Fano Varieties and Fano Polytopes

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Kurzzusammenfassung

Ganz grundsätzlich beschäftigt sich diese Arbeit mit der Fragestellung, wann eine (normale) Gorenstein Fano Varietät zu einer torischen Gorenstein Fano Varietät degeneriert werden kann. Dabei betrachten wir nur solche Degenerierungen, die mit der Wahl eines amplen Linienbündels auf der ursprünglichen Varietät und eines amplen rationalen Cartier-Divisors auf der torischen Varietät *verträglich* sind. Diese Verträglichkeit wird in der Arbeit genauer präzisiert und ist in den Anwendungen der Darstellungstheorie oder auch in der Newton-Okounkov-Theorie stets gegeben.

Im Hauptteil der Arbeit wird die Aussage bewiesen, dass bei solchen verträglichen torischen Degenerierungen von Gorenstein Fano Varietäten das ursprünglich gewählte Linienbündel genau dann isomorph zum antikanonischen Bündel ist, wenn der Divisor auf der torischen Varietät ein antikanonischer Divisor ist. Die Hinrichtung ist bereits seit einiger Zeit bekannt, doch die Rückrichtung noch nicht. Ihr Beweis benötigt verschiedene Methoden aus mehreren Teilgebieten der Mathematik. Wir werden diverse Verschwindungssätze und weitere Resultate aus der algebraischen Geometrie, Methoden aus der polyhedralen Geometrie (insbesondere der Ehrhart-Theorie), Resultate über das Hilbert-Polynom und Erkenntnisse über torische Varietäten verwenden.

Nebenbei etablieren wir einen Zusammenhang zwischen dem Ehrhart Quasipolynom eines rationalen Polytops und der Kohomologie eines assoziierten rationalen Weil-Divisors auf einer torischen Varietät. Bisher war dieser Zusammenhang nur für Polytope mit ganzzahligen Eckpunkten und ganzzahlige Divisoren bekannt. Er erlaubt es, Ehrhart-Macdonald Reziprozität als Spezialfall von Serre-Dualität zu deuten.

Im letzten Kapitel der Arbeit wird gezeigt, dass zu jeder partiellen Fahnenvarietät einer komplexen klassischen Gruppe tatsächlich eine solche verträgliche torische Degenerierung zu einer torischen Gorenstein Fano Varietät existiert. Die Konstruktion erfolgt mit Hilfe der von Littelmann und Berenstein–Zelevinsky etablierten Stringpolytope [47, 8]. Dazu muss eine Klassifizierung der ganzzahligen Stringpolytope bewiesen werden. Der Beweis erfolgt kombinatorisch über eine neuentwickelte diagrammatische Darstellung von so genannten Gelfand-Tsetlin-Mustern in Anlehnung an Hasse-Diagramme von partiell geordneten Mengen.

Abstract

The foundation of this thesis is the problem whether a given (normal) Gorenstein Fano variety can be degenerated to a toric Gorenstein Fano variety. We will only consider those degenerations that are *compatible* with the choice of an ample line bundle on the original variety and an ample rational Cartier divisor on the toric variety. This compatibility will be defined thoroughly and is always granted in applications in representation theory or Newton-Okounkov Theory.

The main matter of this thesis contains the proof that in the setting of these compatible toric degenerations the originally chosen line bundle will be isomorphic to the anti-canonical line bundle if and only if the divisor on the toric variety is anti-canonical. The *if*-part is already known but the *only-if*-part is not. Its proof requires different methods from various areas of mathematical research. We will need multiple vanishing theorems and further results from algebraic geometry, methods from polyhedral geometry (especially Ehrhart theory), results on the Hilbert polynomial and facts about toric varieties.

As a by-product we establish a connection between the Ehrhart quasi-polynomial of a rational polytope and the cohomology of an associated rational Weil divisor on a toric variety. Up until know, this connection was only known for polytopes with integral vertices and integral divisors. It allows us to interpret Ehrhart-Macdonald Reciprocity as a special case of Serre Duality.

In the final chapter of this thesis we will show that there actually exists such a compatible toric degeneration for every partial flag variety of a complex classical group. The construction is done via so called string polytopes that have been established by Littelmann and Berenstein–Zelevinsky [47, 8]. For this purpose we need to prove a classification of integral string polytopes. The proof is done via a newly developed diagrammatic description of so called Gelfand-Tsetlin patterns in spirit of Hasse diagrams of partially ordered sets.

Contents

Chapter 1

Introduction

1.1 Background and Motivation

It has been a common concept throughout the history of mathematics to translate problems in one area of research to another area of research to solve them there.

An especially fruitful example of this translating approach has been the study of *toric varieties*. These are special varieties whose algebraic-geometric properties are completely determined by the combinatorial properties of certain polyhedral objects. This phenomenon has been used by many mathematicians to study more general varieties via flat degenerations to toric varieties. This approach proved especially useful in the field of representation theory. Notable results in this regard have been archived by Gonciulea and Lakshmibai [28], Kogan and Miller [42], Caldero [10], Alexeev and Brion [1] as well as Feigin, Fourier and Littelmann [24].

All of their constructions used polytopes that were already known for other reasons. Most notably, lattice points in certain polyhedral objects correspond to nice bases of Lie algebra representations and allow combinatorial studies of dimensions and branching rules for example.

Starting with the polytopes of Gelfand and Tsetlin in type A_n in [27] Berenstein and Zelevinsky defined *Gelfand-Tsetlin polytopes* for all classical Lie algebras in [8]. A generalization of this approach lead to the construction of so called *string polytopes* for Lie algebras of arbitrary type that were studied by Littelmann in [47] and Berenstein and Zelevinsky in [9]. Recently, Nakashima and Zelevinsky defined a different kind of *string polytope* in [49]. Another approach has been taken by Lusztig in [48], which lead to the introduction of *Lusztig polytopes*. Based on a conjecture by Vinberg, two works of Feigin, Fourier and Littelmann established yet another prominent class of polytopes — so called *Feigin-Fourier-Littelmann-Vinberg polytopes* — in types A_n [22] and C_n [23]. Gornitskii analogously defined *Gornitskii polytopes* in types B_n and D_n [30] as well as G_2 [29].

It turns out that all of these completely different polytopes have a common root.

Okounkov [51, 52], Lazarsfeld and Mustață [46] as well as Kaveh and Khovanskii [38] defined and analyzed convex bodies for arbitrary projective varieties — thereby developing the theory of *Newton-Okounkov bodies*.

It has been shown that most of the formerly mentioned polytopes arise as Newton-Okounkov bodies of flag varieties — for example by Kaveh [37], Kiritchenko [41] and Fujita and Naito [25]. A unified approach has been developed by Fang, Fourier and Littelmann who presented a construction of these polytopes from representation theory via *birational sequences* and connected them to Newton-Okounkov Theory in [21].

Lately, Anderson [2] showed that Newton-Okounkov bodies yield toric degenerations under reasonable technical assumptions, thereby providing a general reason for the existence of the diverse classes of toric degenerations in representation theory mentioned in the beginning.

But there are even more connections. In the field of mirror symmetry, Batyrev [4] developed an interesting approach by constructing mirrors to Calabi-Yau hypersurfaces in toric varieties via *reflexive* polytopes — lattice polytopes whose polar dual is a lattice polytope too. It deems a reasonable hope that his construction can be generalized to other classes of varieties if one were able to associate meaningful reflexive polytopes to those varieties.

In the very same work Batyrev established the connection between reflexive polytopes and (normal) Gorenstein Fano toric varieties — namely, up to isomorphism, they are in one-to-one correspondence. Therefore the following is an important question.

Question 1. *Which Gorenstein Fano varieties admit a flat degeneration to a toric Gorenstein Fano variety?*

The answer to this question might also be interesting in representation theory since reflexivity simplifies calculations in the polytope (see for example [5, Section 4.4]).

However, this question might be too hard to answer. Additionally, it might not even be the correct question to ask with Batyrev's construction in mind. The moral reason is that an arbitrary toric degeneration only determines the normal fan of a convex rational polytope. Since there are infinitely many convex rational polytopes with the same normal fan, we cannot really say that choosing a rather arbitrary polytope in this family is a *meaningful* construction.

The clue lies in another magical connection from toric geometry. While polyhedral fans determine the toric variety, a convex rational polytope determines a variety — via its normal fan — together with an ample (rational) Cartier divisor on the toric variety. In other words, the choice of a polytope determines an embedding. So it is rather useful to consider *embedded* toric degenerations. In standard terminology this embedding is referred to as a *polarization*.

Later on in this thesis we will make precise what this really means. Roughly speaking, we want to consider only those degenerations that respect the choice of an ample line bundle on the original variety and a rational Cartier divisor on the limit variety.

One might ask what reflexivity means in this context. In Batyrev's construction it appears by choosing the anti-canonical divisor as the ample Cartier divisor on the Gorenstein Fano toric variety. So the following question arises naturally.

Question 2. *Does every polarized Gorenstein Fano variety admit a toric degeneration to a polarized toric Gorenstein Fano variety whose polarization is given by its anti-canonical line bundle?*

An initial answer to this question has been given by Rusinko, who found out that the full flag variety in type A_n admits such a toric degeneration — although he did not state this explicitly — by proving that string polytopes in type A_n are reflexive for the weight 2ρ [55]. This very result can be seen as a starting point of this thesis.

1.2 Results

This thesis is divided into two parts that both work towards an answer for Question 2.

In the first part we will focus on a necessary criterion for the — a priori rational — limit divisor on the toric variety to be integral and anticanonical. This criterion can be found in Theorem 5.1.2.

Theorem 1 (Main Theorem)**.** *The limit of a polarized Gorenstein Fano variety under a toric degeneration*[1] *is $\mathbb{Q}$-polarized by its anticanonical divisor if and only if the polarization on the original variety is given by its anticanonical divisor.*

Notice that we have to deviate slightly from our original question by allowing $\mathbb{Q}$-Gorenstein Fano varieties as toric limits. We will give examples of Gorenstein Fano limit varieties in the second part.

The *if*-part of the result is already known (see for example [1, Theorem 3.8]) and can be proved without too much hassle. The *only-if*-part however is completely new and its proof is neither obvious nor straightforward.

[1] See Section 3.2 for our definition of toric degenerations of polarized varieties.

Fitting with the theme of this work — the interplay between algebraic geometry, representation theory and discrete mathematics — we need various intermediate results connecting different concepts in mathematics to complete the proof. A visual sketch of the proof can be found in Figure 5.1.

Essentially, the proof consists of four distinct steps.

Firstly, we will realize in Theorems 2.2.30 and 2.3.40 that the limit toric divisor being anticanonical implies that the rational convex polytope associated to this divisor contains precisely one lattice point in its interior and its polar dual (after proper translation) is a lattice polytope. This combinatorial property is a necessary condition and we will call these polytopes *weakly dual-Fano polytopes* to underline their connection to Fano polytopes that have been studied by Nill [50]. Fano polytopes are in one-to-one correspondence with $\mathbb{Q}$-Gorenstein Fano toric varieties, so it is natural that they appear in this context. The precise statement is the following.

Theorem 2 (Batyrev, Nill)**.** *The polytope associated to the anticanonical divisor of a $\mathbb{Q}$-Gorenstein Fano toric variety is weakly dual-Fano.*

Using Ehrhart Theory and a beautiful result by Hibi (see [34]) we realize that this property is equivalent to a numerical condition on the Ehrhart quasi-polynomial of the rational convex polytope in Theorem 2.2.43.

Theorem 3 (Hibi)**.** *A full-dimensional rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$ is weakly dual-Fano if and only if*

$$\#(n\mathcal{P} \cap \mathbb{Z}^d) = \#(\operatorname{int}(n+1)\mathcal{P} \cap \mathbb{Z}^d)$$

for all $n \in \mathbb{N}$.

The heart of the proof of Theorem 1 might be the observation in Theorem 3.4.1 that this Ehrhart quasi-polynomial is in fact a polynomial and equal to the Hilbert polynomial of the chosen line bundle on the original variety. A different formulation is used in Corollary 3.4.3, which we actually want to present here.

Theorem 4. *Let X be a normal projective complex variety of dimension d and let $\mathcal{L}$ be an ample line bundle over X. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. If the pair $(X, \mathcal{L})$ admits a toric degeneration — in our sense[2] — to the toric variety $(X_\mathcal{P}, D_\mathcal{P})$, then*

$$\chi(X, \mathcal{L}^n) = \#(n\mathcal{P} \cap \mathbb{Z}^d) \textit{ and } \chi(X, \mathcal{L}^n \otimes \omega_X) = \#(\operatorname{int} n\mathcal{P} \cap \mathbb{Z}^d)$$

for all $n \in \mathbb{N}$.

[2]See Section 3.2 for the definition.

This is mostly due to the fact that Euler characteristic behaves well under flat projective degenerations. However, we need to generalize a classical theorem of toric geometry by proving that the number of lattice points in the n-th dilation of the polytope is given by the Euler characteristic of the round down of the n-th multiple of the toric divisor associated to the polytope.

Theorem 5. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. Let $X_{\mathcal{P}}$ denote the associated normal projective toric variety and $D_{\mathcal{P}}$ the associated torus invariant $\mathbb{Q}$-Weil divisor. Then*

$$\#(n\mathcal{P} \cap \mathbb{Z}^d) = \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor))$$

and

$$\#(\operatorname{int} n\mathcal{P} \cap \mathbb{Z}^d) = \chi(X, \mathcal{O}_{X_{\mathcal{P}}}(\lceil nD_{\mathcal{P}} \rceil + K_{X_{\mathcal{P}}}))$$

for all $n \in \mathbb{N}$.

Hence the formerly mentioned numerical property of the Ehrhart polynomial must hold for the Hilbert polynomial too. This observation leads to an equivalence between Ehrhart-Macdonald Reciprocity and Serre Duality in Remark 3.4.4 for a large class of varieties — precisely those having a degeneration to a Gorenstein Fano toric variety in our embedded sense. Previously it had only been known that Serre Duality implies Ehrhart-Macdonald Reciprocity for toric varieties (see for example [13, 11.12.4]) and thus can easily be extended to toric degenerations. But the other implication is new. It is indeed exciting, that one of the most famous results in Algebraic Geometry can be proved by counting lattice points.

Interestingly enough, we did not really need to prove the previous generalization, as in Theorem 3.2.10 it turns out that in our setting all divisors will be Weil divisors, which is an interesting observation in itself.

Theorem 6. *Let X be a normal projective complex variety of dimension d and let D be an ample Cartier divisor on X. Let $X_{\mathcal{P}}$ be the normal projective toric variety associated to a rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$ and $D_{\mathcal{P}}$ the associated torus invariant ample $\mathbb{Q}$-Cartier divisor on $X_{\mathcal{P}}$. Suppose the pair (X, D) admits a toric degeneration — in the sense of Section 3.2 — to the toric pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$. Then $D_{\mathcal{P}}$ is a Weil divisor.*

The final step of our proof is Theorem 5.2.1 where we prove that our numerical condition on the Hilbert polynomial (or rather a reformulation using Ehrhart-Macdonald Reciprocity) is equivalent to the fact that the originally chosen line bundle is isomorphic to the anticanonical one. This result is also interesting in its own right since it shows that the Hilbert polynomial contains even more information about the variety than previously known.

Theorem 7. *Let X be a Gorenstein Fano variety of dimension d that has rational singularities and let $\mathcal{L}$ be an ample line bundle. Then the line bundle $\mathcal{L}$ is isomorphic to the anticanonical line bundle ω_X^{-1} if and only if the Hilbert polynomial $P_{\mathcal{L}}(n) := \chi(X, \mathcal{L}^n)$ of $\mathcal{L}$ fulfills*

$$P_{\mathcal{L}}(n) = (-1)^d P_{\mathcal{L}}(-n-1)$$

for all $n \in \mathbb{N}$.

For the proof of this key result we need several results from algebraic geometry like Serre Duality, Serre Vanishing, Kodaira Vanishing for Rational Singularities and two beautiful theorems by Elkik on rational singularities in flat families (see [19]).

Our statement bears resemblance to a result by Kaveh and Villella in [40] who were able to classify *anticanonical objects* in families of polyhedra associated to flag varieties purely via combinatorial conditions. However, their result needs stronger assumptions like Minkowski property of the occurring polytopes, which we do not need.

It should be noted that there are some delicate details occurring in this setting. For example, the rational polytopes appearing will always have an Ehrhart polynomial instead of an Ehrhart quasi-polynomial and their associated divisor will be Weil instead of $\mathbb{Q}$-Weil. This must mean that the polytope somehow *remembers* that it is associated to the limit of a line bundle. We cannot explain these phenomena better, but we are nevertheless able to state our opinion in Conjecture 5.3.2.

Conjecture 1. *The divisor associated to a full-dimensional rational convex polytope is a Weil divisor on the toric variety associated to the polytope if and only if the polytope is a* quasi-lattice polytope*, i.e. its Ehrhart quasi-polynomial is a polynomial.*

This claim would also prove a missing puzzle piece connecting our definition of weakly dual-Fano polytopes and the formerly known Fano polytopes. We state this connection in Conjecture 5.3.2.

Conjecture 2. *The dual of a convex polytope is Fano if and only if the polytope is a weakly dual-Fano quasi-lattice polytope.*

As a bonus, in Theorem 4.1.1 we give a different proof of our Theorem 1 in the special case of flag varieties to show how the powerful tools of representation theory simplify everything. Most importantly, the Ehrhart quasi-polynomial can be calculated via Weyl's Dimension Formula. This part of the thesis is already available as a preprint [61].

After these considerations Question 2 can be simplified to the following question.

Question 3. *Does every anticanonically polarized Gorenstein Fano variety admit a toric degeneration to a toric Gorenstein Fano variety?*

However, this question is still very hard to answer. In Corollary 5.3.1 we argue that for a given toric degeneration one *just* has to show that the polytope associated to the toric limit divisor is a lattice polytope. But in general this is a heavy task.

The second part of this thesis affirms Question 3 for flag varieties of complex classical groups. The reason is the following result that is stated in Theorem 6.3.1 and was already conjectured in our preprint [61, Conjecture 7.8].

Theorem 8. *Let G be a complex classical group with Lie algebra $\mathfrak{g}$ and let λ be a dominant integral weight of $\mathfrak{g}$. Then the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ (in the sense of [47]) is a lattice polytope if and only if the $\mathfrak{g}$-representation on $V(\lambda)$ integrates to a representation of G.*

Of course, this condition is always satisfied if the algebraic group is of type A_n or C_n, while it gives a parity condition on the dominant integral weight in types B_n and D_n.

This result is especially striking because it is not clear why string polytopes that have been defined solely in terms of Lie algebras should contain information about the underlying algebraic group.

Its proof will be done via a case by case study, using a bijection between these special string polytopes and Gelfand-Tsetlin polytopes as established by Littelmann [47]. In types A_n, B_n and C_n, these polytopes have been interpreted as marked order polytopes of marked posets by Ardila, Bliem and Salazar in [3], which readily yields the claim (alternatively due to a result by Fang and Fourier [20]). However, this result is not applicable in the D_n case. Hence we will give a new proof in the A_n, B_n and C_n cases too, whose underlying idea can be transferred to the D_n case.

The marked order structure of the Gelfand-Tsetlin polytopes allows us to give a diagrammatic condition whether a given point in these polytopes — a so called *Gelfand-Tsetlin pattern* — is a vertex.

In fact, we will define directed graphs — called *identity diagrams* — for arbitrary marked posets and give a combinatorial criterion on these identity diagrams to distinguish vertices from non-vertices in Theorem 6.6.4. Hence we get a full classification of vertices of standard string polytopes via directed graphs, that is notable in its own right.

Theorem 9. *A point in a marked order polytope is a vertex if and only if each connected component in its identity diagram contains a marked element.*

Unfortunately, this approach does not work in type D_n anymore. The reason is that firstly the standard string polytope and the Gelfand-Tsetlin polytope might not be unimodularly equivalent since Littelmann's bijection is only piecewise affine. Secondly, the Gelfand-Tsetlin polytope is not a marked order polytope — or at least such a realization is not known.

We solve these difficulties by altering the definition of Gelfand-Tsetlin polytopes in type D_n slightly, realizing them as polytopes in some affine subspace of a larger vector space. We call these new polytopes *tweaked Gelfand-Tsetlin polytopes* and show that they are actually equivalent to the standard string polytope in Theorem 6.7.6. Even more, due to Littelmann's bijection we still have a classification of the points corresponding to lattice points in the string polytope.

Theorem 10. *Let G be a simple complex algebraic group of type D_n. For every dominant integral weight of the Lie algebra of G there exists an affine bijection between $\mathbb{R}^{n(n-1)}$ and an affine subspace of $\mathbb{R}^{n^2+n-2}$ that sends the standard string polytope onto the tweaked Gelfand-Tsetlin polytope.*

However, by introducing additional coordinates we also made the defining inequalities of the polytope even more complicated. Most of these inequalities will be given by a poset structure but some are not. We will therefore refine our definition of identity diagrams to get *tweaked Gelfand-Tsetlin diagrams.* These diagrams are vertex-colored edge-colored directed graphs that contain much information about a certain element of the tweaked Gelfand-Tsetlin poset.

With some work, we are able to give a criterion for an element of the tweaked Gelfand-Tsetlin polytope to be a vertex in terms of its tweaked Gelfand-Tsetlin diagram in Theorem 6.8.16. The condition is similar to the analogous condition on identity diagrams of elements of marked order polytopes but it needs more special cases[3].

Theorem 11. *A tweaked Gelfand-Tsetlin pattern is a vertex of the tweaked Gelfand-Tsetlin polytope if and only if each of the connected components of its tweaked Gelfand-Tsetlin diagram contains a marked element, contains an anomaly, is a single impurity or is part of a double impurity.*

This condition ensures that the coordinates of a vertex of the standard string polytope in type D_n are always half-integers and they are integers precisely under the desired conditions. This is stated explicitly in Theorem 6.8.25, concluding this thesis.

Theorem 12. *Let G be a simple complex algebraic group of type D_n and let $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i$ be a dominant integral weight of the Lie algebra of G. Then*

[3] See Section 6.8 for the definitions.

each coordinate of every vertex of the tweaked Gelfand-Tsetlin polytope can be written as a $\mathbb{Z}$-linear combination of the λ_i.

Finally, it should be noted that a generalization of this result to arbitrary string polytopes would be very hard. It might however be called an achievement that we were able to disprove a conjecture by Alexeev and Brion (see [1, Conjecture 5.8]), claiming that all string polytopes in type A_n are lattice polytopes, by giving a counter example in rank 5 in Example 6.2.6. This example can even be extended to a non-lattice string polytope for the anticanonical weight of a Grassmannian — the nicest setting imaginable.

In other types the situation is even more difficult — even for the standard word — as Example 6.2.11 shows.

1.3 Structure

This thesis is structured as follows.

In Chapter 2 we recall important concepts and results from algebraic geometry, toric geometry and polyhedral geometry. Nearly all of this chapter is already known, the only exception being our concept of weakly dual-Fano polytopes in Section 2.2.2.

The formerly mentioned concepts and results will be brought together in Chapter 3, bridging the different branches of mathematics. Here we mix previously known results with new insights. Especially, we formalize a nice class of toric degenerations — those that respect embeddings.

Chapter 4 considers the special case of flag varieties and proves our main theorem in this case. This chapter has been part of the preprint [61].

The proof of our Theorem 1 and the heart of this thesis is contained in Chapter 5.

The second part of this thesis as described above is Chapter 6 where we prove Theorem 8 and present some other interesting examples.

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Chapter 2

Foundations

2.1 Algebraic Geometry

In this section we will recall concepts from algebraic geometry that will be vital for the formulation and solution of our problems. A canonical reference for most of the topics is [33].

Notice that we will recall many results not in their most general form since we will be mostly interested in the case of normal projective varieties.

2.1.1 Varieties, Sheaves and Divisors

We will assume that the reader is familiar with the classical notions of sheaves, schemes (especially projective varieties), normality, morphisms of schemes and divisors (and their correspondence with line bundles). This is essentially the content of [33, Chapters 1 and 2].

We will use the following convention.

Definition 2.1.1. A **variety** is an integral (irreducible and reduced), separated scheme of finite type over an algebraically closed field.

Additionally we will fix some notation.

Notation 2.1.2. Let X be a normal variety over the field $\mathbb{K}$. We will denote the structure sheaf of X by $\mathcal{O}_X$ and the field of rational functions by $\mathbb{K}(X)$.

In the following we will always mean *sheaf of $\mathcal{O}_X$-modules* when we simply write *sheaf*.

Definition 2.1.3. Let X be a normal variety.

(i) A **prime divisor** D on X is an subvariety of codimension 1.

(ii) A **Weil divisor** D on X is a formal $\mathbb{Z}$-linear combination $D = \sum a_i D_i$ of prime divisors D_i. The set of all Weil divisors on X is denoted by $\mathrm{Div}(X)$.

(iii) A **$\mathbb{Q}$-Weil divisor** D on X is a formal $\mathbb{Q}$-linear combination $D = \sum a_i D_i$ of prime divisors D_i. The set of all $\mathbb{Q}$-Weil divisors on X is denoted by $\operatorname{Div}_{\mathbb{Q}}(X)$.

(iv) The **round-down** $\lfloor D \rfloor$ of a $\mathbb{Q}$-Weil divisor $D = \sum a_i D_i$ is the Weil divisor $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$. The **round-up** $\lceil D \rceil$ is defined analogously.

Many concepts using Weil divisors can be generalized to $\mathbb{Q}$-Weil divisors.

Definition 2.1.4. Let X be a normal variety.

(i) A ($\mathbb{Q}$-)Weil divisor D on X is called **$\mathbb{Q}$-Cartier** if some positive multiple mD, $m \in \mathbb{Z}_{>0}$, of D is a Cartier Weil divisor.

(ii) A $\mathbb{Q}$-Weil divisor D on X is called **ample** if some positive multiple mD, $m \in \mathbb{Z}_{>0}$, of D is an ample Weil divisor.

(iii) A $\mathbb{Q}$-Weil divisor $D = \sum a_i D_i$ is called **effective** if $a_i \geq 0$ for all i.

We want to recall the correspondence between Cartier divisors and line bundles. For that purpose we need the following proposition from [12, Proposition 4.0.7 (b)].

Proposition 2.1.5. *Let X be a normal variety and D a prime divisor on X. Then the local ring*

$$\mathcal{O}_{X,D} := \{g/h \in \mathbb{K}(X) \mid h|_D \neq 0\}$$

is a discrete valuation ring and $\mathbb{K}(X)$ is its field of fractions.

This result allow us to define the following useful object.

Definition 2.1.6. Let X be a normal variety, D a prime divisor on X and

$$v_D \colon \mathbb{K}(X)^\times \to \mathbb{Z}$$

the corresponding valuation on the field of fractions of $\mathcal{O}_{X,D}$. By construction this implies $v_D(\mathcal{O}_{X,D} \setminus \{0\}) \geq 0$.

(i) For a given rational function $f \in \mathbb{K}(X)^\times$ we will call $v_D(f)$ the **the order of vanishing** of f along D.

(ii) If $v_D(f) > 0$ we say that f has a **zero** of order $v_D(f)$ along D.

(iii) If $v_D(f) < 0$ we say that f has a **pole** of order $-v_D(f)$ along D.

Now we would like to describe a function by its vanishing behavior along all possible prime divisors. The following result from [33, II, Lemma 6.1] allows us to succeed in this task.

Lemma 2.1.7. *Let X be a normal variety and $f \in \mathbb{K}(X)^\times$. Then $v_D(f) = 0$ for all but finitely many prime divisors D on X.*

Let us now associate a divisor to every rational function.

Definition 2.1.8. Let X be a normal variety and $f \in \mathbb{K}(X)^\times$.

(i) The **divisor of** f is defined as $\operatorname{div}(f) := \sum_D v_D(f)D$, where the sum is taken over all prime divisors D on X.

(ii) Any divisor of the above form is called **principal**. The set of all principal divisors on X will be denoted by $\operatorname{Div}_0(X)$.

(iii) Two Weil divisors D and E are called **linearly equivalent**, written $D \sim E$, if their difference $D - E$ is principal.

(iv) The group of Weil divisors modulo linear equivalence is called the **divisor class group** $\operatorname{Cl}(X)$. The group of all Cartier divisors modulo linear equivalence is denoted by $\operatorname{CaCl}(X)$. We will denote the class of a Weil divisor D by $[D]$.

We are now able to define the sheaf associated to a divisor.

Definition 2.1.9. Let X be a normal variety and D a Weil divisor on X. The sheaf $\mathcal{O}_X(D)$ associated to the divisor D is defined by

$$\mathcal{O}_X(D)(U) := \{f \in \mathbb{K}(X)^\times \mid (\operatorname{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

For an easier formulation of the next statement, we will recall some standard notions.

Definition 2.1.10. Let X be a normal variety and $\mathcal{F}$ a coherent sheaf on X.

(i) The sheaf $\mathcal{F}$ is called **reflexive** if it is isomorphic to its second dual $(\mathcal{F}^\vee)^\vee$, often called the **reflexive hull** of $\mathcal{F}$.

(ii) Let $\widetilde{\mathbb{K}(X)}$ denote the constant sheaf on X given by the field of rational functions $\mathbb{K}(X)$. Then the global sections $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \widetilde{\mathbb{K}(X)})$ can be seen as a vector space over $\mathbb{K}(X)$. The dimension of this vector space (over $\mathbb{K}(X)$) is called the **rank** of the sheaf $\mathcal{F}$.

(iii) The sheaf $\mathcal{F}$ is called **divisorial** if it is reflexive and of rank 1.

(iv) The group of all line bundles over X modulo isomorphism is called the **Picard group** of X and denoted by $\operatorname{Pic}(X)$.

Remark 2.1.11. Let X be a normal variety. Notice that the set of divisorial sheaves on X modulo isomorphism need not admit a monoid structure by naively taking tensor products since the tensor product of two reflexive sheaves might not be reflexive. So one usually defines a product structure via $\mathcal{F} \times \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{\vee\vee}$.

The sheaf associated to a divisor fulfills the following properties as proved in [33, Proposition 6.13 and Corollary 6.14] or [12, Theorems 8.0.4 and 8.0.7]. The monoid homomorphism property is proved in [12, Proposition 8.0.6].

Theorem 2.1.12. *Let X be a normal variety.*

(i) The map $D \mapsto \mathcal{O}_X(D)$ induces a monoid isomorphism between the divisor class group $\operatorname{Cl}(X)$ and the monoid of divisorial sheaves on X modulo isomorphism. This implies

$$\mathcal{O}_X(D+E) \simeq (\mathcal{O}_X(D) \otimes \mathcal{O}_X(E))^{\vee\vee}$$

for all pairs of Weil divisors D and E.

(ii) The map $D \mapsto \mathcal{O}_X(D)$ induces a group isomorphism between the group $\operatorname{CaCl}(X)$ of Cartier divisors modulo linear equivalence on X and the Picard group $\operatorname{Pic}(X)$ of X.

The following results can be found in [33, Propositions 6.5 and 6.6]. They will prove useful in our study of toric degenerations.

Proposition 2.1.13. *Let X be a normal variety.*

(i) Let Z be a prime divisor on X and $U := X \setminus Z$. Then the canonical map $\sum a_i D_i \mapsto \sum_{D_i \cap U \neq \emptyset} a_i (D_i \cap U)$ induces an isomorphism

$$\operatorname{Cl}(X)/\mathbb{Z}[Z] \to \operatorname{Cl}(U),$$

where $\mathbb{Z}[Z]$ denotes the subgroup of the divisor class group generated by the equivalence class $[Z]$ of Z.

(ii) The map $D \mapsto D \times \mathbb{A}^1$ induces an isomorphism

$$\operatorname{Cl}(X) \simeq \operatorname{Cl}(X \times \mathbb{A}^1).$$

2.1.2 Sheaf Cohomology and Euler Characteristic

We will now recall the basic principles of sheaf cohomology as this will be an important concept used throughout this thesis. However, we will not repeat the construction of sheaf cohomology itself and rather refer to the literature. Probably the purest approach — as always — is taken in [15, Section 12]. A very fast approach can be found in [33, Chapter III, Sections 1 and 2].

But there is also a different view on things. Since we are only interested in understanding projective varieties — as this seems to be challenging enough — , it is completely sufficient to use Čech cohomology as for projective varieties both concepts of cohomology will associate the very same cohomology groups to every quasi-coherent sheaf (see for example [15, Proposition 1.4.1]).

This version of Čech cohomology was also Serre's original approach in [56], when he introduced many concepts of sheaf theory from analytic into algebraic geometry. Even today this paper still offers one of the most approachable — albeit French — and down-to-earth introductions to this topic. A methodically different presentation of the different versions of cohomology is given in the highly recommendable lecture notes by Vakil [63, Chapter 18].

Whichever version of cohomology one uses, the following statement holds true. In fact, it would still hold in far greater generality. But we will phrase it purely for the cases that we will encounter later on.

Remark 2.1.14. Let X be a variety and $\mathcal{F}$ a quasi-coherent sheaf on X. It follows immediately from the definition(s) that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ which of course is just a different notation for $\mathcal{F}(X)$.

The first statement can be found in [33, Chapter III, Theorem 5.2] and is due to Serre.

Theorem 2.1.15 (Serre)**.** *Let X be a projective variety and $\mathcal{F}$ a quasi-coherent sheaf on X. Then $H^i(X, \mathcal{F})$ is a finite-dimensional vector space over $\mathbb{K}$.*

The following result would also fit in Section 2.1.6 but it is needed for the definition of Euler characterstic. It is presented for example in [33, Chapter III, Theorem 2.7].

Theorem 2.1.16 (Grothendieck Vanishing)**.** *Let X be a variety of dimension d and $\mathcal{F}$ a quasi-coherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > d$.*

Because of these results, the following definition of Euler characteristic makes sense.

Definition 2.1.17. Let X be a projective variety and $\mathcal{F}$ a quasi-coherent sheaf on X.

(i) We denote by $h^i(X, \mathcal{F}) := \dim H^i(X, \mathcal{F})$ the dimension of the i-th cohomology group of $\mathcal{F}$.

(ii) The **Euler characteristic** $\chi(X, \mathcal{F})$ of $\mathcal{F}$ is defined as

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

As a first glimpse on the usefulness of this object, we will state the following result, which can be found in [15, Propositioin 2.5.2]

Proposition 2.1.18 (Additivity of Euler Characteristic)**.** *Let X be a projective variety and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent sheaves on X. Then*

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'').$$

The next sections will give us further insight into this number.

2.1.3 Flat Families

Let us introduce the following shorthand notation.

Notation 2.1.19. Let $\pi\colon X \to Y$ be a morphism of varieties. We will write $X_y := \pi^{-1}(y) \subseteq X$ for the fiber of π over $y \in Y$.

Let us now introduce an important type of morphisms of varieties.

Definition 2.1.20. Let $\pi\colon X \to Y$ be a morphism of varieties.

(i) Then π is called **flat** if for every point $x \in X$ the induced map

$$\pi_x\colon \mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$$

makes the local ring $\mathcal{O}_{X,x}$ into a *flat* $\mathcal{O}_{Y,\pi(x)}$-module, i.e. if for every injective map of $\mathcal{O}_{Y,\pi(x)}$-modules $\phi\colon M \hookrightarrow N$, the induced map

$$M \otimes_{\mathcal{O}_{Y,\pi(x)}} \mathcal{O}_{X,x} \to N \otimes_{\mathcal{O}_{Y,\pi(x)}} \mathcal{O}_{X,x}$$

is injective.

(ii) If $Y = \mathbb{A}^1$ and π is flat, the set $\{X_t\}_{t\in\mathbb{A}^1}$ is called a **flat family** over $\mathbb{A}^1$.

This property might not seem very intuitive at first but it is very helpful in understanding complicated varieties as the fibers of a flat morphism might be quite different while many geometric properties are open under flat morphisms, meaning that if one fiber fulfills a property, then all fibers in an open neighborhood will fulfill the property.

One example is the following standard fact (see for example [63, Proposition 24.5.6]).

Proposition 2.1.21. *All fibers of a flat morphism have the same dimension.*

Another fact is the following consequence of [17] and Serre's criterion for normality from [16, Théorème 5.8.6].

Theorem 2.1.22. *Let $X \to Y$ be a flat morphism of varieties. Suppose that the fiber X_y is normal for some point $y \in Y$. Then there exists an open neighborhood $y \in U \subseteq Y$ such that all the fibers X_z, $z \in U$, are normal.*

Furthermore, if all fibers X_y, $y \in Y$, are normal, then X is normal.

Remark 2.1.23. The last sentence does not really need flatness of the morphism, but we will include it at this point since it fits thematically.

Later on, we will encounter a similar statement by Elkik in Theorems 2.1.56 and 2.1.58. Finally, there is also a result on Euler characteristic that we want to present here directly. It can be found in [63, Theorem 24.7.1] in far greater generality.

Theorem 2.1.24 (Invariance of Euler Characteristic in Flat Families)**.** *Let $\mathcal{X}$ be a variety, $\pi\colon \mathcal{X} \to \mathbb{A}^1$ a flat projective morphism and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. Then $\chi(\mathcal{X}_t, \mathcal{F}|_{\mathcal{X}_t})$ is a constant function in t.*

This result is quite astonishing, since the equivalent statement does not hold for the cohomology groups separately. To be more precise, the functions $t \mapsto h^i(\mathcal{X}_t, \mathcal{F}|_{\mathcal{X}_t})$ would only be upper semi-continuous (see for example [63, Theorem 28.1.1 and Example 28.1.2]).

Remark 2.1.25. It is important to notice that the condition of π being projective can be relaxed to π being proper — but not further (see [63, Theorem 28.2.5]).

2.1.4 The Hilbert Polynomial

We will now introduce a famous polynomial. It will give us a first glance at the fact that algebraic geometry is welcoming combinatorial methods.

The following is a simplification of [15, Theorem 2.5.3].

Theorem 2.1.26 (Hilbert Polynomials)**.** *Let X be a projective variety and $\mathcal{F}$ a coherent sheaf over X. Let $\mathcal{L}$ be an ample line bundle on X. Then there exists a polynomial $P_{\mathcal{L}}(\mathcal{F})(T) \in \mathbb{Q}[T]$ — called* ***Hilbert polynomial*** *— such that*

$$P_{\mathcal{L}}(\mathcal{F})(n) = \chi(X, \mathcal{F} \otimes \mathcal{L}^n)$$

for all $n \in \mathbb{Z}$.

Notation 2.1.27. If $\mathcal{F} = \mathcal{O}_X$ we will simply write $P_{\mathcal{L}}$ for $\mathcal{P}_{\mathcal{L}}(\mathcal{O}_X)$.

The next theorem follows from the fact that Euler characteristic is additive on short exact sequences (see [15, Lemma 2.5.2]) and the fact that tensoring with line bundles is exact (see for example [63, Exercise 13.1.E]).

Theorem 2.1.28 (Additivity of Hilbert Polynomials)**.** *Let X be a projective variety and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of coherent sheaves on X. Then the respective Hilbert polynomials fulfill

$$P_{\mathcal{L}}(\mathcal{F}) = P_{\mathcal{L}}(\mathcal{F}') + P_{\mathcal{L}}(\mathcal{F}'').$$

The existence of these polynomials is very interesting. We will later connect them to other polynomials in polyhedral and toric geometry.

2.1.5 Serre Duality

We will now present one of the most important results of algebraic geometry. Since there are many different versions of Serre Duality and even more formulations of those, we will restrict to the most important case for our purpose.

Probably the fastest self-contained proof of Serre Duality can be found in [33, Chapter III, Sections 6 and 7]. A more detailed and broader approach has been taken in [63, Chapter 30]. The following statement can be found in both references ([33, Chapter III, Corollary 7.7] and [63, Corollary 30.3.10]).

Theorem 2.1.29 (Serre Duality)**.** *Let X be a projective Cohen-Macaulay variety of dimension d. Then there exists a coherent sheaf ω_X° on X — called the* ***dualizing sheaf*** *on X — such that*

$$H^i(X, \mathcal{F}) \simeq H^{d-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)^*$$

for every locally free sheaf $\mathcal{F}$ and $0 \leq i \leq d$.

One can prove that this sheaf is in fact unique (see for example [33, Chapter III, Proposition 7.2] or [63, Proposition 30.1.5]), hence the terminology "*the* dualizing sheaf" in the former theorem was justified.

Proposition 2.1.30. *On any projective variety X, a coherent sheaf ω_X° satisfying the assumptions of Theorem 2.1.29 is — if it exists — unique up to unique isomorphism.*

A useful formula is the following. It can be found for example in [63, Proposition 30.4.8].

Theorem 2.1.31 (Adjunction Formula)**.** *Let X be a projective Cohen-Macaulay variety with dualizing sheaf ω_X° and let D be an effective Cartier divisor on X. Then there exists a dualizing sheaf ω_D° on D and*

$$\omega_D^\circ \simeq (\omega_X^\circ \otimes \mathcal{O}_X(D))|_D.$$

This definition of a dualizing sheaf is rather abstract but luckily it can be traced back to known objects in special cases. In the case of normal varieties, it turns out that the dualizing sheaf is connected to the cotangent bundle of the variety. We will make this precise in the following statements.

The first one can be found in [33, Chapter II, Corollary 4.2].

Proposition 2.1.32. *Let X be a variety. Then the diagonal morphism $\Delta\colon X \to X \times X$ is a closed embedding.*

This image of the diagonal morphism allows us to define a useful object.

Definition 2.1.33. Let X be a variety and consider the image $\Delta(X)$ of X under the diagonal morphism $\Delta\colon X \to X \times X$. Since Δ is a closed embedding, the image is closed and we have an ideal sheaf $\mathcal{J}$ corresponding to $\Delta(X) \subseteq X \times X$. The **sheaf of differentials** Ω_X on X is defined as the pullback under Δ of the quotient sheaf $\mathcal{J}/\mathcal{J}^2$. This sheaf is sometimes called **cotangent sheaf**.

The latter name is due to the following fact that can be found for example in [63, Corollary 21.3.9]

Theorem 2.1.34. *Let X be a variety over a field of characteristic zero and let $x \in X$ be an arbitrary point. Let $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ be the maximal ideal in the local ring $\mathcal{O}_{X,x}$. Then the stalk of Ω_X in x is isomorphic to $\mathfrak{m}_x/\mathfrak{m}_x^2$ — a vector space over the base field called the **Zariski cotangent space**.*

This sheaf of differentials is especially useful in the case of smooth varieties, as the following result shows (see for example [33, Chapter II, Theorem 8.15]).

Theorem 2.1.35. *Let X be a smooth variety of dimension d. Then the sheaf Ω_X of differentials on X is locally free of rank d.*

Since we also want to study (mildly) singular varieties, the following fact might prove useful. It can be found for example in [36, Theorem 4.1.11 (ii) and (iii)]

Theorem 2.1.36. *Let X be a normal variety. The subset of singular points $X_{\mathrm{sing}} \subseteq X$ is a subvariety of codimension greater or equal then 2.*

Now we can construct a very useful and *natural* object.

Definition 2.1.37. Let X be a normal variety of dimension d.

(i) The open subvariety of all smooth points is called the **regular locus** of X, denoted X_{reg}.

(ii) The **canonical sheaf** ω_X of X is defined as the direct image sheaf of the wedge-product $\bigwedge^{\dim X}\Omega_{X_{\mathrm{reg}}}$ under the natural inclusion morphism $X_{\mathrm{reg}} \hookrightarrow X$.

As the notation already suggests, there is a close connection between the dualizing sheaf and the canonical sheaf of a normal variety (see for example [36, Corollary 5.3.9] or [63, Theorem 18.5.1]).

Theorem 2.1.38. *Let X be a normal projective Cohen-Macaulay variety. Then the dualizing sheaf ω_X° on X is isomorphic to the canonical sheaf ω_X on X.*

The following corollary is immediate.

Corollary 2.1.39 (Serre Duality)**.** *Let X be a normal projective Cohen-Macaulay variety of dimension d and $\mathcal{L}$ a line bundle over X. Then*

$$h^i(X, \mathcal{L}) = h^{d-i}(X, \mathcal{L}^{-1} \otimes \omega_X).$$

But there is still more to it. We might be tempted to ask whether the canonical sheaf of a normal projective variety is invertible. This is not always true but a slightly weaker property can be proved (see for example [36, Corollary 5.3.9] once again).

Proposition 2.1.40. *Let X be a normal quasi-projective variety. Then the canonical sheaf ω_X is divisorial.*

Now we remind ourselves of the correspondence between divisorial sheaves and Weil divisors from Theorem 2.1.12.

Definition 2.1.41. Let X be a normal quasi-projective variety and ω_X its canonical sheaf. Any divisor D fulfilling $\mathcal{O}_X(D) \simeq \omega_X$ must be in the same divisor class and is called *the* **canonical divisor** on X, denoted by K_X. Its additive inverse $-K_X$ in the divisor class group $\mathrm{Div}(X)$ is called the **anticanonical divisor** on X and its corresponding divisorial sheaf $\mathcal{O}_X(-K_X)$ the **anticanonical sheaf** on X.

Remark 2.1.42. It is important to notice that — strictly speaking — only the divisor class of the canonical divisor from Definition 2.1.41 is unique and not the divisor itself. Since we are normally not interested in picking a specific representative of this divisor class, we will just call any element of it *the* canonical divisor.

Remark 2.1.43. This definition leads to another famous version of the adjunction formula from Theorem 2.1.31. Let X be a normal quasi-projective variety with canonical divisor K_X and let D be an effective Cartier divisor on X. Then

$$K_D \sim (K_X + D)|_D.$$

Nevertheless for many problems it is necessary to restrict to the case where the canonical sheaf is invertible. Such varieties have a special name.

