Universität zu Köln

Seminar für Wirtschafts- und Sozialstatistik

# Tests of Stochastic Dominance for Time Series Data

## Theory and Empirical Application

Dissertation

zur Erlangung des Doktorgrades Dr. rer. pol.

der Wirtschafts- und Sozialwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Dipl.-Math. Hendrik Kläver

Köln, im Februar 2006

| | |
|---|---|
| Referent: | Prof. Dr. F. Schmid |
| Korreferent: | Prof. Dr. K. Mosler |
| Tag der Promotion: | 10.02.2006 |

# Contents

**Vorwort** **5**

**Abstract** **7**

**1 Introduction and Summary** **11**

**2 Stochastic Dominance as a Decision Criterion** **19**

2.1 Theory of Stochastic Dominance . . . . . . . . . . . . . . . . . . . . 19

2.2 Descriptive Stochastic Dominance . . . . . . . . . . . . . . . . . . . 33

**3 Tests of Stochastic Dominance: A Survey** **37**

**4 GARCH Effects on Statistical Procedures** **49**

4.1 Notations, Definitions and Preliminary Results . . . . . . . . . . . 51

4.2 Procedures for Means . . . . . . . . . . . . . . . . . . . . . . . . . 53

4.3 Procedures for Variances . . . . . . . . . . . . . . . . . . . . . . . 57

4.3.1 Confidence interval for $\sigma^2$ . . . . . . . . . . . . . . . . . . 59

4.3.2 F-test for equality of variances . . . . . . . . . . . . . . . . 60

4.4 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63

**5 Testing for Stochastic Dominance Using Circular Block Methods** **67**

5.1 Tests Based on Resampling Methods . . . . . . . . . . . . . . . . . 69

5.1.1 A Permutation Test from Matched Pairs . . . . . . . . . . . 69

5.1.2 Tests Using Moving Block Methods . . . . . . . . . . . . . . 70

5.2 Simulation Results for the Conventional Tests . . . . . . . . . . . 73
5.3 Circular Block Methods as an Alternative Concept . . . . . . . . 80
5.4 Simulation Results Using Circular Block Methods . . . . . . . . . 84
5.5 Power Investigation . . . . . . . . . . . . . . . . . . . . . . . . . 89
5.6 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 97

**6 A Test in which Stochastic Dominance is the Alternative 99**

6.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . 99
6.2 Previous Tests in which Stochastic Dominance is the Alternative . 100
6.3 Discussion and Modification . . . . . . . . . . . . . . . . . . . . 102
6.4 Simulation Results . . . . . . . . . . . . . . . . . . . . . . . . . 109
6.5 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 129

**7 Empirical Application of SD Tests to German Stock Returns 131**

7.1 Methodology . . . . . . . . . . . . . . . . . . . . . . . . . . . . 131
7.2 Data . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 133
7.3 Results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 135

**Appendix 159**

**List of Symbols 169**

**List of Abbreviations 173**

**Bibliography 175**

# Vorwort

Der Abschluss dieser Dissertation stellt den Schlusspunkt einer sehr interessanten und prägenden Promotionszeit dar. Die drei Jahre im Graduiertenkolleg "Theoretische und empirische Grundlagen des Risikomanagements" waren eine abwechslungsreiche Zeit mit zahlreichen Herausforderungen.

Auf das Promotionsthema aufmerksam wurde ich durch meinen Doktorvater Prof. Dr. Friedrich Schmid, dem mein besonderer Dank gebührt. Er ist ein hervorragender akademischer Lehrer, der mein Interesse in Ökonometrie und Statistik vertieft und mich als kritischer Ratgeber optimal gefördert hat. Prof. Dr. Karl Mosler danke ich für die Übernahme des Zweitgutachtens und viele interessante und fruchtbare Diskussionen. Weiterhin danke ich Prof. Dr. Dieter Hess für den Vorsitz der mündlichen Prüfung.

Besonders wichtig und wesentlich für das Gelingen des Promotionsprojekts waren die Förderung und die wissenschaftliche Arbeit im Graduiertenkolleg. Mein Dank gilt der Deutschen Forschungsgemeinschaft (DFG) für die umfangreiche finanzielle Förderung. Ich danke Prof. Dr. Alexander Kempf, dem Sprecher des Graduiertenkollegs, für die Einrichtung und Organisation, und Silke Ber für die Koordination des Graduiertenkollegs. Weitere Würze erhielt das Graduiertenkolleg durch Interdisziplinarität sowie zahlreiche fachliche und auch nicht-fachliche Diskussionen. Besonders danke ich Daniel Mayston, der mich bei der sprachlichen Überarbeitung der in englischer Sprache geschriebenen Artikel unterstützt hat, und meinem Zimmernachbarn Hendrik Vollrath, mit dem ich, auch durch unseren gemeinsamen Hintergrund als Mathematiker, viele interessante Gespräche geführt habe. Weiter danke ich meinen Kollegiat(inn)en Michaela Bär, Thomas Berry-Stölzle, Felix Müsgens, Peter Scherpereel, Qian Wang und Nikolaus Wrede.

Ich danke dem gesamten Team des Lehrstuhls für Wirtschafts- und Sozialstatistik, die in den aufregenden Monaten der Dissertationsanfertigung und Disputationsvorbereitung angenehme Kollegen waren. Weiterhin gebührt Veronica Weisser und Ruth Esser mein besonderer Dank für die sehr wertvollen und konstruktiven sprachlichen Verbesserungshinweise der auf Englisch abgefassten Dissertationsschrift.

Der Promotion vorausgegangen ist ein Diplomstudium der Mathematik an der Universität Münster. Mein Dank gilt unter verschiedenen Hochschullehrern und Mitarbeitern besonders Prof. Dr. Norbert Schmitz, der in mir das Interesse für Wahrscheinlichkeitstheorie und Statistik geweckt hat und meine Diplomarbeit auf hervorragende Weise betreut hat. Dem Cusanuswerk danke ich für die materielle und ideelle Förderung während meines Mathematikstudiums. Die Angebote und die entstandenen Kontakte und Freundschaften sind eine echte Bereicherung für mein Leben.

Von besonderer Bedeutung bei der Promotion war die Unterstützung durch mein privates Umfeld. Ich danke meinen Eltern Heinz und Hildegard für alles, was sie mir auf den Weg gegeben haben, und meinen Geschwistern Thorsten, Sarah und Ruth. Ebenfalls danke ich meinen Freunden für ihre psychologische Unterstützung. Besonders hervorheben möchte ich Bianca Allendorf, Henrik und Karin Buchholz, Kevin und Maike Buchin, Sanela Čeljo-Hörhager, Stephan Engelhard, Florian Fritzsche, Christoph Heinatsch, Tobias Hessel, Sven Kerr, Robert und Rüdiger Kiauka, Holger Koesters, Marc Kückmann, Irmhild Kühn, Christoph Mönninghoff, Christiane Reher, Mustafa Saidi, Michael Streitbörger, Marcus Wrede und meine Sangesfreunde von "Troubadix Erben". Ganz besonders danke ich meiner Freundin Linda, die mich gerade in schwierigen Phasen immer wieder aufgebaut hat und verlässlich hinter mir stand.

# Abstract

The present thesis investigates the problem of testing for stochastic dominance which is a basic concept of decision theory. We focus on stochastic dominance of first and of second degree which are the most important stochastic dominance decision rules. These decision rules are applied in various branches of economics, e.g. finance and social welfare theory. Statistical tests for stochastic dominance are based on the difference of the empirical distribution or quantile functions. For the determination of the critical region we need the distribution of the test statistic under the null hypothesis. In many tests contemporaneous and serial independence are assumed. However, in many applications the observations do not satisfy these constraints. In particular, financial data usually feature positive correlation between the observations of different samples at the same time and conditional heteroskedasticity within each sample. We confine ourselves to bivariate GARCH (1,1) which has good fit and forecast properties for financial data. In order to get an idea about the impact of conditional heteroskedasticity on statistical tests we analyze its effect on common statistical procedures for means and variances. It turns out that the effect on procedures for variances is tremendous. The main issue of this thesis is the development of tests for stochastic dominance which are robust to these time series properties. Two kinds of tests are considered: on the one hand tests in which dominance is the hypothesis, on the other hand tests with the alternative of dominance. Most of the tests known from literature are not robust to conditional heteroskedasticity. We develop two tests for the first test problem and one test for the second test problem which can be used under the assumption that the underlying processes are strongly mixing. Theoretical analysis and simulations show that the tests using circular subsampling, block permutation and truncation, respectively, have good size and power properties even if conditional heteroskedasticity is prevalent in the data. The developed tests are applied to the daily returns of the stocks of the German stock index DAX.

Die vorliegende Arbeit untersucht Tests auf stochastische Dominanz, welche ein grundlegendes Konzept der Entscheidungstheorie ist. Hierbei konzentrieren wir uns auf stochastische Dominanz erster und zweiter Ordnung. Diese sind die beiden wichtigsten Entscheidungsregeln und finden Anwendung in verschiedenen Bereichen der Wirtschaftswissenschaften, z.B. Finanzwirtschaft und Wohlfahrtstheorie. Tests auf stochastische Dominanz basieren auf der Differenz der empirischen Verteilungs- oder Quantilfunktionen. Für die Bestimmung des kritischen Wertes benötigen wir die Verteilung der Teststatistik unter der Nullhypothese. Obwohl in vielen Tests kontemporäre und serielle Unabhängigkeit der Beobachtungen angenommen werden, sind diese Voraussetzungen bei vielen Anwendungen nicht erfüllt. Insbesondere Finanzmarktdaten weisen positive Korrelation zwischen den Beobachtungen zu derselben Zeit und bedingte Heteroskedastizität innerhalb der jeweiligen Zeitreihen auf. Wir beschränken uns auf bivariaten GARCH (1,1), der bezüglich Anpassung und Vorhersage gut für Finanzmarktdaten geeignet ist. Um einen Eindruck über den Einfluss bedingter Heteroskedastizität auf die Gültigkeit statistischer Tests zu erhalten, untersuchen wir ihren Effekt auf inferenzstatistische Methoden für Erwartungswerte und Varianzen. Es zeigt sich, dass Varianzprozeduren durch GARCH (1,1) stark beeinflusst werden. Hauptthema dieser Arbeit ist die Entwicklung von Tests auf stochastische Dominanz, die robust gegenüber Zeitreiheneigenschaften sind. Zwei

Arten von Tests werden untersucht: einerseits Tests, bei denen stochastische Dominanz die Hypothese ist, andererseits Tests mit Dominanzalternative. Wir entwickeln zwei Tests für das erste Testproblem and einen Test für das zweite Testproblem, die anwendbar sind unter der Bedingung, dass die Prozesse stark mischend sind. Theoretische Untersuchung and Simulationen zeigen, dass die Tests durch die Benutzung von Circular Subsampling, Blockpermutation bzw. geeignetes Abschneiden gute Niveau- und Powereigenschaften erhalten. Dies gilt auch unter dem Einfluss bedingter Heteroskedastizität. Die entwickelten Tests werden angewendet auf die Tagesrenditen der Aktien des deutschen Aktienindex DAX.

# Chapter 1

# Introduction and Summary

The present thesis investigates the problem of testing for stochastic dominance which is a basic concept of decision theory. Since the seminal study of von Neumann/ Morgenstern (1944), maximizing the expected utility is the main approach in the issue of decision making under risk. Due to the fact that utility perception and risk aversion (or risk proneness) differ among different people, it is generally impossible to give universal advice on how to maximize expected utility. However, one can look for a decision rule which is rational for a large class of individuals. The mean-variance approach by Markowitz (1952) proved to be a great contribution, but it also has its drawbacks. It does not consider the full distribution, but only the parameters mean and variance. Hence it neglects a lot of important information and, in some cases, promotes decisions which are obviously not reasonable; an example is given, for instance, by Hanoch/ Levy (1969).

A more universal decision rule and a benchmark for other rules is the concept of stochastic dominance introduced by Quirk/ Saposnik (1962). If one alternative dominates another one in the sense of stochastic dominance, it is preferred by a class of individuals having some similarities in their utility functions. The most important stochastic dominance decision rules are stochastic dominance of first and of second degree, abbreviated by SD1 and SD2, respectively. Let $X$ and $Y$

be random variables standing for payoff. $X$ dominates $Y$ in the sense of SD1 if every individual preferring more over less prefers $X$. This is the most important stochastic dominance rule. SD2 is less restrictive: if $X$ is preferred to $Y$ by every individual who prefers more over less and is risk averse or risk neutral, $X$ dominates $Y$ in the sense of second degree stochastic dominance. It is obvious that SD1 implies SD2. Stochastic dominance decision rules are applied in various branches of economics, e.g. finance and social welfare theory. Furthermore, the concept of stochastic dominance plays an important role in some other sciences, for instance agricultural economics (Langyintuo/ Yiridoe/ Dogbe/ Lowenberg-Debour, 2005) and environmental sciences (Maasoumi/ Millimet, 2005).

For the detection of stochastic dominance there are some useful characterizations concerning the distribution and quantile functions. $X$ dominates $Y$ in the sense of SD1 if and only if $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}$ which is equivalent to $Q_X(p) \geq Q_Y(p)$ for all $p \in (0,1)$. This means that a uniform order of the distribution and quantile functions is equivalent to SD1. Furthermore, SD2 is equivalent to the relations $\int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt$ for all $x \in \mathbb{R}$ and $\int_0^p Q_X(t)dt \geq \int_0^p Q_Y(t)dt$ for all $p \in (0,1)$. These relations are the starting point for the investigation whether one distribution dominates another one. If the distributions are known, the comparison is straightforward with this characterization. However, in empirical applications the distributions are usually unknown and have to be inferred from the data. Due to the strong impact of the standard error on the descriptive comparison, a descriptive comparison is not sufficient for getting meaningful results. Hence we need statistical inference for establishing or rejecting stochastic dominance.

There is a plethora of tests for stochastic dominance. Most of them are based on the empirical distribution function, some on the empirical quantile function. The test statistic is usually the maximum difference or a weighted average of the difference of the empirical distribution or quantile functions. The vast majority of tests consider the null hypothesis of dominance and the alternative of non-dominance. Therefore statistical evidence for stochastic dominance cannot be found. Instead, stochastic dominance can only be rejected or not by most of the tests. The reason for this finding is the mathematical complexity of the set of all distribution pairs without any dominance relation. In particular, the boundary cannot be expressed in closed form. Hence, usually the null hypothesis is the assertion that one random variable dominates the other one.

For the determination of the critical region we need the distribution of the test statistic under the null hypothesis. In many tests contemporaneous and serial independence are assumed, i.e. it is presumed that there is no dependence within each sample and between the samples. However, in many applications the observations do not satisfy these constraints. In particular, financial data usually feature contemporaneous and serial dependence: positive correlation between the observations of different samples at the same time and time series properties, e.g. conditional heteroskedasticity within each sample. In the investigation of conditional heteroskedasticity we confine ourselves to GARCH(1,1) which has good fit and forecast properties for financial data; see e.g. Akgiray (1989), Davis/ Mikosch (2000) and Engle (2001).

In order to get an idea about the impact of conditional heteroskedasticity on tests of stochastic dominance we analyze its effect on common statistical procedures for means and variances, e.g. the t-test and the F-test. The analytical investigation and simulation show that GARCH(1,1) does not have any significant effect on procedures for means. On the contrary, the impact on procedures for variances is tremendous. Hence we advise not to use the usual F-test and the usual confidence interval if conditional heteroskedasticity is prevalent in the data.

The main issue of this thesis is the development of a test for stochastic dominance which is applicable to data featuring conditional heteroskedasticity and contemporaneous correlation. The tests of Schmid/ Trede (1997) for SD2, Xu/ Fisher/ Willson (1997) for SD1 and SD2 and Linton/ Maasoumi/ Whang (2005) for SD1 and SD2, denoted by ST, XFW and LMW, are the starting point. The ST test, based on permutations, captures the correlation whereas it does not consider conditional heteroskedasticity. The XFW and the LMW tests asymptotically keep the size and are consistent if the observations are generated by strongly mixing processes. Due to the fact that GARCH(1,1) processes are strongly mixing, these tests are expected to perform well even if the data feature this dependence structure. The tests are based on block bootstrap methods: XFW on moving block bootstrap, LMW on subsampling.

By means of simulation, we analyze the performance of the tests for finite samples. It turns out that the ST, XFW and LMW are all robust to contemporaneous correlation. On the other hand, the effect of GARCH(1,1) on the size is enormous. The variation of the block length is no remedy for the XFW and the LMW tests. Indeed, the effect decreases with increasing block length, but it does not vanish completely and increases again for very large block length. Hence the original XFW and LMW tests cannot be used for finite samples.

What is the reason for these results? The XFW and LMW tests consider overlapping blocks of a fixed length. Consequently, the data in the middle of the time series are taken into account more than the data at the beginning and at the end. This effect even increases with increasing block length.

Due to the poor performance of the tests we propose some modifications. The permutation principle in the ST test is changed to block permutation. The modified test does not transpose single observation pairs as the original ST test, but whole blocks of them. In this manner it should capture the dependence structure. The XFW test is altered as follows: instead of moving block bootstrap the new test uses circular block bootstrap, introduced by Politis/ Romano (1992). In addition to the blocks of the moving block bootstrap it considers blocks which consist of some observations at the end and continue at the beginning of the sample. According to Lahiri (1999) circular block bootstrap and moving block bootstrap are asymptotically equivalent. The LMW test is modified in a similar way: the distribution of the test statistic is not inferred by usual subsampling, but by circular subsampling. This procedure uses, in addition to the blocks used by usual subsampling, blocks consisting of the first and the last observations. We show that the modified LMW test asymptotically keeps the size and is consistent, as is the original one.

We investigate the performance of the modified tests for finite samples by simulation. The simulation results show the following. On the one hand, the XFW test cannot be improved by the circular block bootstrap. On the other hand, the modified versions of the ST and the LMW tests keep the size for the appropriate block length. We explore the question of the optimal block length for various sample sizes. It turns out that for both tests the optimal block length is of order $\sqrt{n}$ where $n$ is the sample size. Further simulations show that the new tests have good power properties.

As mentioned above, most of the tests developed in recent years test the null hypothesis of dominance against the alternative of non-dominance. The drawback of this approach is that there is no significant evidence of stochastic dominance. Starting from the test of Kaur/ Rao/ Singh (1994), abbreviated by KRS, we look for a remedy to this problem and develop a new test. This test has the alternative of SD2 and the hypothesis of non-SD2. The KRS test does not regard the whole real axis, but only a fixed interval. In addition, it requires the observations to be independent. The new test considers the whole real axis and all data, but appropriately truncates the range for the determination of the infimum. This test asymptotically keeps the size if the truncation value is chosen appropriately. Furthermore, in contrast to the KRS test, for the new test we do not need to assume that the data are independent. It can be applied if the observations are generated by a strongly mixing process and satisfy some moment conditions. In a Monte Carlo study we explore the problem of the appropriate truncation choice for finite samples. We find truncation values in such a way that the test has good size and power properties for the cases we analyze.

Finally we apply the tests developed in this thesis in an empirical study in which we test whether there are stochastic dominance relations among the 30 stocks of the German stock index DAX. We compare the daily returns of a 1-year period and a 10-year period using the modified versions of the LMW test for SD1, of the ST test for SD2 and of the KRS test against SD2. Due to the fact that conditional heteroskedasticity and positive correlation are prevalent in the data, we need these tests which capture the dependence structure. From the test results we determine the efficient sets, i.e. the subsets of the non-dominated stocks. In many comparisons neither the null hypothesis of stochastic dominance nor the null hypothesis of non-dominance is rejected. This is due to the fact that in many cases the empirical distributions are close to each other. The modified LMW and ST tests yield small, the modified KRS test rather large efficient sets. However, as the modified KRS test significantly confirms stochastic dominance, the efficiency results according to this test are more meaningful.

The structure of this thesis is as follows. Chapter 2 illustrates stochastic dominance as a decision criterion. We give a survey on the theoretical results, in particular the definition and some characterizations of stochastic dominance, in section 2.1. In section 2.2 we illustrate the problems of a descriptive comparison.

A survey on various approaches of testing for stochastic dominance is given in chapter 3. Some of the tests will be analyzed in more detail later.

In chapter 4 we investigate the effect of conditional heteroskedasticity on common statistical procedures as the t-test or F-test. We give some definitions and preliminary results in section 4.1. The procedures for means and variances are investigated in the sections 4.2 and 4.3, respectively. Section 4.4 sums up the results of chapter 4.

In chapter 5, we deal with the main issue of this study. We analyze various tests for stochastic dominance which asymptotically capture the dependence structure given by GARCH(1,1) and positive correlation. Having illustrated the tests in section 5.1 we analyze them by means of simulation in section 5.2. Due to poor performance for finite samples, we modify the tests and find analytical results in section 5.3. In section 5.4, we explore the performance of the tests in a Monte Carlo study. Further we investigate the power in section 5.5 and conclude chapter 5 in section 5.6.

We find a test with second degree stochastic dominance in the alternative in chapter 6. After an introduction in section 6.1 we present various approaches in section 6.2, in particular the test of Kaur/ Rao/ Singh (1994). In section 6.3 we analyze and modify the test and prove that the new test asymptotically keeps the size. We explore the performance for various distributions by simulation in section 6.4. In section 6.5 we sum up the results of this chapter.

In chapter 7 we apply the tests developed in chapters 5 and 6 to the daily returns of the stocks of the German stock index DAX. In section 7.1 we explain the methodology which we use in the empirical study. We present and analyze the data in section 7.2. Finally we present and interpret the test results in section 7.3.

# Chapter 2

# Stochastic Dominance as a Decision Criterion

## 2.1 Theory of Stochastic Dominance

The problem of decision making is as old as mankind. The classical theory of rational decisions under risk is based on the von Neumann-Morgenstern utility axioms. Let $\succeq$ denote the binary preference relation "is weakly preferred to". Consider a set $\mathcal{X}$ of real-valued random variables and let $X, Y, Z \in \mathcal{X}$ stand for uncertain, real-valued outcome. Von Neumann/ Morgenstern (1944) make the following assumptions concerning the preferences of a given person. For $X, Y, Z \in \mathcal{X}$ holds:

1. Completeness: $X \succeq Y$ or $Y \succeq X$ holds

2. Transitivity: $X \succeq Y$ and $Y \succeq Z$ yield $X \succeq Z$

3. Monotonicity: $X \geq Y$ a.s. (almost surely) yields $X \succeq Y$

4. Continuity: If $X > Y > Z$ a.s. holds, then there exist $\alpha, \beta \in (0, 1)$ satisfying $\alpha X + (1 - \alpha)Z \succeq Y$ and $Y \succeq \beta X + (1 - \beta)Z$

5. Substitution: $X \succeq Y$ yields $\alpha X + (1-\alpha)Z \succeq \alpha Y + (1-\alpha)Z$ for any $\alpha \in [0,1]$.

If a preference relation $\succeq$ satisfies these axioms and $\mathcal{X}$ is rich enough (contains at least the finite-discrete random variables), then there exists a (von Neumann-Morgenstern) utility function $u$ representing the preference. This means that the inequality $E(u(X)) \geq E(u(Y))$ of the expected utilities holds if and only if $X \succeq Y$. Fishburn (1970) and Zachow/ Schmitz (1977) give necessary and sufficient conditions for the preference order to be equivalent to the expected utility criterion. They also show that the utility function $u$ is unique except for affine transformations. Hence choice of a person between uncertain alternatives depends on their probability distributions and on the individual utility function. The objective of an individual is to maximize his or her expected utility. Due to the fact that the utility function may differ from person to person, their rational decisions differ as well.

It is an important goal of decision theory to find the optimal choice for a large set of utility functions. One approach is the mean-variance analysis of Markowitz (1952). Random variables standing for monetary payoff are compared by their means and variances. $X$ is preferred over $Y$ if and only if $X$ has larger or equal mean and smaller or equal variance. In the case of nonnormal distributions this might yield dissatisfying results. Consider, for example, the case mentioned by Hanoch/ Levy (1969). If $X$ and $Y$ are random variables with the distributions $P(X=1)=0.8$, $P(X=100)=0.2$, $P(Y=10)=0.99$ and $P(Y=1000)=0.01$, we get

$$E(X) = 20.8 > 19.9 = E(Y)$$

and

$$Var(X) = 1468 < 9703 = Var(Y).$$

Hence $X$ is preferred by the Markowitz criterion. However, if a risk averse person has the utility function $u(x) = \log_{10}(x)$, this results in

$$E(u(X)) = 0.4 < 1.02 = E(u(Y)).$$

Hence this person prefers $Y$ which is contrary to the Markowitz criterion. Furthermore, it is easy to find an example where the Markowitz criterion does not lead to a decision in spite of one alternative clearly being preferable. For instance, consider the case where $P(Y = 0) = 1$, $P(X = 0) = 1 - p$, $P(X = x) = p$ for some $0 < p < 1$, $x > 0$. $Y$ has mean and variance zero whereas $X$ has mean $px > 0$ and variance $p(1 - p)x^2 > 0$. In this example, the use of the Markowitz criterion does not lead to a decision for one random variable although the result of $X$ is at least as large as that of $Y$ and larger with positive probability.

This example shows that a criterion which only considers some parameters such as mean or variance in some cases does not lead to an economically meaningful decision. We need a decision criterion which yields a utility maximizing decision for a class of utility functions. This requirement is fulfilled by the concept of *stochastic dominance*. The decision rule for first order stochastic dominance was introduced by Quirk/ Saposnik (1962). Hadar/ Russell (1969) and Hanoch/ Levy (1969) develop stochastic dominance of second degree, Whitmore (1970) of third degree, Rolski (1976) of any positive integer degree and Fishburn (1980) of any real degree $\alpha$ for $\alpha \geq 1$. In this study we confine ourselves to stochastic dominance of positive integer degree which is defined as follows.

**Definition 1.** *Let $X$ and $Y$ be real-valued random variables and $k$ be a positive integer. $X$ weakly dominates $Y$ in the sense of* $k$th degree stochastic dominance (SD$k$) *if and only if $E(u(X)) \geq E(u(Y))$ holds for every utility function $u$ with existing and finite expected values and $(-1)^{j+1}u^{(j)} \geq 0$ for all $j \in \{1, \ldots, k\}$ where $u^{(j)}$ denotes the $j$th derivative of $u$. It is denoted by $X \succeq_k Y$.*

We can illustrate stochastic dominance for lower degrees. $X \succeq_1 Y$ holds if $E(u(X)) \geq E(u(Y))$ for all nondecreasing utility functions $u$; this means that every individual favoring more over less prefers $X$. $X \succeq_2 Y$ means that $E(u(X)) \geq E(u(Y))$ for all nondecreasing and concave utility functions $u$, which stands for risk aversion.

It is obvious from the definition that SD$k$ implies stochastic dominance of any higher degree. SD1 is the strongest stochastic dominance relationship and occurs most rarely. Moreover we see from the definition and the characterizations that stochastic dominance is a partial order of all real-valued random variables. There are, in particular for lower degrees, many pairs of random variables which do not dominate each other in either direction.

Several authors (e.g. Mosler, 1982) generalize the concept of stochastic dominance to probability measures on a measurable space $(E, \mathcal{S})$ where $\mathcal{S}$ is a $\sigma$*-field* on $E$. This means that $\mathcal{S}$ is a non-empty subset of the power set of $E$ satisfying the following properties:

1. $E \in \mathcal{S}$
2. If $A \in \mathcal{S}$, then $E \setminus A \in \mathcal{S}$ where $E \setminus A$ is the relative complement of $A$ in $E$.
3. If $A_k \in \mathcal{S}$ for all $k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{S}$ where $\bigcup_{k \in \mathbb{N}} A_k$ is the union of $A_k$ for all $k \in \mathbb{N}$.

Let $B$ be the set of all measurable functions $u : E \to \mathbb{R}$ and $U \subset B$ be a subset. Suppose that $\nu$ and $\mu$ are probability measures on $\mathcal{S}$. Then $\nu$ dominates $\mu$ regarding $U$ if

$$\int u d\nu \geq \int u d\mu$$

holds for all $u \in U$ with existing and finite integrals. With this notation first degree stochastic dominance can be generalized as follows. Let $(E, \mathcal{S}, \leq)$ be a preordered measurable vector space where $\leq$ is compatible with the addition of vectors. We denote by $U_1$ the set of all nondecreasing real-valued functions, i.e.

$$U_1 = \{u \in B | u(x) \leq u(y) \text{ if } x \leq y, x, y \in E\}.$$

Then $\nu$ dominates $\mu$ in the sense of SD1 if $\int u d\nu \geq \int u d\mu$ holds for all $u \in U_1$ with existing and finite integrals.

Let $X$ and $Y$ be random variables with the *induced probability distributions* $P_X$ and $P_Y$. $X$ is said to dominate $Y$ in the sense of SD1 if and only if $P_X$ dominates $P_Y$ in the sense of SD1. This definition is in accordance with definition 1.

In the case $E = \mathbb{R}^n$ the relation $x \leq y$ means that $x_i \leq y_i$ holds for each component of the vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Stochastic dominance in this setting is called multivariate stochastic dominance. However, in this study we confine ourselves to the case $E = \mathbb{R}$.

For the investigation of stochastic dominance there are useful characterizations, based on the distribution and quantile functions. Let $F_X$, $F_Y$ be the *(cumulative) distribution functions* and $Q_X$, $Q_Y$ be the *quantile functions* of $X$ and $Y$, i.e.

$$F_X(x) = P(X \leq x)$$

for $x \in \mathbb{R}$ and

$$Q_X(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$$

for $p \in (0,1)$, correspondingly for $F_Y$ and $Q_Y$. Define $F_X^{(1)} = F_X$, $Q_X^{(1)} = Q_X$,

$$F_X^{(k+1)}(x) = \int_{-\infty}^{x} F_X^{(k)}(t)dt$$

and

$$Q_X^{(k+1)}(p) = \int_0^p Q_X^{(k)}(t)dt$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}$ and $p \in (0,1)$.

Jean (1984) states a relation between $F_X^{(k)}$ and the *lower partial moment*. The $k$th lower partial moment with reference value $c \in \mathbb{R}$ is a common risk measure which is defined as

$$LPM_X^k(c) = \int_{(-\infty,c)} (c-x)^k dP_X(x).$$

The second lower partial moment is the well known (lower) semivariance

$$SV_X(c) = LPM_X^2(c).$$

An important and useful relation between $F_X^{(k)}$ and $LPM_X^k$ is given by the following proposition. Here and in the following let $\lambda$ denote the Lebesgue measure.

**Proposition 1 (Jean, 1984).**

$$LPM_X^k(c) \quad = \quad k!F_X^{(k+1)}(c) \tag{2.1}$$

*holds for all $k \in \mathbb{N}$.*

*Proof.* We prove the assertion by complete induction.

For $k = 0$ we have

$$LPM_X^0(c) = \int\limits_{(-\infty,c)} (c-x)^0 dP_X(x) = F_X(c) = 0!F_X^{(1)}(c),$$

therefore (2.1) holds for $k = 0$.

Let (2.1) hold for some $k \in \mathbb{N}$. Then we have to prove that

$$LPM_X^{k+1}(c) = (k+1)!F_X^{(k+2)}(c)$$

holds. Due to Fubini's theorem for measure integrals it follows that

$$\begin{aligned}
(k+1)!F_X^{(k+2)}(c) &= \int\limits_{(-\infty,c)} (k+1)k!F_X^{(k+1)}(t)d\lambda(t) \\
&= \int\limits_{(-\infty,c)} \int\limits_{(-\infty,t)} (k+1)(t-x)^k dP_X(x)d\lambda(t) \text{ (Induction hypothesis)} \\
&= \int\limits_{(-\infty,c)} \int\limits_{(x,c)} (k+1)(t-x)^k d\lambda(t)dP_X(x) \text{ (Fubini)} \\
&= \int\limits_{(-\infty,c)} (c-x)^{k+1} dP_X(x) = LPM_X^{k+1}(c)
\end{aligned}$$

which yields the assertion of the induction step and completes the proof. □

It is obvious that the $k$th lower partial moment exists and is finite for every $c \in \mathbb{R}$ if and only if this holds for the $k$th moment. Using the proposition, it follows for any $k \in \mathbb{N}$ that $F_X^{(k)}(x)$ exists and is finite for all $x \in \mathbb{R}$ if and only if $X$ has an existing and finite $(k-1)$th moment.

**Proposition 2.** *Let $k \in \mathbb{N}$ such that $E(X^{k-1})$ and $E(Y^{k-1})$ exist and are finite. Then the following statements are equivalent:*

1. $X \succeq_k Y$
2. $F_X^{(k)}(x) \leq F_Y^{(k)}(x)$ *for all* $x \in \mathbb{R}$

*For $k = 1, 2$ these statements are equivalent to*

3. $Q_X^{(k)}(p) \geq Q_Y^{(k)}(p)$ *for all* $p \in (0, 1)$.

For the proof we need the following auxiliary result.

**Lemma 1.** *For $k \in \mathbb{N}$ arbitrary let $E(X^{k-1})$ and $E(Y^{k-1})$ exist and be finite. Assume that $F_X^{(k)}(x) \leq F_Y^{(k)}(x)$ holds for all $x \in \mathbb{R}$. Then*

$$\lim_{x \to \infty} F_X^{(l)}(x) - F_Y^{(l)}(x) \leq 0$$

*holds for all $l \in \mathbb{N}$, $l < k$.*

*Proof.* Let $G_k(x) := F_X^{(k)}(x)$ and $H_k(x) := F_Y^{(k)}(x)$. First note that for all $k \in \mathbb{N}$ the functions $G_k$ and $H_k$ asymptotically behave like polynomials of degree $k-1$ if $x$ approaches infinity. This holds for $k = 1$ and follows for all $k \in \mathbb{N}$ by complete induction due to the fact that $G_k$ and $H_k$ are some antiderivatives of $G_{k-1}$ and $H_{k-1}$. Therefore

$$\lim_{x \to \infty} G_k(x) - H_k(x)$$

exists – at least in the improper sense – for all $k \in \mathbb{N}$.

We prove the assertion in two steps: First we show that $G_k(x) \leq H_k(x)$ for all $x \in \mathbb{R}$ yields

$$\lim_{x \to \infty} G_{k-1}(x) - H_{k-1}(x) \leq 0,$$

then we see that this results in

$$\lim_{x\to\infty} G_l(x) - H_l(x) \leq 0$$

for all $l < k$.

We prove the first part by contraposition. Assume $\lim_{x\to\infty}(G_{k-1}(x) - H_{k-1}(x)) = c > 0$ or $\lim_{x\to\infty}(G_{k-1}(x) - H_{k-1}(x)) = \infty$. Then there exists an $x_0 \in \mathbb{R}$ satisfying $G_{k-1}(x) - H_{k-1}(x) \geq \frac{c}{2}$ for all $x \geq x_0$. Define

$$d := G_k(x_0) - H_k(x_0) = \int_{-\infty}^{x_0} G_{k-1}(t) - H_{k-1}(t)dt.$$

Then we get for $x > x_0$:

$$G_k(x) - H_k(x) = d + \int_{x_0}^{x} \underbrace{G_{k-1}(t) - H_{k-1}(t)}_{\geq \frac{c}{2}} dt \geq d + \frac{c}{2}(x - x_0).$$

For sufficiently large $x_0$ we get

$$G_k(x) - H_k(x) \geq d + \frac{c}{2}(x - x_0) > 0,$$

which is a contradiction to $G_k(x) \leq H_k(x)$ for all $x \in \mathbb{R}$. This completes the first part of the proof.

We prove the second part by contraposition as well. Assume that

$$\lim_{x\to\infty} G_l(x) - H_l(x) > 0$$

holds for some $l < k - 1$. As in the first part we can deduce that there are $c > 0$ and $d, x_0 \in \mathbb{R}$ satisfying $G_{l+1}(x) - H_{l+1}(x) \geq d + \frac{c}{2}(x - x_0)$ for all $x \geq x_0$. Hence

$$\lim_{x\to\infty} G_{l+1}(x) - H_{l+1}(x) > 0.$$

Then

$$\lim_{x\to\infty} G_{k-1}(x) - H_{k-1}(x) > 0$$

follows by complete induction. □

*Proof of proposition 2.* The equivalence of statements 2 and 3 for $k = 1, 2$ is proved by Levy/ Kroll (1978).

In the following we prove that statements 1 and 2 are equivalent. Let $k \in \mathbb{N}$ be arbitrary and $H^{(k)} := F_X^{(k)} - F_Y^{(k)}$. Note that because of the existence and finiteness of $E(X^{k-1})$ and $E(Y^{k-1})$ the functions $F_X^{(k)}(x)$ and $F_Y^{(k)}(x)$ and therefore $H^{(k)}(x)$ also feature this property for all $x \in \mathbb{R}$.

