On Non-local Boundary Value Problems for Elliptic Operators

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Zusammenfassung

In dieser Arbeit werden nicht-lokale Randwertprobleme für elliptische Differentialoperatoren auf kompakten Mannigfaltigkeiten mit glattem Rand beleuchtet. Dabei wird zunächst die Randwerttheorie für allgemeine elliptische Differentialoperatoren aufgebaut. U.a. wird bewiesen, dass alle schwachen Lösungen, die einer gegebenen Randbedingung genügen, genau dann stark sind, wenn für die starken Lösungen eine gewisse Regularitätsabschätzung gilt. Die hier betrachteten Randbedingungen lassen sich durch Projektionen beschreiben, die folgende Bedingung erfüllen. Der Operator und sein Kommutator mit einem Erzeuger der Sobolev-Skala auf dem Rand ist auf der gesamten Sobolev-Skala beschränkt. Als Erzeuger kann ein beliebiger elliptischer Pseudodifferentialoperator mit skalarem Hauptsymbol gewählt werden. Im Vergleich zur bestehenden Literatur ist diese Klasse von Randbedingungen sehr allgemein und umfasst sowohl spektrale Randbedingungen als auch einige nicht-pseudolokale Randbedingungen. Wie im Anhang bewiesen wird, bilden die Operatoren, die obiger Kommutator-Bedingung genügen, eine lokale C^*-Algebra, was sich als grundlegend für den gesamten Aufbau der Theorie herausstellt.

Im Mittelpunkt der Regularitätstheorie steht der Calderón-Projektor bzw. sein Bild, der Cauchy-Daten-Raum Λ_0. Wir geben hier einen vereinfachten Beweis dafür, dass der Calderón-Projektor eines beliebigen elliptischen Differentialoperators ein klassischer Pseudodifferentialoperator auf dem Rand ist. Zu einer Randbedingung betrachten wir den Raum aller zulässigen Randwerte Λ. Die wichtigsten Eigenschaften einer Fortsetzung, Regularität, Korrektgestelltheit und Selbstadjungiertheit, lassen sich dann über das Raum-Paar (Λ, Λ_0) beschreiben. Im Falle pseudodifferentieller Randbedingungen gelangen wir so zu den bekannten Kriterien. Für alle formal selbst-adjungierten Differentialoperatoren wird gezeigt, dass die Gesamtheit der regulären selbstadjungierten Fortsetzungen durch die zu Λ_0 assoziierte Fredholm-Langrange Grassmannsche parametrisiert wird.

Danach wenden wir die allgemeinen Ergebnisse auf zwei der wichtigsten Klassen geometrischer Differentialoperatoren, Dirac- und Laplace-Operatoren, an. Für eine Reihe von Dirac-Operatoren zeigen wir, wie das Hauptsymbol des Calderón-Projektors aus dem tangentialen Anteil des Dirac-Operators bestimmt werden kann. Auf diese Weise erhalten wir die Regularitätstheorie von Randwertproblemen für den Hodge-de Rham-, Signatur-, Gauß-Bonnet- und den Spin-Dirac-Operator. Die Invarianz des Index eines Dirac-Operators unter Kobordismen wird mit Hilfe elementarer Fredholmtheorie aus den Eigenschaften des Cauchy-Daten-Raums gefolgert.

Für Laplace-Operatoren besprechen wir eine Reihe klassischer Randwertprobleme. Es werden die Randbedingungen, die zu selbstadjungierten Fortsetzungen führen, genauer charakterisiert. Weiter wird ein hinreichendes und notwendiges Kriterium für die Halbbeschränktheit selbstadjungierter Fortsetzungen von Laplace-Operatoren bewiesen.

Abstract

In this thesis we discuss boundary value problems for elliptic differential operators where the boundary conditions belong to a certain class of non-local operators. More precisely, the boundary conditions are given in terms of the standard traces and operators with closed range on the boundary satisfying the following condition: The operator and its commutator with some fixed generator of the Sobolev scale are bounded on the full Sobolev scale of the boundary. This generator is an arbitrary first order elliptic pseudodifferential operator with scalar principal symbol. It is shown that this classe operators form a local C^*-algebra which contains all pseudodifferential operators with closed range.

We establish a regularity theory characterising all boundary conditions which give rise to regular, well-posed and self-adjoint extensions. We show that for a given boundary condition a certain regularity estimate is equivalent to the fact that all weak solutions satisfying this condition are strong ones. The theory is based on the properties of the Cauchy data space, Λ_0, and on the notion of Fredholm pairs. We construct the Calderón projection for any elliptic differential operator avoiding the unique continuation principle. We prove that our construction yields a classical pseudodifferential operator. It follows that the Fredholm-Lagrange-Grassmannian w.r.t. Λ_0 is non-empty and parametrises the space of all regular self-adjoint extensions of a formally self-adjoint operator.

Using only elementary Fredholm theory we deduce cobordism invariance for the index of Dirac operators from the facts that the Cauchy data space is Lagrangian, on the one hand, and that the Calderón projection differs from the positive spectral projection merely by a compact operator, on the other. Many classical examples of Dirac and Laplace operators are revisited. Moreover, for Laplace operators we describe all regular self-adjoint extensions in terms of boundary conditions and establish a necessary and sufficient criterion for semi-boundedness.

Introduction

Boundary value problems are found in many mathematical models of physical systems. Among these are classical mechanical systems such as vibrating strings or clamped plates as well as quantum mechanical systems, e.g. particles in a box and quantum hall systems. Typically, an evolution process, e.g. diffusion or wave propagation, is modelled by a partial differential equation on a manifold with boundary together with a boundary condition. Whereas the partial differential equation models how temperature diffuses or how waves propagate in the interior of a medium, the boundary condition tells us e.g. if the temperature is fixed at the edge, if the boundary is an ideal isolator, or how waves are reflected. Many of the above models contain as its main ingredient a linear elliptic differential operator, so that the question of boundary conditions for these operator naturally arises. However, it should be emphasised that elliptic boundary value problems, e.g. for Dirac operators, are studied in its own right by mathematicians and have important applications in geometry and topology.

In spite of their long history and outstanding role in mathematics and physics, it is hard to find literature on elliptic boundary value problems, that does not make any special assumptions on the shape of the operators and boundary condition. Certainly, the bigger part of the above-mentioned examples does not justify to study general elliptic boundary value problems the more so as natural restrictions on the elliptic operator allow us to employ powerful techniques, e.g. by treating the partial differential equation as a variational problem. Nevertheless, I am convinced that the fundamental questions that one is concerned with when studying an elliptic boundary value problem can be treated in a completely general framework.

Indeed, there are by now successful approaches that aim at a generalisation of the theory of pseudodifferential operators on manifolds without boundary, the most prominent being the early work of L. Boutet de Monvel ([BdM71]). However, the mathematics involved in these theories seems rather complicated and thus out of reach for many non-specialists. In order to bridge this gap the first part of the thesis (Chapter 1 and 2) aims at a description of all boundary value problems having certain regularity and boundedness properties that are essential for most applications. The main theorems proved in these chapters have a long history so that the results, at their heart, may be considered as classical. If here and then the degree of generality is pushed further, this is not the motivation for reproving them. I rather aim at a simplification of the somewhat involved theory of elliptic boundary value problems by changing the point of view from solving PDEs to studying extensions of a closed unbounded operator as subspaces of the set of all possible boundary values, guided by the principle that the analytic properties of such an extension are encoded in the pair formed by the Cauchy data and the space of boundary values subject to the given condition.

To explain this in more detail suppose P is a differential operator on a compact manifold with boundary. In fact, a boundary condition is an extension of the so-called minimal operator $P_{\min}$ whose domain consists by definition of (Sobolev-)regular solutions with vanishing boundary data. Consider the domain $\mathscr{D}(P_{\max})$ of the maximal operator $P_{\max}$, i.e. all functions u such that Pu is square integrable. Since we are interested in solutions u to $Pu = v$ with v square integrable, a boundary condition may thus be viewed as a subspace of the quotient of the maximal by the minimal domain,

$$W(P) = \mathscr{D}(P_{\max})/\mathscr{D}(P_{\min}).$$

Let Λ_0 denote the subspace of all boundary values of solutions u to the homogeneous equation

$$P_{\max}u = 0.$$

The graph scalar product on the domains induces a natural Hilbert space structure on $W(P)$. Motivated by Green's formula, we may define an anti-dual pairing of $W(P)$ with the corresponding Hilbert space of the adjoint operator, P^t, by setting

$$\omega([u],[v]) = \langle Pu, v\rangle - \langle u, P^t v\rangle.$$

As is well-known, if P is symmetric, then this pairing makes $W(P)$ a strongly symplectic Hilbert space. In a purely functional analytic context it is now possible to translate properties of an extension $P \subset P_{\max}$ of $P_{\min}$ into properties of subspaces of $W(P)$ using the following dictionary.

extensions P	subspaces $\Lambda \subset W(P)$
P is closed	Λ is closed
P is (left, right) Fredholm	(Λ, Λ_0) is a (left,right) Fredholm pair
the adjoint of P, P^*	the annihilator of Λ (w.r.t. ω)
P is symmetric	Λ is isotropic
P is self-adjoint	Λ is Lagrangian

Moreover, the annihilator of the Cauchy data space of P is the Cauchy data space of the formal adjoint P^t. In particular, the Cauchy data space is Lagrangian if P is formally self-adjoint. Note that in the above table the last two properties make only sense for a formally self-adjoint P.

It should be emphasised, however, that with the results obtained from abstract extension theory of unbounded operators, of course, we still do not know if a given boundary problem is regular in the sense that weak solutions to $Pu = v$, subject to the boundary condition, are smooth on the boundary. Moreover, the topology on $W(P)$, viewed as a space of functions over the boundary, is indeed rather difficult to describe.

In Chapter 2 we give satisfactory criteria for regularity, at least for boundary conditions with a certain technical property that we will address below. In addition to regularity one is usually interested in existence results. Here, a boundary problem will be called well-posed if the boundary condition is regular and the corresponding extension P has range of finite codimension. The key result states that a boundary problem is well-posed (regular) if and only if the pair formed by the Cauchy data space and the space of all boundary data satisfying the given condition is (left) Fredholm. We recover Seeley's notion of well-posedness (cf. [See69]) in case the boundary condition is pseudodifferential. For the special case of a Dirac operator on a manifold with boundary most theorems of Chapter 2 are well known, see e.g. [BBW93] and [BL01].

Some words are necessary at this stage to make precise what is meant by *simplification* since the analysis of partial differential equations is delicate by nature. Here, the difficulties are hidden, firstly, in the calculus of pseudodifferential operators on closed manifolds that I will make free use of throughout this thesis, and, secondly, in the properties of the Cauchy data space. As is well-known (e.g. [Hör85]) the latter is the range of a classical pseudodifferential projection, called the Calderón projection. The regularity results in Chapter 2 are of no practical relevance unless we know this projection up to compact perturbations. Therefore I will explain in Section 2.3 how to obtain the symbol expansion of this projection.

In the third chapter boundary value problems for Dirac operators are revisited and the well-known regularity theorems are deduced from the results of the preceding chapters. Finally, I give a proof of the cobordism invariance of the index of a Dirac operator. It is remarkable that this proof uses only the fact that the Cauchy data space of a Dirac operator is Lagrangian and that the Calderón projection is a compact perturbation of the positive spectral projection of the Dirac operator on the boundary.

The fourth chapter is devoted to the study of boundary value problems for Laplace operators. There is a series of subsections on a number of classical examples of boundary conditions for Laplacians showing that regularity and well-posedness are easily proven with the machinery established in the first two chapters. Then we discuss the space of all boundary conditions leading to regular self-adjoint extensions. We will prove that essentially all such boundary conditions are of the form

$$\begin{aligned} \Pi_1 \gamma^0 u &= 0, \\ \Pi_2(\gamma^1 u + G\gamma^0 u) &= 0, \end{aligned} \tag{1}$$

where γ^0, γ^1 are the usual traces, Π_i are complementary orthogonal projections and G is a first order self-adjoint operator on the boundary. Such conditions were studied in [Gru03] by Grubb who aimed at rather general assumptions under which the asymptotic expansion of heat and resolvent trace can be obtained. We work out a necessary and sufficient criterion for the realisation to be semi-bounded.

The boundary conditions studied in this thesis are comparatively general. In fact, we only assume that the operators involved in the boundary condition have closed range and satisfy a certain boundedness condition for the commutator with the square root of a Laplacian on the boundary. This condition is satisfied for any pseudodifferential operator but also for some conditions which involve diffeomorphisms on the boundary. The space of such operators acting on the sections of a fixed bundle over a compact manifold is spectrally invariant, and these operators form a local C^*-Algebra. These fundamental facts turn out to be rather crucial for the regularity of such boundary conditions. The proof for spectral invariance and some well-known facts from Fredholm theory are given in the appendix.

In order to fix notation and some conventions we begin with a preparatory chapter where the main objects that we will use are listed. Therefore we will not repeat the setting at the beginning of each section. All chapters start with a short summary and some bibliographic remarks.

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This doctoral thesis is dedicated to my parents, Dr. Dietrich and Dörte Frey.

Chapter 0

Preliminaries and General Assumptions

In this preparatory chapter we pursue two purposes. On the one hand we intend to fix notation and the exact definition of some mathematical vocabulary that we will make frequent use of afterwards. On the other hand we already introduce to the topic of this thesis.

Instead of repeating at the beginning of each chapter or section the objects that we are dealing with we will list the data that is assumed to be given throughout the text. At its heart this is hardly more than a partial differential equation $Pu = v$ on a manifold with boundary, say Ω. However, more geometric data will turn out helpful. This additional information may be present for different reasons. It is either inherent to the special context or constructed by hand. In any case no restrictions are made on the nature of the elliptic equation.

All manifolds in this text are assumed to be smooth. We say that Ω is a manifold with boundary if the image of each chart φ_α, $\alpha \in \mathscr{A}$, is an open subset U of $\mathbb{R}_+ \times \mathbb{R}^{n-1}$, $\mathbb{R}_+ := [0, \infty)$, and if all coordinate transformations $\varphi_\beta \circ \varphi_\alpha^{-1}$, $\alpha, \beta \in \mathscr{A}$, belong to $C^\infty(U_\alpha, U_\beta)$. Here a map from U_α to U_β is called differentiable (smooth) if it has a differentiable (smooth) extension to an open subset of $\mathbb{R}^n$. As a general principle, we will define (almost) all function spaces on manifolds with boundary as quotients. For instance,

$$C^k(A) = C^k(\mathbb{R}^n) / \{u \in C^k(\mathbb{R}^n) \mid u|_A = 0\},$$

if A is a subset of $\mathbb{R}^n$. It follows that all manifolds considered here have smooth boundary. They may always be viewed as closed sets of the form

$$\Omega = \{p \in M \mid f(p) \leq 0\},$$

where M is a smooth manifold (without boundary) and $f : M \to \mathbb{R}$ is a smooth function such that 0 is a regular value of f. Note that Ω contains its boundary $\partial\Omega$. If necessary M may be chosen compact provided that Ω is. $C_c^k(\Omega)$, $C_c^k(M)$ denote the spaces of k times differentiable functions with compact support in Ω, M, resp. If Ω is compact, then $C_c^k(\Omega) = C^k(\Omega)$. The inner of Ω will be denoted by $\Omega^\circ = \Omega \setminus \partial\Omega$. Hence, $C_c^\infty(\Omega^\circ)$ is the space of functions having compact support in the interior, Ω°. Unless otherwise noted all functions are complex-valued.

We will use the standard notation for partial differential equations, i.e. in $\mathbb{R}^n$ we have the differential operators

$$\partial_j := \frac{\partial}{\partial x_j}, \quad D_j := -i\partial_j, \quad D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n.$$

Similarly for $\xi \in \mathbb{R}^n$ we set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

If M carries a Riemannian metric g, then we define the Laplace-Beltrami operator

$$\Delta_0^M := d^t d : C_c^\infty(M) \to C_c^\infty(M),$$

where d^t is the formal adjoint of the exterior derivative $d : C^\infty(M) \to C^\infty(M, \Lambda^1)$. The Friedrichs extension of Δ_0^M, Δ^M, is a positive self-adjoint operator. In fact, one may show that Δ_0^M is essentially self-adjoint (e.g. [Che73]) when M is complete w.r.t. g. This may always be achieved if Ω is complete by gluing Ω with an infinite half-cylinder $(-\infty, 0] \times \partial\Omega$ on which g takes the form $dx_1^2 + g_{\partial\Omega}$ for $x_1 < -1$. Then Δ gives rise to the Sobolev spaces on M

$$H^s(M) := \mathscr{D}((\Delta + 1)^{s/2}), \quad s \geq 0.$$

The corresponding Sobolev space on Ω may again be defined as the quotient

$$H^s(\Omega) := H^s(M)/\big\{u \in H^s(M) \;\big|\; u|_\Omega = 0\big\},$$

an important subspace of which is

$$H_0^s(\Omega) := \overline{C_c(\Omega^\circ)}^{\|\cdot\|_{H^s(\Omega)}}.$$

This definition of $H_0^s(\Omega)$ coincides with that in [LM72] and [Tay96] but is different from the one given in [Gru96] when $s \in \mathbb{N} + 1/2$. For $s > 0$ we define $H^{-s}(M)$ to be the so-called L^2-dual of $H^s(M)$, i.e.

$$H^{-s}(M) = \big\{u \in \mathscr{D}'(M) \;\big|\; \langle v, u\rangle \leq \text{const} \cdot \|v\|_{H^s(M)}\big\}.$$

The Sobolev space $H^{-s}(\Omega)$ with positive s will not be used. Note that the above Sobolev spaces are either domains of a closed operator, quotients, duals or closed subspaces of a Hilbert space and therefore inherit natural Hilbert space structures.

When $U \subset M$ is open the natural "restriction" and "extension by 0" operators are denoted by

$$r_U : C^\infty(M) \to C^\infty(U), \quad e_U : C_c^\infty(U) \to C_c^\infty(M).$$

The above constructions can be generalized for sections of some bundle $E \to M$ carrying an hermitian structure $p \mapsto \langle .,. \rangle_{E_p}$ and a metric connection $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$. More precisely, the Laplacian $d^t d$ can be replaced by the Bochner Laplacian $\nabla^t \nabla$ which is again essentially self-adjoint if M is complete. We set

$$H^s_{\text{loc}}(M, E) := \big\{u \in \mathscr{D}'(M, E) \;\big|\; \chi u \in H^s(M, E) \text{ for all } \chi \in C_c^\infty(M)\big\},$$
$$H^s_{\text{comp}}(M, E) := \big\{u \in H^s(M, E) \;\big|\; \operatorname{supp} u \text{ compact}\big\}.$$

In general, $H^s(M, E)$ depends on the choice of ∇. However, this is of little importance here, since $H^s_{\text{loc}}(M, E)$, $H^s_{\text{comp}}(M, E)$ are independent of ∇ and hence so is $H^s(\Omega, E)$ if Ω is compact.

The notion of a pseudodifferential operator (abbreviated by "ψdo") is employed in the sense of M. A. Shubin (cf. [Shu80]). Recall that

$$T \in \Psi^s(M)$$

if $T : C_c^\infty(M) \mapsto C^\infty(M)$ and for all (not necessarily connected nor small) coordinate charts $\varphi : U \to V$ the restricted operator

$$C_c^\infty(V) \mapsto C^\infty(V), \quad u \mapsto \varphi_* r_U T e_U(\varphi^* u)$$

is in

$$\Psi^s(V) := \Psi^s_{1,0}(V).$$

The space $\Psi^s(V, E, F)$ consisting of ψdos that map sections of some bundle $E \to M$ to sections of some bundle $F \to M$ are likewise defined.

We denote the principal symbol of a ψdo T by $\widehat{T}$, i.e. if $T \in \Psi^s(M; E, F)$, then

$$\widehat{T} \in S^m(\mathrm{Hom}(\pi^* E, \pi^* F))/S^{m-1}(\mathrm{Hom}(\pi^* E, \pi^* F)),$$

where $\pi : T^*M \to M$ denotes the natural projection. T is called a classical ψdo of order $m \in \mathbb{C}$ if for each coordinate chart the full symbol of the restricted ψdo in euclidean space has an asymptotic expansion

$$a(x,\xi) \sim \sum_{j=0}^{\infty} \psi(\xi) a_{m-j}(x,\xi),$$

where ψ as well as all a_{m-j} are smooth, $\psi \equiv 0$ near 0, $\psi \equiv 1$ near ∞, and for all $j \in \mathbb{N}$ the matrices $a_{m-j}(x,\xi)$ are positive homogeneous of degree $m-j$ in ξ. Note that for a classical ψdo A one can identify $\widehat{A}(x,\xi)$ with $a_m(x,\xi)$ since

$$a(x,\xi) - a_m(x,\xi) \in S^{|m|-1}(\mathrm{Hom}(\pi^* E, \pi^* F)).$$

In particular, $\widehat{D}^\alpha = \xi^\alpha$.

Let H, H' be Hilbert spaces and denote by $\mathscr{B}(H, H')$ the space of bounded operators. $Q \in \mathscr{B}(H) = \mathscr{B}(H, H)$ is called a projection if $Q^2 = Q$. Q is called an orthogonal projection if $Q^2 = Q = Q^*$.[1] Whenever V is a closed subspace of H we denote by $\mathrm{pr}_V^{\mathrm{ort}}$ the orthogonal projection onto V. If Q is a projection, then Q^{ort} is to denote $\mathrm{pr}^{\mathrm{ort}}_{\mathrm{ran}\, Q}$, i.e. the orthogonal projection having the same range as Q. The notion of a Fredholm pair of projections is rather crucial throughout the text. For the sake of completeness we present some basic features of Fredholm pairs in the appendix.

We will use scales of Hilbert spaces $(H_s)_{s \in I}$ where $I = \mathbb{R}_+$ or $\mathbb{R}$ in the sense of [BL01, Definition 2.5]. When $I = \mathbb{R}$, set

$$H_\infty := \bigcap_{s \in \mathbb{R}} H_s, \quad H_{-\infty} := \bigcup_{s \in \mathbb{R}} H_s,$$

and endow these spaces with the usual Fréchet topology. $\mathrm{Op}^\mu((H_s)_{s \in \mathbb{R}})$ consists of all operators $T : H_{-\infty} \to H_{-\infty}$ such that for all $s \in \mathbb{R}$, $T|_{H_s}$ is a bounded operator

$$T_s : H_s \to H_{s-\mu}.$$

[1] Some authors call an operator B such that $Q^2 = Q$ "idempotent" whereas projections are sometimes defined to be self-adjoint idempotents.

Let T and T' denote (unbounded) self-adjoint operators in H. We write $T \leq T'$ if $\mathscr{D}(T) \cap \mathscr{D}(T')$ is dense in $\mathscr{D}(T)$ and $\mathscr{D}(T')$ (w.r.t. to graph norms, i.e. $\mathscr{D}(T) \cap \mathscr{D}(T')$ is a core for both T and T') and for all $u \in \mathscr{D}(T) \cap \mathscr{D}(T')$ we have

$$\langle (T' - T)u, u\rangle \geq 0.$$

In particular, if P and Q are orthogonal projections, then $P \leq Q$ means $\operatorname{ran} P \subset \operatorname{ran} Q$, or equivalently, $\ker Q \subset \ker P$.

When T is a normal operator in a Hilbert space and $U \subset \mathbb{C}$, $U \cap \operatorname{spec} T$ Borel measurable,

$$\mathbf{1}_U(T)$$

will denote the spectral projection w.r.t. U, i.e. the spectral measure of $U \cap \operatorname{spec} T$. Similarly, if T is self-adjoint we will write $\mathbf{1}_{>0}(T)$, $\mathbf{1}_{\geq 0}(T)$ etc.

Having specified definitions and terminology so far we can now present the general framework in which we will study elliptic boundary value problems. Let us assume the following:

- (M, g) is a complete Riemannian manifold. The distance function associated to g is denoted by d_g.
- $\Omega \subset M$ is a compact subset with smooth boundary $\partial\Omega =: \Gamma$.
- $E, F \to M$ are hermitian vector bundles, $\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$, $\nabla^F : C^\infty(M, F) \to C^\infty(M, T^*M \otimes F)$ are metric connections. $E' := E\big|_\Gamma$. $E\big|_\Omega$ will still be denoted by E. $F' = F\big|_\Gamma$.
- $P^M : C^\infty(M, E) \to C^\infty(M, F)$ is an elliptic differential operator of order $d \in \mathbb{N}$.

In a tubular neighbourhood of the boundary, say V, the normal coordinate,

$$x_1(p) = \begin{cases} d_g(p, \Gamma) & \text{if } p \in \Omega, \\ -d_g(p, \Gamma) & \text{otherwise,} \end{cases}$$

is smooth and, in V, we can define the inward unit normal field ν by $\operatorname{grad} x_1$.

It is not difficult to see that M, P^M, E and F can be constructed whenever Ω, $E|_\Omega$, $F|_\Omega$ and an elliptic differential operator on Ω,

$$P : C^\infty(\Omega, E) \to C^\infty(\Omega, F),$$

are given. As explained above one can glue Ω with half-cylinders and extend $E|_\Omega$, $F|_\Omega$ by the bundles

$$\pi^*E' \to (-\infty, 0] \times \Gamma, \quad \pi^*F' \to (-\infty, 0] \times \Gamma,$$

where $\pi : (-\infty, 0] \times \Gamma \to \Gamma$ is the natural projection. When extending P to M one has to be slightly more careful in order not to violate the ellipticity condition. First, if P' is any extension of P to M then ellipticity holds at least on some small open neighbourhood of Ω in M. Now, by freezing coefficients along the x_1 coordinate P can be extended to an elliptic operator P'' on M which at $x_1 = 0$ may have non-differentiable coefficients. However, if we use a parition of unity χ_1, χ_2 subordinated to the open cover

$$(-\varepsilon, 0] \times \Gamma \cup_\Gamma \Omega, \quad (-\infty, 0) \times \Gamma,$$

then $P^M = \chi_1 P' + \chi_2 P''$ yields a smooth elliptic operator on M that extends P provided $\varepsilon > 0$ has been chosen sufficiently small.

We continue our list with some natural domains for P.

- $P_0 : C_c^\infty(\Omega^\circ, E) \to C_c^\infty(\Omega^\circ, F)$.
- $P_0^t : C_c^\infty(\Omega^\circ, F) \to C_c^\infty(\Omega^\circ, E)$ where P^t denotes the formally adjoint of P. Note that P^t is again elliptic.
- $P_{\min} := \overline{P_0}$, $P^t_{\min} := \overline{P^t_0}$.
- $P_{\max} := (P^t_0)^* = (P^t_{\min})^*$, i.e. $\mathscr{D}(P_{\max}) = \{u \in L^2(\Omega, E) \mid Pu \in L^2(\Omega, E)\}$. $P^t_{\max}$ is likewise defined.

$P_{\min}$, $P_{\max}$ are called minimal and maximal extensions of P_0.

Chapter 1

Boundary Value Problems

This chapter introduces to the general study of elliptic boundary value problems. It is divided into three sections each of which presents some basic facts about elliptic boundary value problems that we will need afterwards.

In the first section we prove a trace theorem containing the standard trace lemma and the surjectivity of the trace map by a rather explicit construction of a continuous right inverse which is a special case of the construction in [BL01, Sec. 2]. Since in the following we will need a certain property that is true only for a particular choice of this right inverse we repeat the computations. The third part of the theorem is one of the decisive steps towards elliptic boundary value problems for it guarantees the existence of a trace for any weak solution in a certain distribution space. It makes therefore sense to say that u *is a weak solution to* $Pu = v$ *satisfying a given boundary condition. Our proof is based on a certain duality argument that can be found in [LM72, Sec. II.6.5]. Completely different approaches in the case of a Dirac operator are given in [BBW93, Sec. 13] and [BL01, Sec. 2].*

In Section 1.2 we specify the general type of boundary value problems that we will study. We introduce the notions of the realisation associated to a boundary problem and the adjoint boundary condition. Then we define what we will call regularity and well-posedness. A boundary condition will be regular if any weak solution subject to this condition is a strong (i.e. regular) one.

The terminology differs to a certain extent from the standard literature. In particular, Seeley's notion of well-posedness (e.g. [See69]) is expressed as a property of the boundary condition, not as one of the realisation. It will be shown in Chapter 2 that both conditions are equivalent.

It should be emphasised that in this chapter no satisfactory condition for regularity and well-posedness in terms of the boundary condition will be achieved. However, we will establish the following fundamental fact: the validity of a Gårding type estimate (sometimes called coerciveness) is equivalent to regularity. This is a key argument that will enable us to formulate the regularity condition in Fredholm theory. The term "Gårding inequality" is due to estimates first studied by Gårding (cf. [Går53]) a generalisation of which are called a priori-estimates in [LM72, Sec II.5].

In the last section we prove closedness of the ranges of $P_{\min}$ *and* $P_{\max}$ *and, most important, that the image of* $H^d(\Omega, E)$ *under* P *coincides with* $\operatorname{ran} P_{\max}$*. Based on [BBFO01] and [BBF98] we develop some basic results* *for boundary value problems which generalise the theory of symplectic functional analysis for boundary value problems developed there in various directions: First we drop the assumption that P be formally self-adjoint. Consequently, β, which in our notation is $W(P)$, no longer carries a symplectic structure, but only a dual pairing with $W(P^t)$. Furthermore, the operator P may be an elliptic operator of arbitrary order (and shape). As in loc. cit. the abstract Cauchy data space (as a subspace of $W(P)$) plays an important role in the regularity and Fredholm theory worked out in the next chapter.*

As a consequence of the closedness of $\operatorname{ran} P_{\min}$ we obtain another fundamental fact: the Cauchy data space of the adjoint P^t is the annihilator of the Cauchy data space of P. In particular, if P is symmetric, then the Cauchy data space is Lagrangian. The observation that for any densely defined closed unbounded symmetric operator $P_{\min}$ with $\operatorname{ran} P_{\min}$ closed the abstract Cauchy data space is Lagrangian is a standard fact of symplectic functional analysis (see [MS95]).

1.1 Weak Traces for Elliptic Operators

Proposition 1.1.1. *The graph norm of P restricted to $C_c^\infty(\Omega^\circ, E)$ is equivalent to the Sobolev norm $\|.\|_{H^d(\Omega,E)}$. In particular, $\mathscr{D}(P_{\min}) = H_0^d(\Omega, E)$ and $\mathscr{D}(P^t_{\min}) = H_0^d(\Omega, F)$.*

Proof. Set $U = \Omega^\circ$ and let V be an open neighbourhood of Ω such that $U \subset\subset V \subset\subset M$. We have the following interior elliptic estimate ([Tay96, Chap.5 Theorem 11.1]):

$$\|u\|_{H^d(U)} \leq \text{const} \cdot \big(\|Pu\|_{L^2(V,F)} + \|u\|_{L^2(V,E)}\big),$$

for all $u \in H^d_{\text{loc}}(M, E)$. Now, the proposition follows since $\|Pu\|_{L^2(V,F)} + \|u\|_{L^2(V,E)} = \|Pu\|_{L^2(\Omega,F)} + \|u\|_{L^2(\Omega,E)}$ for all $u \in C_c^\infty(\Omega^\circ)$ and $\|Pu\|_{L^2(\Omega,F)} \leq \text{const} \cdot \|u\|_{H^d(\Omega,E)}$. $\square$

Let $\gamma^j : C^\infty(\Omega, E) \to C^\infty(\Gamma, E')$ be the trace map $\gamma^j u := (\nabla^E_\nu)^j u|_\Gamma$. Set $\rho^d := (\gamma^0, ..., \gamma^{d-1})$. Analogously, ∇^F gives rise to trace maps $\gamma^j : C^\infty(\Omega, F) \to C^\infty(\Gamma, F')$. The corresponding maps for F will also be denoted by γ^j, ρ^d, resp.

Proposition 1.1.2. *There exists a (uniquely determined) differential operator $J : C^\infty(\Gamma, E'^d) \to C^\infty(\Gamma, F'^d)$ such that for all $u \in C^\infty(\Omega, E)$, $v \in C^\infty(\Omega, F)$ we have*

$$\langle Pu, v\rangle_{L^2(\Omega,F)} - \langle u, P^t v\rangle_{L^2(\Omega,E)} = \langle J\rho^d u, \rho^d v\rangle_{L^2(\Gamma,F'^d)}. \tag{1.1}$$

J is a matrix of differential operators J_{kj} of order $d-1-k-j$, $0 \leq k, j \leq d-1$ and $J_{kj} = 0$ if $k + j > d - 1$ (J is upper skew-triangular). Moreover, for $j = d-1-k$ we have

$$J_{k(d-1-k)} = i^d(-1)^{d-1-k}\widehat{P}(\nu^\flat).$$

Proof. Let (U, x) be a coordinate system such that $x_1(p) = d(p, \Gamma)$ and $\partial_1 \perp \partial_j$ for $j \geq 2$. We can construct J locally, i.e. find some J on $U \cap \Gamma$ such that (1.1) holds for all u, v supported in U. (1.1) then shows that J is globally defined. U may be chosen such that the restriction of E and F to U is trivial. W.l.o.g. we can assume that $E = F = \mathbb{C} \times M$. Clearly, if u and v have compact support in U, then we may regard u, v as functions in $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1})$.

We write P w.r.t. to the coordinates $x = (x_1, ..., x_n)$. Then

$$P = \sum_{|\alpha|\leq d} p_\alpha(x) D^\alpha = \sum_{k\leq d} A_k(x_1) D_1^k$$

where $A_k(x_1)$ is a differential operator of order $d-k$ in $\mathbb{R}^{n-1}$. (If necessary, P can be extended from $\operatorname{supp} u \cup \operatorname{supp} v$ to a differential operator on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$.) Note that

$$A_d(x_1) = p_{(d,0,\dots,0)}(x) = \widehat{P}(x, dx_1).$$

Green's formula is a consequence of the following partial integration identity

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}^{n-1}} D_1 u(x)\cdot\overline{v(x)}dx = \\ \int_{\mathbb{R}_+\times\mathbb{R}^{n-1}} u(x)\cdot\overline{D_1 v(x)}dx + i\int_{\mathbb{R}^{n-1}}\gamma_0 u(x')\overline{\gamma_0 v(x')}dx'.\end{aligned} \tag{1.2}$$

This rule has to be applied $d-1$ times to the expression

$$\langle Pu, v\rangle = \sum_{k=0}^{d}\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}} D_1^k u(x)\cdot\overline{A_k^t(x_1)v(x)}\sqrt{g(x_1,x')}dx'dx_1, \quad k=0,...,d,$$

where $A_k^t(x_1)$ is the formal adjoint of $A_k(x_1) : C^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1})$ w.r.t. the density $\sqrt{g(x_1,.)}dx'$. Using $\gamma_0 D_1^j = (-i)^j\gamma_j$ it follows from (1.2) that $\langle Pu,v\rangle - \langle u, P^t v\rangle$ can be written in the following form

$$\sum_{k+j\le d-1}\int_{\mathbb{R}^{n-1}}\gamma_j u(x')\cdot\overline{J_{kj}^t\gamma_k v(x')}\sqrt{g(0,x')}dx'$$

where each $J_{kj} : C^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1})$ is a differential operator of order $d-1-k-j$. For the skew-diagonal terms of the matrix J_{kj} we obtain

$$J_{k(d-1-k)} = i^d(-1)^{d-1-k}A_d(0).$$

This proves the proposition since $A_d(0) = \widehat{P}(dx_1)|_\Gamma = \widehat{P}(\nu^\flat)$. □

Remark 1.1.3. We can regard $M \setminus \Omega^\circ$ as a manifold with boundary Γ. Denote by $P_\pm$ the restriction of P to Ω, $M\setminus\Omega$, resp., and define $J_\pm$ accordingly, using in both cases the unit normal field ν. Similarly, denote by $r_\pm$ the restriction onto Ω, $M\setminus\Omega$, resp. Since M is closed

$$\begin{aligned}0 &= \langle Pu, v\rangle_{L^2(M,F)} - \langle u, P^t v\rangle_{L^2(M,F)}\\ &= \langle P_+ r_+ u, r_+ v\rangle_{L^2(\Omega,F)} - \langle r_+ u, P_+^t r_+ v\rangle_{L^2(\Omega,F)}\\ &\quad + \langle P_- r_- u, r_- v\rangle_{L^2(M\setminus\Omega,F)} - \langle r_- u, P_-^t r_- v\rangle_{L^2(M\setminus\Omega,F)}\\ &= \langle J_+\rho^d u, \rho^d v\rangle_{L^2(\Gamma,F'^d)} + \langle J_-\rho^d u, \rho^d v\rangle_{L^2(\Gamma,F'^d)}\end{aligned}$$

for $u \in C_c^\infty(\Omega^\circ, E)$, $v \in C_c^\infty(M, F)$. Since Γ is a smooth submanifold of M the trace operator $\rho^d : C_c^\infty(M, E/F) \to C^\infty(\Gamma, E/F)$ is onto. Hence, $J_+ = -J_-$.

Theorem 1.1.4. (i) *$H^d(\Omega, E) \subset \mathscr{D}(P_{\max})$ is dense.*

(ii) *We have continuous trace maps ρ^d (obtained by continuous extension):*

(a) $\rho^d : H^d(\Omega, E) \to \bigoplus_{j=0}^{d-1} H^{d-j-\frac{1}{2}}(\Gamma, E')$,

(b) $\rho^d : \mathscr{D}(P_{\max}) \to \bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, E')$.

Furthermore, the map (a) is surjective and has a continuous right-inverse, η^d. Green's formula

$$\langle Pu, v\rangle_{L^2(\Omega,F)} - \langle u, P^t v\rangle_{L^2(\Omega,E)} = \langle J\rho^d u, \rho^d v\rangle_{L^2(\Gamma,F'^d)}$$

extends to $\mathscr{D}(P_{\max}) \times H^d(\Omega, F)$ if the right hand side is interpreted as the L^2-dual pairing

$$\bigoplus_{j=0}^{d-1} H^{-d+j+\frac{1}{2}}(\Gamma, F') \;\times\; \bigoplus_{j=0}^{d-1} H^{d-j-\frac{1}{2}}(\Gamma, F') \longrightarrow \mathbb{C}.$$

(iii) *If $u \in \mathscr{D}(P_{\max})$ then $u \in H^d(\Omega, E)$ if and only if*

$$\rho^d u \in H^{d-1/2}(\Gamma, E') \oplus \cdots \oplus H^{1/2}(\Gamma, E').$$

Proof. (ii): Although (a) is standard by now (see [LM72], [See64]) let us give the proof for completeness. Using a partition of unity subordinate to a finite cover of Γ by coordinate charts we can, as in the previous proof, confine ourselves to the case $M = \mathbb{R}^n$, $\Gamma = \mathbb{R}^{n-1}$. W.l.o.g. we may assume again $E = \mathbb{C} \times M$. Let $u \in H^d(\mathbb{R}^n)$. We will have to show that $\partial_1^j u|_{\{x_1=0\}} \in H^{d-j-1/2}(\mathbb{R}^{n-1})$. Since $\partial_1^j u \in H^{d-j}(\mathbb{R}^n)$ we only have to consider the case $j = 0$. $C_c^\infty(\mathbb{R}^n)$ being dense in $H^d(\mathbb{R}^n)$ it suffices to show that for $u \in C_c^\infty(\mathbb{R}^n)$ we have

$$\|u|_{x_1=0}\|_{H^{d-1/2}(\mathbb{R}^{n-1})} \le \text{const} \cdot \|u\|_{H^d(\mathbb{R}^n)}.$$

Set $v = u|_{\{x_1=0\}}$ and let $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Observe that $\widehat{v}(\xi') = \int_{\mathbb{R}} \widehat{u}(\xi_1, \xi') d\xi_1$. Therefore, by the Hölder inequality, we have for all $s > 1/2$

$$\begin{aligned} |\widehat{v}(\xi')|^2 &\le \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 (1 + \|\xi\|^2)^s d\xi_1 \cdot \int_{\mathbb{R}} (1 + \|\xi\|^2)^{-s} d\xi_1 \\ &= C_s \cdot \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 (1 + \|\xi\|^2)^s d\xi_1 \end{aligned}$$

where

$$C_s = \int_{\mathbb{R}} (1 + \|\xi\|^2)^{-s} d\xi_1 = \text{const} \cdot (1 + \|\xi'\|^2)^{-s+1/2}.$$

It follows that

$$\begin{aligned} \|u|_{x_1=0}\|_{H^{d-1/2}(\mathbb{R}^{n-1})} &\le \int_{\mathbb{R}^{n-1}} |\widehat{v}(\xi')|^2 (1 + \|\xi'\|^2)^{d-1/2} d\xi' \\ &\le \text{const} \cdot \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1 + \|\xi\|^2)^s (1 + \|\xi'\|^2)^{d-1/2-s+1/2} d\xi \\ &\le \text{const} \cdot \|u\|_{H^d(\mathbb{R}^n)}, \end{aligned}$$

since

$$(1 + \|\xi\|^2)^s (1 + \|\xi'\|^2)^{d-s} \le (1 + \|\xi\|^2)^d.$$

whenever $d \ge s$.

In order to show surjectivity of the trace map (a) we will construct a right inverse for the trace map $\rho^d : H^d(\mathbb{R}^n) \to H^{d-1/2}(\mathbb{R}^{n-1})$. Set

$$\varphi(\xi') = (\|\xi'\|^2 + 1)^{1/2}, \quad \Phi = (\Delta' + 1)^{1/2}.$$

where $\Delta' = -\sum_{j=0}^{n-1} \partial_j^2$ denotes the standard Laplacian in $\mathbb{R}^{n-1}$. For each $\chi \in C_c^\infty(\mathbb{R})$ one can define an operator $\mu : C_c^\infty(\mathbb{R}^{n-1}) \to C_c^\infty(\mathbb{R}^n)$ by

$$\mu v(x_1, x') = (\chi(x_1\Phi)v)(x').$$

Note that

$$\begin{aligned}\widehat{\mu v}(\xi_1, \xi') &= \tfrac{1}{2\pi} \int_{\mathbb{R}} e^{-ix_1\xi_1} \chi(x_1\varphi(\xi'))\widehat{v}(\xi')dx_1 \\ &= \tfrac{1}{2\pi} \int_{\mathbb{R}} \chi(y_1)e^{-iy_1(\xi_1\varphi(\xi')^{-1})} dy_1\varphi(\xi')^{-1}\widehat{v}(\xi') \\ &= \widehat{\chi}(\xi_1\varphi(\xi')^{-1})\varphi(\xi')^{-1}v(\xi').\end{aligned}$$

Now,

$$\begin{aligned}&\int_{\mathbb{R}^n} |\widehat{\mu v}(\xi_1, \xi')|^2 \varphi(\xi')^{2k}|\xi_1|^{2l}d\xi \\ &\qquad = \int_{\mathbb{R}^n} |\widehat{\chi}(\eta_1)|^2|\eta_1|^{2l}\varphi(\xi')^{-1}|\widehat{v}(\xi')|^2\varphi(\xi')^{2k+2l}d\xi' d\eta_1 \\ &\qquad \leq \text{const} \cdot \|v\|_{H^{d-1/2}(\mathbb{R}^{n-1})},\end{aligned}$$

when $k + l \leq d$. This shows that we have defined a continuous operator

$$\mu : H^{d-1/2}(\mathbb{R}^{n-1}) \to H^d(\mathbb{R}^n).$$

Now, choose functions $\chi_j \in C_c^\infty(\mathbb{R})$ such that

$$\partial_1^l\big|_{x_1=0}\chi_j = \delta_{jl}$$

and set $\mu_j v = \chi_j(x_1\Phi)v$. It follows that $\partial_1^l \mu_j v = \delta_{jl}\Phi^l v$. Hence,

$$\eta^d := \begin{pmatrix}\mu_0 & \mu_1\Phi^{-1} & \cdots & \mu_{-d+1}\Phi^{-d+1}\end{pmatrix} : \bigoplus_{j=0}^{d-1} H^{d-1/2-j}(\mathbb{R}^{n-1}) \to H^d(\mathbb{R}^n)$$

defines a right inverse for ρ^d. Using coordinate charts and a partition of unity one can now glue together such extension operators and thus construct a right inverse for the trace operator on M.

Since ρ^d vanishes on

$$\big\{u \in H^d(\Omega, E) \;\big|\; u|_{M\setminus\Omega} = 0\big\},$$

ρ^d is also well-defined on sections in $H^d(\Omega, E)$. Moreover, a right inverse to the trace map for Ω is simply defined by composing the map η^d constructed above with the restriction to Ω.

(b): By the continuity just proved Green's formula (1.1) extends to elements $u \in C^\infty(\Omega, E), v \in H^d(\Omega, F)$, i.e.

$$\langle Pu, v\rangle_{L^2} - \langle u, P^t v\rangle_{L^2} = \langle J\rho^d u, \rho^d v\rangle_{L^2}.$$

The diagonal elements $J_{j(d+1-j)}$ equal $i^d(-1)^{d-1-j}\widehat{P}(\nu^\flat)$. Therefore, J is invertible and J^{-1} is now lower skew-triangular, i.e. its components, J^{jk}, are differential operators of order $j+k-d+1$ if $j+k>d$ or 0 if $j+k\le d$. In fact, we can compute J^{jk} recursively by using the formulae

$$\sum_{j=0}^{d-1-i} J_{ij}J^{jk}=\delta_{ik}$$

for $i=d-1,d-2,...,0$. In particular, J^{-1} is a continuous operator

$$J^{-1}:\bigoplus_{k=0}^{d-1} H^{-d+k+1/2}\to\bigoplus_{j=0}^{d-1} H^{-j-1/2}.$$

Observe that $(J\circ\rho^d)$ is uniquely determined by the formula

$$\langle (J\circ\rho^d)u,g\rangle=\langle Pu,\eta^d g\rangle-\langle u,P^t\eta^d g\rangle. \tag{1.3}$$

Now, let $u\in\mathscr{D}(P_{\max})$, then by the right hand side of (1.3) we can define a bounded anti-linear functional

$$(J\circ\rho^d)u:H^{d-1/2}(\Gamma,F')\times\cdots\times H^{1/2}(\Gamma,F')\to\mathbb{C},\quad g\mapsto\langle P_{\max}u,\eta^d g\rangle-\langle u,P^t\eta^d g\rangle. \tag{1.4}$$

Furthermore the norm of this functional is bounded by $\|u\|_{\mathscr{D}(P_{\max})}$. We deduce that $J\circ\rho^d$ can be extended to a continuous map

$$J\circ\rho^d:\mathscr{D}(P_{\max})\to\bigoplus_{j=0}^{d-1} H^{-d+j+\frac{1}{2}}(\Gamma,F').$$

Composition with J^{-1} gives a continuous extension of the ordinary trace map to $\mathscr{D}(P_{\max})$ as stated in (b). By construction Green's formula extends to the case where $u\in\mathscr{D}(P_{\max})$.

(iii): The *only if* part is trivial by (ii). To prove the *if* part, assume that $u\in\mathscr{D}(P_{\max})$ and that $\rho^d u$ is in $H^{d-1/2}(\Gamma,E')\times\cdots\times H^{1/2}(\Gamma,E')$. Then, using the analogue of η^d for the complement $M\setminus\Omega$, we can extend u to some $\widetilde{u}$ such that $\widetilde{u}\big|_{M\setminus\Omega}\in H^d_{\mathrm{loc}}(\overline{M\setminus\Omega},E)$ and

$$\rho^d r_-\widetilde{u}=\rho^d r_+\widetilde{u},$$

where the notation is as in the preceding remark. Using Green's formula for Ω and $M\setminus\Omega$ it now follows from Remark 1.1.3 that

$$\langle\widetilde{u},P^t v\rangle_{L^2(M,E)}=\langle Pr_+\widetilde{u},v\rangle_{L^2(\Omega,F)}+\langle Pr_-\widetilde{u},v\rangle_{L^2(M\setminus\Omega,F)}$$

for all $v\in C_c^\infty(M,E)$. Hence, $\widetilde{u}$ is a weak solution of P^M. Since P^M is elliptic $\widetilde{u}$ is in $H^d_{\mathrm{loc}}(M,E)$. It follows that $u\in H^d(\Omega,E)$.

(i): Setting

$$P_S:H^d(\Omega,E)\to L^2(\Omega,E),\quad u\mapsto Pu,$$

we have $P_S\subset P_{\max}$. Hence,

$$(P_S)^*\supset(P_{\max})^*=(P^t_{\min})^{**}=\overline{P^t_{\min}}=P^t_{\min}.$$

We claim that this inclusion is an identity. Let $v\in\mathscr{D}((P_S)^*)$, i.e. there exists $C>0$ such that for all $u\in H^d(\Omega,E)$ we have

$$|\langle Pu,v\rangle_{L^2}|=|\langle u,P^t v\rangle_{L^2}+\langle J\rho^d u,\rho^d v\rangle|\le C\|u\|_{L^2(M,E)}.$$

Using a cut-off function $\chi_\varepsilon \in C^\infty(\Omega)$ such that $d(\operatorname{supp}\chi_\varepsilon, \Gamma) < \varepsilon$, $\chi_\varepsilon \equiv 1$ near Γ, we find

$$|\langle \chi_\varepsilon u, P^t v\rangle + \langle J\rho^d u, \rho^d v\rangle| \leq C\|\chi_\varepsilon u\|_{L^2(M,E)}$$

for $\rho^d \chi_\varepsilon u = \rho^d u$. Since

$$\lim_{\varepsilon\to 0} \chi_\varepsilon u = 0$$

in $L^2(\Omega, E)$ we reach

$$\langle J\rho^d u, \rho^d v\rangle = 0$$

for all $u \in H^d(\Omega, E)$. Using the surjectivity of the map (a) in part (ii) and the surjectivity of J we find that $\rho^d v = 0$. Using (iii) we reach $v \in H^d(\Omega, F)$. Since

$$H_0^d(\Omega, F) = \left\{u \in H^d(\Omega, F) \;\middle|\; \rho^d u = 0\right\},$$

we have shown the claim. It follows that $\mathscr{D}(P_S^*) = \mathscr{D}(P_{\min}^t)$ and therefore $\overline{P_S} = P_{\max}$. □

We have

$$\mathscr{D}(P_{\min}) = \ker\Big(\rho^d : \mathscr{D}(P_{\max}) \to \bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, E')\Big).$$

Let $W(P)$, $S(P)$ be the spaces of boundary values $\rho^d u$ of weak, strong solutions, i.e. $u \in \mathscr{D}(P_{\max})$, $u \in H^d(\Omega, E)$, resp. The following sequences are exact.

$$0 \longrightarrow \mathscr{D}(P_{\min}) \hookrightarrow \mathscr{D}(P_{\max}) \overset{\rho^d}{\twoheadrightarrow} W(P) \longrightarrow 0$$
$$0 \longrightarrow \mathscr{D}(P_{\min}) \hookrightarrow H^d(\Omega, E) \overset{\rho^d}{\twoheadrightarrow} S(P) \longrightarrow 0$$

Hence, the map

$$\overline{\rho^d} : \mathscr{D}(P_{\max})/\mathscr{D}(P_{\min}) \longrightarrow \bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, E'), \quad [u] \mapsto \rho^d u$$

is a continuous injection.