Definition 2.1.44. Let X be a variety.

(i) The variety X is called **Gorenstein** if it is normal and Cohen-Macaulay and the canonical divisor K_X is Cartier.

(ii) The variety X is called $\mathbb{Q}$-**Gorenstein** if it is normal and the canonical divisor K_X is $\mathbb{Q}$-Cartier.

(iii) The variety X is called **Gorenstein Fano** if it is Gorenstein and the anticanonical divisor $-K_X$ is ample,

(iv) The variety X is called $\mathbb{Q}$-**Gorenstein Fano** if it is $\mathbb{Q}$-Gorenstein and the anticanonical divisor $-K_X$ is ample.

We want to close this overview with some remarks.

Remark 2.1.45. Notice that every $\mathbb{Q}$-Gorenstein Fano variety admits a very ample line bundle — namely the sheaf $\mathcal{O}_X(-nK_X)$ of some — properly chosen — positive integer multiple of the anticanonical line bundle. Hence every $\mathbb{Q}$-Gorenstein Fano variety is projective.

Remark 2.1.46. ($\mathbb{Q}$-)Gorenstein Fano varieties are often abbreviated as *Fano varieties.*

Remark 2.1.47. For some years there has been a discussion whether Cohen-Macaulay-ness should be required in the definition of Gorenstein varieties. While the traditional definition always included this hypothesis, some mathematicians brought forward reasonable arguments against this practice — especially the awkward mismatch between the definitions of Gorenstein and $\mathbb{Q}$-Gorenstein, where Cohen-Macaulay-ness is not required. Throughout this thesis we will be interested in applying Serre duality to our varieties, so we chose the classical definition. Otherwise we would have to include the Cohen-Macaulay-property in every theorem as an additional assumption. Hence this choice is neither a moral nor philosophical but a practical one.

2.1.6 Vanishing Theorems

For many reasons it is important to know whether some cohomology groups for a given coherent sheaf on a variety are equal to zero or not. Such results are called vanishing (and non-vanishing) theorems and many mathematicians did tremendous work in proving some of those.

We have already seen one example of a vanishing theorem — namely Grothendieck Vanishing in Theorem 2.1.16. It ensures that for every variety X and every quasi-coherent sheaf $\mathcal{F}$ the i-th cohomology group $H^i(X, \mathcal{F})$ vanishes whenever i is strictly greater than the dimension of X.

Although the hunt for vanishing theorems is an interesting field itself, we will restrict ourselves to some special representatives that will be needed throughout this thesis.

The first vanishing theorem is due to Serre. We will state a version of this theorem that can be found in [15, Proposition 2.2.2] or as a combination of [33, Chapter II, Theorem 5.17] and [33, Chapter III, Proposition III.5.3].

Theorem 2.1.48 (Serre Vanishing)**.** *Let X be a projective variety, let $\mathcal{L}$ be an ample line bundle over X and $\mathcal{F}$ a coherent sheaf on X. Then there exists an integer $n_0 > 0$ such that for every $n > n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated and $h^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$.*

Another important vanishing theorem is due to Kodaira (see for example [43, Theorem 2.47]).

Theorem 2.1.49 (Kodaira Vanishing)**.** *Let X be a smooth complex projective variety and let $\mathcal{L}$ be an ample line bundle over X. Then $h^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$.*

Again we would like to venture beyond the smooth case. For this purpose we need the following definition. We will not define the *higher direct image of a sheaf* but rather refer to the definition given in [33, Chapter III, Section 8].

Definition 2.1.50. Let X be a variety over a field of characteristic zero.

(i) A **resolution of singularities** is a proper birational morphism from a smooth variety to X.

(ii) A resolution of singularities $f\colon Y \to X$ is called **rational** if the higher direct images of the structure sheaf vanish, i.e. $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.

(iii) We say that X **has rational singularities** if X is normal and every resolution of singularities of X is rational.

A very nice property of resolutions was proved by Grauert and Riemenschneider in [31, Satz 2.3]

Theorem 2.1.51 (Grauert-Riemenschneider Vanishing)**.** *Let X be a complex projective variety and $f\colon Y \to X$ a resolution of singularities. Let $\mathcal{L}$ be an ample line bundle over X. Then $R^i f_*(f^*\mathcal{L} \otimes \omega_Y) = 0$ for all $i > 0$.*

We will see later on, that all of the varieties we are interested in have rational singularities. One important classification of such varieties can be found in [43, Theorem 5.10].

Theorem 2.1.52. *Let X be a projective variety over a field of characteristic zero. Then the following are equivalent.*

(i) X has rational singularities.

(ii) X is normal and there exists a rational resolution of singularities of X.

(iii) X is Cohen-Macaulay and for any resolution of singularities $f\colon Y \to X$ we have $f_*\omega_Y^\circ \simeq \omega_X^\circ$.

We will now use this property of rational resolutions to state a singular version of Kodaira Vanishing that follows naturally from Grauert-Riemenschneider Vanishing in Theorem 2.1.51 and the observation that on normal projective Cohen-Macaulay varieties the canonical sheaf is in fact the dualizing sheaf (see Theorem 2.1.38).

Theorem 2.1.53 (Kodaira Vanishing for Rational Singularities)**.** *Let X be a complex projective variety having rational singularities and let $\mathcal{L}$ be an ample line bundle over X. Then $h^i(X, \mathcal{L} \otimes \omega_X) = 0$ for all $i > 0$.*

Equivalently, $h^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$ because of Serre Duality (see Corollary 2.1.39).

Remark 2.1.54. Notice that if X is a complex Gorenstein Fano variety having rational singularities, Theorem 2.1.53 implies $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ since the canonical sheaf is invertible and its inverse sheaf is ample.

There are even more general versions of Kodaira Vanishing. One of the strongest results might be [45, Corollary 6.6]. A version that fits our setting — although we will never apply it directly — can be found in [43, Theorem 2.70].

Theorem 2.1.55 (Kodaira Vanishing for Rational Singularities II)**.** *Let X be a complex projective variety having rational singularities and let D be an nef and big $\mathbb{Q}$-Cartier Weil divisor on X. Then $h^i(X, \mathcal{O}_X(-D)) = 0$ for all $i < \dim X$.*

We will conclude this overview with two beautiful results by Elkik showing that rational singularities behave well under flat degenerations. The first result is a reformulation of [19, Théorème 4].

Theorem 2.1.56 (Elkik)**.** *Let $X \to Y$ be a flat morphism of varieties over a field of characteristic zero. Suppose that the fiber X_y has rational singularities for some point $y \in Y$. Then there exists an open neighborhood $y \in U \subseteq Y$ such that all fibers X_z, $z \in U$, have rational singularities.*

Remark 2.1.57. One also says that the property to have rational singularities is an *open condition in flat families.*

The second result is a reformulation of [19, Théorème 5].

Theorem 2.1.58 (Elkik)**.** *Let $X \to Y$ be a flat morphism of varieties over a field of characteristic zero. Suppose that Y has rational singularities and all fibers X_y, $y \in Y$, have rational singularities. Then the variety X has rational singularities.*

2.2 Polyhedral Geometry

In this section we would like to recall some basic and advanced results from polyhedral geometry and fix some notation. For most results on polytopes we will refer to [64]. For Ehrhart Theory we recommend [5].

2.2.1 Polytopes and Fans

Let us first introduce some basic notation.

Definition 2.2.1. Let $S \subseteq \mathbb{R}^d$ be a (possibly infinite) set.

(i) The **conic hull** of S is defined as

$$\operatorname{cone}(S) := \left\{ \sum_{i=0}^{r} \lambda_i s_i \,\middle|\, \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}, s_1, \ldots, s_r \in S, r \in \mathbb{Z}_{>0} \right\}.$$

(ii) The **affine hull** of S is defined as

$$\operatorname{aff}(S) := \left\{ \sum_{i=0}^{r} \lambda_i s_i \,\middle|\, \lambda_1, \ldots, \lambda_r \in \mathbb{R}, \sum_{i=0}^{r} \lambda_i = 1, s_1, \ldots, s_r \in S, r \in \mathbb{Z}_{>0} \right\}.$$

(iii) The **convex hull** of S is defined as

$$\operatorname{conv}(S) := \left\{ \sum_{i=0}^{r} \lambda_i s_i \,\middle|\, \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}, \sum_{i=0}^{r} \lambda_i = 1, s_1, \ldots, s_r \in S, r \in \mathbb{Z}_{>0} \right\}.$$

We can now define the two main objects of polyhedral geometry.

Definition 2.2.2. (i) A **finitely generated cone** is defined as the conic hull of finitely many points.

(ii) A **convex polytope** is defined as the convex hull of finitely many points.

There is a very well known dual picture to the above descriptions that can be found in [64, Theorems 1.1 and 1.3].

Theorem 2.2.3. *(i) A subset of $\mathbb{R}^d$ is a finitely generated cone if and only if it is a finite intersection of halfspaces through the origin.*

(ii) A subset of $\mathbb{R}^d$ is a convex polytope if and only if it is a bounded finite intersection of halfspaces.

Remark 2.2.4. This equivalent description as finite intersections of halfspaces justifies the term *polyhedral* in this area, hence we will mostly refer to finitely generated cones as *polyhedral cones*.

These polyhedral objects have a natural definition of dimension.

Definition 2.2.5. Let $\mathcal{A} \subseteq \mathbb{R}^d$ be an affine set, let $\mathcal{C} \subseteq \mathbb{R}^d$ be a polyhedral cone and let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope.

(i) The affine set $\mathcal{A}$ can be written as the translation of a linear subspace $L \subseteq \mathbb{R}^d$ by a vector $v \in \mathbb{R}^d$, i.e. $\mathcal{A} = v + L$. The **dimension** of the affine set $\mathcal{A}$ is defined as the dimension of the linear subspace L parallel to $\mathcal{A}$. In other words, let $v \in \mathcal{A}$ be any vector in $\mathcal{A}$. Then

$$\dim \mathcal{A} := \dim(\mathcal{A} - v)$$

(ii) The **dimension** of the polyhedral cone $\mathcal{C}$ is defined as the dimension of the affine hull of $\mathcal{C}$, i.e.

$$\dim \mathcal{C} := \dim \operatorname{aff} \mathcal{C}.$$

(iii) The **dimension** of the convex polytope $\mathcal{P}$ is defined as the dimension of the affine hull of $\mathcal{P}$, i.e.

$$\dim \mathcal{P} := \dim \operatorname{aff} \mathcal{P}.$$

We will now introduce subobjects of cones and polytopes. For their formulation we need the following notation.

Notation 2.2.6. Let $v \in \mathbb{R}^d$ and $b \in \mathbb{R}$. Then we define the halfspaces

$$H_{v,b}^{+} := \{x \in \mathbb{R}^d \mid \langle x, v \rangle \geq b\} \text{ and } H_{v,b}^{-} := \{x \in \mathbb{R}^d \mid \langle x, v \rangle \leq b\}$$

as well as the affine hyperplane

$$H_{v,b} := H_{v,b}^{+} \cap H_{v,b}^{-}.$$

If $b = 0$ we will just write $H_v^+ := H_{v,0}^+$, $H_v^- := H_{v,0}^-$ and $H_v := H_{v,0}$.

We can now pose the definition.

Definition 2.2.7. Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a polyhedral cone and $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope.

(i) A **face** of the polyhedral cone $\mathcal{C}$ is a (possibly empty) subset of the form $\mathcal{C} \cap H_v$ for some vector $v \in \mathbb{R}^d$ such that $\mathcal{C} \subseteq H_v^+$.

(ii) A **face** of the convex polytope $\mathcal{P}$ is a (possibly empty) subset of the form $\mathcal{P} \cap H_{v,b}$ for some vector $v \in \mathbb{R}^d$ and scalar $b \in \mathbb{R}$ such that $\mathcal{P} \subseteq H_{v,b}^-$.

(iii) A zero-dimensional face is called a **vertex**. The set of all vertices of the convex polytope $\mathcal{P}$ is denoted by $\operatorname{vert}(\mathcal{P})$. The same can be used for cones.

(iv) A one-dimensional face of $\mathcal{C}$ or $\mathcal{P}$ is called an **edge**.

(v) A face of $\mathcal{C}$ (or $\mathcal{P}$) of dimension $\dim \mathcal{C} - 1$ (respectively $\dim \mathcal{P} - 1$) is called a **facet**.

Remark 2.2.8. The difference in the sign convention in the former description — although mathematically *irrelevant* — is due to the fact that we would like to write polyhedral cones as

$$\mathcal{C} = \{x \in \mathbb{R}^d \mid Ax \geq 0\}$$

for some matrix $A \in M_{r,d}(\mathbb{R})$ and convex polytopes as

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid Ax \leq b\}$$

for some matrix $A \in M_{r,d}(\mathbb{R})$ and $b \in \mathbb{R}^r$. Expressions of the form $Ax \leq b$ mean that $(Ax)_i \leq b_i$ for all $i = 1, \ldots, r$.

The following properties seem natural but it is not at all trivial to prove them. One would need some machinery such as Fourier-Motzkin Elimination (see [64, Section 1.2]) and/or Farka's Lemma (see [64, Section 1.4]). Although we will not need it directly, we still want to present one of the many versions of Farka's Lemma since it is a beautiful result in itself and morally fits the duality theme of this thesis. The following formulation is [64, Proposition 1.7].

Proposition 2.2.9 (Farka's Lemma)**.** *Let $A \in M_{r,d}(\mathbb{R})$ and $z \in \mathbb{R}^r$. Then one of the following properties hold — but not both!*

(i) *There exists a vector* $x \in \mathbb{R}^d$ *such that* $Ax \leq z$.

(ii) *There exists a vector* $c \in \mathbb{R}^r$ *such that* $c \geq 0, {}^t cA = 0$ *and* ${}^t cz < 0$.

Returning to our study of faces we have the following result from [64, Propositions 2.2 and 2.3].

Proposition 2.2.10. *Let* $\mathcal{P}$ *be a convex polytope.*

(i) *The polytope* $\mathcal{P}$ *is the convex hull of its vertices, i.e.* $\mathcal{P} = \mathrm{conv}(\mathrm{vert}\,\mathcal{P})$.

(ii) *The vertex set* $\mathrm{vert}(\mathcal{P})$ *is finite.*

(iii) *Every face* F *of* $\mathcal{P}$ *is a convex polytope whose faces are precisely the faces of* $\mathcal{P}$ *contained in* F. *Its vertex set is given by* $\mathrm{vert}(F) = \mathrm{vert}(\mathcal{P}) \cap F$.

(iv) *Every intersection of faces in* $\mathcal{P}$ *is a face of* $\mathcal{P}$.

From these statements — and some further, non-trivial considerations — one can prove the following result (see for example [64, Theorem 2.7 (v)]).

Theorem 2.2.11. *Let* $\mathcal{P} = \bigcap_{i=1}^{r} H^-_{\alpha_i,b_i} \subseteq \mathbb{R}^d$ *be a full-dimensional polytope. Then for every vertex* v *one can chose* n *integers* $i_1, \ldots, i_n$ *such that*

$$\{v\} = H_{\alpha_{i_1},b_{i_1}} \cap \ldots \cap H_{\alpha_{i_n},b_{i_n}}.$$

There are still some important notions that we did not introduce. We will try to summarize them in the following definition.

Definition 2.2.12. Let $\mathcal{C}$ be a polyhedral cone and let $\mathcal{P}$ be a convex polytope.

(i) The polyhedral cone $\mathcal{C}$ is called **pointed** (or **strongly convex**) if the origin is a face of $\mathcal{C}$.

(ii) The polyhedral cone $\mathcal{C}$ is called **rational** if it is the conic hull of finitely many rational points or equivalently it can be written as the intersection of finitely many halfspaces through the origin with rational normal vectors.

(iii) The convex polytope $\mathcal{P}$ is called **rational** if it is the convex hull of finitely many rational points or equivalently if all of its vertices are rational points. Equivalently again, it can be written as the bounded intersection of finitely many halfspaces with rational normal vectors and rational scalars.

(iv) A **lattice polytope** is a convex rational polytope whose vertices are *lattice points*, i.e. they have solely integral coordinates.

We will now introduce duality for polyhedral cones and convex polytopes.

Definition 2.2.13. Let $S \subseteq \mathbb{R}^d$ be a set. The **(polar) dual** S^* of the set S is defined as

$$\mathcal{P}^* := \{x \in \mathbb{R}^d \mid \langle x, s\rangle \leq 1 \text{ for all } s \in S\}.$$

Remark 2.2.14. We will mostly omit the word *polar* when speaking about polar duality for polytopes.

Remark 2.2.15. As before, there is another convention regarding polar duality. Many authors — like the ones in [12] — like to define the polar dual of a set $S \subseteq \mathbb{R}^d$ as

$$\{x \in \mathbb{R}^d \mid \langle x, s\rangle \geq -1 \text{ for all } s \in S\}.$$

This is merely a sign convention and does not change mathematics. Since we want to be coherent with our definition of the normal fan of a convex polytope as the *outer* normal fan later on, we chose our convention. This is also consistent with [64].

Remark 2.2.16. If $\mathcal{C}$ is a polyhedral cone one can verify immediately that its polar dual can be constructed via

$$\mathcal{C}^* = \{x \in \mathbb{R}^d \mid \langle x, y\rangle \leq 0 \text{ for all } y \in \mathcal{C}\}.$$

Since this sign convention seems to be a bit awkward, we will introduce the notation

$$\mathcal{C}^\vee := -\mathcal{C}^*$$

for the *other* dual cone. This is coherent with the notation in [12].

Another consequence of Farka's Lemma (see Proposition 2.2.9) is the following result. The presented version is reformulated from [12, Lemma 1.2.13].

Theorem 2.2.17 (Hyperplane Separation)**.** *Let $\sigma_1, \sigma_2 \subseteq \mathbb{R}^d$ be two polyhedral cones whose intersection $\tau = \sigma_1 \cap \sigma_2$ is a face of both cones. Then there exists an $\alpha \in \mathbb{R}^d$ such that*

$$\alpha \in \sigma_1^* \cap (-\sigma_2)^* \text{ and } \tau = \sigma_1 \cap H_\alpha = \sigma_2 \cap H_\alpha.$$

Furthermore, if both cones are rational, then the vector α can be chosen integral.

The following is a collection of facts directly mentioned in or consequences of [64, Theorem 2.11].

Theorem 2.2.18. *Let $\mathcal{P} = \bigcap_{i=1}^r H^-_{\alpha_i,b_i} \subseteq \mathbb{R}^d$ be a convex polytope with vertices $v_1, \dots, v_s$.*

(i) $\mathcal{P} \subseteq (\mathcal{P}^*)^*$ *and equality holds if and only if* $0 \in \mathcal{P}$.

(ii) If $0 \in \operatorname{int} \mathcal{P}$ *(implying that* $\mathcal{P}$ *is full-dimensional), then the* b_i*'s can be chosen non-zero. In this case, the polar dual* $\mathcal{P}^*$ *is a convex polytope and it can be calculated as*

$$\mathcal{P}^* = \operatorname{conv}(b_1^{-1}\alpha_1, \ldots, b_r^{-1}\alpha_r) = \bigcap_{i=1}^{s} H_{v_i,1}^-.$$

(iii) If $0 \in \operatorname{int} \mathcal{P}$ *then* $\mathcal{P}$ *is rational if and only if* $\mathcal{P}^*$ *is rational.*

(iv) For every $\lambda \in \mathbb{R}_{>0}$ *we have* $(\lambda\mathcal{P})^* = \lambda^{-1}\mathcal{P}^*$.

(v) If $\mathcal{P}^*$ *is a convex polytope, there is an inclusion-reversing bijection between the faces of* $\mathcal{P}$ *and the faces of* $\mathcal{P}^*$.

Now we want to introduce the connection between polytopes and cones via the following object.

Definition 2.2.19. Let $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ be a finite collection of non-empty polyhedral cones in $\mathbb{R}^d$.

(i) Then Σ is called a **polyhedral fan** if

  (i) Every non-empty face of a cone in Σ is again a cone in Σ.

  (ii) Every intersection of two cones in Σ is a face of both intersecting cones.

(ii) The fan Σ is called **pointed** if every cone in Σ is pointed.

(iii) The fan Σ is called **rational** if every cone in Σ is rational.

(iv) The **support** $\operatorname{supp}(\Sigma)$ is the union of all cones in Σ, i.e.

$$\operatorname{supp}(\Sigma) := \bigcup_{\sigma \in \Sigma} \sigma.$$

(v) We denote the collection of k-dimensional cones in Σ by $\Sigma(k)$. The elements of $\Sigma(1)$ are called **rays**.

(vi) Suppose Σ is a pointed rational polyhedral fan. Then every ray $\rho \in \Sigma(1)$ can be written as $\mathbb{R}_{\geq 0} v_\rho$ for some rational vector $v_\rho \in \mathbb{Q}^d$. The **primitive ray generator** u_ρ of ρ is defined as the unique vector in $\rho \cap \mathbb{Z}^d \setminus \{0\}$ whose coordinates have no common divisor.

We can now construct the normal fan associated to a convex polytope.

Construction 2.2.20 (Normal Fan of a Polytope)**.** Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope. For every non-empty face F of $\mathcal{P}$ define the set

$$\sigma_F := \{\alpha \in \mathbb{R}^d \mid \exists b \in \mathbb{R} : \mathcal{P} \subseteq H^-_{\alpha,b} \text{ and } F \subseteq H_{\alpha,b}\},$$

which can be shown to be a polyhedral cone. The reason is that it can be realized as the conic hull of the normal vectors of all facets of $\mathcal{P}$ containing $\mathcal{F}$.

We define the **normal fan** $\Sigma_\mathcal{P}$ of the convex polytope $\mathcal{P}$ as

$$\Sigma_\mathcal{P} := \{\sigma_F \mid F \neq \emptyset \text{ face of } \mathcal{P}\}.$$

Remark 2.2.21. Let $\mathcal{P}$ be a convex polytope with normal fan $\Sigma_\mathcal{P}$. The following observations are immediate.

(i) The normal fan $\Sigma_\mathcal{P}$ is a polyhedral fan with $\operatorname{supp}(\Sigma_\mathcal{P}) = \mathbb{R}^d$.

(ii) For every strictly positive integer $k \in \mathbb{Z}_{>0}$ we have $\Sigma_{k\mathcal{P}} = \Sigma_\mathcal{P}$.

(iii) The normal fan $\Sigma_\mathcal{P}$ is pointed if the convex polytope $\mathcal{P}$ is full-dimensional.

(iv) The normal fan $\Sigma_\mathcal{P}$ is rational if $\mathcal{P}$ is rational.

(v) The map $F \mapsto \sigma_F$ induces an inclusion-reversing bijection between the non-empty faces of $\mathcal{P}$ and the cones in $\Sigma_\mathcal{P}$.

Remark 2.2.22. As before, we should mention that there is a sign convention in our construction of the normal fan. To distinguish this version, it is often called *outer* normal fan. In contrast, the *inner* normal fan would be defined with H^+ instead of H^-. Since we are only interested in the outer normal fan, we will just leave out the additional adjective.

The reason behind this naming is that the outer normal fan can be constructed by putting the sun in the *center*—this would have to be defined—and looking at all the rays of sunlight that cross a facet perpendicularly. These rays would be the rays of the outer normal fan.

Finally let us construct a non-standard object.

Construction 2.2.23 (Round-down and Round-up of a Polytope)**.** Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a rational convex polytope and let u_ρ, $\rho \in \Sigma_P(1)$, denote the primitive generators of the rays of its normal fan Σ_P. Then for every ray $\rho \in \Sigma_\mathcal{P}(1)$ there exists a unique rational scalar $b_\rho \in \mathbb{Q}$ such that

$$\mathcal{P} = \bigcap_{\rho \in \Sigma_\mathcal{P}(1)} H^-_{u_\rho, b_\rho} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq b_\rho \text{ for all rays } \rho \text{ in } \Sigma_\mathcal{P}\}.$$

This allows us to define the **round-down** $\lfloor \mathcal{P} \rfloor$ of the rational convex polytope

$\mathcal{P}$ via

$$\lfloor \mathcal{P} \rfloor := \bigcap_{\rho \in \Sigma_{\mathcal{P}}(1)} H^-_{u_\rho, \lfloor b_\rho \rfloor} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq \lfloor b_\rho \rfloor \text{ for all rays } \rho \text{ in } \Sigma_{\mathcal{P}}\}$$

as well as the **round-up** $\lfloor \mathcal{P} \rfloor$ via

$$\lceil \mathcal{P} \rceil := \bigcap_{\rho \in \Sigma_{\mathcal{P}}(1)} H^-_{u_\rho, \lceil b_\rho \rceil} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq \lceil b_\rho \rceil \text{ for all rays } \rho \text{ in } \Sigma_{\mathcal{P}}\}.$$

The following properties give some intuition to these objects.

Proposition 2.2.24. *Let $\mathcal{P}$ be a rational convex polytope.*

(i) *Then the round-down $\lfloor \mathcal{P} \rfloor$ and the round-up $\lceil \mathcal{P} \rceil$ are rational convex polytopes.*

(ii) $\lfloor \mathcal{P} \rfloor \subseteq \mathcal{P} \subseteq \lceil \mathcal{P} \rceil$.

(iii) $\dim \lfloor \mathcal{P} \rfloor \leq \dim \mathcal{P} \leq \dim \lceil \mathcal{P} \rceil$.

(iv) $\lfloor -\mathcal{P} \rfloor = -\lceil \mathcal{P} \rceil$ *and* $-\lfloor \mathcal{P} \rfloor = \lceil -\mathcal{P} \rceil$.

Proof. Observations (i) and (ii) are clear from the definition. Observation (iii) is clear from (ii).

For (iv) notice that

$$\begin{aligned} -\mathcal{P} &= \{x \in \mathbb{R}^d \mid \langle -x, u_\rho \rangle \leq b_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\} \\ &= \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \geq -b_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\} \end{aligned}$$

and $\lfloor -b \rfloor = -\lceil b \rceil$ for every rational number $b \in \mathbb{Q}$.

□

2.2.2 (Dual-)Fano Polytopes

There is one class of polytopes that behaves quite nicely with respect to duality.

Definition 2.2.25. A convex polytope $\mathcal{P}$ is called **reflexive** if both $\mathcal{P}$ itself and its polar dual $\mathcal{P}^*$ are lattice polytopes.

Since this notion is due rigid for our applications we want to look at some weaker properties.

Definition 2.2.26. (i) A convex polytope is called a **Fano** polytope if its vertices are primitive lattice vectors.

(ii) A convex polytope is called a **dual-Fano** polytope if its polar dual is a Fano polytope.

(iii) A convex polytope is called a **weakly dual-Fano** polytope if it contains precisely one lattice point in its interior and its dual polytope is a lattice polytope.

Remark 2.2.27. While the notion of Fano polytopes is quite standard, the notion of a (weakly) dual-Fano polytope is not. But the naming — at least for dual-Fano polytopes — should be quite self-explanatory and it will be a useful shorthand.

Remark 2.2.28. Notice that every (weakly) dual-Fano polytope must contain the origin in its interior by Theorem 2.2.18 since its dual is required to be a convex polytope. By the same theorem, (weakly) dual-Fano polytopes are rational since they are the dual of a lattice (hence rational) polytope.

Before we get into the details, let us look at an example.

Example 2.2.29. In Figure 2.1 we see the sketch of three different polytopes in $\mathbb{R}^2$ and their dual polytopes. We will notice in this example that the three classes of polytopes introduced before are distinct classes.

The first polytope is given by the inequalities

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq -1, y \geq -1, 2x + 3y \leq 1\}$$

and we clearly see from the sketch that it is reflexive, dual-Fano and weakly dual-Fano.

The second polytope is given by the inequalities

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq -1, y \geq -1, x + 3y \leq 1\}$$

and we clearly see from the sketch that it is dual-Fano and weakly dual-Fano. However, since the upper left vertex is not integral, this polytope is not reflexive.

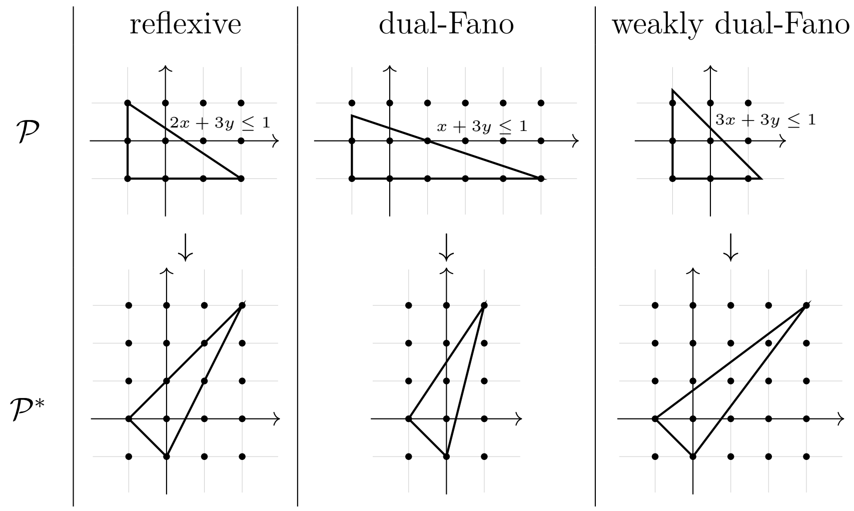
Finally, the third polytope is given by the inequalities

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq -1, y \geq -1, 3x + 3y \leq 1\}.$$

From the sketch we see that it is weakly dual-Fano. But it is not a lattice polytope, so it cannot be reflexive. Additionally the upper right vertex of its dual polytope is the point $(3, 3)$ which is an integral multiple of the lattice point $(1, 1)$. Hence the dual polytope will not be Fano.

Judging from the example we might guess that there is an inclusion relation between the three classes of polytopes and we could even guess how a standard form of these different polytopes would look like. We will formalize this in the next theorem and its corollary.

Figure 2.1: Sketch of the three different polytopes in Example 2.2.29 and their dual polytopes. The first one is reflexive, dual-Fano and weakly dual-Fano; the second one is dual-Fano and weakly dual-Fano but not reflexive; the third one is only dual-Fano.



Theorem 2.2.30. *(i) Every reflexive polytope is dual-Fano.*

(ii) Every dual-Fano polytope is weakly dual-Fano.

(iii) A polytope is reflexive if and only if it is a weakly dual-Fano lattice polytope.

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope. We can assume $\mathcal{P}$ to be rational and of full-dimension, since all properties occurring in the statements imply rationality and full-dimensionality. Let $\Sigma_{\mathcal{P}}$ denote its normal fan with primitive ray generators u_ρ, $\rho \in \Sigma(1)$.

For the first claim, assume that $\mathcal{P}$ is a reflexive polytope. We can find rational numbers b_ρ such that

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq b_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

Because $\mathcal{P}$ is a lattice polytope, the hyperplanes H_{u_ρ, b_ρ} must contain lattice points, which requires all b_ρ to be integers. By Remark 2.2.28 we know that $0 \in \operatorname{int} \mathcal{P}$. So all b_ρ must be strictly positive. By the same theorem we know that vertices of the dual polytope are given by

$$\operatorname{vert} \mathcal{P}^* = \left\{ \frac{u_\rho}{b_\rho} \;\middle|\; \rho \in \Sigma_{\mathcal{P}}(1) \right\}.$$

Since $\mathcal{P}^*$ must be a lattice polytope, we know that all b_ρ must be equal to 1 (because the u_ρ are primitive). So we get

$$\operatorname{vert}\mathcal{P}^* = \{u_\rho \mid \rho \in \Sigma_{\mathcal{P}}(1)\},$$

which means that $\mathcal{P}^*$ is a Fano polytope, i.e. $\mathcal{P}$ is a dual-Fano polytope.

Let us now prove the second claim. Assume that $\mathcal{P}$ is dual-Fano, i.e. $\mathcal{P}^*$ is a lattice polytope whose vertices are primitive. By the same arguments as before, one sees that

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho\rangle \leq b_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}$$

for some strictly positive rational numbers b_ρ and

$$\operatorname{vert}\mathcal{P}^* = \left\{\frac{u_\rho}{b_\rho} \mid \rho \in \Sigma_{\mathcal{P}}(1)\right\}.$$

Since the u_ρ are primitive, this polytope will only be a Fano polytope, if all of the parameters b_ρ are equal to 1. In conclusion we see that

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho\rangle \leq 1 \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

Notice that the origin is the only lattice point x satisfying

$$\langle x, u_\rho\rangle < 1 \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1),$$

so it is the only interior lattice point in $\mathcal{P}$. Obviously $\mathcal{P}^*$ is a lattice polytope since it is Fano, hence $\mathcal{P}$ is weakly dual-Fano.

Notice that claims (i) and (ii) already prove one direction of claim (iii). But the other direction follows immediately from the definitions. □

Interestingly enough, from the proof of Theorem 2.2.30 we get the following descriptions.

Corollary 2.2.31. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope and let u_ρ, $\rho \in \Sigma_{\mathcal{P}}(1)$, denote the primitive ray generators of the normal fan $\Sigma_{\mathcal{P}}$ of $\mathcal{P}$.*

(i) The polytope $\mathcal{P}$ is weakly dual-Fano if and only if there exist strictly positive integers k_ρ, $\rho \in \Sigma_{\mathcal{P}}(1)$, such that

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho\rangle \leq k_\rho^{-1} \textit{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

(ii) The polytope $\mathcal{P}$ is dual-Fano if and only if

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq 1 \textit{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

(iii) The polytope $\mathcal{P}$ is reflexive if and only if it is a lattice polytope and

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq 1 \textit{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

2.2.3 Ehrhart Theory

Given a subset $S \subseteq \mathbb{R}^d$, it is natural to ask, how many lattice points the set S contains, i.e. what the cardinality of $S \cap \mathbb{Z}^d$ is. It turns out that for rational convex polytopes there lies a beautiful theory behind this simple question — called *Ehrhart Theory*. A well-written introduction into this theory is given in [5]. The main object of this theory is the following.

Notation 2.2.32. Let $S \subseteq \mathbb{R}^d$ be an arbitrary subset. Then for every integer $n \in \mathbb{N}$ we will denote the number of lattice points in the n-th dilation of S by $L_S(n)$, i.e.

$$L_S(n) := \#(nS \cap \mathbb{Z}^d).$$

To formulate the birth result in this theory — called *Ehrhart-Macdonald Reciprocity* —, we need the following definition.

Definition 2.2.33. A **quasi-polynomial** over $\mathbb{R}$ is a function $f\colon \mathbb{R} \to \mathbb{R}$ that can be written as

$$f(x) = a_d(x)x^d + a_{d-1}(x)x^{d-1} + \ldots + a_1(x)x + a_0(x)$$

for some periodic functions $a_0, \ldots, a_d$ with integral period and $a_d \not\equiv 0$. We call d the **degree** of f.

Remark 2.2.34. An equivalent definition would be that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called a quasi-polynomial if there exists an integer T and polynomials $f_1, \ldots, f_T \in \mathbb{R}[x]$ such that

$$f(n) = f_i(n) \quad \text{if } i \equiv n \mod T$$

for every integer $n \in \mathbb{N}$.

We can now state a beautiful result that can be found for example in [5, Theorem 4.1].

Theorem 2.2.35 (Ehrhart-Macdonald Reciprocity)**.** *Let $\mathcal{P}$ be a rational convex polytope. Then there exists a quasi-polynomial $l_{\mathcal{P}}$ of degree* $\dim \mathcal{P}$ *— called the* ***Ehrhart quasi-polynomial*** *— such that*

$$l_p(n) = L_{\mathcal{P}}(n) \text{ for all } n \in \mathbb{N}.$$

Any such quasi-polynomial $l_{\mathcal{P}}$ fulfills

$$l_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} L_{\operatorname{int} \mathcal{P}}(n) \text{ for all } n \in \mathbb{N}.$$

Furthermore, if $\mathcal{P}$ is a lattice polytope, $l_{\mathcal{P}}$ can be chosen to be a rational polynomial.

Remark 2.2.36. Since the period of a polynomial is not quite unique and quasi-polynomials are not uniquely determined by their values on integers, for a given polytope $\mathcal{P}$ we have many different quasi-polynomials $l_{\mathcal{P}}$ fulfilling Ehrhart-Macdonald Reciprocity. Thankfully, we are generally only interested in evaluating those polynomials on integers — and these values are unique. So we will just call any quasi-polynomial *the* Ehrhart quasi-polynomial of $\mathcal{P}$ if it fulfills Ehrhart-Macdonald Reciprocity and among all those quasi-polynomials there exists none of strictly smaller period. To simplify notation we will denote this quasi-polynomial by $L_{\mathcal{P}}$ too.

Example 2.2.37. Let $\mathcal{P} = [0, 1/2] \subset \mathbb{R}$. Then $L_{\mathcal{P}}(n) = \lfloor \frac{n}{2} \rfloor$ so one could choose for example

$$l_{\mathcal{P}}(x) = \frac{1}{2}x + 1 - \frac{1}{2}\sin^2\left(\frac{\pi x}{2}\right) \text{ or } l_{\mathcal{P}}(x) = \frac{1}{2}x + \frac{3 + \cos \pi x}{4}.$$

We have seen that every lattice polytope will have an Ehrhart polynomial but there are also non-lattice convex polytopes whose Ehrhart quasi-polynomial is a polynomial. So the following category of polytopes should be quite interesting.

Definition 2.2.38. A **quasi-lattice polytope** is a rational convex polytope whose Ehrhart quasi-polynomial is a polynomial.

Sometimes it is more convenient to work with the generating function of the Ehrhart quasi-polynomial. This will be defined in the following.

Definition 2.2.39. Let $S \subseteq \mathbb{R}^d$ be a set. Then the **Ehrhart series** Ehr_S of S is defined as

$$\operatorname{Ehr}_S(z) := \sum_{n=0}^{\infty} L_S(n) z^n.$$

The following result is due to Stanley [58, Theorem 1.6].

Theorem 2.2.40 (Stanley)**.** *Let $\mathcal{P}$ be a rational convex polytope of dimension d and let $T > 0$ be the smallest integer such that $T\mathcal{P}$ is a lattice polytope. Then there exist non-negative integers $h_i^* \in \mathbb{N}$, $0 \le i \le Td$, such that*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{i=0}^{Td} h_i^* z^i}{(1 - z^T)^{d+1}}$$

as rational functions.

We will conclude this overview with a beautiful results by Hibi [34].

Theorem 2.2.41 (Hibi)**.** *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope and suppose that $0 \in \operatorname{int} \mathcal{P}$. Then the dual polytope $\mathcal{P}^*$ is a lattice polytope if and only if*

$$\mathrm{Ehr}_{\mathcal{P}}(z^{-1}) = (-1)^{d+1} z \, \mathrm{Ehr}_{\mathcal{P}}(z)$$

as rational functions.

Remark 2.2.42. Hibi also remarked in [34, 1.3] that the Ehrhart series of a rational convex polytope $\mathcal{P}$ fulfills

$$\mathrm{Ehr}_{\mathcal{P}}(z^{-1}) = (-1)^{\dim \mathcal{P}+1} \sum_{n=1}^{\infty} L_{\operatorname{int} \mathcal{P}}(n) z^n$$

as rational functions.

So we can reformulate his theorem as follows, additionally introducing our own terminology.

Theorem 2.2.43 (Hibi)**.** *A full-dimensional rational convex polytope $\mathcal{P}$ is weakly dual-Fano if and only if its Ehrhart quasi-polynomial fulfills*

$$L_{\mathcal{P}}(n) = L_{\operatorname{int} \mathcal{P}}(n + 1)$$

for all $n \in \mathbb{N}$.

Remark 2.2.44. Both formulations of Hibi's result are not his original formulation, but especially the last version shows clearer which beautiful magic is actual happening in the background. First of all, Hibi did not use our notion of weakly dual-Fano polytopes. Secondly, for computational purposes it is helpful to notice that by Ehrhart-Macdonald Reciprocity (see Theorem 2.2.35) the condition of Hibi's Theorem is equivalent to the property that

$$L_{\mathcal{P}}(n) = (-1)^d L_{\mathcal{P}}(-n - 1)$$

for all $n \in \mathbb{N}$, a condition that can be verified by studying the Ehrhart quasi-polynomial alone. This was his original statement.

Yet another formulation can be found in [5, Theorem 4.6]. We will never use this result explicitly but it gives a hint that computations can be simplified for reflexive polytopes.

Theorem 2.2.45 (Hibi)**.** *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional lattice polytope and suppose that $0 \in \operatorname{int} \mathcal{P}$. Then $\mathcal{P}$ is a lattice polytope if and only if the coefficients h_i^* from Theorem 2.2.40 fulfill $h_i = h_{d-i}$ for every $0 \leq i \leq d$.*

2.3 Toric Geometry

An important connection between the polyhedral world and the algebraic world can be seen in the case of toric varieties. For the purpose of this thesis we will only consider toric varieties over $\mathbb{C}$ although most of the theory can be done over any algebraically closed field. The results of this chapter will mainly be taken from [12], while occasionally quoting [26].

In contrast with these sources, we will only consider the lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$ since every lattice is isomorphic to a lattice of this form. Hence we will also identify the dual lattice of $\mathbb{Z}^d$ with $\mathbb{Z}^d$ itself via the dual pairing given by the euclidean scalar product on $\mathbb{R}^d$.

2.3.1 Toric Varieties

Let us start by stating the definition of a toric variety.

Definition 2.3.1. A **toric variety** is a complex variety X that contains an algebraic torus $T \simeq (\mathbb{C}^\times)^d$ as a dense open subset such that the multiplication on $T \simeq (\mathbb{C}^\times)^d$ extends to a morphism $T \times X \to X$.

Example 2.3.2. The trivial examples $(\mathbb{C}^\times)^d$, $\mathbb{A}^d_{\mathbb{C}}$ and $\mathbb{P}^d_{\mathbb{C}}$ are toric varieties.

Although this definition is quite easy to state, it does not make the connection to the polyhedral world clear. For that purpose we need some additional work.

We will first introduce the notion of a semigroup and some basic properties.

Definition 2.3.3. Let $\mathsf{S} \subseteq \mathbb{Z}^d$ be a subset.

(i) The set S is called a **semigroup** if $s + t \in \mathsf{S}$ for all $s, t \in \mathsf{S}$.

(ii) The set S is called a **monoid** if it is a semigroup and $0 \in \mathsf{S}$.

(iii) Suppose the set S is a semigroup. It is called **finitely generated** if there exist $s_1, \dots, s_r \in \mathsf{S}$ such that every element of S can be written as an $\mathbb{N}$-linear combination of $s_1, \dots, s_r$.

(iv) Suppose the set S is a semigroup. It is called **saturated** if for any $x \in \mathbb{Z}^d$ the implication

$$(\exists m \in \mathbb{N} \setminus \{0\} : mx \in \mathsf{S}) \implies x \in \mathsf{S}$$

holds.

Let us first recall a standard fact (see for example [12, Proposition 1.2.17]).

Proposition 2.3.4 (Gordan's Lemma)**.** *Let $\sigma \subseteq \mathbb{R}^d$ be a rational polyhedral cone. Then the set $\mathsf{S}_\sigma := \sigma^* \cap \mathbb{Z}^d$ is a finitely generated semigroup.*

Remark 2.3.5. For any rational polyhedral cone $\sigma \subseteq \mathbb{R}^d$ the associated semigroup S_σ is in fact a monoid. But for some reason it is always called a semigroup in the literature.

Remark 2.3.6. Whenever we have a semigroup S we can construct the **semigroup algebra** $\mathbb{C}[\mathsf{S}]$ as the infinite dimensional $\mathbb{C}$-vector space with basis consisting of symbols χ^s for every $s \in \mathsf{S}_\sigma$ and multiplication defined via $\chi^s \cdot \chi^t := \chi^{s+t}$.

We can now define affine toric varieties associated to a rational polyhedral cone.

Definition 2.3.7. Let $\sigma \subseteq \mathbb{R}^d$ be a rational polyhedral cone. The **affine toric variety** U_σ associated to σ is defined as the spectrum of the semigroup algebra of the associated semigroup. In symbols $U_\sigma := \operatorname{Spec} \mathbb{C}[\mathsf{S}_\sigma]$.

This is in fact a toric variety as the following result from [12, Theorems 1.2.18] shows.

Theorem 2.3.8. *Let $\sigma \subseteq \mathbb{R}^d$ be a rational polyhedral cone. Then the associated affine variety U_σ is toric in the sense of Definition 2.3.1. Furthermore, its dimension is equal to d if and only if the cone σ is pointed.*

Remark 2.3.9. Notice that for every rational polyhedral cone $\sigma \in \mathbb{R}^d$ the associated semigroup S_σ is a subsemigroup of $\mathbb{Z}^d$. So we have an inclusion of algebras $\mathbb{C}[\mathsf{S}_\sigma] \hookrightarrow \mathbb{C}[\mathbb{Z}^d]$ which yields a dominant morphism $(\mathbb{C}^\times)^d = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^d] \to \operatorname{Spec} \mathbb{C}[\mathsf{S}_\sigma] = U_\sigma$. Additionally, whenever σ is pointed, its dual cone σ^* is full-dimensional and so it contains a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Hence $\langle \mathsf{S}_\sigma \rangle_\mathbb{Z} = \mathbb{Z}^d$ and the fraction field of $\mathbb{C}[\mathsf{S}_\sigma]$ will be $\mathbb{C}(\mathbb{Z}^d)$. So the formerly mentioned inclusion

$\mathbb{C}[\mathsf{S}_\sigma] \hookrightarrow \mathbb{C}[\mathbb{Z}^d]$ induces an isomorphism of fraction fields. Thus the dominant morphism $(\mathbb{C}^\times)^d \to U_\sigma$ is birational, which realizes the algebraic torus $(\mathbb{C}^\times)^d$ as an open dense subset of the affine variety U_σ. This also proves that every $s \in \mathbb{Z}^d$ defines a rational function $\chi^s \in \mathbb{C}[\mathbb{Z}^d] \hookrightarrow \mathcal{C}(\mathbb{Z}^d) \simeq \mathbb{C}(U_\sigma)$ on U_σ.

