrees with F close to the open ends of B^* corresponding to the exceptional orbits, and such that $\int \Omega = \int F$. The existence of such a volume form is obvious, define for example

$$\Omega := \rho\, F + (1-\rho)\,\lambda\widetilde{\Omega} \; ,$$

with an arbitrary volume form $\widetilde{\Omega}$, a cut-off function ρ with support around E, and a suitable $\lambda \in \mathbb{R}^+$. The difference $F - \Omega$ is equal to zero around the exceptional orbits.

Convert B^* into a closed smooth manifold by gluing disks $D_1, \dots, D_N$ into the open ends corresponding to the exceptional orbits in M. The 2-form $\Omega - F$ can be extended to this compactification by setting it to 0 on the D_j, because the form vanishes around the ends of B^*. Since $\int(\Omega - F) = 0$, there is a 1-form β on the compactification of B^* such that $d\beta = \Omega - F$, but in general β does not need to vanish on the disks D_j that were glued in. Still, β is closed and hence exact on the D_j, so we can choose functions $f : D_j \to \mathbb{R}$ such that $df = \beta$. Finally set $\widetilde{\beta} = \beta - d(\rho f)$.

The 1-form $\alpha = A + \pi^*\widetilde{\beta}$ is a contact form with the desired properties. $\square$

Lemma IV.13. *There is a positive $\mathbb{S}^1$-invariant contact form α without Legendrian orbits on any compact oriented 3-dimensional $\mathbb{S}^1$-manifold M with non-empty boundary and without fixed points.*

Proof. Define on M a generalized connection form A such that $A \wedge dA > 0$ around the exceptional orbits E (Lemma IV.11). To convert A into a contact form on the whole of M, we will add the pull-back of a 1-form on B^*, similarly as in the proof of Corollary IV.12.

A volume form $\widetilde{\Omega}$ on $B^* := (M - E)/\mathbb{S}^1$, can be capped off at the open ends of B^* that correspond to the exceptional orbits by multiplying it with a cut-off function,

$$\Omega := (1-\rho)\,\widetilde{\Omega} \; .$$

Fill the open ends around E by gluing in disks $D_1, \dots, D_N$, and extend Ω to these disks by setting it there to 0. This form is exact, because $\partial M \neq \emptyset$, and thus there is a 1-form β such that $d\beta = \Omega$. Though $\Omega \equiv 0$ at the ends, this need not be true for β. Still, β is exact on the 2-disks that have been glued in. Hence there is a function f defined on the D_j with $df = \beta$, and so the 1-form $\beta - d(\rho f)$ vanishes at the open ends of B^*. The $\mathbb{S}^1$-invariant connection form

$$A + \pi^*\beta - \pi^* df$$

is of contact type on all of M, if one chooses the right sign for β. $\square$

Lemma IV.14. *There is a positive $\mathbb{S}^1$-invariant contact form α without Legendrian orbits on any* 3*-dimensional oriented compact $\mathbb{S}^1$-manifold M with fixed points.*

Proof. To construct such a form, it is possible to take a contact form around the fixed point set F and on the complement of a small neighborhood of F and use a partition of unity argument.

By the lemma above, it is possible to find a contact form α_R on $M - U_F$, where U_F is a small neighborhood of the fixed point set F. Assume that U_F is contained in another open set U_F' that is of the form explained in Section III.1.1, i.e. it is $\mathbb{S}^1$-diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2_\varepsilon$ with the action $e^{i\varphi}(e^{it}, z) = (e^{it}, e^{i\varphi} z)$.

On U_F' define a positive contact form by

$$K\,dt + \frac{1}{2}\Bigl(x\,dy - y\,dx\Bigr) = K\,dt + \frac{r^2}{2}\,d\varphi\ ,$$

where we used polar coordinates. This form is outside the fixed point set $F = \{(e^{it}, 0)\}$ equivalent to

$$\alpha_F := d\varphi + \frac{2K}{r^2}\,dt\ .$$

The contact form α_R is given on $U_F' - U_F$ by

$$\alpha_R = f(t,r)\,dt + d\varphi + g(t,r)\,dr\ ,$$

with smooth functions $f, g : U_F' - U_F \to \mathbb{R}$.

Let $\rho : U_F' \to [0,1]$ be an $\mathbb{S}^1$-invariant cut-off function that only depends on the r-coordinate, such that $\rho(p) \equiv 1$, when $p \in U_F$, and $\rho(p) \equiv 0$ for p close to the boundary of U_F'. Define the $\mathbb{S}^1$-invariant 1-form

$$\alpha = (1-\rho)\,\alpha_R + \rho\,\alpha_F = d\varphi + (1-\rho)\,(f\,dt + g\,dr) + \frac{2K}{r^2}\,\rho\,dt\ .$$

This is a positive contact form, because

$$\begin{aligned}\alpha \wedge d\alpha &= d\varphi \wedge d\bigl((1-\rho)\,\alpha_R + \rho\,\alpha_F\bigr)\\ &= (1-\rho)\,\alpha_R \wedge d\alpha_R + \rho\,\alpha_F \wedge d\alpha_F - d\varphi \wedge d\rho \wedge \alpha_R + d\varphi \wedge d\rho \wedge \alpha_F\\ &= (1-\rho)\,\omega_R + \rho\,\omega_F + \left(\frac{2K}{r^2} - f\right)\,\partial_r\rho\,d\varphi \wedge dr \wedge dt\ ,\end{aligned}$$

with $\omega_R := \alpha_R \wedge d\alpha_R$ and $\omega_F := \alpha_F \wedge d\alpha_F$. Note that the form $dt \wedge dr \wedge d\varphi$ is positive, and hence the last term is also positive if one chooses K large enough, because $\partial_r\rho$ is negative and f is bounded. □

The only case left to prove is that contact forms with some Legendrian orbits exist on any oriented $\mathbb{S}^1$-manifold. This is a consequence of the following Lemma.

Lemma IV.15. *Let M be a closed oriented* 3*-dimensional $\mathbb{S}^1$-manifold with orbit space B. Assume a collection of embedded disjoint loops $\gamma_1, \dots, \gamma_N$ is given on B that do not touch the boundary ∂B or any singular point.*

If it is possible to mark every component B_k of $B - \cup\gamma_j$ with a sign in such a way that at each loop a component marked with "+" touches a component marked with "−", then it is possible to find an $\mathbb{S}^1$-invariant contact form α on M whose set of Legendrian orbits Σ covers the curves γ_j.

PROOF. Let Σ be the set of points that cover $\cup\gamma_j$. To find an invariant contact form for which Σ is the set of Legendrian orbits, proceed like this: On each of the components M_k of $M - \Sigma$ there exists an invariant contact form α_k (according to Lemma IV.13 and IV.14) without Legendrian orbits. Assume that for the generator Z_M of the $\mathbb{S}^1$-action, the function $\alpha_k(Z_M)$ has the same sign as the one that has been attached to the corresponding component of B_k.

If M_k and $M_{k'}$ meet at a component of Σ, we need to connect α_k to $\alpha_{k'}$ by an $\mathbb{S}^1$-invariant contact form such that all the Legendrian orbits of the new form lie in Σ. The standard neighborhood (compare Lemma IV.8) of a component of Σ is given by $U := \mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$, where the circle acts on the first component:

$$\left(e^{i\varphi'}, (e^{i\varphi}, e^{i\vartheta}, s)\right) \mapsto \left(e^{i(\varphi+\varphi')}, e^{i\vartheta}, s\right) .$$

A possible $\mathbb{S}^1$-invariant contact form on U with Legendrian orbits in Σ can be defined by

$$\alpha_0 = s\, d\varphi + K\, d\vartheta ,$$

where $K > 0$ is a number that will be chosen below. The contact form α_k is defined on $\mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, -\varepsilon/2]$, and after rescaling we can assume that it is of the form

$$\alpha_k = -d\varphi + f_1(\vartheta, s)\, d\vartheta + g_1(\vartheta, s)\, ds \ :$$

$\alpha_{k'}$ is defined on $\mathbb{S}^1 \times \mathbb{S}^1 \times [\varepsilon/2, \varepsilon)$, and of the form

$$\alpha_{k'} = d\varphi + f_2(\vartheta, s)\, d\vartheta + g_2(\vartheta, s)\, ds .$$

Note that all of these contact forms are supposed to be positive, which is equivalent to requiring

$$\frac{\partial f_1}{\partial s} - \frac{\partial g_1}{\partial \vartheta} > 0 \quad \text{and} \quad \frac{\partial g_2}{\partial \vartheta} - \frac{\partial f_2}{\partial s} > 0 .$$

By rescaling α_0, we obtain on $U_+ := \mathbb{S}^1 \times \mathbb{S}^1 \times [\varepsilon/2, \varepsilon)$ the form

$$\alpha_0 = d\varphi + \frac{K}{s}\, d\vartheta ,$$

and by using a cut-off function $\rho(s)$, we can consider on U_+ the form

$$\widetilde{\alpha} := \rho\, \alpha_0 + (1-\rho)\, \alpha_{k'} = d\varphi + \frac{K\rho}{s}\, d\vartheta + (1-\rho) f_2\, d\vartheta + (1-\rho) g_2\, ds ,$$

which connects to α_0 on one end and to $\alpha_{k'}$ on the other one. The contact condition for $\widetilde{\alpha}$ is given by

$$\widetilde{\alpha} \wedge d\widetilde{\alpha} = \left(\rho\, \frac{K}{s^2} + (1-\rho)\, (\partial_\vartheta g_2 - \partial_s f_2) - \frac{K}{s}\frac{\partial \rho}{\partial s} + f_2\, \frac{\partial \rho}{\partial s}\right) d\vartheta \wedge ds \wedge d\varphi > 0 .$$

This inequality holds if we choose K large enough, because the sum of the first two terms in the bracket is always positive, and $|f_2|$ is bounded such that we can assure that $K - f_2 > 0$.

The proof for $\mathbb{S}^1 \times \mathbb{S}^1 \times (-\varepsilon, 0]$ is completely analogous. □

The classification can be summarized in the following theorem.

THEOREM IV.16. *Let M be an oriented $\mathbb{S}^1$-manifold of dimension 3 determined by the following numbers*

$$(g, f, s = 0, e, (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N))$$

(as defined in Chapter III). If M has fixed points ($f \neq 0$), or if

$$e \neq -\sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \,,$$

then there is exactly one positive invariant contact structure without Legendrian orbits. If there are no fixed points, and if the Euler number is equal to the term above, then M does not carry invariant contact forms without Legendrian orbits.

For every isomorphism class of graphs Γ_M (as described at the end of Section 3) compatible with M as $\mathbb{S}^1$-manifold, there exists a single positive invariant contact structure, which gives back the graph Γ_M.

5. Overtwisted and fillable $\mathbb{S}^1$-invariant contact structures

In this section we will describe for many $\mathbb{S}^1$-invariant contact structures properties that are also interesting outside the realm of group actions.

LEMMA IV.17. *An $\mathbb{S}^1$-principal bundle P with a Boothby-Wang contact form α has a natural convex filling.*

PROOF. Consider the (complex) line bundle L associated to P, i.e. the bundle obtained from $P \times \mathbb{C}$ by identifying (p, z) with $(pe^{-i\varphi}, e^{i\varphi} z)$ for every $e^{i\varphi} \in \mathbb{S}^1$. The $\mathbb{S}^1$-principal bundle embeds naturally via

$$P \hookrightarrow L, \quad p \mapsto [p, 1] \,.$$

The two forms

$$\frac{1}{2}\left(|z|^2\,\alpha + x\,dy - y\,dx\right) \qquad \text{and} \qquad \frac{1}{2}\,d\alpha$$

on $P \times \mathbb{C}$ induce well-defined forms on L. By adding the differential of the first form to the second one, we obtain a symplectic form

$$\omega := \frac{1}{2}\,d(|z|^2) \wedge \alpha + dx \wedge dy + \frac{1 + |z|^2}{2}\,d\alpha$$

on L, because $2^n\,\omega^n = n\,(1 + |z|^2)^{n-1}\,(d\alpha)^{n-1} \wedge \left(d|z|^2 \wedge \alpha + 2dx \wedge dy\right)$ has only a one-dimensional kernel on $P \times \mathbb{C}$ generated by $-Z_P + x\,\partial_y - y\,\partial_x$.

The following field

$$X := \frac{1 + r^2}{2r}\,\partial_r = \frac{1 + x^2 + y^2}{2\,(x^2 + y^2)}\,(x\,\partial_x + y\,\partial_y)$$

is a Liouville vector field for the manifold (P, α). Hence (L, ω) is a convex filling of P. □

The corollary below is a direct consequence of the statement above, because fillability implies tightness as shown in [**Eli90**], [**Gro85**], and [**Zeh03**].

COROLLARY IV.18. *An 3-dimensional Boothby-Wang fibration (M, α) is tight.*

LEMMA IV.19. *A 3-dimensional closed contact $\mathbb{S}^1$-manifold (M, α) with fixed points and Legendrian orbits is overtwisted.*

Proof. The fixed points project to points on the boundary ∂B of the orbit space, and the tori of Legendrian orbits project onto embedded loops $\gamma_1, \dots, \gamma_N$ in the interior of B. Choose a point $p_1 \in \partial B$, and connect it with a smooth embedded path γ to a point $p_2 \in \gamma_j$ on a Legendrian orbit γ in such a way that γ runs only through points that correspond to regular non-Legendrian orbits (with exception of the starting and end point).

Away from the starting and the end point, the contact form can be considered a connection form of a principal bundle, and this allows us to lift γ to a path that lies in the kernel of α. Close to the end points the path upstairs in M is chosen in such a way that it connects the lifted path smoothly to the fixed point set on one side and to the Legendrian orbits on the other side. In both parts it should be tangential to the contact structure.

The union of all orbits over this curve gives an embedded disk that is overtwisted. ☐

The following lemma shows that overtwisted contact structures are the typical ones (at least if we require $\mathbb{S}^1$-invariance) in the sense that given any invariant contact structure, it is always possible to modify it in a small $\mathbb{S}^1$-neighborhood of a point to obtain the situation described below.

Lemma IV.20. *Let (M, α) be 3-dimensional closed contact $\mathbb{S}^1$-manifold over the orbit space B. Let Γ be the set of embedded circles in B covered by Legendrian orbits in M. If there is a disk D in B bounded by a circle $\gamma_1 \in \Gamma$, such that D is covered only by points with trivial stabilizer, and if D contains a second circle $\gamma_2 \in \Gamma$, then (M, α) is overtwisted.*

Proof. By Remark IV.3 the subset $(\pi^{-1}(D), \alpha)$ is $\mathbb{S}^1$-contactomorphic to $\mathbb{D}^2 \times \mathbb{S}^1 = \{(z, e^{i\varphi})\}$ with contact form

$$\alpha = \cos \frac{3\pi\,|z|}{2}\, d\varphi + \sin \frac{3\pi\,|z|}{2}\, \frac{x\,dy - y\,dx}{|z|}\ .$$

It is easy to check that the set $\{(z, 0) | |z| \leq 2/3\} \subset \mathbb{D}^2 \times \mathbb{S}^1$ is an overtwisted disk. ☐

The lemma has given rise to a construction called **Lutz twist** ([**Geiar**]), which allows to modify a contact form α in a Darboux chart to make α overtwisted.

6. Examples

6.1. Brieskorn manifolds. The Brieskorn manifolds W_k^{2n-1} with the natural $\mathrm{SO}(n)$-action were defined in Section III.6.1.2. In this section we will regard two different invariant contact structures on each of these manifolds.

Lemma IV.21. *The 1-forms*

$$\alpha_+ := k\,(x_0\,dy_0 - y_0\,dx_0) + 2\sum_{j=1}^{n} (x_j\,dy_j - y_j\,dx_j)$$

and

$$\alpha_- := -(k+1)\,(x_0\,dy_0 - y_0\,dx_0) + 2\sum_{j=1}^{n} (x_j\,dy_j - y_j\,dx_j)$$

induce $\mathrm{SO}(n)$-invariant contact structures on W_k^{2n-1}.

Proof. Consider the invariant 1-form

$$\alpha_\lambda := \lambda\,(x_0\,dy_0 - y_0\,dx_0) + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)\ ,$$

and let $f = z_0^k + z_1^2 + \ldots + z_n^2$ be the defining polynomial of W_k^{2n-1}, and $r^2 = \|(z_0,\mathbf{z})\|^2$. The contact condition for a 1-form β on W_k^{2n-1} is equivalent to

$$df \wedge d\bar{f} \wedge d(r^2) \wedge \beta \wedge (d\beta)^{n-1} \neq 0$$

at any point of W_k^{2n-1}. To compute this term note that all differentials in $(d\beta)^{n-1}$ appear in pairs $dx_j \wedge dy_j$ such that any differential that does not come with its corresponding pair in $df \wedge d\bar{f} \wedge d(r^2) \wedge \beta$, vanishes in the end. For $\beta = \alpha_\lambda$, the contact condition is

$$g_\lambda(z_0, z_1, \ldots, z_n) \neq 0$$

with the function

$$g_\lambda(z_0, z_1, \ldots, z_n) = k^2|z_0|^{2(k-1)}|z_i|^2 - \frac{k(\lambda+2)}{2}\,(z_0^k\bar{z}_i^2 + \bar{z}_0^k z_i^2) + 2\lambda\,\Big((|z_0|^2 + |z_i|^2)|z_j|^2 - z_i^2\bar{z}_j^2\Big)\ .$$

By using the equation $r^2 = 2$ and $f = 0$, one can reduce this function to

$$g_\lambda(z_0) = (k(\lambda+2) - k^2 - 2\lambda)\,|z_0|^{2k} + 2k^2\,|z_0|^{2(k-1)} - 4\lambda\,|z_0|^2 + 8\lambda\ .$$

It is easy to check that all α_λ with $\lambda \geq k$ or $\lambda < -k$ satisfy the contact condition. All forms α_λ with $\lambda \geq k$ are equivalent to the canonical contact structure α_+ which was given in [**LM76**]. The forms α_λ for $\lambda < -k$ are all equivalent to α_-. There is always an $L \in [-k, k)$, where the contact condition breaks down for α_L, such that α_+ and α_- need not be equivalent. □

Lemma IV.22. *The set of points in* $(W_k^{2k-1}, \alpha_\pm)$ *which lie on Legendrian orbits is equal to*

$$\{(z_0,\mathbf{z})|\,|z_0| = 1\}\ .$$

Proof. The tangent space $T_p\,\mathrm{Orb}(p)$ of the orbit through $p = (z_0, \mathbf{z})$ is spanned by the infinitesimal generators $X_{W_k^{2n-1}}(p)$ for all $X \in \mathfrak{so}(n)$, which are given by

$$X_{W_k^{2n-1}}(z_0,\mathbf{z}) = (0, X\cdot\mathbf{z})\ .$$

Then one gets $\alpha_\pm(X_{W_k^{2n-1}}) = 4\mathbf{x}^t \cdot X \cdot \mathbf{y}$. The point $(z_0, \mathbf{z})$ lies on a Legendrian orbit, if and only if $\mathbf{x}^t \cdot X \cdot \mathbf{y} = 0$ for all $X \in \mathfrak{so}(n)$. It is easy to check that $\mathbf{x}$ and $\mathbf{y}$ have to be linearly dependent.

In Section III.6.1.2 we already saw that then $|z_0| = 1$, and it follows that the Legendrian orbits are equal to the singular ones, if the dimension $2n-1$ is at least 5. □

Lemma IV.23. *The contact* $\mathbb{S}^1$*-manifolds* $(W_k^3, \alpha_\pm)$ *are all non-equivalent. For a given* k*, the contact form* α_+ *and* α_- *induce opposite orientations. By Lemma III.15,* $(W_k^3, \alpha_\pm)$ *is a principal* $\mathbb{S}^1$*-bundle over* $\mathbb{S}^2$ *with Euler class* $\pm k$ *(the sign of the Euler number depends on the orientation of the manifold; see also Remark IV.1 and IV.4 below). There is a single component of Legendrian orbits* Σ *covering the equator of the base space* $\mathbb{S}^2$*.*

Proof. Most of the claims were already shown above. That α_+ and α_- induce opposite orientations can be easily checked by evaluating the function $g_\lambda(z_0, z_1, z_2)$ defined in the proof of Lemma IV.21 at the point $(z_0, z_1, z_2) = (0, i, 1)$. □

REMARK IV.4. It is very important to measure the $\mathbb{S}^1$-invariants with respect to the orientation induced by the contact form α. If one forgets about the positivity assumption in Theorem IV.16, all invariants of the manifolds (W_1^3, α_+) and (W_1^3, α_-) would be equal, and one could be tricked into believing that both contact manifolds are equivalent. But in fact the two examples are not even contactomorphic, because we have

$$(W_1^3, \alpha_+) \cong (\mathbb{S}^3, \alpha_{\text{tight}}) .$$

This can be seen by finding a filling for the Brieskorn manifold (e.g. the hypersurface $V_f = f^{-1}(0)$ with $f(z_0, z_1, z_2) = z_0 + z_1^2 + z_2^2$ can be desingularized around 0 to give a filling). The contact manifold (W_1^3, α_-) is isomorphic to $(\mathbb{S}^3, \alpha_{\text{OT}})$, where α_{OT} is an overtwisted contact structure. This can be seen by finding an overtwisted disk in the following way:

The manifold W_1^3 is diffeomorphic to the 3-sphere $\mathbb{S}^3$, and can be decomposed into two solid tori V_+ and V_-. This decomposition can be achieved by using the sections $\sigma_\pm$ defined in Section III.6.1.2 to construct diffeomorphisms

$$\Phi_\pm : \mathbb{S}^1 \times \mathbb{D}^2 \to W_1^3,\ (e^{i\varphi}, z) \mapsto e^{i\varphi} \cdot \sigma_\pm(z) .$$

The intersection of the two solid tori V_+ and V_- is the set of Legendrian orbits. The pull-back of the contact form α_- to any of the $V_\pm$ gives

$$\Phi_\pm^* \alpha_- = \pm \frac{A^4 - x^2 - y^2}{A^2}\, d\varphi + \frac{1 - 2A^2}{A^2}\, (x\, dy - y\, dx) ,$$

with $A = \sqrt{2 - r_0^2 + \sqrt{(2 - r_0^2)^2 - r_0^{2k}}}$. We will stretch out a disk that spans through V_+, and has a collar lying in V_-. The standard section in V_+ is a disk $D_+ = \{(1, z)\} \subset V_+$. The induced foliation on D_+ consists of radial rays starting at $z = 0$. The center $z = 0$ is the only singular point, and hence this is not an overtwisted disk, but we will extend D_+ into the torus V_- by attaching an annulus, and consider the foliation there.

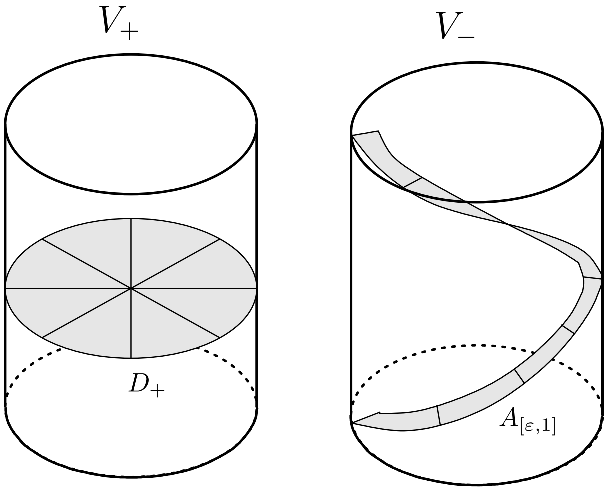


FIGURE 3. W_1^3 decomposes into two solid tori V_+ and V_-. By taking a section in V_+ and extending it far enough into V_-, we obtain an overtwisted disc.