$2 \Rightarrow 1$: Let $H^{(k)}(x) \leq 0$ for all $x \in \mathbb{R}$ and $u \in U_k$ be an arbitrary utility function with existing and finite expectations $E(u(X))$ and $E(u(Y))$. Then the difference

$$E(u(X)) - E(u(Y)) = \int_{(-\infty,\infty)} u(x) dH(x)$$

also exists and is finite. Using $k$-fold partial integration we get

$$\begin{aligned} &\int_{(-\infty,\infty)} u(x) dH(x) \\ &= \sum_{j=0}^{k-1} (-1)^j u^{(j)}(x) H^{(j+1)}(x) \Big|_{-\infty}^{\infty} + (-1)^k \int_{(-\infty,\infty)} u^{(k)}(x) dH^{(k+1)}(x) \\ &= \sum_{j=0}^{k-1} (-1)^j u^{(j)}(x) H^{(j+1)}(x) \Big|_{-\infty}^{\infty} + \int_{(-\infty,\infty)} (-1)^k u^{(k)}(x) H^{(k)}(x) dx. \qquad (2.2) \end{aligned}$$

Due to lemma 1, $H^{(k)}(x) \leq 0$ for all $x \in \mathbb{R}$ yields $\lim_{x\to\infty} H^{(j)}(x) \leq 0$ for all $j \in \{0, \ldots, k-1\}$. Because of $(-1)^j u^{(j)}(x) \leq 0$ for all $j \leq k-1$ and all $x \in \mathbb{R}$ we get

$$\lim_{x\to\infty} (-1)^j u^{(j)}(x) H^{(j+1)}(x) \geq 0$$

for all $j \leq k-1$. Furthermore, note that for all $j \leq k-1$ we have

$$\lim_{x\to-\infty} (-1)^j u^{(j)}(x) H^{(j+1)}(x) = 0.$$

This asymptotic behavior results from the finiteness of

$$\int_{-\infty}^{t} (-1)^j u^{(j+1)}(x) H^{(j+1)}(x) dx$$

for all $t \in \mathbb{R}$. Hence the first part of (2.2) is nonnegative. The nonnegativity of the second part follows from $(-1)^k u^{(k)}(x) \leq 0$ and $H^{(k)}(x) \leq 0$ for all $x \in \mathbb{R}$.

$1 \Rightarrow 2$ (by contraposition): Assume that $H^{(k)}(x_0) > 0$ holds for some $x_0 \in \mathbb{R}$. Due to the continuity from the right of $H^{(k)}$ there exists an $\varepsilon > 0$ satisfying $H^{(k)}(x) > 0$ for all $x$ satisfying $0 < x - x_0 < \varepsilon$. Let $u$ be a utility function with $u^{(k)}(x) = 0$ for all $x$ satisfying $0 < x - x_0 \geq \varepsilon$. Due to the existence and finiteness of the integral $\int_{(-\infty,\infty)} u(x)dH(x)$ we know that

$$\sum_{j=0}^{k-1}(-1)^j u^{(j)}(x)H^{(j+1)}(x)\Big|_{-\infty}^{\infty} =: c$$

also exists and is finite. If we choose $u$ in such a way that $(-1)^k u^{(k)}(x)$ is sufficiently small in the interval $[x, x_0 + \varepsilon]$, using (2.2) we get

$$\int_{(-\infty,\infty)} u(x)dH(x) = c + \int_{(-\infty,\infty)} (-1)^k u^{(k)}(x)H^{(k)}(x)dx < 0$$

which is contrary to 1. $\square$

For $k = 1, 2$ proposition 2 yields:

- $X \succeq_1 Y$ is equivalent to $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}$ and to $Q_X(p) \geq Q_Y(p)$ for all $p \in (0, 1)$.
- $X \succeq_2 Y$ is equivalent to $\int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt$ for all $x \in \mathbb{R}$ and to $\int_0^p Q_X(t)dt \geq \int_0^p Q_Y(t)dt$ for all $p \in (0, 1)$.

This means that first degree stochastic dominance of $X$ over $Y$ is equivalent to $X$ having a uniformly smaller or equal distribution function and a larger or equal quantile function than $Y$. Second degree stochastic dominance is equivalent to a uniform order of the integrals of the distribution and quantile functions from $-\infty$ to $x$ for all $x \in \mathbb{R}$.

In the following let $\mu_X = E(X)$, $\mu_Y = E(Y)$ be the means of $X$ and $Y$, respectively. Define the $k$th central moments $C_X^k = E(X - \mu_X)^k$ and $C_Y^k = E(Y - \mu_Y)^k$. The second central moments are the well-known variances $\sigma_X^2 = C_X^2$ and $\sigma_Y^2 = C_Y^2$.

For the antiderivatives $F_X^{(k)}$ of the distribution function we can state the following for $k = 1, 2, 3$:

- $F_X^{(1)} = F_X$ is nonnegative, continuous from the right, nondecreasing and has the limit properties $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$ as is common knowledge.

- $F_X^{(2)}$ is nonnegative, continuous, convex, nondecreasing and has the limit properties $\lim_{x \to -\infty} F_X^{(2)}(x) = 0$, $\lim_{x \to \infty} F_X^{(2)}(x) - (x - \mu_X) = 1$ (see Ogryczak/ Ruszczynski, 1999).

- $F_X^{(3)}$ is nonnegative, continuous, convex, nondecreasing, differentiable and has the limit properties $\lim_{x \to -\infty} F_X^{(3)}(x) = 0$,

$$\lim_{x \to \infty} F_X^{(3)}(x) - \frac{1}{2}((x - \mu_X)^2 + \sigma_X^2) = 1$$

  (see Gotoh/ Konno, 2000).

Shaked/ Shanthikumar (1994) report the following result which is an equivalent characterization of stochastic dominance of the first two degrees.

**Proposition 3.** *Let $X$ and $Y$ be random variables. Then $X$ dominates $Y$*

- *in the sense of SD1 if and only if there exist random variables $\tilde{X}$ and $\tilde{Y}$, defined on the same probability space, satisfying $P_X = P_{\tilde{X}}$, $P_Y = P_{\tilde{Y}}$ and $P(\tilde{X} \geq \tilde{Y}) = 1$.*

- *in the sense of SD2 if and only if there exist random variables $\tilde{X}$ and $\tilde{Y}$, defined on the same probability space, satisfying $P_X = P_{\tilde{X}}$, $P_Y = P_{\tilde{Y}}$ and $(\tilde{X}, \tilde{Y})$ is a supermartingale, i.e. $E(\tilde{Y}|\tilde{X}) \leq \tilde{X}$ a.s.*

We can interpret these results as follows. If $X$ dominates $Y$ in the sense of SD1, there are some random variables $\tilde{X}$ and $\tilde{Y}$ with the same distributions as $X$ and $Y$, with $\tilde{X}$ almost surely being larger than $\tilde{Y}$. $X$ dominates $Y$ in the sense of SD2 if and only if the mean of $\tilde{Y} - \tilde{X}$, conditional on $\tilde{X}$, is negative.

Since the development of stochastic dominance various necessary conditions have been found. Jean/ Helms (1987) explore a generalization of many moment conditions. Let $b$ be an upper bound for the supports of $P_X$ and $P_Y$. Then a necessary condition for any degree of stochastic dominance of $X$ over $Y$ is $F_X^{(k)}(b) \leq F_X^{(k)}(b)$ and therefore

$$\begin{aligned} &\sum_{j=0}^{k}(-1)^j\binom{k}{j}(b-\mu_X)^{k-j}E((X-\mu_X)^j) \\ \leq\ &\sum_{j=0}^{k}(-1)^j\binom{k}{j}(b-\mu_Y)^{k-j}E((Y-\mu_Y)^j) \end{aligned} \tag{2.3}$$

for all $k \in \mathbb{N}$; see Jean/ Helms (1987).

For $k = 1$ we get the ranking condition $\mu_X \geq \mu_Y$ of the means, for $k = 2$ the mean-variance condition $(\sigma_X^2 - \sigma_Y^2) + (\mu_Y - \mu_X)(b - \mu_X - \mu_Y) \leq 0$. The latter is found by Whitmore (1970) to be a necessary condition for SD3, the result above shows that it is necessary for stochastic dominance of any degree. Furthermore, (2.3) with $k = 3$ yields the following condition for the third central moments $C_X^3$ and $C_Y^3$ which is found by Jean (1984):

$$C_X^3 - 3(b-\mu_X)\sigma_X^2 - (b-\mu_X)^3 \geq C_Y^3 - 3(b-\mu_Y)\sigma_Y^2 - (b-\mu_Y)^3.$$

If the supports of $P_X$ and $P_Y$ have no upper bound $b$, we can generalize (2.3) to

$$\limsup_{b\to\infty}\sum_{j=0}^{k}(-1)^j\binom{k}{j}((b-\mu_X)^{k-j}E((X-\mu_X)^j) - (b-\mu_Y)^{k-j}E((Y-\mu_Y)^j)) \leq 0$$

if $X$ and $Y$ have finite $k$th moments. This yields $\mu_X \geq \mu_Y$ for $k = 1$,

$$\lim_{b\to\infty}(\sigma_X^2 - \sigma_Y^2) + (\mu_Y - \mu_X)(b - \mu_X - \mu_Y) \leq 0$$

for $k = 2$ and

$$\liminf_{b\to\infty} C_X^3 - 3(b-\mu_X)\sigma_X^2 - (b-\mu_X)^3 - C_Y^3 - 3(b-\mu_Y)\sigma_Y^2 - (b-\mu_Y)^3 \geq 0$$

for $k = 3$.

From the above mentioned moment conditions we can deduce the following: If $X$ dominates $Y$ in the sense of any degree and both have equal means, then

$X$ has smaller or equal variance. If, in addition, the variances are equal, $X$ has larger or equal central third moment. We get the following result for the $k$th central moments $C_X^k$, $C_Y^k$: If $X$ dominates $Y$ in the sense of any degree and we have $\mu_X = \mu_Y$ and $C_X^j = C_Y^j$ for $j = 2, \ldots, k-1$, then

$$(-1)^k C_X^k \leq (-1)^k C_Y^k$$

holds.

Furthermore, an ordering of lower partial moments is a necessary condition for certain degrees of stochastic dominance. As stated above, Jean (1984) asserts that

$$LPM_X^k(c) = k! F_X^{(k+1)}(c)$$

holds for all $k \in \mathbb{N}$. Therefore it is obvious that $LPM_X^{k-1}(c) \leq LPM_Y^{k-1}(c)$ is necessary for $X \succeq_k Y$. Due to the fact that SD$k$ implies stochastic dominance of any higher degree we know that $X \succeq_k Y$ yields $LPM_X^l(c) \leq LPM_Y^l(c)$ for all $l \in \mathbb{N}$ satisfying $l \geq k-1$. Remember that the (lower) semivariance is defined by

$$SV_X(c) = LPM_X^2(c).$$

The relation stated above implies that $SV_X(c) \leq SV_Y(c)$ is necessary for first, second and third degree stochastic dominance whereas $SV_X(c) > SV_Y(c)$ for some $c$ does not prevent $X \succeq_k Y$ for $k \geq 4$. This is e.g. the case for $X \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$, $Y \sim \mathcal{N}(0, 1)$ where $\mathcal{U}(a, b)$ denotes the uniform distribution on the interval $(a, b)$ and $\mathcal{N}(\mu, \sigma)$ the normal distribution with mean $\mu$ and variance $\sigma^2$; $X$ dominates $Y$ in the sense of SD4 whereas $SV_X(1) > SV_Y(1)$ holds.

Ogryczak/ Ruszczynski (2001) establish an interesting result for the relationship of mean and semideviation. If $X$ dominates $Y$ in the sense of SD$k$, then $\mu_X \geq \mu_Y$ and

$$\mu_X - (LPM_X^{k-1}(\mu_X))^{\frac{1}{k-1}} \geq \mu_Y - (LPM_Y^{k-1}(\mu_Y))^{\frac{1}{k-1}}$$

hold. If the first inequality is strict, this also holds for the last one. Because of the fact that stochastic dominance of some degree implies stochastic dominance of any higher degree $X \succeq_k Y$ yields

$$\mu_X - (LPM_X^l(\mu_X))^{\frac{1}{l}} \geq \mu_Y - (LPM_Y^l(\mu_Y))^{\frac{1}{l}}$$

for all $l \in \mathbb{N}$ satisfying $l \geq k-1$.

Second degree stochastic dominance is also known as *generalized Lorenz dominance*. Atkinson (1970) introduces the concept of *Lorenz dominance* as a criterion of social inequality. The *Lorenz curve* of a non-negative random variable $X \geq 0$ is defined by

$$L_X(p) = \frac{1}{\mu_X} E(1_{\{X<Q_X(p)\}}X) = \frac{1}{\mu_X}\int_{(-\infty,Q_X(p))} t dP_X(t) = \frac{1}{\mu_X}\int_0^p Q_X(t)dt$$

for $p \in (0,1)$ where $\mu_X$ denotes the mean of $X$. It is easy to see that $\lim_{p\to 0} L_X(p) = 0$ and $\lim_{p\to 1} L_X(p) = 1$ hold. If $X$ has a continuous distribution, then $\frac{1}{\mu_X}Q_X(p)$ is the derivative of $L_X(p)$. $L_X$ is convex because $Q_X$ is monotonically increasing. Therefore we get $L_X(p) \leq p$ for all $p \in (0,1)$. In social welfare investigations the random variable $X$ usually stands for income. The larger the difference $p - L_X(p)$ is, the larger is the inequality in the population. If $L_X(p) \geq L_Y(p)$ holds for all $p \in (0,1)$, then $X$ dominates $Y$ in the sense of Lorenz dominance; it is denoted by $X \succeq_L Y$.

Shorrocks (1983) extends this concept to the *generalized Lorenz curve* and dominance. The generalized Lorenz curve is defined by

$$GL_X(p) = \mu_X L(p) = \int_{(-\infty,Q_X(p))} t dP_X(t) = \int_0^p Q_X(t)dt$$

for $p \in (0,1)$. It has the properties $\lim_{p\to 0} GL_X(p) = 0$ and $\lim_{p\to 1} GL_X(p) = \mu_X$, $\frac{d}{dp}GL_X(p) = Q_X(p)$ and $GL_X(p) \leq \mu_X p$ for all $p \in (0,1)$.

$X$ dominates $Y$ in the sense of generalized Lorenz dominance if

$$GL_X(p) \geq GL_Y(p)$$

holds for all $p \in (0,1)$. Because of $GL_X(p) = Q_X^{(2)}(p)$ it follows from proposition 2 that generalized Lorenz dominance and second degree stochastic dominance are equivalent. If $X$ and $Y$ have the same mean, these dominance relations are also equivalent to Lorenz dominance.

Foster/ Sen (1997) and Zheng/ Formby/ Smith/ Chow (2000) generalize the concept of Lorenz dominance in the same way as stochastic dominance is a generalization of generalized Lorenz dominance. They consider *normalized stochastic dominance* which is defined as follows. Let $X$ and $Y$ be random variables with nonnegative real values. Then $X$ dominates $Y$ in the sense of $k$th degree normalized stochastic dominance if $\frac{X}{\mu_X}$ dominates $\frac{Y}{\mu_Y}$ in the sense of $k$th degree stochastic dominance. Obviously, second degree normalized stochastic dominance is equivalent to Lorenz dominance.

For a survey concerning stochastic dominance see for instance Whitmore/ Findlay (1978), Levy (1992) and Mosler/ Scarsini (1991). Bawa (1982) and Mosler/ Scarsini (1993) give detailed bibliographies for theoretical and applied studies of stochastic dominance.

## 2.2 Descriptive Stochastic Dominance

In this chapter we discuss the pros and cons of a descriptive approach to investigating stochastic dominance. As we stated above, $X \succeq_k Y$ is equivalent to $F_X^{(k)}(x) \leq F_Y^{(k)}(x)$ for all $x \in \mathbb{R}$ and to $Q_X^{(k)}(p) \geq Q_Y^{(k)}(p)$ for all $p \in (0,1)$ if $k \in \{1,2\}$. In applications the compared distributions are usually unknown and have to be inferred from the observations of $X$ and $Y$. One could just compare the $(k-1)$th antiderivatives of the empirical distribution functions $\hat{F}_n^{(k)}$ or of the empirical quantile functions $\hat{Q}_n^{(k)}$. The empirical distribution and quantile functions of the observations $x_1, \ldots, x_n$ of $X$ are defined by

$$\hat{F}_{X,n}(x) = \frac{1}{n} \sum_{k=1}^{n} 1_{(x_k,\infty)}(x)$$

and by

$$\hat{Q}_{X,n}(p) = \inf\{x \in \mathbb{R} : \hat{F}_{X,n}(x) \geq p\},$$

respectively.

For instance, $X(\omega)$ is said to *(descriptively) dominate* $Y(\omega)$ in the sense of SD2 if and only if $\int_{-\infty}^{x} \hat{F}_{X,n}(t)dt \leq \int_{-\infty}^{x} \hat{F}_{Y,n}(t)dt$ holds for all $x \in \mathbb{R}$. The theoretical justification is the well known theorem of Glivenko-Cantelli.

**Theorem 1 (Glivenko-Cantelli).** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables with common distribution function $F_X$. Let $\hat{F}_{X,n}$ be the empirical distribution function of $X_1, \ldots, X_n$. Then*

$$\sup_{x\in\mathbb{R}} |\hat{F}_{X,n}(x) - F_X(x)| \xrightarrow{a.s.} 0.$$

The assertion of this theorem is that almost surely the empirical distribution of an independent, identically distributed sample converges uniformly to the distribution function. According to Yu (1993) the independence assumption can be weakened.

**Theorem 2 (Yu, 1993).** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of identically distributed random variables with a common continuous distribution function $F_X$, and let $\hat{F}_{X,n}$ be the empirical distribution function of $X_1, \ldots, X_n$. Assume that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} Cov(X_n, S_{n-1}) < \infty$$

*where $S_n := \sum_{k=1}^{n} X_k$. Then*

$$\sup_{x\in\mathbb{R}} |\hat{F}_{X,n}(x) - F_X(x)| \xrightarrow{a.s.} 0.$$

However, the main drawback of a descriptive comparison is the strong impact of the standard error. As discussed above, stochastic dominance is a partial order and very restrictive. If the distribution functions (or their antiderivatives, respectively) cross at least once, then there is no dominance relationship between

| | Degree | SD1 | | SD2 | | |
|---|---|---|---|---|---|---|
| Sample | $P_X$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0.1,1)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0.1,1)$ | $\mathcal{N}(0,1)$ |
| size | $P_Y$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0,1.1)$ |
| 250 | | 0.998 | 0.965 | 0.832 | 0.613 | 0.677 |
| 1000 | | 0.999 | 0.960 | 0.865 | 0.554 | 0.630 |
| 4000 | | 1.000 | 0.925 | 0.887 | 0.531 | 0.588 |

Table 2.1: Rate of wrong rejection of stochastic dominance in a descriptive comparison for SD1 and SD2 and various sample sizes and distributions. "$X \succeq_k Y$" is rejected if there exists an $x \in \mathbb{R}$ satisfying $\hat{F}_{X,n}^{(k)}(x) \leq \hat{F}_{Y,n}^{(k)}(x)$. The data are normally distributed and stochastically independent. The number of replications is $R = 1000$.

the random variables. For the empirical distribution functions things are even more involved. Even if $X$ dominates $Y$, the probability of rejecting stochastic dominance, based on the fact that the empirical distributions cross at least once, is very high in many cases.

We investigate the frequency of wrong rejection of stochastic dominance of first and second degree by means of simulation. In our analysis we consider three cases of pairs of normally distributed random variables: in the first case both are standard normally distributed, in the second case $Y$ has a standard normal distribution whereas $X$ has a normal distribution with mean 0.1 and variance 1, in the third case $X$ has a standard normal distribution whereas $Y$ is normally distributed with mean 0 and variance 1.1. In all of the three cases $X$ dominates $Y$ in the sense of SD2, in the first and second case in the sense of SD1 as well. Table 2.1 displays the results. Stochastic dominance is often not detected. This holds particularly for SD1, but the results are also not satisfying for SD2. If $X$ and $Y$ are equally distributed, the rate of wrong rejection is particularly large and increases with increasing sample size.

As explained above, the tendency of the descriptive procedure to reject stochastic dominance too often is not surprising. The multiple comparison is not robust even for small standard errors. For a more detailed analysis of this topic see, for instance, Nelson/ Pope (1991) and Stein/ Pfaffenberger/ Kumar (1983).

Schmid/ Trede (2000) explore the dominance relations in the daily returns of German assets during the 1990s. They show that SD1 is rejected in every comparison. In addition, although SD2 and SD3 can be established in some cases, the efficient sets, i.e. the sets of non-dominated assets, are still large for second and third degree stochastic dominance. This holds in particular for larger periods. In chapter 7 we will also encounter this problem.

These results altogether show that we need statistical tests for surveying a stochastic dominance relationship. In the last two decades many tests have been developed. In the next chapter we will give a review.

# Chapter 3

# Tests of Stochastic Dominance: A Survey

The vast majority of tests for stochastic dominance test the null hypothesis $H_0$ of dominance or equality against the alternative $H_1$ of non-dominance. Therefore in most of the tests stochastic dominance can be rejected or not, but not significantly asserted. This dissatisfying fact results from the complexity of the set of non-dominance. Usually tests are constructed in a way that they just keep the size at the border of the hypothesis. The border of the hypothesis "$X \succeq_k Y$" for any $k$ is the equality of the distributions of $X$ and $Y$. On the contrary, the border of the set "$X \not\succeq_k Y$" cannot be described in such a simple way. Consider e.g. the distributions $P_X = \delta_{0.99}$, i.e. $P(X = 0.99) = 1$, and $P_Y = \mathcal{U}(0,1)$. For the distribution functions we get $F_X = 1_{[0.99,\infty)}$ and $F_Y(x) = x1_{[0,1)}(x) + 1_{[1,\infty)}(x)$. Therefore $F_X(x) \le F_Y(x)$ holds for all $x \notin [0.99, 1)$, we even have

$$\lim_{x \uparrow 0.99} F_X(x) - F_Y(x) = -0.99.$$

On the other hand,

$$F_X(x) = 1 > x = F_Y(x)$$

holds for $x \in [0.99, 1)$. Hence $X$ does not dominate $Y$ in the sense of SD1. The example illustrates the consequence of the fact that $F_X^{(k)}(x) > F_Y^{(k)}(x)$ for one

$x$ suffices to prevent $X$ from dominating $Y$ in the sense of SD$k$. The boundary cannot be described in closed form, hence the construction of a test with non-dominance in the hypothesis is very difficult.

In the following we report on the development of the tests for stochastic dominance. The tests vary in some respects. One aspect is the degree of stochastic dominance they test. Some tests examine SD1, others SD2, others are applicable to various degrees of stochastic dominance. Many tests assume independence of the data, both contemporaneous and serial. In recent years some tests have been developed which permit various kinds of dependence structures. In this study we particularly focus on these tests and extend them.

A further question is the choice of an appropriate test statistic. According to proposition 2 the test statistic can be based on the proper antiderivatives of the empirical distribution functions $\hat{F}^{(k)}_{X,n}$, $\hat{F}^{(k)}_{Y,n}$ or of the empirical quantile functions $\hat{Q}^{(k)}_{X,n}$, $\hat{Q}^{(k)}_{Y,n}$ if $k \in \{1, 2\}$.[1] The differences are derived at some grid points; note that these statistics are multidimensional. In order to get a unidimensional test statistic primarily two kinds of functions are applied to $\hat{F}^{(k)}_{X,n} - \hat{F}^{(k)}_{Y,n}$ and accordingly $\hat{Q}^{(k)}_{X,n} - \hat{Q}^{(k)}_{Y,n}$: area and supremum statistics. An area statistic derives an integral or a weighted average of $\hat{F}^{(k)}_{X,n} - \hat{F}^{(k)}_{Y,n}$ or $\hat{Q}^{(k)}_{X,n} - \hat{Q}^{(k)}_{Y,n}$ at some grid points, whereas a supremum statistic determines the maximal difference.

Presumably the first test of stochastic dominance is the test of Beach/ Davidson (1983). They examine Lorenz dominance and second degree stochastic dominance. The test statistics are derived from the empirical Lorenz and generalized Lorenz curve at some grid points. They consider the covariance structure of the quantile curve ordinates, but implicitly assume that the data are independent. Under this and some regularity assumptions the test statistics are asymptotically $\chi^2$-distributed.

Deshpande/ Singh (1985) also create a test for SD2. They test $H_0 : F = F_0$

[1] In this study we confine ourselves to the case that both samples have the same size $n$.

against $H_1 : F \succeq_2 F_0$ where $F_0$ is known. The test statistic is the double integral

$$T_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{x} (\hat{F}_{X,n}(t) - F_0(t))dt \; dF_0(x).$$

The term is an area statistic, statistical inference is based on its asymptotic normality. Although the test has the alternative of dominance and good efficiency properties, it is of very limited use. A data set can only be tested against a known alternative $F_0$, in addition to this the data have to be independent.

Chow (1989) develops a test for stochastic dominance of any degree. He tests $H_0 : P_X = P_Y$ against the two-sided alternative $H_1 : (X \succeq_2 Y \text{ or } Y \succeq_2 X)$ using a multiple comparison procedure. The difference of the empirical distribution functions is derived at some grid points, in addition to this its empirical covariance matrix contributes to the test statistic. Under the independence assumption the test statistic tends to a Studentized Maximum Modulus (SMM) distribution. Zheng/ Formby/ Smith/ Chow (2000) take up the idea to create a test for normalized stochastic dominance. They develop a corresponding test for normalized stochastic dominance of any order.

Bishop/ Chakraborti/ Thistle (1989) provide a test for SD2 which is asymptotically distribution-free. For the test of $H_0 : X \succeq_2 Y$ against $H_1 : X \not\succeq_2 Y$ they derive the differences $\hat{Q}^{(2)}_{X,n}(p_k) - \hat{Q}^{(2)}_{Y}(p_k)$ at some grid points $0 < p_1 < \cdots < p_n = 1$. The test statistic is

$$T = (\hat{Q}^{(2)}_{X,n}(P) - \hat{Q}^{(2)}_{Y}(P))'\hat{\Omega}^{-1}(\hat{Q}^{(2)}_{X,n}(P) - \hat{Q}^{(2)}_{Y}(P))$$

where $P = (p_1, \dots, p_n)$ and $\hat{\Omega}$ is an estimator of the covariance matrix $\Omega$ of $\hat{Q}^{(2)}_{X,n}(P) - \hat{Q}^{(2)}_{Y}(P)$. If the samples are independent, $T$ is $\chi^2$-distributed.

McFadden (1989) develops tests for SD1 and SD2. For $k = 1, 2$ he tests the null hypothesis $H_0 : X \succeq_k Y$ against the alternative $H_1 : X \not\succeq_k Y$ using the supremum statistic

$$T_k = \sqrt{n} \sup_{x \in [0,1]} (\hat{F}^{(k)}_{X,n}(x) - \hat{F}^{(k)}_{Y,n}(x));$$

$X$ and $Y$ are assumed to have values in [0,1]. The data are assumed to be independent. For $T_1$ he gives the distribution under $P_X = P_Y$ in closed form and asymptotically, whereas he only gives some bounds for the distribution of $T_2$ under $P_X = P_Y$.

Klecan/ McFadden/ McFadden (1991) generalize the procedure to testing *stochastic maximality*. A set is stochastically maximal if no prospect is stochastically dominated by another prospect in the set. In addition to this, they allow for general weak dependence within the processes and *generalized exchangeability* between them.

The weak dependence structure assumed is that the processes are strictly stationary and *strongly mixing* (or *$\alpha$-mixing*) with coefficient $\alpha(k) = O(k^{-\delta})$ for some $\delta > 1$. The *strong mixing coefficient* of two sigma fields $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

A sequence $(Z_k)_{k\in\mathbb{Z}}$ of random variables is strongly mixing if for the generated sigma fields $\mathcal{F}_a^b := \sigma(Z_k : a \leq k < b)$ the following holds:

$$\alpha(m) := \sup\{\alpha(\mathcal{F}_{-\infty}^k, \mathcal{F}_{k+m}^{\infty}) : k \in \mathbb{Z}\} \underset{m\to\infty}{\longrightarrow} 0.$$

The strong mixing coefficient $\alpha(m)$ of $(Z_k)_{k\in\mathbb{Z}}$ is defined for $m \in \mathbb{N}$. Davis/ Mikosch/ Basrak (1999) show that a stationary GARCH process is strongly mixing.

A set of random variables $\{X_1, \ldots, X_n\}$ is *exchangeable* if for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ the tuple $(X_{i_1}, \ldots, X_{i_n})$ has the same distribution as $(X_1, \ldots, X_n)$. Consequently, exchangeable random variables must be identically distributed whereas independence and an identical distribution are sufficient for exchangeability. The set $\{X_1, \ldots, X_n\}$ is *generalized exchangeable* if $\{Y_1, \ldots, Y_n\}$ is exchangeable where $Y_i = F_{X_i}(X_i)$. If $X_1, \ldots, X_n$ have continuous distributions, all $Y_i$ are identically $\mathcal{U}(0, 1)$ distributed. Therefore, in this case independence of $X_1, \ldots, X_n$ is sufficient for generalized exchangeability. However, the assumption of generalized exchangeability is weaker than the assumption of independence.

Altogether we see that the test of Klecan/ McFadden/ McFadden can be applied to more general settings than a test which requires contemporaneous and serial independence of the data.

Schmid/ Trede (1997) propose a similar test for SD2. In chapter 5 we analyze its performance in detail. We find that the test captures the serial dependence very well, but is not robust to the effect of conditional heteroskedasticity. The test of Klecan/ McFadden/ McFadden (1991) is applied e.g. by Maasoumi/ Heshmati (2000) to the comparison of income distributions.

Aly (1991) tests for Lorenz dominance, namely the null hypothesis $H_0 : X \overset{L}{=} Y$ against the alternative $H_1 : (X \succeq_L Y \text{ and } X \overset{L}{\neq} Y)$. He uses the test statistic

$$T = 2 \int_0^1 (\hat{L}_X(p) - \hat{L}_Y(p))dp$$

where

$$\hat{L}_X(p) = \frac{1}{\hat{\mu}_X} \int_0^p \hat{Q}_{X,n}(t)dt$$

is the empirical Lorenz curve of $X$ and $\hat{L}_Y$ is defined analogously. Under the assumption of independence of the data the statistic $\sqrt{\frac{n}{2}} \frac{T}{\hat{\sigma}_{X,Y}}$, where $\hat{\sigma}^2_{X,Y}$ is a variance estimator of the pooled observations of $X$ and $Y$, is asymptotically normal.

Bishop/ Formby/ Thistle (1992) devise a union-intersection test to determine whether the conditional means of the quantile functions differ. As in Chow (1989) and Zheng/ Formby/ Smith/ Chow (2000) the test statistic is SMM distributed under the null hypothesis.

Eubank/ Schechtman/ Yitzhaki (1993) design a test for SD2. They test the equality hypothesis $H_0 : P_X = P_Y$ against the alternative of dominance $H_1 : X \succeq_2 Y$ with the area statistic

$$T = \sqrt{n} \int_{-\infty}^{\infty} (2 - \hat{F}_{X,n}(x) - \hat{F}_{Y,n}(x))(\hat{F}_{X,n}(x) - \hat{F}_{Y,n}(x))dx.$$

If the data are independent, the test statistic asymptotically tends to a normal distribution under $H_0$. Besides the independence assumption the test has another essential drawback. If the test rejects the null hypothesis of equality, it does not significantly confirm that $X$ dominates $Y$. In the case that $F_X^{(2)}$ and $F_Y^{(2)}$ cross the test statistic can still be arbitrarily large.

Kaur/ Rao/ Singh (1994) create a test of

$$H_0 : F_X^{(2)}(x) \geq F_Y^{(2)}(x) \text{ for some } x \in [a, b]$$

against the alternative

$$H_1 : F_X^{(2)}(x) < F_Y^{(2)}(x) \text{ for all } x \in [a, b]$$

where $a$ and $b$ are any real numbers satisfying $a < b$. The alternative $H_1$ is similar to the statement that $X$ dominates $Y$ in the sense of SD2. In contrast to the test of Eubank/ Schechtman/ Yitzhaki (1993) the null hypothesis resembles "$X \not\succeq_2 Y$". Kaur/ Rao/ Singh (1994) use the infimum statistic

$$T = \inf_{x \in [a,b]} \frac{\hat{F}_{Y,n}^{(2)}(x) - \hat{F}_{X,n}^{(2)}(x)}{\sqrt{\frac{1}{n}(S_{X,n}^2(x) + S_{Y,n}^2(x))}}$$

where

$$S_{X,n}^2(x) = \frac{1}{n} \sum_{k=1}^{n} (1_{(x_k,\infty)}(x)(x - x_k)^2) - (\hat{F}_{X,n}^{(2)}(x))^2$$

and $S_{Y,n}^2(x)$ is defined analogously. They show that for the appropriate critical value the test has an upper bound $\alpha$ on the asymptotic size and is consistent. The test of Kaur/ Rao/ Singh is a good starting point for testing for stochastic dominance where dominance is the alternative. However, their approach has two crucial drawbacks. As in many other tests, the observations have to be independent. But the more important disadvantage concerns the shape of the hypotheses. The lower and upper bound $a$ and $b$ of the considered interval are chosen arbitrarily. In the case that the distributions have bounded support $[a, b]$

we get $F_X^{(2)}(a) = F_Y^{(2)}(a) = 0$, hence $H_1$ do not hold even if $X$ dominates $Y$ in the sense of SD2. We will discuss this test and look for a remedy in chapter 6.

Herring (1996a) tests $H_0 : (\mu_X = \mu_Y, X \succeq_2 Y)$ against the alternative $H_1 : (\mu_X = \mu_Y, \sigma_X \geq \sigma_Y)$. The test statistic is the rank sum of $X$ where the observations are ranked by the deviation from the mean. The critical value is easy to determine if the data are independent. The assumption of independence is restrictive, but this holds even more for the assumption of equal means on which the test is based. In particular, in most empirical applications the theoretical mean is unknown.

Herring (1996b) tests $H_0 : P_X = P_Y$ against $H_1 : X \succeq_2 Y$ using the test statistic

$$T = \sum_{k=1}^{2n} (\hat{F}_{Y,n}^{(2)}(z_k) - \hat{F}_{X,n}^{(2)}(z_k))$$

where $(z_1, \ldots, z_{2n})$ is the ordered combined sample of $X$ and $Y$. The critical value is determined by permutations. The data are assumed to be independent. The main drawback is the same as for the test of Eubank/ Schechtman/ Yitzhaki (1993). The alternative is not the whole complement of the hypothesis, hence rejection of the hypothesis does not mean that the alternative of dominance is significantly confirmed.

Anderson (1996) tests for the first three degrees of stochastic dominance. The hypothesis is $H_0 : P_X = P_Y$ in each case, the alternative is $H_1 : X \succeq_k Y$ for $k = 1, 2, 3$. The test is based on modifications of the goodness-of-fit test and composed of a multiple comparison at some grid points. Under the independence assumption the test statistic is $\chi^2$-distributed under $H_0$. As in the tests of Herring (1996b) and Eubank/ Schechtman/ Yitzhaki, the main disadvantage is the fact that rejection of the hypothesis is not equivalent to significant confirmation of the alternative. The reason is that the alternative is not the complement of the hypothesis.

Schmid/ Trede (1996a) develop a test for SD1. For the test of $H_0 : X \succeq_1 Y$

against $H_1 : X \not\succeq_1 Y$ they use the area statistic

$$\begin{aligned} T &= \sqrt{\frac{n}{2}} \int_{-\infty}^{\infty} (\hat{F}_{X,n}(x) - \hat{F}_{Y,n}(x))_+ d\hat{F}_{Y,n}(x) \\ &= \sqrt{\frac{1}{2n}} \sum_{k=1}^{n} (\hat{F}_{X,n}(y_{(k)}) - \frac{k}{n})_+ = \sqrt{\frac{1}{2n}} \sum_{k=1}^{n} (\frac{R(y_{(k)}) - 2k}{n})_+. \end{aligned}$$

$x_+$ denotes the nonnegative part of a real number $x$, i.e. $x_+ = \max\{x, 0\}$, $y_{(1)} \leq \cdots \leq y_{(n)}$ denotes the order statistic of $y_1, \ldots, y_n$, and $R(y_{(k)})$ denotes the rank of $y_{(k)}$ in the combined sample. Schmid/ Trede note that $T$ is an affine transformation of the Wilcoxon-Mann-Whitney statistic $\sum_{k=1}^{n} R(y_{(k)})$. They give the finite sample and asymptotic distribution under the limiting case $P_X = P_Y$ and under the assumption that the samples are independent. Power investigations suggest that the test is an attractive substitute for the Wilcoxon-Mann-Whitney test. However, the performance of the test in the case that the independence assumption is abandoned is not known.