The restriction of $\overline{\rho^d}$ to $S(P)$ is an isomorphism onto $H^{d-1/2}(\Gamma, E') \times \cdots \times H^{1/2}(\Gamma, E')$ for the ordinary trace map has a continuous right inverse. From now on, we identify $W(P)$ and $S(P)$ with the corresponding quotient spaces. In particular, we endow $W(P)$ with the quotient topology. Note that this will (except for the case $\dim M = 1$) differ from the topology induced by the norm of $\bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, E')$. One way to see this immediately is to consider the non-degenerate sesquilinear form

$$\omega'([u], [v]) = \langle J\rho^d u, \rho^d v\rangle_{L^2(\Gamma, F'^d)} \tag{1.5}$$

which extends (by continuity) to $W(P) \times W(P^t)$ whereas $(g, h) \mapsto \langle Jg, h\rangle$ does not extend to

$$\bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, E') \times \bigoplus_{j=0}^{d-1} H^{-j-\frac{1}{2}}(\Gamma, F') :$$

Namely, if $h \mapsto \langle Jg, h\rangle$ is continuous w.r.t. to the correpsonding norm of h, then $(Jg)_j \in H^{j+1/2}(\Gamma, F')$ and hence $g_i \in H^{d-1/2-i}(\Gamma, E')$.

We set $\Delta_\Gamma = \nabla_\Gamma^* \nabla_\Gamma$ where ∇_Γ denotes the restriction of ∇ to Γ. Then,

$$\Phi := (\Delta_\Gamma + 1)^{1/2} \tag{1.6}$$

generates the Sobolev scale $H^s(\Gamma, E')$, i.e. we may assume that the Sobolev norms in $H^s(\Gamma, E')$ are induced by Φ^s. In order to achieve that the whole boundary data is of the same Sobolev order we introduce the matrix

$$\Phi_d := \begin{pmatrix} \Phi^{\frac{d-1}{2}} & 0 & \cdots & 0 \\ 0 & \Phi^{\frac{d-3}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi^{\frac{-d+1}{2}} \end{pmatrix}. \tag{1.7}$$

Note that we could replace Δ_Γ by any second order operator with scalar principal symbol. The exact choice of Φ is of little importance here. However, it will turn out useful to have one fixed Φ and thus one fixed Φ_d at hand throughout this text.

Namely, $\Phi_d \overline{\rho^d}$ maps $W(P)$ into $H^{-d/2}(\Gamma, E'^d)$ and its restriction to $S(P)$ is an isomorphism onto $H^{d/2}(\Gamma, E'^d)$. We set

$$\widetilde{\rho}^d := \Phi_d \rho^d, \quad \widetilde{\eta}^d := \eta^d \Phi_d^{-1}. \tag{1.8}$$

Clearly, $\widetilde{\eta}^d$ is a continuous right inverse to $\widetilde{\rho}^d$.

Analogous constructions for the bundle F lead to $\Delta_\Gamma^{(F)} : C^\infty(\Gamma, F') \to C^\infty(\Gamma, F')$. Whenever this causes no ambiguity we will denote the corresponding matrices, $\Phi^{(F)}$ and $\Phi_d^{(F)}$, again by Φ, Φ_d, resp.

1.2 Basic Properties of Boundary Conditions

Consider a boundary condition of the form

$$B\widetilde{\rho}^d u = 0,$$

where $B : H^{-d/2}(\Gamma, E'^d) \to H'$ is a continuous operator and H' is a Hilbert space. Let Φ be the generator of the Sobolev scale on the boundary introduced in (1.6). If we set $\widetilde{\Phi} := \bigoplus_{j=1}^d \Phi$, then $\widetilde{\Phi}$ induces the Sobolev scale

$$H_t = H_t(\widetilde{\Phi}) = H^t(\Gamma, E'^d).$$

To begin with let us make the following assumption on B.

Assumption 1.2.1. There is a self-adjoint operator $\Psi : H' \supset \mathscr{D}(\Psi) \to H'$ such that:

(i) B is an operator of order μ from the Sobolev scale $H_t = H_t(\widetilde{\Phi})$ to the Sobolev scale $(H'_s := H_s(\Psi))_{s \in \mathbb{R}}$.

(ii) $B : H_s \to H'_{s+\mu}$ has closed range for each $s \in \mathbb{R}$.

(iii) $B\widetilde{\Phi} - \Psi B \in \mathrm{Op}^{\mu-1}((H_s), (H'_s))$.

We will now show that we may always assume that B is a projection. More precisely, we will prove that given B there exists an orthogonal projection B' of order 0, subject to the assumption above, which satisfies

$$\ker(B' : H_s \to H_s) \;=\; \ker(B : H_s \to H'_{s-\mu}),$$

for all $s \in \mathbb{R}$. Here, B' is orthogonal in the sense that $B'^* B' = 0$.

a) We may assume that $\mu = 0$ by substituting B for $\Psi^{-\mu} B$. Each $\Psi^{-\mu} : H'_s \to H'_{s+\mu}$ being an isomorphism, it is straightforward to see that the assumptions (i) - (iii) are preserved by this modification.

b) If $\mu = 0$, then $B^* B \in \mathrm{Op}^0 = \mathrm{Op}^0((H_s), (H_s))$. We have

$$[B^* B, \widetilde{\Phi}] = B^*(B\widetilde{\Phi} - \Psi B) + (B^* \Psi - \widetilde{\Phi} B^*) B \in \mathrm{Op}^0$$

since $B^* \Psi - \widetilde{\Phi} B^* = -(B\widetilde{\Phi} - \Psi B)^* \in \mathrm{Op}^0((H'_s), (H_s))$. For $s \geq 0$ we have

$$\ker(B^* B : H_s \to H_s) = \ker(B : H_s \to H_s)$$

since $\langle B^* B x, x \rangle = \|Bx\|^2$ for any $x \in H_s$. The analogous statement for $-s$, $s > 0$, is less obvious.

Lemma 1.2.2. *Let $T \in \mathrm{Op}^0 = \mathrm{Op}^0(H_s, H'_s)$ be an operator between Sobolev scales generated by Φ, Φ', resp., such that $T\Phi - \Phi' T \in \mathrm{Op}^0$. If $T : H_0 \to H'_0$ has closed range, then $T(H_s) = T(H_0) \cap H'_s$ for all $s > 0$. In particular, $T : H_s \to H'_s$ has closed range for all $s > 0$.*

Proof. For a generator Φ of a Sobolev scale we define

$$\mathscr{A}(\Phi) = \{ T \in \mathrm{Op}^0 \mid [T, \Phi] \in \mathrm{Op}^0 \}.$$

$\mathscr{A}(\Phi)$ is an involutive and spectrally invariant subalgebra of Op^0 (see Proposition A.2.1 and the remarks on p 126 ff).

Since $T(H_0)$ is closed, it follows that $T^*(H'_0)$ is closed. Hence

$$H_0 = T^*(H_0) \oplus (\ker T : H_0 \to H_0)$$

and therefore

$$\operatorname{ran} T = \operatorname{ran} TT^*.$$

TT^* has closed range, i.e. if $0 \in \operatorname{spec} TT^*$, then it is isolated. Let Q be the orthogonal projection onto the kernel of TT^*. Since $TT^* \in \mathscr{A}(\Phi')$ and since $\mathscr{A}(\Phi')$ is spectrally invariant we deduce that Q is again of order 0. Moreover, the operator

$$S = T^*(TT^* + Q)^{-1}$$

is of order 0. Now, if $Tu \in H'_s$, for some $u \in H_0$, then

$$TSTu = TT^*(TT^* + Q)^{-1} Tu = Tu - Q(TT^* + Q)^{-1} Tu.$$

Multiplication by $\mathrm{Id} - Q$ from the left yields

$$Tu = TSTu$$

and since $S \in \mathrm{Op}^0$ we have $STu \in H_s$ and therefore $Tu \in T(H_s)$. $\square$

Note that $B^* : H'_0 \to H_0$ and $B^* B : H_0 \to H_0$ have closed range. Hence, we may apply this lemma to B^* and $B^* B$. Since

$$B^* B(H_0) = B^*(H_0)$$

it follows that

$$B^* B(H_s) = B^* B(H_0) \cap H_s = B^*(H_0) \cap H_s = B^*(H_s),$$

for $s > 0$. For $W \subset H_s$ we define the annihilator $W^\perp$ by

$$W^\perp := \{y \in H_{-s} \mid \langle x, y\rangle = 0 \text{ for all } x \in W\}.$$

Since for an operator of order 0 we have

$$\ker(T : H_{-s} \to H'_{-s}) = (\operatorname{ran} T^* : H_s \to H_s)^\perp,$$

we find

$$\begin{aligned}\ker(B : H_{-s} \to H_{-s}) = (\operatorname{ran}(B^* : & H_s \to H_s))^\perp \\ &= (\operatorname{ran}(B^* B : H_s \to H_s))^\perp = \ker(B^* B : H_{-s} \to H_{-s}).\end{aligned}$$

c) By b) we may assume that B is a non-negative operator with closed range $\operatorname{ran}(B : H_0 \to H_0)$ in the algebra

$$\mathscr{A}(\widetilde{\Phi}) = \{T \in \mathrm{Op}^0 \mid [T, \widetilde{\Phi}] \in \mathrm{Op}^0\},$$

which by Proposition A.2.1 is spectrally invariant. In particular, it is invariant under holomorphic calculus and the spectral projection $\mathbf{1}_{\{0\}}(B)$ is again in $\mathscr{A}(\widetilde{\Phi})$.

It follows that we can restrict attention to the case where B satisfies:

Assumption 1.2.3. (i) B is an operator of order 0 on the Sobolev scale $H^s(\Gamma, E'^d)$.

(ii) $[B, \widetilde{\Phi}]$ is an operator of order 0.

(iii) $B^2 = B$.

In fact, we have seen that B may be chosen self-adjoint, i.e. B is an orthogonal projection.

Proposition 1.2.4. *If B is subject to Assumption 1.2.3, then for all $t \in \mathbb{R}$*

$$[B, \widetilde{\Phi}^t]$$

is an operator of order $t - 1 + \varepsilon$ for any $\varepsilon > 0$ on the Sobolev scale $(H^s(\Gamma, E'^d))_{s\in\mathbb{R}}$. If $t \in \mathbb{Z}$ then this statement also holds with $\varepsilon = 0$.

Proof. Note that Assumption 1.2.3 implies that

$$[B, \widetilde{\Phi}^k] = [B, \widetilde{\Phi}^{k-1}]\widetilde{\Phi} + \widetilde{\Phi}^{k-1}[B, \widetilde{\Phi}]$$

is an operator of order $k - 1$ for all $k \in \mathbb{Z}$, as one can see by induction on k for $k > 1$ and, for negative k, by

$$[B, \widetilde{\Phi}^{-k}] = -\widetilde{\Phi}^{-k}[B, \widetilde{\Phi}^k]\widetilde{\Phi}^{-k}.$$

With the same argument the statement for t is equivalent to that for $-t$. For non-integer $t > 0$ we may write

$$[B, \widetilde{\Phi}^t] = [B, \widetilde{\Phi}^{\lfloor t\rfloor}]\widetilde{\Phi}^{t-\lfloor t\rfloor} + \widetilde{\Phi}^{\lfloor t\rfloor}[B, \widetilde{\Phi}^{t-\lfloor t\rfloor}].$$

Therefore it remains to prove the proposition for $t \in (-1, 0)$. Observe that we can write

$$\widetilde{\Phi}^t = C_\alpha \int_0^\infty (\widetilde{\Phi} + x^\alpha)^{-1} dx,$$

where $\alpha = \frac{1}{1+t} > 1$ and $C_\alpha^{-1} = \int_0^\infty (1+x^\alpha)^{-1} dx$. Hence,

$$[B, \widetilde{\Phi}^t] = -C_\alpha \int_0^\infty (\widetilde{\Phi} + x^\alpha)^{-1} [B, \widetilde{\Phi}] (\widetilde{\Phi} + x^\alpha)^{-1} \; dx.$$

and thus, for $g, h \in H_\infty$, we have

$$|\langle [B, \widetilde{\Phi}^t] g, h\rangle| \le \text{const} \cdot \int_0^\infty \|[B, \widetilde{\Phi}](\widetilde{\Phi} + x^\alpha)^{-1} g\|_s x^\beta \; dx \cdot \sup_{x \in \mathbb{R}_+} \|x^{-\beta}(\widetilde{\Phi} + x^\alpha)^{-1} h\|_{-s},$$

where $\beta < \alpha - 1$. Now,

$$\int_0^\infty \|[B, \widetilde{\Phi}](\widetilde{\Phi} + x^\alpha)^{-1} g\|_s x^\beta dx \le \text{const} \cdot \|g\|_s,$$

for all $s \in \mathbb{R}$. Moreover,

$$(\lambda + x^\alpha)^{-1} x^{-\beta} \le \lambda^{-\frac{\alpha+\beta}{\alpha}}.$$

It follows that

$$\sup_{x \in \mathbb{R}_+} \|(\widetilde{\Phi} + x^\alpha)^{-1} x^{-\beta} h\|_{-s} \le \text{const} \cdot \|h\|_{-s-1-\beta/\alpha} < \infty,$$

and thus

$$[B, \widetilde{\Phi}^t] \in \mathrm{Op}^{-1-\beta/\alpha}.$$

This finishes the proof since $-1 - \frac{\beta}{\alpha} = -1 - \frac{\alpha-1-\delta}{\alpha} = t + \frac{\delta}{\alpha} \le t - 1 + \varepsilon$, when $\delta = \alpha - 1 - \beta \le \varepsilon\alpha$. $\square$

Since $\widetilde{\Phi}$ is a classical pseudodifferential operator with scalar principal symbol every pseudodifferential projection satisfies Assumption 1.2.3. However, there are natural examples for non-pseudolocal boundary conditions B satisfying Assumption 1.2.3. For instance, let $\tau : \Gamma \to \Gamma$ be an isometry such that $\tau^2 = \mathrm{Id}$.[1] Then

$$B = \frac{1}{2}(\mathrm{Id} + \tau^*) : C^\infty(\Gamma) \to C^\infty(\Gamma)$$

defines an orthogonal projection which commutes with the (standard) Laplacian on Γ.

More generally, if $G \times \Gamma \to \Gamma$ is the smooth action of a compact Lie group on Γ, then we may replace the metric Γ by some metric which is invariant under the action of G (see [BtD85]). The corresponding Laplacian Δ_Γ will be invariant under the action of some element $\tau \in G$. It follows that

$$(\tau^* - \mathrm{Id}) : C^\infty(\Gamma) \to C^\infty(\Gamma)$$

is an operator of order 0 which commutes with $(\Delta_\Gamma + 1)^{1/2}$.

This indicates that boundary conditions involving the action by a diffeomorphism may satisfy Assumption 1.2.1 with a suitably chosen Ψ. Such boundary conditions can thus be treated within our framework.

To a boundary condition B we associate its *realisation*

$$P_B : \{u \in H^d(\Omega, E) \;|\; B\widetilde{\rho}^d u = 0\} \longrightarrow L^2(\Omega, F).$$

[1] E.g., let τ be the rotation by π on the circle S^1.

Here, we use the term *regularity* in the following sense. We say B is a *regular* boundary condition for P if and only if all weak solutions u to

$$P_{\max} u = v \in L^2(\Omega, F), \quad B\widetilde{\rho}^d u = 0$$

are strong ones, i.e. $u \in H^d(\Omega, E)$. More precisely, with

$$\mathscr{D}(P_{\max,B}) := \{u \in \mathscr{D}(P_{\max}) \mid B\widetilde{\rho}^d u = 0\}$$

we have

Definition 1.2.5. We call B a *regular* boundary condition if

$$\mathscr{D}(P_B) = \mathscr{D}(P_{\max,B}).$$

B is called *well-posed* if it is regular and $\operatorname{ran} P_B$ has finite codimension.

The form $\omega'([u],[v])$ which was defined (1.5) can be rewritten in terms of the adjusted boundary data $\widetilde{\rho}^d u$, $\widetilde{\rho}^d v$ at least when u or v is in $H^d(\Omega, E)$:

$$\begin{aligned} \omega'([u],[v]) &= \langle J \Phi_d^{-1} \widetilde{\rho}^d u, \Phi_d^{-1} \widetilde{\rho}^d v \rangle_{L^2(\Gamma, F'^d)} \\ &= \langle \widetilde{J} \widetilde{\rho}^d u, \widetilde{\rho}^d v \rangle_{L^2(\Gamma, F'^d)} \end{aligned}$$

where $\widetilde{J} = \big(\Phi_d^{(F)}\big)^{-1} J \big(\Phi_d^{(E)}\big)^{-1}$. It follows that all components of $\widetilde{J}$,

$$\widetilde{J}_{ij} = \Phi^{\frac{2i+1-d}{2}} J_{ij} \Phi^{\frac{2j+1-d}{2}}.$$

are pseudodifferential operators of order $i+j+(1-d)+(d-1)-i-j = 0$. Furthermore, $\widetilde{J}$ is upper skew triangular and has invertible elements on the skew diagonal. Clearly,

Proposition 1.2.6. *Let B be a projection satisfying Assumption 1.2.1. Then, $(P_B)^* = P^t_{\max,B^{\mathrm{ad}}}$, where*

$$B^{\mathrm{ad}} = (\widetilde{J}^*)^{-1}(\operatorname{Id} - B^*)\widetilde{J}^* = (\widetilde{J}(\operatorname{Id} - B)\widetilde{J}^{-1})^*.$$

Proof. $v \in \mathscr{D}((P_B)^*)$ if and only if $v \in \mathscr{D}(P^t_{\max})$ and

$$u \mapsto \langle Pu, v\rangle - \langle u, Pv\rangle = \langle \widetilde{J}\widetilde{\rho}^d u, \widetilde{\rho}^d v\rangle \leq \mathrm{const}\,.\|u\|_{L^2(\Omega,E)}$$

for all $u \in H^d(\Omega, E)$ s.t. $B\widetilde{\rho}^d u = 0$. Let $u \in \mathscr{D}(P_B)$ be such a section. Choose a family of cut-off functions χ_ε as on page 25 and consider $u_\varepsilon(p) = \chi_\varepsilon(p) \cdot u(p)$. The above boundary term remains unchanged if we replace u by u_ε. Since

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2} = 0$$

this shows that $\langle \widetilde{J}\widetilde{\rho}^d u, \widetilde{\rho}^d v\rangle = 0$ for all $u \in H^d(\Omega, E)$ such that $B\widetilde{\rho}^d u = 0$. Hence, by the surjectivity of the ordinary trace map we obtain

$$\langle \widetilde{J}(\operatorname{Id} - B)g, \widetilde{\rho}^d v\rangle = 0$$

for all $g \in H^{d/2}(\Gamma, E'^d)$. Therefore, v is in the domain of the adjoint of P_B if and only if it is in $\mathscr{D}(P^t_{\max})$ and the boundary condition

$$(\operatorname{Id} - B)^* \widetilde{J}^* \widetilde{\rho}^d v = 0$$

is satisfied. From this it follows that $\mathscr{D}((P_B)^*) = \mathscr{D}(P^t_{\max,B^{\mathrm{ad}}})$, $B^{\mathrm{ad}} := (\widetilde{J}^*)^{-1}(\operatorname{Id} - B^*)\widetilde{J}^* = (\widetilde{J}(\operatorname{Id} - B)\widetilde{J}^{-1})^*$. $\square$

We will call B^{ad} the *adjoint boundary condition* to B.

Theorem 1.2.7. *Assume that B is a boundary condition satisfying Assumption 1.2.1. Then B is regular for P if and only if following regularity estimate holds for all $u \in H^d(\Omega, E)$ satisfying $B\widetilde{\rho}^d u = 0$:*

$$\|u\|_{H^d(\Omega,E)} \le \mathrm{const} \cdot \Big(\|u\|_{H^{d-1}(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)} \Big).$$

Proof. The proof of this theorem is surprisingly involved. We will now show two lemmas each of which implying one of the two directions.

Lemma 1.2.8. *Assume that B is a boundary condition satisfying Assumption 1.2.1 such that the following regularity estimate holds for all $u \in H^d(\Omega, E)$ satisfying $B\widetilde{\rho}^d u = 0$:*

$$\|u\|_{H^d(\Omega,E)} \le \mathrm{const} \cdot \Big(\|u\|_{H^{d-1}(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)} \Big).$$

Then B is regular for P.

Proof of the lemma. Let us assume, as we may, that B is a projection. Suppose $u \in L^2(\Omega, E)$, $Pu \in L^2(\Omega, F)$ and that the boundary condition $B\widetilde{\rho}^d u = 0$ is fulfilled. Choose a cut-off function χ_ε as in the proof of Proposition 1.2.6. By elliptic regularity of P in the interior $(1 - \chi_\varepsilon)u$ is in $H^d(\Omega, E)$ a priori. Thus in order to establish the regularity of B it suffices to show that $\chi_\varepsilon u \in H^d(\Omega, E)$. In the proof we may therefore make the assumption that u has compact support in $\Omega_\varepsilon = \{p \in \Omega \,|\, x_1(p) = d(p, \Gamma) < \varepsilon\} \cong [0, \varepsilon) \times \Gamma$. If $\pi : [0, \varepsilon) \times \Gamma \to \Gamma$ denotes the natural projection, then we can find bundle isomorphisms $E\big|_{\Omega_\varepsilon} \cong \pi^* E'$, $F\big|_{\Omega_\varepsilon} \cong \pi^* F'$. Thus, on Ω_ε, P takes the form

$$P = \sum_{j=0}^{d} A_j(x_1) D_1^j$$

where $A_d(x_1)$,...,$A_0(x_1) : C^\infty(\Gamma, E') \to C^\infty(\Gamma, F')$ are differential operators of order $0, 1, ..., d$, resp. We can always achieve $E = F$ and $A_d \equiv \mathrm{Id}$ by multiplying P with $A_d(x_1)^{-1}$. Note that $A_d(x_1) = \widehat{P}(x, dx_1)$ is invertible since P is elliptic.

For sections supported in a collar neighbourhood we can define $\Phi^s u$. Clearly,

$$\widetilde{\rho}^d \Phi^s u = \widetilde{\Phi}^s \widetilde{\rho}^d u, \qquad \text{where } \widetilde{\Phi}^s := \begin{pmatrix} \Phi^s & 0 & \cdots & 0 \\ 0 & \Phi^s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi^s \end{pmatrix}.$$

Assume that B is a boundary condition with an elliptic estimate as above. The regularity of B will follow from a series of statements that we will now prove step by step:

(i) For all $u \in H^d(\Omega, E)$ such that $B\widetilde{\rho}^d u = 0$ we have

$$\|u\|_{H^d(\Omega,E)} \le \mathrm{const} \cdot \Big(\|u\|_{L^2(\Omega,E)} + \|Pu\|_{L^2(\Omega,E)} \Big).$$

(ii) If $u \in H^{d-1}(\Omega, E)$, $Pu \in L^2(\Omega, E)$, $B\widetilde{\rho}^d u = 0$ then $u \in H^d(\Omega, E)$.

(iii) If $u \in H^{d-1}(\Omega, E)$, $Pu \in L^2(\Omega, E)$, $B\widetilde{\rho}^d u \in H^{d/2}(\Gamma, E'^d)$ then $u \in H^d(\Omega, E)$.

(iv) If $u \in \mathscr{D}(P_{\max,B})$ then $u \in H^{d-1}(\Omega, E)$.

From (ii) and (iv) it follows that B is a regular boundary condition.

(i) is clear for the embeddings $H^d(\Omega, E) \hookrightarrow H^{d-1}(\Omega, E)$ and $H^{d-1}(\Omega, E) \hookrightarrow L^2(\Omega, E)$ are compact. Hence, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u\|_{H^{d-1}(\Omega,E)} \leq \varepsilon \|u\|_{H^d(\Omega,E)} + C_\varepsilon \|u\|_{L^2(\Omega,E)}$$

for all $u \in H^d(\Omega, E)$ (cf. [LM72, Theorem 16.3]).

Now, take u as in (ii) and consider $v_n := (\mathrm{Id} + \frac{1}{n}\Phi)^{-1}u$. We have

$$\rho^{d-1}u \in H^{d-3/2} \times \cdots \times H^{1/2}.$$

Now, $D_1^d u = Pu - \sum_{j=0}^{d-1} A_j(x_1) D_1^j u \in L^2((0,\infty), H_{-1})$ and $D_1^{d-1}u \in L^2(\Omega, E)$. It follows that the section $w =: \Phi^{-1} D_1^{d-1} u$ satisfies $-iw' = \Phi^{-1}Pu - \sum_{j=0}^{d-1} \Phi^{-1} A_j(x_1) D_1^j u \in L^2((0,\varepsilon), L^2(\Gamma, E'))$ and $w \in L^2((0,\infty), H_1)$. Therefore $w \in H^1(\Omega, E)$ and $\gamma^0 w = \Phi^{-1}\gamma^0 u \in H^{1/2}(\Gamma, E')$. We conclude that $\widetilde{\rho}^d u \in H^{d/2-1}(\Gamma, E'^d)$.

Hence,

$$\begin{aligned} B\widetilde{\rho}^d v_n &= B(\mathrm{Id} + \tfrac{1}{n}\widetilde{\Phi})^{-1}\widetilde{\rho}^d u \\ &= [B, (\mathrm{Id} + \tfrac{1}{n}\widetilde{\Phi})^{-1}]\widetilde{\rho}^d u \\ &= (\mathrm{Id} + \tfrac{1}{n}\widetilde{\Phi})^{-1}[\widetilde{\Phi}, B](n + \widetilde{\Phi})^{-1}\widetilde{\rho}^d u \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

in $H^{d/2}(\Gamma, E'^d)$ since $(n + \widetilde{\Phi})^{-1}\widetilde{\rho}^d u$ converges to 0 in $H^{d/2}(\Gamma, E')$ and

$$(\mathrm{Id} + \tfrac{1}{n}\widetilde{\Phi})^{-1}[\widetilde{\Phi}, B]$$

is a bounded family of operators in $\mathscr{B}(H^{d/2}(\Gamma, E'^d))$.

We define $u_n := v_n - \widetilde{\eta}^d B\widetilde{\rho}^d v_n$. Note that by what we have just shown it follows that $\widetilde{\eta}^d B\widetilde{\rho}^d v_n \underset{n\to\infty}{\longrightarrow} 0$ in $H^d(\Omega, E)$. Furthermore, $B\widetilde{\rho}^d u_n = 0$ and u_n converges to u in $L^2(\Omega, E)$ for v_n does. In order to show that u_n is a Cauchy sequence in $H^d(\Omega, E)$ it suffices, by (i), to show that Pu_n converges in $L^2(\Omega, E)$. We compute

$$\begin{aligned} Pu_n &= (\mathrm{Id} + \tfrac{1}{n}\Phi)^{-1}Pu + \sum_{j=0}^{d-1}[A_j, (\mathrm{Id} + \tfrac{1}{n}\Phi)^{-1}]D_1^j u \\ &= (\mathrm{Id} + \tfrac{1}{n}\Phi)^{-1}Pu + \sum_{j=0}^{d-1}(\mathrm{Id} + \tfrac{1}{n}\Phi)^{-1}[\Phi, A_j](n + \Phi)^{-1}D_1^j u. \end{aligned}$$

By dominated convergence $(n + \Phi)^{-1}D_1^j u$ converges to 0 in $L^2((0,\varepsilon), H^{d-j}(\Gamma, E'))$, for $j = 0, ..., d-1$. Moreover $(\mathrm{Id} + \frac{1}{n}\Phi)^{-1}[\Phi, A_j]$ is a bounded family of operators in

$$C^\infty\big([0,\varepsilon],\, \Psi^{d-j}(\Gamma, E')\big).$$

We deduce that u is the L^2-limit of a Cauchy sequence in $H^d(\Omega, E)$. Therefore $u \in H^d(\Omega, E)$.[2]

[2]Observe that the proof of the lemma for first order elliptic operators is complete at this stage.

(iii) is a simple consequence of (ii). If $B\widetilde{\rho}^d u \in H^{d/2}(\Gamma, E'^d)$, then $\widetilde{\eta}^d B\widetilde{\rho}^d u \in H^d(\Omega, E)$, i.e. we may consider

$$\widetilde{u} = u - \widetilde{\eta}^d B\widetilde{\rho}^d u \in H^{d-1}(\Gamma, E'^d).$$

Since $P\widetilde{u} \in L^2(\Omega, E)$ and $B\widetilde{\rho}^d u = 0$ it follows from (ii) that $\widetilde{u}$ and thus u are in $H^d(\Omega, E)$.

In order to prove (iv) let us first show the following assertions assuming $u \in \mathscr{D}(P_{\max})$:

(A1) $\Phi^{-d} u \in H^d(\Omega, E)$

(A2) Let $k = 0, 1, .., d-1$. If $\Phi^{-d+k} u \in H^d(\Omega, E)$ then $P\Phi^{-d+k+1} u \in L^2(\Omega, E)$.

(A3) If $\Phi^{-d+k} u \in H^d(\Omega, E)$, then

$$B\widetilde{\rho}^d \Phi^{-d+k+1} u \in H^{d/2}(\Gamma, E'^d).$$

(A1): Here, we have to use operators of the form

$$T(x_1)\partial_1^k : C^\infty((a,b), C^\infty(\Gamma, E')) \to C^\infty((a,b), C^\infty(\Gamma, E'))$$

defined by

$$u(x_1, x') \mapsto \big(T(x_1)\partial_1^k u(x_1, .)\big)(x').$$

where $T(x_1)$ is a smooth family of pseudodifferential operator of order $l - k \in \mathbb{N}$ on Γ, $l \in \mathbb{N}$, $a, b \in \mathbb{R}$. Although $T(x_1)\partial_1^k$ is in general not a pseudodifferential operator, it has similar properties concerning its L^2-continuity.

Lemma 1.2.9. *$T(x_1)\partial_1^k$ extends to a bounded operator*

$$T(x_1)\partial_1^k : H^{l+s}_{\mathrm{comp}}((a,b) \times \Gamma, \pi^* E') \to H^s_{\mathrm{loc}}((a,b) \times \Gamma, \pi^* E')$$

for each $s \in \mathbb{R}$.

Proof. W.l.o.g. we can assume $E' = \mathbb{C} \times \Gamma$. Note that for $s \in \mathbb{N}$ we have (cf. [LM72])

$$H^s(\mathbb{R} \times \Gamma) = \bigcap_{j=0}^{s} H^j(\mathbb{R}, H^{s-j}(\Gamma)),$$

from which we deduce that

$$T(x_1)\partial_1 : H^{s+l}(\mathbb{R} \times \Gamma) \to H^s(\mathbb{R} \times \Gamma),$$

for $s \in \mathbb{N}$, $s \geq k$.

Furthermore, the L^2-adjoint, $\big(T(x_1)\partial_1^k\big)^* = (-\partial_1)^k T(x_1)^*$ is again a sum of operators of the form

$$T'(x_1)\partial_1^{k'},$$

where $k' \leq k$ and $T'(x_1)$ is of order $l - k$. By duality, we deduce that

$$T(x_1)\partial_1^k : H^{-s}(\mathbb{R} \times \Gamma, \pi^* E') \to H^{-s-l}(\mathbb{R} \times \Gamma, \pi^* E')$$

is continuous for $s \in \mathbb{N}$, $s \geq k$. The remaining cases now follow from interpolation theory, since $[H^s(\mathbb{R} \times \Gamma), H^{s'}(\mathbb{R} \times \Gamma)]_\vartheta = H^{s+\vartheta(s'-s)}(\mathbb{R} \times \Gamma)$, for $s, s' \in \mathbb{R}$, $s < s'$, $0 < \vartheta < 1$. $\square$

We have

$$\Phi^{-d}u,\ (\Phi^{-d}P\Phi^{d})\Phi^{-d}u \in L^2(\Omega, E), \quad \widetilde{\rho}^d\Phi^{-d}u = \widetilde{\Phi}^{-d}\widetilde{\rho}^d u \in H^{d/2}(\Omega, E'^d).$$

The operator $\Phi^{-d}P\Phi^{d}$ has the form

$$\sum_{j=0}^{j}\Phi^{-d}A_j(x_1)\Phi^d\partial_1^j = \ P \ + \sum_{j=0}^{j}\Phi^{-d}[A_j(x_1), \Phi^d]\partial_1^j = P + R,$$

where R is a sum of operators of the type discussed above with $l = d-1$. There is an analogue of the trace theorem (Theorem 1.1.4, in particular (ii) and (iii)) for the operator $P + R$ since, on the one hand, we have the following Green's formula

$$\langle \Phi^{-d}P\Phi^d u, v\rangle - \langle u, \Phi^d P^t \Phi^{-d}\rangle = \langle \widetilde{\Phi}^{-d}\widetilde{J}\widetilde{\Phi}^d\widetilde{\rho}^d u, \widetilde{\rho}^d v\rangle.$$

and, on the other, regularity for $P+R$ holds in the following sense: Let P be extended to $(-\varepsilon,\varepsilon)\times\Gamma$, then

$$u \in L^2_{\text{comp}}((-\varepsilon,\varepsilon)\times\Gamma, \pi^*E'), \quad (P+R)u \in H^s_{\text{loc}}((-\varepsilon,\varepsilon)\times\Gamma, \pi^*E') \tag{2.1}$$

implies

$$u \in H^{s+d}_{\text{comp}}((-\varepsilon,\varepsilon)\times\Gamma, \pi^*E').$$

This can be seen as follows: Since $Ru \in H^{-d+1}_{\text{loc}}$, we have $Pu \in H^t_{\text{loc}}$, where $t = \min(-d+1, s)$. If $t = s$, then the regularity of P gives $u \in H^{s+d}_{\text{comp}}$. If $s > t$ then we obtain $u \in H^1_{\text{comp}}$ and thus $Ru \in H^{-d+2}_{\text{loc}}$. Repeating this argument, we finally find $u \in H^{s+d}_{\text{comp}}$.

We conclude that $\Phi^{-d}u \in \mathscr{D}((\Phi^{-d}P\Phi^d)_{\max})$ and $\widetilde{\rho}^d u = 0$, and therefore, by the analogue of Theorem 1.1.4 (iii),

$$\Phi^{-d}u \in H^d([0,\varepsilon)\times\Gamma, \pi^*E').$$

(A2): Let $\Phi^{-d+k}u \in H^d(\Omega, E)$. Then

$$P\Phi\Phi^{-d+k}u = [P,\Phi]\Phi^{-d+k}u + \Phi[P,\Phi^{-d+k}]u + \Phi^{-d+k+1}Pu.$$

Here, the first and third term are sections in $L^2(\Omega, E)$. Let us compute the second term. Note that since $\Phi^{-d+k}u \in H^d(\Omega, E)$ we have $\Phi^{-d+k}u \in H^j((0,\varepsilon), H^{d-j}(\Gamma, E'))$ and therefore

$$D_1^j u \in L^2((0,\varepsilon), H^{-j+k}(\Gamma, E')), \quad j = 0, ..., k.$$

Hence,

$$\Phi[P,\Phi^{-d+k}]u = \sum_{j=0}^{d-1}\underbrace{\Phi[A_j(x_1), \Phi^{-d+k}]}_{\in C^\infty([0,\varepsilon),\Psi^{-j+k}(\Gamma,E'))} D_1^j u$$

is in $L^2(\Omega, E)$.

(A3): Applying the trace theorem to $\Phi^{-d+k}u$ we obtain

$$\widetilde{\rho}^d u = \widetilde{\Phi}^{d-k}\widetilde{\rho}^d\Phi^{-d+k}u \in H^{d/2-d+k}(\Gamma, E'^d).$$

Now, (A3) follows from

$$B\widetilde{\rho}^d\Phi^{-d+k+1}u = \underbrace{[B, \widetilde{\Phi}^{-d+k+1}]}_{\in\Psi^{-d+k}(\Gamma, E'^d)}\widetilde{\rho}^d u.$$

We are now ready to finish the proof of (iv). From (A1)-(A3) it follows that

$$\Phi^{-d+1}u \in H^{d-1}(\Omega, E), \quad \Phi^{-d+1}u \in \mathscr{D}(P_{\max}), \quad B\widetilde{\rho}^d\Phi^{-d+1}u \in H^{d/2}(\Omega, E'^d)$$

Therefore (iii) gives $\Phi^{-d+1}u \in H^d(\Omega, E)$. This together with (A2) and (A3) yields

$$\Phi^{-d+2}u \in H^{d-1}(\Omega, E), \quad \Phi^{-d+2}u \in \mathscr{D}(P_{\max}), \quad B\widetilde{\rho}^d\Phi^{-d+2}u \in H^{d/2}(\Omega, E'^d).$$

Again, (iii) gives $\Phi^{-d+2}u \in H^d(\Omega, E)$. Repeating this argument $d-2$ times we finally reach

$$u \in H^d(\Omega, E).$$

We conclude that B is regular. □

It follows that in order to check regularity it suffices to establish a Gårding type inequality. The next lemma states that such an estimate is in fact necessary.

Lemma 1.2.10. *Let B be regular. Then there exists $C > 0$ such that for all $u \in H^d(\Omega, E)$ we have*

$$\|u\|_{H^d(\Omega,E)} \leq C\big(\|u\|_{L^2(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)} + \|B\widetilde{\rho}^d u\|_{H^{d/2}(\Gamma,E'^d)}\big).$$

Proof of the lemma. First observe that, by the continuity of the trace $\widetilde{\rho}^d : H^d(\Omega, E) \to H^{d/2}(\Gamma, E'^d)$,

$$\big\{u \in H^d(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0\big\}$$

is a closed subspace of $H^d(\Omega, E)$. We claim that it is also complete w.r.t. the graph norm

$$u \mapsto \|u\|_{L^2(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)}. \tag{2.2}$$

Assume that (u_n) is a Cauchy-sequence w.r.t. (2.2). Then (u_n) converges to some section u in $L^2(\Omega, E)$ and (Pu_n) to some section $v \in L^2(\Omega, F)$. Since $P : \mathscr{D}'(\Omega^\circ, E) \to \mathscr{D}'(\Omega^\circ, F)$ is continuous we have $Pu = v$ and therefore $u \in \mathscr{D}(P_{\max})$. By Theorem 1.1.4, it follows that

$$B\widetilde{\rho}^d u = \lim_{n\to\infty} B\widetilde{\rho}^d u_n = 0.$$

Since B is regular, we find

$$u \in \big\{H^d(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0\big\},$$

which proves our claim.

Clearly, there exists $\widetilde{C} > 0$ such that

$$\|u\|_{L^2(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)} \leq \widetilde{C}\|u\|_{H^d(\Omega,E)}.$$

Since $\big\{u \in H^d(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0\big\}$ is complete w.r.t. to both norms we see that, by the open mapping theorem, there exists $C > 0$ s.t.

$$\|u\|_{H^d(\Omega,E)} \leq C\Big(\|u\|_{L^2(\Omega,E)} + \|Pu\|_{L^2(\Omega,F)}\Big)$$

for all $u \in H^d(\Omega, E)$ s.t. $B\widetilde{\rho}^d u = 0$.

For a more general $u \in H^d(\Omega, E)$ we can apply the above estimate to

$$v := u - \widetilde{\eta}^d B\widetilde{\rho}^d u$$

instead of u itself since v satisfies the boundary condition $B\widetilde{\rho}^d v = 0$. It follows that

$$\begin{aligned}\|u\|_{H^d} &\leq \|v\|_{H^d} + \|\widetilde{\eta}^d B\widetilde{\rho}^d u\|_{H^d} \\ &\leq \text{const} \cdot \big(\|v\|_{L^2} + \|Pv\|_{L^2}\big) + \|\widetilde{\eta}^d B\widetilde{\rho}^d u\|_{H^d} \\ &\leq \text{const} \cdot \big(\|u - \widetilde{\eta}^d B\widetilde{\rho}^d u\|_{L^2} + \|Pu - P\widetilde{\eta}^d B\widetilde{\rho}^d u\|_{L^2} + \|B\widetilde{\rho}^d u\|_{H^{d/2}}\big) \\ &\leq \text{const} \cdot \big(\|u\|_{L^2} + \|Pu\|_{L^2} + \|B\widetilde{\rho}^d u\|_{H^{d/2}}\big),\end{aligned}$$

which completes the proof of the lemma. □

Together with Lemma 1.2.8 we have thus finished the proof of Theorem 1.2.7. □

Remark 1.2.11. The proof of Lemma 1.2.8 would considerably simplify if we knew that for any boundary projection the continuous embedding

$$\big\{u \in H^d(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0\big\} \hookrightarrow \mathscr{D}(P_{\max,B})$$

has dense range. Namely, both spaces are complete w.r.t. to their norms. Hence, it would follow that these spaces coincide if and only if their norms are equivalent. It is therefore rather interesting to know if $\mathscr{D}(P_B)$ is dense in $\mathscr{D}(P_{B,\max})$ and, if this is the case, if there as an elegant proof for it.

1.3 Functional Analysis for Boundary Value Problems

Set $H_1 = L^2(\Omega, E)$, $H_2 = L^2(\Omega, F)$. We can regard any realisation P such that $P_{\min} \subset P \subset P_{\max}$ as an unbounded operator

$$P : H_1 \subset \mathscr{D}(P) \longrightarrow H_2.$$

Lemma 1.3.1. $\dim\ker P_{\min} < \infty$. $\operatorname{ran} P_{\max}$, $P_{\max}(H^d(\Omega, E))$ *as well as* $\operatorname{ran} P_{\min}$ *are closed in* H_2*. Moreover,* $\operatorname{ran} P_{\max} = P(H^d(\Omega, E))$ *and these subspaces have finite codimension in* H_2*.*

Proof. We show that

$$P_{\min} : \mathscr{D}(P_{\min}) \to H_2, \;\; P_S : H^d(\Omega, E) \to H_2, \;\; P_{\max} : \mathscr{D}(P_{\max}) \to H_2$$

are semi-Fredholm operators. To this end it suffices to construct a left or right inverses up to compact operators.

Let $Q^M \in \Psi^{-d}(M, F, E)$ be a parametrix for P^M hence

$$\begin{gathered} Q^M : H^s_{\text{comp}}(M, F) \to H^{s+d}_{\text{loc}}(M, E), \\ P^M \circ Q^M - \mathrm{Id} = C_1 \in \Psi^{-\infty}(M, F), \quad Q^M \circ P^M - \mathrm{Id} = C_2 \in \Psi^{-\infty}(M, E). \end{gathered}$$

Let $e_+ : L^2(\Omega, F) \to L^2_{\text{comp}}(M, E)$ denote extension by 0. Observe, that e_+ also maps $H^d_0(\Omega, E)$ into $H^d_{\text{comp}}(M, E)$. Consider the continuous map

$$\pi : H^d_{\text{loc}}(M, E) \to H^d_0(\Omega, E), \quad \pi(u) = u\big|_\Omega - \eta^d \rho^d u$$

which defines a projection onto $H^d_0(\Omega, E)$. Now, if $u \in \mathscr{D}(P_{\min})$ then

$$(\pi \circ Q^M \circ e_+ \circ P_{\min})u = (\pi \circ Q^M \circ P^M \circ e_+)u = u + (\pi \circ C_2 \circ e_+)(u).$$

Note that $\pi \circ C_2 \circ e_+$ is a compact operator in $H_0^d(\Omega, E)$. It follows that $P_{\min}$ is a left Fredholm operator. Hence, $\dim \ker P_{\min} < \infty$ and $\operatorname{ran} P_{\min}$ is closed.

Denote by r_+ the restriction map to Ω. Then,

$$P \circ r_+ \circ Q^M \circ e_+ = r_+ \circ P^M \circ Q^M \circ e_+ = \text{ Id } + r_+ \circ C_1 \circ e_+.$$

Since $r_+ \circ C_1 \circ e_+$ is a compact operator in $L^2(\Omega, F)$ we deduce that P_S as well as $P_{\max}$ are right-Fredholm operators. Hence, both have closed range of finite codimension.

We have seen (Theorem 1.1.4) that P_S is dense in $P_{\max}$. Hence, $\operatorname{ran} P_S$ is dense in $\operatorname{ran} P_{\max}$. Since both ranges are closed they coincide. $\square$

Using only that $P_{\min} : H_1 \supset \mathscr{D}(P_{\min}) \to H_2$, $P_{\max} : H_1 \supset \mathscr{D}(P_{\max}) \to H_2$ are densely defined closed operators with closed range (!) one can show a remarkable amount of properties of the abstract set of all extensions P of $P_{\min}$ such that $P \subset P_{\max}$. Note that in the abstract setting, we define $P^t_{\min}$, $P^t_{\max}$ by $(P_{\max})^*$, $(P_{\min})^*$, resp.

Recall that the space of boundary values of weak solutions to P can be identified with

$$W(P) = \mathscr{D}(P_{\max})/\mathscr{D}(P_{\min}).$$

There is a natural antidual pairing of $W(P)$ and $W(P^t)$,

$$\omega([u],[v]) = \langle P_{\max}u, v\rangle - \langle u, P^t_{\max}v\rangle.$$

$W(P)$ can be mapped into the orthogonal complement of $\mathscr{G}(P_{\min})$ in $H_1 \oplus H_2$, i.e.

$$W(P) \to H_1 \oplus H_2, \quad [u] \mapsto (\operatorname{Id} - \operatorname{pr}^{\text{ort}}_{\mathscr{G}(P_{\min})})(u, P_{\max}u).$$

Note that $\operatorname{pr}^{\text{ort}}_{\mathscr{G}(P_{\min})} : \mathscr{D}(P_{\max}) \to \mathscr{D}(P_{\max})$ since $\mathscr{D}(P_{\min})$ is a subspace of $\mathscr{D}(P_{\max})$.

Proposition 1.3.2. *$W(P)$ is isomorphic (as a Hilbert space) to* $\ker(P^t_{\max}P_{\max} + 1)$.

Proof. Let $u \in \mathscr{D}(P_{\max})$ such that $(u, P_{\max}u) \perp \mathscr{G}(P_{\min})$. Then

$$\langle u, v\rangle + \langle P_{\max}u, P_{\min}v\rangle = 0,$$

for all $v \in \mathscr{D}(P_{\min})$. Hence, $P_{\max}u \in \mathscr{D}(P^t_{\max})$ and $P^t_{\max}P_{\max}u = -u$. On the other hand, if $u \in \ker P^t_{\max}P_{\max} + 1$, then $u \in \mathscr{D}(P_{\max})$ and

$$\langle P_{\max}u, P_{\min}v\rangle = -\langle u, v\rangle,$$

for all $v \in \mathscr{D}(P_{\min})$. It follows that $\mathscr{G}(P_{\max}) \cap \mathscr{G}(P_{\min})^\perp \cong \ker P^t_{\max}P_{\max} + 1$. $\square$

It follows that $W(P^t)$ is naturally isomorphic to $\ker(P_{\max}P^t_{\max} + 1)$. If we view these kernels as subspaces of the graph of $P_{\max}$, $P^t_{\max}$, resp., then the dual pairing of $\ker(P^t_{\max}P_{\max} + 1)$ and $\ker(P_{\max}P^t_{\max} + 1)$ is given by

$$\langle J\, .\,, .\,\rangle_{H_2 \oplus H_1}, \quad J := \begin{pmatrix} 0 & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{pmatrix} : H_1 \oplus H_2 \to H_2 \oplus H_1.$$

Note that J maps $W(P) \cong \ker(P^t_{\max}P_{\max}+1)$ isomorphically onto $W(P^t) \cong \ker(P_{\max}P^t_{\max}+1)$. Define the abstract *Cauchy data spaces* by

$$\Lambda_0(P) := \big\{[u] \in W(P) \;\big|\; u \in \ker P_{\max}\big\},$$
$$\Lambda_0(P^t) := \big\{[v] \in W(P^t) \;\big|\; v \in \ker P^t_{\max}\big\}.$$

Proposition 1.3.3. *The annihilator of $\Lambda_0(P)$ w.r.t. ω equals $\Lambda_0(P^t)$, i.e.*

$$\Lambda_0(P)^\omega = \Lambda_0(P^t).$$

Proof. If $[u] \in \Lambda_0(P)$, $[v] \in \Lambda_0(P^t)$ then, w.l.o.g., $P_{\max} u = 0$, $P^t_{\max} v = 0$. Hence, $\omega([u],[v]) = 0$.

Assume $[v] \in \Lambda_0(P)^\omega$. Then for all $u \in \ker P_{\max}$ we have

$$0 = \omega([u],[v]) = -\langle u, P^t_{\max} v\rangle_{H_1}$$

In other words, $P^t_{\max} v$ is in the orthogonal complement of $\ker P_{\max} = (\operatorname{ran} P^t_{\min})^\perp$. Hence $P^t_{\max} v = P^t_{\min} v_0$ with $v_0 \in \mathscr{D}(P^t_{\min})$, since $P^t_{\min}$ has closed range by Lemma 1.3.1.

It follows that $P^t_{\max}(v - v_0) = 0$. Hence $[v] \in \Lambda_0(P^t)$. □

Set

$$n_P := \dim\ker P_{\min}, \quad d_P := n_{P^t} = \operatorname{codim}\operatorname{ran} P_{\max}.$$

There is a one-to-one correspondence between subspaces of $W(P)$ and extensions P such that $P_{\min} \subset P \subset P_{\max}$. Namely, to $\Lambda \subset W(P)$ we associate

$$P_\Lambda : \mathscr{D}(P_\Lambda) \to H_2, \quad \mathscr{D}(P_\Lambda) = \big\{u \in \mathscr{D}(P_{\max}) \;\big|\; [u] \in \Lambda\big\}.$$

On the other hand, to any extension P_1 of $P_{\min}$ such that $P_1 \subset P_{\max}$ we can associate

$$\Lambda_1 := \mathscr{D}(P_1)/\mathscr{D}(P_{\min}) \subset W(P).$$

Theorem 1.3.4. (i) *P_Λ is a closed unbounded operator if and only if Λ is closed.*

(ii) *P_Λ has closed range if and only if $\Lambda + \Lambda_0(P)$ is closed in $W(P)$.*

(iii) *P_Λ is Fredholm if and only if $(\Lambda, \Lambda_0(P))$ is a Fredholm pair.*

(iv) *P_Λ is semi Fredholm if and only if $(\Lambda, \Lambda_0(P))$ is a semi Fredholm pair.*

(v) *For semi Fredholm, Fredholm operators, resp. we have*

$$\begin{aligned} \dim\ker P_\Lambda &= \operatorname{nul}(\Lambda, \Lambda_0(P)) + n_P, \\ \operatorname{codim}\operatorname{ran} P_\Lambda &= \operatorname{def}(\Lambda, \Lambda_0(P)) + d_P, \\ \operatorname{ind} P_\Lambda &= \operatorname{ind}(\Lambda, \Lambda_0(P)) + n_P - d_P. \end{aligned}$$

For semi Fredholm operators these identities possibly read $\infty = \infty$.

Proof. We begin with a general elementary statement on closed subspaces of a Banach space: Let $A \subset V \subset X$ be subspace of a Banach space X and assume A is closed. Then V is closed if and only if V/A is closed in X/A.

Now, (i) follows since $\mathscr{G}(P_\Lambda)$ is closed if and only if

$$\mathscr{G}(P_\Lambda)/\mathscr{G}(P_{\min}) \cong \Lambda$$

is closed.

(ii) Observe that we have a well-defined map

$$\varphi : W(P)/\Lambda_0(P) \to \operatorname{ran} P_{\max}/\operatorname{ran} P_{\min},$$

given by $[u] + \Lambda_0(P) \mapsto Pu + \operatorname{ran} P_{\min}$. Namely, $\Lambda_0(P)$ is exactly the kernel of the continuous surjective map

$$[P_{\max}] : W(P) \to \operatorname{ran} P_{\max} / \operatorname{ran} P_{\min}, \quad [u] \mapsto [Pu].$$

Since $\operatorname{ran} P_{\max}/\operatorname{ran} P_{\min}$ is closed (Lemma 1.3.1), it follows that φ is an isomorphism.

$(\Lambda_0(P) + \Lambda)/\Lambda_0(P)$ is mapped under φ onto $\operatorname{ran} P_\Lambda / \operatorname{ran} P_{\min}$. On the one hand $\Lambda_0(P) + \Lambda$ is closed in $W(P)$ if and only if $(\Lambda_0(P) + \Lambda)/\Lambda_0(P)$ is closed in the above quotient. On the other, $\operatorname{ran} P_\Lambda / \operatorname{ran} P_{\min}$ is closed if and only if $\operatorname{ran} P_\Lambda$ is closed. This proves (ii).