The gluing map between the two solid tori is given by the equation

$$\Phi_-(e^{i\varphi}, e^{i\vartheta}) = e^{-i\vartheta} \cdot \Phi_+(e^{i\varphi}, e^{i\vartheta})$$

on the boundary of the tori. This can also be written in the form $\Phi_-(e^{i\vartheta}, e^{i\vartheta}) = \Phi_+(1, e^{i\vartheta})$. We will consider the embedding of the annulus $A_{[\varepsilon,1]}$ (which extends the disk D_+) given by

$$A_{[\varepsilon,1]} \hookrightarrow V_-,\ re^{i\vartheta} \mapsto (e^{i\vartheta}, re^{i\vartheta}) \ .$$

The contact form pulled-back to this annulus gives

$$\left(A^2 - 2r^2\right)\, d\vartheta \ .$$

The foliation still runs radially, but there is a circle of singularities on this annulus, because the coefficient in front of $d\vartheta$ is negative for $r = 1$ and positive for small $r > 0$.

Note that it is easy see that all manifolds $(W_k^3, \alpha_\pm)$ decompose into two solid tori with a boundary of Legendrian orbits, and that the proof in Section 3 can be modified to see that all of these solid tori are $\mathbb{S}^1$-contactomorphic to V_+ and V_- given above. Hence the distinction of these contact $\mathbb{S}^1$-manifolds is purely given by the gluing map on the boundary of the tori.

In [**Lut77**] it was proved that every contact structure on the 3-sphere can be represented by an $\mathbb{S}^1$-invariant 1-form.

Remark IV.5. The Milnor fibration together with the SO(2)-action described here convert the manifolds $(W_k^3, \alpha_\pm)$ into toric contact manifolds. This will be explained in more depth in Appendix B. From the discussion there it will be obvious that all (W_k^3, α_-) are overtwisted.

CHAPTER V

The cross-section

Let M be a G-manifold.

DEFINITION. A G-equivariant map $\mu : M \to \mathfrak{g}^*$ is called an **(abstract) moment map**, if for every Lie subgroup $\iota : H \hookrightarrow G$ the map $\mu \circ \iota : M \to \mathfrak{h}^*$ is constant on the components of the fixed point set of H (see [**GGK02**]). If there is such a moment map on M, the G-action is called **Hamiltonian**.

As mentioned in Section II.2, at any point $p \in M$ of a G-manifold M there is a submanifold S_p called a slice. Usually there is a lot of freedom in choosing such a slice, but for the coadjoint action on $\mathfrak{g}^*$, there exists a unique maximal slice at any $\nu \in \mathfrak{g}^*$, which will be denoted by S^*_ν (see [**DK00**]). The maximal slice at a generic point is equal to the (dual of a) Weyl chamber.

EXAMPLE V.1. Consider the SO(3)-structure of $\mathfrak{so}(3)^*$ given by the coadjoint action. The principal orbits are 2-spheres lying concentrically around 0, and $\{0\}$ is the only singular orbit in $\mathfrak{so}(3)^*$. The maximal slice of an element $\nu \in \mathfrak{so}(3)^*$ ($\nu \neq 0$) is $\mathbb{R}^+ \cdot \nu$ and the maximal slice at 0 is the whole of $\mathfrak{so}(3)^*$.

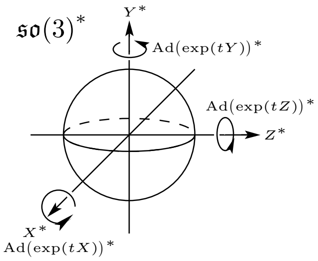


FIGURE 1. The coadjoint action on $\mathfrak{so}(3)^*$ is isomorphic to the standard rotations on $\mathbb{R}^3$.

DEFINITION. For a Hamiltonian G-manifold M with moment map $\mu : M \to \mathfrak{g}^*$, the **cross-section** R **at a point** $\nu \in \mu(M)$ is defined as

$$R := \mu^{-1}(S^*_\nu) \ .$$

The cross-section R is called **principal cross-section**, if it contains no smaller cross-section.

LEMMA V.1. *The cross-section at* ν *is a submanifold with a Hamiltonian* G_ν*-action, where* $G_\nu := \mathrm{Stab}(\nu)$

PROOF. The moment map μ is G-equivariant, and the G-orbit at $\tilde{\nu} \in S^*_\nu$ is transverse to the slice S^*_ν, hence

$$\mu_* T_q M \supseteq \mu_* T_q \operatorname{Orb}(q) = T_{\tilde{\nu}} \operatorname{Orb}(\tilde{\nu}) \pitchfork S^*_\nu$$

for any $q \in \mu^{-1}(\tilde{\nu})$, which shows that μ is transverse to the slice S^*_ν, and the cross-section R is a submanifold.

The G_ν-action on R is just the restriction of the G-action on M, and the moment map μ_R on R is the restriction of the moment μ_M on M, i.e.

$$\mu_R(r) := \mu_M(r) \circ \iota$$

for the natural inclusion $\iota : G_\nu \hookrightarrow G$ and all $r \in R$. □

Remark V.1. Note that the action of G_ν on the cross-section is in general not effective (even if the G-action on M was).

Lemma V.2. *Let M be a G-manifold with moment map $\mu : M \to \mathfrak{g}^*$. Choose a point $\nu \in \mu(M)$, and denote the cross-section at ν by $R := \mu^{-1}(S^*_\nu)$. The product $G \times R$ is a G_ν-bundle over the base space $G \times_{G_\nu} R := \big(G \times R\big)/G_\nu$ with the G_ν-action given by*

$$(h, (g, r)) \mapsto (gh^{-1}, hr)$$

for all $g \in G$, $r \in R$ and $h \in G_\nu$. The following diagram induces a G-equivariant diffeomorphism between $G \times_{G_\nu} R$ and the flow-out $G \cdot R \subset M$.

$$\begin{tikzcd}
G \times R \arrow[r, "\Phi"] \arrow[d, "\pi"'] & G \cdot R \\
G \times_{G_\nu} R \arrow[ur, dotted] &
\end{tikzcd}$$

The original moment map μ can be reconstructed from $\mu_R : R \to \mathfrak{g}^*_\nu$, because with the natural projection $\pi_\nu : \mathfrak{g} \to \mathfrak{g}_\nu$ the equation

$$\mu(\Phi(g, r)) = \mathrm{Ad}(g^{-1})^* \left(\pi^*_\nu \mu_R(r)\right)$$

holds.

Proof. That $G{\cdot}R$ is diffeomorphic to $G{\times_{G_\nu}}R$ has also already been stated in [**LMTW98**], but we will give again the main argument. If $gr = \tilde{g}\tilde{r}$ for some $r, \tilde{r} \in R$ and $g, \tilde{g} \in G$, then $\mu(r) = \mu(g^{-1}\tilde{g}\tilde{r}) = \mathrm{Ad}(\tilde{g}^{-1}g)^*\mu(\tilde{r})$. Since both $\mu(r)$ and $\mu(\tilde{r})$ lie in S^*_ν it follows from the definition of slice that $\tilde{g}^{-1}g \in G_\nu$, and as a consequence:

$$(g, r) \sim (gg^{-1}\tilde{g}, \tilde{g}^{-1}gr) = (\tilde{g}, \tilde{g}^{-1}\tilde{g}\tilde{r}) = (\tilde{g}, \tilde{r}) \ ,$$

as expected. It is not difficult to finish the proof that $G \times_{G_\nu} R \cong G \cdot R$ (for smoothness use that $G \times R$ is a G_ν-bundle, and has local trivializations).

The first step in the reconstruction of the original moment map $\mu : G \cdot R \to \mathfrak{g}^*$ consists in building $\mu : R \to \mathfrak{g}^*$ using $\mu_R : R \to \mathfrak{g}^*_\nu$. The extension from R to $G{\cdot}R$ is then achieved by the G-equivariance. According to Lemma C.2 in Appendix C, there is an embedding $\pi^*_\nu : \mathfrak{g}^*_\nu \hookrightarrow \mathfrak{g}^*$ and $\pi^*_\nu \mu_R = \mu|_R$, because $\mu(R) \subseteq S^*_\nu$. Together this gives the desired equation. □

Corollary V.3. *In the situation above, we can define a 1-form β on $G \times R$ that is invariant under the diagonal G_ν-action by setting*

$$\beta_{(g,r)}(X_g + \dot{r}) := \langle \mu_R(r) | \pi_\nu c_G X_g \rangle \ ,$$

where $(X_g, \dot{r})$ is a vector at $(g, r) \in G \times R$, and $c_G : TG \to \mathfrak{g}$ is the ***Cartan form*** *$c_G(X_g) = g^{-1}_* X_g$.*

Proof. For $h \in G_\nu$ define the map $\psi_h : G \times R \to G \times R$, $(g, r) \mapsto (gh^{-1}, hr)$. The invariance can be seen by the following easy computation (here $\mathcal{R}_g$ denotes right-translation in the Lie algebra)

$$\begin{aligned}\psi_h^* \beta(X_g + \dot{r}) &= \langle \mu_R(hr) | \pi_\nu c_G \mathcal{R}_{h^{-1}} X_g \rangle = \langle \mathrm{Ad}(h^{-1})^* \mu_R(r) | \pi_\nu (gh^{-1})_*^{-1} \mathcal{R}_{h^{-1}} X_g \rangle \\ &= \langle \mu_R(r) | \mathrm{Ad}(h^{-1})^* \pi_\nu h_* \mathcal{R}_{h^{-1}} g_*^{-1} X_g \rangle = \langle \mu_R(r) | \mathrm{Ad}(h^{-1})^* \pi_\nu \,\mathrm{Ad}(h) c_G X_g \rangle \\ &= \beta(X_g + \dot{r}) \; . \qquad \square\end{aligned}$$

In the context of contact and symplectic manifolds, moment maps occur naturally (in fact the notion of abstract moment map is of course a generalization of the symplectic moment map).

Definition. A **moment map** $\mu : M \to \mathfrak{g}^*$ of a contact G-manifold (M, α) is given by

$$\langle \mu(p) | X \rangle := \alpha_p(X_M) \; .$$

For a nowhere vanishing G-invariant function $f : M \to \mathbb{R}$, the two contact forms α and $f\,\alpha$ are equivalent. The corresponding moment maps are μ and $f\,\mu$, i.e. it is possible to rescale a contact moment map by any such function.

Definition. A **moment map** for a symplectic G-manifold (M, ω) is a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that for every $X \in \mathfrak{g}$ the definition $H_X(p) := \langle \mu(p) | X \rangle$ gives a Hamiltonian function of the vector field $X_M(p) := \frac{d}{dt} \exp(tX) \cdot p$, i.e.

$$\iota_{X_M} \omega = -dH_X$$

holds.

For a symplectic G-manifold no moment map needs to exist, but if it does, the moment map is unique up to the addition of elements $\nu \in \mathfrak{g}^*$ that remain invariant under the coadjoint action.

Lemma V.4. *Let (M, ω) (resp. (M, α)) be a symplectic (resp. contact) manifold with a Hamiltonian G-action. Let $\mu_M : M \to \mathfrak{g}^*$ be the moment map, and let R be the cross-section at an element $\nu \in \mu_M(M) \subset \mathfrak{g}^*$. The cross-section becomes in a canonical way a symplectic (contact) submanifold with a Hamiltonian G_ν-action and moment map $\mu_R := \iota^* \mu_M|_R$ (with the natural embedding $\iota : G_\nu \hookrightarrow G$).*

Proof. A proof for symplectic manifolds was given in [**LMTW98**] and one for contact manifolds can be found in [**Wil02**]. We reprove the statement anyway, because the argument used in [**Wil02**] is indirect and would not help in Lemma V.6.

(a) Let (M, α) be a contact manifold. The Reeb field X_{Reeb} is tangent to the cross-section R, because

$$\mathcal{L}_{X_{\mathrm{Reeb}}} \langle \mu | X \rangle = \mathcal{L}_{X_{\mathrm{Reeb}}} \alpha(X_M) = 0 \; .$$

One still needs to show that $V := (T_r R \cap \xi_r, d\alpha)$ is a symplectic vector space for all $r \in R$ (define $\xi_r = \ker \alpha_r$). For any two elements $X, Y \in \mathfrak{g}$, we have

$$d\alpha(X_M, Y_M) = \iota_{Y_M} \iota_{X_M} d\alpha = \iota_{Y_M} \mathcal{L}_{X_M} \alpha - \iota_{Y_M} d\langle \mu | X \rangle = -\langle \mu | [X, Y] \rangle \; .$$

Instead of proving that V itself is symplectic, one shows that the complement of V in ξ_r is symplectic.

It is useful to have a look at Appendix C to understand better the arguments used below. Choose an Ad-invariant metric $\mathfrak{m}$ on $\mathfrak{g}$ and denote the element dual to $\mu(r)$ by Z. Lie algebra and coalgebra have an orthogonal splitting

$$\mathfrak{g} = \operatorname{im}\operatorname{ad}(Z) \oplus \ker\operatorname{ad}(Z) \quad \text{ and } \mathfrak{g}^* = \operatorname{im}\operatorname{ad}(Z)^* \oplus \ker\operatorname{ad}(Z)^* .$$

The slice S^*_ν lies in $\ker\operatorname{ad}(Z)^*$, and hence in particular $\operatorname{ad}(Z)^*\mu(r) = 0$. Every vector $X_M(r)$ for an element $X \in \operatorname{im}\operatorname{ad}(Z)$ lies in ξ_r, because if $X := [Z, \widetilde{X}]$ for $\widetilde{X} \in \mathfrak{g}$, then

$$\alpha(X_M(r)) = \langle\mu(r)|X\rangle = \langle\mu(r)|\operatorname{ad}(Z)\widetilde{X}\rangle = \langle\operatorname{ad}(Z)^*\mu(r)|\widetilde{X}\rangle = 0 .$$

It follows that the set $\{X_M(r)|X \in \operatorname{ad}(Z)\mathfrak{g}\}$ is equal to the complement $V^\perp$ of V in ξ_r.

We want to show that $(V^\perp, d\alpha)$ is a symplectic vector space. This means that for every non-zero $X_M(r) = [Z, \widetilde{X}]_M(r)$, there is an element $Y_M(r) = [Z, \widetilde{Y}]_M(r)$ with $d\alpha(X_M, Y_M) = -\langle\mu(r)|[X, Y]\rangle \neq 0$. If this was not true, it would follow that $\operatorname{ad}(X)^*\mu(r)$ vanishes on $\operatorname{ad}(Z)\mathfrak{g}$, but in fact $\operatorname{ad}(X)^*\mu(r) = 0$ holds even on the whole Lie algebra $\mathfrak{g}$, because for any $W \in \ker\operatorname{ad}(Z)$ we have

$$\begin{aligned}\langle\mu(r)|[X, W]\rangle &= \langle\mu(r)|[[Z, \widetilde{X}], W]\rangle = \langle\mu(r)|[[Z, W], \widetilde{X}] + [Z, [\widetilde{X}, W]]\rangle \\ &= \langle\operatorname{ad}(Z)^*\mu(r)|[\widetilde{X}, W]\rangle = 0 .\end{aligned}$$

The elements $\exp(tX)$ lie in the stabilizer G_ν, because $\operatorname{ad}(X)^*\mu(r) = 0$ and hence $\operatorname{Ad}(\exp(tX))^*\mu(r) = \mu(r)$. By the definition of a slice it follows that $\exp(tX) \in G_\nu$ and $X \in \mathfrak{g}_\nu = \ker(\operatorname{ad}(Z))$, but since the sum $\mathfrak{g} = \operatorname{im}\operatorname{ad}(Z) \oplus \ker\operatorname{ad}(Z)$ is direct, we have $X = 0$.

(b) For symplectic manifolds the equation

$$\omega(X_M, Y_M) = -\iota_{Y_M} d\langle\mu|X\rangle = -\langle\mu|[X, Y]\rangle$$

holds. Thus the argument in (a) can be applied without any major modification. □

One of the reasons why the cross-section is important for symplectic and contact manifolds is that it allows to reconstruct its flow-out (including symplectic or contact structure), and that the flow-out lies open and dense in the original manifold.

LEMMA V.5. *Let R be the principal cross-section in a Hamiltonian G-manifold M with moment map $\mu : M \to \mathfrak{g}^*$. The set R is a connected, open and dense subset.*

PROOF. It is known that $\operatorname{codim} M_{(\mathrm{sing})} \geq 2$, hence it is enough to restrict to the set of regular orbits. For symplectic manifolds the required statement has been proved in [**LMTW98**, Lemma 3.11]. Note that $M_{(\mathrm{prin})}$ in that article denotes the set of regular orbits.

For a contact manifold M, on the other hand, a strategy like in [**Wil02**] can be applied: The symplectization $M \times \mathbb{R}$ is a Hamiltonian G-manifold with moment map $\widetilde{\mu}(p, t) := e^t \mu(p)$. Apply the Lemma to $M \times \mathbb{R}$, and restrict all cross-sections to the set $M \times \{0\}$ to get back to the contact case. □

REMARK V.2. One might wonder whether the principal cross-section always corresponds to a Weyl chamber, or put differently, whether the intersection of an (open) Weyl chamber with the image of the moment map is always non-empty. This is not the case, as the following example shows: The Brieskorn manifolds (W^{2n-1}_k, α_+) with their natural $\mathrm{SO}(n)$-action have principal stabilizer isomorphic to $\mathrm{SO}(n-1)$, which is not abelian for $n \geq 4$. It follows that the moment map does not map any point into the interior of a Weyl chamber. In particular $\mu(W_k^{2n-1})$ lies in a set of codimension 3.

For a symplectic or contact G-manifold it is possible to reconstruct the flow-out from the cross-section (including the corresponding forms). Intuitively this fact can be explained in the following way: If we know the corresponding structure on $TM|_R$ (not just on TR), we are done, because the G-action transports the structure to any point of the flow-out $G \cdot \mathbb{R}$. Now for the contact structure α, the G-orbits are transverse to the cross-section, at each $p \in R$, and we need to find the elements $X \in \mathfrak{g}$ for which $X_M(p)$ lies in the kernel of α and which point out of R. This is not too difficult, since according to Lemma V.2 the moment map $\mu : R \to \mathfrak{g}^*$ is known.

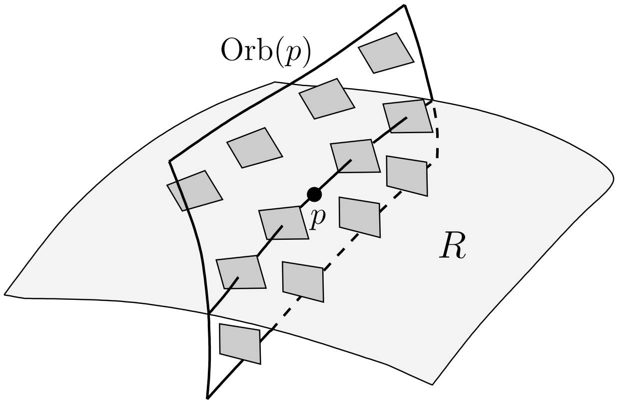


FIGURE 2. Knowing the contact planes in $TM|_R$ is enough because of G-invariance.

LEMMA V.6. *Let (M,ω) (resp. (M,α)) be a symplectic (resp. contact) G-manifold with moment map $\mu_M : M \to \mathfrak{g}^*$, and let R be the cross-section at $\nu \in \mathfrak{g}^*$. With the G-map $\Phi : G \times R \to M$ defined in Lemma V.2 and the 1-form β given in Corollary V.3, the following statements hold:*

If (M,α) is a contact manifold, then $\Phi^*\alpha = \alpha_R + \beta$. Put differently, $\alpha_R + \beta$ induces a G-invariant contact form on $G \times_{G_\nu} R$ that is contactomorphic to the one on $G \cdot R \subseteq M$.

If (M,ω) is a symplectic manifold, then the 2-form $\omega_R + d\beta$ on $G \times R$ induces a G-invariant symplectic form on $G \times_{G_\nu} R$ that is symplectomorphic to the one on $G \cdot R \subseteq M$.

PROOF. To prove the statement, we first need to compute the differential Φ_* of the map $\Phi : G \times R \to M$, $(g,r) \mapsto g \cdot r$. The differential is computed by

$$\Phi_*(X_g + \dot{r}) = \frac{d}{dt}\Phi(g \cdot \exp(tX), r(t)) = \frac{d}{dt}\Big(g \cdot \exp(tX) \cdot r(t)\Big) ,$$

where $X_g \in T_gG$ and $r(t)$ is a path in R, such that $r(0) = r$ and $r'(0) = \dot{r}$. Since Φ_* is linear and by using $\Phi_*(X_g) = g_* X_M(r)$ and $\Phi_*(\dot{r}) = g_*\dot{r}$, one obtains

$$\Phi_*(X_g + \dot{r}) = g_* X_M(r) + g_*\dot{r} .$$

Consider first the case of a contact G-manifold: We need to show that $\Phi^*\alpha = \alpha_R + \beta$. The pull-back of α is

$$(\Phi^*\alpha)(X_g + \dot{r}) = \alpha(g_*\dot{r}) + \alpha(g_* X_M(r)) = \alpha(\dot{r}) + \langle \mu_M(r) | X \rangle .$$

On the other hand, if $X_g + \dot r$ is plugged into $\alpha_R + \beta$ one obtains

$$\alpha_R(X_g + \dot r) + \beta(X_g + \dot r) = \alpha_R(\dot r) + \langle \pi_\nu^* \mu_R | c_G(X_g)\rangle = \alpha_R(\dot r) + \langle \pi_\nu^*\mu_R | X\rangle \ .$$

According to Lemma V.2 the map $\pi_\nu^*\mu_R$ is equal to the restriction of μ_M to R. Since $\dot r$ is tangent to R and α_R is the restriction of α to R it follows that

$$\alpha_R(\dot r) + \langle \pi_\nu^*\mu_R | X\rangle = \alpha(\dot r) + \langle \mu_M(r) | X\rangle \ ,$$

and $\Phi^*\alpha = \alpha_R + \beta$, as we wanted to show.

Let now (M, ω) be a symplectic manifold. We need to show that $\Phi^*\omega = \omega_R + d\beta$. The right-hand side can be expanded (we use that $\pi_\nu^*\mu_R = \mu_M|_R$) to

$$\begin{aligned}(\omega_R + d\beta)(X_g + \dot r, Y_g + \dot s) &= \omega_R(\dot r, \dot s) - \mathcal{L}_{(Y_g+\dot s)}\beta(X_g) + \mathcal{L}_{(X_g+\dot r)}\beta(Y_g)\\ &\quad - \beta\big([X_g + \dot r, Y_g + \dot s]\big)\\ &= \omega_R(\dot r, \dot s) - \mathcal{L}_{(Y_g+\dot s)}\langle \mu_M(r)|X\rangle + \mathcal{L}_{(X_g+\dot r)}\langle \mu_M(r)|Y\rangle\\ &\quad - \beta\big([X_g, Y_g] + [\dot r, \dot s]\big)\\ &= \omega_R(\dot r, \dot s) - \mathcal{L}_{\dot s}\langle \mu_M(r)|X\rangle + \mathcal{L}_{\dot r}\langle \mu_M(r)|Y\rangle - \langle \mu_M(r)|[X,Y]\rangle \ .\end{aligned}$$

For the pull-back of ω one obtains

$$\begin{aligned}(\Phi^*\omega)(X_g + \dot r, Y_g + \dot s) &= \omega(g_*X_M(r) + g_*\dot r,\ g_*Y_M(r) + g_*\dot s)\\ &= \omega(\dot r, \dot s) - \mathcal{L}_{\dot s}\langle \mu_M(r)|X\rangle + \mathcal{L}_{\dot r}\langle \mu_M(r)|Y\rangle + \omega(X_M(r), Y_M(r)).\end{aligned}$$

The last term above is equal to

$$\begin{aligned}\omega(X_M, Y_M) = -\mathcal{L}_{Y_M}\langle \mu_M(r)|X\rangle &= -\frac{d}{dt}\langle \mu_M(\exp(tY)\cdot r)|X\rangle\\ &= -\frac{d}{dt}\langle \mu_M(r)|\mathrm{Ad}(\exp(-tY))X\rangle = -\langle \mu_M(r)|[X,Y]\rangle\end{aligned}$$

and so

$$\begin{aligned}(\Phi^*\omega)(X_g + \dot r, Y_g + \dot s) &= \omega(\dot r, \dot s) - \mathcal{L}_{\dot s}\langle \mu_M(r)|X\rangle\\ &\quad + \mathcal{L}_{\dot r}\langle \mu_M(r)|Y\rangle - \langle \mu_M(r)|[X,Y]\rangle \ .\end{aligned}$$

This shows that the equality $\Phi^*\omega = \omega_R + d\beta$ does indeed hold. □

One can find a symplectic version of the following theorem in [**LMTW98**], the contact version has been described in [**Wil02**].