Schmid/ Trede (1996b) test for SD2 using second degree analogs of some well-known statistics. For the test of $H_0 : X \succeq_2 Y$ against $H_1 : X \not\succeq_2 Y$ they use the supremum statistic

$$T_1 = \sup_{x \in \mathbb{R}} (\hat{F}^{(2)}_{X,n}(x) - \hat{F}^{(2)}_{Y,n}(x)) = \max_{i=1,\ldots,2n} (\hat{F}^{(2)}_{X,n}(z_{(i)}) - \hat{F}^{(2)}_{Y,n}(z_{(i)}))$$

where $z_{(i)}$ denotes the $i$th order statistic of the combined sample $(z_1, \ldots, z_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the area statistic

$$T_2 = \int_{-\infty}^{\infty} (\hat{F}^{(2)}_{X,n}(t) - \hat{F}^{(2)}_{Y,n}(t)) d(\hat{F}_{X,n}(t) + \hat{F}_{Y,n}(t)).$$

$T_1$ is a second degree analog of the one-sided Kolmogorov-Smirnov statistic, $T_2$ of the Wilcoxon statistic. The critical values are determined by permutations. There are $\binom{2n}{n}$ different subsets of order $n$ from $(z_1, \ldots, z_{2n})$, for large $n$ they randomly choose $B = 500$ of them. For every permutation the test statistics are calculated and ordered according to size: $T^{(1)} \leq \cdots \leq T^{(B)}$. Then $c = T^{(B(1-\alpha))}$ is the critical value. Under the assumption that the observations are independent the tests keep the size $\alpha$ and have good power properties.

The independence restrictions of this test are relaxed in the already mentioned test of Schmid/ Trede (1997). They use the test statistic $T_1$ as defined above, but permute matched pairs instead of all observations. With this modification they can capture the dependence between $X_k$ and $Y_k$ for every $k$, but the test is still not robust to serial dependence within each sample. In chapter 5 we will explore this study in more detail.

Schmid/ Trede (1998) confine themselves to the case where one of the compared distributions is completely known. They test $H_0 : X \succeq_2 Y$ against $H_1 : X \not\succeq_2 Y$ where $P_Y$ is a known continuous distribution and $P_X$ has to be inferred from the observed data. If the distribution function $F_Y$ of $Y$ satisfies $F_Y' \geq 0$ and $F_Y'' \leq 0$, i.e. $Y$ has a decreasing density on its support, then $X \succeq_2 Y$ implies $F_Y(X) \succeq_2 F_Y(Y)$. Due to the fact that $F_Y(Y)$ is uniformly distributed on $(0, 1)$, the test can be traced back to the problem $H_0 : Z \succeq_2 \mathcal{U}(0, 1)$ against $H_1 :$ not $H_0$ where $Z = F_Y(X)$. The test statistic is

$$T = \sup_{x \in [0,1]} \sqrt{n} \int_0^x (\hat{F}_{Z,n}(t) - t)dt.$$

If the observations are independent, for the limiting case $Z \sim \mathcal{U}(0, 1)$ of $H_0$, the test statistic converges in distribution to $\sup\limits_{x \in [0,1]} \int_0^x B(t)dt$ where $B$ denotes a Brownian Bridge on $[0, 1]$. They show that the test has better power than the test of Deshpande/ Singh. However, the independence assumption and the fact that one distribution is assumed to be known are very hard restrictions.

Xu/ Fisher/ Willson (1997) create tests for first and second degree stochastic dominance. They test the hypothesis $H_0 : X \succeq_k Y$ against $H_1 : X \not\succeq_k Y$ for $k = 1, 2$ with some test statistics $T_1$ and $T_2$ which are solutions of minimizations under constraints. $T_1$ and $T_2$ are asymptotically distributed as weighted sums of $\chi^2$-variates of various degrees of freedom. The tests are applicable under very general assumptions. If the samples are generated by strongly mixing processes, then the test asymptotically keeps the nominal size. For capturing the dependence structure between proximate observations Xu/ Fisher/ Willson (1997) use the *moving block bootstrap*. We explain and examine the test in chapter 5.

Barrett/ Donald (2003) test for stochastic dominance of any degree. They assume independent samples and test $H_0 : X \succeq_k Y$ against $H_1 : X \not\succeq_k Y$ for any $k \in \mathbb{N}$. The test statistic is

$$T = \sqrt{\frac{n}{2}} \sup_{x \in [0,a]} (\hat{F}^{(k)}_{X,n}(x) - \hat{F}^{(k)}_{Y,n}(x))$$

where $X$ and $Y$ are assumed to be bounded by 0 and $a > 0$. The critical value is determined by bootstrap and Monte Carlo simulation. This approach gives a very general result concerning the distributions of the random variables, but still the assumption of independence is a very hard restriction.

Linton/ Maasoumi/ Whang (2005) develop a test for stochastic dominance of the first and second degree. For a set of random variables $X_1, \ldots, X_n$ they test $H_0 : \exists i, j \in \{1, \ldots, n\}, i \neq j : X_i \succeq_k X_j$, i.e. there is one random variable which is dominated, against the alternative $H_1$ : (not $H_0$). The test statistic

$$T_k = \min_{i \neq j} \sup_{x \in \mathbb{R}} \sqrt{n}(\hat{F}^{(k)}_{X_i,n}(x) - \hat{F}^{(k)}_{X_j,n}(x))$$

can be easily modified by omitting the min operator in the case that we just test $H_0 : X_i \succeq_k X_j$ against $H_1$ : (not $H_0$). A *subsampling* procedure is used for determining the critical value. The test can be applied to strongly mixing processes because the subsampling captures the dependence of proximate observations. The test will be examined in more detail in further chapters.

In most of the tests illustrated in this chapter independence of the data is assumed, both within and between samples. If we interpret the observations as time series data, this means that the data $X_t$ and $Y_t$ are contemporaneously independent for each $t$ and the series $(X_t)_{t \in \mathbb{Z}}$ and $(Y_t)_{t \in \mathbb{Z}}$ are serially independent. There is strong evidence that in many fields of application the assumption of independence is not realistic. The question arises whether the deviations have a strong effect on the performance of the test procedure.

There is a multitude of possible dependence structures in a series of random variables. The most important ones for financial data are contemporaneous correlation – which is usually positive – and conditional heteroskedasticity within each time series. Before investigating the effect on tests of stochastic dominance, we consider standard statistical procedures for means and variances such as the t-test and F-test, respectively. In the next chapter, we present the results of Kläver/ Schmid (2004) concerning the impact of conditional heteroskedasticity and positive correlation on these procedures.

# Chapter 4

# The Effect of Conditional Heteroskedasticity on Common Statistical Procedures for Means and Variances

Commonly used standard statistical procedures for means and variances (such as the t-test for means or the F-test for variances and corresponding confidence procedures) require observations from independent and identically normally distributed variables (standard case). These procedures, however, are routinely applied to data which do not satisfy these constraints. In particular, this is the case for financial data such as daily returns on assets or currencies, which are notoriously nonnormal and show *conditional heteroskedasticity*, hence they are dependent.

Is there any effect of conditional heteroskedasticity on these procedures for means and variances? We will shed some light on this question which might be of interest to statisticians, econometricians and financial analysts. In order to keep the study short we will confine ourselves to the case where data are generated by a

*GARCH(1,1)* process. Though this is a special approach to modelling conditional heteroskedasticity, it is commonly believed that this model is well suited to be used on financial data. Akgiray (1989) concludes in an empirical study of the temporal behavior of daily stock market returns:

"The conditional heteroskedastic processes ... fit to data very satisfactorily. More important, they provide improved forecasts of volatility. Within the class of such models, GARCH(1,1) processes show the best fit and forecast accuracy." For further affirmation of the evidence and usefulness of GARCH(1,1) see e.g. Davis/ Mikosch (2000) and Engle (2001).

In this chapter, we examine the impact of GARCH on the procedures for means and variances. We state some fundamental results, such as strong laws of large numbers (SLLN) and a central limit theorem (CLT) for GARCH(1,1) processes. It is shown that the t-test for a mean is (at least asymptotically) valid for observations generated by a GARCH(1,1) process. The same is true for the corresponding confidence interval for the mean. The effect of conditional heteroskedasticity on confidence and testing procedures for variances is investigated analytically and by simulation. It turns out that the variance estimator $S_n^2$ is still unbiased and consistent, but the variance of $S_n^2$ is larger than in the case of independent random variables. The difference between GARCH(1,1) and the standard case depends on the parameters of the GARCH(1,1) process. Simulations show that the coverage probability of the standard confidence interval and the error probability of the first kind for the F-test differ significantly from their nominal values. This effect increases with increasing GARCH parameters and with an increasing number of observations. The largest effect emerges in the case of an infinite fourth moment of $X_t$, hence infinite variance of $S_n^2$. We further investigate the Levene test (Levene, 1960) which is known to be more robust to deviations from the normal distribution. It yields better results than the F-test, but still does not keep the nominal size.

## 4.1 Notations, Definitions and Preliminary Results

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ denote a sequence of independent and identically $\mathcal{N}(0,1)$-distributed random variables. Let $X_t = \sigma_t\varepsilon_t$ where

$$\sigma_t^2 = \alpha_0 + \sum_{k=1}^{q}\alpha_k X_{t-k}^2 + \sum_{k=1}^{p}\beta_k\sigma_{t-k}^2$$

for $t\in\mathbb{Z}$ and $\alpha_0>0$, $\alpha_k\geq 0, \beta_l\geq 0$ for all $k\in\{1,\dots,q\}$, $l\in\{1,\dots,p\}$. According to Bollerslev (1986) the process $(X_t)_{t\in\mathbb{Z}}$ is called ***G**eneralized **A**uto**R**egressive **C**onditional **H**eteroskedastic process with parameters p and q*, abbreviated by *GARCH(p,q)*. For $p=0$ the process corresponds to the *ARCH(q)* process introduced by Engle (1982).

In the following we consider a GARCH(1,1) process. It is *weakly stationary* with

$$E(X_t) = Cov(X_t, X_s) = 0$$

for $t\neq s$ and

$$\sigma^2 := Var(X_t) = \frac{\alpha_0}{1-\alpha_1-\beta_1}$$

if and only if $\alpha_1+\beta_1<1$.

Further Bollerslev shows that $X_t$ has a finite fourth moment if and only if $3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2<1$; in this case

$$\begin{aligned}\kappa^4 := E(X_t^4) &= \frac{3\alpha_0^2(1+\alpha_1+\beta_1)}{(1-\alpha_1-\beta_1)(1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2)}\\ &= 3(\sigma^2)^2\frac{1-(\alpha_1+\beta_1)^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}\\ &= 3(\sigma^2)^2\left(1+\frac{2\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}\right).\end{aligned} \tag{4.1}$$

From (4.1) we see that $X_t$ is leptokurtic if and only if $\alpha_1>0$.

In a subsequent article Bollerslev (1988) asserts that for the autocorrelations

$\rho_s := Corr(X_t^2, X_{t-s}^2)$ for $t, s \in \mathbb{Z}$, $s > 0$, the following holds:

$$\rho_s = (\alpha_1 + \beta_1)^{s-1} \frac{\alpha_1(1 - \alpha_1\beta_1 - \beta_1^2)}{1 - 2\alpha_1\beta_1 - \beta_1^2} \tag{4.2}$$

and $\rho_{-s} = \rho_s$, due to symmetry. For the covariances $\gamma_s := Cov(X_t^2, X_{t-s}^2)$ and mixed moments $\xi_s := E(X_t^2 X_{t-s}^2)$ this yields

$$\gamma_s = (\sigma^2)^2(\alpha_1 + \beta_1)^{s-1} \frac{2\alpha_1(1 - \alpha_1\beta_1 - \beta_1^2)}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \tag{4.3}$$

$$\xi_s = (\sigma^2)^2 \Big[1 + (\alpha_1 + \beta_1)^{s-1} \frac{2\alpha_1(1 - \alpha_1\beta_1 - \beta_1^2)}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2}\Big] \tag{4.4}$$

and $\gamma_{-s} = \gamma_s$, $\xi_{-s} = \xi_s$.

We will make use of these results when investigating the distribution of the usual variance estimator $S_n^2$.

According to Nelson (1990) the process $(X_t)_{t \in \mathbb{N}}$ is strictly stationary and ergodic, according to White (1984, Prop. 3.36) this implies strict stationarity and ergodicity of $(X_t^2)_{t \in \mathbb{N}}$. Therefore the ergodic theorem yields that $(X_t)_{t \in \mathbb{N}}$ and $(X_t^2)_{t \in \mathbb{N}}$ satisfy the following strong law of large numbers.

**Proposition 4 (SLLN for GARCH(1,1)).** *Let $(X_t)_{t \in \mathbb{Z}}$ be a weakly stationary GARCH(1,1) process, i.e. $X_t = \sigma_t \varepsilon_t$, $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ and $\alpha_1 + \beta_1 < 1$. With $\sigma^2 := Var(X_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$ we have*

$$\frac{1}{n} \sum_{t=1}^{n} X_t \xrightarrow{a.s.} 0 \tag{4.5}$$

$$\frac{1}{n} \sum_{t=1}^{n} X_t^2 \xrightarrow{a.s.} \sigma^2 \tag{4.6}$$

*where "$\xrightarrow{a.s.}$" denotes almost sure convergence.*

In addition to the ergodicity we know that a GARCH(1,1) process is a martingale difference sequence; the result of Hayashi (2000, p. 104) for ARCH processes can easily be extended to GARCH processes. Hence, it follows from a result of Billingsley (1961) that the central limit theorem holds for a GARCH(1,1) process.

**Proposition 5 (CLT for GARCH(1,1)).** *Let $(X_t)_{t\in\mathbb{Z}}$ denote a weakly stationary GARCH(1,1) process with $\sigma^2 := Var(X_t)$. Then*

$$\frac{1}{\sigma\sqrt{n}}\sum_{t=1}^{n} X_t \xrightarrow{d} \mathcal{N}(0,1)$$

*where "$\xrightarrow{d}$" denotes convergence in distribution.*

Note that the existence of fourth moments is not required for proposition 4 and 5 to hold; the only restrictions for $\alpha_1$ and $\beta_1$ are $\alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$.

## 4.2 Procedures for Means

Let $\mu \in \mathbb{R}$ and $Y_t := \mu + X_t$ for $t \in \mathbb{Z}$. The usual t-test statistic for the null hypothesis $H_0 : \mu = \mu_0$ or a corresponding one-sided hypothesis based on observations $Y_1, \ldots, Y_n$ is

$$\tau_n = \frac{\sqrt{n}(\overline{Y}_n - \mu_0)}{\sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \overline{Y}_n)^2}} = \frac{\sqrt{n}(\overline{Y}_n - \mu_0)}{\sqrt{S_n^2}}$$

where $\overline{Y}_n := \frac{1}{n}\sum_{t=1}^{n} Y_t$. $\tau_n$ has a $t$-distribution with $n-1$ degrees of freedom in the standard, i.e. the independent case $\alpha_1 = \beta_1 = 0$. Therefore under $H_0$ the distribution of $\tau_n$ tends to a $\mathcal{N}(0,1)$-distribution for $n \to \infty$ in the standard case. The following proposition states that the latter also holds in the more general case where $(X_t)_{t\in\mathbb{Z}}$ is a weakly stationary GARCH(1,1) process. However, $\tau_n$ does not have a $t_{n-1}$-distribution for finite samples of size $n$.

**Proposition 6.** *Let $(X_t)_{t\in\mathbb{Z}}$ be a weakly stationary GARCH(1,1) process (as defined in section 4.1). Then under $H_0$ we have*

$$\tau_n \xrightarrow{d} \mathcal{N}(0,1).$$

Proposition 6, which follows easily from the SLLN and CLT for GARCH(1,1) and Slutsky's theorem, tells us that asymptotically there is no effect of conditional heteroskedasticity as modelled by a GARCH(1,1) process on the usual t-test.

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.1 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | |
| 0.2 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | |
| 0.3 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | |
| 0.4 | 0.05 | 0.05 | 0.05 | 0.04 | 0.05 | 0.05 | | | | |
| 0.5 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | | | |
| 0.6 | 0.05 | 0.05 | 0.05 | 0.05 | | | | | | |
| 0.7 | 0.05 | 0.04 | 0.05 | | | | | | | |
| 0.8 | 0.05 | 0.05 | | | | | | | | |
| 0.9 | 0.05 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| size | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |

Table 4.1: Rejection probability of the t-test for the nominal value $\alpha = 0.05$. We choose $\alpha_0 = 0.1$, $n = 20$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$.

The simulations show that the distributional convergence to the normal distribution is very fast. Table 4.1 presents the results for $n = 20$. Even for this small length of the time series the deviation from the nominal size $\alpha = 0.05$ is very small and could be a result of the standard error of the simulation. This also holds for $n = 100$ and $n = 1000$ which is presented in table 4.2 and 4.3.

Further, proposition 6 tells us that asymptotically there is no effect of conditional heteroskedasticity on the commonly used confidence interval for $\mu$, i.e. the nominal coverage probability $1 - \alpha$ is clearly kept in the GARCH(1,1) case.

Note that for these findings the existence of fourth moments is not necessary.

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.1 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | |
| 0.2 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | |
| 0.3 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | |
| 0.4 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | | |
| 0.5 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | | | |
| 0.6 | 0.05 | 0.05 | 0.05 | 0.05 | | | | | | |
| 0.7 | 0.05 | 0.05 | 0.05 | | | | | | | |
| 0.8 | 0.05 | 0.05 | | | | | | | | |
| 0.9 | 0.05 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| size | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |

Table 4.2: Rejection probability of the t-test for the nominal value $\alpha = 0.05$. We choose $\alpha_0 = 0.1$, $n = 100$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$.

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 |
| 0.1 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | |
| 0.2 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | | |
| 0.3 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | | | |
| 0.4 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | | | | |
| 0.5 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | | | | | |
| 0.6 | 0.05 | 0.05 | 0.05 | 0.05 | | | | | | |
| 0.7 | 0.05 | 0.05 | 0.05 | | | | | | | |
| 0.8 | 0.05 | 0.05 | | | | | | | | |
| 0.9 | 0.05 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| size | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |

Table 4.3: Rejection probability of the t-test for the nominal value $\alpha = 0.05$. We choose $\alpha_0 = 0.1$, $n = 1000$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$.

## 4.3 Procedures for Variances

The focus of this section is the (unconditional) variance

$$\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

of a weakly stationary GARCH(1,1) process $(X_t)_{t\in\mathbb{Z}}$. For $\mu \in \mathbb{R}$ we define $Y_t := \mu + X_t$. The usual estimator for $\sigma^2$ based on $Y_1, \ldots, Y_n$ is

$$S_n^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \overline{Y}_n)^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \overline{X}_n)^2.$$

$S_n^2$ is *unbiased*, i.e.

$$E(S_n^2) = \sigma^2,$$

and *consistent* for $\sigma^2$, i.e.

$$S_n^2 \xrightarrow{p} \sigma^2,$$

as in the standard (independent) case; "$\xrightarrow{p}$" denotes convergence in probability. The former follows from $Cov(X_t, X_s) = 0$ for $t \neq s$, the latter is a simple consequence of proposition 4 and Slutsky's theorem. Therefore conditional heteroskedasticity (as modelled by GARCH (1,1)) has no effect on unbiasedness and consistency of $S_n^2$. However, it has a tremendous effect on the distribution of $S_n^2$. In the standard case (i.e. $\alpha_1 = \beta_1 = 0$) we have

$$\frac{(n-1)S_n^2}{\sigma^2} \overset{d}{\sim} \chi^2_{n-1}$$

where "$X \overset{d}{\sim} P$" denotes that $X$ has the distribution $P$ and $\chi^2_k$ denotes a chi-squared distribution with k degrees of freedom. Therefore

$$Var(S_n^2) = \frac{2(\sigma^2)^2}{n-1}$$

in the standard case.

Things are much more involved for the more general case under study. If $(X_t)_{t\in\mathbb{Z}}$ is a GARCH(1,1) process, then

$$\frac{(n-1)S_n^2}{\sigma^2}$$

does not have a $\chi^2_{n-1}$-distribution. The computation of $Var(S_n^2)$ is therefore much more sophisticated.

$$S_n^2 = \frac{1}{n-1}\Big(\sum_{t=1}^{n} X_t^2 - n\overline{X}_n^2\Big)$$

yields

$$\begin{aligned}(S_n^2)^2 &= \frac{1}{(n-1)^2}\Big(\sum_{t=1}^{n}\sum_{s=1}^{n} X_t^2X_s^2 - \frac{2}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n} X_t^2X_sX_r \\ &+ \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n} X_tX_sX_rX_q\Big).\end{aligned}$$

Therefore we have to calculate $E(X_t^4)$, $E(X_t^3X_s)$, $E(X_t^2X_s^2)$, $E(X_t^2X_sX_r)$ and $E(X_tX_sX_rX_q)$ for $t$, $s$, $r$, $q$ mutually different. If the fourth moment is infinite, i.e. $3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2 \geq 1$, the variance of $S_n^2$ does not exist. If the fourth moment is finite, i.e. $3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2 < 1$, we know from (4.4) that

$$E(X_t^2X_s^2) = (\sigma^2)^2\Big[1 + (\alpha_1+\beta_1)^{|t-s|-1}\frac{2\alpha_1(1-\alpha_1\beta_1-\beta_1^2)}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}\Big]$$

for $t \neq s$. Further calculation shows

$$E(X_t^3X_s) = E(X_t^2X_sX_r) = E(X_tX_sX_rX_q) = 0$$

for $t$, $s$, $r$, $q$ mutually different. Further calculations lead to

$$(S_n^2)^2 = \frac{1}{(n-1)^2}\Big[\Big(1-\frac{2}{n}+\frac{1}{n^2}\Big)\sum_{t=1}^{n} X_t^4 + \Big(1-\frac{2}{n}+\frac{3}{n^2}\Big)\sum_{\substack{t,s=1\\ t\neq s}}^{n} X_t^2X_s^2 + Z\Big]$$

with $E(Z)=0$ and therefore

$$\begin{aligned}&Var(S_n^2) = E((S_n^2)^2) - (E(S_n^2))^2 \\ &= (\sigma^2)^2\Big\{4\frac{n^2-2n+3}{(n(n-1))^2}\frac{\alpha_1}{1-(\alpha_1+\beta_1)}\Big[n - \frac{1-(\alpha_1+\beta_1)^n}{1-(\alpha_1+\beta_1)}\Big] \\ &\quad \frac{1-\alpha_1\beta_1-\beta_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2} + \frac{3}{n}\frac{1-(\alpha_1+\beta_1)^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2} - \frac{n-3}{n(n-1)}\Big\} \\ &= (\sigma^2)^2\Big\{4\frac{n^2-2n+3}{(n(n-1))^2}\frac{\alpha_1}{1-(\alpha_1+\beta_1)}\Big[n - \frac{1-(\alpha_1+\beta_1)^n}{1-(\alpha_1+\beta_1)}\Big] \\ &\quad \Big(1+\frac{3\alpha_1^2+\alpha_1\beta_1}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}\Big) + \frac{3}{n}\Big(1+\frac{2\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}\Big) - \frac{n-3}{n(n-1)}\Big\}.\end{aligned}$$

In the standard case $\alpha_1 = \beta_1 = 0$ we have $Var(S_n^2) = \frac{2(\sigma^2)^2}{n-1}$ which is consistent with the fact that $\frac{(n-1)S_n^2}{\sigma^2} \overset{d}{\sim} \chi^2_{n-1}$. $Var(S_n^2) = \frac{2(\sigma^2)^2}{n-1}$ even holds for $\alpha_1 = 0$ and $\beta_1 \in [0, 1)$ arbitrary.

$Var(S_n^2)$ increases monotonically in $\alpha_1$ and $\beta_1$, so the variance of the estimator $S_n^2$ increases with the GARCH parameters if $n$ is fixed. The variance increases strictly monotonically with the exception of $\alpha_1 = 0$ and $\beta_1$ increasing, as stated above.

What happens if $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 \to 1$, i.e. if we approach the boundary of the area with finite fourth moment? For $\alpha_1 = 0$ the assertion above yields

$$Var(S_n^2) = \frac{2(\sigma^2)^2}{n-1} \underset{\beta_1 \to 1}{\longrightarrow} \frac{2(\sigma^2)^2}{n-1}.$$

If $\alpha_1 > 0$, we have

$$\frac{\alpha_1^2}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \longrightarrow \infty,$$

hence $Var(S_n^2) \longrightarrow \infty$ for $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 \to 1$.

With these results one might expect that procedures for variances are affected by large values of $\alpha_1$ and $\beta_1$, where the impact of $\alpha_1$ should be larger. The value of $\alpha_0$ should have no effect on the results.

### 4.3.1 Confidence interval for $\sigma^2$

The common $(1-\alpha)$-confidence interval for $\sigma^2$ in the standard case is given by

$$\left[\frac{(n-1)S_n^2}{b}, \frac{(n-1)S_n^2}{a}\right]$$

where $a$ is the $\frac{\alpha}{2}$ quantile and $b$ is the $(1-\frac{\alpha}{2})$ quantile of a $\chi^2_{n-1}$-distribution. The coverage probability is $(1-\alpha)$ for this confidence interval in the standard case.

We cannot compute the true coverage probability if $(X_t)_{t\in\mathbb{Z}}$ is a GARCH(1,1) process. The following simulations show, however, that the true coverage probabilities are much lower than $1-\alpha$; therefore the intervals are much too narrow.

Table 4.4 presents the results for $\alpha_0 = 0.1$, $n = 1000$ and various values of $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$. For the nominal coverage probability we choose $1-\alpha = 0.95$. Simulations with alternative values of $\alpha_0$ show that this variable does not influence the coverage probability. For the process to be weakly stationary we have to choose $\alpha_1$ and $\beta_1$ such that $\alpha_1+\beta_1 < 1$.

Table 4.4 shows: For $\alpha_1 = \beta_1 = 0$ (standard case) the coverage probability is 0.95 as it should theoretically be. This still holds for $\alpha_1 = 0$ and arbitrary values of $\beta_1$; the reason is that in this case the variance of $S_n^2$ does not differ from the standard case. For $\alpha_1 > 0$ we see that the larger $\alpha_1$ and $\beta_1$, the smaller is the coverage probability, where the influence of $\alpha_1$ is much larger than that of $\beta_1$. For $\alpha_1 + \beta_1 = 0.99$ and large $\alpha_1$ the effect of conditional heteroskedasticity is as strong as it can possibly be: The coverage probability is 0.00.

These results are consistent with the fact that $Var(S_n^2)$ increases with $\alpha_1$ and $\beta_1$ and that in particular large values of $\alpha_1$ cause $Var(S_n^2) = \infty$.

Further simulations with different numbers of observations $n$ show that for growing $n$ the effect of conditional heteroskedasticity becomes larger.

### 4.3.2 F-test for equality of variances

In the standard case we have $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ independent, $X_i \overset{d}{\sim} \mathcal{N}(\mu_X, \sigma_X)$, $Y_j \overset{d}{\sim} \mathcal{N}(\mu_Y, \sigma_Y)$. The usual F-test statistic for the null hypothesis $H_0 : \sigma_X = \sigma_Y$ or a corresponding one-sided hypothesis based on observations $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ is

$$T_{n,m} = \frac{S_{X,n}^2}{S_{Y,m}^2}$$

where $S_{X,n}^2$ and $S_{Y,m}^2$ are the usual estimators for $\sigma_X^2$ and $\sigma_Y^2$, respectively. In the standard case we reject $H_0$ if and only if $T_{n,m} < a$ or $T_{n,m} > b$, where $a$ is the $\frac{\alpha}{2}$ quantile and $b$ is the $(1-\frac{\alpha}{2})$ quantile of a $F_{n-1,m-1}$-distribution and $F_{n-1,m-1}$ denotes the $F$-distribution with $n-1$ and $m-1$ degrees of freedom. This test keeps the error probability of the first kind $\alpha$ in the standard case.

Let $(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = (Z_1, \ldots, Z_{n+m})$ where $(Z_t)_{t\in\mathbb{Z}}$ is a GARCH(1,1) process. If we construct the test as above, we cannot compute the true size of

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 |
| 0.1 | 0.92 | 0.91 | 0.90 | 0.90 | 0.89 | 0.87 | 0.85 | 0.80 | 0.66 | |
| 0.2 | 0.86 | 0.85 | 0.83 | 0.81 | 0.78 | 0.73 | 0.63 | 0.43 | | |
| 0.3 | 0.78 | 0.76 | 0.72 | 0.68 | 0.62 | 0.49 | 0.29 | | | |
| 0.4 | 0.68 | 0.63 | 0.58 | 0.50 | 0.38 | 0.21 | | | | |
| 0.5 | 0.55 | 0.49 | 0.41 | 0.29 | 0.14 | | | | | |
| 0.6 | 0.40 | 0.33 | 0.23 | 0.11 | | | | | | |
| 0.7 | 0.27 | 0.18 | 0.08 | | | | | | | |
| 0.8 | 0.14 | 0.07 | | | | | | | | |
| 0.9 | 0.04 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| coverage prob. | 0.95 | 0.12 | 0.05 | 0.02 | 0.01 | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 |

Table 4.4: Coverage probability of the confidence interval for the nominal value $1-\alpha = 0.95$. We choose $\alpha_0 = 0.1$, $n = 1000$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$.

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.1 | 0.09 | 0.09 | 0.09 | 0.10 | 0.11 | 0.12 | 0.15 | 0.19 | 0.33 | |
| 0.2 | 0.13 | 0.14 | 0.16 | 0.18 | 0.21 | 0.26 | 0.36 | 0.53 | | |
| 0.3 | 0.21 | 0.24 | 0.27 | 0.30 | 0.37 | 0.47 | 0.63 | | | |
| 0.4 | 0.31 | 0.35 | 0.39 | 0.47 | 0.56 | 0.69 | | | | |
| 0.5 | 0.41 | 0.46 | 0.53 | 0.61 | 0.72 | | | | | |
| 0.6 | 0.52 | 0.58 | 0.66 | 0.73 | | | | | | |
| 0.7 | 0.61 | 0.69 | 0.75 | | | | | | | |
| 0.8 | 0.71 | 0.76 | | | | | | | | |
| 0.9 | 0.77 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| rejection prob. | 0.05 | 0.78 | 0.83 | 0.83 | 0.84 | 0.83 | 0.83 | 0.82 | 0.82 | 0.81 |

Table 4.5: Rejection probability of the F-test for the nominal value $\alpha = 0.05$. We choose $\alpha_0 = 0.1$, $n = m = 500$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 50000$.

the test. The simulations show that the true size is much higher than $\alpha$.

Table 4.5 presents the results for $\alpha_0 = 0.1$, $n = m = 500$ and various values of $\alpha_1$ and $\beta_1$ satisfying $\alpha_1 + \beta_1 < 1$. The number of Monte Carlo replications is $N = 50000$, the nominal size is $\alpha = 0.05$. Again simulations with alternative values of $\alpha_0$ show that this parameter is not essential for the results. The results in table 4.5 are to some extent analogous to the results concerning confidence intervals: For $\alpha_1 = 0$ there is no GARCH effect. The larger $\alpha_1$ and $\beta_1$ are, the larger is the probability of false rejection; however, the effect of $\alpha_1$ is much stronger. The size is more than 0.8 in the most extreme cases which is sizeable in comparison to the nominal size of $\alpha = 0.05$.

Again these results are consistent with the fact that $Var(S_n^2)$ increases with $\alpha_1$ and $\beta_1$ and that in particular large values of $\alpha_1$ cause $Var(S_n^2) = \infty$.

As it is well known, the Levene test (Levene, 1960) is more robust to deviations from the normal distribution. Let $\sigma_1^2, \dots, \sigma_n^2$ be the variances of the subsamples of a random sample. The Levene test investigates whether these variances are equal. In our setting there are two subsamples of the same size $n$. For the test of $H_0 : \sigma_X = \sigma_Y$ against the alternative $H_1 : \sigma_X \neq \sigma_Y$ the test statistic is

$$T = 2n(n-1)\frac{(\overline{Z}_{1\cdot} - \overline{Z}_{\cdot\cdot})^2 + (\overline{Z}_{2\cdot} - \overline{Z}_{\cdot\cdot})^2}{\sum_{j=1}^{n}(Z_{1j} - \overline{Z}_{1\cdot})^2 + \sum_{j=1}^{n}(Z_{2j} - \overline{Z}_{2\cdot})^2}$$

where $Z_{1j} = |X_j - \overline{X}_n|$, $Z_{2j} = |Y_j - \overline{Y}_n|$, $\overline{Z}_{i\cdot} = \frac{1}{n}\sum_{j=1} Z_{ij}$ and $\overline{Z}_{\cdot\cdot} = \frac{1}{2}(\overline{Z}_{1\cdot} + \overline{Z}_{2\cdot})$. If the observations are independent, $T$ is $F_{1,n-1}$-distributed under the null hypothesis and $T$ tends to be larger if $H_0$ is wrong. Therefore we reject $H_0$ if $T$ is larger than the $(1-\alpha)$ quantile of the $F_{1,n-1}$-distribution.

We examine the effect of conditional heteroskedasticity on the Levene test by means of simulation. Table 4.6 shows the results for data generated by GARCH(1,1). The Levene test performs better than the F-test, but still the true size of the test is much higher than $\alpha$.

## 4.4 Conclusion

The theoretical results and simulations show that the effect of conditional heteroskedasticity as modelled by GARCH(1,1) is very different for procedures regarding means and variances. Procedures for means which are developed for the standard case are still valid if the observations are generated by a GARCH(1,1) process. On the other hand, the GARCH effect on procedures for variances is substantial. The procedures should not be used if conditional heteroskedasticity is prevalent in the data. A topic for further research is to develop adjustments to these procedures to ensure that they keep the nominal error probabilities or coverage probabilities.

| $\alpha_1 \diagdown \beta_1$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 |
| 0.1 | 0.08 | 0.07 | 0.09 | 0.09 | 0.09 | 0.10 | 0.12 | 0.16 | 0.28 | |
| 0.2 | 0.10 | 0.10 | 0.12 | 0.13 | 0.15 | 0.20 | 0.25 | 0.41 | | |
| 0.3 | 0.12 | 0.15 | 0.16 | 0.19 | 0.25 | 0.32 | 0.46 | | | |
| 0.4 | 0.17 | 0.19 | 0.22 | 0.28 | 0.35 | 0.48 | | | | |
| 0.5 | 0.20 | 0.24 | 0.29 | 0.37 | 0.48 | | | | | |
| 0.6 | 0.23 | 0.29 | 0.37 | 0.47 | | | | | | |
| 0.7 | 0.29 | 0.35 | 0.44 | | | | | | | |
| 0.8 | 0.33 | 0.43 | | | | | | | | |
| 0.9 | 0.39 | | | | | | | | | |

| $\alpha_1$ | 0 | 0.11 | 0.22 | 0.33 | 0.44 | 0.55 | 0.66 | 0.77 | 0.88 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.99 | 0.88 | 0.77 | 0.66 | 0.55 | 0.44 | 0.33 | 0.22 | 0.11 | 0 |
| rejection prob. | 0.05 | 0.72 | 0.73 | 0.70 | 0.66 | 0.62 | 0.58 | 0.54 | 0.48 | 0.43 |

Table 4.6: Rejection probability of the Levene test for the nominal value $\alpha = 0.05$. We choose $\alpha_0 = 0.1$, $n = m = 500$ and various values for $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $N = 10000$.

The results of this chapter show that procedures for variances are tremendously affected by GARCH(1,1). This gives us an idea of the impact conditional heteroskedasticity can have. In the next chapter, we will revert to the investigation of tests for stochastic dominance and explore the effect of conditional heteroskedasticity.

# Chapter 5

# Testing for Stochastic Dominance Using Circular Block Methods

In this chapter we investigate various tests for stochastic dominance when conditional heteroskedasticity is prevalent in the data. As financial data often feature this property, we have to consider it for the application of stochastic dominance tests. Various tests developed in the last years asymptotically capture the dependence structure very well, but we still do not know how these tests perform for finite samples. This chapter analyzes this question and proposes some new tests which are asymptotically equivalent and perform better for finite samples. The results of this chapter are from Kläver (2005a).

As we illustrated in chapter 3, many tests for stochastic dominance have been developed in the last two decades. Many of them have restrictive assumptions, in particular concerning the independence of the data. In general, however, economic data do not satisfy these constraints. In particular, this is the case for financial data such as daily returns on assets or currencies, which feature conditional heteroskedasticity. In other words, financial time series feature serial dependence. Furthermore, for every time index $t$ we have some contemporaneous dependence: usually $X_t$ and $Y_t$ are positively correlated.

In chapter 4 we analyzed the effect of conditional heteroskedasticity on common statistical procedures such as the t-test or the F-test. We found that the considered procedures for means are still asymptotically valid if the data are generated by a GARCH(1,1) process whereas the procedures for variances cannot be used in the same way as in the standard case of independent data.

This chapter investigates the performance of various tests for stochastic dominance when conditional heteroskedasticity and contemporaneous correlation are prevalent in the data. We consider the tests of Schmid/ Trede (1997), Xu/ Fisher/ Willson (1997) and Linton/ Maasoumi/ Whang (2005). Hereafter, we denote the tests by ST, XFW and LMW. We investigate these tests because they asymptotically capture a dependence structure which is suitable for financial data. The ST test investigates whether $X$ dominates $Y$ in the sense of SD2. The XFW and LMW tests address SD1 as well as SD2. Schmid/ Trede take $X_t$ and $Y_t$ as matched pairs for each $t$; their test is based on *permutations*. Hence they consider the correlation of $X_t$ and $Y_t$, but no serial dependence. The XFW and LMW tests use block methods for capturing the dependence structure within each time series and the correlation between them: Xu/ Fisher/ Willson use the *moving block bootstrap*, Linton/ Maasoumi/ Whang use a *subsampling* approach. Both papers demonstrate that the tests perform well asymptotically if the data are generated by strongly mixing processes. In particular, GARCH processes are strongly mixing.