This connection allows us to analyze algebraic problems combinatorially. As an example we will state the following normality criteria from [12, Theorem 1.3.5].

Theorem 2.3.10. *Let X be an affine toric variety of dimension d. Then the following are equivalent.*

(i) The variety X is normal.

(ii) There exists a pointed rational polyhedral cone $\sigma \in \mathbb{R}^d$ such that $X \simeq U_\sigma$.

We would like to introduce projective toric varieties next. Roughly speaking we associate toric varieties to rational polyhedral fans or to rational polytopes. Although it might not appear so immediately, these two definitions are connected. But first we need the following observation from [12, Proposition 1.3.16].

Proposition 2.3.11. *Let $\sigma \subseteq \mathbb{R}^d$ be a rational polyhedral cone and τ a face of σ. Then there exists an $s \in \mathsf{S}_\sigma$ such that $\tau = \sigma \cap H_s$ and for any such $s \in \mathsf{S}_\sigma$ the semigroup algebra associated to τ is the localization of the semigroup algebra associated to σ localized at the regular function χ^s, in symbols $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^s}$.*

Construction 2.3.12 (The toric variety associated to a fan)**.** Let $\Sigma \subseteq \mathbb{R}^d$ be a pointed rational polyhedral fan. Let Y_Σ be the set theoretic disjoint union of all affine varieties U_σ, $\sigma \in \Sigma$, in symbols

$$Y_\Sigma := \bigsqcup_{\sigma \in \Sigma} U_\sigma.$$

We now want to glue the U_σ together properly. Let σ_1 and σ_2 be two cones in the fan Σ. Since Σ is a fan, the intersection $\tau = \sigma_1 \cap \sigma_2$ will be a face of both cones. The Hyperplane Separation Theorem (see Theorem 2.2.17) implies that there exists a hyperplane $H_s \subset \mathbb{R}^d$ with $s \in \mathsf{S}_{\sigma_1} \cap \mathsf{S}_{-\sigma_2}$ such that

$$\tau = \sigma \cap H_s = \sigma_2 \cap H_s.$$

By Proposition 2.3.11 we see that U_τ can be seen as the localization of U_{σ_1} at the regular function χ^s as well as the localization of U_{σ_2} at the regular function χ^{-s}. In symbols

$$U_\tau \simeq (U_{\sigma_1})_{\chi^s} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_1}]_{\chi^s})$$

and

$$U_\tau \simeq (U_{\sigma_2})_{\chi^{-s}} = \operatorname{Spec}(\mathbb{C}[S_{\sigma_2}]_{\chi^{-s}}).$$

Notice that the lattice vector s and hence the regular function χ^s might not be unique. However, the localization $\mathbb{C}[S_{\sigma_1}]_{\chi^s}$ and hence the open subset $(U_{\sigma_1})_{\chi^s} \subseteq U_{\sigma_1}$ will be unique by Proposition 2.3.11. We will denote this subset by U_{σ_1,σ_2}.

This means that for every such pair $(\sigma_1, \sigma_2) \in \Sigma \times \Sigma$ we get two inverse isomorphisms of affine varieties

$${}_{\sigma_2}\phi_{\sigma_1} \colon U_{\sigma_1,\sigma_2} \to U_{\sigma_2,\sigma_1} \text{ and } {}_{\sigma_1}\phi_{\sigma_2} \colon U_{\sigma_2,\sigma_1} \to U_{\sigma_1,\sigma_2}$$

between the dense open subsets $U_{\sigma_1,\sigma_2} \subseteq U_{\sigma_1}$ and $U_{\sigma_2,\sigma_1} \subseteq U_{\sigma_2}$.

Notice that every triplet $(\sigma_1, \sigma_2, \sigma_3) \in \Sigma^3$ satisfies the *gluing conditions*

$${}_{\sigma_2}\phi_{\sigma_1}(U_{\sigma_1,\sigma_2} \cap U_{\sigma_1,\sigma_3}) = {}_{\sigma_2}\phi_{\sigma_1}(U_{\sigma_1,\sigma_2}) \cap U_{\sigma_2,\sigma_3}$$

and

$${}_{\sigma_3}\phi_{\sigma_1} = {}_{\sigma_3}\phi_{\sigma_2} \circ {}_{\sigma_2}\phi_{\sigma_1} \text{ on } U_{\sigma_1,\sigma_2} \cap U_{\sigma_1,\sigma_3}.$$

This can be verified by checking the appropriate sequences of localization.

Because of these gluing conditions the following defines an equivalence relation. Let $a, b \in Y_\Sigma$. We say that a is equivalent to b, written $a \sim b$, if and only if there exists $\sigma_1, \sigma_2 \in \Sigma$ such that $a \in U_{\sigma_1}, b \in U_{\sigma_2}$ and $b = {}_{\sigma_2}\phi_{\sigma_1}(a)$.

We can now define the toric variety X_Σ associated to the fan Σ as the quotient space $X_\Sigma := Y_\Sigma / \sim$. Because of the gluing construction this is indeed an (abstract) variety.

Notice that the origin is a face of any cone $\sigma \in \Sigma$ since Σ is pointed. Hence all the affine varieties U_σ contain a torus $\operatorname{Spec}\mathbb{C}[\mathbb{Z}^d]$ and all those tori are identified via the gluing procedure. So the variety X_Σ will naturally contain the torus $U_{\{0\}} \simeq (\mathbb{C}^\times)^d$.

Example 2.3.13. Let us show this construction in an example. Let Σ be the fan in $\mathbb{R}^2$ with maximal cones

$$\sigma_1 := \operatorname{cone}(-e_1, -e_2), \sigma_2 := \operatorname{cone}(-e_1, e_1 + e_2) \text{ and } \sigma_3 := \operatorname{cone}(-e_2, e_1 + e_2).$$

The dual cones to the maximal cones are given by

$$\sigma_1^* := \operatorname{cone}(e_1, e_2), \sigma_2^* := \operatorname{cone}(-e_2, e_1 - e_2) \text{ and } \sigma_3^* := \operatorname{cone}(-e_1, -e_1 + e_2).$$

Including the smaller cones we get the disjoint union

$$\begin{aligned} Y_\Sigma := &\operatorname{Spec}\mathbb{C}[x_1, y_1] \sqcup \operatorname{Spec}\mathbb{C}[x_2 y_2^{-1}, y_2^{-1}] \sqcup \operatorname{Spec}\mathbb{C}[x_3^{-1}, x_3^{-1} y_3] \\ &\sqcup \operatorname{Spec}\mathbb{C}[x_4^{\pm 1}, y_4] \sqcup \operatorname{Spec}\mathbb{C}[x_5, y_5^{\pm 1}] \sqcup \operatorname{Spec}\mathbb{C}[x_6 y_6^{-1}, x_6^{-1} y_6, x_6^{-1} y_6^{-1}] \\ &\sqcup \operatorname{Spec}\mathbb{C}[x_7^{\pm 1}, y_7^{\pm 1}]. \end{aligned}$$

The indexing should emphasize that the rings are a priori not identified as subrings of some greater ring. Although we would not need this rigorous abstractness in this example, it should explain the gluing construction better.

As an example we will now consider the gluing of U_{σ_1} and U_{σ_2}. Notice that $\sigma_1^* \cap (-\sigma_2)^* = \mathbb{R}_{\geq 0} e_2$, so we can chose $s = e_2$. Indeed we have

$$\tau := \sigma_1 \cap \sigma_2 = \sigma_1 \cap H_{e_2} = \sigma_2 \cap H_{e_2} = -\mathbb{R}_{\geq 0} e_1$$

so the gluing sets are

$$U_{\sigma_1, \sigma_2} = \operatorname{Spec}\mathbb{C}[x_1, y_1]_{y_1} = \operatorname{Spec}\mathbb{C}[x_1, y_1^{\pm 1}]$$

and

$$U_{\sigma_2, \sigma_1} = \operatorname{Spec}\mathbb{C}[x_2 y_2^{-1}, y_2^{-1}]_{y_2^{-1}} = \operatorname{Spec}\mathbb{C}[x_2 y_2^{-1}, y_2^{\pm 1}] = \operatorname{Spec}\mathbb{C}[x_2, y_2^{\pm 1}].$$

So in this case the gluing morphisms are induced by the identification of the coordinate rings via

$$\begin{aligned} \mathbb{C}[U_{\sigma_1, \sigma_2}] = \mathbb{C}[x_1, y_1^{\pm 1}] &\leftrightarrow \mathbb{C}[x_2, y_2^{\pm 1}] = \mathbb{C}[U_{\sigma_2, \sigma_1}] \\ x_1 &\leftrightarrow x_2 \\ y_1 &\leftrightarrow y_2 \end{aligned}$$

realizing $U_\tau = \mathbb{C} \times \mathbb{C}^\times$ as a subset of two different affine spaces $\mathbb{A}^2$ glued together along U_τ.

By calculating all the other gluing conditions one might see that the resulting variety $X_\Sigma = Y_\Sigma / \sim$ is nothing else but the projective space $\mathbb{P}^2$ and the affine varieties $U_{\sigma_1}, U_{\sigma_2}$ and U_{σ_3} give in fact the standard affine covering

$$\mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[z_0, z_1, z_2] = \{z_2 \neq 0\} \cup \{z_1 \neq 0\} \cup \{z_0 \neq 0\}$$

Although this definition underlines the strong connection to polyhedral geometry, for some purposes it is too technical. So in some sense there is a dual construction to the one given above.

Construction 2.3.14 (The toric variety associated to a convex polytope)**.** Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a rational convex polytope. Chose any integer $l > 0$. We define the cone over $l\mathcal{P}$ as

$$\mathcal{C}(l\mathcal{P}) := \operatorname{cone}(\{1\} \times l\mathcal{P}) = \{(m,p) \subseteq \mathbb{R} \times \mathbb{R}^d \mid p \in ml\mathcal{P}\} \subseteq \mathbb{R} \times \mathbb{R}^d.$$

Since $\mathcal{P}$ is a rational convex polytope, the cone $\mathcal{C}(l\mathcal{P})$ will be a rational polyhedral cone. We can thus define a saturated and finitely generated semigroup $\mathsf{S}_{l\mathcal{P}}$ as

$$\mathsf{S}_{l\mathcal{P}} := \mathcal{C}(l\mathcal{P}) \cap (\mathbb{N} \times \mathbb{Z}^d).$$

We will remark that this is always a monoid, but somehow it is traditionally known as the semigroup associated to the polytope $l\mathcal{P}$. By construction this semigroup admits a grading by the first coordinate. Hence the semigroup algebra $\mathbb{C}[\mathsf{S}_{l\mathcal{P}}]$ is naturally graded and we can define the toric variety $X_{\mathcal{P}}$ associated to the polytope $\mathcal{P}$ as

$$X_{\mathcal{P}} := \operatorname{Proj} \mathbb{C}[\mathsf{S}_{l\mathcal{P}}]$$

where the Proj is taken with respect to the formerly mentioned grading coming from the first factor. Notice that the ring $\mathbb{C}[\mathsf{S}_{l\mathcal{P}}]$ is just the l-th *Veronese subring* of $\mathbb{C}[S_{\mathcal{P}}]$ (the resulting algebra if we chose $l = 1$). So by [18, Exercise 9.5] the variety $X_{\mathcal{P}}$ is independent of the choice of integer $l > 0$.

Remark 2.3.15. We should remark that in the standard literature (like [12]) this construction is often only given for lattice polytopes. But this limitation is not needed.

Example 2.3.16. Let us show this construction in an example. Consider the lattice polytope

$$\mathcal{P} := \operatorname{conv}(0, e_1, e_2) \subseteq \mathbb{R}^2.$$

Then the cone over $\mathcal{P}$ is

$$\mathcal{C}(\mathcal{P}) = \operatorname{cone}(e_0, e_0 + e_1, e_0 + e_2),$$

where we index the artificially introduced first coordinate by 0. So we get the semigroup

$$\mathsf{S}_{\mathcal{P}} = \mathbb{N}e_0 \oplus \mathbb{N}(e_0 + e_1) \oplus \mathbb{N}(e_0 + e_2)$$

and hence the semigroup algebra

$$\mathbb{C}[\mathsf{S}_{\mathcal{P}}] = \mathbb{C}[t, xt, yt] \simeq \mathbb{C}[x, y, z],$$

where the grading on the first algebra is given by the degree of t, i.e. for any monomial $x^\alpha, y^\beta, t^\gamma$ we set $\deg x^\alpha y^\beta t^\gamma = \gamma$, and the grading on the second algebra is the normal grading on polynomial rings, i.e. $\deg x^\alpha y^\beta z^\gamma = \alpha + \beta + \gamma$.

Hence the isomorphism is in fact an isomorphism of graded algebras. So in this case we can calculate the toric variety $X_{\mathcal{P}}$ as

$$X_{\mathcal{P}} = \operatorname{Proj} \mathbb{C}[t, xt, yt] \simeq \operatorname{Proj} \mathbb{C}[x, y, z] = \mathbb{P}^2$$

thus getting the same variety as in Example 2.3.13. One might notice that the normal fan $\Sigma_{\mathcal{P}}$ of our polytope $\mathcal{P}$ is in fact the fan Σ in that example. We will see in Theorem 2.3.20 that this is no accident.

In the remainder of this section we will state some important facts about the above constructions.

Since normality is a necessary assumption in many results quoted in Section 2.1, the following fact from [12, Theorem 3.1.5] is crucial. It also justifies the term *toric* in Construction 2.3.12.

Theorem 2.3.17. *Let $\Sigma \subseteq \mathbb{R}^d$ be a pointed rational polyhedral fan. Then the variety X_Σ is a normal toric variety with torus given by $U_{\{0\}} = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^d]$.*

Remark 2.3.18. In light of Remark 2.3.9 we notice that the rational functions on any normal toric variety X_Σ corresponding to a pointed rational polyhedral fan Σ are given by the rational functions on the open dense torus, i.e.

$$\mathbb{C}(X_\Sigma) \simeq \mathbb{C}(U_{\{0\}}) = \mathbb{C}(\chi^s \mid s \in \mathbb{Z}^d).$$

Interestingly, in some sense the opposite to the above theorem holds true as can be seen in [12, Corollary 3.1.8], which is a consequence of a result by Sumihiro from [62].

Theorem 2.3.19. *Let X be a normal toric variety of dimension d. Then there exists a pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ such that $X_\Sigma \simeq X$.*

The next result shows the connection between both constructions and can be found in [12, Theorem 7.1.13].

Theorem 2.3.20. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. Then the toric variety $X_{\mathcal{P}}$ associated to the polytope $\mathcal{P}$ is isomorphic to the toric variety $X_{\Sigma_{\mathcal{P}}}$ associated to the normal fan $\Sigma_{\mathcal{P}}$ of $\mathcal{P}$.*

It should be noted at this point that not every rational polyhedral fan can be realized as the normal fan of a convex rational polytope, hence not every normal toric variety is projective. So it is natural to ask whether all projective normal toric varieties can be realized as toric varieties associated to rational convex polytopes. We will affirm this question in Section 2.3.3.

Regarding the singularities of toric varieties, [12, Theorem 11.4.2] gives a very satisfying answer.

Theorem 2.3.21. *Every normal toric variety has rational singularities.*

2.3.2 The Orbit-Cone Correspondence

We would like to make the connection between toric geometry and polyhedral geometry more concrete.

Definition 2.3.22. Let Σ be a pointed rational polyhedral fan and σ a cone in Σ.

(i) The **distinguished point** $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$ corresponding to Σ is defined as the kernel of the homomorphism

$$\phi_\sigma \colon \mathbb{C}[\mathsf{S}_\sigma] \to \mathbb{C} \text{ given by } \phi_\sigma(\chi^s) = \begin{cases} 1 \text{ if } s \in \sigma^\perp \\ 0 \text{ otherwise.} \end{cases}$$

(ii) The **orbit** $O(\sigma) \subseteq U_\sigma \subseteq X_\Sigma$ corresponding to the cone σ is defined as the orbit of the distinguished point γ_σ under the torus action.

There is an alternative way of viewing this orbit, as can be seen in [12, Lemma 3.2.5].

Lemma 2.3.23. *Let Σ be a pointed rational polyhedral fan and σ a cone in Σ. Then the orbit $O(\sigma)$ corresponding to σ is the set of all kernels of homomorphisms $\mathbb{C}[\mathsf{S}_\sigma] \to \mathbb{C}$ that do not vanish on rational functions corresponding to lattice points orthogonal to σ, in symbols*

$$O(\sigma) = \{\operatorname{Ker} f \mid f \colon \mathbb{C}[\mathsf{S}_\sigma] \to \mathbb{C} \textit{ such that } f(\chi^s) \neq 0 \textit{ for all } s \in \sigma^\perp \cap \mathbb{Z}^d\}.$$

The key result about these orbits is the following theorem, reformulated from [12, Theorem 3.2.6 and Proposition 3.2.7].

Theorem 2.3.24 (Orbit-Cone Correspondence)**.** *Let X_Σ be the normal toric variety associated to the pointed rational polyhedral fan Σ.*

(i) The map $\sigma \mapsto O(\sigma)$ induces a bijection between the cones in the fan Σ and the torus orbits in the toric variety X_Σ.

(ii) Let $\sigma \subseteq \mathbb{R}^d$ be a cone in Σ. Then $\dim O(\sigma) = d - \dim \sigma$.

(iii) Every affine open subset $U_\sigma \subseteq X_\Sigma$, $\sigma \in \Sigma$, is the union of all orbits corresponding to faces of σ, in symbols

$$U_\sigma = \bigcup_{\tau \textit{ face of } \sigma} O(\tau).$$

(iv) Let $\sigma, \tau \subseteq \mathbb{R}^d$ be two cones in Σ. Then τ is a face of σ if and only if the orbit $O(\sigma)$ is contained in the orbit closure $\overline{O(\tau)}$. Furthermore, this orbit

closure is a normal toric variety and it is precisely the union of all orbits that it contains. In other words

$$\overline{O(\tau)} = \bigcup_{\tau \text{ face of } \sigma} O(\sigma).$$

2.3.3 Divisors on Toric Varieties

In the following we will recall the theory of torus invariant divisors on toric varieties and outline their combinatorial structure.

Let us first introduce the following notation.

Notation 2.3.25. Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$.

(i) For any ray $\rho \in \Sigma$ let us denote the orbit closure corresponding to ρ by $D_\rho := \overline{O(\rho)}$.

(ii) We will denote the group of torus invariant Weyl divisors on X_Σ by $\mathrm{Div}_T(X_\Sigma)$.

(iii) We will denote the group of torus invariant Cartier Weil divisors on X_Σ by $\mathrm{CaDiv}_T(X_\Sigma)$.

Remark 2.3.26. Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$. Let ρ be a ray in Σ. Notice that by Orbit-Cone Correspondence (see Theorem 2.3.24) the orbit closure $D_\rho \subseteq X_\Sigma$ will be a closed subvariety of codimension one. The irreducible of this subvariety follows from the observation that it contains a torus of dimension $d-1$ as an open dense subset, hence it is the irreducible as the closure of an irreducible set. So the subvariety D_ρ is a prime divisor invariant under the torus action.

Additionally, the irreducible components of the complement of the torus T in X_Σ are precisely the prime divisors D_ρ for $\rho \in \Sigma(1)$.

With these remarks and Remark 2.3.18, the following proposition is immediate. It can be found in [12, Proposition 4.1.2].

Proposition 2.3.27. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ and let $s \in \mathbb{Z}^d$. Then s defines a rational function $\chi^s \in \mathbb{C}(X_\Sigma)$ and its divisor is given by*

$$\mathrm{div}(\chi^s) = \sum_{\rho \in \Sigma(1)} \langle s, u_\rho \rangle D_\rho.$$

The next beautiful result tells us that we can completely restrict our studies to torus invariant divisors. It can be found in [26, Section 3.4, First Proposition].

Theorem 2.3.28. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ and let $s \in \mathbb{Z}^d$. Then the group $\mathrm{Div}_T(X_\Sigma)$ of torus invariant Weil divisors is generated by the prime divisors D_ρ, $\rho \in \Sigma(1)$, and there are natural isomorphisms*

$$\mathrm{Cl}(X_\Sigma) \simeq \mathrm{Div}_T(X_\Sigma)/\mathbb{Z}^d \text{ and } \mathrm{Pic}(X_\Sigma) \simeq \mathrm{CaDiv}_T(X_\Sigma)/\mathbb{Z}^d.$$

Remark 2.3.29. From now on we will *identify* toric and non-toric divisors since we are in general not interested in exact divisors but rather in their divisor classes. And thankfully we can chose a torus invariant representative in every divisor class.

Remark 2.3.30. It is now clear that the round-up and the round-down of a torus invariant divisor is torus invariant.

Of course there is one important divisor that we want to learn more about. [26, First Proposition of Section 4.3] gives us a very nice formula.

Proposition 2.3.31. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$. Then $K_{X_\Sigma} := -\sum_{\rho\in\Sigma(1)} D_\rho$ is a canonical divisor, i.e.*

$$\omega_{X_\Sigma} \simeq \mathcal{O}_{X_\Sigma}\left(-\sum_{\rho\in\Sigma(1)} D_\rho\right).$$

When dealing with projective normal toric varieties, there is another important construction.

Construction 2.3.32. Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$.

(i) Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a torus invariant $\mathbb{Q}$-Weil divisor on X_Σ. We define the polytope $\mathcal{P}_D$ associated to D as

$$\mathcal{P}_D := \{x \in \mathbb{R}^d \mid \langle x, u_\rho\rangle \leq a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

(ii) Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a rational convex polytope such that the fan Σ is a refinement of the normal fan $\Sigma_\mathcal{P}$ of $\mathcal{P}$. It is clear that every ray of $\Sigma_\mathcal{P}$ will be a ray of Σ, hence we can write $\mathcal{P}$ as

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho\rangle \leq b_\rho \text{ for all } \rho \in \Sigma(1)\}$$

for some rational numbers $b_\rho \in \mathbb{Q}, \rho \in \Sigma(1)$. However, for those rays that are not in $\Sigma_\mathcal{P}$, these numbers will not be unique (since the corresponding inequalities are redundant). We will make them unique by requiring them to be minimal among all possible choices. They can be calculated as

$$b_\rho = \max_{x \in \mathcal{P}} \langle x, u_\rho \rangle$$

for all $\rho \in \Sigma(1)$. Then the divisor $D_\mathcal{P}$ associated to $\mathcal{P}$ is defined as

$$D_\mathcal{P} := \sum_{\rho \in \Sigma(1)} b_\rho D_\rho.$$

Let us first remark some direct observations.

Proposition 2.3.33. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$. Let D be a torus invariant $\mathbb{Q}$-Weil divisor on X_Σ. Then the following properties hold.*

(i) $\mathcal{P}_{D_\mathcal{P}} = \mathcal{P}$.

(ii) $D_{\mathcal{P}_D} = D$ *if* $\Sigma_{\mathcal{P}_D} = \Sigma$.

(iii) $\mathcal{P}_{kD} = k\mathcal{P}_D$ *for any* $k \in \mathbb{R}_{\geq 0}$.

(iv) $\mathcal{P}_{D+\mathrm{div}(\chi^s)} = \mathcal{P}_D + s$ *and for all* $s \in \mathbb{Z}^d$.

(v) $\lfloor \mathcal{P}_D \rfloor = \mathcal{P}_{\lfloor D \rfloor}$ *and* $\lceil \mathcal{P}_D \rceil = \mathcal{P}_{\lceil D \rceil}$.

The following fact is a reformulation of [12, Theorem 4.2.8].

Theorem 2.3.34. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ and let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a torus invariant Weil divisor on X_Σ. Then D is Cartier if and only if for every maximal cone $\sigma \in \Sigma$ there exists a $v_\sigma \in \mathbb{Z}^d$ such that*

$$\langle v_\sigma, u_\rho \rangle = a_\rho \textit{ for all rays } \rho \textit{ in } \sigma.$$

Remark 2.3.35. Theorem 2.3.34 implies that on a toric variety X_Σ the $\mathbb{Q}$-Weil divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is $\mathbb{Q}$-Cartier if and only if for every maximal cone $\sigma \in \Sigma$ there exists an $v_\sigma \in \mathbb{Q}^d$ such that

$$\langle v_\sigma, u_\rho \rangle = a_\rho \text{ for all rays } \rho \text{ in } \sigma.$$

The next result can be found in [12, Propositions 4.2.10 and 6.1.10].

Proposition 2.3.36. *Let $X_\mathcal{P}$ be the normal projective toric variety associated to the rational convex polytope $\mathcal{P}$. Then the divisor $D_\mathcal{P}$ is $\mathbb{Q}$-Cartier and ample.*

Remark 2.3.37. Combinatorially it is clear, that a polytope contains more information than its normal fan. Up until now we did not see this in the toric world, since the toric variety associated to a polytope was only depending on the normal fan. But now we have encountered this missing information. Since every very ample divisor on a projective variety defines an embedding into some projective space, the polytope defining our variety gives us — up to some scaling to go from $\mathbb{Q}$-*Cartier and ample* to *Cartier and very ample* — the embedding! So we can think about X_Σ as an abstract variety and $X_\mathcal{P}$ as an embedded variety.

This result allows for a natural classification of embedded projective normal toric varieties. The following is a consequence of [12, Theorem 6.2.1].

Theorem 2.3.38. *The map $\mathcal{P} \mapsto (X_\mathcal{P}, D_\mathcal{P})$ induces a bijection between the set of full-dimensional lattice polytopes and the set of pairs (X, D) where X is a normal toric variety and D is a torus invariant ample Cartier divisor on X. Furthermore, the inverse of this map is given by $(X, D) \mapsto \mathcal{P}_D$.*

Finally, the connection between divisors and polytopes gives us possibilities to answer questions about cohomology. As an example we will state [12, Proposition 4.3.3].

Proposition 2.3.39. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ and let D be a torus invariant Weil divisor on X_Σ. Then*

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \simeq \bigoplus_{s \in \mathcal{P}_D \cap \mathbb{Z}^d} \mathbb{C}\chi^s$$

by identifying global sections as rational functions. Especially,

$$h^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \#(\mathcal{P}_D \cap \mathbb{Z}^d).$$

2.3.4 Gorenstein Fano Toric Varieties

The attentive reader might have noticed that we used the term *Fano* in two separate sections up until now. In Section 2.1.5 we defined ($\mathbb{Q}$-)Gorenstein Fano varieties as normal varieties whose anticanonical divisor is ($\mathbb{Q}$-)Cartier and ample. In Section 2.1.5 we defined Fano polytopes as convex polytopes whose rational vertices are primitive lattice points. The connection between the two notions is the following theorem. It is not a new result in itself but our language is a bit different. The part about Gorenstein Fano toric varieties is quite standard (see for example [12, Theorem 8.3.4]). The claim on

$\mathbb{Q}$-Gorenstein Fano toric varieties can be found in Nill's doctoral thesis [50, Proposition 1.4] and follows directly from a result by Batyrev [4, Proposition 2.2.23]. We will state a proof nonetheless to show the use of our terminology.

Theorem 2.3.40 (Batyrev, Nill)**.** *The toric variety associated to a dual-Fano polytope is $\mathbb{Q}$-Gorenstein Fano and conversely every $\mathbb{Q}$-Gorenstein Fano toric variety is isomorphic to the toric variety associated to a dual-Fano polytope.*

Furthermore, the same statement holds true for Gorenstein Fano toric varieties and reflexive polytopes.

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a dual-Fano polytope. By Corollary 2.2.31 we know that $\mathcal{P}$ can be written as

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq 1 \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\},$$

where u_ρ denotes the primitive ray generator of the normal fan $\Sigma_{\mathcal{P}}$ of $\mathcal{P}$. So by Proposition 2.3.31 we know that the divisor $D_{\mathcal{P}} = \sum_{\rho \in \Sigma_{\mathcal{P}}(1)} D_\rho$ associated to $\mathcal{P}$ is the anticanonical divisor $-K_{X_{\mathcal{P}}}$ on $X_{\mathcal{P}}$. By Proposition 2.3.36 we know that this divisor must be $\mathbb{Q}$-Cartier and ample, hence $X_{\mathcal{P}}$ is $\mathbb{Q}$-Gorenstein Fano. Furthermore, by the classification of torus invariant Cartier divisors in Theorem 2.3.34 we know that this divisor is Cartier if and only if $\mathcal{P}$ is a lattice polytope. Hence $X_{\mathcal{P}}$ is Gorenstein Fano if $\mathcal{P}$ is a dual-Fano lattice polytope. By Theorem 2.2.30 this means that $\mathcal{P}$ is reflexive.

For the other direction let X be a $\mathbb{Q}$-Gorenstein Fano toric variety. By Theorem 2.3.19 there exists a rational polyhedral fan Σ such that $X \simeq X_\Sigma$. Since $-mK_X$ is an ample Cartier divisor on X, Theorem 2.3.38 implies that $\mathcal{P}_{-mK_X}$ is a full-dimensional lattice polytope and $X = X_{\mathcal{P}_{-mK_X}}$. Hence $\mathcal{P}_{-K_X}$ is a full-dimensional rational convex polytope and $X = X_{\mathcal{P}_{-K_X}}$. Let u_ρ denote the primitive ray generators of Σ. Then

$$\mathcal{P}_{-K_X} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq 1 \text{ for all } \rho \in \Sigma(1)\},$$

which implies that $\mathcal{P}_{-K_X}$ is a dual-Fano polytope by Corollary 2.2.31. Furthermore, it is reflexive if it is a lattice polytope, i.e. if $-K_X$ is Cartier. $\square$

Remark 2.3.41. The bijective map from Theorem 2.3.38 is given by

$$\begin{array}{rcl} \{\mathbb{Q}\text{-}\textit{Gorenstein Fano toric varieties}\} & \longleftrightarrow & \{\textit{dual-Fano polytopes}\} \\ X & \longmapsto & \mathcal{P}_{-K_X} \\ X_{\mathcal{P}} & \longleftarrow\!\shortmid & \mathcal{P}. \end{array}$$

Remark 2.3.42. Notice that the bijection in Remark 2.3.41 does not imply that the toric variety $X_{\mathcal{P}}$ associated to an arbitrary full-dimensional rational convex polytope $\mathcal{P}$ is Gorenstein Fano if and only if $\mathcal{P}$ is reflexive. We could easily construct Gorenstein Fano toric varieties by taking translations or dilations of reflexive polytopes since these operations change the polytope but not its normal fan.

However, we can at least say the following.

Proposition 2.3.43. *Let $X_\mathcal{P}$ be the toric variety associated to a full-dimensional rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$. Suppose that $\mathcal{P}$ is translated by a lattice vector to a dual-Fano polytope. Then $X_\mathcal{P}$ is Gorenstein Fano if and only if $\mathcal{P}$ is a lattice polytope.*

Proof. By assumption there exists a lattice vector $p \in \mathbb{Z}^d$ such that $\mathcal{P} - p$ is dual-Fano. If $\mathcal{P}$ is a lattice polytope, then so is $\mathcal{P} - p$. By Theorem 2.2.30 this polytope must be reflexive and hence the toric variety $X_\mathcal{P} = X_{\mathcal{P}-p}$ will be Gorenstein Fano by Remark 2.3.41.

On the other hand, from Remark 2.2.28 we know that

$$\mathcal{P} - p = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq 1 \text{ for all } \rho \in \Sigma_\mathcal{P}(1)\}.$$

By Proposition 2.3.31 this means that $D_{\mathcal{P}-p} = -K_{X_\mathcal{P}}$ and hence $D_\mathcal{P} \sim -K_{X_\mathcal{P}}$ by Proposition 2.3.33. So we know that $\mathcal{P} - p = \mathcal{P}_{-K_{X_\mathcal{P}}}$. By Remark 2.3.41 this polytope is reflexive if $X_\mathcal{P}$ is Gorenstein Fano. □

2.3.5 Toric Serre Duality

Since every normal toric variety has rational singularities by Theorem 2.3.21 we know that every normal toric variety is Cohen-Macaulay. Hence the general version of Serre Duality from Theorem 2.1.29 holds for every normal projective toric variety. However, we would like to apply Serre Duality to sheaves that might not be locally free in general — divisorial sheaves for example. Luckily, there is the following strong version of Serre Duality in toric geometry. A proof can be found in [12, Theorem 9.2.10].

Theorem 2.3.44 (Toric Serre Duality)**.** *Let $X_\mathcal{P}$ be the normal projective toric variety of dimension d associated to the convex rational polytope $\mathcal{P}$ and let D be a $\mathbb{Q}$-Cartier Weil divisor on $X_\mathcal{P}$. Then*

$$H^i(X_\mathcal{P}, \mathcal{O}_{X_\mathcal{P}}(D))^* \simeq H^{d-i}(X_\mathcal{P}, \mathcal{O}_{X_\mathcal{P}}(K_{X_\mathcal{P}} - D))$$

for all $0 \leq i \leq d$, where $K_{X_\mathcal{P}}$ denotes the canonical divisor given by Proposition 2.3.31.

Now this is not quite sufficient for our purpose. So we need the following slight generalization as stated in [12, Exercise 9.3.5].

Theorem 2.3.45 (Toric Serre Duality for $\mathbb{Q}$-Divisors)**.** *Let $X_{\mathcal{P}}$ be the normal projective toric variety of dimension d associated to the convex rational polytope $\mathcal{P}$ and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X_{\mathcal{P}}$. Then*

$$H^i(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor D \rfloor))^* \simeq H^{d-i}(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(K_{X_{\mathcal{P}}} - \lfloor D \rfloor))$$

for all $0 \leq i \leq d$, where $K_{X_{\mathcal{P}}}$ denotes the canonical divisor given by Proposition 2.3.31.

2.3.6 Toric Vanishing Theorems

We will conclude our summary on toric varieties by recalling two nice vanishing results. Of course, since normal toric varieties have rational singularities (see Theorem 2.3.21), Kodaira Vanishing for Rational Singularities (see Theorem 2.1.55) holds for every normal projective toric variety. But again, we would like to apply Kodaira Vanishing to sheaves that might not be locally free. And once again, we are lucky, since there are even stronger vanishing results in the toric world.

Their respective formulations for Weil divisors can be found in [12, Proposition 9.2.3 and Theorem 9.2.7]. The generalization to $\mathbb{Q}$-Weil divisors is done in [12, Theorem 9.3.5].

Theorem 2.3.46 (Demazure and Batyrev-Borisov Vanishing)**.** *Let $X_{\mathcal{P}}$ be the normal projective toric variety associated to the convex rational polytope $\mathcal{P}$ and let D be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X_{\mathcal{P}}$. Then*

$$h^i(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor D \rfloor)) = 0 \textit{ for all } i > 0$$

and

$$h^i(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(-\lceil D \rceil)) = 0 \textit{ for all } i < \dim X_{\mathcal{P}}.$$

The key to proving these $\mathbb{Q}$-versions is the following lemma, which bears a strong result in itself but seems to be overlooked sometimes. So we would like to include its statement at this point. It is proved in [12, Lemma 9.3.4].

Lemma 2.3.47. *Let X_Σ be the normal toric variety corresponding to the pointed rational polyhedral fan $\Sigma \subseteq \mathbb{R}^d$ and let D be a $\mathbb{Q}$-Weil divisor on X_Σ. Let $l > 0$ be an integer such that lD is a Weil divisor. Then there exists an injection*

$$H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(\lfloor D \rfloor)) \hookrightarrow H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(lD))$$

for any integer $i \in \mathbb{N}$.

Chapter 3

The interplay between Algebraic, Polyhedral and Toric Geometry

As we have already seen in the previous chapter, there are numerous connections between algebraic and polyhedral geometry — with toric varieties acting as an intermediary. In this chapter we will explore these connections further.

3.1 Toric Ehrhart Theory

We have seen in Proposition 2.3.39 that the cohomology of a torus invariant Weil divisor on a normal toric variety is given by the number of lattice points in the associated polytope. We will spent this section on proving the following generalization.

Theorem 3.1.1. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. Let $X_{\mathcal{P}}$ denote the associated normal projective toric variety and $D_{\mathcal{P}}$ the associated torus invariant $\mathbb{Q}$-Weil divisor. Then*

$$L_{\mathcal{P}}(n) = \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor))$$

for all $n \in \mathbb{Z}$.

Remark 3.1.2. Theorem 3.1.1 would in theory also prove a method to compute the Ehrhart quasi-polynomial of an arbitrary rational convex polytope via cohomology groups although in practice this might be quite challenging.

Before we can state the proof of this result, we will show the following useful identities.

Proposition 3.1.3. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope and let $X_{\mathcal{P}}$ denote the associated normal projective toric variety. Then* *(i)* $L_{\mathcal{P}}(n) = L_{\lfloor n\mathcal{P} \rfloor}(1)$,

(ii) $L_{\operatorname{int}\mathcal{P}}(n) = L_{\operatorname{int} n\mathcal{P}}(1) = L_{\operatorname{int}\lceil n\mathcal{P}\rceil}(1)$ *and*

(iii) $L_{\operatorname{int}\mathcal{P}_{\lceil nD_{\mathcal{P}}\rceil}}(1) = L_{\mathcal{P}_{\lceil nD_{\mathcal{P}}\rceil + K_{X_{\mathcal{P}}}}}(1)$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. For every ray ρ in the normal fan $\Sigma_{\mathcal{P}}$ let us denote its primitive ray generator by u_ρ. We know that $n\mathcal{P}$ can be written as

$$n\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq nb_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}$$

for some rational numbers $b_\rho \in \mathbb{Q}$, $\rho \in \Sigma_{\mathcal{P}}(1)$.

For the first property, notice that for any lattice point $x \in \mathbb{Z}^d$ and any ray $\rho \in \Sigma_{\mathcal{P}}(1)$, the number $\langle u_\rho, x \rangle$ will be an integer, hence

$$\langle x, u_\rho \rangle \leq nb_\rho \quad \Leftrightarrow \quad \langle x, u_\rho \rangle \leq \lfloor nb_\rho \rfloor,$$

which proves the claim

The first equality of the second property is clear since

$$\begin{aligned} x \in n \operatorname{int}\mathcal{P} \quad &\Leftrightarrow \quad \left\langle \frac{1}{n} \cdot x, u_\rho \right\rangle < b_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1) \\ &\Leftrightarrow \quad \langle x, u_\rho \rangle < nb_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1) \\ &\Leftrightarrow \quad x \in \operatorname{int} n\mathcal{P} \end{aligned}$$

for every $x \in \mathbb{R}^d$.

The second equality follows for the same reason as the first property (on the round-down), since for any $x \in \mathbb{Z}^d$ we have

$$\langle x, u_\rho \rangle < nb_\rho \quad \Leftrightarrow \quad \langle x, u_\rho \rangle < \lceil nb_\rho \rceil$$

for any ray $\rho \in \Sigma_{\mathcal{P}}(1)$ because the scalar products are integers.

For the third property notice that $D_{\mathcal{P}}$ can be written as $D_{\mathcal{P}} = \sum_{\rho \in \Sigma_{\mathcal{P}}(1)} b_\rho D_\rho$ while the canonical divisor is linearly equivalent to $-\sum_{\rho \in \Sigma_{\mathcal{P}}(1)} D_\rho$ by Proposition 2.3.31. So the two polytopes in the equation can be calculated as

$$\mathcal{P}_{\lceil nD_{\mathcal{P}}\rceil} = \{x \in \mathbb{R} \mid \langle x, u_\rho \rangle \leq \lceil nb_\rho \rceil \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}$$

and

$$\mathcal{P}_{\lceil nD_{\mathcal{P}}\rceil + K_{X_{\mathcal{P}}}} = s + \{x \in \mathbb{R} \mid \langle x, u_\rho \rangle \leq \lceil nb_\rho \rceil - 1 \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}$$

for some lattice vector $s \in \mathbb{Z}^d$. So for any $x \in \mathbb{Z}^d$ we get

$$\langle x, u_\rho \rangle < \lceil n b_\rho \rceil \quad \Leftrightarrow \quad \langle x, u_\rho \rangle \leq \lceil n b_\rho \rceil - 1$$

again because the scalar products are integers, which proves the claim. □

We are now able to prove the important formula.

Proof of Theorem 3.1.1. Let us first consider the case where $n \geq 0$. The divisor $nD_{\mathcal{P}}$ will be $\mathbb{Q}$-Cartier and ample, since $D_{\mathcal{P}}$ is $\mathbb{Q}$-Cartier and ample (see Proposition 2.3.36). So by Demazure Vanishing (see Theorem 2.3.46) we know that all higher cohomology groups of the divisor $nD_{\mathcal{P}}$ must vanish. We can thus apply Proposition 2.3.39 to the torus invariant Weil divisor $\lfloor nD \rfloor$ to get

$$\chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)) = h^0(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)) = L_{\mathcal{P}_{\lfloor nD_{\mathcal{P}} \rfloor}}(1).$$

But because of Proposition 2.3.33 this polytope is equal to

$$\mathcal{P}_{\lfloor nD_{\mathcal{P}} \rfloor} = \lfloor \mathcal{P}_{nD_{\mathcal{P}}} \rfloor = \lfloor n\mathcal{P}_{D_{\mathcal{P}}} \rfloor = \lfloor n\mathcal{P} \rfloor,$$

which proves the theorem for positive integers using the observation

$$L_{\lfloor n\mathcal{P} \rfloor}(1) = L_{\mathcal{P}}(n)$$

form Proposition 3.1.3.

Let us now consider the case where $n < 0$. This time the divisor $-nD_{\mathcal{P}}$ will be $\mathbb{Q}$-Cartier and ample. Batyrev-Borisov Vanishing (see Theorem 2.3.46) tells us that all cohomology groups of $-\lceil -nD_{\mathcal{P}} \rceil = \lfloor nD_{\mathcal{P}} \rfloor$ vanish except for the highest one. Since $\mathcal{P} \subseteq \mathbb{R}^d$ is full-dimensional, we have $\dim X_{\mathcal{P}} = \dim \mathcal{P} = d$ by Theorem 2.3.17. So we get

$$\chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)) = (-1)^d h^d(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)).$$

By Toric Serre Duality (see Theorem 2.3.45) we see that

$$\begin{aligned} h^d(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)) &= h^0(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(K_X - \lfloor nD_{\mathcal{P}} \rfloor)) \\ &= h^0(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(K_X - \lfloor nD_{\mathcal{P}} \rfloor)). \end{aligned}$$

Applying Proposition 2.3.39 to the torus invariant Weil divisor $K_{X_{\mathcal{P}}} - \lfloor nD_{\mathcal{P}} \rfloor$ we get

$$h^0(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(K_X - \lfloor nD_{\mathcal{P}} \rfloor)) = L_{\mathcal{P}_{K_{X_{\mathcal{P}}} - \lfloor nD_{\mathcal{P}} \rfloor}}(1) = L_{\mathcal{P}_{K_{X_{\mathcal{P}}} + \lceil -nD_{\mathcal{P}} \rceil}}(1).$$

By our previous observation in Proposition 3.1.3 we know that this number can be rewritten as

$$L_{\mathcal{P}_{K_{X_{\mathcal{P}}}+\lceil -nD_{\mathcal{P}}\rceil}}(1) = L_{\operatorname{int}\mathcal{P}_{\lceil -nD_{\mathcal{P}}\rceil}}(1) = L_{\operatorname{int}\lceil -n\mathcal{P}\rceil}(1) = L_{\operatorname{int}-n\mathcal{P}}(1) = L_{\operatorname{int}\mathcal{P}}(-n)$$

Now Ehrhart-Macdonald Reciprocity (see Theorem 2.2.35) implies that

$$L_{\operatorname{int}\mathcal{P}}(-n) = (-1)^d L_{\mathcal{P}}(n).$$

Plugging everything together, the signs will cancel and we get

$$\chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}}\rfloor)) = L_{\mathcal{P}}(n)$$

which had to be proven. □

3.2 Toric Degenerations

As we have seen, the geometric properties of toric varieties are well understood using combinatorial methods. And we have also seen that geometric properties are sometimes contained in flat families. So the motivation behind the following definition should be clear.

Definition 3.2.1 (First try)**.** Let X be a normal complex variety. We say that the variety X admits a **toric degeneration** to the normal toric variety X_Σ if there exists a complex variety $\mathcal{X}$ and a flat morphism $\mathcal{X} \to \mathbb{A}^1$ such that all fibers $\mathcal{X}_t$, $t \in \mathbb{A}^1$, are isomorphic to X and the special fiber $\mathcal{X}_0$ is isomorphic to X_Σ.

However, this definition would be to weak for our applications. So we need to strengthen it as follows.