THEOREM V.7 (cross-section theorem). *Let (M, α) be a contact G-manifold with moment map $\mu_M : M \to \mathfrak{g}^*$. Let $\nu \in \mathfrak{g}^*$ be an element in the image of the moment map, and let $S_\nu^* \subseteq \mathfrak{g}^*$ be the unique maximal slice at ν.*

Then:

1. *The* ***cross-section*** *$R := \mu_M^{-1}(S_\nu^*)$ is a contact G_ν-submanifold of M, where $G_\nu := \mathrm{Stab}(\nu)$.*
2. *The G-action induces a G-diffeomorphism between the flow-out $G \cdot R \subseteq M$ and $G \times_{G_\nu} R$. The contact form α on the flow-out can be reconstructed from the cross-section and the embedding $\iota : G_\nu \hookrightarrow G$.*

REMARK V.3. The theorem uses the embedding $G_\nu \hookrightarrow G$. If one considers a cross-section R as an abstract H-manifold with $H \cong G_\nu$ and one embeds H in two different ways into G ($\iota_1, \iota_2 : H \hookrightarrow G$), then in general $G \times_{\iota_1 H} R \not\cong G \times_{\iota_2 H} R$. In the case of SO(3)-manifolds, however, the embedding of $\mathbb{S}^1$ into SO(3) is unique up to conjugation, and no problem will arise at this point.

CHAPTER VI

4-dimensional symplectic $\mathrm{SO}(3)$- and $\mathrm{SU}(2)$-manifolds

The classification of closed symplectic 4-manifolds with a Hamiltonian SO(3)-action was given by Iglesias in [**Igl91**]. Later Audin published the corresponding classification result for Hamiltonian SU(2)-actions in [**Aud04**]. In this chapter we will reprove these results using the cross-section theorem. From now on let G denote either SO(3) or SU(2), and let (M, ω) be a 4-dimensional symplectic G-manifold with moment map $\mu : M \to \mathfrak{g}^*$.

Example VI.1. (1) The manifold $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ (where ω_{FB} is the symplectic form associated to the **Fubini-Study metric** [**MS95**]) can be given a Hamiltonian SO(3)-action induced by the standard representation on $\mathbb{R}^3$, which has to be complexified to obtain an action on $\mathbb{C}^3$. The moment map is given by

$$\langle \mu([x_1 + iy_1 : x_2 + iy_2 : x_3 + iy_3]) | X \rangle := \frac{\mathbf{x}^t X \mathbf{y}}{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2} \ ,$$

with $X \in \mathfrak{so}(3)$, $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The point $[1 : 0 : 0]$ has stabilizer isomorphic to O(2), and $\mathrm{Orb}([1 : 0 : 0])$ is an an embedded $\mathbb{RP}^2$. The only other singular orbit is given by $\mathrm{Orb}([1 : i : 0]) \cong \mathbb{S}^2$, with stabilizer isomorphic to $\mathbb{S}^1$. All other points have stabilizer isomorphic to $\mathbb{Z}_2$.

The image of the moment map is a 3-ball. The preimage of every concentric 2-sphere in $\mu(\mathbb{CP}^2)$ is a single orbit. On one end we have the origin whose preimage is $\mathrm{Orb}([1 : 0 : 0])$, and on the other extreme there is the boundary of the ball with preimage $\mathrm{Orb}([1 : i : 0])$.

(2) The manifold $(\mathbb{S}^2 \times \mathbb{S}^2, \lambda_1 \, \omega_{\mathrm{std}} \oplus \lambda_2 \, \omega_{\mathrm{std}})$ is a symplectic manifold (ω_{std} is the SO(3)-invariant volume form on the sphere) with diagonal SO(3)-action, i.e. $(g, (p_1, p_2)) \mapsto (gp_1, gp_2)$. The stabilizer of a point (p_1, p_2) is equal to the intersection $\mathrm{Stab}(p_1) \cap \mathrm{Stab}(p_2)$. Hence the only singular orbits are $\{(p, p) | p \in \mathbb{S}^2\}$ and $\{(p, -p) | p \in \mathbb{S}^2\}$. The principal stabilizer is trivial.

The moment map is given by

$$\begin{aligned} \langle \mu(x_1, y_1, z_1; x_2, y_2, z_2) | X \rangle = \lambda_1 \left(x_1 \, X^* + y_1 \, Y^* + z_1 \, Z^* \right) \\ + \lambda_2 \left(x_2 \, X^* + y_2 \, Y^* + z_2 \, Z^* \right) \end{aligned}$$

(the 2-sphere is equal to a coadjoint SO(3)-orbit and there the moment map is the identity). The image of the moment map is a spherical shell with inner radius $||\lambda_1| - |\lambda_2||$ and outer radius equal to $|\lambda_1| + |\lambda_2|$. The extremal values are taken again on the singular orbits. If $|\lambda_1| = |\lambda_2|$, the image of the moment map is a 3-ball.

(3) The manifold $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ also admits a Hamiltonian SU(2)-action. We let SU(2) act by embedding it in the standard way into the upper left part of SU(3).

The point $[0 : 0 : 1]$ is a discrete fixed point, and the only other singular orbit is $\mathrm{Orb}([1 : 0 : 0]) \cong \mathbb{S}^2$, which has stabilizer isomorphic to $\mathbb{S}^1$.

The image of the moment map is a 3-ball, with the fixed point of $\mathbb{CP}^2$ lying in the preimage of $0 \in \mathfrak{su}(2)^*$, and the other singular orbit is in the preimage of the boundary of the ball.

LEMMA VI.1. *Let (M, ω) be a 4-dimensional symplectic G-manifold with moment map $\mu : M \to \mathfrak{g}^*$. The (principal) cross-section R is a 2-dimensional symplectic $\mathbb{S}^1$-manifold. The principal stabilizer of the G-action on M can only be trivial, isomorphic to $\mathbb{Z}_k$ or to $\mathbb{S}^1$.*

PROOF. The first statement is a direct consequence of the cross-section theorem and Example V.1. If the image of the moment map were completely contained inside $\{0\}$, then the action would be trivial, because for every $X \in \mathfrak{g}$, the Hamiltonian function $H_X := \langle \mu | X \rangle$ would be constant.

Note that $\mathrm{Stab}(p)$ is a subgroup of $\mathrm{Stab}(\mu(p))$ because the moment map is G-equivariant. Thus the principal stabilizer has to be one of the subgroups of $\mathbb{S}^1$, which reduces to one of the cases mentioned above. □

Theorem V.7 says that R together with the embedding of $\mathbb{S}^1 = \mathrm{SO}(2)$ into G determines $G \cdot R$ completely. But the embedding of SO(2) into G is unique (up to inner automorphisms) and thus only R is relevant.

These possible cases will now be inspected separately.

Principal stabilizer isomorphic to SO(2): Such a stabilizer can only occur if $G = \mathrm{SO}(3)$, because the elements $\pm 1 \in \mathrm{SU}(2)$ lie in the intersection of all maximal tori, and an SU(2)-action with such an isotropy group is not effective.

The stabilizer at points of the cross-section R coincides with the $\mathbb{S}^1$ acting on R, hence this group acts trivially on R, and it follows that

$$\mathrm{SO}(3) \cdot R \cong \mathrm{SO}(3) \times_{\mathrm{SO}(2)} R = \mathbb{S}^2 \times R \, .$$

Also, because SO(2) acts trivially, it follows that the differential of the Hamiltonian function $H_Z := \langle \mu_R | Z \rangle$ is zero, and then H_Z is constant on R, which has two consequences: The image $\mu(\mathrm{SO}(3) \cdot R)$ lies in a single orbit in $\mathfrak{g}^*$, and the same then holds for all of $\mu(M)$, in particular the image of M does not contain 0. The manifold M is equal to the flow-out of R and thus

$$M \cong \mathbb{S}^2 \times R \, ,$$

with the standard SO(3)-action on the 2-sphere and trivial action on R. Also, it follows that the symplectic form on M is equal to the sum of an SO(3)-invariant volume form on $\mathbb{S}^2$ and some arbitrary volume form on R. The manifold M is thus determined by the genus of R and the total volumes of the 2-sphere and of R.

Principal stabilizer is discrete: All cyclic groups in $G = \mathrm{SU}(2)$ isomorphic to $\mathbb{Z}_{2k}$ contain the elements $\pm 1 \in \mathrm{SU}(2)$, hence an SU(2)-action is only effective if the order of the principal stabilizer is odd. The SO(2)-action on R has the same stabilizer $\mathbb{Z}_k$ as the G-action on M (i.e. the $\mathbb{S}^1$-action on R is in general not effective).

Assume for now that $0 \notin \mu(M)$. The cross-section is a 2-dimensional toric manifold, i.e. R is simply a 2-sphere with k-fold rotations around a fixed axis. The manifold $M = G \times_{\mathrm{SO}(2)} \mathbb{S}^2$ is an $\mathbb{S}^2$-bundle associated to the $\mathbb{S}^1$-bundle G over $\mathbb{S}^2$. The only two $\mathbb{S}^2$-bundles over $\mathbb{S}^2$ are $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^2$, which is the unique non-trivial bundle obtained e.g. as the projectivization $\mathbb{P}(\mathcal{O}(1) \oplus \mathbb{C})$. Here $\mathcal{O}(1)$ is the dual of the tautological line bundle over $\mathbb{CP}^1$. We denote the k-fold tensor product $\mathcal{O}(1) \otimes \ldots \otimes \mathcal{O}(1)$ by $\mathcal{O}(k)$.

The two sphere-bundles can be distinguished by choosing sections, and measuring the parity of the normal bundles. The trivial $\mathbb{S}^2$-bundle gives even parity, the non-trivial one gives odd parity. In our examples, such a section can be given by $G/\mathbb{S}^1 \hookrightarrow M$, $[g] \mapsto [g, N]$, where N is a fixed point on $\mathbb{S}^2$ (the north pole). The normal bundle of this section is isomorphic to $G \times_{\mathbb{S}^1} \mathbb{C}$ with the k-fold rotations on $\mathbb{C}$. Note that the equivalence relation $(g, z) \sim (ge^{-i\varphi}, e^{ik\varphi}z)$ can be regarded as the one of a k-fold tensor product of a line-bundle, i.e.

$$\begin{aligned}(g, z \otimes 1 \otimes \ldots \otimes 1) = (g, z) \sim (ge^{-i\varphi}, e^{ik\varphi}z) &= (ge^{-i\varphi}, e^{ik\varphi}z \otimes 1 \otimes \ldots \otimes 1)\\ &= \left(ge^{-i\varphi}, \left(e^{i\varphi}z\right) \otimes \left(e^{i\varphi}1\right) \otimes \ldots \otimes \left(e^{i\varphi}1\right)\right).\end{aligned}$$

The normal bundle for the section of an SO(3)-manifold with principal stabilizer $\mathbb{Z}_k$ is isomorphic to $\mathcal{O}(2k)$, and the one of an SU(2)-manifold with stabilizer $\mathbb{Z}_{2k+1}$ is correspondingly isomorphic to $\mathcal{O}(2k+1)$. Hence all of these SO(3)-manifolds are diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, and all SU(2)-manifolds are diffeomorphic to $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^2$.

If $0 \notin \mu(M)$, then M is determined by G, the principal stabilizer $\mathbb{Z}_k$, $\mu_R(\max)$ and $\mu_R(\min)$.

If $0 \in \mu(M)$, the principal cross-section is an open disk $\mathbb{D}^2_R \subset \mathbb{C}$ with an $\mathbb{S}^1$-action that rotates around the origin. The only singular orbit is a fixed point that corresponds to the maximum of μ_R, and there are neither exceptional orbits in R nor in $G \cdot R$. Below we will see that the principal stabilizer has to be trivial or isomorphic to $\mathbb{Z}_2$.

Now we have to check what happens when 0 lies in the image of the moment map. Note that the orbits in $\mu^{-1}(0)$ are all isotropic, because

$$\omega_p(X_M, Y_M) = -\langle \mu(p) | [X, Y] \rangle = 0 \quad \text{for all } X, Y \in \mathfrak{g} \text{ and } p \in \mu^{-1}(0).$$

Thus the orbits in $\mu^{-1}(0)$ have at most dimension 2, and they have to be singular. The possible stabilizer for $p \in \mu^{-1}(0)$ is either G, SO(2) or O(2).

Fixed point: If $G = \mathrm{SO}(3)$, then there are no fixed points, because the representation theory of this group only allows odd-dimensional irreducible submodules, and these are not compatible with the symplectic structure on the tangent space of a fixed point.

If $G = \mathrm{SU}(2)$ there can be at most a single discrete fixed point $p \in M$. The reason for this is that the linearized SU(2)-action on T_pM has to be by the standard matrix representation. The neighborhood of such a fixed point contains only free orbits, and the given case can only occur if the principal stabilizer of M is trivial.

To get the complete manifold M one needs to glue the neighborhood of a fixed point $B_\varepsilon(0) \subset \mathbb{C}^2$ with $\mathrm{SU}(2) \times_{\mathbb{S}^1} R$ in an equivariant way. Up to isotopy there is a unique way to do this. Gluing the two manifolds respecting the symplectic form and the group action is equivalent to gluing the principal cross-sections of these two parts along a neighborhood of the boundary. The cross-sections have to be identified by an $\mathbb{S}^1$-equivariant symplectomorphism on a collar. It can be checked by a short computation that these maps are in a one-to-one correspondence with the maps from $I \to \mathbb{S}^1$, and thus the space of equivariant symplectomorphisms is connected, which means that the gluing is unique.

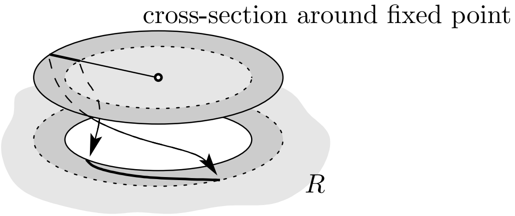


FIGURE 1. Gluing a neighborhood of a fixed point with the rest of the manifold can be done on the cross-section level.

The manifold given in Example VI.1.(3) is a symplectic 4-dimensional SU(2)-manifold which contains 0 in the image of the moment map, and by the result above it is, up to scaling of the symplectic form, the unique one.

Lagrangian orbits: If $p \in M$ lies in $\mu^{-1}(0)$ and $\operatorname{Stab}(p)$ is either SO(2) or O(2), then $\operatorname{Orb}(p)$ is a Lagrangian submanifold, and its neighborhood is symplectomorphic to the cotangent bundle $T^* \operatorname{Orb}(p)$ with the natural symplectic structure (see Example II.1). The only action possible on $T^* \operatorname{Orb}(p)$ is the natural one induced from the action on $\operatorname{Orb}(p)$. This excludes the case $G = \mathrm{SU}(2)$, because this action would not be effective.

If $G = \mathrm{SO}(3)$, both SO(2) and O(2) can occur as stabilizers for singular orbits. Like in the fixed point case above, to produce a closed manifold we need to glue the Lagrangian orbit into the cross-section part conserving the structure, and this is equivalent to gluing the principal cross-sections. The gluing is again unique.

If the stabilizer of the singular orbit is isomorphic to SO(2), then both building blocks are diffeomorphic to $T^*\mathbb{S}^2$, and the gluing respects a bundle structure over $\mathbb{S}^2$. In fact the manifold M is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ described in Example VI.1.(2) with $\lambda_1 = \lambda_2$.

In the case where the singular orbit has stabilizer O(2), one of the building blocks is diffeomorphic to $T^*\mathbb{RP}^2$, and the other one to a line-bundle with Euler class 4 over $\mathbb{S}^2$. The given manifold is $\mathbb{CP}^2$ with the SO(3)-action induced by the standard representation on $\mathbb{R}^3$, which has to be complexified to obtain an action on $\mathbb{C}^3$. The embedded $\mathbb{RP}^2$ is the orbit of $[1:0:0]$ and the singular orbit $\mathbb{S}^2$ is the orbit of $[1:i:0]$. This has been described in Example VI.1.(1).

CHAPTER VII

5-dimensional contact SO(3)-manifolds

This chapter contains the main results of the thesis. It consists of the proof of the theorem below, which describes the classification of 5-dimensional contact SO(3)-manifolds.

Theorem VII.1. *The following list gives a complete set of invariants for a cooriented 5-dimensional closed contact* SO(3)*-manifold M, in the sense that there is an* SO(3)*-contactomorphism between any two manifolds with equal invariants, and there exists a manifold for every choice of invariants from the list.*

- *The principal stabilizer is isomorphic to $\mathbb{Z}_k$ for some $k \in \mathbb{N}$ (including the trivial group, for $k = 1$).*
- *The closure $\overline{R}$ of the cross-section is a compact 3-dimensional contact $\mathbb{S}^1$-manifold without any fixed points or special exceptional orbits. Each boundary component of $\overline{R}$ corresponds to a component of $M_{(\mathrm{sing})}$. The orbits in the boundary are the only Legendrian orbits.*
- *If M has singular orbits, then the principal stabilizer is either isomorphic to $\mathbb{Z}_2$ or trivial. In the first case, all components of $M_{(\mathrm{sing})}$ are isomorphic to $\mathbb{S}^1 \times \mathbb{RP}^2$. If the principal stabilizer is trivial, one has two different types of components in $M_{(\mathrm{sing})}$, which are either copies of $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^2 := \mathbb{R} \times \mathbb{S}^2/\sim$ with the equivalence $(t, p) \sim (t+1, -p)$. The Dehn-Euler number $n(R)$ is an integer, which describes how $M_{(\mathrm{sing})}$ is glued onto $M_{(\mathrm{reg})}$. This Dehn-Euler number satisfies certain arithmetic conditions described in the Definition on page 78.*

Remark VII.1. In Theorem IV.16, we gave the classification of contact 3-dimensional $\mathbb{S}^1$-manifolds. The cross-section R is thus determined by the following invariants:

- If R is closed, it is determined solely by the genus of its orbit space $B := R/\mathbb{S}^1$, the exceptional orbits, and the orbifold Euler number which cannot be zero.
- If R is an open manifold, it is determined by the number of boundary components, the genus of its orbit space B, and its exceptional orbits.

By applying the cross-section theorem, one can reduce the 5-dimensional contact SO(3)-manifold to a 3-manifold.

Corollary VII.2. *Let (M, α) be a 5-dimensional contact* SO(3)*-manifold with moment map $\mu : M \to \mathfrak{so}(3)^*$. The cross-section R is a 3-dimensional contact $\mathbb{S}^1$-manifold without Legendrian orbits or fixed points.*

Conversely, let (R, α) be a 3-dimensional contact $\mathbb{S}^1$-manifold without Legendrian orbits, and without fixed points. Then there is a 5-dimensional contact SO(3)*-manifold M that has R as its cross-section.*

Proof. The first part of the statement is a direct consequence of the cross-section theorem and Example V.1. If R had Legendrian orbits or fixed points, then 0 would be contained in the image $\mu(R)$.

For the second part, the manifold M is given by $\mathrm{SO}(3) \times_{\mathbb{S}^1} R$, with the standard SO(3)-action on the left factor. The contact form on M is constructed by taking $\alpha + \alpha(Z_R) \cdot Z^*$ on $\{e\} \times_{\mathbb{S}^1} R$, and moving it with the SO(3)-action to the rest of M. Here Z^* denotes the dual of Z with respect to the standard basis $\{X, Y, Z\}$ of $\mathfrak{so}(3)$, i.e. to a basis were the Lie bracket of two basis vectors gives the third one. $\square$

In the rest of this chapter we will write **c.p. G-contactormorphism** for coorientation preserving G-contactomorphism.

Lemma VII.3. *Let (M, α) and (M', α') be 5-dimensional contact* SO(3)*-manifolds. A c.p.* SO(3)*-contactomorphism $\Phi : M \to M'$ induces a c.p. $\mathbb{S}^1$-contactomorphism between the cross-sections R and R'.*

Proof. The pull-back $\Phi^*\alpha'$ is equal to $f \cdot \alpha$ with a positive function $f : M \to \mathbb{R}$. For the moment maps, this gives $\mu' \circ \Phi = f \cdot \mu$. The restriction of Φ to R is an $\mathbb{S}^1$-contactomorphism to R'. $\square$

Lemma VII.4. *Let (M, α) and (M', α') be 5-dimensional contact* SO(3)*-manifolds, and let R and R' be their respective cross-sections. A c.p. $\mathbb{S}^1$-contactomorphism $\Phi : R \to R'$ induces an* SO(3)*-contactomorphism between the flow-outs $\mathrm{SO}(3) \cdot R \subset M$ and $\mathrm{SO}(3) \cdot R' \subset M'$.*

Proof. The map is given by

$$\mathrm{SO}(3) \times_{\mathbb{S}^1} R \to \mathrm{SO}(3) \times_{\mathbb{S}^1} R', \quad [g, p] \mapsto [g, \Phi(p)] .$$

One easily checks that the map is well-defined, and respects the contact structures. $\square$

Let (M, α) be a contact 5-manifold and let SO(3) act by contact transformations with moment map μ.

Lemma VII.5. *The principal stabilizer of a contact* SO(3)*-manifold is isomorphic to $\mathbb{Z}_k$ for some $k \in \mathbb{N}$ (including the trivial group, for $k = 1$).*

Proof. Since the moment map μ corresponding to the action is equivariant, $\mathrm{Stab}(p) \leq \mu(\mathrm{Stab}(p))$. The SO(3)-structure of $\mathfrak{so}(3)^*$ was given in Example V.1, and it follows that $\mu \equiv 0$ if the principal stabilizer is not one of $\mathbb{Z}_k$ or $\mathbb{S}^1$. But $\mu \equiv 0$ means that the action is trivial, which in particular contradicts effectiveness.

In fact, the circle $\mathbb{S}^1$ can also be excluded as principal stabilizer. Assume $\exp(tX)$ (for some $X \in \mathfrak{so}(3)$, $X \neq 0$) leaves p fixed, i.e. $\exp(tX) \cdot p = p$, then we have $\mu(p) = \mu(\exp(tX) \cdot p) = \mathrm{Ad}(\exp(-tX))^* \mu(p)$ and as a consequence $\mathrm{ad}(X)^* \mu(p) = 0$. Let now $X, Y, Z \in \mathfrak{so}(3)$ be the standard basis of the Lie algebra. Then, $\langle \mu(p) | Z \rangle = \langle \mu(p) | [X, Y] \rangle = 0$, $\langle \mu(p) | Y \rangle = -\langle \mu(p) | [X, Z] \rangle = 0$ and obviously $\langle \mu(p) | X \rangle = \alpha(X_M(p)) = 0$, i.e. $\mu(p) = 0$.

Not only does this show that $\mathbb{S}^1$ cannot be a principal stabilizer, it also proves that all singular orbits lie in $\mu^{-1}(0)$, and the cross-section has no fixed points. $\square$

The principal cross-section $R = \mu^{-1}(\mathbb{R}^+ Z^*)$ is a contact 3-manifold with a Hamiltonian $\mathbb{S}^1$-action. The $\mathbb{S}^1$-orbits are neither fixed points nor tangent to the contact structure. If $0 \notin \mu(M)$ the cross-section R is a closed subset of M, because $\mathbb{R}^+ Z^* \cap \mu(M)$ is compact, and hence R is a closed manifold and then M, as flow-out of R, is completely determined by R.