Simulations show that all of these tests do not perform very well for finite samples when the data are generated by a GARCH(1,1) process where the sum of the parameters is close to 1. A remedy is found in other blocking methods: the *circular block bootstrap*, its subsampling equivalent and the *block permutation*. From Lahiri (1999) we know that, asymptotically, the circular block bootstrap performs as well as the moving block bootstrap. We show analytically that the asymptotic result of Linton/ Maasoumi/ Whang for usual subsampling also holds for *circular subsampling*. Further simulations indicate that, for a finite sample, circular subsampling performs better than the usual subsampling of Linton/ Maasoumi/ Whang and block permutation performs better than the permutation test of Schmid/ Trede, whereas circular block bootstrap does not improve the performance of the test developed by Xu/ Fisher/ Willson. The choice of the block length is crucial for the modified versions of the tests of Schmid/ Trede and of Linton/ Maasoumi/ Whang. For various values of sample size $n$, in each case with optimal block length, we explore the power of the tests. The main drawback in the investigation is the complexity of the alternative.

In this chapter, we proceed as follows. Section 5.1 presents the tests of Schmid/ Trede, Xu/ Fisher/ Willson and Linton/ Maasoumi/ Whang which use various resampling methods. In section 5.2 we establish the performance of these tests using a simulation study. In section 5.3 we develop some modified tests based on circular block methods. The simulation results for these tests are presented in section 5.4. We examine the power of the tests in section 5.5. Finally we conclude the results of this chapter.

## 5.1 Tests Based on Resampling Methods

### 5.1.1 A Permutation Test from Matched Pairs

Schmid/ Trede test the null hypothesis $H_0 : (X \succeq_2 Y)$ against the alternative $H_1 : (\text{not } H_0)$ and $H_0^* : (X \succeq_2 Y \text{ or } Y \succeq_2 X)$ vs. $H_1^* : (\text{not } H_0^*)$. We confine our investigation to the first testing problem, which can also be written as

$$\begin{aligned} H_0 &: \text{ For all } x \in \mathbb{R} : \; F_X^{(2)}(x) \leq F_Y^{(2)}(x) \\ \text{vs. } H_1 &: \text{ There exists } x' \in \mathbb{R} : \; F_X^{(2)}(x') > F_Y^{(2)}(x'). \end{aligned}$$

Schmid/ Trede use the test statistic

$$T = \sup_{t \in \mathbb{R}} (\hat{F}_{X,n}^{(2)}(t) - \hat{F}_{Y,n}^{(2)}(t)) = \max_i (\hat{F}_{X,n}^{(2)}(z_{(i)}) - \hat{F}_{Y,n}^{(2)}(z_{(i)}))$$

where $z_{(i)}$ denotes the $i$th order statistic of the combined sample $(z_1, \ldots, z_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

$H_0$ is rejected if $T \geq c$ where the critical value $c$ is determined by permutations. There are $2^n$ possibilities of permuting $x_i$ and $y_i$ in the paired sample $(x_1, y_1), \ldots, (x_n, y_n)$. The corresponding values of the test statistics can be arranged according to size: $T^{(1)} \leq \cdots \leq T^{(2^n)}$. The critical value $c$ is determined by $c = T^{((1-\alpha)2^n)}$. Under $P_X = P_Y$ the probability of wrongly rejecting $H_0$ is approximately $\alpha$. As the number of permutations becomes large very quickly with increasing $n$, Schmid/ Trede take only $M$ permutations at random and determine the critical value by $c = T^{((1-\alpha)M)}$. They show in a Monte Carlo study that under the assumption of a bivariate normal distribution with serial independence $M = 500$ permutations are sufficient. Schmid/ Trede do not give any advice on how to decide if there is a tie, i.e. $T = T^{(k)}$ for some $k < (1-\alpha)M$ and some $k \geq (1-\alpha)M$.[1]

In section 5.2 we will investigate the performance of the test by means of simulation for the case that conditional heteroskedasticity is prevalent in the data.

### 5.1.2 Tests Using Moving Block Methods

Xu/ Fisher/ Willson test $H_0^i : (X \succeq_i Y)$ vs. $H_1^i$ : (not $H_0$) for $i = 1, 2$ which can be written as

$$\begin{aligned} H_0^i &: \text{ For all } x \in \mathbb{R}: \ Q_X^{(i)}(x) \geq Q_Y^{(i)}(x) \\ \text{vs. } H_1^i &: \text{ There exists } x' \in \mathbb{R}: \ Q_X^{(i)}(x') < Q_Y^{(i)}(x'). \end{aligned}$$

They compute the difference of the empirical quantile functions $\hat{Q}_n^{(i)}$ $(i = 1, 2)$ at various grid points $p_1, \ldots, p_n$ satisfying $0 < p_1 < \cdots < p_n < 1$ and define $\hat{Q}_n^{(i)}(P) = [\hat{Q}_n^{(i)}(p_1), \ldots, \hat{Q}_n^{(i)}(p_n)]'$.

[1] Indeed, one can practically ignore this problem for the original ST test because in our simulations there is no tie for any replication of the test.

The test statistic is given by

$$T_i = \max_{q \geq 0} [(\hat{Q}^{(i)}_{X,n} - \hat{Q}^{(i)}_{Y,n} - q)' \hat{\Lambda}^{-1} (\hat{Q}^{(i)}_{X,n} - \hat{Q}^{(i)}_{Y,n} - q)]$$

where $\hat{\Lambda}$ is a consistent estimate of the covariance matrix $\Lambda$ of $\hat{Q}^{(i)}_{X,n} - \hat{Q}^{(i)}_{Y,n}$. $T_i$ is asymptotically distributed as a weighted sum of $\chi^2$-variates of various degrees of freedom. The weights are determined by Monte Carlo simulation using nonlinear programming.

In this procedure, the estimation of $\Lambda$ is crucial. Xu/ Fisher/ Willson propose that moving block bootstrap (MBB) captures the dependence structure if the processes $(X_t)_{t \in \mathbb{Z}}$ and $(Y_t)_{t \in \mathbb{Z}}$ are strongly mixing (or $\alpha$-mixing) which is defined in chapter 3. Note that, in particular, a stationary GARCH process is strongly mixing.

MBB is developed by Künsch (1989) and Liu/ Singh (1992). In the last years a multitude of bootstrap methods has been developed which are constructed to capture the dependence structures emerging in time series; see e.g. Härdle/ Horowitz/ Kreiss (2003).

In contrast to the usual bootstrap introduced by Efron (1979) MBB does not resample single observations, but whole blocks of a fixed length $b$. For a sample of observations $(z_1, \ldots, z_n)$ denote the moving blocks as $B_1, \ldots, B_{n-b+1}$, where $B_j = (x_j, x_{j+1}, \ldots, x_{j+b-1})$ stands for the block consisting of $b$ observations starting from $x_j$. One bootstrap resample consists of $k = \lfloor \frac{n}{b} \rfloor$ randomly resampled moving blocks where $\lfloor x \rfloor$ denotes the largest integer equal to or smaller than $x$.

The MBB estimate is consistent if $b(n)$ and $k(n)$ approach infinity with $n$ approaching infinity. For a finite sample the choice of $b$ is vital: on the one hand, a large value of $b$ is necessary to capture strong dependence, while on the other hand, the number of blocks should also be large enough to reproduce the variability of the original sample.

Xu/ Fisher/ Willson proceed as follows: The observations of $X$ and $Y$ are resampled M times by MBB. For every resample Xu/ Fisher/ Willson compute the differences of the empirical quantile functions at the grid points. The empirical covariance matrix $\hat{\Lambda}$ of these vectors is taken as an estimator for $\Lambda$. In an empirical example, Xu/ Fisher/ Willson choose $M = 500$. We follow their example in our investigation.

Linton/ Maasoumi/ Whang test for stochastic maximality of a set of prospects. A set is stochastically maximal if no prospect is stochastically dominated by another prospect in the set. The test can be easily modified in a way that it also tests for stochastic dominance. The test problem is $H_0^i : (X \succeq_i Y)$ vs. $H_1^i$ : (not $H_0$) for $i = 1, 2$ as in Xu/ Fisher/ Willson, but in contrast Linton/ Maasoumi/ Whang use the test statistic

$$T_{n,i} = \sup_{x \in \mathbb{R}} \sqrt{n}(\hat{F}_{X,n}^{(i)}(x) - \hat{F}_{Y,n}^{(i)}(x)).$$

In the study of Linton/ Maasoumi/ Whang $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are errors in a linear regression model. For the investigation of stochastic dominance some regularity conditions have to be satisfied. If we do not assume a regression model, the only persisting regularity condition is that $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are strongly mixing with $\alpha(m) = O(m^{-3})$. If $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are generated by a strictly stationary GARCH process with innovations $\varepsilon_t$ satisfying $E|\varepsilon_t|^\delta < \infty$ for some $\delta > 0$, then they are strongly mixing with a geometric rate, i.e. $\alpha(m) = O(a^m)$ for some $a \in (0, 1)$ (Davis/ Mikosch/ Basrak, 1999); thus $\alpha(m) = O(m^{-3})$ holds.

For the approximation of the distribution of $T_{n,i}$ under $H_0^i$ Linton/ Maasoumi/ Whang use a subsampling method developed by Politis/ Romano (1994). An overview of resampling methods for various situations, e.g. stationary observations, is given by Politis/ Romano/ Wolf (1999). We outline the procedure for the test of SD1. Let

$$d_n(W_1, \ldots, W_n) = \frac{1}{\sqrt{n}} T_{n,1}$$

and $d_{n,b,k} = d_b(W_k, W_{k+1}, \ldots, W_{k+b-1})$ for $k = 1, \ldots, n-b+1$ be the transformed test statistic for the subsample $(W_k, W_{k+1}, \ldots, W_{k+b-1})$ of size $b$. Suppose that $g_{n,b}$ is the empirical quantile function of $\{\sqrt{b} d_{n,b,k} : k = 1, \ldots, n-b+1\}$ and $g$ the quantile function of the asymptotic distribution of $T_{n,1}$ under $H_0^1$. Assume that $b(n) \underset{n\to\infty}{\longrightarrow} \infty$ and $\frac{b(n)}{n} \underset{n\to\infty}{\longrightarrow} 0$ and that the mixing condition stated above holds. For example, this will be the case for a stationary GARCH process.

Then under the subcase $P_X = P_Y$ of $H_0^1$ we have $g_{n,b}(1-\alpha) \xrightarrow{p} g(1-\alpha)$ and

$$P(T_{n,1} > g_{n,b}(1-\alpha)) \underset{n\to\infty}{\longrightarrow} \alpha.$$

Under $H_1^1$ the test is consistent, i.e.

$$P(T_{n,1} > g_{n,b}(1-\alpha)) \underset{n\to\infty}{\longrightarrow} 1.$$

The result concerning SD2 is analogous.

The described tests in this section are robust to contemporaneous correlation between the processes. Moreover, the XFW and the LMW tests are asymptotically robust to serial dependence within the processes if they are strongly mixing. An important theoretical topic and problem for applications is the performance of these tests for finite samples if the data are dependent, in particular if they are conditionally heteroskedastic. This will be investigated in the next section.

## 5.2 Simulation Results for the Conventional Tests

By simulation we investigate the effect of some dependence structures on the size of the tests described in the previous section. The nominal size in each test is $\alpha = 0.05$. Unless stated differently, the sample size is $n = 1000$ and the number of replications is $R = 500$ for the ST test and $R = 1000$ for the XFW test and the LMW test. For our research concerning contemporaneous dependence we determine the size of the tests for various values of the correlation coefficient $\rho$. In exploring the effects of serial dependence we confine ourselves to GARCH(1,1), a kind of conditional heteroskedasticity. As already mentioned in chapter 4, this kind of process is an appropriate approach to modelling conditional heteroskedasticity for financial data. Akgiray (1989), Davis/ Mikosch (2000) and Engle (2001) affirm the evidence and usefulness of GARCH(1,1).

We consider the following situations:

- For every $t$ the vector $(X_t, Y_t)$ has a bivariate normal distribution with mean $\underline{0} := (0 \quad 0)$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The process $(X_t, Y_t)_{t\in\mathbb{Z}}$ is serially independent.

- For every $t$ the random variables $X_t$ and $Y_t$ are independent. Both random variables follow a GARCH(1,1) process as we have defined in chapter 4: Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ denote a sequence of independent and identically $\mathcal{N}(0,1)$-distributed random variables. Let $X_t = \sigma_t \varepsilon_t$ where

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

  for $t \in \mathbb{Z}$ and $\alpha_0 > 0$, $\alpha_1, \beta_1 \geq 0$. Then $(X_t)_{t\in\mathbb{Z}}$ is called a GARCH(1,1) process. As mentioned in chapter 4, $(X_t)_{t\in\mathbb{Z}}$ is weakly stationary if and only $\alpha_1 + \beta_1 < 1$.

- $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a *bivariate GARCH(1,1)* process: Let $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ be independent and identically $\mathcal{N}_2(0, I_2)$-distributed random vectors for all $t \in \mathbb{Z}$; here $I_n$ is the $n$-dimensional identity matrix. For $i = 1, 2$ define the conditional variances

$$h_{ii,t} = \alpha_0 + \alpha_1 Z_{i,t-1}^2 + \beta_1 h_{ii,t-1}$$

  where $Z_{1,t} = X_t$, $Z_{2,t} = Y_t$. Let

$$h_{12,t} = \rho\alpha_0 + \alpha_1 Z_{1,t-1} Z_{2,t-1} + \beta_1 h_{12,t-1}$$

  be the conditional covariance and $H_t := \begin{pmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{pmatrix}$ be the conditional covariance matrix. If $H_0$ is positive definite, this holds for all $H_t$, $t \in \mathbb{N}$. This follows from a more general result of Engle/ Kroner (1995).
  Let $(X_t, Y_t)' = C_t \varepsilon_t$, where $C_t$ is a positive definite matrix satisfying $C_t^2 = H_t$, i.e. $C_t$ is a root of $H_t$. Then $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ follow a GARCH(1,1)

| $\rho$ | $-0.5$ | $0$ | $0.5$ |
|---|---|---|---|
| size | 0.05 | 0.05 | 0.04 |

Table 5.1: Rejection probability of the ST test for the nominal value $\alpha = 0.05$. The processes are serially independent and contemporaneously correlated with coefficient $\rho$. The sample size is $n = 1000$, the number of Monte Carlo replications is $R = 500$.

process as described above. In particular, $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are weakly stationary if and only $\alpha_1+\beta_1 < 1$. For every $t$ the unconditional correlation between $X_t$ and $Y_t$ is $\rho$. However, the conditional correlation

$$\frac{h_{12,t}}{\sqrt{h_{11,t}h_{22,t}}}$$

depends on $t$. For a more general approach and details concerning *multivariate GARCH*, see Engle/ Kroner (1995).

First we investigate the ST test. As recommended by Schmid/ Trede, we choose $M = 500$ as the number of permutations. As expected, the test is robust to contemporaneous correlation. We simulate two samples $(X_t : t = 1, \dots, n)$ and $(Y_t : t = 1, \dots, n)$. For every time $t$ the observation pairs $(X_t, Y_t)$ follow a bivariate normal distribution with correlation coefficient $\rho$. The observation pairs are independent of each other. The true size ranges from $\alpha = 0.05$ to $\alpha = 0.04$ if the correlation varies from $\rho = -0.5$ to $\rho = 0.5$ (see table 5.1). Hence the correlation does not have a significant effect on the size of the test. This was already asserted by Schmid and Trede.

Concerning conditional heteroskedasticity within the samples things are much more involved. Table 5.2 presents the results for various values of the GARCH(1,1) parameters $\alpha_1$ and $\beta_1$. The size of the test increases slowly in $\alpha_1$ and $\beta_1$; however, the increase becomes faster for larger values of $\alpha_1$ and $\beta_1$. If $\alpha_1+\beta_1$ is close to 1, the true size of the test is much larger than the nominal size of the test. For $\alpha_1 = 0.14$ and $\beta_1 = 0.85$, which is an appropriate choice for financial data,

| $\alpha_1$ | $\beta_1$ | $\alpha_1 + \beta_1$ | size |
|---|---|---|---|
| 0.1 | 0.8 | 0.9 | 0.07 |
| 0 | 0.99 | 0.99 | 0.04 |
| 0.1 | 0.89 | 0.99 | 0.30 |
| 0.14 | 0.85 | 0.99 | 0.30 |

Table 5.2: Rejection probability of the ST test for the nominal value $\alpha = 0.05$. The processes are generated by contemporaneously independent GARCH(1,1) processes with parameters $\alpha_0 = 0.1$, $\alpha_1$ and $\beta_1$. The sample size is $n = 1000$, the number of Monte Carlo replications is $R = 500$.

| $\rho$ | $-0.8$ | $-0.5$ | 0 | 0.5 | 0.8 |
|---|---|---|---|---|---|
| size | 0.06 | 0.06 | 0.06 | 0.07 | 0.05 |

Table 5.3: Rejection probability of the XFW test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and block length $b = 1$. The processes are serially independent and contemporaneously correlated with parameter $\rho$. The number of grid points is $K - 1 = 9$, the number of Monte Carlo replications is $R = 1000$.

the size is $\alpha = 0.30$. Hence the ST test should not be used if the data follow a GARCH(1,1) process with large parameters.

In our investigation of the XFW test we choose the grid points $(\frac{1}{K}, \ldots, \frac{K-1}{K})$ with $K = 10$ and $K = 20$. The performance depends on the length $b$ of the blocks, whereas the number of grid points $K$ does not have a significant effect. Hence we only report the results for $K = 10$. In the case of serial independence within each sample the test works very well, even for $b = 1$, i.e. usual bootstrap. For various values of contemporaneous correlation $\rho$ the size ranges between $\alpha = 0.05$ and $\alpha = 0.07$ (see table 5.3).

However, if the observations are generated by a GARCH(1,1) process, there is a notable effect on the size of the test. We focus on the parameter constellation $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. For $K = 10$ and $b = 1$ we get $\alpha = 0.62$,

| $b$ \ $\rho$ | $-0.5$ | 0 | 0.5 |
|---|---|---|---|
| 1 | 0.38 | 0.62 | 0.47 |
| 10 | 0.29 | 0.44 | 0.34 |
| 50 | 0.18 | 0.21 | 0.15 |
| 100 | 0.14 | 0.13 | 0.17 |
| 200 | 0.20 | 0.21 | 0.24 |

Table 5.4: Rejection probability of the XFW test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various values of block length $b$. The processes are generated by bivariate GARCH(1,1) processes with correlation parameter $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of grid points is $K - 1 = 9$. The number of Monte Carlo replications is $R = 1000$.

the size decreases for increasing $b$ as can be seen in table 5.4. However, for larger values of $b$ the size increases, therefore, for $b = 200$ the size is larger than for $b = 100$.

If the data are generated by a bivariate GARCH(1,1) process, the results are similar. For $\alpha_0 = 0.1$, $\alpha_1 = 0.14$, $\beta_1 = 0.85$ and $\rho = \pm 0.5$ we choose various values of the block length $b$; the results are also reported in table 5.4. As in the case of independent univariate GARCH(1,1) processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$, the size first decreases with increasing block length, but increases after the block length has exceeded a critical border. We observe the following results: For sample size $n = 1000$ we cannot manage to keep the nominal rejection probability $\alpha$ by adjusting the size of the block length. Hence the asymptotic result of Xu/ Fisher/ Willson is not useful if the number of observations is equal to 1000 or even lower!

For the LMW test the results are similar. Even for small block length the test performs very well in the case of serial independence. The block length $b = 1$ is, by construction, no reasonable choice. Table 5.5 shows the results for various block lengths and correlation parameters. For $b = 10$ the size lies between

| $b$ \ $\rho$ | $-0.8$ | $-0.5$ | 0 | 0.5 | 0.8 |
|---|---|---|---|---|---|
| 10 | 0.07 | 0.06 | 0.08 | 0.09 | 0.06 |
| 20 | 0.07 | 0.06 | 0.04 | 0.06 | 0.06 |
| 40 | 0.04 | 0.04 | 0.04 | 0.04 | 0.06 |

Table 5.5: Rejection probability of the LMW test for the nominal value $\alpha = 0.05$, serial independence, contemporaneous correlation $\rho$, sample size $n = 1000$ and block length $b$. The number of Monte Carlo replications is $R = 1000$.

| $\alpha_1$ | $\beta_1$ | $\alpha_1 + \beta_1$ | size |
|---|---|---|---|
| 0 | 0.99 | 0.99 | 0.05 |
| 0.1 | 0.8 | 0.9 | 0.08 |
| 0.1 | 0.89 | 0.99 | 0.35 |
| 0.14 | 0.85 | 0.99 | 0.28 |

Table 5.6: Rejection probability of the LMW test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and block length $b = 20$. The processes are generated by contemporaneously independent GARCH(1,1) processes with parameters $\alpha_0 = 0.1$, $\alpha_1$ and $\beta_1$. The number of Monte Carlo replications is $R = 1000$.

$\alpha = 0.06$ and $\alpha = 0.09$ for various values of the correlation coefficient $\rho$. If we choose $b = 40$, the nominal size is kept well: the size ranges from $\alpha = 0.04$ to $\alpha = 0.06$.

However, if the data are generated by a GARCH(1,1) process, there is a significant effect on the size of the test. Table 5.6 shows the same effect as in the ST test and the XFW test: If $\alpha_1 + \beta_1$ approaches 1, the nominal size is not kept any longer.

Is the variation of the block length a remedy? No! As we see in table 5.7, for the GARCH parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$ and correlation coefficient $\rho = 0$ or $\rho = \pm 0.5$, the size first decreases with increasing block length,

| $b$ \ $\rho$ | $-0.5$ | $0$ | $0.5$ |
|---|---|---|---|
| 10 | 0.31 | 0.38 | 0.37 |
| 20 | 0.25 | 0.28 | 0.30 |
| 40 | 0.16 | 0.20 | 0.19 |
| 100 | 0.10 | 0.10 | 0.11 |
| 200 | 0.07 | 0.09 | 0.09 |
| 250 | 0.07 | 0.07 | 0.07 |
| 300 | 0.09 | 0.07 | 0.08 |
| 500 | 0.08 | 0.13 | 0.11 |

Table 5.7: Rejection probability of the LMW test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various values of block length $b$. The processes are generated by bivariate GARCH(1,1) processes with correlation parameter $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

but then increases again. We do not find any such block length that the nominal size is kept. The LMW test should not be used if conditional heteroskedasticity is prevalent in the data and the number of observations is 1000 or less.

In many applications in finance a GARCH(1,1) process with large parameters, i.e. $\alpha_1 + \beta_1$ close to 1, is a good fit of the time series. The simulations indicate that the dominance tests of Schmid/ Trede, Xu/ Fisher/ Willson and Linton/ Maasoumi/ Whang are not useful in this case if there are less than 1000 observations.

## 5.3 Circular Block Methods as an Alternative Concept

Due to the moderate success of the moving block methods we propose alternatively investigating the use of circular block methods. The circular block bootstrap (CBB) is developed by Politis/ Romano (1992). As the MBB method, CBB resamples overlapping blocks of observations which are of a fixed length $b$. One problem of MBB is that the observations at the beginning and the end of the time series are less considered. CBB solves this problem as follows. The collection of blocks from which it is resampled consists of the blocks $B_1, \ldots, B_{n-b+1}$ of the MBB and additionally of the blocks $B_{n-b+2}, \ldots, B_n$ of the form $B_k = (x_k, \ldots, x_n, x_1, \ldots, x_{k+b-n-1})$. Lahiri (1999) investigates the asymptotic behavior of some block bootstrap methods and found that MBB and CBB are asymptotically equivalent. We apply CBB to the XFW test and investigate by simulation whether this improves the size of the test.

The modification of the subsampling method of Linton/ Maasoumi/ Whang is analogous. The distribution of $T_{n,i}$ under $H_0^i$ is approximated by $\sqrt{b}d_{n,b,k}$ where

$$d_{n,b,k} = \begin{cases} d_b(W_k, W_{k+1}, \ldots, W_{k+b-1}) & \text{for} \quad k = 1, \ldots, n-b+1, \\ d_b(W_k, \ldots, W_n, W_1, \ldots, W_{k+b-n-1}) & \text{for} \quad k = n-b+2, \ldots, n. \end{cases}$$

By some modification of the proofs we can show that the main results of Linton/ Maasoumi/ Whang still hold if we use the modified subsampling method.

**Theorem 3.** *Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be strongly mixing with $\alpha(m) = O(m^{-3})$. Assume $b(n) \to \infty$ and $\frac{b(n)}{n} \to 0$ as $n \to \infty$. Let $\alpha \in (0,1)$, $g_{n,b}$ be the empirical quantile function of $\{\sqrt{b}d_{n,b,k} : k = 1, \ldots, n\}$ and $g$ the quantile function of the asymptotic distribution of $T_{n,1}$ under $H_0^1$. Then:*

1. *Under the subcase $P_X = P_Y$ of $H_0^1$ we have $g_{n,b}(1-\alpha) \xrightarrow{p} g(1-\alpha)$ and $P(T_{n,1} > g_{n,b}(1-\alpha)) \underset{n\to\infty}{\longrightarrow} \alpha$, i.e. the test keeps the size $\alpha$ asymptotically.*

*2. Under $H_1^1$ we have $P(T_{n,1} > g_{n,b}(1-\alpha)) \underset{n\to\infty}{\longrightarrow} 1$, i.e. the test is consistent.*

*Proof.* Let us first prove part 1. Let $\hat{G}_{n,b}$ be the empirical distribution function of $\{\sqrt{b}d_{n,b,k} : k = 1,\ldots,n\}$ and $G$ the distribution function of the asymptotic distribution of $T_{n,1}$ under $H_0^1$. As Linton/ Maasoumi/ Whang state, $G$ is absolutely continuous according to Lifshits (1982), theorem 1. Therefore, to prove part 1, it suffices to show

$$\hat{G}_{n,b}(w) \underset{n\to\infty}{\overset{p}{\longrightarrow}} G(w)\ \forall w \in \mathbb{R}.$$

By definition of $G$

$$G_b(w) := P(\sqrt{b}d_{n,b,1} \le w) \underset{b\to\infty}{\longrightarrow} G(w)$$

holds for all $w \in \mathbb{R}$. Hence we have to show $\hat{G}_{n,b}(w) \underset{n\to\infty}{\longrightarrow} G_b(w)$ for all $w \in \mathbb{R}$; note that $b \to \infty$ as $n \to \infty$. Let $I_k = 1(\sqrt{b}d_{n,b,k} \le w)$ for $k = 1,\ldots,n$. $E(I_k) = P(\sqrt{b}d_{n,b,k} \le w) = P(\sqrt{b}d_{n,b,1} \le w) = G_b(w)$ holds for $k = 1,\ldots,n-b+1$. This yields

$$E(\hat{G}_{n,b}(w)) = \frac{1}{n}\sum_{k=1}^{n} E(I_k) = \frac{n-b+1}{n}G_b(w) + \frac{1}{n}\sum_{k=n-b+2}^{n} E(I_k)$$

and therefore

$$|E(\hat{G}_{n,b}(w)) - G_b(w)| = \left|\frac{1}{n}\sum_{k=n-b+2}^{n} E(I_k) - \frac{b-1}{n}G_b(w)\right| \le \frac{b-1}{n} \underset{n\to\infty}{\longrightarrow} 0. \quad (5.1)$$

Furthermore,

$$\begin{aligned} Var(\hat{G}_{n,b}(w)) &= \frac{1}{n^2}\Big(\sum_{k=1}^{n} Var(I_k) + 2\sum_{1\le k<l\le n} Cov(I_k, I_l)\Big) \\ &= \frac{1}{n^2}\Big(\sum_{k=1}^{n} Var(I_k) + 2\sum_{m=1}^{n-1}\sum_{k=1}^{n-m} Cov(I_k, I_{k+m})\Big) \\ &= S_{n,0} + 2\sum_{m=1}^{n-1} S_{n,m} \end{aligned}$$

where

$$S_{n,m} = \frac{1}{n^2}\sum_{k=1}^{n-m} Cov(I_k, I_{k+m}).$$

$|I_k| \leq 1$ for all $k$ yields $|Cov(I_k, I_{k+m})| \leq 1$ and so $|S_{n,m}| \leq \frac{1}{n}$ for all $m$. Therefore

$$\left| S_{n,0} + 2\sum_{m=1}^{b-1} S_{n,m} + 2\sum_{m=n-b+1}^{n-1} S_{n,m} \right| \leq O\Big(\frac{b}{n}\Big) = o(1).$$

In addition, we have

$$\begin{aligned}
\left| 2\sum_{m=b}^{n-b} S_{n,m} \right| &= \frac{2}{n^2}\left| \sum_{m=b}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{n-m} Cov(I_k, I_{k+m}) + \sum_{m=\lfloor \frac{n}{2} \rfloor+1}^{n-b} \sum_{k=1}^{n-m} Cov(I_k, I_{k+m}) \right| \\
&\leq \frac{8}{n^2}\left| \sum_{m=b}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{n-m} \alpha(m-b+1) + \sum_{m=\lfloor \frac{n}{2} \rfloor+1}^{n-b} \sum_{k=1}^{n-m} \alpha(n-m-b+1) \right| \qquad (5.2) \\
&\leq \frac{8}{n}\left| \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor-b+1} \alpha(m) + \sum_{m=1}^{\lceil \frac{n}{2} \rceil-b} \alpha(m) \right| \\
&\leq O(n^{-1}) = o(1) \qquad (5.3)
\end{aligned}$$

where (5.2) holds by Hall/ Heyde (1980), theorem A.5, and (5.3) holds by the assumption that $\alpha(m) = O(m^{-3})$. Hence we have shown

$$\lim_{n\to\infty} Var(\hat{G}_{n,b}(w)) = 0. \qquad (5.4)$$

(5.1) and (5.4) yield $\hat{G}_{n,b}(w) \xrightarrow[n\to\infty]{p} G_b(w)$ and therefore $\hat{G}_{n,b}(w) \xrightarrow[n\to\infty]{p} G(w)$. This establishes part 1 of the theorem.

For the proof of part 2 first note that under $H_1^1$ we have

$$d := \sup_{x\in\mathbb{R}} (F_X(x) - F_Y(x)) > 0.$$

Analogously to Linton/ Maasoumi/ Whang $d_n(W_1, \ldots, W_n) \xrightarrow[n\to\infty]{p} d$ holds. Let $\hat{G}^0_{n,b}$ and $g^0_{n,b}$ be the empirical distribution and quantile function of $\{d_{n,b,k} : k = 1, \ldots, n\}$ and $G^0_b$ the distribution function of $d_{n,b,1}$. Due to the mixing condition the convergence $\hat{G}^0_{n,b}(w) \xrightarrow[n\to\infty]{p} G^0_b(w)$ holds; this can be shown analogously to part 1. With $d_b(W_1, \ldots, W_b) \xrightarrow[n\to\infty]{p} d$ this yields $\hat{G}^0_{n,b}(w) \xrightarrow[n\to\infty]{d} \delta_d$ where $\delta_d$ denotes the Dirac distribution in $d$. Therefore we have $g^0_{n,b}(1-\alpha) \xrightarrow[n\to\infty]{p} d$. Because of $g_{n,b}(1-\alpha) = \sqrt{b}g^0_{n,b}(1-\alpha)$ this yields

$$P(T_{n,1} > g_{n,b}(1-\alpha)) \xrightarrow[n\to\infty]{} 1$$

as in Linton/ Maasoumi/ Whang. □

By simulation we investigate if the modified subsampling method improves the performance of the LMW test for finite samples.

In contrast to the XFW and the LMW test, the ST test does not consider any serial dependence at all. In this section, we modify the permutation test to a block permutation test. Like the block bootstrap and subsampling methods, the block permutation reproduces the dependence structure of the observations.

The block permutation method is performed as follows. We consider the random variable $U = |\{i \in \{1, \ldots, n\} : X_i \text{ and } Y_i \text{ are transposed}\}|$. In the permutations of Schmid/ Trede, $U$ follows a binomial distribution with parameters $n$ and $\frac{1}{2}$. Therefore, in the modified test we first generate for every permutation the number $u$ of the transposed pairs. If the given block length is $b$, we choose by chance $\lfloor \frac{u}{b} \rfloor$ blocks of length $b$ and one block of length $u - b\lfloor \frac{u}{b} \rfloor$ for which $X_i$ and $Y_i$ are transposed. The test is further performed as described in Schmid/ Trede. We will investigate by simulation whether this block permutation improves the ST test in finite samples with conditional heteroskedasticity.

In this section, we developed some modifications of the tests of Schmid/ Trede, Xu/ Fisher/ Willson and Linton/ Maasoumi/ Whang. We find that the circular block methods are asymptotically equivalent to their moving block counterparts. For finite samples, there are two opposed effects. On the one hand, the observations at the beginning and at the end of the time series are considered as well as the observations in the middle. This is an improvement to moving block bootstrap and subsampling. On the other hand, some blocks we build are no reasonable construction in terms of reproduction of the dependence structure. In strongly mixing processes the observations with a large time lag are nearly independent. Therefore a block consisting of the first and the last observations does not seem to make sense. But, on the other hand, a block of the first $k$ and the last $b - k$ observations is just a combination of two blocks with a strong dependence struc-

| $\rho$ | $-0.5$ | $0$ | $0.5$ |
|---|---|---|---|
| size | 0.03 | 0.03 | 0.03 |

Table 5.8: Rejection probability of the STm test for the nominal value $\alpha = 0.05$ and sample size $n = 1000$. The processes are serially independent and contemporaneously correlated with coefficient $\rho$. The block length is $b = 100$, the number of Monte Carlo replications is $R = 500$.

ture within both of them. In other words, the resample consists of blocks with different lengths and a strong dependence within each block. Hence we suppose that the advantages of our modification outweigh the disadvantages.

On the basis of these considerations, one can expect that the circular block methods improve the performance of the tests. In the next section, we investigate this by means of simulation.

## 5.4 Simulation Results Using Circular Block Methods

In this section we report on the simulation results of the modified tests. We refer to the modified tests as STm test, XFWm test and LMWm test.

The modification described in the previous section is a remedy for the weaknesses of the ST test. As can be seen in table 5.8, the modification does not destroy the good result for data which are serially independent, but contemporaneously correlated. The nominal size $\alpha = 0.05$ is kept well, the test is just a bit too conservative.

But the modification is a real improvement. In section 5.2 we stated that the ST test does not keep the nominal size if the data are generated by a GARCH(1,1) process with parameters whose sum is close to 1. Table 5.9 shows that the STm test keeps the nomimal size if the block length is chosen appropriately. This also

| $b$ \ $\rho$ | $-0.5$ | 0 | 0.5 |
|---|---|---|---|
| 100 | 0.09 | 0.14 | 0.13 |
| 200 | 0.07 | 0.08 | 0.08 |
| 300 | 0.04 | 0.05 | 0.06 |
| 500 | 0.00 | 0.01 | 0.00 |

Table 5.9: Rejection probability of the STm test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various values of block length $b$. The processes are generated by bivariate GARCH(1,1) with correlation coefficient $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 500$.

| $\rho$ | $-0.8$ | $-0.5$ | 0 | 0.5 | 0.8 |
|---|---|---|---|---|---|
| size | 0.06 | 0.06 | 0.05 | 0.07 | 0.05 |

Table 5.10: Rejection probability of the XFWm test for the nominal value $\alpha = 0.05$ and sample size $n = 1000$. The processes are serially independent and contemporaneously correlated with parameter $\rho$. The number of grid points is $K - 1 = 9$, the block length $b = 1$. The number of Monte Carlo replications is $R = 1000$.

holds for the bivariate GARCH(1,1) process with correlation coefficient $\rho$. The choice $b = 200$ is not sufficient to keep the level; for $b = 500$ the test is too conservative. According to table 5.9, $b = 300$ seems to be a good choice.

The XFW test cannot be improved significantly by modifying it with circular block bootstrap. At least, like the original test, the modified version still keeps the size if the observations are serially independent, but contemporaneously correlated. Table 5.10 presents the simulation results for $K = 10$, block length $b = 1$ and various values of the correlation coefficient $\rho$.

However, if the data are generated by a GARCH(1,1) process, the circular block bootstrap is no remedy for the weaknesses of the XFW test. The simulation

| $b$ \ $\rho$ | $-0.5$ | 0 | 0.5 |
|---|---|---|---|
| 10 | 0.28 | 0.44 | 0.34 |
| 50 | 0.17 | 0.21 | 0.17 |
| 100 | 0.16 | 0.17 | 0.17 |
| 200 | 0.19 | 0.19 | 0.22 |
| 500 | 0.42 | 0.49 | 0.52 |

Table 5.11: Rejection probability of the XFWm test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various values of block length $b$. The processes are generated by bivariate GARCH(1,1) with correlation parameter $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of grid points is $K - 1 = 9$. The number of Monte Carlo replications is $R = 1000$.

results (see table 5.11) show that there is no block length for which the nominal size is kept. As for the original versions of the XFW and LMW test, for small block length $b$ the size decreases with increasing $b$, but is always significantly higher than the nominal size. For $b > 100$ the size increases with increasing $b$. This holds for both independent GARCH(1,1) processes and bivariate GARCH(1,1) processes with contemporaneous correlation.