Definition 3.2.2. Let X be a normal projective complex variety of dimension d and let D be an ample Weil divisor on X. Let $X_{\mathcal{P}}$ be the normal projective toric variety associated to the rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$. Let $D_{\mathcal{P}}$ denote the ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $X_{\mathcal{P}}$ associated to $\mathcal{P}$ (see Proposition 2.3.36 for these properties of $D_{\mathcal{P}}$). We say that the pair (X, D) admits a **toric degeneration** to the pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$ if there exists a complex variety $\mathcal{X}$ and a morphism

$$\pi\colon \mathcal{X} \to \mathbb{A}^1$$

such that

(i) π is projective and flat,

(ii) $\mathcal{X}_t \simeq X$ for all $t \neq 0$ and $\mathcal{X}_0 \simeq X_{\mathcal{P}}$, and

(iii) for every $n \in \mathbb{N}$ there exists a divisorial sheaf $\mathcal{F}^{(n)}$ on $\mathcal{X}$ such that $\mathcal{F}^{(n)}|_{\mathcal{X}_t} \simeq \mathcal{O}_X(nD)$ and $\mathcal{F}^{(n)}|_{\mathcal{X}_0} \simeq \mathcal{O}_{X_\mathcal{P}}(\lfloor nD_\mathcal{P} \rfloor)$.

The variety $X_\mathcal{P}$ is called the **limit** of X under the degeneration π.

Remark 3.2.3. Whenever we say that *a variety admits a toric degeneration*, we mean this in the sense of Definition 3.2.2 and not in the standard sense! We acknowledge that our terminology is not standard. Yet, we will see that basically all known examples of toric degenerations in the usual sense are also toric degenerations in our sense. The main reason behind this change is basically that we want to keep track of the embedding of our varieties. So one could think of our degenerations as *embedded toric degenerations*.

Remark 3.2.4. We will use the analogue definition of toric degenerations for pairs $(X, \mathcal{L})$ where $\mathcal{L}$ is an ample line bundle on the normal projective variety X.

Remark 3.2.5. This definition should not come completely unexpected. It has been proven by Anderson that toric degenerations in our sense arise naturally in the setting of Newton-Okounkov bodies (see Theorem 3.3.10).

In the case of flag varieties there is an even broader formalism. Fang, Fourier and Littelmann constructed toric degenerations via so called *birational sequences* in [21]. They proved in [21, Theorem 6] that their toric degenerations satisfy the properties of Definition 3.2.2. Additionally, they were able to show that every birational sequence defines a valuation. Hence ever toric degeneration via birational sequences can be seen as a toric degeneration associated to a Newton-Okounkov body. It is noticeable that basically all polytopes in representation theory can be constructed via birational sequences and hence are Newton-Okounkov bodies. We will make use of this fact in Chapter 4.

Definition 3.2.6. Let $\mathcal{X} \to \mathbb{A}^1$ be a toric degeneration. The variety $\mathcal{X}$ is called the **degeneration space**. The fiber $\mathcal{X}_0$ is called the **special fiber** and the fiber $\mathcal{X}_1$ — being isomorphic to any $\mathcal{X}_t$, $t \neq 0$, — is called the **general fiber**.

We will collect some useful facts of the degeneration space $\mathcal{X}$. It should be noted that the following three propositions also hold under the usual definition of toric degenerations.

Proposition 3.2.7. *Let $\mathcal{X} \to \mathbb{A}^1$ be a toric degeneration in the sense of Definition 3.2.2. Then $\mathcal{X}$ is quasi-projective, normal and has rational singularities.*

Proof. $\mathcal{X}$ is clearly quasi-projective since it is the domain of a projective morphism. Normality follows directly from Serre's criterion of normality since every fiber $\mathcal{X}_t$, $t \in \mathbb{R}$, is normal (see our formulation in Theorem 2.1.22). Rationality of singularities follows from the fact that normal toric varieties have rational singularities (see Theorem 2.3.21). So by Elkik's first theorem (see Theorem 2.1.56) there exists an open neighborhood $0 \in U \subseteq \mathbb{A}^1$ such that all fibers $\mathcal{X}_t$, $t \in U$, have rational singularities. But all fibers are isomorphic (except for the special fiber), hence all fibers have rational singularities. This implies that $\mathcal{X}$ has rational singularities by Elkik's second theorem (see Theorem 2.1.58). □

Proposition 3.2.8. *Let $\mathcal{X} \to \mathbb{A}^1$ be a toric degeneration in the sense of Definition 3.2.2. Then $\mathcal{X}_0$ is a principal prime divisor on $\mathcal{X}$.*

Proof. By [33, Corollary 9.6] we know that $\mathcal{X}_0$ is a subvariety of codimension 1. The irreducibility is clear since $\mathcal{X}_0 \simeq X_{\mathcal{P}}$. Notice that $\pi \in \mathcal{O}_{\mathcal{X}}(\mathcal{X}) \subseteq \mathbb{C}(\mathcal{X})$, so π is a rational function on $\mathcal{X}$ and $\mathcal{X}_0 = \operatorname{div}(\pi)$ is principal. □

Proposition 3.2.9. *Let $\mathcal{X} \to \mathbb{A}^1$ be a toric degeneration in the sense of Definition 3.2.2 and let X be isomorphic to the general fiber. Then for any $t \neq 0$ the restriction map $\mathcal{D} \mapsto \mathcal{D}|_{\mathcal{X}_t}$ induces an isomorphism of divisor class groups*

$$\mathrm{Cl}(\mathcal{X}) \simeq \mathrm{Cl}(X).$$

Proof. Consider the variety $X \times \mathbb{A}^1$ and the natural projection $\tau\colon X \times \mathbb{A}^1 \to \mathbb{A}^1$. Let $Z := \tau^{-1}(0)$. Then the open set $U := \mathcal{X} \setminus \mathcal{X}_0$ is isomorphic to the open set $V := (X \times \mathbb{A}^1) \setminus Z$.

By Proposition 3.2.8 we know that the special fiber $\mathcal{X}_0$ is a principal prime divisor on $\mathcal{X}$. By the same arguments Z is a principal prime divisor on $X \times \mathbb{A}^1$. So repeated application of Proposition 2.1.13 gives a sequence of isomorphisms

$$\begin{array}{l} \mathrm{Cl}(\mathcal{X}) \to \mathrm{Cl}(\mathcal{X})/\mathbb{Z}[\mathcal{X}_0] \to \mathrm{Cl}(U) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Cl}(V) \to \mathrm{Cl}(X \times \mathbb{A}^1)/\mathbb{Z}[Z] \to \mathrm{Cl}(X \times \mathbb{A}^1) \to \mathrm{Cl}(X), \end{array}$$

where each step is either a restriction, identity or induced by the isomorphism between X and the general fiber. □

Although we allowed our limit divisor $D_{\mathcal{P}}$ to be a $\mathbb{Q}$-Weil divisor we will now prove that this divisor will in fact be a Weil divisor if the Weil divisor on our original variety was Cartier. This in return restricts the polytopes that could appear in our setting. We will try to explain these restrictions in Section 5.3.

Theorem 3.2.10. *Let X be a normal projective complex variety of dimension d and let D be an ample Cartier Weil divisor on X. Let $X_{\mathcal{P}}$ be the normal* *projective toric variety associated to the rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$. Suppose the pair (X, D) admits a toric degeneration to the toric pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$. Then $D_{\mathcal{P}}$ is a Weil divisor.*

Proof. Notice that $\mathcal{F}^{(n)}$ is divisorial for every $n \in \mathbb{N}$, so there exist divisors $\mathcal{D}_n$ on the degeneration space $\mathcal{X}$, $n \in \mathbb{N}$, such that $\mathcal{F}^{(n)} \simeq \mathcal{O}_{\mathcal{X}}(\mathcal{D}_n)$. Fix $t \neq 0$ and $n \in \mathbb{N}$. Since D is Cartier, we know that

$$\mathcal{F}^{(n)}|_{\mathcal{X}_t} \simeq \mathcal{O}_X(nD) \simeq \mathcal{O}_X(D)^{\otimes n} \simeq (\mathcal{F}^{(1)})^n|_{\mathcal{X}_t}.$$

Translated to divisors this means

$$\mathcal{D}_n|_{\mathcal{X}_t} \sim n\mathcal{D}_1|_{\mathcal{X}_t}.$$

By Proposition 3.2.9 this is only possible if $\mathcal{D}_n \sim n\mathcal{D}_1$.

So there exists a Weil divisor $\mathcal{D} := \mathcal{D}_1$ on $\mathcal{X}$ such that $\mathcal{F}^{(n)} \simeq \mathcal{O}_{\mathcal{X}}(n\mathcal{D})$ for every $n \in \mathbb{N}$. Let E denote the divisor on $X_{\mathcal{P}}$ that is isomorphic to $\mathcal{D}|_{\mathcal{X}_0}$ via the isomorphism $\mathcal{X}_0 \simeq X_{\mathcal{P}}$.

Since $\mathcal{F}^{(n)}|_{\mathcal{X}_0} \simeq \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)$ for every $n \in \mathbb{N}$, we have

$$\lfloor nD_{\mathcal{P}} \rfloor \sim nE$$

for every $n \in \mathbb{N}$. Chose $l \in \mathbb{N}$ such that $lD_{\mathcal{P}}$ is Weil. Then

$$lD_{\mathcal{P}} \sim lE$$

i.e. there exist rational functions $f, g \in \mathbb{C}(X_{\mathcal{P}})^{\times}$ such that

$$\lfloor D_{\mathcal{P}} \rfloor = E + \operatorname{div}(f) \quad \text{and} \quad lD_{\mathcal{P}} = lE + \operatorname{div}(g).$$

We want to show that $g = \lambda f^l$ for some $\lambda \in \mathbb{C}$. Notice that

$$D_{\mathcal{P}} = E + l^{-1}\operatorname{div}(g),$$

hence

$$E + \operatorname{div}(f) = \lfloor D_{\mathcal{P}} \rfloor = \lfloor E + l^{-1}\operatorname{div}(g) \rfloor = E + \lfloor l^{-1}\operatorname{div}(g) \rfloor.$$

So in conclusion we know that $\operatorname{div}(f) = \lfloor l^{-1}\operatorname{div}(g) \rfloor$. Since every coefficient of $\lfloor l^{-1}\operatorname{div}(g) \rfloor$ is smaller than or equal to the corresponding coefficient of $l^{-1}\operatorname{div}(g)$, we see that $l^{-1}\operatorname{div}(g) - \operatorname{div}(f)$ is an effective divisor. Equivalently, the difference $\operatorname{div}(g) - l\operatorname{div}(f) = \operatorname{div}(f^{-l}g)$ is an effective divisor.

But this implies that $f^{-l}g$ is a rational function on $X_{\mathcal{P}}$ whose valuation $v_{D_i}(f^{-l}g)$ on every prime divisor D_i is positive. This is only possible if $f^{-l}g$

is regular. But since $X_{\mathcal{P}}$ is projective, every regular functions is constant, so there exist $\lambda \in \mathbb{C}^\times$ such that $f^{-l}g = \lambda$—or equivalently $g = \lambda f^l$.

In conclusion we see that $\operatorname{div}(g) = \operatorname{div}(\lambda f^l) = \operatorname{div}(f^l)$ and thus

$$D_{\mathcal{P}} = E + l^{-1} \operatorname{div}(g) = E + l^{-1} \operatorname{div}(f^l) = E + \operatorname{div}(f)$$

is a Weil divisor on $X_{\mathcal{P}}$. □

We will return to this observation in Section 5.3. An important feature of toric degenerations is the following. A version formulated for spherical varieties can be found for example in [1, Theorem 3.8].

Theorem 3.2.11. *Suppose the Gorenstein Fano variety X admits a toric degeneration of (X, ω_X^{-1}) to the pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$. Then $D_{\mathcal{P}}$ must be an anticanonical (Weil) divisor on $X_{\mathcal{P}}$ and $X_{\mathcal{P}}$ is $\mathbb{Q}$-Gorenstein Fano.*

Proof. By Theorem 3.2.10 we know that $D_{\mathcal{P}}$ must be a Weil divisor.

Notice that the isomorphism from Proposition 3.2.9 yields $\mathcal{F}^{(1)} \simeq \mathcal{O}_{\mathcal{X}}(-K_{\mathcal{X}})$ and recall that $\mathcal{X}_0$ is a principal prime divisor on $\mathcal{X}$ by Proposition 3.2.8. So the adjunction formula (see Theorem 2.1.31 or Remark 2.1.43) shows that

$$K_{\mathcal{X}_0} = (K_{\mathcal{X}} + \mathcal{X}_0)|_{\mathcal{X}_0} \sim K_{\mathcal{X}}|_{\mathcal{X}_0}$$

which implies

$$\mathcal{O}_{\mathcal{X}}(D_{\mathcal{P}}) \simeq \mathcal{F}^{(1)}|_{\mathcal{X}_0} = \mathcal{O}_{\mathcal{X}_0}(-K_{\mathcal{X}_0})$$

and thus $D_{\mathcal{P}} \sim -K_{X_{\mathcal{P}}}$.

Now notice that by Proposition 2.3.36 the divisor $-K_X \sim D_{\mathcal{P}}$ is $\mathbb{Q}$-Cartier and ample, hence the toric limit variety $X_{\mathcal{P}}$ is $\mathbb{Q}$-Gorenstein Fano. □

3.3 Newton-Okounkov Bodies

We have already seen some of the benefits of finding toric degenerations. However, this is in principal a very difficult task. So one would like to have some kind of standard procedure that generates toric degenerations. This lead to the theory of Newton-Okounkov bodies. We want to present the general construction and some of the main examples in this chapter.

Before we start with the construction, let us introduce a rather technical notion.

Definition 3.3.1. A total order $\leq$ on $\mathbb{Z}^d$ is called **monoidal** if

$$a \leq b \quad \Rightarrow \quad a + c \leq b + c$$

for all $a, b, c \in \mathbb{Z}^d$.

Let us now start with our most important ingredient.

Definition 3.3.2. Let A be a $\mathbb{K}$-algebra and assume that A is an integral domain. Fix a monoidal total ordering $\leq$ on $\mathbb{Z}^d$. Let $v\colon A \setminus \{0\} \to \mathbb{Z}^d$ be a map.

The map v is called a $\mathbb{Z}^d$**-valuation** on A if it satisfies the following properties.

(i) $v(cf) = v(f)$ for all $c \in \mathbb{K}^\times$ and $f \in A \setminus \{0\}$.

(ii) $v(fg) = v(f) + v(g)$ for all $f, g \in A \setminus \{0\}$.

(iii) $v(f+g) \geq \min\{v(f), v(g)\}$ for all $f, g \in A \setminus \{0\}$ such that $f + g \neq 0$.

Remark 3.3.3. Beware that there is a sign convention in condition (iii). While half of the authors like to require that a sum is *bigger or equal than the minimum*, the other half prefer that a sum should be *smaller or equal then the maximum*. The following construction and results hold for both versions, but their statements might have to be tweaked.

Remark 3.3.4. Condition (iii) yields the implication

$$v(f+g) > \min\{v(f), v(g)\} \quad \Rightarrow \quad v(f) = v(g)$$

for all $f, g \in A \setminus \{0\}$ such that $f + g \neq 0$. Indeed suppose that $v(f) \neq v(g)$ for some $f, g \in A \setminus \{0\}$. Without loss of generality let $v(f) < v(g)$. We thus write $v(f) = v((f+g)+(-g)) \geq \min\{v(f+g), v(g)\} \geq \min\{\min\{v(f), v(g)\}, v(g)\} = v(f)$. Hence we have $v(f) = \min\{v(f+g), v(g)\}$. Since $v(f) < v(g)$ this implies $v(f+g) = v(f) = \min\{v(f), v(g)\}$.

Definition 3.3.5. Let $v\colon A \to \mathbb{Z}^d$ be a valuation on the $\mathbb{K}$-algebra A with respect to the monoidal total order $\leq$ on $\mathbb{Z}$.

(i) By slight abuse of notation we will denote the **valuation image** of v as $\operatorname{Im} v := v(A \setminus \{0\})$.

(ii) We say that v has **full rank** if the dimension of the $\mathbb{R}$-linear span of the valuation image $\operatorname{Im} v \subseteq \mathbb{R}^d$ equals the Krull dimension of A.

(iii) Let $s \in \mathbb{Z}^d$. Then we define the ideals

$$A_{\geq s} := \{f \in A \setminus \{0\} \mid v(f) \geq s\} \cup \{0\}$$

and

$$A_{>s} := \{f \in A \setminus \{0\} \mid v(f) > s\} \cup \{0\}.$$

The **leaf** A_s is defined as the vector space

$$A_s := A_{\geq s}/A_{>s}.$$

(iv) We say that v has **at most one-dimensional leaves** if

$$\dim_{\mathbb{K}} A_s \leq 1 \text{ for all } s \in \mathbb{Z}^d.$$

We can now construct Newton-Okounkov bodies.

Construction 3.3.6 (Newton-Okounkov Body)**.** Let X be a normal projective variety of dimension d and let $\mathcal{L}$ be an ample line bundle over X. Let

$$R(X, \mathcal{L}) := \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{L}^m)$$

denote the associated ring of global sections and let $v\colon R(X, \mathcal{L}) \setminus \{0\} \to \mathbb{Z}^d$ be a valuation with respect to some monoidal order $\leq$ on $\mathbb{Z}^d$. We consider the graded monoid

$$\Gamma(X, \mathcal{L}, v) := \bigcup_{m \in \mathbb{N}} \{(m, v(f)) \mid f \in H^0(X, \mathcal{L}^m)\} \subseteq \mathbb{N} \times \mathbb{Z}^d,$$

often called the **valuation monoid** or **valuation semigroup** with respect to X, $\mathcal{L}$ and v. It is indeed a monoid because for every $f \in H^0(X, \mathcal{L}^m)$ and $g \in H^0(X, \mathcal{L}^n)$ we have $fg \in H^0(X, \mathcal{L}^{m+n})$ and

$$(m + n, v(fg)) = (m + n, v(f) + v(g)) = (m, v(f)) + (n, v(g)).$$

Let

$$\mathcal{C}(X, \mathcal{L}, v) := \overline{\operatorname{cone} \Gamma(X, \mathcal{L}, v)} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^d$$

denote the closed cone over Γ. The **Newton-Okounkov body** $NO(X, \mathcal{L}, v)$ associated to X, $\mathcal{L}$ and v is then defined as the intersection

$$\{1\} \times NO(X, \mathcal{L}, v) := \{(1, x) \mid x \in \mathbb{R}^d\} \cap \mathcal{C}(X, \mathcal{L}, v).$$

We will now state some important results about valuations and Newton-Okounkov bodies. The first one is due to Kaveh and Manon and can be found in [39, Theorem 2.3].

Theorem 3.3.7. *Every full-rank valuation has at most one-dimensional leaves.*

During our construction we introduced an interesting ring that has a remarkable property, as found in [14, Théorème 4.5.2].

Theorem 3.3.8. *Let X be a normal projective variety and $\mathcal{L}$ an ample line bundle over X. Then $X \simeq \operatorname{Proj} R(X, \mathcal{L})$.*

The next observations are quite standard.

Proposition 3.3.9. *Let X be a normal projective variety of dimension d, let $\mathcal{L}$ be an ample line bundle over X and let $v\colon R(X,\mathcal{L}) \setminus \{0\} \to \mathbb{Z}^d$ be a valuation on the ring of global sections $R(X, \mathcal{L})$ with respect to some monoidal order $\leq$ on $\mathbb{Z}^d$.*

(i) The Newton-Okounkov Body $NO(X, \mathcal{L}, v)$ is a convex body.

(ii) If the valuation v has full rank, the Newton-Okounkov body $NO(X, \mathcal{L}, v)$ has dimension d.

(iii) If the semigroup $\Gamma(X, \mathcal{L}, v)$ is finitely generated, the Newton-Okounkov body $NO(X, \mathcal{L}, v)$ is a rational convex polytope. If the semigroup is generated in degree 1, i.e. by elements of the form $(1, v(f))$ with $f \in H^0(X, \mathcal{L})$, it is a lattice polytope.

(iv) If the valuation v has one-dimensional leaves and the semigroup $\Gamma(X, \mathcal{L}, v)$ is finitely generated and saturated, then

$$\#(NO(X, \mathcal{L}, v) \cap \mathbb{Z}^d) = \dim H^0(X, \mathcal{L}).$$

(v) (Lazarsfeld, Mustață) *Assume that the semigroup $\Gamma(X, \mathcal{L}, v)$ is finitely generated and saturated. Then*

$$n \cdot NO(X, \mathcal{L}, v) = NO(X, \mathcal{L}^n, v)$$

for every positive integer $n \in \mathbb{N}$, where we realize $R(X, \mathcal{L}^n)$ as a subring of $R(X, \mathcal{L})$, so v is a valuation on both rings.

Proof. We will prove the claims separately.

(i) Notice that the Newton-Okounkov body is a convex body since it is defined as the intersection of two convex sets.

(ii) By Theorem 3.3.8 we know that $\dim_{\text{Krull}} R(X, \mathcal{L}) = \dim X + 1 = d + 1$. Since the valuation image $\operatorname{Im} v \subseteq \mathbb{R}^d$ has dimension d, we know that there exist d linearly independent vectors $v(f_1), \ldots, v(f_d)$ for $f_i \in H^0(X, \mathcal{L}^{m_i})$. Then the vectors $(1, 0), (m_1, v(f_1)), \ldots, (m_d, v(f_1)) \in \Gamma(X, \mathcal{L}, v)$ must be linearly independent because any linear combination

$$a_0(1, 0) + a_1(m_1, v(f_1)) + \ldots + a_d(m_d, v(f_d)) = 0$$

would lead to the linear combination

$$a_1 v(f_1) + \dots a_d v(f_d) = 0$$

under the projection onto the last d coordinates. So $a_1 = \dots = a_d = 0$ and hence also $a_0 = 0$. So by rescaling we get a basis of $\mathbb{R} \times \mathbb{R}^d$ given by

$$(1, 0), (1, m_1^{-1} v(f_1)), \dots, (1, m_d^{-1} v(f_d))$$

whose convex hull C is a subset of $\{1\} \times NO(X, \mathcal{L}, v)$. We will now show that the affine hull of these generators is equal to the hyperplane

$$H := \{(1, x) \mid x \in \mathbb{R}^d\}.$$

This will then imply that the affine hull of $\{1\} \times NO(X, \mathcal{L}, v)$ has dimension d since

$$C \subseteq NO(X, \mathcal{L}, v) \subseteq H \Rightarrow H = \operatorname{aff} C \subseteq \operatorname{aff} NO(X, \mathcal{L}, v) \subseteq H$$

and $\dim H = d$.

Let $(1, x) \in H$. Since our vectors define a basis of $\mathbb{R} \times \mathbb{R}^d$ we find coefficients $\lambda_0, \dots, \lambda_d \in \mathbb{R}$ such that

$$(1, x) = \lambda_0(1, 0) + \lambda_1(1, m_1^{-1} v(f_1)) + \dots + \lambda_d(1, m_d^{-1} v(f_d)).$$

By analyzing the initial coordinate we see that

$$\lambda_0 + \lambda_1 + \dots + \lambda_d = 1$$

hence $(1, x) \in \operatorname{aff} C$ which proves our claim.

(iii) If $\Gamma(X, \mathcal{L}, v)$ is finitely generated, we find sections $f_1, \dots, f_r$ such that the semigroup is generated by $(m_1, v(f_1)), \dots, (m_r, v(f_r))$, where $f_i \in H^0(X, \mathcal{L}^{m_i})$. These vectors also generate the cone over the semigroup, which will hence be closed already. By rescaling these vectors and by similar arguments as before we see that

$$NO(X, \mathcal{L}, v) = \operatorname{conv}\{m_1^{-1} v(f_1), \dots, m_r^{-1} v(f_r)\}$$

hence the Newton-Okounkov body is a rational polytope. For the second claim, if we can chose all $f_i \in H^0(X, \mathcal{L})$, we see that the Newton-Okounkov body will be a lattice polytope.

(iv) We will prove that

$$v(H^0(X,\mathcal{L})) = NO(X,\mathcal{L},v) \cap \mathbb{Z}^d.$$

The claim then follows from the fact that

$$\#(v(H^0(X,\mathcal{L}))) = \dim H^0(X,\mathcal{L})$$

by [38, Proposition 2.6]. Notice that this Proposition only holds true for valuations with at most one-dimensional leaves.

The inclusion $v(H^0(X,\mathcal{L})) \subseteq NO(X,\mathcal{L},v) \cap \mathbb{Z}^d$ is obvious. So let p be a lattice point in the Newton-Okounkov body. By our previous results there exist coefficients $a_1,\dots,a_r \geq 0$ such that

$$p = a_1 m_1^{-1} v(f_1) + \dots a_r m_r^{-1} v(f_r) \text{ and } a_1 + \dots + a_r = 1$$

for some global sections $f_i \in H^0(X,\mathcal{L}^{m_i})$. Since finding the a_i corresponds to solving a system of linear equations over $\mathbb{Q}$, we can chose the a_i to be rational. Fix $b_1,\dots,b_r \in \mathbb{Z}_{\geq 0}$ and $c_1,\dots,c_r \in \mathbb{Z}_{>0}$ such that $a_i := \frac{b_i}{c_i}$ for every $1 \leq i \leq r$. Now consider the point

$$\begin{aligned}\left(\prod_{i=1}^r c_i m_i\right)(1,p) &= \left(\prod_{i=1}^r c_i m_i\right)\sum_{i=1}^r a_i(1, m_i^{-1} v(f_i)) \\ &= \sum_{i=1}^r \left(\prod_{j\neq i} c_j m_j\right) b_i(m_i, v(f_i)).\end{aligned}$$

Since this is a $\mathbb{Z}_{\geq 0}$-linear combination of elements of $\Gamma(X,\mathcal{L},v)$, it is an element of $\Gamma(X,\mathcal{L},v)$. But this means that $(1,p) \in \Gamma(X,\mathcal{L},v)$ since $\Gamma(X,\mathcal{L},v)$ is saturated. Thus $p \in v(H^0(X,\mathcal{L}) \setminus \{0\})$.

(v) The dilation property is proven in [46, Proposition 4.1].

□

One crucial property of Newton-Okounkov bodies was proved by Anderson in [2, Theorem 1, Corollary 2 and Lemma 3].

Theorem 3.3.10 (Anderson)**.** *Let X be a normal projective complex variety of dimension d, let $\mathcal{L}$ be an ample line bundle over X and let $v\colon R(X,\mathcal{L}) \setminus \{0\} \to \mathbb{Z}^d$ be a valuation on the ring of global sections $R(X,\mathcal{L})$ with respect to some monoidal order $\leq$ on $\mathbb{Z}^d$. If the semigroup $\Gamma(X,\mathcal{L},v)$ is finitely generated and saturated, the pair (X,L) admits a toric degeneration (in the sense of Definition 3.2.2) to the toric variety associated to $NO(X,\mathcal{L},v)$.*

This theorem justifies our harsh assumptions on toric degenerations. As we will see later, almost all toric degenerations known in applications have been — sometimes a posteriori — realized as degenerations stemming from the Newton-Okounkov machinery.

3.4 The Two Polynomials

The careful reader will have noticed, that during Chapter 2 we have encountered two polytopes. In Section 2.1.4 the Hilbert-Polynomial $P_{\mathcal{L}}$ associated to the projective variety X and ample line bundle $\mathcal{L}$ over X was defined via

$$P_{\mathcal{L}}(n) = \chi(X, \mathcal{L}^n)$$

for every $n \in \mathbb{Z}$.

On the other hand we found the Ehrhart (quasi-)polynomial $L_{\mathcal{P}}$ associated to the convex rational polytope $\mathcal{P} \subseteq \mathbb{R}^d$ via

$$L_{\mathcal{P}}(n) = \#(n\mathcal{P} \cap \mathbb{Z}^d)$$

for every $n \in \mathbb{Z}$.

We will now see the connection between the two.

Theorem 3.4.1 (Theorem of the Two Polynomials)**.** *Let X be a normal projective complex variety of dimension d and let $\mathcal{L}$ be an ample line bundle over X. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. If the pair $(X, \mathcal{L})$ admits a toric degeneration — in our sense — to the toric variety $(X_{\mathcal{P}}, D_{\mathcal{P}})$, the Hilbert polynomial associated to X and $\mathcal{L}$ coincides with the Ehrhart quasi-polynomial associated to $\mathcal{P}$, i.e.*

$$P_{\mathcal{L}}(n) = L_{\mathcal{P}}(n)$$

for all $n \in \mathbb{Z}$.

Proof. By Theorem 3.2.10 we know that $D_{\mathcal{P}}$ is a Weil divisor on $X_{\mathcal{P}}$.

By our assumption there exist divisorial sheaves $\mathcal{F}^{(n)}$ for every $n \in \mathbb{N}$ such that

$$\mathcal{F}^{(n)}|_{\mathcal{X}_t} \simeq \mathcal{L}^n \text{ for all } t \neq 0 \text{ and } \mathcal{F}^{(n)}|_{\mathcal{X}_0} \simeq \mathcal{O}_{X_{\mathcal{P}}}(nD_{\mathcal{P}}).$$

By Theorem 3.1.1 we have

$$L_{\mathcal{P}}(n) = \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(nD_{\mathcal{P}})).$$

We have seen that Euler characteristic is constant in flat projective families (see Theorem 2.1.24). Hence we conclude

$$L_{\mathcal{P}}(n) = \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(nD_{\mathcal{P}})) = \chi(X, \mathcal{L}^n) = P_{\mathcal{L}}(n)$$

for every $n \in \mathbb{N}$.

Now notice that the Ehrhart quasi-polynomial coincides with the Hilbert polynomial (which is indeed a polynomial due to Theorem 2.1.26) on all positive integers, hence it must be a polynomial itself. Furthermore, since both polynomials coincide on all positive integers, they coincide on all integers. $\square$

The following consequence is immediate.

Corollary 3.4.2. *Let $\mathcal{P}$ be a rational convex polytope. Then the associated pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$ can only be the limit of a normal projective variety and an ample line bundle under a toric degeneration if the Ehrhart quasi-polynomial of $\mathcal{P}$ is a polynomial.*

We will return to this observation in Section 5.3. Another consequence is the following.

Corollary 3.4.3. *Let X be a normal projective complex variety of dimension d and let $\mathcal{L}$ be an ample line bundle over X. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a full-dimensional rational convex polytope. If the pair $(X, \mathcal{L})$ admits a toric degeneration — in our sense — to the toric variety $(X_{\mathcal{P}}, D_{\mathcal{P}})$, then*

$$\chi(X, \mathcal{L}^n) = \#(n\mathcal{P} \cap \mathbb{Z}^d) \quad \text{and} \quad \chi(X, \mathcal{L}^n \otimes \omega_X) = \#(\operatorname{int} n\mathcal{P} \cap \mathbb{Z}^d)$$

for all $n \in \mathbb{N}$.

Proof. The first equality is clear. The second equality follows from Ehrhart-Macdonald reciprocity (see Theorem 2.2.35) and Serre Duality (see Theorem 2.1.29), yielding

$$\chi(X, \mathcal{L}^n \otimes \omega_X) = (-1)^d \chi(X, \mathcal{L}^{-n}) = (-1)^d L_{\mathcal{P}}(-n) = L_{\operatorname{int} \mathcal{P}}(n)$$

for every $n \in \mathbb{N}$. $\square$

Remark 3.4.4. Philosophically speaking, Ehrhart-Macdonald Reciprocity *is* Serre Duality. This can be seen in the Figure 3.1.

Figure 3.1: Let X be a normal projective complex variety, $\mathcal{L}$ an ample line bundle over X and $\mathcal{P}$ a rational convex polytope such that $(X, \mathcal{L})$ admits a toric degeneration to $(X, D_{\mathcal{P}})$. Then the sketched equalities hold for every $n \in \mathbb{N}$.

$$\begin{tikzcd}
\chi(X, \mathcal{L}^{-n}) \arrow[r, equal, "\text{Serre}", "\text{Duality}"'] \arrow[d, equal] & (-1)^d \chi(X, \mathcal{L}^n \otimes \omega_X) \arrow[d, equal] \\
L_{\mathcal{P}}(-n) \arrow[r, equal, "\text{Ehrhart-Macdonald}", "\text{Reciprocity}"'] & (-1)^d L_{\operatorname{int} \mathcal{P}}(n)
\end{tikzcd}$$

Chapter 4

The Flag Variety Case

In this chapter we will prove a version of our main theorem in the case of Newton-Okounkov bodies associated to partial flag varieties of complex simple algebraic groups. Most of this chapter has already been presented in the preprint [61]. We will present the content nonetheless since the proof does not need all of the sophisticated results from algebraic geometry that we previously introduced. Instead, we have methods from representation theory that help us in our work.

4.1 Exposition

The main theorem is the following.

Theorem 4.1.1 (Main Theorem for Flag Varieties)**.** *If the valuation semigroup $\Gamma(G/P, \mathcal{L}_\lambda, v)$ associated to a partial flag variety G/P via the P-regular dominant integral weight λ and full-rank valuation v is finitely generated and saturated, the following properties of the Newton-Okounkov body $NO(G/P, \mathcal{L}_\lambda, v)$ are equivalent.*

(i) $\mathcal{L}_\lambda$ is the anticanonical line bundle over G/P.

(ii) $NO(G/P, \mathcal{L}_\lambda, v)$ contains exactly one lattice point $p_{G/P,\mathcal{L},v}$ in its interior.

Furthermore, in this case the dual of the translated Newton-Okounkov body

$$\widetilde{NO}(X, \mathcal{L}_\lambda, v) := NO(G/P, \mathcal{L}_\lambda, v) - p_{G/P,\mathcal{L}_\lambda,v}$$

is a lattice polytope, hence the Newton-Okounkov body $NO(X, \mathcal{L}_\lambda, v)$ is translated to a weakly dual-Fano polytope.

There are numerous consequences that we want to address.

Corollary 4.1.2. *If the valuation semigroup $\Gamma(G/P, \mathcal{L}_\lambda, v)$ associated to a partial flag variety G/P via the P-regular dominant integral weight λ and full-rank* *valuation v is finitely generated and saturated, the associated Newton-Okounkov body $NO(G/P, \mathcal{L}_\lambda, v)$ is reflexive (after translation by a lattice vector) if and only if it is a lattice polytope and $\mathcal{L}_\lambda$ is the anticanonical line bundle over G/P.*

The polytopes in the following corollary will not be explained in this chapter, since we will discuss them in detail in Chapter 6.

Corollary 4.1.3. *Let G/P be a partial flag variety and let λ be a P-regular dominant integral weight. Let $\mathcal{P}(\lambda)$ be*

(i) the Gelfand Tsetlin polytope $GT(\lambda)$ or the Feigin-Fourier-Littelmann-Vinberg polytope $FFLV(\lambda)$ if G is of type A_n,

(ii) the Feigin-Fourier-Littelmann-Vinberg polytope $FFLV(\lambda)$ if G is of type C_n or

(iii) the Gornitskii polytope $G(\lambda)$ if G is of type G_2.

Then $\mathcal{P}(\lambda)$ is reflexive (after translation by a lattice vector) if and only if λ is the weight of the anticanonical line bundle over G/P.

Corollary 4.1.4. *Let G/P be a partial flag variety and let λ be a P-regular dominant integral weight. Then the pair $(G/P, \mathcal{L}_\lambda)$ does not admit a toric degeneration (in the sense of Definition 3.2.2) to a Gorenstein Fano toric variety if $\mathcal{L}_\lambda$ is not the anticanonical line bundle over G/P.*

4.2 Famous Results

We need the following two famous results from representation theory.

The first one can be found in [6, Theorem V].

Theorem 4.2.1 (Borel-Weil)**.** *Let G/B be a full flag variety and λ a dominant integral weight for the Lie algebra $\mathfrak{g}$ of G. Then the cohomology group $H^0(G/B, \mathcal{L}_\lambda)$ is an irreducible $\mathfrak{g}$-representation, isomorphic to the dual of the irreducible $\mathfrak{g}$-representation $V(\lambda)$ with highest weight λ, i.e.*

$$H^0(G/B, \mathcal{L}_\lambda) \simeq V(\lambda)^*.$$

The following consequence of Weyl's Chracter Formula can be found in [32, Section 10.5].

Theorem 4.2.2 (Weyl's Dimension Formula)**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let λ be a dominant integral weight. Then the dimension of the irreducible representation $V(\lambda)$ of highest weight λ is given by*

$$\dim V(\lambda) = \prod_{\beta \in \Phi^+} \frac{\langle \lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}.$$

Although the first results was stated only for the full flag variety, Kostant proved the following generalization for partial flag varieties in [44, Corollary 5.14].

Theorem 4.2.3 (Kostant)**.** *Let G/B be a full flag variety and let λ be a dominant integral weight for the Lie algebra $\mathfrak{g}$ of G. Then the cohomology group $H^0(G/P, \mathcal{L}_\lambda)$ is an irreducible $\mathfrak{g}$-representation, isomorphic to the dual of the irreducible $\mathfrak{g}$-representation $V(\lambda)$ with highest weight λ. Hence*

$$\dim H^0(G/P, \mathcal{L}_\lambda) = \prod_{\beta \in \Phi_P^+} \frac{\langle \lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}.$$

4.3 Notation

Before we get to the technical work, we need to fix some notation.

Let G be a simple algebraic group of rank r with Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G and B a Borel subgroup of G containing T. Let P be a parabolic subgroup of G containing B and let L be the Levi subgroup of P containing T. Let $\mathcal{W} := N_G(T)/T$ denote the Weyl group of G.

Let Φ be the set of roots of G and let Φ^+ be the subset of positive roots with respect to B. Denote the set of simple roots by $S = \{\alpha_1, \dots, \alpha_r\}$. Let N be the number of positive roots.

Let Λ be the lattice of integral weights of G and Λ^+ the subset of dominant integral weights with respect to B. Let $\omega_i \in \Lambda^+$ be the fundamental weight corresponding to $\alpha_i \in S$ and $\rho := \frac{1}{2} \sum_{\beta \in \Phi^+} \beta = \sum_{i=1}^r \omega_i$.

We know (see [57, Theorem 8.4.3]) that there exists a set of simple roots $I \subseteq S$ such that $P = \bigcup_{w \in \mathcal{W}_I} B\tilde{w}B$, where $\mathcal{W}_I \subseteq \mathcal{W}$ is the Weyl group generated by the simple reflections $\{s_\alpha \,|\, \alpha \in I\}$ and $\{\tilde{w} \in N_G(T) \,|\, w \in \mathcal{W}\}$ is a set of representatives for the Weyl group elements. We will use the shorthand notations $\langle I \rangle := \Phi \cap \{\sum_{\alpha \in I} m_\alpha \alpha \,|\, m_\alpha \in \mathbb{Z}\}$ and $\langle I \rangle^+ := \langle I \rangle \cap \Phi^+$. We define $\Lambda_P := \{\lambda \in \Lambda \,|\, \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I\}$ and $\Lambda_P^+ := \Lambda_P \cap \Lambda^+$ as well as $\Phi_P^+ := \Phi^+ \setminus \langle I \rangle^+$. Let N_P be the cardinality of Φ_P^+.

A dominant weight $\lambda \in \Lambda$ extends to a character of P if and only if $\lambda \in \Lambda_P$. For every such λ we define the one-dimensional vector space $\mathbb{C}_{-\lambda}$ with P-action given by $p.x := \lambda(p)^{-1}x$. We will consider the line bundle

$$\mathcal{L}_{P,\lambda} := G \times_P \mathbb{C}_{-\lambda} = (G \times \mathbb{C}_{-\lambda})/P$$

over G/P where the P-action on $G \times \mathbb{C}_{-\lambda}$ is given by $p.(g, x) := (gp, p^{-1}.x)$. We know that for a dominant weight $\lambda \in \Lambda_P^+$ the line bundle $\mathcal{L}_{P,\lambda}$ is ample if and only if λ is P-regular, i. e. $\lambda \in \Lambda_P^+$ and $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in S \setminus I$. We will just write $\mathcal{L}_\lambda$ for $\mathcal{L}_{P,\lambda}$ if the parabolic is fixed. We will always implicitly exclude the trivial case $I = S$.

4.4 Technical Lemmata

The following lemmata state important results on the Weyl group $\mathcal{W}_I \subseteq \mathcal{W}$ corresponding to P. Let $w_I \in \mathcal{W}_I$ denote the longest word of $\mathcal{W}_I$.

Lemma 4.4.1. *$w_I(\Phi_P^+) = \Phi_P^+$ and $w_I(\langle I\rangle^+) = -\langle I\rangle^+$.*

Proof. Since $\mathcal{W}_I$ is generated by all simple reflections $\{s_\alpha \,|\, \alpha \in I\}$ we know that $w_I(\langle I\rangle) = \langle I\rangle$. Since $w_I \in \mathcal{W}$ we also have $w_I(\Phi) = \Phi$, thus $w_I(\Phi_P^+) \subseteq \Phi_P^+ \sqcup -\Phi_P^+$. Notice that for every $\beta = \sum_{\alpha\in S} m_\alpha \alpha \in \Phi_P^+$ there is at least one $\alpha \in S \setminus I$ such that $m_\alpha > 0$. Since $w_I \in \langle s_\alpha \,|\, \alpha \in I\rangle$ this sign cannot be changed by w_I. This yields $w_I(\Phi_P^+) = \Phi_P^+$.

The second part follows from the fact that w_I is the longest word of the Weyl group $\mathcal{W}_I$ corresponding to the Levi L_I, so it sends positive roots in $\langle I\rangle$ onto negative roots and vice versa. $\square$

Lemma 4.4.2. *The weight of the anticanonical line bundle over G/P is*

$$\lambda_{G/P} = \rho + w_I(\rho).$$

Proof. We know that the anticanonical bundle is the dual of the highest wedge power of the tangent space of G/P whose highest weight as a $\mathfrak{g}$-representation is exactly $\sum_{\beta\in\Phi_P^+}\beta$. On the other hand we have

$$\begin{aligned}\rho + w_I(\rho) &= \frac{1}{2}\sum_{\beta\in\Phi^+}\beta + \frac{1}{2}\left(\sum_{\beta\in\langle I\rangle^+} w_I(\beta) + \sum_{\beta\in\Phi_P^+} w_I(\beta)\right)\\ &= \frac{1}{2}\sum_{\beta\in\langle I\rangle^+}\beta + \frac{1}{2}\sum_{\beta\in\Phi_P^+}\beta - \frac{1}{2}\sum_{\beta\in\langle I\rangle^+}\beta + \frac{1}{2}\sum_{\beta\in\Phi_P^+}\beta = \sum_{\beta\in\Phi_P^+}\beta\end{aligned}$$

since w_I permutes all elements of Φ_P^+ and sends all elements of $\langle I\rangle^+$ onto elements of $-\langle I\rangle^+$ bijectively as we proved in Lemma 4.4.1. $\square$

The following lemma on root systems seems rather technical, but it is crucial to the proof of our main theorem.

Lemma 4.4.3. *Let $\lambda \in \Lambda_P^+$ be P-regular. Suppose there exists $\beta \in \Phi_P^+$ such that $\langle \lambda - \rho, \beta^\vee\rangle < 0$. Then there exists $\tilde\beta \in \Phi_P^+$ such that $\langle \lambda - \rho, \tilde\beta^\vee\rangle = 0$.*

To prove the lemma we need the following two lemmata.

Lemma 4.4.4. *Let* $\beta = \sum_{i=1}^{r} m_i\alpha_i \in \Phi^+$ *and* $\operatorname{ht}\beta > 1$*. For every integer* $i \in \{1, \ldots, r\}$ *such that* $m_i = 1$ *there exists* $j \neq i$ *such that* $\beta - \alpha_j \in \Phi$*.*

Proof. We will prove the lemma by induction on $\operatorname{ht}\beta$.

For $\operatorname{ht}\beta = 2$ we have nothing to prove since $\beta = \alpha_i + \alpha_j$ for some integers $i, j \in \{1, \ldots, r\}$.

Now suppose that $\operatorname{ht}\beta > 2$. Fix an $i \in \{1, \ldots, r\}$ such that $m_i = 1$. If $\langle\beta, \alpha_i^\vee\rangle \leq 0$, we again have nothing to prove because the proof of [35, Lemma A of 10.2] ensures that there exists at least one $j \in \{1, \ldots, r\}$ such that $\langle\beta, \alpha_j^\vee\rangle > 0$ which cannot be equal to i by assumption. By [35, Lemma 9.4] this j would then possess the desired property.

So we only have to prove the case where $\langle\beta, \alpha_i^\vee\rangle > 0$. Because of [35, Lemma 9.4] this means that $\beta - \alpha_i$ is a (necessarily positive) root.

Hence we know that the support of $\beta - \alpha_i$ is connected in the Dynkin diagram of $\mathfrak{g}$. But because $m_i = 1$ we know that this support does not contain α_i. This means that there exists only one simple root in the support of β that is adjacent to α_i, because otherwise the removal of α_i would result in a disconnected subgraph. Denote this adjacent simple root by α_j. So for every $k \in \{1, \ldots, r\} \setminus \{i, j\}$ with $m_k > 0$ we have $\langle\alpha_k, \alpha_i^\vee\rangle = 0$. From $\langle\alpha_j, \alpha_i^\vee\rangle \leq -1$ and

$$0 < \langle\beta, \alpha_i^\vee\rangle = m_i\langle\alpha_i, \alpha_i^\vee\rangle + m_j\langle\alpha_j, \alpha_i^\vee\rangle \leq 2 - m_j$$

we conclude that $m_j < 2$ and thus $m_j = 1$. So we can use the induction hypothesis on $\beta - \alpha_i$ and get a $k \neq j$ such that $\beta - \alpha_i - \alpha_k$ is a root. Because $\beta - \alpha_i$ does not contain α_i in its support, we know that $k \neq i$. Thus we conclude

$$\langle\beta - \alpha_i - \alpha_k, \alpha_i^\vee\rangle = m_j\langle\alpha_j, \alpha_i^\vee\rangle - \langle\alpha_k, \alpha_i^\vee\rangle \leq -m_j - 0 = -1 < 0$$

and [35, Lemma 9.4] shows that $\beta - \alpha_k = \beta - \alpha_i - \alpha_k + \alpha_i$ is a (positive) root. $\square$

Lemma 4.4.5. *Let* $\beta \in \Phi_P^+$*. There exists a sequence* $(i_j)_{j\in\{1,\ldots,\operatorname{ht}\beta\}}$ *in* $\{1, \ldots, r\}$ *such that* $\beta = \sum_{j=1}^{\operatorname{ht}\beta} \alpha_{i_j}$ *and* $\sum_{j=1}^{k} \alpha_{i_j} \in \Phi_P^+$ *for every* $k \in \{1, \ldots, \operatorname{ht}\beta\}$*.*

Proof. We will prove the lemma by induction on $\operatorname{ht}\beta$.