Lemma VII.6. *Let (M, α) be a 5-dimensional contact* SO(3)*-manifold. Then $M_{(\mathrm{sing})} = \mu^{-1}(0)$.*

Proof. The preimage $\mu^{-1}(0)$ is the union of SO(3)-orbits tangent to $\ker\alpha$, i.e. a collection of isotropic submanifolds. But isotropic submanifolds of a 5-dimensional contact manifold have at most dimension 2, and hence these orbits have to be singular. On the other hand, the proof of Lemma VII.5 shows that all singular orbits lie in $\mu^{-1}(0)$. □

Furthermore a stabilizer of an exceptional orbit is isomorphic to some $\mathbb{Z}_m$ and these orbits lie discrete surrounded by principal orbits.

1. Examples

In this section we will introduce a few examples. They will be used during the rest of the chapter to apply the theory while it is being developed.

Example VII.1. The standard contact structure on the 5-sphere $\mathbb{S}^5 \subset \mathbb{C}^3$ is given at a point (z_1, z_2, z_3) by

$$\alpha_+ = \sum_{j=1}^{3} \big(x_j\, dy_j - y_j\, dx_j\big) ,$$

with $z_j = x_j + iy_j$. This contact form is invariant under the SO(3)-action induced by the standard matrix representation.

The stabilizer of a point $\mathbf{x} + i\mathbf{y} \in \mathbb{S}^5$ with $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ is the intersection of the stabilizer of $\mathbf{x}$ and that of $\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, we have $\operatorname{Stab}(\mathbf{x} + i\mathbf{y}) = \{e\}$, and $\operatorname{Stab}(\mathbf{x} + i\mathbf{y}) \cong \mathbb{S}^1$ otherwise.

For any matrix $A \in \mathfrak{so}(3)$, the moment map is given by $\langle \mu_+(\mathbf{x} + i\mathbf{y}) | A\rangle = 2\mathbf{x}^t A\mathbf{y}$. The cross-section is then the set

$$R = \{\mathbf{x} + i\mathbf{y} \in \mathbb{S}^5 | x_1y_3 - y_1x_3 = x_2y_3 - y_2x_3 = 0 \text{ and } x_1y_2 - y_1x_2 > 0\} .$$

The condition $x_1y_2 - y_1x_2 > 0$ implies that the other two equations, regarded as a linear system in (x_3, y_3), have the unique solution $(x_3, y_3) = 0$. Hence the cross-section is given by

$$R = \{(z_1, z_2, 0) \in \mathbb{S}^5 | x_1y_2 - y_1x_2 > 0\} .$$

The $\mathbb{S}^1$-action on R is given by simultaneous rotations in the (x_1, x_2)- and (y_1, y_2)-plane. Its orbit space $R/\mathbb{S}^1$ lies in a natural way in $\mathbb{CP}^1$ with the projection $\pi : R \to R/\mathbb{S}^1$ given by $\pi(x_1 + iy_1, x_2 + iy_2, 0) = [x_1 + ix_2 : y_1 + iy_2]$. Note that the equation $x_1y_2 - x_2y_1 = 0$ is well-defined in $\mathbb{CP}^1$ and its solutions are given by the standard embedding of $\mathbb{RP}^1$. Hence $R/\mathbb{S}^1$ is diffeomorphic to an open disk and $R \cong \mathbb{D}^2_{<1} \times \mathbb{S}^1$.

Another SO(3)-invariant contact form on $\mathbb{S}^5$ can be given by

$$\begin{aligned}\alpha_- = i &\sum_{j=1}^{3} \big(z_j\, d\bar z_j - \bar z_j\, dz_j\big)\\ &- i\big((z_1^2 + z_2^2 + z_3^2)\, d(\bar z_1^2 + \bar z_2^2 + \bar z_3^2) - (\bar z_1^2 + \bar z_2^2 + \bar z_3^2)\, d(z_1^2 + z_2^2 + z_3^2)\big) .\end{aligned}$$

Note that the first part of the form is identical to the standard form α_+. It is easy to check that the second term does not give any contribution to the moment map, and hence $\mu_+ = \mu_-$. The cross-section for α_+ and α_- are then of course also equal.

The example will be continued at the end of the next section.

Example VII.2. Other interesting SO(3)-manifolds are the Brieskorn spheres $(W_k^5, \alpha_\pm)$, which were already introduced in Section III.6.1.2 and IV.6.1. As we will see later, these examples cover all the simply connected contact SO(3)-manifolds with singular orbits of dimension 5. The open book decomposition of these manifolds is closely related to the SO(3)-symmetry (see [**vKNar**] and Appendix E).

It is well-known that W_k^5 is diffeomorphic to $\mathbb{S}^5$ for k odd, and to $\mathbb{S}^2 \times \mathbb{S}^3$ for k even, but Ustilovsky ([**Ust99**]) showed by using contact homology that all of the contact manifolds $(W_{2n+1}^5, \alpha_+) \cong \mathbb{S}^5$ are different, so in particular we cannot expect them to be SO(3)-contactomorphic.

The moment map for $(W_k^5, \alpha_\pm)$ was already computed in the proof of Lemma IV.22. The infinitesimal generators of the SO(3)-action have a trivial z_0-component. Hence the moment maps $\mu_k(z_0, z_1, z_2, z_3)$ for both α_k and α_{-k} have to be equal. They are given by

$$\langle \mu_k | X\rangle = 4(x_3y_2 - x_2y_3), \quad \langle \mu_k | Y\rangle = 4(x_1y_3 - x_3y_1)\ , \text{ and } \langle \mu_k | Z\rangle = 4(x_2y_1 - x_1y_2)\ .$$

It can be seen with a similar computation as in Example VII.1 that the cross-section R is given by the points $(z_0, z_1, z_2, 0) \in W_k^5$ with $x_2y_1 - x_1y_2 > 0$.

The map $(z_0, z_1, z_2, 0) \mapsto z_0$ from R to the open unit disk is the projection of R onto its quotient space (see [**HM68**]). The cross-section is $\mathbb{S}^1$-diffeomorphic to $\mathbb{D}^2_{<1} \times \mathbb{S}^1$.

The example will be continued at the end of the next section.

2. Singular orbits

In this section, we will show that each component of $M_{(\mathrm{sing})}$ corresponds to one of three possible models.

Lemma VII.7. *Let (M, α) be a 5-dimensional closed contact SO(3)-manifold. Recall from Lemma VII.5 that the principal stabilizer H is either trivial or isomorphic to $\mathbb{Z}_k$.*

If $H \cong \mathbb{Z}_k$ with $k \geq 3$, then M does not have any singular orbits.

If $H \cong \mathbb{Z}_2$, then any component of $M_{(\mathrm{sing})}$ has a neighborhood that is SO(3)-diffeomorphic to a neighborhood of the zero-section in $\mathbb{S}^1 \times T\mathbb{RP}^2$, with trivial action on the first part and natural action on the second one.

If H is trivial, any component of $M_{(\mathrm{sing})}$ has a neighborhood that is SO(3)-diffeomorphic to a neighborhood of the zero-section in the vertical bundle VE_{triv} or VE_{twist}, where E_{triv} is the trivial $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ and E_{twist} is the twisted $\mathbb{S}^2$-bundle over $\mathbb{S}^1$.

In all of these cases, there is up to SO(3)-contactomorphism a unique invariant contact form on sufficiently small neighborhoods of $M_{(\mathrm{sing})}$.

In the rest of this section we will describe all possible cases, and show the claims of the lemma.

One of the conclusions will be that the closure of the cross-section of a 5-dimensional contact SO(3)-manifold M is a compact 3-dimensional contact $\mathbb{S}^1$-manifold with boundary. The interior points of R lie in regular SO(3)-orbits, while ∂R lies in $M_{(\mathrm{sing})}$. The $\mathbb{S}^1$-orbits at the boundary are Legendrian.

Lemma VII.8 (Equivariant Weinstein Theorem). *Let (M, α) be a contact G-manifold, and let $\mathrm{Orb}(p) \hookrightarrow M$ be a Legendrian G-orbit. Then a neighborhood of $\mathrm{Orb}(p)$ is G-contactomorphic to a neighborhood of the zero-section in $(\mathbb{R} \oplus T^* \mathrm{Orb}(p), dt + \lambda_{\mathrm{can}})$, where G acts by $g \cdot (t, v) = (t, g_*^{-1} v)$.*

Proof. There is a G-invariant almost complex structure J on the contact structure $\xi = \ker\alpha$ such that

$$T_q \operatorname{Orb}(p) \cap J \cdot (T_q \operatorname{Orb}(p)) = \{0\} \text{ for all } q \in \operatorname{Orb}(p).$$

The trivial line bundle ε^1 spanned by the Reeb vector field of α is also G-invariant. This implies that the normal bundle of $T \operatorname{Orb}(p)$ in M can be equivariantly identified with $\varepsilon^1 \oplus T \operatorname{Orb}(p) \cong \varepsilon^1 \oplus T^* \operatorname{Orb}(p)$. The contact form restricts to $dt + c\,\lambda_{\mathrm{can}}$ on the zero-section, and rescaling the fiber gives the desired form $dt + \lambda_{\mathrm{can}}$. The differential of α is a positive multiple of $d\lambda_{\mathrm{can}}$ on the orbit. This allows us to apply Theorem A.3, which finishes the proof. □

By looking at the different stabilizers that can occur, it will be seen that all singular orbits are either isomorphic to $\mathbb{S}^2$ with stabilizer $\mathbb{S}^1$ or to $\mathbb{RP}^2$ with stabilizer $\mathrm{O}(2)$.

2.1. Fixed points. The irreducible representations of $\mathrm{SO}(3)$ are all odd-dimensional. This implies that 5-dimensional contact $\mathrm{SO}(3)$-manifolds do not have fixed points by the following argument. The vector space spanned by the Reeb field is a trivial submodule of T_pM, and the contact plane (ξ_p, J_p) is a complex 2-dimensional $\mathrm{SO}(3)$-module, which also has to be trivial. That means the action on T_pM is trivial, which contradicts effectiveness.

2.2. Stabilizer $\mathrm{O}(2)$**.** The neighborhood of an orbit with stabilizer $\mathrm{O}(2)$ is $\mathrm{SO}(3)$-equivariant to $\mathbb{R} \times T^* \operatorname{Orb}(p)$ with $\operatorname{Orb}(p) \cong \mathbb{RP}^2$. The stabilizer of any non-zero element in $T^*\mathbb{RP}^2$ is isomorphic to $\mathbb{Z}_2$, which is then the principal stabilizer.

A connected component of

$$M_{(\mathrm{O}(2))} := \{p \in M \,|\, \operatorname{Stab}(p) \text{ is conjugate to } \mathrm{O}(2)\}$$

is an $\mathbb{RP}^2$-bundle over $\mathbb{S}^1$ (the closure $\overline{M_{(\mathrm{O}(2))}}$ is a closed submanifold, possibly containing points with larger stabilizer than $O(2)$, but we proved that M has no fixed points, and hence $\overline{M_{(\mathrm{O}(2))}} = M_{(\mathrm{O}(2))}$). The structure group of a (G/H)-bundle with the standard G-action on the fibers is just the group of G-equivariant diffeomorphisms from G/H to itself. It is not very difficult to see that this is given by $N(H)/H$ (see [**Bre93**]). In our case $N(\mathrm{O}(2))/\mathrm{O}(2) = \mathrm{O}(2)/\mathrm{O}(2) = \{e\}$, and hence every component of $M_{(\mathrm{O}(2))}$ is of the form $\mathbb{S}^1 \times \mathbb{RP}^2$. The neighborhood of such a component is $\mathrm{SO}(3)$-diffeomorphic to $\mathbb{S}^1 \times T^*\mathbb{RP}^2$ with the standard $\mathrm{SO}(3)$-action on the second part. A possible invariant contact form is given by $dt + \lambda_{\mathrm{can}}$, where λ_{can} is the canonical 1-form on $T^*\mathbb{RP}^2$.

In fact, the contact form above is the only one in a small neighborhood of the singular orbit up to $\mathrm{SO}(3)$-contactomorphisms. This can be proved in a similar way as Lemma VII.8: After pulling back the form to $\mathbb{S}^1 \times T^*\mathbb{RP}^2$, one has $\alpha = f(t)\,dt + r(t)\lambda_{\mathrm{can}}$ on the singular orbits. One can divide by $f(t)$ and then rescale the fibers to obtain the standard form $dt+\lambda_{\mathrm{can}}$, which allows us to use [**LW01**, Theorem 5.2], which states that there is a neighborhood of the orbit $\mathrm{SO}(3)$-contactomorphic to the normal bundle.

In Sections 3 and 4, it will be important to know what the cross-section looks like in a neighborhood of the singular orbits. We compute the cross-section close to $M_{(\mathrm{O}(2))}$ in a coordinate description.

A chart of $\mathbb{RP}^2$ around $[1:0:0]$ is given by $\mathbb{R}^2 \to \mathbb{RP}^2$, $(q_1, q_2) \mapsto [1 : q_1 : q_2]$, and the $\mathrm{SO}(3)$-action is induced by the standard matrix representation. Let X, Y, Z be the standard basis of $\mathfrak{so}(3)$, where each element generates the rotation around the corresponding axis of

$\mathbb{R}^3$. For Y, for example, the action looks like

$$\begin{aligned}\exp(tY)\cdot[1:q_1:q_2] &= [\cos t + q_2\sin t : q_1 : q_2\cos t - \sin t]\\ &= \left[1 : \frac{q_1}{\cos t + q_2\sin t} : \frac{q_2\cos t - \sin t}{\cos t + q_2\sin t}\right]\end{aligned}$$

The infinitesimal generators of the action are given in this chart by

$$\begin{aligned}X_{\mathbb{RP}^2}([1:q_1:q_2]) &= q_2\,\partial_{q_1} - q_1\,\partial_{q_2}\ ,\\ Y_{\mathbb{RP}^2}([1:q_1:q_2]) &= -q_1q_2\,\partial_{q_1} - (1+q_2^2)\,\partial_{q_2}\ ,\\ Z_{\mathbb{RP}^2}([1:q_1:q_2]) &= -(1+q_1^2)\,\partial_{q_1} - q_1q_2\,\partial_{q_2}\ ,\end{aligned}$$

and the moment map is

$$\begin{aligned}\langle\mu(t,q_1,q_2,p_1,p_2)|X\rangle &= q_2p_1 - q_1p_2\ ,\\ \langle\mu(t,q_1,q_2,p_1,p_2)|Y\rangle &= -q_1q_2p_1 - (1+q_2^2)\,p_2\ ,\\ \langle\mu(t,q_1,q_2,p_1,p_2)|Z\rangle &= -(1+q_1^2)\,p_1 - q_1q_2p_2\ .\end{aligned}$$

Elements of $\mu^{-1}(\mathbb{R}^+Z^*)$ have $p_1\neq 0$ or $p_2\neq 0$, and for such elements $q_2p_1 - q_1p_2 = 0$ and $-q_1q_2p_1 - (1+q_2^2)\,p_2 = 0$ hold. These two equations can be read as a linear system in p_1 and p_2, and there are only non-trivial solutions if the corresponding determinant vanishes, that is, if $-q_2\,(1+q_2^2) - q_1^2q_2 = -q_2\,(1+q_1^2+q_2^2) = 0$. If this is the case, then $q_2 = 0$, and from this it follows that $p_2 = 0$. The cross-section R consists of vectors in $T\mathbb{RP}^2$ tangent to $\mathbb{RP}^1$, but pointing only in positive direction (with the embedding of $\mathbb{RP}^1$ in $\mathbb{RP}^2$ given by $[a:b]\mapsto[a:b:0]$).

The restriction of the contact form on R is given in the chart above by $dt + p_1\,dq_1$. Hence α is a contact form even on the boundary of $\overline{R}$, and the orbits of the $\mathbb{S}^1$-action are Legendrian on $\partial\overline{R}\cong\mathbb{S}^1\times\mathbb{S}^1$.

A collar neighborhood of $\partial\overline{R}$ is of the form $\mathbb{S}^1\times[0,\varepsilon)\times\mathbb{S}^1$ with contact form $dt + r\,d\varphi$ and action $e^{i\vartheta}\cdot(t,r,\varphi) = (t,r,\varphi+2\vartheta)$. The embedding of this neighborhood into M is given by

$$(t,r,\varphi)\mapsto\big(t,[\cos(\varphi/2):\sin(\varphi/2):0],-r\sin(\varphi/2)\,\partial_1 + r\cos(\varphi/2)\,\partial_2\big)\ ,$$

and the points $(t,0,0)\in\partial\overline{R}$ all have equal stabilizer in SO(3).

2.3. Stabilizer $\mathbb{S}^1$.

The neighborhood of an orbit with stabilizer $\mathbb{S}^1$ is SO(3)-diffeomorphic to $\mathbb{R}\times T\mathbb{S}^2$ with trivial action on the first, and standard action on the second component. The principal stabilizer is trivial. A connected component of

$$M_{(\mathrm{SO}(2))} := \{p\in M\,|\,\mathrm{Stab}(p)\text{ is conjugate to }\mathrm{SO}(2)\}$$

is a closed manifold, because no fixed points or points with stabilizer O(2) do exist, and hence $M_{(\mathrm{SO}(2))}$ is diffeomorphic to an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. The structure group of such a bundle is $N(\mathrm{SO}(2))/\mathrm{SO}(2)\cong\mathbb{Z}_2$, hence the only two $\mathbb{S}^2$-bundles over $\mathbb{S}^1$ are the trivial one E_{triv} and the twisted one E_{twist}. They can be described as $\mathbb{R}\times\mathbb{S}^2$ under the equivalence relations $(t,p)\sim(t+1,p)$ and $(t,p)\sim(t+1,-p)$ (with $t\in\mathbb{R}$ and $p\in\mathbb{S}^2$) respectively. A neighborhood of a component of $M_{(\mathrm{sing})}$ is diffeomorphic to the corresponding vertical bundle. The SO(3)-action on the second component of $\mathbb{R}\times\mathbb{S}^2$ is compatible with these identifications, and one obtains an action on either vertical bundle VE_{triv} and VE_{twist}.

A possible invariant contact form is given by $dt+\lambda_{\mathrm{can}}$ on $\mathbb{R}\times T^*\mathbb{S}^2$, where $T^*\mathbb{S}^2$ is identified with $T\mathbb{S}^2$ via an invariant metric. This form descends to VE_{triv} and also to VE_{twist}, because the reflection in the construction of E_{twist} is induced by a diffeomorphism of $\mathbb{S}^2$, and λ_{can} on T^*N remains invariant under maps induced by diffeomorphisms of the base space N.

In a small neighborhood of $M_{(\mathrm{SO}(2))}$, every invariant contact form is SO(3)-contactomorphic to $dt+\lambda_{\mathrm{can}}$. The proof of this fact is completely analogous to the one for orbits with stabilizer O(2) above, and will be omitted.

Now we will describe what the cross-section looks like in a neighborhood of the singular orbits. The moment map μ is given in the neighborhood of a singular orbit by

$$\langle \mu(t,q,p)|X\rangle = p^tXq$$

with $(t;q,p)\in\mathbb{R}^1\times T^*\mathbb{S}^2\subseteq\mathbb{R}^1\times\mathbb{R}^3\times\mathbb{R}^3$ and $X\in\mathfrak{so}(3)$ in its standard matrix representation. One easily checks that the cross-section is the set of points (t,q,p) where q lies in the equator of the sphere and p is a vector tangent to the equator at q, with all these vectors oriented the same way. The $\mathbb{S}^1$-action on the cross-section is induced by rotations around the z-axis of the sphere.

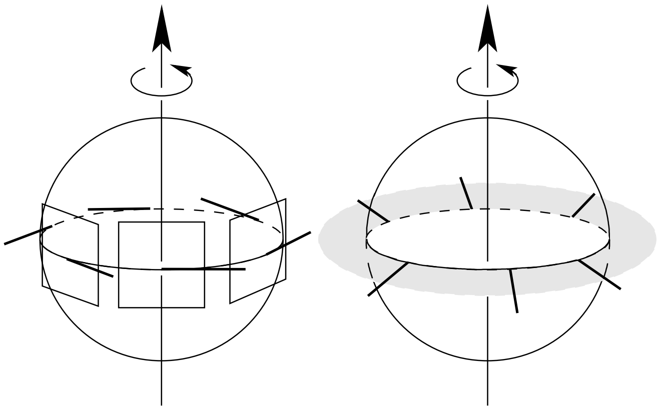


FIGURE 1. On the left the cross-section around an exceptional orbit is displayed: It consists of vectors at the equator pointing into positive direction. The picture on the right displays a model more accessible to the imagination: The cross-section sits as a ring around the equator of the sphere. Vectors pointing into the cross-section are normal to the sphere.

For E_{triv}, a collar neighborhood of the boundary $\partial\overline{R}$ can be given by $\mathbb{S}^1\times[0,\varepsilon)\times\mathbb{S}^1$, while for components of type E_{twist}, the form $\mathbb{R}\times[0,\varepsilon)\times\mathbb{S}^1/\sim$ with the equivalence relation $(t,r,\varphi)\sim(t+1,r,\varphi+\pi)$ will be used. The contact form in both cases is $dt+r\,d\varphi$, and the $\mathbb{S}^1$-action is $e^{i\vartheta}\cdot(t,r,\varphi)=(t,r,\varphi+\vartheta)$. The embedding of $\overline{R}$ into the neighborhood of $M_{(\mathrm{sing})}$ is given by

$$(t,r,\varphi)\mapsto\big(t;(\cos\varphi,\sin\varphi,0);r\cdot(-\sin\varphi,\cos\varphi,0)\big)\ .$$

With this embedding, the points $(t,0,0)$ and $(t,0,\pi)$ in $\partial\overline{R}$ all have equal stabilizer.

This concludes the description of all cases of singular orbits, and the proof of Lemma VII.7.

EXAMPLE VII.1 (cont.). As described above, the singular orbits of $\mathbb{S}^5$ are composed of all points $\mathbf{x}+i\mathbf{y}$ where $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)$ are linearly dependent. The singular orbits are 2-spheres, and we have to decide whether the component of ${\mathbb{S}^5}_{(\mathrm{sing})}$ is equal to E_{triv}

or to E_{twist}. This of course is independent of the contact structure. The only points invariant under rotations around the z_3-axis are $(0,0,e^{i\varphi})$ with $0 \le \varphi < 2\pi$. But since $(0,0,1)$ and $(0,0,-1)$ both lie in $\operatorname{Orb}(0,0,1)$, we have $\mathbb{S}^5{}_{(\text{sing})} \cong E_{\text{twist}}$.

Example VII.2 (cont.). Now we will determine the type of the singular orbits of W_k^5. This of course does not depend on the contact structure. As we said above, a point $(z_0,z_1,z_2,z_3) \in W_k^5$ lies on a singular orbit if and only if $\mathbf{x}$ is parallel to $\mathbf{y}$, where $\mathbf{x} = (x_1,x_2,x_3)$ and $\mathbf{y} = (y_1,y_2,y_3)$. In particular, consider the points that are invariant under rotations around the z_1-axis. They are given by $\big\{\big(e^{i\varphi}, \pm i e^{\frac{ki}{2}\varphi}, 0, 0\big) \,|\, 0 \le \varphi < 2\pi\big\}$. For k odd, all points lie on a single path, but for k even there are two connected components. Hence, one obtains $\big(W_k^5\big)_{(\text{sing})} \cong E_{\text{twist}}$ for k odd, and $\big(W_k^5\big)_{(\text{sing})} \cong E_{\text{triv}}$ for k even.

So far all invariants found for $(W_k^5, \alpha_{\pm k})$, and $(W_{k'}^5, \alpha_{\pm k'})$ are equal if $k \equiv k' \mod 2$. But at the end of the next section, a last invariant will be computed that allows us to distinguish all of the $(W_k^5, \alpha_{\pm k})$.

3. Equivalence between contact SO(3)-manifolds

In this section, the necessary and sufficient conditions for the existence of an SO(3)-equivariant contactomorphism $\Phi : M \to M'$ between two 5-dimensional contact SO(3)-manifolds (M,α) and (M',α') will be given.

If there are no singular orbits on M, then $0 \notin \mu(M)$ and the whole manifold is determined according to Theorem V.7 by its cross-section. Two contact 5-manifolds with an SO(3)-action without singular orbits are thus equivalent if and only if their cross-sections are. The possible cross-sections, being closed contact 3-manifold with $\mathbb{S}^1$-actions, can be found in the Classification Theorem IV.16.