The LMWm test keeps the size for appropriate block length. Table 5.12 shows the simulation results for serially independent time series and various values of contemporaneous correlation $\rho$. The block length is $b = 40$. As for the original LMW test, the nominal size $\alpha = 0.05$ is kept well. Therefore, the performance of the modified LMW test is at least not worse than that of the original test.

But the modified version is even better. With the appropriate block length it keeps the nominal size even if the data feature conditional heteroskedasticity. The simulation results for a bivariate GARCH(1,1) process with parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$ and correlation $\rho = 0$ and $\rho = \pm 0.5$ are reported in table 5.13. In contrast to the original LMW test, the size decreases monotonically with

| $\rho$ | $-0.8$ | $-0.5$ | $0$ | $0.5$ | $0.8$ |
|---|---|---|---|---|---|
| size | 0.04 | 0.04 | 0.04 | 0.05 | 0.06 |

Table 5.12: Rejection probability of the LMWm test for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and block length $b = 40$. The processes are serially independent and contemporaneously correlated with parameter $\rho$. The number of Monte Carlo replications is $R = 1000$.

| $\rho$ / $b$ | $-0.5$ | $0$ | $0.5$ |
|---|---|---|---|
| 100 | 0.09 | 0.11 | 0.09 |
| 200 | 0.05 | 0.07 | 0.06 |
| 300 | 0.05 | 0.05 | 0.05 |
| 500 | 0.03 | 0.03 | 0.04 |

Table 5.13: Rejection probability of the LMWm test for the nominal value $\alpha = 0.05$ and various values of block length $b$. The processes are generated by bivariate GARCH(1,1) with correlation parameter $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

increasing $b$. If we choose block length $b = 300$, the size is kept well, whereas for $b = 500$ the test is too conservative. As for the ST test, $b = 300$ seems to be a good choice.

We have seen that for sample size $n = 1000$ the STm and LMWm tests keep the size if we choose the appropriate block length. If the sample size varies, which block length is the best choice? We explore this question with the help of some further simulations. Tables 5.14 and 5.15 show the corresponding results for the STm and the LMWm tests.

For the STm test, the optimal block length seems to be proportional to $\sqrt{n}$. The block length $b = 150$ is a good choice for $n = 250$ whereas the size cannot be kept for $b = 100$ or $b = 200$. With increasing sample size the block length has

| | $\rho$ | $-0.5$ | 0 | 0.5 |
|---|---|---|---|---|
| $n$ | $b$ | | | |
| 250 | 100 | 0.14 | 0.12 | 0.18 |
| 250 | 150 | 0.06 | 0.03 | 0.03 |
| 250 | 200 | 0.12 | 0.17 | 0.20 |
| 2500 | 300 | 0.12 | 0.12 | 0.11 |
| 2500 | 500 | 0.05 | 0.05 | 0.04 |

Table 5.14: Rejection probability of the STm test for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The processes are generated by bivariate GARCH(1,1) with correlation coefficient $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 500$.

to increase. The block length $b = 300$ yields a bad result for $n = 2500$, but the performance is much better for $b = 500$. This suggests an optimal block length of approximately $b(n) \approx 10\sqrt{n}$.

We see that for the sample size $n = 4000$ the LMWm test performs well if we choose the block length $b = 300$. This result suggests that for the considered dependence structure and a sample size larger than $n = 1000$ the increase of the optimal block length is very slow. Furthermore, the range of block lengths with good performance becomes larger with increasing sample size: $b = 600$ is still a reasonable choice for $n = 4000$. However, for smaller block lengths the choice of the block length is more critical. For $n = 250$ the block length $b = 150$ is a good choice whereas $b = 100$ and $b = 200$ yield rather poor results. These results suggest that at least for smaller sample sizes the optimal block length is proportional to $\sqrt{n}$, the optimal block length is approximately $b(n) \approx 10\sqrt{n}$, as for the STm test.

Summing up the results of this section, the modifications of the considered tests are successful for the tests of Schmid/ Trede and of Linton/ Maasoumi/ Whang, whereas it does not improve the performance of the test of Xu/ Fisher/

| $\rho$ | | $-0.5$ | 0 | 0.5 |
|---|---|---|---|---|
| $n$ | $b$ | | | |
| 250 | 100 | 0.09 | 0.08 | 0.10 |
| 250 | 150 | 0.04 | 0.04 | 0.05 |
| 250 | 200 | 0.01 | 0.01 | 0.01 |
| 500 | 220 | 0.04 | 0.06 | 0.06 |
| 4000 | 300 | 0.04 | 0.05 | 0.05 |
| 4000 | 600 | 0.04 | 0.03 | 0.05 |

Table 5.15: Rejection probability of the LMWm test for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The processes are generated by bivariate GARCH(1,1) with correlation parameter $\rho$ and parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

Willson. This test cannot be improved significantly by moving block bootstrap, but block permutation makes the permutation test of Schmid/ Trede robust to conditional heteroskedasticity, and circular subsampling improves the performance of the test of Linton/ Maasoumi/ Whang for finite samples. The choice of the appropriate block length is crucial. For both the modified ST and LMW tests the optimal block length is approximately $b(n) = 10\sqrt{n}$.

## 5.5 Power Investigation

In the previous sections we modified the tests of Schmid/ Trede for SD2 and Linton/ Maasoumi/ Whang for SD1 successfully. We investigated the optimal block lengths for various sample sizes.

In this section we explore the power of the tests. The main problem is the shape of the alternative $H_1$. There are many combinations of distribution functions $F_X$, $F_Y$ so that $F_X(x) > F_Y(x)$ for at least one $x \in \mathbb{R}$ (analogous for $F^{(2)}$). Hence the alternative $H_1 : X \not\succeq Y$ is very complex.

We start with the investigation of some scale alternatives. For both tests we consider the alternative

$$H_1(\sigma_X) : P_X = \mathcal{N}(0, \sigma_X), P_Y = \mathcal{N}(0, 1)$$

and vary $\sigma_X$ from 1.1 to 1.5 in 0.1 steps. Furthermore, we analyze the alternative

$$H_1(\sigma_Y) : P_X = \mathcal{N}(0, 1), P_Y = \mathcal{N}(0, \sigma_Y)$$

with $\sigma_Y = 1.1, 1.2, \ldots, 1.5$ only for the LMWm test of SD1. Note that $P_X = \mathcal{N}(0, 1)$, $P_Y = \mathcal{N}(0, \sigma_Y)$ with $\sigma_Y > 1$ is in the null hypothesis for SD2. For both tests we consider the sample sizes $n = 250, 1000, 2500$ and the values of the block length $b$ which are found to be optimal in the previous section. The samples are generated by contemporaneously and serially independent processes. The number of Monte Carlo replications is $R = 100$. This small number causes a large standard error of the simulation results, but we can at least interpret them as a tendency.

Tables 5.16, 5.17 and 5.18 display the results. As one might expect, the power increases with increasing standard deviation $\sigma_Z$ $(Z = X, Y)$ and increasing sample size $n$. The larger $\sigma_Z$, the larger is the distance to $H_0$. Hence the violation of the null hypothesis is more likely to be detected for larger $\sigma_Z$. With larger sample size the consistency of the tests becomes stronger. For $n = 250$ the results are not that satisfactory whereas for $n = 2500$ the power tends toward 1 very fast with growing $\sigma_Z$.

Then we explore the power of the tests for the location alternative $H_1 : P_X = \mathcal{N}(0, 1), P_Y = \mathcal{N}(0.1, 1)$ where the observations are contemporaneously and serially independent. Here $Y$ dominates $X$ in the sense of SD1 and SD2, hence the dominance of $X$ has to be rejected. The results are presented in table 5.19. For $n = 2500$ the power is very high, for the STm test for SD2 $n = 1000$ suffices to give good results. The low power for smaller sample size is caused by the fact that the deviation from equality, which is a limiting case of stochastic dominance, is small.

| | $\sigma_X$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|
| $n$ | $b$ | | | | | |
| 250 | 150 | 0.15 | 0.24 | 0.39 | 0.48 | 0.64 |
| 1000 | 300 | 0.16 | 0.52 | 0.85 | 0.96 | 1.00 |
| 2500 | 500 | 0.42 | 0.94 | 1.00 | 1.00 | 1.00 |

Table 5.16: Power of the STm test for SD2 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $H_1(\sigma_X) : P_X = \mathcal{N}(0, \sigma_X), P_Y = \mathcal{N}(0, 1)$ where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

| | $\sigma_X$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|
| $n$ | $b$ | | | | | |
| 250 | 150 | 0.03 | 0.06 | 0.08 | 0.21 | 0.27 |
| 1000 | 300 | 0.11 | 0.43 | 0.67 | 0.92 | 0.97 |
| 2500 | 300 | 0.29 | 0.96 | 1.00 | 1.00 | 1.00 |

Table 5.17: Power of the LMWm test for SD1 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $H_1(\sigma_X) : P_X = \mathcal{N}(0, \sigma_X), P_Y = \mathcal{N}(0, 1)$ where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

| | $\sigma_Y$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|
| $n$ | $b$ | | | | | |
| 250 | 150 | 0.00 | 0.01 | 0.08 | 0.10 | 0.19 |
| 1000 | 300 | 0.07 | 0.45 | 0.73 | 0.94 | 0.99 |
| 2500 | 300 | 0.24 | 0.89 | 1.00 | 1.00 | 1.00 |

Table 5.18: Power of the LMWm test for SD1 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $H_1(\sigma_Y) : P_X = \mathcal{N}(0,1), P_Y = \mathcal{N}(0, \sigma_Y)$ where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

| | Test | LMWm SD1 | STm SD2 |
|---|---|---|---|
| $n$ | $b$ | | |
| 250 | 150 | 0.05 | 0.39 |
| 1000 | 300 | 0.29 | 0.69 |
| 2500 | 300 | 0.79 | – |
| 2500 | 500 | – | 0.97 |

Table 5.19: Power of the LMWm test for SD1 and STm test for SD2 – hence also SD1 – for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $H_1 : P_X = \mathcal{N}(0,1), P_Y = \mathcal{N}(0.1, 1)$ where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

Furthermore, we analyze the power of the tests if the observations are contemporaneously and serially independent and normally distributed and differ in both mean and variance. We consider the cases $P_X = \mathcal{N}(0,1)$, $P_Y = \mathcal{N}(1,2)$, $P_X = \mathcal{N}(0,1)$, $P_Y = \mathcal{N}(-1,2)$ and vice versa. $P_X = \mathcal{N}(0,1)$, $P_Y = \mathcal{N}(-1,2)$ is the only combination of distributions where $X$ dominates $Y$ in the sense of SD2, whereas SD1 does not hold in any of these cases.

Tables 5.20 and 5.21 display the results. For both tests and all alternatives the power is very good for the sample size $n = 2500$; for the LMWm test $n = 1000$ already gives satisfactory results. In addition to this, both tests have good power even for $n = 250$ if $P_X = \mathcal{N}(-1,2)$, $P_Y = \mathcal{N}(0,1)$ or if $P_X = \mathcal{N}(0,1)$, $P_Y = \mathcal{N}(1,2)$ hold. In these cases, $X$ has a smaller mean and the dominance of $X$ in the sense of SD1 or SD2 can therefore be clearly rejected. The case $P_X = \mathcal{N}(1,2)$, $P_Y = \mathcal{N}(0,1)$ is most critical. The LMWm test has low power for $n = 250$, the STm test even for $n = 1000$. If $X$ and $Y$ are distributed like this, $F_X(x) \leq F_Y(x)$ holds for $x \geq -1$. The fact that $X$ takes on very small values with larger probability than $Y$ is the reason why $X$ does not dominate $Y$ in the sense of SD1 or SD2. However, in small samples the number of extreme observations is very small. Hence the violation of SD is often not detected whereas for larger samples this problem becomes more and more negligible. In the case $P_X = \mathcal{N}(0,1)$, $P_Y = \mathcal{N}(-1,2)$, $X$ dominates $Y$ in the sense of SD2, but not of SD1. $X$ does not dominate $Y$ in the sense of SD1 because $Y$ takes very large values with larger probability than $Y$. The LMWm test does not detect the violation of SD1 for $n = 250$ because there are only few observations belonging to the right tail for this small sample. For $n = 1000$ and $n = 2500$ this problem does not occur any more.

So far we have investigated the power of the LMWm test for SD1 and the STm test for SD2 in various settings, but under the assumption of contemporaneous and serial independence. As mentioned earlier, this is not a realistic presumption for financial data. Due to the fact that contemporaneous correlation does not have

| | | $P_X$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(1,2)$ | $\mathcal{N}(-1,2)$ |
|---|---|---|---|---|---|
| | | $P_Y$ | $\mathcal{N}(1,2)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(0,1)$ |
| $n$ | $b$ | | | | |
| 250 | 150 | | 1.00 | 0.01 | 1.00 |
| 1000 | 300 | | 1.00 | 0.04 | 1.00 |
| 2500 | 500 | | 1.00 | 0.64 | 1.00 |

Table 5.20: Power of the STm test for SD2 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is the combination of two normal distributions with different means and variances where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

| | | $P_X$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(1,2)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(-1,2)$ |
|---|---|---|---|---|---|---|
| | | $P_Y$ | $\mathcal{N}(1,2)$ | $\mathcal{N}(0,1)$ | $\mathcal{N}(-1,2)$ | $\mathcal{N}(0,1)$ |
| $n$ | $b$ | | | | | |
| 250 | 150 | | 1.00 | 0.08 | 0.11 | 0.99 |
| 1000 | 300 | | 1.00 | 0.88 | 0.87 | 1.00 |
| 2500 | 300 | | 1.00 | 1.00 | 1.00 | 1.00 |

Table 5.21: Power of the LMWm test for SD1 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is the combination of two normal distributions with different means and variances where the processes $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are contemporaneously and serially independent. The number of Monte Carlo replications is $R = 100$.

a significant effect on the performance of the tests, we only report the impact of serial dependence caused by GARCH(1,1). We consider the following settings. Let $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ be two independent univariate GARCH(1,1) processes. As in previous sections, we choose the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$ and $\beta = 0.85$. We consider the scale alternatives $X_t = \sigma_X A_t$, $Y_t = B_t$ (case 1) and $X_t = A_t$, $Y_t = \sigma_Y B_t$ (case 2) where we choose $\sigma_X = 1.1, 1.5$ and $\sigma_Y = 1.1, 1.5$. In both cases $X$ does not dominate $Y$ in the sense of SD1 whereas $X$ dominates $Y$ in the sense of SD2 in case 2. This can be proved as follows. In case 1 we have $F_X(x) = F_Y(\frac{x}{\sigma})$ and $F_X^{(2)}(x) = \sigma F_Y^{(2)}(\frac{x}{\sigma})$ and therefore $F_X(x) > F_Y(x)$ for some $x < 0$, $F_X(x) < F_Y(x)$ for some $x > 0$ and $F_X^{(2)}(x) \geq F_Y^{(2)}(x)$ for all $x \in \mathbb{R}$, in case 2 this holds with reversed roles of $X$ and $Y$.

As we have seen, case 2 is in the null hypothesis that $X$ dominates $Y$ in the sense of SD2. Hence we only explore the power of the LMWm test for SD1 in case 2. Furthermore, we explore the power of the LMWm test for SD1 and of the STm test for SD2 in case 1. In all settings we choose the sample sizes $n = 250, 1000, 2500$. The results, which are displayed in table 5.22, are similar to those in the case of independent observations. The power increases with increasing $\sigma_X$ or $\sigma_Y$, respectively, and with increasing sample size. The main difference to the case of independent observations is the speed of convergence. For $\sigma_X = 1.5$ or $\sigma_Y = 1.5$ the power is lower than in the independent case, in particular for smaller sample sizes.

Finally we analyze the location alternative $X_t = A_t$, $Y_t = B_t + \mu_Y$ where $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are two independent univariate GARCH(1,1) processes as described above. If we choose the values $\mu_Y = 0.1, 0.5$, $X$ does obviously not dominate $Y$ in the sense of SD1 or SD2. We investigate the power of the LMWm test for SD1 and the STm test for SD2. As above, we choose the sample sizes $n = 250, 1000, 2500$. Table 5.23 displays the results. For $\mu_Y = 0.5$, which implies a larger deviation from dominance of $X$, the power is high. However, for $\mu_Y = 0.1$ the power increases very slowly with increasing sample size. Even for $n = 2500$

| | | Test | LMWm SD1 | | | | STm SD2 | |
|---|---|---|---|---|---|---|---|---|
| | | $\sigma_X$ | 1.1 | 1.5 | 1.0 | 1.0 | 1.1 | 1.5 |
| $n$ | $b$ | $\sigma_Y$ | 1.0 | 1.0 | 1.1 | 1.5 | 1.0 | 1.0 |
| 250 | 150 | | 0.07 | 0.15 | 0.07 | 0.11 | 0.19 | 0.55 |
| 1000 | 300 | | 0.05 | 0.31 | 0.12 | 0.43 | 0.21 | 0.66 |
| 2500 | 300 | | 0.12 | 0.88 | 0.10 | 0.79 | – | – |
| 2500 | 500 | | – | – | – | – | 0.24 | 0.86 |

Table 5.22: Power of the LMWm test for SD1 and of the STm test for SD2 – hence also SD1 – for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = \sigma_X A_t$ and $Y_t = \sigma_Y B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 100$.

we do not get satisfactory results. Nevertheless, the main tendency is, as in the case of independent observations, that the power increases with increasing $\mu_Y$ and sample size. Altogether we see that conditional heteroskedasticity does not have a strong effect on the power of the tests which we have developed.

In this section we analyze the power of the tests we have developed in this chapter. Among the large number of alternatives we first confine ourselves to the case of independent, normally distributed observations. We see that the STm test has good power for $n = 2500$, for the LMWm test we get satisfactory results even for $n = 1000$. The larger the distance to $H_0$, the higher is the power. For some alternatives the power is close to one even for sample size $n = 250$. Furthermore, we analyze the power if conditional heteroskedasticity is prevalent in the data. It turns out that there is a weak effect of GARCH(1,1), but that the power is still satisfactory with a sufficiently large sample size.

| | | Test | LMWm SD1 | | STm SD2 | |
|---|---|---|---|---|---|---|
| $n$ | $b$ | $\mu_Y$ | 0.1 | 0.5 | 0.1 | 0.5 |
| 250 | 150 | | 0.08 | 0.19 | 0.27 | 0.68 |
| 1000 | 300 | | 0.15 | 0.61 | 0.30 | 0.90 |
| 2500 | 300/500 | | 0.15 | 0.99 | 0.27 | 0.98 |

Table 5.23: Power of the LMWm test for SD1 and of the STm test for SD2 for the nominal value $\alpha = 0.05$ and various values of sample size $n$ and block length $b$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = B_t + \mu_Y$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.14$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 100$.

## 5.6 Conclusion

We investigate the tests for stochastic dominance of Schmid/ Trede, Xu/ Fisher/ Willson and Linton/ Maasoumi/ Whang using a simulation study. They determine the critical values of the tests using subsampling methods. Schmid/ Trede use the permutation principle, Xu/ Fisher/ Willson the moving block bootstrap and Linton/ Maasoumi/ Whang subsampling estimation. Simulations show that all these tests perform rather poorly for finite samples if the data are generated by GARCH(1,1) processes, which is an appropriate choice for financial data.

We develop several modifications to overcome these shortcomings. Our modification consists of using circular block methods: the modified ST test uses circular block permutation, the modified XFW test circular block bootstrap and the modified LMW test circular subsampling.

We show analytically that the modifications of the XFW and the LMW tests are asymptotically equivalent to the original tests. We argue that the circular block methods are more appropriate for resampling the dependence structure. By Monte Carlo simulation we show that for finite samples the modifications make the ST and the LMW tests robust to conditional heteroskedasticity. In contrast, the modifications do not improve the XFW test.

For the modified ST and LMW tests the appropriate block length has to be chosen. We find that for both tests the optimal block length is proportional to $\sqrt{n}$ where $n$ is the sample size. For the LMWm test the range of suitable block lengths is broader for larger sample sizes whereas this is is not true for the STm test.

Then we investigate the power of the developed tests. For various combinations of normally distributed observations we see that the power increases with increasing sample size and with increasing distance to the null hypothesis. If the data are generated by bivariate GARCH(1,1), the power is lower than in the case of serially independent observations. However, the power still increases with increasing sample size and increasing distance to $H_0$. For a sufficiently large sample size (e.g. $n = 2500$) we get satisfactory power results.

In the Monte Carlo simulations for investigating size and power, we consider certain settings which are only small parts of the null hypothesis of dominance and the alternative of non-dominance, respectively. In further research one could explore size and power under different assumptions concerning distribution and dependence structure, e.g. stochastic volatility or non-Gaussian copulas.

Furthermore, one could search for a test with non-dominance in the null hypothesis and dominance in the alternative. In investigating this topic similar complexity problems as in the power investigation arise. In the next chapter we will explore this topic in more detail and find a test which asymptotically keeps the size.

# Chapter 6

# A Test in which Stochastic Dominance is the Alternative

## 6.1 Introduction

As we stated in previous chapters, the vast majority of the tests for stochastic dominance test the hypothesis of dominance against the alternative of non-dominance. The reason is that the set of all pairs of random variables without a dominance relationship is mathematically complex. However, there is a need for a test which can significantly assert stochastic dominance.

In this chapter, we approach the problem of finding a test in which stochastic dominance is the alternative. The starting point is the test of Kaur/ Rao/ Singh (1994). The main drawback is that this test only regards a fixed interval which has to be a proper subset of the support of the distributions. We propose some modifications of the test statistic as a remedy to this problem. The new test which we develop appropriately truncates the range for the determination of the infimum. This test asymptotically keeps the size if the truncation value is chosen appropriately. In a Monte Carlo study we explore the problem of the right truncation choice for finite samples. The results of this chapter are based on Kläver (2005b).

It is obvious that SD$k$ in a descriptive sense, i.e. $\hat{F}^{(k)}_{X,n}(x) \leq \hat{F}^{(k)}_{Y,n}(x)$ for all $x \in \mathbb{R}$, is a necessary condition for a significant conclusion of SD$k$. Due to the fact that SD1 is rejected in the descriptive sense in most of the empirical comparisons whereas SD2 is asserted more often, there is more need for a test for SD2. Hence in this chapter we focus on a test for SD2.

The structure of this chapter is as follows. The next section gives a survey on the problem and on approaches for the construction of an appropriate test. Furthermore, the test of Kaur/ Rao/ Singh is presented. We discuss their test and propose a modified one in section 6.3. Section 6.4 shows the simulation results for various settings in order to investigate the size and power properties of the proposed test. Section 6.5 draws conclusions from the results of this chapter.

## 6.2 Previous Tests in which Stochastic Dominance is the Alternative

In most of the tests for stochastic dominance the hypothesis of dominance is tested against the alternative of non-dominance. In such a procedure stochastic dominance cannot be significantly asserted, but only be rejected or not. The reason for the asymmetric development of tests for stochastic dominance is the mathematical complexity of the hypothesis of non-dominance. Whereas the limiting case of the dominance hypothesis is just equality of distributions, the boundary of the non-dominance hypothesis cannot be expressed in closed form. In the approaches of Eubank/ Schechtman/ Yitzhaki (1993), Anderson (1996) and Herring (1996) the hypothesis of equality of the distributions is tested against the alternative of stochastic dominance. The critical point is that in these tests rejection of the hypothesis is not equivalent to significant assertion of stochastic dominance. Equality of distributions can also be rejected in favor of reversed stochastic dominance or of incomparableness of the distributions.

The test of Kaur/ Rao/ Singh (1994) is a good starting point for testing for second degree stochastic dominance. They test the null hypothesis

$$H_0 : F_X^{(2)}(x) \geq F_Y^{(2)}(x) \text{ for some } x \in [a, b]$$

against the alternative

$$H_1 : F_X^{(2)}(x) < F_Y^{(2)}(x) \text{ for all } x \in [a, b]$$

where $a$ and $b$ are any real numbers satisfying $-\infty < a < b < \infty$. They use the infimum statistic $\tilde{T}^*_{n,m} = \inf_{x\in[a,b]} \tilde{T}_{n,m}(x)$ where

$$\tilde{T}_{n,m}(x) = \frac{\hat{F}^{(2)}_{Y,m}(x) - \hat{F}^{(2)}_{X,n}(x)}{\sqrt{\frac{1}{n}S^2_{X,n}(x) + \frac{1}{m}S^2_{Y,m}(x)}}$$

and

$$S^2_{X,n}(x) = \frac{1}{n}\sum_{k=1}^{n}((x - x_k)_+ - (\hat{F}^{(2)}_{X,n}(x)))^2 = \frac{1}{n}\sum_{k=1}^{n}(x - x_k)_+^2 - (\hat{F}^{(2)}_{X,n}(x))^2$$

and $S^2_{Y,m}(x)$ is defined analogously. Remember that $x_+$ denotes the nonnegative part of a real number $x$, i.e. $x_+ = \max\{x, 0\}$. In case the numerator and the denominator are both zero, $\tilde{T}_{n,m}(x)$ is defined to be zero. Under the assumption that the observations are independent Kaur/ Rao/ Singh state the following result for the critical value $c_\alpha := \Phi(1 - \alpha)$ where $\Phi$ is the distribution function of the standard normal distribution.

**Theorem 4 (Theorem 2.2 in Kaur/ Rao/ Singh).** *Let* $\lim_{n,m\to\infty} \frac{m}{m+n} = \kappa \in [0, 1)$ *and* $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ *independent random variables with the distributions* $P_X$ *and* $P_Y$*, respectively. Let* $E(X^2)$ *and* $E(Y^2)$ *be finite and* $F_X$ *and* $F_Y$ *be continuous. Then the following holds:*

1. *If* $(P_X, P_Y) \in H_0$*, then*

$$\limsup_{n,m\to\infty} P(\tilde{T}^*_{n,m} > c_\alpha) \leq \alpha.$$

*2. If there is a non-empty subset $M \subset (a,b)$ such that*

$$F_X^{(2)}(x) = F_Y^{(2)}(x) \neq 0 \textit{ for all } x \in M$$

*and*

$$F_X^{(2)}(x) < F_Y^{(2)}(x) \textit{ for all } x \in [a,b] \backslash M,$$

*then*

$$\lim_{n,m\to\infty} P(\tilde{T}^*_{n,m} > c_\alpha) = \alpha.$$

*3. If $(P_X, P_Y) \in H_1$, then*

$$\lim_{n,m\to\infty} P(\tilde{T}^*_{n,m} > c_\alpha) = 1.$$

Kaur/ Rao/ Singh propose a test which rejects $H_0$ in favor of $H_1$ if and only if $\tilde{T}^*_{n,m} > c_\alpha$. Hereafter, we denote the test by KRS. Theorem 4 establishes that the KRS test has an upper bound $\alpha$ on the asymptotic size and is consistent.

The KRS test is similar to a test for SD2, but there are some critical distinctions. We will discuss them in the next section.

## 6.3 Discussion and Modification

In this section, we discuss and modify the KRS test. As in previous chapters, let $n = m$ in the following for the sake of simplicity. First we modify the function $\tilde{T}_{n,m}(x)$ for relaxing the independence assumption. With the notation

$$W_k(x) = ((x - Y_k)_+ - (x - X_k)_+)$$

we get $\bar{W}_{(n)}(x) = (\hat{F}^{(2)}_{Y,n}(x) - \hat{F}^{(2)}_{X,n}(x))$ and $E(W_k(x)) = F_Y^{(2)}(x) - F_X^{(2)}(x)$ for $k = 1, \ldots, n$ where $\bar{x}_{(n)}$ denotes the average of a sample $x_1, \ldots, x_n$. We consider the function

$$T_n(x) = \frac{\sqrt{n}(\hat{F}^{(2)}_{Y,n}(x) - \hat{F}^{(2)}_{X,n}(x))}{S_{X,Y,n}(x)}$$

where

$$S^2_{X,Y,n}(x) = \frac{1}{n}\sum_{k=1}^{n}(V_k(x))^2 + \sum_{l=1}^{\lfloor\sqrt{n}\rfloor}\frac{2}{n-l}\sum_{k=1}^{n-l}V_k(x)V_{k+l}(x)$$

and $V_k(x) = W_k(x) - \bar{W}_{(n)}(x)$. We define $T_n(x)$ to be zero, when both numerator and denominator are zero. The following proposition is similar to theorem 2.1 in Kaur/ Rao/ Singh, but the data are not required to be independent. The result will help us to construct a new test which is robust to some deviations from independence.

**Proposition 7.** *Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be nondegenerate, strictly stationary, ergodic, strongly mixing sequences of random variables satisfying $E|X_0|^{2+\delta} < \infty$, $E|Y_0|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose that the mixing coefficients $\alpha_X(k)$ and $\alpha_Y(k)$ satisfy*

$$\sum_{k=1}^{\infty}\alpha_X(k)^{\frac{\delta}{2+\delta}}, \sum_{k=1}^{\infty}\alpha_Y(k)^{\frac{\delta}{2+\delta}} < \infty.$$

*Let*

$$T_n(x) = \frac{\sqrt{n}(\hat{F}^{(2)}_{Y,n}(x) - \hat{F}^{(2)}_{X,n}(x))}{S_{X,Y,n}(x)}$$

*as defined above. If $Var(W_1(x)) > 0$, then*

$$P(T_n(x) > c_\alpha) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if} \quad F^{(2)}_X(x) > F^{(2)}_Y(x) \\ \alpha & \text{if} \quad F^{(2)}_X(x) = F^{(2)}_Y(x) \\ 1 & \text{if} \quad F^{(2)}_X(x) < F^{(2)}_Y(x) \end{cases} .$$

*If $Var(W_1(x)) = 0$, then $T_n(x) = 0$ a.s. and so $P(T_n(x) > c_\alpha) = 0$ for all $n$.*

*Proof.* Let

$$T_n(x) = \frac{A_n(x)}{S_{X,Y,n}(x)} + \frac{B_n(x)}{S_{X,Y,n}(x)} \tag{6.1}$$

where

$$\begin{aligned} A_n(x) &= \sqrt{n}((\hat{F}^{(2)}_{Y,n}(x) - \hat{F}^{(2)}_{X,n}(x)) - (F^{(2)}_Y(x) - F^{(2)}_X(x))) \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(W_k(x) - E(W_1(x))), \\ B_n(x) &= \sqrt{n}(F^{(2)}_Y(x) - F^{(2)}_X(x)) = \sqrt{n}E(W_1(x)). \end{aligned}$$

If $Var(W_1(x)) = 0$, then $(x - X_1)_+$ and $(x - Y_1)_+$ are degenerate. As $X_1$ and $Y_1$ are nondegenerate by assumption, $X_1$ and $Y_1$ have support in $[x, \infty)$ and therefore $T_n(x) = 0$ a.s.

In the following let $Var(W_1(x)) > 0$. By assumption, $(X_t)_{t \in \mathbb{Z}}$ and $(Y_t)_{t \in \mathbb{Z}}$ are strictly stationary and ergodic. According to Stout (1974, pp. 170, 182), this also holds for $(W_k(x))_{k \in \mathbb{Z}}$. Furthermore, for the mixing coefficient $\alpha_W(k)$ of $(W_k(x))_{k \in \mathbb{Z}}$ we have $\alpha_W(k) \leq \max\{\alpha_X(k), \alpha_Y(k)\}$. Hence

$$\sum_{k=1}^{\infty} \alpha_W(k)^{\frac{\delta}{2+\delta}} < \infty.$$

Therefore, according to Jones (2004), $A_n(x)$ converges in distribution to a $\mathcal{N}(0, \sigma(x))$ distribution where

$$\sigma^2(x) = Var(W_1(x)) + 2 \sum_{l=1}^{\infty} Cov(W_1(x), W_{1+l}(x)) < \infty.$$

Furthermore the ergodic theorem yields that $S^2_{X,Y,n}(x)$ satisfies the strong law of large numbers:

$$\begin{aligned} S^2_{X,Y,n}(x) \quad &\xrightarrow{a.s.} \quad E(V_1^2(x)) + 2 \sum_{l=1}^{\infty} E(V_1(x) V_{1+l}(x)) \\ &= Var(W_1(x)) + 2 \sum_{l=1}^{\infty} Cov(W_1(x), W_{1+l}(x)). \end{aligned}$$

It follows by Slutsky's theorem that

$$\frac{A_n(x)}{S_{X,Y,n}(x)} \quad \xrightarrow{d} \quad \mathcal{N}(0,1). \tag{6.2}$$

Furthermore, by definition of $B_n(x)$, we have

$$B_n(x) \underset{n \to \infty}{\longrightarrow} \begin{cases} -\infty & \text{if} \quad F_X^{(2)}(x) > F_Y^{(2)}(x) \\ 0 & \text{if} \quad F_X^{(2)}(x) = F_Y^{(2)}(x) \\ \infty & \text{if} \quad F_X^{(2)}(x) < F_Y^{(2)}(x) \end{cases}. \tag{6.3}$$

(6.1), (6.2) and (6.3) yield the assertion. □

Now let us have a closer look at the hypothesis and the alternative of the KRS test. The modification

$$H_1^{**} : F_X^{(2)}(x) \leq F_Y^{(2)}(x) \text{ for all } x \in \mathbb{R}$$

of $H_1$ is equivalent to second degree stochastic dominance of $X$ over $Y$. There are two differences between $H_1$ and $H_1^{**}$: Kaur/ Rao/ Singh still rank $X$ and $Y$ among the null hypothesis if $F_X^{(2)}(x) = F_Y^{(2)}(x)$ for some $x \in \mathbb{R}$. Furthermore they only consider the bounded interval $[a, b]$ instead of $\mathbb{R}$. The first difference only affects some limiting cases where $F_X^{(2)}(x) \leq F_Y^{(2)}(x)$ for all $x$ and $F_X^{(2)}(x) = F_Y^{(2)}(x)$ for some $x$. However, the second difference is more serious. $H_1$ does not assure SD2 because outside the interval $[a, b]$ the inequality does not need to hold. The values $a$ and $b$ cannot be chosen in a way that $[a, b]$ is the support of $P_X$ and $P_Y$. In this case, $F_X^{(2)}(a) = F_Y^{(2)}(a) = 0$ which is contrary to $H_1$.

In the following, we modify the KRS test. If the supports of $P_X$ and $P_Y$ have a lower bound $a$ and therefore $F_X(x) = F_Y(x) = 0$ for all $x < a$, we get $F_X^{(2)}(x) = F_Y^{(2)}(x) = 0$ for all $x \leq a$. Further, if $X$ and $Y$ have the same mean, $\lim_{x\to\infty}(F_X^{(2)}(x) - F_Y^{(2)}(x)) = 0$ (see e.g. Ogryczak/ Ruszczynski, 1999). If the supports have an upper bound $b$, we even get $F_X^{(2)}(b) = F_Y^{(2)}(b)$. Therefore we propose the following modification of the KRS test. We test

$$H_0^* : F_X^{(2)}(x) \geq F_Y^{(2)}(x) \text{ for some } x \in \mathbb{R} \text{ satisfying } 0 < F_X(x) + F_Y(x) < 2$$

against

$$H_1^* : F_X^{(2)}(x) < F_Y^{(2)}(x) \text{ for all } x \in \mathbb{R} \text{ satisfying } 0 < F_X(x) + F_Y(x) < 2.$$

The alternative $H_1^*$ is still not equivalent to SD2. Indeed, the set of the pairs of random variables $(X, Y)$ satisfying $H_1^*$ is a subset of the pairs satisfying $X \succeq_2 Y$. Why does this relation hold? First note that $F_X^{(2)}(x) = F_Y^{(2)}(x) = 0$ holds for all $x$ satisfying $F_X(x) = F_Y(x) = 0$. Second, if $F_X^{(2)}(x) < F_Y^{(2)}(x)$ holds for all $x$ satisfying $0 < F_X(x) + F_Y(x) < 2$, then $F_X^{(2)}(x) \leq F_Y^{(2)}(x)$ for all $x$ satisfying

$F_X(x) + F_Y(x) = 2$, due to the continuity of $F_X^{(2)}$ and $F_Y^{(2)}$. Furthermore $H_1^*$ is a proper subset of the pairs $(X, Y)$ satisfying $X \succeq_2 Y$. The relative complement consists of the cases where $F_X^{(2)}(x) \leq F_Y^{(2)}(x)$ for all $x$ and $F_X^{(2)}(x) = F_Y^{(2)}(x)$ for some $x$ satisfying $0 < F_X(x) + F_Y(x) < 2$.

How can we perform an appropriate test? At first sight the test statistic $T_n^* = \inf_{x \in \mathbb{R}} T_n(x)$ where $T_n(x)$ is defined as in proposition 7 seems to be suitable. In contrast to the KRS test, the infimum of $T_n(x)$ is taken over all values $x \in \mathbb{R}$.