For extensions with closed range the other statements now follow easily from dimension counting. More precisely, we deduce from the above identifications that

$$\begin{aligned}\operatorname{codim} \operatorname{ran} P_\Lambda &= \operatorname{codim}(\operatorname{ran} P_\Lambda / \operatorname{ran} P_{\min}) + \operatorname{codim} \operatorname{ran} P_{\max} \\ &= \dim W(P)/(\Lambda_0(P) + \Lambda) + d_P,\end{aligned}$$

where $\operatorname{ran} P_\Lambda / \operatorname{ran} P_{\min}$ is viewed as a subspace of $\operatorname{ran} P_{\max}/\operatorname{ran} P_{\min}$. Clearly,

$$\begin{aligned}\Lambda_0(P) \cap \Lambda = \big\{[x] \in \Lambda_0 \;\big|\; [x] \in \Lambda\big\} &\cong \big\{x \in \ker P_{\max} \;\big|\; x \in \mathscr{D}(P_\Lambda)\big\}/\ker P_{\min} \\ &= \ker P_\Lambda / \ker P_{\min}.\end{aligned}$$

Hence, $\dim \ker P_\Lambda = \dim(\Lambda \cap \Lambda_0(P)) + \dim \ker P_{\min} = \dim(\Lambda \cap \Lambda_0(P)) + n_p$. □

Moreover, we have an abstract version of Proposition 1.2.6.

Proposition 1.3.5. $(P_\Lambda)^* = P_{\Lambda^\omega}$.

Proof. Let $u \in \mathscr{D}((P_\Lambda)^*)$, i.e. $u \in \mathscr{D}(P^t_{\max})$ and for all $v \in \mathscr{D}(P_{\max})$ such that $[v] \in \Lambda$ we have

$$\langle Pv, u \rangle \le \text{const} \cdot \|v\|.$$

By $\langle Pv, u\rangle = \langle v, P^t u\rangle + \omega([v],[u])$, it follows that

$$\omega([v],[u]) \le \text{const} \cdot \|v\|,$$

for all $[v] \in \Lambda$. Since $\mathscr{D}(P_{\min}) \subset H_1$ is dense, it follows that for all $v \in \mathscr{D}(P_{\max})$, there exists a series $v_n \in \mathscr{D}(P_{\min})$ such that $\lim\limits_{n\to\infty} v - v_n = 0$ in H_1. It follows that

$$\omega([v],[u]) = \omega([v - v_n],[u]) \longrightarrow 0$$

as $n \to \infty$. Hence, $[u] \in \Lambda^\omega$.

On the other hand, if $u \in \mathscr{D}(P_{\max})$ and $[u] \in \Lambda^\omega$, then

$$\langle Pv, u\rangle = \langle v, P^t u\rangle,$$

for all $v \in \mathscr{D}(P_\Lambda)$. Hence $u \in \mathscr{D}((P_\Lambda)^*)$. □

Chapter 2

Regularity and Well-posedness

This chapter is devoted to the study of elliptic regularity for boundary value problems. We study regularity and well-posedness and express these properties in terms of the so-called "Calderón projection". Here, the notion of a Fredholm pair of projections naturally comes into play. For instance, well-posedness is expressed as the Fredholm property of a pair formed by the boundary condition and the Calderón projection.

If B is pseudodifferential, then well-posedness is translated into a condition on the principal symbol of B. We will see that this is exactly Seeley's well-posedness condition (see [See69]) which itself is a slight generalisation of the Shapiro-Lopatinskii condition (sometimes called "covering" condition, cf. [LM72, Def. II.1.5]). The latter was first introduced for differential operators B and later generalized for pseudodifferential (cf. [Hör85, Sec. 20.1]).

The Calderón projection was introduced by A.P.Calderón (cf. [Cal63]) and then used by Palais et al., see [Pal65]. Seeley showed in [See66] that the Calderón projection of a general elliptic operator is in fact pseudodifferential.

If P satisfies the so-called "transmission property" then, as worked out in [Gru96, Section 1.3.5] and [Gru99] , this follows from the general calculus of "Poisson", pseudodifferential and so-called "trace" operators. More precisely, the Calderón projection can be written as a composition of operators of the three above-mentioned types and this is shown by Grubb to yield a pseudodifferential operator on the boundary.

Since many of our arguments rest on properties of the Calderón projection we will give a direct approach to it, here. However, we prefer to postpone its construction as a pseudodifferential operator until Section 2.3 and draw some immediate conclusions first.

In the second section, we show how to pass from regularity to estimates of higher Sobolev norms and to corresponding higher regularity theorems. In particular, it follows for a regular boundary condition that every solution u to $Pu = v$ (satisfying the boundary condition) is smooth if v is.

In the third section we construct the Calderón projection step by step. We show that it is a pseudodifferential operator and discuss its behaviour under variations of P.

Many authors construct the Calderón projection using the so-called "invertible double", i.e. one has to choose M compact and find extensions of E, F and P such that $P^M : C^\infty(M, E) \to C^\infty(M, F)$ is an invertible el-

liptic operator on a compact manifold. However, even if P is a Dirac type operator the existence of such an extension is non-trivial (cf. [BBW93, Sec. I.9]). For a general P one can find at least an elliptic extension of P that is pseudodifferential outside Ω (cf. Appendix of [See69]). We suggest here an alternative approach to the Calderón projection: Similar to [Hör85, Sec. 20.1] we take a parametrix for P^M, say Q^M, on the open manifold M. The difference here is the special choice for Q^M which seems more suitable for our purposes since we obtain directly a projection, not a "projection up to smoothing operators" as in loc. cit. If P satisfies the unique continuation property on M, i.e.

$$\ker P^M_{\max} \cap H^d_0(M, E) = \{0\},$$

then some of the (rather technical) arguments are obsolete and the proof can be considerably simplified.

We begin Section 2.4 by briefly reviewing Bojarski's Theorem formulated for arbitrary elliptic operators on manifolds with a smooth closed hypersurface which separates M into two pieces. Bojarski already showed in [Boj60] how the computation of the index of a generalized Riemann-Hilbert problem localises to the hypersurface. It was conjectured in [Boj79] and finally proved in [BW86] that the index of the global operator can be expressed as a pair index of Cauchy data spaces, at least when the operator is of Dirac type. Mitrea proved a generalisation of Bojarski's Theorem in the context of L^p-spaces for first order operators and non-smooth hypersurfaces, cf. [Mit99].

Related to this is the Agranovič-Dynin formula which makes a precise if abstract statement about how the index depends on the choice of a boundary condition if one of them is a compact perturbation of the Calderón projection. Agranovič and Dynin established a version of this theorem for integrodifferential boundary conditions, cf. [AD62].

All statements about indices in Section 2.4 require the unique continuation principle (UCP) which is well-known for many elliptic operators such as Dirac- and Laplace-operators (and hence for their powers), scalar differential operators, operators with real analytic coefficients, see [BBW93, Chap. 8] and the references there. For Laplace operators UCP can be traced back to the early papers [Cor56] and [Aro57].

We specialise the observations made so far to the case where P is formally self-adjoint. It will be shown that given a formally self-adjoint boundary condition B the realisation P_B is self-adjoint if and only if B is regular for P. This is a generalisation of [BL01, Theorem 1.3] where the proof for Dirac operators is given.

For a realisation corresponding to a non-formally self-adjoint boundary condition we then show, assuming UCP, that its index only depends on the boundary condition and the shape of P near the boundary. This can be used when computing the index of a boundary value problem for a Laplacian on a surface: Since the boundary is a disjoint union of circles we may restrict attention to a disk, cf. Section 4.3.2.

Moreover, when P is formally self-adjoint $W(P)$ becomes a (strongly) symplectic Hilbert space and the abstract Cauchy data space $\Lambda_0 = \Lambda_0(P)$ is Lagrangian. Hence, the so-called "Fredholm-Lagrange Grassmannian" associated to Λ_0 parametrises the space of all self-adjoint Fredholm realisations.

For instance, any projection with range Λ_0 whose kernel is Lagrangian is an example of a boundary condition that gives rise to a self-adjoint Fredholm realisation. We conclude that there are always regular self-adjoint realisations defined by pseudodifferential projections. This has far-reaching consequences such as the cobordism invariance of the index of Dirac operators (Theorem 3.4.5) which will be proved in next chapter.

It should be emphasized that there are essentially self-adjoint realisations with domain in $H^d(\Omega, E)$ that are not given by a regular boundary conditions. The example in Remark 2.5.9 is due to Brüning and Lesch [BL01, Proposition 4.18].

In the following section we consider continuous families of elliptic operators P_s and boundary conditions B_s where s is a real parameter. Stability results for regularity and well-posedness are established for small perturbations of the coefficients of P and the projections B. The difficulty in proving such results stems from the fact that the domains of the realisations may vary.

Finally, when $E = F$ and P_{s,B_s} is a curve of self-adjoint realisations we give sufficient conditions for P_{s,B_s} to form a continuous curve w.r.t. the gap metric. This generalises the results summarized in [BBLP02]. A more detailed discussion of the gap topology can be found in [BBLP01] where the authors also give a rigourous definition of the spectral flow for paths of Fredholm operators that are continuous in this topology. The gap metric itself (giving rise to the gap topology) dates back to Cordes and Labrousse (see [CL63] ,[Lab66] and [Kat76]).

2.1 The Regularity Condition

Recall the definition of the space of boundary values of strong solutions as the set of boundary values of sections in $H^d(\Omega, E)$, i.e.

$$S(P) \cong H^{d/2}(\Gamma, E'^d) \cong H^d(\Omega, E)/\mathscr{D}(P_{\min}),$$

where we have replaced the usual trace map ρ^d by

$$\widetilde{\rho}^d = \Phi_d \rho^d.$$

Since we are interested only in extensions P such that all solutions u of $Pu = v$ are *regular* we now discuss extensions P_Λ such that

$$P_{\min} \subset P_\Lambda \subset P_{\max}, \quad \Lambda = \mathscr{D}(P_\Lambda)/\mathscr{D}(P_{\min}) \subset S(P),$$

which means $\mathscr{D}(P) \subset H^d(\Omega, E)$ (by Theorem 1.1.4 (iii)). Suppose we are given an abstract boundary condition

$$\Lambda_{\max,B} = \big\{[x] \in W(P) \;\big|\; B[x] = 0\big\},$$

where B is some boundary condition subject to Assumption 1.2.3. Then, in Section 1.2, we defined the operators $P_B, P_{\max,B}$, resp. by

$$\mathscr{D}(P_B) = \big\{x \in H^d(\Omega, E) \;\big|\; [x] \in \Lambda_{\max,B}\big\},$$
$$\mathscr{D}(P_{\max,B}) = \big\{x \in \mathscr{D}(P_{\max}) \;\big|\; [x] \in \Lambda_{\max,B}\big\}.$$

The Cauchy data space was defined by

$$\Lambda_0(P) = \{\widetilde{\varrho}^d u \mid u \in \mathscr{D}(P_{\max}),\ Pu = 0\}.$$

One can consider the intersection of $\Lambda_0(P)$ with any $H^s(\Omega, E)$ yielding closed subspaces of $H^{s-d/2}(\Omega, E)$ (cf. [BBW93, Sec. 13]). Here, we need only the special case

$$\Lambda_0^S(P) = \{\widetilde{\varrho}^d u \mid u \in H^d(\Omega, E),\ Pu = 0\} = \Lambda_0(P) \cap S(P).$$

$\Lambda_0^S(P)$ is closed in $S(P)$ since $S(P)$ is continuously embedded into $W(P)$. Let us denote the restriction of $\omega : W(P) \times W(P^t) \to \mathbb{C}$ to $S(P) \times S(P^t)$ by ω_S, i.e. via the identification of $S(P)$, $S(P^t)$ with $H^{d/2}(\Gamma, E'^d)$, $H^{d/2}(\Gamma, F'^d)$, we have

$$\omega_S(g, h) = \langle \widetilde{J}g, h\rangle_{L^2(\Gamma, F'^d)}.$$

Since the pairing on the right hand side is not the natural scalar product in $H^{d/2}(\Gamma, E'^d)$, the pairing ω_S has an unpleasant property: it is only weakly non-degenerate. However, we have the following analogue of Proposition 1.3.3.

Proposition 2.1.1. *The annihilator of the strong Cauchy data space $\Lambda_0^S(P)$ w.r.t. ω_S is the strong Cauchy data space of the adjoint, i.e.*

$$\left(\Lambda_0^S(P)\right)^{\omega_S} = \Lambda_0^S(P^t).$$

Proof. Since ω_S vanishes on $\Lambda_0^S(P) \times \Lambda_0^S(P^t)$ it is clear that

$$\left(\Lambda_0^S(P)\right)^{\omega_S} \supset \Lambda_0^S(P^t).$$

Assume $h \in \left(\Lambda_0^S(P)\right)^{\omega_S}$. That means $h = \widetilde{\varrho}^d v$ for some $v \in H^d(\Omega, F)$ and for all $u \in H^d(\Omega, E)$ such that $Pu = 0$ we have

$$0 = \omega_S(u, v) = -\langle u, P^t v\rangle.$$

Hence, $P^t v \perp \ker(P : H^d(\Omega, E) \to L^2(\Omega, F))$. We have seen in the proof of Theorem 1.1.4 (i) that the adjoint of

$$P : H^d(\Omega, E) \to L^2(\Omega, F)$$

is $P^t_{\min}$, which by Proposition 1.3.1 has closed range. We conclude

$$P^t v \in \overline{\operatorname{ran} P^t_{\min}} = \operatorname{ran} P^t_{\min}.$$

Hence, there exists $\widetilde{v} \in H_0^d(\Omega, F)$ such that

$$P^t(v - \widetilde{v}) = 0.$$

Since $\widetilde{\varrho}^d(v - \widetilde{v}) = \widetilde{\varrho}^d v = h$, we find $h \in \Lambda_0^S(P^t)$. □

Proposition 2.1.2. *Let B be a projection that satisfies Assumption 1.2.3. The following conditions are equivalent*

(i) *B is a well-posed (regular) boundary condition.*

(ii) *The realisation $P_B : \mathscr{D}(P_B) \to L^2(\Omega, F)$ is (left) Fredholm.*

(iii) *The pair*

$$\begin{pmatrix} P \\ B\widetilde{\rho}^d \end{pmatrix} : H^d(\Omega, E) \to L^2(\Omega, F) \oplus \operatorname{ran}(B : H^{d/2}(\Omega, E'^d) \to H^{d/2}(\Omega, E'^d))$$

is (left) Fredholm.

(iv) *The operator*

$$B : \Lambda_0^S(P) \to \operatorname{ran}(B : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d))$$

is (left) Fredholm for some $s \in \mathbb{R}$.

(v) $\Big(\Lambda_0^S(P)\ , \ker(B : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)\Big)$ *is a (left) Fredholm pair in* $S(P)$.

Proof. (i) ⇔ (ii): Provide $X = \mathscr{D}(P_B)$ with the Sobolev-norm $\|.\|_{H^d(\Omega,E)}$ and set $Y = L^2(\Omega, F)$, $Z = L^2(\Omega, E)$. Then, by Lemma 1.2.8 and Lemma 1.2.10, B is regular if and only if a Gårding type inequality

$$\|x\|_X \leq C\big(\|x\|_Z + \|Px\|_Y\big)$$

holds for all $x \in X$. By the Rellich theorem, the embedding $X \hookrightarrow Z$ is compact. By Proposition A.1.2 and A.1.4 such an estimate is equivalent to P_B being left Fredholm.

It follows that P_B is left Fredholm if and only if B is a regular boundary condition. Moreover, P_B is Fredholm if and only if $\operatorname{ran} P_B$ has finite codimension, i.e. if B is well-posed.

(ii) ⇒ (iii): For a moment let us denote the $H^{d/2}$-range of B simply by $\operatorname{ran} B$. Let Q be a left parametrix of P_B. Then

$$R : L^2(\Omega, F) \oplus \operatorname{ran} B \to H^d(\Omega, E), \quad R(v, g) := Q(v - P\widetilde{\eta}^d g) + \widetilde{\eta}^d g$$

defines a left parametrix for $(P, B\widetilde{\rho}^d)$. Namely,

$$\begin{aligned} R\begin{pmatrix} Pu \\ B\widetilde{\rho}^d u \end{pmatrix} &= Q(P(\underbrace{u - \widetilde{\eta}^d B\widetilde{\rho}^d u}_{\in \mathscr{D}(P_B)})) + \widetilde{\eta}^d B\widetilde{\rho}^d u \\ &= (\operatorname{Id} + C_1)(u - \widetilde{\eta}^d B\widetilde{\rho}^d u) + \widetilde{\eta}^d B\widetilde{\rho}^d u \\ &= u + C_1(u - \widetilde{\eta}^d B\widetilde{\rho}^d u) \end{aligned}$$

for $u \in H^d(\Omega, E)$, where $C_1 : H^d(\Omega, E) \to H^d(\Omega, E)$ is a compact operator.

If Q is a right parametrix, then

$$\begin{aligned} \begin{pmatrix} P \\ B\widetilde{\rho}^d \end{pmatrix} R(v, g) &= \begin{pmatrix} (\operatorname{Id} + C_2)(v - P\widetilde{\eta}^d g) + P\widetilde{\eta}^d g \\ \underbrace{B\widetilde{\rho}^d Q}_{=0}(v - P\widetilde{\eta}^d g) + B\widetilde{\rho}^d\widetilde{\eta}^d g \end{pmatrix} \\ &= \begin{pmatrix} v + C_2(v - P\widetilde{\eta}^d g) \\ g \end{pmatrix}, \end{aligned}$$

for $v \in L^2(\Omega, E)$, $g \in H^{d/2}(\Gamma, E'^d) \cap \operatorname{ran} B$, where $C_2 : L^2(\Omega, F) \to L^2(\Omega, F)$ is compact. Thus R is a right parametrix of $(P, B\widetilde{\eta}^d)$.

(iii) ⇒ (ii): If R is a left parametrix of the pair above, then we may define a left parametrix Q for P_B by

$$Q := (\operatorname{Id} - \widetilde{\eta}^d B\widetilde{\rho}^d) \circ R \circ \begin{pmatrix} \operatorname{Id} \\ 0 \end{pmatrix},$$

since

$$
\begin{aligned}
(\mathrm{Id} - \widetilde{\eta}^d B \widetilde{\rho}^d) R \begin{pmatrix} Pu \\ 0 \end{pmatrix} &= (\mathrm{Id} - \widetilde{\eta}^d B \widetilde{\rho}^d) R \begin{pmatrix} Pu \\ B\widetilde{\rho}^d u \end{pmatrix} \\
&= (\mathrm{Id} - \widetilde{\eta}^d B \widetilde{\rho}^d)(\mathrm{Id} + C_3) u = u + (\mathrm{Id} - \widetilde{\eta}^d B \widetilde{\rho}^d) C_3 u,
\end{aligned}
$$

for $u \in \mathscr{D}(P_B)$, where $C_3 : H^d(\Omega, E) \to H^d(\Omega, E)$ is a compact operator.

Similarly, if R is a right parametrix, then

$$
P(\mathrm{Id} - \widetilde{\eta}^d B \widetilde{\rho}^d) R \begin{pmatrix} v \\ 0 \end{pmatrix} = v + C_4 v - P \widetilde{\eta}^d B C_5 v,
$$

for $v \in L^2(\Omega, F)$, where $C_4 : L^2(\Omega, F) \to L^2(\Omega, F)$ and $C_5 : L^2(\Omega, F) \to \operatorname{ran} B \subset H^{d/2}(\Gamma, E'^d)$ are compact operators.

(iii) $\Leftrightarrow$ (iv) Consider the closed subspaces of $H^d(\Omega, E)$ defined by

$$
\begin{aligned}
V_1 &:= \{ u \in H^d(\Omega, E) \mid \widetilde{\rho}^d u \perp_{H^{d/2}} \Lambda_0^S(P) \}, \\
V_2 &:= \{ u \in H^d(\Omega, E) \mid Pu = 0,\ u \perp_{L^2} \ker P_{\min} \}.
\end{aligned}
$$

Clearly,

$$
V_1 \cap V_2 = \{ u \in H^d(\Omega, E) \mid \widetilde{\rho}^d u \perp_{H^{d/2}} \Lambda_0^S(P),\ u \perp_{L^2} \ker P_{\min},\ Pu = 0 \} = \{0\},
$$

and if $u \in H^d(\Omega, E)$ is given we can take the $H^{d/2}$-orthogonal decomposition $\widetilde{\rho}^d u = g_1 + g_2 \in \Lambda_0^S(P)^{\perp_{H^{d/2}}} \oplus \Lambda_0^S(P)$. Then, we find $u_2 \in H^d(\Omega, E)$ such that

$$
g_2 = \widetilde{\rho}^d u_2,\ Pu_2 = 0,\ u_2 \perp_{L^2} \ker P_{\min},
$$

and $g_1 = \widetilde{\rho}^d u_1 \perp_{H^{d/2}} \Lambda_0^S(P)$ holds with $u_1 = u - u_2 \in V_1$. Hence,

$$
H^d(\Omega, E) = V_1 \oplus V_2.
$$

Observe that $\widetilde{\rho}^d : V_2 \to \Lambda_0^S(P)$ is a continuous bijection and $P : V_1 \to L^2(\Omega, F)$ is a bounded operator with kernel $\ker P_{\min}$ and closed range of finite codimension, since

$$
P(V_1) = P(H^d(\Omega, E))
$$

is of finite codimension by Lemma 1.3.1. Hence,

$$
\widetilde{\rho}^d : V_2 \to \Lambda_0^S(P), \quad P : V_1 \to L^2(\Omega, F)
$$

are Fredholm operators. Note that w.r.t. to this decomposition $(P, B\widetilde{\rho}^d)$ has the form

$$
\begin{pmatrix} P|_{V_1} & 0 \\ * & B\widetilde{\rho}^d \end{pmatrix} : V_1 \oplus V_2 \to L^2(\Omega, F) \oplus \operatorname{ran} B.
$$

It follows that $(P, B\widetilde{\rho}^d)$ is (left) Fredholm if and only if

$$
B\widetilde{\rho}^d : \operatorname{ran} \Lambda_0^S(P) \to \operatorname{ran} B
$$

is (left) Fredholm. This shows (iii) $\Leftrightarrow$ (v).

That (iv) and (v) are equivalent is a direct application of Proposition A.1.12 (ii). $\square$

Proposition 2.1.3. *Let $B : C^\infty(\Gamma, E'^d) \to H'_\infty$ be a an operator of order μ as in Assumption 1.2.1. Then $B\widetilde{\rho}^d u = 0$ is a regular boundary condition for P if and only if*

$$B : \Lambda_0^S(P) \to \operatorname{ran}\big(B : H'_{d/2} \to H'_{d/2-\mu}\big)$$

is left Fredholm.

Proof. We have seen in Section 1.2 that B can be replaced by an equivalent condition

$$B'\widetilde{\rho}^d u = 0$$

where B' satisfies Assumption 1.2.3, i. e. B' is an orthogonal projection. Equivalent means that

$$\ker(B' : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)) = \ker(B : H^s(\Gamma, E'^d) \to H'_{s-\mu}).$$

By Proposition 2.1.2 (v) $B\widetilde{\rho}^d u = 0$ is a regular boundary condition if and only if

$$\big(\ker(B' : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)), \Lambda_0^S(P)\big)$$

is a left Fredholm pair. Since

$$\ker(B' : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)) = \ker(B : H^{d/2}(\Gamma, E'^d) \to H'_{d/2-\mu}).$$

this is the case, by Lemma A.1.11, if and only if

$$B|_{\Lambda_0^S(P)} : \Lambda_0^S(P) \to H'_{d/2-\mu}$$

is left Fredholm. □

We will show in Theorem 2.3.5 that $\Lambda_0(P)$ is the range of a pseudodifferential projection

$$C_+(P) : H^{-d/2}(\Gamma, E'^d) \to H^{-d/2}(\Gamma, E'^d),$$

the so-called *Calderón projection*. In particular $C_+(P)$ is itself an operator satisfying Assumption 1.2.3. Let us now draw some consequences of the above regularity theorem using the fact that $C_+(P)$ is pseudodifferential. First of all it follows that

$$\begin{aligned}\Lambda_0^S(P) &= H^{d/2}(\Gamma, E'^d) \cap \operatorname{ran}\big(C_+(P) : H^{-d/2}(\Gamma, E'^d) \to H^{-d/2}(\Gamma, E'^d)\big)\\ &= \operatorname{ran}\big(C_+(P) : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)\big).\end{aligned}$$

Theorem 2.1.4. (i) *Let B be an operator satisfying Assumption 1.2.1. Then $B\widetilde{\rho}^d u = 0$ is a regular boundary condition if and only if for some (and hence for all) $s \in \mathbb{R}$*

$$B : \big(\operatorname{ran} C_+(P) : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)\big) \longrightarrow H'_{s-\mu} \tag{1.1}$$

is left Fredholm. In this case $B\widetilde{\rho}^d u = 0$ is a well-posed boundary condition if and only if

$$B : \big(\operatorname{ran} C_+(P) : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)\big) \longrightarrow \operatorname{ran}(B : H'_s \to H'_{s-\mu})$$

has finite dimensional cokernel for some (and hence for all) $s \in \mathbb{R}$.

(ii) *If* $B : C^\infty(\Gamma, E'^d) \to C^\infty(\Gamma, G)$ *is also pseudodifferential, then regularity holds if and only if for all* $q \in \Gamma$, $\xi \in T_q^*\Gamma$

$$\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to G|_p$$

is injective. Here, $\widehat{C}_+(\xi)$ *denotes the principal symbol of* $C_+ = C_+(P)$*. Moreover, well-posedness holds if and only if for all* $q \in \Gamma$, $\xi \in T_q^*\Gamma$

$$\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to \operatorname{ran} \widehat{B}(\xi)$$

is invertible.

Proof. (i) Using Proposition 2.1.3 it suffices to show that the above regularity criterion does not depend on $s \in \mathbb{R}$. Observe first that the condition is fulfilled if and only if

$$\begin{pmatrix} BC_+(P) \\ \widetilde{\Phi}^\mu(\operatorname{Id} - C_+(P)) \end{pmatrix} : H^s(\Gamma, E'^d) \to H'_{s-\mu} \oplus H^{s-\mu}(\Gamma, E'^d) \tag{1.2}$$

is left Fredholm. Now, applying Proposition A.2.2, we see that this condition is independent of $s \in \mathbb{R}$.

Let us fix again $s = d/2$. If regularity holds then well-posedness is equivalent to

$$(\ker B, \operatorname{ran} C_+(P))$$

being also right Fredholm, which means that

$$\ker B \; + \; \operatorname{ran} C_+(P) \subset H^{d/2}(\Gamma, E'^d)$$

has finite codimension. Let

$$H^{d/2}(\Gamma, E'^d) = \big(\ker B \; + \; \operatorname{ran} C_+(P)\big) \oplus V,$$

and thus

$$B(H^{d/2}(\Gamma, E'^d)) = B\big(\ker B \; + \; \operatorname{ran} C_+(P)\big) \oplus B(V).$$

It follows that

$$B(\operatorname{ran} C_+(P) : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)) = B(\ker B \; + \; \operatorname{ran} C_+(P))$$

has finite codimension in $\operatorname{ran} B$ if and only if (1.1) is also right Fredholm for $s = d/2$. Now, in order to establish (i) for general $s \in \mathbb{R}$ we can proceed as follows. As explained in Section 1.2 there exists an orthogonal projection $Q : H'_s \to H'_s$ subject to Assumption 1.2.3 (adapted to the scale (H'_s)) such that

$$\ker(Q : H'_s \to H'_s) = \ker(B^* : H'_s \to H^{s-\mu}(\Gamma, E'^d)),$$

where B^* denotes the H'_0-dual of B. Taking the annihilators w.r.t. to the pairings

$$H'_s \times H'_{-s} \to \mathbb{C},$$

we find

$$\operatorname{ran}\big(Q : H_{-s} \to H_{-s}\big) = \operatorname{ran}\big(B : H^{-s+\mu}(\Gamma, E'^d) \to H'_{-s}\big).$$

Now,

$$BC_+ : H^s(\Gamma, E'^d) \to \operatorname{ran}(B : H^s(\Gamma, E'^d) \to H'_{s-\mu})$$

is right Fredholm if and only if

$$\begin{pmatrix} BC_+ & (\operatorname{Id}-Q)|\Psi|^\mu \end{pmatrix} : H^s(\Gamma, E'^d) \oplus H'_s \to H'_{s-\mu} \tag{1.3}$$

is right Fredholm. That this condition does not depend on the choice of s follows from Proposition A.2.2 again.

(ii) When B is pseudodifferential, then (1.2) is pseudodifferential and $H'_{s-\mu} = H^{s-\mu}(\Gamma, G)$. Since Γ is compact, by [Hör85, Thm. 19.5.1], any pseudodifferential operator $T \in \Psi^\mu(\Gamma, G_1, G_2)$ is injectively elliptic if and only if

$$T : H^t(\Gamma, G_1) \to H^{t-\mu}(\Gamma, G_2)$$

is left Fredholm for some $t \in \mathbb{R}$. But

$$\begin{pmatrix} BC_+(P) \\ \widetilde{\Phi}^\mu(\operatorname{Id}-C_+(P)) \end{pmatrix}$$

is injectively elliptic if and only if

$$\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to G|_p$$

is injective for all $\xi \in T^*_p\Gamma$, $q \in \Gamma$.

Note that if B is pseudodifferential then the constructions in Section 1.2 show that the operator Q can be chosen pseudodifferential, too. Analogously, (1.3) is a right Fredholm operator for some s (and hence for all s) if and only if

$$\begin{pmatrix} \widehat{B}(\xi)\widehat{C}_+(\xi) & (\operatorname{Id}-\widehat{Q}(\xi))\|\xi\|^\mu \end{pmatrix}$$

is surjective for all $\xi \in T^*\Gamma$, cf. [Hör85, Thm. 19.5.2]. Since

$$\operatorname{ran} \widehat{B}(\xi) \perp \operatorname{ran}(\operatorname{Id}-\widehat{Q}(\xi)),$$

we see that well-posedness holds if and only if

$$\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to \operatorname{ran} \widehat{B}(\xi).$$

is invertible for all $\xi \in T^*\Gamma$. □

Corollary 2.1.5. *Let B satisfy Assumption 1.2.3. If P_B is a well-posed realisation of P, then $P^t_{B^{\mathrm{ad}}}$ is a well-posed realisation of P^t.*

Proof. W.l.o.g. we may assume that B is an orthogonal projection. By Proposition 2.1.1 we have $\Lambda_0^S(P^t) = (\Lambda_0^S(P))^{\omega_S}$. $\Lambda_0^S(P)$ is the kernel of

$$\operatorname{Id}-C_+^{\mathrm{ort}}(P) : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)$$

where C_+^{ort} is the orthogonal projection onto $\operatorname{ran} C_+(P)$. Note that, by Remark 2.3.10, $C_+^{\mathrm{ort}}(P)$ is still pseudodifferential. The annihilator of the strong Cauchy data space w.r.t. ω_s equals

$$\begin{aligned} &\ker\left((\widetilde{J}^*)^{-1} C_+^{\mathrm{ort}}(P)\widetilde{J}^* : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)\right) \\ &\qquad = \operatorname{ran}\left((\widetilde{J}^*)^{-1}(\operatorname{Id}-C_+^{\mathrm{ort}}(P))\widetilde{J}^* : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)\right). \end{aligned}$$

By Proposition 1.2.6, B^{ad} is given by

$$(\widetilde{J}^*)^{-1}(\mathrm{Id} - B)\widetilde{J}^*.$$

By Theorem 2.1.4, B^{ad} is well-posed for P^t if and only if

$$(\widetilde{J}^*)^{-1}(\mathrm{Id} - B)\widetilde{J}^* : \operatorname{ran}(\widetilde{J}^*)^{-1}(\mathrm{Id} - C_+^{\mathrm{ort}}(P))\widetilde{J}^* \to \operatorname{ran}(\widetilde{J}^*)^{-1}(\mathrm{Id} - B)\widetilde{J}^*$$

is Fredholm. This is equivalent to

$$(\mathrm{Id} - B) : \operatorname{ran}(\mathrm{Id} - C_+^{\mathrm{ort}}(P)) \to \operatorname{ran} \mathrm{Id} - B,$$

and by Proposition A.1.12 (i) to

$$\pm \mathrm{Id} - B + C_+^{\mathrm{ort}} : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d),$$

being Fredholm. □

Remark 2.1.6. The well-posedness condition given in Theorem 2.1.4 (ii) may be viewed as a generalisation of the Shapiro-Lopatinskii condition, cf. [See69, Chap. VI].

Suppose $\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to G|_q$ is bijective for all $\xi \in T_q\Gamma$, $q \in \Gamma$. Then the system (P, B) is called *elliptic*, cf. loc. cit. If merely the map

$$\widehat{B}(\xi) : \operatorname{ran} \widehat{C}_+(\xi) \to \operatorname{ran} \widehat{B}(\xi).$$

is bijective, then Seeley calls B well-posed for P. Hence the terminology chosen here is consistent with Seeley's, provided B is pseudodifferential.

Recall that the ellipticity condition for a constant coefficient homogenous operator $P(D)$ in euclidean space states that $\widehat{P}(\xi)u(\xi) = v(\xi)$ is uniquely solvable for all $\xi \neq 0$. Similarly the Shapiro-Lopatinskii condition makes a statement about the model equation near each point $q \in \Gamma$, which then is the partial differential equation obtained upon freezing coefficients in a coordinate chart. However, instead of replacing $D = (-i\partial_1, ..., -i\partial_n)$ by $\xi = (\xi_1, ..., \xi_n)$ one takes the Fourier transform in the tangential direction only, i.e. one substitutes D for (D_1, ξ'). Thus one arrives at an ordinary differential equation

$$\widehat{P}((D_1, \xi'))u(x_1, \xi') = v(x_1, \xi'), \tag{1.4}$$

which is required to possess a unique exponentially decreasing solution for each

$$(v(0, \xi'), \partial_1 v(0, \xi'), ..., \partial_1^{d-1} v(0, \xi'))$$

such that

$$\widehat{B}(\xi')(v(0, \xi'), \partial_1 v(0, \xi'), ..., \partial_1^{d-1} v(0, \xi')) = 0,$$

whenever $\xi' \neq 0$. This point of view is widely explained by M. E. Taylor (cf. [Tay96, Sec. V.11]) who, in contrast to us, uses the term "regular boundary condition" for what is called a well-posed boundary condition here. The principal symbol of the Calderón projection, $\widehat{C}_+(\xi')$, is a projection onto the space of Cauchy data of solutions to (1.4) with $v = 0$, as shown in [Hör85, Thm. 20.1.3].

2.2 Higher Regularity

Let B be a regular boundary condition for P. In this section we seek for higher regularity theorems of the following form: If $u \in L^2(\Omega, E)$, $Pu \in H^s(\Omega, F)$ and $B\widetilde{\rho}^d u = 0$ then $u \in H^{s+d}(\Omega, E)$.

So far we have only dealt with the case $s = 0$. This, by definition, means that B is regular for P. We will show that the statement for general $s \in \mathbb{Z}_+$ follows from the regularity of B.

Theorem 2.2.1. *Let $s \in \mathbb{Z}_+$ and suppose $B\widetilde{\rho}^d u = 0$ is a regular boundary condition for P where B is a projection satisfying Assumption 1.2.3. Assume that $u \in L^2(\Omega, E)$ satisfies*

$$Pu \in H^s(\Omega, F), \quad B\widetilde{\rho}^d u \in H^{d/2+s}(\Gamma, E'^d).$$

Then, $u \in H^{s+d}(\Omega, E)$.

Moreover, there are corresponding regularity estimates for s, i.e. there exists $C > 0$ such that

$$\|u\|_{H^{s+d}(\Omega,E)} \leq C \cdot \Big(\|u\|_{L^2(\Omega,E)} + \|Pu\|_{H^s(\Omega,F)} + \|B\widetilde{\rho}^d u\|_{H^{d/2+s}(\Gamma,E'^d)} \Big).$$

Proof. We proceed by induction on s. The regularity statement for the case $s = 0$ is nearly trivial, since if $B\widetilde{\rho}^d u \in H^{d/2}(\Gamma, E'^d)$, then we may consider

$$\widetilde{u} = u - \widetilde{\eta}^d B\widetilde{\rho}^d u$$

instead of u. The estimate follows immediately from Lemma 1.2.10.

Assume now $s \geq 1$ and that the theorem has been proved for $s - 1$. Take a solution $u \in L^2(\Omega, E)$ to

$$Pu \in H^s(\Omega, F), \quad B\widetilde{\rho}^d u \in H^{d/2+s}(\Gamma, E'^d).$$

By interior regularity, $u \in H^{s+d}_{\text{loc}}(\Omega^\circ, E)$. Hence, as in the proof of Lemma 1.2.8, we only have to deal with a collar neighbourhood Ω_ε. More precisely, we will have to show that

$$u|_{\Omega_\varepsilon} \in H^{s+d}(\Omega_\varepsilon, E).$$

Using a cut-off function with support near Γ, we may assume w.l.o.g. that $\operatorname{supp} u \subset \Omega_\varepsilon$. Identify $E|_{\Omega_\varepsilon}$ with $\pi^* E$, where $\pi : \Omega_\varepsilon \cong [0, \varepsilon) \times \Gamma \to \Gamma$ denotes the natural projection. Over the collar, E and F are isomorphic via $\widehat{P}(dx_1)$. Hence, we may assume for simplicity that $E = F$. As in the proof of Lemma 1.2.8 consider Φ as an operator acting on sections over Ω_ε.

Recall that, near the boundary, P takes the form

$$P = \sum_{j=0}^{d} A_j(x_1) D_1^j.$$

and hence, as in Lemma 1.2.8, it follows that

$$[P, \Phi] : H^{d+s-1}(\Omega_\varepsilon, E) \to H^{s-1}(\Omega_\varepsilon, E)$$

is bounded.

The case $s - 1$ gives $u \in H^{d+s-1}(\Omega, E)$. Moreover, since u satisfies

$$P\Phi u = \Phi P u + [P, \Phi] u \in H^{s-1}(\Omega_\varepsilon, F)$$

as well as

$$B\widetilde{\rho}^d\Phi u = B\widetilde{\Phi}\widetilde{\rho}^d u = \big(\widetilde{\Phi}B + [B,\widetilde{\Phi}]\big)\widetilde{\rho}^d u \in H^{d/2+s-1}(\Gamma, E'^d).$$

we infer again from the case $s-1$ that Φu is in $H^{d+s-1}(\Omega_\varepsilon, E)$. Hence,

$$u \in \bigcap_{j=0}^{d+s-1} H^j((0,\varepsilon), H^{d+s-j}(\Gamma, E')).$$

Moreover,

$$D_1^{d+s}u = D_1^s\Big(A_d(x_1)^{-1}\big(Pu - \sum_{j=0}^{d-1} A_j(x_1)D_1^j u\big)\Big)$$

and we have just proved that the right hand side is in

$$L^2((0,\varepsilon), L^2(\Gamma, E')).$$

We conclude that

$$u \in \bigcap_{j=0}^{d+s} H^j((0,\varepsilon), H^{d+s}(\Gamma, E')) = H^{d+s}(\Omega_\varepsilon, E)$$

and thus $u \in H^{d+s}(\Omega, E)$.

Applying the estimate for $s-1$ to Φu we obtain from the above

$$\begin{aligned}
\|u\|_{H^{d+s}(\Omega_\varepsilon,E)} &\le \text{const}\cdot\big(\|\Phi u\|_{H^{d+s-1}(\Omega_\varepsilon,E)} + \|D_1^{d+s}u\|_{L^2(\Omega_\varepsilon,E)}\big)\\
&\le \text{const}\cdot\big(\|\Phi u\|_{H^{d+s-2}(\Omega_\varepsilon,E)} + \|P\Phi u\|_{H^{s-1}(\Omega_\varepsilon,E)}\\
&\qquad + \|B\widetilde{\Phi}\widetilde{\rho}^d u\|_{H^{d/2+s-1}(\Gamma,E'^d)} + \|Pu\|_{H^s(\Omega_\varepsilon,E)}\big)\\
&\le \text{const}\cdot\big(\|\Phi u\|_{H^{d+s-2}(\Omega_\varepsilon,E)} + \|\Phi Pu\|_{H^{s-1}(\Omega_\varepsilon,E)}\\
&\qquad + \|[P,\Phi]u\|_{H^{s-1}(\Omega_\varepsilon,E)} + \|B\widetilde{\Phi}\widetilde{\rho}^d u\|_{H^{d/2+s-1}(\Gamma,E'^d)} + \|Pu\|_{H^s(\Omega_\varepsilon,E)}\big)\\
&\le \text{const}\cdot\big(\|u\|_{H^{s+d-1}(\Omega_\varepsilon,E)} + \|Pu\|_{H^s(\Omega_\varepsilon,F)} + \|B\widetilde{\rho}^d u\|_{H^{d/2+s}(\Gamma,E'^d)}\big),
\end{aligned}$$

where we have used once more that

$$[P,\Phi] : H^{d+s-1}(\Omega_\varepsilon, E) \to H^{s-1}(\Omega_\varepsilon, E)$$

is bounded. Together with an interior elliptic estimate, i.e. a corresponding inequality for a compact subset of Ω°, this gives a corresponding inequality with Ω_ε replaced by Ω. Finally, by the Peter-Paul inequality,

$$\|u\|_{H^{s+d-1}(\Omega,E)} \le \delta\|u\|_{H^{s+d}(\Omega,E)} + C_\delta\|u\|_{L^2(\Omega,E)},$$

we obtain the desired estimate if δ is chosen sufficiently small. □

Recall that $C^\infty(\Omega, E)$ consists of all restrictions of smooth sections of E over M. By the Sobolev embedding theorem (cf. [LM72, Sec. I.9]), we have

$$C^\infty(\Omega, E) = \bigcap_{k\ge 0} H^k(\Omega, E).$$

Corollary 2.2.2. *Let B be a regular boundary condition for P and assume $u \in L^2(\Omega, E)$ is a solution to*

$$Pu \;=\; v, \quad B\widetilde{\rho}^d u \;=\; g,$$

where $v \in C^\infty(\Omega, F)$ and $g \in C^\infty(\Gamma, E'^d)$. Then $u \in C^\infty(\Omega, E)$.

Proof. We can apply Theorem 2.2.1 for all $s \in \mathbb{N}$. □

Together with Corollary 2.1.5 this shows that all sections in

$$\big(\operatorname{ran} P_B\big)^{\perp} = \ker(P_B)^* = \ker P^t_{B^t}$$

are smooth when B is a well-posed boundary condition.

We now want to apply interpolation theory. Recall that the Sobolev scale for manifolds with smooth boundary,

$$\big(H^s(\Omega, E)\big)_{s\in\mathbb{R}}$$

has the interpolation property for $s \geq 0$, i.e.

$$[H^s(\Omega, E), H^{s+t}(\Omega, E)]_\vartheta = H^{s+\vartheta t}(\Omega, E),$$

where $s, t \geq 0$, $\vartheta \in (0, 1)$. A proof of this fact can be found in [LM72, Chap. I].

Theorem 2.2.3. *When B is also a well-posed boundary condition, then Theorem 2.2.1 holds for all $s \in \mathbb{R}_+$.*

Proof. Let P_B be a well-posed realisation and consider the operators

$$P_B^{(s)} : \big\{u \in H^{d+s}(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0\big\} \to H^s(\Omega, F), \quad u \mapsto P_B u.$$

and

$$\begin{aligned} P_{B,0}^{(s)} : &\big\{u \in H^{d+s}(\Omega, E) \;\big|\; B\widetilde{\rho}^d u = 0, u \perp \ker P_B\big\} \\ &\to \big\{u \in H^s(\Omega, F) \;\big|\; u \in \operatorname{ran} P_B\big\}, \quad u \mapsto P_B u. \end{aligned}$$

The latter is a continuous bijection for each $s \in \mathbb{Z}_+$, by Theorem 2.2.1. Now,

$$\big\{u \in H^s(\Omega, F) \;\big|\; u \in \operatorname{ran} P_B\big\} = \operatorname{ran} P_B \cap H^s(\Omega, F)$$

is closed in $H^s(\Omega, F)$ since $\operatorname{ran} P_B$ is closed in $L^2(\Omega, F)$. Hence $P_{B,0}^{(s)}$ has a bounded inverse for $s \in \mathbb{Z}_+$. Denote by pr_2 the L^2-orthogonal projection onto $(\operatorname{ran} P_B)^{\perp}$. Since $(\operatorname{ran} P_B)^{\perp}$ is finite dimensional and consists of smooth sections only, this projection defines bounded operators

$$\mathrm{pr}_2^{(s)} : H^s(\Omega, E) \to H^s(\Omega, E) \cap (\operatorname{ran} P_B)^{\perp}$$

for each $s \in \mathbb{R}_+$. It follows that

$$Q^{(s)} := (P_{B,0}^{(s)})^{-1} \circ (\mathrm{Id} - \mathrm{pr}_2^{(s)}) : H^s(\Omega, F) \to H^{d+s}(\Omega, E),$$

is continuous for each $s \in \mathbb{Z}_+$. By interpolation it is continuous for all $s \in \mathbb{R}_+$. Moreover,

$$P_B^{(s)} \circ Q^{(s)} = \mathrm{Id} - \mathrm{pr}_2^{(s)},$$

and

$$Q^{(s)} \circ P_B^{(s)} = \mathrm{Id} - \mathrm{pr}_1^{(s)},$$

where $\mathrm{pr}_1^{(s)} : H^{d+s}(\Omega, E) \to H^{d+s}(\Omega, E)$ denotes the L^2-orthogonal projection onto $\ker P_B$. This shows that all $P_B^{(s)}$ are Fredholm. Using Proposition A.1.2 we obtain the regularity estimates. □

2.3 The Calderón Projection

Let us assume that M itself is a compact manifold with smooth boundary ∂M. E.g. we can replace any open manifold $M \supset \Omega$ by $\{p \in M \mid p \in \Omega \text{ or } d(p,\Gamma) \leq \varepsilon\}$ which has smooth boundary provided ε is chosen sufficiently small. To start with let us list the ingredients of the construction:

(i) Consider the (modified) trace map

$$\widetilde{\rho}^d_M : H^d_{\mathrm{loc}}(M^\circ, E) \to H^{d/2}(\Gamma, E'^d).$$

whose dual,

$$(\widetilde{\rho}^d_M)^* : H^{-d/2}(\Gamma, E'^d) \to H^{-d}_{\mathrm{comp}}(M^\circ, E),$$

is given by

$$\langle u, (\widetilde{\rho}^d_M)^* g\rangle = \langle \widetilde{\rho}^d_M u, g\rangle, \quad u \in H^d_{\mathrm{loc}}(M^\circ, E),\ g \in H^{-d/2}(\Gamma, E'^d).$$

(ii) Consider the Cauchy data space of $P^M : C^\infty(M,E) \to C^\infty(M,F)$,

$$\Lambda_0(P^M) := \{g \in W(P^M) \mid \exists u \in \ker P^M_{\max} :\ \widetilde{\rho}^d u = g\}.$$

In the following let us denote by $\widetilde{\rho}^d_{\partial M}$ the trace map from $\mathscr{D}(P^M_{\max})$ to $H^{-d/2}(\partial M, E^d|_{\partial M})$. Let $\Lambda_0(P^M)^\perp$ denote the orthogonal complement of the Cauchy data space in $W(P^M) \subset L^2(M,E) \oplus L^2(M,F)$, (cf. Section 1.3). Since

$$(\Lambda_0(P^M), \Lambda_0(P^M)^\perp)$$

is a Fredholm pair in $W(P)$ it follows that

$$P^M_0 : \{u \in \mathscr{D}(P^M_{\max}) \mid \widetilde{\rho}^d_{\partial M} u \perp \Lambda_0(P^M)\} \to L^2(M,F)$$

is a Fredholm realisation of P^M. The operator

$$\widetilde{P}^M_0 : \mathscr{D}(P^M_0) \cap (\ker P^M_0)^\perp \to \operatorname{ran} P^M_0$$

has a continuous inverse which, composed with the orthogonal projection onto $\operatorname{ran} P^M_0$, yields a bounded operator

$$Q^M : L^2(M,F) \to H^d(M,E).$$

Denote by pr_1, pr_2 the orthogonal projections onto the finite-dimensional spaces

$$\ker P^M_0, \quad (\operatorname{ran} P^M_0)^\perp,$$

resp. Note that the adjoint boundary condition of $\Lambda_0(P^M)^\perp$ is given by

$$\big(\Lambda_0(P^M)^\perp\big)^\omega = (J\Lambda_0(P^M)^\perp)^\perp = \big(\Lambda_0(P^M)^\omega\big)^\perp = \Lambda_0(P^{M,t})^\perp.$$

Hence, $(\operatorname{ran} P^M_0)^\perp = \ker(P^M_0)^* = \ker P^{M,t}_0$ where we use analogous notations for $P^{M,t}$. Since all sections in $\ker P^M_0$ and $P^{M,t}_0$ are *inner* solutions it follows that

$$\mathrm{pr}_1 = \sum_{j=1}^{n_M} e_j \otimes e_j^*, \quad \mathrm{pr}_2 = \sum_{j=1}^{d_M} f_j \otimes f_j^*$$

where e_j, f_j are suitable smooth sections of E, F, resp. Moreover, we have $Q^M P_0^M u = (\mathrm{Id} - \mathrm{pr}_1)u$ and $P^M Q^M v = (\mathrm{Id} - \mathrm{pr}_2)v$ for $u \in \mathscr{D}(P_0^M)$, $v \in L^2(M, F)$. Observe that these lines are obsolete if P and P^t satisfy the (weak) unique continuation principle (UCP) in which case $\mathrm{pr}_1 = 0$, $\mathrm{pr}_2 = 0$. If UCP does not hold in Ω then we may still assume that all sections u in $\ker P_{\min}^M$ have support in Ω, i.e., by slight abuse of notation, $\ker P_{\min}^M = \ker P_{\min}$. More precisely, if ε is chosen sufficiently small, then we have

$$u \in \ker P^M,\ d(\operatorname{supp} u, \Omega) \le \varepsilon \ \Rightarrow\ \operatorname{supp} u \subset \Omega.$$

To see this assume the contrary. Then we would find a series $\varepsilon_n \searrow 0$, and $u_n \in \ker P_{\min}^M$ such that

$$d(\Omega \cup \operatorname{supp} u_n, \Omega) < \varepsilon_n,\ u_n \perp \ker P_{\min},\ d(\Omega \cup \operatorname{supp} u_n, \Omega) > \varepsilon_{n+1}.$$

This way we would obtain a series of linearly independent inner solutions to $P_{\min}^M u = 0$ in contradiction to $\dim \ker P_{\min}^M < \infty$. Hence, if ∂M is sufficiently "close" to Γ we have $\ker P_{\min}^M = \ker P_{\min}$. Analogously, we may assume $\ker P_{\min}^{M,t} = \ker P_{\min}^t$. It follows that

$$\widetilde{\rho}^d \circ \mathrm{pr}_1 = 0, \quad \widetilde{\rho}^d \circ \mathrm{pr}_2 = 0.$$

From interior regularity it follows that if $v \in L^2(M, F)$ is smooth on M° then so is $Q^M v$.