If $\operatorname{ht}\beta = 1$ there is nothing to prove.

So let $h \in \mathbb{N}$, $h > 1$, and suppose the lemma is true for every positive root $\beta' \in \Phi_P^+$ with $\operatorname{ht}\beta' < h$. Let us now assume $\beta \in \Phi_P^+$ with $\operatorname{ht}\beta = h$. If no such β exists we have nothing to prove.

We know that there exists $\alpha \in S$ such that $\beta - \alpha \in \Phi^+$. If $\beta - \alpha \notin \Phi_P^+$ then β must be of the form $\beta = \alpha + \sum_{\alpha'\in I} m_{\alpha'}\alpha'$. In this case Lemma 4.4.4 assures us that there exists another $\alpha' \in S$ such that $\beta - \alpha' \in \Phi$ and furthermore this root has to be in Φ_P^+.

So we can always find $\alpha \in S$ such that $\beta - \alpha \in \Phi_P^+$. By applying the induction hypothesis on that root we find the correct sequence $(i_j)_{j\in\{1,\dots,h-1\}}$ in $\{1,\dots,r\}$ for $\beta - \alpha$. Defining i_h by $\alpha_{i_h} = \alpha$ will yield the desired sequence for β. $\square$

We can now prove our last key lemma and finish our preparations.

Proof of Lemma 4.4.3. Let $\beta \in \Phi_P^+$ such that $\langle \lambda - \rho, \beta^\vee \rangle < 0$. Let $h := \operatorname{ht} \beta$.

Notice that $h > 1$ since for every simple root $\alpha \in \Phi_P^+$, i.e. $\alpha \in S \setminus I$, we have $\langle \lambda - \rho, \alpha^\vee \rangle \geq 0$ because λ is P-regular.

By Lemma 4.4.5 we find a sequence of indices $(i_j)_{j\in\{1,\dots,h\}}$ in $\{1,\dots,r\}$ such that $\beta = \sum_{j=1}^h \alpha_{i_j}$ and $\beta_k := \sum_{j=1}^k \alpha_{i_j} \in \Phi_P^+$ for every $k \in \{1,\dots,h\}$.

Since $\langle \lambda - \rho, \alpha_{i_1}^\vee \rangle \geq 0$, there exists an index $k \in \{1,\dots,h\}$ such that

$$\langle \lambda - \rho, \beta_{k-1}^\vee \rangle \geq 0 \quad \text{and} \quad \langle \lambda - \rho, \beta_k^\vee \rangle < 0.$$

We have

$$0 \leq \langle \lambda - \rho, \beta_{k-1}^\vee \rangle = 2 \cdot \frac{\langle \lambda - \rho, \beta_k \rangle - \langle \lambda - \rho, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} < -\frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \langle \lambda - \rho, \alpha_{i_k}^\vee \rangle.$$

Since λ is P-regular this is only possible if $\alpha_{i_k} \in I$, i. e. $\langle \lambda, \alpha_{i_k}^\vee \rangle = 0$, and thus

$$0 \leq \langle \lambda - \rho, \beta_{k-1}^\vee \rangle < \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle}.$$

This shows that there are only three possible values for $\langle \lambda - \rho, \beta_{k-1}^\vee \rangle$ since the fraction on the right side must be an element of $\{\frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$.

If $\langle \lambda - \rho, \beta_{k-1}^\vee \rangle = 0$ we have found the desired root $\tilde{\beta} = \beta_{k-1}$.

If $\langle \lambda - \rho, \beta_{k-1}^\vee \rangle = 1$ we must have $\frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \in \{2, 3\}$. Set

$$\tilde{\beta} := \alpha_{i_k} + \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \beta_{k-1}$$

as an element of the root lattice. We have

$$\begin{aligned}
\langle \lambda - \rho, \tilde{\beta} \rangle &= \langle \lambda - \rho, \alpha_{i_k} \rangle + \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \langle \lambda - \rho, \beta_{k-1} \rangle \\
&= \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{2} \langle \lambda - \rho, \alpha_{i_k}^\vee \rangle + \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \cdot \frac{\langle \beta_{k-1}, \beta_{k-1} \rangle}{2} \langle \lambda - \rho, \beta_{k-1}^\vee \rangle
\end{aligned}$$

$$= -\frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{2} + \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \cdot \frac{\langle \beta_{k-1}, \beta_{k-1} \rangle}{2} = 0.$$

We still have to show that $\tilde{\beta}$ is actually a root. By expanding $\langle \beta_k - \alpha_{i_k}, \beta_k - \alpha_{i_k} \rangle$ we find that

$$\langle \beta_k, \alpha_{i_k}^\vee \rangle = 1 + \frac{\langle \beta_k, \beta_k \rangle}{\langle \alpha_{i_k}, \alpha_{i_k} \rangle} - \frac{\langle \beta_{k-1}, \beta_{k-1} \rangle}{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}.$$

Since the last summand is not an integer, we know that the second summand must not be an integer too. But this means that β_k and β_{k-1} must have the same length because only two root lengths are allowed to occur in any irreducible root system ([35, Lemma C of 10.4]). We conclude that $\langle \beta_k, \alpha_{i_k}^\vee \rangle = 1$ and thus $\langle \beta_{k-1}, \alpha_{i_k}^\vee \rangle = -1$. This yields

$$\langle \alpha_{i_k}, \beta_{k-1}^\vee \rangle = \frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \langle \beta_{k-1}, \alpha_{i_k}^\vee \rangle = -\frac{\langle \alpha_{i_k}, \alpha_{i_k} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle},$$

which implies that $\tilde{\beta}$ is a root because it is the reflection of α_{i_k} under the reflection orthogonal to β_{k-1} (alternatively see [35, 9.4]).

The last possible case $\langle \lambda - \rho, \beta_{k-1}^\vee \rangle = 2$ can only occur if the root system is G_2, α_{i_k} is the long simple root and β_{k-1} is a short positive root. Since their sum must again be a root, we know that β_{k-1} has to be the short simple root. In that case we set $\tilde{\beta} = 2\alpha_{i_k} + 3\beta_{k-1} \in \Phi_P^+$ and calculate

$$\begin{aligned} \langle \lambda - \rho, \tilde{\beta} \rangle &= \frac{3}{2} \langle \beta_{k-1}, \beta_{k-1} \rangle \langle \lambda - \rho, \beta_{k-1}^\vee \rangle + \langle \alpha_{i_k}, \alpha_{i_k} \rangle \langle \lambda - \rho, \alpha_{i_k}^\vee \rangle \\ &= 3 \langle \beta_{k-1}, \beta_{k-1} \rangle - \langle \alpha_{i_k}, \alpha_{i_k} \rangle = 0, \end{aligned}$$

which concludes the proof. □

4.5 Ehrhart Theory in Representation Theory

Before we prove Theorem 4.1.1, let us state the following lemma.

Lemma 4.5.1 (Key Lemma). *Let $NO(G/P, \mathcal{L}_\lambda, v)$ denote the Newton-Okounkov body associated to the partial flag variety G/P, a P-regular dominant integral weight λ and a full-rank valuation v on $R(G/P, \mathcal{L}_\lambda)$. Assume that the valuation semigroup $\Gamma(G/P, \mathcal{L}_\lambda, v)$ is finitely generated and saturated. Then*

$$L_{NO(G/P, \mathcal{L}_\lambda, v)}(n) = \prod_{\beta \in \Phi_P^+} \frac{\langle n\lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}$$

for all $n \in \mathbb{Z}$.

Proof. By Proposition 3.3.9 (or equivalently [46, Theorem 4.1]) we know that

$$n \cdot NO(G/P, \mathcal{L}_\lambda, v) = NO(G/P, \mathcal{L}_\lambda^n, v)$$

and hence

$$L_{NO(G/P,\mathcal{L}_\lambda,v)}(n) = h^0(G/P, \mathcal{L}_\lambda^n)$$

for every $n \in \mathbb{N}$. We want to show that this is equal to $\dim V(n\lambda)$. Consider the n-fold product map

$$H^0(G/P, \mathcal{L}_\lambda) \times \ldots \times H^0(G/P, \mathcal{L}_\lambda) \to H^0(G/P, \mathcal{L}_\lambda^n).$$

Notice that $\mathcal{L}_\lambda^n$ is ample, so $H^0(G/P, \mathcal{L}_\lambda^n)$ will be an irreducible $\mathfrak{g}$-representation. Since this product map is $\mathfrak{g}$-equivariant, its image must be a subrepresentation. The image is obviously not empty, so the product map is surjective.

Let $f_\lambda \in H^0(G/P, \mathcal{L}_\lambda) \simeq V(\lambda)^*$ be the global section corresponding to the lowest weight. Then the product $f_\lambda^n \in H^0(G/P, \mathcal{L}_\lambda^n)$ must be the lowest weight vector of $H^0(G/P, \mathcal{L}_\lambda^n)$. Since its weight is $-n\lambda$ we see that $H^0(G/P, \mathcal{L}_\lambda^n)$ is isomorphic to $V(n\lambda)^*$.

Now the claim for positive integers follows from Kostant's version of Weyl's Dimension Formula in Theorem 4.2.3. Notice that the right hand side of

$$L_{NO(G/P,\mathcal{L}_\lambda,v)}(n) = \prod_{\beta \in \Phi_P^+} \frac{\langle n\lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}$$

can be seen as the evaluation of a polynomial at positive integers. Hence the Ehrhart quasi-polynomial must be a polynomial and since the two polynomials coincide on all positive integers, they coincide on all integers. □

Example 4.5.2. For the full-flag variety and its anticanonical weight 2ρ we get

$$L_{NO(G/B,\mathcal{L}_{2\rho},v)}(n) = (2n+1)^N$$

for all $n \in \mathbb{Z}^d$ where N denotes the number of positive roots.

We will now state the complete proof of the main theorem for flag varieties.

Proof of Theorem 4.1.1. By Lemma 4.5.1 we know that

$$L_{NO(G/P,\mathcal{L}_\lambda,v)}(n) = \prod_{\beta \in \Phi_P^+} \frac{\langle n\lambda + \rho, \beta^\vee \rangle}{\langle \rho, \beta^\vee \rangle}$$

for all $n \in \mathbb{Z}$.

Now suppose that $NO(G/P,\mathcal{L}_\lambda,v)$ contains one unique lattice point in its interior. By Ehrhart-Macdonald Reciprocity (see Theorem 2.2.35) we have

$$1 = L_{\operatorname{int} NO(G/P,\mathcal{L}_\lambda,v)}(1) = (-1)^{N_P} L_{NO(G/P,\mathcal{L}_\lambda,v)}(-1) = \prod_{\beta\in\Phi_P^+} \frac{\langle \lambda-\rho,\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle}.$$

This implies that $\langle \lambda-\rho,\beta^\vee\rangle \neq 0$ for every $\beta\in\Phi_P^+$ and by Lemma 4.4.3 this actually means that $\langle \lambda-\rho,\beta^\vee\rangle > 0$ for all $\beta\in\Phi_P^+$. From Lemma 4.4.1 we know that the longest word $w_I \in \mathcal{W}_I \subseteq \mathcal{W}$ permutes the elements of Φ_P^+. Since it is a reflection, it leaves the scalar product invariant and by reshuffling factors we have

$$1 = \prod_{\beta\in\Phi_P^+} \frac{\langle \lambda-\rho,\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle} = \prod_{\beta\in\Phi_P^+} \frac{\langle \lambda-\rho,(w_I\beta)^\vee\rangle}{\langle \rho,\beta^\vee\rangle} = \prod_{\beta\in\Phi_P^+} \frac{\langle w_I(\lambda-\rho),\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle}.$$

Consider the integral weight $\mu = \sum_{i=1}^r \mu_i\omega_i := w_I(\lambda-\rho)$. Every coefficient μ_i is strictly positive since $\langle \lambda-\rho,(w_I\beta)^\vee\rangle > 0$ for every $\beta\in\Phi_P^+$—especially for every $\alpha\in S\setminus I$—and $\langle \lambda-\rho,(w_I\alpha)^\vee\rangle = -\langle \rho,(w_I\alpha)^\vee\rangle > 0$ for every $\alpha\in I$ because $w_I(\alpha)\in -\langle I\rangle^+$ by Lemma 4.4.1.

This observation allows us to use the weighted inequality of arithmetic and geometric means to calculate

$$\begin{aligned} 1 &= \prod_{\beta\in\Phi_P^+} \frac{\langle \mu,\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle} = \prod_{\beta\in\Phi_P^+} \frac{\sum_{i=1}^r \langle \omega_i,\beta^\vee\rangle \mu_i}{\langle \rho,\beta^\vee\rangle} \\ &\geq \prod_{\beta\in\Phi_P^+} \left(\prod_{i=1}^r \mu_i^{\langle \omega_i,\beta^\vee\rangle} \right)^{\frac{1}{\langle \rho,\beta^\vee\rangle}} = \prod_{i=1}^r \left(\mu_i^{\sum_{\beta\in\Phi_P^+} \frac{\langle \omega_i,\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle}} \right). \end{aligned}$$

Since $\langle \omega_i,\beta^\vee\rangle \geq 0$ for all $\beta\in\Phi_P^+$ with strict inequality at least once for every $i\in\{1,\dots,r\}$, we have strictly positive coefficients $a_1,\dots,a_r\in\mathbb{R}_{>0}$ such that

$$1 \geq \mu_1^{a_1}\cdots\mu_r^{a_r}.$$

Since all of the μ_i are strictly positive integers, this inequality can only hold if $\mu_i = 1$ for all $i\in\{1,\dots,r\}$ and then it is in fact an equality. But this means that $w_I(\lambda-\rho) = \mu = \sum_{i=1}^r \omega_i = \rho$ and thus $\lambda = \rho + w_I(\rho)$. By Lemma 4.4.2 this is the weight of the anticanonical line bundle over G/P, which proves the first direction.

In fact we also proved the other direction on the way because we noticed that μ is equal to ρ if λ is the weight of the anticanonical bundle, which yields $L_{\operatorname{int} NO(G/P,\mathcal{L}_\lambda,v)}(1) = \prod_{\beta\in\Phi_P^+} \frac{\langle \mu,\beta^\vee\rangle}{\langle \rho,\beta^\vee\rangle} = 1$ if we apply the above calculations in opposite order.

Let us now prove the final implication of the theorem. Let $\lambda = \rho + w_I(\rho)$ be the weight of the anticanonical line bundle over G/P. We calculate

$$\begin{aligned}(-1)^{N_P} L_{NO(G/P,\mathcal{L}_\lambda,v)}(-n-1) &= \prod_{\beta\in\Phi_P^+} \frac{\langle (n+1)\lambda - \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle} \\ &= \prod_{\beta\in\Phi_P^+} \frac{\langle n\rho + \rho + nw_I(\rho) + w_I(\rho) - \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle} \\ &= \prod_{\beta\in\Phi_P^+} \frac{\langle n(\rho + w_I(\rho)) + w_I(\rho), (w_I\beta)^\vee\rangle}{\langle \rho, \beta^\vee\rangle} \\ &= \prod_{\beta\in\Phi_P^+} \frac{\langle n(w_I(\rho) + \rho) + \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle} \\ &= \prod_{\beta\in\Phi_P^+} \frac{\langle n\lambda + \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle} = L_{NO(G/P,\mathcal{L}_\lambda,v)}(n)\end{aligned}$$

for all $n \in \mathbb{N}$. It is clear that the Ehrhart polynomial of a polytope is invariant under translation of the polytope via a lattice vector. Hence Hibi's Theorem in Theorem 2.2.43 concludes the proof. $\square$

Chapter 5

Anticanonically Polarized Degenerations

This chapter contains the heart of this thesis. We will state a precise criterion whether the toric limit of a Gorenstein Fano variety is polarized anticanonically, thereby simplifying Question 2 from the introduction.

5.1 A necessary and sufficient condition, ...

For a shorter formulation of the theorem, we will introduce the following standard notation.

Definition 5.1.1. Let X be a normal variety and D an ample Cartier Weil divisor on X. This implies that X is projective. We call the pair (X, D) a **polarized** normal variety. More generally, if D is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X, we call the pair (X, D) **$\mathbb{Q}$-polarized**.

We can now formulate the main result of this thesis.

Theorem 5.1.2. *The limit of a polarized Gorenstein Fano variety under a toric degeneration is $\mathbb{Q}$-polarized by its anticanonical divisor if and only if the polarization on the original variety is given by its anticanonical divisor.*

5.2 ... its proof, ...

We would like to encourage the reader to have a glance at Figure 5.1 before reading the actual proof as the scheme behind the proof is sketched there.

We will first prove the following ingredient separately, since its statement is interesting on its own and it needs a lot of powerful tools of algebraic geometry for its proof. In fact, we think of this "lemma" as a theorem instead.

Theorem 5.2.1 (Main "Lemma")**.** *Let X be a Gorenstein Fano variety of dimension d that has rational singularities and $\mathcal{L}$ an ample line bundle. Then the line bundle $\mathcal{L}$ is isomorphic to the anticanonical line bundle ω_X^{-1} if and only if the Hilbert polynomial of $\mathcal{L}$ fulfills*

$$P_{\mathcal{L}}(n) = (-1)^d P_{\mathcal{L}}(-n-1)$$

for all $n \in \mathbb{N}$.

Proof. Notice first that Serre Duality (see Theorem 2.1.29) implies that

$$P_{\mathcal{L}}(-n-1) = \chi(X, \mathcal{L}^{-n-1}) = (-1)^d \chi(X, \mathcal{L}^{n+1} \otimes \omega_X).$$

So the first implication is obvious, since

$$P_{\omega_X^{-1}}(-n-1) = (-1)^d \chi(X, \omega_X^{-n-1} \otimes \omega_X) = (-1)^d \chi(X, \omega_X^{-n}) = (-1)^d P_{\omega_X^{-1}}(n)$$

for every $n \in \mathbb{N}$.

For the other implication let us notice that Kodaira Vanishing for Rational Singularities (see Theorem 2.1.53) implies that

$$\chi(X, \mathcal{L}^{n+1} \otimes \omega_X) = h^0(X, \mathcal{L}^{n+1} \otimes \omega_X)$$

for every $n \in \mathbb{N}$. So for the special case $n = 0$ the assumption on the Hilbert Polynomial implies that

$$\begin{aligned} h^0(X, \mathcal{L} \otimes \omega_X) &= \chi(X, \mathcal{L} \otimes \omega_X) = (-1)^d P_{\mathcal{L}}(-1) \\ &= P_{\mathcal{L}}(0) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1. \end{aligned}$$

Chose a non-zero section $s \in H^0(X, \mathcal{L} \otimes \omega_X)$ and consider the natural morphism $f \colon \mathcal{O}_X \to \mathcal{L} \otimes \omega_X$ given by

$$f(U) \colon \mathcal{O}_X(U) \to (\mathcal{L} \otimes \omega_X)(U), \quad \xi \mapsto \xi \cdot s|_U,$$

on every open $U \subseteq X$.

Since $\mathcal{L} \otimes \omega_X$ is a line-bundle, it is torsion-free. So the maps $f(U)$ and hence the morphism f are injective. Thus we get a short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{f} \mathcal{L} \otimes \omega_X \to \mathcal{C} \to 0$$

for some coherent sheaf $\mathcal{C}$. Additivity of Hilbert Polynomials (see Theorem 2.1.28) yields

$$P_{\mathcal{L}}(\mathcal{C})(n) = P_{\mathcal{L}}(\mathcal{L} \otimes \omega_X)(n) - P_{\mathcal{L}}(\mathcal{O}_X)(n)$$

$$= \chi(X, \mathcal{L}^{n+1} \otimes \omega_X) - \chi(X, \mathcal{L}^n)$$
$$= (-1)^d P_{\mathcal{L}}(-n-1) - P_{\mathcal{L}}(n) = 0$$

for every $n \in \mathbb{N}$.

By the Serre Vanishing Theorem (see Theorem 2.1.48) there exists an integer n_0 such that for every integer $n > n_0$ the sheaf $\mathcal{C} \otimes \mathcal{L}^n$ is globally generated and $h^i(X, \mathcal{C} \otimes \mathcal{L}^n) = 0$ for all $i > 0$. Fix any $n > n_0$. We conclude

$$h^0(X, \mathcal{C} \otimes \mathcal{L}^n) = \chi(X, \mathcal{C} \otimes \mathcal{L}^n) = P_{\mathcal{L}}(\mathcal{C})(n) = 0.$$

Since $\mathcal{C} \otimes \mathcal{L}^n$ is globally generated, this is only possible if $\mathcal{C} \otimes \mathcal{L}^n \simeq 0$, hence $\mathcal{C} \simeq 0$.

In conclusion we see that $\mathcal{L} \otimes \omega_X \simeq \mathcal{O}_X$ holds — or equivalently $\mathcal{L} \simeq \omega_X^{-1}$, which proves the claim. □

We are now able to prove our main result.

Proof of Theorem 5.1.2. Notice that the first implication of the theorem is given by Theorem 3.2.11, which also proves that the toric limit of a Gorenstein Fano variety is $\mathbb{Q}$-Gorenstein Fano.

The other direction will be proved in several steps by applying results previously presented in this thesis.

Let $(X, \mathcal{L})$ be a polarized Gorenstein Fano variety and suppose that it admits a toric degeneration to the pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$, where $\mathcal{P} \subseteq \mathbb{R}^d$ denotes a full-dimensional rational convex polytope, $X_{\mathcal{P}}$ the associated toric variety and $D_{\mathcal{P}}$ the associated torus invariant ample $\mathbb{Q}$-Cartier Weil divisor on $X_{\mathcal{P}}$ (see Proposition 2.3.36 and Theorem 3.2.10).

By our assumption $D_{\mathcal{P}}$ is linearly equivalent to the anticanonical divisor on $X_{\mathcal{P}}$. By Proposition 2.3.27 and Theorem 2.3.28 we know that two torus invariant divisors are linearly equivalent if their difference is given by the divisor of a rational function χ^s, $s \in \mathbb{Z}^d$, so let

$$D_{\mathcal{P}} = -K_{X_{\mathcal{P}}} + \operatorname{div}(\chi^s)$$

for some $s \in \mathbb{Z}^d$. By Proposition 2.3.33 we know that

$$\mathcal{P} = \mathcal{P}_{D_{\mathcal{P}}} = \mathcal{P}_{-K_{X_{\mathcal{P}}} + \operatorname{div}(\chi^s)} = \mathcal{P}_{-K_{X_{\mathcal{P}}}} + s,$$

so we see that $\mathcal{P}$ is a translation of $P_{-K_{X_{\mathcal{P}}}}$ by the lattice vector s.

Since the toric variety $X_{\mathcal{P}}$ is $\mathbb{Q}$-Gorenstein Fano, we know that the polytope $\mathcal{P}_{-K_{X_{\mathcal{P}}}}$ is dual-Fano by Theorem 2.3.40. This implies (see Theorem 2.2.30) that $\mathcal{P}_{-K_{X_{\mathcal{P}}}}$ is weakly dual-Fano, i.e. it contains exactly one interior lattice point (the origin) and its dual polytope is a lattice polytope. Hence $\mathcal{P}$ is translated via a lattice vector to a weakly dual-Fano polytope.

The following reasoning depends solely on the Ehrhart quasi-polynomial of $\mathcal{P}$, which is invariant under lattice translation. So without loss of generality let us assume that $\mathcal{P}$ is weakly dual-Fano.

By Hibi's Theorem (see Theorem 2.2.43) we know that the Ehrhart quasi-polynomial of $\mathcal{P}$ suffices

$$L_{\mathcal{P}}(n) = L_{\operatorname{int}\mathcal{P}}(n+1)$$

for all $n \in \mathbb{N}$. Equivalently, by Ehrhart-Macdonald reciprocity (see Theorem 2.2.35), this means that

$$L_{\mathcal{P}}(n) = (-1)^d L_{\mathcal{P}}(-n-1)$$

for all $n \in \mathbb{N}$.

We have seen in Theorem 3.1.1 that the value of the Ehrhart quasi-polynomial in $n \in \mathbb{N}$ is given by the Euler characteristic of $nD_{\mathcal{P}}$, i.e.

$$L_{\mathcal{P}}(n) = \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(nD_{\mathcal{P}})).$$

Notice that in this case we can omit the rounding of $nD_{\mathcal{P}}$ since it is already a Weil divisor.

Furthermore, we were able to connect this to the Hilbert polynomial of $\mathcal{L}$ on X via the Theorem of the Two Polynomials (recall Theorem 3.4.1 or Corollary 3.4.3), which ensures that

$$L_{\mathcal{P}}(n) = \chi(X, \mathcal{L}^n) \quad \text{and} \quad L_{\operatorname{int}\mathcal{P}}(n) = \chi(X, \mathcal{L}^n \otimes \omega_X)$$

for every $n \in \mathbb{N}$. In conclusion, we know that

$$\chi(X, \mathcal{L}^n) = \chi(X, \mathcal{L}^{n+1} \otimes \omega_X)$$

for every $n \in \mathbb{N}$. Equivalently, by Serre Duality (see Theorem 2.1.29) the Hilbert polynomial fulfills

$$P_{\mathcal{L}}(n) = (-1)^d P_{\mathcal{L}}(-n-1)$$

for every $n \in \mathbb{N}$. Since $X_{\mathcal{P}}$ has rational singularities (recall Theorem 2.3.21), Elkik's Theorem (see Theorem 2.1.56) proves that X has rational singularities as well. So we can apply Theorem 5.2.1, which implies that $\mathcal{L}$ is isomorphic to ω_X^{-1}. This concludes the proof. $\square$

Figure 5.1: Sketch of the proof of Theorem 5.1.2. Notation as in the formulation of the Theorem. The equations are supposed to hold for every $n \in \mathbb{N}$.

$$
\begin{tikzcd}
\mathcal{L} \simeq \omega_X^{-1} \arrow[r, Rightarrow, "\text{Theorem 3.2.11}"'] \arrow[dddd, Leftrightarrow, "\substack{\text{Main “Lemma”} \\ \text{Serre Duality} \\ \text{Elkik's Theorem} \\ \text{Kodaira Vanishing for} \\ \text{Rational Singularities} \\ \text{Serre Vanishing}}"', "\substack{\text{Theorem 5.2.1} \\ \text{Theorem 2.1.29} \\ \text{Theorem 2.1.56} \\ \text{Theorem 2.1.53} \\ \\ \text{Theorem 2.1.48}}"] & D_{\mathcal{P}} \sim -K_{X_{\mathcal{P}}} \arrow[d, Leftrightarrow, "\substack{\text{Proof of} \\ \text{Theorem 2.3.40}}"] \\
 & \mathcal{P}\ \textit{is}^{1}\text{dual-Fano} \arrow[d, Rightarrow, "\text{Theorem 2.2.30}"] \\
 & \mathcal{P}\ \textit{is}^{1}\ \text{weakly dual-Fano} \arrow[d, Leftrightarrow, "\text{Hibi's Theorem}"', "\text{Theorem 2.2.43}"] \\
 & \#(n\mathcal{P} \cap \mathbb{Z}^d) = \#(\mathrm{int}(n+1)\mathcal{P} \cap \mathbb{Z}^d) \arrow[d, Leftrightarrow, "\substack{\text{Ehrhart-Macdonald} \\ \text{Reciprocity}}"', "\text{Theorem 2.2.35}"] \\
P_{\mathcal{L}}(n) = (-1)^d P_{\mathcal{L}}(-n-1) \arrow[r, Leftrightarrow, "\substack{\text{Theorem of the} \\ \text{Two Polynomials}}", "\text{Theorem 3.4.1}"'] \arrow[d, Leftrightarrow, "\text{Theorem 2.1.26}"] & L_{\mathcal{P}}(n) = (-1)^d L_{\mathcal{P}}(-n-1) \arrow[d, Leftrightarrow, "\text{Theorem 3.1.1}"] \\
\chi(X, \mathcal{L}^n) = (-1)^d \chi(X, \mathcal{L}^{-n-1}) \arrow[r, Leftrightarrow, "\substack{\text{Invariance of} \\ \text{Euler Characteristic}}", "\substack{\text{Theorem 2.1.24} \\ \text{Corollary 3.4.3}}"'] & \substack{\chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor nD_{\mathcal{P}} \rfloor)) = \\ (-1)^d \chi(X_{\mathcal{P}}, \mathcal{O}_{X_{\mathcal{P}}}(\lfloor -(n+1)D_{\mathcal{P}} \rfloor))}
\end{tikzcd}
$$

5.3 ...and its consequences

The following consequence follows directly. It might prove useful in the hunt for mirror symmetry since it yields a necessary condition for reflexive polytopes to appear in the setting of toric degenerations.

Corollary 5.3.1. *Let X be a Gorenstein Fano variety and $\mathcal{L}$ an ample line bundle over X. Suppose that $(X, \mathcal{L})$ admits a toric degeneration to the pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$ associated to the rational convex polytope $\mathcal{P}$. Then $\mathcal{P}$ is reflexive (up to translation by a lattice vector) if and only if $\mathcal{P}$ is a lattice polytope and $\mathcal{L}$ is isomorphic to the anticanonical line bundle over X.*

There is another, more delicate consequence. To find it, let us review the equivalences and implications of our proof schematically in Figure 5.1. Es-

[1] up to translation by a lattice vector

sentially we have proved that every weakly dual-Fano polytope appearing in the context of toric degenerations of polarized Gorenstein Fano varieties must already be a dual-Fano polytope. So somehow the polytope must know that it is the limit of a polarized variety under a toric degeneration. We already noticed in Theorem 3.2.10 that not every polytope can appear as the polytope associated to the toric limit divisor since that divisor must be Weil.

Philosophically speaking, this information should be contained in the combinatorics of the polytope. In fact, there is one on-the-fly result we obtained but never used in later arguments. In Corollary 3.4.2 we showed that every polytope whose associated toric pair is the limit of a polarized Gorenstein Fano variety must be a quasi-lattice polytope, i.e. its Ehrhart quasi-polynomial must be a polynomial.

By this reason — and from examples — we reach the following conjecture.

Conjecture 5.3.2. *A convex polytope is dual-Fano if and only if it is a weakly dual-Fano quasi-lattice polytope.*

In other words this conjecture can be formulated as follows (by Corollary 2.2.31).

Conjecture 5.3.3. *Let $\mathcal{P}$ be a full-dimensional rational convex polytope and suppose for every ray ρ in the normal fan $\Sigma_{\mathcal{P}}$ of $\mathcal{P}$ there exist primitive lattice vectors u_ρ and a strictly positive integer $k_\rho > 0$ such that*

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq k_\rho^{-1} \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

Then the Ehrhart quasi-polynomial$L_{\mathcal{P}}$ of $\mathcal{P}$ is a polynomial if and only if all integers k_ρ, $\rho \in \Sigma_{\mathcal{P}}(1)$, are equal to 1.

This conjecture would immediately follow from the following one.

Conjecture 5.3.4. *The divisor associated to a full-dimensional rational convex polytope is a Weil divisor on the toric variety associated to the polytope if and only if the polytope is a quasi-lattice polytope.*

Again, we can give a more combinatorial formulation of this conjecture.

Conjecture 5.3.5. *Let $\mathcal{P}$ be a full-dimensional rational convex polytope such that $0 \in \operatorname{int} \mathcal{P}$. Then the Ehrhart quasi-polynomial $L_{\mathcal{P}}$ of $\mathcal{P}$ is a polynomial if and only if there exist strictly positive integers $m_\rho > 0$ for every ray $\rho \in \Sigma_{\mathcal{P}}(1)$ such that*

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid \langle x, u_\rho \rangle \leq m_\rho \text{ for all } \rho \in \Sigma_{\mathcal{P}}(1)\}.$$

Here u_ρ denotes the primitive ray generator of the ray $\rho \in \Sigma_{\mathcal{P}}(1)$.

Chapter 6

Gorenstein Fano Degenerations and Representation Theory

It is a natural question to ask whether a given Gorenstein Fano variety admits a toric degeneration to a toric Gorenstein Fano variety. However, this question is very hard to answer.

We have seen in Corollary 5.3.1 — alternatively in Corollary 4.1.2 for the case of flag varieties and Newton-Okounkov bodies — that the rational convex polytope $\mathcal{P}$ associated to the toric limit $(X_{\mathcal{P}}, D_{\mathcal{P}})$ of a Gorenstein Fano variety $(X, \mathcal{L})$ is reflexive (up to translation by a lattice vector) if and only if $\mathcal{P}$ is a lattice polytope and $\mathcal{L} \simeq \omega_X^{-1}$.

By Theorem 2.3.40 and Remark 2.3.41 we know that $X_{\mathcal{P}}$ must be Gorenstein Fano if $\mathcal{P}$ is reflexive (up to translation by a lattice vector). Sadly, the converse does not hold as we have seen in Remark 2.3.42, simply because we can translate and dilate reflexive polytopes without changing their normal fan and hence their toric variety.

Nevertheless it seems reasonable to search for reflexive polytopes that appear in the setting of toric degenerations of Gorenstein Fano varieties because this would already give us a lot of examples. The advantage here is that we can restrict our studies to toric degenerations of Gorenstein Fano varieties with their anticanonical polarization. So we might restrict ourselves to the question, whether every Gorenstein Fano pair (X, ω_X^{-1}) admits a toric degeneration to a pair $(X_{\mathcal{P}}, D_{\mathcal{P}})$ such that $\mathcal{P}$ is a lattice polytope.

Still, this is a very hard question to answer since even for very easy polytopes it might be quite challenging to actually compute their vertices.

So we will finally restrict ourselves to the the case of flag varieties, where many different polytopes are known and we have more tools to calculate their vertices.

Throughout this chapter we will use notation from Chapter 4.

6.1 Some Sufficient Conditions

We know from Theorem 3.3.10 that every *reasonably nice* Newton-Okounkov body will yield a toric degeneration in our sense. So it is important to notice which polytopes in representation theory can be described as Newton-Okounkov bodies. The same holds true for polytopes constructed via birational sequences (see [21, Theorem 6]).

We will now introduce many polytopes known in representation theory without actually defining them. Instead we will give references to the papers where they were originally constructed.

Notation 6.1.1. Let G be a complex simple algebraic group and let $\lambda \in \Lambda^+$ be a dominant integral weight of the Lie algebra $\mathfrak{g}$ of G.

(i) If G is of type A_n, the **Gelfand-Tsetlin polytope** introduced in [27] will be denoted by $GT(\lambda)$.

(ii) The **Berenstein-Littelmann-Zelevinsky string polytope** associated to the reduced decomposition $\underline{w_0}$ of the longest word w_0 of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ introduced in [47] and [9] will be denoted by $\mathcal{Q}_{\underline{w_0}}(\lambda)$.

(iii) The **Nakashima-Zelevinsky string polytope** associated to the reduced decomposition $\underline{w_0}$ of the longest word w_0 of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ introduced in [49] will be denoted by $NZ_{\underline{w_0}}(\lambda)$.

(iv) The **Lusztig polytope** associated to the reduced decomposition $\underline{w_0}$ of the longest word w_0 of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ introduced in [48] will be denoted by $L_{\underline{w_0}}(\lambda)$.

(v) If G is of type A_n or C_n, the **Feigin-Fourier-Littelmann-Vinberg polytope** constructed in [22] and [23] following a conjecture by Vinberg will be denoted by $FFLV(\lambda)$.

(vi) If G is of type G_2, the **Gornitskii polytope** introduced in [29] will be denoted by $G(\lambda)$.

Remark 6.1.2. Whenever we say *string polytope*, we usually refer to the Berenstein-Littelmann-Zelevinsky string polytope.

Remark 6.1.3. Littelmann proved in [47, Corollary 5.2] that $GT(\lambda)$ is unimodularly equivalent to a string polytope. In fact, he proved even more. We will recall further results in Section 6.4.

Remark 6.1.4. We will use the following convention. Let G/P be a partial flag variety and suppose that $\lambda \in \Lambda_P^+$. Let $\mathcal{P}(\lambda)$ be one of the polytopes in Notation 6.1.1. Then $0 \in \mathcal{P}(\lambda)$, so its affine hull will just be the linear subspace generated by $\mathcal{P}(\lambda)$. Additionally, $\dim \mathcal{P}(\lambda) = \dim G/P$ but technically this polytope might be defined as a polytope in $\mathbb{R}^{\dim G/B}$. To avoid technicalities, we will identify the polytope $\mathcal{P}(\lambda)$ with its image under the natural projection onto the subspace generated by $\mathcal{P}(\lambda)$. Hence we can always assume that our polytope is full-dimensional.

The following summary of known results justifies that the main setting of this thesis applies to all of these polytopes. It follows directly from the results in [21] — especially [21, Theorem 6].

Theorem 6.1.5. *Let G/P be a partial flag variety and $\lambda \in \Lambda_p^+$. Let $\mathcal{P}(\lambda)$ be one of the polytopes from Notation 6.1.1. Then the pair $(G/P, \mathcal{L}_\lambda)$ admits a toric degeneration in our sense to the toric pair $(X_{\mathcal{P}(\lambda)}, D_{\mathcal{P}(\lambda)})$.*

Remark 6.1.6. It should be noted that one could also get this result from Anderson's Theorem (see Theorem 3.3.10). For that purpose one needs to show that all polytopes in question can be realized as Newton-Okounkov bodies via a valuation that is *nice enough* (i.e. it should have full-rank and the associated valuation semigroup should be finitely generated and saturated). This has been done by Okounkov for $GT(\lambda)$ [51], Kaveh for $\mathcal{Q}_{\underline{w_0}}(\lambda)$ [44], Fujita and Naito for $NZ(\lambda)$ [25], Kiritchenko for $FFLV(\lambda)$.

From Theorem 6.1.5 and Corollary 5.3.1 we get the following consequence.

Corollary 6.1.7. *Let G/P be a partial flag variety and let $\lambda \in \Lambda_P^+$ be a P-regular dominant integral weight. Let $\mathcal{P}(\lambda)$ be one of the polytopes in Notation 6.1.1. Then $\mathcal{P}(\lambda)$ is reflexive up to translation by a lattice vector if and only if it is a lattice polytope and λ is the weight of the anticanonical line bundle over G/P.*

Definition 6.1.8. Let G/P be a partial flag variety. We will call the weight of the anticanonical line bundle over G/P the **anticanonical weight** for G/P and denote it by $\lambda_{G/P}$.

So the question remains, which of the formerly mentioned polytopes are lattice polytopes (at least for $\lambda_{G/P}$).

Ardila, Bliem and Salazar proved that the polytopes $GT(\lambda)$ and $FFLV(\lambda)$ can be realized as so called *marked order* and *marked chain polytopes*, thereby proving the following in [3, Lemma 3.5].

Theorem 6.1.9. *The polytopes $GT(\lambda)$ and $FFLV(\lambda)$ are lattice polytopes for every $\lambda \in \Lambda^+$.*

So we get the immediate corollary (see also Corollary 4.1.3).

Corollary 6.1.10. *Let G/P be a partial flag variety of type A_n or C_n and $\lambda \in \Lambda_P^+$. The polytopes $GT(\lambda)$ and $FFLV(\lambda)$ are reflexive after translation by a lattice vector if and only if the weight λ is the anticanonical weight $\lambda_{G/P}$.*

6.2 String Polytopes

We will now focus our attention on string polytopes and try to find as much lattice string polytopes as possible. In the following we will see that this is not an easy task — even in supposedly easy cases. We try to answer the following question.

Question. *Let G/P be a partial flag variety and let $\underline{w_0}$ be a reduced decomposition of the longest word in the Weyl group. Is $\mathcal{Q}_{\underline{w_0}}(\lambda_{G/P})$ a lattice polytope?*

The examples in this section have been made public before in [61].

We will start with the *easiest* case of standard reduced decompositions.

Definition 6.2.1. The **standard reduced decomposition** $\underline{w_0}^{\mathrm{std}}$ of the longest word of the Weyl group of a simple Lie algebra is the reduced decomposition described in [47]. The corresponding string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ will be called the **standard string polytope** for $\lambda \in \Lambda^+$.

However, we will not use the enumeration of positive roots from [47] but rather stick to the more common notation from [7].

Example 6.2.2. If G is of type A_n, we have

$$\underline{w_0}^{\mathrm{std}} = (s_1) \cdot (s_2 s_1) \cdots (s_n s_{n-1} \cdots s_1).$$

6.2.1 Examples in Type A_n

As mentioned before, these standard string polytopes are actually known in type A_n. The following result can be found in [47, Corollary 5.2].

Theorem 6.2.3. *In type A_n the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ is unimodularly equivalent to the Gelfand-Tsetlin polytope $GT(\lambda)$ for every $\lambda \in \Lambda^+$.*

From Theorem 6.1.9 we see the following.

Corollary 6.2.4. *In type A_n the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ is a lattice polytope for every $\lambda \in \Lambda^+$.*

So it is natural to consider generalizations of this observation either to arbitrary reduced decompositions or to other types. Let us start by studying other reduced decompositions.

The following conjecture has been posed by Alexeev and Brion in [1, Conjecture 5.8]. They were able to prove it for $n \leq 4$.

Conjecture 6.2.5 (Alexeev–Brion)**.** *For G of type A_n and any reduced decomposition $\underline{w_0}$, the string polytope $Q_{\underline{w_0}}(\lambda)$ is a lattice polytope for every $\lambda \in \Lambda^+$.*

This conjecture has been generally believed to be true for quite some time. But it actually turns out to be false as the following example shows.

Example 6.2.6. Let $G = \mathrm{SL}_6$ and consider the Grassmannian $G/P = \mathrm{Gr}(3,6)$. Choose the reduced decomposition $\underline{w_0} = s_1s_3s_2s_1s_3s_2s_4s_3s_2s_1s_5s_4s_3s_2s_1$. Notice that this reduced decomposition arises from the standard reduced decomposition of [47] by applying two 3-moves (and two 2-moves). Hence we have multiple ways of calculating the string polytopes in addition to the construction by Berenstein and Zelevinsky in [9, Theorem 3.14]. We find that the vertices of $\mathcal{Q}_{\underline{w_0}}(\omega_3)$ are the rows of the matrix

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

So there exists a non-integral vertex! Luckily the non-integral vertex has half-integral coordinates, so the corresponding string polytope for the anticanonical weight $\lambda_{\mathrm{Gr}(3,6)} = 6\omega_3$ is again a lattice polytope.

But this magic trick does not happen every time, since we can enlarge this example in A_5 to a whole class of examples for arbitrary rank by using the reduced decomposition $\underline{w_0} = (s_1s_3s_2s_1s_3s_2)(s_4s_3s_2s_1)(s_5s_4s_3s_2s_1)\cdots(s_ns_{n-1}\cdots s_2s_1)$. The respective string polytope $\mathcal{Q}_{\underline{w_0}}(\omega_3)$ will never be a lattice polytope for $n \geq 5$. In particular for $n = 6$ we can calculate that $\mathcal{Q}_{\underline{w_0}}(\omega_3)$ has half-integral vertices. Thus even for the anticanonical weight $\lambda_{\mathrm{Gr}(3,7)} = 7\omega_3$ over $\mathrm{Gr}(3,7)$ the string polytope $Q_{\underline{w_0}}(7\omega_3) = 7 \cdot Q_{\underline{w_0}}(\omega_3)$ will not be a lattice polytope.

Remark 6.2.7. It seems that this observation is connected to the fact that the string polytopes for the reduced decomposition $\underline{w_0} = s_1s_3s_2s_1s_3s_2$ in A_3 do not fulfill the *Minkowski property* (also called *Integral Decomposition Property*), i.e. for arbitrary $\lambda, \mu \in \Lambda^+$ the lattice points in the string polytope $\mathcal{Q}_{\underline{w_0}}(\lambda+\mu)$ cannot be written as sums of lattice points from $\mathcal{Q}_{\underline{w_0}}(\lambda)$ and $\mathcal{Q}_{\underline{w_0}}(\mu)$. This implies that there exists $\lambda \in \Lambda^+$ such that $\mathcal{Q}_{\underline{w_0}}(\lambda)$ contains lattice points that are not sums of lattice points of the fundamental string polytopes. And although A_3 and A_4 are too small to create non-integral string polytopes, this already foreshadows that something interesting might happen for higher rank.

We will briefly return to Minkowski properties in Section 6.5.

Remark 6.2.8. In [54] Rietsch and Williams constructed Newton-Okounkov bodies for Grassmannians using plabic graphs. In some cases their construction leads to non-integral polytopes — the first two appearing for the same Grassmannian $\mathrm{Gr}(3, 6)$. Both of these polytope have a single non-integral vertex as well.

I want to thank Valentin Rappel for pointing out this remarkable similarity.