However, a test based on this statistic does not give meaningful results. Let $x_{(k)}$ and $y_{(k)}$ be the $k$th order statistics of the samples in ascending order and $(z_1, \ldots, z_{2n})$ be the ordered combined sample. Because of $\hat{F}_{Y,n}^{(2)}(z_1) = \hat{F}_{X,n}^{(2)}(z_1) = 0$ we get $T_n(z_1) = 0$ and therefore $T_n^* \leq 0$. The problem cannot be solved simply by redefining the term $\frac{0}{0}$. For the second smallest value $z_{(2)}$ we get

$$T_n(x) \approx \begin{cases} \sqrt{\frac{n}{n-2\sqrt{n}}} & \text{if} \quad z_1 = y_{(1)} \\ -\sqrt{\frac{n}{n-2\sqrt{n}}} & \text{if} \quad z_1 = x_{(1)} \end{cases}$$

which is close to $\pm 1$ for large $n$. Therefore, if the critical value is larger than one, the test will never reject the hypothesis of non-dominance, i.e. the power is zero.

In addition to this, $F_X^{(2)}(x)$ and $F_Y^{(2)}(x)$ have the same asymptote for $x \to \infty$ if $X$ and $Y$ have the same mean, as mentioned above. Hence, if $X \succeq_2 Y$ and $\mu_X = \mu_Y$, $\hat{F}_{X,n}^{(2)}(x)$ and $\hat{F}_{Y,n}^{(2)}(x)$ are expected to be close to each other for large $x \in \mathbb{R}$. Tse/ Zhang (2004) find that the KRS test is too conservative which is in accordance with the assertions above.

Which remedy can be found to solve these problems? We propose the following procedure. The statistic $T_n(x)$ is determined for all $x \in \mathbb{R}$. For each interval $[z_k, z_{k+1}]$ for $k = 1, \ldots, 2n-1$ the indicator functions contained in $T_n(x)$ are constants, and the infimum

$$\inf_{x \in [z_k, z_{k+1}]} T_n(x)$$

can be derived using standard procedures.

Due to the fact that $F_X^{(2)}(x)$ and $F_Y^{(2)}(x)$ are equal for very small $x$ and, if $X$ and $Y$ have equal mean, close to each other for very large $x$, we do not take the infimum of $T_n(x)$ over all $x \in \mathbb{R}$. Instead, we only consider all $x \in [z_{a_n}, z_{2n-b_n}]$ where $a_n$ and $b_n$ have to be chosen appropriately. In other words, the test statistic is

$$T_n^* = \inf\{T_n(x) : \hat{F}_{X,n}(x) + \hat{F}_{Y,n}(x) \in [\frac{a_n}{n}, 2 - \frac{b_n}{n}]\}.$$

The null hypothesis $H_0^*$ is rejected if and only if $T_n^* > c_\alpha$.

If

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{b_n}{n} = 0$$

holds, then the test asymptotically keeps the size $\alpha$. This is stated by the following theorem.

**Theorem 5.** *Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be nondegenerate, strictly stationary, ergodic, strongly mixing sequences of random variables satisfying $E|X_0|^{2+\delta} < \infty$, $E|Y_0|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose that the mixing coefficients $\alpha_X(k)$ and $\alpha_Y(k)$ satisfy*

$$\sum_{k=1}^{\infty} \alpha_X(k)^{\frac{\delta}{2+\delta}}, \sum_{k=1}^{\infty} \alpha_Y(k)^{\frac{\delta}{2+\delta}} < \infty. \tag{6.4}$$

*Further assume that the sequences $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are either serially independent or serially uncorrelated with continuous distribution functions $F_X$ and $F_Y$.*

*Let*

$$T_n^* = \inf\{T_n(x) : \hat{F}_{X,n}(x) + \hat{F}_{Y,n}(x) \in [\frac{a_n}{n}, 2 - \frac{b_n}{n}]\}$$

*be the test statistic for testing the null hypothesis*

$$H_0^* : F_X^{(2)}(x) \geq F_Y^{(2)}(x) \text{ for some } x \in \mathbb{R} \text{ satisfying } 0 < F_X(x) + F_Y(x) < 2$$

*against the alternative*

$$H_1^* : F_X^{(2)}(x) < F_Y^{(2)}(x) \text{ for all } x \in \mathbb{R} \text{ satisfying } 0 < F_X(x) + F_Y(x) < 2.$$

*If* $(P_X, P_Y) \in H_0^*$ *and*

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{b_n}{n} = 0,$$

*then*

$$\limsup_{n\to\infty} P(T_n^* > c_\alpha) \le \alpha.$$

*Proof.* Let $(P_X, P_Y) \in H_0^*$. Then there exists an $x_0 \in \mathbb{R}$ satisfying $0 < F_X(x_0) + F_Y(x_0) < 2$ and $F_X^{(2)}(x_0) \ge F_Y^{(2)}(x_0)$. Let $d := F_X(x_0) + F_Y(x_0)$. Due to

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{b_n}{n} = 0$$

there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ satisfying $d \in (\frac{a_n}{n} + \delta, 2 - \frac{b_n}{n} - \delta]$ for all $n \ge n_0$. Then we get

$$\begin{aligned} P(T_n^* > c_\alpha) &= P(\inf\{T_n(x) : \hat{F}_{X,n}(x) + \hat{F}_{Y,n}(x) \in [\frac{a_n}{n}, 2 - \frac{b_n}{n}]\} > c_\alpha) \\ &\le P(\inf\{T_n(x) : F_X(x) + F_Y(x) \in [\frac{a_n}{n} + \delta, 2 - \frac{b_n}{n} - \delta]\} > c_\alpha) \qquad (6.5) \\ &\le P(T_n(x_0) > c_\alpha) \end{aligned}$$

for sufficiently large $n$ where (6.5) holds due to Yu's (1993) extension of the theorem of Glivenko-Cantelli (proposition 2 in this thesis). Using proposition 7 we get

$$\limsup_{n\to\infty} P(T_n^* > c_\alpha) \le \limsup_{n\to\infty} P(T_n(x_0) > c_\alpha) \le \alpha$$

which yields the assertion. □

To which settings can we apply this theorem? Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be GARCH(1,1) processes as defined in chapter 4. As we stated in chapter 5, GARCH (1,1) processes are strongly mixing with a geometric rate, i.e. $\alpha(m) = O(a^m)$ for some $a \in (0,1)$; for more details see Davis/ Mikosch/ Basrak (1999). If $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ have finite fourth moments, they satisfy the assumptions of theorem 5. According to Bollerslev (1986) $X_t$ and $Y_t$ have finite fourth moments for every $t \in \mathbb{Z}$ if and only if $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$. Hence we can conclude that the test developed in this chapter asymptotically keeps the size if $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are GARCH(1,1) processes satisfying $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$.

Concerning consistency things are more involved. Kaur/ Rao/ Singh consider an interval $[a, b]$ of the domain of the distribution functions, i.e. the support of the random variables. This interval has a fixed length. In contrast to this, in the new test the length of the considered interval varies with $n$. In addition to this, the interval is part of the range of the distribution functions. Therefore, the asymptotic behavior of $T_n^*$ depends on how fast $\frac{a_n}{n}$ and $\frac{b_n}{n}$ tend to zero. We suppose that the test is consistent if these terms converge sufficiently slowly.

The main assertion in this section is that the proposed test asymptotically keeps the size $\alpha$ if $\frac{a_n}{n} \underset{n\to\infty}{\longrightarrow} 0$ and $\frac{b_n}{n} \underset{n\to\infty}{\longrightarrow} 0$ hold for the truncation sequences. Furthermore we guess that for sufficiently slow convergence the test is consistent. This involves a tradeoff concerning the choice of $a_n$ and $b_n$, i.e. the truncation. How do we choose $a_n$ and $b_n$ appropriately for various finite values of $n$? We will investigate this question in a Monte Carlo study in the next section.

## 6.4 Simulation Results

In the following we analyze the performance of the proposed test by means of simulation. We consider the strictly stationary bivariate process $(X_t, Y_t)_{t\in\mathbb{Z}}$ which implies that $(X_t, Y_t)$ has the same distribution for all $t \in \mathbb{Z}$. The marginal probability measures are denoted by $P_X$ and $P_Y$, respectively, and the corresponding distribution functions by $F_X$ and $F_Y$. We explore two kinds of settings: for size investigation we examine various settings of $(X_t, Y_t)$ satisfying $H_0^*$, for power investigation we have a look at $(X_t, Y_t)$ satisfying $H_1^*$. For all considered cases we choose the nominal size $\alpha = 0.05$.

First we consider a limiting case of $H_0^*$ (case 1). The observations are independent within each sample and between the samples. $X_t$ takes on the values $-\sqrt{\frac{2}{\pi}}$ and $\sqrt{\frac{2}{\pi}}$ each with probability $\frac{1}{2}$, i.e. $P_X = \frac{1}{2}(\delta_{-\sqrt{\frac{2}{\pi}}} + \delta_{\sqrt{\frac{2}{\pi}}})$ where $\delta_x$ stands for the Dirac distribution in $x$. In the following we abbreviate this by just saying $P_X = dirac$. It follows that

$$F_X^{(2)}(x) = \begin{cases} 0 & : \quad x < -\sqrt{\frac{2}{\pi}} \\ \frac{1}{2}(x + \sqrt{\frac{2}{\pi}}) & : \quad -\sqrt{\frac{2}{\pi}} \leq x < \sqrt{\frac{2}{\pi}} \\ x & : \quad x \geq \sqrt{\frac{2}{\pi}} \end{cases} .$$

$Y_t$ is standard normally distributed, i.e. $P_Y = \mathcal{N}(0,1)$. Due to (2.1) in chapter 2 we get

$$F_Y^{(2)}(x) = LPM_Y^1(x) = \int_{(-\infty,x)} (x-t)dP_Y(t) = x\Phi(x) + \phi(x)$$

where $\phi$ is the density of the standard normal distribution. Altogether this yields $F_X^{(2)}(0) = \frac{1}{\sqrt{2\pi}} = F_Y^{(2)}(0)$. We show that

$$F_X^{(2)}(x) < F_Y^{(2)}(x) \tag{6.6}$$

holds for all $x \neq 0$.

$$\frac{d}{dx}(F_Y^{(2)}(x) - F_X^{(2)}(x)) = \Phi(x) - \frac{1}{2} \begin{cases} < 0 & : \quad -\sqrt{\frac{2}{\pi}} < x < 0 \\ > 0 & : \quad 0 < x < \sqrt{\frac{2}{\pi}} \end{cases}$$

yields (6.6) for all $x$ satisfying $|x| < \sqrt{\frac{2}{\pi}}$ and $x \neq 0$. It is obvious that

$$F_X^{(2)}(x) = 0 < \int_{-\infty}^{x} \Phi(t)dt = F_Y^{(2)}(x)$$

for all $x \leq -\sqrt{\frac{2}{\pi}}$. Further, due to

$$\lim_{x\to\infty} F_Y^{(2)}(x) - F_X^{(2)}(x) = \lim_{x\to\infty} x\Phi(x) + \phi(x) - x = 0$$

and

$$\frac{d}{dx}(F_Y^{(2)}(x) - F_X^{(2)}(x)) = \Phi(x) - 1 < 0,$$

(6.6) also holds for $x \geq \sqrt{\frac{2}{\pi}}$.

Altogether we have shown that this setting belongs to a limiting case of $H_0^*$. The inequality $F_X^{(2)}(x) < F_Y^{(2)}(x)$ holds for all but one $x \in \mathbb{R}$, in this exceptional case at least equality holds, which already follows from the continuity of $F_X^{(2)}$ and $F_Y^{(2)}$.

The second setting, which is denoted by case 2, also belongs to the null hypothesis $H_0^*$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of independent random variables which have a normal distribution with mean $\underline{0}$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. We choose $\rho = 0, \pm 0.5$. With this setting, $F_X^{(2)}(x) = F_Y^{(2)}(x)$ holds for all $x \in \mathbb{R}$. Compared to the first case, $(X, Y)$ is located further in the interior of the null hypothesis. This suggests that the size $\alpha$ will not be completely exploited here.

Whereas in cases 1 and 2 the observations are serially independent, in case 3 we deal with the effect of conditional heteroskedasticity on the size of the test. We consider a bivariate GARCH(1,1) process $(X_t, Y_t)_{t\in\mathbb{Z}}$ as defined in chapter 5. In our Monte Carlo study we choose the GARCH parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$ and $\beta_1 = 0.85$ and the correlation parameters $\rho = 0, \pm 0.5$. Due to $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 = 0.9942 < 1$ these parameters imply that $X_t$ and $Y_t$ have finite fourth moments. Hence, due to theorem 5, the test asymptotically keeps the size if the truncations satisfy $\lim\limits_{n\to\infty} \frac{a_n}{n} = \lim\limits_{n\to\infty} \frac{b_n}{n} = 0$.

After the investigation of the size we have a look at the power. First we analyze case 4 where $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of independent, identically normally distributed random vectors $(X_t, Y_t)$ with mean $\underline{0}$ and covariance matrix $\begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}$ with $\sigma > 1$ and various values of $\rho \in (-1, 1)$. Then we have $F_X^{(2)}(x) = \int_{-\infty}^{x} \Phi(t)dt$ and $F_Y^{(2)}(x) = \sigma F_X^{(2)}(\frac{x}{\sigma})$. It is well known and can be shown easily that the inequality $F_X^{(2)}(x) < F_Y^{(2)}(x)$ holds for all $x \in \mathbb{R}$. Due to the fact that $X_t$ and $Y_t$ both have mean zero, for the limit behavior we get $\lim\limits_{x\to\infty}(F_Y^{(2)}(x) - F_X^{(2)}(x)) = 0$. We choose the values $\sigma = 1.1$ and $\sigma = 2$ for the standard deviation of $Y_t$ and $\rho = 0, \pm 0.5$ for the correlation between $X_t$ and $Y_t$.

Furthermore we consider another case which is part of the alternative: in case 5 $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of independent, identically normally distributed random vectors $(X_t, Y_t)$ with mean $(0.1,\ 0)$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. In this setting $X_t$ dominates $Y_t$ even in the sense of SD1. Furthermore we have the asymptotic behavior $\lim_{x\to\infty}(F_Y^{(2)}(x) - F_X^{(2)}(x)) = 0.1 > 0$. Hence it may be expected that in this case even without upper truncation the power will be good if the lower truncation is large enough.

Finally we analyze the power if conditional heteroskedasticity is prevalent in the data. Let $(A_t, B_t)_{t\in\mathbb{Z}}$ be a bivariate GARCH(1,1) process with a constant correlation coefficient $\rho$. We choose the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$ and $\beta = 0.85$ and the correlation coefficient $\rho = 0, \pm 0.5$. In case 6 we consider the scale alternative $X_t = A_t$, $Y_t = \sigma B_t$ and choose $\sigma = 1.1, 2$ as in case 4. Finally we have a look at the location alternative $X_t = A_t + \mu$, $Y_t = B_t$ where we choose $\mu = 0.1$. It is denoted by case 7.

For each setting, we choose the sample sizes $n = 250, 1000, 4000$. This means that the empirical distributions are based on $2n$ observations $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. For $n = 250$ we vary the truncations $a_n$ and $b_n$ from 0 to 250 in steps of 10, for $n = 1000$ and $n = 4000$ we vary $a_n$ and $b_n$ from 0 to 500 in steps of 10 and from 500 to 1000 and to 4000 respectively in steps of 100. For all cases the number of replications is $R = 1000$. The truncations from below and above are smaller than or equal to the sample size in each case, i.e. $a_n \le n$, $b_n \le n$. The reason for this choice is that in order to get reasonable results the sum of the truncations should be smaller than the number of observations the empirical distribution is based on. If $a_n = 0$, the test statistic is zero in any case. Hence the rejection rate of the null hypothesis is zero whatever the data are. Therefore we have to choose $a_n > 0$.

In the following we report and analyze the results. It turns out that in all considered cases the correlation $\rho$ between $X_t$ and $Y_t$ does not have a significant effect. Therefore we only report the settings where $X_t$ and $Y_t$ are uncorrelated. In cases 1 and 2 the size is kept for all sample sizes and all considered truncations.

| $b_n$ / $a_n$ | 0 | 10 | 100 | 200 | 250 |
|---|---|---|---|---|---|
| 10 | 0.03 | 0.05 | 0.05 | 0.05 | 0.06 |
| 100 | 0.03 | 0.05 | 0.05 | 0.05 | 0.06 |
| 200 | 0.03 | 0.05 | 0.05 | 0.05 | 0.06 |
| 250 | 0.04 | 0.05 | 0.06 | 0.06 | · |

Table 6.1: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = dirac$ and $P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

This even holds if $a_n$ and $b_n$ are large simultaneously, i.e. if $a_n + b_n$ is close to $2n$. In the limiting case of the hypothesis (case 1), where $P_X = dirac$, $P_Y = \mathcal{N}(0, 1)$, the size is more exploited than in case 2 which is more in the interior of $H_0^*$. In case 2 the size still grows with the truncation if the truncation is larger. However, the main result of the size investigation for serially independent observations is that the size is kept for any truncation satisfying $a_n < n$ and $b_n < n$. For detailed results see tables 6.1, 6.2 and 6.3 (limiting case $P_X = dirac$, $P_Y = \mathcal{N}(0, 1)$) and 6.4, 6.5 and 6.6 (equal distributions $P_X = P_Y = \mathcal{N}(0, 1)$).

If the data feature conditional heteroskedasticity, the choice of the truncation values is more crucial. Tables 6.7, 6.8 and 6.9 display the simulation results of case 3 where $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are independent GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$ and $\beta_1 = 0.85$. As we see in the tables, in this setting the upper truncation $b_n$ has a stronger impact on the size than the lower truncation $a_n$. The size $\alpha = 0.05$ is kept for truncation values up to $(a_n, b_n) \in \{(10, 50), (30, 40), (100, 30)\}$ for sample size $n = 250$, up to $(a_n, b_n) \in \{(10, 500), (200, 150), (1000, 100)\}$ for $n = 1000$, and up to

| $a_n$ \ $b_n$ | 0 | 10 | 100 | 500 | 1000 |
|---|---|---|---|---|---|
| 10 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 |
| 100 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 |
| 500 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 |
| 1000 | 0.04 | 0.04 | 0.05 | 0.05 | · |

Table 6.2: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = dirac$ and $P_Y = \mathcal{N}(0,1)$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 50 | 100 | 2000 | 4000 |
|---|---|---|---|---|---|
| 10 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 |
| 2000 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 |
| 4000 | 0.04 | 0.05 | 0.05 | 0.05 | · |

Table 6.3: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = dirac$ and $P_Y = \mathcal{N}(0,1)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 10 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 |
| 50 | 0.01 | 0.01 | 0.02 | 0.02 | 0.02 | 0.02 | 0.03 |
| 100 | 0.02 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 |
| 150 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 |
| 200 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 | 0.04 | 0.05 |
| 250 | 0.03 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | · |

Table 6.4: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 200 | 0.01 | 0.01 | 0.01 | 0.02 | 0.02 | 0.02 |
| 400 | 0.02 | 0.02 | 0.02 | 0.03 | 0.03 | 0.03 |
| 600 | 0.02 | 0.02 | 0.03 | 0.03 | 0.03 | 0.04 |
| 800 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 | 0.05 |
| 1000 | 0.03 | 0.03 | 0.04 | 0.04 | 0.05 | · |

Table 6.5: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 | 4000 |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 500 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| 1000 | 0.01 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 1500 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| 2000 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 |
| 2500 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 |
| 3000 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 | 0.04 | 0.04 |
| 3500 | 0.03 | 0.03 | 0.03 | 0.03 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 |
| 4000 | 0.03 | 0.04 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | · |

Table 6.6: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 10 | 20 | 30 | 40 | 50 | 100 | 200 | 250 |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.01 | 0.02 | 0.03 | 0.03 | 0.04 | 0.05 | 0.07 | 0.09 | 0.10 |
| 30 | 0.02 | 0.03 | 0.03 | 0.04 | 0.05 | 0.06 | 0.08 | 0.13 | 0.14 |
| 50 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.09 | 0.14 | 0.15 |
| 100 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.10 | 0.15 | 0.16 |
| 200 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.11 | 0.17 | 0.19 |
| 250 | 0.03 | 0.05 | 0.05 | 0.07 | 0.08 | 0.09 | 0.12 | 0.18 | · |

Table 6.7: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

$(a_n, b_n) \in \{(10, 4000), (100, 500), (4000, 200)\}$ for $n = 4000$.

In the Monte Carlo simulations we analyzed various settings. They only cover, however, a small part of the null hypothesis. Hence it is difficult to give clear advice concerning the lower and upper truncations. However, we infer from the simulation results that $a_n = b_n = 0.5n^{\frac{3}{4}}$ is an appropriate choice for sample size $n$. Using this formula we get

$$a_n = b_n \approx \begin{cases} 31 & : \quad n = 250 \\ 89 & : \quad n = 1000 \\ 251 & : \quad n = 4000 \end{cases}$$

which is in accordance with the simulation results. Note that the formula is just a rule of thumb which we inferred from the simulation results concerning the size. Nonetheless, we will use these truncation values in the following and refer to them as the recommended values.

In the following we will have a look at the power results. For the scale alternative with independent observations (case 4) the choice of the upper truncation

| $a_n$ \ $b_n$ | 0 | 10 | 50 | 100 | 150 | 200 | 300 | 400 | 500 | 1000 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.01 | 0.01 | 0.02 | 0.02 | 0.03 | 0.04 | 0.04 | 0.05 | 0.06 |
| 50 | 0.01 | 0.01 | 0.02 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.12 |
| 100 | 0.01 | 0.01 | 0.02 | 0.04 | 0.05 | 0.07 | 0.08 | 0.10 | 0.11 | 0.14 |
| 200 | 0.01 | 0.01 | 0.03 | 0.04 | 0.05 | 0.07 | 0.08 | 0.10 | 0.11 | 0.15 |
| 500 | 0.01 | 0.01 | 0.03 | 0.04 | 0.06 | 0.07 | 0.09 | 0.11 | 0.12 | 0.17 |
| 1000 | 0.02 | 0.02 | 0.03 | 0.05 | 0.06 | 0.08 | 0.10 | 0.12 | 0.12 | · |

Table 6.8: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 100 | 200 | 500 | 1000 | 4000 |
|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.01 | 0.01 | 0.01 | 0.02 | 0.03 |
| 100 | 0.02 | 0.03 | 0.03 | 0.05 | 0.07 | 0.10 |
| 500 | 0.02 | 0.04 | 0.04 | 0.06 | 0.09 | 0.13 |
| 1000 | 0.02 | 0.04 | 0.04 | 0.06 | 0.10 | 0.14 |
| 4000 | 0.03 | 0.04 | 0.05 | 0.07 | 0.11 | · |

Table 6.9: Rejection probability of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 50 | 100 | 250 |
|---|---|---|---|---|
| 10 | 0.06 | 0.41 | 0.80 | 0.99 |
| 100 | 0.06 | 0.41 | 0.81 | 1.00 |
| 250 | 0.06 | 0.41 | 0.81 | · |

Table 6.10: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0,1)$ and $P_Y = \mathcal{N}(0,2)$. The number of Monte Carlo replications is $R = 1000$.

$b_n$ is more crucial than of the lower truncation $a_n$. In case 4 with $\sigma = 2$ the power is high if the truncation parameters are sufficiently large. Tables 6.10, 6.11 and 6.12 display the results. If we choose the truncation parameters as recommended above, the power is about 0.4.

If we take a pair of distributions with smaller distance to $H_0^*$, the power is not as good any more. If we choose $\sigma = 1.1$ in case 4, the power increases very slowly with increasing truncation values $a_n$ and $b_n$. The results are reported in tables 6.13, 6.14 and 6.15. The power for the recommended truncation values is only about 0.06. Hence, in this setting, this test in many cases fails to detect the second degree stochastic dominance of $X$ over $Y$ if we apply it to smaller samples, but this problem becomes less important the more the sample size increases.

In case 5 we compare $P_X = \mathcal{N}(0.1,1)$ and $P_Y = \mathcal{N}(0,1)$, i.e. two normal distributions with the same variance, but different means. Tables 6.16, 6.17 and 6.18 display the results. As expected, the value of the upper truncation $b_n$ is not important. The power is quite high in this setting where $X$ has higher mean and dominates $Y$ in the sense of SD1, although the difference between their means is not very large. If we choose the recommended truncation values, the power increases with increasing sample size from less than 0.1 for $n = 250$ to about 0.4

| $b_n$ / $a_n$ | 0 | 100 | 200 | 300 | 1000 |
|---|---|---|---|---|---|
| 10 | 0.05 | 0.38 | 0.85 | 0.96 | 0.99 |
| 1000 | 0.05 | 0.38 | 0.86 | 0.99 | · |

Table 6.11: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0,1)$ and $P_Y = \mathcal{N}(0,2)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 100 | 200 | 300 | 400 | 500 | 1000 | 4000 |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.06 | 0.15 | 0.37 | 0.63 | 0.84 | 0.95 | 1.00 | 1.00 |
| 4000 | 0.06 | 0.15 | 0.37 | 0.64 | 0.85 | 0.96 | 1.00 | · |

Table 6.12: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0,1)$ and $P_Y = \mathcal{N}(0,2)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| 10 | 0.01 | 0.03 | 0.04 | 0.05 | 0.06 | 0.08 |
| 50 | 0.03 | 0.06 | 0.08 | 0.09 | 0.12 | 0.14 |
| 100 | 0.04 | 0.07 | 0.10 | 0.12 | 0.15 | 0.17 |
| 250 | 0.05 | 0.09 | 0.11 | 0.14 | 0.18 | · |

Table 6.13: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0, 1)$ and $P_Y = \mathcal{N}(0, 1.1)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 100 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|---|
| 10 | 0.01 | 0.02 | 0.03 | 0.05 | 0.06 | 0.08 | 0.11 |
| 100 | 0.04 | 0.06 | 0.08 | 0.14 | 0.19 | 0.25 | 0.32 |
| 200 | 0.05 | 0.07 | 0.09 | 0.16 | 0.22 | 0.29 | 0.38 |
| 1000 | 0.06 | 0.08 | 0.11 | 0.18 | 0.25 | 0.34 | · |

Table 6.14: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0, 1)$ and $P_Y = \mathcal{N}(0, 1.1)$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 500 | 1000 | 1500 | 2000 | 4000 |
|---|---|---|---|---|---|---|
| 10 | 0.02 | 0.06 | 0.10 | 0.14 | 0.21 | 0.31 |
| 100 | 0.06 | 0.15 | 0.24 | 0.34 | 0.49 | 0.77 |
| 500 | 0.06 | 0.16 | 0.27 | 0.38 | 0.55 | 0.90 |
| 4000 | 0.06 | 0.16 | 0.27 | 0.39 | 0.56 | · |

Table 6.15: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0,1)$ and $P_Y = \mathcal{N}(0,1.1)$. The number of Monte Carlo replications is $R = 1000$.

for $n = 4000$.

Finally we analyze the power in the settings where conditional heteroskedasticity is prevalent in the data. The results for the scale and location alternatives for conditionally heteroskedastic data are similar to the corresponding results for serially independent observations. The results for the scale alternative (case 6) are displayed in tables 6.19, 6.20 and 6.21 for $\sigma = 2$ and in tables 6.22, 6.23 and 6.24 for $\sigma = 1.1$. The power is about 0.4 for $\sigma = 2$ and about 0.1 for $\sigma = 1.1$ if we choose the truncation values which we recommended in the size investigation.

For the location alternative (case 7) we also see that conditional heteroskedasticity does not have a strong impact on the power. Tables 6.25, 6.26 and 6.27 display the results. As in case 5 where the observations are independent, the power increases with increasing sample size if we take the recommended truncation values. It increases from about 0.09 for $n = 250$ to 0.16 for $n = 4000$.

The simulation results in this study have various implications. In cases 1 and 2 where we investigate the size for serially independent observations, we see that the test keeps the size even if the truncations $a_n$ and $b_n$ are close to the sample size. However, this does not hold in case 3 where the data are generated

| $b_n$ / $a_n$ | 0 | 50 | 100 | 250 |
|---|---|---|---|---|
| 10 | 0.04 | 0.04 | 0.04 | 0.05 |
| 50 | 0.09 | 0.09 | 0.09 | 0.11 |
| 100 | 0.13 | 0.13 | 0.14 | 0.16 |
| 250 | 0.21 | 0.22 | 0.24 | · |

Table 6.16: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t \in \mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0.1, 1)$ and $P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 200 | 400 | 1000 |
|---|---|---|---|---|
| 10 | 0.05 | 0.05 | 0.05 | 0.06 |
| 100 | 0.18 | 0.19 | 0.19 | 0.20 |
| 200 | 0.28 | 0.28 | 0.28 | 0.29 |
| 400 | 0.39 | 0.40 | 0.40 | 0.42 |
| 1000 | 0.58 | 0.59 | 0.59 | · |

Table 6.17: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t \in \mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0.1, 1)$ and $P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 2000 | 4000 |
|---|---|---|---|
| 10 | 0.07 | 0.07 | 0.07 |
| 100 | 0.26 | 0.26 | 0.26 |
| 200 | 0.39 | 0.39 | 0.39 |
| 500 | 0.62 | 0.62 | 0.62 |
| 1000 | 0.80 | 0.80 | 0.80 |
| 4000 | 0.99 | 0.99 | · |

Table 6.18: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. $(X_t, Y_t)_{t\in\mathbb{Z}}$ is a sequence of serially independent random vectors. The marginal distributions of $X_t$ and $Y_t$, which are uncorrelated for every $t$, are $P_X = \mathcal{N}(0.1, 1)$ and $P_Y = \mathcal{N}(0, 1)$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 50 | 100 | 250 |
|---|---|---|---|---|
| 10 | 0.06 | 0.32 | 0.53 | 0.64 |
| 50 | 0.08 | 0.44 | 0.71 | 0.88 |
| 100 | 0.08 | 0.44 | 0.72 | 0.90 |
| 250 | 0.08 | 0.44 | 0.72 | · |

Table 6.19: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 2B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 100 | 200 | 300 | 400 | 1000 |
|---|---|---|---|---|---|---|
| 10 | 0.03 | 0.29 | 0.52 | 0.60 | 0.62 | 0.62 |
| 100 | 0.05 | 0.42 | 0.80 | 0.94 | 0.97 | 0.97 |
| 1000 | 0.05 | 0.42 | 0.80 | 0.94 | 0.98 | · |

Table 6.20: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 2B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 100 | 200 | 300 | 400 | 500 | 4000 |
|---|---|---|---|---|---|---|---|
| 10 | 0.04 | 0.15 | 0.27 | 0.43 | 0.51 | 0.54 | 0.56 |
| 100 | 0.06 | 0.23 | 0.47 | 0.75 | 0.89 | 0.95 | 0.98 |
| 4000 | 0.06 | 0.23 | 0.47 | 0.75 | 0.89 | 0.96 | · |

Table 6.21: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 2B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 50 | 100 | 250 |
|---|---|---|---|---|
| 10 | 0.01 | 0.06 | 0.09 | 0.13 |
| 50 | 0.02 | 0.09 | 0.13 | 0.21 |
| 100 | 0.03 | 0.10 | 0.14 | 0.23 |
| 250 | 0.04 | 0.11 | 0.16 | · |

Table 6.22: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 1.1B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 100 | 200 | 300 | 400 | 1000 |
|---|---|---|---|---|---|---|
| 10 | 0.00 | 0.03 | 0.06 | 0.08 | 0.09 | 0.11 |
| 100 | 0.02 | 0.07 | 0.12 | 0.15 | 0.17 | 0.24 |
| 200 | 0.02 | 0.07 | 0.13 | 0.16 | 0.18 | 0.27 |
| 1000 | 0.03 | 0.09 | 0.15 | 0.19 | 0.21 | · |

Table 6.23: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 1.1B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 200 | 500 | 1000 | 2000 | 4000 |
|---|---|---|---|---|---|---|
| 10 | 0.01 | 0.04 | 0.06 | 0.08 | 0.10 | 0.11 |
| 100 | 0.02 | 0.07 | 0.13 | 0.19 | 0.25 | 0.27 |
| 500 | 0.03 | 0.08 | 0.16 | 0.26 | 0.35 | 0.40 |
| 1000 | 0.03 | 0.09 | 0.17 | 0.27 | 0.37 | 0.43 |
| 4000 | 0.03 | 0.09 | 0.18 | 0.28 | 0.39 | · |

Table 6.24: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t$ and $Y_t = 1.1B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $b_n$ / $a_n$ | 0 | 30 | 50 | 100 | 250 |
|---|---|---|---|---|---|
| 10 | 0.04 | 0.07 | 0.07 | 0.09 | 0.12 |
| 30 | 0.05 | 0.09 | 0.10 | 0.13 | 0.18 |
| 50 | 0.06 | 0.10 | 0.11 | 0.14 | 0.19 |
| 100 | 0.07 | 0.11 | 0.13 | 0.16 | 0.22 |
| 250 | 0.09 | 0.14 | 0.16 | 0.20 | · |

Table 6.25: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 250$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t + 0.1$ and $Y_t = B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 100 | 200 | 400 | 1000 |
|---|---|---|---|---|---|
| 10 | 0.03 | 0.07 | 0.07 | 0.08 | 0.09 |
| 100 | 0.08 | 0.15 | 0.17 | 0.18 | 0.20 |
| 200 | 0.09 | 0.16 | 0.18 | 0.20 | 0.22 |
| 400 | 0.10 | 0.18 | 0.20 | 0.22 | 0.25 |
| 1000 | 0.13 | 0.21 | 0.24 | 0.26 | · |

Table 6.26: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 1000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t + 0.1$ and $Y_t = B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

| $a_n$ \ $b_n$ | 0 | 100 | 2000 | 4000 |
|---|---|---|---|---|
| 10 | 0.04 | 0.05 | 0.05 | 0.05 |
| 100 | 0.10 | 0.13 | 0.14 | 0.14 |
| 200 | 0.13 | 0.16 | 0.17 | 0.17 |
| 500 | 0.15 | 0.18 | 0.20 | 0.20 |
| 1000 | 0.18 | 0.21 | 0.24 | 0.24 |
| 2000 | 0.22 | 0.25 | 0.28 | 0.28 |
| 4000 | 0.29 | 0.33 | 0.37 | · |

Table 6.27: Power of the modified KRS test for SD2 for the nominal value $\alpha = 0.05$, sample size $n = 4000$ and various truncation values $a_n$ and $b_n$. The alternative considered is $(X_t, Y_t)_{t\in\mathbb{Z}}$ where $X_t = A_t + 0.1$ and $Y_t = B_t$ and $(A_t)_{t\in\mathbb{Z}}$ and $(B_t)_{t\in\mathbb{Z}}$ are independent univariate GARCH(1,1) processes with the parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.13$, $\beta_1 = 0.85$. The number of Monte Carlo replications is $R = 1000$.

by a GARCH(1,1) process. From the simulation results we infer the formula $a_n = b_n = 0.5n^{\frac{3}{4}}$ for appropriate truncation values where $n$ is the sample size. In the course of this study we refer to this formula as providing the recommended truncation values.

In the power investigation we see that conditional heteroskedasticity does not have a strong impact neither on the scale nor on the location alternative. The power is low for the scale alternative if the deviation from the null hypothesis is not very large ($\sigma = 1.1$), but we get good power results for $\sigma = 2$. For the location alternative the power is high even in the case of a small distance to the null hypothesis. In this setting, the power increases strongly with increasing sample size. Altogether we suppose that the developed test is consistent for a large part of the alternative.

## 6.5 Conclusion

The goal of this chapter was to find a test of the hypothesis of non-dominance against the alternative that one random variable dominates another one in the sense of second degree stochastic dominance. An appropriate starting point for the construction of such a test procedure is the test of Kaur/ Rao/ Singh. However, this test faces the problem that in the comparison it only considers an interval with arbitrarily chosen bounds. Furthermore their test requires the data to be contemporaneously and serially independent. If we replace the bounded interval by the set of all real numbers, the test completely loses its power. We try to remedy this problem by a truncation of the range over which we take the infimum. If the truncations grow more slowly than the sample size, the test asymptotically keeps the size. Furthermore, we modify the test statistic in a way that the test can be applied to strongly mixing processes.

In a Monte Carlo study, we investigate the question of the appropriate truncation for finite samples. We find that for sample size $n$ the truncation values

$a_n = b_n = 0.5n^{\frac{3}{4}}$ are an appropriate choice in order to keep the size of the test. Furthermore the test has good power properties for this choice.

An open question is how the test performs if the distributions differ from the situation we analyzed by simulation. This is an interesting question requiring further research. However, we still have the result that the test at least asymptotically keeps the size under certain conditions. Furthermore it is of interest whether there are truncation values which would make the test consistent.

In the next chapter, we analyze if there are any stochastic dominance relations among the daily returns of the stocks in the German stock index DAX. In addition to the modified KRS test which we designed in this chapter, we apply the modified versions of the ST and LMW tests developed in chapter 5.

# Chapter 7

# Empirical Application: Testing for Stochastic Dominance in German Stock Returns

## 7.1 Methodology

There are various fields of economics in which stochastic dominance decision rules are of use, e.g. in social welfare theory and financial economics. In social welfare theory, stochastic dominance is a criterion for comparing two income distributions without making strict assumptions concerning the social welfare function. In financial economics, the stochastic dominance decision rules make an assertion whether the return distribution of one asset is preferred to another one by a specified group of decision makers.