Proposition 2.3.1. *$Q^M : C_c^\infty(M^\circ, F) \to C^\infty(M^\circ, E)$ is a pseudodifferential parametrix for P^M.*

Proof. First of all, we see that the adjoint, $Q^{M,t}$, is given by

$$Q^{M,t} = (\widetilde{P}_0^{M,t})^{-1} \circ (\mathrm{Id} - \mathrm{pr}_1)$$

such that $\langle Q^{M,t}u, v\rangle = \langle u, Q^M v\rangle$ for all $u \in C_c^\infty(M, E)$, $v \in C_c^\infty(M, F)$. By interior regularity, $Q^{M,t} : C_c^\infty(M, E) \to C^\infty(M, F)$ is bounded and hence Q^M can be extended to distributions $v \in \mathscr{E}'(M, F)$ by

$$\langle Q^M v, u\rangle = \langle v, Q^{M,t} u\rangle, \quad u \in C_c^\infty(M^\circ, E),$$

yielding a bounded operator

$$Q^M : \mathscr{E}'(M^\circ, F) \to \mathscr{D}'(M^\circ, E).$$

Let Q' be a properly supported pseudodifferential parametrix of P^M, i.e.

$$Q' : C_c^\infty(M^\circ, F) \to C_c^\infty(M^\circ, E)$$

is bounded and $Q'P^M - \mathrm{Id}$ as well as $P^M Q' - \mathrm{Id}$ are given by integral operators with kernels[1]

$$R_1 \in C^\infty(M^\circ \times M^\circ, E \boxtimes E^*), \quad R_2 \in C^\infty(M^\circ \times M^\circ, F \boxtimes F^*).$$

[1]Here, $G_1 \boxtimes G_2$ denotes the tensor product of the lifted bundles $\pi_1^* G_1 \to M^\circ \times M^\circ$ and $\pi_2^* G_2 \to M^\circ \times M^\circ$ where π_i denotes the natural projection $\pi_i^* : M^\circ \times M^\circ \to M^\circ$, $(p_1, p_2) \mapsto p_i$.

It follows that

$$P^M(Q' - Q^M)v = (R_2 + \mathrm{pr}_2)v \in C^\infty(M^\circ, F)$$

for any $v \in \mathscr{E}'(M^\circ, E)$. Hence

$$Q'P^M(Q' - Q^M) = Q' - Q^M + R_1 \circ (Q' - Q^M) : \mathscr{E}'(M^\circ, F) \to C^\infty(M^\circ, E),$$

is continuous. Hence, Q^M differs from Q' merely by a smoothing operator. □

Note that

$$Q^M : H^{-d}_{\mathrm{comp}}(M^\circ, F) \to L^2_{\mathrm{loc}}(M, E).$$

(iii) Denote by $r_+ : L^2_{\mathrm{loc}}(M, E) \to L^2(\Omega, E)$ the restriction operator to Ω. Consider the continuous map

$$r_+Q^M(\widetilde{\rho}^d)^* : H^{-d/2}(\Gamma, F'^d) \to L^2(\Omega, E).$$

We have

$$P\, r_+Q^M(\widetilde{\rho}^d)^* = r_+P^MQ^M(\widetilde{\rho}^d)^* = r_+(\mathrm{Id} - \mathrm{pr}_2)(\widetilde{\rho}^d)^* = r_+\mathrm{pr}_2(\widetilde{\rho}^d)^*,$$

since $r_+(\widetilde{\rho}^d)^* = 0$. (The second r_+ denotes the corresponding restriction $L^2(M, F) \to L^2(\Omega, F)$, of course.) Now, $r_+\mathrm{pr}_2(\widetilde{\rho}^d)^* : H^{-d/2}(\Gamma, E') \to C^\infty(\Omega, E)$ equals 0 for $\widetilde{\rho}^d\mathrm{pr}_2 = 0$ and $(\widetilde{\rho}^d\mathrm{pr}_2)^* = \mathrm{pr}_2(\widetilde{\rho}^d)^*$.

We thus obtain a continuous operator

$$r_+Q^M(\widetilde{\rho}^d)^* : H^{-d/2}(\Gamma, F'^d) \to \ker P_{\max} \subset \mathscr{D}(P_{\max}).$$

Now, using the weak trace $\widetilde{\rho}^d : \mathscr{D}(P_{\max}) \to H^{-d/2}(\Gamma, E'^d)$ defined in Theorem 1.1.4 and (1.8), we can write down the Calderón projection.

Definition 2.3.2. The operator

$$C_+ : H^{-d/2}(\Gamma, E'^d) \to H^{-d/2}(\Gamma, E'^d), \quad C_+g := -\widetilde{\rho}^d r_+Q^M(\widetilde{\rho}^d_M)^*\widetilde{J}g$$

is called the *Calderón projection.*

Remark 2.3.3. Since C_+ depends on the choice of P^M it is, strictly speaking, not correct to call C_+ *the* Calderón projection. However, since we think of the enveloping manifold M and the extension P^M as being fixed, we refrain from adding "with respect to M and P^M" every time we refer to C_+.

Remark 2.3.4. Assume $d = 1$ and P is of order 1, i.e.

$$P = c(\partial_1)\left(\frac{\partial}{\partial x_1} + A(x_1)\right)$$

on a collar neighbourhood of Γ. Then the above formula for the Calderón projection yields

$$C_+ = \gamma^0 r_+Q^M(\gamma^0_M)^*c(0).$$

for $J = -c(\partial_1)$.

Theorem 2.3.5. (i) *C_+ is a projection onto the Cauchy data space*

$$\Lambda_0(P) \subset H^{-d/2}(\Gamma, E'^d).$$

(ii) $C_+ \in \Psi^0_{\mathrm{cl}}(\Gamma, E'^d)$.

Proof. (i) Let $h = C_+ g = -\widetilde{\rho}^d r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g$ and set $u := -r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g$. Then $u \in L^2(\Omega, E)$ and

$$Pu = -r_+ P^M Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g = -r_+ (\mathrm{Id} - \mathrm{pr}_2)(\widetilde{\rho}^d_M)^* \widetilde{J} g = 0,$$

since $\mathrm{pr}_2 (\widetilde{\rho}^d_M)^* = (\widetilde{\rho}^d_M \mathrm{pr}_2)^* = 0$ and $r_+(\widetilde{\rho}^d_M)^* \widetilde{J} g = 0$. Thus, $u \in \ker P_{\max}$ and $h = \widetilde{\rho}^d u \in \Lambda_0(P)$. We conclude that $\operatorname{ran} C_+ \subset \Lambda_0(P)$.

Now, let $g \in \Lambda_0(P)$. Then, there exists $u \in \ker P_{\max}$, such that $\widetilde{\rho}^d u = g$. We want to compute $(\widetilde{\rho}^d_M)^* \widetilde{J} g$. For any test function $v \in H^d_{\mathrm{comp}}(M, F)$ we have

$$\begin{aligned}\langle (\widetilde{\rho}^d_M)^* \widetilde{J} g, v\rangle &= \langle \widetilde{J} g, \widetilde{\rho}^d_M v\rangle \\ &= \langle Pu, r_+ v\rangle - \langle u, P^t r_+ v\rangle = -\langle u, P^t r_+ v\rangle.\end{aligned}$$

Hence, for any $v \in C^\infty_c(M^\circ, E)$,

$$\begin{aligned}\langle Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, v\rangle &= \langle (\widetilde{\rho}^d_M)^* \widetilde{J} g, Q^{M,t} v\rangle \\ &= -\langle u, P^t r_+ Q^{M,t} v\rangle = -\langle u, r_+ P^{M,t} Q^{M,t} v\rangle \\ &= -\langle u, r_+ (Q^M P^M_0)^t v\rangle = -\langle (\mathrm{Id} - \mathrm{pr}_1) e_+ u, v\rangle.\end{aligned}$$

Therefore,

$$-\widetilde{\rho}^d r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g = \widetilde{\rho}^d r_+ (\mathrm{Id} - \mathrm{pr}_1) e_+ u = \widetilde{\rho}^d u = g,$$

for $\widetilde{\rho}^d r_+ \mathrm{pr}_1 = 0$. We have thus proved

$$C_+^2 = C_+, \quad \operatorname{ran} C_+ = \Lambda_0(P)$$

(ii) Since $\widetilde{J}$ is a 0-th order elliptic pseudodifferential operator it suffices to show that

$$\widetilde{\rho}^d r_+ Q^M (\widetilde{\rho}^d_M)^*$$

is a pseudodifferential operator of order 0. Recall that

$$\widetilde{\rho}^d = \left(\Phi^{\frac{d-1-2i}{2}} \gamma^i\right)_{i=0}^{d-1} : \mathscr{D}(P_{\max}) \to H^{-d/2}(\Gamma, E'^d)$$

and similarly

$$(\widetilde{\rho}^d_M)^* = \sum_{j=0}^{d-1} (\gamma^j_M)^* \Phi^{\frac{d-1-2j}{2}} : H^{-d/2}(\Gamma, F'^d) \to H^{-d}_{\mathrm{comp}}(M, F).$$

Hence, we have to prove that

$$\gamma^i r_+ Q^M (\gamma^j_M)^* : H^{-d+j+1/2}(\Gamma, F') \to H^{-i-1/2}(\Gamma, E') \tag{3.1}$$

is a ΨDO of order $-d+1+j+i$.[2] In order to localise the situation we cover M with charts and choose a subordinate partition of unity, say (h_α). Restriction to Γ gives a partition of unity on Γ. If $\operatorname{supp} h_\alpha \cap \operatorname{supp} h_\beta = \emptyset$ it is easy to see that

$$h_\alpha \gamma^i r_+ Q^M (\gamma^j_M)^* h_\beta$$

[2] One might guess that an analogous statement holds for all ψdos Q^M. However, when Q does not have the so-called "transmission property" (3.1) might not be pseudodifferential (cf. [Hör85, Sec. 18.2]). In our case Q inherits the transmission property from P and the analysis of loc. cit. applies to our case. For the sake of brevity we avoid a lengthy discussion of the transmission property here and give direct arguments following the ideas presented in [BBW93].

is a smoothing operator. Namely, for all $g \in \mathscr{D}'(\Gamma, F')$

$$(\gamma_M^j)^* h_\beta g \in \mathscr{E}'(M, F), \quad \operatorname{supp}(\gamma_M^j)^* h_\beta g \subset (\operatorname{supp} h_\beta \cap \Gamma),$$

hence

$$\big(Q(\gamma_M^j)^* h_\beta g\big)\big|_{\operatorname{supp} h_\alpha}$$

is smooth. Therefore we only have to show that the restriction of $\gamma^i r_+ Q(\gamma_M^j)^*$ to one of our coordinate systems is a ψdo on an open set $V \subset \mathbb{C}^n$. We may assume that V is relatively compact and that the image of Γ is $V \cap \{x_1 = 0\}$. Furthermore, for sufficiently small charts the bundles $E|_V$ and $F|_V$ can be identified with $\mathbb{R}^N \times V$ in such a way that γ_M^i corresponds to $\gamma_M^0(\partial_1)^i$. Hence, we have to study

$$(-1)^j \gamma^0 r_+ \,\partial_1^i Q \partial_1^j\, (\gamma_V^0)^*,$$

where, for simplicity, we write Q for Q^M. Let us further assume that P has been extended to an elliptic operator on $\mathbb{R}^n$ such that all coefficients of P are constant outside a compact subset. We may assume that Q is a parametrix for this P (see Lemma 2.3.6 below).

Let $\chi \in C_c^\infty(\mathbb{R})$ be a function such that

$$\operatorname{supp} \chi \subset (-1, 0), \qquad \int_{-\infty}^{\infty} \chi(x_1) dx_1 = 0,$$

and set $\chi_p(x_1) = p \cdot \chi(p \cdot x_1)$, $p \in \mathbb{N}$. Then

$$\langle (\gamma_V^0)^* g, u \rangle = \langle g, \gamma_V^0 u \rangle = \lim_{p \to \infty} \langle \chi_p \otimes g, u \rangle,$$

for all $g \in \mathscr{D}'(\mathbb{R}^{n-1})$, $u \in C^\infty(\mathbb{R}^n)$. In other words $(\gamma_V^0)^* g$ is the weak limit of $\chi_p \otimes g$. When $g \in \mathscr{S}(\mathbb{R}^{n-1})$ the Fourier transform of $\chi_p \otimes g$ equals

$$\widehat{\chi_p \otimes g} = \widehat{\chi_p} \otimes \widehat{g} = \widehat{\chi}\big(\tfrac{\cdot}{p}\big) \otimes \widehat{g}.$$

In order to distinguish the index i from the square root of -1 we denote the latter by $\mathbf{i}$. If $q(x, \xi)$ denotes the symbol of Q, i.e.

$$Qu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathbf{i}\langle x, \xi \rangle} q(x, \xi) \widehat{u}(\xi) d\xi,$$

then we may write

$$\Big(\partial_1^i Q \partial_1^j (\gamma_V^0)^* g\Big)(x) = \lim_{p \to \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathbf{i}\langle x, \xi \rangle} \widetilde{q}(x, \xi)\ \widehat{\chi}\big(\tfrac{\xi_1}{p}\big)\ \widehat{g}(\xi')\ d\xi$$

where the limit $p \to \infty$ is taken in the weak sense and $\widetilde{q}$ denotes the symbol of $\widetilde{Q} := \partial_1^i Q \partial_1^j$.

Lemma 2.3.6. *Let $R \in \Psi^m(\mathbb{R}^n)$, $m \le -2$. Then*

$$\gamma^0 R (\gamma^0)^* : C_c(\mathbb{R}^{n-1}) \to C_c^\infty(\mathbb{R}^{n-1}).$$

is a pseudodifferential operator of order $m - 1$ in $\mathbb{R}^{n-1}$.

Proof. Denoting the symbol of R by $r(x,\xi)$ we have

$$\gamma^0 R(\gamma^0)^* g(0,x') = \lim_{p\to\infty}(2\pi)^{-n}\int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\langle x',\xi'\rangle}\int_{\mathbb{R}} r(0,x',\xi_1,\xi')\widehat{\chi}(\tfrac{\xi_1}{p})d\xi_1\,\widehat{g}(\xi')d\xi'.$$

We have

$$(\widehat{\chi}(\tfrac{\xi_1}{p})-1)(1+\xi_1^2)^{-1/2}\underset{p\to\infty}{\longrightarrow}0$$

in $L^2(\mathbb{R})$ since $\widehat{\chi}$ is a bounded smooth function, and

$$\xi_1\mapsto r(0,x',\xi_1,\xi')(1+\xi_1^2)^{1/2}$$

is bounded in $L^2(\mathbb{R})$ since R is of order at most -2. It follows that

$$\gamma^0 R(\gamma^0)^* g(0) = (2\pi)^{-n}\int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\langle x',\xi'\rangle}\int_{\mathbb{R}} r(0,x',\xi_1,\xi')d\xi_1\,\widehat{g}(\xi')d\xi'.$$

From the estimates

$$\begin{aligned}\left|D^\alpha_{x'}D^\beta_{\xi'}\int_{\mathbb{R}} r(0,x',\xi_1,\xi')d\xi_1\right| &\le \text{const}\cdot\int_{\mathbb{R}}(1+|\xi_1|^2+|\xi'|^2)^{\frac{m-|\beta|}{2}}d\xi_1\\ &\le \text{const}\cdot(1+|\xi'|^2)^{\frac{m-|\beta|+1}{2}},\end{aligned}$$

for $\alpha,\beta\in\mathbb{Z}_+^{n-1}$, we obtain that $\gamma^0R(\gamma^0)^*$ is a pseudodifferential operator of order $m+1$. $\square$

Now, Q is a parametrix for the elliptic differential operator P. Hence, the symbol $q(x,\xi)$ has an expansion

$$q(x,\xi)\sim\sum_{k\le -d} q_k(x,\xi),$$

where $q_k(x,\xi)\in S^k(\mathbb{R}^n)$ is homogeneous of degree k in ξ, for $\|\xi\|>1$ i.e.

$$q_k(x,\lambda\xi)=\lambda^k q_k(x,\lambda)$$

for $\lambda\in\mathbb{R}\setminus(-1,1)$. We obtain an analogous expansion for $\widetilde{q}(x,\xi)$ of the form

$$\widetilde{q}(x,\xi)\sim\sum_{k\le -d+i+j}\widetilde{q}_k(x,\xi),$$

Now assume that $\widetilde{Q}=Q_1+R$ where the symbol of Q_1 equals

$$\sum_{k=m+1}^{-d+i+j}\widetilde{q}_k(x,\xi)$$

and R is an operator of order $m\le -2$. By the above lemma it remains to show that $\gamma^0 r_+ Q_1(\gamma^0)^*$ is a classical pseudodifferential operator, since $m\le-2$ is arbitrary. The proof is thus finished once we show that

$$(T^k g)(0,x') = (2\pi)^{-n}\lim_{p\to\infty}\int_{\mathbb{R}^n} e^{\mathbf{i}\langle x',\xi'\rangle}\widetilde{q}_k(0,x',\xi)\widehat{\chi}(\tfrac{\xi_1}{p})\widehat{g}(\xi')d\xi.$$

is a classical ψdo of order $k+1$ on $\mathbb{R}^{n-1}$.

To start with we decompose the integral again using Fubini's theorem:

$$(T_p^k g)(x_1, x') := (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\langle x', \xi' \rangle} \int_{\mathbb{R}} e^{\mathbf{i}x_1\xi_1} \widetilde{q}_k(x_1, x', \xi_1, \xi') \widehat{\chi}(\tfrac{\xi_1}{p}) \, d\xi_1 \, \widehat{g}(\xi') d\xi'$$

Let us study

$$\int_{\mathbb{R}} e^{\mathbf{i}x_1\xi_1} \widetilde{q}_k(x_1, x', \xi_1, \xi') \widehat{\chi}(\tfrac{\xi_1}{p}) \, d\xi_1$$

for $x_1 > 0$ as $p \to \infty$. For each p, this integral equals, up to a factor $2\pi\mathbf{i}$, the sum of the residues of

$$z \to e^{\mathbf{i}x_1 z} \widetilde{q}_k(x_1, x', z, \xi') \widehat{\chi}(\tfrac{z}{p})$$

in the upper half-plane $\{\operatorname{Im} z > 0\}$. Namely,

$$\widehat{\chi}(\tfrac{z}{p}) = p \cdot \int_{-1}^{0} e^{-\mathbf{i}y_1 z} \chi(py_1) dy_1 = p^{n+1} \cdot \int_{-1}^{0} (\tfrac{1}{\mathbf{i}z})^n e^{-\mathbf{i}y_1 z} \chi^{(n)}(py_1) dy_1.$$

Therefore $\widehat{\chi}(\frac{z}{p})$ extends to $\{\operatorname{Im} z > 0\}$ as a holomorphic function bounded by $C_N(1 + |z|^2)^{-N}$ for all $N > 0$. Moreover, since $x_1 \geq 0$, $z \mapsto e^{\mathbf{i}x_1 z} \widetilde{q}_k(x, z, \xi')$ is a meromorphic function on $\{\operatorname{Im} z > 0\}$ which is polynomially bounded for large $|z|$. Therefore, for $\xi' \in K \subset \mathbb{R}^{n-1} \setminus \{0\}$, K compact, we may write

$$\int_{\mathbb{R}} e^{\mathbf{i}x_1\xi_1} \widetilde{q}_k(x_1, x', \xi_1, \xi') \widehat{\chi}(\tfrac{\xi_1}{p}) \, d\xi_1 = \int_{\gamma_K} e^{\mathbf{i}x_1 z} \widetilde{q}_k(x_1, x', z, \xi') \widehat{\chi}(\tfrac{z}{p}) \, dz$$

where γ_K is a closed path in $\{\operatorname{Im} z > 0\}$ that encircles all poles of the meromorphic function

$$z \mapsto \widetilde{q}_k(x_1, x', z, \xi')$$

for all $\xi' \in K$ in the positive sense. To see that such a path always exists we claim that the set of all these poles is a compact subset of $\{\operatorname{Im} z > 0\}$ if K is compact. First note that $\widetilde{q}_k(x_1, x', z, \xi')$ has a pole at z_0 if and only if $\widehat{P}(x_1, x', z_0, \xi')$ is not invertible. The zeros of the polynomial $z \mapsto \det \widehat{P}(x_1, x', z, \xi')$, however, are bounded by some continuous function of the coefficients[3]. Since $\widehat{P}(x, \xi)$ is constant in the x-variables outside a compact subset, the union of all zeros is a bounded subset of $\mathbb{C}$ when ξ' runs over a compact subset of $\mathbb{R}^n$.

Now, since γ_K is a closed path in $\mathbb{C}$

$$e^{\mathbf{i}x_1 z} \widetilde{q}_k(x_1, x', z, \xi') \widehat{\chi}(\tfrac{z}{p})$$

converges to $e^{\mathbf{i}x_1 z} \widetilde{q}_k(x_1, x', z, \xi')$ uniformly on $\operatorname{im} \gamma_K$ as $p \to \infty$. We conclude that

$$(T^k g)(x_1, x') = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\langle x', \xi' \rangle} t_k(x_1, x', \xi') \widehat{g}(\xi') d\xi',$$

where

$$\begin{aligned} t_k(x_1, x', \xi') &= (2\pi)^{-1} \int_{\gamma_K} \widetilde{q}_k(x_1, x', z, \xi') \, dz \\ &= \mathbf{i} \sum_{\operatorname{Im} w > 0} \operatorname{Res}_w \big(z \mapsto \widetilde{q}_k(x_1, x', z, \xi') \big). \end{aligned} \tag{3.2}$$

---

[3]Note, for instance, that all zeros of the polynomial $\sum_{k=0}^{l} a_k z^k$ lie within a disc of radius $\max(|a_l|^{-1}(|a_{l-1}| + \ldots + |a_0|), 1)$.

for $\xi' \in K$. In order to compute $t_k(x_1, x', \xi')$ for all (x_1, x', ξ') we may, of course, choose different compact sets K and encircling paths γ_K. Using again that $\operatorname{im} \gamma_K$ is compact we easily see that $t_k(x_1, x', \xi')$ is a smooth function on $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. To see that $t_k(0, x', \xi')$ is positive homogeneous in ξ' let $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \mathbb{R}_+$ be given and choose K such that $\xi', \lambda\xi' \in K$. Find a path γ_K such that $\lambda^{-1}\gamma_K$ as well as γ_K encircle all poles. Then

$$\begin{aligned} t_k(0, x', \lambda\xi') &= (2\pi)^{-1} \int_{\gamma_K} \widetilde{q}_k(0, x', z, \lambda\xi')\ dz \\ &= (2\pi)^{-1} \int_{\lambda^{-1}\gamma_K} \widetilde{q}_k(0, x', \lambda(w, \xi'))\ \lambda dw \\ &= (2\pi)^{-1} \int_{\lambda^{-1}\gamma_K} \widetilde{q}_k(0, x', w, \xi')\ \lambda^{k+1} dw \\ &= \lambda^{k+1} t_k(0, x', \xi'). \end{aligned} \tag{3.3}$$

Hence $T^k\big|_{x_1=0}$ is a classical ψdo of order $k+1$ on $\mathbb{R}^{n-1}$. □

An immediate consequence of the proof is the following observation.

Corollary 2.3.7. *Let Q'^M be any pseudodifferential parametrix of P^M. Then*

$$-\widetilde{\rho}^d r_+ Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J}$$

differs from $C_+(P)$ by a smoothing operator on Γ.

Proof. $Q'^M - Q^M$ is smoothing. Hence, by Lemma 2.3.6

$$\widetilde{\rho}^d r_+ (Q^M - Q'^M)(\widetilde{\rho}^d)^*$$

is smoothing. □

Moreover, part (ii) of the proof shows that, in principal, we are now able to compute the symbol expansion of $C_+(P)$.

Theorem 2.3.8. *The Calderón projection $C_+(P)$ is a matrix of classical pseudodifferential operators,*

$$\big(C_+(P)_{jk}\big)_{0 \le j,k \le d-1},$$

where

$$C_+(P)_{jk} = -\sum_{l=0}^{d-1} (-1)^l \Phi^{\frac{d-1-2j}{2}} T_{jl} J_{lk} \Phi^{-\frac{d-1-2k}{2}} \in \Psi^0_{\mathrm{cl}}(\Gamma, E').$$

and

$$T_{jl} \in \Psi^{-d+j+l+1}_{\mathrm{cl}}(\Gamma, E')$$

is given by

$$T_{jl} = \gamma^0 r_+ (\nabla_1^j Q^M \nabla_1^l)(\gamma^0_M)^*.$$

The principal symbol of $C_+(P)_{jk}$ equals

$$-i\|\xi'\|^{k-j} \sum_{l=0}^{d-1} \sum_{\operatorname{Im} w > 0} (iw)^{j+l} (-1)^l \operatorname{Res}_w(z \mapsto (\widehat{P}(0, x', z, \xi'))^{-1})\ \widehat{J}_{lk}(x', \xi').$$

Proof. We have seen in the proof of Theorem 2.3.5 that

$$\begin{aligned} C_+(P)_{jk} &= -\sum_{l=0}^{d-1}(-1)^l \Phi^{\frac{d-1-2j}{2}}\gamma^0 r_+ \nabla_1^j Q \nabla_1^l (\gamma_M^0)^* \Phi^{\frac{d-1-2l}{2}} \widetilde{J}_{lk} \\ &= -\sum_{l=0}^{d-1}(-1)^l \Phi^{\frac{d-1-2j}{2}}\gamma^0 r_+ \nabla_1^j Q \nabla_1^l (\gamma_M^0)^* J_{lk} \Phi^{-\frac{d-1-2k}{2}} \end{aligned}$$

since

$$\widetilde{J} = \Phi_d^{-1} \widetilde{J} \Phi_d^{-1},$$

and thus

$$\widetilde{J}_{lk} = \Phi^{-\frac{d-1-2l}{2}} J_{lk} \Phi^{-\frac{d-1-2k}{2}}.$$

For the symbol computation note that the principal symbol of ∇_1 equals $i\xi_1$, Hence,

$$\operatorname{Res}_w \widehat{\nabla_1^j}(z)(\widehat{P}(0,x',z,\xi')^{-1}\widehat{\nabla_1^l}(z) = (iw)^{j+l} \operatorname{Res}_w \widehat{P}(0,x',z,\xi'))^{-1}$$

for all $w \in \mathbb{C}$. Now, the theorem follows using $\widehat{\Phi}(\xi') = \|\xi'\|$. $\square$

Let us now describe the kernel of $C_+(P)$. Let M be a compact manifold with (possibly empty!) smooth boundary and suppose that P^M is an elliptic extension of P to M and with a well-posed boundary condition

$$B\widetilde{\rho}^d_{\partial M} u = 0.$$

Then we may consider the subspace of $H^{-d/2}(\Gamma, E'^d)$ spanned by all $\widetilde{\rho}^d u$ where

$$u \in L^2(M \setminus \Omega, E), \quad B\widetilde{\rho}^d_{\partial M} = 0, \quad P^M u = 0.$$

Assuming the boundary condition B to be fixed, let us call this space the *Cauchy data space* of $M \setminus \Omega^\circ$.

Theorem 2.3.9. *Assume that all sections in* $\ker P_B^M$ *have support in* Ω. *Then* C_+ *is the projection onto the Cauchy data space of* Ω *along the Cauchy data space w.r.t.* $M \setminus \Omega^\circ$.

At the beginning of this section we showed that such an extension P_B^M does exist. Observe that the assumptions are trivially satisfied if M is closed and P^M is invertible. Such an *invertible double* can be constructed in some applications.

Proof. To begin with let us assume for simplicity that P_B^M is invertible and let Q^M denote its inverse.

Note that the corresponding $\widetilde{J}$-operator which appears in Green's formula when integrating over $M \setminus \Omega$ is $-\widetilde{J}$ if we use the same trace operators as for Ω (coming from the outward (!) unit normal field at Γ, cf. Remark 1.1.3). Clearly, the proof of Theorem 2.3 (i) shows that

$$C_-(P) = \widetilde{\rho}^d r_- Q^M (\widetilde{\rho}^d_M)^* \widetilde{J}$$

is a projection onto the Cauchy data space w.r.t. $M \setminus \Omega$.

Let $g \in H^{-d/2}(\Gamma, E'^d)$, $h \in H^{d/2}(\Gamma, F'^d)$ and choose $v \in H^d_{\mathrm{comp}}(M^\circ, F)$ such that $\widetilde{\rho}^d v = h$. Then

$$\begin{aligned} \langle \widetilde{J}(C_+(P) + C_-(P))g, h\rangle &= \langle -\widetilde{J}\widetilde{\rho}^d r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, h\rangle + \langle \widetilde{J}\widetilde{\rho}^d r_- Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, h\rangle \\ &= \langle r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^t r_+ v\rangle + \langle r_+ Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^t r_- v\rangle \\ &= \langle Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^{M,t} v\rangle_{L^2(M,E)} \qquad (3.4) \\ &= \langle P^M Q^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, v\rangle \\ &= \langle \widetilde{J} g, h\rangle, \end{aligned}$$

since $P^M r_+ Q^M (\widetilde{\rho}^d_M)^* = 0$ and $P^M r_- Q^M (\widetilde{\rho}^d_M)^* = 0$. It follows that

$$C_+(P) + C_-(P) = \mathrm{Id},$$

and therefore

$$\operatorname{ran}(C_-(P) : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)) = \ker(C_+(P) : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)),$$

and vice versa.

Otherwise, when P^M_B has inner solutions, recall that Q^M satisfies

$$Q^M P^M u = (\mathrm{Id} - \mathrm{pr}_1)u, \quad P^M Q^M = (\mathrm{Id} - \mathrm{pr}_2)v$$

for all $u \in C^\infty_c(M^\circ, E)$, $v \in C^\infty_c(M^\circ, F)$, where pr_1, pr_2 are the orthogonal projections onto the finite-dimensional spaces of inner solutions of P and P^t. This shows that

$$P^M Q^M (\widetilde{\rho}^d_M)^* = (\widetilde{\rho}^d_M)^*,$$

so that the above argument is still valid in this case. □

The preceding theorem uniquely determines $C_+(P)$ whenever an invertible realisation of P^M is given.

Remark 2.3.10. Finally, one might want instead of $C_+ = C_+(P)$ an orthogonal projection onto the Cauchy data space. The orthogonal projection onto $\operatorname{ran} C_+$, which we denote by C^{ort}_+ here, is given by the well-known formula (cf. [BBW93, Lemma 12.8])

$$C^{\mathrm{ort}}_+ := C_+ C^*_+ (C_+ C^*_+ + (\mathrm{Id} - C^*_+)(\mathrm{Id} - C_+))^{-1}.$$

Since $C_+ C^*_+ + (\mathrm{Id} - C^*_+)(\mathrm{Id} - C_+)$ is elliptic we infer that C^{ort}_+ is still a classical ΨDO of order 0.

2.4 Bojarski's Theorem and the Agranovič-Dynin-Formula

Let us assume that $P^M : C^\infty(M, E) \to C^\infty(M, F)$ is an elliptic operator on a closed manifold which is cut into two pieces by the closed hypersurface Σ, i.e.

$$M = \Omega_+ \cup_\Sigma \Omega_-,$$

where Ω_+ and Ω_- are manifolds with boundary Σ. In order to define a trace

$$\widetilde{\rho}^d : H^d(M, E) \to H^{d/2}(\Gamma, E'^d)$$

let us assume that ν is the unit normal vector field which points into Ω_+, i.e. Ω_+ takes the rôle of Ω and Ω_- that of $M \setminus \Omega^\circ$. Denote by $r_\pm$, $e_\pm$ the corresponding restriction, resp. "extension by zero" operators. We write $\widetilde{\eta}^d_\pm$ for the continuous right inverses to the trace maps $\widetilde{\rho}^d_\pm : H^d(\Omega_\pm, G) \to H^{d/2}(\Sigma, G'^d)$ and set $\widetilde{\eta}^d_M = e_+ \widetilde{\eta}^d_+ + e_- \widetilde{\eta}^d_-$, where $G = E$ or F. Clearly, $\widetilde{\eta}^d_M$ is a continuous right inverse to $\widetilde{\rho}^d_M : H^d(M, G) \to H^{d/2}(\Sigma, G'^d)$.

Denote by $P_\pm$ the restriction of P to $\Omega_\pm$. Let $\Lambda^S_0(P_\pm)$ denote the corresponding strong Cauchy data spaces, i.e.

$$\Lambda^S_0(P_\pm) = \left\{ \widetilde{\rho}^d u \;\middle|\; u \in H^d(\Omega_\pm, E),\ P_\pm u = 0 \right\}.$$

Theorem 2.4.1 (B. Bojarski). $(\Lambda_0^S(P_+), \Lambda_0^S(P_-))$ *is a Fredholm pair in* $S(P)$. *Moreover, if* P *satisfies the unique continuation principle, then*

$$\operatorname{ind} P = \operatorname{ind}\big(\Lambda_0^S(P_+), \Lambda_0^S(P_-)\big).$$

Proof. Let $C_\pm(P)$ denote the corresponding Calderón projections which are classical pseudodifferential operators. Let Q'^M be a pseudodifferential parametrix for P, such that

$$Q'^M P^M = \mathrm{Id} - R_1, \quad P^M Q'^M = \mathrm{Id} - R_2,$$

where R_1 and R_2 are the orthogonal projections onto the kernel and cokernel of P^M, resp. Then, by Corollary 2.3.7 we can approximate the Calderón projections $C_\pm(P)$ by

$$C'_+ := -\widetilde{\rho}^d r_+ Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J}, \quad C'_- := \widetilde{\rho}^d r_- Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J},$$

resp., up to operators in $\Psi^{-\infty}(\Sigma, E'^d)$. In particular, a computation analogous to (3.4) shows that for all $g \in H^{-d/2}(\Sigma, E'^d)$, $h \in H^{d/2}(\Sigma, F'^d)$ we have

$$\begin{aligned}
\langle \widetilde{J}(C'_+ + C'_-)g, h\rangle &= -\langle r_+ P^M Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, \widetilde{\eta}^d_+ h\rangle + \langle r_+ Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^{M,t} \widetilde{\eta}^d_+ h\rangle \\
&\quad + \langle r_- P^M Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, \widetilde{\eta}^d_- h\rangle - \langle r_- Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^{M,t} \widetilde{\eta}^d_- h\rangle \\
&= +\langle r_+ R_2 (\widetilde{\rho}^d_M)^* \widetilde{J} g, \widetilde{\eta}^d_+ h\rangle + \langle r_- R_2 (\widetilde{\rho}^d_M) \widetilde{J} g, \widetilde{\eta}^d_- h\rangle \\
&\quad + \langle Q'^M (\widetilde{\rho}^d_M)^* \widetilde{J} g, P^{M,t} \widetilde{\eta}^d_M h\rangle \\
&= \langle R_2 (\widetilde{\rho}^d_M)^* \widetilde{J} g, \widetilde{\eta}^d_M h\rangle + \langle \widetilde{J} g, \widetilde{\rho}^d_M \widetilde{\eta}^d_M h\rangle - \langle R_2 (\widetilde{\rho}^d_M)^* \widetilde{J} g, \widetilde{\eta}^d_M h\rangle \\
&= \langle \widetilde{J} g, h\rangle.
\end{aligned}$$

Hence, $C'_+ + C'_- = \mathrm{Id}$. C'_+ and C'_- approximate the true Calderón projections up to smoothing operators. In particular, $C_+(P)$ and $C_-(P)$ satisfy

$$C_+(P) + C_-(P) \equiv \mathrm{Id} \mod \Psi^{-\infty}(\Sigma, E'^d),$$

By Proposition A.1.12 (iv) it follows that

$$(\operatorname{ran} C_+(P), \operatorname{ran} C_-(P)) = (\Lambda_0^S(P), \Lambda_0^S(P))$$

is a Fredholm pair, where $\operatorname{ran} C_\pm(P)$ denote the range of $C_\pm(P) : H^{d/2}(\Sigma, E'^d) \to H^{d/2}(\Sigma, E'^d)$.

When unique continuation holds, then we have isomorphisms

$$\ker P \cong \big\{\widetilde{\rho}^d_M u \in H^{d/2}(\Sigma, E'^d) \;\big|\; u \in \ker P : H^d(M, E) \to H^d(M, F)\big\},$$

hence

$$\ker P \cong \Lambda_0^S(P_+) \cap \Lambda_0^S(P_-),$$

and analogously,

$$\ker P^t \cong \Lambda_0^S(P^t_+) \cap \Lambda_0^S(P^t_-).$$

By Proposition 2.1.1

$$\begin{aligned}
\operatorname{coker} P \cong (\operatorname{ran} P)^\perp \cong \ker P^t &\cong \Lambda_0^S(P^t_+) \cap \Lambda_0^S(P^t_-) \\
&= \Lambda_0^S(P_+)^{\omega_S} \cap \Lambda_0^S(P_-)^{\omega_S} \\
&= \big(\Lambda_0^S(P_+) + \Lambda_0^S(P_-)\big)^{\omega_S} \\
&\cong S(P)/\big(\Lambda_0^S(P_+) + \Lambda_0^S(P_-)\big).
\end{aligned}$$

Taking the difference of the dimensions gives the formula for the index. □

Let us now return to the standard situation, i.e. consider a manifold with boundary Ω, $\Gamma := \partial\Omega$, and an elliptic operator $P : C^\infty(\Omega, E) \to C^\infty(\Omega, F)$. For simplicity we make the following convention for the remainder of this section. Whenever we write $\ker B$, $\operatorname{ran} B$ for an operator of order 0 acting on the scale $((H^s(\Gamma, G'^d))$, this stands for the kernel resp. range of

$$B : H^{d/2}(\Gamma, G'^d) \to H^{d/2}(\Gamma, G'^d),$$

where G is either E or F.

Theorem 2.4.2 (Agranovič-Dynin Formula). *If $B\widetilde{\rho}^d u = 0$ is a well-posed boundary condition where B satisfies Assumption 1.2.1 and if UCP holds for P then*

$$\operatorname{ind} P_B = \operatorname{ind}(\ker B, \Lambda_0(P)).$$

If moreover, we are given another well-posed projection $\widetilde{B}$ and if B or $\widetilde{B}$ happens to differ from the Calderón projection merely by a compact perturbation, then

$$\operatorname{ind} P_B - \operatorname{ind} P_{\widetilde{B}} = \operatorname{ind}(\ker B, \operatorname{ran} \widetilde{B}) = \operatorname{ind}(B : \operatorname{ran} \widetilde{B} \to \operatorname{ran} B).$$

Proof. By Theorem 2.1.4, B is a well-posed if and only if $(\ker B, \operatorname{ran} C_+(P))$ is Fredholm. By UCP, any section $u \in H^d(\Omega, E)$ such that $Pu = 0$ is uniquely determined by its trace $\widetilde{\rho}^d u \in \Lambda_0^S(P)$. Together with the same argument for P^t this gives

$$\begin{aligned}
\operatorname{ind} P_B &= \dim\ker P_B - \dim\ker P^t_{B^{\mathrm{ad}}} \\
&= \dim\left(\Lambda_0^S(P) \cap \ker B\right) - \dim\left(\Lambda_0^S(P^t) \cap \ker B^{\mathrm{ad}}\right) \\
&= \dim\left(\Lambda_0^S(P) \cap \ker B\right) - \dim\left(\Lambda_0^S(P)^{\omega_S} \cap (\ker B)^{\omega_S}\right) \\
&= \dim\left(\Lambda_0^S(P) \cap \ker B\right) - \dim\left(\Lambda_0^S(P) + \ker B\right)^{\omega_S} \\
&= \operatorname{ind}(\Lambda_0^S(P), \ker B).
\end{aligned}$$

As for the second statement we have to consider the difference

$$\begin{aligned}
\operatorname{ind}(\ker B,&\Lambda_0^S(P)) - \operatorname{ind}(\ker \widetilde{B}, \Lambda_0^S(P)) \\
&= \operatorname{ind}(B : \operatorname{ran} C_+(P) \to \operatorname{ran} B) + \operatorname{ind}(C_+(P)\widetilde{B} : \operatorname{ran} \widetilde{B} \to \operatorname{ran} C_+(P)) \\
&= \operatorname{ind}(BC_+(P)\widetilde{B} : \operatorname{ran} \widetilde{B} \to \operatorname{ran} B).
\end{aligned}$$

Whenever B or $\widetilde{B}$ is a compact perturbation of $C_+(P)$, $B(\mathrm{Id} - C_+(P))\widetilde{B}$ is compact and the index is thus given by $\operatorname{ind}(B : \operatorname{ran} \widetilde{B} \to \operatorname{ran} B)$. □

Remark 2.4.3. When B and $\widetilde{B}$ are allowed to differ from $C_+(P)$ by non-compact operators the above Agranovič-Dynin formula for the difference of the indices no longer holds. In fact, $(\ker B, \operatorname{ran} \widetilde{B})$ may not even be Fredholm. Consider, for instance, Dirichlet and Neumann boundary conditions for the Laplacian on functions. They are given by complementary projections, cf. the examples in Section 4.3.

Remark 2.4.4. All pairs we have considered in this section are pairs of subspaces of

$$S(P) = H^{d/2}(\Gamma, E'^d).$$

Since $C_+(P)$ is pseudodifferential all subspaces are given as ranges or kernels of projections satisfying Assumption 1.2.3. For such operators,

$$(\ker B_1, \operatorname{ran} B_2)$$

is a Fredholm pair if and only if

$$\pm \operatorname{Id} - B_1 + B_2 : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d).$$

is Fredholm. By Proposition A.2.2, the essential spectrum of

$$T : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d)$$

does not depend on $s \in \mathbb{R}$. Hence, $(\ker B_1, \operatorname{ran} B_2)$ is Fredholm if and only if

$$\pm \operatorname{Id} - B_1 + B_2 : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^d).$$

for any $s \in \mathbb{R}$. Moreover, the dimension of the kernels are independent of s. It follows, that

$$(\ker B_1|_{H^s(\Gamma, E'^d)}, \operatorname{ran} B_2|_{H^s(\Gamma, E'^d)})$$

is a Fredholm pair for all $s \in \mathbb{R}$ if it is for one s, and the index is independent of s.

By Proposition A.1.12, $(\ker B_1, \operatorname{ran} B_2)$ is left Fredholm if and only if

$$B_1 B_2 : \operatorname{ran} B_1 \to \operatorname{ran} B_1$$

is Fredholm. This is the case, by Proposition A.2.2, if and only if

$$B_1 B_2 + (\operatorname{Id} - B_1) : H^s(\Gamma, E'^d) \to H^s(\Gamma, E'^s),$$

is Fredholm for some $s \in \mathbb{R}$.

We conclude that, in order to check the (left) Fredholm condition we might restrict attention to the corresponding L^2-spaces. Furthermore, we may compute nullity and deficiency of the corresponding (left) Fredholm pair in $L^2(\Gamma, E'^d)$. This can be interesting in applications of Theorem 2.4.2.

2.5 The Formally Self-adjoint Case

Let $E = F$ and consider a *formally self-adjoint* operator $P : C^\infty(M, E) \to C^\infty(M, E)$, i.e. $P^t = P$. Note, that $P_{\max} = (P^t_{\min})^* = (P_{\min})^* = P^t_{\max}$. In particular, we see that $W(P)$ is a symplectic Hilbert space (with the symplectic structure ω), in the sense that

$$\omega(g, h) = -\overline{\omega(h, g)},$$

for all $g, h \in W(P)$. Let us reformulate, in a purely functional analytic context, the statements proved in Section 1.3. For instance, by Proposition 1.3.5, we have:

Proposition 2.5.1. *The adjoint of the realisation corresponding to a subspace Λ is the realisation that corresponds to the symplectic complement of Λ. In particular, P_Λ is symmetric (self-adjoint) if and only if Λ is isotropic (Lagrangian).*

Consider $\Lambda_0 = \Lambda_0(P) = \Lambda_0(P^t)$. Proposition 1.3.3 now reads as follows.

Proposition 2.5.2. *The abstract Cauchy data space Λ_0 is a Lagrangian subspace of $W(P)$.*

Remark 2.5.3. Hence, if M is a compact manifold with boundary and P is a formally self-adjoint operator, then there is no obstruction to the existence of self-adjoint extensions. The extension defined by Λ_0 is called the *soft* extension.

If M is not compact, then $P_{\min}$ does not necessarily have closed range. In this case, $\Lambda_0(P)$ need not be Lagrangian. In general, it is only isotropic. Moreover, there may not even be a single self-adjoint extension of $P_{\min}$. The classical counterexample is the operator $P = -i\frac{\partial}{\partial x}$ on $\Omega = [0, \infty)$. Then $\Lambda_0 = \{0\}$, $W(P) \cong \mathbb{C}$ and the deficiency indices are 0 and 1.

Theorem 2.5.4. *The space of self-adjoint Fredholm extensions is parametrised by the Fredholm-Lagrange Grassmannian w.r.t.* Λ_0*,*

$$\mathscr{F}\mathscr{L}(\Lambda_0) := \big\{\Lambda \subset W(P) \;\big|\; (\Lambda, \Lambda_0) \text{ is Fredholm, } \Lambda_0 \text{ is Lagrangian}\big\}.$$

This space is non-empty and contains a Λ*, such that* $\ker P_\Lambda$ *consists of inner solutions only.*

Proof. From Proposition 2.5.1 we deduce that the space of self-adjoint extensions is parametrised by the Lagrange Grassmannian in $W(P)$. Combined with Theorem 1.3.4 we obtain the above characterisation of the self-adjoint Fredholm extensions.

By Proposition 2.5.2, Λ_0 is Lagrangian. Recall from Section 1.3 that we can view $W(P)$ as a subspace of $H \oplus H$, where $H = L^2(\Omega, E)$. Then it is identified with the graph of $P_{\max}$ restricted to $\ker(P_{\max}^2 + 1)$. We have

$$\omega([u],[v]) = \Big\langle J_{H\oplus H}\begin{pmatrix} u \\ P_{\max}u\end{pmatrix}, \begin{pmatrix} v \\ P_{\max}v\end{pmatrix}\Big\rangle,$$

where

$$J_{H\oplus H} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

In particular, $J_{H\oplus H}^* = -J_{H\oplus H} = J_{H\oplus H}^{-1}$. It follows that the orthogonal complement, $\Lambda_0^\perp$ of Λ_0, regarded as a subspace of

$$\mathscr{G}(P_{\max}|_{\ker(P_{\max}^2+1)}),$$

is a Lagrangian subspace of $W(P)$. By construction $(\Lambda_0, \Lambda_0^\perp)$ is a Fredholm pair of Lagrangian subspace and hence $P_{\Lambda_0^\perp}$ is self-adjoint and Fredholm. Since $\Lambda_0 \cap \Lambda_0^\perp = \emptyset$, by Theorem 1.3.4, the kernel of P_{Λ_0} equals $\ker P_{\min} = (\operatorname{ran} P_{\max})^\perp$. ☐

Note that the self-adjoint Fredholm extension constructed in the proof above is still unsatisfactory. For instance, it is not known to be regular. However, with this extension, we will be able to construct a well-posed self-adjoint boundary condition given by a pseudodifferential projection.

Let us now digress from this abstract setting and come to the question of boundary conditions that are given by projections satisfying Assumption 1.2.3. Since ω is symplectic it follows that $\widetilde{J}$ is skew-adjoint, i.e.

$$\langle \widetilde{J}g, h\rangle = \omega(g,h) = -\overline{\omega(h,g)} = -\overline{\langle \widetilde{J}h, g\rangle} = -\langle \widetilde{J}^*g, h\rangle.$$

Let now B be a projection satisfying Assumption 1.2.3. Then Proposition 1.2.6 gives:

Proposition 2.5.5. *The adjoint boundary condition of B is given by*

$$B^{\mathrm{ad}} = \widetilde{J}^{-1}(\mathrm{Id} - B^*)\widetilde{J}.$$

In particular, P_B is symmetric if and only if $\ker B \subset \ker \widetilde{J}^{-1}(\mathrm{Id} - B^*)\widetilde{J}$.

Definition 2.5.6. If $\ker B = \ker \widetilde{J}^{-1}(\mathrm{Id} - B^*)\widetilde{J}$, then we call the realisation P_B *formally self-adjoint.*

Another consequence of Proposition 1.2.6 is the following:

Theorem 2.5.7. (i) *Let P_B be formally self-adjoint. Then P_B is self-adjoint if and only if B is a regular boundary condition.*

(ii) *All regular self-adjoint realisations have discrete spectrum.*

Proof. (i) If P_B is formally self-adjoint then

$$(P_B)^* = P^t_{\max, B^{\mathrm{ad}}} = P_{\max, B}.$$

Hence, $(P_B)^*$ equals P_B if and only if B is regular for P.

(ii) If P_B is self-adjoint, then the resolvent set is non-empty. By the Rellich Theorem the embedding

$$H^d(\Omega, E) \to L^2(\Omega, E)$$

and thus the resolvents

$$(P_B - \lambda)^{-1} : L^2(\Omega, E) \to L^2(\Omega, E), \quad \lambda \in \rho(P_B)$$

are compact. Hence, P_B has discrete spectrum. □

We will now address the question of existence of self-adjoint Fredholm extensions given by pseudodifferential boundary conditions. Hence, we are looking for a pseudodifferential projection

$$B : C^\infty(\Gamma, E'^d) \to C^\infty(\Gamma, E'^d),$$

such that $\ker B = \ker \widetilde{J}^{-1}(\mathrm{Id} - B)\widetilde{J}$ and such that

$$(B, C_+(P))$$

is a Fredholm pair of projections. By the above theorem and Theorem 2.1.4, it would follow that

$$P_B : \{u \in H^d(\Omega, E) \mid B\widetilde{\rho}^d u = 0\} \to L^2(\Omega, E)$$

is a self-adjoint Fredholm extension.

We have a good candidate for such a Lagrangian subspace given by a pseudodifferential projection: the Cauchy data space of a formally self-adjoint operator. More precisely, Proposition 2.1.1 tells us that $\Lambda_0^S(P)$ is Lagrangian in $S(P)$ if $P = P^t$. By Theorem 2.3.9 (and the preceding remarks), the kernel of C_+ is the Cauchy data space w.r.t $M \setminus \Omega$, i.e.

$$\ker C_+ = \{\widetilde{\rho}^d u \in H^{-d/2}(\Gamma, E'^d) \mid u \in L^2(M \setminus \Omega),\ P^M u = 0,\ \widetilde{\rho}^d_{\partial M} u \perp \Lambda_0(P^M)\}.^4$$

[4] "$\perp$" means orthogonal in $\mathscr{G}(P_{\min})^\perp \subset L^2(M, E) \oplus L^2(M, E)$, see Section 2.3. This will be crucial in the following.

Let us assume that P^M is formally self-adjoint. Otherwise one can symmetrise P^M, i.e. substitute P^M for $\frac{1}{2}(P^M + P^{M,t})$, which is elliptic on an open neighbourhood of Ω. So, after shrinking M to a small tubular neighbourhood of Ω, $\frac{1}{2}(P^M + P^{M,t})$ is a formally self-adjoint elliptic extension.

Since P^M is formally self-adjoint there exists, by Theorem 2.5.4, a self-adjoint Fredholm extension

$$P_0^M : L^2(M,E) \supset \mathscr{D}(P_0^M) \to L^2(M,E)$$

of P^M. Consider its restriction to the closed subspace

$$\mathscr{D}(P_{\min}^{M\setminus\Omega}) := \{u \in \mathscr{D}(P_0^M) \mid u|_\Omega = 0\}$$

as a symmetric(!) unbounded operator in $L^2(M \setminus \Omega, E)$. This is possible, since if $u|_\Omega = 0$, then $P^M u|_\Omega = 0$. Since P_0^M is Fredholm the range of this restriction is closed. Its adjoint, $P_{\max}^{M\setminus\Omega}$ is given by

$$\mathscr{D}(P_{\max}^{M\setminus\Omega}) := \{u \in L^2(M \setminus \Omega) \mid Pu \in L^2(M \setminus \Omega),\ \chi u \in \mathscr{D}(P_0^M)\}$$

where $\chi \in C_c^\infty(M \setminus \Omega)$ equals 1 near ∂M. Observe that for $u, v \in \mathscr{D}(P_{\max}^M)$ we have

$$\langle P^M u, v\rangle - \langle u, P^M v\rangle = -\omega(\widetilde{\rho}^d u, \widetilde{\rho}^d v).$$

Now, by Proposition 1.3.3, we find that

$$\Lambda_0(P_-) = \Lambda_0(P^{M\setminus\Omega}) = \{\widetilde{\rho}^d u \mid u \in \mathscr{D}(P_{\max}^{M\setminus\Omega}),\ Pu = 0\}.$$

is Lagrangian in $(W(P), -\omega)$. It follows that

$$\ker C_+ = \ker \widetilde{J}^{-1}(\operatorname{Id} - C_+^*)\widetilde{J}$$

and therefore

$$P_{C_+} : \{H^d(\Omega, E) \mid C_+ \widetilde{\rho}^d u = 0\} \to L^2(\Omega, E)$$

defines a regular self-adjoint extension. We have thus proved.