On the other hand, if $0 \in \mu(M)$, then $M = M_{(\text{reg})} \cup M_{(\text{sing})}$, but there are several ways to glue both parts. The flow-out $\mathrm{SO}(3) \cdot R \cong \mathrm{SO}(3) \times_{\mathbb{S}^1} R$ is determined by R, but for the whole of M the problem is that $p \in \partial\overline{R}$ does not "remember" as point in the $\mathbb{S}^1$-manifold $\overline{R}$, which stabilizer $\operatorname{Stab}(p) \le \mathrm{SO}(3)$ it had in M.

The solution lies in choosing an arbitrary point $p_0 \in \partial\overline{R}$ and marking all other points p in the boundary with $\operatorname{Stab}(p) = \operatorname{Stab}(p_0) \le \mathrm{SO}(3)$. The marked points form curves in $\partial\overline{R}$. If the boundary component corresponds to E_{triv}, these curves are given by two sections to the $\mathbb{S}^1$-action that are related to each other by a 180°-rotation. If the component corresponds to E_{twist}, the marked points lie on a single curve, which intersects each $\mathbb{S}^1$-orbit twice. If the singular orbits have stabilizer isomorphic to O(2), then the marked points form a single section.

Another way to describe the situation is the following: Gluing $M_{(\text{sing})}$ onto $M_{(\text{reg})}$ can be achieved by gluing R onto the cross-section in the neighborhood of $M_{(\text{sing})}$. This means that one has to identify two tori. The generators of the homology in ∂R are given by an $\mathbb{S}^1$-orbit and a section σ to the $\mathbb{S}^1$-action in $\overline{R}$. The generators of the homology of $\overline{R} \cap M_{(\text{sing})}$ can be described by an $\mathbb{S}^1$-orbit, and by a curve of marked points as fixed above. The $\mathbb{S}^1$-obits have to coincide in both parts, and the only freedom when gluing consists in choosing the relative position of the other two homology classes.

Lemma VII.9. *Let (M,α) and (M',α') be two 5-dimensional contact* SO(3)*-manifolds with principal cross-sections (R,α) and (R',α'). Assume there is an $\mathbb{S}^1$-contactomorphism ψ between $\overline{R}$ and $\overline{R'}$ that maps the marked curves $\gamma_1,\dots,\gamma_n$ in $\partial\overline{R}$ onto the marked curves in $\partial\overline{R'}$, i.e. $\psi \circ \gamma_i = \gamma_i'$. Then there is an* SO(3)*-equivariant contactomorphism $\Psi : M \to M'$.*

PROOF. Over the flow-out $\mathrm{SO}(3)\cdot R$ and $\mathrm{SO}(3)\cdot R'$ the claim holds. Hence if $M_{(\mathrm{sing})}=\emptyset$, then the statement is true. The problem for $\partial\overline{R}\neq\emptyset$ is that ψ extends to an SO(3)-homeomorphism on M, but this map is in general not smooth at the singular orbits. Hence we will need to deform ψ in a neighborhood of $\partial\overline{R}$.

Choose a component K of $M_{(\mathrm{sing})}$. The image $\psi(K)$ in ${M'}_{(\mathrm{sing})}$ is of the same type: If the principal stabilizer of R is isomorphic to $\mathbb{Z}_2$, then every component in $M_{(\mathrm{sing})}$ and ${M'}_{(\mathrm{sing})}$ is diffeomorphic to $\mathbb{S}^1\times\mathbb{RP}^2$, and if the principal stabilizer of R is trivial, then the two types of component in $M_{(\mathrm{sing})}$ and ${M'}_{(\mathrm{sing})}$ can be distinguished by the curves of marked points.

Now one can represent the neighborhood of K and $\psi(K)$ by the standard models described at the end of Section 2.2 and 2.3. The cross-section is either given by $(\mathbb{R}\times[0,c)\times\mathbb{S}^1/\sim, dt+r\,d\varphi)$ for E_{twist} or by $(\mathbb{S}^1\times[0,c)\times\mathbb{S}^1, dt+r\,d\varphi)$ for the other two types of singular orbits.

The map ψ is $\mathbb{S}^1$-equivariant, thus

$$\psi(t,r,\varphi)=\big(T(t,r),R(t,r),\varphi+\Phi(t,r)\big)\ .$$

Furthermore it rescales the form $\alpha=dt+r\,d\varphi$ by a function $f(t,r)>0$, i.e.

$$f(t,r)\,dt+rf(t,r)\,d\varphi=f\alpha=\psi^*\alpha=\left(\frac{\partial T}{\partial t}+R\cdot\frac{\partial\Phi}{\partial t}\right)dt+R\,d\varphi+\left(\frac{\partial T}{\partial r}+R\cdot\frac{\partial\Phi}{\partial r}\right)dr\ .$$

The consequences are $R(t,r)=rf(t,r)$, $\partial_tT(t,r)+rf(t,r)\cdot\partial_t\Phi(t,r)=f(t,r)$, and $\partial_rT(t,r)+rf(t,r)\cdot\partial_r\Phi(t,r)=0$. The boundary is mapped onto the boundary, i.e. $R(t,0)=0$. We can assume $T(0,0)=0$ and $\Phi(0,0)=0$. Also, all of the three cases E_{triv}, E_{twist}, and $\mathbb{S}^1\times\mathbb{RP}^2$ lead to $\Phi(t,0)=0$, because the γ_i are mapped onto the γ_i'.

Let $\rho_\varepsilon:\mathbb{R}^+\to[0,1]$ be the smooth map

$$\rho_\varepsilon(r)=\begin{cases}0 & \text{for } r\le\varepsilon/2\\ N(\varepsilon)\cdot\int_{\varepsilon/2}^{r}\exp\frac{\varepsilon^2}{4(x-\varepsilon/2)(x-\varepsilon)}\,dx & \text{for } \varepsilon/2<r<\varepsilon\\ 1 & \text{for } r\ge\varepsilon\end{cases}$$

with $N(\varepsilon)$ the reciprocal value of $\int_{\varepsilon/2}^{\varepsilon}\exp\frac{\varepsilon^2}{4(x-\varepsilon/2)(x-\varepsilon)}\,dx$. The maximum of the derivative of this function is $N(\varepsilon)\cdot\exp(-4)=N(1)e^{-4}/\varepsilon$. One can now replace the original map ψ by

$$\widehat{\psi}(t,r,\varphi):=\big(T(t,r),R(t,r),\varphi+\rho_\varepsilon(r)\cdot\Phi(t,r)\big)\ .$$

It is easy to check that $\widehat{\psi}$ is well-defined on the cross-section R: The relations $\psi(t+2\pi a,r,\varphi+2\pi b)=\psi(t,r,\varphi)+(2\pi a,0,2\pi b)$ carry over to $\widehat{\psi}$.

The map $\widehat{\psi}$ is equal to $\big(T(t,r),rf(t,r),\varphi\big)$ for points with $r\le\varepsilon/2$ and equal to ψ for points with $r\ge\varepsilon$. It is also an $\mathbb{S}^1$-diffeomorphism. The determinant of the differential $d\widehat{\psi}$ is equal to the one of $d\psi$. The injectivity and surjectivity follow easily from the same properties of ψ. For example to show that (t',r',φ') lies in the image of $\widehat{\psi}$, use that there is a (t,r,φ) with $\psi(t,r,\varphi)=(t',r',\varphi')$. Then $\widehat{\psi}\big(t,r,\varphi+(1-\rho_\varepsilon(r))\cdot\Phi(t,r)\big)=(t',r',\varphi')$.

There is now an SO(3)-diffeomorphism $\widehat{\Psi}$ on M extending $\widehat{\psi}$. Away from the singular orbits, the map $\widehat{\Psi}$ is given as in the proof of Lemma VII.4. In the neighborhood of $M_{(\mathrm{sing})}$ one can use the standard model for E_{triv} and E_{twist}, where the map $\widehat{\Psi}$ is given by

$$\widehat{\Psi}:\ (t;p,v)\mapsto(T(t,\|v\|);\,p,\,f(t,\|v\|)\,v)\ ,$$

for $p\in\mathbb{S}^2$ and for $v\in T_p^*\mathbb{S}^2$ with $\|v\|<\varepsilon/2$. If the component of $M_{(\mathrm{sing})}$ was diffeomorphic to $\mathbb{S}^1\times\mathbb{RP}^2$ the map is given by the projectivization of $\widehat{\Psi}$ defined above. These maps clearly define SO(3)-equivariant diffeomorphisms in the neighborhood of a singular orbit, but one still needs to check that this definition is compatible with the map given in the proof of Lemma VII.4. Because both maps are SO(3)-equivariant, it is enough to check that these maps agree on the cross-section R. But $\widehat{\Psi}$ restricted to R gives back the map $\widehat{\psi}$. This shows that $\widehat{\Psi}$ is a globally-defined map.

The map $\widehat{\Psi}$ is an SO(3)-diffeomorphism, but it is only a contactomorphism far away from the singular orbits. All of the SO(3)-invariant 1-forms in the family $\alpha_s := (1-s)\,\alpha + s\,\widehat{\Psi}^*\alpha$ on M satisfy the contact condition. This can easily be checked in a small neighborhood of the singular orbits by using the local form given above. On $M_{(\mathrm{princ})}$, one checks the contact condition along R (by choosing ε small enough) and then uses SO(3)-invariance. The equivariant Gray stability shows that $\widehat{\Psi}$ deforms to an SO(3)-contactomorphism Ψ. □

Of course, the next question is how to find maps with the properties required in Lemma VII.9. For this, we need to define a last invariant for the cross-section.

Let $\overline{R}$ be a compact oriented 3-dimensional $\mathbb{S}^1$-manifold with non-empty boundary. Denote the components of $\partial\overline{R}$ by $\partial\overline{R}_j$ $(j = 1, \ldots, N)$ and assume that on each of the boundary components a smooth closed curve γ_j is given that intersects the $\mathbb{S}^1$-orbits transversely. Orient the curves in such a way that $\dot{\gamma}_j$ followed by the infinitesimal generator Z_R of the $\mathbb{S}^1$-action gives the orientation of $\partial\overline{R}_j$.

The γ_j should be of the same form as the marked points described above, i.e. if the principal stabilizer is isomorphic to $\mathbb{Z}_2$, assume γ_j intersects each $\mathbb{S}^1$-orbit in $\partial\overline{R}_j$ exactly once. If the principal stabilizer of R is trivial, the curves are either sections or intersect each orbit twice.

On the boundary of a small tubular neighborhood of the exceptional orbits one can define standard sections (see III.1.2), which can be extended to a global section σ of $\overline{R} \to \overline{R}/\mathbb{S}^1$. Let σ be oriented in such a way that the tangent space to the image of σ followed by the positive $\mathbb{S}^1$-direction gives the positive orientation of $\overline{R}$.

Definition. Denote the intersection number of two oriented loops α and β in an oriented torus by $\iota(\alpha, \beta)$. If the principal stabilizer in R is trivial define the **Dehn-Euler-number** $n(R, \gamma_1, \ldots, \gamma_N) \in \mathbb{Z}$ by

$$n(R, \gamma_1, \ldots, \gamma_N) := 2\sum_{j=0}^{m} \iota(\gamma_j, \partial\sigma) + \sum_{j=m+1}^{N} \iota(\gamma_j, \partial\sigma)\ ,$$

where we assume the first m curves to be sections to the $\mathbb{S}^1$-action, and the other curves to intersect each orbit twice. Note that the first term is a sum over even numbers and the second term is a sum over odd numbers.

If the principal stabilizer is isomorphic to $\mathbb{Z}_2$ define the Dehn-Euler number by

$$n(R, \gamma_1, \ldots, \gamma_N) := \sum_{j=1}^{N} \iota(\gamma_j, \partial\sigma)\ .$$

In this case $n(R, \gamma_1, \ldots, \gamma_N)$ can be any integer.

The Dehn-Euler number is very similar to the Euler invariant for an $\mathbb{S}^1$-manifold. To see that $n(R, \gamma_1, \ldots, \gamma_N)$ is independent of the section chosen, one can copy the proof of Lemma III.10. Note also that the coorientation of the contact structure has no effect on this definition.

Remark VII.2. In Lemma VII.4, it was shown that the cross-section R (as contact $\mathbb{S}^1$-manifold) is an invariant of a 5-dimensional contact manifold M. It has just been proved that the number $n(R, \gamma_1, \ldots, \gamma_m)$ is also an invariant of M, because under an SO(3)-contactomorphism the marked curves are mapped onto each other. Below we will finish the proof that a manifold M is completely determined by the invariants mentioned in Theorem VII.1 (i.e. cross-section, singular orbits and $n(R)$).

The 3-manifolds in the following lemma are cross-sections of 5-manifolds.

Lemma VII.10. *Let (R, α) and (R', α') be two $\mathbb{S}^1$-diffeomorphic 3-dimensional contact $\mathbb{S}^1$-manifolds without fixed points, but both with N boundary components. Assume that the orbits in the boundary are the only ones that are Legendrian. Assume further that on each of the boundary components ∂R_j and $\partial R_i'$, curves γ_j and γ_i' are specified such that for both manifolds the first k curves ($k \leq N$) are sections to the $\mathbb{S}^1$-action and the other curves intersect each orbit exactly twice. Then there is an $\mathbb{S}^1$-contactomorphism $\Phi : R \to R'$ such that $\Phi \circ \gamma_j = \gamma_j'$, if and only if $n(R, \gamma_1, \ldots, \gamma_N) = n(R', \gamma_1', \ldots, \gamma_N')$.*

Proof. The basic strategy is to find diffeomorphic sections with certain properties in R and R'. With these sections one can construct an $\mathbb{S}^1$-diffeomorphism between the 3-manifolds that maps the boundary curves in R onto the ones in R'. Afterwards this map is deformed to a contactomorphism.

According to Lemma IV.5, the contact form around an exceptional orbit is locally unique up to $\mathbb{S}^1$-contactomorphisms. Thus one can start the construction of Φ by taking an $\mathbb{S}^1$-contactomorphism from a small neighborhood of the exceptional orbits in R to a neighborhood of the orbits of the same type in R'. Choose also, for each $j \in \{1, \ldots, N-1\}$, an $\mathbb{S}^1$-diffeomorphism from a neighborhood of ∂R_j to a neighborhood of $\partial R_j'$ that maps γ_j onto γ_j'
.

The standard sections to the $\mathbb{S}^1$-action around the exceptional orbits extend to a global section σ on $R_{(\mathrm{princ})}$. In R', construct a section in the following way: Take σ in the neighborhood of the exceptional orbits and in the neighborhood of ∂R_j for $1 \leq j \leq N-1$ and map it with Φ to R'. Now extend the image of σ to a global section σ' on $R'_{(\mathrm{princ})}$.

By the assumptions of the lemma, we know that $n(R, \gamma_1, \ldots, \gamma_N) = n(R', \gamma_1', \ldots, \gamma_N')$, and by our construction $\iota(\sigma, \gamma_j) = \iota(\sigma', \gamma_j')$ for all $1 \leq j \leq N-1$. It follows that the intersection numbers $\iota(\sigma, \gamma_N)$ and $\iota(\sigma', \gamma_N')$ are also equal. Hence one can homotope σ' in such a way that its position with respect to γ_N' is the same as the one of σ with respect to γ_N.

One can map σ onto σ' and by using the $\mathbb{S}^1$-action, we obtain an $\mathbb{S}^1$-diffeomorphism $\Phi : R \to R'$ such that $\Phi \circ \gamma_j = \gamma_j'$ for all $j = 1, \ldots, N$.

To transform the map above into a contactomorphism we need to sharpen Remark IV.3 to avoid that the Moser trick moves the curves on the boundaries. The neighborhoods of the boundary components are of the form $\mathbb{S}^1 \times [0, \delta) \times \mathbb{S}^1$ with coordinates (t, r, φ), and the circle action on the last coordinate. Assume one contact form to be $\alpha = dt + r\, d\varphi$ and the other one $\alpha' = g(t, r)\, dt + h(t, r)\, dr + f(t, r)\, d\varphi$. The orbits in the boundary are Legendrian, hence $f(t, 0) = 0$ and $\partial_t f(t, 0) = 0$. Thus the contact condition along such an orbit becomes $g(t, 0) \neq 0$, and we can divide the whole form by the function g to obtain the equivalent form $dt + h(t, r)\, dr + f(t, r)\, d\varphi$ (with new functions f and h).

Define now a map $\Psi : R \to R$ by

$$(t, r, \varphi) \mapsto (t - (1 - \rho_\varepsilon(r)) r h(t, 0), r, \varphi)$$

for points with $r < \varepsilon$ and the identity otherwise. Here ρ_ε is the cut-off function defined in the proof of Lemma VII.9.

The map Ψ is an $\mathbb{S}^1$-diffeomorphism. It is surjective, because it is the identity on the two tori $\mathbb{S}^1 \times \{0\} \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \{\varepsilon\} \times \mathbb{S}^1$. The map is a local diffeomorphism because $\det(d\Psi) = 1 - r\,(1-\rho_\varepsilon(r))\,\partial_t h(t,0)$ does not vanish if we choose ε small enough. Injectivity relies on a similar argument: If $\Psi(t,r,\varphi) = \Psi(t',r',\varphi')$, then clearly $\varphi = \varphi'$ and $r = r'$. Finally $t - t' = r\,(1-\rho_\varepsilon(r))\,(h(t,0) - h(t',0))$. With the mean value theorem one sees that if $t \neq t'$, one has $1 = r\,(1-\rho_\varepsilon(r))\,\partial_t h(\hat{t},0)$ with $\hat{t} \in (t,t')$, which is not possible if ε is chosen small enough.

For $r = 0$ the forms α and $\Psi^*\alpha'$ are equal, hence the linear interpolation $\alpha_s = (1-s)\,\alpha + s\,\Psi^*\alpha'$ consists of $\mathbb{S}^1$-invariant contact forms. To apply the Moser trick one considers the vector field X_s that is the solution to the equations

$$\iota_{X_s}\alpha_s = 0 \quad \text{and} \quad \iota_{X_s} d\alpha_s = \lambda_s\,\alpha_s - \dot{\alpha}_s\ ,$$

with the function $\lambda_s := \iota_{Y_s}\dot{\alpha}_s$, where Y_s is the Reeb field of the contact form α_s. The solution X_s vanishes on ∂R, and X_s has a time-1 flow in a small neighborhood of the boundary. Hence one has constructed an $\mathbb{S}^1$-diffeomorphism between R and R' that maps the boundary curves onto each other, and respects the contact forms close to the boundaries and in the neighborhood of the exceptional orbits.

The proof is now finished by applying the Moser trick a second time, but now in the interior of the manifold. The vector field generates a global isotopy, because the two contact forms are identical close to the boundary components, and the vector field has compact support. $\square$

Example VII.1 (cont.). The Dehn-Euler number $n(R,\gamma)$ is the last invariant that needs to be computed to find $(\mathbb{S}^5,\alpha_\pm)$ in the classification scheme. The path γ can be taken to be $(e^{i\varphi},0,0)$ with $0 \le \varphi < 2\pi$, and a section in $R = \{(z_1,z_2,0) \in \mathbb{S}^5 | x_1y_2 > x_2y_1\}$ can be found by

$$\sigma: \quad \{z \in \mathbb{C}|\,\operatorname{Im} z > 0\} \hookrightarrow R \subset \mathbb{S}^5, \quad z \mapsto \frac{1}{\sqrt{2+2|z|^2}}\bigl(1+z, z-1, 0\bigr)\ .$$

The boundary of σ is composed of two segments $1/\sqrt{2}\cdot(e^{i\varphi},e^{i\varphi},0)$ with $\varphi \in [0,\pi]$ and $1/\sqrt{2+2x^2}\cdot(x+1,x-1,0)$ with $x \in (-\infty,\infty)$. The boundary can be smoothed at the points where the two components meet, but this has no effect on the intersection number, because the only intersection point of $\partial\sigma$ and γ is given by $(1,0,0)$, and hence $n(R,\gamma) = \pm 1$. The cross-section R has opposite orientations for α_+ and α_-, thus $n_+(R,\gamma) = 1$ and $n_-(R,\gamma) = -1$.

The complete set of invariants for $(\mathbb{S}^5,\alpha_\pm)$ is: The principal stabilizer is trivial, ${\mathbb{S}^5}_{(\mathrm{sing})}$ has a single component that is isomorphic to E_{twist}, the cross-section is $\mathbb{D}^2_{<1} \times \mathbb{S}^1$, and the Dehn-Euler number $n(R)$ equals ± 1.

Example VII.2 (cont.). Above, we already saw that the cross-section of any W^5_k is $\mathbb{S}^1$-diffeomorphic to $\mathbb{D}^2_{<1} \times \mathbb{S}^1$, and $\bigl(W^5_k\bigr)_{(\mathrm{sing})}$ is isomorphic to E_{triv} for k even and E_{twist} for k odd.

Now we will compute $n(R,\gamma)$ for (W^5_k,α_k) and (W^5_k,α_{-k}). The curve $\gamma(\varphi)$ is given by $(e^{i\varphi}, +ie^{\frac{k}{2}i\varphi},0,0)$ with $\varphi \in [0,2\pi]$ for k even and with $\varphi \in [0,4\pi]$ for k odd.

In III.6.1.2 we already found a section for the cross-section R: Set $r_0 = |z_0|$ and $A = \sqrt{2 - r_0^2 + \sqrt{(2-r_0^2)^2 - r_0^{2k}}}$. The map below is a section of $\overline{R}$:

$$\sigma: \quad \mathbb{D}^2 \hookrightarrow \overline{R}, \quad z_0 \mapsto \left(z_0, \frac{iz_0^k}{2A} + \frac{iA}{2}, -\frac{z_0^k}{2A} + \frac{A}{2}, 0 \right) .$$

The restriction of σ to $\partial\overline{R}$ is $\sigma(\varphi) = \bigl(e^{i\varphi}, \frac{i}{2}(1 + e^{ik\varphi}), \frac{1}{2}(1 - e^{ik\varphi}), 0\bigr)$.

The intersection of γ and $\partial\sigma$ is given by the equations $2e^{ik\varphi/2} = 1 + e^{ik\varphi}$ and $1 - e^{ik\varphi} = 0$, and hence $k\varphi = 4\pi n$ with $n \in \mathbb{Z}$. For $k = 0$, every point of $\partial\sigma$ lies in the curve of marked points, but by shifting the section a bit along the $\mathbb{S}^1$-action, one obtains $n(R, \gamma) = 0$. For k even, the curve γ is parametrized by $\varphi \in [0, 2\pi)$, and there are $k/2$ intersection points, for k odd, the curve γ closes for $\varphi \in [0, 4\pi)$, and there are k intersection points.

The calculations so far did not depend on the contact form, but one can check that R has different orientations for α_k and α_{-k}. This changes the orientation of $\partial\sigma$ and γ, but also of ∂R, and hence for (W_k^5, α_k) we have $n(R, \gamma) = k$, and for (W_k^5, α_{-k}) we have $n(R, \gamma) = -k$.

The complete set of invariants for $(W_k^5, \alpha_{\pm k})$ is: The principal stabilizer is trivial, $\bigl(W_k^5\bigr)_{(\mathrm{sing})}$ is isomorphic to E_{twist} for k odd and to E_{triv} for k even, the cross-section is $\mathbb{D}^2_{<1} \times \mathbb{S}^1$, and $n(R) = \pm k$. In particular it follows that the 5-sphere $(\mathbb{S}^5, \alpha_+)$ in Example VII.1 is equivalent to (W_1^5, α_{+1}), and $(\mathbb{S}^5, \alpha_-)$ is equivalent to (W_1^5, α_{-1}).