In previous chapters, we developed various tests for stochastic dominance. In chapter 5, we analyzed and modified some tests of the null hypothesis $H_0 : X \succeq_i Y$ $(i = 1, 2)$ which means that $X$ dominates $Y$ in the sense of $i$th degree stochastic dominance (SD$i$) against the alternative $H_1 : X \not\succeq_i Y$. It turns out that modified versions of the Linton/ Maasoumi/ Whang (LMWm) and of the Schmid/

Trede (STm) tests asymptotically keep the size and have good power properties even if conditional heteroskedasticity is prevalent in the data. In chapter 6 we developed a test where the hypotheses are reversed to the case described above. The modified test of Kaur/ Rao/ Singh (KRSm) asymptotically keeps the size and has good power properties. It can be applied to strongly mixing processes, in particular to time series featuring conditional heteroskedasticity. The main contribution of this test is that it can significantly assert SD2.

In this chapter, we analyze the stochastic dominance relations among the daily returns of the 30 stocks contained in the German stock index DAX. First we investigate whether stochastic dominance can be established in a descriptive sense, as described in chapter 2.2. In order to get significant results for the hypothesis of dominance against the alternative of non-dominance, we apply the LMWm test for SD1 and the STm test for SD2 to the data. Furthermore we test the hypothesis of non-SD2 against the alternative of SD2 using the KRSm test.

From the obtained test results we determine the *efficient* sets. For a given set $\mathcal{A}$ of random variables, the *efficient* set is the subset of random variables which are not dominated by another random variable in $\mathcal{A}$. If the distributions of the random variables are known, the determination of stochastic dominance and therefore of the efficient set is straightforward. The same holds for the determination of the efficient set if we compare the empirical distributions in a descriptive sense, as described in chapter 2.2.

However, if we infer stochastic dominance relations using tests, things are more involved. First we have a look at the LMWm and STm tests where stochastic dominance is the null hypothesis. It can – and does – happen that for some random variables $X$ and $Y$ stochastic dominance is not rejected in neither direction, i.e. neither $X \succeq_i Y$ nor $Y \succeq_i X$ are rejected. We proceed as follows. SD$i$ of $X$ against $Y$ is established if and only if $Y \succeq_i X$ is rejected whereas $X \succeq_i Y$ is not and $X$ has a larger mean than $Y$. The LMWm and STm efficient sets consist of the stocks which are not dominated in this sense. The condition concerning the means is required in order to prevent the paradox result that a stock dominates another one with larger mean. A necessary condition for stochastic dominance of any degree is that the mean of the dominant random variable has to be at least as large as the mean of the dominated one.

Note that we have to take the efficiency results for these tests with a pinch of salt. The considered tests do not assert significantly that one random variable dominates another one, instead, the tests do or do not reject the hypothesis of dominance. In many cases the empirical distributions are very close to each other, therefore the test can often not reject stochastic dominance.

In the following we consider the KRSm test where stochastic dominance is the alternative. In some comparisons, the null hypothesis of non-SD2 is rejected and therefore the alternative of SD2 significantly asserted. The KRSm efficient set consists of the stocks which are not significantly dominated by another stock. Due to the fact that the KRSm test significantly confirms stochastic dominance, the identification of the efficient set is more justifiable for this test than for the LMWm and the STm tests.

Nevertheless, we have to be aware of the fact that even a dominated stock can be a useful member of a portfolio. Diversification diminishes risk, and this effect can be stronger than the one caused by stochastic dominance. Hence, in many cases, a dominated stock should not be eliminated from a portfolio. Post (2003) and Kuosmanen (2004) study the problem of stochastic dominance efficiency of a portfolio.

## 7.2 Data

In our study we examine the daily returns of the 30 stocks listed in the German stock index DAX. The return at day $t$ is defined by $r_t = \ln(\frac{p_t}{p_{t-1}})$ where $p_t$ is the daily spot stock price. We consider the 10-year period between 16 September 1994 and 15 September 2004 and the 1-year period between 16 September 2003 and 15 September 2004. The data are taken from Datastream. There are 2,522 observations for the 10-year period and 255 observations for the 1-year period. For the 10-year period we only consider the 22 stocks which were listed in the DAX for the entire period.

Tables 7.1 and 7.2 display some descriptive statistics of the stock returns. More precisely, they present the annualized means and standard deviations, skewness and kurtosis for all stocks and both periods considered in this study. We annualize the empirical mean $\bar{x}$ and standard deviation $s$ by multiplying them by $d$ and $\sqrt{d}$, respectively, where $d$ is the average number of trading days per year. For the 1-year period we have $d = 255$, for the 10-year period $d = 252.2$. The skewness and kurtosis are the empirical third and fourth central moments, respectively, divided by $s^3$ and $s^4$, respectively.

In tables 7.1 and 7.2, the abbreviations of the companies, which are also used in the other tables, are specified. 8 out of 30 stocks have negative mean returns in the 1-year period, this number diminishes to 4 out of 22 for the 10-year period. The stocks of the travel agency TUI had the strongest decline which might have been caused by the tourism crisis after 11 September 2001. Most of the standard deviations range between 20 and 30 percent for the 1-year period and between 30 and 40 percent for the 10-year period. The majority of the stocks have positive skewed daily returns, i.e. the third central moment is positive. This holds for both periods considered in this study, but in many cases the absolute value of the skewness is not large. However, for the kurtosis we see a stronger tendency. Remember that the kurtosis of a normally distributed random variable is 3. If the kurtosis is larger than 3, the distribution is *leptokurtic*. As we see in tables 7.1 and 7.2, in the 10-year period all stocks have leptokurtic daily returns whereas in the 1-year period the only exception is Infineon whose returns have a kurtosis of 2.838. In the 1-year period, SAP has the largest kurtosis (10.116), Bayer has the largest kurtosis (26.805) in the 10-year period. These results suggest that the returns have fat tails, i.e. the probability of very large and of very small returns is larger than it would be in the case of a normal distribution.

As mentioned in the previous chapters, financial data feature contemporaneous correlation and conditional heteroskedasticity. The estimated correlations are summarized in tables 7.3 and 7.4 for the 1-year period and in tables 7.5 and 7.6 for the 10-year period. The stocks are all positively correlated with each other, the correlations range from 0.11 to 0.76 for the 1-year period and from 0.15 to 0.74 for the 10-year period. The estimates for the parameters $\alpha_1$ and $\beta_1$ in the GARCH model $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ are presented in table 7.7 for the 1-year and the 10-year periods. For many stocks the sum $\alpha_1 + \beta_1$ is close to unity. This phenomenon is stronger for the 10-year period.

## 7.3 Results

We start the investigation of stochastic dominance with a descriptive comparison. $X$ descriptively dominates $Y$ in the sense of SD$i$ if for the empirical distribution functions $\hat{F}_{X,n}^{(i)}(x) \leq \hat{F}_{Y,n}^{(i)}(x)$ holds for all $x \in \mathbb{R}$. Descriptively, no dominance relationship in the sense of SD1 can be established between any pair of return series, neither for the 1-year nor for the 10-year period. Every pair of empirical distribution functions crosses at least once. Hence all stocks are SD1 efficient in a descriptive sense.

Concerning SD2 the findings are different. The results are reported in tables 7.8 and 7.9 for the 1-year period and in table 7.10 for the 10-year period. In the descriptive sense, SD2 can be established in 187 of 870 comparisons for the 1-year period and in 91 of 462 comparisons for the 10-year period. For the 1-year period there are only 4 out of 30 stocks which are efficient: Adidas-Salomon, BASF, Continental, RWE. 8 out of 22 stocks are efficient for the 10-year period: Altana, BASF, Continental, Eon, Henkel, RWE, SAP, Schering. Table 7.20 displays the descriptive efficiency results; see the second column for each period. For a larger sample size, a descriptive dominance relationship is harder to establish. Hence the fraction of the SD2 efficient stocks becomes larger the longer the period is. The problem of too large efficient sets has already been mentioned by Nelson/ Pope (1991) and Stein/ Pfaffenberger/ Kumar (1983).

In order to get significant results concerning stochastic dominance we apply the test which we developed in this study to the data. First we apply the tests which test the null hypothesis of dominance against the alternative of non-dominance. We use the LMWm test for SD1 and the STm test for SD2. For the 1-year period we choose the block length $b = 150$ for both tests. Due to the fact that $b = 300$ is a good choice for the LMWm test and both $n = 1000$ and $n = 4000$, we choose $b = 300$ for the application of the LMWm test to the 10-year period. For this period we choose $b = 500$ for the STm test because the simulations show that this is an appropriate block length for $n = 2500$. Furthermore $b = 500$ is the recommended block length if we apply the rule $b(n) = 10\sqrt{n}$ for the appropriate block length. The MATLAB programs used are given in the appendix.

Tables 7.11, 7.12, 7.13 and 7.14 display the test results for the 1-year period, tables 7.17 and 7.18 for the 10-year period. We find that in most cases dominance cannot be rejected at a size of $\alpha = 0.05$. For many comparisons this holds in both directions. For instance, the application of the LMWm test for SD1 for the 1-year period (see table 7.11) yields no rejection for dominance of Adidas-Salomon against Allianz and vice versa. SD1 cannot be rejected in 777 out of 870 comparisons for the 1-year period and in 361 out of 462 comparisons for the 10-year period, SD2 is not rejected in 665 out of 870 comparisons for the 1-year period and in 365 out of 462 comparisons for the 10-year period. These findings suggest that in many cases the empirical distributions are very close to each other, hence stochastic dominance cannot be rejected in either direction.

From the test results we determine the efficient sets, i.e. the sets of non-dominated stocks, as described above. Table 7.20 summarizes the results. For the 1-year period Adidas-Salomon, Continental, Eon and RWE are in the SD1 and SD2 efficient sets whereas Altana, Deutsche Bank, Deutsche Post, Metro and Siemens are only in the SD1 efficient set. The SD1 efficient set for the 10-year period consists of Altana, BASF, Continental, Henkel, SAP, Schering and Siemens. These stocks are also contained in the SD2 efficient set, as well as Eon, Linde and RWE. It seems to be paradox that for the 10-year period some stocks are found to be SD2 efficient, but not SD1 efficient. Indeed, SD2 is a necessary condition for SD1. Hence the SD2 efficient set is a subset of the SD1 efficient set. This is a contradiction to the result we get when applying the tests to the data of the 10-year period. The reason for this finding is the fact that the tests are affected by sampling errors.

Furthermore we apply the KRSm test for SD2 which tests the null hypothesis of non-dominance against the alternative of dominance. In chapter 6, we recommended the truncation values $a_n = b_n = 0.5n^{\frac{3}{4}}$ for the sample size $n$. This yields $a_n = b_n \approx 32$ for $n = 255$ (1-year period) and $a_n = b_n \approx 178$ for $n = 2522$. We choose these truncation values in our empirical analysis.

Table 7.15 displays the test results for the 1-year period, tables 7.16 and 7.19 for the 10-year period. In the vast majority of the pairwise comparisons the null hypothesis of non-dominance cannot be rejected. SD2 is significantly confirmed in 70 out of 870 comparisons for the 1-year period and in 15 out of 462 comparisons for the 10-year period. The reason for this finding is that in many cases the distributions are very close to each other, hence second degree stochastic dominance cannot be confirmed in either direction. But there are still some comparisons where stochastic dominance is significantly confirmed by the KRSm test which is a stronger assertion than the establishment of stochastic dominance in a descriptive comparison.

As described above, we determine the efficient sets from the test results. Table 7.20 displays the results. Due to the fact that in most of the comparisons non-dominance cannot be rejected, the efficient sets are large. For the 1-year period, 18 out of 30 stocks are not dominated by another stock and therefore belong to the efficient set. For the 10-year period, this holds for 16 out of 22 stocks. As we see in table 7.20, the SD2 efficient set according to the STm test is a proper subset of the SD2 efficient set according to the KRSm test. This holds for the 1-year period and for the 10-year period.

In this chapter, we investigated the stochastic dominance relations among daily returns of the DAX30 stocks. In a descriptive analysis SD1 is rejected in every case whereas SD2 is confirmed in some cases. In order to get significant results, we applied the tests which we developed in previous chapters to the data. We use the modified tests of Linton/ Maasoumi/ Whang for SD1 and of Schmid/ Trede for SD2 in order to test the hypothesis of dominance against the alternative of non-dominance. Conversely, we test the hypothesis of non-dominance against the alternative of dominance using the modified test of Kaur/ Rao/ Singh for SD2. In many comparisons neither SD$k$ ($k = 1, 2$) nor non-SD2 can be rejected by the developed tests. The reason for this finding is that in many cases the empirical distributions of the different stock returns are close to each other. We determined the efficient sets, i.e. the sets of stocks which are not dominated by another stock. The LMWm and STm tests yield small efficient sets. However, in these tests stochastic dominance is not significantly confirmed. In contrast, in the KRSm test stochastic dominance is the alternative. We get larger efficient sets, but some stocks are significantly dominated by another stock.

| Name of Stock | | Mean ×100 | Std.dev. ×100 | Skewness | Kurtosis |
|---|---|---|---|---|---|
| Adidas-Salomon | ADS | 37.102 | 19.653 | 0.313 | 4.523 |
| Allianz | ALL | 1.428 | 27.287 | 0.031 | 3.591 |
| Altana | ALT | −13.082 | 22.955 | 0.148 | 5.354 |
| BASF | BAS | 10.990 | 20.491 | 0.209 | 4.247 |
| Bayer | BAY | 9.588 | 26.676 | 0.176 | 4.549 |
| Bay. Hypo-Vereinsbank | BHV | 9.869 | 37.097 | −0.112 | 4.732 |
| BMW | BMW | −2.270 | 23.468 | −0.145 | 4.309 |
| Commerzbank | CBK | 9.384 | 29.970 | 0.186 | 3.562 |
| Continental | CON | 63.495 | 26.080 | −0.120 | 3.227 |
| Daimler-Chrysler | DAC | 4.386 | 24.242 | 0.161 | 3.903 |
| Deutsche Bank | DBK | 4.208 | 24.454 | 0.516 | 5.147 |
| Deutsche Boerse | DBO | −6.987 | 23.102 | −0.720 | 9.405 |
| Deutsche Lufthansa | DLH | −15.810 | 28.634 | 0.099 | 3.546 |
| Deutsche Post | DPO | 10.685 | 26.732 | −0.054 | 3.420 |
| Deutsche Telekom | DTL | 8.772 | 20.466 | −0.087 | 3.111 |
| Eon | EON | 27.285 | 19.744 | −0.284 | 3.989 |
| Fresenius | FRE | 17.162 | 21.091 | 0.166 | 3.617 |
| Henkel | HEN | 4.080 | 20.561 | −0.395 | 5.092 |
| Infineon | INF | −41.718 | 35.534 | −0.112 | 2.838 |
| Linde | LIN | 17.162 | 22.209 | 0.199 | 3.968 |
| MAN | MAN | 32.334 | 30.329 | −0.068 | 4.165 |
| Metro | MET | 17.416 | 25.109 | 0.288 | 4.534 |
| Muenchner Rueckvers. | MRV | −11.271 | 26.067 | 0.332 | 4.841 |
| RWE | RWE | 46.308 | 23.150 | 0.070 | 3.485 |
| SAP | SAP | 11.297 | 31.008 | 1.200 | 10.116 |
| Schering | SCH | 22.797 | 21.511 | 0.170 | 4.703 |
| Siemens | SIE | 9.894 | 25.184 | −0.215 | 4.160 |
| Thyssen-Krupp | TYK | 17.416 | 28.330 | −0.136 | 3.590 |
| TUI | TUI | −5.075 | 32.008 | 0.533 | 4.260 |
| Volkswagen | VW | −28.229 | 24.105 | 0.024 | 3.674 |

Table 7.1: Descriptive statistics of the *annualized* daily returns of DAX stocks for the 1-year period.

| Name of Stock | | Mean ×100 | Std.dev. ×100 | Skewness | Kurtosis |
|---|---|---|---|---|---|
| Allianz | ALL | −2.093 | 36.778 | −0.016 | 7.546 |
| Altana | ALT | 16.065 | 36.972 | −0.408 | 13.763 |
| BASF | BAS | 10.441 | 28.955 | 0.039 | 5.238 |
| Bayer | BAY | 1.639 | 34.344 | 0.881 | 26.805 |
| Bay. Hypo-Vereinsbank | BHV | −1.412 | 41.449 | 0.097 | 7.738 |
| BMW | BMW | 10.113 | 35.729 | −0.070 | 6.676 |
| Commerzbank | CBK | −0.530 | 35.098 | 0.146 | 8.596 |
| Continental | CON | 13.114 | 32.427 | 0.072 | 6.296 |
| Deutsche Bank | DBK | 5.321 | 33.996 | −0.101 | 6.065 |
| Deutsche Lufthansa | DLH | 0.378 | 36.791 | −0.024 | 6.385 |
| Eon | EON | 7.566 | 29.132 | 0.106 | 5.355 |
| Henkel | HEN | 7.188 | 29.902 | 0.057 | 6.722 |
| Linde | LIN | 1.009 | 29.750 | 0.004 | 5.722 |
| MAN | MAN | 2.926 | 34.809 | 0.052 | 5.224 |
| Muenchner Rueckvers. | MRV | 2.623 | 38.763 | −0.096 | 8.217 |
| RWE | RWE | 5.120 | 29.702 | 0.323 | 6.130 |
| SAP | SAP | 21.992 | 51.177 | 0.068 | 8.983 |
| Schering | SCH | 11.324 | 29.480 | −0.053 | 6.023 |
| Siemens | SIE | 9.685 | 36.505 | 0.079 | 5.535 |
| Thyssen-Krupp | TYK | 0.303 | 34.034 | −0.036 | 6.391 |
| TUI | TUI | −4.111 | 35.894 | 0.098 | 7.286 |
| Volkswagen | VW | 3.354 | 34.490 | −0.273 | 5.622 |

Table 7.2: Descriptive statistics of the *annualized* daily returns of DAX stocks for the 10-year period.

| | ADS | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DAC | DBK | DBO | DLH | DPO | DTL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | | 36 | 18 | 37 | 34 | 30 | 27 | 33 | 29 | 32 | 34 | 19 | 25 | 25 | 39 |
| ALL | 36 | | 27 | 64 | 63 | 61 | 62 | 64 | 52 | 67 | 66 | 33 | 63 | 52 | 66 |
| ALT | 18 | 27 | | 17 | 17 | 22 | 25 | 22 | 15 | 18 | 23 | 13 | 20 | 12 | 19 |
| BAS | 37 | 64 | 17 | | 71 | 46 | 59 | 52 | 50 | 60 | 62 | 24 | 50 | 50 | 63 |
| BAY | 34 | 63 | 17 | 71 | | 46 | 56 | 52 | 52 | 54 | 56 | 20 | 50 | 42 | 61 |
| BHV | 30 | 61 | 22 | 46 | 46 | | 46 | 67 | 43 | 50 | 52 | 31 | 48 | 43 | 49 |
| BMW | 27 | 62 | 25 | 59 | 56 | 46 | | 49 | 56 | 69 | 53 | 22 | 52 | 40 | 51 |
| CBK | 33 | 64 | 22 | 52 | 52 | 67 | 49 | | 46 | 52 | 60 | 33 | 49 | 49 | 56 |
| CON | 29 | 52 | 15 | 50 | 52 | 43 | 56 | 46 | | 55 | 46 | 14 | 42 | 43 | 48 |
| DAC | 32 | 67 | 18 | 60 | 54 | 50 | 69 | 52 | 55 | | 54 | 26 | 53 | 39 | 53 |
| DBK | 34 | 66 | 23 | 62 | 56 | 52 | 53 | 60 | 46 | 54 | | 32 | 49 | 53 | 57 |
| DBO | 19 | 33 | 13 | 24 | 20 | 31 | 22 | 33 | 14 | 26 | 32 | | 32 | 31 | 32 |
| DLH | 25 | 63 | 20 | 50 | 50 | 48 | 52 | 49 | 42 | 53 | 49 | 32 | | 49 | 52 |
| DPO | 25 | 52 | 12 | 50 | 42 | 43 | 40 | 49 | 43 | 39 | 53 | 31 | 49 | | 49 |
| DTL | 39 | 66 | 19 | 63 | 61 | 49 | 51 | 56 | 48 | 53 | 57 | 32 | 52 | 49 | |
| EON | 27 | 49 | 27 | 61 | 50 | 41 | 47 | 44 | 43 | 46 | 46 | 21 | 42 | 39 | 47 |
| FRE | 24 | 32 | 26 | 40 | 32 | 27 | 36 | 33 | 31 | 37 | 29 | 15 | 27 | 28 | 29 |
| HEN | 27 | 44 | 27 | 35 | 38 | 34 | 37 | 39 | 34 | 35 | 39 | 19 | 35 | 36 | 30 |
| INF | 31 | 55 | 25 | 49 | 47 | 50 | 44 | 49 | 32 | 45 | 52 | 29 | 55 | 45 | 52 |
| LIN | 33 | 49 | 19 | 52 | 48 | 40 | 47 | 48 | 39 | 47 | 50 | 31 | 47 | 46 | 44 |
| MAN | 26 | 58 | 16 | 51 | 53 | 41 | 60 | 46 | 46 | 51 | 47 | 28 | 58 | 42 | 49 |
| MET | 27 | 58 | 25 | 58 | 46 | 47 | 46 | 48 | 44 | 50 | 48 | 22 | 43 | 44 | 46 |
| MRV | 36 | 76 | 21 | 63 | 61 | 58 | 54 | 63 | 49 | 60 | 58 | 25 | 56 | 46 | 60 |
| RWE | 34 | 48 | 19 | 52 | 44 | 37 | 45 | 38 | 37 | 42 | 45 | 22 | 37 | 42 | 46 |
| SAP | 28 | 50 | 12 | 50 | 51 | 46 | 46 | 47 | 37 | 43 | 50 | 30 | 47 | 44 | 52 |
| SCH | 21 | 31 | 36 | 30 | 35 | 33 | 33 | 39 | 34 | 30 | 28 | 14 | 21 | 21 | 29 |
| SIE | 39 | 74 | 24 | 70 | 66 | 55 | 61 | 59 | 53 | 63 | 66 | 31 | 61 | 54 | 65 |
| TYK | 37 | 66 | 19 | 58 | 55 | 56 | 60 | 52 | 58 | 60 | 52 | 17 | 58 | 49 | 58 |
| TUI | 30 | 56 | 11 | 49 | 46 | 38 | 46 | 41 | 39 | 48 | 46 | 18 | 52 | 44 | 54 |
| VW | 33 | 65 | 20 | 59 | 57 | 55 | 75 | 50 | 61 | 69 | 59 | 27 | 57 | 49 | 51 |

Table 7.3: Correlation coefficients ($\times$ 100) of the daily returns of DAX stocks for the 1-year period.

| | EON | FRE | HEN | INF | LIN | MAN | MET | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | 27 | 24 | 27 | 31 | 33 | 26 | 27 | 36 | 34 | 28 | 21 | 39 | 37 | 30 | 33 |
| ALL | 49 | 32 | 44 | 55 | 49 | 58 | 58 | 76 | 48 | 50 | 31 | 74 | 66 | 56 | 65 |
| ALT | 27 | 26 | 27 | 25 | 19 | 16 | 25 | 21 | 19 | 12 | 36 | 24 | 19 | 11 | 20 |
| BAS | 61 | 40 | 35 | 49 | 52 | 51 | 58 | 63 | 52 | 50 | 30 | 70 | 58 | 49 | 59 |
| BAY | 50 | 32 | 38 | 47 | 48 | 53 | 46 | 61 | 44 | 51 | 35 | 66 | 55 | 46 | 57 |
| BHV | 41 | 27 | 34 | 50 | 40 | 41 | 47 | 58 | 37 | 46 | 33 | 55 | 56 | 38 | 55 |
| BMW | 47 | 36 | 37 | 44 | 47 | 60 | 46 | 54 | 45 | 46 | 33 | 61 | 60 | 46 | 75 |
| CBK | 44 | 33 | 39 | 49 | 48 | 46 | 48 | 63 | 38 | 47 | 39 | 59 | 52 | 41 | 50 |
| CON | 43 | 31 | 34 | 32 | 39 | 46 | 44 | 49 | 37 | 37 | 34 | 53 | 58 | 39 | 61 |
| DAC | 46 | 37 | 35 | 45 | 47 | 51 | 50 | 60 | 42 | 43 | 30 | 63 | 60 | 48 | 69 |
| DBK | 46 | 29 | 39 | 52 | 50 | 47 | 48 | 58 | 45 | 50 | 28 | 66 | 52 | 46 | 59 |
| DBO | 21 | 15 | 19 | 29 | 31 | 28 | 22 | 25 | 22 | 30 | 14 | 31 | 17 | 18 | 27 |
| DLH | 42 | 27 | 35 | 55 | 47 | 58 | 43 | 56 | 37 | 47 | 21 | 61 | 58 | 52 | 57 |
| DPO | 39 | 28 | 36 | 45 | 46 | 42 | 44 | 46 | 42 | 44 | 21 | 54 | 49 | 44 | 49 |
| DTL | 47 | 29 | 30 | 52 | 44 | 49 | 46 | 60 | 46 | 52 | 29 | 65 | 58 | 54 | 51 |
| EON | | 42 | 41 | 33 | 46 | 36 | 45 | 53 | 69 | 34 | 36 | 46 | 52 | 44 | 49 |
| FRE | 42 | | 23 | 26 | 25 | 30 | 30 | 29 | 32 | 30 | 31 | 34 | 37 | 29 | 39 |
| HEN | 41 | 23 | | 30 | 45 | 41 | 38 | 45 | 32 | 29 | 33 | 41 | 37 | 26 | 41 |
| INF | 33 | 26 | 30 | | 39 | 45 | 42 | 53 | 29 | 66 | 25 | 67 | 50 | 43 | 49 |
| LIN | 46 | 25 | 45 | 39 | | 48 | 44 | 47 | 43 | 42 | 36 | 51 | 49 | 41 | 50 |
| MAN | 36 | 30 | 41 | 45 | 48 | | 39 | 49 | 37 | 47 | 28 | 61 | 59 | 44 | 59 |
| MET | 45 | 30 | 38 | 42 | 44 | 39 | | 52 | 37 | 36 | 29 | 54 | 48 | 36 | 50 |
| MRV | 53 | 29 | 45 | 53 | 47 | 49 | 52 | | 47 | 47 | 29 | 66 | 58 | 47 | 59 |
| RWE | 69 | 32 | 32 | 29 | 43 | 37 | 37 | 47 | | 33 | 32 | 41 | 45 | 39 | 45 |
| SAP | 34 | 30 | 29 | 66 | 42 | 47 | 36 | 47 | 33 | | 26 | 63 | 47 | 42 | 49 |
| SCH | 36 | 31 | 33 | 25 | 36 | 28 | 29 | 29 | 32 | 26 | | 34 | 25 | 25 | 33 |
| SIE | 46 | 34 | 41 | 67 | 51 | 61 | 54 | 66 | 41 | 63 | 34 | | 67 | 52 | 66 |
| TYK | 52 | 37 | 37 | 50 | 49 | 59 | 48 | 58 | 45 | 47 | 25 | 67 | | 52 | 61 |
| TUI | 44 | 29 | 26 | 43 | 41 | 44 | 36 | 47 | 39 | 42 | 25 | 52 | 52 | | 48 |
| VW | 49 | 39 | 41 | 49 | 50 | 59 | 50 | 59 | 45 | 49 | 33 | 66 | 61 | 48 | |

Table 7.4: Correlation coefficients (× 100) of the daily returns of DAX stocks for the 1-year period.

| | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DBK | DLH | EON |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | | 23 | 49 | 48 | 58 | 45 | 58 | 37 | 61 | 46 | 39 |
| ALT | 23 | | 23 | 21 | 19 | 19 | 21 | 17 | 23 | 17 | 21 |
| BAS | 49 | 23 | | 68 | 44 | 50 | 46 | 37 | 51 | 48 | 41 |
| BAY | 48 | 21 | 68 | | 41 | 45 | 44 | 34 | 48 | 41 | 37 |
| BHV | 58 | 19 | 44 | 41 | | 40 | 62 | 32 | 59 | 40 | 32 |
| BMW | 45 | 19 | 50 | 45 | 40 | | 45 | 40 | 48 | 44 | 36 |
| CBK | 58 | 21 | 46 | 44 | 62 | 45 | | 36 | 64 | 45 | 34 |
| CON | 37 | 17 | 37 | 34 | 32 | 40 | 36 | | 37 | 36 | 29 |
| DBK | 61 | 23 | 51 | 48 | 59 | 48 | 64 | 37 | | 47 | 39 |
| DLH | 46 | 17 | 48 | 41 | 40 | 44 | 45 | 36 | 47 | | 33 |
| EON | 39 | 21 | 41 | 37 | 32 | 36 | 34 | 29 | 39 | 33 | |
| HEN | 31 | 18 | 39 | 37 | 29 | 37 | 30 | 31 | 32 | 32 | 31 |
| LIN | 39 | 19 | 45 | 43 | 35 | 41 | 39 | 32 | 40 | 37 | 31 |
| MAN | 43 | 17 | 44 | 42 | 40 | 41 | 43 | 35 | 44 | 41 | 30 |
| MRV | 74 | 25 | 47 | 45 | 53 | 44 | 53 | 35 | 55 | 41 | 39 |
| RWE | 41 | 24 | 40 | 40 | 35 | 37 | 34 | 26 | 39 | 31 | 58 |
| SAP | 37 | 19 | 34 | 32 | 32 | 31 | 38 | 25 | 42 | 34 | 23 |
| SCH | 35 | 24 | 38 | 38 | 31 | 35 | 33 | 25 | 35 | 29 | 33 |
| SIE | 52 | 23 | 47 | 44 | 44 | 44 | 48 | 36 | 56 | 43 | 34 |
| TYK | 42 | 17 | 46 | 45 | 37 | 44 | 42 | 33 | 43 | 39 | 32 |
| TUI | 45 | 15 | 43 | 38 | 38 | 41 | 43 | 32 | 45 | 44 | 32 |
| VW | 48 | 20 | 54 | 48 | 44 | 61 | 48 | 44 | 52 | 48 | 40 |

Table 7.5: Correlation coefficients ($\times$ 100) of the daily returns of DAX stocks for the 10-year period.

| | HEN | LIN | MAN | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | 31 | 39 | 43 | 74 | 41 | 37 | 35 | 52 | 42 | 45 | 48 |
| ALT | 18 | 19 | 17 | 25 | 24 | 19 | 24 | 23 | 17 | 15 | 20 |
| BAS | 39 | 45 | 44 | 47 | 40 | 34 | 38 | 47 | 46 | 43 | 54 |
| BAY | 37 | 43 | 42 | 45 | 40 | 32 | 38 | 44 | 45 | 38 | 48 |
| BHV | 29 | 35 | 40 | 53 | 35 | 32 | 31 | 44 | 37 | 38 | 44 |
| BMW | 37 | 41 | 41 | 44 | 37 | 31 | 35 | 44 | 44 | 41 | 61 |
| CBK | 30 | 39 | 43 | 53 | 34 | 38 | 33 | 48 | 42 | 43 | 48 |
| CON | 31 | 32 | 35 | 35 | 26 | 25 | 25 | 36 | 33 | 32 | 44 |
| DBK | 32 | 40 | 44 | 55 | 39 | 42 | 35 | 56 | 43 | 45 | 52 |
| DLH | 32 | 37 | 41 | 41 | 31 | 34 | 29 | 43 | 39 | 44 | 48 |
| EON | 31 | 31 | 30 | 39 | 58 | 23 | 33 | 34 | 32 | 32 | 40 |
| HEN | | 35 | 33 | 32 | 32 | 19 | 28 | 28 | 30 | 27 | 36 |
| LIN | 35 | | 44 | 38 | 34 | 28 | 30 | 36 | 43 | 39 | 43 |
| MAN | 33 | 44 | | 40 | 32 | 32 | 28 | 44 | 45 | 40 | 45 |
| MRV | 32 | 38 | 40 | | 42 | 32 | 36 | 46 | 39 | 41 | 47 |
| RWE | 32 | 34 | 32 | 42 | | 23 | 34 | 35 | 32 | 33 | 37 |
| SAP | 19 | 28 | 32 | 32 | 23 | | 25 | 50 | 29 | 30 | 38 |
| SCH | 28 | 30 | 28 | 36 | 34 | 25 | | 31 | 28 | 31 | 33 |
| SIE | 28 | 36 | 44 | 46 | 35 | 50 | 31 | | 41 | 42 | 49 |
| TYK | 30 | 43 | 45 | 39 | 32 | 29 | 28 | 41 | | 43 | 45 |
| TUI | 27 | 39 | 40 | 41 | 33 | 30 | 31 | 42 | 43 | | 44 |
| VW | 36 | 43 | 45 | 47 | 37 | 38 | 33 | 49 | 45 | 44 | |

Table 7.6: Correlation coefficients (× 100) of the daily returns of DAX stocks for the 10-year period.

| Name of Stock | | 1-year period | | 10-year period | |
|---|---|---|---|---|---|
| | | $\alpha_1$ | $\beta_1$ | $\alpha_1$ | $\beta_1$ |
| Adidas-Salomon | ADS | 0.0000 | 0.0051 | | |
| Allianz | ALL | 0.1131 | 0.5529 | 0.1037 | 0.8921 |
| Altana | ALT | 0.2275 | 0.4364 | 0.0561 | 0.9439 |
| BASF | BAS | 0.0802 | 0.8882 | 0.0991 | 0.8741 |
| Bayer | BAY | 0.0827 | 0.7675 | 0.0713 | 0.9193 |
| Bay. Hypo-Vereinsbank | BHV | 0.0861 | 0.8268 | 0.1290 | 0.8710 |
| BMW | BMW | 0.0732 | 0.8407 | 0.0927 | 0.9073 |
| Commerzbank | CBK | 0.1038 | 0.8351 | 0.1469 | 0.8530 |
| Continental | CON | 0.0847 | 0.8184 | 0.0828 | 0.8806 |
| Daimler-Chrysler | DAC | 0.0870 | 0.8453 | | |
| Deutsche Bank | DBK | 0.0207 | 0.7580 | 0.0909 | 0.9070 |
| Deutsche Boerse | DBO | 0.0503 | 0.0000 | | |
| Deutsche Lufthansa | DLH | 0.0536 | 0.7385 | 0.0655 | 0.9180 |
| Deutsche Post | DPO | 0.0287 | 0.9681 | | |
| Deutsche Telekom | DTL | 0.0375 | 0.8183 | | |
| Eon | EON | 0.0185 | 0.9781 | 0.0793 | 0.9118 |
| Fresenius | FRE | 0.0154 | 0.9816 | | |
| Henkel | HEN | 0.0776 | 0.7292 | 0.0681 | 0.9188 |
| Infineon | INF | 0.0495 | 0.8289 | | |
| Linde | LIN | 0.0621 | 0.8004 | 0.0563 | 0.9390 |
| MAN | MAN | 0.0977 | 0.7602 | 0.0538 | 0.9381 |
| Metro | MET | 0.0912 | 0.8217 | | |
| Muenchner Rueckvers. | MRV | 0.1203 | 0.7791 | 0.1013 | 0.8944 |
| RWE | RWE | 0.1148 | 0.3936 | 0.0694 | 0.9249 |
| SAP | SAP | 0.1187 | 0.7656 | 0.1603 | 0.7917 |
| Schering | SCH | 0.0642 | 0.8161 | 0.0700 | 0.9092 |
| Siemens | SIE | 0.0000 | 0.9991 | 0.0554 | 0.9435 |
| Thyssen-Krupp | TYK | 0.0293 | 0.9670 | 0.0937 | 0.9001 |
| TUI | TUI | 0.1079 | 0.4465 | 0.0935 | 0.9044 |
| Volkswagen | VW | 0.0337 | 0.9288 | 0.0731 | 0.9180 |

Table 7.7: Estimated GARCH parameters of the daily returns of DAX stocks for the 1-year and the 10-year periods.

| | ADS | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DAC | DBK | DBO | DLH | DPO | DTL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALL | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ALT | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| BAS | 1 | 0 | 0 | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| BAY | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BHV | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BMW | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| CBK | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| CON | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | | 1 | 1 | 1 | 1 | 0 | 1 |
| DAC | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 0 | 1 | 1 |
| DBK | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 0 | 1 | 1 |
| DBO | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 |
| DLH | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 |
| DPO | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 |
| DTL | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | |
| EON | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| FRE | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| HEN | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| INF | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| LIN | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| MAN | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| MET | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| MRV | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| RWE | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| SAP | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| SCH | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| SIE | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| TYK | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| TUI | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| VW | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 7.8: Descriptive results for stochastic dominance (2nd order) of the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of dominance).

| | EON | FRE | HEN | INF | LIN | MAN | MET | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALL | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| ALT | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BAS | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| BAY | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| BHV | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BMW | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| CBK | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| CON | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |
| DAC | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| DBK | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| DBO | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| DLH | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| DPO | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| DTL | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| EON | | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| FRE | 1 | | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| HEN | 1 | 1 | | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| INF | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| LIN | 1 | 1 | 1 | 0 | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| MAN | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| MET | 1 | 1 | 1 | 0 | 1 | 1 | | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| MRV | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| RWE | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | | 0 | 1 | 0 | 0 | 0 | 0 |
| SAP | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 0 | 1 |
| SCH | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | | 0 | 0 | 0 | 0 |
| SIE | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 |
| TYK | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 |
| TUI | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 |
| VW | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |

Table 7.9: Descriptive results for stochastic dominance (2nd order) of the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of dominance).

| | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DBK | DLH | EON | HEN | LIN | MAN | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ALT | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BAS | 0 | 1 | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| BAY | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BHV | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| BMW | 1 | 1 | 1 | 1 | 0 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| CBK | 1 | 1 | 1 | 1 | 0 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| CON | 0 | 1 | 1 | 0 | 0 | 0 | 0 | | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| DBK | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| DLH | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| EON | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| HEN | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| LIN | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| MAN | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| MRV | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| RWE | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | | 1 | 1 | 1 | 0 | 0 | 0 |
| SAP | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 |
| SCH | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | | 1 | 0 | 0 | 0 |
| SIE | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 |
| TYK | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 0 | 1 |
| TUI | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 |
| VW | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | |

Table 7.10: Descriptive results for stochastic dominance (2nd order) of the daily returns of DAX stocks for the 10-year period ($1 \sim$ rejection of dominance).

| | ADS | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DAC | DBK | DBO | DLH | DPO | DTL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALL | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 1 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAY | 1 | 0 | 0 | 0 | | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 1 | 1 | 0 | 1 | 0 | | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| BMW | 1 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| CBK | 1 | 0 | 0 | 1 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CON | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| DAC | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| DBO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | | 0 | 0 | 0 |
| DLH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 1 |
| DPO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 |
| DTL | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | |
| EON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FRE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| INF | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| MET | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| TYK | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| TUI | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| VW | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |

Table 7.11: LMWm test results on stochastic dominance (1st order) of the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of dominance).

| | EON | FRE | HEN | INF | LIN | MAN | MET | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| ALL | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| BAY | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BMW | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| CBK | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| CON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DAC | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| DBO | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DLH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DPO | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DTL | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| EON | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| FRE | 0 | | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| INF | 1 | 0 | 1 | | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| MAN | 1 | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MET | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 1 | 0 | 1 | 1 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 1 | 0 |
| TYK | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 |
| TUI | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 |
| VW | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | |

Table 7.12: LMWm results on stochastic dominance (1st order) between the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of dominance).

| | ADS | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DAC | DBK | DBO | DLH | DPO | DTL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALL | 1 | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| ALT | 1 | 0 | | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 1 | 0 | 0 | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAY | 1 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| BHV | 1 | 1 | 0 | 1 | 0 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| BMW | 1 | 1 | 0 | 1 | 0 | 0 | | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| CBK | 1 | 0 | 0 | 1 | 0 | 0 | 0 | | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| CON | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| DAC | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 0 | 0 |
| DBK | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | 0 | 0 | 0 | 0 |
| DBO | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 |
| DLH | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | | 0 | 1 |
| DPO | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | 1 |
| DTL | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | |
| EON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FRE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| INF | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| LIN | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| MET | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| RWE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| SIE | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| TYK | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| TUI | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| VW | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |

Table 7.13: STm results on stochastic dominance (2nd order) between the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of dominance).

| | EON | FRE | HEN | INF | LIN | MAN | MET | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALL | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| ALT | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAY | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| BMW | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| CBK | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| CON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DAC | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| DBK | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| DBO | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| DLH | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| DPO | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| DTL | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| EON | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| FRE | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| INF | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| LIN | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 1 | 0 | 1 | 0 | 1 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| MET | 1 | 1 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 1 | 1 | 1 | 0 | 1 | 1 | 1 | | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | 1 | 0 | 0 | 0 | 0 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| TYK | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 |
| TUI | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | | 0 |
| VW | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | |

Table 7.14: STm results on stochastic dominance (2nd order) between the daily returns of DAX stocks for the 1-year period (1 $\sim$ rejection of dominance).

| | ADS | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DAC | DBK | DBO | DLH | DPO | DTL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| ALL | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | 0 | | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| BAY | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BMW | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CON | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | 1 | 0 | 0 |
| DAC | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| DBO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| DLH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 |
| DPO | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 |
| DTL | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | |
| EON | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| FRE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| INF | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| MAN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MET | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| RWE | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| SAP | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| TYK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| TUI | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| VW | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 7.15: KRSm results on stochastic dominance (2nd order) between the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of non-dominance).

| | EON | FRE | HEN | INF | LIN | MAN | MET | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ADS | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| ALL | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| BAY | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BMW | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CBK | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CON | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| DAC | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DBK | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DBO | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DLH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DPO | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DTL | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| EON | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| FRE | 0 | | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| HEN | 0 | 0 | | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| INF | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 1 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 0 | 0 | 0 | 1 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MET | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 0 | 0 | 0 | 1 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | 1 | 0 | 0 | 0 | 1 | 0 |
| SAP | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| SCH | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| TYK | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 |
| TUI | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 |
| VW | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |

Table 7.16: KRSm results on stochastic dominance (2nd order) between the daily returns of DAX stocks for the 1-year period ($1 \sim$ rejection of non-dominance).

| | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DBK | DLH | EON | HEN | LIN | MAN | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| BAY | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| BHV | 0 | 0 | 0 | 0 | | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| BMW | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| CBK | 1 | 0 | 0 | 0 | 1 | 1 | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| CON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| DLH | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| EON | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| HEN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| LIN | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| MAN | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| MRV | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | | 1 | 0 | 0 | 0 | 0 | 1 |
| SAP | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | | 1 | 0 | 1 | 0 | 1 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| TYK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | | 0 | 0 |
| TUI | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | | 0 |
| VW | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | |

Table 7.17: LMWm results on stochastic dominance (1st order) between the daily returns of DAX stocks for the 10-year period ($1 \sim$ rejection of dominance).

| | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DBK | DLH | EON | HEN | LIN | MAN | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| ALT | 0 | | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| BAS | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAY | 0 | 0 | 1 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| BHV | 1 | 0 | 1 | 1 | | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| BMW | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| CBK | 0 | 0 | 1 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| CON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| DLH | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| EON | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| HEN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| MRV | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 |
| SCH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| TYK | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | | 0 | 0 |
| TUI | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 |
| VW | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | |

Table 7.18: STm results on stochastic dominance (2nd order) of the daily returns of DAX stocks for the 10-year period ($1 \sim$ rejection of dominance).

| | ALL | ALT | BAS | BAY | BHV | BMW | CBK | CON | DBK | DLH | EON | HEN | LIN | MAN | MRV | RWE | SAP | SCH | SIE | TYK | TUI | VW |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ALL | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ALT | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BAS | 1 | 0 | | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| BAY | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BHV | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| BMW | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CBK | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CON | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| DBK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| DLH | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| EON | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| HEN | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| LIN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MAN | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| MRV | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| RWE | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 | 0 |
| SAP | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| SCH | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 1 | 0 |
| SIE | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 |
| TYK | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 | 0 |
| TUI | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | 0 |
| VW | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |

Table 7.19: KRSm results on stochastic dominance (2nd order) of the daily returns of DAX stocks for the 10-year period ($1 \sim$ rejection of non-dominance).

| Abbr. | 1-year period | | | | 10-year period | | | |
|---|---|---|---|---|---|---|---|---|
| of | LMWm | descr. | STm | KRSm | LMWm | descr. | STm | KRSm |
| stock | SD1 | SD2 | SD2 | SD2 | SD1 | SD2 | SD2 | SD2 |
| ADS | • | • | • | • | – | – | – | – |
| ALL | | | | | | | | |
| ALT | • | | | | • | • | • | • |
| BAS | | • | | • | • | • | • | • |
| BAY | | | | • | | | | • |
| BHV | | | | | | | | |
| BMW | | | | | | | | • |
| CBK | | | | | | | | |
| CON | • | • | • | • | • | • | • | • |
| DAC | | | | | – | – | – | – |
| DBK | • | | | • | | | | • |
| DBO | | | | • | – | – | – | – |
| DLH | | | | | | | | • |
| DPO | • | | | • | – | – | – | – |
| DTL | | | | • | – | – | – | – |
| EON | • | | • | • | | • | • | • |
| FRE | | | | • | – | – | – | – |
| HEN | | | | • | • | • | • | • |
| INF | | | | | – | – | – | – |
| LIN | | | | • | | | • | • |
| MAN | | | | • | | | | • |
| MET | • | | | • | – | – | – | – |
| MRV | | | | | | | | |
| RWE | • | • | • | • | | • | • | • |
| SAP | | | | | • | • | • | • |
| SCH | | | | • | • | • | • | • |
| SIE | • | | | • | • | | • | • |
| TYK | | | | • | | | | • |
| TUI | | | | | | | | |
| VW | | | | | | | | • |

Table 7.20: Efficiency results concerning the application of the considered tests on stochastic dominance to the daily returns of DAX stocks for the 1-year and the 10-year periods. The efficient stocks are denoted with a bullet. The stocks not considered for the longer period are denoted with a hyphen.

# Appendix

In the following we give the MATLAB programs which apply the modified versions of the tests of Linton/ Maasoumi/ Whang and of Schmid/ Trede to the daily returns of the DAX30 stocks. Lines beginning with “%” are comments and are not executed by the program. Comments can be found behind the lines they refer to.

## LMWm test

```
tic
% The test for SD1 of Linton/Maasoumi/Whang with circular
% subsampling is applied to the daily returns of the DAX stocks.
lengthX = 2522;
% We consider the data of 10 years, i.e. 2522 observations.
load Returns.mat;
% The data set is loaded.
Returns = Returns((end-lengthX+1):end,:);
index = find(sum(isnan(Returns))==0);
Returns = Returns(:,index);
% Incomplete datasets are excluded.
block = 500;
% We fix the block length of our choice.
```

```
alpha = 0.05;
% The size of the test is fixed.
for m = 1:length(index)
  for l = 1:(m-1)
    data = Returns(:,[m l]);
    data1 = data(:,1); data2 = data(:,2);
    s = prod(size(data));
    dataset = sort(data(1:s));
    for j = 1:s
      distrdiff(j) = distrfct(dataset(j),data1)...
        - distrfct(dataset(j),data2);
    end
    d(m,l) = sqrt(lengthX)*max(distrdiff);
    d(l,m) = sqrt(lengthX)*max(-distrdiff);
    % d(m,l) and d(l,m) are the test statistics for testing for
    % stochastic dominance of stock m against stock l and vice versa.
    for i = 1:lengthX
      for j = 1:block
        B1(i,j) = data1(mod(i+j-2,lengthX)+1);
        B2(i,j) = data2(mod(i+j-2,lengthX)+1);
        % The blocks for circular subsampling are created.
      end
      B(i,:)  = [B1(i,:)  B2(i,:)];
      blockset(i,:)  = sort(B(i,:));
      for k = 1:(2*block)
        blockdiff(i,k) = distrfct(blockset(i,k),B1(i,:))
        - distrfct(blockset(i,k),B2(i,:));
      end
```

```
        db1(i) = sqrt(block)*max(blockdiff(i,:));
        db2(i) = sqrt(block)*max(-blockdiff(i,:));
        % db1 and db2 are the test statistics for the
        % circular subsampling blocks.
      end
    q(m,l) = mean(d(m,l)>db1)+.5.*mean(d(m,l)==db1);
    q(l,m) = mean(d(l,m)>db2)+.5.*mean(d(l,m)==db2);
    % The test statistics for the whole sample are compared to
    % the test statistics of the circular subsampling blocks.
    domin(m,l) = (q(m,l) >= 1-alpha);
    domin(l,m) = (q(l,m) >= 1-alpha);
    % The decision is made.  ``1'' means rejection of the
    % dominance hypothesis, ``0'' means no rejection.
    save test.mat;
    % The results are buffered after each comparison.
  end
end
toc
time = toc;
% The computation time is determined.
save SD1-lmw-block500-length2522.mat;
% The results are saved.
```

## STm test

```
tic
lengthX = 2522;
```

```
% We consider the data of 10 years, i.e. 2522 observations.
load Returns.mat;
% The data set is loaded.
Returns = Returns((end-lengthX+1):end,:);
index = find(sum(isnan(Returns))==0);
Returns = Returns(:,index);
% Incomplete datasets are excluded.
M = 500;
% The number of permutations is as recommended by Schmid and Trede.
block = 500;
% We fix the block length of our choice.
alpha = 0.05;
% The size of the test is fixed.
for n = 1:length(index)
  for l = 1:(n-1)
    data = Returns(:,[n l]);
    data1 = data(:,1); data2 = data(:,2);
    s = prod(size(data));
    dataset = sort(data(1:s));
    for j = 1:s
      distrdiff(j) = distrfctint(dataset(j),data1)...
        - distrfctint(dataset(j),data2);
    end
    d(n,l) = max(distrdiff);
    d(l,n) = max(-distrdiff);
    % d(n,l) and d(l,n) are the test statistics for testing for
    % stochastic dominance of stock n against stock l and vice versa.
    datadouble = [data; data];
```

```
for m = 1:M
  permnumber = binornd(lengthX,.5);
  % The number of transposed observations pairs is chosen
  % randomly.
  startindex = unidrnd(lengthX,1,1);
  dataturn = datadouble(startindex:(startindex+lengthX-1),:);
  blocknumber = ceil(permnumber./block);
  turnindex = randperm(lengthX-permnumber+blocknumber);
  turnindex = sort(turnindex(1:blocknumber));
  % The blocks of observation pairs which are transposed
  % are chosen randomly.
  dataperm = dataturn;
  if blocknumber>0
    for k = 1:(blocknumber-1)
      dataperm((turnindex(k)+(k-1).*(block-1))...
        :(turnindex(k)+k.*(block-1)),:)...
        = fliplr(dataperm((turnindex(k)+(k-1).*(block-1))...
        :(turnindex(k)+k.*(block-1)),:));
    end
    dataperm((turnindex(blocknumber)...
      +(blocknumber-1).*(block-1))...
      :(turnindex(blocknumber)+permnumber-blocknumber),:)...
      = fliplr(dataperm((turnindex(blocknumber)...
      +(blocknumber-1).*(block-1))...
      :(turnindex(blocknumber)+permnumber-blocknumber),:));
    % The data are transposed for the chosen blocks.
  end
  dataperm1 = dataperm(:,1); dataperm2 = dataperm(:,2);
```

```
      for k = 1:s
        tperm(k) = distrfctint(dataset(k),dataperm1)...
          - distrfctint(dataset(k),dataperm2);
      end
      Tperm1(m) = max(tperm); Tperm2(m) = max(-tperm);
      % Tperm1 and Tperm2 are the test statistics for the
      % permuted data testing in either direction.
    end
    rank1(n,l) = sum(Tperm1>d(n,l));
    rank2(n,l) = sum(Tperm1>=d(n,l));
    rank1(l,n) = sum(Tperm2>d(l,n));
    rank2(l,n) = sum(Tperm2>=d(l,n));
    % The test statistics for the original data are compared to
    % the test statistics for the permutations.
    domin(n,l) = ((rank1(n,l)+rank2(n,l))./2 < alpha.*M);
    domin(l,n) = ((rank1(l,n)+rank2(l,n))./2 < alpha.*M);
    % The decision is made.  ''1'' means rejection of the
    % dominance hypothesis, ''0'' means no rejection.
    save test.mat;
    % The results are buffered after each comparison.
  end
end
toc
time = toc;
% The computation time is determined.
save SD2-st-block500-length2522.mat;
% The results are saved.
```

## KRSm test

```
tic;
lengthX = 255;
% We consider the data of 10 years, i.e. 2522 observations.
load Returns.mat;
% The data set is loaded.
Returns = Returns((end-lengthX+1):end,:);
index = find(sum(isnan(Returns))==0);
Returns = Returns(:,index);
lindex = length(index);
% Incomplete datasets are excluded.
alpha = 0.05; q = norminv(1-alpha);
% The size of the test is fixed.
for m = 1:lindex
  for l = 1:lindex;
    if m  = l
      data = Returns(:,[m l]);
      data1 = data(:,1);
      data2 = data(:,2);
      s = prod(size(data));
      datenset = sort(data(1:s));
      for j = 1:s
        Z(j,m,l) = testkrs(dataset(j),data1,data2);
      end
      % The auxiliary function for the test statistic is derived
      % at all data points.
      save test.mat;
    end
```

```
  end
end
Trunc10 = zeros(51,51,lindex,lindex);
for k = 1:51
  for j = 1:51
    if 10*(k-1)+1 <= size(Z,1)-10*(j-1)
      Trunc10(k,j,:,:)  = min(Z((10*(k-1)+1):(end-10*(j-1)),:,:));
    else
      Trunc10(k,j,:,:)  = 10*ones(1,1,lindex,lindex);
    end
  end
end
% The value of the test statistic is derived for various
% truncation values.
decis10 = (Trunc10>q);
% The test decision is made, it is dependent on the truncation
% values.
save test.mat;
toc
time = toc;
% The computation time is determined.
save SD2-krscov-laen255-mod.mat;
% The results are saved.
```

The auxiliary function testkrs is defined as follows.

```
function t = testkrs(x,dat1,dat2);
lenda = length(dat1);
covar = covardistrfctint(x,dat1,dat2,1:lenda);
```

```
for k = 1:floor(lenda^(1/2))
  covarcross(k) = mean(covar(1:(lenda-k)).*covar((k+1):lenda));
end
numerator = sqrt(lenda).*(distrfctint(x,dat1)-distrfctint(x,dat2));
denominator = sqrt(max(mean(covar.^2) + 2.*sum(covarcross),0));
if denominator ~= 0
  t = numerator./denominator;
elseif numerator == 0
  t = 0;
else
  t = sign(numerator)./eps;
end
```

## Basic Functions

The functions distrfct, distrfctint and covardistrfctint which we use in the programs are created by ourselves. They are defined as follows.

```
function q = distrfct(x,data)
q = sum(data<=x)./length(data);
% distrfct is the empirical distribution function F̂n^(1).
```

```
function q = distrfctint(x,data)
q = sum((data<=x).*(x-data))./length(data);
% distrfctint is the antiderivative F̂n^(2) of the empirical
% distribution function.
```

```
function q = covardistrfctint(x,data1,data2,index)
q = (data1(index)<=x).*(x-data1(index))-(data2(index)<=x)...
```

```
.*(x-data2(index))-distrfctint(x,data1)+distrfctint(x,data2);
% covardistrfctint is the empirical covariance function of the
% antiderivative F^(2) of the distribution function.
```

# List of Symbols

| | |
|---|---|
| $X \succeq Y$ | $X$ is weakly preferred to $Y$ by an individual |
| $X \succeq_k Y$ | $X$ dominates $Y$ in the sense of $k$th degree stochastic dominance |
| $X \succeq_L Y$ | $X$ dominates $Y$ in the sense of Lorenz dominance |
| $\underline{0}$ | zero vector |
| $1_A$ | indicator function of a set $A$ |
| $C_X^k$ | $k$th central moment of a random variable $X$ |
| $Cov(X, Y)$ | covariance of two random variables $X$ and $Y$ |
| $E(X)$ | expected value or mean of a random variable $X$ |
| $F_X$ | cumulative distribution function (c.d.f.) of the random variable $X$ |
| $\hat{F}_{X,n}$ | empirical distribution function of a sample of size $n$ generated by a random variable $X$ |
| $F_X^{(k)}$ | higher order antiderivative of $F_X$, defined recursively by $F_X^{(1)} = F_X$, $F_X^{(k+1)}(x) = \int_{-\infty}^{x} F_X^{(k)}(t)dt$ for all $k \in \mathbb{N}$ |
| $\hat{F}_{X,n}^{(k)}$ | empirical equivalent of $F_X^{(k)}$ |
| $GL_X$ | generalized Lorenz curve of a nonnegative random variable $X$ |
| $H_0$ | null hypothesis |
| $H_1$ | alternative |
| $I_d$ | $d$-dimensional identity matrix |
| $L_X$ | Lorenz curve of a nonnegative random variable $X$ |
| $LPM_X^k(c)$ | $k$th lower partial moment of a random variable $X$ with reference |

| | |
|---|---|
| | value $c$ |
| $P_X$ | probability measure of a random variable $X$ |
| $Q_X$ | quantile function of the random variable $X$ |
| $\hat{Q}_{X,n}$ | empirical quantile function of a sample of size $n$ generated by a random variable $X$ |
| $Q_X^{(k)}$ | higher order antiderivative of $Q_X$, defined recursively by $Q_X^{(1)} = Q_X$, $Q_X^{(k+1)}(x) = \int_{-\infty}^{x} Q_X^{(k)}(t)dt$ for all $k \in \mathbb{N}$ |
| $\hat{Q}_{X,n}^{(k)}$ | empirical equivalent of $Q_X^{(k)}$ |
| SD$k$ | $k$th degree stochastic dominance |
| $SV_X(c)$ | lower semivariance of a random variable $X$ with reference value $c$ $(= LPM_X^2(c))$ |
| $u$ | utility function |
| $u^{(k)}$ | $k$th derivative of the utility function $u$ |
| $U_k$ | set of all utility functions satisfying $(-1)^j u^{(j)} \leq 0$ for all $j = 1, \ldots, k$ |
| $Var(X)$ | variance of a random variable $X$ |
| $x_{(k)}$ | $k$th order statistic of a sample $(x_1, \ldots, x_n)$ |
| $\lfloor x \rfloor$ | largest integer equal to or smaller than $x$ |
| $x_+$ | nonnegative part of a real number $x$, i.e. $x_+ = \max\{x, 0\}$ |
| $\bar{x}_{(n)}$ | average of a sample $x_1, \ldots, x_n$, i.e. $\bar{x}_{(n)} = \frac{1}{n}\sum_{k=1}^{n} x_k$ |
| $\alpha(\mathcal{A}, \mathcal{B})$ | strong mixing coefficient of two $\sigma$-fields: $\alpha(\mathcal{A}, \mathcal{B}) = \sup\{\lvert P(A \cap B) - P(A)P(B)\rvert : A \in \mathcal{A}, B \in \mathcal{B}\}$ |
| $\alpha(m)$ | strong mixing coefficient of a process: $\alpha(m) := \sup\{\alpha(\mathcal{F}_{-\infty}^{k}, \mathcal{F}_{k+m}^{\infty}) : k \in \mathbb{Z}\}$ |
| $\alpha_i, \beta_i$ | parameters in the GARCH model $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_t^2$ |
| $\delta_x$ | Dirac distribution in $x$ |
| $\mu_X$ | expected value or mean of a random variable $X$ |
| $\sigma_X^2$ | variance of a random variable $X$ |
| $\sigma_t^2$ | conditional variance in $t \in \mathbb{Z}$ |

| | |
|---|---|
| $\mathcal{F}_a^b$ | $\sigma$-field generated by a process $(X_t)_{t\in\mathbb{Z}}$: $\mathcal{F}_a^b := \sigma(X_t : a \leq t < b)$ |
| $\mathcal{N}(\mu, \sigma)$ | univariate normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $\mathcal{N}_d(\mu, \Sigma)$ | $d$-variate normal distribution with location vector $\mu$ and covariance matrix $\Sigma$ |
| $\mathcal{U}(a, b)$ | uniform distribution on the interval $(a, b)$ |
| $\mathbb{N}$ | set of positive integers |
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{R}^n$ | $n$-dimensional real space |
| $\mathbb{Z}$ | set of integers |

# List of Abbreviations

| | |
|---|---|
| a.s. | almost surely |
| CBB | Circular Block Bootstrap |
| CLT | Central Limit Theorem |
| DAX | German stock index |
| e.g. | exempli gratia (for example) |
| GARCH | generalized autoregressive conditional heteroskedasticity |
| i.e. | id est (that is) |
| KRS | test for stochastic dominance of Kaur, Rao and Singh (1994) |
| KRSm | modified KRS test |
| LMW | test for stochastic dominance of Linton, Maasoumi and Whang (2005) |
| LMWm | modified LMW test |
| MBB | Moving Block Bootstrap |
| SLLN | strong law of large numbers |
| ST | test for stochastic dominance of Schmid and Trede (1997) |
| STm | modified ST test |
| XFW | test for stochastic dominance of Xu, Fisher and Willson (1997) |
| XFWm | modified XFW test |

# Bibliography


Akgiray, V. (1989): Conditional Heteroskedasticity in Time Series of Stock Returns: Evidence and Forecasts. *Journal of Business* 62, 55–80.

Aly, E.-E. (1991): On Testing for Lorenz Ordering. *Metrika* 38, 117–124.

Anderson, G. (1996): Nonparametric Tests of Stochastic Dominance in Income Distributions. *Econometrica* 64(5), 1183–1193.

Atkinson, A.B. (1970): On the Measurement of Inequality. *Journal of Economic Theory* 2, 244–263.

Barrett, G., Donald, S. (2003): Consistent Tests for Stochastic Dominance. *Econometrica* 71(1), 71–104.

Bawa, V. (1982): Stochastic Dominance: A Research Bibliography. *Management Science* 28(6), 698–713.

Beach, C.M., Davidson, R. (1983): Distribution-Free Statistical Inference with Lorenz Curves and Income Shares. *Review of Economic Studies* 50, 723–735.

Billingsley, P. (1961): The Lindeberg-Lévy Theorem for Martingales. *Proceedings of the American Mathematical Society* 12, 788–792.

Bishop, J.A., Chakraborti, S., Thistle, P.D. (1989): Asymptotically Distribution-Free Statistical Inference for Generalized Lorenz Curves. *Review of Eco-*

*nomics and Statistics* 71(4), 725–727.

Bishop, J.A., Formby, J., Thistle, P.D. (1992): Convergence of the South and Non-South Income Distribution, 1969–1979. *American Economic Review* 82, 262–272.

Bollerslev, T. (1986): Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics* 31, 307–327.

Bollerslev, T. (1988): On the Correlation Structure for the Generalized Autoregressive Conditional Heteroskedastic Process. *Journal of Time Series Analysis* 9(2), 121–131.

Chow, K.V. (1989): Statistical Inference for Stochastic Dominance: A Distribution Free Approach. Dissertation, University of Alabama.

Davis, R.A., Mikosch, T. (2000): The Sample Autocorrelations of Financial Time Series Models. In *Nonlinear and Nonstationary Signal Processing* (W.J. Fitzgerald, R.L. Smith, A.T. Walden, P. Young, eds.), 247–274. Cambridge University Press.

Davis, R.A., Mikosch, T., Basrak, B. (1999): Sample ACF of Multivariate Stochastic Recurrence Equations With Application to GARCH. Technical Report, University of Groningen.

Deshpande, J., Singh, P. (1985): Testing for Second Order Stochastic Dominance. *Communications in Statistics, Theory and Methods* 14(4), 887–893.

Efron, B. (1979): Bootstrap Methods: Another Look at the Jackknife. *The Annals of Statistics* 7(1), 1–16.

Engle, R.F. (1982): Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflation. *Econometrica* 50(4), 987–1008.

Engle, R.F. (2001): GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics. *Journal of Economic Perspectives* 15(4), 157–168.

Engle, R.F., Kroner, K. (1995): Multivariate Simultaneous Generalized ARCH. *Econometric Theory* 11, 122–150.

Eubank, R., Schechtman, E., Yitzhaki, S. (1993): A Test for Second Order Stochastic Dominance. *Communications in Statistics, Theory and Methods* 22(7), 1893–1905.

Fishburn, P.C. (1970): *Utility Theory for Decision Making.* Wiley, New York.

Fishburn, P.C. (1980): Continua of Stochastic Dominance Relations for Unbounded Probability Distributions. *Journal of Mathematical Economics* 7, 271–285.

Foster, J.E., Sen, A. (1997): On Economic Inequality after a Quarter Century. In *On Economic Inequality* (A. Sen, ed.), 107–219. Clarendon Press, Oxford.

Gotoh, J., Konno, H. (2000): Third Degree Stochastic Dominance and Mean-Risk Analysis. *Management Science* 46(2), 289–301.

Hadar, J., Russell, W.R. (1969): Rule for Ordering Uncertain Prospects. *American Economic Review* 59, 25–34.

Härdle, W., Horowitz, J., Kreiss, J.-P. (2003): Bootstrap Methods for Time Series. *International Statistical Review* 71(2), 435–459.

Hall, P., Heyde, C.C. (1980): *Martingale Limit Theory and its Applications.* Academic Press, New York.

Hanoch, G., Levy, H. (1969): The Efficiency Analysis of Choices Involving Risk. *Review of Economic Studies* 36, 335–346.

Hayashi, F. (2000): *Econometrics.* Princeton University Press.

Herring, S.K. (1996a): A Permutation Test for Second-Order Stochastic Dominance. *Journal of Applied Statistical Science* 3(1), 45–51.

Herring, S.K. (1996b): A Test of Variances for Second Order Stochastic Dominance. *Journal of Applied Statistical Science* 3(2/3), 129–134.

Jean, W.H. (1984): The Harmonic Mean and Other Necessary Conditions for Stochastic Dominance. *Journal of Finance* 39, 527–534.

Jean, W.H., Helms, B.P. (1987): The Identification of Stochastic Dominance Efficient Sets by Moment Combination Orderings. *Journal of Banking and Finance* 12, 243–253.

Jones, G.L. (2004): On the Markov Chain Central Limit Theorem. *Probability Surveys* 1, 299–320.

Kaur, A., Rao, B.L.S.P., Singh, H. (1994): Testing for Second Order Stochastic Dominance of Two Distributions. *Econometric Theory* 10, 849–866.

Kläver, H. (2005a): Testing for Stochastic Dominance Using Circular Block Methods. Working Paper, University of Cologne.

Kläver, H. (2005b): A Test in which Stochastic Dominance is the Alternative. Working Paper, University of Cologne.

Kläver, H., Schmid, F. (2004): The Effect of Conditional Heteroskedasticity on Common Statistical Procedures for Means and Variances. *Allgemeines Statistisches Archiv* 88, 397–407.

Klecan, L., McFadden, R., McFadden, D. (1991): A Robust Test for Stochastic Dominance. Working Paper, MIT.

Künsch, H.R. (1989): The Jackknife and the Bootstrap for General Stationary Observations. *The Annals of Statistics* 17(3), 1217–1241.

Kuosmanen, T. (2004): Efficient Diversification According to Stochastic Dominance Criteria. *Management Science* 50(10), 1390–1406.

Lahiri, S.N. (1999): Theoretical Comparisons of Block Bootstrap Methods. *The Annals of Statistics* 27(1), 386–404.

Langyintuo, A.S., Yiridoe, E.K., Dogbe, W., Lowenberg-Debour, J. (2005): Yield and Risk-Efficiency Analysis of Alternative Systems for Rice Production in the Guinea Savannah of Northern Ghana. *Agricultural Economics* 32, 141–150.

Levene, H. (1960): Robust Tests for Equality of Variances. In *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling* (I. Olkin, S.G. Ghurye, W. Hoeffding, W.G. Madow, H.B. Mann, eds.), 278–292. Stanford University Press.

Levy, H., Kroll, Y. (1978): Ordering Uncertain Options with Borrowing and Lending. *Journal of Finance* 33, 553–573.

Levy, H. (1992): Stochastic Dominance and Expected Utility: Survey and Analysis. *Management Science* 38, 555–593.

Lifshits, M.A. (1982): On the Absolute Continuity of Distributions of Functionals of Random Processes. *Theory of Probability and Its Applications* 27, 600–607.

Linton, O., Maasoumi, E., Whang, Y.-J. (2005): Consistent Testing for Stochastic Dominance under General Sampling. *Review of Economic Studies* 72(3), 735–765.

Liu, R.Y., Singh, K. (1992): Moving Blocks Jackknife and Bootstrap Capture Weak Dependence. In *Exploring the Limits of Bootstrap* (R. LePage, L. Billard, eds.), 225–248. Wiley & Sons, New York.

Maasoumi, E., Heshmati, A. (2000): Stochastic Dominance amongst Swedish Income Distributions. *Econometric Reviews* 19(3), 287–320.

Maasoumi, E., Millimet, D.L. (2005): Robust Inference Concerning Recent Trends in US Environmental Quality. *Journal of Applied Econometrics* 20(1), 55–77.

Markowitz, H.M. (1952): Portfolio Selection. *Journal of Finance* 7, 77–91.

McFadden, D. (1989): Testing for Stochastic Dominance. *Studies in the Economics of Uncertainty: In Honor of Josef Hadar*, Springer, 113–133.

Mosler, K. (1982): *Entscheidungsregeln bei Risiko: Multivariate stochastische Dominanz.* Springer, Berlin.

Mosler, K., Scarsini, M. (1991): *Some Theory of Stochastic Dominance. IMS Lecture Notes – Monograph Series* 19, 261–284.

Mosler, K.; Scarsini, M. (1993): *Stochastic Orders and Applications: A Classified Bibliography.* Springer, Berlin.

Nelson, D.B. (1990): Stationarity and Persistence in the GARCH(1,1) Model. *Econometric Theory* 6, 318–334.

Nelson, R.D., Pope, R.D. (1991): Bootstrapped Insights into Empirical Applications of Stochastic Dominance. *Management Science* 37(9), 1182–1194.

Ogryczak, W., Ruszczynski, A. (1999): From Stochastic Dominance to Mean-Risk Models: Semideviations as Risk Measures. *European Journal of Operational Research* 116, 33–50.

Ogryczak, W., Ruszczynski, A. (2001): On Consistency of Stochastic Dominance and Mean-Semideviation Models. *Mathematical Programming B* 89, 217–232.

Politis, D.N., Romano, J.P. (1992): A Circular Block-Resampling Procedure for Stationary Data. In *Exploring the Limits of Bootstrap* (R. LePage, L. Billard, eds.), 263–270. Wiley & Sons, New York.

Politis, D.N., Romano, J.P. (1994): Large Sample Confidence Regions Based on Subsamples under Minimal Assumptions. *The Annals of Statistics* 22(4), 2031–2050.

Politis, D.N., Romano, J.P., Wolf, M. (1999): *Subsampling.* Springer, New York.

Post, T. (2003): Empirical Tests for Stochastic Dominance Efficiency. *Journal of Finance* 58(5), 1905–1931.

Quirk, J.P., Saposnik, R. (1962): Admissibility and Measurable Utility Functions. *Review of Economic Theory* 29, 140–146.

Rolski, T. (1976): *Order Relations in the Set of Probability Distribution Functions and their Applications in Queuing Theory.* Dissertationes Mathematicae, Polish Academy of Sciences, Warsaw.

Shaked, M., Shanthikumar, J.G. (1994): *Stochastic Orders and their Applications.* Academic Press, San Diego.

Schmid, F., Trede, M. (1996a): Testing for First Order Stochastic Dominance: A New Distribution-Free Test. *The Statistician* 45(3), 371–380.

Schmid, F., Trede, M. (1996b): Nonparametric Inference for Second Order Stochastic Dominance. Discussion Paper in Statistics and Econometrics 2/96, Seminar of Economic and Social Statistics, University of Cologne.

Schmid, F., Trede, M. (1997): Nonparametric Inference for Second Order Stochastic Dominance from Paired Observations: Theory and Empirical Application. *Wirtschafts- und Sozialstatistik heute, Theorie und Praxis; Festschrift für Walter Krug.*

Schmid, F., Trede, M. (1998): A Kolmogorov-Type Test for Second-Order Stochastic Dominance. *Statistics and Probability Letters* 37, 183–193.

Schmid, F., Trede, M. (2000): Stochastic Dominance in German Asset Returns: Empirical Evidence from the 1990s. *Jahrbücher für Nationalökonomie und Statistik* 220, 315-326.

Shorrocks, A.F. (1983): Ranking Income Distributions. *Economica* 50, 3–17.

Stein, W., Pfaffenberger, R., Kumar, P.C. (1983): On the Estimation Risk in First-Order Stochastic Dominance: A Note. *Journal of Financial and Quantitative Analysis* 18(4), 471–476.

Stout, W.F. (1974): *Almost Sure Convergence.* Academic Press, New York.

Tse, Y.K., Zhang, X. (2004): A Monte Carlo Investigation of Some Tests for Stochastic Dominance. *Journal of Statistical Computation and Simulation* 74(5), 361–378.

Von Neumann, J., Morgenstern, O. (1944): *Theory of Games and Economic Behavior.* Princeton University Press.

White, H. (1984): *Asymptotic Theory for Econometricians.* Academic Press, Orlando.

Whitmore, G. (1970): Third-Degree Stochastic Dominance. *American Economic Review* 60, 457–459.

Whitmore, G., Findlay, M.C. (1978): *Stochastic Dominance: An Approach to Decision-Making under Risk.* Heath, Lexington.

Xu, K., Fisher, G., Willson, D. (1997): Tests for Stochastic Dominance Based on $\alpha$-Mixing with an Application in Finance. Working Paper, Dalhousie University.

Yu, H. (1993): A Glivenko-Cantelli Lemma and Weak Convergence for Empirical Processes of Associated Sequences. *Probability Theory and Related Fields* 95, 357–370.

Zachow, E.-W., Schmitz, N. (1977): Eine Axiomatisierung des erwarteten Nutzens. In *Mathematical Economics and Game Theory. Essays in Honor of Oskar Morgenstern* (R. Henn, O. Möschlin, eds.), 250–264. Springer, Berlin.

Zheng, B., Formby, J.P., Smith, W.J., Chow, K.V. (2000): Inequality Orderings, Normalized Stochastic Dominance, and Statistical Inference. *Journal of Business and Economic Statistics* 18(4), 479–488.