Theorem 2.5.8. *There exists a well-posed pseudodifferential boundary projection B such that P_B is self-adjoint.*

Remark 2.5.9. Part (i) of Theorem 2.5.7 makes no statement about essential self-adjointness. There are non-regular boundary projections B such that P_B is essentially self-adjoint. For instance, the realisation with domain

$$\mathscr{D}(P_{\Lambda_0^s}) := \{u \in H^d(\Omega, E) \mid [u] \in \Lambda_0\},$$

is given by the boundary condition $(\operatorname{Id} - C_+)\widetilde{\rho}^d u = 0$ and corresponds to the isotropic space

$$\Lambda_0^S := \Lambda_0 \cap S.$$

$S(P) \subset W(P)$ is dense and we have

$$\Lambda_0^S = C_+(H^{d/2}(\Gamma, E'^d)) = C_+(S(P)),$$

hence Λ_0^S is dense in Λ_0. By Proposition 1.3.4 this means that P_{Λ_0} is the closure of $P_{\Lambda_0^s}$. However, Λ_0 corresponds to the soft extension whose kernel, unless $\dim \Omega = 1$, is infinite dimensional. Hence, Λ_0^S corresponds to a non-regular boundary condition whose realisation is essentially self-adjoint.

We will now discuss non-formally self-adjoint extensions of a formally self-adjoint operator P. It will turn out that the index of such a realisation, if it is Fredholm, only depends on the boundary condition B and the germ of the operator P at Γ.

More precisely, let Ω_1 and Ω_2 be different manifolds with the same boundary Γ and with elliptic operators $P_i : C^\infty(\Omega_i, E_i) \to C^\infty(\Omega_i, E_i)$, $i = 1, 2$. Assume that there are collar neighbourhoods of Γ, say N_1, N_2, such that

$$\Omega_1 \supset N_1 \cong [0, \delta) \times \Gamma \cong N_2 \subset \Omega_2,$$

so $N_1 \cong N_2$. Let us further assume that $E_1|_{N_1} \cong E_2|_{N_2}$ and that P_1 coincides with P_2 over Γ w.r.t. to these identifications. Assume that P and $\widetilde{P}$ are formally self-adjoint operators satisfying UCP.

Theorem 2.5.10. *Let B be a well-posed boundary condition for P (and hence for $\widetilde{P}$). We have*

$$\operatorname{ind} P_{1,B} = \operatorname{ind} P_{2,B}.$$

Proof. Since P and $\widetilde{P}$ are formally self-adjoint the associated Cauchy data spaces are Lagrangian. In particular

$$\operatorname{ind}\big(C_+(P_1) : \operatorname{ran} C_+(P_2) \to \operatorname{ran} C_+(P_1)\big) = 0.$$

Moreover, since the principal symbols of P_1 and P_2 coincide over Γ the principal symbols of the Calderón projection are the same by Theorem 2.3.8. In particular, $(\operatorname{Id} - C_+(P_1))C_+(P_2)$ is compact. We deduce that

$$\begin{aligned}
\operatorname{ind} P_{1,B} :&= \operatorname{ind}\big(B : \operatorname{ran} C_+(P_1) \to \operatorname{ran} B\big)\\
&= \operatorname{ind}\big(BC_+(P_1) : \operatorname{ran} C_+(P_2) \to \operatorname{ran} B\big)\\
&= \operatorname{ind}\big(BC_+(P_1)C_+(P_2) : \operatorname{ran} C_+(P_2)B \to \operatorname{ran} B\big)\\
&= \operatorname{ind}(B : \operatorname{ran} C_+(P_2) \to \operatorname{ran} B)\\
&= \operatorname{ind} P_{2,B},
\end{aligned}$$

since $(\operatorname{Id} - C_+(P_1))C_+(P_2)$ is compact. □

2.6 Perturbation Theory for Well-posed Boundary Problems

Let B be a well-posed boundary condition for P. We will first show that regularity and well-posedness are stable under "small" perturbations of P and B.

Theorem 2.6.1. *Let B be regular (well-posed) for P. There exists $\varepsilon > 0$ such that for all P', B' satisfying*

$$\|P - P'\|_{\mathscr{B}(H^d(\Omega,E),L^2(\Omega,F))} + \|B - B'\|_{\mathscr{B}(H^{d/2}(\Gamma,E'^d))} < \varepsilon \tag{6.1}$$

B' is regular (well-posed) for P'. Moreover, if P_B is invertible then there exists $\varepsilon > 0$ such that (6.1) implies that $P'_{B'}$ is also invertible.

Proof. By Theorem 2.1.2, B is regular (well-posed) for P if and only if the mapping pair

$$\big(P, B\widetilde{\varrho}^d\big) : H^d(\Omega, E) \to L^2(\Omega, E) \oplus \big(\operatorname{ran} B \cap H^{d/2}(\Gamma, E'^d)\big).$$

is left-Fredholm (Fredholm). Compose the mapping pair of a perturbed boundary problem (P', B') with the projection B in the second component, i.e. we consider

$$(P', BB'\widetilde{\rho}^d) : H^d(\Omega, E) \to L^2(\Omega, F) \oplus \big(\operatorname{ran} B \cap H^{d/2}(\Gamma, E'^d)\big)$$

By Proposition A.1.12, $B'B : \operatorname{ran} B \to \operatorname{ran} B'$ is invertible if and only if $\operatorname{Id} - B' + B$ is an isomorphism. We infer that if

$$\|B' - B\|_{\mathscr{B}(H^{d/2}(\Gamma, E'))} < 1,$$

then $B'B$ is invertible. Consider the the map $(P', BB'\widetilde{\rho}^d)$ as a small perturbation of

$$(P, B\widetilde{\rho}^d).$$

Note that $L^2(\Omega, F) \oplus \big(\operatorname{ran} B \cap H^{d/2}(\Gamma, E'^d)\big)$ is a fixed Banach space, since $\operatorname{ran} B : H^{d/2}(\Gamma, E'^d) \to H^{d/2}(\Gamma, E'^d)$ has closed range.

Hence, if ε is chosen sufficiently small and (6.1) holds, then B' is regular (well-posed) for P'. Analogously, for small perturbations (P', BB') is invertible if (P, B) is. This implies that (P', B') is invertible which in turn gives the invertibility of $P'_{B'}$. □

Now, we consider a family of formally self-adjoint operators

$$P_s : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$$

and of boundary conditions

$$B_s \widetilde{\rho}^d u = 0$$

where $-\varepsilon < s < \varepsilon$. We assume that

$$P_s : H^d(\Omega, E) \to L^2(\Omega, E) \tag{6.2}$$

forms a continuous curve of bounded operators. For instance, let Ω be covered by coordinate charts $(U, \varphi) \in \mathscr{A}$ and local trivialisations of $E|_U$. (6.2) is a continuous family in s if for all $(U, \varphi) \in \mathscr{A}$ and all compact subsets $K \subset U$ the coefficient matrices vary continuously w.r.t. to the $C(K)$-norm.

We require $\ker B_s$ to form a continuous family of subspaces in $S(P_s) = S(P_0) = H^{d/2}(\Gamma, E'^d)$ in the following sense: $(B_s)_{-\varepsilon<s<\varepsilon}$ is a curve of orthogonal projections which is continuous in $\mathscr{B}(H^{d/2}(\Gamma, E'^d))$. Moreover, we suppose that for each s the pair $(B_s, C_+(P_s))$ is Fredholm, i.e. we assume P_{s,B_s} is a curve of well-posed self-adjoint extensions.

Definition 2.6.2. Let $T_1 : H_1 \supset \mathscr{D}(T_1) \to H_2$, $T_2 : H_1 \supset \mathscr{D}(T_2) \to H_2$ be unbounded operators between Hilbert spaces H_1 and H_2. The gap metric $\delta(T_1, T_2)$ is given by

$$\|P_1 - P_2\|_{\mathscr{B}(H_1 \oplus H_2)}$$

where P_i denotes the orthogonal projection in $H_1 \oplus H_2$ onto the graph $\mathscr{G}(T_i)$. The topology defined by this metric is called the gap topology.

Theorem 2.6.3. *P_{s,B_s} is a continuous curve w.r.t. the gap topology.*

Proof. It suffices to show continuity close to 0. Since all P_{s,B_s} are self-adjoint we see that

$$(P_s + i)_{B_s} : \mathscr{D}(P_{s,B_s}) \to L^2(\Omega, E)$$

is a family of invertible operators. Hence,

$$\big(P_s + i, B_s\widetilde{\rho}^d\big) : H^d(\Omega, E) \to L^2(\Omega, E) \oplus \big(\operatorname{ran} B_s \cap H^{d/2}(\Gamma, E'^d)\big)$$

is a family of invertible operators. Again by Proposition A.1.12, $B_{s'}B_s$ is invertible if

$$\|B_{s'} - B_s\|_{\mathscr{B}(H^{d/2}(\Gamma, E'))} < 1.$$

This proves that for s close to 0 we obtain a contiuous curve of bounded invertible operators between fixed Hilbert spaces by composition with $\big(\mathrm{Id}, B_0B_s\big)$. Therefore,

$$\big(P_s + i, B_0B_s\widetilde{\rho}^d\big)^{-1} \circ (\mathrm{Id}, 0) : L^2(\Omega, E) \to H^d(\Omega, E)$$

is continuous w.r.t. the operator norm. Since $B_0B_s\widetilde{\rho}^d u = 0$ if and only if $B_s\widetilde{\rho}^d u = 0$ it follows that

$$\big(P_s + i, B_0B_s\widetilde{\rho}^d\big)^{-1}\big(\mathrm{Id}, 0\big)$$

equals the resolvent $(P_{s,B_s} + i)^{-1}$. In particular, the resolvents form a continuous curve of bounded operators in $L^2(\Omega, E)$. By [BBLP01, Theorem 1.1], this means that P_{s,B_s} is continuous w.r.t. the gap topology, for s close to 0. ☐

Chapter 3

Operators of Dirac type

This section is devoted to the study of boundary value problems for Dirac operators, i.e. to operators that are associated to a Dirac bundle. To start with, we show that a Dirac bundle on the manifold M induces a natural Dirac bundle on the hypersurface $\Gamma \subset M$. Then we discuss canonical forms of Dirac operators over a collar of the boundary. Since the analytic properties of a boundary value problem for a Dirac operator depend on the so-called tangential operator it is useful to make the ingredients of the canonical form precise, i.e. relate it to well-known geometric operators on the boundary.

Then we compute the Calderón projection for a general first order elliptic differential operator placing emphasis on its principal symbol. These computations are applied in the case of a Dirac type operator.

Finally, we give a direct proof for the cobordism invariance of the index on compact manifolds. Roughly speaking, the principal symbol of a Dirac operator D over an odd-dimensional manifold with boundary, evaluated on the unit normal 1-form $\nu^\flat$ defines a $\mathbb{Z}_2$-grading such that the tangential part of D is odd w.r.t. the corresponding splitting and has vanishing index. As in [Pal65] our proof is based on the Calderón projection. However, we translate it into an equivalent problem of symplectic functional analysis: The positive spectral projection of the tangential part defines an isotropic subspace in the L^2-space of boundary values provided with the symplectic form given by Green's formula. The vanishing of the above index is equivalent to the existence of a Lagrangian extension. The latter problem is solved by elementary Fredholm theory for pairs of isotropic subspaces.

In the original proof of the Atiyah-Singer Index Theorem (cf. [Pal65] and [AS63]) cobordism invariance was one of the main ingredients. It has also direct consequences when applied to special Dirac operators. For instance, for the odd signature operator, cobordism invariance shows that the signature of an even-dimensional manifold vanishes if it is the boundary of a compact oriented manifold (cf. [Les92]).

3.1 Dirac Bundles on Manifolds with Boundary

Consider a Riemannian manifold (M, g). Let $\mathrm{Cl}(M) = \mathrm{Cl}(TM)$ denote the Clifford bundle of M and assume that $E \to M$ is a Dirac bundle (sometimes called Clifford module) in the following sense:

- E is a complex vector bundle with an hermitian structure $\langle ., . \rangle_E$.

- $c : \mathrm{Cl}(M) \to \mathrm{End}(E)$ is a bundle homomorphism of algebras such that $c(1) = \mathrm{Id}$. For $\psi \in E_p$, $v \in \mathrm{Cl}(M)_p$, $p \in M$, we will sometimes abbreviate $c(v)\psi$ by $v \cdot \psi$.
- c is unitary in the sense that $c(v)^* = -c(v)$ for any $v \in TM$.
- $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ is a metric connection such that $\nabla c = 0$, meaning that
$$\nabla_X (Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi$$
for all $X, Y \in C^\infty(M, TM)$, $\psi \in C^\infty(M, E)$.

Now, let Γ be a cooriented hypersurface of M, i.e. there exists a unit normal vector field $\nu \in C^\infty(\Gamma, TM|_\Gamma)$. Recall that the Clifford bundle of Γ is identified with the even part of the Clifford bundle of M restricted to Γ:

$$\mathrm{Cl}(\Gamma)|_q = \mathrm{Cl}^{\mathrm{ev}}(\Gamma)|_q \oplus \mathrm{Cl}^{\mathrm{odd}}(\Gamma)|_q \to \mathrm{Cl}^{\mathrm{ev}}(M)|_q, \quad v = v_0 + v_1 \mapsto v_0 + v_1 \cdot \nu,$$

where $v_0 \in \mathrm{Cl}^{\mathrm{ev}}(\Gamma)$, $v_1 \in \mathrm{Cl}^{\mathrm{odd}}(\Gamma)$. We can now define on $E' := E|_\Gamma$ a natural Dirac bundle structure. Since $\mathrm{Cl}(\Gamma) \hookrightarrow \mathrm{Cl}(M)|_\Gamma$ is an algebra homomorphism the action

$$c' : \mathrm{Cl}(\Gamma) \to \mathrm{End}(E'), \quad c'(v_0 + v_1) = c(v_0 + v_1 \cdot \nu)$$

defines a representation. Moreover,

$$c(v_0 + v_1 \cdot \nu)^* = c(v_0)^* + c(\nu)^* c(v_1)^* = -c(v_0) - c(v_1)c(\nu) = -c(v_0 + v_1 \cdot \nu),$$

so c' is unitary. Denote for a moment by ∇^M, ∇^Γ the Levi-Civita connection on M, resp. Γ. For all vector fields X, Y on Γ we have

$$\nabla^M_X Y - \nabla^\Gamma_X Y = \langle A_\nu(X), Y \rangle \nu$$

where $A_\nu : C^\infty(T\Gamma) \to C^\infty(T\Gamma)$ is the Weingarten map[1] w.r.t. ν. For $X \in C^\infty(\Gamma, T\Gamma)$ define

$$\nabla^{E'}_X \psi := \nabla^E_X \psi - \frac{1}{2} A_\nu(X) \cdot \nu \cdot \psi.$$

Proposition 3.1.1. *$\nabla^{E'}$ is a metric connection and $\nabla^{E'}$ is compatible with c'. Moreover, the clifford action of ν is $\nabla^{E'}$-parallel.*

Proof. That $\nabla^{E'}$ is metric follows from

$$\begin{aligned} X\langle \psi, \eta \rangle &= \langle \nabla^E \psi, \eta \rangle + \langle \psi, \nabla^E_X \eta \rangle \\ &= \langle \nabla^{E'} \psi, \eta \rangle + \langle \psi, \nabla^{E'}_X \eta \rangle \\ &\quad + \tfrac{1}{2}\big(\langle A_\nu(X) \cdot \nu \cdot \psi, \eta \rangle + \langle \psi, A_\nu(X) \cdot \nu \cdot \eta \rangle \big) \\ &= \langle \nabla^{E'} \psi, \eta \rangle + \langle \psi, \nabla^{E'}_X \eta \rangle, \end{aligned}$$

for $X, Y \in C^\infty(\Gamma, T\Gamma)$, $\psi, \eta \in C^\infty(M, E)$, since $\langle A_\nu(X), \nu \rangle = 0$.

To check that $\nabla^{E'}$ is compatible with c' we have to show that

$$\nabla^{E'}(Y \cdot \nu \cdot \psi) = \nabla^\Gamma_X Y \cdot \nu \cdot \psi + Y \cdot \nu \cdot \nabla^\Gamma_X \psi.$$

Since $\nabla^M \nu = -A_\nu(X)$ the left hand side yields

$$\nabla^M_X Y \cdot \nu \cdot \psi - Y \cdot A_\nu(X) \cdot \psi + Y \cdot \nu \cdot \nabla^E_X \psi - \tfrac{1}{2} A_\nu(X) \cdot Y \cdot \psi$$

[1] Recall that $A_\nu(X)$ is obtained by a metric type change from the second fundamental form which takes values in the trivial normal bundle of Γ.

whereas for the right hand side we obtain

$$\nabla^M_X Y \cdot \nu \cdot \psi + \langle A_\nu(X), Y\rangle \cdot \psi + Y \cdot \nu \cdot \nabla^E_X \psi - \tfrac{1}{2} Y \cdot A_\nu(X) \cdot \psi,$$

which proves compatibility for $-\frac{1}{2}A_\nu(X) \cdot Y = \frac{1}{2} Y \cdot A_\nu(X) + \langle A_\nu(X), Y\rangle$. For the last statement, observe that

$$\nabla^{E'}_X(c(\nu)\psi) = \nabla^E_X(c(\nu)\psi) + \tfrac{1}{2}c(A_\nu(X))\psi = -c(A_\nu(X))\psi + c(\nu)\nabla^E_X\psi + \tfrac{1}{2}c(A_\nu(X))\psi.$$

Since $-\frac{1}{2}c(A_\nu(X))\psi = -\frac{1}{2}c(\nu \cdot A_\nu(X) \cdot \nu)\psi$, it follows that $\nabla^{E'}_X(c(\nu)\psi) = c(\nu)\nabla^{E'}_X\psi$. □

To the Dirac bundles E and E' we associate the Dirac operators

$$D := \sum_{j=1}^n c(e_j)\nabla^E_j, \quad D^\Gamma := \sum_{j=2}^n c'(e_j)\nabla^{E'}_j, \tag{1.1}$$

where $(e_1, e_2, ..., e_n)$, $(e_2, ..., e_n)$ are local orthonormal frames of M, Γ, resp. Here, $\nabla_j := \nabla_{e_j}$.

Proposition 3.1.2. *Let $H : \Gamma \to \mathbb{R}$ denote the mean curvature of Γ w.r.t. ν. Then, on Γ, we have*

$$D = c(\nu)\big(\nabla_\nu + D^\Gamma - \tfrac{n-1}{2}H\big).$$

Proof. Let $(e_1, ..., e_n)$ be a local orthonormal frame such that $e_1|_\Gamma = \nu$. Then, on Γ

$$\begin{aligned} D\psi &= c(e_1)(\nabla_1 + \sum_{j=2}^n c(-e_1)c(e_j)\nabla_j)\psi \\ &= c(\nu)\big(\nabla_\nu + \sum_{j=2}^n c'(e_j)\nabla^{E'}_j + \tfrac{1}{2}\sum_{j=2}^n c(e_j)c(A_\nu(e_j))\big)\psi \end{aligned}$$

Now, since A_ν is symmetric,

$$\sum_{j=2}^n e_j \cdot A_\nu(e_j) = \sum_{i,j=2}^n \langle A_\nu(e_j), e_i\rangle e_j \cdot e_i = -\operatorname{Tr} A_\nu = -(n-1)H.$$

□

We call $D^\Gamma - \frac{n-1}{2}H$ the tangential part of D.

Let us now consider the $\mathbb{Z}_2$-graded case, i.e. assume $\alpha : E \to E$ is a bundle map such that $\alpha^2 = 1$ and $\nabla^{E\otimes E^*}\alpha = 0$ and $[\alpha, c(X)] = 0$ for all $X \in C^\infty(M, TM)$. Then $E = E^+ \oplus E^-$, where $E^\pm = \ker(\alpha \mp 1)$ and D is odd w.r.t. this splitting, i.e.

$$D\alpha + \alpha D = 0.$$

Then D decomposes into two parts

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E\mp),$$

and $(D^+)^t = D^-$. By

$$[\nabla^E_X, \alpha] = 0, \quad [c(X \cdot Y), \alpha] = 0,$$

if follows that $[D^\Gamma, \alpha] = 0$ and hence

$$D^\Gamma = \begin{pmatrix} D^{\Gamma,+} & 0 \\ 0 & D^{\Gamma,-} \end{pmatrix}$$

w.r.t. the splitting $E' = E'^+ \oplus E'^-$. Using ν we can define a natural isomorphism

$$E'^+ := E^+|_\Gamma \xrightarrow{\sim} E'^- = E^-|_\Gamma, \quad \psi \mapsto \nu \cdot \psi.$$

for ν anticommutes with α. Observe that

$$D^{\Gamma,-}(\nu \cdot \psi) = -\nu \cdot D^{\Gamma,+}\psi,$$

since $c(\nu)$ is $\nabla^{E'}$-parallel and $c(\nu)$ anti-commutes with $c'(X)$ for all $X \in T\Gamma$. Hence, in the graded case, D takes the form

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\nabla_1 + \begin{pmatrix} D^{\Gamma,+} - \frac{n-1}{2}H & 0 \\ 0 & -D^{\Gamma,+} - \frac{n-1}{2}H \end{pmatrix} \right). \tag{1.2}$$

Finally, let Ω be a manifold with smooth boundary Γ where Ω is viewed again as compact subset in M. We will now discuss canonical forms for a Dirac operator D on a tubular neighbourhood of Γ. For sufficiently small $\delta > 0$ and all $\varepsilon \in (-\delta, \delta)$

$$\Gamma(\varepsilon) = \{ p \in \Omega \mid x_1(p) = \varepsilon \}$$

is a smooth hypersurface in M. Moreover, using the geodesic flow through ν we can identify $\Gamma(\varepsilon)$ with Γ and by parallel transport along the corresponding geodesics we obtain isometries $E|_\Gamma \to E|_{\Gamma(\varepsilon)}$. Setting $M_\delta = \{ p \in M \mid d(p, \Gamma) < \delta \}$ we obtain

$$C^\infty(M_\delta, E) \cong C^\infty((-\delta, \delta), C^\infty(\Gamma, E|_\Gamma)).$$

∇_ν is now translated into ∂_1. Clearly, Proposition 3.1.2 applies to each $\Gamma(\varepsilon)$. Hence, for sections over Ω_ε we have

$$D = c(\nu)\big(\partial_1 + D^{\Gamma(x_1)} - \tfrac{n-1}{2} H(x_1)\big).$$

Here, $H(x_1)$ denotes the mean curvature of the hypersurface $\Gamma(x_1)$. Observe that the corresponding identification of sections over Γ and $\Gamma(\varepsilon)$ is not an L^2-isometry since the first variation of the volume need not vanish. However, if Ω_ε and the Dirac bundle E is of product type near Γ, then then the above formula for D simplifies to

$$D = c(\nu)\big(\partial_1 + D^\Gamma\big).$$

Given a Dirac operator D on M, we will now seek for interpretations of D^Γ in terms of natural operators on Γ.

3.1.1 The Gauß-Bonnet and Signature Operator

Consider the Dirac bundle of complex-valued differential forms $E = \Lambda^\bullet T^*M = \Lambda^\bullet_{\mathbb{C}} T^*M$, with the hermitian structure

$$\langle ., . \rangle_g : \Lambda^\bullet T^*M \times \Lambda^\bullet T^*M \to \mathbb{C},$$

the Levi-Cività connection on differential forms, and the Clifford multiplication

$$c(X)\omega = X^\flat \wedge \omega - \iota_X \omega, \quad \omega \in \Lambda^\bullet T^*M_p,\ X \in T_pM,\ p \in M,$$

where ι denotes inner multiplication. It is well-known (e.g. [LM89]) that the associated Dirac operator is $d + d^t$ where d denotes the exterior derivative. We will now relate $d + d^t$ to the corresponding operator on Γ, $(d + d^t)^\Gamma$.

Using the orthogonal projection $TM|_\Gamma \to T\Gamma$ we can embed $T^*\Gamma$ into T^*M and thus $\Lambda^\bullet T^*\Gamma$ into $\Lambda^\bullet T^*M|_\Gamma$. Moreover, there is a natural identification of $E' = \Lambda T^*M|_\Gamma$ with $\Lambda^\bullet T^*\Gamma \oplus \Lambda^\bullet T^*\Gamma$. Namely, we can define

$$\Lambda^\bullet T^*M|_\Gamma \to \Lambda^\bullet T^*\Gamma \oplus \Lambda^\bullet T^*\Gamma, \quad \omega_1 + \nu^\flat \wedge \omega_2 \ \mapsto \ i^*\omega_1 + i^*\omega_2.$$

where $\iota_\nu \omega_1 = \iota_\nu \omega_2 = 0$. Then, Clifford multiplication with ν is given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Unfortunately the Clifford connection $\nabla^{E'}$ constructed in the previous section does not necessarily coincide with the Levi-Cività connection on $\Lambda^\bullet T^*\Gamma$. For instance,

$$\tfrac{1}{2} c(A_\nu(X) \cdot \nu) f = -\tfrac{1}{2} f \nu \wedge A_\nu(X)^\flat,$$

for a function f and a vector field X on Γ. Instead, we have

$$\nabla^M_X \omega = \nabla^\Gamma_X \omega + \nu^\flat \wedge \ \iota_{A_\nu(X)} \omega,$$

for any $\omega \in C^\infty(\Gamma, \Lambda^\bullet T^*\Gamma)$, $X \in C^\infty(\Gamma, T\Gamma)$. Let $(e_1, ..., e_n)$ denote a local orthonormal frame of TM along Γ such that $e_1 = \nu$ and $e_2, ...e_n$ are principal curvature vectors of the hypersurface Γ, i.e.

$$A_\nu(e_i) = \lambda_i e_i,$$

where λ_i denote the principal curvatures of Γ. Moreover define for any tangent vector $v \in TM$ the absolute and relative projections

$$Q_a(v) := \mathrm{ext}_v \circ \iota_v, \quad Q_r(v) := \iota_v \circ \mathrm{ext}_v \,.$$

where ext_v denotes exterior multiplication by $v^\flat$. Then

$$\begin{aligned} \sum_{j=2}^n c(e_j) c(e_1) \nabla^M_j \omega_1 &= -e_1^\flat \wedge \sum_{j=2}^n c(e_j) \nabla^\Gamma_j \omega_1 - \sum_{j=2}^n c(e_j) \iota_{A_\nu(e_j)} \omega_1 \\ &= -e_1^\flat \wedge (d + d^t)^\Gamma \omega_1 - \sum_{j=2}^n \lambda_j Q_a(e_j) \omega_1, \end{aligned}$$

and

$$\begin{aligned} \sum_{j=2}^n c(e_j) c(e_1) \nabla^M_j (\nu^\flat \wedge \omega_2) &= -\sum_{j=2}^n c(e_j) \nabla^\Gamma_j \omega_2 + \sum_{j=2}^n c(e_j) c(e_1) (-A_\nu(e_j))^\flat \wedge \omega_2 \\ &= -(d + d^t)^\Gamma \omega_2 - e_1^\flat \sum_{j=2} \lambda_j Q_r(e_j) \omega_2, \end{aligned}$$

for $\omega_i \in C^\infty(\Gamma, \Lambda^\bullet T^*\Gamma$, $i = 1, 2$. Hence, at Γ, the *Hodge-de Rham operator*

$$d + d^t : C^\infty(M, \Lambda^\bullet T^*M) \to C^\infty(M, \Lambda^\bullet T^*M)$$

has the form

$$d + d^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\nabla_1 + \begin{pmatrix} -\sum_{j=2}^n \lambda_j Q_a(e_j) & -(d + d^t)^\Gamma \\ -(d + d^t)^\Gamma & -\sum_{j=2}^n \lambda_j Q_r(e_j) \end{pmatrix} \right). \tag{1.3}$$

There are two important $\mathbb{Z}_2$-gradings on the Dirac bundle $\Lambda^\bullet T^*M$. The first one, given by

$$\alpha\omega = (-1)^p \omega, \quad \omega \in \Lambda^p T^*M,$$

exists for all dimensions, and the corresponding even-to-odd part of $d + d^t$, $(d + d^t)^{\mathrm{ev}}$ is sometimes called the Gauß-Bonnet operator (for its index theorem is the (Chern-)Gauß-Bonnet index theorem). The even part of $\Lambda^\bullet T^*M|_\Gamma$ w.r.t. α is

$$\Lambda^{\mathrm{ev}} T^*M|_\Gamma \cong \Lambda^{\mathrm{ev}} T^*\Gamma \oplus \Lambda^{\mathrm{odd}} T^*\Gamma \cong \Lambda^\bullet T^*\Gamma.$$

Moreover, the even-to-odd part of $d + d^t$ is given by

$$\nabla_1 + \begin{pmatrix} -\sum_{j=2}^n Q_a(e_j)|_{\Lambda^{\mathrm{ev}} T^*\Gamma} & -(d + d^t)^{\Gamma,\mathrm{odd}} \\ -(d + d^t)^{\Gamma,\mathrm{ev}} & -\sum_{j=2}^n \lambda_j Q_r(e_j)|_{\Lambda^{\mathrm{odd}} T^*\Gamma} \end{pmatrix},$$

i.e.

$$(d + d^t)^{\mathrm{ev}} = \nabla_1 - (d + d^t)^\Gamma + R^\Gamma,$$

where R^Γ is a 0-th order symmetric operator given by the principal curvatures of Γ (as explained above).

Let now M be oriented and assume $n = 4k$, $k \in \mathbb{N}$. Define the complex volume element by

$$\tau = (-1)^k e_1 \cdots e_n,$$

where $(e_1, ..., e_n)$ is a local orthonormal frame of TM. Recall that τ is parallel w.r.t. the Levi-Cività connection on forms and that τ is related to the Hodge $\star$-operator by

$$\tau \cdot \varphi = (-1)^{k + p(p-1)/2} \star \varphi$$

for $\varphi \in \Lambda^p T^*M$ (see [LM89]). We set

$$\Lambda^\pm T^*M := \ker(\tau - \mp 1)$$

$d + d^t$ anticommutes with left multiplication by τ, i.e. D is odd w.r.t. the corresponding splitting

$$\Lambda^\bullet T^*M = \Lambda^+ T^*M \oplus \Lambda^- T^*M.$$

Over Γ a form

$$\omega = \omega_1 + dx_1 \wedge \omega_2$$

is invariant under τ if and only if

$$(-1)^k e_1 \cdots e_n \cdot \omega_1 = dx_1 \wedge \omega_2,$$

i.e. $\omega_2 = (-1)^k e_2 \cdots e_n \cdot \omega_1 = (-1)^{k + p(p+1)/2} \star_\Gamma \omega_1$, where $\star_\Gamma$ denotes the Hodge-$\star$ operator of Γ. It follows that

$$\Lambda^+ T^*M|_\Gamma \cong \Lambda^\bullet \Gamma \cong \Lambda^- T^*M|_\Gamma.$$

Now, over Γ we may write the so-called *signature* operator

$$(d + d^t)^+ : C^\infty(M, \Lambda^+ T^*M) \to C^\infty(M, \Lambda^- T^*M)$$

as in (1.2). The computations for $d + d^t$ in (1.3) imply that

$$(d + d^t)^+ = \nabla_1 + (-1)^{k + p(p+1)/2 + 1} (d + d^t)^\Gamma \star_\Gamma - \sum_{j=2}^n \lambda_j Q_a(e_j).$$

over Γ. Since Γ is $4k-1$-dimensional we find

$$d^{t,\Gamma}\omega = (-1)^{\deg\omega} \star_\Gamma d^\Gamma \star_\Gamma, \quad \star_\Gamma^2 = 1,$$

and therefore

$$(d+d^t)^+ = \nabla_\nu + B$$

where

$$B := (-1)^{k+p(p+1)/2+1}\big(d^\Gamma \star_\Gamma - (-1)^p \star_\Gamma d^\Gamma\big) - \sum_{j=2}^n \lambda_j Q_a(e_j).$$

Now, $C^\infty(\Gamma, \Lambda^{\mathrm{ev/odd}} T^*\Gamma)$ are invariant under B so that $B = B^{\mathrm{ev}} \oplus B^{\mathrm{odd}}$. Moreover, since B and $\star_\Gamma$ commute B^{ev} and B^{odd} are isomorphic.

As an example let us describe B^{ev} in case M is 4-dimensional. Moreover, assume for simplicity Γ is totally geodesic, which means that its Weingarten map vanishes. Identifying $\Lambda^1 T^*\Gamma$ and $\Lambda^2 T^*\Gamma$ via $\star_\Gamma$, it follows that

$$B^{\mathrm{ev}} : C^\infty(\Gamma, \Lambda^{0+1} T^*\Gamma) \to C^\infty(\Gamma, \Lambda^{0+1} T^*\Gamma), \quad B^{\mathrm{ev}}(f+\omega) = -d^t\tau - (df + \star d\omega),$$

where we have omitted the lower and upper Γ. Hence, up to sign B^{ev} equals the so-called *odd signature operator*.

Note that we have used the first-inward convention, i.e. $(\nu, e_2, ..., e_n)$ is oriented in M when $(e_2, ..., e_n)$ is oriented in Γ and ν is considered as the inward pointing unit vector field. If M is even-dimensional then considering ν as the *last* vector means reversing the orientation on Γ, and thus reversing the sign of $\star_\Gamma$. Working with outward unit normal fields means subsituting ∇_ν for $-\nabla_\nu$ and thus reversing the sign of the whole tangential part and the orientation on Γ.

3.1.2 The Spin Dirac Operator

Let (M, g) be an oriented spin manifold with spin structure

$$\xi : P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M).$$

There is a natural embedding $P_{\mathrm{SO}}(\Gamma) \hookrightarrow P_{\mathrm{SO}}(M)|_\Gamma$ by inserting ν at the first position. Restricting ξ to the preimage of $P_{\mathrm{SO}}(\Gamma)$ gives a spin structure on Γ. Recall that the Clifford bundle of Γ is identified with the even part of the Clifford bundle of M restricted to Γ:

$$\mathrm{Cl}(\Gamma)|_q = \mathrm{Cl}^{\mathrm{ev}}(\Gamma)|_q \oplus \mathrm{Cl}^{\mathrm{odd}}(\Gamma)|_q \to \mathrm{Cl}^{\mathrm{ev}}(M)|_q, \quad v = v_0 + v_1 \mapsto v_0 + v_1 \cdot \nu.$$

Recall also that a complex spinor bundle over the spin manifold M is simply a complex vector bundle $\Sigma M \to M$ together with a (smoothly varying) complex irreducible representation

$$c : \mathrm{Cl}(TM)_p \to \mathrm{End}_{\mathbb{C}}(\Sigma M)_p$$

at each point $p \in M$. There is an embedding of the complex spinor bundle(s) $\Sigma\Gamma$ over Γ into the spinor bundle(s) ΣM over M which we will now make precise. Note that one has to distinguish between the even and odd-dimensional case. In the following $(e_1, ..., e_n)$ will always denote a local oriented orthonormal frame of TM, i.e. a section in $C^\infty(U, P_{\mathrm{SO}}(M))$ for some open $U \subset M$.

a) Let $n = \dim M = 2k$. Then (up to isomorphisms) the only complex spinor bundle is

$$\Sigma M := P_{\mathrm{Spin}}(M) \times_c \Delta.$$

where $\Delta = \Lambda^\bullet_{\mathbb{C}}\mathbb{C}^k$ and c is the complex spin representation given by the restriction of

$$c : \mathrm{Cl}(\mathbb{R}^n) \to \mathrm{End}(\Delta), \quad c(v)\alpha := -\iota_{v^\flat}\alpha + v \wedge \alpha, \quad v \in \mathbb{R}^n \cong \mathbb{C}^k.$$

The complex volume element $\tau = i^k e_1 \cdot \ldots \cdot e_n$ defines an involution which gives rise to the splitting

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M.$$

Here, $\Sigma^\pm M$ denote the ± 1 eigenspaces of τ. All vectors $v \in TM$ anticommute with τ. Since elements in $\mathrm{Cl}^{\mathrm{ev}}(M)$ commute with τ we obtain two representations

$$c : \mathrm{Cl}(\Gamma)_q \cong \mathrm{Cl}^{\mathrm{ev}}(M)_q \to \mathrm{End}_{\mathbb{C}}(\Sigma^\pm M)_q.$$

In fact, $\Sigma^+ M|_\Gamma$ and $\Sigma^- M|_\Gamma$ are the two different spinor bundles over Γ, since $i^k c'(e_2 \cdots e_n)^2 = \tau$.

b) Let $n = 2k+1$ and let ΣM be one of the two spinor bundles on M. Set $\Sigma\Gamma := \Sigma M|_\Gamma$. Again, $\mathrm{Cl}(\Gamma) \cong \mathrm{Cl}^{\mathrm{ev}}(M)|_\Gamma$ and this way $\Sigma\Gamma$ becomes an irreducible module over $\mathrm{Cl}(\Gamma)$. Since $\nu^2 = -1$, Clifford action on $\Sigma\Gamma$ by ν gives rise to a splitting into the $\pm i$ eigenspaces of ν. Setting $\Sigma^\pm\Gamma = \ker(\nu \pm i)$ (or vice versa) we obtain the two possible irreducible complex graded representations of the module $\mathrm{Cl}(\Gamma)$:

$$\mathrm{Cl}(\Gamma) \to \mathrm{End}_{\mathbb{C}}(\Sigma^+ M \oplus \Sigma^- M), \quad (v, \psi) \mapsto v \cdot \nu \cdot \psi.$$

Let us now compare the spin connections on Γ and M. Denote by ∇^M, ∇^Γ the Levi-Cività connection on M, resp. Γ.

Proposition 3.1.3. *Let $\nabla^{s,M}$, $\nabla^{s,\Gamma}$ be the spin connections on M, Γ, resp. Then*

$$(\nabla^{s,M}_X - \nabla^{s,\Gamma}_X)\psi = \frac{1}{2} A_\nu(X) \cdot \nu \cdot \psi$$

Proof. Let $\mathscr{E} = (e_1, \ldots, e_n)$ be a local oriented orthonormal frame of $U \subset M$ and let $\mathscr{F} \in C^\infty(U, P_{\mathrm{Spin}}(M))$ be one of the two lifts to the spin structure. Consider a spinor of the form $\psi(p) = (\mathscr{F}(p), \alpha) \in C^\infty(U, \Sigma M)$ where $\alpha \in \Delta$.

We have the following formula for the spin connection (cf. [LM89, Chap. II, Thm. 4.14]):

$$\nabla^{s,M}_X \psi = \frac{1}{2} \sum_{i<j} \langle \nabla_X e_i, e_j \rangle e_i \cdot e_j \cdot \psi.$$

Assume that $U \cap \Gamma \neq 0$ and that $e_1 = \nu$ on Γ. The analogous formula for $\nabla^{s,\Gamma}$ gives

$$\begin{aligned}
(\nabla^{s,M}_X - \nabla^{s,\Gamma}_X)\psi =& \frac{1}{2} \sum_{2 \le i < j} \langle (\nabla^M_X - \nabla^\Gamma_X) e_i, e_j \rangle e_i \cdot e_j \cdot \psi \\
&+ \frac{1}{2} \sum_{j=2}^n \langle \nabla^M_X e_1, e_j \rangle e_1 \cdot e_j \cdot \psi \\
=& 0 + \frac{1}{2} \sum_{j=2}^n \langle \nabla^M_X e_j, e_1 \rangle e_j \cdot e_1 \cdot \psi \\
=& \frac{1}{2} \sum_{j=2}^n \langle A_\nu(X), e_j \rangle e_j \cdot e_1 \cdot \psi = \frac{1}{2} A_\nu(X) \cdot \nu \cdot \psi.
\end{aligned}$$

Any spinor $\psi \in \Sigma M_p$ can be extended to a spinor of the form above and this proves the claimed formula since the difference of two connections is a bundle homomorphism. $\square$

In the even-dimensional case $n = 2k$ w.r.t. the splitting $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ We obtain

$$D^\Gamma = \begin{pmatrix} D^{\Gamma,+} & 0 \\ 0 & D^{\Gamma,-} \end{pmatrix},$$

where $D^{\Gamma,\pm}$ are the two spin Dirac operators on Σ. By $\psi \mapsto \nu \cdot \psi$ we may identify in a natural way $\Sigma^+ M|_\Gamma$ with $\Sigma^- M|_\Gamma$ for ν anticommutes with τ. Observe that

$$D^{\Gamma,-}(\nu \cdot \psi) = -\nu \cdot D^{\Gamma,+}\psi.$$

In this sense we obtain $D^{\Gamma,-} = -D^{\Gamma,+}$ Clearly, w.r.t. the identification $\Sigma^+ M \cong \Sigma^- M$ (near Γ), $c(\nu)$ itself takes the form

$$\nu \cdot \psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

Now, let $n = 2k + 1$ and let $\Sigma^\pm \Gamma_p = \ker(c(\nu) \mp i)$. D^Γ anticommutes with $c(\nu)$ since $c(\nu)$ is $\nabla^{s,\Gamma}$-parallel. Hence,

$$D^\Gamma = \begin{pmatrix} 0 & D^{\Gamma,-} \\ D^{\Gamma,+} & 0 \end{pmatrix}$$

Since $n - 1$ is even and $i^k c(\nu \cdot e_2 \cdots e_n) = \pm 1$ we have

$$i^k c'(e_2 \cdots e_n) = i^k c(e_2 \cdots e_n) = \pm i^k c(\nu),$$

for a local orthonormal oriented frame $(e_2, ..., e_n)$. Therefore $\Sigma\Gamma = \Sigma^+\Gamma \oplus \Sigma^-\Gamma$ is exactly the natural splitting of the spinor bundle into positive and negative spinors over the even dimensional manifold Γ. W.r.t. its own eigenspace decomposition $c(\nu)$ decomposes

$$c(\nu) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Theorem 3.1.4. *Over Γ the spin Dirac operator on M, D, may be decomposed as follows:*

(i) *If $n = 2k$, then $\Sigma M|_\Gamma = \Sigma^+ M|_\Gamma \oplus \Sigma^- M|_\Gamma \cong \Sigma^+\Gamma \oplus \Sigma^+\Gamma$ and*

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\nabla_\nu + \begin{pmatrix} D^\Gamma & 0 \\ 0 & -D^\Gamma \end{pmatrix} - \tfrac{n-1}{2} H \right).$$

where $\pm D^\Gamma$ are the two Dirac operators associated to the induces spin structure on Γ.

(ii) *If $n = 2k + 1$ then $\Sigma M|_\Gamma \cong \Sigma^+\Gamma \oplus \Sigma^-\Gamma$ and*

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\nabla_\nu + \begin{pmatrix} 0 & D^{\Gamma,-} \\ D^{\Gamma,+} & 0 \end{pmatrix} - \tfrac{n-1}{2} H \right).$$

where $D^{\Gamma,\pm} : C^\infty(\Gamma, \Sigma^\pm\Gamma) \to C^\infty(\Gamma, \Sigma^\mp\Gamma)$ is the positive (negative) part of the spin Dirac operator associated to the induced spin structure on Γ w.r.t. the decomposition into positive and negative spinors.

Observe that in (ii) the ambiguity in choosing a spinor bundle on the odd-dimensional manifold M is reflected in the ambiguity in choosing a $\mathbb{Z}_2$-grading of $\Sigma\Gamma$ into positive and negative spinors. Namely, switching to the other complex spin representation means swapping $\Sigma^+\Gamma$ and $\Sigma^-\Gamma$.

3.2 The Calderón Projection of a First Order Elliptic Operator

Let

$$D : C^\infty(M,E) \to C^\infty(M,F)$$

be a first order elliptic operator and let $\Omega \subset M$ be a compact subset with smooth boundary. Consider the tubular neighbourhood of the boundary

$$M_\delta = \{p \in M \mid d(p,\Gamma) < \delta\}.$$

Define $\pi : \{p \in \Omega \mid d(p,\Gamma) < \varepsilon_0\} \to \Gamma$ by $(x_1,x') \mapsto (0,x')$. Identify E and F near Γ with $\pi^*(E|_\Gamma)$ and $\pi^*(F|_\Gamma)$. Then, D can be written in the form

$$D = -J(x_1)\left(\frac{d}{dx_1} + A(x_1)\right)$$

where $A(x_1) : C^\infty(\Gamma,E') \to C^\infty(\Gamma,F')$ is a smooth family of elliptic first order operators and $J(x_1)$ is a smooth family of bundle isomorphisms from E to F. The minus sign is not accidentally chosen for

$$\langle Du,v\rangle_{L^2(\Omega,F)} - \langle u,Dv\rangle_{L^2(\Omega,F)} = \langle J(0)u|_\Gamma, v|_\Gamma\rangle_{L^2(\Gamma,F)},$$

which is consistent with the notation of the previous chapters. Since $d=1$ the formulas for the Calderón projection and its symbol considerably simplify. Let R be a parametrix of

$$\frac{d}{dx_1} + A(x_1)$$

on an open neighbourhood of Γ in M. Note that

$$\langle J(x_1)^{-1}(\gamma_M^0)^* J(0)g, v\rangle = \langle J(0)g, J(0)^{-1}\gamma_M^0 v\rangle,$$

hence $J(x_1)^{-1}(\gamma_M^0)^*J(0) = (\gamma_M^0)^*$. From Theorem 2.3.5 (and Remark 2.3.4) it follows that C_+ is a compact perturbation of

$$\gamma^0 r_+ R J^{-1}(\gamma_M^0)^* J(0) = \gamma^0 r_+ R(\gamma_M^0)^*.$$

By Corollary 2.3.8 we obtain

$$\widehat{C}_+(x',\xi') = i\sum_{\operatorname{Im} w>0} \operatorname{Res}_w (iz + \widehat{A}(0,x',\xi'))^{-1} \tag{2.1}$$

where $\widehat{A}(0,x',\xi')$ denotes the principal symbol of $A(0)$. Consider the decomposition into Jordan blocks of $\widehat{A}(0,x',\xi')$.

Proposition 3.2.1. *Let $S \in \operatorname{End}(V)$ where V is some finite-dimensional vector space and assume $\operatorname{spec} S \cap i\mathbb{R} = \emptyset$. Denote by $E(\lambda)$ the generalised eigenspace of S to the eigenvalue λ. Then*

$$i\sum_{\operatorname{Im} w>0} \operatorname{Res}_w (iz+S)^{-1} = \mathbf{1}_{\{\operatorname{Re} z>0\}}(S),$$

where $\mathbf{1}_{\{\operatorname{Re} z>0\}}(S)$ denotes the projection $(v_+,v_-) \mapsto (v_+,0)$ w.r.t. the decomposition

$$V = \Big(\bigoplus_{\operatorname{Re}\lambda>0} E(\lambda)\Big) \oplus \Big(\bigoplus_{\operatorname{Re}\lambda\le 0} E(\lambda)\Big).$$

Proof. Let us consider one Jordan block i.e. the restriction of S to one invariant subspace V_0 associated to the eigenvalue λ. Then

$$i(iz+S)^{-1}|_{V_0} = \begin{pmatrix} (z-i\lambda)^{-1} & -(z-i\lambda)^{-2} & 0 & \cdots & 0 \\ 0 & (z-i\lambda)^{-1} & -(z-i\lambda)^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (z-i\lambda)^{-1} \end{pmatrix},$$

w.r.t. a suitable basis. Hence, for any $v \in V_0$, we have

$$i\,\mathrm{Res}_w(iz+S)^{-1}v = \begin{cases} v, & w = i\lambda \\ 0, & \text{otherwise.} \end{cases}$$

□

(2.1) together with the above proposition now gives the following characterisation of the Calderón projection.

Theorem 3.2.2. *The Calderón projection of an elliptic first order differential operator D which has the form*

$$D = -J(x_1)\big(\partial_1 + A(x_1)\big)$$

on a collar of the boundary is a pseudodifferential projection whose principal symbol is given by

$$\mathbf{1}_{\{\mathrm{Re}\, z>0\}}(\widehat{A}(0)).$$

Lemma 3.2.3. *If $A(x_1) : C^\infty(\Gamma, E') \to C^\infty(\Gamma, E')$ is a smooth family of differential operators such that $\partial_1 + A(x_1) : C^\infty((-\varepsilon,\varepsilon)\times\Gamma, E') \to C^\infty((-\varepsilon,\varepsilon)\times\Gamma, E')$ is elliptic, then $A_0 = A(0)$ is elliptic and has discrete spectrum. Moreover, there exists $\delta > 0$ such that*

$$(0,\delta)\times i\mathbb{R} \;\cap\; \mathrm{spec}\, A_0 \;=\; \emptyset$$

Proof. Since $\partial_1 + A(0)$ is elliptic it follows that

$$i\lambda + \widehat{A}_0(\xi')$$

is invertible for all $\lambda \in \mathbb{R}$, $\xi' \in T^*\Gamma$. Hence, the spectrum of $\widehat{A}_0(\xi')$ is contained in the complement of a fixed cone

$$\Lambda := \big\{z \in \mathbb{C} \;\big|\; \pi/2 - \mu < \arg z < \pi/2\big\}, \quad \mu > 0,$$

for all $\xi' \in T^*\Gamma$, $\|\xi'\| = 1$. Since $i\lambda + \widehat{A}_0(\xi')$ is an elliptic differential operator with parameter $\lambda \in i\Lambda$. It follows (see [Shu80, Sec. 9.3]) that the resolvent set of A_0 contains

$$\Lambda_R = \Lambda \cap \{|z| > R\}$$

for some $R > 0$. In particular $\rho(A_0) \neq \emptyset$. Since $A_0 : H^1(\Gamma, E) \to L^2(\Gamma, E)$ is compact, A_0 has discrete spectrum. Moreover,

$$\inf\big\{r > 0 \;\big|\; r = \mathrm{Re}\,\lambda, \quad \lambda \in \mathrm{spec}\, A(0)\big\}$$

is positive. □

Consider the special case where A_0 is self-adjoint in $L^2(\Gamma, E)$. Then, the spectral projection

$$\mathbf{1}_{>0}(A_0)$$

is a classical pseudodifferential projection. Namely, if we set $T_0 := A_0(\operatorname{Id} + A_0^2)^{-1/2}$, then

$$\mathbf{1}_{>0}(A_0) = \mathbf{1}_{>0}(T_0) = \tfrac{1}{2\pi i} \int_\gamma (\lambda - T_0)^{-1} d\lambda,$$

where $\gamma : S^1 \to \mathbb{C}$ is smooth closed path such that for the winding number $\operatorname{wind}(\gamma, z)$ of a point $z \in \sigma(T_0)$ we have

$$\operatorname{wind}(\gamma, z) = \begin{cases} 1, & \text{if } \operatorname{Re} z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Such a curve γ exists since the spectrum of T_0 accumulates only at -1 and 1.