Note also that every 5-dimensional simply connected contact SO(3)-manifolds with singular orbits is SO(3)-contactomorphic to one of the Brieskorn examples $(W_k^5, \alpha_{\pm k})$. The reason is that the orbit space $M/\operatorname{SO}(3)$ of M has to be simply connected ([**Bre93**]), and must have non-empty boundary. Hence $M/\operatorname{SO}(3)$ is a 2-disk, and $M_{(\mathrm{sing})}$ has a single component. From this it follows that the cross-section is isomorphic to $\mathbb{D}^2_{<1} \times \mathbb{S}^1$. The principal stabilizer cannot be isomorphic to $\mathbb{Z}_2$, since then it follows by applying the Theorem of Seifert-van Kampen that $\pi_1(M) \cong \mathbb{Z}_2$. Thus, the principal stabilizer has to be trivial, and all cases are covered by the $(W_k^5, \alpha_{\pm k})$.

4. Construction of 5-manifolds

In this section, we will construct a manifold M for each of the possible combinations of invariants given in Theorem VII.1.

4.1. $M_{(\mathrm{sing})} = \emptyset$**.** The classification given in Theorem IV.16 shows that there is an $\mathbb{S}^1$-invariant contact structure without Legendrian orbits on any closed 3-dimensional contact $\mathbb{S}^1$-manifold R with non-vanishing orbifold Euler number and such that R has no special exceptional orbits or fixed points.

The 5-manifold M is then given by $M \cong \operatorname{SO}(3) \times_{\mathbb{S}^1} R$, where the circle on R acts with k-fold speed to get the desired stabilizer on M.

On the other hand, it follows from Lemma VII.6 that $0 \notin \mu(M)$, and thus R cannot have Legendrian orbits. It is also clear that R cannot have fixed points.

4.2. $M_{(\mathrm{sing})} \neq \emptyset$ **and trivial principal stabilizer.** Let $\overline{R}$ be any 3-dimensional $\mathbb{S}^1$-manifold without fixed points and without special exceptional orbits, but with non-empty boundary ∂R. By the requirement that only the $\mathbb{S}^1$-orbits on the boundary are Legendrian, the contact structure on $\overline{R}$ is uniquely determined (Remark IV.3).

Over the interior of $\overline{R}$, the 5-manifold $M^* = \operatorname{SO}(3) \times_{\mathbb{S}^1} (\overline{R} - \partial R)$ is a contact SO(3)-manifold. Now one has to glue in the singular orbits, in such a way as to get the chosen combination of components of type E_{triv} and E_{twist} and the Dehn-Euler number $n(R)$. First we will show how to glue in the standard model for E_{triv}; for this, we need to have a standard form for a neighborhood of ∂R.

Let σ be any section in $\overline{R}$ that is compatible with the standard sections around the exceptional orbits. In Lemma IV.8 it has been shown that any contact form around ∂R is equivalent to a standard form: Denote the coordinates of a collar $\mathbb{S}^1 \times [0,\varepsilon) \times \mathbb{S}^1$ around a boundary component by $(e^{it}, r, e^{i\varphi})$ and let the $\mathbb{S}^1$-action be $e^{i\vartheta} \cdot (e^{it}, r, e^{i\varphi}) = (e^{it}, r, e^{i(\varphi+\vartheta)})$. Every invariant contact form is up to an $\mathbb{S}^1$-contactomorphism equal to $dt + r\,d\varphi$. In general the section σ will not be of the form $\sigma(e^{it}, r) = (e^{it}, r, 1)$ in the collar though, but it is not very difficult to arrange the model neighborhood in this way. Let $[t]$ and $[\varphi] \in H_1(M, \mathbb{Z})$ be the classes given by $\mathbb{S}^1 \times \{0\} \times \{1\}$ and $\{1\} \times \{0\} \times \mathbb{S}^1$, respectively. The section σ represents an element $[t] + a\,[\varphi]$, and there is a linear map $A \in \mathrm{SL}(2,\mathbb{Z})$ that induces an $\mathbb{S}^1$-diffeomorphism such that σ represents $[t]$ in the new coordinates. The contact form becomes $(1+ar)\,dt + r\,d\varphi$, which after dividing by $1 + ar$ and rescaling in the r-direction can be transformed back into $dt + r\,d\varphi$. Now by deforming σ, one obtains a collar for the boundary where the action, the contact form, and the section are all in standard form.

The standard way of gluing is to consider $\mathbb{S}^1 \times T^*\mathbb{S}^2$ with SO(3)-action on the second factor and with the contact form $dt + \lambda_{\mathrm{can}}$. The cross-section of $\mathbb{S}^1 \times T^*\mathbb{S}^2$ looks exactly like the neighborhood of the boundary components of $\overline{R}$, which allows us to identify both. Since the cross-section determines the 5-manifold lying over it, this gives a gluing of $\mathbb{S}^1 \times T^*\mathbb{S}^2$ to M^*. In the boundary, the section σ and the curve of marked points are identical, but one can push σ a bit along the $\mathbb{S}^1$-action to avoid having any intersection points. Thus the contribution of this gluing to $n(R)$ is zero.

To construct a general M, i.e. an M with $n(R) \neq 0$ or with $E_{\mathrm{twist}} \subset M_{(\mathrm{sing})}$, we need to change the construction.

Assume first that we want to glue in a component of type E_{triv}, which adds $2c$ to the Dehn-Euler number. The neighborhood of ∂R was chosen above to be $\mathbb{S}^1 \times [0,\varepsilon) \times \mathbb{S}^1$ with contact form $dt + r\,d\varphi$ and with a section σ of the form $\sigma(e^{it}, r) = (e^{it}, r, 1)$. The matrix

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

induces a diffeomorphism, which can be isotoped as above to obtain a new model for the neighborhood of ∂R, where σ represents the homology class $[t] + c\,[\varphi]$, and where the contact form is still in standard form. Gluing E_{triv} along the cross-section $\overline{R}$ works again without any problem. The intersection number between the section σ and the curve of marked points gives now c.

To glue in a component of type E_{twist}, recall that the cross-section around E_{twist} could be described by $\mathbb{R} \times [0,\varepsilon) \times \mathbb{S}^1/\sim$ with the equivalence relation $(t, r, e^{i\varphi}) \sim (t+1, r, e^{i(\varphi+\pi)})$ and contact form $\alpha = dt + r\,d\varphi$. The curve of marked points was given by $\{(t,0,1)\}$ and $\{(t,0,-1)\}$. There is now a diffeomorphism $\Phi : \mathbb{S}^1 \times [0,\varepsilon) \times \mathbb{S}^1 \to \mathbb{R} \times [0,\varepsilon) \times \mathbb{S}^1/\sim$, $(e^{2\pi it}, r, e^{i\varphi}) \mapsto (t, r, e^{i(\varphi+\pi t/2)})$. The curve of marked points pulls back to $\{(e^{2\pi it}, 0, e^{-\pi it})\}$, and $\Phi^*\alpha = (1 + \pi r/2)\,dt + r\,d\varphi$, which can be isotoped into standard form. The model for the cross-section close to E_{twist} and close to ∂R looks identical, and it is possible to glue both parts. The Dehn-Euler number $n(R)$ can be arranged in the desired way as above.

4.3. $M_{(\mathrm{sing})} \neq \emptyset$ **and principal stabilizer is** $\mathbb{Z}_2$**.** If the principal stabilizer is isomorphic to $\mathbb{Z}_2$, then all components of $M_{(\mathrm{sing})}$ are equivalent to $\mathbb{S}^1 \times \mathbb{RP}^2$. The gluing occurs completely analogous to the way it was done above: Choose identical charts for a neighborhood of ∂R, and for the cross-section around $M_{(\mathrm{sing})}$, and glue along these.

5. Relation between the Dehn-Euler number and generalized Dehn twists

In Appendix D, a short introduction to **Dehn twists** is given. In this section we want to show that the Dehn-Euler number $n(R)$ counts the number of Dehn twists needed to glue in the singular orbits.

Assume a 5-dimensional contact SO(3)-manifold (M, α) is given whose principal stabilizer is trivial, and which has singular orbits of type E_{triv}. Above it was shown how to glue in new singular orbits by attaching them at the cross-section R in a way to arrange any Dehn-Euler number $n(R)$. The neighborhood of a component of $M_{(\mathrm{sing})}$ is SO(3)-contactomorphic to $(\mathbb{S}^1 \times T^*\mathbb{S}^2, dt + \lambda_{\mathrm{can}})$. Write points in $T^*\mathbb{S}^2$ as $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^6$ with $|\mathbf{q}| = 1$ and $\mathbf{q} \perp \mathbf{p}$. The cross-section is

$$R = \Big\{\big(t; (x, y, 0), (ry, -rx, 0)\big)\Big\} ,$$

and assume the section σ was of the form $\sigma(r, t) = (t, (1, 0, 0), (0, -r, 0))$.

Now cut out a small neighborhood of E_{triv}, and glue in a mapping torus $(M^{\pm k}_{\mathrm{Dehn}}, \beta^{\pm}_k)$ as described in Appendix D. It is easy to check that this respects the SO(3)-action. The component of the singular orbits in $M^{\pm k}_{\mathrm{Dehn}}$ correspond to E_{triv} if k is even, and to E_{twist} otherwise. In fact, it is known ([**Sei98**]) that on $T^*\mathbb{S}^2$ the Dehn twist $\tau^{\pm}_{2n}$ is isotopic to id and $\tau^{\pm}_{2n+1}$ is isotopic to τ^{+}_1 (both in the space of diffeomorphisms with compact support), hence the diffeomorphism type of M does not change after gluing in $M^{\pm k}_{\mathrm{Dehn}}$ if k is even.

Now, it only remains to see what effect this has on the Dehn-Euler number $n(R)$. The contact form on the mapping torus is

$$\beta^{\pm}_k = h^{\pm}_k(|\mathbf{p}|)\, dt + \lambda_{\mathrm{can}} \mp t|\mathbf{p}|\, d\big(f_k(|\mathbf{p}|)\big) .$$

The cross-section R in $\mathbb{R} \times T^*\mathbb{S}^2$ is equal to the one for the standard contact form,

$$R = \Big\{\big(t; (x, y, 0), (ry, -rx, 0)\big)\Big\}/\!\sim ,$$

because the last term of $\beta^{\pm}_k$ does not change the moment map ($\iota_{X_M} df_k = \mathcal{L}_{X_M} f_k$, but f_k only changes in radial direction).

To compute the local contribution to $n(R)$, notice that the section

$$\sigma(t, r) = (t; (1, 0, 0), (0, -r, 0))$$

to the $\mathbb{S}^1$-action in $\big\{\big(t; (x, y, 0), (ry, -rx, 0)\big)\big\} \subset \mathbb{R} \times T^*\mathbb{S}^2$ does not descend to a continuous section in the mapping torus. Instead one could replace σ by

$$\sigma(t, r) = \Big(t; \big(\cos(\pm t g_k(r)), -\sin(\pm t g_k(r)), 0\big), \big(-r\, \sin(\pm t g_k(r)), -r\, \cos(\pm t g_k(r)), 0\big)\Big) .$$

Since σ remains unchanged far away from the singular orbits, it extends to the unmodified section, and it is easy to check that σ induces a continuous section on $M^{\pm k}_{\mathrm{Dehn}}$.

The intersections of σ with the curve of marked points is given by

$$(\cos(\pm t g_k(0)), -\sin(\pm t g_k(0)), 0) = (\pm 1, 0, 0) ,$$

i.e. $\cos \pi k t = \pm 1$ and $\sin \pi k t = 0$, and then $kt \in \mathbb{Z}$. There are k points on ∂R, where σ intersects the marked set of points.

If k is odd, the boundary corresponds to E_{twist}. Then there is only a single curve of marked points and the contribution of this boundary to $n(R)$ is k. If k is even, then there are two disjoint curves of marked points, and there are only $k/2$ intersection points with the first one. But since for singular orbits of type E_{triv} this number is multiplied by 2, the contribution to $n(R)$ is again k.

Thus the Dehn-Euler number $n(R)$ counts the number of Dehn twists applied at $M_{(\text{sing})}$.

All constructions on $\mathbb{S}^1 \times \mathbb{S}^2$ in Appendix D are $\mathbb{Z}_2$-equivariant, and this allows us to build manifolds with principal stabilizer $\mathbb{Z}_2$ and arbitrary $n(R)$.

APPENDIX A

Equivariant Gray stability

The Moser trick is a powerful method for showing that two contact forms α_0 and α_1 on a manifold M are related by a contactomorphism. In a first step, one tries to find a smooth 1-parameter family of contact forms α_t on M with $t \in [0,1]$ connecting the two forms given above. Note that this is often relatively easy to accomplish, e.g. if α_0 and α_1 are sufficiently similar (C^1-close) then the linear interpolation will give the desired family. Once this family has been found the following arguments are applied.

Assume there is a smooth isotopy $\Phi_t : M \to M$ generated as the flow of a vector field X_t, i.e.

$$\left.\frac{d}{dt}\right|_{t=t_0} \Phi_t(p) = X_{t_0} \circ \Phi_{t_0}(p) \ ,$$

with $f_t \neq 0$, such that

$$\Phi_t^* \alpha_t = f_t \, \alpha_0 \ . \tag{2}$$

Below we will deduce equations for the field X_t. One can then consider these equations without the a priori assumption of having a smooth isotopy, and then try to construct one from the solutions of these equations.

Taking the derivative of (2) yields

$$\left.\frac{d}{dt}\right|_{t=t_0} (\Phi_t^* \alpha_t) = \dot{f}_{t_0} \, \alpha_0 \ .$$

The left side is equal to (see [**Geiar**])

$$\frac{d}{dt}(\Phi_t^* \alpha_t) = \Phi_t^* \left(\mathcal{L}_{X_t} \alpha_t + \dot{\alpha}_t\right) = \Phi_t^* \big(d(\alpha_t(X_t)) + \iota_{X_t} d\alpha_t + \dot{\alpha}_t\big) \ .$$

On the right side one can eliminate α_0 using equation (2), and one obtains

$$\Phi_t^* \left(d(\alpha_t(X_t)) + \iota_{X_t} d\alpha_t + \dot{\alpha}_t\right) = \frac{\dot{f}_t}{f_t} \cdot \Phi_t^* \alpha_t = \left(\frac{d}{dt} \ln(f_t)\right) \cdot \Phi_t^* \alpha_t \ .$$

By our assumption, Φ_t is a diffeomorphism, and we can apply its inverse to get

$$d(\alpha_t(X_t)) + \iota_{X_t} d\alpha_t + \dot{\alpha}_t = \left(\left(\frac{d}{dt} \ln(f_t)\right) \circ \Phi_t^{-1}\right) \alpha_t$$

If we further assume X_t to lie in the contact structure $\xi_t = \ker \alpha_t$, then the equation reduces to

$$\iota_{X_t} d\alpha_t + \dot{\alpha}_t = \left(\left(\frac{d}{dt} \ln(f_t)\right) \circ \Phi_t^{-1}\right) \alpha_t \tag{3}$$

The problem is that the right side depends on the flow Φ_t, which we are trying to compute. But by plugging the Reeb field Y_t of the contact form α_t, i.e. the unique vector field that satisfies $\alpha_t(Y_t) = 1$ and $\iota_{Y_t} d\alpha_t = 0$, into equation (3), we obtain

$$\dot\alpha_t(Y_t) = \left(\frac{d}{dt}\ln(f_t)\right) \circ \Phi_t^{-1} ,$$

which allows us to eliminate the term containing Φ_t. That means, if we find a solution X_t to the equations

$$\iota_{X_t}\alpha_t = 0 \quad \text{and} \quad \iota_{X_t} d\alpha_t = h_t \cdot \alpha_t - \dot\alpha_t ,$$

where $h_t = \dot\alpha_t(Y_t)$, such that X_t has a globally defined flow Φ_t, then Φ_t will have the desired property $\Phi_t^*\alpha_t = f_t\,\alpha_0$.

If we restrict the second equation to the contact structure ξ_t, then we can use that the 2-form $d\alpha_t$ is non-degenerate on ξ_t, and we find a solution $X_t \in \xi_t$. Note that X_t also solves the second equation on TM, because both sides vanish if we plug in the Reeb field Y_t.

The Moser trick is usually applied for closed manifolds, because there every vector field has a globally defined flow. In this thesis we are interested in equivariant applications.

LEMMA A.1. *Let G be a connected Lie group that acts smoothly on a manifold M, and let Φ_t be the flow of a time-dependent vector field X_t. If the Lie bracket $[X_M, X_t]$ vanishes for every $X \in \mathfrak{g}$, then the maps Φ_t are G-equivariant.*

PROOF. To show that Φ_t commutes with the action of any element $g \in G$ note that g can be written as a finite product $g = g_1 \cdots g_n$, with $g_j = \exp(X_j)$ and $X_j \in \mathfrak{g}$. Hence it is enough to show $\Phi_t \circ \exp(X) = \exp(X) \circ \Phi_t$ for small $X \in \mathfrak{g}$.

It is well-known that the flows Φ_s^Y and Φ_t^Z of time-independent vector fields commute if the bracket $[Y, Z]$ vanishes ([**KMS93**, Corollary I.3.15]). We will make use of this result by constructing time-independent flows related to X_t and X_M.

Define on $M \times I$ with $I = [0, 1]$ the vector fields $Y(p, t) := X_t(p) + \partial_t$, and $Z(p, t) := X_M(p)$. The Lie bracket $[Y, Z] = [X_t, X_M] + [\partial_t, X_M]$ vanishes. The flow of Y is given by $\Phi_s^Y(p, t) = (\Phi_{s+t}(p), s + t)$, and the flow of Z is just $\Phi_s^Z(p, t) = (\exp(sX) \cdot p, t)$. Both flows commute, and one has

$$\begin{aligned}(\Phi_{s_1+t}(\exp(s_2X) \cdot p), s_1 + t) &= \Phi_{s_1}^Y \circ \Phi_{s_2}^Z(p, t) = \Phi_{s_2}^Z \circ \Phi_{s_1}^Y(p, t) \\ &= (\exp(s_2X) \cdot \Phi_{s_1+t}(p), s_1 + t) .\end{aligned}$$

This gives the desired equality $\exp(X) \cdot \Phi_t(p) = \Phi_t(\exp(X) \cdot p)$. □

LEMMA A.2. *Let G be a connected Lie group that acts smoothly on a manifold M. Assume there is a 1-parameter family α_t of G-invariant contact forms (with $0 \le t \le 1$) on M, and that the vector field X_t, defined as solution of the equations*

$$\iota_{X_t}\alpha_t = 0 \quad \textit{and} \quad \iota_{X_t} d\alpha_t = h_t \cdot \alpha_t - \dot\alpha_t ,$$

with Y_t the Reeb field of α_t and $h_t = \dot\alpha_t(Y_t)$, has a flow Φ_t that exists for all $t \in [0, 1]$. Then the maps Φ_t are G-equivariant contactomorphisms, and all α_t are equivalent contact forms.

PROOF. First note that for every t the Reeb field Y_t of α_t commutes with the infinitesimal generators of the action X_M for all $X \in \mathfrak{g}$, which means that the Lie bracket $[X_M, Y_t]$ vanishes. The Reeb field is the unique solution of the equations

$$\alpha_t(Y_t) = 1 \quad \text{and} \quad \iota_{Y_t} d\alpha_t = 0 .$$

With the Leibniz rules for the Lie derivative one obtains

$$0 = \mathcal{L}_{X_M}\alpha_t(Y_t) = \iota_{Y_t}\mathcal{L}_{X_M}\alpha_t + \alpha_t([X_M, Y_t]) = \alpha_t([X_M, Y_t])$$

and

$$0 = \mathcal{L}_{X_M}(\iota_{Y_t} d\alpha_t) = \iota_{Y_t}\mathcal{L}_{X_M} d\alpha_t + \iota_{[X_M,Y_t]} d\alpha_t = \iota_{[X_M,Y_t]} d\alpha_t \; .$$

These two equations together show that $[X_M, Y_t] = 0$.

To prove that $[X_M, X_t] = 0$, apply the Lie derivative to the defining equation for X_t:

$$0 = \mathcal{L}_{X_M}\alpha_t(X_t) = \alpha_t([X_M, X_t]) \quad \text{and} \quad \mathcal{L}_{X_M}(\iota_{X_t} d\alpha_t) = \iota_{[X_M,X_t]} d\alpha_t = (\mathcal{L}_{X_M} h_t)\cdot\alpha_t \; .$$

The last term vanishes, because the function h_t is given by $h_t = \dot\alpha_t(X_t)$. Together with Lemma A.1 this shows that all Φ_t are G-equivariant. ☐

THEOREM A.3. *Let G be a connected Lie group that acts smoothly on a manifold M. Assume there are two G-invariant contact forms α_0 and α_1 such that the equations*

$$\alpha_0 = \alpha_1 \quad \textit{and} \quad d\alpha_0 = d\alpha_1$$

both hold at a point $p \in M$. Then the two forms are contactomorphic in a small neighborhood of the orbit $\mathrm{Orb}(p)$.

PROOF. Since both forms are G-invariant, it is clear that the equations hold on the whole orbit $\mathrm{Orb}(p)$. It is also clear that the convex span $\alpha_t = (1-t)\,\alpha_0 + t\,\alpha_1$ consists of contact forms in a small neighborhood of the orbit, because $\alpha_t = \alpha_0$ on $\mathrm{Orb}(p)$, and the contact condition is open.

The Moser equations reduce on the orbit to

$$\iota_{X_t}\alpha_t = 0 \quad \text{and} \quad \iota_{X_t} d\alpha_t = h_t\cdot\alpha_t - \dot\alpha_t = 0 \; ,$$

and hence the vector field X_t vanishes on $\mathrm{Orb}(p)$. There is a small neighborhood of $\mathrm{Orb}(p)$, where the flow is defined for all $t \in [0,1]$. ☐

APPENDIX B

3-dimensional contact toric manifolds

As shown in Section IV.6.1, there are two commuting contact $\mathbb{S}^1$-actions on the manifolds $(W_k^3, \alpha_\pm)$. We want to show in this appendix how these spaces fit into the classification scheme for toric contact manifolds.

Definition. A $(2n-1)$-dimensional **toric contact manifold** (M, α) is a contact manifold with an n-torus $\mathbb{T}^n$ acting effectively through contactomorphisms.

Toric contact manifolds have been classified by Lerman in [**Ler03**]. For manifolds of dimension 5 or larger the classification is given basically by the **moment polytope**, i.e. by the cone over the image of the moment map (which happens to be a convex polytope). Here we are only interested in $\mathbb{T}^2$-actions on $(W_k^3, \alpha_\pm)$ (see Section IV.6.1). In the 3-dimensional case, the image of the moment map represents a curve that can run more than once around the origin, and hence the moment polytope alone does not classify the 3-dimensional toric manifolds. For this case one has instead to normalize the moment map such that its image lies in the unit circle of $\mathfrak{t}^* \cong \mathbb{R}^2$. Measure the angle φ_1 of the starting ray, and the total angle φ_2 that is traced out by the moment map on $\mathbb{S}^1 \subset \mathfrak{t}^*$.

Theorem B.1 (Lerman). *Closed connected co-oriented (!) 3-dimensional toric contact manifolds (M, α) fall into one of the following cases:*

1. *If the action of $\mathbb{T}^2$ is free, then M is diffeomorphic to $\mathbb{T}^3 \cong \mathbb{T}^2 \times \mathbb{S}^1$. With $(\varphi_1, \varphi_2, t)$ the coordinates of $\mathbb{T}^2 \times \mathbb{S}^1$, the contact form α is given by*

$$\alpha = \cos(nt)\, d\varphi_1 + \sin(nt)\, d\varphi_2$$

   for some $n \in \mathbb{N}$.