In [61] we posed the question whether the string polytope from Example 6.2.6 and the respective Rietsch-Williams polytopes are actually unimodularly equivalent. Joint with Lara Boßinger we were actually able to show that this is in fact true for one of the two Rietsch Williams polytopes but not true for the other one.

It would be very interesting to understand the reason behind this sporadic equivalence.

6.2.2 Examples in Other Types

So we have seen that in type A_n only non-standard reduced decomposition can — and indeed will — give rise to non-integral string polytopes. In other types the situation is even more challenging since the standard reduced decompositions of [47] will already provide those as we will see in the next example.

Example 6.2.9. Let G be of type B_2 and choose $\underline{w_0}$ to be the standard reduced decomposition from [47, Section 6], which is $\underline{w_0} = s_2 s_1 s_2 s_1$, where α_2 denotes the short root. Let $\lambda = \omega_2$. The corresponding string polytope is then given by

$$Q_{\underline{w_0}}(\omega_2) = \left\{ x \in \mathbb{R}^4 \,\middle|\, A \cdot \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0 \right\},$$

where A is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 2 & -2 & 2 \\ 0 & 0 & -1 & 1 & -2 \\ 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The irreducible $\mathfrak{g}$-representation $V(\omega_2)$ is 4-dimensional and indeed we find four adapted strings — i.e. lattice points in the string polytope — given by the rows of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

But the rank of this matrix is equal to 2, which is strictly smaller than the dimension $\dim \mathcal{Q}_{\underline{w_0}}(\omega_2) = \dim G/P(\alpha_1) = 3$. Thus the string polytope cannot be a lattice polytope because its lattice points only span a proper subspace of the affine hull of $\mathcal{Q}_{\underline{w_0}}(\omega_2)$.

Indeed one can calculate that the vertices are the rows of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Since these vertices have at worst half-integral coordinates, we see that the string polytope for the weight of the anticanonical bundle $\lambda_{G/P(\alpha_1)} = 4\omega_2$ over $G/P(\alpha_1)$ will be a lattice polytope and by our theorem reflexive after translation by the lattice vector $(1, 2, 3, 0)^T$.

In contrast to our previous example in type A_n, this observation seems to holds for arbitrary rank.

Remark 6.2.10. Example 6.2.9 contradicts [1, Theorem 4.5], which claims that the string polytope for any (co)minuscule weight and any reduced decomposition must be a lattice polytope. Peter Littelmann and Michel Brion were able to solve this contradiction by finding a fault in the proof of said claim. Essentially the problem arises by applying a result of Caldero and Littelmann on standard monomials. In the proof of [1, Theorem 4.5], the authors construct a sequence of subwords of the longest word of the Weyl group of the form

$$\underline{w_0} = s_{i_1} \cdots s_{i_N} \geq s_{i_{j_1}} \cdots s_{i_N} \geq \ldots \geq s_{i_{j_n}} \cdots s_{i_N}$$

but the result of Caldero and Littelmann would actually require a sequence of the form

$$\underline{w_0} = s_{i_1} \cdots s_{i_N} \geq s_{i_1} \cdots s_{i_{k_1}} \geq \ldots \geq s_{i_1} \cdots s_{i_{k_n}}.$$

I want to thank Peter Littelmann and Michel Brion for explaining this problem.

In the exceptional cases the situation is even more unclear as we will see in our final example.

Example 6.2.11. Let G be of type G_2. Consider the anticanonical bundle over the full flag variety G/B. We choose $\underline{w_0} = \underline{w_0}^{\text{std}} = s_1s_2s_1s_2s_1s_2$ starting with the short root. Following [47, Section 2] and in analogous notation to Example 6.2.9 the string polytope $\mathcal{Q}_{\underline{w_0}}(2\rho)$ is given by the matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 3 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & -3 & 0 & 0\\
0 & 0 & 0 & 0 & 3 & -2 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -3\\
0 & 0 & 0 & 0 & 0 & 0 & 1\\
2 & -1 & 3 & -2 & 3 & -2 & 3\\
2 & 0 & -1 & 1 & -2 & 1 & -2\\
2 & 0 & 0 & -1 & 3 & -2 & 3\\
2 & 0 & 0 & 0 & -1 & 1 & -2\\
2 & 0 & 0 & 0 & 0 & -1 & 3\\
2 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.$$

One calculates that the vertices of $\mathcal{Q}_{\underline{w_0}}(2\rho)$ are the rows of the matrix

$$\begin{bmatrix}
0 & 2/3 & 2 & 4/3 & 2 & 0\\
0 & 10/3 & 10 & 4 & 2 & 0\\
0 & 10/3 & 10 & 6 & 8 & 2\\
0 & 8/3 & 8 & 2 & 0 & 0\\
0 & 2/3 & 2 & 0 & 0 & 0\\
0 & 8/3 & 8 & 4 & 6 & 2\\
0 & 8/3 & 8 & 16/3 & 8 & 2\\
10 & 4 & 2 & 0 & 0 & 0\\
4 & 4 & 8 & 4 & 6 & 2\\
2 & 10/3 & 8 & 16/3 & 8 & 2\\
8 & 2 & 0 & 0 & 0 & 0\\
8 & 10/3 & 2 & 4/3 & 2 & 0\\
8 & 6 & 10 & 4 & 2 & 0\\
2 & 2 & 6 & 4 & 6 & 2\\
4 & 2 & 6 & 4 & 2 & 0\\
1 & 3 & 9 & 6 & 8 & 2\\
5 & 1 & 3 & 2 & 0 & 0\\
2 & 4 & 10 & 6 & 8 & 2\\
10 & 6 & 8 & 2 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & 2 & 0 & 0 & 0\\
0 & 4 & 8 & 4 & 6 & 2\\
0 & 10/3 & 8 & 16/3 & 8 & 2\\
0 & 2 & 0 & 0 & 0 & 0\\
0 & 10/3 & 2 & 4/3 & 2 & 0\\
0 & 6 & 10 & 4 & 2 & 0\\
0 & 2 & 6 & 4 & 6 & 2\\
0 & 2 & 6 & 4 & 2 & 0\\
0 & 3 & 9 & 6 & 8 & 2\\
0 & 1 & 3 & 2 & 0 & 0\\
0 & 4 & 10 & 6 & 8 & 2\\
0 & 6 & 8 & 2 & 0 & 0
\end{bmatrix}.$$

Hence $\mathcal{Q}_{\underline{w_0}}(2\rho)$ is not a lattice polytope and thus not reflexive even after translation by the unique interior lattice point $(1,2,5,3,4,1)^T$.

In fact one calculates easily that for all but one combination of parabolics and reduced decompositions, the respective anticanonical string polytope will not be a lattice polytope. The only exception is the lattice polytope $\mathcal{Q}_{s_2s_1s_2s_1s_2s_1}(2\rho)$.

6.2.3 A Conjecture on Standard String Polytopes

Based on the above examples and further calculations, we have established the following conjecture publicly in [61, Conjecture 7.9].

Conjecture 6.2.12. *Let G be a complex classical group, let $\lambda \in \Lambda^+$ and let $\underline{w_0}^{\mathrm{std}}$ be the standard reduced decomposition of the longest word of the Weyl group of G as stated in [47]. Then the string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ is a lattice polytope if and only if one of the following conditions holds.*

(i) *G is of type A_n,*

(ii) *G is of type B_n and $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$,*

(iii) *G is of type C_n or*

(iv) *G is of type D_n and $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$ or $n < 4$.*

Notice the slight rank restriction in contrast to our original conjecture. The remainder of this thesis is dedicated to proving this conjecture.

6.3 When is a Standard String Polytope a Lattice Polytope?

Without further ado, we will state our theorem.

Theorem 6.3.1. *Let G be a complex classical group with Lie algebra $\mathfrak{g}$ and $\lambda \in \Lambda^+$. The standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ (in the sense of [47]) is a lattice polytope if and only if the $\mathfrak{g}$-representation on $V(\lambda)$ integrates to a representation of G.*

Before proving this theorem, let us show two consequences.

Corollary 6.3.2. *Conjecture 6.2.12 holds.*

Proof. If G is simply connected — i.e. G is of type A_n or C_n — it is known (see for example [53, Chapter 10, Theorem 6.1]) that the irreducible representations of the Lie algebra $\mathfrak{g}$ of G are in one-to-one correspondence with the irreducible representations of G. So there are no further restrictions on λ in these types.

If G however is not simply connected, i.e. $G = \mathrm{SO}_n$, it is known that not every irreducible representation of $\mathfrak{so}_n$ integrates to a representation of SO_n. Instead, in general it integrates only to a representation of the spin group Spin_n as the universal covering of SO_n. However, in some cases $V(\lambda)$ will still integrate to a representation of SO_n. By [53, Chapter 11, Theorem 6.6] these cases are precisely the ones listed in Conjecture 6.2.12. □

From this proof we see that the claims in Conjecture 6.2.12 and Theorem 6.3.1 are equivalent. In fact we will prove the theorem by proving the conjecture in Section 6.6.

Another important consequence is the following.

Corollary 6.3.3. *Let G be a complex classical group with Lie algebra $\mathfrak{g}$ and $\lambda \in \Lambda^+$. The standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ (in the sense of [47]) is a reflexive polytope after translation by a lattice vector if and only if λ is the anticanonical weight $\lambda_{G/P}$ of some partial flag variety G/P.*

Proof. Notice that the anticanonical line bundle over G/P can be realized as the highest wedge power of the cotangent bundle over G/P, i.e.

$$\mathcal{L}_{\lambda_{G/P}} = \bigwedge^{\dim G/P} (\mathfrak{g}/\mathfrak{p})^*.$$

From this it is clear that $V(\lambda_{G/P})^* \simeq H^0(G/P, \mathcal{L}_{\lambda_{G/P}})$ carries the structure of a G-representation. So by Theorem 6.3.1 we know that $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda_{G/P})$ must be a lattice polytope.

The claim now follows directly from Corollary 6.1.7. □

Remark 6.3.4. It should be noted that morally it is absolutely not clear why the string polytope should now *anything* about the representations of the underlying algebraic group. Firstly, its definition and many of its explicit descriptions are done purely from the perspective of the Lie algebra (see [47] and [9]). Secondly, we have already seen in Example 6.2.6 that this connection does not hold for arbitrary reduced decompositions. So the connection between the standard reduced decomposition and representations of the algebraic group remains mysterious.

6.4 Cones, Crystals, Patterns

Before proving Theorem 6.3.1 we will recall so called *(Generalized) Gelfand-Tsetlin Patterns* introduced by Berenstein and Zelevinsky in [8]. We will widely stick to the notation in [47] although we will make slight adjustments.

We will use the following shorthand notation.

Notation 6.4.1. For two numbers $a, b \in \mathbb{R}$ the inequality $a \geq b$ will be written graphically as

$$\begin{array}{cc} a & \\ & b \end{array} \quad \text{or} \quad \begin{array}{cc} & b \\ a & \end{array}.$$

We will now define Gelfand-Tsetlin patterns for all classical types.

Definition 6.4.2 (Gelfand-Tsetlin Patterns in Type A_n)**.** Let G be of type A_n and $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \Lambda^+$. A **Gelfand-Tsetlin pattern** of type λ is a tuple $(y_{i,j}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$, $1 \leq i \leq n$, $i \leq j \leq n$, such that the coordinates fulfill the relations in Figure 6.1.

Figure 6.1: Inequalities of Gelfand-Tsetlin Patterns in type A_n.

$$\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \cdots & & \cdots & & \lambda_n & & 0 \\
& y_{1,1} & & y_{1,2} & & \cdots & & \cdots & & y_{1,n-1} & & y_{1,n} & \\
& & y_{2,2} & & y_{2,3} & & \cdots & & y_{2,n-1} & & y_{2,n} & & \\
& & & \ddots & & \ddots & & \iddots & & \iddots & & & \\
& & & & y_{n-2,n-2} & & y_{n-2,n-1} & & y_{n-2,n} & & & & \\
& & & & & y_{n-1,n-1} & & y_{n-1,n} & & & & & \\
& & & & & & y_{n,n} & & & & & &
\end{array}$$

Remark 6.4.3. Notice that in Littelmann's definition of type A_n Gelfand-Tsetlin patterns, the top row would be included in the tuple $(y_{i,j})$ as the initial row (i.e. $y_{0,0} = \lambda_1, \ldots, y_{0,n-1} = \lambda_n, y_{0,n} = 0$). However, for fixed λ this does only change the embedding of the pattern and not the pattern itself. So we adapted the definition to embed our patterns in a vector space whose dimension equals the number of positive roots of the algebraic group G. Additionally, these entries have a different character (we will call these entries a *marking* later on), so we would like to treat them separately.

Definition 6.4.4 (Gelfand-Tsetlin Patterns in Types B_n and C_n)**.** Let G be of type B_n or C_n and $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \Lambda^+$. A **Gelfand-Tsetlin pattern** of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq n$, $i \leq j \leq n$, and $\mathbf{z} = (z_{i,j}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$, $1 \leq i \leq n$, $i \leq j \leq n$, such that the coordinates fulfill the relations in Figure 6.2.

Remark 6.4.5. Notice that in a Gelfand-Tsetlin pattern $(\mathbf{y}, \mathbf{z})$ of type B_n or C_n the first row as well as the zeroes in the last column are not actually part of

Figure 6.2: Inequalities of Gelfand-Tsetlin patterns in types B_n and C_n.

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \dots & & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n & & 0 \\
& z_{1,1} & & z_{1,2} & & \dots & & z_{1,n-2} & & z_{1,n-1} & & z_{1,n} & \\
& & y_{2,2} & & y_{2,3} & & \dots & & y_{2,n-1} & & y_{2,n} & & 0 \\
& & & z_{2,2} & & z_{2,3} & & \dots & & z_{2,n-1} & & z_{2,n} & \\
& & & & y_{3,3} & & y_{3,4} & & \dots & & y_{3,n} & & 0 \\
& & & & & z_{3,3} & & z_{3,4} & & \dots & & z_{3,n} & \\
& & & & & & \ddots & & \ddots & & \ddots & & \\
& & & & & & & z_{n-2,n-2} & & z_{n-2,n-1} & & z_{n-2,n} & \\
& & & & & & & & y_{n-1,n-1} & & y_{n-1,n} & & 0 \\
& & & & & & & & & z_{n-1,n-1} & & z_{n-1,n} & \\
& & & & & & & & & & y_{n,n} & & 0 \\
& & & & & & & & & & & z_{n.n} & \\
& & & & & & & & & & & & 0
\end{array}
$$

the tuple $(\mathbf{y}, \mathbf{z})$. In Littelmann's definition, the first row would be included as $y_{1,1} = \lambda_1, \dots, y_{1,n} = \lambda_n$. The reasons for our change of definition are the same as in type A_n. Otherwise we want to stick with his notation, which yields to the awkward fact that our tuple $(\mathbf{y})$ starts with the index $i = 2$. However, we will not need these indices explicitly, so this should not become a problem.

Definition 6.4.6 (Gelfand-Tsetlin Patterns in Types D_n)**.** Let G be of type D_n and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. A **Gelfand-Tsetlin pattern** of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq n$, $i \leq j \leq n$, and $\mathbf{z} = (z_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $1 \leq i \leq n-1$, $i \leq j \leq n-1$, such that

$$
\begin{aligned}
z_{1,n-1} &\leq \lambda_n + y_{2,n} + \min\{\lambda_{n-1}, y_{2,n-1}\}, \\
z_{i,n-1} &\leq y_{i,n} + y_{i+1,n} + \min\{y_{i,n-1}, y_{i+1,n-1}\} \text{ for all } 2 \leq i \leq n-2, \\
z_{n-1,n-1} &\leq y_{n-1,n} + y_{n,n} + y_{n-1,n-1},
\end{aligned}
$$

and the coordinates fulfill the relations in Figure 6.3.

Remark 6.4.7. As before we deviate from Littelmann's notation by not including the initial row containing the λ_j in our notion of a Gelfand-Tsetlin pattern. For unifying notation of the additional inequalities, it is understood

Figure 6.3: Inequalities of Gelfand-Tsetlin patterns in type D_n.

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{n-3} & & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n \\
& z_{1,1} & & z_{1,2} & & \cdots & & z_{1,n-3} & & z_{1,n-2} & & z_{1,n-1} & \\
& & y_{2,2} & & y_{2,3} & & \cdots & & y_{2,n-2} & & y_{2,n-1} & & y_{2,n} \\
& & & z_{2,2} & & z_{2,3} & & \cdots & & z_{2,n-2} & & z_{2,n-1} & \\
& & & & y_{3,3} & & y_{3,4} & & \cdots & & y_{3,n-1} & & y_{3,n} \\
& & & & & z_{3,3} & & z_{3,4} & & \cdots & & z_{3,n-1} & \\
& & & & & & \ddots & & \ddots & & \ddots & & \\
& & & & & & & z_{n-3,n-3} & & z_{n-3,n-2} & & z_{n-3,n-1} & \\
& & & & & & & & y_{n-2,n-2} & & y_{n-2,n-1} & & y_{n-1,n} \\
& & & & & & & & & z_{n-2,n-2} & & z_{n-2,n-1} & \\
& & & & & & & & & & y_{n-1,n-1} & & y_{n,n} \\
& & & & & & & & & & & z_{n-1,n-1} & \\
& & & & & & & & & & & & y_{n,n}
\end{array}
$$

that we mean λ_j if we write $y_{1,j}$.

One can see quite easily that the set of all Gelfand-Tsetlin patterns for a given weight is a polytope.

Definition 6.4.8. Let G be a complex classical group of type X_n and let $\lambda \in \Lambda^+$. Let N denote the number of positive roots of G. The set of all possible Gelfand-Tsetlin patterns of type λ is called the **Gelfand-Tsetlin polytope** $GT_{\mathsf{X}_n}(\lambda) \subseteq \mathbb{R}^N$ of X_n and λ. We will sometimes omit the subscript X_n if this is clear from the context.

Sometimes we might be interested to use the original definitions instead. So we introduce the following notation.

Definition 6.4.9. Let G be of type X_n and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. Let N denote the number of positive roots of G. Let $x \in GT_{\mathsf{X}_n}(\lambda) \subseteq \mathbb{R}^n$. The **extended Gelfand-Tsetlin pattern** $\hat{x}$ of x is defined as

$$
\hat{x} = \begin{cases} \{(\lambda_1, \ldots, \lambda_n, 0)\} \times x \in \mathbb{R}^{n+1} \times \mathbb{R}^N & \text{if } \mathsf{X}_n = \mathsf{A}_n \\ \{(\lambda_1, \ldots, \lambda_n)\} \times x \in \mathbb{R}^n \times \mathbb{R}^N & \text{else.} \end{cases}
$$

We will now recall the connection between Gelfand-Tsetlin polytopes and standard string polytopes. For that purpose we will define a non-standard terminology.

Definition 6.4.10. Let G be a complex classical group. A Gelfand-Tsetlin pattern x is called **standard** if one of the following conditions hold.

(i) G is of type A_n and all coordinates of $\hat{x}$ are integral.

(ii) G is of type B_n, the $z_{1,n}, \ldots, z_{n,n}$ are in $\frac{1}{2}\mathbb{Z}$ and the other coordinates of $\hat{x}$ are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.

(iii) G is of type C_n and all coordinates of $\hat{x}$ are integral.

(iv) G is of type D_n and the coordinates of $\hat{x}$ are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.

The following is a combination of [47, Corollary 5 and Corollary 7].

Theorem 6.4.11 (Littelmann)**.** *Let G be a complex classical group of type $\mathsf{X}_n \neq \mathsf{D}_n$ and let N denote the number of positive roots. For each dominant integral weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ there exists an affine bijection $\phi_\lambda \colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\phi_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)) = GT_{\mathsf{X}_n}(\lambda)$.*

Furthermore, ϕ_λ induces a bijection between the lattice points in $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ and the standard X_n-Gelfand-Tsetlin patterns of type λ.

Remark 6.4.12. Interestingly, in type A_n Cho, Kim, Lee and Park gave a combinatorial classification of all reduced decompositions whose string polytope is unimodularly equivalent to the Gelfand-Tsetlin polytope in [11].

In type D_n the situation is more delicate as can be seen in [47, Corollary 9]. In this case the map ϕ_λ will only be piecewise affine.

Theorem 6.4.13 (Littelmann)**.** *Let $G = SO_{2n}$ and let N denote the number of positive roots. For each dominant integral weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ there exists a piecewise affine bijection $\phi_\lambda \colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\phi_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)) = GT_{\mathsf{D}_n}(\lambda)$.*

Furthermore, ϕ_λ induces a bijection between the lattice points in $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ and the standard D_n-Gelfand-Tsetlin patterns of type λ.

The non-affineness will lead to difficulties in proving Theorem 6.3.1, which we will overcome by defining new Gelfand-Tsetlin polytopes for type D_n in Section 6.8.

6.5 Marked Order Polytopes

We will now introduce generalized versions of Stanley's order polytopes. The definition of the order polytope associated to a poset is due to Stanley [59]. A *marking* on the poset lead to a generalization by Ardila, Bliem and Salazar in [3]. These polytopes have been studied and further generalized by Fang and Fourier in [20].

Definition 6.5.1. Let $(P, \leq)$ be a finite poset, i.e. P is a finite set with a partial order $\leq$ on P.

(i) The **Hasse diagram** of P is a directed graph whose set of nodes is P and there is an arrow $p \to q$ whenever $p < q$ and there exists no r with $p < r < q$.

(ii) A **marking** on P is a pair (A, λ) where A is a subset of P containing all minimal and maximal elements of P and $\lambda = (\lambda_a)_{a\in A} \in \mathbb{R}^A$ is a real vector such that $\lambda_a \leq \lambda_b$ whenever $a \leq b$. The triplet (P, A, λ) is called a **marked poset**. We will call the elements of A **marked elements**.

(iii) Let (A, λ) be a marking on P. The **marked order polytope** $\mathcal{O}_{P,A}(\lambda)$ associated to (P, A, λ) is defined as

$$\mathcal{O}_{P,A}(\lambda) := \left\{ x \in \mathbb{R}^{P\setminus A} \;\middle|\; \begin{array}{c} x_p \leq x_q \text{ for all } p \leq q, \\ \lambda_a \leq x_p \text{ for all } a \leq p, \\ x_p \leq \lambda_b \text{ for all } p \leq b \end{array} \right\}.$$

Remark 6.5.2. The Gelfand-Tsetlin polytopes of types A_n, B_n and C_n are marked order polytopes where the marking is given by the dominant integral weight and some zeroes. The Gelfand-Tsetlin polytopes of type D_n however are not marked order polytopes because of the additional four-term inequalities in their definition.

The following theorem is due to Ardila, Bliem and Salazar. A different proof was given by Fang and Fourier. It can be found in [3, Lemma 3.5] and [20, Corollary 2.2].

Theorem 6.5.3 (Ardila–Bliem–Salazar, Fang–Fourier)**.** *Let (P, A, λ) be any marked poset. If $\lambda \in \mathbb{Z}^A$, the marked order polytope $\mathcal{O}_{P,A}(\lambda)$ is a lattice polytope.*

Fang's and Fourier's proof uses the so called *integral decomposition property*— short *IDP*. This property says that for any integer $m > 0$ we can write every lattice point in $\mathcal{O}_{P,A}(m\lambda)$ as a sum of m (possibly distinct) lattice points in $\mathcal{O}_{P,A}(\lambda)$. It should be noted that this property does only hold for integral and not arbitrary markings. However, their proofs can be adapted to an arbitrary group $l^{-1}\mathbb{Z}$, $l \in \mathbb{Z}_{>0}$, by using $(l^{-1}\mathbb{Z})^{P\setminus A}$ instead of $\mathbb{Z}^{P\setminus A}$ as our lattice. This would then result in the following statement.

Corollary 6.5.4. *Let (P, A, λ) be a marked poset. If $\lambda \in (l^{-1}\mathbb{Z})^A$, $l \in \mathbb{Z}_{>0}$, every vertex of the marked order polytope $\mathcal{O}_{P,A}(\lambda)$ lies in $(l^{-1}\mathbb{Z})^{P\setminus A}$.*

Another consequence is the following result on Gefand-Tsetlin polytopes.

Corollary 6.5.5. *Let G be a complex classical group of type $\mathsf{X}_n \neq \mathsf{D}_n$ and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. Then the Gelfand-Tsetlin polytope $GT_{\mathsf{X}_n}(\lambda)$ is a lattice polytope if $\mathsf{X}_n = \mathsf{A}_n$ or C_n or if $\mathsf{X}_n = \mathsf{B}_n$ and $\langle \lambda, \alpha_n^\vee \rangle \in 2\mathbb{Z}$.*

Proof. By Theorem 6.5.3 it is clear that $GT_{\mathsf{X}_n}(\lambda)$ will be a lattice polytope if all coordinates of the marking vector are integral. The set of non-zero coordinates is precisely $\{\lambda_1, \dots, \lambda_n\}$, so we need to know when the coefficients of a dominant integral weight written in the ϵ_i are integral. By [53, Chapter 10, Theorem 6.1, and Chapter 11, Theorem 6.6] this is always the case in types A_n and C_n. In type B_n however we could get half-integral coefficients. To be more precise, each λ_i can be written as the sum of some integers plus $\frac{\langle \lambda, \alpha_n^\vee \rangle}{2}$. So we see that the λ_i are integers if and only if $\langle \lambda, \alpha_n^\vee \rangle$ is an even integer, which concludes the proof. □

This corollary would allow us to prove one implication of the claims of Conjecture 6.2.12 and hence Theorem 6.3.1 for types A_n, B_n and C_n via Littelmann's affine bijection from Theorem 6.4.11. The reason is that the inverse of Littelmann's map sends vertices to vertices and lattice points in the Gelfand-Tsetlin patterns (notice that these are a only a proper subset of the standard Gelfand-Tsetlin patterns in type B_n) to lattice points of the standard string polytope.

Sadly, we cannot prove the *only-if*-part in type B_n directly and we simply cannot use these methods in the case D_n. Firstly, the Gelfand-Tsetlin polytope in type D_n is not a marked order polytope. Secondly, since Littelmann's bijection of Theorem 6.4.13 is only piecewise affine, it need not send vertices to vertices. Some vertices could be send to non-vertices and vice-versa.

From examples I reached the following conjecture which would at least solve this problem. But I could not find a proof.

Conjecture 6.5.6. *Let ϕ_λ be the map from Theorem 6.4.13, i.e. the piecewise affine bijection with $\phi_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)) = GT_{\mathsf{D}_n}(\lambda)$. Then ϕ_λ induces a bijection between* $\operatorname{vert} \mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ *and* $\operatorname{vert} GT_{\mathsf{D}_n}(\lambda)$.

So we will develop a tweaked version of Gelfand-Tsetlin patterns in type D_n that can be studied more easily. Additionally we will introduce a new method to classify vertices of these tweaked Gelfand-Tsetlin polytopes via diagrammatic combinatorics in Section 6.8.

We will also apply these methods to the other classical types, thereby reproving Corollary 6.5.5 and additionally the missing second implication of Conjecture 6.2.12 in type B_n.

6.6 Identity Diagrams

We will state our definitions for arbitrary marked posets. The reductions to the Gelfand-Tsetlin cases A_n, B_n and C_n are obvious.

Definition 6.6.1. Let (P, A, λ) be a marked poset and let $x \in \mathcal{O}_{P,A}(\lambda)$. The **identity diagram** $\mathcal{D}^{\lambda}_{P,A}(x)$ associated to (P, A, λ, x) is a graph that contains all nodes and arrows of the Hasse diagram of P. Additionally, we draw an arrow $q \to p$ between two nodes p and q whenever there exists an arrow $p \to q$ in the Hasse diagram of P and $x_p = x_q$ (if $p, q \neq A$) or $\lambda_p = x_q$ (if $p \in A$) or $x_p = \lambda_q$ (if $q \in A$).

Whenever we draw these identity diagrams, for simplicity we will represent double arrows $p \rightleftarrows q$ by straight lines and omit single arrows. From this practice we get the following non-standard terminology.

Definition 6.6.2. Let (P, A, λ) be a marked poset, let $x \in \mathcal{O}_{P,A}(\lambda)$ and let $\mathcal{D}^{\lambda}_{P,A}(x)$ be the associated identity diagram. A subset $\mathcal{C}$ of nodes is called **connected** if it is connected via double arrows, i.e. for any two nodes p and q in $\mathcal{C}$ there exists a (possibly empty) sequence $p_1, \ldots, p_t \in \mathcal{C}$ such that

$$p \rightleftarrows p_1 \rightleftarrows \ldots \rightleftarrows p_t \rightleftarrows q.$$

The maximal (with respect to inclusion) connected subsets are called **connected components**.

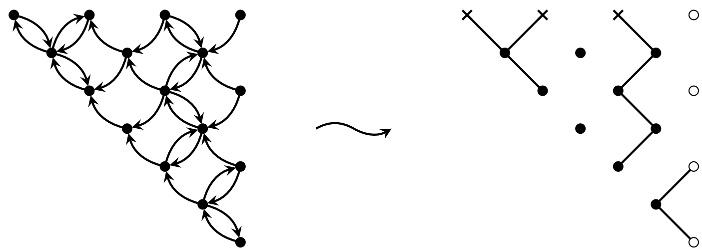
Additionally, when drawing identity diagrams for Gelfand-Tsetlin patterns we will represent the nodes corresponding to marked elements as follows. The zeros in the rightmost column will be drawn as small circles, while the nodes corresponding to the λ_i in the first row will be drawn as small crosses. This change is made for easier readability as the following example shows.

Example 6.6.3. Let G be of type C_4 and $\lambda = 2\epsilon_1 + 2\epsilon_2 + \epsilon_3 + 0\epsilon_4$. The pattern

$$\begin{array}{ccccccc}
2 & & 2 & & 1 & & 0 \\
& 2 & & \frac{3}{2} & & 1 & \\
& & 2 & & 1 & & 0 \\
& & & \frac{3}{2} & & 1 & \\
& & & & 1 & & 0 \\
& & & & & 0 & \\
& & & & & & 0
\end{array}$$

admits the identity diagram and its visually more appealing drawing depicted in Figure 6.4.

Figure 6.4: Identity diagram and visually more appealing drawing of the Gelfand-Tsetlin pattern in Example 6.6.3.



These diagrams give us an easy way to *draw* vertices of marked order polytopes as the following result shows.

Theorem 6.6.4. *Let (P, A, λ) be a marked poset. A point $x \in \mathcal{O}_{P,A}(\lambda)$ is a vertex of the marked order polytope if and only if every connected component of the associated diagram $\mathcal{D}^{\lambda}_{P,A}(x)$ contains a marked element, i.e. an element of A.*

Although this theorem sounds quite technical, it is actually quite practical, as the following consequence shows.

Corollary 6.6.5. *A point in the Gelfand-Tsetlin polytope is a vertex, if each entry of its Gelfand-Tsetlin pattern is equal to its upper left or upper right neighbor.*

To prove this result we need the following useful trick.

Lemma 6.6.6. *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polytope. Then $x \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only there does not exist a vector $v \in \mathbb{R}^d$, $v \neq 0$, such that $x + v \in \mathcal{P}$ and $x - v \in \mathcal{P}$.*

Proof. Let $\mathcal{P} = \{y \in \mathbb{R}^d \mid Ay \leq b\}$ for some matrix $A \in M_{r,d}(\mathbb{R})$ and vector $b \in \mathbb{R}^r$. Assume that x is a vertex of $\mathcal{P}$. Since every vertex can be written as the intersection of n facets (recall Theorem 2.2.11), there exists a (necessarily invertible) submatrix $\tilde{A} \in M_d(\mathbb{R})$ of A and a subvector $\tilde{b} \in \mathbb{R}^d$ of b such that

$$\{y \in \mathbb{R}^d \mid \tilde{A}y = \tilde{b}\} = \{x\}.$$

Now let $v \in \mathbb{R}^d$ and assume that $x + v \in \mathcal{P}$ and $x - v \in \mathcal{P}$. This implies that $\tilde{A}(x+v) \leq \tilde{b}$ and $\tilde{A}(x-v) \leq \tilde{b}$. Hence $\tilde{A}v \geq 0$ and $-\tilde{A}v \geq 0$, which is only possible if $\tilde{A}v = 0$. Since $\tilde{A}$ is invertible, this implies $v = 0$.

For the other implication, assume there exists a vector $v \in \mathbb{R}^d$, $v \neq 0$, such that $x + v \in \mathcal{P}$ and $x - v \in \mathcal{P}$. Let $\tilde{A}$ be the submatrix of A consisting of all rows α_i of A such that $\alpha_i \cdot x = b_i$ and let $\tilde{b}$ denote the corresponding subvector of b. Since $x + v \in \mathcal{P}$ and $x - v \in \mathcal{P}$ we know that $\tilde{A}(x+v) \leq \tilde{b}$ and $\tilde{A}(x-v) \leq \tilde{b}$. This implies $\tilde{A}v \leq 0$ and $-\tilde{A}v \leq 0$, i.e. $\tilde{A}v = 0$. Since $v \neq 0$, the rank of the matrix $\tilde{A}$ must be strictly smaller than d. Hence the set $\{y \in \mathbb{R}^d \mid \tilde{A}y = \tilde{b}\}$ has affine dimension strictly larger than zero, so x cannot be a vertex of $\mathcal{P}$. □

We will now prove our theorem on the vertices of marked order polytopes.

Proof of Theorem 6.6.4. Let x be a point in the marked order polytope $\mathcal{O}_{P,A}(\lambda)$. Suppose there exists a vector $v \in \mathbb{R}^{P \setminus A}$ such that $x + v$ and $x - v$ both lie in the marked order polytope. Let $p, q \in P \setminus A$, $p \leq q$, be two nodes of the identity diagram $\mathcal{D}^{\lambda}_{P,A}(x)$ such that $p \leftrightarrows q$. Hence we know that $x_p = x_q$.

Now since $x + v$ and $x - v$ lie in the marked order polytope we must have

$$x_p + v_p \leq x_q + v_q \quad \text{and} \quad x_p - v_p \leq x_q - v_q.$$

This implies that $v_p = v_q$. Continuing this argument yields $v_p = v_q$ for any two nodes p and q of the identity diagram lying in the same connected component.

Now let $\mathcal{C}$ be a connected component of $\mathcal{D}^{\lambda}_{P,A}(x)$. If $\mathcal{C}$ contains an element $a \in A$, we know that $x_p = \lambda_a$ for all $p \in \mathcal{C}$. Suppose that $\mathcal{C}$ is not completely contained in A. Then there exists a pair $p \leftrightarrows a$ with $p \in \mathcal{C}$ and $a \in A$. Without loss of generality let us assume that $p \leq a$. Then we know that

$$x_p + v_p \leq \lambda_a \quad \text{and} \quad x_p - v_p \leq \lambda_a,$$

which implies that $v_p = 0$.

In conclusion we see that $v_p = 0$ for all $p \in P \setminus A$ such that the connected component of p contains an element of A. By Lemma 6.6.6 this implies that x is a vertex whenever each connected component of $\mathcal{D}^{\lambda}_{P,A}(x)$ contains a marked element.

For the other implication let us assume there exists a connected component $\mathcal{C}$ of $\mathcal{D}^{\lambda}_{P,A}(x)$ that does not contain any marked element. By definition of identity diagrams this means that $x_p < x_q$ for any $p \in \mathcal{C}$ and $q \in \mathcal{P} \setminus (\mathcal{C} \cup A)$ such that $p \leq q$. Additionally $x_p < \lambda_a$ for any $p \in \mathcal{C}$ and $a \in A$ such that $p \leq a$. Analogous statements hold if $p \geq q$ or $p \geq a$. Since the poset is finite we can find $\epsilon > 0$ such that

$$\begin{aligned}
&x_p \pm \epsilon < x_q \text{ for all } p \in \mathcal{C} \text{ and } q \in \mathcal{P} \setminus (\mathcal{C} \cup A) \text{ such that } p \leq q,\\
&x_p \pm \epsilon > x_q \text{ for all } p \in \mathcal{C} \text{ and } q \in \mathcal{P} \setminus (\mathcal{C} \cup A) \text{ such that } p \geq q,\\
&x_p \pm \epsilon < \lambda_a \text{ for all } p \in \mathcal{C} \text{ and } a \in A \text{ such that } p \leq a,\\
&x_p \pm \epsilon > \lambda_a \text{ for all } p \in \mathcal{C} \text{ and } a \in A \text{ such that } p \geq a.
\end{aligned}$$

Consider the vector $v \in \mathbb{R}^{P \setminus A}$ defined by

$$v_p := \begin{cases} \epsilon & \text{if } p \in \mathcal{C} \\ 0 & \text{else.} \end{cases}$$

Then it is clear that $x + v$ and $x - v$ both lie in $\mathcal{O}_{P,A}(\lambda)$. By Lemma 6.6.6 this implies that x is not a vertex of $\mathcal{O}_{P,A}(\lambda)$. □

This description of the vertices yields the following implication.

Corollary 6.6.7. *Let (P, A, λ) be a marked poset. The coordinates of every vertex of the marked order polytope $\mathcal{O}_{P,A}(\lambda)$ must lie in the set $\{\lambda_a \mid a \in A\}$.*

Proof. Let $x \in \mathbb{R}^{P \setminus A}$ be a vertex of $\mathcal{O}_{P,A}(\lambda)$. From Theorem 6.6.4 we know that every connected component $\mathcal{C}$ of the identity diagram $\mathcal{D}^{\lambda}_{P,A}(x)$ of x contains an element $a \in A$. By definition this means that that $x_p = \lambda_a$ for all $p \in \mathcal{C}$, hence every coordinate of x must be equal to one of the λ_a. □

This gives another proof of Theorem 6.5.3.

We can now finally prove Theorem 6.3.1 in three types.

Proof of Theorem 6.3.1 in Types A_n, B_n and C_n. Let G be of types A_n, B_n or C_n. By Theorem 6.4.11 we know that the vertices of the standard string polytopes are in one-to-one correspondence with the vertices of the Gelfand-Tsetlin polytopes. We have seen that the Gelfand-Tsetlin polytopes are marked order polytopes. For $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \Lambda^+$ their marking is given by a vector whose coordinates are precisely the λ_i and some zeros.

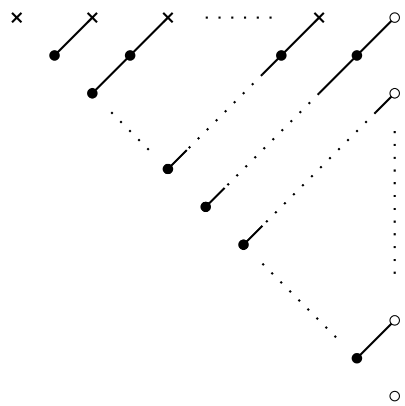
In types A_n and C_n we know that $\lambda_i \in \mathbb{Z}$ for every $1 \leq i \leq n$. By Corollary 6.6.7 (or alternatively Theorem 6.5.3) we know that the vertices of the corresponding Gelfand-Tsetlin polytope have integral coordinates, so via Littelmann's map in Theorem 6.4.11 they correspond to lattice points in the standard string polytope.

In type B_n the same argument holds if $\langle\lambda, \alpha_n^\vee\rangle \in 2\mathbb{Z}$. However, if $\langle\lambda, \alpha_n^\vee\rangle$ is an odd integer, we know that $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all $1 \leq i \leq n$. Now it is enough to notice that the pattern

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \dots & & \lambda_n & & 0 \\
& \lambda_2 & & \dots & & \lambda_n & & 0 & \\
& & \ddots & & \iddots & & \iddots & & 0 \\
& & & \lambda_n & & 0 & & & \\
& & & & 0 & & & & \vdots \\
& & & & & 0 & & & \\
& & & & & & \ddots & & 0 \\
& & & & & & & 0 & \\
& & & & & & & & 0
\end{array}$$

lies in $GT_{\mathsf{B}_n}(\lambda)$ for every $\lambda \in \Lambda^+$. Its identity diagram is drawn in Figure 6.5.

Figure 6.5: Identity diagram of the B_n-Gelfand-Tsetlin pattern described in the proof of Theorem 6.3.1 with the usual drawing conventions for readability.



We see that every connected component of the identity diagram contains a node corresponding to a marked element of the poset, hence this pattern must be a vertex of $GT_{\mathsf{B}_n}(\lambda)$ by Theorem 6.6.4.

By construction and since $n > 1$, the coordinate $y_{1,n} = \lambda_n$ of this pattern lies in $\frac{1}{2} + \mathbb{Z}$ while the coordinate $y_{2,n} = 0$ lies in $\mathbb{Z}$. This shows that the pattern is not standard, hence by Theorem 6.4.11 its preimage under Littelmann's affine bijection is not a lattice point. So we found a non-integral vertex of the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ in type B_n for every $\lambda \in \Lambda^+$ such that $\langle \lambda, \alpha_n^\vee \rangle$ is odd, which concludes our proof in types A_n, B_n and C_n. □

By studying identity diagrams abstractly, we can even give a complete classification of vertices of marked order polytope as the following construction in the Gelfand-Tsetlin case shows.

Construction 6.6.8. We want to construct vertices of Gelfand-Tsetlin polytopes diagrammatically. Fix a dominant integral weight λ and let us consider the poset $(P, A, \tilde{\lambda})$ such that $\mathcal{O}_{P,A}(\tilde{\lambda}) = GT(\lambda)$. Notice that the marking $\tilde{\lambda}$ is completely determined by the weight λ.

An important step in the construction will be the following *completion procedure*. Let $\mathcal{G}$ be a directed graph with node set P that contains every arrow of the Hasse diagram of P. Assume that for every arrow $p \to q$ in $\mathcal{G}$ we have either an arrow $p \to q$ or an arrow $q \to p$ in the Hasse diagram. Additionally assume that $\mathcal{G}$ does not contain double arrows in the same direction. By this we mean that $p \leftrightarrows q$ is allowed but $p \rightrightarrows q$ is forbidden. We say that $\mathcal{G}$ is **complete** if the following two conditions hold.

(i) Whenever there exists a set of arrows $p \leftrightarrows r \leftrightarrows q$ and $p \to s \to q$, there exists a set of arrows $p \leftarrow s \leftarrow q$ as well.

(ii) Whenever there exists a sequence of arrows $a \to p_1 \to \ldots \to p_t \to b$ with $a, b \in A$ such that $\tilde{\lambda}_a = \tilde{\lambda}_b$, there exists a reverse sequence of arrows $a \leftarrow p_1 \leftarrow \ldots \leftarrow p_t \leftarrow b$ as well.

Notice that we can complete any graph with the mentioned assumptions by repeatedly adding new arrows — but only those that are strictly necessary — until the graph is complete. Of course, we might have to check every set of arrows repeatedly since we are constantly introducing new arrows in this process. However, since we will never produce a double arrow $p \rightrightarrows q$ and P is finite, this algorithm will eventually stop. Additionally, the completion will be unique, i.e. it does not depend on the order in which we check for and — if necessary — add arrows.

Coming back to our Gelfand-Tsetlin patterns, we can now describe a process to construct vertices of Gelfand-Tsetlin polytopes.

We start with the Hasse diagram of the corresponding poset. First of all, the graph might not be complete. So whenever there exists an arrow $a \to p \to b$ for some $a, b \in A$ and $p \in P$ we must check whether $\tilde{\lambda}_a = \tilde{\lambda}_b$. Whenever this is the case, we must add the two arrows $b \to p \to a$ to the Hesse diagram. The resulting graph might not be complete after this initial step, so finish the completion procedure.

Since we want to create a vertex, we must add more arrows. We can freely introduce new arrows $p \to q$ whenever there exists an arrow $q \to p$ and there does not already exist an arrow $p \to q$. However, we must not add an arrow between two nodes p and q if p is connected (via a possibly empty sequence of double arrows) to an element $a \in A$ and q is connected (via a possibly empty sequence of double arrows) to an element $b \in A$ such that $\tilde{\lambda}_a \neq \tilde{\lambda}_b$. After adding an arrow, we must always complete the graph.

We must keep adding new arrows until we can no longer legally add new arrows. At that point, every vertex will be connected (via a sequence of double arrows) to at least one marked element, i.e. every connected component of the resulting diagram will contain at least one marked element.

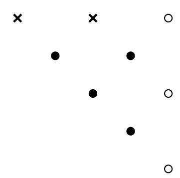
It is clear that we will always reach this stage. But of course the resulting graph is not unique. By adding different arrows, we will in general terminate in a different graph.

By construction, every terminal graph in our algorithm will be the identity graph of a Gelfand-Tsetlin pattern. Every coordinate x_p of the pattern is given by $x_p = \tilde{\lambda}_a$, where a is a marked element in the connected component of p.

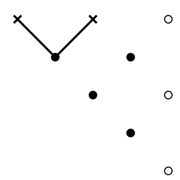
Because of Theorem 6.6.4 this pattern must be a vertex of the Gelfand-Tsetlin polytope $GT(\lambda)$. Additionally, we are able to reach every vertex of $GT(\lambda)$ by this procedure (although admittedly it might take some time).

Let us apply this procedure in an example.

Example 6.6.9. Let G be of type B_2 and consider the weight $\lambda = \omega_2 = \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2$. We will use our usual convention to not draw the arrows of the Hasse diagram but remember their existence by careful positioning of the nodes. Then the Hasse diagram is drawn in the following way.



As an initial step we must search for arrows $a \to p \to b$ with $a, b \in A$ such that the marking of a and b coincides. Since $\lambda_1 = \lambda_2 = \frac{1}{2}$ we have one such path in the upper left corner. Hence we must add arrows in the opposite direction. With our usual convention to draw $\leftrightarrows$ as straight lines we get the following diagram.