A similar definition of $\mathbf{1}_{\{\operatorname{Re} z>0\}}(A_0)$ is possible when A_0 is merely normal, in the sense that

$$A_0 A_0^* = A_0^* A_0$$

both sides considered as operators from $H^2(\Gamma, E)$ to $L^2(\Gamma, E)$. Then, T_0 has to be defined by $A_0(\operatorname{Id} + A_0^* A_0)^{-1/2}$. Observe that in both cases the whole spectrum is mapped into the disk $\mathbb{D} \subset \mathbb{C}$ and the spectrum has only finitely many points inside the cone Λ, as defined in the previous lemma. In particular,

$$\overline{\{z \in \operatorname{spec} T_0 \mid \operatorname{Re} z > 0\}} \cap \{\operatorname{Re} z \leq 0\} = \emptyset$$

Note that by [Shu80, Sec. II.11] T_0 is a classical pseudodifferential operator since $\operatorname{Id} + A_0^* A_0$ is positive. So the Cauchy integral above yields a classical pseudodifferential operator.

Theorem 3.2.4. *Let D be a first order elliptic operator with normal principal symbol such that on a collar of Γ we have $D = -J(x_1)(\partial_1 + A(x_1))$. The Calderón projection of D differs from the spectral projection $\mathbf{1}_{\{\operatorname{Re} z>0\}}(A(0))$ merely by a classical pseudodifferential operator of order -1.*

Proof. The principal symbol of $\mathbf{1}_{\{\operatorname{Re} z>0\}}(A_0)$ is given by

$$\tfrac{1}{2\pi i} \int_\gamma (\lambda - \widehat{T_0}(\xi'))^{-1} d\lambda, = \mathbf{1}_{\{\operatorname{Re} z>0\}}(\widehat{T_0}(\xi')).$$

Now,

$$\mathbf{1}_{\{\operatorname{Re} z>0\}}(\widehat{T_0}(\xi')) = \mathbf{1}_{\{\operatorname{Re} z>0\}}(\widehat{A_0}(\xi')).$$

Thus we have shown that the principal symbols of C_+ and $\mathbf{1}_{\{\operatorname{Re} z>0\}}$ coincide. ☐

3.3 The Calderón Projection of a Dirac Operator

The results of the previous section combined with the computations of the tangential parts now yield interpretations of the Calderón projection of the classical Dirac operators. By Theorem 3.2.4 and Proposition 3.1.2 the Calderón projection of a Dirac operator D associated to a Dirac bundle on M equals

$$\mathbf{1}_{>0}(D^\Gamma),$$

up to an operator in $\Psi^{-1}_{\mathrm{cl}}(\Gamma, E)$. We have seen that the principal symbol of the positive spectral projection only depends on the principal symbol of D^Γ. That is why we may drop the additional curvature terms.

In the $\mathbb{Z}_2$-graded case we have seen that

$$D^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$$

takes the form

$$-\partial_1 + D^{\Gamma(x_1),+} - \tfrac{n-1}{2} H(x_1),$$

over a collar of the boundary. Thus $C_+(D^+) \equiv \mathbf{1}_{>0}(D^{\Gamma,+}) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, E)$. For the Hodge-de Rham, Gauß-Bonnet and signature operator we obtain:

Theorem 3.3.1. *For $\omega \in C^\infty(\Omega, \Lambda^\bullet T^*\Omega)$ we have the decomposition $\gamma^0\omega = \omega_1 + \nu^\flat \wedge \omega_2$ with $\omega_1, \omega_2 \in C^\infty(\Gamma, \Lambda^\bullet T^*\Gamma)$.*

(i) *Let* sign : $\mathbb{R} \to \mathbb{R}$ *denote the sign function.*

$$C_+(d + d^t) \equiv \frac{1}{2}\begin{pmatrix} \mathrm{Id} & -\operatorname{sign}((d+d^t)^\Gamma) \\ -\operatorname{sign}((d+d^t)^\Gamma) & \mathrm{Id} \end{pmatrix} \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Lambda^\bullet T^*\Gamma \oplus \Lambda^\bullet T^*\Gamma).$$

(ii) *When $\omega \in \Lambda^{\mathrm{ev}} T^*\Gamma$ then $\gamma_0\omega$ is uniquely determined by $\omega_1 + \omega_2 \in \Lambda^\bullet T^*\Gamma$.*

$$C_+((d + d^t)^{\mathrm{ev}}) \equiv \mathbf{1}_{>0}((d + d^t)^\Gamma) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Lambda^\bullet T^*\Gamma).$$

(iii) *Recall that if $\tau\omega = \omega$, then $\gamma^0\omega$ is uniquely determined by ω_1. We have*

$$C_+((d + d^t)^+) = \mathbf{1}_{>0}(B) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Lambda^\bullet T^*\Gamma),$$

where

$$B\omega_1 = (-1)^{k+p(p+1)/2+1}(d^\Gamma \star_\Gamma - (-1)^p \star_\Gamma d^\Gamma)\omega_1,$$

for $\omega_1 \in C^\infty(\Gamma, \Lambda^p T^*\Gamma)$.

Proof. Observe that for a self-adjoint Fredholm operator $A : H \supset \mathscr{D}(A) \to H$ we have

$$\mathbf{1}_{\geq 0}\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \operatorname{sign} A \\ \operatorname{sign} A & \mathrm{Id} \end{pmatrix}.$$

Hence, the three statements are just special cases of Theorem 3.2.4. □

From the decomposition of the spin Dirac operator near the boundary (Theorem 3.1.4) we ontain:

Theorem 3.3.2. *Denote by $D : C^\infty(\Omega, \Sigma\Omega) \to C^\infty(\Omega, \Sigma\Omega)$ the spin Dirac operator and by D^Γ the spin Dirac operator on Γ. If n is even, then D^Γ is to denote the Dirac operator on the spinor bundle such that $i^k c'(e_2 \cdots e_n) = 1$ for an oriented orthonormal frame $(e_2, \cdots, e_n)$.*

(i) *Let $n = 2k$. Recall that $\Sigma\Omega|_\Gamma = \Sigma\Gamma \oplus \Sigma\Gamma$. We have*

$$C_+(D) \equiv \mathbf{1}_{>0}(D^\Gamma) \oplus \mathbf{1}_{<0}(D^\Gamma) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Sigma^\Gamma \oplus \Sigma^\Gamma),$$

and

$$C_+(D^\pm) \equiv \mathbf{1}_{\gtrless 0}(D^\Gamma) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Sigma^\Gamma).$$

(ii) *Let $n = 2k + 1$.*

$$C_+(D) \equiv \mathbf{1}_{>0}(D^\Gamma) \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, \Sigma^\Gamma).$$

3.4 Cobordism Invariance of the Index

We make some general observations on symplectic functional analysis first. In infinite dimensions one has to distinguish between strong and weak symplectic structures.

Let $(H, \langle .,. \rangle, \omega)$ be a (weakly) symplectic complex Hilbert space, i.e.

$$\omega : H \times H \to \mathbb{C}$$

is a bounded non-degenerate sesquilinear form on H. Here, non-degeneracy means that the bounded operator $J : H \to H$ defined by

$$\langle Jx, y \rangle_H = \omega(x, y)$$

is injective. If J is an isomorphism, then $(H, \langle .,. \rangle, \omega)$ is called a strongly symplectic Hilbert space (cf. [CM74]).

From now on we assume that ω is a strong symplectic structure. We will now address the following questions:

(i) Is any isotropic subspace contained in a Lagrangian?

(ii) Are there Lagrangian subspaces at all?

There may be no non-trivial isotropic subspaces at all. For instance, if $\omega(x, y) = \langle ix, y \rangle$, then $\omega(x, y) = -\overline{\omega(y, x)}$ but

$$\omega(x, x) \neq 0$$

for all $x, y \in H$. In particular, there exists no Lagrangian.

In finite dimensions such pathologic examples could be excluded by requiring that the signature of the hermitian form iJ vanishes. To cover the infinite-dimensional case as well we could just make the existence of a Lagrangian subspace part of the definition. However, even in this case the problem raised in (i) is non-trivial.

Example 3.4.1. Let $H = \ell^2 \oplus \ell^2$ and let $S : \ell^2 \to \ell^2$ be an isometric operator which is not surjective, e.g.

$$Se_j = e_{2j},$$

where $(e_j)_{j \in \ell^2}$ is the standard Hilbert basis of ℓ^2. Define

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda := \big\{ (x, Sx) \bigm| x \in \ell^2 \big\},$$

and set $\omega(v, w) = \langle Jv, w \rangle$ for $v, w \in H$. It follows that

$$\lambda^\omega = \big\{ (S^* y, y) \bigm| y \in \ell^2 \big\}.$$

Then $\lambda \subsetneq \lambda^\omega$, but $\lambda^\perp = \big\{ (S^* y, -y) \bigm| y \in \ell^2 \big\}$, hence

$$\lambda^\omega \cap \lambda^\perp = \{0\} \oplus (\operatorname{ran} S)^\perp.$$

Since $\omega(v, v) \neq 0$ for any non-zero $v \in \lambda^\omega \cap \lambda^\perp$, λ is maximally isotropic but not Lagrangian. Clearly, H has Lagrangian subspaces, e.g. the diagonal $\{(x, x) | \; x \in \ell^2\}$.

Proposition 3.4.2. *If* (λ_1, λ_2) *is a left Fredholm pair of isotropic subspaces, then* $\operatorname{ind}(\lambda_1, \lambda_2) \leq 0$.

Proof. This follows from the inclusion

$$\lambda_1 \cap \lambda_2 \ \subset \ \lambda_1^\omega \cap \lambda_2^\omega \ = \ (\lambda_1 + \lambda_2)^\omega \tag{4.1}$$

for $\operatorname{codim}(\lambda_1 + \lambda_2) = \dim(\lambda_1 + \lambda_2)^\omega$ whenever (λ_1, λ_2) is a left-Fredholm pair. $\square$

Note that

$$\operatorname{codim}(\lambda_1 + \lambda_2) = \operatorname{codim} J(\lambda_1 + \lambda_2) = \dim(J(\lambda_1 + \lambda_2))^\perp = \dim(\lambda_1 + \lambda_2)^\omega$$

is only valid if $J(\lambda_1 + \lambda_2)$ is closed and if J is bijective. Here, we need that (λ_1, λ_2) is a left Fredholm pair and that J is an isomorphism, i.e. (H, ω) is *strongly* symplectic.

Proposition 3.4.3. *Let λ_0 be Lagrangian and λ an isotropic subspace and assume (λ_0, λ) is a left Fredholm pair.*

(i) *If* $\operatorname{ind}(\lambda_0, \lambda) < 0$*, then λ is not maximally isotropic.*

(ii) *If* $\operatorname{ind}(\lambda_0, \lambda) = 0$*, then λ is Lagrangian.*

Proof. (i) The corresponding inclusion (4.1) is now strict, i.e.

$$\lambda \cap \lambda_0 \subsetneq \lambda^\omega \cap \lambda_0^\omega = \lambda^\omega \cap \lambda_0.$$

We may thus take some $x \in \big(\lambda^\omega \cap \lambda_0\big) \setminus \big(\lambda \cap \lambda_0\big)$ and define

$$\widetilde{\lambda} := \lambda \oplus \operatorname{span}\{x\}.$$

Since $x \in \lambda_0$ we have $\omega(x, x) = 0$ and hence

$$\omega(u + \zeta_1 x, v + \zeta_2 x) = 0$$

for all $u, v \in \lambda$, $\zeta_1, \zeta_2 \in \mathbb{C}$. It follows that $\widetilde{\lambda}$ is an isotropic extension of λ. $\square$

Corollary 3.4.4. *Assume λ_0 is a Lagrangian subspace. Then, every maximally isotropic subspace $\lambda \in \mathscr{F}(\lambda_0)$ is Lagrangian.*

Let us now apply the theory of strongly symplectic Hilbert spaces to $L^2(\Gamma, \Sigma\Gamma)$ where Γ is the boundary of an odd-dimensional spin manifold.

Observe that in the odd-dimensional case, the Dirac operator on the boundary is odd w.r.t. the splitting $\Sigma\Gamma = \Sigma^+\Gamma \oplus \Sigma^-\Gamma$ induced by Clifford action of ν (or, equivalently, by the complex volume element on Γ, τ_Γ). As for any even-dimensional spin manifold, the spin Dirac operator on Γ thus takes the form

$$D^\Gamma = \begin{pmatrix} 0 & D^{\Gamma,-} \\ D^{\Gamma,+} & 0 \end{pmatrix}.$$

Clearly, $D^{\Gamma,\pm}$ are elliptic operators on a compact manifold and hence Fredholm. The following theorem states that $\operatorname{ind} D^{\Gamma,+} = -\operatorname{ind} D^{\Gamma,-} = 0$.

More generally, let $D : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ be a first order formally self-adjoint elliptic operator such that D takes the form

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\frac{d}{dx_1} + \begin{pmatrix} 0 & A^*(x_1) \\ A(x_1) & 0 \end{pmatrix} \right) + V(x_1) \tag{4.2}$$

over a collar of the boundary. Here, $E = E_+ \oplus E_-$ near the boundary and $A(x_1) : C^\infty(\Gamma, E_+) \to C^\infty(\Gamma, E_-)$ is a smooth family of elliptic differential operators and $V(x_1)$ is a smooth family of bundle endomorphisms. For instance, as we have seen in Section 3.1 such a decomposition holds for any twisted Dirac operator on a an odd-dimensional spin manifold with boundary. Let $(W, \langle .,. \rangle_W, \nabla^W)$ denote an auxiliary bundle and set $E = \Sigma\Omega \otimes W$ (with the induced hermitian structure, affine connection etc.). Near the boundary there is a natural splitting

$$E = E_+ \oplus E_- := (\Sigma^+\Omega \otimes W) \oplus (\Sigma^-\Omega \otimes W)$$

so that the twisted Dirac operator $D = \sum_{j=0}^{2k+1} e_j \cdot \nabla_j^{\Sigma\Omega\otimes W}$ takes the form (4.2).

Theorem 3.4.5 (Cobordism invariance of the index). $\operatorname{ind} A(0) = 0$.

Proof. Set

$$D^\Gamma = \begin{pmatrix} 0 & A^*(0) \\ A(0) & 0 \end{pmatrix}, \quad J = -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We have $\Sigma^\pm\Gamma = \ker(iJ \mp 1)$ and therefore

$$\ker A(0) = \ker(iJ - 1)\big|_{\ker D^\Gamma}, \quad \operatorname{coker} A(0) = \ker(iJ + 1)\big|_{\ker D^\Gamma}.$$

In other words, it suffices to show that

$$\operatorname{sign} iJ\big|_{\ker D^\Gamma} = 0.$$

Since $(\ker D^\Gamma, J|_{\ker D^\Gamma})$ is a finite-dimensional (complex) symplectic Hilbert space, this is equivalent to the existence of a Lagrangian subspace.

Set $H = L^2(\Gamma, E) = L^2(\Gamma, E_+ \oplus E_-)$. As usual, define the symplectic structure $\omega(x, y) = \langle Jx, y \rangle$. Note that $\Lambda_{<0} := \ker \mathbf{1}_{\geq 0}(D^\Gamma)$ is an isotropic subspace of (H, ω). Namely, D^Γ anticommutes with J. Therefore J maps the $\ker \mathbf{1}_{\geq 0}(D^\Gamma)$ onto $\ker \mathbf{1}_{\leq 0}(D^\Gamma)$ and vice versa.

Consider a maximally isotropic extension of $\Lambda_{<0}$:

$$\Lambda' = \Lambda_{<0} \oplus_\perp \lambda.$$

Then $\lambda \subset \ker D^\Gamma$ for

$$\lambda \subset \big(\Lambda_{<0} \oplus_\perp \lambda\big)^\omega \subset \big(\Lambda_{<0}\big)^\omega = \ker \mathbf{1}_{>0}(D^\Gamma).$$

Let $\Lambda_0 := \operatorname{ran}(C_+ : L^2(\Gamma, E) \to L^2(\Gamma, E))$. Since C_+ is a compact perturbation of $\mathbf{1}_{>0}(D^\Gamma)$, it follows that (Λ_0, Λ') is a Fredholm pair. By Corollary 3.4.4, we infer that Λ' is Lagrangian. We have

$$\Lambda_{<0} \oplus_\perp \lambda = \Lambda' = (\Lambda')^\omega = \big(\Lambda_{<0}\big)^\omega \cap \lambda^\omega = \ker \mathbf{1}_{>0}(D^\Gamma) \cap \lambda^\omega = \Lambda_{<0} \oplus_\perp \big(\lambda^\omega \cap \ker D^\Gamma\big).$$

Hence, λ is a Lagrangian subspace of $\big(\ker D^\Gamma, \omega|_{\ker D^\Gamma}\big)$. $\square$

Chapter 4

Operators of Laplace Type

We study boundary value problems for Laplace type operators, i.e. differential operators whose principal symbol is given by the metric tensor. We are mainly interested in the formally self-adjoint case, though it would take little effort to generalise most of our statements to the case of a general Laplace type operator.

Our first aim is to show that, over a collar neighbourhood of the boundary, one can transform any Laplace operator into one of the form $-\partial_1^2 + \Delta'(x_1)$. We determine exact formulas for the corresponding transformation of boundary conditions.

After translating the general theory of boundary value problems established so far to the special case of Laplace operators we study in detail some of the most prominent examples, in particular Dirichlet and (modified) Neumann conditions. Some features of Dirichlet and Neumann realisations will be used in the theory of more general boundary conditions for Laplacians. For surfaces with boundary we give a proof of a theorem due to Vekua which determines the index of a boundary value problem given by an arbitrary vector field along the boundary. Then absolute and relative boundary conditions are discussed. We prove Hodge decomposition for manifolds with boundary and show how elements of the relative and absolute (rational) cohomology are represented by harmonic forms.

Moreover, we discuss the Dirichlet-to-Neumann operator which associates to given Dirichlet data the corresponding Neumann data. We use our results on the Calderón projection of a Laplacian to show that the Dirichlet-to-Neumann operator is in fact a classical pseudodifferential operator. Moreover, considering $\Delta + \mu^2$ instead of Δ, we obtain a classical parameter dependent pseudodifferential operator (in the sense of [Shu80]).

Whereas the results for Laplacians mentioned so far are either classical or well-known (cf. e.g. [Gru96], [Gru03]) in the last section we tackle the problem of characterising all "good" boundary value problems for Laplacians within the framework of Assumption 1.2.1. Here, a boundary condition is considered a "good" one if it gives rise to a well-posed, self-adjoint and semi-bounded extension. Observe that the solvability of the initial value problem given by the heat equation

$$(\partial_t + \Delta)u = 0, \quad u|_{t=0} = u_0$$

and a boundary condition for Δ depends, above all, on the semi-boundedness of the corresponding realisation.

When determining the numerical range of a well-posed realisation of a Laplacian Lemma 4.5.6 together with the simple observation that Dirichlet and Friedrichs' extension of a Laplacian coincide turns out to be very useful. This was already observed by Grubb [Gru68, Chapter II].

As a consequence we obtain a criterion for semi-boundedness of self-adjoint realisations in terms of the boundary condition and the Dirichlet-to-Neumann operator, $T_{DN,\mu}$, for the operator $\Delta + \mu^2$. However this is of practical relevance only after the principal behaviour of this parameter dependent operator for large μ has been studied. The principal symbol of $T_{DN,\mu}$ equals, up to a minus sign, that of the square root of the Laplace resolvent on the boundary, as shown by Grubb ([Gru03]). Here, this is deduced from a refined analysis of the Calderón projection of $\Delta + \mu^2$, similar to that in [Gru99]. After reducing a general well-posed self-adjoint boundary condition to the form

$$\Pi_1\gamma^0 u = 0,$$
$$\Pi_2(\gamma^1 u + G\gamma^0 u) = 0,$$

with Π_1, Π_2 projections and G a self-adjoint operator, we are finally able to express lower-boundedness in terms of G and Π_2.

The fact that there exist self-adjoint Fredholm realisations whose spectrum is unbounded below has an interesting consequence. Recall that, by Theorem 2.5.4, the mapping

$$\mathscr{F}\mathscr{L}(\Lambda_0) \to \mathscr{C}\mathscr{F}^{sa}, \quad \Lambda \mapsto \Delta_\Lambda,$$

parametrises all self-adjoint Fredholm extensions of $\Delta_{\min}$, where

$$\mathscr{D}(\Delta_\Lambda) = \big\{u \in \mathscr{D}(\Delta_{\max}) \;\big|\; \widetilde{\rho}^d u \in \Lambda\big\},$$

and $\mathscr{C}\mathscr{F}^{sa}$ denotes the space of all possibly unbounded self-adjoint Fredholm operators. Recall that in Section 2.6 we introduced a topology on $\mathscr{F}\mathscr{L}(\Lambda_0)$ such that the above map was continuous into the space $\mathscr{C}\mathscr{F}^{sa}$, endowed with the gap metric. Since the Riesz topology is striclty finer than the gap topology it is an interesting question whether continuity also holds w.r.t. the Riesz metric, see e.g. [Les05], where the corresponding question for Dirac operators is raised. Now, the Riesz topology is so strong that the Fredholm realisations with spectrum unbounded below lie in a different connected component than the semi-bounded ones. Since, by [BL01, Sec. 3], the pseudodifferential Fredholm-Lagrange Grassmannian of Λ_0, as a subspace of all orthogonal projections, is connected, we see that at least for Laplace operators continuity w.r.t. the Riesz metric cannot hold unless $\mathscr{F}\mathscr{L}(\Lambda_0)$ is given a strictly finer topology.

Assume that $d = 2$ and that Δ is a formally self-adjoint operator of *Laplace type*, i.e. Δ has the following form (in local coordinates and trivialisations of E):

$$\Delta = -\sum_{i,j} g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + 1^{\mathbf{st}} \text{ order terms.}$$

Recall that this means that for all $f \in C^\infty_{\mathbb{R}}(M)$ we have

$$\sigma^L(\Delta)(df) = -\tfrac{1}{2}[[\Delta, f], f]\big|_p = \|df\|_g^2 \cdot \mathrm{Id}_E\,.$$

Observe that

$$-[[\Delta, f_1], f_2] = 2\langle \operatorname{grad} f_1, \operatorname{grad} f_2\rangle \tag{1}$$

which can be seen as follows. By the Jacobi identity both sides define symmetric bilinear forms on $C^\infty_{\mathbb{R}}(M)$. Since Δ is of Laplace type they coincide whenever $f_1 = f_2$. Polarisation gives the identity in general.

4.1 Reduction to the Model Cylinder

Recall that an affine connection

$$\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$$

extends to the bundle $T^*M \otimes \cdots \otimes T^*M \otimes E$ by requiring the Leibniz-rule $\nabla(\omega \otimes v) = \nabla\omega \otimes v + \omega \otimes \nabla v$ for $\omega \in C^\infty(M, T^*M)$, $v \in C^\infty(M, T^*M \otimes \cdots \otimes T^*M \otimes E)$. $\nabla\omega$ is the Levi-Civita connection on 1-forms, i.e. $\nabla\omega = \left(\nabla\omega^\sharp\right)^\flat$.

To ∇ one can associate the *Bochner Laplacian* acting on $C^\infty(M, E)$,

$$\Delta^\nabla = -\mathbf{C}_{12} \circ \nabla^{T^*M\otimes E} \circ \nabla^E.$$

Here, $\mathbf{C}_{12}$ denotes metric contraction of the first two slots, i.e.

$$\mathbf{C}_{12} : T^*M \otimes T^*M \otimes E \to E, \quad \xi \otimes \eta \otimes u \mapsto \sum_{i,j} g^{ij} \cdot \xi_i \cdot \eta_j \cdot u.$$

When ∇ is unitary, i.e. compatible with the hermitian structure, then we find $\Delta^\nabla = \nabla^*\nabla$. Namely,

$$\nabla^* = -\mathbf{C}_{12} \circ \nabla^{T^*M\otimes E}.$$

To see this let $\omega \in \Omega^1(M)$, $u, v \in C^\infty_c(M, E)$ have compact support in a contractible open set where we chose a local orthonormal frame $(e_j)_{1\le j\le n}$ for TM and set $\nabla_j = \nabla_{e_j}$. Then

$$\begin{aligned}
\int_M \langle -\mathbf{C}_{12} \circ \nabla(\omega \otimes u), v\rangle \operatorname{vol}_g &= -\sum_j \int_M \langle \nabla_j(\omega \otimes u), e_j^\flat \otimes v\rangle \operatorname{vol}_g \\
&= \sum_j \int_M \Big(\langle \omega \otimes u, \nabla_j(e_j^\flat \otimes v)\rangle - e_j\langle \omega \otimes u, e_j^\flat \otimes v\rangle\Big) \operatorname{vol}_g \\
&= \int_M \langle \omega \otimes u, \nabla v\rangle \operatorname{vol}_g + \sum_j \int_M \Big(\omega(\nabla_j e_j)\langle u, v\rangle - (d^t e_j^\flat)\omega(e_j)\langle u, v\rangle\Big) \operatorname{vol}_g .
\end{aligned}$$

Since $\sum_j (d^t e_j^\flat)\omega(e_j) = \sum_j \omega(\nabla_j e_j)$ the above formula for ∇^* follows.

Observe that one has

$$[\Delta^\nabla, f]u = (\Delta_0 f)u - 2\nabla_{\operatorname{grad} f} u \tag{1.1}$$

for all $f \in C^\infty(M)$ and $u \in C^\infty(E)$. Here, Δ_0 denotes the Laplace-Beltrami operator on functions.

Theorem 4.1.1 (Weitzenböck formula). *Let $\Delta : C^\infty(M, E) \to C^\infty(M, E)$ be a formally self-adjoint operator of Laplace-type. Then there exists a unique unitary connection $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ such that*

$$\Delta = \Delta^\nabla + H$$

where H is a symmetric bundle endomorphism.

Proof. If ∇ exists then (1.1) also holds for Δ since H commutes with any function. One can use (1.1) to define a connection. Namely, for $X \in T_pM$, $p \in M$ let

$$(\nabla_X u) := \frac{1}{2}\Big(-[\Delta, f] + (\Delta_0 f)\Big)u(p)$$

where $f \in C^\infty(M)$ is any function such that $\operatorname{grad} f(p) = X$. When f_1, f_2 are two such functions then there exist $c \in \mathbb{R}$, $m \in C^\infty(M)$ such that $f_1 = f_2 + c + m^2$ and $m(p) = 0$. It follows that

$$\begin{aligned}\Big(-[\Delta, f_1 - f_2] + \Delta_0(f_1 - f_2)\Big)\Big|_p &= \Big(-[[\Delta, m], m] + \Delta_0(m^2)\Big)\Big|_p \\ &= 2\|\operatorname{grad} m(p)\|^2 - [\Delta_0, m^2]\big|_p = 0\end{aligned}$$

since Δ_0 has principal symbol of metric type. It follows that ∇ is well-defined.

Using (1) we see that for any $\varphi \in C^\infty(M)$

$$[\nabla_{\operatorname{grad} f}, \varphi] = -\tfrac{1}{2}[[\Delta, f], \varphi] = \langle \operatorname{grad} f, \operatorname{grad} \varphi \rangle.$$

Therefore, ∇ is an affine connection. By (1.1) and the definition of ∇ we have $[\Delta^\nabla - \Delta, \varphi] = 0$ for any $\varphi \in C^\infty(M)$. We infer that $\Delta^\nabla - \Delta$ is a bundle homomorphism.

It remains to show that ∇ is compatible with the hermitian structure. For all $f \in C_c^\infty(M, \mathbb{R})$, $u, v \in C_c^\infty(M, E)$, we have

$$\int_M \big(\langle \nabla_{\operatorname{grad} f} u, v\rangle + \langle u, \nabla_{\operatorname{grad} f} v\rangle\big) \operatorname{vol}_g = \int_M (\Delta_0 f)\langle u, v\rangle \operatorname{vol}_g = \int_M (\operatorname{grad} f)\langle u, v\rangle \operatorname{vol}_g,$$

since $[\Delta, f]$ is skew-symmetric. Now, let $p \in M$, $X \in T_pM$ and choose $f \in C_c^\infty(M, \mathbb{R})$ such that $\operatorname{grad} f(p) = X$. Furthermore, let $(\varphi_n)_{n \in \mathbb{N}} \subset C_c^\infty(M)$ be a Dirac series that converges to δ_p. Replacing v by $\varphi_n v$ yields

$$\begin{aligned}\int_M \varphi_n \langle \nabla_{\operatorname{grad} f} u, v\rangle + \langle u, \nabla_{\operatorname{grad} f} \varphi_n v\rangle &= \frac{1}{2}\int_M 2(\Delta_0 f)\langle u, \varphi_n v\rangle \operatorname{vol}_g \\ &= \int_M (\operatorname{grad} f)\langle u, \varphi_n v\rangle \operatorname{vol}_g\end{aligned}$$

and thus

$$\begin{aligned}\langle \nabla_X u, v\rangle + \langle u, \nabla_X v\rangle + \lim_{n\to\infty}\int_M \langle \operatorname{grad} f, \operatorname{grad}\varphi_n\rangle\langle u, v\rangle \operatorname{vol}_g = \\ X\langle u, v\rangle + \lim_{n\to\infty}\int_M \langle \operatorname{grad} f, \operatorname{grad}\varphi_n\rangle\langle u, v\rangle \operatorname{vol}_g.\end{aligned}$$

after taking the limit $n \to \infty$. It follows that ∇ is compatible with the hermitian structure on E. □

Remark 4.1.2. The first part of the proof shows that the statement remains valid when Δ is of Laplace type but not necessarily formally self-adjoint. However, the induced connection ∇ would in general not be compatible with the metric and H would no longer be symmetric.

Let us give a formula for the Bochner-Laplacian of a compatible connection in terms of local coordinates. Let $(U, \psi = (x_1, ..., x_n))$ be a chart of M and denote $\nabla_j = \nabla_{\frac{\partial}{\partial j}}$.

Suppose that (U,ψ) is oriented. When u,v are smooth sections of E with compact support in U and $\xi \in C^\infty(M,T^*M)$ then

$$\begin{aligned}\int_M \langle \nabla^*(\xi\otimes u), v\rangle \operatorname{vol} &= \int_{\mathbb{R}^n} \langle \xi\otimes u, \nabla v\rangle \sqrt{g}dx_1\cdots dx_n\\ &= \sum_j \int_M \langle \xi, dx_j\rangle \langle u, \nabla_j v\rangle \sqrt{g} dx_1\cdots dx_n\\ &= \sum_{i,j}\int_{\mathbb{R}^n} \xi_i g^{ij}\big(\langle -\nabla_j u, v\rangle + \partial_j\langle u,v\rangle\big)\sqrt{g}dx_1\cdots dx_n\\ &= \int_M \langle -\nabla_{\xi^\sharp} u, v\rangle \operatorname{vol} - \int_{\mathbb{R}^n}\langle u,v\rangle\big(\frac{1}{\sqrt{g}}\sum_{i,j}\partial_j(\xi_i g^{ij}\sqrt{g})\big)\sqrt{g}dx_1\cdots dx_n.\end{aligned}$$

Now, $-\frac{1}{\sqrt{g}}\sum_{i,j}\partial_i(\xi_i g^{ij}\sqrt{g}) = d^t\xi$. We conclude

$$\nabla^*(\xi\otimes u) = -\nabla_{\xi^\sharp}u + d^t\xi\cdot u.$$

For the Bochner-Laplacian we obtain

$$\begin{aligned}\nabla^*\nabla u &= \sum_j \nabla^*(dx_j\otimes\nabla_j u)\\ &= \sum_j\big(-\nabla_{dx_j^\sharp}\nabla_j u + (d^t dx_j)\nabla_j u\big) \qquad (1.2)\\ &= \sum_j\big(-\nabla_{dx_j^\sharp}\nabla_j u + \Delta_0 x_j\nabla_j u\big).\end{aligned}$$

In particular, when (U,ψ) is normal at p, then (1.2) yields

$$\nabla^*\nabla = -\sum_j \nabla_j^2.$$

Now, we study Δ on the collar $M_\delta = \{p\in M \mid d(p,\Gamma)<\delta\}$. The geodesic flow through ν induces a diffeomorphism of $M_\delta \cong (0,\delta)\times\Gamma$. Let $\pi:\Omega_\delta\to(-\delta,\delta)\times\Gamma$ be the natural projection onto the boundary, i.e. $\pi(x_1,q)\mapsto q$.

In the following we compare $E|_{M_\delta}$ with $\pi^*(E|_\Gamma)$. Via parallel transport along the geodesics through ν, i.e. $t\mapsto(t,q)$, ∇ gives rise to a bundle isomorphism $\wp:\pi^*(E|_\Gamma)\to E|_{M_\delta}$. Since ∇ is compatible, this identification preserves the hermitian structure. We can transform sections in $C^\infty(M_\delta,E)$ to sections in $C^\infty(M_\delta,\pi^*(E|_\Gamma))$ accordingly:

$$\Psi_1 : u(x_1,q)\mapsto \widetilde{u}(x_1,q) := \wp^{-1}u(x_1,q).$$

Let $\varphi: M_\delta\to\mathbb{R}_+$ be defined by

$$\varphi^2(x_1,q) = \frac{\operatorname{vol}_\Omega(x_1,q)}{dx_1\wedge\pi^*(\operatorname{vol}_\Gamma)(x_1,q)}.$$

Using φ we obtain an isometry

$$\Psi: L^2(M_\delta, E|_{M_\delta})\to L^2((-\delta,\delta),L^2(\Gamma,E')),\quad u\mapsto\varphi\cdot\Psi_1 u.$$

Namely,

$$\begin{aligned}\|\Psi u\|^2_{L^2((\delta,\delta),L^2(\Gamma,E'))} &= \int_{M_\delta}\varphi^2(x_1,q)\|\wp^{-1}(u(x_1,q))\|^2_{E'_q}dx_1\wedge\operatorname{vol}_\Gamma\\ &= \int_{M_\delta}\|u(x_1,q)\|^2_{E_{(x_1,q)}}\operatorname{vol}_\Omega = \|u\|^2_{L^2(M_\delta,E)}.\end{aligned}$$

Partial differential operators over M_δ are transformed by Ψ into operators on the model space $(0,\delta)\times\Gamma$ by $(\Psi_* P)u := \Psi P \Psi^{-1}u$. E.g. for a vector field $X\in C^\infty(M_\delta)$ we obtain $(\Psi_*\nabla_X) = \nabla_X - \varphi^{-1}(X\varphi)$. Since Ψ preserves the L^2-structure Ψ_* preserves formal self-adjointness.

Assume now that, as above, ∇ is a compatible connection on E. Let us now compute $\Psi_*\Delta^\nabla$ by using (1.2). Let $x_2,...,x_n$ be arbitrary coordinates of Γ. By $x_j := x_j\circ\pi$ we may extend x_j for $j=2,...,n$ to coordinates on M_δ. Note that $g_{1j}=g^{1j}=\delta_{1j}$ since ∂_1 is orthogonal to the submanifolds $\Gamma(\varepsilon)=\{x_1(p)=\varepsilon\}$.

In view of (1.2) let us compute $\Delta_0 x_1$:

$$\Delta_0 x_1 = -\tfrac{1}{\sqrt{g}}\sum_{i,j}\partial_i\big(\sqrt{g}g^{ij}\partial_j x_1\big) = -\tfrac{1}{\sqrt{g}}\partial_1\sqrt{g}.$$

On the other hand, we have $\mathrm{vol}_\Gamma(x_1,q) = \sqrt{g(x_1,q)}\,dx_1\wedge\cdots\wedge dx_n = \varphi^2(x_1,q)\sqrt{g(0,q)}\,dx_1\wedge\cdots\wedge dx_n$ from which we deduce that $\partial_1\varphi^2(x_1,x') = -\varphi^2(x_1,x')\cdot\Delta_0 x_1$ and hence

$$\Delta_0 x_1 = -\tfrac{\partial_1\varphi^2(x_1,x')}{\varphi^2(x_1,x')} = -2\tfrac{\partial_1\varphi}{\varphi}(x_1,x').$$

It follows that $\nabla^*\nabla$ has the form

$$-\nabla_1^2 - \tfrac{2\partial_1\varphi}{\varphi}\nabla_1 + P(x_1),$$

where $P(x_1): C^\infty(\Gamma,E')\to C^\infty(\Gamma,E')$ is a smoothly varying family of differential operators. So, in order to compute the x_1-derivatives of $\Psi_*(\nabla^*\nabla)$, we have to consider

$$M_\varphi\Big(-\nabla_1^2 - \tfrac{2\partial_1\varphi}{\varphi}\nabla_1\Big)M_\varphi^{-1}$$

where M_φ denotes multiplication by φ. We have

$$-M_\varphi\nabla_1^2 M_\varphi^{-1} = -\nabla_1^2 + 2M_{\varphi^{-1}\partial_1\varphi}\nabla_1 + M_{-2\varphi^{-2}(\partial_1\varphi)^2+\varphi^{-1}\partial_1^2\varphi},$$

and

$$M_\varphi\Big(-\tfrac{2\partial_1\varphi}{\varphi}\nabla_1\Big)M_\varphi^{-1} = -2M_{\varphi^{-1}\partial_1\varphi}\nabla_1 + M_{2\varphi^{-2}(\partial_1\varphi)^2}.$$

Note that $\Psi_1\nabla_1\Psi_1^{-1}=\partial_1$. Moreover,

$$\partial_1\varphi = -\tfrac12(\Delta_0 x_1)\varphi = -\tfrac12(n-1)H^\Gamma(x_1)\varphi$$

where $H^\Gamma(\varepsilon)$ denotes the mean curvature of $\Gamma(\varepsilon)=\{x_1(p)=\varepsilon\}$. Namely,

$$\begin{aligned}\Delta_0 x_1 = d^t dx_1 &= -\sum_{i,j=2}^n g^{ij}\langle\nabla_i dx_1, dx_j\rangle\\ &= \sum_{i,j=2}^n \langle\nabla_i dx_j, dx_1\rangle g^{ij} = \mathrm{Tr}\,A_\nu = (n-1)H^\Gamma(x_1).\end{aligned}$$

It follows that

$$\varphi^{-1}\partial_1^2\varphi = -\tfrac12(n-1)\partial_1 H^\Gamma(x_1) + \tfrac14(n-1)^2H^\Gamma(x_1)^2.$$

Set $\Delta'(x_1) = M_\varphi P(x_1)M_{\varphi^{-1}} + M_{-\frac12(n-1)H^\Gamma(x_1)+\frac14(n-1)^2H^\Gamma(x_1)^2}$. We finally reach

Proposition 4.1.3. *Let $\Delta = \nabla^*\nabla + H$ be a formally self-adjoint operator of Laplace type and suppose that Ψ is constructed as above. Then*

$$\Psi_*\Delta = -\frac{\partial^2}{\partial x_1^2} + \Delta'(x_1)$$

where $\Delta'(x_1)$ is a smooth family of differential operators

$$\Delta'(x_1) : C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma).$$

Each $\Delta'(x_1)$ is formally self-adjoint and its principal symbol equals

$$g^{ij}(t,q)\xi_i'\xi_j' \cdot \mathrm{Id}_E$$

in local coordinates $(x_2, ..., x_n)$ at $q \in \Gamma$, $\xi' = \sum_{j=2}^{n} \xi_j' dx_j$. In particular, $\Delta'(0)$ has principal symbol given by the metric tensor on Γ.

Example 4.1.4 *(The Laplacian on the Euclidean ball).* Consider the open ball $B = \{p \in \mathbb{R}^n \mid \|p\| < 1\}$ as a manifold with boundary $S = S^{n-1}$. For $q \in S^{n-1}$, $x_1 \in (0,1)$ we identify $(x_1, q) \approx (1 - x_1)q$. With respect to these coordinates we can write the Euclidean Laplace operator in the form

$$\Delta = -\frac{\partial^2}{\partial x_1^2} + \frac{n-1}{1-x_1}\frac{\partial}{\partial x_1} + \frac{1}{(1-x_1)^2}\Delta_S, \tag{1.3}$$

where Δ_S denotes the Laplacian on the unit sphere S^{n-1}. The dilatation φ is given by

$$\varphi(x_1, q) = (1 - x_1)^{\frac{n-1}{2}}$$

since $dx_1 \wedge ... \wedge dx_n = r^{n-1}dr \wedge \mathrm{vol}_S$. Let Ψ be defined as above. Hence,

$$\Psi : L^2(B^n \setminus \{0\}) \longrightarrow L^2((0,1) \times S).$$

Noting that $H^\Gamma(x_1) = (1 - x_1)^{-1}$ and that parallel transport is trivial here, we obtain

$$\Psi_*\Delta = -\partial_1^2 + \tfrac{1}{(1-x_1)^2}\Delta_S + \tfrac{(n-1)(n-3)}{4}\tfrac{1}{(1-x_1)^2}.$$

4.2 Regularity and Well-posedness

Assume we are given a semi-homogeneous boundary value problem for Δ of the form

$$\Delta u = v, \quad B\widetilde{\rho}^2 u = 0,$$

where B is subject to Assumption 1.2.1. Near the boundary we can transform this system by Ψ. Over the collar the new boundary value problem reads

$$(\Psi_*\Delta)\Psi u = 0, \quad B\widetilde{\rho}^2(\varphi^{-1}\Psi u) = 0,$$

which is equivalent to

$$\big(\tfrac{\partial^2}{\partial x_1^2} + \Delta'(x_1)\big)\Psi u = 0, \quad B'\widetilde{\rho}^2\Psi u = 0.$$

Let H^Γ denote the mean curvature of Γ. Since

$$\gamma^1\varphi^{-1}u = -\varphi^2\partial_1\varphi|_\Gamma\gamma^0 u + \gamma^1 u = \tfrac{n-1}{2}H^\Gamma\gamma^0 u + \gamma^1,$$

B' is given by

$$B' = B\Phi_2\begin{pmatrix}\mathrm{Id} & 0\\ \frac{n-1}{2}H^\Gamma & \mathrm{Id}\end{pmatrix}\Phi_2^{-1}.$$

Since $\Phi^{-1/2}H^\Gamma\Phi^{-1/2}, \in \Psi^{-1}(\Gamma, E')$, B' is a perturbation of oder -1 of B. In particular B' satisfies Assumption 1.2.1 if and only if B does. For convenience, we may assume that B' satisfies Assumption 1.2.3, i.e. B' is supposed to be a projection. One of the advantages of transforming our boundary value problem by Ψ is that the symplectic structure on $W(\Psi_*\Delta)$ becomes highly simple. Namely, for $u, v \in C_c^\infty([0,\delta), C^\infty(\Gamma, E'))$ we have

$$\langle(\Psi_*\Delta)u, v\rangle - \langle u, (\Psi_*\Delta)v\rangle = \langle\widetilde{J}\widetilde{\rho}^d u, \widetilde{\rho}^d v\rangle$$

where

$$\widetilde{J} = \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}.$$

A boundary condition for Δ is now a matrix $B = \left(\begin{smallmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{smallmatrix}\right)$. Let

$$B_0 = \frac{1}{2}\begin{pmatrix}\mathrm{Id} & -\mathrm{Id}\\ -\mathrm{Id} & \mathrm{Id}\end{pmatrix}$$

be the projection onto the diagonal $\{x = -y\}$. In fact B_0 differs from the true Calderón projector by a pseudodifferential operator of order -1 thus by a compact operator.

Theorem 4.2.1. *The principal symbol of the Calderón projection is given by*

$$\widehat{C}_+(\Delta) = \frac{1}{2}\begin{pmatrix}\mathrm{Id} & -\mathrm{Id}\\ -\mathrm{Id} & \mathrm{Id}\end{pmatrix}.$$

Proof. This is a special case of the statement in Corollary 2.3.8. Since $g = dx_1^2 + g'(x_1)$, the principal symbol of P at $x_1 = 0$ is given by

$$\|\xi\|^2 = \xi_1^2 + \|\xi'\|^2,$$

hence

$$q_{-2}(0, x', z, \xi') = \frac{1}{z^2 + \|\xi'\|^2}.$$

This function has one pole in the upper half-plane at $z = \mathbf{i}\|\xi'\|$, $\mathbf{i}$ denoting the square root of -1, and

$$(-\|\xi'\|)^{i+j}\operatorname{Res}_{\mathbf{i}\|\xi'\|}\frac{1}{z^2 + \|\xi'\|^2} = \frac{1}{2\mathbf{i}}(-1)^{i+j}\|\xi'\|^{i+j-1}.$$

For the principal symbol of C_+ this yields

$$\widehat{C}_+(\xi') = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \qquad \square$$

In order to check regularity or well-posedness of a boundary condition B' one only has to check the (semi-)Fredholm property for the pair (B_0, B'). Note also, that for this purpose it suffices to consider the pair (B_0, B). We summarize these observations in the following theorem, using of the one-to-one correspondence between boundary conditions B for Δ and B' for $\Psi_*\Delta$.

Theorem 4.2.2. *Let B be a boundary condition for Δ satisfying Assumption 1.2.3.*

(i) Δ_B *is formally self-adjoint if and only if* $\ker B'$ *is isotropic in* $(L^2(\Gamma, E' \oplus E'), \omega)$ *where* ω *is the standard symplectic form*

$$\omega(g,h) = \langle \widetilde{J} g, h\rangle = \langle g_1, h_2\rangle - \langle g_2, h_1\rangle .$$

(ii) *The adjoint condition of B is given by* $(B^{\mathrm{ad}})' = -\widetilde{J}(\mathrm{Id} - B')\widetilde{J}$.

(iii) *B is regular for Δ if and only if (B_0, B) is a left-Fredholm-pair, i.e. $B : \operatorname{ran} B_0 \to$ $\operatorname{ran} B$ is a left-Fredholm operator.*

(iv) *B is well-posed for Δ if and only if (B_0, B) is a Fredholm-pair.*

(v) *Δ_B is self-adjoint if and only if $B' \in \mathscr{F}\mathscr{L}(B_0)$, the latter denoting the Fredholm-Lagrange Grassmannian of B_0.*

Proof. Since $C_+(\Delta)$ is compact perturbation of B_0 this follows immediately from Theorem 2.1.4, Proposition 2.5.5 and Theorem 2.5.4. □

Let us study pseudodifferential boundary value problems now. Assume $S_1 \in \Psi^s(\Gamma, E, F)$, $S_2 \in \Psi^{s-1}(\Gamma, E, F)$ and consider the boundary condition

$$S_1\gamma^0 + S_2\gamma^1 u = 0.$$

Suppose

$$\begin{pmatrix} S_1 & S_2\Phi \end{pmatrix}$$

is surjectively elliptic. E.g. this is the case when S_1 or S_2 are surjectively elliptic. Then the boundary condition is equivalent to $B\widetilde{\rho}^d u = 0$, where

$$B = \begin{pmatrix} S_1\Phi^{-1/2} & S_2\Phi^{1/2} \end{pmatrix}.$$

B has closed range and satisfies Assumption 1.2.1.

Proposition 4.2.3. *Let S_1, S_2 be as above. Then $S_1\gamma^0 u + S_1\gamma^1 u = 0$ is a well-posed boundary condition for Δ if and only if $S_1 - S_2\Phi$ is elliptic, i.e. if*

$$\widehat{S}_1(\xi') - \widehat{S}_2(\xi')\|\xi'\| : E_p \to F_p,$$

is invertible for all $\xi' \in T_p\Gamma$, $p \in \Gamma$.

Proof. By Theorem 2.1.4 we have to check for all $\xi' \in T^*\Gamma$ if

$$\widehat{B}(\xi') : \operatorname{ran} \widehat{C}_+(\xi') \to \operatorname{ran} \widehat{B}(\xi')$$

is an isomorphism. Since $\begin{pmatrix} S_1 & S_2\Phi \end{pmatrix}$ is surjectively elliptic, it follows that $\widehat{B}(\xi')$ is surjective. Moreover,

$$\operatorname{ran} \widehat{C}_+(\xi') = \left\{ (u, -u) \;\middle|\; u \in E_{\pi(\xi')} \right\}.$$

Hence, B is well-posed for Δ if and only if

$$\widehat{S}_1(\xi')\|\xi'\|^{-1/2} - \widehat{S}_2(\xi')\|\xi'\|^{1/2} : E_{\pi(\xi')} \to F_{\pi(\xi')}$$

is an isomorphism for all $\xi' \in T^*\Gamma$. □

4.3 Examples of Boundary Value Problems for Laplacians

4.3.1 Dirichlet and Neumann Boundary Conditions

The Dirichlet boundary condition is given by the projection

$$B'_{Dir} = B_{Dir} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $B'_{Dir} \in \mathscr{F}\mathscr{L}(B_0)$, the corresponding realisation is a regular self-adjoint extension of $\Delta_{\min}$ (as is well-known). Since

$$\Delta_{\min} = (\nabla^*\nabla + H)_{\min} \geq C := -\|H\|$$

one can also apply Friedrichs' extension process (cf. [Fri34]) to $\Delta_{\min}$. More precisely, we have

$$\langle(\Delta + C + 1)u, u\rangle = \langle\nabla u, \nabla u\rangle + \langle(H + C + 1)u, u\rangle.$$

Thus $u \mapsto \langle(\Delta + C + 1)u, u\rangle^{1/2}$ induces a norm on $\mathscr{D}(\Delta_{\min})$ which is equivalent to $\|.\|_{H^1(\Omega,E)}$. The completion of $\mathscr{D}(\Delta_{\min})$ yields $H^1_0(\Omega, E)$. Let T denote the embedding

$$T : H^1_0(\Omega, E) \to L^2(\Omega, E).$$

TT^* is a bounded injective self-adjoint map with dense range (since $\overline{\operatorname{ran} TT^*} = \ker T^{**}T^* = \ker TT^* = \{0\}$). Its inverse is given by

$$\mathscr{D}((TT^*)^{-1}) = \big\{u \in H^1_0(\Omega, E) \bigm| u \in \operatorname{ran} T^*\big\}, \quad (TT^*)^{-1}u = (\Delta + C + 1)u.$$

Now, $u \in \operatorname{ran} T^*$ if and only if there exists $w \in L^2(\Omega, E)$ such that

$$\langle(\Delta + C + 1)u, v\rangle = \langle w, u\rangle.$$

In other words, $\Delta^F = (TT^*)^{-1} - C - 1$ is a self-adjoint extension of $\Delta_{\min}$ with domain

$$\mathscr{D}(\Delta^F) = \big\{u \in H^1_0(\Omega, E) \bigm| (\Delta + C + 1)u \in L^2(\Omega, E)\big\} = \mathscr{D}(\Delta_{Dir}),$$

by the regularity of the Dirichlet condition. Assume that Δ_{Dir} is invertible. Then we can uniquely solve the Dirichlet problem, i.e. there exists exactly one solution $u \in H^2(\Omega, E)$ to the system

$$\Delta u = 0 \tag{3.1}$$
$$\gamma^0 u = g. \tag{3.2}$$

for any $g \in H^{3/2}(\Gamma, E')$. Namely, if $\widetilde{u} \in H^2(\Omega, E)$ satisfies $\gamma^0 g$ then

$$u := (\mathrm{Id} - \Delta^{-1}_{Dir}\Delta)\widetilde{u}$$

gives the unique solution. By interpolation, the so-called Poisson operator,

$$K_{Dir} : H^{3/2+s}(\Gamma, E') \to H^{2+s}(\Omega, E), \quad g \to u,$$

is continuous for all $s \in \mathbb{R}_+$. In fact, the Dirichlet problem is uniquely solvable even when the boundary data is merely in $H^{-1/2}(\Gamma, E')$.

Theorem 4.3.1. *Let Δ_{Dir} be invertible. Then for all $u \in L^2(\Omega, E)$ such that $\Delta u = 0$ we have*

$$\gamma^0 u \in H^{-1/2}(\Gamma, E')$$

and the map

$$\gamma^0 : \{u \in L^2(\Omega, E) \mid \Delta u = 0\} \longrightarrow H^{-1/2}(\Gamma, E')$$

is an isomorphism.

Proof. The operator

$$\gamma^0 : \{u \in L^2(\Omega, E) \mid \Delta u = 0\} \to H^{-1/2}(\Gamma, E') \tag{3.3}$$

is well-defined and continuous by Theorem 1.1.4. It is injective since any solution $u \in L^2(\Omega, E)$ to $\Delta u = 0$ such that $\gamma^0 u = 0$ is regular and thus in $\ker \Delta_{Dir} = \{0\}$.