2. *If the $\mathbb{T}^2$-action is not free, then M is diffeomorphic to a lens space (including $\mathbb{S}^1 \times \mathbb{S}^2$). As contact toric manifold, M is classified by two real numbers φ_1, φ_2 with $0 \le \varphi_1 < 2\pi$, $\varphi_1 < \varphi_2$ such that both $\tan\varphi_1$ and $\tan\varphi_2$ are rational.*

The Milnor $\mathbb{S}^1$-action defined in Section III.6.1.1 and the SO(2)-action defined in Section III.6.1.2 commute, and give $(W_k^3, \alpha_\pm)$ the structure of a contact toric manifold. Denote the generator of the Milnor action by Y, and the one of the SO(2)-action by Z. The $\mathbb{T}^2$-action is not free, because the two circles of points

$$\{(0, e^{i\varphi}, ie^{i\varphi}) | e^{i\varphi} \in \mathbb{S}^1\} \quad \text{and} \quad \{(0, e^{i\varphi}, -ie^{i\varphi}) | e^{i\varphi} \in \mathbb{S}^1\}$$

have non-trivial stabilizer in $\mathbb{T}^2$. Points in the first set remain fixed under elements generated by $Y - (\operatorname{lcm}(k,2)/2)\, Z$, points in the second set are fixed by the circle generated by $Y + (\operatorname{lcm}(k,2)/2)\, Z$. All the toric manifolds $(W_k^3, \alpha_\pm)$ thus fall into the second case of Lerman's classification theorem.

Note that, unfortunately, for k odd the $\mathbb{T}^2$-action is not effective. We will first treat the manifolds W_{2n}^3, because there the torus does not have to be modified.

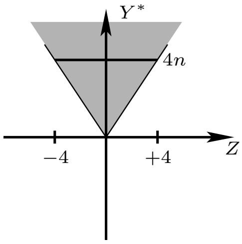


FIGURE 1. Moment polytope of (W^3_{2n}, α_+)

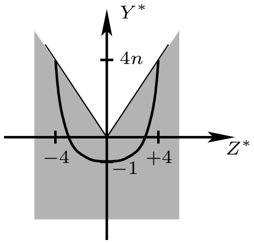


FIGURE 2. Moment polytope of (W^3_{2n}, α_-)

LEMMA B.2. *The image of the moment map for the manifolds* $(W^3_{2n}, \alpha_\pm)$ *is displayed in Figure 1 and 2. The angles* φ_1 *and* φ_2 *can easily be read off.*

PROOF. Remember that the Milnor action had to be rescaled to give an effective circle action, hence the generator of the action is

$$Y_{W^3_{2n}}(z_0, z_1, z_2) := \big(-y_0, x_0, -ny_1, nx_1, -ny_2, nx_2\big)\ .$$

The moment map gives

$$\langle \mu_+ | Y\rangle = 4n \quad\text{ and }\quad \langle \mu_- | Y\rangle = 4n - (4n+1)\,|z_0|^2\ .$$

The generator $Z_{W^3_{2n}}$ for the SO(2)-action and its moment map were already given in Section IV.6.1. The moment map evaluates to

$$\langle \mu_\pm | Z\rangle = 4\,(x_1 y_2 - x_2 y_1)\ .$$

We would like to express the function $\langle \mu_+ | Z\rangle$ as a map that only depends on the z_0-coordinate. Note that the moment map is invariant under the torus action. Hence, instead of considering the image of the moment map for all points (z_0, z_1, z_2), it is enough to consider only points of the form (r_0, x_1, iy_2), where $r_0 \in [0,1]$ and $x_1 \geq 0$. In particular, we have

$$x_1 = \sqrt{\frac{2 - r_0^2 - r_0^{2n}}{2}} \quad\text{ and }\quad y_2 = \pm\sqrt{\frac{2 - r_0^2 + r_0^{2n}}{2}}\ ,$$

and

$$\langle \mu_\pm(z_0, z_1, z_2) | Z\rangle = \pm 2\sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n}}\ . \qquad \square$$

For the manifolds W^3_{2n+1}, the combination of the Milnor and the SO(2)-action is not an effective torus action. The element $e^{\pi i}$ acts under the Milnor action (with $\operatorname{lcm}(2, 2n+1) = 2(2n+1)$) as

$$(e^{i\pi}, (z_0, z_1, z_2)) \mapsto (e^{2k\pi i/k} z_0, e^{2k\pi i/2} z_1, e^{2k\pi i/2} z_1) = (z_0, -z_1, -z_2)\ ,$$

and this map is equal to the SO(2)-action of $e^{i\pi}$. To make the action effective, we have to quotient out the torus $\mathbb{T}^2$ by the subgroup generated by $(e^{i\pi}, e^{i\pi})$. This is equivalent to adding to the lattice $\mathbb{Z}^2 = \langle (1,0), (0,1)\rangle$ the points generated by $(1/2, 1/2)$. The new lattice can be represented by as $\langle (1,0), (1/2, 1/2)\rangle$.

In our case, the infinitesimal generator for the effective $\mathbb{T}^2$-action corresponds thus to the unmodified generator $Z_{W^3_{2n+1}}$ of the SO(2)-action, and the second vector is given as

$$Y_{W^3_{2n+1}}(z_0, z_1, z_2) := \Big(-y_0, x_0, -\frac{2n+1}{2}\,y_1, \frac{2n+1}{2}\,x_1, -\frac{2n+1}{2}\,y_2, \frac{2n+1}{2}\,x_2\Big) + \frac{1}{2}\,Z_{W^3_{2n+1}}\ .$$

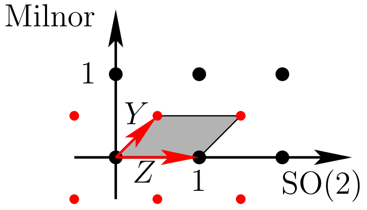


FIGURE 3. The vectors $(1,0)$, $(1/2,1/2)$ generate the refined lattice.

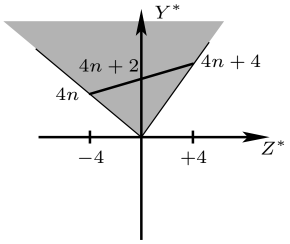


FIGURE 4. Moment polytope of (W^3_{2n+1}, α_+)

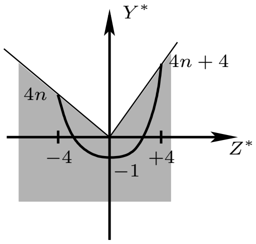


FIGURE 5. Moment polytope of (W^3_{2n+1}, α_-)

LEMMA B.3. *The image of the moment map for the manifolds* $(W^3_{2n+1}, \alpha_\pm)$ *is displayed in Figure 4 and 5.*

PROOF. The moment map for the vector $Z_{W^3_{2n+1}}$ is the same as for k even, and is

$$\langle \mu_\pm | Z \rangle = 4\left(x_1 y_2 - x_2 y_1\right) ,$$

and after rewriting the dependence in z_0, we obtain again

$$\langle \mu_\pm(z_0, z_1, z_2) | Z \rangle = \pm 2 \sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n+2}} \, .$$

The moment map for the vector $Y_{W^3_{2n+1}}$ defined above gives

$$\langle \mu_+ | Y \rangle = 4n + 2 + \frac{1}{2} \langle \mu_+ | Z \rangle = 4n + 2 \pm \sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n+2}}$$

and

$$\begin{aligned} \langle \mu_- | Y \rangle &= 4n + 2 - (4n+3)\,|z_0|^2 + \frac{1}{2} \langle \mu_+ | Z \rangle \\ &= 4n + 2 - (4n+3)\,|z_0|^2 \pm \sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n+2}} \, . \end{aligned}$$

□

Finally, we would like to check that all manifolds (W^3_k, α_-) are overtwisted. The argument will be based on finding a suitable circle lying inside the torus $\mathbb{T}^2$ to which Lemma IV.19 can be applied.

LEMMA B.4. *The contact structure defined by* α_+ *on* W^3_k *is fillable, and the one defined by* α_- *is overtwisted.*

PROOF. The manifolds (W_k^3, α_+) have a filling given by the desingularization of the corresponding Brieskorn variety $V_f = f^{-1}(0)$ with $f(z_0, z_1, z_2) = z_0^k + z_1^2 + z_2^2$.

In the examples W_{2n}^3 with even index, the circle sitting in the torus $\mathbb{T}^2$ that is generated by $Y - n\,Z$ has fixed point set $\{(0, e^{i\varphi}, ie^{i\varphi})\}$, and the points in the zero set of

$$f_\pm(z_0) = \alpha_-\bigl(Y_{W_{2n}^3} - n\,Z_{W_{2n}^3}\bigr) = 4n - (4n+1)\,|z_0|^2 \pm 2n\sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n}}$$

lie on Legendrian orbits. The function f_- only vanishes for $z_0 = 0$, because the derivative of f_- is negative. The points $(0, z_1, z_2)$ are fixed points and do not give Legendrian orbits, but the function f_+ has a zero, because $f_+(0) = 8n$ and $f_+(1) = -1$. Hence we have found an $\mathbb{S}^1$-action both with Legendrian orbits and fixed points, and Lemma IV.19 shows that (W_{2n}^3, α_-) is overtwisted.

For the manifolds W_{2n+1}^3 with odd index, the stabilizer for all points in $\{(0, e^{i\varphi}, ie^{i\varphi})\}$ is generated by $Y - (n+1)\,Z$. Legendrian orbits of this action correspond to zeros of the functions

$$\begin{aligned} f_\pm(z_0) &= \alpha_-\bigl(Y_{W_{2n}^3} - (n+1)\,Z_{W_{2n}^3}\bigr) \\ &= 4n + 2 - (4n+3)\,|z_0|^2 \pm (2n+1)\sqrt{(2 - |z_0|^2)^2 - |z_0|^{4n+2}}\,. \end{aligned}$$

Again the function f_- only vanishes at the fixed point in zero, but f_+ has a zero, because $f_+(0) = 8n + 4$ and $f_+(1) = -1$. By the same lemma as above it follows that these manifolds are also overtwisted. □

REMARK B.1. It would be interesting to look into the proof of Lerman's classification theorem to check the following conjecture: There seem to be two different circles in the torus which have fixed points. Each of these circles is generated by elements orthogonal to the rays enclosing the moment polytope. When the moment polytope spans more than 180°, it appears that the argument above can be applied to show that the contact structure is overtwisted.

APPENDIX C

Remarks on Lie algebras and Lie coalgebras

Let $V \leq W$ be vector spaces and let $\iota : V \hookrightarrow W$ be the inclusion. In general there is no natural embedding $\varphi : V^* \hookrightarrow W^*$ such that $\iota^* \circ \varphi = \mathrm{id}_{V^*}$.

Such embeddings φ can be constructed by choosing a metric on W and taking the orthogonal splitting $W = V \oplus V^\perp$. The orthogonal projection $\pi : W \to V$ with respect to the chosen metric induces then a map $\varphi = \pi^*$. For general vector spaces such an embedding depends on the metric, but it will be shown below that for a compact Lie algebra $\mathfrak{g} = W$ and certain subalgebras $\mathfrak{h} = V$, the orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ is independent of the Ad-invariant metric chosen, and in this sense there is then a natural embedding $\pi^* : \mathfrak{h}^* \hookrightarrow \mathfrak{g}^*$.

In Chapter V this is used to reconstruct moment maps in Lemma V.2.

Let G always be a connected, compact Lie group, and let $\mathfrak{g}$ be its Lie algebra. The natural G-action on $\mathfrak{g}$ is the adjoint one, and on $\mathfrak{g}^*$ it is the coadjoint action. An $\mathrm{Ad}(G)$-invariant metric induces a G-equivariant diffeomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, but this diffeomorphism is, in general, not canonical. Still, most known results from the adjoint action on $\mathfrak{g}$ carry over to $\mathfrak{g}^*$. An interesting presentation of the G-manifold structure of $\mathfrak{g}$ can be found in [**DK00**].

At every $X \in \mathfrak{g}$ there is a unique maximal slice $S_X \subseteq \mathfrak{g}$, and this slice is an open set in $\mathfrak{g}_X = \ker \mathrm{ad}(X)$ (in particular $X \in \mathfrak{g}_X$), which is the Lie algebra of the stabilizer $G_X \leq G$ of X. For every $X \in \mathfrak{g}$ there is a splitting

$$\mathfrak{g} = \mathfrak{g}_X \oplus \mathrm{ad}(X)\mathfrak{g}.$$

The splitting is natural in the sense that $\mathfrak{g}_X$ and $\mathrm{ad}(X)\mathfrak{g}$ are orthogonal to each other with respect to any Ad-invariant metric $\mathfrak{m}$:

$$\mathfrak{m}(\mathfrak{g}_X, \mathrm{ad}(X)\mathfrak{g}) = -\mathfrak{m}(\mathrm{ad}(X)\mathfrak{g}_X, \mathfrak{g}) = 0 \ .$$

Lemma C.1. *Let $\mathfrak{g}$ be a Lie algebra of a compact Lie group G, and let $\mathfrak{h} \leq \mathfrak{g}$ be a subalgebra for which there is an element $X \in \mathfrak{g}$ with $\mathfrak{h} = \ker \mathrm{ad}(X)$.*

Then there is a natural splitting (which does not depend on the particular X)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \ .$$

Because an $\mathrm{Ad}(G)$-invariant metric induces a G-equivariant diffeomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, it follows that at each point $\nu \in \mathfrak{g}^*$ there is single maximal slice $S^*_\nu \subseteq \mathfrak{g}^*$ (for otherwise there would also be several slices at points in $\mathfrak{g}$).

Let $\nu \in \mathfrak{g}^*$ be an arbitrary element. If $X \in \mathfrak{g}$ is an element dual to ν (with respect to some $\mathrm{Ad}(G)$-invariant metric $\mathfrak{m}$), then the slice S^*_ν lies inside $\ker\big(\mathrm{ad}(X)^*\big)$, as can be seen from $\mathrm{ad}(X)^* S^*_\nu = \mathfrak{m}(S_X, \mathrm{ad}(X)\,\cdot) = -\mathfrak{m}(\mathrm{ad}(X)S_X, \,\cdot\,) = 0$.

Lemma C.2. *Let G be a connected, compact Lie group, and let $\nu \in \mathfrak{g}^*$ be an element with stabilizer $G_\nu \leq G$.*

Lemma C.1 gives a natural projection $\pi : \mathfrak{g} = \mathfrak{g}_\nu \oplus \mathfrak{g}_\nu^\perp \to \mathfrak{g}_\nu$, and this map induces an embedding $\pi^* : \mathfrak{g}_\nu^* \hookrightarrow \mathfrak{g}^*$.

Let $\tilde{\nu} \in S_\nu^*$ be an arbitrary element in the slice at ν. The element $\iota^* \tilde{\nu}$ lies in $\mathfrak{g}_\nu^*$ (here $\iota : G_\nu \to G$ is the natural embedding). Now the original element $\tilde{\nu}$ can be recovered from $\iota^* \tilde{\nu}$ with π^*, i.e.

$$\tilde{\nu} = \pi^* \iota^* \tilde{\nu} \ .$$

Proof. Let $Y \in \mathfrak{g}_\nu$ be an arbitrary element. One easily sees that

$$(\pi^* \iota^* \tilde{\nu})(Y) = (\iota^* \tilde{\nu})(\pi Y) = (\iota^* \tilde{\nu})(Y) = \tilde{\nu}(Y) \ .$$

Let now $Z \in \mathfrak{g}_\nu^\perp$ be another element. For the left side it follows

$$(\pi^* \iota^* \tilde{\nu})(Z) = (\iota^* \tilde{\nu})(\pi_{\mathfrak{h}} Z) = (\iota^* \tilde{\nu})(0) = 0 \ .$$

Denote the element dual to ν by $\nu^\dagger$. For the right side it is known that $\mathfrak{g}_\nu^\perp = \mathrm{ad}(\nu^\dagger)\mathfrak{g}$, i.e. $Z = \mathrm{ad}(\nu^\dagger)\tilde{Z}$ with some $\tilde{Z} \in \mathfrak{g}$, thus we have $\tilde{\nu}(Z) = (\mathrm{ad}(\nu^\dagger)^* \tilde{\nu})(\tilde{Z})$. But we remarked above that the slice at ν lies in $\ker\big(\mathrm{ad}(\nu^\dagger)^*\big)$. $\square$

APPENDIX D

Generalized Dehn twists in contact topology

In relation with open book decompositions, Giroux proposed a method for obtaining new contact manifolds from given ones by removing a certain open set, and gluing in some other set. First we will describe the sets that are cut out: Let (M, α) be a $(2n+1)$-dimensional contact manifold, and assume there is an embedding $\mathbb{S}^1 \times \mathbb{S}^n \hookrightarrow M$ or $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n \hookrightarrow M$ (where $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n = \{(t,p) \in \mathbb{R} \times \mathbb{S}^n\}/\sim$ with $(t,p) \sim (t+1,-p)$) such that the image of every n-sphere is Legendrian, and such that the $\mathbb{S}^1$-direction is always transverse to the contact structure. Note that for n odd, the antipodal map is isotopic to the identity, and in that case it is enough to consider embeddings of $\mathbb{S}^1 \times \mathbb{S}^n$.

By the Weinstein Theorem, the neighborhood of $\mathbb{S}^1 \times \mathbb{S}^n$ is contactomorphic to a neighborhood of the zero section in $(\mathbb{S}^1 \times T^*\mathbb{S}^n, dt + \lambda_{\mathrm{can}})$, and the neighborhood of $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n$ is contactomorphic to a neighborhood of the zero section in $(\mathbb{S}^1 \widetilde{\times} T^*\mathbb{S}^n, dt + \lambda_{\mathrm{can}})$. If the contact structure on M was invariant under an $\mathrm{SO}(n+1)$-action, and if every Legendrian sphere was an $\mathrm{SO}(n+1)$-orbit, then we can even apply the equivariant Weinstein Theorem (Lemma VII.8).

There exists embeddings of $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n$ into any Darboux chart, and hence into any contact manifold (M, α), such that every n-sphere $\{\star\} \times \mathbb{S}^n$ is Legendrian. A map of this type can be constructed as follows: Begin with the embedding

$$\begin{aligned} \mathbb{S}^1 \widetilde{\times} \mathbb{S}^n &\hookrightarrow (\mathbb{S}^{2n+1}, \alpha = \sum x_j\, dy_j - y_j\, dx_j) \\ (e^{i\varphi}, p) &\mapsto e^{i\varphi} \cdot p\,. \end{aligned}$$

One can delete a point from $\mathbb{S}^{2n+1}$ to obtain ([**Geiar**])

$$(\mathbb{S}^{2n+1} - \{\star\}, \alpha) \cong (\mathbb{R}^{2n+1}, \alpha_0 = dt + \sum x_j\, dy_j - y_j\, dx_j)\,,$$

and hence there is an embedding $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n \hookrightarrow (\mathbb{R}^{2n+1}, \alpha_0)$. By rescaling

$$(t, x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (\lambda^2 t, \lambda x_1, \dots, \lambda x_n, \lambda y_1, \dots, \lambda y_n)$$

one can push $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^n$ into an arbitrarily small chart.

On the other hand, there are sometimes obvious obstructions for embedding $\mathbb{S}^1 \times \mathbb{S}^n$ into an arbitrary manifold. For example, in dimension 5 the intersection number of $\mathbb{S}^1 \times \mathbb{S}^2$ with a sphere $\{\star\} \times \mathbb{S}^2$ in $\mathbb{S}^1 \times T^*\mathbb{S}^2$ is 2. Hence $H_2(M)$ and $H_3(M)$ cannot vanish, if we want to embed $\mathbb{S}^1 \times \mathbb{S}^2$ in the desired way into M.

Let U_1 be a neighborhood of the sphere bundle of $(\mathbb{S}^1 \times T^*\mathbb{S}^n, dt + \lambda_{\mathrm{can}})$, and let U_2 be a neighborhood of the sphere bundle of $(\mathbb{S}^1 \widetilde{\times} T^*\mathbb{S}^n, dt + \lambda_{\mathrm{can}})$. Note that the map

$$\begin{aligned} \Phi : U_1 &\to U_2 \\ (t, \mathbf{q}, \mathbf{p}) &\mapsto (t, \mathbf{q}\, \cos(\pi t) + \mathbf{p}/|\mathbf{p}|\, \sin(\pi t), -|\mathbf{p}|\mathbf{q}\, \sin(\pi t) + \mathbf{p}\, \cos(\pi t)) \end{aligned}$$

can be isotoped into a contactomorphism. Hence it is the same sets that can be glued in, in either of the two cases.

1. Symplectic Dehn twists

A **Dehn twist** τ_k^- or τ_k^+ (with $k \in \mathbb{N}_0$) is a diffeomorphism from $T^*\mathbb{S}^{n-1}$ to itself constructed in the following way. Write points in $T^*\mathbb{S}^{n-1}$ as $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n}$ with $|\mathbf{q}| = 1$ and $\mathbf{q} \perp \mathbf{p}$.

Set

$$\tau_k^{\pm}(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \cos\bigl(\pm g_k(\mathbf{p})\bigr) & |\mathbf{p}|^{-1}\sin\bigl(\pm g_k(\mathbf{p})\bigr) \\ -|\mathbf{p}|\sin\bigl(\pm g_k(\mathbf{p})\bigr) & \cos\bigl(\pm g_k(\mathbf{p})\bigr) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

If g_k was equal to $g_k(r) = r$, then $\tau_k^{\pm}$ would just be the standard geodesic flow. Instead, here we choose $g_k(\mathbf{p}) = \pi k + f_k(|\mathbf{p}|)$, where f_k is a smooth function that increases monotonously from 0 to πk on an interval that will be specified later. Outside this interval, f_k will be identically equal to 0 or πk. Though the details do not matter for the Dehn twist itself, our computations will turn out to put some constraints on f_k.

For small $|\mathbf{p}|$, the map $\tau_k^{\pm}$ equals $(-1)^k \,\mathrm{id}$, while for large $|\mathbf{p}|$ it equals the identity map.

Definition. The map τ_k^+ ($k \in \mathbb{N}$) is called a **k-fold right-handed Dehn twist**. The map τ_k^- is called a **k-fold left-handed Dehn twist**.

We will now construct a mapping torus of $T^*\mathbb{S}^{n-1}$ using these Dehn twists following the construction of Giroux and Mohsen [**Gir02b**]. The canonical 1-form $\lambda_{\mathrm{can}} = \mathbf{p} \cdot d\mathbf{q}$ on $T^*\mathbb{S}^{n-1}$ transforms like

$$\bigl(\tau_k^{\pm}\bigr)^* \lambda_{\mathrm{can}} = \lambda_{\mathrm{can}} \pm |\mathbf{p}|\, d\bigl(f_k(|\mathbf{p}|)\bigr) .$$

Note that the difference $\lambda_{\mathrm{can}} - \bigl(\tau_k^{\pm}\bigr)^* \lambda_{\mathrm{can}}$ is exact. This implies in particular that the Dehn twists are symplectomorphisms of $(T^*\mathbb{S}^{n-1}, d\lambda_{\mathrm{can}})$. As a primitive of the difference $\lambda_{\mathrm{can}} - \bigl(\tau_k^{\pm}\bigr)^* \lambda_{\mathrm{can}}$ we will take

$$h_k^{\pm}(|\mathbf{p}|) := 1 \mp \int_0^{|\mathbf{p}|} s f_k'(s)\, ds .$$

For left-handed Dehn twists τ_k^-, the function h_k^- is always positive, but for right-handed Dehn twists τ_k^+, the function h_k^+ can be assumed to be positive by choosing a suitable interval where f_k increases. To be more explicit, choose a smooth function f that is identically 0 on the interval $[0, 1]$, on the interval $[1, 2]$ it increases monotonically from 0 to 1, and f is identically 1 on the interval $[2, \infty)$. Furthermore, we may assume that the derivative f' is bounded by 2. Then we can take $f_k(x) := k\pi f(c_k x)$ with $c_k > 3k\pi$. We have

$$\int_0^{|\mathbf{p}|} s f_k'(s)\, ds \le \int_0^{\infty} k\pi c_k s f'(c_k s)\, ds \le k\pi \int_0^{\infty} y f'(y)\, dy / c_k \le \frac{k\pi}{c_k} \int_1^2 y 2\, dy = \frac{3k\pi}{c_k} ,$$

where we have substituted $y = c_k s$ and used that $f'(y) = 0$ outside the interval $[1, 2]$ and that f' is bounded by 2. Our choice of c_k ensures for $k > 0$ that this integral is indeed smaller than 1, so h_k^+ is positive.