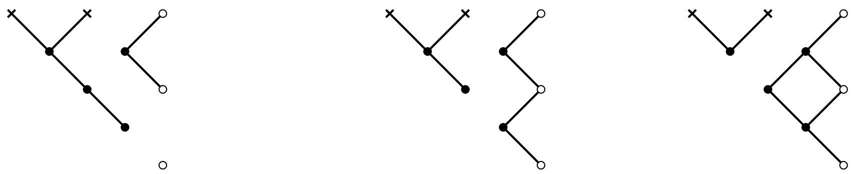


Now we can add new arrows as opposites of already existing arrows. As an example, let us add an arrow from the middle vertex of the top row to its right bottom neighbor. Below is the resulting diagram and its completion.

For our next arrow we have three possible choices (all towards the bottom). Two possibilities give the same diagram after completion. The other possibility gives a different diagram. The two distinct complete diagrams are shown below.

Both complete diagrams terminate the procedure. They correspond to the vertices $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ of $GT_{\mathsf{B}_2}(\omega_2)$.

However, there are other possibilities by choosing a different arrow in the first addition. They lead to the following three complete diagrams.



Those are the identity diagrams of the vertices $(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0, 0)$ respectively. So we have found a visual way to calculate the 5 vertices of $GT_{\mathsf{B}_2}(\omega_2)$.

Alternatively, we could use the idea behind the proof of Theorem 6.3.1 to compute the unique interior lattice point for the *anticanonical* string polytopes. Basically, we just have to find the Gelfand-Tsetlin pattern with the least amount of *black* double arrows possible in its identity diagram. We will show this idea in the following examples.

Example 6.6.10. Let us first consider the case $G = \mathrm{SL}_3$ and the full flag variety G/B. Then the anticanonical weight is given by $\lambda_{G/B} = 2\rho$. From Corollary 5.3.1 we know that the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(2\rho)$ will be translated to a weakly dual-Fano polytope. Hence it contains a unique lattice point in its interior. We can find this lattice point by finding the unique Gelfand-Tsetlin pattern of type 2ρ such that every entry is neither equal to its left upper neighbor nor its right upper neighbor. This pattern is

$$\begin{array}{ccccc} 4 & & 2 & & 0 \\ & 3 & & 1 & \\ & & 2 & & \end{array}$$

and its preimage under Littelmann's bijection $\phi_{2\rho}$ is the point $(1,2,1)$.

Example 6.6.11. For $G = \mathrm{SO}_5$ and the partial flag variety $G/P(\alpha_1)$. The anticanonical weight is given by $\lambda_{G/P(\alpha_1)} = 4\omega_2 = 2\epsilon_1 + 2\epsilon_2$. Since the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(4\omega_2)$ is not full-dimensional, we will only find points in its relative interior. We can also see this diagrammatically. Every identity diagram in this setting will have three connected nodes in the upper left corner since $2 = \lambda_1 \geq z_{1,1} \geq \lambda_2 = 2$. But if we re-embed the string polytope such that it is indeed full-dimensional, it will contain a unique interior lattice point. This point is associated to the pattern

$$\begin{array}{ccccc} 2 & & 2 & & 0 \\ & 2 & & \frac{1}{2} & \\ & & 1 & & 0 \\ & & & \frac{1}{2} & \\ & & & & 0 \end{array}$$

whose preimage under Littelmann's bijection $\phi_{4\omega_2}$ is the point $(1,2,3,0)$.

6.7 Tweaked Gelfand-Tsetlin Patterns

Let us now consider the type D_n case. Before stating our new construction, let us first underline where the problems arise when trying to copy the previous proof of Theorem 6.3.1.

In slight deviation of Littlemann's notation in [47], we will enumerate the simple roots of SO_{2n} as

$$\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_{n-1} + \epsilon_n.$$

Additionally we will consider the reduced decomposition

$$\underline{w_0}^{\mathrm{std}} = (s_{n-1}s_n)(s_{n-2}s_{n-1}s_n s_{n-2}) \cdots (s_1 s_2 \cdots s_{n-2}s_{n-1}s_n s_{n-2} \cdots s_2 s_1)$$

of the longest word of the Weyl group as the standard one. Notice that this does not completely correspond to Littelmann's standard decomposition since we swap the positions of the (commuting) reflections corresponding to $\epsilon_{n-1}-\epsilon_n$ and $\epsilon_{n-1}+\epsilon_n$. However, since the two reflections commute, the string polytopes will be the same after permutation of some coordinates. So we will sloppily say that these two polytopes are *the same*.

We know that the polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ will be a subset of $\mathbb{R}^{n(n-1)}$. We will denote the coordinates of a vector $a \in \mathbb{R}^{n(n-1)}$ as

$$\begin{aligned} a = \big(&a_{n-1,n-1}, a_{n-1,n}, a_{n-2,n-2}, a_{n-2,n-1}, a_{n-2,n}, a_{n-2,n+1}, \dots \\ &\dots, a_{1,1}, a_{1,2}, \dots, a_{1,n-2}, a_{1,n-1}, a_{1,n}, a_{1,n+1}, \dots, a_{1,2n-2}\big). \end{aligned}$$

It is understood that $a_{i,j} = 0$ if any of the indices is outside of its allowed range. For every tuple $(a_{i,j})$ with $1 \leq i \leq n-1$ and $i \leq j \leq 2n-1-i$ we will use the notation $\overline{a_{i,j}} := a_{i,2n-1-j}$.

We think of these coordinates as entries of the following triangle.

$$\begin{array}{ccccccccc} a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & \overline{a_{1,n-1}} & \overline{a_{1,n-2}} & \cdots & \overline{a_{1,2}} & \overline{a_{1,1}} \\ & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & \overline{a_{2,n-1}} & \overline{a_{2,n-2}} & \cdots & \overline{a_{2,2}} & \\ & & \ddots & & & & & \iddots & & \\ & & & a_{n-2,n-2} & a_{n-2,n-1} & \overline{a_{n-2,n-1}} & \overline{a_{n-2,n-2}} & & & \\ & & & & a_{n-1,n-1} & \overline{a_{n-1,n-1}} & & & & \end{array}$$

Notice that our $(n-1)$-st column is Littelmann's n-th column and our n-th column is Littelmann's $(n-1)$-st column. The reason for this change is that the j-th column corresponds to the reflection s_j for $j \leq n-2$. So it is more intuitive if the $(n-1)$-st row corresponds to the simple reflection s_{n-1} — and not s_n. However, these changes are rather cosmetic.

The following is a combination of [47, Theorem 7.1 and Corollary 8].

Theorem 6.7.1 (Littelmann)**.** *Let G be of type D_n and $\lambda = \sum_{i=1}^n \lambda_i \omega_i \in \Lambda^+$. A tuple $(a_{i,j}) \in \mathbb{R}^{n(n-1)}$ is an element of $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ if and only if the following two sets of conditions hold.*

$$a_{i,i} \geq a_{i,i+1}, \geq \dots \geq a_{i,n-2} \geq \left\{ \begin{array}{c} a_{i,n-1} \\ \overline{a_{i,n-1}} \end{array} \right\} \geq \overline{a_{i,n-2}} \geq \dots \geq \overline{a_{i,i+1}} \geq \overline{a_{i,i}} \geq 0,$$
$$a_{n-1,n-1} \geq 0, \quad \overline{a_{n-1,n-1}} \geq 0,$$

for every $1 \le i \le n-2$ *and*

$$\begin{aligned} a_{i,j} &\le \lambda_j + a_{i,j+1} + \overline{a_{i,j+1}} - 2\overline{a_{i,j}} + \overline{a_{i,j-1}} \\ &\quad + \sum_{k=1}^{i-1}(a_{k,j-1} - 2a_{k,j} + a_{k,j+1} + \overline{a_{k,j+1}} - 2\overline{a_{k,j}} + \overline{a_{k,j-1}}), \\ \overline{a_{i,j}} &\le \lambda_j + \overline{a_{i,j-1}} + \sum_{k=1}^{i-1}(a_{k,j-1} - 2a_{k,j} + a_{k,j+1} + \overline{a_{k,j+1}} - 2\overline{a_{k,j}} + \overline{a_{k,j-1}}), \\ a_{i,n-1} &\le \lambda_{n-1} + \overline{a_{i,n-2}} + \sum_{k=1}^{i-1}(a_{k,n-2} - 2a_{k,n-1} + \overline{a_{k,n-2}}), \\ \overline{a_{i,n-1}} &\le \lambda_n + \overline{a_{i,n-2}} + \sum_{k=1}^{i-1}(a_{k,n-2} - 2\overline{a_{k,n-1}} + \overline{a_{k,n-2}}). \end{aligned}$$

for every $1 \le i \le n-1$ *and* $i \le j \le n-2$.

Now we can describe an adapted version of Littelmann's piecewise affine map directly. Notice that we have to make slight adjustments because of our change of reduced decomposition. Let $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. Notice the base change from ω_i to ϵ_i in contrast to Theorem 6.7.1. Fix a point $(a_{i,j}) \in \mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$. This point is sent via the piecewise affine bijection ϕ_λ to a Gelfand-Tsetlin pattern $x = (\mathbf{y}, \mathbf{z})$ in $\mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^{\frac{n(n-1)}{2}}$. By our convention the row index of $\mathbf{y} = (y_{i,j})$ starts with $i = 2$. For easier notation we will set $y_{1,j} := \lambda_j$. Again, in our terminology this row is not actually part of the pattern x. The other rows can be computed reciprocally as

$$\begin{aligned} y_{i,j} &= y_{i-1,j} + a_{i-1,j-1} - a_{i-1,j} - \overline{a_{i-1,j}} + \overline{a_{i,j-1}} \text{ and} \\ y_{i,n} &= y_{i-1,n} + a_{i-1,n-1} - \overline{a_{i-1,n-1}} \end{aligned}$$

for every $2 \le i \le n$ and $i \le j \le n-1$. For the z-coordinates we have the formulae

$$\begin{aligned} z_{i,j} &= y_{i,j} + \overline{a_{i,j-1}} - \overline{a_{i,j}}, \\ z_{i,n-1} &= y_{i,n} \ + \min\{a_{i,n-2} - \overline{a_{i,n-1}}, a_{i,n-1} - \overline{a_{i,n-2}}\} \text{ and} \\ z_{n-1,n-1} &= y_{i,n} + a_{n-1,n-1} \end{aligned}$$

for every $1 \le i \le n-2$ and $i \le j \le n-2$.

So we see that the non-affine part appears in the coordinates $z_{i,n-1}$. Since ϕ_λ is a bijection we are not loosing any information when applying ϕ_λ but the minimum function makes it appear that way.

Our goal now is to embed the Gelfand-Tsetlin pattern in a subspace of a larger vector space to keep track of both values $a_{i,n-2} - \overline{a_{i,n-1}}$ and $a_{i,n-1} - \overline{a_{i,n-2}}$.

For that purpose we will introduce new coordinates $z^{\uparrow}_{i,n-1}$ and $z^{\downarrow}_{i,n-1}$ to replace the *bad* coordinate $z_{i,n-1}$.

For easier presentation we will use the following notation.

Notation 6.7.2. Let a, b, c, d, e and f be some real numbers. We will write

$$\begin{array}{ccc} a & & b \\ & c & \\ & d & \\ e & & f \end{array}$$

if the numbers fulfill the conditions

$$\begin{aligned} a \geq c \geq \left\{ \begin{array}{c} b \\ f \end{array} \right\} \quad &\text{and} \quad c \leq a + b + f, \\ e \geq d \geq \left\{ \begin{array}{c} b \\ f \end{array} \right\} \quad &\text{and} \quad d \leq e + b + f. \end{aligned}$$

Notice and beware of the asymmetry in this notation! We do not for example require $a \geq d$!

We can now define a modified version of D_n-Gelfand-Tsetlin patterns.

Definition 6.7.3 (Tweaked Gelfand-Tsetlin Patterns in Type D_n)**.** Let G be of type D_n and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. A **tweaked Gelfand-Tsetlin pattern** of type λ is a pair $(\mathbf{y}, \mathbf{z})$ of tuples $\mathbf{y} = (y_{i,j}) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, $2 \leq i \leq n$, $i \leq j \leq n$, and

$$\mathbf{z} = \left(z_{1,1}, \ldots, z_{1,n-2}, z^{\uparrow}_{1,n-1}, z^{\downarrow}_{1,n-1}, \ldots, z_{n-2,n-2}, z^{\uparrow}_{n-2,n-1}, z^{\downarrow}_{n-2,n-1}, z_{n-1,n-1}\right)$$

such that

$$y_{i,n-1} - y_{i+1,n-1} = z^{\uparrow}_{i,n-1} - z^{\downarrow}_{i,n-1} \quad \text{for all } 1 \leq i \leq n-2 \tag{6.7.1}$$

and the coordinates fulfill the relations in Figure 6.6.

To simplify notation we will sometimes write $z^{\uparrow}_{n-1,n-1}$ for $z_{n-1,n-1}$.

As in the usual definition, these patterns will define a polytope.

Definition 6.7.4. Let G be of type D_n and $\lambda \in \Lambda^+$. The **tweaked Gelfand-Tsetlin polytope** $\widetilde{GT}(\lambda)$ is defined as the set of all tweaked Gelfand-Tsetlin patterns of type λ.

Figure 6.6: Inequalities of tweaked Gelfand-Tsetlin patterns.

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \dots & & \lambda_{n-3} & & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n \\
& z_{1,1} & & z_{1,2} & & \dots & & z_{1,n-3} & & z_{1,n-2} & & \substack{z^{\uparrow}_{1,n-1} \\ z^{\downarrow}_{1,n-1}} & \\
& & y_{2,2} & & y_{2,3} & & \dots & & y_{2,n-2} & & y_{2,n-1} & & y_{2,n} \\
& & & z_{2,2} & & z_{2,3} & & \dots & & z_{2,n-2} & & \substack{z^{\uparrow}_{2,n-1} \\ z^{\downarrow}_{2,n-1}} & \\
& & & & y_{3,3} & & y_{3,4} & & \dots & & y_{3,n-1} & & y_{3,n} \\
& & & & & z_{3,3} & & z_{3,4} & & \dots & & \substack{z^{\uparrow}_{3,n-1} \\ z^{\downarrow}_{3,n-1}} & \\
& & & & & & \ddots & & \ddots & & \ddots & & \\
& & & & & & & z_{n-3,n-3} & & z_{n-3,n-2} & & \substack{z^{\uparrow}_{n-3,n-1} \\ z^{\downarrow}_{n-3,n-1}} & \\
& & & & & & & & y_{n-2,n-2} & & y_{n-2,n-1} & & y_{n-1,n} \\
& & & & & & & & & z_{n-2,n-2} & & \substack{z^{\uparrow}_{n-2,n-1} \\ z^{\downarrow}_{n-2,n-1}} & \\
& & & & & & & & & & y_{n-1,n-1} & & y_{n,n} \\
& & & & & & & & & & & z_{n-1,n-1} & \\
& & & & & & & & & & & & y_{n,n}
\end{array}
$$

The relation between usual Gelfand-Tsetlin patterns and tweaked Gelfand-Tsetlin patterns is given by the following observation.

Let $\mathcal{V}_\lambda$ denote the linear subspace of $\mathbb{R}^{n(n-1)+(n-2)}$ defined by the relations in Equation (6.7.1).

Theorem 6.7.5. *For every $\lambda \in \Lambda^+$ there exists a bijection $\psi_\lambda \colon \mathcal{V}_\lambda \to \mathbb{R}^{n(n-1)}$ given by $(z^{\uparrow}_{i,n-1}, z^{\downarrow}_{i,n-1}) \mapsto \min\{z^{\uparrow}_{i,n-1}, z^{\downarrow}_{i,n-1}\}$ for all $1 \le i \le n-2$ and identity on the other coordinates. Its inverse is given by*

$$
\begin{aligned}
z^{\uparrow}_{i,n-1} &:= z_{i,n-1} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\
z^{\downarrow}_{i,n-1} &:= z_{i,n-1} + y_{i+1,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\}
\end{aligned}
$$

for every $1 \le i \le n-2$ and identity on the other coordinates. Furthermore, ψ_λ induces a bijection between the tweaked Gelfand-Tsetlin patterns of type λ and the usual Gelfand-Tsetlin patterns of type λ.

Proof. Let ρ_λ denote the proposed inverse. It is clear that its image is in $\mathcal{V}_\lambda$.

Furthermore, notice that

$$\begin{aligned}\min\{z^{\uparrow}_{i,n-1}, z^{\downarrow}_{i,n-1}\} &= \min\left\{\begin{array}{c} z_{i,n-1} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\}, \\ z_{i,n-1} + y_{i+1,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\}\end{array}\right\} \\ &= z_{i,n-1} + \min\{y_{i,n-1}, y_{i+1,n-1}\} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &= z_{i,n-1}\end{aligned}$$

so $\psi_\lambda \circ \rho_\lambda$ is just the identity. Conversely, for an element $(\mathbf{y}, \mathbf{z}) \in \mathcal{V}_\lambda$ we must compute

$$\min\{z^{\uparrow}_{i,n-1}, z^{\downarrow}_{i,n-1}\} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\}.$$

By Equation (6.7.1) we can substitute $z^{\downarrow}_{i,n-1} = z^{\uparrow}_{i,n-1} + y_{i+1,n-1} - y_{i,n-1}$ and get

$$\begin{aligned}&\min\{z^{\uparrow}_{i,n-1}, z^{\downarrow}_{i,n-1}\} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &\quad = \min\{z^{\uparrow}_{i,n-1}, z^{\uparrow}_{i,n-1} + y_{i+1,n-1} - y_{i,n-1}\} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &\quad = z^{\uparrow}_{i,n-1} + \min\{y_{i,n-1}, y_{i+1,n-1}\} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &\quad = z^{\uparrow}_{i,n-1}.\end{aligned}$$

The same works for $z^{\downarrow}_{i,n-1}$, so we see that $\rho_\lambda \circ \psi_\lambda$ is the identity on $\mathcal{V}_\lambda$. Thus ψ_λ is bijective.

Let $x = (\mathbf{y}, \mathbf{z}) \in \mathcal{V}_\lambda$. We now have to show that $x \in \widetilde{GT}(\lambda)$ if and only if $\psi_\lambda(x) \in GT_{\mathsf{D}_n}(\lambda)$. Notice that most coordinates remain invariant, so we only have to compare the inequalities containing $z_{i,n-1}$ and their counterparts respectively. Notice also that

$$\left\{\begin{array}{c} z^{\uparrow}_{i,n-1} \\ z^{\downarrow}_{i,n-1}\end{array}\right\} \geq \left\{\begin{array}{c} y_{i,n} \\ y_{i+1,n}\end{array}\right\} \quad \Leftrightarrow \quad z_{i,n-1} \geq \left\{\begin{array}{c} y_{i,n} \\ y_{i+1,n}\end{array}\right\},$$

so these inequalities will transfer correctly. Let us now check the remaining few.

For the first implication notice that ψ_λ just takes the minimum of two coordinates. Hence we see that

$$\begin{aligned} y_{i,n-1} \geq z^{\uparrow}_{i,n-1} \quad &\Rightarrow \quad y_{i,n-1} \geq z_{i,n-1}, \\ y_{i+1,n-1} \geq z^{\downarrow}_{i,n-1} \quad &\Rightarrow \quad y_{i+1,n-1} \geq z_{i,n-1}, \\ z^{\uparrow}_{i,n-1} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n} \quad &\Rightarrow \quad z_{i,n-1} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n}, \\ z^{\downarrow}_{i,n-1} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n} \quad &\Rightarrow \quad z_{i,n-1} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}.\end{aligned}$$

For the second implication we calculate

$$y_{i,n-1} - z^{\uparrow}_{i,n-1} = \min\{y_{i,n-1}, y_{i+1,n-1}\} - z_{i,n-1} \geq 0$$

because $y_{i,n-1} \geq z_{i,n-1}$ and $y_{i+1,n-1} \geq z_{i,n-1}$. An analogous computation yields $y_{i+1,n-1} - z^{\downarrow}_{i,n-1} \geq 0$. Finally notice that

$$y_{i,n-1} + y_{i,n} + y_{i+1,n} - z^{\uparrow}_{i,n-1} = \min\{y_{i,n-1}, y_{i+1,n-1}\} + y_{i,n} + y_{i+1,n} - z_{i,n} \geq 0$$

since $z_{i,n} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n}$ and $z_{i,n-1} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$. The same argument shows that $y_{i+1,n-1} + y_{i,n} + y_{i+1,n} - z^{\downarrow}_{i,n-1} \geq 0$. This concludes the proof. $\square$

We can now state an analogue of Theorem 6.4.11 for D_n.

Theorem 6.7.6. *Let G be of type D_n and $\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \Lambda^+$. The map $\widetilde{\phi}_\lambda := \psi_\lambda^{-1} \circ \phi_\lambda \colon \mathbb{R}^{n(n-1)} \to \mathcal{V}_\lambda$ is an affine bijection and $\widetilde{\phi}_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)) = \widetilde{GT}(\lambda)$. Furthermore, $a \in \mathbb{R}^{n(n-1)}$ is a lattice point if and only if the coordinates of $\widetilde{\phi}_\lambda(a)$ — including the first row $y_{1,j} = \lambda_j$ — are either all integral or all are in $\frac{1}{2} + \mathbb{Z}$.*

Proof. Since ϕ_λ and ψ_λ^{-1} are piecewise affine bijections, the same holds true for $\widetilde{\phi}_\lambda$. Since $\phi_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}) = GT_{\mathsf{D}_n}(\lambda)$ by Theorem 6.4.13 and $\psi_\lambda(\widetilde{GT}(\lambda)) = GT_{\mathsf{D}_n}(\lambda)$ by Theorem 6.7.5, we have $\widetilde{\phi}_\lambda(\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)) = \widetilde{GT}(\lambda)$. The claim on lattice points is clear from Theorem 6.4.13 and the definition of ψ_λ.

It remains to show that $\widetilde{\phi}_\lambda$ and $\widetilde{\phi}_\lambda^{-1}$ are in fact affine. We only have to check the coordinates $z^{\uparrow}_{i,n-1}$ and $z^{\downarrow}_{i,n-1}$ since the map is affine in all other coordinates. Let $x = (\mathbf{y}, \mathbf{z})$ be the image of $(a_{i,j})$ under $\widetilde{\phi}_\lambda$. We calculate

$$\begin{aligned} z^{\uparrow}_{i,n-1} &= z_{i,n-1} + y_{i,n-1} - \min\{y_{i,n-1}, y_{i+1,n-1}\} \\ &= z_{i,n-1} - \min\{0, y_{i+1,n-1} - y_{i,n-1}\} \\ &= y_{i,n} + \min\{a_{i,n-2} - \overline{a_{i,n-1}}, a_{i,n-1} - \overline{a_{i,n-2}}\} \\ &\qquad - \min\{0, a_{i,n-2} - a_{i,n-1} - \overline{a_{i,n-1}} + \overline{a_{i,n-2}}\} \\ &= y_{i,n} + a_{i,n-1} - \overline{a_{i,n-2}}. \end{aligned}$$

This implies that $z^{\uparrow}_{i,n-1}$ is actually a linear combination of the coordinates of $(a_{i,j})$ plus λ_n. The same holds true for $z^{\downarrow}_{i,n-1}$ by analogous computation. Hence the map $\widetilde{\phi}_\lambda$ is affine, i.e. the concatenation of a linear map and a translation. Since the inverse of a linear map is linear and the inverse of a translation is a translation we know that $\widetilde{\phi}_\lambda^{-1}$ must be affine too. This concludes the proof. $\square$

Example 6.7.7. Let $G = \mathrm{SO}_6$ and let $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. The image of the point $(a, b, c, d, e, f) \in \mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ under the map $\widetilde{\phi}_\lambda$ is drawn in Figure 6.7.

Figure 6.7: The point $\widetilde{\phi}_\lambda(a, b, c, d, e, f)$ from Example 6.7.7.

$$\begin{array}{ccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 \\
& \lambda_1 - e + f & & \begin{matrix}\lambda_3 + d - f \\ \lambda_3 + c - e\end{matrix} & \\
& & \lambda_2 + c - d - e + f & & \lambda_3 + d - e \\
& & & \lambda_3 + a + d - e & \\
& & & & \lambda_3 + a - b + d - e
\end{array}$$

6.8 Tweaked Gelfand-Tsetlin Diagrams

We will now define an analogue of identity diagrams of elements of marked order polytopes for tweaked Gelfand-Tsetlin patterns. For that purpose we want to define a poset that describes as much of the tweaked Gelfand-Tsetlin polytope as possible.

Construction 6.8.1 (Tweaked Gelfand-Tsetlin Poset)**.** Let G be of type D_n. Let $\mathcal{GT}_n$ be the set of symbols

(i) $\xi_{i,j}$ for $1 \leq i \leq n$ and $i \leq j \leq n$,

(ii) $\zeta_{i,j}$ for $1 \leq i \leq n-2$ and $i \leq j \leq n-2$,

(iii) $\zeta^{\uparrow}_{i,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ for $1 \leq i \leq n-2$, and

(iv) $\zeta_{n-1,n-1}$.

For easier notation we sometimes write $\zeta^{\uparrow}_{n-1,n-1}$ for $\zeta_{n-1,n-1}$.

We define a partial order on $\mathcal{GT}_n$ via

(i) $\xi_{1,j} \geq \zeta_{1,j}$ for all $1 \leq j \leq n-1$,

(ii) $\xi_{1,n-1} \geq \zeta^{\uparrow}_{1,n-1}$,

(iii) $\xi_{i,j} \geq \zeta_{i-1,j}$ and $\xi_{i,j} \geq \zeta_{i,j}$ for all $2 \leq i \leq n-2$ and $i \leq j \leq n-2$,

(iv) $\xi_{i,n-1} \geq \zeta^{\downarrow}_{i-1,n-1}$ and $\xi_{i,n-1} \geq \zeta^{\uparrow}_{i,n-1}$ for all $2 \leq i \leq n-2$,

(v) $\xi_{n-1,n-1} \geq \zeta^{\downarrow}_{n-2,n-1}$ and $\xi_{n-1,n-1} \geq \zeta_{n-1,n-1}$,

(vi) $\zeta_{i,j} \geq \xi_{i,j+1}$ and $\zeta_{i,j} \geq \xi_{i+1,j+1}$ for all $1 \leq i \leq n-2$ and $i \leq j \leq n-2$,

(vii) $\zeta^{\uparrow}_{i,n-1} \geq \xi_{i,n}$ and $\zeta^{\uparrow}_{i,n-1} \geq \xi_{i+1,n}$ for all $1 \leq i \leq n-2$,

(viii) $\zeta^{\downarrow}_{i,n-1} \geq \xi_{i,n}$ and $\zeta^{\downarrow}_{i,n-1} \geq \xi_{i+1,n}$ for all $1 \leq i \leq n-2$, and

(ix) $\zeta_{n-1,n-1} \geq \xi_{n-1,n}$ and $\zeta_{n-1,n-1} \geq \xi_{n,n}$.

The poset $\mathcal{GT}_n$ will be called the **tweaked Gelfand-Tsetlin poset**.

We usually call the coordinates of $x \in \mathbb{R}^{\mathcal{GT}_n}$ by the coordinates of the respective Gelfand-Tsetlin patterns like $x_{\xi_{i,j}} =: y_{i,j}$ and $x_{\zeta_{i,j}} =: z_{i,j}$.

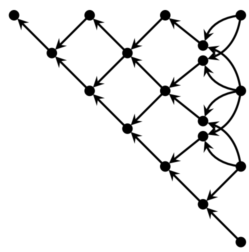
Notice that this describes almost all relations defining tweaked Gelfand-Tsetlin patterns. The ones missing are the four-term relations of type

$$\begin{aligned} z^{\uparrow}_{i,n-1} - z^{\downarrow}_{i,n-1} &= y_{i,n-1} - y_{i+1,n-1}, \\ z^{\uparrow}_{i,n-1} &\leq y_{i,n-1} + y_{i,n} + y_{i+1,n} \text{ and} \\ z^{\downarrow}_{i,n-1} &\leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}. \end{aligned}$$

We will return to them later.

Example 6.8.2. The Hasse diagram of the tweaked Gelfand-Tsetlin poset $\mathcal{GT}_4$ is depicted in Figure 6.8.

Figure 6.8: Hasse diagram of the tweaked Gelfand-Tsetlin poset $\mathcal{GT}_4$.



We would like to define an analogue of the marked order polytope. For this we will use the following non-standard definition.

Definition 6.8.3. A **pseudo-marking** on a poset P is a pair (A, λ) where A is a subset of P and $\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A$ is a real vector such that $\lambda_a \leq \lambda_b$ whenever $a \leq b$. The triplet (P, A, λ) is called a **pseudo-marked poset**. We will call the elements of A **marked elements**.

Notice that in contrast to Definition 6.5.1 we do not require that all minimal and all maximal elements of the poset are marked. As a consequence, the following definition might not give a polytope.

Definition 6.8.4. Let (P, A, λ) be a pseudo-marked poset. The **marked order polyhedron** $\mathcal{O}_{P,A}(\lambda)$ associated to (P, A, λ) is defined as

$$\mathcal{O}_{P,A}(\lambda) := \left\{ x \in \mathbb{R}^{P \setminus A} \;\middle|\; \begin{array}{l} x_p \leq x_q \text{ for all } p \leq q, \\ \lambda_a \leq x_p \text{ for all } a \leq p, \\ x_p \leq \lambda_b \text{ for all } p \leq b \end{array} \right\}.$$

The following observation is clear by construction.

Proposition 6.8.5. *Let G be of type D_n. Set $A := \{\xi_{1,1}, \ldots, \xi_{1,n}\} \subseteq \mathcal{GT}_n$. Let $\lambda \in \mathbb{R}^A$ and denote by $\tilde{\lambda} := \sum_{i=1}^n \lambda_{\xi_{1,i}} \epsilon_i$ the associated weight (not necessarily dominant nor integral). Then*

$$\widetilde{GT}(\tilde{\lambda}) := \mathcal{O}_{\mathcal{GT}_n,A}(\lambda) \cap \mathcal{V}_{\tilde{\lambda}} \cap \left\{ \begin{array}{l} z^{\uparrow}_{i,n-1} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n} \\ z^{\downarrow}_{i,n-1} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n} \end{array} \;\middle|\; 1 \leq i \leq n-1 \right\}.$$

Notice that we were a bit sloppy with our notation here since $\widetilde{GT}(\tilde{\lambda})$, $\mathcal{O}_{\mathcal{GT}_n,A}(\lambda)$ and $\mathcal{V}_{\tilde{\lambda}}$ are defined as subsets of $\mathbb{R}^{\mathcal{GT}_n \setminus A}$. So we want to understand the fourth polyhedron as a subset of $\mathbb{R}^{\mathcal{GT}_n \setminus A}$ too.

Additionally, we simplify notation by using λ for the weight and for the marking simultaneously.

Remark 6.8.6. Because of the defining relations of $\mathcal{V}_\lambda$ it is clear that the two inequalities $z^{\uparrow}_{i,n-1} \leq y_{i,n-1} + y_{i,n} + y_{i+1,n}$ and $z^{\downarrow}_{i,n-1} \leq y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$ are in fact equivalent. So when checking whether a given point is a tweaked Gelfand-Tsetlin pattern, it is sufficient to just verify one of those inequalities for every i. Furthermore, if one of these inequalities happens to be an equality, the other one will be too.

We will now define an analogue of identity diagrams for these special posets.

Construction 6.8.7. Let (A, λ) be the pseudo-marking on the tweaked Gelfand-Tsetlin poset $\mathcal{GT}_n$ from Proposition 6.8.5 and let $x \in \widetilde{GT}(\tilde{\lambda})$. Let $\mathcal{H}_n$ denote the Hasse diagram of $\mathcal{GT}_n$. The **tweaked Gelfand-Tsetlin pre-diagram** $\mathrm{pre}\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ associated to x is the colored directed graph whose nodes are labeled by the elements of the tweaked Gelfand-Tsetlin poset and whose arrows are given by the following construction.

(i) Add a *black* arrow $p \to q$ if there exists an arrow $p \to q$ in $\mathcal{H}_n$ between the corresponding nodes.

(ii) Add a *black* arrow $p \to q$ if there exists an opposite arrow $q \to p$ in $\mathcal{H}_n$ between the corresponding nodes and $x_p = x_q$.

(iii) For every $1 \leq i \leq n-1$ add six (only three if $i = n-1$) *red* arrows

$$\begin{tikzcd}[row sep=small]
\xi_{i,n-1} \arrow[r, dashed] & \zeta^{\uparrow}_{i,n-1} \arrow[r, dashed] \arrow[dr, dashed] & \xi_{i,n} \\
\xi_{i+1,n-1} \arrow[r, dashed] & \zeta^{\downarrow}_{i,n-1} \arrow[r, dashed] \arrow[ur, dashed] & \xi_{i+1,n}
\end{tikzcd}$$

if $z^{\uparrow}_{i,n-1} = y_{i,n-1} + y_{i,n} + y_{i+1,n}$ (or equivalently $z^{\downarrow}_{i,n-1} = y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$).

For reasons of readability we will always draw *red* arrows as black but dashed arrows.

However, this diagram can be simplified for our purposes. The reason is the following observation.

Remark 6.8.8. Let x be a tweaked Gelfand-Tsetlin pattern. Let $1 \leq i \leq n-1$ be an index such that $z^{\uparrow}_{i,n-1} = y_{i,n-1} + y_{i,n} + y_{i+1,n}$. Then the following implications hold.

(i) $y_{i,n-1} = y_{i,n} = y_{i+1,n} \Rightarrow y_{i,n-1} = y_{i,n} = y_{i+1,n} = z^{\uparrow}_{i,n-1} = 0$.

(ii) $z^{\uparrow}_{i,n-1} = y_{i,n-1} \Rightarrow y_{i,n} = -y_{i+1,n}$.

(iii) $z^{\uparrow}_{i,n-1} = y_{i,n} \Rightarrow y_{i,n-1} = -y_{i+1,n}$.

(iv) $z^{\uparrow}_{i,n-1} = y_{i+1,n} \Rightarrow y_{i,n-1} = -y_{i,n}$.

The analogue statements hold true for $z^{\downarrow}_{i,n-1}$ and $y_{i+1,n-1}$.

We will use these observations to adapt our pre-diagrams. The goal is to indicate whether two entries must be additive inverses of each other because of an equation of the form $z^{\uparrow}_{i,n-1} = y_{i,n-1} + y_{i,n} + y_{i+1,n}$ or $z^{\downarrow}_{i,n-1} = y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$.

Construction 6.8.9. Let x be a tweaked Gelfand-Tsetlin pattern and let $\mathrm{pre}\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ be its tweaked Gelfand-Tsetlin pre-diagram. We will replace *red* arrows as follows.

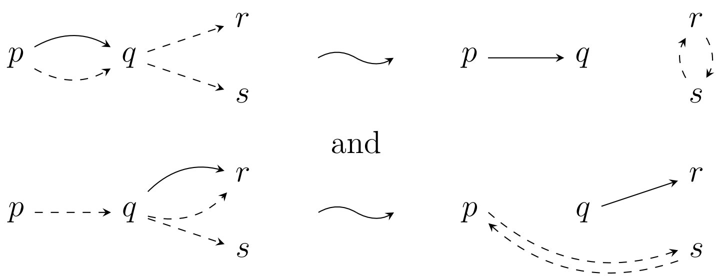
For all triplets of *red* arrows

$$\begin{tikzcd}[row sep=tiny]
 & & r \\
p \arrow[r, dashed] & q \arrow[ur, dashed] \arrow[dr, dashed] & \\
 & & s
\end{tikzcd}$$

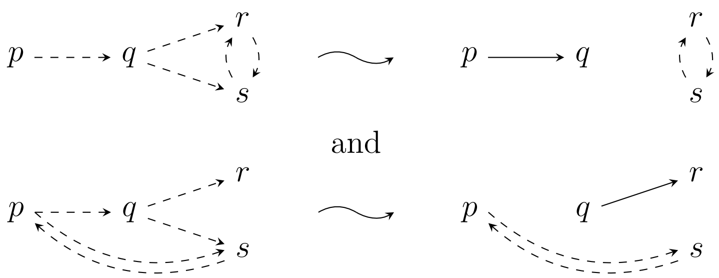
do the following replacements. (We do not specify whether $r = \xi_{i,n}$ and $s = \xi_{i+1,n}$ or $r = \xi_{i+1,n}$ and $s = \xi_{i,n}$. Both possibilities are allowed!)

If there exist *black* arrows $p \to q$, $q \to r$ and $q \to s$, delete all three red arrows but color the four nodes differently. By default we will call every node *black*. But in this case we chose to color these nodes *white* instead. We will draw them as empty circles, while our normal *black* nodes are filled circles.

For all remaining triplets of *red* arrows do the following two replacements if possible. If both are applicable to a certain triplet, do only one (it does not matter which one, though the first one is usually preferred due to readability).



Our replacement procedure could have produced *red* triangles. We will replace those as follows.



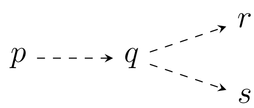
Finally, for every $1 \leq i \leq n-1$ we will add a pair of *black* arrows $\xi_{i,n-1} \leftrightarrows \xi_{i+1,n-1}$ if one of the following two conditions hold.

(i) There exist two *black* arrows $\zeta^{\uparrow}_{i,n-1} \to \xi_{i,n}$ and $\zeta^{\downarrow}_{i,n-1} \to \xi_{i,n}$.

(ii) There exist two *black* arrows $\zeta^{\downarrow}_{i,n-1} \to \xi_{i+1,n}$ and $\zeta^{\downarrow}_{i,n-1} \to \xi_{i+1,n}$.

The resulting graph is called the **tweaked Gelfand-Tsetlin diagram** $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ of x.

Remark 6.8.10. It is clear by construction that $x_p = x_q$ whenever there exists a pair of *black* arrows $p \to q$ and $q \to p$. Analogously, $x_p = -x_q$ whenever there exists a pair of *red* arrows $p \overset{red}{\to} q$ and $q \overset{red}{\to} p$. Additionally for every subgraph



we have $x_q = x_p + x_r + x_s$. So our construction really visualizes which of the defining inequalities of $\widetilde{GT}(\lambda)$ are actually equalities for the pattern x.

Furthermore, $x_p = 0$ for every *white* node p. The reason is the following. Every white node is part of a quadruplet $\{p, q, r, s\}$ such that $x_p = x_q = x_r = x_s$ and $x_q = x_p + x_r + x_s$ (after proper renaming). But this implies that $x_q = 3x_q$ and hence $0 = x_q = x_p = x_q = x_r = x_s$.

For readability of our drawings, sometimes we do not draw single arrows (we will remember their existence from the positions of the nodes), replace *black* double arrows $p \to q$ and $q \to p$ by a straight line and represent *red* double arrows $p \overset{red}{\to} q$ and $q \overset{red}{\to} p$ by a double straight line. We also usually omit *red* double arrows between white nodes, since they do not contain any new information. Single *red* arrows will be drawn as a dashed line.

Definition 6.8.11. Let x be a tweaked Gelfand-Tsetlin pattern and $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ its tweaked Gelfand-Tsetlin diagram. A subset $\mathcal{C}$ of nodes of this diagram is called **connected**, if it is connected via double-*black* and double-*red* arrows, i.e. for any two nodes p and q in $\mathcal{C}$ there exists a sequence $p_1, \ldots, p_t \in \mathcal{C}$ such that $p = p_1$, $q = p_t$ and for every i there either exist two *black* arrows $p_i \to p_{i+1}$ and $p_{i+1} \to p_i$ or there exist two *red* arrows $p_i \overset{red}{\to} p_{i+1}$ and $p_{i+q} \overset{red}{\to} p_i$. The sequence $(p_1, \ldots, p_t)$ is called a **connecting sequence** between p and q.

The maximal (with respect to inclusion) connected subsets are called **connected components**.

One might be inclined to think that a tweaked Gelfand-Tsetlin pattern is a vertex of the tweaked Gelfand-Tsetlin polytope if and only if each of the connected components in its tweaked Gelfand-Tsetlin diagram contains a marked element. However, this is not true in contrast to types A_n, B_n and C_n. There are more possibilities as the following examples show.

Example 6.8.12. Let $G = \mathrm{SO}_6$ and $\lambda = \rho = 2\epsilon_1 + \epsilon_2 \in \Lambda^+$. Then the pattern in Figure 6.9 is a vertex of $\widetilde{GT}(\lambda)$. However, its tweaked Gelfand-Tsetlin diagram contains an isolated node ($\zeta^{\downarrow}_{1,2}$).

Example 6.8.13. Let $G = \mathrm{SO}_8$ and $\lambda = \omega_1 + \omega_2 + \omega_3 + 3\omega_4 = 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4$. Then the pattern in Figure 6.10 is a vertex of $\widetilde{GT}(\lambda)$. We see two things happening in this example. First of all we see a triangular pattern of *white* notes. Secondly we see two nodes ($\zeta^{\uparrow}_{1,3}$ and $\zeta^{\downarrow}_{1,3}$) that are not connected to any other node — although they are connected via single *red* arrows. But these single arrows do not count as connections in our sense.

We will systematize these deviations from the A_n, B_n and C_n cases as follows.

Figure 6.9: Vertex of the tweaked Gelfand-Tsetlin polytope $\widetilde{GT}(2\epsilon_1 + \epsilon_2)$ and its tweaked Gelfand-Tsetlin diagram in type D_3.

$$\begin{array}{ccccc}
2 & & 1 & & 0 \\
 & 2 & & \begin{smallmatrix}0\\1\end{smallmatrix} & \\
 & & 2 & & 0 \\
 & & & 2 & \\
 & & & & 2
\end{array}$$

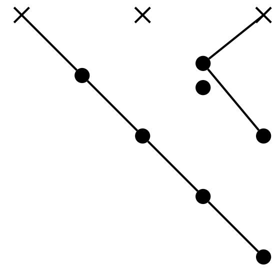
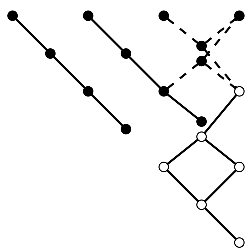


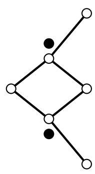
Figure 6.10: Vertex of the tweaked Gelfand-Tsetlin polytope $\widetilde{GT}(4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4)$ and its tweaked Gelfand-Tsetlin diagram in type D_4.

$$\begin{array}{ccccccc}
4 & & 3 & & 2 & 1 & -1 \\
 & 4 & & 3 & & 2 & \\
 & & 4 & & 3 & 3 & 0 \\
 & & & 4 & & 0 & \\
 & & & & 0 & & 0 \\
 & & & & & 0 & \\
 & & & & & & 0
\end{array}$$



Definition 6.8.14. Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and let $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ be its tweaked Gelfand-Tsetlin diagram.

(i) An **anomaly** is a triangle of white nodes of the following form.



(ii) A **single impurity** is a node $\zeta^{\uparrow}_{i,n-1}$ or $\zeta^{\downarrow}_{i,n-1}$ with $i < n-1$ that is not connected to any other node but the other node $\zeta^{\downarrow}_{i,n-1}$ (resp. $\zeta^{\uparrow}_{i,n-1}$) is connected to another node.

(iii) A **double impurity** is a pair of nodes $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ with $i < n-1$ such that both nodes are not connected to any other node but they are incident to *red* arrows.

(iv) A **triviality** is a pair of nodes $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ with $i < n-1$ such that both nodes are not connected to any other node and they are not incident to *red* arrows.

Notice that we do not allow single impurities to happen at $\zeta_{n-1,n-1}$. The reason for this will become clear later.

The size of the anomalies is due to the following fact.

Lemma 6.8.15. *Every* white *node in a tweaked Gelfand-Tsetlin pattern is part of an anomaly or its connected component contains a marked element.*

Proof. Let $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ be the tweaked Gelfand-Tsetlin diagram of some $x \in \widetilde{GT}(\lambda)$. By construction, white nodes can only appear as part of a connected *white* quadruplet $(\xi_{i,n-1}, \zeta^{\uparrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n})$ or $(\xi_{i+1,n-1}, \zeta^{\downarrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n})$ for $1 \leq i \leq n-1$.

If $i = 1$, both possible quadruplets contain a marked element which proves the claim.

Let us now consider the first quadruplet for $i > 1$. The arguments for the second quadruplet are completely analogous. We need to show that $\zeta^{\downarrow}_{i-1,n-1}$ and $\xi_{i-1,n}$ are *white* nodes as well.

We know that $y_{i,n-1} = y_{i,n} = 0$. Since $\xi_{i,n-1} \geq \zeta^{\downarrow}_{i-1,n-1} \geq \xi_{i,n}$ we have

$$0 = y_{i,n} \leq z^{\downarrow}_{i-1,n-1} \leq y_{i,n-1} = 0$$

and hence $z^{\downarrow}_{i-1,n-1} = 0$. Additionally $y_{i-1,n} \leq z^{\downarrow}_{i-1,n-1} = 0$ and

$$0 = z^{\downarrow}_{i-1,n-1} \leq y_{i,n-1} + y_{i-1,n} + y_{i,n} = y_{i-1,n},$$

which implies that $y_{i-1,n} = 0$.

So all six coordinates must be equal to zero. In our construction we would have labeled the corresponding nodes as *white*, which — by definition — constitutes an anomaly. □

Our goal is to prove the following classification of vertices of tweaked Gelfand-Tsetlin patterns.