By the open mapping theorem it remains to show that (3.3) is surjective. Its range is dense for it contains $H^{3/2}(\Gamma, E')$. Let $g \in H^{-1/2}(\Gamma, E')$ and choose a series $(g_n) \subset H^{3/2}(\Gamma, E')$ that converges to g w.r.t. $\|.\|_{H^{-1/2}(\Gamma,E')}$. Let $u_n \in H^2(\Omega, E)$ be the unique solutions to $\gamma^0 u_n = g_n$ such that $\Delta u_n = 0$. In order to show that u_n converges in $L^2(\Omega, E)$ we compute for all $v \in \mathscr{D}(\Delta_{Dir})$

$$\begin{aligned}
\langle u_n - u_m, \Delta v\rangle &= \langle \gamma^0(u_n - u_m), \gamma^1 v\rangle \\
&\le \text{const} \cdot \|g_n - g_m\|_{H^{-1/2}} \|\gamma^1 v\|_{H^{1/2}} \\
&\le \text{const} \cdot \|g_n - g_m\|_{H^{-1/2}} \|v\|_{H^2} \\
&\le \text{const} \cdot \|g_n - g_m\| \|\Delta v\|_{L^2}
\end{aligned}$$

which shows that $\|u_n - u_m\|_{L^2} \le \text{const} \cdot \|g_n - g_m\|_{H^{-1/2}}$, since $\Delta_{Dir} : \mathscr{D}(\Delta_{Dir}) \to L^2(\Omega, E)$ is surjective. Hence, $u_n \to u$ w.r.t $\|.\|_{L^2}$ for some $u \in \ker \Delta_{\max}$. By the continuity of the weak trace it follows that

$$\gamma^0 u = \lim_{n\to\infty} \gamma^0 u_n = \lim_{n\to\infty} g_n = g,$$

where the limits are taken in $H^{-1/2}(\Gamma, E')$. □

By interpolation theory we can extend the Poisson operator to the intermediate Sobolev spaces,

$$K_{Dir} : H^{-1/2+s}(\Gamma, E') \to H^s(\Omega, E'),$$

for all $s \ge 0$.

The second "classical" boundary condition is the Neumann boundary condition

$$(\nabla_\nu u)|_\Gamma = 0.$$

More generally, one can consider boundary conditions given by *oblique* vector fields along Γ, i.e.

$$(\nabla_X u)|_\Gamma = 0, \quad X \in C^\infty(\Gamma, T\Omega), \quad X \pitchfork \Gamma.$$

We may assume $X = \nu + Y$, $Y \in C^\infty(\Gamma, T\Gamma)$. The boundary condition thus takes the form

$$\begin{pmatrix} S_1 \Phi^{-1/2} & S_2 \Phi^{1/2} \end{pmatrix} \widetilde{\rho}^2 u = 0$$

where $S_2 = \text{Id}$, $S_1 = \nabla_Y$. The prinipal symbol of the operator

$$S_1 - S_2 \Phi = \nabla_Y - \Phi$$

is given by

$$\xi' \mapsto (i\xi'(Y) - \|\xi'\|) \cdot \text{Id}.$$

Hence, by Proposition 4.2.3, we have

Proposition 4.3.2. *When $X \in C^\infty(\Gamma, T\Omega|_\Gamma)$ is everywhere transversal to Γ, then $Xu|_\Gamma = 0$ is a well-posed boundary condition for Δ. The corresponding realisation, Δ_X, is self-adjoint if and only if $Y = 0$.*

Proof. Since Δ_X is well-posed, it remains to study self-adjointness. By Theorem 2.5.7 Δ_X is self-adjoint if and only if Δ_X is formally self-adjoint. Observe that $X = \nu + Y$ induces a similar problem for $\Psi_*\Delta$. Namely, $\nabla_Y \gamma^0 u + \gamma^1 u = 0$, if and only if $(Y + \frac{n-1}{2} H^\Gamma)\gamma^0 u + \gamma^1(\varphi^{-1}u) = 0$. The boundary condition for $\Psi_*\Delta$ thus reads

$$\begin{pmatrix} S_1'\Phi^{-1/2} & \Phi^{1/2} \end{pmatrix} \widetilde{\rho}^d u = 0,$$

with $S_1' = \nabla_Y + \frac{n-1}{2} H^\Gamma$. The adjoint boundary condition is given by

$$\begin{pmatrix} S_1'^*\Phi^{-1/2} & \Phi^{1/2} \end{pmatrix} \widetilde{\rho}^d u = 0.$$

Now $S_1'^* = -\nabla_Y - \operatorname{div} Y + \frac{n-1}{2} H^\Gamma$, hence it follows that Δ_X is formally self-adjoint if and only if Y vanishes. □

When $X = \nu$, i.e. $Y = 0$, then these conditions are called Neumann boundary conditions. The realisation Δ_X is denoted by Δ_{Neu} in this case.

The proof also shows that we could replace S_1 by any first order operator. However, the well-posedness condition, i.e.

$$S_1 - \Phi$$

elliptic, may not be satisfied. For instance, a real function $f \in C^\infty(\Gamma)$ and a vector field Y on Γ with vanishing divergence give rise to the condition

$$(i\nabla_Y + f)\gamma^0 + \gamma^1 u = 0.$$

When $Y \equiv 0$ these boundary conditions are sometimes called *Robin boundary conditions.* The induced realisation is formally self-adjoint for

$$S_1'^* = i\nabla_Y + i \operatorname{div} Y + \tfrac{n-1}{2} H^\Gamma + f,$$

but $i\nabla_Y + f - \Phi$ is elliptic if and only $\|Y(q)\| \neq 1$ for all $q \in \Gamma$. As a counterexample consider the boundary condition

$$\frac{\partial u}{\partial \bar z}\Big|_\Gamma = 0$$

on a compact subset $\Omega \subset \mathbb{C}$ with smooth boundary. Observe that

$$\frac{\partial}{\partial \bar z} = \tfrac{1}{2}(\nu + i\tfrac{\partial}{\partial\vartheta})$$

where $\vartheta : \mathbb{R} \to \mathbb{C}$ parametrises the boundary (in the positive sense). Since any holomorphic function on Ω is harmonic this condition is far from being well-posed for the Laplacian on Ω. In fact, $\Phi \equiv |i\frac{\partial}{\partial\vartheta}|$ up a pseudodifferential operator of order 0. Hence, the ellipticity condition is violated at each point of Γ.

4.3.2 Rotating Vector Fields for Surfaces with Boundary: A Theorem of I. N. Vekua

When Ω is a surface with boundary any vector field along Γ induces a well-posed boundary condition. More precisely, consider

$$(\nabla_X u)|_\Gamma = 0, \quad X \in C^\infty(\Gamma, T\Omega),$$

dropping the assumption of transversality. As before, we set

$$\mathscr{D}(\Delta_X) = \big\{ u \in H^2(\Omega, E) \bigm| \nabla_X u = 0 \text{ on } \Gamma \big\}.$$

Proposition 4.3.3. *Let* $\dim \Omega = 2$. *If* X *is nowhere vanishing then* Δ_X *is a well-posed realisation.*

Proof. Each component of Γ may be parametrized by $\vartheta \in [0, L)$, $L > 0$, so that X may be written
$$X = -(\cos\varphi)\nu + (\sin\varphi)\partial_\vartheta, \quad \varphi \in C^\infty(\mathbb{R}/L\mathbb{Z}, \mathbb{R}/2\pi\mathbb{Z}),$$
on each component since, w.l.o.g., we may assume that $\|X\| \equiv 1$. Consider $S_1 = (\sin\varphi)\partial_\vartheta$, and the multiplication operator $S_2 u = -(\cos\varphi)u$. Now,
$$S_1 - S_2\Phi \equiv (\sin\varphi)\partial_\vartheta - (\cos\varphi)|i\partial_\vartheta| \mod \Psi^0(\Gamma, E),$$
and thus $S_1 - S_2\Phi$ is elliptic even at those points where $\cos\varphi = 0$. By Proposition 4.2.3, Δ_X is well-posed. □

Proposition 4.3.4. *If for each connected component* Γ_i *of* Γ *there exists* $q \in \Gamma_i$ *such that* $X(q) \pitchfork T_q\Gamma$, *then*
$$\operatorname{ind} \Delta_X = \operatorname{ind}\left(\Delta \quad \nabla_X.|_\Gamma\right).$$

Proof. Our first aim is to show that there exists a continuous right inverse to $\nabla_X.|_\Gamma$. Here it is necessary to assume that X is transversal to each Γ_i at least on a non-empty open set.[1]

We will show that the operator
$$\rho^X := \nabla_X.|_\Gamma : H^2(\Omega, E) \to H^{1/2}(\Gamma, E') \tag{3.4}$$
is surjective. Let $g \in H^{1/2}(\Gamma, E')$ be given and assume, for simplicity, that $\Gamma \cong S^1$. As before we may write $X = -(\cos\varphi)\nu + (\sin\varphi)\partial_\vartheta$. The two open sets $\{\cos\varphi > 0\}$, $\{\sin\varphi > 0\}$ cover Γ. Choose a subordinated partition of unity, say χ_1, χ_2. Since X is not everywhere tangential, we may assume $\chi_1 \neq 0$. Therefore, we find some $h \in C^\infty(S^1, E')$ such that
$$\frac{\chi_2 g}{\sin\varphi} + \chi_1 h \in \operatorname{ran}\left(\nabla_\vartheta : H^{3/2}(S^1, E') \to H^{1/2}(S^1, E')\right).$$
Namely, the cokernel of this map consists of the finite-dimensional subspace of globally parallel, hence nowhere vanishing sections. (Alternatively, it is elementary to see that we find some h such that the solutions to the ordinary differential equation
$$\nabla_\vartheta f(\vartheta) = \frac{\chi_2(\vartheta)}{\sin\varphi(\vartheta)} + \chi_1(\vartheta)h(\vartheta), \quad \vartheta \in \mathbb{R}$$
are 2π-periodic.) Hence, we find $f \in H^{3/2}(S^1, E')$ such that
$$(\sin\varphi)\nabla_\vartheta f = \chi_2 g + (\sin\varphi)\chi_1 h.$$
Now, by Theorem 1.1.4 there exists $u \in H^2(\Omega, E)$ such that
$$u|_\Gamma = f, \quad \nabla_\nu u|_\Gamma = -\frac{\chi_1}{\cos\varphi}(g - (\sin\varphi)h).$$

[1] Indeed, this explains the exceptional role played by vector fields that are everywhere tangential to one of the components of Γ. Note that the Neumann vector field $-\nu$ for the disk can be continuously deformed into the tangential field ∂_ϑ. The variational arguments below show, however, that the index of $(\Delta, \nabla_X.|_\Gamma)$ depends on the homotopy type of the vector field X only. Since the index of the first (self-adjoint) Neumann realisation is 0 and that of the latter is 1 this may seem contradictory. The deeper reason, of course, is that one can never prescribe the tangential derivative along the whole circle for $H^1_{dR}(S^1) \neq 0$!

It follows that

$$\begin{aligned}\nabla_X u|_\Gamma &= -(\cos\varphi)\nabla_\nu u|_\Gamma + (\sin\varphi)\nabla_\vartheta f \\ &= (\sin\varphi)\chi_1 h - (\sin\varphi)\chi_1 h + \chi_1 g + \chi_2 g = g.\end{aligned}$$

We conclude, that the map (3.4) is a continuous surjective map between Hilbert spaces. Hence, there exists a continuous right-inverse, say η^X.

Now, let us prove the proposition. Observe that

$$\begin{pmatrix}\mathrm{Id} - \eta^X \rho^X \\ \rho^X\end{pmatrix} : H^2(\Omega, E) \to \mathscr{D}(\Delta_X) \oplus H^{1/2}(\Gamma, E')$$

is an isomorphism, its inverse is given by $(u, g) \mapsto u + \eta^X g$. On the other hand,

$$\begin{pmatrix}\Delta \\ \rho^X\end{pmatrix} = \begin{pmatrix}\Delta_X & \Delta\eta^X \\ 0 & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\mathrm{Id} - \eta^X \rho^X \\ \rho^X\end{pmatrix}. \tag{3.5}$$

Now, the proposition follows, since $\operatorname{ind}\begin{pmatrix}\Delta_X & \Delta\eta^X \\ 0 & \mathrm{Id}\end{pmatrix} = \operatorname{ind}\Delta_X$. □

Proposition 4.3.5. *Let $\Delta_t : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ be a family of second order differential operators with positive scalar principal symbol such that the coefficient matrices vary continuously w.r.t. to $\|.\|_\infty$ and let X_t be a continuous family of non-vanishing vector fields along Γ. Then*

$$\begin{pmatrix}\Delta_t \\ \nabla_{X_t}\cdot|_\Gamma\end{pmatrix}$$

is a continuous family of Fredholm operators.

Proof. The family

$$\begin{pmatrix}\Delta_t \\ \nabla_{X_t}\cdot|_\Gamma\end{pmatrix} : H^2(\Omega, E) \to L^2(\Omega, E) \oplus H^{1/2}(\Gamma, E').$$

is a continuous family of bounded operators. Since Δ_{t,X_t} is a well-posed realisation for each t the operator identity (3.5) shows that it is a family of Fredholm operators. □

For any surface Ω with smooth boundary, $\Gamma \cong \dot{\coprod}_{1\le j\le N} S^1$. If X is normalised, i.e. $\|X\| \equiv 1$, then we can regard X as a map

$$X : \bigcup_{1\le j\le N} S^1 \longrightarrow S^1.$$

Namely, let Γ_j be one of the components of Γ and let $\vartheta : \Gamma_j \to \mathbb{R}/2\pi\mathbb{Z}$ be a parametrization of Γ_j. Hence, $T\Omega|_{\Gamma_i} \cong \mathbb{C} \times S^1$ where we identify $-\nu$ with 1, and ∂_ϑ with i. Let $p_j + 1$ denote the winding number of $X|_{\Gamma_j} : S^1 \to S^1$. This way we assign to the outward unit normal field $-\nu$ the winding number 1.

Theorem 4.3.6 (I. N. Vekua). *Let p_i be the winding numbers of X. If for each connected component of Γ there exists $q \in \Gamma$ such that $X(q) \pitchfork T_q\Gamma$, then*

$$\operatorname{ind}\Delta_X = 2(\operatorname{rk} E)\sum_{j=1}^N (1 - p_j).$$

Proof. Observe first that $E|_\Gamma$ is trivial for $E|_\Gamma$ is a complex vector bundle over a disjoint union of circles.[2] Replace Ω by N disks, i.e. let $\widetilde{\Omega} = \mathbb{D}\dot{\cup}\ldots\dot{\cup}\mathbb{D}$. Equip $\widetilde{\Omega}$ by a metric such that Ω and $\widetilde{\Omega}$ have isometric collars. Since $\widehat{\Delta}(\xi) = \|\xi\|^2$ we find a Laplacian $\widetilde{\Delta} : C^\infty(\widetilde{\Omega}, \mathbb{C}^{\operatorname{rk} E})$ whose coefficients coincide near Γ with those of P. We deduce from Corollary 2.5.10, that

$$\operatorname{ind} \Delta_X = \operatorname{ind} \widetilde{\Delta}_X.$$

Using Proposition 4.3.4 and 4.3.5, we may deform $\widetilde{\Delta}$ continuously into the standard Laplacian

$$\bigoplus_{i=1}^{N} \Delta_{\mathbb{D}} : C^\infty(\mathbb{D}, \mathbb{C}^{\operatorname{rk} E}) \to C^\infty(\mathbb{D}, \mathbb{C}^{\operatorname{rk} E})$$

and X into the union of the vector fields

$$X_{p_j}(\vartheta) := -(\cos((p_j - 1)\vartheta))\nu + (\sin((p_j - 1)\vartheta))\partial_\vartheta,$$

without changing the index. It remains to show that

$$\operatorname{ind} \Delta_{\mathbb{D}, X_{k+1}} = 2k.$$

For this purpose, let us compute the Cauchy data space of the Laplacian on the disk. Here, it is convenient to work with direct sums of Sobolev spaces of different order, avoiding the adjusting isomorphism Φ_2.

Lemma 4.3.7.

$$\left\{\rho^2 u \;\middle|\; \Delta u = 0\right\} = \left\{(g, h) \in H^{-1/2}(S^1) \oplus H^{-3/2}(S^1) \;\middle|\; h = -|D| g\right\}$$

where $D = i\partial_\vartheta$.

Proof of the lemma. Assume $u \in L^2(\mathbb{D})$, $\Delta u = 0$. Let $(a_k) \in \ell^2(\mathbb{Z}, (1 + |k|)^{-1} d\#) \cong H^{-1/2}(S^1)$ such that

$$\gamma^0 u(\vartheta) = \sum_{k\geq 0} a_k e^{ik\vartheta} + \sum_{k>0} a_{-k} e^{-ik\vartheta}. \tag{3.6}$$

It follows that

$$u(r, \vartheta) = \sum_{k\geq 0} r^k a_k e^{ik\vartheta} + \sum_{k>0} a_{-k} r^k e^{-ik\vartheta}.$$

Similarly, let $(b_k) \in \ell^2(\mathbb{Z}, (1 + |k|)^{-3} d\#) \cong H^{-3/2}(S^1)$ such that

$$\gamma^1 u(\vartheta) = \sum_{k\geq 0} b_k e^{ik\vartheta} + \sum_{k>0} b_{-k} e^{-ik\vartheta}. \tag{3.7}$$

It follows that

$$b_k = \begin{cases} -k a_k & k \geq 0, \\ k a_k & k < 0. \end{cases}$$

Observe that $|D| e^{ik\vartheta} = |k| e^{ik\vartheta}$. □

[2] However, if E is real then we may complexify the bundle and the operator. The complex index of the complex operator equals the real index of the real operator. Hence, one may apply the proof below.

Let us compute the dimension of $\ker \Delta_{\mathbb{D},X_{k+1}} \cong \{\rho^2 u|\ \Delta u = 0\} \cap \{\rho^2 u|\ X_{k+1}u|_\Gamma = 0\}$. Consider $g, h \in C^\infty(S^1)$ such that

$$(\sin k\vartheta)\partial_\vartheta g - (\cos k\vartheta)h = 0, \quad h = -|D|g.$$

Introduce the shift operators with respect to D, $s_l := e^{-il\vartheta}$. Then, $\cos k\vartheta = \frac{1}{2}(s_{-k}+s_k)$, $\sin k\vartheta = \frac{1}{2i}(s_{-k} - s_k)$ and $\partial_\vartheta = -iD$, thus

$$-(s_{-k} - s_k)Dg + (s_{-k} + s_k)|D|g = 0, \quad h = -|D|g.$$

and therefore

$$-h = |D|g \in \ker\big(s_k\mathbf{1}_{(0,\infty)}(D) + s_{-k}\mathbf{1}_{(-\infty,0)}(D)\big).$$

If follows that this kernel merely consists of $\ker |D|$ when $k \geq 0$. Hence[3],

$$\dim\ker \Delta_{\mathbb{D},X_{k+1}} = 1, \quad k \geq 0.$$

When $k = -l < 0$, let (b_k) be the Fourier coefficients of h as in (3.7). We obtain $b_j = 0$ for all $|j| > 2l$. Among the remaining $4l + 1$ Fourier coefficients there are $2l + 1$ linearly independent conditions:

$$b_{2l-j} = b_{-j}, \quad j = 0, 1..., 2l$$

However, since $h = -|D|g$, we have $b_0 = 0$. So there are $2l + 2$ constraints for the $4l + 1$ coefficients $b_{-2l}, ..., b_{2l}$. Together with the constant solution $g = a_0$ we obtain

$$\dim\ker \Delta_{\mathbb{D},X_{k+1}} = -2k, \quad k < 0.$$

In order to compute the cokernel let us consider the adjoint. The boundary values satisfying the adjoint boundary condition are exactly those that are in the symplectic complement of

$$\ker\big(i(\sin(k\vartheta)D \ \ \cos(k\vartheta)\big).$$

Multiplying by J and taking the orthogonal complement yields that the boundary values satisfying the adjoint boundary condition are given by

$$\operatorname{ran}\begin{pmatrix} s_k + s_{-k} \\ D(s_{-k} - s_k) \end{pmatrix}.$$

In order to compute the dimension of the cokernel we have to take the intersection of this range with $\ker\big(|D| \quad \mathrm{Id}\big)$. Hence, we have to look for all u in

$$\ker \ \ |D|\big(\mathbf{1}_{(0,\infty)}(D)s_{-k} + \mathbf{1}_{(-\infty,0)}(D)s_k\big).$$

It follows that $\mathbf{1}_{(0,\infty)}(D)s_{-k}u = 0$ and $\mathbf{1}_{(-\infty,0)}(D)s_k u = 0$. In particular, when $k < 0$ then $u = 0$ thus the cokernel is trivial. If $k \geq 0$, then $u \in \operatorname{ran} \mathbf{1}_{[-k,k]}(D)$. Therefore g is the Dirichlet boundary data of a solution to $\ker(\Delta_{\mathbb{D},X_{k+1}})^*$ if and only if

$$g \in \operatorname{ran}(s_{-k} + s_k)\mathbf{1}_{[-k,k]}(D).$$

The dimension of this space is $2k + 1$.

Summarizing these computations, we have

$$\operatorname{ind} \Delta_{\mathbb{D},X_p} = 1 - (2(p-1)+1) = 2 - 2p$$

if $p \geq 1$ and

$$\inf \Delta_{\mathbb{D},X_p} = -2(p-1) - 0 = 2 - 2p,$$

if $p < 1$. □

In fact we have computed the index of a Toeplitz operator,

$$\operatorname{ind}\big((s_k\mathbf{1}_{(0,\infty)}(D) - s_{-k}\mathbf{1}_{(-\infty,0)}(D))|D|\big) = \operatorname{ind}\big(\mathbf{1}_{<0}(D)\ s_{-2k}\ \mathbf{1}_{<0}(D)\big).$$

[3]The constant function on $\mathbb{D}$ is always a trivial solution!

4.3.3 Absolute and relative boundary condition

Consider the bundle $E = \Lambda^\bullet := \Lambda^\bullet T^*\Omega$ of exterior forms. Recall the definition of the Hodge-de Rham operator

$$D := d + d^t : C^\infty(\Omega, \Lambda^\bullet) \to C^\infty(\Omega, \Lambda^\bullet),$$

cf. Section 3.1.1. We are now going to introduce boundary conditions for D and its square, $\Delta := (d + d^t)^2$, the Hodge-Laplacian. As usual, let x_1 be the normal coordinate near Γ, i.e. $x_1(p) := d(p, \Gamma)$. If $\delta > 0$ is chosen sufficiently small, then (x_1, π) defines a diffeomorphism of $U := \{d(p, \Gamma) < \delta\}$ onto $[0, \delta) \times \Gamma$ where $\pi(p)$ is the projection of a point along the geodesic ray through ν onto Γ.

Every differential form ω admits a decomposition near Γ

$$\omega|_U = \omega_1 + \nu^\flat \wedge \omega_2,$$

where $\omega_1, \omega_2 \in C^\infty([0, \delta), C^\infty(\Gamma, \Lambda^\bullet T^*\Gamma))$. Note that since ν is perpendicular to each hypersurface $\Gamma(\varepsilon) = \{p \in M \mid x_1(p) = \varepsilon\}$, this decomposition is orthogonal. Relative and absolute boundary conditions are given by

$$\omega_1|_\Gamma = 0, \quad \omega_2|_\Gamma = 0,$$

resp. and may be expressed by the orthogonal projections $Q_r = Q_r(\nu)$ and $Q_a = Q_a(\nu)$ where

$$Q_r := \iota_\nu \circ \operatorname{ext}_{\nu^\flat}, \quad Q_a := \operatorname{ext}_{\nu^\flat} \circ \iota_\nu.$$

In particular,

$$Q_r + Q_a = -c(\nu)^2 = \operatorname{Id}.$$

Consider the corresponding realisations of D,

$$\mathscr{D}(D_{r/a}) := \{\omega \in H^1(\Omega, \Lambda^\bullet) \mid \omega_{1/2}|_\Gamma = 0\}.$$

Theorem 4.3.8. *D_r and D_a are well-posed self-adjoint realisations of $D = d + d^t$.*

Proof. Consider the splitting

$$\Lambda^\bullet T^*\Omega|_\Gamma = \operatorname{ran} Q_r \oplus \operatorname{ran} Q_a.$$

W.r.t. this splitting, the Calderón projection of $d + d^t$ is given by (cf. Theorem 3.3.1 (i))

$$C_+(d + d^t) \equiv \frac{1}{2} \begin{pmatrix} \operatorname{Id} & -\operatorname{sign}((d + d^t)^\Gamma) \\ -\operatorname{sign}((d + d^t)^\Gamma) & \operatorname{Id} \end{pmatrix},$$

up to a perturbation in $\Psi^{-1}(\Gamma, \Lambda^\bullet T^*\Gamma \oplus \Lambda^\bullet T^*\Gamma)$. Hence,

$$(C_+(d + d^t), Q_{r/a})$$

are obviously Fredholm pairs.

Since

$$J(\omega_1, \omega_2) = -c(\nu)(\omega_1 + \nu^\flat \wedge \omega_2) = \omega_2 - \nu^\flat \wedge \omega_1,$$

it follows that

$$J = \begin{pmatrix} 0 & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{pmatrix}.$$

We deduce that $Q_a^{\mathrm{ad}} = J(\operatorname{Id} - Q_a)J^* = Q_a$. Hence, Q_a and Q_r induce self-adjoint realisations of D. $\square$

Considering $(D_a)^2$ and $(D_r)^2$ one obtains natural boundary conditions for the p-th Hodge-Laplacian

$$\Delta_p := (d + d^t)^2 : C^\infty(\Omega, \Lambda^p) \to C^\infty(\Lambda^p).$$

Absolute boundary conditions are given by

$$\gamma^0 \omega_2 = 0, \quad \gamma^1 \omega_1 - \sum_{j=2}^{n} \lambda_j Q_a(e_j) \gamma^0 \omega_1 = 0,$$

and relative boundary conditions by

$$\gamma^0 \omega_1 = 0, \quad \gamma^1 \omega_2 - \sum_{j=2}^{n} \lambda_j Q_r(e_j) \gamma^0 \omega_2 = 0,$$

cf. Section 3.1.1. Calling ω_1 the tangential part and $\nu^\flat \wedge \omega_2$ the perpendicular part of a form $\omega = \omega_1 + \nu^\flat \wedge \omega_2 \in \Lambda^\bullet|_\Gamma$ we may characterise these conditions as follows: Absolute boundary conditions are Dirichlet boundary conditions on the perpendicular part and (modified) Neumann boundary conditions on the tangential part. Relative boundary conditions are Dirichlet boundary conditions on the tangential part and (modified) Neumann boundary conditions on the perpendicular part.

Being squares of well-posed self-adjoint realisations of D

$$\Delta_a = (D_a)^2, \quad \Delta_r = (D_r)^2$$

are well-posed self-adjoint realisation of $\Delta = \Delta_0 \oplus \cdots \oplus \Delta_n$. In particular, any harmonic square-integrable p-form ω such that $Q_{a/r}\omega|_\Gamma = Q_{a/r} D\omega|_\Gamma = 0$ is immediately in

$$\mathscr{H}^p_{a/r}(\Omega) := \big\{ \omega \in C^\infty(\Omega, \Lambda^p) \;\big|\; D\omega = 0,\ Q_{a/r}\omega|_\Gamma = 0 \big\}.$$

To the self-adjoint operators D_a and D_r there is a Hodge decomposition generalising the situation on closed manifolds. Let us first introduce some notation:

$$\begin{aligned}
C^\infty_{a/r}(\Omega, \Lambda^p) &:= \big\{ \omega \in C^\infty(\Omega, \Lambda^p) \;\big|\; Q_{a/r}\omega|_\Gamma = 0 \big\}, \\
H^k_{a/r}(\Omega, \Lambda^p) &:= \big\{ \omega \in H^k(\Omega, \Lambda^p) \;\big|\; Q_{a/r}\omega|_\Gamma = 0 \big\}, \\
\mathscr{Z}^p(\Omega) &:= \big\{ \omega \in C^\infty(\Omega, \Lambda^p) \;\big|\; du = 0 \big\}, \\
\mathscr{Z}^p_r(\Omega) &:= \big\{ \omega \in C^\infty_r(\Omega, \Lambda^p) \;\big|\; du = 0 \big\}, \\
\mathscr{E}^p(\Omega) &:= \big\{ \omega \in C^\infty(\Omega, \Lambda^p) \;\big|\; \exists \eta \in C^\infty(\Omega, \Lambda^{p-1}) : d\eta = \omega \big\}, \\
\mathscr{E}^p_r(\Omega) &:= \big\{ \omega \in C^\infty_r(\Omega, \Lambda^p) \;\big|\; \exists \eta \in C^\infty_r(\Omega, \Lambda^{p-1}) : d\eta = \omega \big\}, \\
H^p_{dR}(\Omega) &:= \mathscr{Z}^p(\Omega) / \mathscr{E}^p(\Omega), \\
H^p_{dR}(\Omega, \Gamma) &:= \mathscr{Z}^p_r(\Omega) / \mathscr{E}^p_r(\Omega).
\end{aligned}$$

They correspond to smooth, Sobolev regular, closed and exact forms, respectively. The two quotients define usual and relative de Rham cohomology groups. Note that $d(\omega_1 + \nu^\flat \wedge \omega_2)|_\Gamma = -\nu^\flat \wedge d^\Gamma \omega_2|_\Gamma$ if $\omega = \omega_1 + \nu^\flat \wedge \omega_2$ satisfies $Q_r\omega|_\Gamma = 0$. Thus d preserves relative boundary condition and the last quotient is well-defined. It is in fact the p-th cohomology group assosciated to the relative de Rham complex

$$\cdots C^\infty_r(\Omega, \Lambda^p) \xrightarrow{d} C^\infty_r(\Omega, \Lambda^{p+1}) \cdots$$

Theorem 4.3.9 (Hodge decomposition). *We have L^2-orthogonal decompositions*

$$H^k(\Omega,\Lambda^p) = \mathscr{H}^p_{a/r}(\Omega) \oplus dH^{k+1}_{a/r}(\Omega,\Lambda^{p-1}) \oplus d^t H^{k+1}_{a/r}(\Omega,\Lambda^{p+1}).$$

In particular, all factors are closed w.r.t. the H^k-norm.

Proof. Consider the self-adjoint operator

$$D_{a/r} : H^1_{a/r}(\Omega,\Lambda^\bullet) \to L^2(\Omega,\Lambda^\bullet).$$

By the regularity of relative and absolute boundary conditions these two operators have closed range. Thus we have the L^2-orthogonal decomposition $L^2(\Omega,\Lambda^\bullet) = \ker D_{a/r} \oplus \operatorname{ran} D_{a/r}$. By higher regularity it follows that $\ker D_{a/r} = \mathscr{H}^\bullet_{a/r}(\Omega)$. This space is finite dimensional and consists of smooth forms only.

As for the range note that, again by regularity,

$$H^k(\Omega,\Lambda^p) \cap \operatorname{ran} D_{a/r} = dH^{k+1}_{a/r}(\Omega,\Lambda^{p-1}) + d^t H^{k+1}_{a/r}(\Omega,\Lambda^{p+1}).$$

In order to show that the sum on the right hand side is L^2-direct, let us compute

$$\langle d\omega, d^t\eta\rangle = \langle (d\circ d)\omega,\eta\rangle - \langle d\omega|_\Gamma, \iota_\nu\eta|_\Gamma\rangle_{L^2(\Gamma,\Lambda^\bullet)},$$

which vanishes whenever η satisfies absolute boundary conditions. If ω satisfies relative boundary conditions, then we have

$$\langle d\omega, d^t\eta\rangle = \langle \omega, (d^t\circ d^t)\eta\rangle + \langle \operatorname{ext}_{\nu^\flat}\omega|_\Gamma, \eta|_\Gamma\rangle_{L^2(\Gamma,\Lambda^\bullet)} = 0,$$

Hence, we have established Hodge decomposition in both cases. □

Together with the Sobolev Lemma we obtain:

Corollary 4.3.10 (Smooth Hodge decomposition). *We have (topologically) direct sum decompositions*

$$C^\infty(\Omega,\Lambda^p) = \mathscr{H}^p_{a/r}(\Omega) \oplus dC^\infty_{a/r}(\Omega,\Lambda^{p-1}) \oplus d^t C^\infty_{a/r}(\Omega,\Lambda^{p+1}).$$

In particular, all factors are closed w.r.t. the locally convex topology given by all $C^k(\Omega,\Lambda^\bullet)$-norms.

The Hodge decomposition theorem has an important consequence.

Corollary 4.3.11. *There are natural isomorphisms:*

(i) $H^p(\Omega) \cong \mathscr{H}^p_a(\Omega)$,

(ii) $H^p(\Omega,\Gamma) \cong \mathscr{H}^p_r(\Omega)$.

Proof. (i) Consider the Hodge decomposition of some closed form $\omega \in C^\infty(\Omega,\Lambda^p)$ with absolute boundary condition,

$$\omega = \omega_1 + d\omega_2 + d^t\omega_3.$$

By Theorem 4.3.10, all ω_i are smooth. Since $d\omega = dd^t\omega_3 = 0$ we have

$$0 = \langle dd^t\omega_3,\omega_3\rangle = \langle d^t\omega_3, d^t\omega_3\rangle + \langle \gamma^0 d^t\omega_3, \gamma^0\iota_\nu\omega_3\rangle = \langle d^t\omega_3, d^t\omega_3\rangle.$$

Hence, $d^t\omega_3 = 0$ for all closed forms. If $\omega = d\alpha \in \mathscr{E}^p(\Omega)$, then

$$\langle \omega, \omega_1 \rangle = \langle d\alpha, \omega_1 \rangle = \langle \alpha, d^t\omega_1 \rangle + \langle \gamma^0 \alpha, \gamma^0 \iota_{\partial_1} \omega_1 \rangle = 0.$$

In particular, the harmonic component in the absolute Hodge decomposition of an arbitrary smooth exact form vanishes. So, we have a well-defined map

$$H^p_{dR}(\Omega) \to \mathscr{H}^p_a(\Omega), \quad [\omega] \mapsto \omega_1.$$

Since, $[\omega_1] \mapsto \omega_1$ this mapping is clearly onto. Assume that $[\omega]$ is in its kernel. This means $\omega = d\omega_2$, so ω is exact.

(ii) Let $[\omega] \in H^p_{dR}(\Omega, \Gamma)$, i.e. $\omega \in C^\infty_r(\Omega, \Lambda^p)$ and $d\omega = 0$. Consider the Hodge decomposition of ω with relative boundary conditions:

$$\omega = \omega_1 + d\omega_2 + d^t\omega_3.$$

By Theorem 4.3.9, all ω_i are smooth. Since ω_1, ω and $d\omega_2$ satisfy relative boundary conditions, so does $d^t\omega_3$. It follows that

$$0 = \langle dd^t\omega_3, \omega_3 \rangle = \langle d^t\omega_3, d^t\omega_3 \rangle + \langle \gamma^0 \operatorname{ext}_{\nu^\flat} d^t\omega_3, \gamma^0\omega_3 \rangle = \|d^t\omega_3\|^2_{L^2}.$$

If $\omega = d\alpha$ with $\alpha \in C^\infty_r(\Omega, \Lambda^p)$ then the Hodge decomposition of ω is simply $\omega = 0 + d\alpha + 0$, so that the map

$$H^p_{dR}(\Omega, \Gamma) \mapsto \mathscr{H}^p_r(\Omega), \quad [\omega] \mapsto \omega_1$$

is well-defined. On the other hand, if $\omega = d\alpha$ with $\alpha \in C^\infty_r(\Omega, \Lambda^p)$, then $[\omega] = 0$. The mapping is onto for $[\omega_1] \mapsto \omega_1$ whenever ω_1 is a harmonic form satisfying relative boundary conditions. We conclude that it is an isomorphism. □

The Euler characteristics associated to the complexes

$$\cdots \longrightarrow C^\infty(\Omega, \Lambda^p) \xrightarrow{d} C^\infty(\Omega, \Lambda^{p+1}) \longrightarrow \cdots,$$
$$\cdots \longrightarrow C^\infty_r(\Omega, \Lambda^p) \xrightarrow{d} C^\infty_r(\Omega, \Lambda^{p+1}) \longrightarrow \cdots,$$

are given by

$$\chi(\Omega) = \sum_{p=0}^{n} (-1)^p \dim H^p_{dR}(\Omega), \quad \chi(\Omega, \Gamma) = \sum_{p=0}^{n} (-1)^p \dim H^p_{dR}(\Omega, \Gamma),$$

respectively. By the above corollary they coincide with the indices of (the even parts of) D_a and D_r since

$$\operatorname{ind} D^{\mathrm{ev}}_a = \dim \mathscr{H}^{\mathrm{ev}}_a(\Omega) - \dim \mathscr{H}^{\mathrm{odd}}_a(\Omega), \quad \operatorname{ind} D^{\mathrm{ev}}_r = \dim \mathscr{H}^{\mathrm{ev}}_r(\Omega) - \dim \mathscr{H}^{\mathrm{odd}}_r(\Omega).$$

4.4 The Dirichlet-to-Neumann Operator

Assume $\Delta_{Dir} > 0$. Since

$$\langle \Delta u, u \rangle = \langle \nabla u, \nabla u \rangle + \langle Hu, u \rangle$$

for all u satisfying Dirichlet boundary conditions this can always be achieved by adding to Δ some constant $\mu \in \mathbb{R}_+$. Let $g \in C^\infty(\Gamma, E')$. Then the unique solution to the semi-homogeneous Dirichlet problem,

$$\Delta u = 0, \tag{4.1}$$

$$\gamma^0 u = g, \tag{4.2}$$

is given by

$$u := (\mathrm{Id} - \Delta_{Dir}^{-1}\Delta)\widetilde{u},$$

where $\widetilde{u}$ is any section in $H^2(\Omega, E)$ such that $\gamma^0\widetilde{u} = g$, e.g. $\widetilde{u} = \eta^0 g$.

Definition 4.4.1. Let $g \in C^\infty(\Gamma, E')$. The Dirichlet-to-Neumann operator, $T_{DN} : C^\infty(\Gamma, E') \to C^\infty(\Gamma, E')$ is defined by

$$T_{DN} g := \gamma^1 u,$$

where u is the unique solution to (4.1) and (4.2).

Example 4.4.2. From Lemma 4.3.7 we infer that $T_{DN} g := -|D|$ when Δ is the standard Laplacian on a disk.

Since

$$\langle T_{DN} g, h\rangle - \langle g, T_{DN} h\rangle = \langle \Delta u, v\rangle - \langle u, \Delta v\rangle = 0$$

where $u, v \in \ker \Delta_{\max}$ such that $\gamma^0 u = g$, $\gamma^0 v = h$, it follows that T_{DN} is a densely defined symmetric operator in $L^2(\Gamma, E')$.

Proposition 4.4.3. (i) *If $\Delta_{Neu} > 0$ then T_{DN} is injective.*

(ii) *T_{DN} is a classical elliptic pseudodifferential operator of order* 1. *Its principal symbol is given by $\widehat{T}_{DN}(\xi') := -\|\xi'\|$.*

(iii) *In particular, $T_{DN} : C^\infty(\Gamma, E') \to C^\infty(\Gamma, E')$ is an unbounded essentially self-adjoint operator in $L^2(\Gamma, E')$.*

Proof. (i) If $\Delta_{Neu} > 0$, then $\gamma^1 u$ uniquely determines any section $u \in C^\infty(\Omega, E)$ such that $\Delta u = 0$. Otherwise, if there were two different solutions $u, v \in \ker \Delta_{\max}$ such that $\gamma^1 u = \gamma^1 v$, then $u - v \in \ker \Delta_{Neu}$.

(ii) Let $C_+(\Delta)$ be the orthogonal projection onto the Cauchy data space of Δ. By Remark 2.3.10 and Theorem 4.2.1 it follows that the principal symbol of $C_+(\Delta)$ is the projection onto the diagonal $\{(x, -x) |\ x \in E'\}$. Let

$$C_+(\Delta) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad c_{ij} \in \Psi^0(\Gamma, E').$$

Then $\widehat{c}_{ij}(\xi') = \frac{1}{2}(-1)^{i+j}\,\mathrm{Id}$. Observe that

$$(\mathrm{Id} - C_+^\perp(\Delta))\Phi_2 \begin{pmatrix} \mathrm{Id} \\ T_{DN} \end{pmatrix} = 0.$$

Hence, $c_{11}\Phi^{1/2} + c_{22}\Phi^{-1/2} T_{DN} = 0$. Since Φ and c_{ij}, $i, j = 1, 2$, are elliptic we infer that T_{DN} is an elliptic pseudodifferential operator of order 1. Moreover, its principal symbol is given by

$$\widehat{T}_{DN}(\xi') = -\|\xi'\|.$$

Now (iii) follows since we have shown that T_{DN} is a formally self-adjoint elliptic pseudodifferential operator of order 1. $\square$

Even without the knowledge about the principal symbol of the Calderón projection the preceding proposition is rather plausible. Suppose Ω is a half-cylinder, $[0,\infty)\times\Gamma$, and let $\Delta'(x_1) : C^\infty(\Gamma, E') \to C^\infty(\Gamma, E')$ be non-negative and constant, i.e.

$$\Delta = -\frac{\partial^2}{\partial x_1^2} + \Delta'.$$

Then the unique square-integrable solution to

$$\Delta u = 0, \quad u\big|_{x_1=0} = g$$

for $g \in C^\infty(\Gamma, E')$ is given by

$$u = e^{-x_1\sqrt{\Delta'}}g.$$

We deduce that in this model situation

$$T_{DN} = -\sqrt{\Delta'}.$$

Let us now drop the condition $\Delta_{Dir} > 0$ and consider the parameter-dependent Laplacian

$$\Delta + \mu^2, \quad \mu \in \mathbb{C}.$$

Note that the Dirichlet realisation for $\Delta + \mu^2$ will be invertible when

$$\mu \in \Lambda_{R,\vartheta} = \big\{\mu \in \mathbb{C} \;\big|\; |\arg\mu| < \vartheta,\ |\mu| > R\big\}$$

for some $R, \vartheta \in \mathbb{R}_+$ since Δ_{Dir} is lower bounded. Let $T_{DN,\mu}$ denote the Dirichlet-to-Neumann operator associated to $\Delta + \mu^2$.

Theorem 4.4.4. *For $\mu \in \Lambda_{R,\vartheta}$, the Dirichlet-to-Neumann operator,*

$$T_{DN,\mu} \in \Psi^1(\Gamma, E'),$$

is a classical pseudodifferential operator with parameter (in the sense of [Shu80, Chapter II], i.e. $T_{DN,\mu} \in \Psi^1_{\mathrm{cl}}(\Gamma, E', \Lambda_{R,\vartheta})$.[4])

Its principal symbol is given by $-\|(\xi', \mu)\| = -\sqrt{\|\xi'\|^2 + \mu^2}$.

Proof. We can carry out the computations of the Calderón projection for the parameter dependent operator $\Delta + \mu^2$. Recall that we had to extend $\Delta + \mu^2$ to a slightly larger manifold with boundary M such that $\Omega \subset M^\circ$. Clearly, we can extend Δ first and then add μ^2 in order to obtain a parameter dependent extension $\Delta^M + \mu^2$. Then, imposing e.g. Dirichlet boundary conditions on ∂M yields an invertible family for $\mu \in \Lambda_{R,\vartheta}$.

The resolvent family

$$Q^M_\mu := (\Delta^M_{Dir} + \mu^2)^{-1}$$

defines a parameter dependent parametrix of order -2. Note that, in principle, this is the only ingredient of the Calderón projection where the parameter μ enters. Arguing as in the proof of Theorem 2.3.5 (ii) and always treating μ like an additional cotangent variable ξ_{n+1}, the operator

$$-\rho^2 r_+ Q^M_\mu (\rho^2_M)^* J \tag{4.3}$$

turns out to be a matrix of classical parameter dependent ψdos.

The next step was to make all the entries operators of order 0 by conjugation with Φ_2. However, since Φ_2 is *not* a parameter-dependent ψdo (in the sense mentioned above!), some care has to be taken at this step.

[4] This class of operators is sometimes called strongly polyhomogeneous, cf. [GS95].

We could replace Φ_2 by an analogous diagonal matrix of parameter-dependent operators, say $\Phi_{2,\mu}$. Simply replace $\Phi = (\Delta' + 1)^{1/2}$ by $(\Delta' + \mu^2)^{1/2}$. Thus one obtains a classical pseudodifferential projection with parameter. This method is used in [Gru99, Chapter 7].

Alternatively, we could content ourselves with the projection (4.3) which is a bounded operator on the following product of Sobolev spaces

$$H^{s-1/2}(\Gamma, E') \times H^{s-3/2}(\Gamma, E'), \quad s \in \mathbb{R}.$$

By the analysis carried out in the proof of Theorem 2.3.5, the (i,j)-component, say $c_{ij,\mu}$, is a classical ψdo with parameter of order $i-j$. Since (4.3) is a projection onto the non-adjusted Cauchy data space, we have

$$c_{11,\mu} - c_{12,\mu} T_{DN,\mu} = 0.$$

Now, the principal symbol of the $c_{ij,\mu}$ can be computed from the principal symbol of $\Delta + \mu^2$ exactly as in the proof of Theorem 4.2.1. Thus

$$\widehat{c_{ij,\mu}}(\xi') = (-1)^{i+j} \sqrt{\|\xi'\|^2 + \mu^2}^{\,i-j}.$$

It follows that $T_{DN,\mu}$ is a classical parameter-dependent ψdo with principal symbol

$$-\sqrt{\|\xi'\|^2 + \mu^2},$$

which finishes the proof. $\square$

4.5 Self-adjoint Realisations and Semi-boundedness

4.5.1 A Characterisation of Well-posed Self-adjoint Realisations and Canonical Forms of Boundary Conditions

First we aim for a characterisation of all well-posed self-adjoint realisations Δ_B. Recall that this is equivalent to characterising all well-posed self-adjoint realisations $\Psi_* \Delta_{B'}$. W.l.o.g. we may assume that B' is an orthogonal projection subject to Assumption 1.2.3.

Theorem 4.5.1. *Δ_B is a formally self-adjoint extension of $\Delta_{\min}$ if and only if*

$$B' = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & \mathrm{Id} - b_{11} \end{pmatrix}$$

where b_{11}, b_{12} are bounded self-adjoint operators such that

$$0 \le b_{11} \le 1, \ -1/2 \le b_{12} \le 1/2, \ [b_{11}, b_{12}] = 0, \ b_{11}(\mathrm{Id} - b_{11}) = b_{12}^2.$$

If B' is as above then B' is well-posed if and only if

$$1/2 \notin \operatorname{spec}_{\mathrm{ess}} b_{12}.$$

Proof. Let

$$B' = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be self-adjoint and $\widetilde{J}$-symmetric, i.e. $-\widetilde{J}(\mathrm{Id}-B)\widetilde{J}=B$. Then

$$b_{11}^*=b_{11},\quad ,b_{12}^*=b_{21},\quad b_{22}=1-b_{11},\quad b_{21}=b_{12}.$$

Using that B' is a projection, we see

$$b_{11}^2+b_{12}^2=b_{11},\quad b_{11}b_{12}-b_{12}b_{11}=0.$$

The operator inequalities for b_{11}, b_{12} follow immediately.

B is well-posed for Δ if and only if (B',B_0) is Fredholm, which, by Proposition A.1.12, means that

$$\pm\,\mathrm{Id}+B'-B_0$$

are Fredholm.

Consider

$$B'-B_0+\mathrm{Id}=\begin{pmatrix} b_{11}+1/2 & b_{12}+1/2\\ b_{12}+1/2 & 3/2-b_{11}\end{pmatrix}.$$

Since $b_{11}\geq 0$, $|b_{12}|\leq 1/2$, $b_{11}+1/2$ and $b_{12}^2+3/4$ are invertible. Moreover, the inverses are related by

$$(b_{11}+1/2)^{-1}=-(b_{11}-3/2)(b_{12}^2+3/4)^{-1}.$$

It follows that $B'-B_0+\mathrm{Id}$ is Fredholm if and only if

$$3/2-b_{11}+(b_{11}-3/2)(b_{12}^2+3/4)^{-1}(b_{12}+1/2)^2=(b_{11}-3/2)(b_{12}-1/2)(b_{12}+3/4)^{-1}$$

is. Since $b_{11}-3/2$ is invertible this is the case if and only if $b_{12}-1/2$ is Fredholm. Since B_0 and B' are $\widetilde{J}$-symmetric we have

$$J^*(B'-B_0+\mathrm{Id})J=B_0-B'+\mathrm{Id},$$

so that $B'-B_0\pm\mathrm{Id}$ have the same (essential) spectrum. □

We note that our technical assumption that $[\widetilde{\Phi},B']$ is bounded on each Sobolev space $H^s(\Gamma,E'\oplus E')$ is satisfied if and only if

$$[b_{11},\Phi],\quad [b_{12},\Phi]$$

are bounded operators on the Sobolev scale $H^s(\Gamma,E')$.

Assume we are given a self-adjoint well-posed realisation Δ_B. Transform the operator near the boundary into one of the form (cf. Proposition 4.1.3)

$$\Psi^*\Delta=-\partial_1^2+\Delta'(x_1).$$

We would like to transform the boundary condition

$$B'\widetilde{\rho}^2u=0$$

into a form which is more common in practice. Let $\widetilde{\rho}^2u=(g_1,g_2)$. Then the boundary conditions read

$$b_{11}g_1+\quad b_{12}g_2\quad=0, \tag{5.1}$$

$$b_{12}g_1+\quad(1-b_{11})g_2\quad=0. \tag{5.2}$$

Let $\Pi_1' := \mathbf{1}_{\{1\}}(b_{11})$ act on (5.1). Note that $(\Pi_1' b_{12})^2 = \Pi_1'(1-b_{11})b_{11} = 0$. Set $\Pi_2' = \mathrm{Id} - \Pi_1'$. We obtain

$$\begin{aligned} \Pi_1' g_1 & & &= 0, & (5.3)\\ \Pi_2' b_{11} g_1 + & \quad \Pi_2' b_{12} g_2 & &= 0, & (5.4)\\ \Pi_2' b_{12} g_1 + & \quad \Pi_2'(1-b_{11}) g_2 & &= 0, & (5.5) \end{aligned}$$

for $\Pi_1'(1-b_{11}) = \Pi_1' b_{12} = 0$. Note also that (5.4) follows from (5.5) by applying b_{12}, since $b_{12}^2 = b_{11}(1-b_{11})$ and $1-b_{11}$ is bijective on $\operatorname{ran}\Pi_2'$.

Since (5.5) determines $\Pi_2' g_2$ uniquely let us introduce the (possibly unbounded) operator G' whose graph is given by

$$\mathscr{G}(G') := \big\{(x,y) \in L^2(\Gamma, E') \oplus L^2(\Gamma, E') \;\big|\; \Pi_2' b_{12} x + \Pi_2'(1-b_{11})y = 0. \quad \Pi_1' y = 0\big\}.$$

In fact, we have

$$G' = \Pi_2' b_{12}(1 - b_{11} + \Pi_1')^{-1}.$$

Since

$$\operatorname{ran} \mathbf{1}_{[0,1]\setminus(1-\varepsilon,1)}(b_{11}) \subset \operatorname{ran} 1 - b_{11} + \Pi_1'$$

for all $\varepsilon > 0$ it follows that $1 - b_{11} + \Pi_1'$ has always dense range. Consequently, $(1 - b_{11} + \Pi_1')^{-1}$ is unbounded self-adjoint and since $1 - b_{11} + \Pi_1'$ commutes with $\Pi_2' b_{12}$ it follows that the domain of $(1 - b_{11} + \Pi_1')^{-1}$ is invariant under $\Pi_2' b_{12}$. We conclude that G' is also an unbounded self-adjoint operator in $L^2(\Gamma, E')$.