2. The mapping torus

Consider the map

$$\begin{aligned} \varphi_k^{\pm} :\ \mathbb{R} \times T^*\mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \times T^*\mathbb{S}^{n-1}, \\ (t; \mathbf{q}, \mathbf{p}) &\longmapsto \bigl(t + h_k^{\pm}(|\mathbf{p}|); \tau_k^{\pm}(\mathbf{q}, \mathbf{p})\bigr) . \end{aligned}$$

This map preserves the contact form $dt+\lambda_{\mathrm{can}}$ on $\mathbb{R}\times T^*\mathbb{S}^{n-1}$, so we obtain an induced contact structure on $\mathbb{R}\times T^*\mathbb{S}^{n-1}/\varphi_k^\pm$.

The manifolds $(\mathbb{R}\times T^*\mathbb{S}^{n-1}/\varphi_k^\pm, dt+\lambda_{\mathrm{can}})$ are a bit inconvenient, because it is not possible to recognize easily the fibers in t-direction. To make computations easier, we will use instead a different mapping torus that is contactomorphic to the manifold above. Let $\mathbb{R}\times T^*\mathbb{S}^{n-1}/\sim_k^\pm$ be the mapping torus obtained by identifying $(t;\mathbf{q},\mathbf{p}) \sim_k^\pm (t+1;\tau_k^\pm(\mathbf{q},\mathbf{p}))$. We can define a diffeomorphism

$$\mathbb{R}\times T^*\mathbb{S}^{n-1}/\sim_k^\pm \longrightarrow \mathbb{R}\times T^*\mathbb{S}^{n-1}/\varphi_k^\pm$$

by sending $(t;\mathbf{q},\mathbf{p})$ to $(h_k^\pm(|\mathbf{p}|)\,t;\mathbf{q},\mathbf{p})$. The pull-back $\beta_k^\pm$ of the described contact form under this diffeomorphism is given by

$$\beta_k^\pm = h_k^\pm(|\mathbf{p}|)\,dt + \lambda_{\mathrm{can}} \mp t|\mathbf{p}|\,d\big(f_k(|\mathbf{p}|)\big)\ .$$

We will denote this last mapping torus $(M_{\mathrm{Dehn}}^{\pm k}, \beta_k^\pm)$ by

$$M_{\mathrm{Dehn}}^{\pm k} := \mathbb{R}\times T^*\mathbb{S}^{n-1}/\sim_k^\pm\ .$$

Far away from the zero section, the mapping torus $M_{\mathrm{Dehn}}^{\pm k}$ is diffeomorphic to a trivial product, and the contact form is just $\beta_k^\pm = h_k^\pm(\infty)\,dt + \lambda_{\mathrm{can}}$. After rescaling the fiber direction, the contact structure is in standard form, and hence it is possible to substitute small neighborhoods of embedded $\mathbb{S}^n$-bundles $\mathbb{S}^1\times\mathbb{S}^n$ or $\mathbb{S}^1\widetilde{\times}\mathbb{S}^n$ as described at the beginning of this appendix with such mapping tori.

APPENDIX E

Open book decompositions

This appendix contains results obtained together with Otto van Koert, and described in [**vKNar**]. The only changes done here are that definitions already given at some other point of this thesis have been removed, and additionally to the right-handed Dehn-twists considered in the article, also left-handed Dehn-twists are explained.

0. Introduction

At the ICM of 2002 Giroux announced some of his results concerning a correspondence between contact structures on manifolds and open book structures on them. In one direction this correspondence is relatively easy. We are given a compact Stein manifold M (i.e. a compact subset of a Stein manifold where the boundary is a level set of a plurisubharmonic function on it) and a symplectomorphism ψ of M that is the identity near the boundary of M. It can be shown that this symplectomorphism gives rise to a mapping torus that inherits a contact structure. Furthermore, the boundary of the mapping torus will always look like $\mathbb{S}^1 \times \partial M$, so the binding $\mathbb{D}^2 \times \partial M$ with the obvious contact structure can be glued in to give a compact contact manifold.

Although Giroux announced much more than just this, it is already interesting to see how this construction turns out in a few simple cases. As a Stein manifold we will take $T^*\mathbb{S}^{n-1}$ with its canonical symplectic form. For the symplectomorphisms used for the monodromy of the mapping torus we will be using so-called generalized Dehn twists, a symplectomorphism that can be written down explicitly. Seidel has shown [**Sei98**] that these Dehn twists generate the symplectomorphism group of $T^*\mathbb{S}^2$ up to isotopy. Furthermore his results show that Dehn twists of $T^*\mathbb{S}^2$ are of order 2 diffeomorphically, but not symplectically. This means that many of these Dehn twists are isotopic to each other, but not symplectically so.

We will show that the above construction using $T^*\mathbb{S}^{n-1}$ with a k-fold Dehn twist yields the Brieskorn manifold W_k^{2n-1}. In particular, this shows that the Ustilovsky spheres (special Brieskorn spheres with non-isomorphic contact structures) can all be written in terms of open book decompositions with Dehn twists as their monodromy. It also shows that Dehn twists cannot be of order 2 in all dimensions (this is never true for n even). Namely, among the Brieskorn spheres (these correspond to n and k odd) are exotic spheres as well as standard ones. As the binding is always glued in in the same way, the Dehn twists corresponding to a standard and an exotic sphere cannot be isotopic.

1. Notation & Definitions

1.1. Open books.

The following definitions are taken from [**Gir02a**].

Definition. An **open book** on M is given by a codimension-2 submanifold $B \hookrightarrow M$ with trivial normal bundle, and a bundle $\vartheta : (M - B) \to \mathbb{S}^1$. The neighborhood of B should have a trivialization $B \times \mathbb{D}^2$, where the angle coordinate on the disk agrees with the map ϑ.

The manifold B is called the **binding** of the open book and a fiber $P = \vartheta^{-1}(\varphi_0)$ is called a **page**.

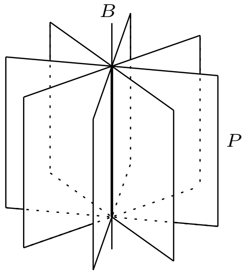


Figure 1. $\mathbb{S}^1$-action with fixed points

Remark E.1. The closure of a page P is a compact codimension-1 submanifold, whose boundary is B.

Remark E.2. The open set $M - B$ is a bundle over $\mathbb{S}^1$, hence it is diffeomorphic to $\mathbb{R} \times P/\sim$, where $\sim$ identifies $(t,p) \sim (t+1, \Phi(p))$ for some diffeomorphism Φ of P.

Definition. A contact structure $\xi = \ker \alpha$ on M is said to be **supported by an open book** (B, ϑ) of M, if

1. (B, α) is a contact manifold,
2. $d\alpha$ is a symplectic form on any page $\overline{P}$, and
3. the natural orientation of (B, α) coincides with the one as boundary of $(\overline{P}, d\alpha)$.

Remark E.3. Note that if the binding is connected, point (3) of the definition above holds automatically, because

$$0 < \int_P d\alpha^n = \int_B \alpha \wedge d\alpha^{n-1} ,$$

by Stokes theorem. Hence the orientation of B as boundary of P agrees with the one given by the contact form.

1.2. Dehn twists. Dehn twists were introduced in Appendix D, and we will use the notation given there.

2. Open books for the contact structure $\alpha_\pm$ on the Brieskorn manifolds W_k^{2n-1}

The **Brieskorn manifolds** $W_k^{2n-1} \subset \mathbb{C}^{n+1}$ (with $k \in \mathbb{N}_0$) are defined as the intersection of the sphere $\mathbb{S}^{2n+1}$ with the zero set of the polynomial $f(z_0, z_1, \ldots, z_n) = z_0^k + z_1^2 + \cdots + z_n^2$. To make computations easier, assume that the radius of the $(2n+1)$-sphere is $\sqrt{2}$.

The orthogonal group $\mathrm{SO}(n)$ acts linearly on $\mathbb{C}^{n+1}$ by leaving the first coordinate of $(z_0, z_1, \ldots, z_n)$ fixed and multiplying the last n coordinates with $\mathrm{SO}(n)$ in its standard matrix representation, i.e. $A \cdot (z_0, z_1, \ldots, z_n) := (z_0, A \cdot (z_1, \ldots, z_n))$. This action restricts to W_k^{2n-1}, because the polynomial f can be written as $z_0^k + \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 + 2i\langle \mathbf{x} | \mathbf{y} \rangle$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

It was shown in Lemma IV.21 that the two 1-forms

$$\alpha_+ := k\,(x_0\,dy_0 - y_0\,dx_0) + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)$$

and

$$\alpha_- := -(k+1)\,(x_0\,dy_0 - y_0\,dx_0) + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)$$

induce $\mathrm{SO}(n)$-invariant contact structures on W_k^{2n-1}. Note that the standard form given by Lutz and Meckert [**LM76**] has positive sign in front of the first term.

It is well-known that all W_k^{2n-1} are $(n-2)$-connected, and some of these Brieskorn manifolds are spheres [**Bri66**], [**HM68**]. Ustilovsky [**Ust99**] showed that among them there are diffeomorphic but non-contactomorphic manifolds. Namely if $2n-1 = 1 \mod 4$, then all W_k^{2n-1} with $k = \pm 1 \mod 8$ are standard spheres with inequivalent contact structures.

In the remainder of this paper will we show that the contact structures $\alpha_\pm$ on Brieskorn manifolds W_k^{2n-1} are supported by an open book whose monodromy is given by a k-fold right-handed Dehn twist for α_+ and a k-fold left-handed Dehn twist for α_-. We define the binding B of the open book by the set in W_k^{2n-1} with $z_0 = 0$. We have the fibration $\vartheta : (W_k^{2n-1} - B) \to \mathbb{S}^1$, given by $(z_0, z_1, \dots, z_n) \mapsto z_0/|z_0|$.

2.1. The binding.

The only stabilizers of the $\mathrm{SO}(n)$-action on the Brieskorn manifold that occur are $\mathrm{SO}(n-1)$ and $\mathrm{SO}(n-2)$. The projection onto the orbit space is given by

$$\begin{array}{ccc} W_k^{2n-1} & \longrightarrow & \mathbb{D}^2 \\ (z_0, z_1, \dots, z_n) & \longmapsto & z_0. \end{array}$$

Points $(z_0, \dots, z_n)$ lying over the interior of the disk (i.e. $|z_0| \neq 1$) have principal stabilizer, points over $\partial\mathbb{D}^2$ lie on singular orbits. The orbit $B = \mathrm{Orb}\,(0,1,0,\dots,0) \cong \mathrm{SO}(n)/\,\mathrm{SO}(n-2)$ is the binding of the open book. It is naturally contactomorphic to W_2^{2n-3}. In fact, $W_2^{2n-3} = \mathrm{SO}(n)/\,\mathrm{SO}(n-2)$ is diffeomorphic to the unit sphere bundle $\mathbb{S}(T^*\mathbb{S}^n)$. This shows that part (1) of Definition 1.1 is satisfied.

The symplectic normal bundle of the binding is trivial, because for $k \neq 1$ we have a symplectic basis

$$\frac{1}{\sqrt{2k}}(1, 0, \dots, 0), \quad \frac{1}{\sqrt{2k}}(i, 0, \dots, 0),$$

and for $k = 1$ we have the basis

$$\sqrt{\frac{2}{5}}(1, -\frac{\bar{z}_1}{4}, \dots, -\frac{\bar{z}_n}{4}), \quad \sqrt{\frac{2}{5}}(i, -\frac{i\bar{z}_1}{4}, \dots, -\frac{i\bar{z}_n}{4}).$$

The neighborhood theorem for contact submanifolds [**Geiar**] then shows that there is a neighborhood of the binding that is contactomorphic to $(B \times \mathbb{D}^2, \alpha_+|_B + r^2 d\vartheta)$, where (r, ϑ) are polar coordinates on the disk.

2.2. The pages. In this section, we will prove that $W^{2n-1}_k - B$ is contactomorphic to $\mathbb{R}\times T^*\mathbb{S}^{n-1}/\sim_k$, the mapping torus of a k-fold Dehn twist. To obtain this final contactomorphism, we will combine several maps that will be described in this chapter. The following diagram is meant as a reference:

$$(M^{\pm k}_{\text{Dehn}}, \beta^{\pm}_k) \xleftarrow{\Psi^{\pm}_k} M_k \xleftarrow{S^{\pm}_k} \mathbb{R}\times T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1} \xrightarrow{\Phi_k} (W^{2n-1}_k - B, \alpha_{\pm})$$

The $\mathbb{R}$-action on $W^{2n-1}_k - B$, given by

$$e^{it}(z_0, z_1, \dots, z_n) = (e^{it}z_0, e^{\frac{ki}{2}t}z_1, \dots, e^{\frac{ki}{2}t}z_n).$$

induces a diffeomorphism between the pages $\vartheta^{-1}(1)$ and $\vartheta^{-1}(e^{it})$.

Let us define an auxiliary mapping torus to make computations more convenient. Define

$$M_k := \mathbb{R}\times T^*\mathbb{S}^{n-1}/\sigma_k,$$

where

$$\sigma_k(t, \mathbf{q}, \mathbf{p}) = (t+1, (-1)^k\mathbf{q}, (-1)^k\mathbf{p}).$$

We will now give an explicit map to show that P is diffeomorphic to $T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1}$. Here $T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1}$ denotes the open unit disk bundle associated with the cotangent bundle of $\mathbb{S}^{n-1}$. A point $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{S}^{n-1} \subset \mathbb{R}^n\times\mathbb{R}^n$ with $|\mathbf{q}| = 1$, $|\mathbf{p}| \leq 1$, and $\mathbf{q}\perp\mathbf{p}$ is mapped to

$$(\mathbf{q}, \mathbf{p}) \mapsto \Big(1-|\mathbf{p}|^2, F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q}\Big)$$

with $F(r) = \sqrt{\frac{2-(1-r^2)^2-(1-r^2)^k}{2r^2}}$ and $G(r) = \sqrt{\frac{2-(1-r^2)^2+(1-r^2)^k}{2}}$.

Together with the $\mathbb{R}$-action this gives a map

$$\begin{aligned} \Phi_k : \mathbb{R}\times T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1} &\longrightarrow W^{2n-1}_k \\ (t, \mathbf{q}, \mathbf{p}) &\longmapsto \Big(e^{2\pi it}(1-|\mathbf{p}|^2), e^{\pi kit}(F(|\mathbf{p}|)\mathbf{p} + iG(|\mathbf{p}|)\mathbf{q})\Big). \end{aligned}$$

This descends to a diffeomorphism of the subset of M_k with $|\mathbf{p}| < 1$ to $W^{2n-1}_k - B$. For k even, one obtains $\Phi_k(t+1, \mathbf{q}, \mathbf{p}) = \Phi_k(t, \mathbf{q}, \mathbf{p})$, so that $W^{2n-1}_k - B \cong \mathbb{S}^1\times T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1}$, and for k odd, one obtains $\Phi_k(t+1, \mathbf{q}, \mathbf{p}) = \Phi_k(t, -\mathbf{q}, -\mathbf{p})$, so that $W^{2n-1}_k - B$ is a non-trivial $T^*_{|\mathbf{p}|<1}\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$.

The pull-back of the contact form $\alpha_{\pm}$ to M_k under Φ_k gives

$$\begin{aligned} \Phi_k^*\alpha_+ &= 2\pi k\big((1-|\mathbf{p}|^2)^2 + |\mathbf{p}|^2F^2 + G^2\big)\,dt + 4FG\,\lambda_{\text{can}} = 4\pi k\,dt + 4FG\,\lambda_{\text{can}} \\ \Phi_k^*\alpha_- &= 2\pi\big(k|\mathbf{p}|^2F^2 + kG^2 - (k+1)(1-|\mathbf{p}|^2)^2\big)\,dt + 4FG\,\lambda_{\text{can}} \\ &= 2\pi\Big(2k - (2k+1)(1-|\mathbf{p}|^2)^2\Big)\,dt + 4FG\,\lambda_{\text{can}}\,. \end{aligned}$$

Next, we construct a diffeomorphism $\Psi^{\pm}_k$ from M_k to the mapping torus $M^{\pm k}_{\text{Dehn}}$ by defining

$$\begin{aligned} \Psi^{\pm}_k(t; \mathbf{q}, \mathbf{p}) = \Big[t; \mathbf{q}\cdot\cos\big(\pm tf_k(|\mathbf{p}|)\big) + \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\sin\big(\pm tf_k(|\mathbf{p}|)\big), \\ \mathbf{p}\cdot\cos\big(\pm tf_k(|\mathbf{p}|)\big) - |\mathbf{p}|\mathbf{q}\cdot\sin\big(\pm tf_k(|\mathbf{p}|)\big)\Big]\,. \end{aligned}$$

The map is well-defined, because $\Psi_k\circ\sigma_k(t; \mathbf{q}, \mathbf{p})$ is identified with $\Psi^{\pm}_k(t; \mathbf{q}, \mathbf{p})$ in the mapping torus $M^{\pm k}_{\text{Dehn}}$. In order to show that the composition $\Phi_k\circ\big(\Psi^{\pm}_k\big)^{-1}$ is a contactomorphism, we will show that the pull-back of $\alpha_\pm$ under Φ_k is contactomorphic to the pull-back of $\beta_k^\pm$ under $\Psi_k^\pm$.

We now compute the pull-back of $\beta_k^\pm$ under $\Psi_k^\pm$, noting that the norm of $\mathbf{p}$ is invariant under $\Psi_k^\pm$ (we do not write the dependence of $h_k^\pm$ and f_k on $|\mathbf{p}|$):

$$\left(\Psi_k^\pm\right)^* \beta_k^\pm = \left(h_k^\pm \pm |\mathbf{p}| f_k\right)\, dt + \lambda_{\text{can}}\ .$$

Using partial integration to get the equation $h_k^\pm(y) = 1 \mp y f_k(y) \pm \int_0^y f_k(s)\, ds$, we find

$$\left(\Psi_k^\pm\right)^* \beta_k^\pm = \left(1 \pm \int_0^{|\mathbf{p}|} f_k(s)\, ds\right) dt + \lambda_{\text{can}}\ .$$

Note that $\Phi_k^* \alpha_+$ has a very similar form. We make the following ansatz for a contactomorphism of $(M_k|_{|\mathbf{p}|<1}, \Phi_k^*\alpha_\pm)$ to $(M_k, (\Psi_k^\pm)^*\beta_k^\pm)$:

$$S_k^\pm : (t,\mathbf{q},\mathbf{p}) \quad \mapsto \quad (t,\mathbf{q},\pm\frac{g(|\mathbf{p}|)}{|\mathbf{p}|}\mathbf{p}).$$

With this ansatz we find what $\mathbf{p}$ should map to in order for the map to be a contactomorphism. For right-handed Dehn twists, we are just rescaling $\mathbf{p}$, while for left-handed ones, we are also applying a reflection. The pull-back under this map of $(\Psi_k^\pm)^*\beta_k^\pm$ is given by

$$\left(1 \pm \int_0^{g(|\mathbf{p}|)} f_k(s)\, ds\right) dt \pm \frac{g(|\mathbf{p}|)}{|\mathbf{p}|}\lambda_{\text{can}}\ .$$

Since we want this to be a multiple of $\Phi_k^*\alpha_\pm$, we need to solve the following equation:

$$\frac{g(|\mathbf{p}|)}{1+\int_0^{g(|\mathbf{p}|)} f_k(s)\, ds} = \frac{|\mathbf{p}|FG}{k\pi}$$

for right-handed Dehn twists, and

$$\frac{\int_0^{g(|\mathbf{p}|)} f_k(s)\, ds - 1}{g(|\mathbf{p}|)} = \frac{\pi(2k-(2k+1)(1-|\mathbf{p}|^2)^2)}{2|\mathbf{p}|FG}$$

for left-handed Dehn twists. Define auxiliary functions

$$h^+(y) := \frac{y}{1+\int_0^y f_k(s)\, ds} \qquad \text{and} \qquad h^-(y) := \frac{\int_0^y f_k(s)\, ds - 1}{y}\ .$$

The above equations becomes

$$h^+\left(g(|\mathbf{p}|)\right) = \frac{|\mathbf{p}|FG}{k\pi}$$

for right-handed Dehn twists, and

$$h^-\left(g(|\mathbf{p}|)\right) = \frac{\pi(2k-(2k+1)(1-|\mathbf{p}|^2)^2)}{2|\mathbf{p}|FG}$$

for left-handed Dehn twists.

We will solve for $g(|\mathbf{p}|)$ by inverting $h^{\pm}$. This can be done by the following considerations. The derivatives of $h^{\pm}$ are given by

$$\frac{dh^{+}(y)}{dy} = \frac{1 - \int_0^y s f_k'(s)\, ds}{\left(1 + \int_0^y f_k(s)\, ds\right)^2} = \frac{h_k^{+}(y)}{\left(1 + \int_0^y f_k(s)\, ds\right)^2} > 0$$

and

$$\frac{dh^{-}(y)}{dy} = \frac{1 + \int_0^y s f_k'(s)\, ds}{y^2} = \frac{h_k^{-}(y)}{y^2} > 0 \ ,$$

where we used that the $h_k^{\pm}$ are positive by our choices in Appendix D.

Let us first consider the problem for right-handed Dehn twists. Since h^{+} is strictly increasing, we observe that the function h^{+} maps $[0, \infty)$ bijectively onto $[0, \frac{1}{k\pi})$. This can be seen by noting that $f_k(s) = k\pi$ for s sufficiently large, again due to our choice of h_k^{+}. It also means that h^{+} can be inverted, when restricted to a suitable range. One easily checks that the right-hand side of the above equation, $\frac{|\mathbf{p}|FG}{k\pi}$, has positive derivative and is therefore strictly increasing on the interval $[0, 1)$. Moreover it has the same range as h^{+}, namely $[0, \frac{1}{k\pi})$. Therefore we can find a smooth solution to $g(|\mathbf{p}|)$ by applying the inverse of h to $\frac{|\mathbf{p}|FG}{k\pi}$.

For left-handed Dehn twists, we find that h^{-} is also strictly increasing, and it maps the interval $(0, \infty)$ to $(-\infty, k\pi)$. The right-hand side, $\frac{\pi(2k-(2k+1)(1-|\mathbf{p}|^2)^2)}{2|\mathbf{p}|FG}$, can be shown to be monotonously increasing and maps $(0, 1)$ onto $(-\infty, k\pi)$. On the interval $(0, 1)$ there is a smooth solution to $g(|\mathbf{p}|)$ given by applying the inverse of h^{-} to $\frac{\pi(2k-(2k+1)(1-|\mathbf{p}|^2)^2)}{2|\mathbf{p}|FG}$.

This shows that the open book (B, ϑ) on $\left(W_k^{2n-1}, \alpha_{\pm}\right)$ has page $T^*\mathbb{S}^{n-1}$ with monodromy given by either a right-handed or left-handed k-fold Dehn twist. The contactomorphism that achieves this is

$$C_k := \Phi_k \circ \left(S_k^{\pm}\right)^{-1} \circ \left(\Psi_k^{\pm}\right)^{-1} : (M_{\text{Dehn}}^{\pm k}, \beta_k^{\pm}) \to (W_k^{2n-1} - B, \alpha_{\pm}) \ .$$

Note that this contactomorphism also respects the projection to $\mathbb{S}^1$, because the $\mathbb{S}^1$-coordinate is invariant under C_k.

2.3. The contact structure on W_k^{2n-1} is supported by the open book. Part (1) of the Definition 1.1 was already checked in Section 2.1. Note that $\Phi^*\alpha_+$ restricts to the same form on each page as $\Phi^*\alpha_-$, hence it is enough to show part (2) only for α_+. The Milnor fibration, which is transverse to the pages, is the Reeb field of α_+, hence $d\alpha_+$ cannot have non-trivial kernel on the page.

By Remark E.3, point (3) follows immediately if $2n - 1 \geq 5$, because the binding is connected. If $2n - 1 = 3$ the binding has two components B_1 and B_2, but by symmetry considerations, one can easily see that integrating α over B_1 yields up to sign the same value as integrating over B_2. If at least one of the two signs was negative, then the inequality in Remark E.3 would be false.

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