Theorem 6.8.16. *A tweaked Gelfand-Tsetlin pattern is a vertex of the tweaked Gelfand-Tsetlin polytope if and only if each of the connected components of its tweaked Gelfand-Tsetlin diagram contains a marked element, contains an anomaly, is a single impurity or is part of a double impurity.*

The first implication will follow directly from the following observation.

Lemma 6.8.17. *Let $x \in \widetilde{GT}(\lambda) \subseteq \mathbb{R}^{\mathcal{GT}_n \setminus A} \hookrightarrow \mathbb{R}^{\mathcal{GT}_n}$ and let $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ be its tweaked Gelfand-Tsetlin diagram. Suppose there exists $v \in \mathbb{R}^{\mathcal{GT}_n}$ such that $x + v \in \widetilde{GT}(\lambda)$ and $x - v \in \widetilde{GT}(\lambda)$. Then the following properties hold.*

(i) $v_{\xi_{i,n-1}} - v_{\xi_{i+1,n-1}} = v_{\zeta^{\uparrow}_{i,n-1}} - v_{\zeta^{\downarrow}_{i,n-1}}$ *for all* $1 \leq i \leq n-2$.

(ii) $v_q = v_p + v_r + v_s$ *if* (p, q, r, s) *is equal to* $(\xi_{i,n-1}, \zeta^{\uparrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n})$ *or* $(\xi_{i+1,n-1}, \zeta^{\downarrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n})$ *and* $x_q = x_p + x_r + x_s$.

(iii) $v_p = v_q$ *if there exist two* black *arrows* $p \to q$ *and* $q \to p$ *in* $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$.

(iv) $v_p = -v_q$ *if there exist two* red *arrows* $p \overset{red}{\to} q$ *and* $q \overset{red}{\to} p$ *in* $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$.

(v) $v_p = 0$ *if* p *is a marked node or a* white *node.*

Proof. For the first claim let us consider the set

$$\mathcal{V} := \{a \in \mathbb{R}^{\mathcal{GT}_n} \mid a_{\xi_{i,n-1}} - a_{\xi_{i+1,n-1}} = a_{\zeta^{\uparrow}_{i,n-1}} - a_{\zeta^{\downarrow}_{i,n-1}} \text{ for all } 1 \le i \le n-2\}.$$

This set is in fact a vector space and $\widetilde{GT}(\lambda) \subseteq \mathcal{V}$ (after properly embedding $\widetilde{GT}(\lambda) \subseteq \mathbb{R}^{\mathcal{GT}_n}$). Since $x \in \mathcal{V}$ and $x + v \in \mathcal{V}$ we must have $v \in \mathcal{V}$ which proves observation (i).

For the second claim, let (p, q, r, s) be a quadruplet with the desired properties. Since $x \pm v \in \widetilde{GT}(\lambda)$ we must have

$$(x \pm v)_q \le (x \pm v)_p + (x \pm v)_r + (x \pm v)_s.$$

Since $x_q = x_p + x_r + x_s$ this is equivalent to

$$v_q \le v_q + v_r + v_s \quad \text{and} \quad -v_q \le -(v_p + v_r + v_s).$$

This is only possible if $v_q = v_p + v_r + v_s$, proving (ii).

Notice that (iii) is true by the same argument that was used in (ii) whenever the *black* arrows $p \leftrightarrows q$ are drawn between vertices that are comparable in the partial order. The only other *black* double arrows appear vertically between $\xi_{i,n-1}$ and $\xi_{i+1,n-1}$ whenever $\zeta^{\uparrow}_{i,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ are connected via *black* double arrows to the same node. So for these nodes we know that $v_{\zeta^{\uparrow}_{i,n-1}} = v_{\zeta^{\downarrow}_{i,n-1}}$. The claim follows because (i) implies that

$$v_{\xi_{i,n-1}} - v_{\xi_{i+1,n-1}} = v_{\zeta^{\uparrow}_{i,n-1}} - v_{\zeta^{\downarrow}_{i,n-1}} = 0.$$

Claim (iv) follows from the observation that *red* double arrows between two nodes p and q can only appear if there exist other nodes r and s that are connected via a *black* double arrow and $x_s = x_p + x_q + x_r$ (after possibly swapping r and s). By (ii) this implies $v_s = v_p + v_q + v_r$ and by (iii) this reduces to $v_p = -v_q$ since $v_r = v_s$.

For the final claim notice that $(x \pm v)_p = \lambda_p = x_p$ whenever p is a marked node, hence $v_p = 0$. If p is a *white* node, by construction (or Lemma 6.8.15) there exist three other *white* nodes q, r and s that are connected to p via *black* double arrows such that $\{p, q, r, s\}$ is equal to $\{\xi_{i,n-1}, \zeta^{\uparrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n}\}$

or $\{\xi_{i+1,n-1}, \zeta^{\downarrow}_{i,n-1}, \xi_{i,n}, \xi_{i+1,n}\}$. In either case we know by (iii) that $v_p = v_q = v_r = v_s$. Additionally, (ii) implies $v_q = v_p + v_r + v_s = 3v_q$ (after proper renaming). This yields $v_q = 0$ and hence $v_p = v_q = v_r = v_s = 0$, which concludes the proof. $\square$

We can now prove the first implication of Theorem 6.8.16.

Proof of Theorem 6.8.16 (First Implication). Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and suppose that each connected component of its tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ contains a marked element, contains an anomaly, is a single impurity or is part of a double impurity. Let v be a vector such that $x+v \in \widetilde{GT}(\lambda)$ and $x-v \in \widetilde{GT}(\lambda)$. By Lemma 6.8.17 we know that $v_p = 0$ if the connected component of p contains a marked element or an anomaly. So almost all coordinates of v must vanish. Let us now consider the remaining two cases.

Suppose that p is a single impurity. Let us assume that $p = \zeta^{\uparrow}_{i,n-1}$. Then $\zeta^{\downarrow}_{i,n-1}$ cannot be an impurity, so by the arguments above we know that the coordinates of v at the nodes $\xi_{i,n-1}$, $\xi_{i,n}$ and $\zeta^{\downarrow}_{i,n-1}$ vanish. Now Lemma 6.8.17 implies that $v_p = v_{\zeta^{\uparrow}_{i,n-1}} = v_{\zeta^{\downarrow}_{i,n-1}} + v_{\xi_{i,n-1}} - v_{\xi_{i+1,n-1}} = 0$. The claim for $p = \zeta^{\downarrow}_{i,n-1}$ follows analogously.

Finally let p be part of a double impurity. Assume that $p = \zeta^{\uparrow}_{i,n-1}$. By definition we know that $x_p = x_{\xi_{i,n-1}} + x_{\xi_{i,n}} + x_{\xi_{i+1,n}}$. Notice that the latter three nodes are obviously not impurities so the corresponding coordinates of v must vanish. Again, Lemma 6.8.17 implies that $v_p = v_{\xi_{i,n-1}} + v_{\xi_{i,n}} + v_{\xi_{i+1,n}} = 0$. The claim for $p = \zeta^{\downarrow}_{i,n-1}$ follows analogously.

So we see that the only vector satisfying both $x + v \in \widetilde{GT}(\lambda)$ and $x - v \in \widetilde{GT}(\lambda)$ is the trivial vector $v = 0$. Thus x is a vertex by Lemma 6.6.6. $\square$

For the poof of the converse we want to assume that a tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ contains a node p whose connected component $\mathcal{C}$ does not fall into one of the four categories of Theorem 6.8.16. We then want to explicitly construct a vector $v \neq 0$ such that $x \pm v \in \widetilde{GT}(\lambda)$, hence showing that x is no vertex.

Our naive guess is the following: Start by setting $v_p = 1$ for said node p. The conditions of Lemma 6.8.17 now require us to fix $v_r = v_s$ whenever two nodes r and s are connected via two *black* arrows $r \to s$ and $s \to r$ and $v_r = -v_s$ whenever r and s are connected via two *red* arrows $r \overset{red}{\to} s$ and $s \overset{red}{\to} r$.

This will inductively determine many coordinates of v — we will show that this procedure is actually well-defined — and by setting all other coordinates of v to be zero, we get a natural candidate. Unfortunately, we have to slightly modify this construction, but the idea will remain the same.

We will start by formalizing this procedure.

Notation 6.8.18. Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and let p and q be two nodes in its tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$. We set

$$(-1)^{(p,q)} := \begin{cases} 1 \text{ if there exist two } \textit{black} \text{ arrows } p \to q \text{ and } q \to p, \\ -1 \text{ if there exist two } \textit{red} \text{ arrows } p \overset{red}{\to} q \text{ and } q \overset{red}{\to} p, \\ 0 \text{ else.} \end{cases}$$

Notice that this sign is actually well defined because in our construction of tweaked Gelfand-Tsetlin diagrams we only drew *red* double arrow between nodes $\xi_{i,n-1}$ and $\xi_{j,n}$ or between nodes $\xi_{i,n}$ and $\xi_{i+1,n}$. But neither pair could possibly be connected directly via two *black* arrows.

We will now extend this local sign to a more global sign. For that purpose we need the following result.

Lemma 6.8.19. *Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and let $\mathcal{C}$ be a connected component of its tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ that does not contain a* white *node. Let p and q be two nodes in $\mathcal{C}$ and let $(p_1, \dots, p_r)$ and $(q_1, \dots, q_s)$ be two connecting sequences between p and q. Then*

$$\prod_{i=1}^{r-1} (-1)^{(p_i,p_{i+1})} = \prod_{j=1}^{s-1} (-1)^{(q_j,q_{j+1})}.$$

Proof. The claim is trivial if $\#\mathcal{C} = 1$. So let us assume that $\#\mathcal{C} > 1$.

Let k be the number of *red* double arrows appearing between the nodes in the sequence $(p_1, \dots, p_r)$ and let l denote the analogue number for the other sequence. Then clearly

$$\prod_{i=1}^{r-1} (-1)^{(p_i,p_{i+1})} = (-1)^k \quad \text{and} \quad \prod_{j=1}^{s-1} (-1)^{(q_j,q_{j+1})} = (-1)^l.$$

So we just have to show that k and l have the same parity.

We will assume that their parity is different and show that this contradicts the assumption that $\mathcal{C}$ contains no *white* node.

Without loss of generality let us assume that k is even while l is odd. Then we know via the first sequence that $x_p = x_q$ while the second sequence implies $x_p = -x_q$. So we conclude that $x_p = x_q = 0$ and hence $x_t = 0$ for all $t \in \mathcal{C}$.

Let j be an index such that $(-1)^{(q_j,q_{j+1})} = -1$. There are four possible cases for the pair $\{q_j, q_{j+1}\}$. It could be equal to $\{\xi_{m,n-1}, \xi_{m,n}\}$ or $\{\xi_{m,n-1}, \xi_{m+1,n}\}$ or $\{\xi_{m,n-1}, \xi_{m-1,n}\}$ or $\{\xi_{m,n-1}, \xi_{m+1,n-1}\}$ for some m.

Let us first assume that $\{q_j, q_{j+1}\} = \{\xi_{m,n-1}, \xi_{m,n}\}$. Since $y_{m,n-1} = y_{m,n} = 0$ and $\xi_{m,n-1} \geq \zeta^{\uparrow}_{m,n-1} \geq \xi_{m,n}$, we have

$$0 = y_{m,n} \leq z^{\uparrow}_{m,n-1} \leq y_{m,n-1} = 0$$

and hence $z^{\uparrow}_{m,n-1} = 0$. So we just have to show that $y_{m+1,n} = 0$ because then by construction all four nodes would be *white*. Indeed we have

$$0 = z^{\uparrow}_{m,n-1} \leq y_{m,n-1} + y_{m,n} + y_{m+1,n} = y_{m+1,n}$$

and $\xi_{m+1,n} \leq \zeta^{\uparrow}_{m,n-1}$, implying that $y_{m+1,n} \leq z^{\uparrow}_{m,n-1} = 0$. Thus $y_{m+1,n} = 0$, so all four nodes must be *white* contradicting the assumption on $\mathcal{C}$.

The cases $\{q_j, q_{j+1}\} = \{\xi_{m,n-1}, \xi_{m+1,n}\}$ and $\{q_j, q_{j+1}\} = \{\xi_{m,n-1}, \xi_{m-1,n}\}$ can be considered completely analogous by just exchanging symbols in our former reasoning, so we will not state the arguments for these cases again.

The final case $\{q_j, q_{j+1}\} = \{\xi_{m,n}, \xi_{m+1,n}\}$ however needs different arguments.

Notice that the concatenated sequence $(p = p_1, \ldots, p_r = q = q_s, \ldots, q_1 = p)$ forms a (connected) circle in the graph $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$. Since the nodes q_j and q_{j+1} are in the rightmost column, there must exist a node in this circle that is *left* of q_j and q_{j+1}. To be more precise, there must exist a node t in this circle such that $t \geq q_j$ and $t \geq q_{j+1}$. Since $x_t = \pm x_{q_j} = 0$, we know that every node $u \in \mathcal{GT}_n$ such that $t \geq u \geq q_j$ must fulfill $x_u = 0$.

Because $t \geq \xi_{m,n}$ and $t \geq \xi_{m+1,n}$, the structure of $\mathcal{GT}_n$ demands that $t \geq \zeta^{\uparrow}_{m,n-1}$ or $t \geq \zeta^{\downarrow}_{m,n-1}$. The following arguments can be applied to both possibilities, so without loss of generality let us assume that $t \geq \zeta^{\uparrow}_{m,n-1}$. By our previous considerations we know that $z^{\uparrow}_{m,n-1} = y_{m,n} = y_{m+1,n} = 0$. Now because there exists a *red* double arrow between $\xi_{m,n}$ and $\xi_{m+1,n}$, we know that

$$0 = z^{\uparrow}_{m,n-1} = y_{m,n-1} + y_{m,n} + y_{m,n+1} = y_{m,n-1},$$

so the coordinates corresponding to all four nodes $\xi_{m,n-1}$, $\xi_{m,n}$, $\xi_{m+1,n}$ and $\zeta^{\uparrow}_{m,n-1}$ are zero. Hence we would have labeled these nodes *white* in our construction, contradicting the assumption on $\mathcal{C}$. This concludes the proof. □

This result ensures that the following object is well-defined.

Definition 6.8.20. Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and let $\mathcal{C}$ be a connected component of its tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ that does not contain a *white* node. Let p and q be two nodes in $\mathcal{C}$ and let $(p_1, \ldots, p_t)$ be any connecting sequence between p and q. Then the **sign**

$\operatorname{sgn}(p, q)$ of p and q is defined as

$$\operatorname{sgn}(p,q) := \prod_{i=1}^{t-1} (-1)^{(p_i, p_{i+1})}.$$

To complete our proof of Theorem 6.8.16, we will need the following two lemmata.

Lemma 6.8.21. *Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and suppose its tweaked Gelfand-Tsetlin diagram contains a subgraph of the form*

$$\begin{tikzcd}[row sep=small]
 & & r \arrow[dl, bend right] \\
p \arrow[r, dashed] & q \arrow[l, bend left] \arrow[ur, dashed] \arrow[dr, dashed] & \\
 & & s \arrow[ul, bend left]
\end{tikzcd}$$

of pairwise different nodes with no other arrows between them. Then the nodes p, q and r (equivalently s) lie in pairwise different connected components. The nodes r and s lie in different connected components if $x_r \neq x_s$.

Proof. Let us first notice that the connected component of q must be just q itself since q can only be connected to the nodes p, r or s and no other node. So the claim is trivial concerning q.

Now notice that by definition we have $x_p > x_r$ because when $x_p = x_r$ we would have drawn *black* arrows $p \to q$ and $q \to r$ since $x_p = x_q$ and $x_q = x_r$. But these do not exist. So p and r could only be in the same connected component if $x_p = -x_r$. But then from the equality $x_q = x_p + x_r + x_s$ we would get $x_q = x_s$, which cannot be true since there does not exist a *black* arrow $q \to s$. The same argument works for s instead of r. So the connected component of p does not contain any of the other nodes.

Finally, suppose $x_r \neq x_s$. Then r and s could only lie in the same connected component if $x_r = -x_s$. This would imply (by the same equality as above) that $x_p = x_q$, which is not true (since we have no *black* arrow $p \to q$). ☐

Lemma 6.8.22. *Let $x \in \widetilde{GT}(\lambda)$ be a tweaked Gelfand-Tsetlin pattern and $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ be its tweaked Gelfand-Tsetlin diagram. Then for every $1 \leq i \leq n-2$ one of the following conditions hold. (The last two are not exclusive.)*

(i) The pair $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ is a triviality.

(ii) The pair $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ is a double impurity.

(iii) The node $\zeta^{\uparrow}_{i,n-1}$ is a single impurity.

(iv) The node $\zeta^{\downarrow}_{i,n-1}$ is a single impurity.

(v) There exist two pairs of black *arrows $\xi_{i,n-1} \leftrightarrows \zeta^{\uparrow}_{i,n-1}$ and $\xi_{i+1,n-1} \leftrightarrows \zeta^{\downarrow}_{i,n-1}$.*

(vi) There exists a pair of black *arrows* $\xi_{i,n-1} \leftrightarrows \xi_{i+1,n-1}$.

Proof. Let us assume that $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ is neither a triviality nor does it contain an impurity of any kind. Then both nodes must be connected (via *black* double arrows) to some other node(s).

Notice that the defining relations of $\widetilde{GT}(\lambda)$ require that

$$y_{i,n-1} - z^{\uparrow}_{i,n-1} = y_{i+1,n-1} - z^{\downarrow}_{i,n-1}.$$

This implies that $y_{i,n-1} = z^{\uparrow}_{i,n-1}$ if and only if $y_{i+1,n-1} = z^{\downarrow}_{i,n-1}$. So whenever there exists a pair of *black* arrows between $\xi_{i,n-1}$ and $\zeta^{\uparrow}_{i,n-1}$ there must exist a pair of *black* arrows between $\xi_{i+1,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ and vice versa.

So let us finally assume that these two pairs of *black* arrows do not exist. By our assumption we know that $\zeta^{\uparrow}_{i,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ must be connected to something. If they are connected to the same node, we are done since our construction would have produced a pair of *black* arrows between $\xi_{i,n-1}$ and $\xi_{i+1,n-1}$. We will now show that this is the only possibility.

Recall again that the nodes $\zeta^{\uparrow}_{i,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ must be connected to something, so let us assume that there exist two pairs of *black* arrows $\zeta^{\uparrow}_{i,n-1} \leftrightarrows \xi_{i,n}$ and $\zeta^{\downarrow}_{i,n-1} \leftrightarrows \xi_{i+1,n}$. The only other possible case $\zeta^{\uparrow}_{i,n-1} \leftrightarrows \xi_{i+1,n}$ and $\zeta^{\downarrow}_{i,n-1} \leftrightarrows \xi_{i,n}$ will follow analogously by swapping symbols.

From the poset structure of $\mathcal{GT}_n$ we know that $\zeta^{\uparrow}_{i,n-1} \geq \xi_{i+1,n}$ and $\zeta^{\downarrow}_{i,n-1} \geq \xi_{i,n}$. Thus

$$z^{\uparrow}_{i,n-1} \geq y_{i+1,n} = z^{\downarrow}_{i,n-1} \geq y_{i,n} = z^{\uparrow}_{i,n-1},$$

which implies that all four coordinates must take the same value. So there exist the additional two pairs of *black* arrows $\zeta^{\uparrow}_{i,n-1} \leftrightarrows \xi_{i+1,n}$ and $\zeta^{\downarrow}_{i,n-1} \leftrightarrows \xi_{i,n}$. Hence our construction would have produced the desired pair of *black* arrows $\xi_{i,n-1} \leftrightarrows \xi_{i+1,n-1}$, which concludes the proof. □

Let us introduce the following notation.

Notation 6.8.23. For every node $p \in \mathcal{GT}_n$ we define the vector $\chi_p \in \mathbb{R}^{\mathcal{GT}_n \setminus A}$ via

$$(\chi_p)_q = \delta_{p,q}.$$

We are now finally able to conclude the proof of Theorem 6.8.16.

Proof of Theorem 6.8.16 (Second Implication). We will explicitly construct a vector v such that $x \pm v \in \widetilde{GT}(\lambda)$ whenever the tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ contains a connected component $\mathcal{C}$ that is neither a single impurity nor part of a double impurity and additionally does not contain a marked element nor an anomaly.

It is clear that whenever we find a vector v such that $x \pm v$ lies on the same faces of $\widetilde{GT}(\lambda)$ as x — i.e. whenever one of the defining inequalities of $\widetilde{GT}(\lambda)$ is not fulfilled strictly by x — then we can find an $\epsilon > 0$ such that $x \pm \epsilon v \in \widetilde{GT}(\lambda)$. The reason is that $H^-_{\alpha,b} \setminus H_{\alpha,b}$ is a convex open set.

So we just have to show that there exists a vector $v \neq 0$ such that $x \pm v$ lies on the same faces as x since the rest is just a matter of scaling.

Let us first notice that whenever there exists a triviality $(\zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ we can chose $v = \chi_{\zeta^{\uparrow}_{i,n-1}} + \chi_{\zeta^{\downarrow}_{i,n-1}}$. This vector fulfills the desired properties after rescaling.

So let us assume that $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ contains no triviality.

Let p_0 be a node that is neither connected to a marked element nor an anomaly and suppose that it is neither a single impurity nor part of a double impurity. Let $\mathcal{C}$ denote its connected component. For every $p \in \mathcal{C}$ set

$$v_p := \operatorname{sgn}(p_0, p)$$

Additionally set $v_p = 0$ whenever $p \neq \mathcal{C}$ is not an impurity (neither single nor contained in double). We leave the coordinates at the impurities open for the moment.

It is clear that the vector v — independent of its coordinates at the impurities — will fulfill $(x \pm v)_p = (x \pm v)_q$ whenever there exists a pair of *black* arrows between p and q. This deals with all the *poset like* inequalities defining $\widetilde{GT}(\lambda)$. Additionally, whenever x fulfills an equality of type

$$z^{\uparrow}_{i,n-1} = y_{i,n-1} + y_{i,n} + y_{i+1,n} \quad \text{or} \quad z^{\downarrow}_{i,n-1} = y_{i+1,n-1} + y_{i,n} + y_{i+1,n}$$

and there exists a pair of *black* arrows between any two of the respective nodes — in terms of our construction this means that we would have drawn some *red* double arrows — the analogue equation is fulfilled by v and hence by $x + v$ and $x - v$ too.

So it remains to chose the coordinates at the impurities correctly such that $v \in \mathcal{V}_\lambda$ and for every quadruplet

$$\begin{tikzcd}[row sep=small] & & r \\ p \arrow[r, dashed] & q \arrow[ur, dashed] \arrow[dr, dashed] & \\ & & s \end{tikzcd}$$

we have $v_q = v_p + v_r + v_s$. Let us start with the second set of equalities.

By Lemma 6.8.21 we have three different cases. If none of the occurring nodes lies in $\mathcal{C}$ we set $v_q := 0$. If precisely one of the nodes p, r or s lies in $\mathcal{C}$, set $v_q := v_t$ where t is the unique adjacent node of q lying in $\mathcal{C}$. If r and s lie in $\mathcal{C}$ we know that $x_r = x_s$, hence $v_r = v_s$. In this case we set $v_q = 2v_r$. With these choices it is clear that $v_q = v_p + v_r + v_s$ as desired.

It is clear that the vector v fulfills

$$v_{\xi_{i,n-1}} - v_{\xi_{i+1,n-1}} = v_{\zeta^{\uparrow}_{i,n-1}} - v_{\zeta^{\downarrow}_{i,n-1}} \tag{$\star$}$$

automatically whenever it fulfills

$$v_{\zeta^{\uparrow}_{i,n-1}} = v_{\xi_{i,n-1}} + v_{\xi_{i,n}} + v_{\xi_{i+1,n}} \quad \text{and} \quad v_{\zeta^{\downarrow}_{i,n-1}} = v_{\xi_{i+1,n-1}} + v_{\xi_{i,n}} + v_{\xi_{i+1,n}}.$$

By our construction these two equalities hold whenever the analogue equalities hold for x. This especially contains the case of double impurities and single impurities incident to *red* arrows.

For every single impurity $\zeta^{\uparrow}_{i,n-1}$ or $\zeta^{\downarrow}_{i,n-1}$ that is not incident to *red* arrows, we will simply define the corresponding coordinate of v to be the unique real number fulfilling ($\star$). This is possible since three of these values are already determined by our previous construction.

With this step we have set all coordinates of v. We have already seen that $x \pm v$ fulfills most of the equalities that are fulfilled by x. We only have to verify some defining equalities of $\mathcal{V}_\lambda$ — namely those where none of the coordinates corresponds to an impurity.

So let us finally consider an arbitrary quadruplet $(\xi_{i,n-1}, \xi_{i+1,n-1}, \zeta^{\uparrow}_{i,n-1}, \zeta^{\downarrow}_{i,n-1})$ that neither contains a single nor a double impurity. Let us assume one of these nodes actually lies in $\mathcal{C}$ because ($\star$) holds trivially otherwise.

Now Lemma 6.8.22 implies that there are only two possibilities. There exist two pairs of *black* arrows $\xi_{i,n-1} \leftrightarrows \zeta^{\uparrow}_{i,n-1}$ and $\xi_{i+1,n-1} \leftrightarrows \zeta^{\downarrow}_{i,n-1}$ or there exists one pair of *black* arrows $\xi_{i,n-1} \leftrightarrows \xi_{i+1,n-1}$. In both cases we see that ($\star$) holds since the first case implies that

$$v_{\xi_{i,n-1}} - v_{\zeta^{\uparrow}_{i,n-1}} = 0 = v_{\xi_{i+1,n-1}} - v_{\zeta^{\downarrow}_{i,n-1}}$$

while the second case implies that both nodes $\zeta^{\uparrow}_{i,n-1}$ and $\zeta^{\downarrow}_{i,n-1}$ are connected via *black* double arrows to the very same node. So their coordinates are equal and thus

$$v_{\xi_{i,n-1}} - v_{\xi_{i+1,n-1}} = 0 = v_{\zeta^{\uparrow}_{i,n-1}} - v_{\zeta^{\downarrow}_{i,n-1}}.$$

In conclusion, we have constructed a vector v such that $x \pm \epsilon v \in \widetilde{GT}(\lambda)$ for $\epsilon > 0$ small enough. By Lemma 6.6.6 this implies that x is not a vertex of $\widetilde{GT}(\lambda)$, which finally concludes our proof. $\square$

We will now come back to string polytopes. For that purpose we introduce the following notation.

Notation 6.8.24. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ let $\Gamma(\lambda) \subseteq \mathbb{R}$ denote the free abelian group generated by the coefficients $\lambda_1, \ldots, \lambda_n$, i.e.

$$\Gamma(\lambda) := \mathbb{Z}\lambda_1 + \ldots \mathbb{Z}\lambda_n.$$

This will allow us to state the following crucial result.

Theorem 6.8.25. *Let $\lambda \in \mathbb{R}^n$ and let $x \in \widetilde{GT}(\sum_{i=1}^n \lambda_i \epsilon_i)$ be a vertex of the tweaked Gelfand-Tsetlin polytope. Then every coordinate of x lies in $\Gamma(\lambda)$.*

Proof. Let $p \in \mathcal{GT}_n$. We want to calculate x_p via the tweaked Gelfand-Tsetlin diagram $\mathcal{D}^\lambda_{\mathsf{D}_n}(x)$.

By Theorem 6.8.16 we know that p is connected to a marked element, is connected to an anomaly, is a single impurity or is part of a double impurity.

By construction of $\mathcal{D}^\lambda_{\mathsf{D}_n}(x)$ we know that $x_p = \pm\lambda_j$ if p is connected to $\xi_{1,j}$. If p is connected to an anomaly, we know that $x_p = 0$. So in both cases x_p lies in $\{\pm\lambda_1, \ldots, \pm\lambda_n\} \cup \{0\} \subseteq \Gamma(\lambda)$.

Let us finally assume that p is a single impurity or part of a double impurity. We will only state the proof for $p = \zeta^\uparrow_{i,n}$ because the proof for $\zeta^\downarrow_{i,n-1}$ is completely analogous.

If $p = \zeta^\uparrow_{i,n-1}$ is incident to *red* arrows, we know that $x_p = z^\uparrow_{i,n-1}$ can be calculated as $z^\uparrow_{i,n-1} = y_{i,n-1} + y_{i,n} + y_{i+1,n}$. Since the latter three coordinates lie in $\Gamma(\lambda)$, the same holds true for $z^\uparrow_{i,n-1} = x_p$.

If $p = \zeta^\uparrow_{i,n-1}$ is not incident to *red* arrows, we know that $\zeta^\downarrow_{i,n-1}$ cannot be an impurity too. So the value $x_p = z^\uparrow_{i,n-1}$ is uniquely determined by the equation $z^\uparrow_{i,n-1} = z^\downarrow_{i,n-1} + y_{i,n-1} - y_{i+1,n-1}$. Since the latter three coordinates lie in $\Gamma(\lambda)$, the same holds true for $z^\uparrow_{i,n-1} = x_p$. This concludes our proof. □

During this proof we have actually shown the following special case.

Corollary 6.8.26. *Let $\lambda \in (\frac{1}{2} + \mathbb{Z})^n$ and let $x \in \widetilde{GT}(\sum_{i=1}^n \lambda_i \epsilon_i)$ be a vertex of the tweaked Gelfand-Tsetlin polytope. Then*

(i) $x_p = 0$ *if p is connected to an anomaly,*

(ii) $x_p \in \frac{1}{2}\mathbb{Z}$ *if p is a single impurity or part of a double impurity, and*

(iii) $x_p \in \frac{1}{2} + \mathbb{Z}$ *for any other p.*

We can now prove the following useful translation.

Corollary 6.8.27. *Let G be of type D_n and $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. Let x be a vertex of the standard string polytope $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$. Then x is a lattice point if and only if* *(i)* $\lambda_i \in \mathbb{Z}$ *for all* i *or*

(ii) $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ *for all* i *and* $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(\widetilde{\phi}_\lambda(x))$ *does not contain any anomaly.*

Proof. By Theorem 6.7.6 we know that x is a vertex of $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ if and only if $\widetilde{\phi}_\lambda(x)$ is a vertex of $\widetilde{GT}(\lambda)$. Additionally, x will be a lattice point if and only if the coordinates of $\widetilde{\phi}_\lambda(x)$ are either all integers or all in $\frac{1}{2} + \mathbb{Z}$. The claim now follows directly from Theorem 6.8.25 if $\lambda_i \in \mathbb{Z}$ for all i.

Let us consider the other case $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all i. Notice that most of the coordinates will be in $\frac{1}{2} + \mathbb{Z}$ by Corollary 6.8.26. If $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$ contains an anomaly, there will be coordinates equal to zero. So x cannot be a vertex in this case. So assume that there does not exist an anomaly in $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$. The only nodes whose corresponding coordinate could potentially be not in $\frac{1}{2} + \mathbb{Z}$ would be single or double impurities. But the proof of Theorem 6.8.25 shows that these coordinates can be calculated as a sum of three coordinates in $\frac{1}{2}+\mathbb{Z}$. Hence they must be in $\frac{1}{2} + \mathbb{Z}$ as well. □

We can now finally prove the last remaining case of Theorem 6.3.1.

Proof of Theorem 6.3.1 in Type D_n. Let G be of type D_n and fix a dominant weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda^+$. If $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle$ is an even integer, we know that $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq n$. So $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ will be a lattice polytope by Corollary 6.8.27.

If $\langle \lambda, \alpha_{n-1}^\vee \rangle + \langle \lambda, \alpha_n^\vee \rangle$ is an odd integer, we know that $\lambda_i \in \frac{1}{2} + \mathbb{Z}$ for all $1 \leq i \leq n$. Since Theorem 6.7.6 yields a bijection between the vertices of $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ and the vertices of $\widetilde{GT}(\lambda)$, it is necessary and sufficient to find a vertex x of $\widetilde{GT}(\lambda)$ that contains an anomaly in its tweaked Gelfand-Tsetlin diagram $\mathcal{D}^{\lambda}_{\mathsf{D}_n}(x)$. Then Corollary 6.8.27 implies that the vertex $\widetilde{\phi}_\lambda^{-1}(x)$ of $\mathcal{Q}_{\underline{w_0}^{\mathrm{std}}}(\lambda)$ will not be a lattice point.

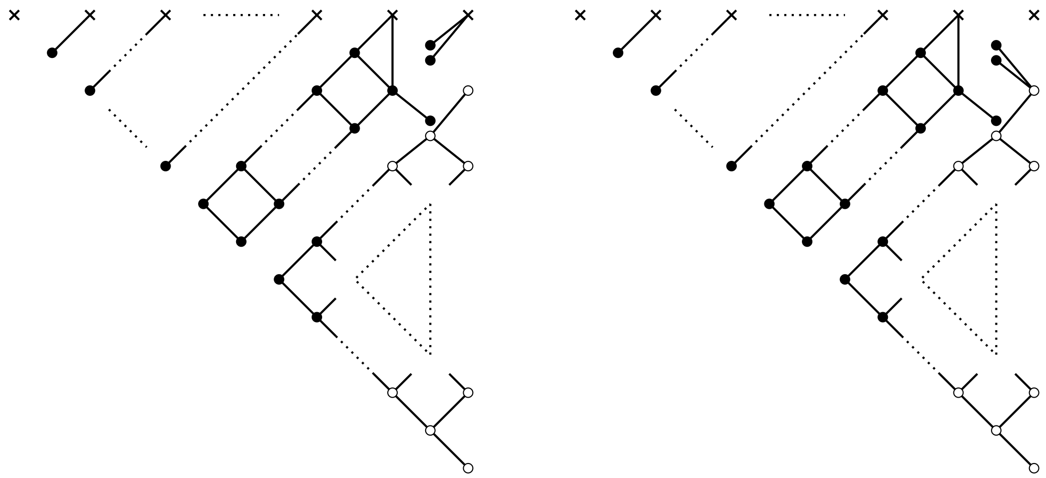
Notice that for $n < 4$ any tweaked Gelfand-Tsetlin diagram cannot contain an anomaly if $\lambda_n \neq 0$. So for small ranks all standard string polytopes must be lattice polytopes.

For $n \geq 4$ consider the pattern in Figure 6.11. We can verify quite easily that this pattern is indeed a tweaked Gelfand-Tsetlin pattern for λ. The only nontrivial (in)equality to verify is $\lambda_{n-1} + \lambda_n \geq 0$. But this is true for any dominant integral weight in type D_n. The tweaked Gelfand-Tsetlin diagram of this pattern is drawn in Figure 6.12. We see that every connected component contains a marked element or an anomaly. Hence this pattern is indeed a vertex of the tweaked Gelfand-Tsetlin polytope containing an anomaly in its tweaked Gelfand-Tsetlin diagram. This concludes the proof. □

Figure 6.11: A special tweaked Gelfand-Tsetlin pattern in type D_n for an arbitrary weight λ.

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \dots & & \dots & & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n \\
 & \lambda_2 & & & & & & & & \lambda_{n-1} & & \substack{\max\{\lambda_n,0\} \\ \max\{\lambda_n,0\}} & \\
 & & \ddots & & & & & & \lambda_{n-1} & & \lambda_{n-1} & & 0 \\
 & & & \ddots & & & & \iddots & & \lambda_{n-1} & & \substack{\lambda_{n-1} \\ 0} & \\
 & & & & \lambda_{n-2} & & \lambda_{n-1} & & \iddots & & 0 & & 0 \\
 & & & & & \lambda_{n-1} & & \lambda_{n-1} & & \iddots & & & \\
 & & & & & & \lambda_{n-1} & & 0 & & & & \vdots \\
 & & & & & & & 0 & & & & & \\
 & & & & & & & & 0 & & & & \vdots \\
 & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & \ddots & & 0 \\
 & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & 0
\end{array}
$$

Figure 6.12: Tweaked Gelfand-Tsetlin diagrams of the tweaked Gelfand-Tsetlin pattern from Figure 6.11. Left hand side for $\lambda_n > 0$, right hand side for $\lambda_n < 0$. We decided to only draw the case where $\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > \lambda_n$ since any equality between the coefficients of λ would only result in more pairs of *black* arrows connecting formerly disjoint connected components, thus not changing any of our arguments.



Bibliography

[1] V. Alexeev, M. Brion, *Toric degeneration of spherical varieties*. Selecta Math., Vol. 10, No. 4, pp. 453 – 478, 2004.

[2] D. Anderson, *Okounkov bodies and toric degenerations*. Math. Ann. 356, no. 3, 1183 – 1202, 2013.

[3] F. Ardila, T. Bliem, D. Salazar, *Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes*. Journal of Combinatorial Theory, Series A 118.8, pp. 2454 – 2462, 2011.

[4] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*. J. Algebraic Geom., Vol. 3, pp. 493 – 535, 1994.

[5] M. Beck, S. Robins, *Counting the Continuous Discretely*. Springer, 2009.

[6] R. Bott, *Homogeneous vector bundles*. Ann. of Math., No. 66, 203 – 248, 1957.

[7] N. Bourbaki, *Algèbre de Lie VI-VII*. Chap. 4 – 6, Hermann, Paris, 1968, in French.

[8] A. Berenstein, A. Zelevinsky, *Tensor product multiplicities and convex polytopes in partition space*. J. Geom. Phys., Vol. 5, pp. 453 – 472, 1989.

[9] A. Berenstein, A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*. Invent. Math., Vol. 143, No. 1, pp. 77 – 128, 2001.

[10] P. Caldero, *Toric degenerations of Schubert varieties*. Transformation Groups, Vol. 7, No. 1, pp. 51 – 60, 2002.

[11] Y. Cho, Y. Kim, E. Lee, K. Park, *On the combinatorics of string polytopes*. Preprint, arXiv:1904.00130, 2019.

[12] D. Cox, J.B. Little, H.K. Schenck, *Toric varieties*. Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[13] V. I. Danilov (1978), *The Geometry of Toric Varieties.* Russian Mathematical Surveys, Vol. 33., pp. 97 – 154, 1978.

[14] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes.* Inst. Hautes Études Sci. Publ. Math. No. 8, 1961, in French.

[15] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.* Inst. Hautes Études Sci. Publ. Math. No. 11, 1961, in French.

[16] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II.* Inst. Hautes Études Sci. Publ. Math. No. 24, 1965, in French.

[17] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III.* Inst. Hautes Études Sci. Publ. Math. No. 28, 1966, in French.

[18] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150, Springer, New York, 1995.

[19] R. Elkik, *Singularités rationnelles et deformations.* Invent. Math., No. 47, pp. 139 – 147, 1978, in French.

[20] X. Fang, G. Fourier, *Marked chain-order polytopes.* European Journal of Combinatorics, No. 58, pp. 267 – 282, 2016.

[21] X. Fang, G. Fourier, P. Littelmann, *Essential bases and toric degenerations arising from birational sequences.* Advances in Mathematics, Vol. 312, pp. 107 – 149, 2017.

[22] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for irreducible modules in type A_n.* Transformation Groups, Vol. 16, No. 1, pp. 71 – 89, 2011.

[23] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for symplectic Lie algebras.* International Mathematics Research Notices, Vol. 24, pp. 5760 – 5784, 2011.

[24] E. Feigin, G. Fourier, P. Littelmann, *Favourable modules: Filtrations, polytopes, Newton-Okounkov bodies and flat degenerations.* Transformation Groups, Vol. 22, No. 2, pp. 321 – 352, 2017.

[25] N. Fujita, S. Naito, *Newton-Okounkov convex bodies of Schubert varieties and polyhedral realizations of crystal bases*. Mathematische Zeitschrift, Vol. 285, No. 1 – 2, pp. 325 – 352, 2017.

[26] W. Fulton, *Introduction to Toric Varieties*. Princeton University Press, 1993.

[27] I. Gelfand, M. Tsetlin, *Finite-dimensional representations of the group of unimodular matrices*. Dokl. Akad. Nauk SSSR, Vol. 71, pp. 825 – 828, 1950.

[28] N. Gonciulea, V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*. Transformation Groups, Vol. 1, No. 3, pp. 215 – 248, 1996.

[29] A. Gornitskii, *Essential signatures and canonical bases in irreducible representations of the group G_2*. Diploma thesis, 2011 (in Russian).

[30] A. Gornitskii, *Essential Signatures and Canonical Bases for B_n and D_n*. Preprint, arXiv:1611.07381, 2016.

[31] H. Grauert, O. Riemenschneider, *Verschwindungssätze für analytische Kohomologie auf komplexen Räumen*. Invent. Math., No. 11, pp. 263 – 297, 1970, in German.

[32] B. Hall, *Lie groups, Lie algebras, and representations: An elementary introduction*. Graduate Texts in Mathematics, No. 222 (2nd ed.), Springer 2015.

[33] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[34] T. Hibi, *Dual polytopes of rational convex polytopes*. Combinatorica, Vol. 12, No. 2, pp. 237 – 240, 1992.

[35] J. Humphries, *Introduction to Lie Algebras and Representation Theory*. Springer, 1972.

[36] S. Ishii, *Introduction to singularities*. Springer, 2014.

[37] K. Kaveh, *Crystal bases and Newton-Okounkov bodies*. Duke Math. J., Vol. 164, No. 13, pp. 2461 – 2506, 2015.

[38] K. Kaveh, A. G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*. Ann. of Math., Vol. 176, No. 2, pp. 925 – 978, 2012.

[39] K. Kaveh, C. Manon, *Khovanskii bases, higher rank valuations and tropical geometry.* SIAM J. Appl. Algebra Geom. Vol. 3, No. 2, 292 – 336., 2019.

[40] K. Kaveh, E. Villella, *On a notion of anticanonical class for families of convex polytopes.* Preprint, arXiv:1802.06674, 2018.

[41] V. Kiritchenko, *Newton-Okounkov polytopes of flag varieties.* Transformation Groups, Vol. 22, No. 2, pp. 387 – 402, 2017.

[42] M. Kogan, E. Miller, *Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes.* Advances in Mathematics, Vol. 193, No. 1, pp. 1-17, 2015.

[43] J. Kollár, S. Mori, *Birational geometry of algebraic varieties.*. Cambridge Texts in Mathematics, No. 134, Cambridge University Press, 1998.

[44] B. Kostant, *Lie Algebra Cohomology and the Generalized Borel-Weil Theorem.* Ann. of Math., Vol. 74, No. 2, pp. 329 – 387, 1961.

[45] S. J. Kovács, K. Schwede, K. E. Smith, *The canonical sheaf of Du Bois singularities.* Adv. Math., Vol. 224, No. 4, pp. 1618 – 1640, 2010.

[46] R. Lazarsfeld, M. Mustață, *Convex bodies associated to linear series.* Ann. de l' ENS 42, pp. 783 – 835, 2009.

[47] P. Littelmann, *Cones, Crystals and Patterns.* Transformation Groups, Vol. 3, No. 2, pp. 145 – 179, 1998.

[48] G. Lusztig, *Canonical bases arising from quantized enveloping algebras.* J. Amer. Math. Soc., Vol. 3, No. 2, pp. 447-498, 1990.

[49] T. Nakashima, A. Zelevinsky, *Polyhedral realizations of crystal bases for quantized Kac-Moody algebras.* Adv. Math., Vol. 131, pp. 253 – 278, 1997.

[50] B. Nill, *Gorenstein toric Fano varieties.* Manuscripta Math., No. 116, no. 2, pp. 183 – 210, 2005.

[51] A. Okounkov, *Brunn-Minkowski inequality for multiplicities.* Invent. Math., Vol. 125, No. 3, pp. 405 – 411, 1996.

[52] A. Okounkov, *Multiplicities and Newton polytopes.* Kirillov's seminar on representation theory, in Amer. Math. Soc. Transl. Ser. 2, Vol. 181, pp. 231 – 244, 1998.

[53] C. Procesi, *Lie Groups: An Approach Through Invariants and Representations*. Universitext, Springer, New York, 2006.

[54] K. Rietsch, L. Williams, *Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians*. Preprint, arXiv:1712.00447, 2017.

[55] J. Rusinko, *Equivalence of mirror families constructed from toric degenerations of flag varieties*. Transformation Groups, Vol. 13, No. 1, pp. 173 – 194, 2008.

[56] J. P. Serre, *Faisceaux algébriques cohérents.* The Annals of Mathematics, 2nd Ser., Vol. 61, No. 2, pp. 197 – 278, 1955, in French.

[57] T. Springer, *Linear Algebraic Groups*. Second Edition, Birkhäuser, 1998.

[58] R. Stanley, *Decompositions of Rational Convex Polytopes*. Annals of Discrete Mathematics, No. 6, pp. 333 – 342, 1980.

[59] R. Stanley, *Two poset polytopes*. Discrete Comput. Geom., No. 1, pp. 9 – 23, 1986.

[60] W. A. Stein et al., *Sage Mathematics Software*. Version 9.0, The Sage Development Team, http://www.sagemath.org, 2020.

[61] C. Steinert, *Reflexivity of Newton-Okounkov Bodies of Partial Flag Varieties*. Preprint, arXiv:1902.07105v1, 2019.

[62] H. Sumihiro, *Equivariant completion I, II.* J. Math. Kyoto Univ., No. 14, pp. 1 – 28, 1974 and No. 15, pp. 573 – 605, 1975.

[63] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*. Preprint, November 18, 2017 version, http://math.stanford.edu/ vakil/216blog/index.html.

[64] G. Ziegler, *Lectures on Polytopes*. Graduate Texts in Mathematics, No. 152, 7th edition, Springer, 2007.

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Christian Steinert