We obtain

$$\begin{aligned} \Pi_1' g_1 &= 0,\\ \Pi_2'(g_2 + G' g_1) &= 0. \end{aligned} \tag{5.6}$$

Proposition 4.5.2. *Boundary conditions of the this form induce well-posed self-adjoint realisations Δ_B if and only if $1 \notin \operatorname{spec}_{\mathrm{ess}} G'$*

Proof. We only have to compute b_{11} and b_{12} when Π_1', Π_2' and G' are given. By $b_{12}^2 = b_{11}(1-b_{11})$, $G'(1-b_{11}) = b_{12}$ and $\mathbf{1}_{\{1\}}(b_{11}) = \Pi_1'$, one easily finds that

$$b_{12} := G'(\mathrm{Id} + G'^2)^{-1}, \quad b_{11} := G'^2(\mathrm{Id} + G'^2) + \Pi_1'.$$

By the spectral mapping theorem $1 \in \operatorname{spec}_{\mathrm{ess}} \in G'$ if and only if $1/2 \in \operatorname{spec}_{\mathrm{ess}} b_{12}$ which, by Theorem 4.5.1 means that B is well-posed for Δ. ☐

If we replace $g_1 = \Phi^{1/2}\gamma^0 u$, $g_2 = \Phi^{-1/2}\gamma^1 u$, then our boundary condition is equivalent to

$$\left\{ \begin{aligned} \Pi_1 \gamma^0 u &= 0\\ \Pi_2(\gamma^1 u + G\gamma^0 u) &= 0 \end{aligned} \right\}, \tag{5.7}$$

where

$$\Pi_1 = \Phi^{-1/2}\Pi_1'\Phi^{1/2}, \quad \Pi_2 = \Phi^{1/2}\Pi_2'\Phi^{-1/2}$$

and $G = \Phi^{1/2} G' \Phi^{1/2}$, with domain

$$G : \Phi^{-1/2}\mathscr{D}(G') \longrightarrow H^{-1/2}(\Gamma, E').$$

The system (5.7) is exactly the type of boundary conditions that is considered by Grubb in [Gru03] and [Gru05] where it is additionally assumed that Π_1, Π_2 and G are pseudodifferential of order 0, 1, respectively.

We notice that when $u \in \mathscr{D}(\Psi_* \Delta_{B'})$ then

$$\Phi^{1/2}\gamma^0 u \in \mathscr{D}(G').$$

The projections Π_1, Π_2 may in general be rather unpleasant. For instance, they act as bounded operators on the Sobolev spaces $H^{1/2}(\Gamma, E')$ and $H^{-1/2}(\Gamma, E')$. It is not clear whether they are operators of order 0 on the full Sobolev scale. However, we know that their kernels are perpendicular, i.e.

$$\ker_{L^2} \Pi_1 \perp_{L^2} \ker(\Pi_2 \cap L^2(\Gamma, E'))$$

since $\big(\ker \Pi_1' \Phi^{1/2}\big)^{\perp} = \operatorname{ran} \Phi^{1/2}\Pi_1' = \ker \Pi_2' \Phi^{-1/2}$. Similarly,

$$G : \big\{ g \in H^{1/2}(\Gamma, E') \;\big|\; \Phi^{1/2} g \in \mathscr{D}(G') \big\} \to H^{-1/2}(\Gamma, E')$$

need not be self-adjoint. However, we have the following symmetry property

$$\langle Gg, h \rangle = \langle g, Gh \rangle, \quad g, h \in \mathscr{D}(G)$$

in the distributional sense.

Example 4.5.3. When B' represents absolute/relative boundary conditions, then

$$\Pi_1 = Q_{a/r}, \quad \Pi_2 = Q_{r/a}.$$

G is given by the endomorphisms

$$G_{a/r} = -\sum_{j=2}^{n} \lambda_j Q_{a/r}(e_j) Q_{r/a},$$

cf. the remarks following Theorem 4.3.8. Observe that, since $\nu \perp e_j$, $Q_{a/r}(e_j)$ commutes with $Q_{r/a} = Q_{r/a}(\nu)$.

Note that if 1 is not isolated in the spectrum of b_{11}, then G' is unbounded. Thus G is not an operator of order 1 on the Sobolev scale $H^s(\Gamma, E')$. Furthermore, not every spectral projection of an operator of order 0 is again of order 0.

Example 4.5.4. Let $b : S^1 \to [0,1]$ be a smooth function such that $b^{-1}(1)$ equals some small neighbourhood K of 1. Set

$$b_{11} = M_b, \quad b_{12} := \sqrt{b_{11}(1 - b_{11})} = M_{\sqrt{b(1-b)}},$$

where M_f denotes multiplication by $f \in C(S^1)$. Let $\Omega = \mathbb{D}$ and let B' be defined as in Theorem 4.5.1. Then Δ_B is a well-posed self-adjoint extension of the standard Laplacian on the disk. However, for the spectral projection Π_1' we obtain

$$\Pi_1' = M_{\mathbf{1}_K} : L^2(\Gamma) \to L^2(\Gamma),$$

i.e. Π_1' is the multiplication operator of the characteristic function of K. It follows that Π_1' is not of order 0 on the scale $H^s(\Gamma)$.

Note that in fact G may be replaced by any $\widetilde{G}$ such that

$$\Pi_2 \widetilde{G} \Pi_2 = G.$$

Example 4.5.5 *(Natural boundary conditions for the Dirac-Laplacian).* Let Q denote some pseudodifferential orthogonal projection. Assume D is a formally self-adjoint Dirac type operator which takes the form

$$\Psi_* D = -J(\partial_1 + D^\Gamma(x_1))$$

near Γ. Set $D^\Gamma := D^\Gamma(0)$ and write D instead of $\Psi_* D$.

Then we may consider

$$\Delta = D^2, \quad \Delta_B := (D_Q)^t D_Q = D_{Q^{\mathrm{ad}}} D_Q.$$

If $D_Q : \mathscr{D}(D_Q) \to L^2(\Omega, E)$ is closed, then $(D_Q)^* D_Q$ is self-adjoint (by abstract functional analysis). Now, with the methods of Section 1.2, one can show that

$$\overline{D_Q} = D_{\max,Q}.$$

Hence, D_Q is closed if and only if Q is regular for D. Note that regularity for Q does not imply regularity of Q^{ad}.

Let us consider the operator

$$(D_Q)^t D_Q = \Delta_{B'} : \mathscr{D}(\Delta'_B) \to L^2(\Omega, E).$$

Since $Q^{\mathrm{ad}} = -J(\mathrm{Id} - Q)J$ the boundary conditions are:

$$\begin{aligned} Q\gamma^0 u \quad &= 0 \\ (\mathrm{Id} - Q)(\gamma^1 u + D^\Gamma \gamma^0 u) \quad &= 0 \end{aligned} \tag{5.8}$$

We may set $\Phi = ((D^\Gamma)^2 + 1)^{1/2}$. Then $\Pi_2 = (\mathrm{Id} - Q)$. Hence, G' is given by

$$G' = \Phi^{-1/2} \Pi_2 D^\Gamma \Pi_2 \Phi^{-1/2}.$$

We obtain

$$G' \equiv (\mathrm{Id} - Q)(\operatorname{sign} D^\Gamma)(\mathrm{Id} - Q) \mod \Psi^{-1}(\Gamma, E')$$

since $[\Phi^{-1/2}, \Pi_2]$ is an operator of order $-3/2$ and $D(D^2+1)^{-1/2}$ differs from $\operatorname{sign} D^\Gamma$ by an operator in $\Psi^{-1}(\Gamma, E')$. Here, $\operatorname{sign} D^\Gamma$ may be defined arbitrarily on $\ker D^\Gamma$.

By Proposition 4.5.2 Δ_B is well-posed if and only if $1 \notin \operatorname{spec}_{\mathrm{ess}} G'$ which is the case if and only if

$$(\mathrm{Id} - Q)(\operatorname{sign} D^\Gamma - \mathrm{Id})(\mathrm{Id} - Q)$$

is Fredholm. Since $\operatorname{sign} D^\Gamma - \mathrm{Id} = -2P_{\leq 0}(D^\Gamma)$ (up to an operator of finite rank) it follows from Proposition A.1.12(iii) that Δ_Q is a self-adjoint Fredholm realisation if and only if

$$(\ker P_{\leq 0}(D^\Gamma), \operatorname{ran}(\mathrm{Id} - Q)) = (\operatorname{ran} P_{>0}(D^\Gamma), \ker Q)$$

is a left Fredholm pair. By Theorem 2.1.4 and Theorem 3.2.4 this is equivalent to the regularity of Q for D.

An important example of a regular but not well-posed boundary condition for D is given by $\gamma^0 u = 0$. The realisation is D_{Id}, its adjoint D_0 and the corresponding Laplacian

$$D_0 D_{\mathrm{Id}}$$

is the Dirichlet realisation of D^2.

4.5.2 Criteria for Lower Boundedness

We will now seek for precise criteria for boundary conditions to induce self-adjoint lower bounded realisations of Δ.

Lemma 4.5.6. *Assume $\Delta_{Dir} > 0$.*

(i) *Every $u \in H^2(\Omega, E)$ admits a unique decomposition*

$$u = u_D + u_h$$

such that u_h is harmonic and u_D satisfies Dirichlet boundary conditions, i.e. $\Delta u_h = 0$, $\gamma^0 u_D = 0$.

(ii) $\langle \Delta u, u \rangle = \langle \Delta u_D, u_D \rangle + \langle \gamma^1 u - T_{DN} \gamma^0 u, \gamma^0 u \rangle$.

Proof. (i) Since Δ_{Dir} is invertible u_h is uniquely determined by requiring $\Delta u_D = \Delta u$ and $\gamma^0 u_D = 0$. Setting $u_h := u - u_D$ defines a harmonic function.

(ii) We have

$$\begin{aligned} \langle \Delta u, u \rangle &= \langle \Delta u_D, u_D + u_h \rangle \\ &= \langle \Delta u_D, u_D \rangle + \langle \gamma^1 u_D, \gamma^0 u_h \rangle - \langle \gamma^0 u_D, \gamma^1 u_h \rangle \\ &= \langle \Delta u_D, u_D \rangle + \langle \gamma^1 u - T_{DN} \gamma^0 u, \gamma^0 u \rangle, \end{aligned}$$

since $\gamma^0 u_D = 0$, $\gamma^1 u_D = \gamma^1 u - \gamma^1 u_h$, $\gamma^1 u_h = T_{DN} \gamma^0 u_h$ and $\gamma^0 u_h = \gamma^0 u$. □

Recall from the previous section that $-G' + \mathrm{Id}$ is a self-adjoint Fredholm operator when B is a well-posed boundary condition for Δ.

Theorem 4.5.7. *Let Δ_B be a well-posed self-adjoint realisation of Δ. Let $G' = -\Phi^{-1/2} G \Phi^{-1/2}$ and Π_1', Π_2' be defined as above.*

(i) *Assume $\Delta_{Dir} > 0$. Δ_B is non-negative if and only if*

$$-G' + \Pi_2' \Phi^{-1/2} T_{DN} \Phi^{-1/2} \Pi_2' \geq 0,$$

(ii) *If $-G' + \mathrm{Id}$ is unbounded below or has essential spectrum on the negative real axis, then Δ_B is unbounded below. Otherwise, if $\mathrm{spec}(-G' + \mathrm{Id}) \cap \mathbb{R}_-$ is finite (counted with multiplicities), then Δ_B is bounded.*

Proof. (i) Let $u \in \mathscr{D}(\Delta_B)$. By Lemma 4.5.6 we have $u = u_D + u_h$ and

$$\langle \Delta u, u \rangle = \langle \Delta u_D, u_D \rangle + \langle \gamma^1 u - T_{DN} \gamma^0 u, \gamma^0 u \rangle.$$

It is obviously sufficient for Δ_B to be non-negative that

$$\langle \gamma^1 u - T_{DN} \gamma^0 u, \gamma^0 u \rangle \geq 0$$

for all $u \in \mathscr{D}(\Delta_B)$. We claim that it is also necessary. Let $u \in \mathscr{D}(\Delta_B)$. Recall that the Friedrichs extension of $\Delta_{\min}$ coincides with Δ_{Dir}. Since $u_D \in \mathscr{D}(\Delta_{Dir})$ we find $u_n \in H_0^2(\Omega, E)$, $n \in \mathbb{N}$, such that[5]

$$\langle \Delta(u_n - u_D), u_n - u_D \rangle \xrightarrow[n \to \infty]{} 0.$$

[5] In other words, we simply use that $\mathscr{D}(\Delta_{Dir})$ is contained in the H^1-completion of $H_0^2(\Omega, E)$.

Note that $u - u_n \in \mathscr{D}(\Delta_B)$ and $\gamma^0(u - u_n) = \gamma^0 u$. Moreover, the above decomposition for $u - u_n$ reads

$$u - u_n = \ (u_D - u_n) \ + \ u_h.$$

Hence

$$\langle \gamma^1 u - T_{DN}\gamma^0 u, \gamma^0 u\rangle = \langle \Delta_B(u - u_n), (u - u_n)\rangle - \langle \Delta(u_D - u_n), (u_D - u_n)\rangle.$$

Since, $\langle \Delta(u_D - u_n), (u_D - u_n)\rangle$ converges to 0 as $n \to \infty$ we conclude that

$$\langle \gamma^1 u - T_{DN}\gamma^0 u, \gamma^0 u\rangle \geq 0$$

for all $u \in \mathscr{D}(\Delta_B)$ if $\Delta_B \geq 0$.

As in the previous section let $\widetilde{\rho}^2 u = (g_1, g_2)$, i.e.

$$g_1 = \Phi^{1/2}\gamma^0 u, \quad g_2 = \Phi^{-1/2}\gamma^1 u.$$

Now, g_1 ranges over $\operatorname{ran}\Pi_1' \cap \mathscr{D}(G')$ and $\Pi_2' g_2 = -G' g_1$. Hence, for all $g \in \mathscr{D}(G')$ we have

$$\begin{aligned}\langle \gamma^1 u - T_{DN}\gamma^0 u, \gamma^0 u\rangle &= \langle g_2 - \Phi^{-1/2}T_{DN}\Phi^{-1/2}g_1, g_1\rangle \\ &= \langle -G' g - \Pi_2'\Phi^{-1/2}T_{DN}\Phi^{-1/2}\Pi_2' g, g\rangle\end{aligned} \tag{5.9}$$

if $\Pi_2' g = g_1$. Note that

$$\Pi_2'\Phi^{-1/2}T_{DN}\Phi^{-1/2}\Pi_2'$$

is bounded. Hence,

$$-G' - \Pi_2'\Phi^{-1/2}T_{DN}\Phi^{-1/2}\Pi_2'$$

is a (possibly unbounded) self-adjoint operator which is positive if and only if Δ_B is.

(ii) It follows from (i) that Δ_B is bounded below if and only if there exists μ such that

$$-G' - \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2' \geq 0$$

for some $\mu > 0$. Note that for all $\mu > 0$

$$R_1(\mu) := \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2' + \Pi_2'$$

is an operator of order -1 for

$$\Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2' + \Pi_2' = \Pi_2'\Phi^{-1}\big(-(\Phi^2 + \mu^2)^{1/2} + \Phi\big)\Pi_2' \mod \Psi^{-1}_{\mathrm{cl}}(\Gamma, E').$$

In particular, $R_1(\mu)$ is compact and it follows that

$$-G' - \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2' : \operatorname{ran}\Pi_2' \to \operatorname{ran}\Pi_2'$$

is Fredholm (by Proposition 4.5.2). Now, there are two cases we have to distinguish:

a) Suppose $-G' - \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2'$ has essential spectrum on the negative real axis or is even unbounded below. Then this is the case for all $\mu' > \mu$ since

$$\Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2' \equiv -\Pi_2'$$

up to compact operators. It follows that $\Delta_B + \mu^2$ is positive for no $\mu \in \mathbb{R}$. This means Δ_B is unbounded below.

b) Suppose that for some μ (in particular, $\mu = 0$ is allowed)

$$-G' + \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2'$$

has finitely many eigenvalues in $\mathbb{R}_-$. More precisely, we assume that

$$V := \operatorname{ran} \mathbf{1}_{(-\infty,0)}(-G' - \Pi_2'\Phi^{-1/2}T_{DN,\mu}\Phi^{-1/2}\Pi_2')$$

is finite dimensional. Then it follows that

$$\langle \gamma^1 u - T_{DN}\gamma^0 u, \gamma^0 u \rangle \geq 0$$

if $u \in \mathscr{D}(\Delta_B)$ and $\Phi^{1/2}\gamma^0 u \perp V$. Hence,

$$\langle \Delta_B u, u \rangle \geq 0$$

for all $u \in W$ where $W \subset \mathscr{D}(\Delta_B)$ is a closed subspace of finite codimension. If Δ_B were unbounded below then

$$\operatorname{ran} \mathbf{1}_{(-\infty,0)}(\Delta_B) \cap W \neq \{0\}$$

which is impossible since for all non-zero $u \in \mathscr{D}(\Delta_B) \cap \operatorname{ran} \mathbf{1}_{(-\infty,0)}(\Delta_B)$ we have

$$\langle \Delta_B u, u \rangle < 0.$$

It follows that Δ_B is bounded below.

This proves part (ii) of the theorem since the essential spectra of

$$-G' - \Pi_2'\Phi^{-1/2}T_{DN}\Phi^{-1/2}\Pi_2' : \operatorname{ran}\Pi_2' \to \operatorname{ran}\Pi_2'$$

and $-G' + \mathrm{Id}$ differ at most by $\{1\}$. Similarly, if one of them is unbounded below, then so is the other. □

Let us now characterise lower boundness in terms of Π_1, Π_2 and G. For this it seems indispensable to restrict attention to boundary conditions such that G, Π_1 and Π_2 have certain regularity properties. Recall that $\Phi = \sqrt{\Delta' + \mu^2}$ for some $\mu \geq 0$.

Assumption 4.5.8. Suppose that Π_1 and Π_2 are orthogonal projections in $L^2(\Gamma, E')$ such that $\Pi_1 + \Pi_2 = \mathrm{Id}$. Moreover, assume:

(i) $\Pi_1, \Pi_2, [\Pi_1, \Phi], [\Pi_2, \Phi] \in \mathrm{Op}^0((H^s(\Gamma, E'))_{s\in\mathbb{R}})$.

(ii) $G,\ [G, \Phi] \in \mathrm{Op}^1((H^s(\Gamma, E'))_{s\in\mathbb{R}})$.

(iii) G is symmetric, i.e. for all $g, h \in H^1(\Gamma, E')$, $\langle Gg, h\rangle = \langle g, Gh\rangle$.

Theorem 4.5.9. *The system*

$$\Pi_1\gamma^0 u = 0 \tag{5.10}$$
$$\Pi_2(\gamma^1 u + G\gamma^0 u) = 0 \tag{5.11}$$

defines a well-posed boundary condition for Δ if and only if

$$\Pi_2 G \Pi_2 - \Phi$$

is elliptic (in the sense of Remark A.2.3). The induced realisation, Δ_B, is bounded below if and only if $\Pi_2 G \Pi_2 - \Phi$ is bounded above.

Proof. As before, set $\Pi_1'' = \Phi^{1/2}\Pi_1\Phi^{-1/2}$, and $\Pi_2'' := \Phi^{-1/2}\Pi_2\Phi^{1/2}$ and

$$G' := \Phi^{-1/2}\Pi_2 G \Pi_2 \Phi^{-1/2}.$$

Note that G' is bounded and self-adjoint and that Π_1'' and Π_2'' differ from the corresponding orthogonal projections with the same kernel, Π_1', Π_2', by operators in $\mathrm{Op}^{-1+\varepsilon}$ with $\varepsilon > 0$ arbitrary small. Namely, by Proposition 1.2.4 and the Assumptions on Π_1, Π_2,

$$\Pi_1'' - \Pi_1''^*, \quad \Pi_2'' - \Pi_2''^* \in \mathrm{Op}^{-1+\varepsilon}$$

for any $\varepsilon > 0$. It follows that the boundary condition is equivalent to the system (5.6) where Π_1', Π_2' are orthogonal projections, $\Pi_1' + \Pi_2' = 1$ and $G'\Pi_2' = G'$. By Proposition 4.5.2, Δ_B is well-posed if and only if

$$-G' + \mathrm{Id} = \Phi^{-1/2}(-\Pi_2 G \Pi_2 + \Phi)\Phi^{-1/2}$$

is a bounded Fredholm operator on $L^2(\Omega, E)$. But this is equivalent to

$$-\Pi_2 G \Pi_2 + \Phi$$

being an elliptic element in Op^1 in the sense of Remark A.2.3. In this case, since the embedding $H^1(\Gamma, E') \to L^2(\Gamma, E')$ is compact, $\Pi_2 G \Pi_2 + \Phi$ is discrete. If it is bounded below then

$$\langle(-G' + \mathrm{Id})g, g\rangle = \langle(-\Pi_2 G \Pi_2 + \Phi)\Phi^{-1/2}g, \Phi^{-1/2}g\rangle \geq 0$$

on a subspace of $L^2(\Omega, E)$ of finite codimension. Hence, $\mathrm{spec}(-G' + \mathrm{Id}) \cap \mathbb{R}_-$ is finite and Δ_B is bounded below, by Theorem 4.5.7.

Otherwise, if $-\Pi_2 G \Pi_2 + \Phi$ is unbounded below, then

$$\langle(-G' + \mathrm{Id})g, g\rangle = \langle(-\Pi_2 G \Pi_2 + \Phi)\Phi^{-1/2}g, \Phi^{-1/2}g\rangle < 0$$

on an infinite dimensional subspace of $L^2(\Gamma, E')$ and Δ_B is unbounded below. □

Example 4.5.10. Let Π_1, Π_2 be pseudodifferential projections and set $G = \lambda D^\Gamma$. If $-\lambda\Pi_2(\mathrm{sign}\, D^\Gamma)\Pi_2 + 1$ is Fredholm then Δ_B is well-posed self-adjoint. However

$$\mathrm{spec}_{\mathrm{ess}}\big(-\lambda\Pi_2(\mathrm{sign}\, D^\Gamma)\Pi_2 + 1\big) \cap \mathbb{R}_-$$

may be non-empty. For instance, take $\Pi_1 = 0$, $\Pi_2 = \mathrm{Id}$ and let $\lambda \neq 1$. Then if $|\lambda| > 1$ it follows that the (seemingly innocent) boundary condition

$$\gamma^1 u + \lambda D^\Gamma \gamma^0 u = 0$$

induces a well-posed self-adjoint realisation whose spectrum is unbounded below!

Example 4.5.11. We finally illustrate the semi-boundedness condition given in Theorem 4.5.9 by a trivial example. Let Q be a regular (not necessarily well-posed) pseudodifferential boundary condition for some Dirac-type operator D with product structure

$$D = -J(\partial_1 + D^\Gamma)$$

near Γ. As we have seen above $(D_Q)^* D_Q$ is a well-posed self-adjoint realisation of $\Delta = D^2$. Of course it is non-negative! The boundary condition for this Dirac-Laplacian is

$$Q\gamma^0 u = 0, \tag{5.12}$$
$$(\mathrm{Id} - Q)(\gamma^1 u + D^\Gamma \gamma^0 u) = 0. \tag{5.13}$$

Hence, we are in the situation of the previous example, but with $\lambda = 1$, $\Pi_1 = Q$ and $G = (\mathrm{Id} - Q)D^\Gamma(\mathrm{Id} - Q)$. But up to a compact perturbation we have

$$\begin{aligned}\Pi_2 G \Pi_2 - \Phi &\equiv (\mathrm{Id} - Q)(D^\Gamma - |D^\Gamma|)(\mathrm{Id} - Q) - Q|D^\Gamma|Q \\ &= -2(\mathrm{Id} - Q)\mathbf{1}_{<0}(D^\Gamma)|D^\Gamma|(\mathrm{Id} - Q) - Q|D^\Gamma|Q.\end{aligned}$$

Here, $\Pi_2 G \Pi_2 - \Phi$ is obviously bounded above so that the semi-boundedness criterion is trivially satisfied.

Appendix A

Functional Analysis

A.1 Fredholm Theory

In the following X, Y, Z will always denote Hilbert spaces. Let $T : X \to Y$ be a continuous operator.

Definition A.1.1. T is called left Fredholm if $\operatorname{ran} T$ is closed and $\dim \ker T < \infty$. T is called right Fredholm if $\operatorname{ran} T$ is closed and $\operatorname{codim} \operatorname{ran} T < \infty$. T is called semi Fredholm if it is left or right Fredholm. T is called Fredholm if it is left and right Fredholm.

Proposition A.1.2. *Let $X \hookrightarrow Z$ be a continuous embedding. If T is left Fredholm then there exists $C > 0$ such that for all $x \in X$ we have*

$$\|x\|_X \leq C\big(\|x\|_Z + \|Px\|_Y\big)$$

Proof. The proposition is a consequence of the following lemma.

Lemma A.1.3. *Let $S \subset X$ be a finite-dimensional subspace. Set $\widetilde{X} = X/S$. Then, for all $x \in X$ we have the estimate $\|x\|_X \leq \operatorname{const}\big(\|[x]\|_{\widetilde{X}} + \|x\|_Z\big)$.*

Proof of the lemma. We have $Z = S \oplus S^{\perp_Z}$ and

$$V := X \cap S^{\perp_Z} = \big\{x \in X \;\big|\; x \perp_Z S\big\}$$

is closed in X and $X = S \oplus V$. Now, decompose $x = x_1 + x_2 \in X$ w.r.t. S and V. Then, $\|x\|_X \leq \|x_1\|_X + \|x_2\|_X$. Hence,

$$\|x\|_X \leq \operatorname{const}(\|x_1\|_Z + \|[x_2]\|_{\widetilde{X}})$$

for S is finite dimensional and $V \to \widetilde{X}, \quad x \mapsto [x]$ is an isomorphism. □

Let $S := \ker T$, $\widetilde{X} := X/S$ and define $\widetilde{T} : \widetilde{X} \to Y$ by $\widetilde{T}[x] = Tx$. Note that $\widetilde{T}$ is continuous with closed range. By the open mapping theorem T is bounded away from 0. From the lemma above it follows that

$$\|x\|_X \leq \operatorname{const}\big(\|x\|_Z + \|[x]\|_{\widetilde{X}}\big) \leq \operatorname{const}\big(\|x\|_Z + \|\widetilde{T}[x]\|_Y\big).$$

This proves the estimate since $\widetilde{T}[x] = Tx$. □

Proposition A.1.4. *If $X \hookrightarrow Z$ is a compact embedding and there exists $C > 0$ such that for all $x \in X$ we have*

$$\|x\|_X \leq C\big(\|x\|_Z + \|Tx\|_Y\big),$$

then T is left Fredholm.

Proof. Assume the above inequality holds. It follows that $\ker T$ is finite-dimensional since, otherwise, there would be an orthonormnal sequence in $\ker T$ that converges in Z. From the estimate it follows that such a sequence is a Cauchy sequence in X.

As before set $\widetilde{T} : X/\ker T \to Y$, $\widetilde{T}[x] = Tx$. We claim $\widetilde{T}$ is bounded away from 0.

Namely, assume there exists a series $([x_n])$ in $\widetilde{T}X$ such that Tx_n converges to 0. Then, by adding $s_n \in \ker S$ to x_n, we can assume that (x_n) is bounded in X. Moreover, by taking a subsequence we can assume that (x_n) converges in Z. By the inequality

$$\|x_n - x_m\|_X \leq \big(\|x_n - x_m\|_Z + \|T(x_n - x_m)\|_Y\big)$$

we see that x_n converges in X. Hence, $[x_n]$ converges to some $[y]$. Since Tx_n tends to 0, we have $[y] = 0$, in contradiction to the assumption that $[x_n] = 1$.

Hence, $\widetilde{T}$ is bounded away from 0 and therefore $\operatorname{ran} T = \operatorname{ran} \widetilde{T}$ is closed in Y. $\square$

Proposition A.1.5. *If $\operatorname{ran} T$ is finite codimensional then T is semi Fredholm.*

Proof. If $\operatorname{ran} T$ has finite codimension then we find $y_1, ..., y_k$ such that $Y = \operatorname{ran} T \oplus \operatorname{span}\{y_1, ..., y_k\}$ in the algebraic sense. Then consider

$$\widetilde{T} : X/\ker T \to \widetilde{Y} := Y/\operatorname{span}\{y_1, ..., y_k\}, \quad [x] \to [Tx].$$

$\widetilde{T}$ is continuous and bijective. Hence, $\widetilde{T}^{-1}$ is bounded. Set $\overline{T} : X/\ker T \to Y, [x] \mapsto Tx$ and $\pi : Y \to \widetilde{Y}$ the natural projection. Then

$$\operatorname{ran} T = \ker\,\big(\operatorname{Id} - \overline{T} \circ \widetilde{T}^{-1} \circ \pi\big).$$

Hence $\operatorname{ran} T$ is closed and T is right Fredholm. $\square$

Proposition A.1.6. *$T : X \to Y$ is left Fredholm if and only if T^*T is Fredholm.*

Proof. We have $\ker T = \ker T^*T$. Moreover, T is left Fredholm if and only if $\dim\ker T < \infty$ and

$$\|x\|_X \leq C\|Tx\|_Y,$$

for all $x \perp \ker T$ and some constant $C > 0$. This is equivalent to

$$\langle x, x\rangle_X \leq C^2\langle T^*Tx, x\rangle_X$$

for all $x \perp \ker T^*T$. Equivalently, $T^*T|_{(\ker T)^\perp} : (\ker T)^\perp \to (\ker T)^\perp$ is bounded away from 0. Together with the fact that $\dim\ker T^*T < \infty$ this means that T^*T is Fredholm. $\square$

The following definition goes back to [Kat76, Section IV.4].

Definition A.1.7. Let Λ_1, Λ_2 be linear subspaces of X. We say that (Λ_1, Λ_2) is a left Fredholm pair if its nullity is finite, i.e.

$$\operatorname{nul}(\Lambda_1, \Lambda_2) = \dim \Lambda_1 \cap \Lambda_2 < \infty,$$

and $\Lambda_1 + \Lambda_2$ is closed. (Λ_1, Λ_2) is called a right Fredholm pair if its deficiency is finite, i.e.

$$\operatorname{def}(\Lambda_1, \Lambda_2) = \operatorname{codim} \Lambda_1 + \Lambda_2 < \infty,$$

and $\Lambda_1 + \Lambda_2$ is closed. (Λ_1, Λ_2) is a Fredholm pair if it is left and right Fredholm. A Fredholm pair (Λ_1, Λ_2) is called invertible if its nullity and deficiency are trivial.

Proposition A.1.8. *Let Λ_1, Λ_2 be Hilbert spaces that are continuously embedded into X (not necessarily with closed image!). (Λ_1, Λ_2) is a right Fredholm pair in X if and only if*

$$\operatorname{def}(\Lambda_1, \Lambda_2) < \infty.$$

Proof. Consider the topological direct sum $\Lambda_1 \oplus_{\text{top}} \Lambda_2$ with the norm

$$\|(x,y)\|^2_{\Lambda_1 \oplus_{\text{top}} \Lambda_2} = \|x\|^2_{\Lambda_1} + \|y\|^2_{\Lambda_2}$$

and define $T : \Lambda_1 \oplus_{\text{top}} \Lambda_2 \to X, \quad T(x,y) = x + y.$

Applying Proposition A.1.5 to T we obtain that $\Lambda_1 + \Lambda_2$ is closed if $\operatorname{ran} T = \Lambda_1 + \Lambda_2$ has finite codimension. □

Remark A.1.9. For (Λ_1, Λ_2) to form a right Fredholm pair it is not necessary that Λ_i be closed in X. For instance, set $\Lambda_1 = X$ and let Λ_2 be any dense proper subset of X. More interesting and less trivial counterexamples are given in Section 2.

Finally, there is a notion of Fredholm pairs of projections, say $P, Q \in \mathscr{L}(X)$.

Definition A.1.10. We say (P, Q) is a (left/right) Fredholm pair (of projections) if $(\ker P, \operatorname{ran} Q)$ is a (left/right) Fredholm pair of subspaces. A Fredholm pair (P, Q) is called invertible if $(\ker P, \operatorname{ran} Q)$ is invertible.

The following criterion turns out useful for subspaces that are given as kernels or ranges.

Lemma A.1.11. *Let $T : X \to Y$, $S : Y \to Z$ be bounded operators with closed range. Then*

$$(\ker S, \operatorname{ran} T)$$

is a left Fredholm pair if and only if

$$S|_{\operatorname{ran} T} : \operatorname{ran} T \to \operatorname{ran} S$$

is left Fredholm.

Proof. We have

$$\ker S|_{\operatorname{ran} T} = \ker S \cap \operatorname{ran} T.$$

So it remains to show that $\ker S + \operatorname{ran} T$ is closed if $S|_{\operatorname{ran} T}$ has closed range. If $\ker S + \operatorname{ran} T$ is closed then

$$S(\operatorname{ran} T) = S(\ker S + \operatorname{ran} T) = S(\{u \in \ker S + \operatorname{ran} T \mid u \perp \ker S\})$$

which is closed since

$$\{u \in \ker S + \operatorname{ran} T \mid u \perp \ker S\} \subset (\ker S)^\perp$$

is closed and $S|_{\ker S^\perp} : (\ker S)^\perp \to \operatorname{ran} S$ is an isomorphism.

On the other hand if $S(\operatorname{ran} T)$ is closed then

$$\big\{u \in \ker S + \operatorname{ran} T \;\big|\; u \perp \ker S\big\} = \big(S|_{\ker S^\perp}\big)^{-1}(S(\operatorname{ran} T))$$

is also closed. Hence the direct sum

$$\ker S + \operatorname{ran} T = \big\{u \in \ker S + \operatorname{ran} T \;\big|\; u \perp \ker S\big\} \oplus_\perp \ker S$$

is closed. □

The Fredholm pair condition is not(!) symmetric in P and Q since $(\ker P, \operatorname{ran} Q)$ may be a Fredholm pair even when $\operatorname{ran} P = \ker Q$. However, this cannot happen when P and Q are symmetric.

Proposition A.1.12. *Let P, Q be orthogonal projections.*

(i) *(P, Q) is a Fredholm pair if and only if*

$$\pm \operatorname{Id} + P - Q : X \to X$$

are Fredholm. This, is turn, is equivalent to

$$P : \operatorname{ran} Q \to \operatorname{ran} P$$

being Fredholm.

(ii) *(P, Q) is left/right Fredholm if and only if*

$$P : \operatorname{ran} Q \to \operatorname{ran} P$$

is left/right Fredholm.

(iii) *(P, Q) is left Fredholm if and only if*

$$QP : \operatorname{ran} Q \to \operatorname{ran} Q$$

is Fredholm.

(iv) *(P, Q) is an invertible pair if and only if*

$$\pm \operatorname{Id} + P - Q : X \to X$$

are isomorphisms. This, is turn, is equivalent to

$$P : \operatorname{ran} Q \to \operatorname{ran} P$$

being isomorphisms.

Proof. The proof of (i), (ii), (iv) can be found [ASS94]. For completeness we indicate the proofs.

(i) and (ii) follow from Lemma A.1.11 noting that

$$(P : \operatorname{ran} Q \to \operatorname{ran} P)^* = \big(Q : \operatorname{ran} P \to \operatorname{ran} Q\big).$$

It remains to consider $\operatorname{Id} + P - Q$. Clearly, $x \in \ker \operatorname{Id} + P - Q$ if and only

$$\langle Px, x\rangle + \langle(\operatorname{Id} - Q)x, x\rangle = 0,$$

hence $\ker(\operatorname{Id}+P-Q) = \ker P \cap \operatorname{ran} Q$. Similarly, $\ker(-\operatorname{Id}+P-Q) = \ker Q \cap \ker P$. In order to finish the proofs for (i) and (iv) we have to show that $\operatorname{ran}(\operatorname{Id}+P-Q)$ is closed if and only if $\ker P + \operatorname{ran} Q$ is closed. Observe that, w.r.t. to the decompositions $H = \ker Q \oplus \operatorname{ran} Q$, $H = \ker P \oplus \operatorname{ran} P$, the operator $\operatorname{Id}+P-Q$ takes the form

$$\operatorname{Id}-Q+P = \begin{pmatrix} (\operatorname{Id}-P)|_{\ker Q} & 0 \\ 2P|_{\ker Q} & P|_{\operatorname{ran} Q} \end{pmatrix},$$

and $(\operatorname{Id}-P)|_{\ker Q} = \big((\operatorname{Id}-Q)|_{\ker P}\big)^*$ has closed range if and only if $\operatorname{ran} Q + \ker P$ is closed (cf. the proof of Lemma A.1.11).

For (iii) note once again that the adjoint of

$$T : \operatorname{ran} Q \to \operatorname{ran} P, \quad Tu = Pu$$

is given by $T^* : \operatorname{ran} P \to \operatorname{ran} Q$, $T^* = Qu$. The composition of these yields

$$T^*T : \operatorname{ran} P \to \operatorname{ran} P, \quad T^*Tu = PQu.$$

Hence, (iii) follows from Proposition A.1.6.

(iv): In the proof of (i) we have seen that $\pm\operatorname{Id}+P-Q$ are Fredholm and that $\ker(\operatorname{Id}+P-Q) = \ker P \cap \operatorname{ran} Q$ and $\ker(-\operatorname{Id}+P-Q) = \operatorname{ran} P \cap Q$. Hence, (P,Q) is invertible if and only if $\pm\operatorname{Id}+P-Q$ are isomorphisms. $\square$

A.2 The Spectral Invariance of a Certain Spectral Triple

Let $\Lambda : H \supset \mathscr{D}(\Lambda) \to H$ be an unbounded self-adjoint operator and set

$$H_\infty := \bigcap_{k\geq 0} \mathscr{D}(|\Lambda|^k), \quad H_t := H_t(\Lambda) = \overline{H}_\infty^{\|\cdot\|_t}, \quad H_{-\infty} = \bigcup_{t\in\mathbb{R}} H_t,$$

where

$$\|x\|_s := \|(\Lambda^2+1)^{s/2}x\|_H.$$

For simplicity we write Op^μ for $\operatorname{Op}^\mu((H_t)_{t\in\mathbb{R}})$. Observe that for $T_i \in \operatorname{Op}^{\mu_i}$, $i=1,2$ we have

$$[T_1,T_2] = T_1T_2 - T_2T_1 \in \operatorname{Op}^{\mu_1+\mu_2}.$$

To Λ we associate the following operator algebra

$$\mathscr{A} := \mathscr{A}(\Lambda) = \big\{T \in \operatorname{Op}^0 \;\big|\; [T,|\Lambda|] \in \operatorname{Op}^0\big\}.$$

Clearly, $(H, |\Lambda|, \mathscr{A})$ is a spectral triple (cf., e.g., [Con95]).

Proposition A.2.1. *$\mathscr{A}$ is a spectrally invariant $*$-subalgebra of $\mathscr{B}(H)$.*

Proof. Clearly, $\mathscr{A}$ is a $*$-subalgebra since

$$[T_1T_2, |\Lambda|] = T_1[T_2,|\Lambda|] - [T_1,|\Lambda|]T_2 \in \operatorname{Op}^0$$

and

$$[T^*, |\Lambda|]x = -[T,|\Lambda|]^*x$$

for all $T, T_1, T_2 \in \mathscr{A}$ and $x \in H_\infty$. We have used that Op^0 is involutive, cf. [BL01, Prop. 2.2].

Let us now show that $\mathscr{A}$ is spectrally invariant. Assume T is invertible in $\mathscr{B}(H_s, H_s)$ for some $s \in \mathbb{R}$. We can form the operators

$$(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1} T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|), \quad (\mathrm{Id} + \tfrac{1}{n}|\Lambda|) T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1},$$

which are operators of order 0 and satisfy

$$(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1} T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|) = T + \tfrac{1}{n}(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}[T, |\Lambda|] \tag{2.1}$$
$$(\mathrm{Id} + \tfrac{1}{n}|\Lambda|) T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1} = T + \tfrac{1}{n}[|\Lambda|, T](\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}. \tag{2.2}$$

Since

$$\begin{aligned}\|(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}\|_{s,s} &= \|(1+\Lambda^2)^{s/2}(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}(1+\Lambda^2)^{-s/2}\|_{0,0} \\ &\leq \sup_{\lambda \in \mathbb{R}} \big|(1+\lambda^2)^{s/2}(\mathrm{Id} + \tfrac{1}{n}|\lambda|)^{-1}(1+\lambda^2)^{-s/2}\big| \leq 1,\end{aligned}$$

we find that for sufficiently large $n \in \mathbb{N}$

$$(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1} T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|), \quad (\mathrm{Id} + \tfrac{1}{n}|\Lambda|) T (\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}$$

are invertible operators in $\mathscr{B}(H_s)$ since $\mathrm{GL}(H_s)$ is open in $\mathscr{B}(H_s)$. But for every $n \in \mathbb{N}$,

$$(\mathrm{Id} + \tfrac{1}{n}|\Lambda|)^{-1}, \quad (\mathrm{Id} + \tfrac{1}{n}|\Lambda|)$$

are in Op^{-1}, Op^{1}, resp. It follows that, $T \in \mathrm{GL}(H_{s+1})$ and $T \in \mathrm{GL}(H_{s-1})$ and hence, by induction, T^{-1} is in $\mathscr{B}(H_{s+k}, H_{s+k})$ for all $k \in \mathbb{Z}$. From interpolation theory it follows that $T^{-1} \in \mathrm{Op}^0$. Since T^{-1} is in Op^0 we finally see that

$$[T^{-1}, |\Lambda|] = T^{-1}[|\Lambda|, T]T^{-1}$$

is in Op^0. Hence $T^{-1} \in \mathscr{A}$. □

It is well-known that spectral invariance implies invariance under holomorphic calculus, i.e. if $T \in \mathscr{A}$ and $f : \operatorname{spec} T \to \mathbb{C}$ extends to a holomorphic function on an open neighbourhood of $\operatorname{spec} T$, then $f(T) \in \mathscr{A}$. In particular, if $\operatorname{spec} T$ decomposes into two components U and V, such that U and V are open and closed in $\operatorname{spec} T$, then $\mathbf{1}_U(T) \in \mathscr{A}$.

Namely, by choosing a suitable smooth curve $\gamma : [0, 2\pi] \to \rho(B)$ that encircles U once (but no points of V) we may define a projection in Op^0 by[1]

$$\mathbf{1}_U(B) = \frac{1}{2\pi i}\int_\gamma (\lambda - B)^{-1} d\lambda.$$

It is straightforward to check the boundedness condition for the commutator with Φ since

$$\begin{aligned}[\mathbf{1}_U(B), \Phi] &= \frac{1}{2\pi i}\int_\gamma [(\lambda - B)^{-1}, \Phi] d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma (\lambda - B)^{-1}[B, \Phi](\lambda - B)^{-1} d\lambda \in \mathrm{Op}^0.\end{aligned}$$

More generally, if f is holomorphic on an open neighbourhood of $\operatorname{spec} T$, then since $\operatorname{spec} T$ is compact it can be covered by finitely many connected open subsets U_j of $\mathbb{C}$

[1] Note that $(\gamma(t) - B)^{-1} : H_s \to H_s$ is a continuous path of operators for all $s \in \mathbb{R}$.

such that f is holomorphic on the union of these connected open subsets. Now for each U_j we may define

$$f_j(B) = \mathbf{1}_{U_j}(B)f(B) := \frac{1}{2\pi i}\int_{\gamma_i} f(\lambda)(\lambda - B)^{-1}d\lambda,$$

with suitably chosen closed path $\gamma_j : [a,b] \to U_j$ encircling $U_j \cap \operatorname{spec} T$ once. Now,

$$f(B) = \sum_j f_j(B)$$

can be shown to yield an operator in $\mathscr{A}$ again since

$$\begin{aligned}[f(B),\Phi] &= \frac{1}{2\pi i}\sum_j \int_{\gamma_j} f_j(\lambda)[(\lambda - B)^{-1},\Phi]d\lambda \\ &= \frac{1}{2\pi i}\sum_j \int_{\gamma_j} f_j(\lambda)(\lambda - B)^{-1}[B,\Phi](\lambda - B)^{-1}d\lambda \in \operatorname{Op}^0.\end{aligned}$$

Clearly, $\mathscr{A} \subset \mathscr{B}(H_0)$. Hence, the operator norm $\|.\|_{\mathscr{B}(H_0)}$ defines a pre-C^*-norm on $\mathscr{A}$. We have seen that $\mathscr{A}$ is in fact a *local C^*-Algebra*, i.e. it is an pre-C^*-Algebra which is invariant under holomorphic calculus.

We can make similar statements for operators acting between scales of Hilbert spaces $(H_s)_{s\in\mathbb{R}}$ and $(H'_s)_{s\in\mathbb{R}}$ induced by discrete operators Λ and Λ', say.

Proposition A.2.2. *Let $T \in \operatorname{Op}^\mu((H_s),(H'_s))$ such that*

$$T|\Lambda| - |\Lambda'|T \in \operatorname{Op}^\mu((H_s),(H'_s)).$$

Then, if

$$T : H_s \to H'_{s-\mu}$$

is (left, right) Fredholm for some $s \in \mathbb{R}$, then it is for all $s \in \mathbb{R}$. Moreover, for a left Fredholm operator T and for all t, t', $t' < t + \mu$ we have estimates

$$\|u\|_{t+\mu} \le \operatorname{const}\cdot\big(\|u\|_{t'} + \|Tu\|_t\big).$$

In the left Fredholm case, if $u \in H_{-\infty}$ and $Tu \in H_t$, then $u \in H_{t+\mu}$.

Remark A.2.3. The (left, right) Fredholm operators may be considered as the (injectively, surjectively) elliptic elements of our calculus, in view of the results on elliptic pseudodifferential operators, cf. [Hör85, Sec. 19.5]. In particular, the kernels of injectively elliptic elements are finite dimensional subspaces of H_∞.

Proof. Let T be as above and assume

$$T : H_s \to H'_{s-\mu}$$

is (left, right) Fredholm for some $s \in \mathbb{R}$. Analogously to (2.1) and (2.2) we have

$$(\operatorname{Id} + \tfrac{1}{n}|\Lambda'|)T(\operatorname{Id} + \tfrac{1}{n}|\Lambda|)^{-1} = T + \tfrac{1}{n}(|\Lambda'|T - T|\Lambda|)(\operatorname{Id} + \tfrac{1}{n}|\Lambda|)^{-1},$$

and

$$(\operatorname{Id} + \tfrac{1}{n}|\Lambda'|)^{-1}T(\operatorname{Id} + \tfrac{1}{n}|\Lambda|) = T + \tfrac{1}{n}(\operatorname{Id} + \tfrac{1}{n}|\Lambda'|)^{-1}(T|\Lambda| - |\Lambda'|T).$$

By the estimates for $\|(\mathrm{Id}+\frac{1}{n}|\Lambda|)^{-1}\|_{s,s}$ and $\|(\mathrm{Id}+\frac{1}{n}|\Lambda'|)^{-1}\|_{s,s}$ it now follows inductively that

$$T : H_{s+k} \to H'_{s+k-\mu}$$

is (left, right) Fredholm for all $k \in \mathbb{Z}$ with the same index since the space of bounded (left, right) Fredholm operators (of a given index) is open in $\mathscr{B}(H_{s+k}, H'_{s+k-\mu})$. Let us indicate how the same method can be used for the remaining $t \in \mathbb{R}$. For simplicity assume now $(H_s) = (H'_s)$. For real $s \in \mathbb{R}$ we have

$$\begin{aligned}(1+\tfrac{1}{n}|\Lambda|^s)T(1+\tfrac{1}{n}|\Lambda|^s)^{-1} &= T + \tfrac{1}{n}[|\Lambda|^s, T](1+\tfrac{1}{n}|\Lambda|^s)^{-1}\\ &= T + \tfrac{1}{n}[|\Lambda|^s, T](1+|\Lambda|)^{-s+1-\varepsilon}(1+|\Lambda|)^{s-1+\varepsilon}(1+\tfrac{1}{n}|\Lambda|^s)^{-1},\end{aligned}$$

and by Proposition 1.2.4,

$$[|\Lambda|^s, T](1+|\Lambda|)^{-s+1-\varepsilon}$$

is an operator of order μ for any $\varepsilon > 0$. Furthermore, for all $t \in \mathbb{R}$ we can estimate

$$\begin{aligned}\|\tfrac{1}{n}(1+|\Lambda|)^{s-1+\varepsilon}(1+\tfrac{1}{n}|\Lambda|^s)^{-1}\|_{t,t} &\le \tfrac{1}{n}\sup_{\lambda>1}\frac{(1+\lambda)^{s-1+\varepsilon}}{1+\frac{1}{n}\lambda^s}\\ &\le \sup_{\lambda>1}\frac{(1+\lambda)^{s-1+\varepsilon}}{n+\lambda^s},\end{aligned}$$

which tends to 0 as $n \to \infty$. It follows that if

$$T : H_t \to H_{t-\mu}$$

is (left, right) Fredholm, then so is

$$(\mathrm{Id}+\tfrac{1}{n}|\Lambda|^s)T(\mathrm{Id}+\tfrac{1}{n}|\Lambda|^s)^{-1} : H_t \to H_{t-\mu}$$

for sufficiently large $n \in \mathbb{N}$ and hence

$$T : H_{t+s} \to H_{t+s-\mu}$$

is (left, right) Fredholm.

Now, if $T : H_s \to H'_{s-\mu}$ is left Fredholm then, by Proposition A.1.2 we have estimates

$$\|u\|_s \le \mathrm{const}\cdot\big(\|u\|_{s-1} + \|Tu\|_{s-\mu}\big) \tag{2.3}$$

for all $u \in H_s$. Let now $u \in H_{s-1}$ and $Tu \in H'_{s-\mu}$. Then

$$u_n := (\mathrm{Id}+\tfrac{1}{n}|\Lambda|)^{-1}u$$

defines a series in H_s that converges to u in H_{s-1}. Since $(n+|\Lambda|)^{-1}u$ converges to 0 in H_s it follows that

$$T(\mathrm{Id}+\tfrac{1}{n}|\Lambda|)^{-1}u = (\mathrm{Id}+\tfrac{1}{n}|\Lambda'|)^{-1}Tu + (\mathrm{Id}+\tfrac{1}{n}|\Lambda|)^{-1}(T|\Lambda| - |\Lambda'|T)(n+|\Lambda|)^{-1}u$$

is a convergent series in $H'_{s-\mu}$. By (2.3) (u_n) is thus a Cauchy sequence in H_s. Its limit in H_s has to coincide with u hence $u \in H_s$. Now, it follows inductively that $\ker T : H_{-\infty} \to H_{-\infty}$ is finite dimensional and consists of elements in H_∞ only.

Finally, observe that $T : H_t \to H'_{t-\mu}$ is left Fredholm if and only if

$$T^* : H'_{-t+\mu} \to H_{-t}$$

is right Fredholm. □

Remark A.2.4. That the index of a Fredholm operator of order μ between the scales (H_s) and (H'_s) is stable is a non-trivial property. In fact, by [Her89], if

$$T : H_0 \to H'_\mu, \quad T : H_1 \to H'_{\mu+1}$$

are Fredholm operators of the same index, then T is Fredholm on all intermediate spaces H_s, $0 < s < 1$. If the indices differ, then there exists $s \in (0, 1)$ such that $T|_{H_s}$ is not Fredholm. Moreover, semi Fredholmness is in general not stable under interpolation, see loc. cit. for a number of counterexamples.

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